이학 박사 학위논문

Rabinowitz Floer homology and Coisotropic intersections

(라비노위츠 플로어 호몰로지와 여등방성 교차)

2014 년 2 월

서울대학교 대학원

수리과학부

강 정 수

Rabinowitz Floer homology and Coisotropic intersections

(라비노위츠 플로어 호몰로지와 여등방성 교차)

지도교수 Urs Frauenfelder

이 논문을 이학 박사 학위논문으로 제출함

2013 년 10 월

서울대학교 대학원

수리과학부

강 정 수

강 정 수의 이학 박사 학위논문을 인준함

2013 년 12 월

위 원 장	______________________	(인)
부 위 원 장	______________________	(인)
위 원	______________________	(인)
위 원	______________________	(인)
위 원	______________________	(인)

Rabinowitz Floer homology and Coisotropic intersections

A dissertation
submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
to the faculty of the Graduate School of
Seoul National University

by

Jungsoo Kang

Dissertation Director : Professor Urs Frauenfelder

Department of Mathematical Science
Seoul National University

February 2014

Abstract

Rabinowitz Floer homology and Coisotropic intersections

Jungsoo Kang

Department of Mathematical Sciences
The Graduate School
Seoul National University

Rabinowitz Floer homology theory was developed by Kai Cieliebak and Urs Frauenfelder using a Lagrange multiplier action functional, which was introduced by Paul Rabinowitz in order to detect periodic orbits of autonomous Hamiltonian systems.

In this thesis, we study a generalized Rabinowitz action functional with several Lagrange multipliers, which is well suited for exploring dynamics on coisotropic submanifolds of arbitrary codimensions. Using this, we investigate among others, the existence problem of leafwise coisotropic intersection points, displaceability of coisotropic submanifolds, and Rabinowitz Floer homology for coisotropic submanifolds. We also derive a Künneth formula for the Rabinowitz Floer homology of product coisotropic submanifolds, and this enables us to find a class of coisotropic submanifolds which have infinitely many leafwise coisotropic intersection points. This study will serve as a crucial tool for exploring autonomous dynamical systems with several integrals.

Key words: Rabinowitz Floer homology, Hamiltonian dynamics, First integral, Coisotropic submanifold, Leafwise intersection.
Student Number: 2008-20276

Contents

CONTENTS

Chapter 1

Preliminaries on symplectic geometry

A symplectic form on a smooth manifold M is a closed nondegenerate 2-form $\omega \in \Omega^2(M)$. We call such a pair (M, ω) **symplectic manifold**. By nondegeneracy, every symplectic manifold is of even dimension and orientable. In particular, $\omega^{\wedge n}$ is a volume form of M if $\dim M = 2n$. The easiest example of a symplectic manifold is a Euclidean space with the standard symplectic structure $\left(\mathbb{R}^{2n}, \sum_{i=1}^n dx_i \wedge dy_i\right)$. In fact, every symplectic manifold is locally equivalent to this standard Euclidean space by Darboux's theorem. Thus in order to construct invariants of symplectic manifolds, one has to go beyond local considerations. The constructions of most global invariants in symplectic geometry, such as Floer-type homologies and Gromov-Witten invariants, use the fact that every symplectic manifold admits a family of compatible almost complex structure. An almost complex structure J on M is a complex structure on the tangent bundle, explicitly $J \in \mathrm{End}(TM)$ and $J^2 = -\mathbb{1}_{TM}$. A symplectic form $\omega \in \Omega^2(M)$ is called **compatible** with J if $g(\cdot, \star) := \omega(\cdot, J\star)$ defines a Riemannian metric on M such that $g(\cdot, \star) = g(J\cdot, J\star)$.

1.1 Hamiltonian diffeomorphisms

For any time-dependent smooth function $F \in C^\infty(S^1 \times M)$, the vector field X_F defined implicitly by

$$i_{X_F}\omega = dF$$

is called the **Hamiltonian vector field** associated to the Hamiltonian function F. The flow of the Hamiltonian vector field X_F is denoted by ϕ_F^t. The time one map $\phi_F = \phi_F^1$ of a Hamiltonian flow is called a **Hamiltonian diffeomorphism**. The set $\mathrm{Ham}(M, \omega)$ of all Hamiltonian diffeomorphisms is a group with respect to composition. We are interested in the subgroup $\mathrm{Ham}_c(M, \omega)$ which consists of Hamiltonian diffeomorphisms generated by compactly supported Hamiltonian functions. Next, we briefly recall the **Hofer norm** which gives rise to a unique nondegenerate bi-invariant Finsler metric on the group $\mathrm{Ham}_c(M, \omega)$.

Definition 1.1.1. Let $F \in C_c^\infty(S^1 \times M, \mathbb{R})$ be a compactly supported Hamiltonian function. Consider the L^∞-norm of F defined by

$$||F|| := ||F||_+ + ||F||_-.$$

where

$$||F||_+ := \int_0^1 \max_{x \in M} F(t,x)dt, \qquad ||F||_- := -\int_0^1 \min_{x \in M} F(t,x)dt = ||-F||_+.$$

For $\phi \in \mathrm{Ham}_c(M, \omega)$, the Hofer norm is

$$||\phi|| := \inf\{||F|| \mid \phi = \phi_F,\ F \in C_c^\infty(S^1 \times M, \mathbb{R})\}.$$

As mentioned above, the function d on $\mathrm{Ham}_c(M, \omega) \times \mathrm{Ham}_c(M, \omega)$ defined by $d(\phi, \psi) = ||\phi^{-1} \circ \psi||$ is the unique bi-invariant Finsler metric. The existence of the Hofer bi-invariant metric shows that $\mathrm{Ham}_c(M,\omega)$ is an infinite dimensional Lie group.

The following easy lemma will be useful in our story.

Lemma 1.1.2. [AF1] For all $\phi \in \mathrm{Ham}_c(M,\omega)$,

$$||\phi|| = |||\phi||| := \inf\{||F|| \mid \phi = \phi_F,\ F(t,\cdot) = 0\ \ \forall t \in [\tfrac{1}{2}, 1]\}\ .$$

1.2 Coisotropic submanifolds

Definition 1.2.1. A submanifold Σ in (M,ω) is said to be **coisotropic** if the symplectic orthogonal bundle

$$T\Sigma^\omega := \{(x,\xi) \in TM \,|\, \omega_x(\xi,\zeta) = 0 \ \text{ for all } \ \zeta \in T_x\Sigma\}$$

is a subbundle of $T\Sigma$. By definition,

$$0 \leq \operatorname{codim}\Sigma \leq \frac{1}{2}\dim M.$$

Example 1.2.2. Any hypersurface in (M,ω) is coisotropic. A submanifold $L \subset (M,\omega)$ is called **Lagrangian** if $TL = TL^\omega$ (or equivalently $\omega|_L \equiv 0$) and clearly every Lagrangian submanifold is coisotropic.

Since ω is closed, the symplectic orthogonal bundle $T\Sigma^\omega$ is integrable, and thus Σ is foliated by leaves of the characteristic foliation. We denote by L_x the **leaf through** x. In the extremal case that a connected coisotropic submanifold is Lagrangian, it is foliated by a single leaf.

Coisotropic submanifolds naturally arise in autonomous Hamiltonian systems with several integrals. Let (M,ω) be a $2n$-dimensional symplectic manifold. We denote by the **Hamiltonian tuple** $\mathcal{G} := (G_1,\ldots,G_k)$ for time-independent Hamiltonian functions $G_i \in C^\infty(M)$, $i \in \{1,\ldots,k\}$ for $1 \leq k \leq n$. We often regard $\mathcal{G}$ as an element of $C^\infty(M,\mathbb{R}^k)$.

Definition 1.2.3. Given two Hamiltonian functions F and G in $C^\infty(M)$, the **Poisson bracket**

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$$

is defined by $\{F, G\} := \omega(X_F, X_G)$. A Hamiltonian tuple $\mathcal{G}$ is said to be **Poisson-commuting** if $\{G_i, G_j\} = 0$ for any $1 \leq i, j \leq k$.

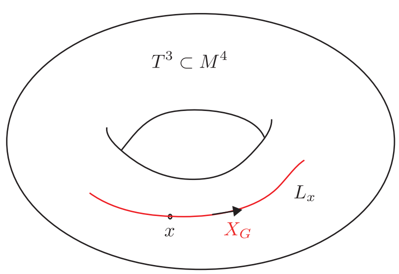
If a Hamiltonian tuple $\mathcal{G} \in C^\infty(M, \mathbb{R}^k)$ Poisson-commutes and $c \in \mathbb{R}^k$ is a regular value of $\mathcal{G}$, then an invariant submanifold $\mathcal{G}^{-1}(c)$ is a smooth coisotropic submanifold of codimension k in (M, ω) with

$$T\mathcal{G}^{-1}(c)^\omega = \langle X_{G_1}, \ldots, X_{G_k} \rangle.$$

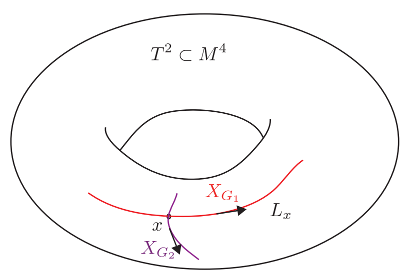
In this case the leaf L_x through $x \in \mathcal{G}^{-1}(c)$ can be written by

$$L_x = \left\{ \phi_{G_1}^{t_1} \circ \phi_{G_2}^{t_2} \circ \cdots \circ \phi_{G_k}^{t_k}(x) \,|\, t_1, \ldots t_k \in \mathbb{R} \right\}.$$

Note that dimension of leaves equals $\dim M - \dim \mathcal{G}^{-1}(c) = k$, see pictures below.



$L_x = \{\phi_G^t(x) \,|\, t \in \mathbb{R}\}$



$L_x = \{\phi_{G_1}^{t_1} \circ \phi_{G_2}^{t_2}(x) \,|\, t_1,\, t_2 \in \mathbb{R}\}$

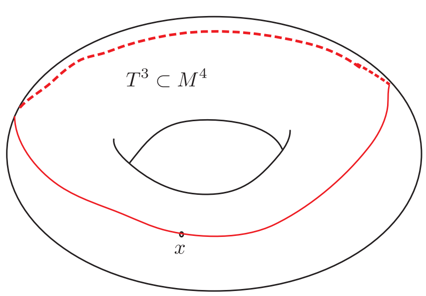
We briefly explain why such Hamiltonian systems are of great importance. A function $F \in C^\infty(M)$ is called an **integral** for a Hamiltonian system $\partial_t z = X_G(z(t))$ if F is constant along the solutions of $\partial_t z = X_G(z(t))$. It is easy to check that this condition is equivalent to $\{F, G\} = 0$. Hence, the motion of a Hamiltonian system $\partial_t z = X_G(z(t))$ with k independent Poisson commuting Hamiltonian integrals $G_1 = G, \ldots, G_k$ is confined to a $(2n-k)$-dimensional invariant submanifold $\bigcap_{1\le i\le k} G_i^{-1}(c_i)$, $c_i \in \mathbb{R}$.

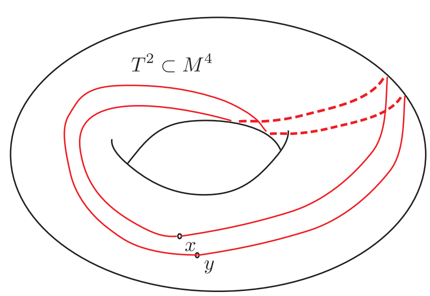
Remark 1.2.4. A 2n-dimensional Hamiltonian system is called **integrable** if there exist n independent Poisson commuting integrals $G_1, \ldots, G_n$. According to Liouville-Arnold, compact connected invariant submanifolds of integrable Hamiltonian systems are diffeomorphic to torus, i.e. $\bigcap_{1\le i\le n} G_i^{-1}(c_i) \cong T^n$, $c_1, \ldots, c_n \in \mathbb{R}$. Moreover integrable Hamiltonian systems admit the so-called **action-angle coordinates** and this coordinates are described explicitly sometimes, e.g. Delaunay coordinates in the Kepler problem.

A periodic orbit $v : S^1 = \mathbb{R}/\mathbb{Z} \to \mathcal{G}^{-1}(c)$ lying on a leaf

$$\frac{d}{dt}v(t) = \sum_{i=1}^{k} \eta_i X_{G_i}(v(t)), \quad \eta_1, \ldots, \eta_k \in \mathbb{R} \tag{1.2.1}$$

is a key player of this thesis. Note that constant loops in $\mathcal{G}^{-1}(c)$ are trivial solutions of (1.2.1) with $\eta_1 = \cdots = \eta_k = 0$. Note that if $\mathcal{G}^{-1}(c)$ is a hypersurface, i.e. $k = 1$, a periodic orbit exists if and only if a leaf closes up.





We remark that if there is a periodic solution v of (1.2.1) on a leaf L_x, the leaf L_x is foliated by periodic solutions of (1.2.1). To see this, let x be a periodic point, i.e. $\phi_{G_1}^{t_1} \circ \cdots \circ \phi_{G_k}^{t_k}(x) = x$ for some $t_1, \ldots, t_k \in \mathbb{R}$. For any $y \in L_x$, there exists $r_1, \ldots, r_k \in \mathbb{R}$ such that $\phi_{G_1}^{r_1} \circ \cdots \circ \phi_{G_k}^{r_k}(x) = y$. Then

$$\begin{aligned}
\phi_{G_1}^{t_1} \circ \cdots \circ \phi_{G_k}^{t_k}(y) &= \phi_{G_1}^{t_1} \circ \cdots \circ \phi_{G_k}^{t_k} \circ \phi_{G_1}^{r_1} \circ \cdots \circ \phi_{G_k}^{r_k}(x) \\
&= \phi_{G_1}^{r_1} \circ \cdots \circ \phi_{G_k}^{r_k} \circ \phi_{G_1}^{t_1} \circ \cdots \circ \phi_{G_k}^{t_k}(x) \\
&= \phi_{G_1}^{r_1} \circ \cdots \circ \phi_{G_k}^{r_k}(x) \\
&= y.
\end{aligned}$$

Here we used the fact that the Hamiltonian flows commute due to Poisson commutativity. Therefore there is a periodic solution of (1.2.1) passing through any $y \in L_x$ provided the existence of a periodic solution of (1.2.1) on the leaf L_x.

Let us consider a single time-independent Hamiltonian function $G \in C^\infty(M)$. Suppose that a level hypersurface $G^{-1}(c)$ for $c \in \mathbb{R}$ is regular. From a simple computation

$$dG(X_G) = \omega(X_G, X_G) = 0,$$

we know that the Hamiltonian vector field X_G is tangent to the level hypersurface $G^{-1}(c)$. In general it is difficult to understand or foresee the dynamics of X_G on the given level surface $G^{-1}(c)$. For instance, even in $\mathbb{R}^4$ there is a time-independent Hamiltonian function such that at least one of its level surfaces has no periodic orbits which disproves the Hamiltonian Seifert conjecture, see [GG]. For this reason, we usually require an additional structure on a level hypersurface.

Definition 1.2.5. A hypersurface S in (M, ω) is called of **contact type** if there exists a 1-form $\alpha \in \Omega^1(S)$ such that $d\alpha = \omega|_S$ and $\omega|_S$ is nondegenerate on the hyperplane field TS^ω. There exists a unique vector field R on a contact hypersurface (S, α) such that

$$i_R d\alpha = 0, \qquad i_R \alpha = 1.$$

This vector field is called the **Reeb vector field** on (S, α).

The Reeb dynamics on contact hypersurfaces and the intersection problems for Lagrangian submanifolds have been widely studied. In contrast, coisotropic submanifolds have so far received little attention. The aim of this thesis is to study dynamics on a contact coisotropic submanifold, which is a natural generalization of a contact hypersurface. The notions of stable, contact, and restricted contact type for coisotropic submanifolds were introduced by Philippe Bolle [Bo1, Bo2].

Definition 1.2.6. A coisotropic submanifold Σ of codimension k in (M, ω) is called **stable** if there exist 1-forms $\alpha = (\alpha_1, \dots, \alpha_k)$ on Σ which satisfy

1. $\ker d\alpha_i \supset T\Sigma^\omega$ for $i = 1, \dots, k$;
2. $\alpha_1 \wedge \dots \wedge \alpha_k \wedge (\omega|_\Sigma)^{n-k} \neq 0$.

We say that Σ is of **contact type** if $\alpha_1, \dots, \alpha_k$ are primitives of $\omega|_\Sigma$. If there are 1-forms $\lambda = (\lambda_1, \dots, \lambda_k)$ on M such that $d\lambda_i = \omega$ and $\lambda_i|_\Sigma = \alpha_i$ for all $i = 1, \dots, k$, Σ is said to be of **restricted** contact type.

Examples of stable/contact/restricted contact coisotropic submanifolds will be treated in the following section.

Definition 1.2.7. Let (Σ, α) be a stable coisotropic submanifold in (M, ω). The unique vector fields $R_1, \dots, R_k$ on Σ characterized by

$$\alpha_i(R_j) = \delta_{ij}, \quad R_i \in \ker \omega|_\Sigma, \quad i, j \in \{1, \dots, k\}$$

are called the **Reeb vector fields** associated with the stable structure (Σ, α). Here δ_{ij} stands for the Kronecker delta.

When a level surface $\mathcal{G}^{-1}(c)$ is stable, a periodic solution of 1.2.1 corresponds to a periodic solution $v \in C^\infty(S^1, \mathcal{G}^{-1}(c))$ of

$$\partial_t v(t) = \sum_{i=1}^{k} \eta_i R_i(v(t)), \quad \eta_1, \dots, \eta_k \in \mathbb{R}. \tag{1.2.2}$$

since
$$T\mathcal{G}^{-1}(c)^{\omega} = \langle R_1, \ldots, R_k \rangle = \langle X_{G_1}, \ldots, X_{G_k} \rangle.$$

Note that the normal bundle of a stable coisotropic submanifold $(\Sigma, \alpha) \subset (M, \omega)$ is trivial, i.e. $N\Sigma \cong \Sigma \times \mathbb{R}^k$ and from the Weinstein neighborhood theorem, we have the following proposition.

Proposition 1.2.8 ([Bo1, Bo2]). *Let (Σ, α) be a closed stable coisotropic submanifold of codimension k in (M, ω). Then there exist $r > 0$, a neighborhood V of Σ which is symplectomorphic by $\psi : U_r \to V$ to*

$$U_r := \{(q, \mathfrak{p}) = (q, p_1, \ldots, p_k) \in \Sigma \times \mathbb{R}^k \,|\, |p_i| < r, \text{ for all } i = 1, \ldots, k\}$$

with $\psi^*\omega = \omega|_\Sigma + \sum_{i=1}^k d(p_i \alpha_i)$.

Here we use the same symbols $\omega|_\Sigma$ and α_i for differential forms in Σ and for their pullback to $\Sigma \times \mathbb{R}^k$. We set

$$\delta_0 := \max\left\{ r \in \mathbb{R} \,\middle|\, \text{there exists a symplectic embedding } \psi : U_r \hookrightarrow M \right\}$$

and let $\psi : U_{\delta_0} \hookrightarrow M$ be a maximal symplectic embedding. Henceforth, we identify U_δ with $\psi(U_\delta)$ for all $0 < \delta \leq \delta_0$. We have $X_{p_i} \in \ker \omega|_\Sigma$, $dp_j(X_{p_i}) = 0$ and $\alpha_j(X_{p_i}) = \delta_{ij}$ on Σ for $1 \leq i, j \leq k$ since $i_{X_{p_i}}\omega = dp_i$. Moreover the (local) Hamiltonian tuple $\mathfrak{p} = (p_1, \ldots, p_k)$ Poisson-commutes since $\{X_{p_1}, \ldots, X_{p_k}\}$ forms a basis for $\ker \omega|_\Sigma$.

We note that $X_{p_1}, \ldots, X_{p_k}$ correspond to $R_1, \ldots, R_k$ via the identification ψ_0. From now on, we choose an almost complex structure J on M which splits on U_ϵ with respect to

$$TU_{\delta_0} = \Big(\underbrace{\ker \omega|_\Sigma}_{=:\xi} \Big) \bigoplus \underbrace{\left(T\Sigma^\omega \oplus \frac{\partial}{\partial p_1} \oplus \cdots \oplus \frac{\partial}{\partial p_k} \right)}_{=:\xi^\omega}. \tag{1.2.3}$$

i.e. $J|_{\xi^\omega}$ is an almost complex structure which interchanges the Reeb vector fields R_i with $\frac{\partial}{\partial p_i}$ for $1 \leq i \leq k$; strictly speaking $JR_i = \frac{\partial}{\partial p_i}$ and $J\frac{\partial}{\partial p_i} = -R_i$.

1.3 Examples of contact coisotropic submanifolds

Although the contact condition is restrictive, we still have the following classes of contact coisotropic submanifolds.

(i) A coisotropic submanifold which is C^1-close to a contact coisotropic submanifold is also of contact type.

(ii) A Lagrangian torus is of contact type with contact one forms $d\theta_1, \ldots, d\theta_n$ where $\theta_1, \ldots, \theta_n$ are angular coordinates on the n-dimensional torus. Indeed it turns out that a closed Lagrangian submanifold of contact type is necessarily a torus.

(iii) Let $\Sigma \subset (M_1, \omega_1)$ be a contact coisotropic submanifold and $T^{n_2} \subset (M_2, \omega_2)$ be a Lagrangian torus. Then a coisotropic submanifold $\Sigma \times T^{n_2}$ in $(M_1 \times M_2, \omega_1 \oplus \omega_2)$ is of contact type. In particular, the stabilization of $\Sigma \subset (M, \omega)$, $\Sigma \times S^1 \subset (M \times T^*S^1, \omega \oplus d\theta \wedge dt)$ is of (restricted) contact type whenever Σ is of (restricted) contact type. Here θ is the base coordinate and t is the fiber coordinate.

(iv) Consider the Hopf fibration $\pi : S^{2n-1} \to \mathbb{C}P^{n-1}$. According to Marsden-Weinstein-Meyer reduction, we know that there is a canonical symplectic form $\omega_{\mathbb{C}P^{n-1}}$ on $\mathbb{C}P^{n-1}$ satisfying $\pi^*\omega_{\mathbb{C}P^{n-1}} = \omega_{\mathbb{R}^{2n}}|_{S^{2n-1}}$ where $\omega_{\mathbb{R}^{2n}}$ is the standard symplectic form on $\mathbb{R}^{2n}$. For a contact hypersurface $(\Delta, \alpha) \subset \mathbb{C}P^{n-1}$, $\pi^{-1}(\Delta)$ is a contact submanifold in $\mathbb{R}^{2n}$ of codimension 2.

Let (M,ω) be a closed symplectic manifold with an integral symplectic form $[\omega] \in \mathrm{H}^2(M;\mathbb{Z})$. For each $N \in \mathbb{N}$, there exists a complex line bundle $p : E^N \to M$ with the first Chern class $c_1(E^N) = -N[\omega]$. We note that S^1 acts on the bundle E^N by

$$\begin{aligned} S^1 \times E^N &\longrightarrow E^N \\ (t,v) &\longmapsto e^{2\pi i t} v. \end{aligned}$$

Thus by the Boothby-Wang theorem, there exists a connection 1-form α on $E^N \setminus E_0$ where E_0 is the zero section of the complex line bundle $E^N \xrightarrow{p} M$; moreover it holds that $p^*F_\alpha = d\alpha$ for the curvature 2-form $F_\alpha = N\omega$. We abbreviate $r = |e|$ for $e \in E^N$ and define $q : \mathbb{R} \to \mathbb{R}$ by $q(r) = \pi r^2 + 1/N$. Then the following two form gives a symplectic structure on E^N:

$$\Omega_E := q'(r) dr \wedge \alpha + q(r) N p^*\omega.$$

It is easy to check that $\Omega_E|_{E_0} = p_1^*\omega$ and $\Omega_E|_{E\setminus E_0} = d(q(r)\alpha)$. Furthermore, for all $c > 1/N$, the following submanifold

$$\Sigma_c := \{q(r) = c\}$$

is of contact type. We perform this construction once again. We choose a complex line bundle $p' : F^K \to M$ with the first Chern class $c_1(F^K) = -K[\omega]$. As before, there is a connection 1-form β on $F^K \setminus F_0$ where F_0 is the zero section of the bundle $F^K \xrightarrow{p'} M$ such that its curvature 2-form F_β satisfies $F_\beta = K\omega$. We set the function $h(s) = \pi s^2 + 1/K$ for $s = |f| \in \mathbb{R}$ where $f \in F^K$, then

$$\Omega_F := h'(s) ds \wedge \beta + h(s) K p'^*\omega$$

is a symplectic form on F^K. Next, we consider the Whitney sum of E^N and F^K, $E^N \oplus F^K$ and let $\pi_1 : E^N \oplus F^K \to E^N$ and $\pi_2 : E^N \oplus F^K \to F^K$ be the projection maps to the first factor and the second factor respectively. We abbreviate $\tilde{\omega} := (p \circ \pi_1)^* \omega = (p' \circ \pi_2)^* \omega$, and use the same symbols r, s, $g(r)$, $h(s)$, α, and β for their pull-backs to $E^N \oplus F^K$. Then the following 2-form

$$\Omega_{E \oplus F} := h'(s) ds \wedge \beta + q'(r) dr \wedge \alpha + (q(r)N + h(s)K)\tilde{\omega}$$

becomes a symplectic form on $E^N \oplus F^K$. We have

(v) For any $c > 1/N$ and $d > 1/K$, set

$$\Delta_{c,d} := \{q(r) = c,\, h(s) = d\}.$$

Since $\Omega_{E \oplus F}|_{\Delta_{c,d}} = (cN + dK)\tilde{\omega}$, $\Delta_{c,d}$ with 1-forms $\frac{cN+dK}{N}\alpha$ and $\frac{cN+dK}{K}\beta$ is a contact coisotropic submanifold in $(E^N \oplus F^K, \Omega_{E \oplus F})$ of codimension 2.

Proposition 1.3.1. *Let $\mathcal{G} \in C^\infty(M, \mathbb{R}^k)$ be a Poisson-commuting Hamiltonian tuple such that $c = (c_1, \dots c_k) \in \mathbb{R}^k$ is a regular value of $\mathcal{G}$. Suppose that there is Liouville vector fields $Y_1, \dots, Y_k$ (i.e. $\mathcal{L}_{Y_1}\omega = \cdots \mathcal{L}_{Y_k}\omega = \omega$) such that the matrix*

$$[dG_i(Y_j)]_{1 \le i,j, \le k} = \begin{pmatrix} dG_1(Y_1) & \cdots & dG_1(Y_k) \\ \vdots & \ddots & \vdots \\ dG_k(Y_1) & \cdots & dG_k(Y_k) \end{pmatrix}$$

on $T\mathcal{G}^{-1}(c)$ is nonsingular. Then $\mathcal{G}^{-1}(c)$ is a contact coisotropic submanifold with contact forms $i_{Y_1}\omega, \dots, i_{Y_k}\omega$.

Proof. Indeed, each $\alpha_j = i_{Y_j}\omega$ is a primitive of ω:

$$d\alpha_j = di_{Y_j}\omega = \mathcal{L}_{Y_j}\omega = \omega, \quad 1 \le j \le k.$$

Note that

$$T\mathcal{G}^{-1}(c)^\omega = \langle X_{G_1}, \dots, X_{G_k} \rangle \subset T\mathcal{G}^{-1}(c),$$

and for all $1 \leq i \leq k$,

$$\omega(X_{G_i}, v) = dG_i(v) = 0, \quad \forall v \in T\mathcal{G}^{-1}(c).$$

We denote by

$$\xi := \{(x, v) \in T\mathcal{G}^{-1}(c) \,|\, \omega_x(Y_1, v) = \cdots = \omega_x(Y_k, v) = 0\}.$$

Since $[dG_i(Y_j)]_{1 \leq i,j, \leq k}$ is nonsingular, we have the splitting

$$T\mathcal{G}^{-1}(c) = T\mathcal{G}^{-1}(c)^{\omega} \oplus \xi.$$

Moreover ξ is a symplectic complement of $\langle Y_1, \ldots, Y_k \rangle \oplus T\mathcal{G}^{-1}(c)^{\omega}$. Hence

$$\alpha_1 \wedge \cdots \wedge \alpha_k \wedge \omega|_{T\mathcal{G}^{-1}(c)} \neq 0$$

by nonsingularity of $[dG_i(Y_j)]_{1 \leq i,j, \leq k}$ again. □

Dynamical problems, such as the (rotating) Kepler problem or Euler's three-body problem, sometimes admit several integrals. It is tempting to show whether such a problem has a (restricted) contact structure using the previous proposition.

Remark 1.3.2. [Bo2, Gi] Let Σ be a closed contact coisotropic submanifold in (M, ω). Then a 1-form $\lambda = a_1\lambda_1 + \cdots + a_k\lambda_k$ with $a_1 + \cdots + a_k = 0$ is closed and represents an element in $\mathrm{H}^1_{\mathrm{dR}}(\Sigma)$. In addition, $\lambda \neq 0$ is not exact; otherwise $\lambda = df$ for some $f \in C^1(\Sigma)$, $\lambda(x) = 0$ at a critical point x of f, but condition (ii) yields that $\lambda_1, \ldots, \lambda_k$ are linearly independent on Σ; thus $\lambda_1(x) = \cdots \lambda_k(x) = 0$. As a result, $\dim \mathrm{H}^1_{\mathrm{dR}}(\Sigma) \geq k - 1$. It imposes a restriction on the contact condition that a product of contact type coisotropic submanifolds is not necessarily of contact type; for instance, $S^3 \times S^3$ is not of contact type in $\mathbb{R}^8$.

Remark 1.3.3. Furthermore, a connected sum of a contact coisotropic submanifold is not of contact type in general; for instance, a connected sum of Lagrangian tori is not a torus any more, hence cannot be of contact type. Different from the contact case, however, a product of stable coisotropic submanifolds is of stable type again.

Chapter 2

Statement of the results

The coisotropic intersection problems were first studied in depth by Viktor Ginzburg [Gi], and have been recently explored by many mathematicians, see Section 2.7. Rabinowitz Floer homology theory, which was developed by Kai Cieliebak and Urs Frauenfelder [CF] using the Rabinowitz action functional [Ra], is one of the effective methods to study the intersection problems for hypersurfaces. By generalizing the Rabinowitz Floer homology theory, we investigate the intersection problems of coisotropic submanifolds.

Throughout this thesis, we deal with a symplectic manifold (M,ω) which is symplectically aspherical and geometrically bounded. The condition that (M,ω) is **symplectically aspherical** means $\int_{\pi_2(M)}\omega = 0$. We call (M,ω) **geometrically bounded** if there exists an ω-compatible almost complex structure J with the property that the Riemannian metric $g(\cdot,\star)=\omega(\cdot,J\star)$ is complete, has injective radius bounded away from zero, and has bounded sectional curvature.

2.1 Assumptions on manifolds

In this thesis, we deal with the following classes of manifolds.

i) A closed coisotropic submanifold Σ in (M, ω) is stable or of contact type or of restricted contact type.

ii) A symplectic manifold (M, ω) is symplectically aspherical and geometrically bounded.

If Σ is a restricted contact coisotropic submanifold, (M, ω) is automatically symplectically aspherical (due to $\int_{\pi_2(M)} \omega = \int_{\pi_2(M)} d\lambda_i = 0$) but never closed. Thus if this is the case, (M, ω) is only assumed to be geometrically bounded. On the other hand, if (M, ω) is stable or of contact type, M can be closed. In this case, (M, ω) is obviously geometrically bounded and we only need to assume symplectic asphericity of (M, ω).

To define Rabinowitz Floer homology we need an additional assumption on stable/contact/restricted contact coisotropic submanifolds. In this thesis we focus on coisotropic submanifolds which are regular level sets of Poisson-commuting Hamiltonian tuples. Suppose that a stable coisotropic submanifold (Σ, α) is a regular level set of a Poisson-commuting Hamiltonian tuple $\mathcal{G} = (G_1, \cdots, G_k) \in C^\infty(M, \mathbb{R}^k)$, say $\mathcal{G}^{-1}(0) = \Sigma$. Then since both the Reeb vector fields of $\alpha = (\alpha_1, \ldots, \alpha_k)$ and the Hamiltonian vector fields of $\mathcal{G}$ span the symplectic orthogonal bundle, i.e.

$$T\Sigma^\omega = \langle R_1, \ldots, R_k \rangle = \langle X_{G_1}, \ldots, X_{G_k} \rangle,$$

there exists a map from $\mathcal{G}^{-1}(0)$ to the set of $k \times k$ matrices

$$\Phi = (\Phi_{i,j}) : \mathcal{G}^{-1}(0) \to \mathrm{Mat}(k \times k)$$

such that

$$X_{G_j}(x) = \sum_{i=1}^{k} \Phi_{i,j}(x) R_i(x).^1$$

Note that $\Phi(x)$ for any $x \in \mathcal{G}^{-1}(0)$ is an invertible matrix. However in order for Rabinowitz Floer homology to be defined, we further require $\Phi(x)$ to have the following property.

iii) Σ is a regular level set of a Poisson-commuting Hamiltonian tuple $\mathcal{G} \in C^\infty(M, \mathbb{R}^k)$. For any $v \in C^\infty(S^1, \Sigma)$ contractible in M,

$$\int_{S^1} \Phi(v(t))dt \in \mathrm{Mat}(k \times k)$$

is invertible.

Remark 2.1.1. We choose a function $\chi(t) : S^1 \to [0, \infty)$ with $\int_{S^1} \chi(t)dt = 1$. Such a function will be used in Section 3. If $\int_{S^1} \Phi(v(t))dt$ is invertible for any $v \in C^\infty(S^1, \Sigma)$, so is $\int_{S^1} \chi(t)\Phi(v(t))dt$. Indeed, we can reparametrize a given $v \in C^\infty(S^1, \Sigma)$ to $v_\chi(t) = v \circ \int_0^t \chi(s)ds : S^1 \to [0, \infty)$ so that

$$\int_{S^1} \Phi(v_\chi(t))dt = \int_{S^1} \chi(t)\Phi(v(t))dt.$$

Note that an S^1-family of definite or diagonal matrices meets this third assumption. The assumption on the existence of "global coordinates" in [Ka3] is a special case of this assumption iii).

In order to find one leafwise coisotropic intersection point or one periodic orbit (Theorems A and D), we do not need the last assumption as Rabinowitz Floer homology is not directly involved. However, the last assumption is still indispensable to define Rabinowitz Floer homology and results using Rabinowitz Floer homology (Theorems B, C, E, F, and G).

[1] Strictly speaking, $\Phi(x)$ is an automorphism on $T_x\Sigma^\omega$, but here we tacitly assume $T\Sigma^\omega \cong \Sigma \times \mathbb{R}^k$ to have been trivialized.

Remark 2.1.2. All the above three assumptions appear in Rabinowitz Floer homology theory for hypersurfaces (see [CF]) as well. In particular, the last assumption matches with a **separating condition** for stable hypersurfaces. The separating condition means that a hypersurface Σ separates M into two connected components of which one is relatively compact. With the separating condition, it is possible to find a Hamiltonian function $G \in C^\infty(M)$ of Σ such that $G^{-1}(0) = \Sigma$. Moreover since Σ is of codimension 1, $\langle R \rangle = \langle X_G \rangle$ which in turn implies the assumption iii).

2.2 Main theorem

Let $\mathcal{L} \subset C^\infty(S^1, M)$ be the space of contractible loops in M. Let $\mathcal{G} = (G_1, \dots, G_k) \in C^\infty(M, \mathbb{R}^k)$ be a Poisson-commuting Hamiltonian tuple which has $0 \in \mathbb{R}^k$ (for simplicity) as a regular value. We also choose a compactly supported time-dependent Hamiltonian function $F \in C_c^\infty(S^1 \times M)$. For $\eta = (\eta_1, \dots, \eta_k) \in \mathbb{R}^k$, the generalized (perturbed) Rabinowitz action functional $\mathcal{A}_F^{\mathcal{G}} : \mathcal{L} \times \mathbb{R}^k \to \mathbb{R}$ is defined by

$$\mathcal{A}_F^{\mathcal{G}}(v, \eta) := -\int_{D^2} \bar{v}^* \omega - \sum_{i=1}^{k} \eta_i \int_0^1 G_i(v(t))dt - \int_0^1 F(t, v(t))dt.$$

where $\bar{v}$ is any filling disk of v, i.e. $\bar{v}|_{\partial D^2}(t) = v(t)$ for $t \in S^1$. The symplectic asphericity condition implies that the value of the above action functional is independent of the choice of filling discs. Then in Theorem 3.2.8, we will prove the following compactness result under the assumptions on $(M, \omega, \Sigma, \alpha)$ described in the previous section.

Main theorem. *Let $\{w^\nu\}_{\nu \in \mathbb{N}}$ be a sequence of gradient flow lines of $\mathcal{A}_F^{\mathcal{G}}$ for which there exist $a \leq b$ such that*

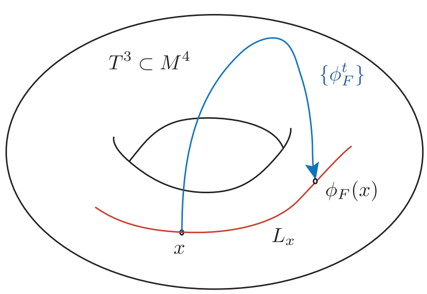
$$a \leq \mathcal{A}_F^{\mathcal{G}}(w^\nu(s)) \leq b, \quad \text{for all } \nu \in \mathbb{N},\ s \in \mathbb{R}. \tag{2.2.1}$$

Then for every reparametrization sequence $\sigma_\nu \in \mathbb{R}$ the sequence $w^\nu(\cdot + \sigma_\nu)$ has a convergent subsequence in the C^∞_{loc}-topology. That is, $\{w^\nu\}_{\nu\in\mathbb{N}}$ has a subsequence which converges with all derivatives on every compact subset to a gradient flow line $w \in C^\infty(\mathbb{R}\times S^1, M)\times C^\infty(\mathbb{R},\mathbb{R}^k)$.

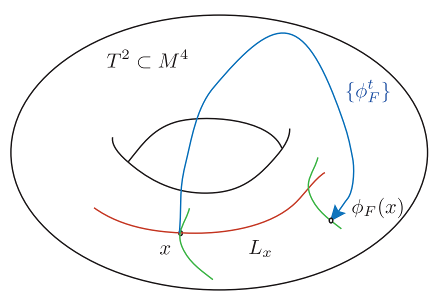
We refer to the next sections for a detailed and precise statement. Once we prove this compactness theorem, all the applications of Rabinowitz Floer homology to stable/contact/restricted contact hypersurfaces extend to corresponding results of stable/contact/restricted contact coisotropic submanifolds with minor modifications. For the sake of completeness, we include (sketches of) some applications, [AF1, AMo, CFP, Ka2, Ka3].

2.3 Leafwise coisotropic intersections

Let (M,ω) be a $2n$-dimensional symplectic manifold and Σ be a closed coisotropic submanifold of codimension k. Recall that Σ is foliated by leaves of $T\Sigma^\omega$ and L_x is the leaf through $x\in\Sigma$. A point $x\in\Sigma$ is called a **leafwise coisotropic intersection point** of $\phi_F \in \mathrm{Ham}_c(M,\omega)$ if $\phi(x)_F \in L_x$, see pictures below. In the extremal case $k=n$, a leafwise coisotropic intersection point is nothing but a Lagrangian intersection point.



$L_x = \{\phi^t_G(x)\,|\,t\in\mathbb{R}\}$



$L_x = \{\phi^{t_1}_{G_1}\circ\phi^{t_2}_{G_2}(x)\,|\,t_1,\,t_2\in\mathbb{R}\}$

Definition 2.3.1. We denote by $\wp(\Sigma) > 0$ the **minimal symplectic area** of all solutions of (1.2.2) contractible in M. To be more exact,

$$\wp(\Sigma) := \inf\left\{|\Omega(v) > 0| \,\middle|\, v \in C^\infty(S^1, \Sigma) \text{ solving (1.2.2) and contractible in } M\right\}.$$

Here $\Omega : \mathcal{L} \to \mathbb{R}$ stands for the symplectic area functional, i.e.

$$\Omega(v) = \int_{D^2} \bar{v}^*\omega$$

where $\bar{v} \in C^\infty(D^2, M)$ is a filling disk of v, i.e. $\bar{v}|_{\partial D^2}(t) = v(t)$ for $t \in S^1$. The symplectic asphericity condition guarantees that the value of $\Omega(v)$ is independent of the choice of a filling disk. If there are no solutions of (1.2.2), we set $\wp(\Sigma) = \infty$ by convention.

Theorem A. *Let Σ be a closed restricted contact coisotropic submanifold in a symplectic manifold (M, ω) being geometrically bounded. If $||\phi_F|| < \wp(\Sigma)$, there exists a leafwise coisotropic intersection point for $\phi_F \in \mathrm{Ham}_c(M, \omega)$.*

The assumption on the Hofer norm of ϕ_F is sharp. For instance $\wp(S^{2n-1})$ equals the displacement energy of S^{2n-1} inside $(\mathbb{R}^{2n}, d\mathbf{x} \wedge d\mathbf{y})$.

Remark 2.3.2. Basak Gürel [Gü] also proved Theorem A using a different method. We cannot entirely drop the restricted contact condition in Theorem A, see [Gi, Example 7.2] and [Gü, Remark 1.4].

Even if a coisotropic submanifold Σ is of contact type, we still can find a leafwise intersection point for a restricted class of perturbations. In this case our ambient symplectic manifold need not to be exact and can be closed; so we have more examples. Recall that

$$U_r = \left\{(q, \mathfrak{p}) = (q, p_1, \ldots, p_k) \in \Sigma \times \mathbb{R}^k \,\middle|\, |p_i| < r, \text{ for all } i = 1, \ldots, k\right\}$$

and $\psi : U_{\delta_0} \hookrightarrow M$ is a maximal symplectic embedding. For a time dependent Hamiltonian function $F \in C^\infty_c(S^1 \times M)$, we define the support of the

Hamiltonian vector field X_F by

$$\mathrm{Supp} X_F := \{x \in M \,|\, X_F(t,x) \neq 0 \text{ for some } t \in S^1\}.$$

We call a Hamiltonian function $F \in C_c^\infty(S^1 \times M)$ **admissible** if F is constant outside of $\psi(U_{\delta_0})$, i.e. $\mathrm{Supp} X_F \subsetneq \psi(U_{\delta_0})$. We denote by $\mathfrak{F}$ the set of all admissible Hamiltonian functions:

$$\mathfrak{F} := \{F \in C_c^\infty(S^1 \times M) \,|\, \mathrm{Supp} X_F \subsetneq \psi(U_{\delta_0})\}.$$

Then Theorem A holds even for (not necessarily restricted) contact coisotropic submanifolds with $F \in \mathfrak{F}$.

Theorem A$^+$. *Let Σ be a closed contact coisotropic submanifold in a symplectically aspherical symplectic manifold (M,ω) which is geometrically bounded (M can be closed). Then ϕ_F for $F \in \mathfrak{F}$ has a leafwise coisotropic intersection point provided $||F|| < \wp(\Sigma)$.*

In fact, the assumptions in Theorem A is not sufficient to define a Rabinowitz Floer homology for Σ. That is one reason why we can find only one leafwise coisotropic intersection point. However if we additionally assume that Σ is given by a regular level set of a Poisson-commuting Hamiltonian tuple $\mathcal{G} \in C^\infty(M, \mathbb{R}^k)$ which is compatible with the Reeb vector fields on (Σ, α) in the sense of the assumption iii), we obtain a Morse-type estimate and a relative cup-length estimate for leafwise coisotropic intersection points.

Theorem B. *Let (M,ω) be geometrically bounded and Σ be a closed regular level set of a Poisson-commuting Hamiltonian tuple $\mathcal{G} \in C^\infty(M,\mathbb{R}^k)$. Suppose that Σ is of restricted contact type, and $\int_{S^1} \Phi(v)dt$ is invertible for any $v \in C^\infty(S^1, \Sigma)$ contractible in M. Then the number of leafwise coisotropic intersection points for a generic $\phi \in \mathrm{Ham}_c(M,\omega)$ with $||\phi|| < \wp(\Sigma)$ is bounded below by the sum of $\mathbb{Z}/2$-Betti numbers of Σ.*

Theorem B$^+$. *Let (M,ω) be geometrically bounded (M can be closed) and symplectically aspherical, and Σ be a closed regular level set of a Poisson-commuting Hamiltonian tuple $\mathcal{G} \in C^\infty(M,\mathbb{R}^k)$. Suppose that Σ is of contact type, and $\int_{S^1}\Phi(v)dt$ is invertible for any $v \in C^\infty(S^1,\Sigma)$ contractible in M. Then the number of leafwise coisotropic intersection points for a generic $\phi_F \in \mathrm{Ham}_c(M,\omega)$ with $F \in \mathfrak{F}$ and with $||F|| < \wp(\Sigma)$ is bounded below by the sum of $\mathbb{Z}/2$-Betti numbers of Σ.*

The genericity assumption on $\phi_F \in \mathrm{Ham}_c(M,\omega)$ in the above theorems comes from the Morse property of the Rabinowitz action functional perturbed by F. We are able to remove this assumption by the following cup-length estimate as usual.

Definition 2.3.3. The **relative cup-length** of Σ in M is defined by

$$\mathrm{cl}(\Sigma, M) := \max\{k \in \mathbb{N} \,|\, \exists a_1,\ldots,a_k \in H^{\geq 1}(M;\mathbb{Z}/2) \text{ with } (a_1\cup\cdots\cup a_k)|_\Sigma \neq 0\}.$$

Theorem C. *Let (M,ω) be geometrically bounded and Σ be a closed regular level set of a Poisson-commuting Hamiltonian tuple $\mathcal{G} \in C^\infty(M,\mathbb{R}^k)$. Suppose that Σ is of restricted contact type, and $\int_{S^1}\Phi(v)dt$ is invertible for any $v \in C^\infty(S^1,\Sigma)$ contractible in M. Then the number of leafwise coisotropic intersection points for any $\phi \in \mathrm{Ham}_c(M,\omega)$ with $||\phi|| < \wp(\Sigma)$ is bounded below by $\mathrm{cl}(\Sigma,M)+1$.*

Theorem C$^+$. *Let (M,ω) be geometrically bounded (M can be closed) and symplectically aspherical, and Σ be a closed regular level set of a Poisson-commuting Hamiltonian tuple $\mathcal{G} \in C^\infty(M,\mathbb{R}^k)$. Suppose that Σ is of contact type, and $\int_{S^1}\Phi(v)dt$ is invertible for any $v \in C^\infty(S^1,\Sigma)$ contractible in M. Then the number of leafwise coisotropic intersection points for any $\phi_F \in \mathrm{Ham}_c(M,\omega)$ with $F \in \mathfrak{F}$ and with $||\phi|| < \wp(\Sigma)$ is bounded below by $\mathrm{cl}(\Sigma,M)+1$.*

We do not include the proofs of theorems with "+" but these immediately follow from the proofs of the corresponding theorems (without "+") together with arguments in [Ka2].

Theorems A and B were proved by Peter Albers and Urs Frauenfelder [AF1], and Theorem C was proved by Peter Albers and Al Momin [AMo] for separating restricted contact hypersurfaces. As mentioned, once we obtain the main theorem in the previous section, such applications immediately follow with minor modifications. It is noteworthy that we succeed in removing the separating condition in Theorem A by a simple approximation argument.

2.4 Leafwise displacement energy

A coisotropic submanifold Σ in a symplectic manifold (M,ω) is said to be **leafwisely displaceable** if there exists a Hamiltonian diffeomorphism $\phi_F \in \mathrm{Ham}_c(M,\omega)$ such that $\phi_F(L_x) \cap L_x = \emptyset$ for all $x \in \Sigma$. The **leafwise displacement energy** of Σ in M is defined by

$$e(\Sigma) := \inf \big\{ ||F|| \,\big|\, F \in C_c^\infty(S^1 \times M),\ \phi_F(L_x) \cap L_x = \emptyset,\ \forall x \in \Sigma \big\}.$$

We set $e(\Sigma) = \infty$ for the infimum of the empty set; that is, the leafwise displacement energy of a leafwisely nondisplaceable coisotropic submanifold is infinity.

Theorem D. *Let Σ be a closed stable coisotropic submanifold leafwisely displaceable inside (M,ω) which is geometrically bounded (M can be closed) and symplectically aspherical. Then there exists a periodic orbit $v \in C^\infty(S^1,\Sigma)$,* *i.e. a solution of* (1.2.2)*, contractible in M, such that*

$$0 < |\Omega(v)| \leq e(\Sigma). \tag{2.4.1}$$

Remark 2.4.1. The estimate (2.4.1) is sharp. The unit sphere S^{2n-1} in $(\mathbb{R}^{2n}, d\mathbf{x} \wedge d\mathbf{y})$ has $e(S^{2n-1}) = \pi = \Omega(v)$ where v is a periodic Reeb orbit of the standard contact structure on S^{2n-1}. For displaceable closed restricted contact coisotropic submanifolds, Theorem D was proved by Viktor Ginzburg [Gi]. A similar result was also proved by Kai Cieliebak, Urs Frauenfelder, and Gabriel Paternain [CFP] for stable separating hypersurfaces using Rabinowitz Floer theory. Making use of their proof, we slightly improve their theorem.

2.5 Rabinowitz Floer homology

We introduced the Rabinowitz action functional $\mathcal{A}_F^{\mathcal{G}} : \mathcal{L} \times \mathbb{R}^k \to \mathbb{R}$. With $F \equiv 0$, the action functional $\mathcal{A}^{\mathcal{G}}$ is generically Morse-Bott. The chain complex for Floer homology of $\mathcal{A}^{\mathcal{G}}$ is generated by critical points of an auxiliary Morse function on the solution space of (1.2.2) and the boundary map is defined by counting gradient flow lines of the Morse function with gradient flow lines (cascades) of $\mathcal{A}^{\mathcal{G}}$ (based on Urs Frauenfelder's Morse-Bott homology [Fr]). On the other hand, $\mathcal{A}_F^{\mathcal{G}}$ with nonzero F is Morse for generic $F \in C^\infty(S^1 \times M, \mathbb{R})$. Up to reparametrization of time supports of $\mathcal{G}$ and F (see Chapter 3), the chain complex for Floer homology of $\mathcal{A}_F^{\mathcal{G}}$ is generated by leafwise coisotropic intersection points and the boundary map is defined by counting gradient flow lines of $\mathcal{A}_F^{\mathcal{G}}$. Here gradient flow lines of $\mathcal{A}^{\mathcal{G}}$ resp. $\mathcal{A}_F^{\mathcal{G}}$ are solutions of a nonlinear elliptic PDE.

One of the power of Floer homology is the invariance property. Two Floer homologies obtained by $\mathcal{A}^{\mathcal{G}}$ and $\mathcal{A}_F^{\mathcal{G}}$ are isomorphic due to the standard continuation argument in Floer theory, see Section 5. Thus we name

Rabinowitz Floer homology for both and denote by

$$\mathrm{RFH}(\Sigma, M) := \mathrm{HF}(\mathcal{A}^{\mathcal{G}}) \cong \mathrm{HF}(\mathcal{A}^{\mathcal{G}}_F).$$

We should mention that $\mathrm{RFH}(\Sigma, M)$ does not depend on the choice of $\mathcal{G} \in C^\infty(M, \mathbb{R}^k)$ the defining Hamiltonian tuple for Σ (up to canonical isomorphism).

Remark 2.5.1. Though we only deal with restricted contact coisotropic submanifolds, it is possible to define $\mathrm{HF}(\mathcal{A}^{\mathcal{G}})$ in the stable case or $\mathrm{HF}(\mathcal{A}^{\mathcal{G}}_F)$ with $F \in \mathfrak{F}$ in the contact case. The assertions (i) and (ii) in Theorem E continue to hold for contact coisotropic submanifolds if we restrict the class of perturbations to $\mathfrak{F}$ and (iii) holds true for stable coisotropic submanifolds.

The following theorem is an immediate consequence of the construction and invariance property of Rabinowitz Floer homology.

Theorem E. *Let (M, ω) be geometrically bounded and Σ be a closed regular level set of a Poisson-commuting Hamiltonian tuple $\mathcal{G} \in C^\infty(M, \mathbb{R}^k)$. Suppose that Σ is of restricted contact type, and $\int_{S^1} \Phi(v)dt$ is invertible for any $v \in C^\infty(S^1, \Sigma)$ contractible in M.*

(i) *If Rabinowitz Floer homology does not vanish, there exists a leafwise coisotropic intersection point for every $\phi \in \mathrm{Ham}_c(M, \omega)$. In particular, if Σ is displaceable inside M, $\mathrm{RFH}(\Sigma, M) = 0$.*

(ii) *There exists a nonconstant solution of* (1.2.2) *contractible in M, provided that Σ is displaceable inside M.*

(iii) *If Σ carries no nonconstant solution of* (1.2.2) *contractible in M,*

$$\mathrm{RFH}(\Sigma, M) \cong \mathrm{H}(\Sigma; \mathbb{Z}/2).$$

In the extremal case, the assertions (i) and (iii) can be interpreted as:

(iv) *Let Σ be a Lagrangian torus, i.e. $k = n$. If $i_\# : \pi_1(\Sigma) \to \pi_1(M)$ is injective for the natural embedding $i : \Sigma \hookrightarrow M$,*[2]

$$\mathrm{RFH}(\Sigma, M) \cong \mathrm{H}(T^n; \mathbb{Z}/2).$$

2.6 Künneth formula

Here we only deal with the restricted contact case, but the same Künneth formulas for stable/contact coisotropic manifolds can be derived exactly the same way.

Theorem F. *Let (Σ_1, λ_1) and (Σ_2, λ_2) be restricted contact hypersurfaces in symplectic manifolds (M_1, ω_1) and (M_2, ω_2) respectively. Assume that Σ_1 resp. Σ_2 bounds a compact region in M_1 resp. M_2 and that M_1 and M_2 are geometrically bounded. Then,*

$$\mathrm{RFH}_n(\Sigma_1 \times \Sigma_2, M_1 \times M_2) \cong \bigoplus_{p=0}^{n} \mathrm{RFH}_p(\Sigma_1, M_1) \otimes \mathrm{RFH}_{n-p}(\Sigma_2, M_2).$$

Remark 2.6.1. Unfortunately we are only able to prove a compactness theorem for gradient flow lines of the unperturbed Rabinowitz action functional on $(\Sigma_1 \times \Sigma_2, M_1 \times M_2)$. Thus we cannot study about leafwise coisotropic intersection points except the case that $\Sigma_1 \times \Sigma_2$ is of restricted contact type again.

In Theorem G we do not consider Σ_2, and M_2 need to be closed.

[2] This implies that every solution of (1.2.2) is not contractible even in M.

Theorem G. *Let $(\Sigma_1, \lambda_1) \subset (M_1, \omega_1)$ be as in Theorem F above. Assume that (M_2, ω_2) is a closed symplectically aspherical symplectic manifold. Then,*

(G1) *$\Sigma_1 \times M_2$ has a leafwise coisotropic intersection point for $\phi \in \mathrm{Ham}_c(M_1 \times M_2, \omega_1 \oplus \omega_2)$ with Hofer-norm $||\phi|| < \wp(\Sigma_1, \lambda_1)$ even if Σ_1 does not bound a compact region in M_1.*

(G2) *The Rabinowitz Floer homology $\mathrm{RFH}(\Sigma_1 \times M_2, M_1 \times M_2) \cong \mathrm{HF}(\mathcal{A}_F^{\mathcal{G}})$ is defined for a generic $F \in C_c^\infty(M_1 \times M_2)$. Moreover, we have the Künneth formula:*

$$\mathrm{RFH}_n(\Sigma_1 \times M_2, M_1 \times M_2) \cong \bigoplus_{p=0}^{n} \mathrm{RFH}_p(\Sigma_1, M_1) \otimes \mathrm{H}_{n-p}(M_2).$$

Since we have not assumed any contact structure on $\Sigma_1 \times M_2$, we need a special version of isoperimetric inequality, see Lemma (6.3.1), in order to prove Theorem G.

Remark 2.6.2. It is worth emphasizing that $\Sigma_1 \times M_2$ is **never** of restricted contact type since M_2 is closed. Nevertheless, interestingly enough, we can achieve compactness of gradient flow lines of the perturbed Rabinowitz action functional for a generic (Morse property) perturbation $\phi_F \in \mathrm{Ham}_c(M_1 \times M_2, \omega_1 \oplus \omega_2)$.

Using the Künneth formulas and a result of [AF2], we are able to find infinitely many leafwise coisotropic intersection points on some coisotropic submanifolds.

Corollary F. *Let N be a closed Riemannian manifold of $\dim N \geq 2$ with $\dim \mathrm{H}_*(\Lambda N) = \infty$ where ΛN is the free loop space of N. Then there exists infinitely many leafwise coisotropic intersection points for a generic $\phi \in \mathrm{Ham}_c(T^*S^1 \times T^*N)$ on $(S^*S^1 \times S^*N, T^*S^1 \times T^*N)$.*

Remark 2.6.3. Since $(S^*S^1 \times S^*N, T^*S^1 \times T^*N)$ is of restricted contact type (see Lemma 7.1.3), ϕ in Corollary F is not necessarily of product type.

Corollary G. *Let N be as in Corollary F above, and (M, ω) be a closed symplectically aspherical symplectic manifold. Then a generic $\phi \in \mathrm{Ham}_c(T^*N \times M)$ has infinitely many leafwise coisotropic intersection points on $(S^*N \times M, T^*N \times M)$.*

2.7 List of related results

- **On Rabinowitz Floer homology theory:** [AF1, AF2, AF3, AF4, AF5, AF6, AFMe, AMe1, AMe2, AMo, AS, BF, CF, CFO, CFP, FS, Ka1, Ka2, Ka3, Ka4, Me1, Me2, MP, MMP].
- **On leafwise (coisotropic) intersections:** [AF1, AF2, AF4, AMo, AMe1, AMc, Ba, Dr, EH, Gi, Gü, Ho, Ka2, Ka3, Ka4, Mo, Me2, MMP, Zi1, Zi2].
- **On (Leafwise) displacement energy:** [Bo1, Bo2, Gi, Ka3, Ke, Us].

Chapter 3

The Rabinowitz action functional with several Lagrange multipliers

This chapter is devoted to the proof of the main theorem, which proves a compactness result for gradient flow lines of the Rabinowitz action functional, and to the proof of Theorem A.

3.1 The Rabinowitz action functional for coisotropic submanifolds

Let $\eta = (\eta_1, \dots, \eta_k) \in \mathbb{R}^k$ be a k-tuple of Lagrange multipliers. We denote by $\mathcal{L} \subset C^\infty(S^1, M)$ the space of contractible loops in M. For an arbitrary Hamiltonian tuple $\mathcal{G} = (G_1, \dots, G_k) \in C^\infty(M, \mathbb{R}^k)$ which has $0 \in \mathbb{R}^k$ as a regular value, and which is Poisson-commuting near $\bigcup_{i=1}^k G_i^{-1}(0)$, the generalized Rabinowitz action functional $\mathcal{A}^{\mathcal{G}} : \mathcal{L} \times \mathbb{R}^k \to \mathbb{R}$ is defined as follows:

$$\mathcal{A}^{\mathcal{G}}(v, \eta) := -\int_{D^2} \bar{v}^* \omega - \sum_{i=1}^{k} \eta_i \int_0^1 G_i(v(t)) dt \tag{3.1.1}$$

where $\bar{v}$ is any filling disk of v, i.e. $\bar{v}|_{\partial D^2}(t) = v(t)$ for $t \in S^1$. The symplectic asphericity condition implies that the value of the above action functional is independent of the choice of filling discs. Using the standard scalar product $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^k$, we can express (3.1.1) by

$$\mathcal{A}^{\mathcal{G}}(v, \eta) = -\int_{D^2} \bar{v}^* \omega - \int_0^1 \langle \eta, \mathcal{G} \rangle (v(t)) dt.$$

A critical point of the Rabinowitz action functional, $(v, \eta) \in \mathrm{Crit}\mathcal{A}^{\mathcal{G}}$ satisfies the following equations.

$$\left.\begin{aligned} \partial_t v(t) = \sum_{i=1}^k \eta_i X_{G_i}(v(t)), \quad t \in S^1 \\ \int_0^1 G_i(v(t)) dt = 0, \quad i \in \{1, \dots, k\} \end{aligned}\right\} \tag{3.1.2}$$

Proposition 3.1.1. *If $(v, \eta) \in \mathrm{Crit}\mathcal{A}^{\mathcal{G}}$, $v(t) \in \mathcal{G}^{-1}(0)$ for all $t \in S^1$.*

Proof. Assume by contradiction that $G_j(v(t_0)) > 0$ for some $t_0 \in S^1$ and $j \in \{1, \dots, k\}$. Then to satisfy the second equation in (3.1.2), there exists $t_1 \in S^1$ such that $G_j(v(t_1)) < 0$ and hence $v(t_2) \in G_j^{-1}(0)$ for some $t_2 \in S^1$. Using the first equation in (3.1.2), we have

$$\frac{d}{dt} G_i(v(t)) = dG_i(v(t))[\partial_t v] = dG_i\bigg(\sum_{j=1}^k \eta_j X_{G_j}(v(t))\bigg) = \sum_{j=1}^k \eta_j \{G_i, G_j\}(v(t))$$

which implies $G_i(v(t))$ is stationary whenever $v(t) \in G_j^{-1}(0)$ due to Poisson-commutativity of $\mathcal{G}$ near $\bigcup_{i=1}^k G_i^{-1}(0)$. Since $v(t_2) \in G_j^{-1}(0)$, $G_j(v(t)) = 0$ for all $t \in S^1$. This contradiction proves the proposition. $\square$

3.2 The perturbed Rabinowitz action functional

Let $\mathcal{G} \in C^\infty(M, \mathbb{R}^k)$ be as in the subsection. We choose a smooth function $\chi \in C^\infty(S^1, \mathbb{R})$ such that $\chi(t) \geq 0$, $\int_0^1 \chi(t)dt = 1$, and $\mathrm{Supp}\chi \subset (1/2, 1)$. Using χ, we define a time-dependent Hamiltonian $H_i : S^1 \times M \to \mathbb{R}$ by $H_i(t,x) = \chi(t)G_i(x)$ for $1 \leq i \leq k$, i.e.

$$\mathcal{H}(t,x) := \chi(t)\mathcal{G}(x) \in C^\infty(S^1 \times M, \mathbb{R}^k).$$

Let $F \in C_c^\infty(S^1 \times M)$ be an arbitrary time-dependent Hamiltonian function. Thanks to Lemma 1.1.2, we assume that F has time support in $(0, \frac{1}{2})$. We note that the time support of $\mathcal{H}$ and the time support of F are **disjoint**. With these Hamiltonian functions, the perturbed Rabinowitz action functional $\mathcal{A}_F^{\mathcal{H}} : \mathcal{L} \times \mathbb{R}^k \to \mathbb{R}$ is defined by

$$\mathcal{A}_F^{\mathcal{H}}(v, \eta) := -\int_{D^2} \bar{v}^*\omega - \int_0^1 F(t, v(t))dt - \int_0^1 \langle \eta, \mathcal{H} \rangle (t, v(t))dt.$$

where $\bar{v} : D^2 \to M$ is any filling disk of v. A critical point of the perturbed Rabinowitz action functional, $(v, \eta) \in \mathrm{Crit}\mathcal{A}_F^{\mathcal{H}}$ satisfies the following equations.

$$\left.\begin{aligned} &\partial_t v(t) = X_F(t,v) + \sum_{i=1}^k \eta_i X_{H_i}(t, v(t)), \quad && t \in S^1 \\ &\int_0^1 H_i(t, v(t))dt = 0, && i \in \{1, \ldots, k\} \end{aligned}\right\} \tag{3.2.1}$$

In the next proposition, we observe that a critical point of $\mathcal{A}_F^{\mathcal{H}}$ gives rise to a leafwise coisotropic intersection point. Albers-Frauenfelder [AF1] proved the following proposition when Σ is a hypersurface. Their proof continues to work for coisotropic submanifolds with minor modifications.

Definition 3.2.1. A leafwise coisotropic intersection point $x \in \Sigma$ is called **periodic** if the leaf L_x contains a solution of (1.2.2).

Proposition 3.2.2. *If $(v,\eta) \in \mathrm{Crit}\mathcal{A}_F^{\mathcal{H}}$, $v(0) \in \Sigma$ is a leafwise coisotropic intersection point. Moreover, the map*

$$\mathrm{Crit}\mathcal{A}_F^{\mathcal{H}} \longrightarrow \big\{\textit{leafwise coisotropic intersections}\big\}$$

is injective unless there is no periodic leafwise coisotropic intersection.

PROOF. Since the time support of F is $(0,1/2)$, for $t \geq 1/2$ and for all $i = 1,\dots,k$,

$$\frac{d}{dt}G_i(v(t)) = dG_i(v(t))[\partial_t v] = dG_i(v(t))\big[\underbrace{X_F(t,v)}_{=0} + \sum_{j=1}^{k}\chi(t)\eta_j X_{G_j}(v)\big]$$

As in the proof of Proposition 3.1.1, the second equation in (3.2.1) implies $v(t) \in \mathcal{G}^{-1}(0) = \Sigma$ for $t \in (1/2,1)$. On the other hand, v solves $\partial_t v = X_F(t,v)$ on $(0,1/2)$ so that $v(1/2) = \phi_F^{1/2}(v(0)) = \phi_F^1(v(0))$ since $F = 0$ for $t \geq 1/2$. For $t \in (1/2,1)$, it holds that $\partial_t v = \sum_{i=1}^k \eta_i X_{H_i}(t,v)$ and thus $v(0) = v(1) \in L_{v(1/2)}$. Thus we conclude that $v(0) \in L_{\phi_F(v(0))}$ which is equivalent to $\phi_F(v(0)) \in L_{v(0)}$. $\square$

From now on, we allow s-dependence on F as follows. Let $\{F_s\}_{s\in\mathbb{R}}$ be a family of Hamiltonian functions varying only on a finite interval in $\mathbb{R}$. More specifically, we assume $F_s(t,x) = F_-(t,x)$ for $s \leq -1$ and $F_s(t,x) = F_+(t,x)$ for $s \geq 1$. We also choose a family of compatible almost complex structures $\{J(s,t)\}_{(s,t)\in\mathbb{R}\times S^1}$ on M such that $J(s,t)$ is invariant outside of the interval $[-1,1] \subset \mathbb{R}$ and they still split as in (1.2.3).

On the tangent space $T_{(v,\eta)}(\mathcal{L}\times\mathbb{R}^k) = T_v\mathcal{L}\times T_\eta\mathbb{R}^k$ for $(v,\eta) \in \mathcal{L}\times\mathbb{R}^k$, we define the metric m as follows:

$$m_{(v,\eta)}\big((\hat{v}^1,\hat{\eta}^1),(\hat{v}^2,\hat{\eta}^2)\big) := \int_0^1 g_v(\hat{v}^1,\hat{v}^2)dt + \langle\hat{\eta}^1,\hat{\eta}^2\rangle.$$

Recall that $g(\cdot,\cdot) = \omega(\cdot,J\cdot)$ is a metric on M. Here $\hat{\eta}^1$ and $\hat{\eta}^2$ are elements in $T_\eta \mathbb{R}^k \cong \mathbb{R}^k$ and $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^k$.

Definition 3.2.3. A map $w \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^k)$ which solves

$$\partial_s w(s) + \nabla_m \mathcal{A}^{\mathcal{H}}_{F_s}(w(s)) = 0. \tag{3.2.2}$$

is called a **gradient flow line** of $\mathcal{A}^{\mathcal{H}}_{F_s}$ with respect to the metric m.

According to Floer's interpretation, the gradient flow equation (3.2.2) can be interpreted as $w = (u, \tau) = (u, \tau_1, \ldots, \tau_k)$ with $u(s,t) : \mathbb{R} \times S^1 \to M$ and $\tau_i(s) : \mathbb{R} \to \mathbb{R}$, solving

$$\left.\begin{aligned} &\partial_s u + J(s,t,u)\big(\partial_t u - \sum_{i=1}^k \tau_i X_{H_i}(t,u) - X_{F_s}(t,u)\big) = 0 \\ &\partial_s \tau_i - \int_0^1 H_i(t,u)dt = 0, \qquad\qquad 1 \le i \le k \end{aligned}\right\} \tag{3.2.3}$$

Definition 3.2.4. The **energy** of a map $w \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^k)$ is defined as

$$E(w) := \int_{-\infty}^{\infty} ||\partial_s w||_m^2 ds.$$

Lemma 3.2.5. Let $w \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^k)$ be a gradient flow line of $\mathcal{A}^{\mathcal{H}}_{F_s}$ with finite energy. Then we have the following estimate.

$$E(w) \le \mathcal{A}^{\mathcal{H}}_{F_-}(w_-) - \mathcal{A}^{\mathcal{H}}_{F_+}(w_+) + \int_{-\infty}^{\infty} ||\partial_s F_s||_- ds \tag{3.2.4}$$

where $w_\pm := \lim_{s \to \pm\infty} w(s) \in \mathrm{Crit}\mathcal{A}^{\mathcal{H}}_{F_s}$. Moreover, equality holds if $\partial_s F_s = 0$.

PROOF. The following computation proves the lemma.

$$\begin{aligned} E(w) &= -\int_{-\infty}^{\infty} d\mathcal{A}^{\mathcal{H}}_{F_s}\big(w(s)\big)[\partial_s w(s)]ds \\ &= -\int_{-\infty}^{\infty} \frac{d}{ds}\Big(\mathcal{A}^{\mathcal{H}}_{F_s}\big(w(s)\big)\Big)ds + \int_{-\infty}^{\infty} \big(\partial_s \mathcal{A}^{\mathcal{H}}_{F_s}\big)\big(w(s)\big)ds \\ &= \mathcal{A}^{\mathcal{H}}_{F_-}(w_-) - \mathcal{A}^{\mathcal{H}}_{F_+}(w_+) - \int_{-\infty}^{\infty}\int_0^1 \partial_s F_s(w)dtds \\ &\leq \mathcal{A}^{\mathcal{H}}_{F_-}(w_-) - \mathcal{A}^{\mathcal{H}}_{F_+}(w_+) + \int_{-\infty}^{\infty} ||\partial_s F_s||_- ds\ . \end{aligned}$$

□

Remark 3.2.6. We note that $\int_{-\infty}^{\infty} ||\partial_s F_s||_- ds$ has a finite value since $\partial_s F_s$ has a compact support by construction.

Proposition 3.2.7. $\mathcal{A}^{\mathcal{H}}_{F_s}$ *has a uniform bound along gradient flow lines.*

PROOF. For any gradient flow line $w \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^k)$ of $\mathcal{A}^{\mathcal{H}}_F$ and $s_1 < s_2 \in \mathbb{R}$, we calculate

$$\begin{aligned} 0 &\leq \int_{s_1}^{s_2} ||\partial_s w||^2_m\, ds \\ &= -\int_{s_1}^{s_2} d\mathcal{A}^{\mathcal{H}}_{F_s}(w(s))(\partial_s w)ds \\ &= \mathcal{A}^{\mathcal{H}}_{F_{s_1}}(w(s_1)) - \mathcal{A}^{\mathcal{H}}_{F_{s_2}}(w(s_2)) - \int_{s_1}^{s_2}\int_0^1 \partial_s F_s(t,v)dtds \\ &\leq \mathcal{A}^{\mathcal{H}}_{F_{s_1}}(w(s_1)) - \mathcal{A}^{\mathcal{H}}_{F_{s_2}}(w(s_2)) + \int_{s_1}^{s_2} ||\partial_s F_s||_- ds. \end{aligned}$$

From the above inequality we obtain

$$\mathcal{A}^{\mathcal{H}}_{F_{s_2}}(w(s_2)) \leq \mathcal{A}^{\mathcal{H}}_{F_-}(w_-) + \int_{-\infty}^{\infty} ||\partial_s F_s||_- ds,$$
$$\mathcal{A}^{\mathcal{H}}_{F_{s_1}}(w(s_1)) \geq \mathcal{A}^{\mathcal{H}}_{F_+}(w_+) - \int_{-\infty}^{\infty} ||\partial_s F_s||_- ds.$$

This proves the proposition. □

3.2.1 Compactness

In this subsection, we prove Theorem 3.2.8 which is a vital ingredient for all our results. Here, Σ is assumed to be a closed restricted contact coisotropic submanifold. However for a perturbation $F \in \mathfrak{F}$, adapting an idea in [Ka2] we are able to prove the theorem in the contact case as well. We also need the assumptions ii) and iii).

Recall that $\Sigma = \mathcal{G}^{-1}(0)$. For compactness, we cut-off $\mathcal{G}$ to be constant away from Σ. More precisely, $M \setminus G_i^{-1}(0)$ consists of two parts M_i^+ and M_i^- such that $\pm G_i|_{M_i^\pm} > 0$ for $1 \leq i \leq k$. Therefore we are able to modify G_i so that for a small $\epsilon > 0$,

$$G_i = \begin{cases} \textbf{\textit{unchanged}} & \text{on } G_i^{-1}(-\epsilon, \epsilon), \\ \textbf{\textit{constant}} & \text{near infinity.} \end{cases}$$

for all $1 \leq i \leq k$. Note that $\mathcal{G}$ is **still** Poisson-commuting on $\bigcup_{i=1}^k G_i^{-1}(-\epsilon, \epsilon)$ after such a modification and thus Proposition 3.1.1 and Proposition 3.2.2 remain true.

Theorem 3.2.8. *Let $\{w^\nu = (u^\nu, \tau^\nu)\}_{\nu \in \mathbb{N}}$ be a sequence of gradient flow lines of $\mathcal{A}^{\mathcal{H}}_{F_s}$ for which there exist $a \leq b$ such that*

$$a \leq \mathcal{A}^{\mathcal{H}}_{F_s}(w^\nu(s)) \leq b, \quad \text{for all } \nu \in \mathbb{N},\ s \in \mathbb{R}. \tag{3.2.5}$$

Then for every reparametrization sequence $\sigma_\nu \in \mathbb{R}$ the sequence $w^\nu(\cdot + \sigma_\nu)$ has a convergent subsequence in the C^∞_{loc}-topology. That is, $\{w^\nu\}_{\nu\in\mathbb{N}}$ has a subsequence which converges with all derivatives on every compact subset to a gradient flow line $w \in C^\infty(\mathbb{R}\times S^1, M)\times C^\infty(\mathbb{R},\mathbb{R}^k)$.

PROOF. Once we prove Theorem 3.2.11 which is a new feature of Rabinowitz Floer theory, the rest of the proof is established by the following steps which are standard by now in Floer theory.

1. Since (M,ω) is geometrically bounded and we have modified $\mathcal{G}$ so that $\mathcal{G}$ is constant near infinity, we have a uniform bound on images of u^ν, see [AL] (also see [Mc, Lemma 2.4] for the convex at infinity case).

2. Due to Lemma 3.2.5 and Proposition 3.2.7, we have a uniform energy bound on u^ν and this implies a uniform bound on $\partial_s u^\nu$ except finitely many points.

3. On such finitely many points where the derivative $\partial_s u^\nu$ explodes, we can detect nonconstant J-holomorphic spheres, see [McS, Chapter 4.2]. However this so-called bubbling-off phenomenon does not occur due to symplectic asphericity.

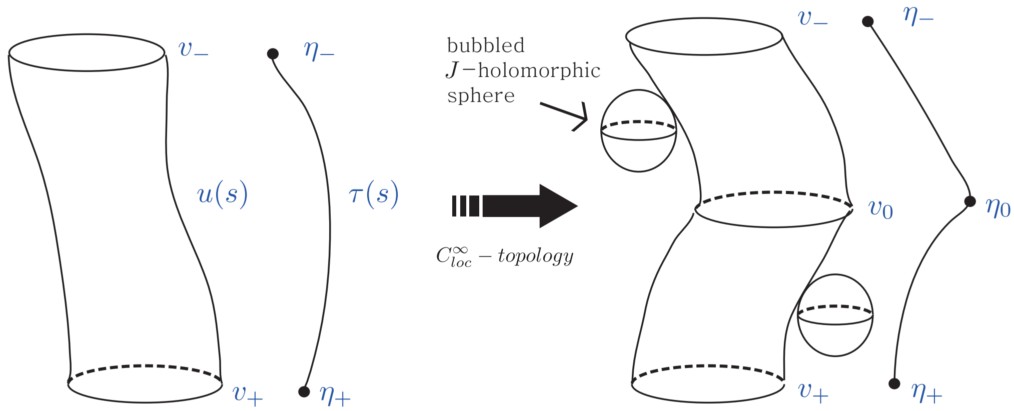
4. By Theorem 3.2.11, we have a uniform bound on $\tau_1^\nu,\dots,\tau_k^\nu$. From the gradient flow equation

$$\partial_s u^\nu + J(t,u^\nu)\Big(\partial_t u^\nu - \sum_{i=1}^k \tau_i^\nu(s) X_{G_i}(u^\nu)\Big) = 0,$$

   we obtain a uniform bound on $\partial_t u^\nu$ as well.

5. Now we can apply the elliptic bootstrapping argument in Floer theory, see [McS, Theorem B.4.2] and hence the assertion follows.

□



We first prove the following fundamental lemma which is a key step in proving Theorem 3.2.11.

Lemma 3.2.9. There exist $\epsilon > 0$ and $C > 0$ such that for $(v, \eta) \in \mathcal{L} \times \mathbb{R}^k$,

$$||\nabla_m \mathcal{A}^{\mathcal{H}}_{F_s}(v,\eta)||_m < \epsilon \quad \text{implies} \quad |\eta_i| \leq C\big(|\mathcal{A}^{\mathcal{H}}_{F_s}(v,\eta)| + 1\big) \ \text{ for all } 1 \leq i \leq k.$$

Proof. The proof proceeds in three steps.

Step 1: There exists a small constant $\delta \in (0, \delta_0)$ satisfying the following. Assume $v(t) \in U_\delta$ for $t \in (1/2, 1)$. Then there exists $C_0 > 0$ such that

$$|\eta_i| \leq C_0\big(|\mathcal{A}^{\mathcal{H}}_{F_s}(v,\eta)| + ||\nabla_m \mathcal{A}^{\mathcal{H}}_{F_s}(v,\eta)||_m + 1\big), \quad i = 1, \ldots, k.$$

Proof of Step 1. Recall that there exists a family of definite matrices

$$\Phi = (\Phi_{i,j}) : \mathcal{G}^{-1}(0) \to \mathrm{Mat}(k \times k)$$

such that

$$X_{G_i} = \Phi R_i, \quad 1 \leq i \leq k$$

and we have assumed

$$\int_{S^1} \chi(t)\Phi(v(t))dt \in \mathrm{Mat}(k \times k)$$

is invertible for any $v \in C^\infty(S^1, \Sigma)$ contractible in M, see Remark 2.1.1. For each $j = 1, \ldots, k$,

$$\begin{aligned}
\mathcal{A}^{\mathcal{H}}_{F_s}(v, \eta) =& -\int_0^1 v^*\lambda_j - \sum_{i=1}^k \eta_i \int_0^1 H_i(t, v)dt - \int_0^1 F_s(t, v)dt \\
=& -\int_0^1 \lambda_j\big(\partial_t v - \sum_{i=1}^k \eta_i X_{H_i}(t, v) - X_{F_s}(t, v)\big)dt - \sum_{i=1}^k \int_0^1 \lambda_j\big(\eta_i X_{H_i}(t, v)\big)dt \\
& -\int_0^1 \lambda_j(X_{F_s}(t, v))dt - \sum_{i=1}^k \eta_i \int_0^1 H_i(t, v)dt - \int_0^1 F_s(t, v)dt \\
=& -\int_0^1 \lambda_j\big(\nabla_m \mathcal{A}^{\mathcal{H}}_{F_s}(v, \eta)\big)dt - \sum_{i=1}^k \eta_i \int_0^1 \chi(t)\lambda_j\big(\sum_{\ell=1}^k \Phi_{i,\ell} R_\ell(v)\big)dt \\
& -\int_0^1 \lambda_j(X_{F_s}(t, v))dt - \sum_{i=1}^k \eta_i \int_0^1 H_i(t, v)dt - \int_0^1 F_s(t, v)dt \\
=& -\int_0^1 \lambda_j\big(\nabla_m \mathcal{A}^{\mathcal{H}}_{F_s}(v, \eta)\big)dt - \sum_{i=1}^k \eta_i \int_0^1 \chi(t)\Phi_{i,j}(v)dt \\
& -\int_0^1 \lambda_j(X_{F_s}(t, v))dt - \sum_{i=1}^k \eta_i \int_0^1 \chi(t)G_i(v)dt - \int_0^1 F_s(t, v)dt
\end{aligned}$$

Thus we have

$$-\sum_{i=1}^k \eta_i \chi(t) \int_0^1 \big(\Phi_{i,j} + G_i\big)(v)dt = \mathcal{A}^{\mathcal{H}}_{F_s}(v, \eta) + \int_0^1 \lambda_j\big(\nabla_m \mathcal{A}^{\mathcal{H}}_{F}(v, \eta) + X_{F_s}(t, v)\big) + F_s(t, v)dt$$

and

$$\Gamma(v)\begin{pmatrix}\eta_1\\ \vdots\\ \eta_k\end{pmatrix}=\begin{pmatrix}\mathcal{A}^{\mathcal{H}}_{F_s}(v,\eta)+\int_0^1\lambda_1\big(\nabla_m\mathcal{A}^{\mathcal{H}}_{F_s}(v,\eta)+X_{F_s}(t,v)\big)+F_s(t,v)dt\\ \vdots\\ \mathcal{A}^{\mathcal{H}}_{F_s}(v,\eta)+\int_0^1\lambda_k\big(\nabla_m\mathcal{A}^{\mathcal{H}}_{F_s}(v,\eta)+X_{F_s}(t,v)\big)+F_s(t,v)dt\end{pmatrix}$$

where $\Gamma(v)$ is a $k\times k$ matrix defined by

$$\Gamma(v):=\left[-\int_0^1\chi(t)\big(\Phi_{i,j}+G_i\big)(v)dt\right]_{1\le i,j\le k}$$

We choose small $\delta>0$ so that $\Gamma(v)$ is still invertible for any $v\subset U_\delta:=\mathcal{G}^{-1}(-\delta,\delta)$. Therefore,

$$\begin{pmatrix}\eta_1\\ \vdots\\ \eta_k\end{pmatrix}=\Gamma(v)^{-1}\begin{pmatrix}\mathcal{A}^{\mathcal{H}}_{F_s}(v,\eta)+\int_0^1\lambda_1\big(\nabla_m\mathcal{A}^{\mathcal{H}}_{F_s}(v,\eta)+X_{F_s}(t,v)\big)+F_s(t,v)dt\\ \vdots\\ \mathcal{A}^{\mathcal{H}}_{F_s}(v,\eta)+\int_0^1\lambda_k\big(\nabla_m\mathcal{A}^{\mathcal{H}}_{F_s}(v,\eta)+X_{F_s}(t,v)\big)+F_s(t,v)dt\end{pmatrix}.$$

Since

$$||\lambda_i||_{L^\infty(U_\delta)},\ ||(\Phi_{i,j}+G_i)||_{L^\infty(U_\delta)},\ ||F_s||_{L^\infty(U_\delta)},\ ||X_{F_s}||_{L^\infty(U_\delta)}<\infty,$$

there exists a constant $C_0>0$ such that

$$|\eta_j|\le C_0\big(|\mathcal{A}^{\mathcal{H}}_{F_s}(v,\eta)|+||\nabla_m\mathcal{A}^{\mathcal{H}}_{F_s}(v,\eta)||_m+1\big),\quad \forall j=1,\dots,k.$$

Step 2: If there is $t\in(\frac{1}{2},1)$ such that $v(t)\notin U_\delta$ then $||\nabla_m\mathcal{A}^{\mathcal{H}}_{F_s}(v,\eta)||_m\ge\epsilon$.

Proof of Step 2. The assumption $v(t)\notin U_\delta$ means that there exists $i\in\{1,\dots,k\}$ such that $v(t)\notin U^i_\delta:=G_i^{-1}(-\delta,\delta)$. If in addition, $v(t)\in M-U^i_{\delta/2}$ for all $t\in(\frac{1}{2},1)$ then we easily have

$$||\nabla_m\mathcal{A}^{\mathcal{H}}_{F_s}(v,\eta)||_m\ge\left|\int_0^1 H_i(t,v(t))dt\right|=\left|\int_{1/2}^1\chi(t)G_i(v(t))dt\right|\ge\frac{\delta}{2}.$$

Otherwise there exists $t' \in (\frac{1}{2}, 1)$ such that $v(t') \in U^i_{\delta/2}$. Thus we can find $t_0, t_1 \in (\frac{1}{2}, 1)$ such that

$$v(t_0) \in \partial U^i_{\delta/2}, \;\; v(t_1) \in \partial U^i_{\delta}, \;\; v(t) \in U^i_{\delta} - U^i_{\delta/2}, \quad \forall t \in [t_0, t_1],$$

or

$$v(t_1) \in \partial U^i_{\delta}, \;\; v(t_0) \in \partial U^i_{\delta/2}, \;\; v(t) \in U^i_{\delta} - U^i_{\delta/2}, \quad \forall t \in [t_1, t_0].$$

We treat only the first case. The latter case is analogous. With

$$\mathfrak{P} := \max_{x \in U_\delta} ||\nabla_g G_i(x)||_g < \infty$$

we estimate,

$$\begin{aligned}
\mathfrak{P}||\nabla_m \mathcal{A}^{\mathcal{H}}_{F_s}(v,\eta)||_m &\geq \mathfrak{P}||\partial_t v - \sum_{j=1}^{k} \eta_j X_{H_j}(t,v) - X_{F_s}(t,v)||_{L^2} \\
&\geq \mathfrak{P}||\partial_t v - \sum_{j=1}^{k} \eta_j X_{H_j}(t,v) - X_{F_s}(t,v)||_{L^1} \\
&\geq \int_{t_0}^{t_1} ||\partial_t v - \sum_{j=1}^{k} \eta_j X_{H_j}(t,v) - X_{F_s}(t,v)||_g ||\nabla_g G_i(v(t))||_g dt \\
&\geq \Big| \int_{t_0}^{t_1} \big\langle \nabla_g G_i(v(t)), \partial_t v(t) - \sum_{j=1}^{k} \eta_j X_{H_j}(t,v) - X_{F_s}(t,v) \big\rangle_g dt \Big| \\
&= \Big| \int_{t_0}^{t_1} dG_i(v(t)) \big(\partial_t v(t) - \sum_{j=1}^{k} \eta_j X_{H_j}(t,v) - \underbrace{X_{F_s}(t,v)}_{=0}\big) dt \Big| \\
&= \Big| \int_{t_0}^{t_1} \frac{d}{dt} G_i(v(t)) dt - \underbrace{dG_i(v)\big(\sum_{j=1}^{k} \eta_j X_{H_j}(t,v)\big)}_{=0} \Big| \\
&\geq \big|G_i(v(t_1))\big| - \big|G_i(v(t_0))\big| \\
&= \frac{\delta}{2}.
\end{aligned} \tag{3.2.6}$$

Thus Step 2 follows with $\epsilon = \min\big\{\frac{\delta}{2}, \frac{\delta}{2\mathfrak{P}}\big\}$.

Step 3: Proof of the lemma.

Proof of Step 3. According to Step 2, $v(t) \in U_\delta$ for all $t \in (\frac{1}{2}, 1)$. Then Step 1 completes the proof of the lemma with $C = C_0 + \epsilon + 1$. $\square$

For a given gradient flow line w of $\mathcal{A}^{\mathcal{H}}_{F_s}$ and $\sigma \in \mathbb{R}$, we define

$$\begin{aligned} o(\sigma, w, \epsilon) &:= \inf\big\{\tau \geq 0 \,\big|\, ||\nabla_m \mathcal{A}^{\mathcal{H}}_{F_s}(w(\sigma+\tau))||_m \leq \epsilon\big\}, \\ C_F &:= \int_{-\infty}^{\infty}\int_0^1 \max_{x\in M} ||\partial_s F_s(t,x)||_g dtds \;<\; \infty. \end{aligned} \tag{3.2.7}$$

Lemma 3.2.10. For a gradient flow line w of $\mathcal{A}^{\mathcal{H}}_{F_s}$ with $\lim_{s\to\pm\infty} w(s) = w_\pm$,

$$o(\sigma, w, \epsilon) \leq \frac{\mathcal{A}^{\mathcal{H}}_{F_s}(w_-) - \mathcal{A}^{\mathcal{H}}_{F_s}(w_+) + C_F}{\epsilon^2}.$$

Proof. We compute

$$\begin{aligned} \epsilon^2 o(\sigma, w, \epsilon) &\leq \int_\sigma^{\sigma+o(\sigma,w,\epsilon)} \big|\big|\nabla_m \mathcal{A}^{\mathcal{H}}_{F_s}(w)\big|\big|_m^2 ds \\ &\leq \int_{-\infty}^{\infty} -d\mathcal{A}^{\mathcal{H}}_{F_s}(w)(\partial_s w)ds - C_F + C_F \\ &\leq \int_{-\infty}^{\infty} -\frac{d}{ds}\big(\mathcal{A}^{\mathcal{H}}_{F_s}(w(s))\big)ds + C_F \\ &= \mathcal{A}^{\mathcal{H}}_{F_s}(w_-) - \mathcal{A}^{\mathcal{H}}_{F_s}(w_+) + C_F \end{aligned}$$

We obtain a bound on $o(\sigma, w, \epsilon)$ by dividing ϵ^2 in the above inequality. □

Theorem 3.2.11. *Assume that $w = (u, \tau) \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^k)$ is a gradient flow line of $\mathcal{A}^{\mathcal{H}}_{F_s}$ for which there exist $a \leq b$ such that*

$$a \leq \mathcal{A}^{\mathcal{H}}_{F_s}(w(s)) \leq b, \quad \text{for all } s \in \mathbb{R}. \tag{3.2.8}$$

Then the L^∞-norms of τ_i's are uniformly bounded.

As we have mentioned, Theorem 3.2.11 completes the proof of Theorem 3.2.8.

Proof. Using Lemma 3.2.9 and Lemma 3.2.10, we obtain

$$\begin{aligned}|\tau_i(\sigma)| &\leq |\tau_i(\sigma + o(\sigma,w,\epsilon))| + \int_{\sigma}^{\sigma+o(\sigma,w,\epsilon)} |\partial_s \tau_i(s)| ds \\ &\leq C\big(\big|\mathcal{A}^{\mathcal{H}}_{F_s}(w(\sigma + o(\sigma,w,\epsilon)))\big| + 1\big) + o(\sigma,w,\epsilon)||H_i||_{L^\infty} \\ &\leq C(\max\{|a|,|b|\}+1) + \left(\frac{|b-a|+C_F}{\epsilon^2}\right)||H_i||_{L^\infty}.\end{aligned}$$

□

3.3 Proof of Theorem A

The proof proceeds in two steps. In Step 1, we prove Theorem A under the assumption that Σ is a regular level set of a Poisson commuting Hamiltonian tuple $\mathcal{G}$ satisfying the assumption iii) as before. Then we remove this additional assumption in Step 2.

Step 1. There exists a critical point (v,η) of $\mathcal{A}^{\mathcal{H}}_F$ if $||F|| < \wp(\Sigma)$ and Σ is of restricted contact type with $\Phi : \Sigma \to \mathrm{Mat}_{\mathrm{Def}}(k \times k)$. Moreover the action value of that critical point is uniformly bounded as below:

$$-\,||F|| \leq \mathcal{A}^{\mathcal{H}}_F(v,\eta) \leq ||F||. \tag{3.3.1}$$

Proof of Step 1. We mainly follow the proof of Theorem A in [AF1] which made use of the "stretching the neck" argument. For $0 \leq r$, we choose a smooth family of functions $\varphi_r \in C^\infty(\mathbb{R},[0,1])$ satisfying

1. for $r \geq 1$: $\varphi_r'(s)\cdot s \leq 0$ for all $s \in \mathbb{R}$, $\varphi_r(s) = 1$ for $|s| \leq r-1$, and $\varphi_r(s) = 0$ for $|s| \geq r$,
2. for $r \leq 1$: $\varphi_r(s) \leq r$ for all $s \in \mathbb{R}$ and $\mathrm{Supp}\varphi_r \subset [-1,1]$,

We note that $\varphi_\infty \equiv 1$ is the limit of φ_r with respect to C^∞_{loc}-topology.

We fix a point $p \in \Sigma$ and consider the moduli space

$$\mathcal{M} := \left\{ (r,w) \in [0,\infty) \times C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^k) \;\middle|\; \begin{array}{c} w \text{ is a gradient flow line of } \mathcal{A}^{\mathcal{H}}_{\varphi_r F} \text{ with} \\ \lim_{s\to-\infty} w(s) = (p,0),\ \lim_{s\to\infty} w(s) \in \Sigma \times \{0\} \end{array} \right\}.$$

Assume on the contrary that there is **no** leafwise coisotropic intersection point of ϕ_F for $||F|| < \wp(\Sigma)$. For $(r,w) \in \mathcal{M}$ with $w_- = (p,0)$ and $w_+ = (q,0)$ in $\Sigma \times \{0\}$, we estimate

$$\begin{aligned}
E(w) &= -\int_{-\infty}^{\infty} d\mathcal{A}^{\mathcal{H}}_{\varphi_{r(s)}F}(w(s))(\partial_s w) ds \\
&\le \mathcal{A}^{\mathcal{H}}_0(p,0) - \mathcal{A}^{\mathcal{H}}_0(q,0) + \int_{-\infty}^{\infty} ||\partial_s \varphi_r F||_- ds \\
&= \int_{-\infty}^{\infty} ||\varphi_r'(s) F||_- ds \\
&= \int_{-\infty}^{0} \varphi_r'(s)||F||_- ds - \int_0^{\infty} \varphi_r'(s)||F||_+ ds \\
&= \varphi_r(0)\big(||F||_- + ||F||_+\big) \\
&\le ||F||.
\end{aligned}$$

Accordingly we can also estimate,

$$-||F|| \le \mathcal{A}^{\mathcal{H}}_{\varphi_{r_n}F}(w_n(s)) \le ||F||, \qquad (r_n, w_n) \in \mathcal{M}. \tag{3.3.2}$$

Due to the action bound, Theorem 3.2.8 yields that a sequence $\{w_n\}_{n\in\mathbb{N}}$ for $(r_n, w_n) \in \mathcal{M}$ has a convergent subsequence (still denoted w_n) in C^∞_{loc}-topology. We denote by x the limit gradient flow line (which can be a constant gradient flow line). We want to show that $\mathcal{M}$ is compact and so assume by contradiction that $x_+ \notin \Sigma \times \{0\}$ where $x_\pm$ are asymptotic ends of x, i.e. $x_\pm = \lim_{s\to\pm\infty} x(s)$.

Case 1. r_n is bounded.

There is no loss of generality in assuming that $r_n \to r$ as $n \to \infty$. Let $U \in \mathcal{L} \times \mathbb{R}^k$ be an open set containing only the constant critical points of $\mathcal{A}^{\mathcal{H}}_{\varphi_r F}$. Since $x_+ \notin \Sigma \times \{0\}$, we can take for large n, $\sigma_n \in \mathbb{R}$ the last U-entry time of w_n, i.e. $w_n(\sigma_n) \notin U$ and $w_n(s) \in U$ for $s > \sigma_n$. We note that $\sigma_n \to \infty$ as $n \to \infty$ and that the reparametrized sequence $\sigma_n^* w_n$ is a gradient flow line of $\mathcal{A}^{\mathcal{H}}_{\sigma_n^*\varphi_{r_n} F}$ where $\sigma_n^* w_n(\cdot) := w_n(\cdot + \sigma_n)$ and $\sigma_n^*\varphi_{r_n}(\cdot) := \varphi_{r_n}(\cdot + \sigma_n)$. The new sequence $\sigma_n^* w_n$ also has a C^∞_{loc}-convergent subsequence by Theorem 3.2.8 again and we denote by z the limit gradient flow line. Since $r_n \to r$ and $\sigma_n \to \infty$, $\sigma_n^*\varphi_{r_n}$ C^∞_{loc}-converges to the zero function, and thus z is the gradient flow line of $\mathcal{A}^{\mathcal{H}}$. Since $\sigma_n^* w_n \to z$ in C^∞_{loc}-topology, we have

$$E(z) = \int_{-\infty}^{\infty} ||\partial_s z||_m^2 ds = \lim_{T\to\infty} \int_{-T}^{T} ||\partial_s z||_m^2 ds \le \lim_{T\to\infty} \limsup_{n\in\mathbb{N}} E(w_n) = \limsup_{n\in\mathbb{N}} E(w_n).$$

We observe that $z(0) \notin U$ and the positive asymptotic end $z_+ \in \Sigma \times \{0\}$ since $\Sigma \times \{0\}$ is a Morse-Bott component of $\mathrm{Crit}\mathcal{A}^{\mathcal{H}}$ (see [AF1, Lemma 2.12]) and hence z is a non-constant gradient flow line of $\mathcal{A}^{\mathcal{H}}$. Thus the negative asymptotic end z_- is a critical point of $\mathcal{A}^{\mathcal{H}}$; moreover it is not a constant loop since otherwise z is a non-constant gradient flow line with zero energy $E(z) = 0$. But this case is ruled out by the assumption that $||F|| < \wp(\Sigma)$ as well. To be precise, with $z_- = (v, \eta)$, we can derive the following estimate which contradicts the definition of $\wp(\Sigma)$.

$$0 < |\Omega(v)| = |\mathcal{A}^{\mathcal{H}}_0(z_-)| = E(z) \le \limsup_{n\in\mathbb{N}} E(w_n) \le ||F|| < \wp(\Sigma).$$

<u>Case 2</u>. r_n is unbounded.

Without loss of generality, we assume that $r_n \to \infty$ as $n \to \infty$. The limit of $\{w_n\}_{n\in\mathbb{N}}$ is a gradient flow line of $\mathcal{A}^{\mathcal{H}}_F$ since $\beta_\infty \equiv 1$. Then the asymptotic ends of the limit are critical points of $\mathcal{A}^{\mathcal{H}}_F$ which give rise to a leafwise coisotropic intersection point of ϕ_F. It contradicts our assumption and Case 2 is ruled out.

With σ_n the first U-exit time of w_n, the case $x_- \notin \Sigma \times \{0\}$ is analogous. If $x_- = (q, 0) \in \Sigma$ with $q \neq p$, as Case 1, there exists a gradient flow line of $\mathcal{A}^{\mathcal{H}}$ with asymptotic ends $(q, 0)$ and $(p, 0)$. But this cannot occur. Therefore we conclude that the moduli space $\mathcal{M}$ is compact.

Next, we regard the moduli space $\mathcal{M}$ as the zero set of a Fredholm section with index 1 of a Banach bundle over a Banach manifold as in (5.1.1). Moreover, the Fredholm section is already transversal at the $(0, p, 0)$ since Σ is a Morse-Bott component by [AF1, Lemma 2.12]. Therefore we can perturb the Fredholm section away from $(0, p, 0)$ (even if varying J, $(0, p, 0)$ still solves the gradient flow equation) to obtain a transverse Fredholm section whose zero set is a compact one-dimensional smooth manifold with boundary $(0, p, 0)$. But there is no one-dimensional manifold with a single boundary point. This finishes the proof of Claim 1. $\square$

Step 2. End of the proof of Theorem A.

Proof of Step 2. In Step 2, our restricted contact coisotropic submanifold Σ is not necessarily of the form $\Sigma = \mathcal{G}^{-1}(0)$. Recall that on the open neighborhood $U_{\delta_0} \cong \{(q, p_1, \ldots, p_k) \in \Sigma \times D_r^k\}$ of Σ, $\omega|_{U_{\delta_0}} = \omega|_\Sigma + \sum_{i=1}^k d(p_i \alpha_i)$ and $X_{p_i} = R_i$ for all $i = 1, \ldots, k$.

We consider a family of Hamiltonian tuples $\mathcal{H}_\nu(t, x) = \chi(t)\mathcal{G}_\nu(x)$, $\nu \in \mathbb{N}$ where $\mathcal{H}_\nu = (H_{1,\nu}, \ldots, H_{k,\nu})$ and $\mathcal{G}_\nu = (G_{1,\nu}, \ldots, G_{k,\nu})$ such that

1. $0 < \epsilon_\nu < \delta$ converges to zero as ν goes to infinity,

2. $G_{i,\nu}|_{U_{\delta_0}} = g_i(p_i)$ for some $g_i \in C^\infty(\mathbb{R})$,

3. for $(x, \mathfrak{p}) \in \Sigma \times (-\delta_0, \delta_0)^k \cong U_{\delta_0}$,

$$G_{i,\nu}|_{U_{2\epsilon_\nu} - U_{\epsilon_\nu/2}}(x, \mathfrak{p}) = \begin{cases} p_i - \epsilon_\nu & \text{if } p_i > 0 \\ -p_i - \epsilon_\nu & \text{if } p_i < 0, \end{cases} \tag{3.3.3}$$

4. $G_{i,\nu}|_{M - U_{\delta_0}} = constant$,

5. $\mathcal{G}_\nu^{-1}(0) = \bigcup_{2^k} \Sigma \times (\pm\epsilon_\nu, \dots, \pm\epsilon_\nu)$.

We note that

$$X_{G_{i,\nu}}|_{\Sigma\times(\pm\epsilon_\nu,\dots,+\epsilon_\nu,\dots,\pm\epsilon_\nu)} = +X_{p_i}, \quad X_{G_{i,\nu}}|_{\Sigma\times(\pm\epsilon_\nu,\dots,-\epsilon_\nu,\dots,\pm\epsilon_\nu)} = -X_{p_i}.$$

By construction, $\mathcal{H}_\nu$ Poisson-commutes and Step 1 guarantees the existence of critical points (v_ν, η_ν) lying on $\mathcal{G}_\nu^{-1}(0)$ for sufficiently large ν because $||F|| < \wp(\Sigma \times \{(\pm\epsilon_\nu, \dots, \pm\epsilon_\nu)\})$ for large $\nu \in \mathbb{N}$. For $(v_\nu, \eta_\nu) \in \mathrm{Crit}\mathcal{A}_F^{\mathcal{H}_\nu}$, v_ν lies on one of the components of $\mathcal{G}_\nu^{-1}(0)$, say $v_\nu \subset \Sigma \times (\epsilon_\nu, \dots, \epsilon_\nu)$. According to Proposition 3.2.2, it holds that

$$\phi_F^1\big(v_\nu(1/2)\big) = v_\nu(0) = \phi_{H_{1,\nu}}^{-\eta_{1,\nu}} \circ \dots \circ \phi_{H_{k,\nu}}^{-\eta_{k,\nu}}\big(v_\nu(1/2)\big).$$

Then the estimate (3.3.1) in Step 1 implies the following lemma.

Lemma 3.3.1. For $(v_\nu, \eta_\nu) \in \mathrm{Crit}\mathcal{A}_F^{\mathcal{H}_\nu}$, $\eta_{1,\nu}, \dots, \eta_{k,\nu}$ are uniformly bounded in terms of $\lambda_1, \dots, \lambda_k$ and F.

PROOF. We estimate as in (3.3.1): For all $i \in \{1, \ldots, k\}$,

$$\begin{aligned}
||F|| &\geq \left|\mathcal{A}_F^{\mathcal{H}_\nu}(v_\nu, \eta_\nu)\right| \\
&= \Big| \int_0^1 v^*\lambda_i + \int_0^1 \langle \eta, \mathcal{H}_\nu \rangle (t, v_\nu(t))dt + \int_0^1 F(t, v_\nu(t))dt \Big| \\
&= \Big| \int_0^1 \lambda_i(v_\nu)\big(\sum_{j=1}^k \eta_{j,\nu} X_{H_{j,\nu}}(v_\nu) + X_F(t, v_\nu)\big)dt + \int_0^1 F(t, v_\nu(t))dt \Big| \\
&= \frac{3}{4}|\eta_{i,\nu}| - \frac{1}{4(k-1)} \sum_{j \neq i}^k |\eta_{j,\nu}| - \Big| \int_0^1 \lambda_i(v_\nu)\big(X_F(t, v_\nu)\big) + \int_0^1 F(t, v_\nu(t))dt \Big|.
\end{aligned}$$

Therefore we conclude

$$\frac{1}{2}\sum_{i=1}^k |\eta_{i,\nu}| \leq k\big(||F|| + \max_{1 \leq i \leq k} ||\lambda_{i|U_{\delta_0/2}}||_{L^\infty} ||X_F||_{L^\infty} + ||F||_{L^\infty}\big).$$

□

The two sequences of points $\{v_\nu(0)\}_{\nu \in \mathbb{N}}$ and $\{v_\nu(1/2)\}_{\nu \in \mathbb{N}}$ converge up to taking a subsequence (still denoted by $v_\nu(0)$ and $v_\nu(1/2)$) and we denote by

$$x_0 := \lim_{\nu \to \infty} v_\nu(0), \quad x_{1/2} := \lim_{\nu \to \infty} v_\nu\big(1/2\big).$$

Obviously x_0 and $x_{1/2}$ are points in Σ. Moreover we know that

$$x_0 = \lim_{\nu \to \infty} v_\nu(0) = \lim_{\nu \to \infty} \phi_F^1(v_\nu(1/2)) = \phi_F^1(\lim_{\nu \to \infty} v_\nu(1/2)) = \phi_F^1(x_{1/2}). \quad (3.3.4)$$

Furthermore, due to Lemma 3.3.1, the limit $\{\eta_{i,\nu}\}_{\nu \in \mathbb{N}}$ exists for all i, say

$$\mathfrak{n}_i := \lim_{\nu \to \infty} \eta_{i,\nu}.$$

Thus we conclude that x_0 and $x_{1/2}$ lie on the same leaf:

$$x_0 = \lim_{\nu\to\infty} v_\nu(0) = \lim_{\nu\to\infty} \phi_{H_{1,\nu}}^{-\eta_{1,\nu}} \circ \cdots \circ \phi_{H_{k,\nu}}^{-\eta_{k,\nu}}(v_\nu(1/2)) = \phi_{H_1}^{-\mathfrak{n}_1} \circ \cdots \circ \phi_{H_k}^{-\mathfrak{n}_k}(x_{1/2}). \tag{3.3.5}$$

It directly follows

$$\phi_{H_1}^{-\mathfrak{n}_1} \circ \cdots \circ \phi_{H_k}^{-\mathfrak{n}_k}(x_{1/2}) = \phi_F^1(x_{1/2})$$

from (3.3.4) together with (3.3.5). This completes the proof of Theorem A. $\square$

Chapter 4

The existence of a periodic orbit and the leafwise displacement energy

In this chapter, we study the existence of a periodic orbit, i.e. a solution of (1.2.2), together with a relation between its symplectic area and the leafwise displacement energy in the stable case. This proves Theorem D which were proved by Kai Cieliebak, Urs Frauenfelder, and Gabriel Paternain [CFP] for separating stable hypersurfaces. Adapting their idea, we can extend (and slightly improve) their result to stable coisotropic submanifolds.

Let Σ be a closed stable coisotropic submanifold in a symplectically aspherical symplectic manifold (M, ω) which is geometrically bounded. As in Theorem A we first assume that $\Sigma = \mathcal{G}^{-1}(0)$ for some Poisson commuting Hamiltonian tuple $\mathcal{G} \in C^\infty(M, \mathbb{R}^k)$, but this additional assumption will be removed in the second step. Suppose that Σ is **displaced by** $F \in C_c^\infty(S^1 \times M)$, i.e. $\phi_F(\Sigma) \cap \Sigma = \emptyset$. We consider again the smooth family of functions $\varphi_r \in C^\infty(\mathbb{R}, [0, 1])$ defined in the proof of Theorem A. As before, we fix a point $p \in \Sigma$ and consider the moduli space $\mathcal{M}$ defined by

$$\mathcal{M} = \left\{ (r,w) \in [0,\infty) \times C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^k) \;\middle|\; \begin{array}{c} w \text{ is a gradient flow line of } \mathcal{A}^{\mathcal{H}}_{\varphi_r F} \text{ with} \\ \lim\limits_{s\to-\infty} w(s) = (p,0), \ \lim\limits_{s\to\infty} w(s) \in \Sigma \times \{0\} \end{array} \right\}.$$

Theorem 4.0.2. *For $(r,w) \in \mathcal{M}$ where $w = (u,\tau)$, τ and r are uniformly bounded.*

In the previous sections we showed how Rabinowitz Floer theory for hypersurfaces can be generalized to our set-up. Since the proof of Theorem 4.0.2 needs several technical lemmas and auxiliary action functionals as in the contact case [Ka2],we refer the reader to [CFP, Section 4.3] or [Ka3] instead of giving a proof .

4.1 Proof of Theorem D

Step 1. We know that a sequence $\{(r_n, w_n)\}_{n\in\mathbb{N}}$ in $\mathcal{M}$ has a C^∞_{loc}-convergent subsequence due to Theorem 4.0.2 together with the argument in the proof of Theorem 3.2.8. We denote by (r,w) the limit which is a gradient flow line of $\mathcal{A}^{\mathcal{H}}_{\varphi_r F}$. Again by compactness, w asymptotically converges to $w_\pm = (v_\pm, \eta_\pm) \in \mathrm{Crit}\mathcal{A}^{\mathcal{H}}$ since $\varphi_r(\pm\infty) = 0$. If $(r,w) \in \mathcal{M}$, the moduli space $\mathcal{M}$ is a one dimensional compact manifold with a single boundary point $\{(0,p,0)\}$ (after perturbing a Fredholm section as in the proof of Theorem A). However such a manifold does not exist and therefore one of the asymptotic ends $w_\pm$ of w is a nontrivial solution of (1.2.2). For simplicity, let us assume $w_+ \notin \Sigma \times \{0\}$. Following the notation from the proof of Theorem A, we consider $\sigma_n \in \mathbb{R}$ the last U-entry time. Then $\sigma_n^* w_n$ is a gradient flow line of $\mathcal{A}^{\mathcal{H}}_{\sigma_n^*\varphi_{r_n}F}$ and C^∞_{loc}-converges to a non-constant gradient flow line z of $\mathcal{A}^{\mathcal{H}}$ with $z(0) \notin$

U and $z_+ \in \Sigma\times\{0\}$.[1] By compactness and the energy estimate, $z_- = (v,\eta) \in \mathrm{Crit}\mathcal{A}^{\mathcal{H}}$ and z_- is a nontrivial solution of (1.2.2). Moreover, by (3.3.2), we have

$$-||F|| \leq \mathcal{A}^{\mathcal{H}}_{\sigma_n^*\varphi_{r_n}F}(\sigma_n^* w_n(s)) \leq ||F||, \quad \forall s \in \mathbb{R}.$$

As n goes to infinity, it holds that

$$-\,||F|| \leq \Omega(v) = \mathcal{A}^{\mathcal{H}}(z_-) \leq ||F|| \tag{4.1.1}$$

for every Hamiltonian function $F \in C_c^\infty(S^1\times M)$ displacing Σ. Since $\mathcal{A}^{\mathcal{H}}(z_+) = 0$ and the action value of $\mathcal{A}^{\mathcal{H}}$ decreases along z,

$$\big|\Omega(v)\big| = \big|\mathcal{A}^{\mathcal{H}}(z_-)\big| > 0. \tag{4.1.2}$$

(4.1.1) and (4.1.2) prove Theorem E provided that Σ is a level set of some Poisson-commuting Hamiltonian tuple.

Step 2. Now we consider the situation that Σ is not necessarily a level set of some Poisson-commuting Hamiltonian tuple. We choose a family of Hamiltonian tuples $\mathcal{H}_\nu(t,x) = \chi(t)\mathcal{G}_\nu(x)$, $\nu \in \mathbb{N}$ where $\mathcal{H}_\nu = (H_{1,\nu},\dots,H_{k,\nu})$ and $\mathcal{G}_\nu = (G_{1,\nu},\dots,G_{k,\nu})$ such that

1. $0 < \epsilon_\nu < \min\{1/4k, \delta_0/2, \delta_1\}$ converges to zero as ν goes to infinity,
2. $G_{i,\nu}|_{U_{\delta_0}} = g_i(p_i)$ for some $g_i \in C^\infty(\mathbb{R})$,
3. for $(x,\mathfrak{p}) \in \Sigma \times (-\delta_0,\delta_0)^k \cong U_{\delta_0}$,

$$G_{i,\nu}|_{U_{2\epsilon_\nu}-U_{\epsilon_\nu/2}}(x,\mathfrak{p}) = \begin{cases} p_i - \epsilon_\nu & \text{if } p_i > 0 \\ -p_i - \epsilon_\nu & \text{if } p_i < 0, \end{cases}$$

[1] Honestly speaking, we did not prove C^∞_{loc}-convergence of $(r_n, \sigma_n^* w_n)$; but it follows from the proof of Theorem 4.0.2.

4. $G_{i,\nu}|_{M-U_{\delta_0}} = constant$,
5. $\mathcal{G}_\nu^{-1}(0) = \bigcup_{2^k} \Sigma \times (\pm\epsilon_\nu, \dots, \pm\epsilon_\nu)$.

With this defining Hamiltonian tuple $\mathcal{H}_\nu$, the argument in Step 1 still works and thus there exists $v_\epsilon \in \mathcal{G}_\nu^{-1}(0)$ a solution of (1.2.2) satisfying $0 < \Omega(v_\epsilon) \leq e(\mathcal{G}_\nu^{-1}(0))$. Since $\mathcal{G}_\nu^{-1}(0)$ is disconnected, v_ϵ lies in one of its connected components, say $v_\epsilon \subset \Sigma_\epsilon$. Since there is a diffeomorphism ψ_ϵ between Σ_ϵ and Σ, $\psi_\epsilon(v_\epsilon)$ is a loop solving (1.2.2), contractible in M with $\Omega(\psi_\epsilon(v_\epsilon)) = \Omega(v_\epsilon) > 0$. Moreover if we have chosen sufficiently large ν, $e(\Sigma) = e(\mathcal{G}_\nu^{-1}(0))$. For simplicity, let us assume that $e(\Sigma) + \varepsilon < e(\mathcal{G}_\nu^{-1}(0))$ for some small $\varepsilon > 0$ and for all $\nu \in \mathbb{N}$; it means that there is $F \in C_c^\infty(S^1 \times M)$ such that $||F|| \in (e(\Sigma), e(\Sigma) + \varepsilon)$ such that $\phi_F(\Sigma) \cap \Sigma = \emptyset$; but if ν is big enough, ϕ_F also displaces $\mathcal{G}_\nu^{-1}(0)$ and it contradicts $||F|| < e(\mathcal{G}_\nu^{-1}(0))$. Hence, we have proved that

$$0 < \Omega(\psi_\epsilon(v_\epsilon)) = \Omega(v_\epsilon) \leq e(\mathcal{G}_\nu^{-1}(0)) = e(\Sigma).$$

□

Remark 4.1.1. If one succeeds in proving compactness of gradient flow lines of the perturbed Rabinowitz action functional in the stable case, Theorem D is an immediate consequence of the invariance property of Rabinowitz Floer homology.

Chapter 5

Rabinowitz Floer homology

In the hypersurface case, [CFP, AF1] proved that the (perturbed) Rabinowitz action functional is generically Morse-Bott (Morse). Their argument undeniably continues to hold in our set-up. That is, $\mathcal{A}^{\mathcal{G}}$ is Morse-Bott and $\mathcal{A}^{\mathcal{H}}_F$ is Morse for a generic perturbation $F \in C^\infty_c(S^1 \times M)$. Furthermore, we know that gradient flow lines of the Rabinowitz action functional are compact modulo breaking (see (F1) and (F2) below) for restricted contact coisotropic submanifolds due to Theorem 3.2.8. Therefore we can define Floer homologies of $\mathcal{A}^{\mathcal{G}}$ and $\mathcal{A}^{\mathcal{H}}_F$ as usual.[1] As one expects, these two Floer homologies are isomorphic by the standard continuation method in Floer theory. Here we only treat the restricted contact case and refer to Remark 2.5.1 for other cases. As before, (M, ω) is an exact symplectic manifold being geometrically bounded with a family of ω-compatible almost complex structures $J = J(s,t)$.

[1] $\mathcal{A}^{\mathcal{G}}$ is never Morse since there is a S^1-symmetry coming from time-shift on the critical points set. However $\mathcal{A}^{\mathcal{G}}$ is Morse-Bott for a generic coisotropic submanifold, thus we can define Morse-Bott homology of $\mathcal{A}^{\mathcal{G}}$ by counting gradient flow lines with cascades, see [Fr]. Since Rabinowitz Floer homology is invariant under homotopies there is no loss of generality in assuming $\mathcal{A}^{\mathcal{H}}$ is Morse-Bott, see [CFP].

5.1 Boundary Operator

We can assign some index to critical points of $\mathcal{A}_F^{\mathcal{H}}$, namely the transverse Conley-Zehnder index.[2] But we omit the definition, referring the reader to [BO2, CF, MP]. We denote the index by

$$\mu : \mathrm{Crit}\mathcal{A}_F^{\mathcal{H}} \longrightarrow \mathbb{Z}.$$

Here we assumed that the first Chern class c_1 vanishes over $\pi_2(M)$ for simplicity; otherwise the index μ is well defined modulo $2N$ where N is the minimal Chern number of (M,ω).

Let $\mathcal{M}_J(w_-,w_+)$ be the moduli space of gradient flow lines of $\mathcal{A}_F^{\mathcal{H}}$ with asymptotic ends $w_\pm \in \mathrm{Crit}\mathcal{A}_F^{\mathcal{H}}$.

$$\mathcal{M}_J(w_-,w_+) := \left\{ (u,\tau) \in C^\infty(\mathbb{R}\times S^1, M) \times C^\infty(\mathbb{R},\mathbb{R}^k) \;\middle|\; \begin{array}{r} (u,\tau) \text{ solves (3.2.3)}, \\ \lim\limits_{s\to\pm\infty}(u,\tau) = w_\pm \end{array} \right\}.$$

In order to show that $\mathcal{M}_J(w_-,w_+)$ is a finite dimensional smooth manifold, we interpret it as the zero set of a Fredholm section of a Banach bundle over a Banach space. Let $\mathcal{P}(w_-,w_+)$ be the Banach manifold given by

$$\mathcal{P}(w_-,w_+) := \big\{ (u,\tau) \in W^{1,2}(\mathbb{R}\times S^1, M) \times W^{1,2}(\mathbb{R},\mathbb{R}^k) \;\big|\; \lim_{s\to\pm\infty}(u,\tau) = w_\pm \big\}$$

and $\mathcal{E}$ be the Banach bundle over $\mathcal{P}(w_-,w_+)$ whose fibre at $(u,\tau)\in\mathcal{P}(w_-,w_+)$ is

$$\mathcal{E}_{(u,\tau)} := L^2(\mathbb{R}\times S^1, u^*TM \times \tau^*T\mathbb{R}^k).$$

Then the moduli space $\mathcal{M}(w_-,w_+)$ is the zero set of the section

$$s_J : \mathcal{P}(w_-,w_+) \longrightarrow \mathcal{E}, \quad s_J(u,\tau) = \big(\bar\partial_{\mathcal{H},F,J}(u), \bar\partial_1(\tau_1), \cdots, \bar\partial_k(\tau_k)\big) \qquad (5.1.1)$$

[2] We can define Floer homology of $\mathcal{A}_F^{\mathcal{H}}$ without this index.

defined by

$$\left.\begin{aligned} \bar\partial_{\mathcal{H},F,J}(u) &= \partial_s u + J(s,t,u)\Big(\partial_t u - \sum_{i=1}^k \eta_i X_{H_i}(t,u) - X_{F_s}(t,u)\Big) \\ \bar\partial_i(\tau_i) &= \partial_s \tau_i - \int_0^1 H_i(t,u)dt, \qquad 1 \le i \le k \end{aligned}\right\}$$

where $\tau = (\tau_1, \dots, \tau_k)$. It turns out that this section is Fredholm. Then we regard the moduli space as the zero set of this section, $\mathcal{M}_J(w_-, w_+) = s_J^{-1}(0)$. Let

$$Ds_J(u,\tau) : T_{(u,\tau)}\mathcal{P}(w_-, w_+) \longrightarrow \mathcal{E}_{(u,\tau)}$$

be the vertical differential of s_J at (u,τ). It is known that $Ds_J(u,\tau)$ is surjective for a generic ω-compatible almost complex structure J and for any $(u,\tau) \in s_J^{-1}(0)$, see [FHS, Section 5] and [BO1]. This transversality issues (surjectivity of $Ds_J(u,\tau)$) can now also be settled using the framework of polyfolds developed by Hofer-Wysocki-Zehnder [HWZ1, HWZ2, HWZ3]. Thus we perturb the section s_J (varying J slightly) so that $Ds_J(u,\tau)$ is surjective and the implicit function theorem yields that $s_J^{-1}(0) = \mathcal{M}_J(w_-, w_+)$ is a smooth finite dimensional manifold. Moreover the dimension of the moduli space $\mathcal{M}_J(w_-, w_+)$ coincides with the dimension of the kernel of Ds_J which in turn is the same as the Fredholm index of s_J since it is surjective; besides, the Fredholm index of s_J can be computed in terms of the indices of $\mu(w_-)$ and $\mu(w_+)$ using the spectral flow [RS, BO2, CF]. In conclusion, we have the identity

$$\dim \mathcal{M}_J(w_-, w_+) = \mu(w_-) - \mu(w_+), \quad w_\pm \in \mathrm{Crit}\mathcal{A}_F^{\mathcal{H}}.$$

We suppress the subindex J in $\mathcal{M}_J(w_-, w_+)$ for notational convenience. We divide out the $\mathbb{R}$-action on $\mathcal{M}(w_-, w_+)$ defined by shifting the gradient flow lines in the s-variable. Then we obtain the moduli space of unparametrized gradient flow lines which we denote by

$$\widehat{\mathcal{M}}(w_-, w_+) := \mathcal{M}(w_-, w_+)/\mathbb{R}.$$

For the compactification of the moduli space $\mathcal{M}(w_-, w_+)$, we recall the **Floer-Gromov convergence**. A sequence $\{(u^\nu, \tau^\nu)\}_{\nu\in\mathbb{N}}$ in $\mathcal{M}(w_-, w_+)$ is said to Floer-Gromov converge to a broken gradient flow lines $\{(u_j, \tau_j)\}_{j=1}^m$ where $z_0, \dots, z_m \in \mathrm{Crit}\mathcal{A}^{\mathcal{H}}_{F_s}$ with $z_0 = w_-$ and $z_m = w_+$, and

$$(u_j, \tau_j) \in \mathcal{M}(z_{j-1}, z_j), \quad j \in \{1, \dots, m\}$$

if there exist $\sigma_j^\nu \in \mathbb{R}$ such that reparametrized sequences $(u^\nu, \tau^\nu)(\sigma_j^\nu + \cdot)$ converge to (u_j, τ_j) for all $j \in \{1, \dots, m\}$ in the C^∞_{loc}-topology. The following statements are the key ingredients for boundary operators of various Floer homologies, including Rabinowitz Floer homology.

(F1) The moduli space $\mathcal{M}(w_-, w_+)$ is a one dimensional compact smooth manifold with respect to the topology of Floer-Gromov convergence when $\mu(w_-) - \mu(w_+) = 1$.[3] Accordingly, $\widehat{\mathcal{M}}(w_-, w_+)$ is a finite set.

(F2) Let $\widehat{\mathcal{M}}_c(w_-, w_+)$ be the compactification of $\widehat{\mathcal{M}}(w_-, w_+)$ with respect to the topology of Floer-Gromov convergence. If $\mu(w_-) - \mu(w_+) = 2$, $\widehat{\mathcal{M}}_c(w_-, w_+)$ is a compact one-dimensional manifold whose boundary is

$$\partial\widehat{\mathcal{M}}_c(w_-, w_+) = \bigcup_z \widehat{\mathcal{M}}(w_-, z) \times \widehat{\mathcal{M}}(z, w_+) \tag{5.1.2}$$

where the union runs over $z \in \mathrm{Crit}\mathcal{A}^{\mathcal{H}}_F$ with $\mu(w_-) - \mu(z) = 1$.

(F1) follows from the elliptic bootstrapping argument as discussed in Theorem 3.2.8, see also Floer's beautiful paper [Fl2]. (F2) is proved by Floer's gluing theorem [Fl1].

[3] Without help of the Conley-Zehnder index, we can rephrase that the one dimensional component of $\mathcal{M}(w_-, w_+)$ is a compact smooth manifold.

We denote by $\mathrm{Crit}_q\mathcal{A}_F^{\mathcal{H}}$ the set of critical point of $\mathcal{A}_F^{\mathcal{H}}$ of index $q \in \mathbb{Z}$, i.e. $\mu((v,\eta)) = q$ for $(v,\eta) \in \mathrm{Crit}_q\mathcal{A}_F^{\mathcal{H}}$. We define a $\mathbb{Z}/2$-vector space

$$\mathrm{CF}_q(\mathcal{A}_F^{\mathcal{H}}) := \Big\{\xi = \sum_{(v,\eta)\in\mathrm{Crit}_q\mathcal{A}_F^{\mathcal{H}}} \xi_{(v,\eta)}(v,\eta) \,\Big|\, \xi_{(v,\eta)} \in \mathbb{Z}/2\Big\}$$

where $\xi_{(v,\eta)}$ satisfies the finiteness condition:

$$\#\big\{(v,\eta) \in \mathrm{Crit}_q\mathcal{A}_F^{\mathcal{H}} \,\big|\, \xi_{(v,\eta)} \neq 0,\ \mathcal{A}_F^{\mathcal{H}}(v,\eta) \geq \kappa\big\} < \infty, \quad \forall \kappa \in \mathbb{R}.$$

We denote by $n(w_-, w_+)$ be the parity of elements of the finite set $\widehat{\mathcal{M}}(w_-, w_+)$ when $\mu(w_-) - \mu(w_+) = 1$, see (F1) above. Then the boundary operators $\{\partial_q\}_{\{q\in\mathbb{Z}\}}$ are defined by

$$\begin{aligned} \partial_q : \mathrm{CF}_q(\mathcal{A}_F^{\mathcal{H}}) &\longrightarrow \mathrm{CF}_{q-1}(\mathcal{A}_F^{\mathcal{H}}) \\ w_- \in \mathrm{Crit}_q\mathcal{A}_F^{\mathcal{H}} &\longmapsto \sum_{w_+\in\mathrm{Crit}_{q-1}\mathcal{A}_F^{\mathcal{H}}} n(w_-, w_+) \cdot w_+. \end{aligned}$$

Due to (F2), we know $\partial_{q-1} \circ \partial_q = 0$ (in $\mathbb{Z}/2$) so that $(\mathrm{CF}_\bullet(\mathcal{A}_F^{\mathcal{H}}), \partial_\bullet)$ is a chain complex indeed. We define **Rabinowitz Floer homology** by

$$\mathrm{HF}_q(\mathcal{A}_F^{\mathcal{H}}) := \mathrm{H}_q(\mathrm{CF}_\bullet(\mathcal{A}_F^{\mathcal{H}}), \partial_\bullet), \quad \mathrm{RFH}_q(\Sigma, M) := \mathrm{HF}_q(\mathcal{A}^{\mathcal{G}}).$$

To be exact, since $\mathcal{A}^{\mathcal{G}}$ is Morse-Bott, $\mathrm{HF}(\mathcal{A}^{\mathcal{G}})$ is defined by Frauenfelder's Morse-Bott homology [Fr, Appendix A]. We note that $\mathrm{Crit}\mathcal{A}^{\mathcal{G}}$ consists of Σ and circles. We pick a Morse function f on $\mathrm{Crit}\mathcal{A}^{\mathcal{G}}$ and then the boundary operator for $\mathrm{HF}(\mathcal{A}^{\mathcal{G}})$ is defined by counting gradient flow lines of $\mathcal{A}^{\mathcal{G}}$ (called cascades) together with gradient flow lines of f. Note that if there is no nonconstant solution of (1.2.2), $\mathrm{Crit}\mathcal{A}^{\mathcal{G}} \cong \Sigma$ and thus there are no cascades since the energy of each cascade is positive. Thus if this is the case, $\mathrm{HF}(\mathcal{A}^{\mathcal{G}}) \cong \mathrm{H}(\Sigma; \mathbb{Z}/2)$.

5.2 Continuation Homomorphism

Given any two Hamiltonian functions F and K in $C_c^\infty(S^1 \times M)$, we consider the homotopies $D_s^\pm \in C^\infty(S^1 \times M)$, $s \in \mathbb{R}$,

$$D_s^+(t,x) := K(t,x) + \varphi_+(s)\big(F(t,x) - K(t,x)\big)$$

and

$$D_s^-(t,x) := K(t,x) + \varphi_-(s)\big(F(t,x) - K(t,x)\big)$$

where $\varphi_\pm \in C^\infty(\mathbb{R}, [0,1])$ are cut-off functions defined by

$$\varphi_+(s) = \begin{cases} 0 & s \leq -1 \\ 1 & s \geq 1 \end{cases} \qquad \varphi_-(s) = \begin{cases} 1 & s \leq -1 \\ 0 & s \geq 1. \end{cases}$$

We consider the time-dependent version of the gradient flow equation:

$$\left.\begin{aligned} &\partial_s u + J_s(t,u)\Big(\partial_t u - \sum_{i=1}^k \tau_i X_{H_i}(t,u) - X_{D_s^+}(t,u)\Big) = 0 \\ &\partial_s \tau_i - \int_0^1 H_i(t,u)dt = 0, \qquad 1 \leq i \leq k. \end{aligned}\right\} \tag{5.2.1}$$

The solutions of (5.2.1) with an asymptotic condition form the following moduli space:

$$\mathcal{M}(w_K, w_F) := \left\{ w \in C^\infty(\mathbb{R} \times S^1, M) \times C^\infty(\mathbb{R}, \mathbb{R}^k) \;\middle|\; \begin{array}{l} w = (u,\tau) \text{ solves (5.2.1) with} \\ \lim\limits_{s \to \pm\infty} w(s) = w_{F/K} \in \mathrm{Crit}\mathcal{A}^{\mathcal{H}}_{F/K} \end{array} \right\}.$$

As we discussed in the previous subsection, it is also a well-known fact in Floer theory that the moduli space $\mathcal{M}(w_K, w_F)$ is a smooth manifold of dimension $\mu(w_K) - \mu(w_F)$ for a generic homotopy. In particular, it is known that $\mathcal{M}(w_K, w_F)$ is a finite set when w_K and w_F have the same index and thus we denote the parity of $\mathcal{M}(w_K, w_F)$ by $n(w_K, w_F)$ if this is the case. Then we define the continuation homomorphism as follows.

$$\begin{aligned} \Phi_K^F : \mathrm{CF}_q(\mathcal{A}_K^{\mathcal{H}}) &\longrightarrow \mathrm{CF}_q(\mathcal{A}_F^{\mathcal{H}}) \\ w_K \in \mathrm{Crit}_q \mathcal{A}_K^{\mathcal{H}} &\longmapsto \sum_{w_F \in \mathrm{Crit}_q \mathcal{A}_F^{\mathcal{H}}} n(w_K, w_F) \cdot w_F. \end{aligned}$$

In the same way, we also define

$$\Phi_F^K : \mathrm{CF}_q(\mathcal{A}_F^{\mathcal{H}}) \longrightarrow \mathrm{CF}_q(\mathcal{A}_K^{\mathcal{H}})$$

using the other homotopy D_s^-. Then we obtain the invariance property of Rabinowitz Floer homology via the continuation homomorphisms using a homotopy of homotopies $D_s^r(t,x) := K(t,s) + \varphi_r(s)(F(t,x) - K(t,x))$ where $\varphi_r : \mathbb{R} \to [0,1]$, $r \in \mathbb{R}$ and $\varphi_r = \varphi_\pm$ if $\pm r \geq 1$, see [Sa, Section 3.4] [4]:

Theorem 5.2.1. *Rabinowitz Floer homology is independent of the choice of perturbations up to canonical isomorphism. In particular, it holds that*

$$\mathrm{RFH}(\Sigma, M) \cong \mathrm{HF}(\mathcal{A}_F^{\mathcal{H}}), \quad F \in C_c^\infty(S^1 \times M).$$

For the later purpose, we compare the action values of $\mathcal{A}_K^{\mathcal{H}}$ and $\mathcal{A}_F^{\mathcal{H}}$:

Proposition 5.2.2. *If the moduli space $\mathcal{M}(w_K, w_F)$ is not empty,*

$$\mathcal{A}_F^{\mathcal{H}}(w_F) \leq \mathcal{A}_K^{\mathcal{H}}(w_K) + ||F - K||_-.$$

[4] Here we again make use of Floer-Gromov compactness and Floer's gluing theorem.

Proof. We pick $w \in \mathcal{M}(w_K, w_F)$ and estimate its energy:

$$
\begin{aligned}
0 &\leq E(w) \\
&= -\int_{-\infty}^{\infty} d\mathcal{A}^{\mathcal{H}}_{D_s^+}(w(s))[\partial_s w]ds \\
&= -\int_{-\infty}^{\infty} \frac{d}{ds}\big(\mathcal{A}^{\mathcal{H}}_{D_s^+}(w(s))\big)ds - \int_{-\infty}^{\infty}\int_0^1 {\varphi_+}'(s)\big(F(t,w(s)) - K(t,w(s))\big)dtds \\
&\leq \mathcal{A}^{\mathcal{H}}_{D^+_{-\infty}}(w_K) - \mathcal{A}^{\mathcal{H}}_{D^+_{\infty}}(w_F) - \int_{-\infty}^{\infty} {\varphi_+}'(s)\int_0^1 \big(F(t,w(s)) - K(t,w(s))\big)dtds \\
&\leq \mathcal{A}^{\mathcal{H}}_K(w_K) - \mathcal{A}^{\mathcal{H}}_F(w_F) + ||F-K||_-.
\end{aligned}
$$

□

5.3 Proof of Theorem E

Suppose that there are no leafwise coisotropic intersection points for some $\phi_F \in \mathrm{Ham}_c(M,\omega)$. Then the set $\mathrm{Crit}\mathcal{A}^{\mathcal{H}}_F$ is empty since otherwise a critical point of $\mathcal{A}^{\mathcal{H}}_F$ gives rise to a leafwise coisotropic intersection point. Thus $\mathrm{HF}(\mathcal{A}^{\mathcal{H}}_F) = 0$ and Theorem 5.2.1 proves (i).

If there are only constant solutions of (1.2.2), no cascades appear in the boundary operator of Morse-Bott homology. Thus the Rabinowitz Floer homology of (Σ, M) is isomorphic to the Morse homology of Σ and hence to the singular homology of Σ. This proves (iii).

Suppose there are only constant solutions of (1.2.2). Due to (iii), we know that the Rabinowitz Floer homology of (Σ, M) is isomorphic to the singular homology of Σ. While the singular homology of Σ never vanishes, the Rabinowitz Floer homology of (Σ, M) vanishes by (i) since Σ is displaceable. This contradiction proves (ii). □

5.4 Filtered Rabinowitz Floer Homology

For $a < b \in \mathbb{R}$ which are not critical values of $\mathcal{A}_F^{\mathcal{H}}$, we define the $\mathbb{Z}/2$-vector space

$$\mathrm{CF}_q^{(a,b)}(\mathcal{A}_F^{\mathcal{H}}) := \mathrm{Crit}_q^{(a,b)}(\mathcal{A}_F^{\mathcal{H}}) \otimes \mathbb{Z}/2$$

where

$$\mathrm{Crit}_q^{(a,b)}(\mathcal{A}_F^{\mathcal{H}}) := \big\{(v,\eta) \in \mathrm{Crit}_q \mathcal{A}_F^{\mathcal{H}} \,\big|\, \mathcal{A}_F^{\mathcal{H}}(v,\eta) \in (a,b)\big\}.$$

Then $\big(\mathrm{CF}_*^{(-\infty,b)}(\mathcal{A}_F^{\mathcal{H}}), \partial_*^b\big)$ is a sub-complex of $\big(\mathrm{CF}_*(\mathcal{A}_F^{\mathcal{H}}), \partial_*\big)$ since (negative) gradient flow lines of $\mathcal{A}_F^{\mathcal{H}}$ flow downhill. Here $\partial_*^b := \partial_*|_{\mathrm{CF}_*^{(-\infty,b)}}$. There are canonical homomorphisms

$$i_a^{b,c} : \mathrm{CF}_q^{(a,b)}(\mathcal{A}_F^{\mathcal{H}}) \longrightarrow \mathrm{CF}_q^{(a,c)}(\mathcal{A}_F^{\mathcal{H}}), \qquad a \le b \le c$$

and

$$\pi_{a,b}^c : \mathrm{CF}_q^{(a,c)}(\mathcal{A}_F^{\mathcal{H}}) \longrightarrow \mathrm{CF}_q^{(b,c)}(\mathcal{A}_F^{\mathcal{H}}), \qquad a \le b \le c.$$

$i_a^{b,c}$ is a natural inclusion and $\pi_{a,b}^c$ is a projection along $\mathrm{CF}_q^{(a,b)}(\mathcal{A}_F^{\mathcal{H}})$. We note that

$$\mathrm{CF}_q^{(a,c)}(\mathcal{A}_F^{\mathcal{H}}) = \mathrm{CF}_q^{(a,b)}(\mathcal{A}_F^{\mathcal{H}}) \oplus \mathrm{CF}_q^{(b,c)}(\mathcal{A}_F^{\mathcal{H}}),$$

We suppress the indices a, b, and c if there is no confusion. The short exact sequence

$$0 \longrightarrow \mathrm{CF}_q^{(-\infty,a)}(\mathcal{A}_F^{\mathcal{H}}) \xrightarrow{\ i\ } \mathrm{CF}_q^{(-\infty,b)}(\mathcal{A}_F^{\mathcal{H}}) \xrightarrow{\ \pi\ } \mathrm{CF}_q^{(a,b)}(\mathcal{A}_F^{\mathcal{H}}) \longrightarrow 0$$

gives rise to a boundary operator ∂_{a*}^b on $\mathrm{CF}_*^{(a,b)}(\mathcal{A}_F^{\mathcal{H}})$ and this induces a homology group, namely the **filtered Rabinowitz Floer homology**:

$$\mathrm{HF}_q^{(a,b)}(\mathcal{A}_F^{\mathcal{H}}) = \mathrm{H}_q(\mathrm{CF}_\bullet^{(a,b)}(\mathcal{A}_F^{\mathcal{H}}), \partial_{a\bullet}^b).$$

More generally for $a \le b \le c$, we have

$$0 \longrightarrow \mathrm{CF}_q^{(a,b)}(\mathcal{A}_F^{\mathcal{H}}) \xrightarrow{i} \mathrm{CF}_q^{(a,c)}(\mathcal{A}_F^{\mathcal{H}}) \xrightarrow{\pi} \mathrm{CF}_q^{(b,c)}(\mathcal{A}_F^{\mathcal{H}}) \longrightarrow 0.$$

The canonical homomorphisms i, π, and the boundary map ∂ are compatible with each other so that they induce canonical homomorphisms on the homology level. Thus we have

$$\cdots \xrightarrow{\delta} \mathrm{HF}_q^{(a,b)}(\mathcal{A}_F^{\mathcal{H}}) \xrightarrow{i_*} \mathrm{HF}_q^{(a,c)}(\mathcal{A}_F^{\mathcal{H}}) \xrightarrow{\pi_*} \mathrm{HF}_q^{(b,c)}(\mathcal{A}_F^{\mathcal{H}}) \xrightarrow{\delta} \mathrm{HF}_{q-1}^{(a,b)}(\mathcal{A}_F^{\mathcal{H}}) \xrightarrow{i_*} \cdots.$$

where δ is the connecting homomorphism.

Corollary 5.4.1. *In the filtered case, the canonical homomorphism is given by*

$$(\Phi_K^F)_* : \mathrm{HF}_q^{(a,b)}(\mathcal{A}_K^{\mathcal{H}}) \longrightarrow \mathrm{HF}_q^{(a-||F-K||_-,b+||F-K||_-)}(\mathcal{A}_F^{\mathcal{H}}).$$

Proof. This is a well-known fact in Floer theory; it follows from the comparison of the action values of $\mathcal{A}_K^{\mathcal{H}}$ and $\mathcal{A}_F^{\mathcal{H}}$, see Proposition 5.2.2. □

5.5 Proof of Theorem B

All of the lemmas and the propositions in this subsection were established for hypersurfaces in [AF1]. Without doubt, their arguments continue to hold in our situation, but we outline the arguments for the sake of completeness.

For $||F|| < \wp(\Sigma)$, we define

$$\mathrm{Crit}_{\mathrm{loc}}(\mathcal{A}_F^{\mathcal{H}}) := \left\{ (v,\eta) \in \mathrm{Crit}\mathcal{A}_F^{\mathcal{H}} \,\middle|\, -||F||_+ \le \mathcal{A}_F^{\mathcal{H}}(v,\eta) \le ||F||_- \right\}.$$

We note that the set $\mathrm{Crit}_{\mathrm{loc}}(\mathcal{A}_F^{\mathcal{H}})$ is finite. This follows from the Arzela-Ascoli theorem since the Lagrange multipliers η_i's are uniformly bounded according to Theorem 3.2.11. We define the finite dimensional $\mathbb{Z}/2$ vector space

$$\mathrm{CF}_{\mathrm{loc}}(\mathcal{A}_F^{\mathcal{H}}) := \mathrm{Crit}_{\mathrm{loc}}(\mathcal{A}_F^{\mathcal{H}}) \otimes \mathbb{Z}/2\ .$$

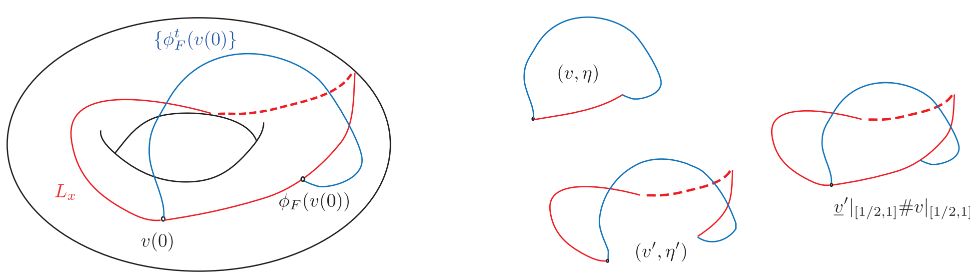
$(\mathrm{CF}_{\mathrm{loc}}(\mathcal{A}_F^{\mathcal{H}}), \partial_{\mathrm{loc}})$ is a chain complex and the **local Rabinowitz Floer homology** is defined by

$$\mathrm{HF}_{\mathrm{loc}}(\mathcal{A}_F^{\mathcal{H}}) := \mathrm{H}(\mathrm{CF}_{\mathrm{loc}}(\mathcal{A}_F^{\mathcal{H}}), \partial_{\mathrm{loc}}).$$

Proposition 5.5.1. *For $F \in C_c^\infty(S^1, \mathcal{M})$ with $||F|| < \wp(\Sigma)$, the following inequalities hold.*

$$\#\left\{\begin{matrix} \textit{Leafwise coisotropic} \\ \textit{intersection points of } \phi_F \end{matrix}\right\} \geq \dim \mathrm{CF}_{\mathrm{loc}}(\mathcal{A}_F^{\mathcal{H}}) \geq \dim \mathrm{HF}_{\mathrm{loc}}(\mathcal{A}_F^{\mathcal{H}})\ .$$

Proof. We briefly sketch the proof and refer to [AF1, Lemma 2.19] for details. The last inequality is obvious. For the first inequality, it suffices to show that different critical points of $\mathcal{A}_F^{\mathcal{H}}$ give rise to different leafwise coisotropic intersection points. If two distinct critical points (v, η), $(v', \eta') \in \mathrm{Crit}_{\mathrm{loc}}\mathcal{A}_F^{\mathcal{H}}$ give rise to the same leafwise coisotropic intersection point, then $\gamma := \underline{v}'|_{[1/2,1]}\#v|_{[1/2,1]}$, where $\underline{v}(t) = v(1-t)$ and $\#$ is the path catenation operator, is a periodic orbit solving (1.2.2), see pictures below. Moreover a close look at γ reveals that $\Omega(\gamma) \leq ||F|| < \wp(\Sigma)$ which contradicts the definition of $\wp(\Sigma)$. $\square$



Proposition 5.5.2. *The local Rabinowitz Floer homology of $\mathcal{A}^{\mathcal{H}}$ is isomorphic to the singular homology of Σ, i.e.*

$$\mathrm{H}(\Sigma;\mathbb{Z}/2) \overset{\Theta}{\cong} \mathrm{HF}_{\mathrm{loc}}(\mathcal{A}^{\mathcal{H}})\ .$$

Proof. The set $\mathrm{Crit}_{\mathrm{loc}}\mathcal{A}^{\mathcal{H}}$ consists of critical points of $\mathcal{A}^{\mathcal{H}}$ whose action values are zero which in turn implies $\mathrm{Crit}_{\mathrm{loc}}\mathcal{A}^{\mathcal{H}} \cong \Sigma$. Therefore no cascades appear in the boundary operator and $\mathrm{HF}_{\mathrm{loc}}(\mathcal{A}^{\mathcal{H}})$ is isomorphic to Morse homology of Σ. $\square$

The lemma below directly follows from the definition of $\wp(\Sigma)$.

Lemma 5.5.3. For any $(a,b)\subset(-\wp(\Sigma),\wp(\Sigma))$, we have an isomorphism

$$\mathrm{HF}^{(a,b)}(\mathcal{A}^{\mathcal{H}}) \cong \mathrm{HF}_{\mathrm{loc}}(\mathcal{A}^{\mathcal{H}}).$$

Proposition 5.5.4. *If $||F||<\wp(\Sigma)$, there exists an injective homomorphism*

$$\iota:\mathrm{H}(\Sigma;\mathbb{Z}/2)\longrightarrow \mathrm{HF}_{\mathrm{loc}}(\mathcal{A}^{\mathcal{H}}_F)\ .$$

In particular, $\dim\mathrm{HF}_{\mathrm{loc}}(\mathcal{A}^{\mathcal{H}}_F)\geq\dim\mathrm{H}(\Sigma;\mathbb{Z}/2)$.

Proof. We pick $a\in\mathbb{R}$ with $0<a<||F||<\wp(\Sigma)$ then using the continuation homomorphism in Corollary 5.4.1, we obtain

$$(\Phi_0^F)_*:\mathrm{HF}_{\mathrm{loc}}(\mathcal{A}^{\mathcal{H}})\cong\mathrm{HF}^{(-a,0)}(\mathcal{A}^{\mathcal{H}})\longrightarrow\mathrm{HF}^{(-a+||F||_-,|F||_-)}(\mathcal{A}^{\mathcal{H}}_F)\cong\mathrm{HF}_{\mathrm{loc}}(\mathcal{A}^{\mathcal{H}}_F).$$

On the other hand, we also have

$$(\Phi_F^0)_*:\mathrm{HF}^{(-a+||F||_-,|F||_-)}(\mathcal{A}^{\mathcal{H}}_F)\longrightarrow\mathrm{HF}^{(-a+||F||,||F||)}(\mathcal{A}^{\mathcal{H}})\cong\mathrm{RFH}_{\mathrm{loc}}(\Sigma,M).$$

Using a homotopy of homotopies $D_s^r(t,x)=\varphi_r(s)F(t,x)$, we deduce

$$(\Phi_F^0)_*\circ(\Phi_0^F)_*=\mathrm{id}_{\mathrm{HF}_{\mathrm{loc}}(\mathcal{A}^{\mathcal{H}})}.$$

Therefore $(\Phi_0^F)_*$ is injective and the proposition follows with

$$\iota := (\Phi_0^F)_* \circ \Theta.$$

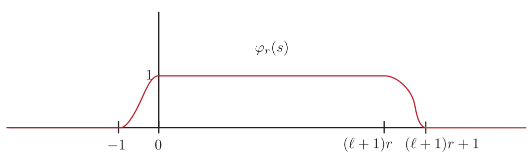
$\square$

Proof of Theorem B. It directly follows from Proposition 5.5.1 and Proposition 5.5.4. $\square$

5.6 Proof of Theorem C

We give a sketch of the proof here and refer to [AMo] for details.[5]

As before, $F \in C_c^\infty(S^1 \times M)$ with $||F|| < \wp(\Sigma)$. Let $\ell \in \mathbb{N}$. For $r \geq 0$, we choose a smooth family of functions $\varphi_r \in C^\infty(\mathbb{R}, [0,1])$.



We consider the following moduli space.

$$\mathcal{M}(r) := \left\{ w \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^k) \;\middle|\; \begin{array}{c} w \text{ is a gradient flow line of} \\ \mathcal{A}^{\mathcal{H}}_{\varphi_r F} \text{ with } \lim\limits_{s \to \pm\infty} w(s) \in \Sigma \times \{0\} \end{array} \right\}.$$

Note that $\mathcal{M}(0) \cong \Sigma$. Moreover one can show that $\mathcal{M}(r)$ is compact in the sense of Theorem 3.2.8.[6]

[5]We tacitly assume all transversality conditions of evaluation maps and Fredholm sections involved (or hidden) in the proof. These conditions are true up to small perturbations, as a matter of fact.

[6]The proof is similar to the corresponding part of the proof of Theorem A.

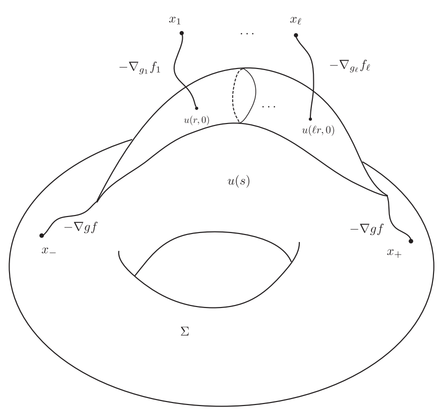
Now we consider the evaluation map

$$\begin{aligned} ev_r : \mathcal{M}(r) &\longrightarrow M^{\times \ell} \\ w = (u,\tau) &\longmapsto \big(u(r,0), \dots u(\ell r, 0)\big). \end{aligned}$$

For generic Morse functions f_i and Riemannian metrics g_i on M and f, g on Σ and for any $x = (x_1, \dots, x_\ell, x_-, x_+) \in \mathrm{Crit} f_1 \times \cdots \times \mathrm{Crit} f_\ell \times \mathrm{Crit} f \times \mathrm{Crit} f$,

$$\mathcal{M}(r,x) := \left\{ w = (u,\tau) \in \mathcal{M}(r) \,\middle|\, \begin{array}{c} \lim\limits_{s\to\pm\infty} u(s) \in W^{u/s}(x_\pm, f) \\ ev_r(u) \in W^s(x_1, f_1) \times \cdots \times W^s(x_\ell, f_\ell) \end{array} \right\}$$

is a smooth manifold. The map defined by



$(u,\tau) \in \mathcal{M}(r,x)$

$$\begin{aligned} \theta_r : CM^*(f_1) \otimes \cdots \otimes CM^*(f_\ell) \otimes CM_*(f) &\longrightarrow CM_*(f) \\ (x_1 \otimes \cdots \otimes x_\ell) \otimes x_- &\longmapsto \sum_{x_+ \in \mathrm{Crit} f} \#_2 \mathcal{M}(r,x) \cdot x_+. \end{aligned}$$

is a chain map. Since $\mathcal{M}(r,x)$ is chain homotopy equivalent to $\mathcal{M}(0,x)$ via the moduli space $\mathcal{M}[0,r] := \{(e,w) \,|\, e \in [0,r],\, w \in \mathcal{M}(r)\}$, θ_r is chain ho-motopic to θ_0. The map θ_0 induces the cohomology operation

$$\begin{aligned}\Theta : H^*(M)^{\otimes \ell} \otimes H_*(\Sigma) &\longrightarrow H_*(\Sigma),\\ (a_1 \otimes \cdots \otimes a_\ell) \otimes b &\longmapsto (a_1 \cup \cdots \cup a_\ell)|_\Sigma \cap b.\end{aligned}$$

Let $\ell = \mathrm{cl}(\Sigma, M)$ so that the cohomology operation Θ is nonzero, and hence $\mathcal{M}(r, x) \neq \emptyset$ for some $x \in \mathrm{Crit} f_1 \times \cdots \times \mathrm{Crit} f_\ell \times \mathrm{Crit} f \times \mathrm{Crit} f$ and for all $r \in \mathbb{R}$. We may assume that Morse functions $f, f_1, \ldots, f_\ell$ and Riemannian metrics $g, g_1, \ldots g_\ell$ satisfy the following generic condition.

- $W^s(x_i, f_i)$ does not intersect with the set of leafwise coisotropic intersection points for $x_i \in \mathrm{Crit} f_i$ with nonzero Morse index.

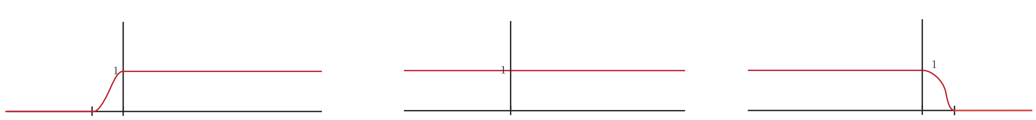
We choose a sequence $w^n = (u^n, \tau^n) \in \mathcal{M}(n, x)$, $n \in \mathbb{N}$. That is,

$$\begin{cases} \partial_s u^n(s,t) + J(s,t,u^n)\big(\partial_t u^n - \displaystyle\sum_{i=1}^k \tau_i^n(s) X_{H_i}(t, u^n) - \varphi_n(s) X_F(t, u^n)\big) = 0, \\ \partial_s \tau_i^n - \displaystyle\int_0^1 \mathcal{H}(t, u^n) dt = 0, \qquad\qquad 1 \leq \forall i \leq k. \end{cases}$$

Consider the following $\ell + 2$ sequences of maps:

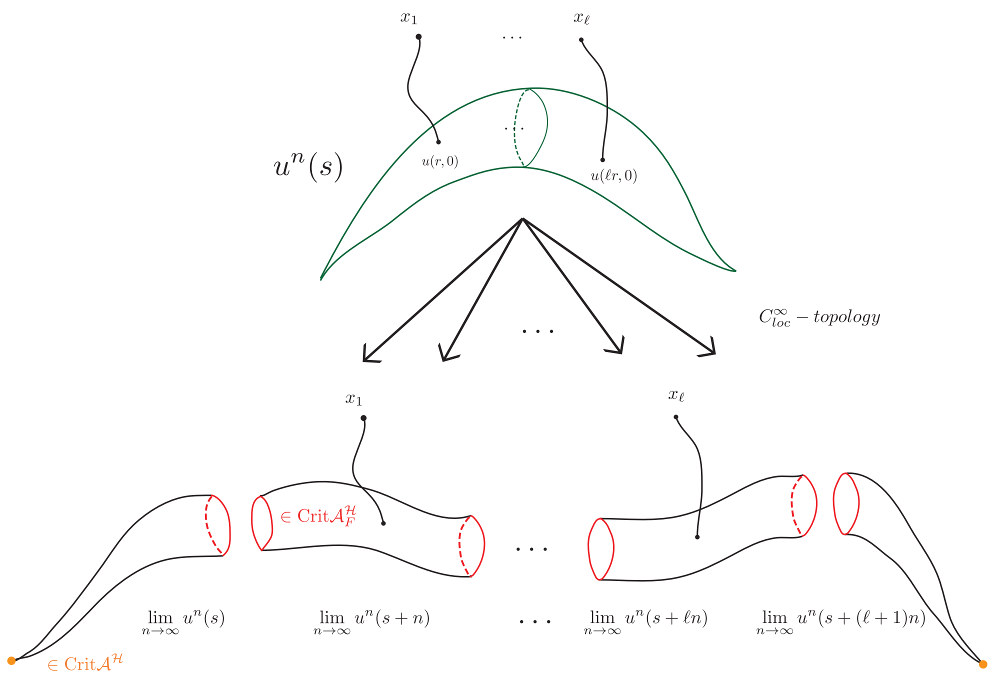
$$w^n(s + jn), \quad j \in \{0, \ldots \ell + 1\}.$$

The limits of $\varphi_n(s + jn)$, $0 \leq j \leq \ell + 1$ in the C^∞_{loc}-topology look like as pictures below and in particular $\varphi_n(s + jn)F$ converges to F for $1 \leq j \leq \ell$. By applying Theorem 3.2.8, $w^n(s + jn)$ converges (up to subsequence) to



$\lim_{n\to\infty} \varphi_n(s)$ $\qquad$ $\lim_{n\to\infty} \varphi_n(s + jn), \quad 1 \leq j \leq \ell$ $\qquad$ $\lim_{n\to\infty} \varphi_n(s + (\ell+1)n)$

some map $\widehat{w}_j$ in the C^∞_{loc}-topology for $0 \leq j \leq \ell+1$. Note that $\widehat{w}_j$ is a gradient flow line of $\mathcal{A}^{\mathcal{H}}_F$ for $1 \leq j \leq \ell$ and in particular $\widehat{w}_j(\pm\infty) \in \mathrm{Crit}\mathcal{A}^{\mathcal{H}}_F$ for $1 \leq j \leq \ell$. Since we have assumed that $W^s(x_i, f_i)$ does not intersect with



the set of leafwise coisotropic intersection points for $x_i \in \mathrm{Crit} f_i$ with nonzero Morse index, $\widehat{w}_j$, $1 \leq j \leq \ell$ are not constant gradient flow lines. Therefore $\ell+1$ critical points

$$\widehat{w}_1(-\infty),\ \widehat{w}_2(-\infty), \cdots, \widehat{w}_\ell(-\infty),\ \widehat{w}_\ell(\infty)$$

of $\mathcal{A}^{\mathcal{H}}_F$ are distinct. Moreover as in the proof of Theorem A, the assumption $||F|| < \wp(\Sigma)$ guarantees that they give rise to distinct leafwise coisotropic intersection points. This shows the existence of $\mathrm{cl}(\Sigma, M)+1$ leafwise coisotropic intersection points. $\square$

Chapter 6

Künneth formula in Rabinowitz Floer homology

In this chapter, we analyze the Rabinowitz Floer action functional for a product of restricted contact hypersurfaces in a product of symplectic manifolds and derive a Künneth formula for Rabinowitz Floer homology. Consider restricted contact hypersurfaces (Σ_1, λ_1) resp. (Σ_2, λ_2) in exact symplectic manifolds $(M_1, \omega_1 = d\lambda_1)$ resp. $(M_2, \omega_2 = d\lambda_2)$. Moreover we assume that Σ_1 resp. Σ_2 bounds a compact region in M_1 resp. M_2 and that those M_1 and M_2 are geometrically bounded. We introduce projection maps $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$; then $(M_1 \times M_2, \omega_1 \oplus \omega_2)$ admits the symplectic structure $\omega_1 \oplus \omega_2 = \pi_1^* \omega_1 + \pi_2^* \omega_2$.

6.1 Rabinowitz action functional for product manifolds

Since Σ_1 and Σ_2 are restricted contact hypersurfaces, there exist associated Liouville vector fields Y_1 resp. Y_2 on M_1 resp. M_2 such that $\mathcal{L}_{Y_i}\omega_i = \omega_i$ and $Y_i \pitchfork \Sigma_i$ for $i = 1, 2$. We denote by $\phi_{Y_i}^t$ the flow of Y_i and fix $\delta > 0$ such that $\phi^t_{Y_i}|_{\Sigma_i}$ is defined for $|t| < \delta$. Since Σ_1 resp. Σ_2 bounds a compact region in M_1 resp. M_2, we are able to define Hamiltonian functions $G_1 \in C^\infty(M_1)$ and $G_2 \in C^\infty(M_2)$ so that

1. $G_1^{-1}(0) = \Sigma_1$ and $G_2^{-1}(0) = \Sigma_2$ are regular level sets;
2. dG_1 and dG_2 have compact supports;
3. $G_i(\phi^t_{Y_i}(x_i)) = t$ for all $x_i \in \Sigma_i$, $i = 1, 2$, and $|t| < \delta$;

We extend G_1, G_2 to be defined on the whole of $M_1 \times M_2$:

$$\begin{aligned} \widetilde{G}_i := \pi_i^* G_i : M_1 \times M_2 &\longrightarrow \mathbb{R}, \qquad i = 1, 2 \\ (x_1, x_2) &\longmapsto G_i(x_i). \end{aligned}$$

We denote by $\mathcal{L} = \mathcal{L}_{M_1 \times M_2} \subset C^\infty(S^1, M_1 \times M_2)$ the space of contractible loops in $M_1 \times M_2$. The perturbed Rabinowitz action functional $\mathcal{A}_F^{\widetilde{G}_1, \widetilde{G}_2}(v, \eta_1, \eta_2) : \mathcal{L} \times \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$\mathcal{A}_F^{\widetilde{G}_1, \widetilde{G}_2}(v, \eta_1, \eta_2) = -\int_0^1 v^*(\lambda_1 \oplus \lambda_2) - \eta_1 \int_0^1 \widetilde{G}_1(v)dt - \eta_2 \int_0^1 \widetilde{G}_2(v)dt$$

where $\lambda_1 \oplus \lambda_2 := \pi_1^*\lambda_1 + \pi_2^*\lambda_2$. The real numbers η_1 and η_2 can be thought of as Lagrange multipliers as before. A critical point $(v, \eta_1, \eta_2) \in \mathrm{Crit}\mathcal{A}_F^{\widetilde{G}_1, \widetilde{G}_2}$ satisfies

$$\left.\begin{aligned} &\partial_t v = \eta_1 X_{\widetilde{G}_1}(v) + \eta_2 X_{\widetilde{G}_2}(v), \\ &\int_0^1 \widetilde{G}_1(v)dt = 0, \\ &\int_0^1 \widetilde{G}_2(v)dt = 0. \end{aligned}\right\} \tag{6.1.1}$$

We choose a compatible almost complex structure J_1 on M_1 and define the metric on (M_1, ω_1) by $g_1(\cdot, \cdot) = \omega_1(\cdot, J_1 \cdot)$. Analogously we also define the metric $g_2(\cdot, \cdot) = \omega_2(\cdot, J_2 \cdot)$ on (M_2, ω_2). Then $g = g_1 \oplus g_2$ which is the metric on $(M_1 \times M_2, \omega_1 \oplus \omega_2)$ induces a metric m on the tangent space $T_{(v,\eta_1,\eta_2)}(\mathcal{L} \times \mathbb{R}^2) \cong T_v\mathcal{L} \times \mathbb{R}^2$ as follows:

$$m_{(v,\eta_1,\eta_2)}\big((\hat{v}^1, \hat{\eta}_1^1, \hat{\eta}_2^1), (\hat{v}^2, \hat{\eta}_1^2, \hat{\eta}_2^2)\big) := \int_0^1 g_v(\hat{v}^1, \hat{v}^2)dt + \hat{\eta}_1^1\hat{\eta}_1^2 + \hat{\eta}_2^1\hat{\eta}_2^2 \,.$$

In this set-up, the gradient flow equation

$$\partial_s w(s) + \nabla_m \mathcal{A}_F^{\widetilde{G}_1,\widetilde{G}_2}(w(s)) = 0, \quad w \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^2)$$

can be interpreted as maps $u(s,t) : \mathbb{R} \times S^1 \to M_1 \times M_2$ and $\tau_1(s)$, $\tau_2(s) : \mathbb{R} \to \mathbb{R}$ solving

$$\left.\begin{aligned} &\partial_s u + J(t,u)\big(\partial_t v - \tau_1 X_{\widetilde{G}_1}(u) - \tau_2 X_{\widetilde{G}_2}(u)\big) = 0, \\ &\partial_s \tau_1 - \int_0^1 \widetilde{G}_1(u)dt = 0, \\ &\partial_s \tau_2 - \int_0^1 \widetilde{G}_2(u)dt = 0. \end{aligned}\right\} \tag{6.1.2}$$

6.1.1 Compactness

In order to define Rabinowitz Floer homology, we prove the compactness theorem for gradient flow lines of the Rabinowitz action functional in this subsection.

We introduce two auxiliary action functionals $\mathcal{A}_1, \mathcal{A}_2 : \mathcal{L}_{M_1 \times M_2} \times \mathbb{R}^2 \to \mathbb{R}$:

$$\begin{aligned} \mathcal{A}_1(v, \eta_1, \eta_2) &:= \int_0^1 v^*\pi_1^*\lambda_1 - \eta_1 \int_0^1 G_1(v)dt \\ \mathcal{A}_2(v, \eta_1, \eta_2) &:= \int_0^1 v^*\pi_2^*\lambda_2 - \eta_2 \int_0^1 G_2(v)dt. \end{aligned}$$

Lemma 6.1.1. Let $w = (v, \eta_1, \eta_2) \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^2)$ be a gradient flow line of $\mathcal{A}_F^{\widetilde{G}_1,\widetilde{G}_2}$ with asymptotic ends $w_- = (v_-, \eta_{1-}, \eta_{2-})$ and $w_+ = (v_+, \eta_{1+}, \eta_{2+})$.

Then the action values of $\mathcal{A}_1$ and $\mathcal{A}_2$ are bounded along w in terms of the asymptotic data:

$$\text{(i)}\quad \mathcal{A}_1(w(s)) \leq 2|\mathcal{A}_1(w_-)| + |\mathcal{A}_1(w_+)|, \quad \forall s \in \mathbb{R};$$

$$\text{(ii)}\quad \mathcal{A}_2(w(s)) \leq 2|\mathcal{A}_2(w_-)| + |\mathcal{A}_2(w_+)|, \quad \forall s \in \mathbb{R}.$$

Proof. We only show the first inequality, the latter one is proved in a similar way. Since it holds that $i_{X_{\widetilde{G}_2}} \pi_1^* \omega_1 = 0$, we compute

$$\begin{aligned}
\frac{d}{ds}\mathcal{A}_1(w(s)) &= d\mathcal{A}_1(w(s))[\partial_s w(s)] \\
&= \int_0^1 \pi_1^*\omega_1\big(\partial_t v, \partial_s v\big) - \int_0^1 \omega_1 \oplus \omega_2\big(\eta_1 X_{\widetilde{G}_1}(v), \partial_s v\big) - \Big(\int_0^1 \widetilde{G}_1(v)dt\Big)^2 \\
&= \int_0^1 \pi_1^*\omega_1\big(\partial_t v - \eta_1 X_{\widetilde{G}_1}(v), \partial_s v\big)dt - \Big(\int_0^1 \widetilde{G}_1(v)dt\Big)^2 \\
&= -\int_0^1 \pi_1^*\omega_1(\partial_s v, J\partial_s v)dt - \Big(\int_0^1 \widetilde{G}_1(v)dt\Big)^2.
\end{aligned}$$

Integrating the above equality from $-\infty$ to any $s_0 \in \mathbb{R}$, we have

$$\begin{aligned}
\mathcal{A}_1(w(s_0)) - \mathcal{A}_1(w_-) &= \int_{-\infty}^{s_0} \frac{d}{ds}\mathcal{A}_1(w(s))ds \\
&= -\int_{-\infty}^{s_0}\int_0^1 \pi_1^*\omega_1(\partial_s v, J\partial_s v)dtds - \int_{-\infty}^{s_0}\Big(\int_0^1 \widetilde{G}_1(v)dt\Big)^2 ds.
\end{aligned}$$

We set

$$\mathbf{B}(s) := \int_0^1 \pi_1^*\omega_1(\partial_s v, J\partial_s v)dt + \Big(\int_0^1 \widetilde{G}_1(v)dt\Big)^2.$$

Therefore the following estimate can be derived for any $s_0 \in \mathbb{R}$

$$|\mathcal{A}_1(w(s_0))| \leq |\mathcal{A}_1(w_+)| + \Big|\int_{-\infty}^{s_0} \mathbf{B}(s)ds\Big|,$$

and it remains to find a bound for $|\int_{-\infty}^{s_0} \mathbf{B}(s)ds|$. Since $\mathbf{B}(s)$ is nonnegative, we are able to estimate as the following. By setting $s_0 = \infty$, we have

$$\mathcal{A}_1(w_+) - \mathcal{A}_1(w_-) = -\int_{-\infty}^{\infty} \mathbf{B}(s)ds$$

Using the above formula, we obtain

$$\Big|\int_{-\infty}^{s_0} \mathbf{B}(s)ds\Big| \leq \Big|\int_{-\infty}^{\infty} \mathbf{B}(s)ds\Big| \leq |\mathcal{A}_1(w_+)| + |\mathcal{A}_1(w_-)|.$$

Thus we finally deduce

$$|\mathcal{A}_1(w(s_0))| \leq |\mathcal{A}_1(w_+)| + 2|\mathcal{A}_1(w_-)|, \quad \forall s_0 \in \mathbb{R}.$$

□

Lemma 6.1.2. Assume that $v \subset U_\delta := \widetilde{G}_1^{-1}(-\delta,\delta) \cap \widetilde{G}_2^{-1}(-\delta,\delta)$ with $0 < 2\delta < \min\{1, \delta_0\}$. Then there exists $C_i > 0$ satisfying

$$|\eta_i| \leq C_i\Big(|\mathcal{A}_i(v,\eta)| + ||\nabla_m \mathcal{A}^{\widetilde{G}_1,\widetilde{G}_2}||_m + 1\Big), \qquad i = 1, 2.$$

Proof. We estimate

$$\begin{aligned}
|\mathcal{A}_i(v,\eta_1,\eta_2)| &= \Big|\int_0^1 v^*\pi_i^*\lambda_i + \eta_i \int_0^1 \widetilde{G}_i(v)dt\Big| \\
&\geq \Big|\eta_i \int_0^1 \pi_i^*\lambda_i(v)\big(X_{\widetilde{G}_i}(v)\big)dt\Big| - \Big|\eta_i \int_0^1 \widetilde{G}_i(v)dt\Big|| \\
&\quad - \Big|\int_0^1 \pi_i^*\lambda_i(v)\big(\partial_t v - \eta_1 X_{\widetilde{G}_1}(v) - \eta_2 X_{\widetilde{G}_2}(v)\big)dt\Big| \\
&\geq |\eta_i| - \delta|\eta_i| - C_{i,\delta}||\partial_t v - \eta_1 X_{\widetilde{G}_1}(v) - \eta_2 X_{\widetilde{G}_2}(v))||_{L^1} \\
&\geq |\eta_i| - \delta|\eta_i| - C_{i,\delta}||\nabla_m \mathcal{A}^{\widetilde{G}_1,\widetilde{G}_2}||_m
\end{aligned}$$

where $C_{i,\delta} := ||\pi_i^*\lambda_i|_{U_\delta}||_{L^\infty}$. The second inequality holds since $\pi_i^*\lambda_i(X_{\widetilde{G}_j}) = 0$

if $i \neq j$. This estimate finishes the lemma with

$$C_i := \max\left\{\frac{1}{1-\delta}, \frac{C_{i,\delta}}{1-\delta}, \right\}, \qquad i = 1,\, 2.$$

$\square$

Along arguments in Chapter 3, one can easily show the following fundamental lemma using previous two lemmas.

Lemma 6.1.3. For a gradient flow line $w = (u, \tau_1, \tau_2) \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^2)$ of $\mathcal{A}^{\widetilde{G}_1, \widetilde{G}_2}$, the following assertion holds for $i = 1,\, 2$ with some $C, \epsilon > 0$.

$$|\tau_i| \leq C\big(|\mathcal{A}_i(w_-)| + |\mathcal{A}_i(w_+)| + 1\big) \quad \text{ if } \quad ||\nabla_m \mathcal{A}^{\widetilde{G}_1, \widetilde{G}_2}(u, \tau_1, \tau_2)||_m < \epsilon.$$

The following compactness theorem immediately follows from the fundamental lemma as before, see Chapter 3.

Theorem 6.1.4. *Let $\{w_n\}_{n \in \mathbb{N}}$ be a sequence of gradient flow lines of $\mathcal{A}^{\widetilde{G}_1, \widetilde{G}_2}$ for which there exist $a < b$ such that*

$$a \leq \mathcal{A}^{\widetilde{G}_1, \widetilde{G}_2}(w_n(s)) \leq b, \qquad \textit{for all } s \in \mathbb{R}.$$

Then for every reparametrization sequence $\sigma_n \in \mathbb{R}$, the sequence $w_n(\cdot + \sigma_n)$ has a subsequence which is converges in $C^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{L} \times \mathbb{R}^2)$.

This theorem enables us to define the Rabinowitz Floer homology

$$\mathrm{RFH}(\Sigma_1 \times \Sigma_2, M_1 \times M_2) = \mathrm{H}\big(\mathrm{CF}(\mathcal{A}^{\widetilde{G}_1, \widetilde{G}_2}), \partial^{1,2}\big).$$

6.2 Proof of Theorem F

Thanks to the previous section, we are ready to define Rabinowitz Floer homology of $(\Sigma_1 \times \Sigma_2, M_1 \times M_2)$ and to prove Theorem F. Consider the Rabinowitz action functionals $\mathcal{A}^{G_1} : \mathcal{L}_{M_1} \times \mathbb{R} \to \mathbb{R}$ and $\mathcal{A}^{G_2} : \mathcal{L}_{M_2} \times \mathbb{R} \to \mathbb{R}$:

- $\mathcal{A}^{G_1}(v_1, \eta_1) = -\displaystyle\int_0^1 v_1^* \lambda_1 - \eta_1 \int_0^1 G_1(v_1) dt,$
- $\mathcal{A}^{G_2}(v_2, \eta_2) = -\displaystyle\int_0^1 v_2^* \lambda_2 - \eta_2 \int_0^1 G_2(v_2) dt.$

Recall for $i = 1, 2$ that $(v_i, \eta_i) \in \mathrm{Crit}\mathcal{A}^{G_i}$ if and only if

$$\partial_t v_i = \eta_i X_{G_i}(v_i), \quad \int_0^1 G_1(v_i) dt = 0, \tag{6.2.1}$$

and $w_i(s,t) = \big(u_i(s,t), \tau_i(s)\big) : \mathbb{R} \times S^1 \to M_i \times \mathbb{R}$ is a gradient flow line of $\mathcal{A}^{G_i}$ if and only if

$$\partial_s u_i + J_i(t, u_i)\big(\partial_t u_i - \eta_i X_{G_i}(u_i)\big) = 0, \quad \partial_s \tau_i - \int_0^1 G_i(u_i) dt = 0. \tag{6.2.2}$$

Then we define chain complexes $\mathrm{CF}(\mathcal{A}^{G_1})$, $\mathrm{CF}(\mathcal{A}^{G_2})$ and their boundary operators ∂^1, ∂^2 analogously as before and denote their Floer homologies by

$$\mathrm{RFH}(\Sigma_1, M_1) = \mathrm{H}\big(\mathrm{CF}(\mathcal{A}^{G_1}), \partial^1\big), \quad \mathrm{RFH}(\Sigma_2, M_2) = \mathrm{H}\big(\mathrm{CF}(\mathcal{A}^{G_2}), \partial^2\big).$$

Next, for a Künneth formula, we define the tensor product of chain complexes by

$$\big(\mathrm{CF}_*(\mathcal{A}^{G_1}) \otimes \mathrm{CF}_*(\mathcal{A}^{G_2})\big)_n := \bigoplus_{i=0}^n \mathrm{CF}_i(\mathcal{A}^{G_1}) \otimes \mathrm{CF}_{n-i}(\mathcal{A}^{G_2}).$$

together with the boundary operator $\partial_n^{\otimes}$ given by

$$\partial_n^{\otimes}\big((v_1, \eta_1)_i \otimes (v_2, \eta_2)_{n-i}\big) = \partial_i^1 (v_1, \eta_1)_i \otimes (v_2, \eta_2)_{n-i} + (v_1, \eta_1)_i \otimes \partial_{n-i}^2 (v_2, \eta_2)_{n-i}.$$

Comparing the critical point equations (6.1.1) and (6.2.1), we easily notice that $\big((v_1, v_2), \eta_1, \eta_2\big) = (v, \eta_1, \eta_2) \in \mathrm{Crit}\mathcal{A}^{G_1,G_2}$ if and only if $(v_1, \eta_1) \in \mathrm{Crit}\mathcal{A}^{G_1}$ and $(v_2, \eta_2) \in \mathrm{Crit}\mathcal{A}^{G_2}$ where $v_1 = \pi_1 \circ v : S^1 \to M_1$ and $v_2 = \pi_2 \circ v : S^1 \to M_2$ for the projections π_1, π_2. Here, $(v_1, v_2) \in C^\infty(S^1, M_1 \times M_2)$ is defined by

$$\begin{aligned} (v_1, v_2) : S^1 &\longrightarrow M_1 \times M_2, \\ t &\longmapsto (v_1(t), v_2(t)). \end{aligned}$$

Moreover since the Conley-Zehnder index behaves additively, we have

$$\mathrm{Crit}_n(\mathcal{A}^{\widetilde{G}_1,\widetilde{G}_2}) = \bigcup_{i+j=n} \mathrm{Crit}_i(\mathcal{A}^{G_1}) \times \mathrm{Crit}_j(\mathcal{A}^{G_2}),$$

and we are able to define a chain homomorphism:

$$\begin{aligned} P_n : \big(\mathrm{CF}_*(\mathcal{A}^{G_1}) \otimes \mathrm{CF}_*(\mathcal{A}^{G_2})\big)_n &\longrightarrow \mathrm{CF}_n(\mathcal{A}^{\widetilde{G}_1,\widetilde{G}_2}), \\ (v_1, \eta_1) \otimes (v_2, \eta_2) &\longmapsto \big((v_1, v_2), \eta_1, \eta_2\big). \end{aligned}$$

To verify that P_n is a chain homomorphism, we need to show that

$$\partial_n^{1,2} \circ P_n = P_{n-1} \circ \partial_n^{\otimes}.$$

For $w_{1-} = (v_{1-}, \eta_{1-}) \in \mathrm{Crit}\mathcal{A}^{G_1}$ and $w_{2-} = (v_{2-}, \eta_{2-}) \in \mathrm{Crit}\mathcal{A}^{G_2}$, we compute

$$\begin{aligned}
\partial_n^{1,2} \circ P_n(w_{1-} \otimes w_{2-}) &= \partial_n^{1,2} \underbrace{\big((v_{1-}, v_{2-}), \eta_{1-}, \eta_{2-}\big)}_{=:w_-} \\
&= \sum_{\substack{w_+ \in \mathrm{Crit}\mathcal{A}^{\widetilde{G}_1, \widetilde{G}_2};\\ \mu(w_+)=\mu(w_-)-1}} \#_2 \mathcal{M}\{w_-, w_+\} w_+ \\
&= \sum_{\substack{(v_{1+}, \eta_{1+}) \in \mathrm{Crit}\mathcal{A}^{G_1};\\ \mu(w_{1+})=\mu(w_{1-})-1}} \#_2 \mathcal{M}\big\{w_-, ((v_{1+}, v_{2-}), \eta_{1+}, \eta_{2-})\big\}\big((v_{1+}, v_{2-}), \eta_{1+}, \eta_{2-}\big) \\
&\quad + \sum_{\substack{(v_{2+}, \eta_{2+}) \in \mathrm{Crit}\mathcal{A}^{G_2};\\ \mu(w_{2+})=\mu(w_{2-})-1}} \#_2 \mathcal{M}\big\{w_-, ((v_{1-}, v_{2+}), \eta_{1-}, \eta_{2+})\big\}\big((v_{1-}, v_{2+}), \eta_{1-}, \eta_{2+}\big) \\
&= \sum_{\substack{(v_{1+}, \eta_{1+}) \in \mathrm{Crit}\mathcal{A}^{G_1};\\ \mu(w_{1+})=\mu(w_{1-})-1}} \#_2 \mathcal{M}\big\{w_{1-}, w_{1+}\big\} P_{n-1}(w_{1+} \otimes w_{2-}) \\
&\quad + \sum_{\substack{(v_{2+}, \eta_{2+}) \in \mathrm{Crit}\mathcal{A}^{G_2};\\ \mu(w_{2+})=\mu(w_{2-})-1}} \#_2 \mathcal{M}\big\{w_{2-}, w_{2+}\big\} P_{n-1}(w_{1-} \otimes w_{2+}) \\
&= P_{n-1}(\partial_i^1 w_{1-} \otimes w_{2-}) + P_{n-1}(w_{1-} \otimes \partial_{n-i}^2 w_{2-}) \\
&= P_{n-1} \circ \partial_n^{\otimes}(w_{1-} \otimes w_{2-}).
\end{aligned}$$

where $\mathcal{M}\big\{w_{1-}, w_{1+}\big\}$ resp. $\mathcal{M}\big\{w_{2-}, w_{2+}\big\}$ is the moduli space which consists of gradient flow lines with cascades of $\mathcal{A}^{G_1}$ resp. $\mathcal{A}^{G_2}$. The fourth equality follows by comparing (6.1.2) together with (6.2.2). Therefore we have an isomorphism

$$(P_\bullet)_* : \mathrm{H}_\bullet\big(\mathrm{CF}(\mathcal{A}^{G_1}) \otimes \mathrm{CF}(\mathcal{A}^{G_2})\big) \xrightarrow{\cong} \mathrm{H}_\bullet(\mathrm{CF}(\mathcal{A}^{\widetilde{G}_1, \widetilde{G}_2})) = \mathrm{RFH}_\bullet(\Sigma_1 \times \Sigma_2, M_1 \times M_2).$$

Finally, the algebraic Künneth formula enable us to derive the desired (topological) Künneth formula in Rabinowitz Floer homology.

$$\mathrm{RFH}_n(\Sigma_1 \times \Sigma_2, M_1 \times M_2) \cong \bigoplus_{p=0}^{n} \mathrm{RFH}_p(\Sigma_1, M_1) \otimes \mathrm{RFH}_{n-p}(\Sigma_2, M_2).$$

6.3 Proof of Theorem G

In this section, we do not consider Σ_2 and let (M_2, ω_2) be closed and symplectically aspherical, i.e. $\omega_2|_{\pi_2(M_2)} = 0$. To prove Statement (G1) in Theorem G, we need a compactness theorem for gradient flow lines of the perturbed Rabinowitz action functional on $(\Sigma_1 \times M_2, M_1 \times M_2)$ with an arbitrary perturbation $F \in C_c^\infty(S^1 \times M_1 \times M_2)$. For that reason, we analyze the Rabinowitz action functional again. Once we establish the fundamental lemma, then the remaining steps are exactly same as before. We assume that $\Sigma_1 \times M_2$ bounds a compact region in $M_1 \times M_2$ for Statement (G2). As before, we choose a defining Hamiltonian function $G \in C^\infty(M_1)$ so that

1. $G^{-1}(0) = \Sigma_1$ is a regular level set and dG has a compact support.
2. $G_i(\phi_Y^t(x)) = t$ for all $x \in \Sigma_i$, and $|t| < \delta$;

where Y is the Liouville vector field for $\Sigma_1 \subset M_1$. We define $\widetilde{G} \in C^\infty(M_1 \times M_2)$ by $\widetilde{G}(x_1, x_2) = G(x_1)$ so that $\widetilde{G}$ is a defining Hamiltonian function for $\Sigma_1 \times M_2$. We let $\widetilde{H}(t,x) = \chi(t)\widetilde{G}(x) \in C^\infty(S^1 \times M_1 \times M_2)$ for $\chi \in C^\infty(S^1, \mathbb{R}_{\geq 0})$ with $\int_0^1 \chi(t)dt = 1$ and $\mathrm{Supp}\chi \subset (1/2, 1)$. With a perturbation $F \in C_c^\infty(S^1 \times M_1 \times M_2)$ satisfying $F(t, \cdot) = 0$ for $t \in (1/2, 1)$, the perturbed Rabinowitz action functional $\mathcal{A}_F^{\widetilde{H}} : \mathcal{L} \times \mathbb{R} \to \mathbb{R}$ is given by

$$\mathcal{A}_F^{\widetilde{H}}(v, \eta) = -\int_{D^2} \bar{v}^* \omega_1 \oplus \omega_2 - \eta \int_0^1 \widetilde{H}(t, v)dt - \int_0^1 F(t, v)dt$$

where $\mathcal{L} = \mathcal{L}_{M_1 \times M_2} \subset C^\infty(S^1, M_1 \times M_2)$ is the space of contractible loops in $M_1 \times M_2$ and $\bar{v} : D^2 \to M_1 \times M_2$ is a filling disk of v.

We prove the following key lemma using a kind of isoperimetric inequality.

Lemma 6.3.1. Let $w(s,t) = (v(s,t), \eta(s)) \in C^\infty(\mathbb{R} \times S^1, M_1 \times M_2) \times C^\infty(\mathbb{R}, \mathbb{R})$ be a gradient flow line of $\mathcal{A}_F^{\widetilde{H}}$. We set $\gamma(t) = v(s_0, t) \in C^\infty(S^1, M_1 \times M_2)$ for some fixed $s_0 \in \mathbb{R}$. Then $\int_{D^2} \bar{\gamma}^* \pi_2^* \omega_2$ is uniformly bounded provided

$$||\nabla_m \mathcal{A}_F^{\widetilde{H}}(v(s_0, \cdot), \eta(s_0))||_m < \epsilon$$

for some $\epsilon > 0$:

$$\Big| \int_{D^2} \bar{\gamma}^* \pi_2^* \omega_2 \Big| \leq \max_{x \in \widetilde{M_2}} \big\{ ||\lambda_{\widetilde{M_2}}(x)||_{\tilde{g}_2} \,\big|\, d_{\tilde{g}_2}(x, \widetilde{M}_\star) < \epsilon + ||X_F||_{L^\infty} \big\} \big(\epsilon + ||X_F||_{L^\infty}\big). \tag{6.3.1}$$

where $\widetilde{M_2}$ is the universal covering of M_2; $\tilde{g}_2$ is the lifting of the metric $g_2(\cdot,\cdot) = \omega_2(\cdot, J_2 \cdot)$ on M_2; $\widetilde{M}_\star$ is a fundamental domain in $\widetilde{M_2}$; $d_{\tilde{g}_2}(x, \widetilde{M}_\star)$ is the distance between x and $\widetilde{M}_\star$; the value on the right hand side of (6.3.1) is finite since $\widetilde{M}_\star \cong M_2$ is compact.

Proof. We write $v(s,t)$ as $v(s,t) = (v_1, v_2)(s,t)$ where $v_1 : \mathbb{R} \times S^1 \to M_1$ and $v_2 : \mathbb{R} \times S^1 \to M_2$. Let $\gamma \in C^\infty(S^1, M_1 \times M_2)$ be defined by $\gamma(t) = v(s_0, t)$ for some $s_0 \in \mathbb{R}$. Since γ is contractible and M_2 is symplectically aspherical, the value of $\int_{D^2} \bar{\gamma}^* \pi_2^* \omega_2$ is well-defined. Let $\gamma_2 := \pi_2 \circ \gamma$. We also consider $(\widetilde{M_2}, \widetilde{\omega_2})$ the universal cover of M_2 where $\widetilde{\omega_2}$ is the lift of ω_2 and we also lift the metric g_2 on M_2 which we write as $\tilde{g}_2$. Since we have assumed the symplectically asphericity of (M_2, ω_2), there exists a primitive one form $\lambda_{\widetilde{M_2}}$ of $\widetilde{\omega_2}$. Let $\widetilde{M}_\star (\cong M_2)$ be one of the fundamental domains in $\widetilde{M_2}$ and $\tilde{v}(s,t) : \mathbb{R} \times S^1 \to M_1 \times \widetilde{M_2}$ be the lift of v such that $\tilde{v}(s_0, t) = \tilde{\gamma}(t)$ intersects $M_1 \times \widetilde{M}_\star$. Now, we can show the following kind of isoperimetric inequality.

This inequality concludes the proof.

$$\begin{aligned}
\Big|\int_{D^2}\bar{\gamma}^*\pi_2^*\omega_2\Big| &= \Big|\int_{D^2}(\tilde{\bar{\gamma}}_2)^*\widetilde{\omega_2}\Big| = \Big|\int_0^1\tilde{\gamma}_2^*\lambda_{\widetilde{M}_2}\Big| \\
&\leq ||\lambda_{\widetilde{M}_2}|_{\gamma_2(S^1)}||_{L^\infty}\int_0^1||\partial_t\tilde{\gamma}_2||_{\tilde{g}_2}dt \\
&= ||\lambda_{\widetilde{M}_2}|_{\gamma_2(S^1)}||_{L^\infty}\int_0^1||\partial_t\gamma_2||_{g_2}dt \\
&= ||\lambda_{\widetilde{M}_2}|_{\gamma_2(S^1)}||_{L^\infty}\int_0^1||J\partial_s\gamma_2+\pi_{2*}X_F(t,\gamma_2)||_{g_2}dt \\
&\leq \lambda_{\mathrm{Max}}\big(||\nabla_m\mathcal{A}_F^{\widetilde{H}}(v(s_0,\cdot),\eta(s_0))||_m+||X_F||_{L^\infty}\big).
\end{aligned}$$

where

$$\begin{aligned}
\lambda_{\mathrm{Max}} &:= \max_{x\in\widetilde{M}_2}\Big\{||\lambda_{\widetilde{M}_2}(x)||_{\tilde{g}_2}\;\Big|\;d_{\tilde{g}_2}(x,\widetilde{M}_\star)<\int_0^1||\partial_t\gamma_2||_{g_2}dt\Big\} \\
&\leq \max_{x\in\widetilde{M}_2}\big\{||\lambda_{\widetilde{M}_2}(x)||_{\tilde{g}_2}\;\big|\;d_{\tilde{g}_2}(x,\widetilde{M}_\star)<||\nabla_m\mathcal{A}_F^{\widetilde{H}}(v(s_0,\cdot),\eta(s_0))||_m+||X_F||_{L^\infty}\big\}.
\end{aligned}$$

□

The following two lemmas can be proved similarly as before.

Lemma 6.3.2. We assume that for $(v,\eta)\in C^\infty(S^1,M_1\times M_2)\times\mathbb{R}$, $v(t)\in U_\delta:=\widetilde{G}^{-1}(-\delta,\delta)$ for all $t\in(\frac{1}{2},1)$ with $0<2\delta<\min\{1,\delta_0\}$. Then there exists $C>0$ satisfying

$$|\eta|\leq C\Big(|\mathcal{A}_F^{\widetilde{H}}(v,\eta)|+||\nabla_m\mathcal{A}_F^{\widetilde{H}}(v,\eta)||_m+\Big|\int_{D^2}\bar{v}^*\pi_2^*\omega_2\Big|+1\Big).$$

Lemma 6.3.3. For $(v,\eta)\in C^\infty(S^1,M_1\times M_2)\times\mathbb{R}$ if there exists $t\in[\frac{1}{2},1]$ such that $v(t)\notin U_\delta$, then $||\nabla_m\mathcal{A}_F^{\widetilde{H}}(v,\eta)||_m>\epsilon$ for some $\epsilon=\epsilon_\delta$.

Due to the three previous lemmas, we are able to deduce the fundamental lemma in the situation of Theorem G, and thus we obtain a uniform L^∞-

bound on the Lagrange multiplier η.

Lemma 6.3.4. For a gradient flow line $w(s) = (v,\eta)(s) \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R})$, the following assertions holds with some $C, \epsilon > 0$. If $||\nabla_m \mathcal{A}_F^{\widetilde{H}}(v,\eta)||_m < \epsilon$,

$|\eta| \leq C\big(|\mathcal{A}_F^{\widetilde{H}}(w_-)|+|\mathcal{A}_F^{\widetilde{H}}(w_+)|+\epsilon+\Xi_\epsilon+1\big)$ provided that $||\nabla_m \mathcal{A}_F^{\widetilde{H}}(v,\eta)||_m < \epsilon$

where $\Xi_\epsilon = \max\big\{||\lambda_{\widetilde{M_2}}(x)||_{\tilde{g}_2} \,|\, d_{\tilde{g}_2}(x, \widetilde{M}_\star) < \epsilon + ||X_F||_{L^\infty}\big\}\big(\epsilon + ||X_F||_{L^\infty}\big) < \infty$.

Proof. The proof is almost same as the proof of Lemma 6.1.3. Since

$$||\nabla_m \mathcal{A}_F^{\widetilde{H}}(v,\eta)||_m < \epsilon,$$

$v(t) \subset U_\delta$ for $t \in (\frac{1}{2}, 1)$ by Lemma 6.3.3. Thus Lemma 6.3.1 and Lemma 6.3.2 prove the lemma. □

This fundamental lemma proves compactness of gradient flow lines and enables us to find a leafwise intersection points. Let $\phi \in \mathrm{Ham}_c(M_1 \times M_2, \omega_1 \oplus \omega_2)$ be a Hamiltonian diffeomorphism with the Hofer norm less than $\wp(\Sigma_1, \lambda_1)$. Then there exists a leafwise coisotropic intersection point even if $\Sigma_1 \times M_2$ does not bound a compact region in $M_1 \times M_2$, see the proof of Theorem A.

Next, we define the Rabinowitz Floer homology for $(\Sigma_1 \times M_2, M_1 \times M_2)$ in the same way as before and derive the Künneth formula in this situation. We consider another two action functionals $\mathcal{A}^H : \mathcal{L}_{M_1} \times \mathbb{R} \to \mathbb{R}$ and $\mathcal{A} : \mathcal{L}_{M_2} \to \mathbb{R}$ defined by

$$\mathcal{A}^H(v_1,\eta) := -\int_0^1 v_1^*\lambda_1 - \eta \int_0^1 H(t,v)dt, \quad \mathcal{A}(v_2) := -\int_{D^2} \bar{v}_2^*\omega_2.$$

where $H(t,x) = \chi(t)G(x) \in C^\infty(S^1 \times M_1)$. As in the proof of Theorem F, we compare critical points of $\mathcal{A}^{\widetilde{H}}$ and critical points of $\mathcal{A}^H$ as follows.

$$\mathrm{Crit}_n(\mathcal{A}^{\widetilde{H}}) = \bigcup_{i+j=n} \mathrm{Crit}_i(\mathcal{A}^H) \times \mathrm{Crit}_j(\mathcal{A}).$$

Since Crit$\mathcal{A}$ consists of one component M_2, any gradient flow line with cascades of $\mathcal{A}$ necessarily has zero cascades, and hence is simply a gradient flow line of an additional Morse function $f \in C^\infty(M_2)$. Thus the chain group for the Morse-Bott homology of $\mathcal{A}$ is given by $\mathrm{CF}(\mathcal{A}, f) = \mathrm{CM}(f)$. Here CM stands for the Morse complex. The following map is a chain isomorphism, which can be verified using the methods of the previous subsection.

$$\begin{aligned} P_n : \big(\mathrm{CF}_*(\mathcal{A}^H) \otimes \mathrm{CM}_*(f)\big)_n &\longrightarrow \mathrm{CF}_n(\mathcal{A}^{\widetilde{H}}), \\ (v_1, \eta) \otimes v_2 &\longmapsto \big((v_1, v_2), \eta\big). \end{aligned}$$

Therefore it induces an isomorphism on the homology level

$$(P_\bullet)_* : \mathrm{H}_\bullet\big(\mathrm{CF}(\mathcal{A}^H) \otimes \mathrm{CM}(f)\big) \xrightarrow{\cong} \mathrm{H}_\bullet\big(\mathrm{CF}(\mathcal{A}^{\widetilde{H}})\big) = \mathrm{RFH}_\bullet(\Sigma_1 \times M_2, M_1 \times M_2)$$

and the Künneth formula for $(\Sigma_1 \times M_2, M_1 \times M_2)$ directly follows:

$$\mathrm{RFH}_n(\Sigma_1 \times M_2, M_1 \times M_2) \cong \bigoplus_{p=0}^{n} \mathrm{RFH}_p(\Sigma_1, M_1) \otimes \mathrm{H}_{n-p}(M_2).$$

Chapter 7

Infinitely many leafwise coisotropic intersection points

As we have mentioned, we do not have a compactness theorem for the perturbed Rabinowitz action functional on product manifolds in general. For that reason, the existence problem of leafwise coisotropic intersection points for a product of restricted contact hypersurfaces is still open. However if a product of restricted contact hypersurfaces is of restricted contact type again, we have proved the compactness theorem in Chapter 3. Therefore we are able to find leafwise coisotropic intersection points using the Künneth formula derived in the previous chapter on restricted contact coisotropic submanifolds of product type. In particular, we find a class of restricted contact coisotropic submanifolds which have infinitely many leafwise coisotropic intersection points for a generic perturbations using the Künneth formula.

7.1 Proofs of Corollary F and Corollary G

Since the Rabinowitz action functional can be defined for each homotopy class of loops, we can define the Rabinowitz Floer homology $\mathrm{RFH}(\Sigma, M, \gamma)$

for $\gamma \in [S^1, M]$. Note that $\mathrm{RFH}(\Sigma, M)$ considered so far, equals $\mathrm{RFH}(\Sigma, M, x)$, $x \in M$. We also can define Rabinowitz Floer homology on the full loop space $\Lambda N := C^\infty(S^1, M)$ and denote it by $\mathbf{RFH}(\Sigma, M)$. Then we have

$$\mathbf{RFH}_*(\Sigma, M) = \bigoplus_{\gamma \in [S^1, M]} \mathrm{RFH}_*(\Sigma, M, \gamma).$$

Theorem 7.1.1. *[CFO, AS] For a unit cotangent bundle S^*N over a closed Riemannian manifold N,*

$$\mathbf{RFH}_*(S^*N, T^*N) \cong \begin{cases} \mathrm{H}_*(\Lambda N), & * > 1, \\ \mathrm{H}^{-*+1}(\Lambda N), & * < 0. \end{cases}$$

Since the Künneth formula obviously holds for **RFH** as well, the following corollary directly follows.

Corollary 7.1.2. *Let Σ_1 be a restricted contact hypersurface in (M_1, ω_1) bounding a compact region. If $\mathbf{RFH}_*(\Sigma_1, M_1) \neq 0$, and $\dim \mathrm{H}_*(\Lambda N) = \infty$ then*

$$\dim \mathbf{RFH}_*(\Sigma_1 \times S^*N, M_1 \times T^*N) = \infty.$$

Accordingly, if $\Sigma_1 \times S^*N$ is of contact type again, $\Sigma_1 \times S^*N$ has infinitely many leafwise coisotropic intersection points or a periodic leafwise coisotropic intersection point for a generic perturbation $\phi_F \in \mathrm{Ham}_c(M_1 \times M_2)$.

From now on, we investigate leafwise coisotropic intersection points on $(S^*S^1 \times S^*N, T^*S^1 \times T^*N)$.

Lemma 7.1.3. $S^*S^1 \times S^*N$ is a contact submanifold of codimension two in $T^*S^1 \times T^*N$.

Proof. $(T^*S^1, \omega_{\mathrm{S}^1,\mathrm{can}}) \cong (S^1 \times \mathbb{R}, d\theta \wedge dr)$ where θ is the angular coordinate on S^1 and r is the coordinate on $\mathbb{R}$. Then $d\theta \wedge dr$ has two global primitives $-rd\theta$ and $-rd\theta + d\theta$. We can easily check that $S^*S^1 \times S^*N$ carries a contact structure with $-rd\theta \oplus \lambda_{\mathrm{N,can}}$ and $(-rd\theta + d\theta) \oplus \lambda_{\mathrm{N,can}}$ where $\lambda_{\mathrm{N,can}}$ is the canonical one form on T^*N. $\square$

To exclude periodic leafwise coisotropic intersection points, we consider the loop space Ω defined by

$$\Omega := \big\{ v = (v_1, v_2) \in C^\infty(S^1, T^*S^1 \times T^*N) \,\big|\, v_1 \text{ is contractible in } T^*S^1 \big\}.$$

Then we consider the Rabinowitz action functional on this loop space, $\mathcal{A}^{\widetilde{G}_1, \widetilde{G}_2} : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ which defines the Rabinowitz Floer homology $\mathrm{RFH}(S^*S^1 \times S^*N, T^*S^1 \times T^*N, \Omega)$. Moreover the following type of the Künneth formula holds.

$$\mathrm{RFH}_n(S^*S^1 \times S^*N, T^*S^1 \times T^*N, \Omega) \cong \bigoplus_{p=0}^{n} \mathrm{RFH}_p(S^*S^1, T^*S^1) \otimes \mathbf{RFH}_{n-p}(S^*N, T^*N).$$

Therefore $\mathrm{RFH}(S^*S^1 \times S^*N, T^*S^1 \times T^*N, \Omega)$ is of infinite dimensional whenever $\dim \mathrm{H}_*(\Lambda N) = \infty$ and Lemma 7.1.4 below yields that there are infinitely many leafwise coisotropic intersection points for a generic perturbation $\phi_F \in \mathrm{Ham}_c(T^*S^1 \times T^*N)$ if $\dim N \geq 2$. This proves Corollary F.

In order to prove that there is generically no periodic leafwise coisotropic intersection points, we use an argument in [AF2]. Consider $\mathcal{A}_F^{\widetilde{H}_1, \widetilde{H}_2} : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ where $\widetilde{H}_i(t, x) = \chi(t) G_i(x) \in C^\infty(S^1 \times M_1 \times M_2)$, $i = 1, 2$ and where $F \in C_c^\infty(S^1 \times M_1 \times M_2)$ with $F(t, \cdot) = 0$ for $t \in (1/2, 1)$. We denote by $\mathcal{R}$ the set of periodic Reeb orbits in T^*N which has dimension one. It is convenient to introduce the following sets:

$$\mathcal{F}^j := \big\{ F \in C_c^j(S^1 \times T^*S^1 \times T^*N) \,\big|\, F(t, \cdot) = 0,\ \forall t \in \big[\frac{1}{2}, 1\big] \big\}, \quad \mathcal{F} := \bigcap_{j=1}^{\infty} \mathcal{F}^j.$$

Lemma 7.1.4. If $\dim N \geq 2$, the following set is dense in $\mathcal{F}$.

$$\mathcal{F}_{S^*S^1\times S^*N} := \left\{ F \in \mathcal{F} \;\middle|\; \begin{array}{c} \mathcal{A}_F^{\widetilde{H}_1,\widetilde{H}_2} \text{ is Morse, } v(0)\cap(S^*S^1\times R)=\emptyset \\ \text{for all } (v,\eta_1,\eta_2)\in \mathrm{Crit}\mathcal{A}_F^{\widetilde{H}_1,\widetilde{H}_2},\ R\in\mathcal{R}. \end{array} \right\}.$$

Proof. We denote by

$$\Omega^{1,2} := \left\{ v=(v_1,v_2)\in W^{1,2}(S^1,T^*S^1\times T^*N) \,\middle|\, v_1 \text{ is contractible in } T^*S^1 \right\}.$$

the loop space which is indeed a Hilbert manifold. Let $\mathcal{E}$ be the L^2-bundle over $\Omega^{1,2}$ with $\mathcal{E}_v = L^2(S^1, v^*T(S^*S^1\times S^*N))$. We consider the section

$$S:\Omega^{1,2}\times\mathbb{R}^2\times\mathcal{F}^j \longrightarrow \mathcal{E}^\vee\times\mathbb{R}^2 \quad \text{defined by} \quad S(v,\eta_1,\eta_2,F) := d\mathcal{A}_F^{\widetilde{H}_1,\widetilde{H}_2}(v,\eta_1,\eta_2).$$

Here the symbol $\vee$ represents the dual space. At $(v,\eta_1,\eta_2,F)\in S^{-1}(0)$, the vertical differential

$$DS : T_{(v,\eta_1,\eta_2,F)}\Omega^{1,2}\times\mathbb{R}^2\times\mathcal{F}^j \longrightarrow \mathcal{E}_v^\vee\times\mathbb{R}^2$$

is given by the pairing

$$\left\langle DS_{(v,\eta_1,\eta_2,F)}[\hat{v}^1,\hat{\eta}_1^1,\hat{\eta}_2^1,\hat{F}],[\hat{v}^2,\hat{\eta}_2^1,\hat{\eta}_2^2]\right\rangle = \mathcal{H}_{\mathcal{A}_F^{\widetilde{H}_1,\widetilde{H}_2}}[(\hat{v}^1,\hat{\eta}_1^1,\hat{\eta}_2^1),(\hat{v}^2,\hat{\eta}_1^2,\hat{\eta}_2^2)] + \int_0^1 \hat{F}(t,v)dt.$$

where $\mathcal{H}_{\mathcal{A}_F^{\widetilde{H}_1,\widetilde{H}_2}}$ is the Hessian of $\mathcal{A}_F^{\widetilde{H}_1,\widetilde{H}_2}$. As shown in [AF1], we know that for $(v,\eta_1,\eta_2,F)\in S^{-1}(0)$, $DS_{(v,\eta_1,\eta_2,F)}$ is surjective on the space

$$\mathcal{V} := \left\{ (\hat{v},\hat{\eta}_1,\hat{\eta}_2,\hat{F})\in T_{(v,\eta_1,\eta_2,F)}(\Omega^{1,2}\times\mathbb{R}^2\times\mathcal{F}^j) \,\middle|\, \hat{v}(0)=0 \right\}.$$

Next, we consider the evaluation map

$$\begin{aligned} \mathrm{ev} : \mathcal{M} &\longrightarrow S^*S^1 \times S^*N, \\ (v, \eta_1, \eta_2, F) &\longmapsto v(0). \end{aligned}$$

The surjectivity of $DS_{(v,\eta_1,\eta_2,F)}|_{\mathcal{V}}$ implies that ev is a submersion, see a lemma due to Salamon [AF2, Lemma 3.5]. Then $\mathcal{M}_{\mathcal{R}} := \mathrm{ev}^{-1}(S^*S^1 \times \mathcal{R})$ is a submanifold in $\mathcal{M}$ of

$$\operatorname{codim}(\mathcal{M}_{\mathcal{R}}/\mathcal{M}) = \operatorname{codim}(S^*S^1 \times \mathcal{R}/S^*S^1 \times S^*N).$$

We consider the projections $\Pi : \mathcal{M} \to \mathcal{F}^j$ and $\Pi_{\mathcal{R}} := \Pi_{|\mathcal{M}_{\mathcal{R}}}$. Then $\mathcal{A}_F^{\widetilde{H}_1,\widetilde{H}_2}$ is Morse if and only if F is a regular value of Π, which is a generic property by Sard-Smale theorem (for j large enough). The set $\Pi^{-1}(F)$ of leafwise coisotropic intersection points for F is manifold of required dimension zero since it is a critical set of $\mathcal{A}_F^{\widetilde{H}_1,\widetilde{H}_2}$. On the other hand, $\Pi_{\mathcal{R}}^{-1}(F)$ is a manifold of dimension

$$0 + \dim \mathcal{M}_{\mathcal{R}} - \dim \mathcal{M} = -\operatorname{codim}(\mathcal{M}_{\mathcal{R}}/\mathcal{M}) < 0$$

since we have assumed $\dim N \geq 2$. Therefore ev does not intersect $S^*S^1 \times \mathcal{R}$, so the set

$$\mathcal{F}^j_{S^*S^1 \times S^*N} := \mathcal{F}_{S^*S^1 \times S^*N} \cap \mathcal{F}^j$$

is dense in $\mathcal{F}$ for all $j \in \mathbb{N}$. Since $\mathcal{F}_{S^*S^1 \times S^*N}$ is the countable intersection of $\mathcal{F}^j_{S^*S^1 \times S^*N}$ for $j \in \mathbb{N}$, it is dense again in $\mathcal{F}$ and the lemma is proved. □

In the case of Theorem G, we consider the Rabinowitz action functional $\mathcal{A}_F^{\widetilde{H}} : \Omega_{M_2} \times \mathbb{R} \to \mathbb{R}$ by where

$$\Omega_{M_2} : \left\{ v = (v_1, v_2) \in C^\infty(S^1, M_1 \times M_2) \,\middle|\, v_2 \text{ is contractible in } M_2 \right\}.$$

In a similar vein as above, we are able to prove Corollary G.

Corollary 7.1.5. *Let* (M_2, ω_2) *be a closed symplectically aspherical symplectic manifold. If a closed manifold* N *has* $\dim \mathrm{H}_*(\Lambda N) = \infty$,

$$\dim \mathrm{RFH}_*(S^*N \times M_2, T^*N \times M_2, \Omega_{M_2}) = \infty.$$

Therefore, if $\dim N \geq 2$, $S^*N \times M_2$ *has infinitely many leafwise coisotropic intersection points for a generic perturbation.*

Remark 7.1.6. Corollary F and Corollary G still holds when we deal with a fiber-wise star shaped hypersurface in T^*N instead of S^*N, see [AF2].

Bibliography

[AF1] P. Albers, U. Frauenfelder, *Leaf-wise intersections and Rabinowitz Floer homology,* Journal of Topology and Analysis **2** (2010), no. 1, 77–98.

[AF2] P. Albers, U. Frauenfelder, *Infinitely many leaf-wise intersection points on cotangent bundles,* Expositiones Mathematicae **30** (2012), no. 2, 168–182.

[AF3] P. Albers, U. Frauenfelder, *Spectral invariants in Rabinowitz Floer homology and global Hamiltonian perturbation*, Journal of Modern dynamics **4** (2010), no. 2, 329–357.

[AF4] P. Albers, U. Frauenfelder, *A remark on a theorem by Ekeland-Hofer,* Israel Journal of Mathematics **187** (2012), no. 1, 485–491.

[AF5] P. Albers, U. Frauenfelder, *Rabinowitz Floer homology : A survey*, Global Differential Geometry, Springer Proceedings in Mathematics (2012), 437–461.

[AF6] P. Albers, U. Frauenfelder, *A variational approach to Givental's nonlinear Maslov index*, Geometric and Functional Analysis (GAFA), (2012), 1033–1050.

[AL] M. Audin and J. Lafontaine, editors. "Holomorphic curves in symplectic geometry", Progress Math. **117**, Birkhäuser Verlag , 1994.

BIBLIOGRAPHY

[AMo] P. Albers, A. Momin, *Cup-length estimates for leaf-wise intersections*, Mathematical Proceedings of the Cambridge Philosophical Society **149** (2010), 539–551.

[AMe1] P. Albers, W. Merry, *Translated points and Rabinowitz Floer homology*, Journal of Fixed Point Theory and Applications, (2013), 201–214.

[AMe2] P. Albers, W. Merry, *Orderability, contact non-squeezing, and Rabinowitz Floer homology*, (2013) arXiv:1302.6576.

[AFMe] P. Albers, U. Fuchs, W. Merry, *Orderability and the Weinstein Conjecture*, (2013) arXiv:1310.0786.

[AMc] P. Albers, M. McLean, Non-displaceable contact embeddings and infinitely many leaf-wise intersections, Journal of Symplectic Geometry **9** (2011), no. 3, 271–284.

[AS] A. Abbondandolo, M. Schwarz, *Estimates and computations in Rabinowitz-Floer homology,* Journal of Topology and Analysis **1** (2009), no. 4, 307–405.

[Ba] A. Banyaga, *On fixed points of symplectic maps,* Inventiones Mathematicae, **50** (1980), no. 3, 215–229.

[Bo1] P. Bolle, *Une condition de contact pour les sous-variétés coïsotropes d'une variété symplectique,* Comptes Rendus de l'Académie des Sciences - Series I **322** (1996), 83–86.

[Bo2] P. Bolle, *A contact condition for p-dimensional submanifolds of a symplectic manifold (*$2 \leq p \leq n$*),* Mathematische Zeitschrift **227** (1998), 211–230.

[BF] Y. Bae, U. Frauenfelder, *Continuation homomorphism in Rabinowitz Floer homology for symplectic deformations*, Math. Proc. Camb. Phil. Soc. (2011), **151** 471–502.

BIBLIOGRAPHY

[BO1] F. Bourgeois, A. oancea, *Fredholm theory and transversality for the parametrized and for the S^1-invariant symplectic action,* Journal of the European Mathematical Society **12** (2010), no. 5, 1181–1229.

[BO2] F. Bourgeois, A. oancea, *The index of Floer moduli problems for parametrized action functionals,* Geometriae Dedicata **165** (2013), no. 1, 5–24.

[CF] K. Cieliebak, U. Frauenfelder, *A Floer homology for exact contact embeddings*, Pacific Journal of Mathematics **249** (2009), no. 2, 251–316.

[CFO] K. Cieliebak, U. Frauenfelder, A. Oancea, *Rabinowitz Floer homology and symplectic homology,* Annales scientifiques de l'ENS **43** (2010), fasc. 6, 957–1015.

[CFP] K. Cieliebak, U. Frauenfelder, G.P. Paternain, *Symplectic topology of Mañé's critical values,* Geometry & Topology **14** (2010) 1765–1870.

[Dr] D. L. Dragnev, Symplectic rigidity, symplectic fixed points and global perturbations of Hamiltonian systems. Comm. Pure Appl. Math, **(**61) (2008), 346–370.

[EH] I. Ekeland, H. Hofer, *Two symplectic fixed-point theorems with applications to Hamiltonian dynamics,* J. Math. Pures Appl. **68** (1989), no. 4, 467–489.

[FHS] A. Floer, H. Hofer, D. Salamon, *Transversality in elliptic Morse theory for the symplectic action*, Duke Math. J. **80** (1995), no.1, 251–292.

[Fl1] A. Floer, *Morse theory for Lagrangian intersections*, J. Diff. Geom. **28** (1988), 513–547.

[Fl2] A. Floer, *The unregularizaed gradient flow of the symplectic action*, Comm. Pure Appl. Math. **41** (1988), 775–813.

BIBLIOGRAPHY

[Fr] U. Frauenfelder, *The Arnold-Givental conjecture and moment Floer homology,* Int. Math. Res. Not. **42** (2004), 2179–2269.

[FS] U. Frauenfelder, F. Schlenk, *S^1-equivariant Rabinowitz-Floer homology*, (2013) arXiv:1312.5201.

[Gi] V. L. Ginzburg, *Coisotropic intersections,* Duke Math. J. **140** (2007), no. 1, 111–163.

[Gü] B. Gürel, *Leafwise coisotropic intersections,* Int. Math. Res. Not. (2010), no. 5, 914–931.

[GG] V. L. Ginzburg, B. Gürel, *A C^2-smooth counterexample to the Hamiltonian Seifert conjecture in R^4*, Ann. of Math. **158** (2003), 953–976.

[Ho] H. Hofer, *On the topological properties of symplectic maps,* Proc. Roy. Soc. Edinburgh Sect. A **115** (1990), no. 1-2, 25–38.

[HWZ1] H. Hofer, K.Wysocki, E. Zehnder, *A general Fredholm theory I: A splicing-based differential geometry*, J. Eur. Math. Soc. **9** (2007), 841–876.

[HWZ2] H. Hofer, K.Wysocki, E. Zehnder, *A general Fredholm theory II: Implicit function theorems*, GAFA. **19** (2009), 206–293.

[HWZ3] H. Hofer, K.Wysocki, E. Zehnder, *A general Fredholm theory III: Fredholm functors and Polyfolds*, Geometry and Topology **13** (2009), 2279–2387.

[Ka1] J. Kang, *Survival of infinitely many critical points for the Rabinowitz action functional,* J. Modern Dynamics **4** (2010), no. 4, 733–739.

[Ka2] J. Kang, *Existence of leafwise intersection points in the unrestricted case,* Israel Journal of Mathematics, (2012) **190**, no. 1, 111–134.

BIBLIOGRAPHY

[Ka3] J. Kang, *Generalized Rabinowitz Floer homology and coisotropic intersections*, Int. Math. Res. Notices, (2013), no. 10, 2271–2322.

[Ka4] J. Kang, *Kuñneth formula in Rabinowitz Floer homology,* Cal. Var. Partial Differential Equations, (2013) **47**, No. 1, 763–787.

[Ke] E. Kerman, *Displacement energy of coisotropic submanifolds and Hofer's geometry,* J. Modern dynamics, (2008) **2** no. 3, 471–497.

[McS] D. McDuff, D. Salamon, "J-holomorphic curves and symplectic topology", AMS Colloquium Publication, **52**, 2004.

[MS] C.C. Moore, C. Schochet, "Global analysis on foliated spaces", Mathematical Sciences Research Institute Publications, **9**, 1988.

[MP] W. Merry, G. Paternain, *Index computations in Rabinowitz Floer homology*, J. Fixed Point Theory Appl. **10** (2011), No. 1, 88–111.

[MMP] L. Macarini, W. Merry, G. Paternain, *On the growth rate of leaf-wise intersections*, J. Symplectic Geom. **12** (2012), No. 4, 601–653.

[Mc] D. McDuff, *Symplectic manifolds with contact type boundaries*, Invent. Math. **103** (1991), no.3, 651–671.

[Me1] W. Merry, *On the Rabinowitz Floer homology of twisted cotangent bundles*, Calc. Var. Partial Differential Equations, **42** (2011), 355–404.

[Me2] W. Merry, *Lagrangian Rabinowitz Floer homology and twisted cotangent bundles*, Geom. Ded., (2013) DOI 10.1007/s10711-013-9903-9.

[Mo] J. Moser, *A fixed point theorem in symplectic geometry,* Acta Math. **141** (1978), no. 1-2, 17–34.

[Oa] A. Oancea, The Künneth formula in Floer homology for manifolds with restricted contact type boundary, Math. Ann.**334** (2006), 51–91.

BIBLIOGRAPHY

[Po] L. Polterovich, "The geometry of the group of symplectic diffeomorphisms", Lectures in Mathematics–Birkhäuser, (2001).

[Ra] P.H. Rabinowitz, *Periodic solutions of Hamiltonian systems,* Comm. Pure. Appl. Math **31** (1978). 157–184.

[RS] J.W. Robbin, D.A. Salamon *The spectral flow and the Maslov index*, Bulletin of the LMS **27** (1995), 1–33.

[Sa] D.A. Salamon, *Lectures on Floer homology,* in "Symplectic Geometry and Topology", Eds: Y. Eliashberg and L. Traynor, IAS/Park City Mathematics series **7** (1999), 143–230.

[Sc] M. Schwarz, "Morse homology", Birkhäuser, (1993)

[SV] D. Sullivan, M. Vigué-Poirrier, *The homology theory of the closed geodesic problem*, J. Differential Geometry **11** (1976), no. 4, 633–644.

[Us] M. Usher, *Boundary depth in Floer theory and its applications to Hamiltonian dynamics and coisotropic submanifolds,* Israel Journal of Mathematics **184** (2011), no. 1, 1–57.

[Zi1] F. Ziltener, *Coisotropic Submanifolds, leafwise fixed points, and presymplectic embeddings,* Journal of Symplectic Geom. **8** (2010), no. 1, 1–24.

[Zi2] F. Ziltener, *A Maslov map for coisotropic submanifolds, leaf-wise fixed points and presymplectic non-embeddings,* (2009), arXiv:0911.1460.

국문초록

Urs Frauenfelder와 Kai Cieliebak은 Paul Rabinowitz가 자율적 해밀턴 시스템에서 주기궤도들 찾기 위해 제안한 라그랑즈 승수 함수를 사용하여 Rabinowitz Floer homology 이론을 개발하였다.

이 논문에서는 우리는 임의의 여차원을 가지는 여등방성 부분다양체 위의 역학구조를 분석하는데 적합한 여러개의 Lagrange 상수들을 가지는 일반화된 Rabinowitz 함수를 연구할 것이다. 우리는 일반화된 Rabinowitz 함수를 사용하여 여등방성 궤적 교차점, 여등방성 부분 다양체의 전치가능성, 그리고 여등방성 부분다양체의 Rabinowitz Floer homology 등에 관해 연구할 것이다. 우리는 또한 Rabinowitz Floer homology의 Künneth 공식을 유도하여 무한개의 여등방 궤적 교차점을 가지는 여등방성 부분다양체들을 찾을 것이다. 이 연구는 여러 개의 운동 상수 (보존량) 를 가지는 운동 시스템을 연구하는데 중요한 역할을 할 것이다.

주요어휘: 라비노위츠 플로어 호몰로지, 해밀턴 역학, 보존량, 여등방성 부분다양체, 여등방 궤적교차점.

학번: 2008-20276

감사의 글

5 년이란 긴 시간동안 저를 아낌없이 지도해주신 Urs Frauenfelder 교수님께 감사드립니다. 어떻게 수학이란 학문을 공부하고 연구하여야하는지 배울 수 있었던 소중한 시간이었습니다. 저를 사교기하의 길로 인도해주시고 아낌없는 가르침과 조언을 주시는 조철현 교수님께 진심으로 감사의 말씀을 올립니다. Otto van Koert 교수님과의 토론 역시 폭 넓은 지식의 밑거름이 되었음에 감사드립니다. 1 년동안 Münster 대학에서 연구할 수 있도록 해주시며 많은 도움을 주신 Peter Albers 교수님께도 다시 한번 감사의 인사를 올립니다. 이 학위논문 심사를 비롯하여 많은 도움과 가르침을 주신 박종일 교수님과 이재혁 교수님께도 감사드립니다.

그리고 같이 공부하고 토론하며 즐거운 대학원 생활을 보내도록 해준 저의 모든 선후배, 친구들에게도 고맙습니다. 앞으로도 같은 길을 걸어가며 서로에게 든든한 동료가 되기를 바랍니다.

마지막으로 저의 든든한 후원자인 가족들에게 고마움과 사랑을 전합니다. 항상 제 옆을 지켜주는 혜선에게는 특별한 ♡ 을 보냅니다.

이학 박사 학위논문

Rabinowitz Floer homology and Coisotropic intersections

(라비노위츠 플로어 호몰로지와 여등방성 교차)

2014 년 2 월

서울대학교 대학원

수리과학부

강 정 수

Rabinowitz Floer homology and Coisotropic intersections

(라비노위츠 플로어 호몰로지와 여등방성 교차)

지도교수 Urs Frauenfelder

이 논문을 이학 박사 학위논문으로 제출함

2013 년 10 월

서울대학교 대학원

수리과학부

강 정 수

강 정 수의 이학 박사 학위논문을 인준함

2013 년 12 월

위 원 장 ______________________ (인)

부 위 원 장 ______________________ (인)

위 원 ______________________ (인)

위 원 ______________________ (인)

위 원 ______________________ (인)

Rabinowitz Floer homology and Coisotropic intersections

A dissertation
submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
to the faculty of the Graduate School of
Seoul National University

by

Jungsoo Kang

Dissertation Director : Professor Urs Frauenfelder

Department of Mathematical Science
Seoul National University

February 2014

Abstract

Rabinowitz Floer homology and Coisotropic intersections

Jungsoo Kang

Department of Mathematical Sciences
The Graduate School
Seoul National University

Rabinowitz Floer homology theory was developed by Kai Cieliebak and Urs Frauenfelder using a Lagrange multiplier action functional, which was introduced by Paul Rabinowitz in order to detect periodic orbits of autonomous Hamiltonian systems.

In this thesis, we study a generalized Rabinowitz action functional with several Lagrange multipliers, which is well suited for exploring dynamics on coisotropic submanifolds of arbitrary codimensions. Using this, we investigate among others, the existence problem of leafwise coisotropic intersection points, displaceability of coisotropic submanifolds, and Rabinowitz Floer homology for coisotropic submanifolds. We also derive a Künneth formula for the Rabinowitz Floer homology of product coisotropic submanifolds, and this enables us to find a class of coisotropic submanifolds which have infinitely many leafwise coisotropic intersection points. This study will serve as a crucial tool for exploring autonomous dynamical systems with several integrals.

Key words: Rabinowitz Floer homology, Hamiltonian dynamics, First integral, Coisotropic submanifold, Leafwise intersection.
Student Number: 2008-20276

Contents

CONTENTS

Chapter 1

Preliminaries on symplectic geometry

A symplectic form on a smooth manifold M is a closed nondegenerate 2-form $\omega \in \Omega^2(M)$. We call such a pair (M, ω) **symplectic manifold**. By nondegeneracy, every symplectic manifold is of even dimension and orientable. In particular, $\omega^{\wedge n}$ is a volume form of M if $\dim M = 2n$. The easiest example of a symplectic manifold is a Euclidean space with the standard symplectic structure $\left(\mathbb{R}^{2n}, \sum_{i=1}^{n} dx_i \wedge dy_i\right)$. In fact, every symplectic manifold is locally equivalent to this standard Euclidean space by Darboux's theorem. Thus in order to construct invariants of symplectic manifolds, one has to go beyond local considerations. The constructions of most global invariants in symplectic geometry, such as Floer-type homologies and Gromov-Witten invariants, use the fact that every symplectic manifold admits a family of compatible almost complex structure. An almost complex structure J on M is a complex structure on the tangent bundle, explicitly $J \in \mathrm{End}(TM)$ and $J^2 = -\mathbb{1}_{TM}$. A symplectic form $\omega \in \Omega^2(M)$ is called **compatible** with J if $g(\cdot, \star) := \omega(\cdot, J\star)$ defines a Riemannian metric on M such that $g(\cdot, \star) = g(J\cdot, J\star)$.

1.1 Hamiltonian diffeomorphisms

For any time-dependent smooth function $F \in C^\infty(S^1 \times M)$, the vector field X_F defined implicitly by

$$i_{X_F}\omega = dF$$

is called the **Hamiltonian vector field** associated to the Hamiltonian function F. The flow of the Hamiltonian vector field X_F is denoted by ϕ_F^t. The time one map $\phi_F = \phi_F^1$ of a Hamiltonian flow is called a **Hamiltonian diffeomorphism**. The set $\mathrm{Ham}(M,\omega)$ of all Hamiltonian diffeomorphisms is a group with respect to composition. We are interested in the subgroup $\mathrm{Ham}_c(M,\omega)$ which consists of Hamiltonian diffeomorphisms generated by compactly supported Hamiltonian functions. Next, we briefly recall the **Hofer norm** which gives rise to a unique nondegenerate bi-invariant Finsler metric on the group $\mathrm{Ham}_c(M,\omega)$.

Definition 1.1.1. Let $F \in C_c^\infty(S^1 \times M, \mathbb{R})$ be a compactly supported Hamiltonian function. Consider the L^∞-norm of F defined by

$$||F|| := ||F||_+ + ||F||_-.$$

where

$$||F||_+ := \int_0^1 \max_{x \in M} F(t,x)dt, \qquad ||F||_- := -\int_0^1 \min_{x \in M} F(t,x)dt = ||-F||_+.$$

For $\phi \in \mathrm{Ham}_c(M,\omega)$, the Hofer norm is

$$||\phi|| := \inf\{||F|| \mid \phi = \phi_F,\ F \in C_c^\infty(S^1 \times M, \mathbb{R})\}.$$

As mentioned above, the function d on $\mathrm{Ham}_c(M,\omega) \times \mathrm{Ham}_c(M,\omega)$ defined by $d(\phi,\psi) = ||\phi^{-1} \circ \psi||$ is the unique bi-invariant Finsler metric. The existence of the Hofer bi-invariant metric shows that $\mathrm{Ham}_c(M,\omega)$ is an infinite dimensional Lie group.

The following easy lemma will be useful in our story.

Lemma 1.1.2. [AF1] For all $\phi \in \mathrm{Ham}_c(M,\omega)$,

$$||\phi|| = |||\phi||| := \inf\{||F|| \mid \phi = \phi_F,\ F(t,\cdot) = 0\ \ \forall t \in [\tfrac{1}{2}, 1]\}\ .$$

1.2 Coisotropic submanifolds

Definition 1.2.1. A submanifold Σ in (M,ω) is said to be **coisotropic** if the symplectic orthogonal bundle

$$T\Sigma^\omega := \{(x,\xi) \in TM \,|\, \omega_x(\xi,\zeta) = 0 \ \text{ for all } \ \zeta \in T_x\Sigma\}$$

is a subbundle of $T\Sigma$. By definition,

$$0 \leq \operatorname{codim} \Sigma \leq \frac{1}{2} \dim M.$$

Example 1.2.2. Any hypersurface in (M,ω) is coisotropic. A submanifold $L \subset (M,\omega)$ is called **Lagrangian** if $TL = TL^\omega$ (or equivalently $\omega|_L \equiv 0$) and clearly every Lagrangian submanifold is coisotropic.

Since ω is closed, the symplectic orthogonal bundle $T\Sigma^\omega$ is integrable, and thus Σ is foliated by leaves of the characteristic foliation. We denote by L_x the **leaf through** x. In the extremal case that a connected coisotropic submanifold is Lagrangian, it is foliated by a single leaf.

Coisotropic submanifolds naturally arise in autonomous Hamiltonian systems with several integrals. Let (M,ω) be a $2n$-dimensional symplectic manifold. We denote by the **Hamiltonian tuple** $\mathcal{G} := (G_1,\ldots,G_k)$ for time-independent Hamiltonian functions $G_i \in C^\infty(M)$, $i \in \{1,\ldots,k\}$ for $1 \leq k \leq n$. We often regard $\mathcal{G}$ as an element of $C^\infty(M,\mathbb{R}^k)$.

Definition 1.2.3. Given two Hamiltonian functions F and G in $C^\infty(M)$, the **Poisson bracket**

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$$

is defined by $\{F,G\} := \omega(X_F, X_G)$. A Hamiltonian tuple $\mathcal{G}$ is said to be **Poisson-commuting** if $\{G_i, G_j\} = 0$ for any $1 \leq i,j \leq k$.

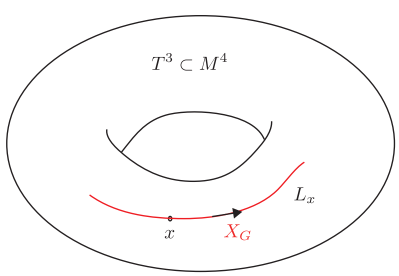
If a Hamiltonian tuple $\mathcal{G} \in C^\infty(M, \mathbb{R}^k)$ Poisson-commutes and $c \in \mathbb{R}^k$ is a regular value of $\mathcal{G}$, then an invariant submanifold $\mathcal{G}^{-1}(c)$ is a smooth coisotropic submanifold of codimension k in (M,ω) with

$$T\mathcal{G}^{-1}(c)^\omega = \langle X_{G_1}, \dots, X_{G_k} \rangle.$$

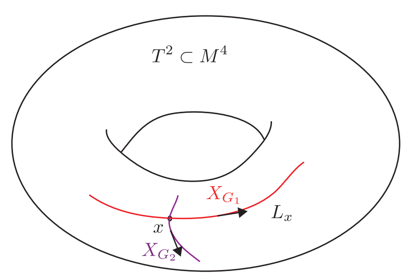
In this case the leaf L_x through $x \in \mathcal{G}^{-1}(c)$ can be written by

$$L_x = \left\{ \phi^{t_1}_{G_1} \circ \phi^{t_2}_{G_2} \circ \cdots \circ \phi^{t_k}_{G_k}(x) \,|\, t_1, \dots t_k \in \mathbb{R} \right\}.$$

Note that dimension of leaves equals $\dim M - \dim \mathcal{G}^{-1}(c) = k$, see pictures below.



$L_x = \{\phi^t_G(x) \,|\, t \in \mathbb{R}\}$



$L_x = \{\phi^{t_1}_{G_1} \circ \phi^{t_2}_{G_2}(x) \,|\, t_1,\, t_2 \in \mathbb{R}\}$

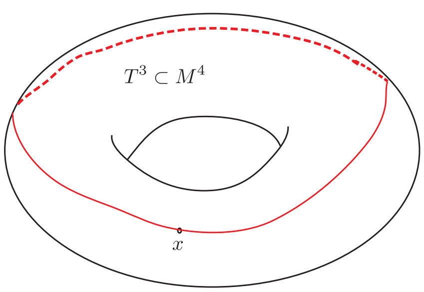
We briefly explain why such Hamiltonian systems are of great importance. A function $F \in C^\infty(M)$ is called an **integral** for a Hamiltonian system $\partial_t z = X_G(z(t))$ if F is constant along the solutions of $\partial_t z = X_G(z(t))$. It is easy to check that this condition is equivalent to $\{F, G\} = 0$. Hence, the motion of a Hamiltonian system $\partial_t z = X_G(z(t))$ with k independent Poisson commuting Hamiltonian integrals $G_1 = G, \ldots, G_k$ is confined to a $(2n-k)$-dimensional invariant submanifold $\bigcap_{1 \le i \le k} G_i^{-1}(c_i)$, $c_i \in \mathbb{R}$.

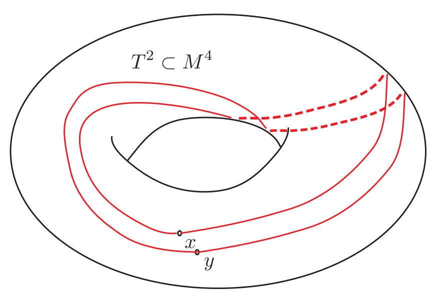
Remark 1.2.4. A 2n-dimensional Hamiltonian system is called **integrable** if there exist n independent Poisson commuting integrals $G_1, \ldots, G_n$. According to Liouville-Arnold, compact connected invariant submanifolds of integrable Hamiltonian systems are diffeomorphic to torus, i.e. $\bigcap_{1 \le i \le n} G_i^{-1}(c_i) \cong T^n$, $c_1, \ldots, c_n \in \mathbb{R}$. Moreover integrable Hamiltonian systems admit the so-called **action-angle coordinates** and this coordinates are described explicitly sometimes, e.g. Delaunay coordinates in the Kepler problem.

A periodic orbit $v : S^1 = \mathbb{R}/\mathbb{Z} \to \mathcal{G}^{-1}(c)$ lying on a leaf

$$\frac{d}{dt} v(t) = \sum_{i=1}^{k} \eta_i X_{G_i}(v(t)), \quad \eta_1, \ldots, \eta_k \in \mathbb{R} \tag{1.2.1}$$

is a key player of this thesis. Note that constant loops in $\mathcal{G}^{-1}(c)$ are trivial solutions of (1.2.1) with $\eta_1 = \cdots = \eta_k = 0$. Note that if $\mathcal{G}^{-1}(c)$ is a hypersurface, i.e. $k = 1$, a periodic orbit exists if and only if a leaf closes up.





We remark that if there is a periodic solution v of (1.2.1) on a leaf L_x, the leaf L_x is foliated by periodic solutions of (1.2.1). To see this, let x be a periodic point, i.e. $\phi_{G_1}^{t_1} \circ \cdots \circ \phi_{G_k}^{t_k}(x) = x$ for some $t_1, \ldots, t_k \in \mathbb{R}$. For any $y \in L_x$, there exists $r_1, \ldots, r_k \in \mathbb{R}$ such that $\phi_{G_1}^{r_1} \circ \cdots \circ \phi_{G_k}^{r_k}(x) = y$. Then

$$\begin{aligned}
\phi_{G_1}^{t_1} \circ \cdots \circ \phi_{G_k}^{t_k}(y) &= \phi_{G_1}^{t_1} \circ \cdots \circ \phi_{G_k}^{t_k} \circ \phi_{G_1}^{r_1} \circ \cdots \circ \phi_{G_k}^{r_k}(x) \\
&= \phi_{G_1}^{r_1} \circ \cdots \circ \phi_{G_k}^{r_k} \circ \phi_{G_1}^{t_1} \circ \cdots \circ \phi_{G_k}^{t_k}(x) \\
&= \phi_{G_1}^{r_1} \circ \cdots \circ \phi_{G_k}^{r_k}(x) \\
&= y.
\end{aligned}$$

Here we used the fact that the Hamiltonian flows commute due to Poisson commutativity. Therefore there is a periodic solution of (1.2.1) passing through any $y \in L_x$ provided the existence of a periodic solution of (1.2.1) on the leaf L_x.

Let us consider a single time-independent Hamiltonian function $G \in C^\infty(M)$. Suppose that a level hypersurface $G^{-1}(c)$ for $c \in \mathbb{R}$ is regular. From a simple computation

$$dG(X_G) = \omega(X_G, X_G) = 0,$$

we know that the Hamiltonian vector field X_G is tangent to the level hypersurface $G^{-1}(c)$. In general it is difficult to understand or foresee the dynamics of X_G on the given level surface $G^{-1}(c)$. For instance, even in $\mathbb{R}^4$ there is a time-independent Hamiltonian function such that at least one of its level surfaces has no periodic orbits which disproves the Hamiltonian Seifert conjecture, see [GG]. For this reason, we usually require an additional structure on a level hypersurface.

Definition 1.2.5. A hypersurface S in (M, ω) is called of **contact type** if there exists a 1-form $\alpha \in \Omega^1(S)$ such that $d\alpha = \omega|_S$ and $\omega|_S$ is nondegenerate on the hyperplane field TS^ω. There exists a unique vector field R on a contact hypersurface (S, α) such that

$$i_R d\alpha = 0, \qquad i_R \alpha = 1.$$

This vector field is called the **Reeb vector field** on (S, α).

The Reeb dynamics on contact hypersurfaces and the intersection problems for Lagrangian submanifolds have been widely studied. In contrast, coisotropic submanifolds have so far received little attention. The aim of this thesis is to study dynamics on a contact coisotropic submanifold, which is a natural generalization of a contact hypersurface. The notions of stable, contact, and restricted contact type for coisotropic submanifolds were introduced by Philippe Bolle [Bo1, Bo2].

Definition 1.2.6. A coisotropic submanifold Σ of codimension k in (M, ω) is called **stable** if there exist 1-forms $\alpha = (\alpha_1, \dots, \alpha_k)$ on Σ which satisfy

1. $\ker d\alpha_i \supset T\Sigma^\omega$ for $i = 1, \dots, k$;
2. $\alpha_1 \wedge \cdots \wedge \alpha_k \wedge (\omega|_\Sigma)^{n-k} \neq 0$.

We say that Σ is of **contact type** if $\alpha_1, \dots, \alpha_k$ are primitives of $\omega|_\Sigma$. If there are 1-forms $\lambda = (\lambda_1, \dots, \lambda_k)$ on M such that $d\lambda_i = \omega$ and $\lambda_i|_\Sigma = \alpha_i$ for all $i = 1, \dots, k$, Σ is said to be of **restricted** contact type.

Examples of stable/contact/restricted contact coisotropic submanifolds will be treated in the following section.

Definition 1.2.7. Let (Σ, α) be a stable coisotropic submanifold in (M, ω). The unique vector fields $R_1, \dots, R_k$ on Σ characterized by

$$\alpha_i(R_j) = \delta_{ij}, \quad R_i \in \ker \omega|_\Sigma, \quad i, j \in \{1, \dots, k\}$$

are called the **Reeb vector fields** associated with the stable structure (Σ, α). Here δ_{ij} stands for the Kronecker delta.

When a level surface $\mathcal{G}^{-1}(c)$ is stable, a periodic solution of 1.2.1 corresponds to a periodic solution $v \in C^\infty(S^1, \mathcal{G}^{-1}(c))$ of

$$\partial_t v(t) = \sum_{i=1}^{k} \eta_i R_i(v(t)), \quad \eta_1, \dots, \eta_k \in \mathbb{R}. \tag{1.2.2}$$

since

$$T\mathcal{G}^{-1}(c)^{\omega} = \langle R_1, \ldots, R_k \rangle = \langle X_{G_1}, \ldots, X_{G_k} \rangle.$$

Note that the normal bundle of a stable coisotropic submanifold $(\Sigma, \alpha) \subset (M, \omega)$ is trivial, i.e. $N\Sigma \cong \Sigma \times \mathbb{R}^k$ and from the Weinstein neighborhood theorem, we have the following proposition.

Proposition 1.2.8 ([Bo1, Bo2]). *Let (Σ, α) be a closed stable coisotropic submanifold of codimension k in (M, ω). Then there exist $r > 0$, a neighborhood V of Σ which is symplectomorphic by $\psi : U_r \to V$ to*

$$U_r := \{(q, \mathfrak{p}) = (q, p_1, \ldots, p_k) \in \Sigma \times \mathbb{R}^k \,|\, |p_i| < r, \text{ for all } i = 1, \ldots, k\}$$

with $\psi^*\omega = \omega|_{\Sigma} + \sum_{i=1}^k d(p_i \alpha_i)$.

Here we use the same symbols $\omega|_{\Sigma}$ and α_i for differential forms in Σ and for their pullback to $\Sigma \times \mathbb{R}^k$. We set

$$\delta_0 := \max\big\{ r \in \mathbb{R} \,\big|\, \text{there exists a symplectic embedding } \psi : U_r \hookrightarrow M \big\}$$

and let $\psi : U_{\delta_0} \hookrightarrow M$ be a maximal symplectic embedding. Henceforth, we identify U_δ with $\psi(U_\delta)$ for all $0 < \delta \leq \delta_0$. We have $X_{p_i} \in \ker \omega|_{\Sigma}$, $dp_j(X_{p_i}) = 0$ and $\alpha_j(X_{p_i}) = \delta_{ij}$ on Σ for $1 \leq i, j \leq k$ since $i_{X_{p_i}}\omega = dp_i$. Moreover the (local) Hamiltonian tuple $\mathfrak{p} = (p_1, \ldots, p_k)$ Poisson-commutes since $\{X_{p_1}, \ldots, X_{p_k}\}$ forms a basis for $\ker \omega|_{\Sigma}$.

We note that $X_{p_1}, \ldots, X_{p_k}$ correspond to $R_1, \ldots, R_k$ via the identification ψ_0. From now on, we choose an almost complex structure J on M which splits on U_ϵ with respect to

$$TU_{\delta_0} = \Big(\underbrace{\ker \omega|_{\Sigma}}_{=:\xi} \Big) \bigoplus \underbrace{\Big(T\Sigma^{\omega} \oplus \frac{\partial}{\partial p_1} \oplus \cdots \oplus \frac{\partial}{\partial p_k} \Big)}_{=:\xi^{\omega}}. \tag{1.2.3}$$

i.e. $J|_{\xi^\omega}$ is an almost complex structure which interchanges the Reeb vector fields R_i with $\frac{\partial}{\partial p_i}$ for $1 \leq i \leq k$; strictly speaking $JR_i = \frac{\partial}{\partial p_i}$ and $J\frac{\partial}{\partial p_i} = -R_i$.

1.3 Examples of contact coisotropic submanifolds

Although the contact condition is restrictive, we still have the following classes of contact coisotropic submanifolds.

(i) A coisotropic submanifold which is C^1-close to a contact coisotropic submanifold is also of contact type.

(ii) A Lagrangian torus is of contact type with contact one forms $d\theta_1, \ldots, d\theta_n$ where $\theta_1, \ldots, \theta_n$ are angular coordinates on the n-dimensional torus. Indeed it turns out that a closed Lagrangian submanifold of contact type is necessarily a torus.

(iii) Let $\Sigma \subset (M_1, \omega_1)$ be a contact coisotropic submanifold and $T^{n_2} \subset (M_2, \omega_2)$ be a Lagrangian torus. Then a coisotropic submanifold $\Sigma \times T^{n_2}$ in $(M_1 \times M_2, \omega_1 \oplus \omega_2)$ is of contact type. In particular, the stabilization of $\Sigma \subset (M, \omega)$, $\Sigma \times S^1 \subset (M \times T^*S^1, \omega \oplus d\theta \wedge dt)$ is of (restricted) contact type whenever Σ is of (restricted) contact type. Here θ is the base coordinate and t is the fiber coordinate.

(iv) Consider the Hopf fibration $\pi : S^{2n-1} \to \mathbb{C}P^{n-1}$. According to Marsden-Weinstein-Meyer reduction, we know that there is a canonical symplectic form $\omega_{\mathbb{C}P^{n-1}}$ on $\mathbb{C}P^{n-1}$ satisfying $\pi^*\omega_{\mathbb{C}P^{n-1}} = \omega_{\mathbb{R}^{2n}}|_{S^{2n-1}}$ where $\omega_{\mathbb{R}^{2n}}$ is the standard symplectic form on $\mathbb{R}^{2n}$. For a contact hypersurface $(\Delta, \alpha) \subset \mathbb{C}P^{n-1}$, $\pi^{-1}(\Delta)$ is a contact submanifold in $\mathbb{R}^{2n}$ of codimension 2.

Let (M, ω) be a closed symplectic manifold with an integral symplectic form $[\omega] \in \mathrm{H}^2(M;\mathbb{Z})$. For each $N \in \mathbb{N}$, there exists a complex line bundle $p : E^N \to M$ with the first Chern class $c_1(E^N) = -N[\omega]$. We note that S^1 acts on the bundle E^N by

$$\begin{aligned} S^1 \times E^N &\longrightarrow E^N \\ (t, v) &\longmapsto e^{2\pi i t} v. \end{aligned}$$

Thus by the Boothby-Wang theorem, there exists a connection 1-form α on $E^N \setminus E_0$ where E_0 is the zero section of the complex line bundle $E^N \xrightarrow{p} M$; moreover it holds that $p^* F_\alpha = d\alpha$ for the curvature 2-form $F_\alpha = N\omega$. We abbreviate $r = |e|$ for $e \in E^N$ and define $q : \mathbb{R} \to \mathbb{R}$ by $q(r) = \pi r^2 + 1/N$. Then the following two form gives a symplectic structure on E^N:

$$\Omega_E := q'(r) dr \wedge \alpha + q(r) N p^* \omega.$$

It is easy to check that $\Omega_E|_{E_0} = p_1^* \omega$ and $\Omega_E|_{E \setminus E_0} = d(q(r)\alpha)$. Furthermore, for all $c > 1/N$, the following submanifold

$$\Sigma_c := \{q(r) = c\}$$

is of contact type. We perform this construction once again. We choose a complex line bundle $p' : F^K \to M$ with the first Chern class $c_1(F^K) = -K[\omega]$. As before, there is a connection 1-form β on $F^K \setminus F_0$ where F_0 is the zero section of the bundle $F^K \xrightarrow{p'} M$ such that its curvature 2-form F_β satisfies $F_\beta = K\omega$. We set the function $h(s) = \pi s^2 + 1/K$ for $s = |f| \in \mathbb{R}$ where $f \in F^K$, then

$$\Omega_F := h'(s) ds \wedge \beta + h(s) K p'^* \omega$$

is a symplectic form on F^K. Next, we consider the Whitney sum of E^N and F^K, $E^N \oplus F^K$ and let $\pi_1 : E^N \oplus F^K \to E^N$ and $\pi_2 : E^N \oplus F^K \to F^K$ be the projection maps to the first factor and the second factor respectively. We abbreviate $\tilde{\omega} := (p \circ \pi_1)^*\omega = (p' \circ \pi_2)^*\omega$, and use the same symbols r, s, $g(r)$, $h(s)$, α, and β for their pull-backs to $E^N \oplus F^K$. Then the following 2-form

$$\Omega_{E\oplus F} := h'(s)ds \wedge \beta + q'(r)dr \wedge \alpha + (q(r)N + h(s)K)\tilde{\omega}$$

becomes a symplectic form on $E^N \oplus F^K$. We have

(v) For any $c > 1/N$ and $d > 1/K$, set

$$\Delta_{c,d} := \{q(r) = c,\, h(s) = d\}.$$

Since $\Omega_{E\oplus F}|_{\Delta_{c,d}} = (cN + dK)\tilde{\omega}$, $\Delta_{c,d}$ with 1-forms $\frac{cN+dK}{N}\alpha$ and $\frac{cN+dK}{K}\beta$ is a contact coisotropic submanifold in $(E^N \oplus F^K, \Omega_{E\oplus F})$ of codimension 2.

Proposition 1.3.1. *Let $\mathcal{G} \in C^\infty(M, \mathbb{R}^k)$ be a Poisson-commuting Hamiltonian tuple such that $c = (c_1, \ldots c_k) \in \mathbb{R}^k$ is a regular value of $\mathcal{G}$. Suppose that there is Liouville vector fields $Y_1, \ldots, Y_k$ (i.e. $\mathcal{L}_{Y_1}\omega = \cdots \mathcal{L}_{Y_k}\omega = \omega$) such that the matrix*

$$[dG_i(Y_j)]_{1\le i,j,\le k} = \begin{pmatrix} dG_1(Y_1) & \cdots & dG_1(Y_k) \\ \vdots & \ddots & \vdots \\ dG_k(Y_1) & \cdots & dG_k(Y_k) \end{pmatrix}$$

on $T\mathcal{G}^{-1}(c)$ is nonsingular. Then $\mathcal{G}^{-1}(c)$ is a contact coisotropic submanifold with contact forms $i_{Y_1}\omega, \ldots, i_{Y_k}\omega$.

Proof. Indeed, each $\alpha_j = i_{Y_j}\omega$ is a primitive of ω:

$$d\alpha_j = di_{Y_j}\omega = \mathcal{L}_{Y_j}\omega = \omega, \quad 1 \le j \le k.$$

Note that

$$T\mathcal{G}^{-1}(c)^\omega = \langle X_{G_1}, \ldots, X_{G_k}\rangle \subset T\mathcal{G}^{-1}(c),$$

and for all $1 \leq i \leq k$,

$$\omega(X_{G_i}, v) = dG_i(v) = 0, \quad \forall v \in T\mathcal{G}^{-1}(c).$$

We denote by

$$\xi := \{(x,v) \in T\mathcal{G}^{-1}(c) \,|\, \omega_x(Y_1, v) = \cdots = \omega_x(Y_k, v) = 0\}.$$

Since $[dG_i(Y_j)]_{1\leq i,j,\leq k}$ is nonsingular, we have the splitting

$$T\mathcal{G}^{-1}(c) = T\mathcal{G}^{-1}(c)^\omega \oplus \xi.$$

Moreover ξ is a symplectic complement of $\langle Y_1, \ldots, Y_k\rangle \oplus T\mathcal{G}^{-1}(c)^\omega$. Hence

$$\alpha_1 \wedge \cdots \wedge \alpha_k \wedge \omega|_{T\mathcal{G}^{-1}(c)} \neq 0$$

by nonsingularity of $[dG_i(Y_j)]_{1\leq i,j,\leq k}$ again. □

Dynamical problems, such as the (rotating) Kepler problem or Euler's three-body problem, sometimes admit several integrals. It is tempting to show whether such a problem has a (restricted) contact structure using the previous proposition.

Remark 1.3.2. [Bo2, Gi] Let Σ be a closed contact coisotropic submanifold in (M,ω). Then a 1-form $\lambda = a_1\lambda_1 + \cdots + a_k\lambda_k$ with $a_1 + \cdots + a_k = 0$ is closed and represents an element in $\mathrm{H}^1_{\mathrm{dR}}(\Sigma)$. In addition, $\lambda \neq 0$ is not exact; otherwise $\lambda = df$ for some $f \in C^1(\Sigma)$, $\lambda(x) = 0$ at a critical point x of f, but condition (ii) yields that $\lambda_1, \ldots, \lambda_k$ are linearly independent on Σ; thus $\lambda_1(x) = \cdots \lambda_k(x) = 0$. As a result, $\dim \mathrm{H}^1_{\mathrm{dR}}(\Sigma) \geq k-1$. It imposes a restriction on the contact condition that a product of contact type coisotropic submanifolds is not necessarily of contact type; for instance, $S^3 \times S^3$ is not of contact type in $\mathbb{R}^8$.

Remark 1.3.3. Furthermore, a connected sum of a contact coisotropic submanifold is not of contact type in general; for instance, a connected sum of Lagrangian tori is not a torus any more, hence cannot be of contact type. Different from the contact case, however, a product of stable coisotropic submanifolds is of stable type again.

Chapter 2

Statement of the results

The coisotropic intersection problems were first studied in depth by Viktor Ginzburg [Gi], and have been recently explored by many mathematicians, see Section 2.7. Rabinowitz Floer homology theory, which was developed by Kai Cieliebak and Urs Frauenfelder [CF] using the Rabinowitz action functional [Ra], is one of the effective methods to study the intersection problems for hypersurfaces. By generalizing the Rabinowitz Floer homology theory, we investigate the intersection problems of coisotropic submanifolds.

Throughout this thesis, we deal with a symplectic manifold (M, ω) which is symplectically aspherical and geometrically bounded. The condition that (M, ω) is **symplectically aspherical** means $\int_{\pi_2(M)} \omega = 0$. We call (M, ω) **geometrically bounded** if there exists an ω-compatible almost complex structure J with the property that the Riemannian metric $g(\cdot, \star) = \omega(\cdot, J\star)$ is complete, has injective radius bounded away from zero, and has bounded sectional curvature.

2.1 Assumptions on manifolds

In this thesis, we deal with the following classes of manifolds.

i) A closed coisotropic submanifold Σ in (M, ω) is stable or of contact type or of restricted contact type.

ii) A symplectic manifold (M, ω) is symplectically aspherical and geometrically bounded.

If Σ is a restricted contact coisotropic submanifold, (M, ω) is automatically symplectically aspherical (due to $\int_{\pi_2(M)} \omega = \int_{\pi_2(M)} d\lambda_i = 0$) but never closed. Thus if this is the case, (M, ω) is only assumed to be geometrically bounded. On the other hand, if (M, ω) is stable or of contact type, M can be closed. In this case, (M, ω) is obviously geometrically bounded and we only need to assume symplectic asphericity of (M, ω).

To define Rabinowitz Floer homology we need an additional assumption on stable/contact/restricted contact coisotropic submanifolds. In this thesis we focus on coisotropic submanifolds which are regular level sets of Poisson-commuting Hamiltonian tuples. Suppose that a stable coisotropic submanifold (Σ, α) is a regular level set of a Poisson-commuting Hamiltonian tuple $\mathcal{G} = (G_1, \cdots, G_k) \in C^\infty(M, \mathbb{R}^k)$, say $\mathcal{G}^{-1}(0) = \Sigma$. Then since both the Reeb vector fields of $\alpha = (\alpha_1, \dots, \alpha_k)$ and the Hamiltonian vector fields of $\mathcal{G}$ span the symplectic orthogonal bundle, i.e.

$$T\Sigma^\omega = \langle R_1, \dots, R_k \rangle = \langle X_{G_1}, \dots, X_{G_k} \rangle,$$

there exists a map from $\mathcal{G}^{-1}(0)$ to the set of $k \times k$ matrices

$$\Phi = (\Phi_{i,j}) : \mathcal{G}^{-1}(0) \to \mathrm{Mat}(k \times k)$$

such that

$$X_{G_j}(x) = \sum_{i=1}^{k} \Phi_{i,j}(x) R_i(x).^1$$

Note that $\Phi(x)$ for any $x \in \mathcal{G}^{-1}(0)$ is an invertible matrix. However in order for Rabinowitz Floer homology to be defined, we further require $\Phi(x)$ to have the following property.

iii) Σ is a regular level set of a Poisson-commuting Hamiltonian tuple $\mathcal{G} \in C^\infty(M, \mathbb{R}^k)$. For any $v \in C^\infty(S^1, \Sigma)$ contractible in M,

$$\int_{S^1} \Phi(v(t))dt \in \mathrm{Mat}(k \times k)$$

is invertible.

Remark 2.1.1. We choose a function $\chi(t) : S^1 \to [0, \infty)$ with $\int_{S^1} \chi(t)dt = 1$. Such a function will be used in Section 3. If $\int_{S^1} \Phi(v(t))dt$ is invertible for any $v \in C^\infty(S^1, \Sigma)$, so is $\int_{S^1} \chi(t)\Phi(v(t))dt$. Indeed, we can reparametrize a given $v \in C^\infty(S^1, \Sigma)$ to $v_\chi(t) = v \circ \int_0^t \chi(s)ds : S^1 \to [0, \infty)$ so that

$$\int_{S^1} \Phi(v_\chi(t))dt = \int_{S^1} \chi(t)\Phi(v(t))dt.$$

Note that an S^1-family of definite or diagonal matrices meets this third assumption. The assumption on the existence of "global coordinates" in [Ka3] is a special case of this assumption iii).

In order to find one leafwise coisotropic intersection point or one periodic orbit (Theorems A and D), we do not need the last assumption as Rabinowitz Floer homology is not directly involved. However, the last assumption is still indispensable to define Rabinowitz Floer homology and results using Rabinowitz Floer homology (Theorems B, C, E, F, and G).

[1]Strictly speaking, $\Phi(x)$ is an automorphism on $T_x\Sigma^\omega$, but here we tacitly assume $T\Sigma^\omega \cong \Sigma \times \mathbb{R}^k$ to have been trivialized.

Remark 2.1.2. All the above three assumptions appear in Rabinowitz Floer homology theory for hypersurfaces (see [CF]) as well. In particular, the last assumption matches with a **separating condition** for stable hypersurfaces. The separating condition means that a hypersurface Σ separates M into two connected components of which one is relatively compact. With the separating condition, it is possible to find a Hamiltonian function $G \in C^\infty(M)$ of Σ such that $G^{-1}(0) = \Sigma$. Moreover since Σ is of codimension 1, $\langle R \rangle = \langle X_G \rangle$ which in turn implies the assumption iii).

2.2 Main theorem

Let $\mathcal{L} \subset C^\infty(S^1, M)$ be the space of contractible loops in M. Let $\mathcal{G} = (G_1, \dots, G_k) \in C^\infty(M, \mathbb{R}^k)$ be a Poisson-commuting Hamiltonian tuple which has $0 \in \mathbb{R}^k$ (for simplicity) as a regular value. We also choose a compactly supported time-dependent Hamiltonian function $F \in C_c^\infty(S^1 \times M)$. For $\eta = (\eta_1, \dots, \eta_k) \in \mathbb{R}^k$, the generalized (perturbed) Rabinowitz action functional $\mathcal{A}_F^{\mathcal{G}} : \mathcal{L} \times \mathbb{R}^k \to \mathbb{R}$ is defined by

$$\mathcal{A}_F^{\mathcal{G}}(v, \eta) := -\int_{D^2} \bar{v}^*\omega - \sum_{i=1}^{k} \eta_i \int_0^1 G_i(v(t))dt - \int_0^1 F(t, v(t))dt.$$

where $\bar{v}$ is any filling disk of v, i.e. $\bar{v}|_{\partial D^2}(t) = v(t)$ for $t \in S^1$. The symplectic asphericity condition implies that the value of the above action functional is independent of the choice of filling discs. Then in Theorem 3.2.8, we will prove the following compactness result under the assumptions on $(M, \omega, \Sigma, \alpha)$ described in the previous section.

Main theorem. *Let $\{w^\nu\}_{\nu \in \mathbb{N}}$ be a sequence of gradient flow lines of $\mathcal{A}_F^{\mathcal{G}}$ for which there exist $a \leq b$ such that*

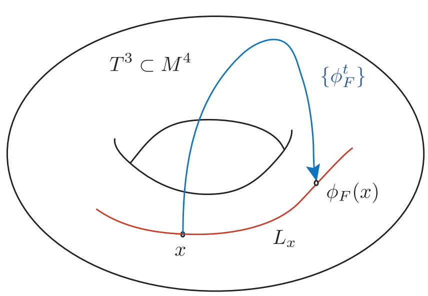
$$a \leq \mathcal{A}_F^{\mathcal{G}}(w^\nu(s)) \leq b, \quad \text{for all } \nu \in \mathbb{N},\ s \in \mathbb{R}. \tag{2.2.1}$$

Then for every reparametrization sequence $\sigma_\nu \in \mathbb{R}$ *the sequence* $w^\nu(\cdot + \sigma_\nu)$ *has a convergent subsequence in the* C^∞_{loc}*-topology. That is,* $\{w^\nu\}_{\nu\in\mathbb{N}}$ *has a subsequence which converges with all derivatives on every compact subset to a gradient flow line* $w \in C^\infty(\mathbb{R}\times S^1, M) \times C^\infty(\mathbb{R},\mathbb{R}^k)$*.*

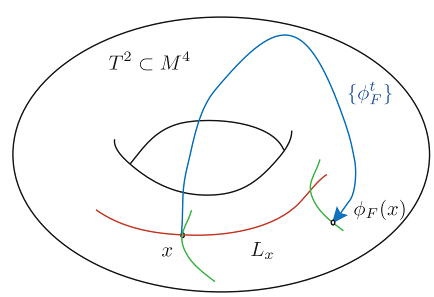
We refer to the next sections for a detailed and precise statement. Once we prove this compactness theorem, all the applications of Rabinowitz Floer homology to stable/contact/restricted contact hypersurfaces extend to corresponding results of stable/contact/restricted contact coisotropic submanifolds with minor modifications. For the sake of completeness, we include (sketches of) some applications, [AF1, AMo, CFP, Ka2, Ka3].

2.3 Leafwise coisotropic intersections

Let (M,ω) be a $2n$-dimensional symplectic manifold and Σ be a closed coisotropic submanifold of codimension k. Recall that Σ is foliated by leaves of $T\Sigma^\omega$ and L_x is the leaf through $x \in \Sigma$. A point $x \in \Sigma$ is called a **leafwise coisotropic intersection point** of $\phi_F \in \mathrm{Ham}_c(M,\omega)$ if $\phi(x)_F \in L_x$, see pictures below. In the extremal case $k = n$, a leafwise coisotropic intersection point is nothing but a Lagrangian intersection point.



$L_x = \{\phi^t_G(x) \,|\, t \in \mathbb{R}\}$



$L_x = \{\phi^{t_1}_{G_1} \circ \phi^{t_2}_{G_2}(x) \,|\, t_1,\, t_2 \in \mathbb{R}\}$

Definition 2.3.1. We denote by $\wp(\Sigma) > 0$ the **minimal symplectic area** of all solutions of (1.2.2) contractible in M. To be more exact,

$$\wp(\Sigma) := \inf\left\{|\Omega(v) > 0| \,\middle|\, v \in C^\infty(S^1, \Sigma) \text{ solving (1.2.2) and contractible in } M\right\}.$$

Here $\Omega : \mathcal{L} \to \mathbb{R}$ stands for the symplectic area functional, i.e.

$$\Omega(v) = \int_{D^2} \bar{v}^* \omega$$

where $\bar{v} \in C^\infty(D^2, M)$ is a filling disk of v, i.e. $\bar{v}|_{\partial D^2}(t) = v(t)$ for $t \in S^1$. The symplectic asphericity condition guarantees that the value of $\Omega(v)$ is independent of the choice of a filling disk. If there are no solutions of (1.2.2), we set $\wp(\Sigma) = \infty$ by convention.

Theorem A. *Let Σ be a closed restricted contact coisotropic submanifold in a symplectic manifold (M, ω) being geometrically bounded. If $||\phi_F|| < \wp(\Sigma)$, there exists a leafwise coisotropic intersection point for $\phi_F \in \mathrm{Ham}_c(M, \omega)$.*

The assumption on the Hofer norm of ϕ_F is sharp. For instance $\wp(S^{2n-1})$ equals the displacement energy of S^{2n-1} inside $(\mathbb{R}^{2n}, d\mathbf{x} \wedge d\mathbf{y})$.

Remark 2.3.2. Basak Gürel [Gü] also proved Theorem A using a different method. We cannot entirely drop the restricted contact condition in Theorem A, see [Gi, Example 7.2] and [Gü, Remark 1.4].

Even if a coisotropic submanifold Σ is of contact type, we still can find a leafwise intersection point for a restricted class of perturbations. In this case our ambient symplectic manifold need not to be exact and can be closed; so we have more examples. Recall that

$$U_r = \left\{(q, \mathfrak{p}) = (q, p_1, \ldots, p_k) \in \Sigma \times \mathbb{R}^k \,\middle|\, |p_i| < r, \text{ for all } i = 1, \ldots, k\right\}$$

and $\psi : U_{\delta_0} \hookrightarrow M$ is a maximal symplectic embedding. For a time dependent Hamiltonian function $F \in C_c^\infty(S^1 \times M)$, we define the support of the

Hamiltonian vector field X_F by

$$\mathrm{Supp}X_F := \{x \in M \,|\, X_F(t,x) \neq 0 \text{ for some } t \in S^1\}.$$

We call a Hamiltonian function $F \in C_c^\infty(S^1 \times M)$ **admissible** if F is constant outside of $\psi(U_{\delta_0})$, i.e. $\mathrm{Supp}X_F \subsetneq \psi(U_{\delta_0})$. We denote by $\mathfrak{F}$ the set of all admissible Hamiltonian functions:

$$\mathfrak{F} := \{F \in C_c^\infty(S^1 \times M) \,|\, \mathrm{Supp}X_F \subsetneq \psi(U_{\delta_0})\}.$$

Then Theorem A holds even for (not necessarily restricted) contact coisotropic submanifolds with $F \in \mathfrak{F}$.

Theorem A$^+$. *Let Σ be a closed contact coisotropic submanifold in a symplectically aspherical symplectic manifold (M,ω) which is geometrically bounded (M can be closed). Then ϕ_F for $F \in \mathfrak{F}$ has a leafwise coisotropic intersection point provided $||F|| < \wp(\Sigma)$.*

In fact, the assumptions in Theorem A is not sufficient to define a Rabinowitz Floer homology for Σ. That is one reason why we can find only one leafwise coisotropic intersection point. However if we additionally assume that Σ is given by a regular level set of a Poisson-commuting Hamiltonian tuple $\mathcal{G} \in C^\infty(M,\mathbb{R}^k)$ which is compatible with the Reeb vector fields on (Σ,α) in the sense of the assumption iii), we obtain a Morse-type estimate and a relative cup-length estimate for leafwise coisotropic intersection points.

Theorem B. *Let (M,ω) be geometrically bounded and Σ be a closed regular level set of a Poisson-commuting Hamiltonian tuple $\mathcal{G} \in C^\infty(M,\mathbb{R}^k)$. Suppose that Σ is of restricted contact type, and $\int_{S^1}\Phi(v)dt$ is invertible for any $v \in C^\infty(S^1,\Sigma)$ contractible in M. Then the number of leafwise coisotropic intersection points for a generic $\phi \in \mathrm{Ham}_c(M,\omega)$ with $||\phi|| < \wp(\Sigma)$ is bounded below by the sum of $\mathbb{Z}/2$-Betti numbers of Σ.*

Theorem B$^+$. *Let (M, ω) be geometrically bounded (M can be closed) and symplectically aspherical, and Σ be a closed regular level set of a Poisson-commuting Hamiltonian tuple $\mathcal{G} \in C^\infty(M, \mathbb{R}^k)$. Suppose that Σ is of contact type, and $\int_{S^1} \Phi(v)dt$ is invertible for any $v \in C^\infty(S^1, \Sigma)$ contractible in M. Then the number of leafwise coisotropic intersection points for a generic $\phi_F \in \mathrm{Ham}_c(M, \omega)$ with $F \in \mathfrak{F}$ and with $||F|| < \wp(\Sigma)$ is bounded below by the sum of $\mathbb{Z}/2$-Betti numbers of Σ.*

The genericity assumption on $\phi_F \in \mathrm{Ham}_c(M, \omega)$ in the above theorems comes from the Morse property of the Rabinowitz action functional perturbed by F. We are able to remove this assumption by the following cup-length estimate as usual.

Definition 2.3.3. The **relative cup-length** of Σ in M is defined by

$$\mathrm{cl}(\Sigma, M) := \max\{k \in \mathbb{N} \,|\, \exists a_1, \dots, a_k \in H^{\geq 1}(M; \mathbb{Z}/2) \text{ with } (a_1 \cup \cdots \cup a_k)|_\Sigma \neq 0\}.$$

Theorem C. *Let (M, ω) be geometrically bounded and Σ be a closed regular level set of a Poisson-commuting Hamiltonian tuple $\mathcal{G} \in C^\infty(M, \mathbb{R}^k)$. Suppose that Σ is of restricted contact type, and $\int_{S^1} \Phi(v)dt$ is invertible for any $v \in C^\infty(S^1, \Sigma)$ contractible in M. Then the number of leafwise coisotropic intersection points for any $\phi \in \mathrm{Ham}_c(M, \omega)$ with $||\phi|| < \wp(\Sigma)$ is bounded below by $\mathrm{cl}(\Sigma, M) + 1$.*

Theorem C$^+$. *Let (M, ω) be geometrically bounded (M can be closed) and symplectically aspherical, and Σ be a closed regular level set of a Poisson-commuting Hamiltonian tuple $\mathcal{G} \in C^\infty(M, \mathbb{R}^k)$. Suppose that Σ is of contact type, and $\int_{S^1} \Phi(v)dt$ is invertible for any $v \in C^\infty(S^1, \Sigma)$ contractible in M. Then the number of leafwise coisotropic intersection points for any $\phi_F \in \mathrm{Ham}_c(M, \omega)$ with $F \in \mathfrak{F}$ and with $||\phi|| < \wp(\Sigma)$ is bounded below by $\mathrm{cl}(\Sigma, M) + 1$.*

We do not include the proofs of theorems with "+" but these immediately follow from the proofs of the corresponding theorems (without "+") together with arguments in [Ka2].

Theorems A and B were proved by Peter Albers and Urs Frauenfelder [AF1], and Theorem C was proved by Peter Albers and Al Momin [AMo] for separating restricted contact hypersurfaces. As mentioned, once we obtain the main theorem in the previous section, such applications immediately follow with minor modifications. It is noteworthy that we succeed in removing the separating condition in Theorem A by a simple approximation argument.

2.4 Leafwise displacement energy

A coisotropic submanifold Σ in a symplectic manifold (M,ω) is said to be **leafwisely displaceable** if there exists a Hamiltonian diffeomorphism $\phi_F \in \mathrm{Ham}_c(M,\omega)$ such that $\phi_F(L_x)\cap L_x = \emptyset$ for all $x \in \Sigma$. The **leafwise displacement energy** of Σ in M is defined by

$$e(\Sigma) := \inf\big\{||F||\;\big|\;F \in C_c^\infty(S^1\times M),\ \phi_F(L_x)\cap L_x = \emptyset,\ \forall x \in \Sigma\big\}.$$

We set $e(\Sigma) = \infty$ for the infimum of the empty set; that is, the leafwise displacement energy of a leafwisely nondisplaceable coisotropic submanifold is infinity.

Theorem D. *Let Σ be a closed stable coisotropic submanifold leafwisely displaceable inside (M,ω) which is geometrically bounded (M can be closed) and symplectically aspherical. Then there exists a periodic orbit $v \in C^\infty(S^1,\Sigma)$,* *i.e. a solution of* (1.2.2)*, contractible in M, such that*

$$0 < |\Omega(v)| \leq e(\Sigma). \tag{2.4.1}$$

Remark 2.4.1. The estimate (2.4.1) is sharp. The unit sphere S^{2n-1} in $(\mathbb{R}^{2n}, d\mathbf{x} \wedge d\mathbf{y})$ has $e(S^{2n-1}) = \pi = \Omega(v)$ where v is a periodic Reeb orbit of the standard contact structure on S^{2n-1}. For displaceable closed restricted contact coisotropic submanifolds, Theorem D was proved by Viktor Ginzburg [Gi]. A similar result was also proved by Kai Cieliebak, Urs Frauenfelder, and Gabriel Paternain [CFP] for stable separating hypersurfaces using Rabinowitz Floer theory. Making use of their proof, we slightly improve their theorem.

2.5 Rabinowitz Floer homology

We introduced the Rabinowitz action functional $\mathcal{A}_F^{\mathcal{G}} : \mathcal{L} \times \mathbb{R}^k \to \mathbb{R}$. With $F \equiv 0$, the action functional $\mathcal{A}^{\mathcal{G}}$ is generically Morse-Bott. The chain complex for Floer homology of $\mathcal{A}^{\mathcal{G}}$ is generated by critical points of an auxiliary Morse function on the solution space of (1.2.2) and the boundary map is defined by counting gradient flow lines of the Morse function with gradient flow lines (cascades) of $\mathcal{A}^{\mathcal{G}}$ (based on Urs Frauenfelder's Morse-Bott homology [Fr]). On the other hand, $\mathcal{A}_F^{\mathcal{G}}$ with nonzero F is Morse for generic $F \in C^\infty(S^1 \times M, \mathbb{R})$. Up to reparametrization of time supports of $\mathcal{G}$ and F (see Chapter 3), the chain complex for Floer homology of $\mathcal{A}_F^{\mathcal{G}}$ is generated by leafwise coisotropic intersection points and the boundary map is defined by counting gradient flow lines of $\mathcal{A}_F^{\mathcal{G}}$. Here gradient flow lines of $\mathcal{A}^{\mathcal{G}}$ resp. $\mathcal{A}_F^{\mathcal{G}}$ are solutions of a nonlinear elliptic PDE.

One of the power of Floer homology is the invariance property. Two Floer homologies obtained by $\mathcal{A}^{\mathcal{G}}$ and $\mathcal{A}_F^{\mathcal{G}}$ are isomorphic due to the standard continuation argument in Floer theory, see Section 5. Thus we name

Rabinowitz Floer homology for both and denote by

$$\mathrm{RFH}(\Sigma, M) := \mathrm{HF}(\mathcal{A}^{\mathcal{G}}) \cong \mathrm{HF}(\mathcal{A}_F^{\mathcal{G}}).$$

We should mention that $\mathrm{RFH}(\Sigma, M)$ does not depend on the choice of $\mathcal{G} \in C^\infty(M, \mathbb{R}^k)$ the defining Hamiltonian tuple for Σ (up to canonical isomorphism).

Remark 2.5.1. Though we only deal with restricted contact coisotropic submanifolds, it is possible to define $\mathrm{HF}(\mathcal{A}^{\mathcal{G}})$ in the stable case or $\mathrm{HF}(\mathcal{A}_F^{\mathcal{G}})$ with $F \in \mathfrak{F}$ in the contact case. The assertions (i) and (ii) in Theorem E continue to hold for contact coisotropic submanifolds if we restrict the class of perturbations to $\mathfrak{F}$ and (iii) holds true for stable coisotropic submanifolds.

The following theorem is an immediate consequence of the construction and invariance property of Rabinowitz Floer homology.

Theorem E. *Let (M, ω) be geometrically bounded and Σ be a closed regular level set of a Poisson-commuting Hamiltonian tuple $\mathcal{G} \in C^\infty(M, \mathbb{R}^k)$. Suppose that Σ is of restricted contact type, and $\int_{S^1} \Phi(v)dt$ is invertible for any $v \in C^\infty(S^1, \Sigma)$ contractible in M.*

(i) *If Rabinowitz Floer homology does not vanish, there exists a leafwise coisotropic intersection point for every $\phi \in \mathrm{Ham}_c(M, \omega)$. In particular, if Σ is displaceable inside M, $\mathrm{RFH}(\Sigma, M) = 0$.*

(ii) *There exists a nonconstant solution of* (1.2.2) *contractible in M, provided that Σ is displaceable inside M.*

(iii) *If Σ carries no nonconstant solution of* (1.2.2) *contractible in M,*

$$\mathrm{RFH}(\Sigma, M) \cong \mathrm{H}(\Sigma; \mathbb{Z}/2).$$

In the extremal case, the assertions (i) and (iii) can be interpreted as:

(iv) *Let* Σ *be a Lagrangian torus, i.e.* $k = n$. *If* $i_\# : \pi_1(\Sigma) \to \pi_1(M)$ *is injective for the natural embedding* $i : \Sigma \hookrightarrow M$,[2]

$$\mathrm{RFH}(\Sigma, M) \cong \mathrm{H}(T^n; \mathbb{Z}/2).$$

2.6 Künneth formula

Here we only deal with the restricted contact case, but the same Künneth formulas for stable/contact coisotropic manifolds can be derived exactly the same way.

Theorem F. *Let* (Σ_1, λ_1) *and* (Σ_2, λ_2) *be restricted contact hypersurfaces in symplectic manifolds* (M_1, ω_1) *and* (M_2, ω_2) *respectively. Assume that* Σ_1 *resp.* Σ_2 *bounds a compact region in* M_1 *resp.* M_2 *and that* M_1 *and* M_2 *are geometrically bounded. Then,*

$$\mathrm{RFH}_n(\Sigma_1 \times \Sigma_2, M_1 \times M_2) \cong \bigoplus_{p=0}^{n} \mathrm{RFH}_p(\Sigma_1, M_1) \otimes \mathrm{RFH}_{n-p}(\Sigma_2, M_2).$$

Remark 2.6.1. Unfortunately we are only able to prove a compactness theorem for gradient flow lines of the unperturbed Rabinowitz action functional on $(\Sigma_1 \times \Sigma_2, M_1 \times M_2)$. Thus we cannot study about leafwise coisotropic intersection points except the case that $\Sigma_1 \times \Sigma_2$ is of restricted contact type again.

In Theorem G we do not consider Σ_2, and M_2 need to be closed.

[2] This implies that every solution of (1.2.2) is not contractible even in M.

Theorem G. *Let $(\Sigma_1, \lambda_1) \subset (M_1, \omega_1)$ be as in Theorem F above. Assume that (M_2, ω_2) is a closed symplectically aspherical symplectic manifold. Then,*

(G1) *$\Sigma_1 \times M_2$ has a leafwise coisotropic intersection point for $\phi \in \mathrm{Ham}_c(M_1 \times M_2, \omega_1 \oplus \omega_2)$ with Hofer-norm $||\phi|| < \wp(\Sigma_1, \lambda_1)$ even if Σ_1 does not bound a compact region in M_1.*

(G2) *The Rabinowitz Floer homology $\mathrm{RFH}(\Sigma_1 \times M_2, M_1 \times M_2) \cong \mathrm{HF}(\mathcal{A}_F^{\mathcal{G}})$ is defined for a generic $F \in C_c^\infty(M_1 \times M_2)$. Moreover, we have the Künneth formula:*

$$\mathrm{RFH}_n(\Sigma_1 \times M_2, M_1 \times M_2) \cong \bigoplus_{p=0}^{n} \mathrm{RFH}_p(\Sigma_1, M_1) \otimes \mathrm{H}_{n-p}(M_2).$$

Since we have not assumed any contact structure on $\Sigma_1 \times M_2$, we need a special version of isoperimetric inequality, see Lemma (6.3.1), in order to prove Theorem G.

Remark 2.6.2. It is worth emphasizing that $\Sigma_1 \times M_2$ is **never** of restricted contact type since M_2 is closed. Nevertheless, interestingly enough, we can achieve compactness of gradient flow lines of the perturbed Rabinowitz action functional for a generic (Morse property) perturbation $\phi_F \in \mathrm{Ham}_c(M_1 \times M_2, \omega_1 \oplus \omega_2)$.

Using the Künneth formulas and a result of [AF2], we are able to find infinitely many leafwise coisotropic intersection points on some coisotropic submanifolds.

Corollary F. *Let N be a closed Riemannian manifold of $\dim N \geq 2$ with $\dim \mathrm{H}_*(\Lambda N) = \infty$ where ΛN is the free loop space of N. Then there exists infinitely many leafwise coisotropic intersection points for a generic $\phi \in \mathrm{Ham}_c(T^*S^1 \times T^*N)$ on $(S^*S^1 \times S^*N, T^*S^1 \times T^*N)$.*

Remark 2.6.3. Since $(S^*S^1 \times S^*N, T^*S^1 \times T^*N)$ is of restricted contact type (see Lemma 7.1.3), ϕ in Corollary F is not necessarily of product type.

Corollary G. *Let N be as in Corollary F above, and (M, ω) be a closed symplectically aspherical symplectic manifold. Then a generic $\phi \in \mathrm{Ham}_c(T^*N \times M)$ has infinitely many leafwise coisotropic intersection points on $(S^*N \times M, T^*N \times M)$.*

2.7 List of related results

- **On Rabinowitz Floer homology theory:** [AF1, AF2, AF3, AF4, AF5, AF6, AFMe, AMe1, AMe2, AMo, AS, BF, CF, CFO, CFP, FS, Ka1, Ka2, Ka3, Ka4, Me1, Me2, MP, MMP].
- **On leafwise (coisotropic) intersections:** [AF1, AF2, AF4, AMo, AMe1, AMc, Ba, Dr, EH, Gi, Gü, Ho, Ka2, Ka3, Ka4, Mo, Me2, MMP, Zi1, Zi2].
- **On (Leafwise) displacement energy:** [Bo1, Bo2, Gi, Ka3, Ke, Us].

Chapter 3

The Rabinowitz action functional with several Lagrange multipliers

This chapter is devoted to the proof of the main theorem, which proves a compactness result for gradient flow lines of the Rabinowitz action functional, and to the proof of Theorem A.

3.1 The Rabinowitz action functional for coisotropic submanifolds

Let $\eta = (\eta_1, \ldots, \eta_k) \in \mathbb{R}^k$ be a k-tuple of Lagrange multipliers. We denote by $\mathcal{L} \subset C^\infty(S^1, M)$ the space of contractible loops in M. For an arbitrary Hamiltonian tuple $\mathcal{G} = (G_1, \ldots, G_k) \in C^\infty(M, \mathbb{R}^k)$ which has $0 \in \mathbb{R}^k$ as a regular value, and which is Poisson-commuting near $\bigcup_{i=1}^k G_i^{-1}(0)$, the generalized Rabinowitz action functional $\mathcal{A}^{\mathcal{G}} : \mathcal{L} \times \mathbb{R}^k \to \mathbb{R}$ is defined as follows:

$$\mathcal{A}^{\mathcal{G}}(v, \eta) := -\int_{D^2} \bar{v}^* \omega - \sum_{i=1}^{k} \eta_i \int_0^1 G_i(v(t))dt \tag{3.1.1}$$

where $\bar{v}$ is any filling disk of v, i.e. $\bar{v}|_{\partial D^2}(t) = v(t)$ for $t \in S^1$. The symplectic asphericity condition implies that the value of the above action functional is independent of the choice of filling discs. Using the standard scalar product $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^k$, we can express (3.1.1) by

$$\mathcal{A}^{\mathcal{G}}(v, \eta) = -\int_{D^2} \bar{v}^* \omega - \int_0^1 \langle \eta, \mathcal{G} \rangle (v(t)) dt.$$

A critical point of the Rabinowitz action functional, $(v, \eta) \in \mathrm{Crit} \mathcal{A}^{\mathcal{G}}$ satisfies the following equations.

$$\left.\begin{aligned} \partial_t v(t) &= \sum_{i=1}^{k} \eta_i X_{G_i}(v(t)), \quad t \in S^1 \\ \int_0^1 G_i(v(t)) dt &= 0, \quad i \in \{1, \dots, k\} \end{aligned}\right\} \tag{3.1.2}$$

Proposition 3.1.1. *If $(v, \eta) \in \mathrm{Crit} \mathcal{A}^{\mathcal{G}}$, $v(t) \in \mathcal{G}^{-1}(0)$ for all $t \in S^1$.*

Proof. Assume by contradiction that $G_j(v(t_0)) > 0$ for some $t_0 \in S^1$ and $j \in \{1, \dots, k\}$. Then to satisfy the second equation in (3.1.2), there exists $t_1 \in S^1$ such that $G_j(v(t_1)) < 0$ and hence $v(t_2) \in G_j^{-1}(0)$ for some $t_2 \in S^1$. Using the first equation in (3.1.2), we have

$$\frac{d}{dt} G_i(v(t)) = dG_i(v(t))[\partial_t v] = dG_i \left(\sum_{j=1}^{k} \eta_j X_{G_j}(v(t)) \right) = \sum_{j=1}^{k} \eta_j \{G_i, G_j\}(v(t))$$

which implies $G_i(v(t))$ is stationary whenever $v(t) \in G_j^{-1}(0)$ due to Poisson-commutativity of $\mathcal{G}$ near $\bigcup_{i=1}^{k} G_i^{-1}(0)$. Since $v(t_2) \in G_j^{-1}(0)$, $G_j(v(t)) = 0$ for all $t \in S^1$. This contradiction proves the proposition. $\square$

3.2 The perturbed Rabinowitz action functional

Let $\mathcal{G} \in C^\infty(M, \mathbb{R}^k)$ be as in the subsection. We choose a smooth function $\chi \in C^\infty(S^1, \mathbb{R})$ such that $\chi(t) \geq 0$, $\int_0^1 \chi(t)dt = 1$, and $\mathrm{Supp}\chi \subset (1/2, 1)$. Using χ, we define a time-dependent Hamiltonian $H_i : S^1 \times M \to \mathbb{R}$ by $H_i(t,x) = \chi(t)G_i(x)$ for $1 \leq i \leq k$, i.e.

$$\mathcal{H}(t,x) := \chi(t)\mathcal{G}(x) \in C^\infty(S^1 \times M, \mathbb{R}^k).$$

Let $F \in C_c^\infty(S^1 \times M)$ be an arbitrary time-dependent Hamiltonian function. Thanks to Lemma 1.1.2, we assume that F has time support in $(0, \frac{1}{2})$. We note that the time support of $\mathcal{H}$ and the time support of F are **disjoint**. With these Hamiltonian functions, the perturbed Rabinowitz action functional $\mathcal{A}_F^{\mathcal{H}} : \mathcal{L} \times \mathbb{R}^k \to \mathbb{R}$ is defined by

$$\mathcal{A}_F^{\mathcal{H}}(v, \eta) := -\int_{D^2} \bar{v}^*\omega - \int_0^1 F(t, v(t))dt - \int_0^1 \langle \eta, \mathcal{H}\rangle(t, v(t))dt.$$

where $\bar{v} : D^2 \to M$ is any filling disk of v. A critical point of the perturbed Rabinowitz action functional, $(v, \eta) \in \mathrm{Crit}\mathcal{A}_F^{\mathcal{H}}$ satisfies the following equations.

$$\left.\begin{aligned} &\partial_t v(t) = X_F(t, v) + \sum_{i=1}^k \eta_i X_{H_i}(t, v(t)), \quad && t \in S^1 \\ &\int_0^1 H_i(t, v(t))dt = 0, && i \in \{1, \dots, k\} \end{aligned}\right\} \tag{3.2.1}$$

In the next proposition, we observe that a critical point of $\mathcal{A}_F^{\mathcal{H}}$ gives rise to a leafwise coisotropic intersection point. Albers-Frauenfelder [AF1] proved the following proposition when Σ is a hypersurface. Their proof continues to work for coisotropic submanifolds with minor modifications.

Definition 3.2.1. A leafwise coisotropic intersection point $x \in \Sigma$ is called **periodic** if the leaf L_x contains a solution of (1.2.2).

Proposition 3.2.2. *If $(v,\eta) \in \mathrm{Crit}\mathcal{A}^{\mathcal{H}}_F$, $v(0) \in \Sigma$ is a leafwise coisotropic intersection point. Moreover, the map*

$$\mathrm{Crit}\mathcal{A}^{\mathcal{H}}_F \longrightarrow \big\{\textit{leafwise coisotropic intersections}\big\}$$

is injective unless there is no periodic leafwise coisotropic intersection.

Proof. Since the time support of F is $(0,1/2)$, for $t \geq 1/2$ and for all $i = 1,\dots,k$,

$$\frac{d}{dt}G_i(v(t)) = dG_i(v(t))[\partial_t v] = dG_i(v(t))\big[\underbrace{X_F(t,v)}_{=0} + \sum_{j=1}^{k}\chi(t)\eta_j X_{G_j}(v)\big]$$

As in the proof of Proposition 3.1.1, the second equation in (3.2.1) implies $v(t) \in \mathcal{G}^{-1}(0) = \Sigma$ for $t \in (1/2,1)$. On the other hand, v solves $\partial_t v = X_F(t,v)$ on $(0,1/2)$ so that $v(1/2) = \phi_F^{1/2}(v(0)) = \phi_F^1(v(0))$ since $F = 0$ for $t \geq 1/2$. For $t \in (1/2,1)$, it holds that $\partial_t v = \sum_{i=1}^k \eta_i X_{H_i}(t,v)$ and thus $v(0) = v(1) \in L_{v(1/2)}$. Thus we conclude that $v(0) \in L_{\phi_F(v(0))}$ which is equivalent to $\phi_F(v(0)) \in L_{v(0)}$. $\square$

From now on, we allow s-dependence on F as follows. Let $\{F_s\}_{s\in\mathbb{R}}$ be a family of Hamiltonian functions varying only on a finite interval in $\mathbb{R}$. More specifically, we assume $F_s(t,x) = F_-(t,x)$ for $s \leq -1$ and $F_s(t,x) = F_+(t,x)$ for $s \geq 1$. We also choose a family of compatible almost complex structures $\{J(s,t)\}_{(s,t)\in\mathbb{R}\times S^1}$ on M such that $J(s,t)$ is invariant outside of the interval $[-1,1] \subset \mathbb{R}$ and they still split as in (1.2.3).

On the tangent space $T_{(v,\eta)}(\mathcal{L}\times\mathbb{R}^k) = T_v\mathcal{L}\times T_\eta\mathbb{R}^k$ for $(v,\eta) \in \mathcal{L}\times\mathbb{R}^k$, we define the metric m as follows:

$$m_{(v,\eta)}\big((\hat{v}^1,\hat{\eta}^1),(\hat{v}^2,\hat{\eta}^2)\big) := \int_0^1 g_v(\hat{v}^1,\hat{v}^2)dt + \langle\hat{\eta}^1,\hat{\eta}^2\rangle.$$

Recall that $g(\cdot,\cdot) = \omega(\cdot,J\cdot)$ is a metric on M. Here $\hat{\eta}^1$ and $\hat{\eta}^2$ are elements in $T_\eta \mathbb{R}^k \cong \mathbb{R}^k$ and $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^k$.

Definition 3.2.3. A map $w \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^k)$ which solves

$$\partial_s w(s) + \nabla_m \mathcal{A}^{\mathcal{H}}_{F_s}(w(s)) = 0. \tag{3.2.2}$$

is called a **gradient flow line** of $\mathcal{A}^{\mathcal{H}}_{F_s}$ with respect to the metric m.

According to Floer's interpretation, the gradient flow equation (3.2.2) can be interpreted as $w = (u, \tau) = (u, \tau_1, \ldots, \tau_k)$ with $u(s,t) : \mathbb{R} \times S^1 \to M$ and $\tau_i(s) : \mathbb{R} \to \mathbb{R}$, solving

$$\left.\begin{aligned} &\partial_s u + J(s,t,u)\big(\partial_t u - \sum_{i=1}^k \tau_i X_{H_i}(t,u) - X_{F_s}(t,u)\big) = 0 \\ &\partial_s \tau_i - \int_0^1 H_i(t,u)dt = 0, \qquad\qquad 1 \le i \le k \end{aligned}\right\} \tag{3.2.3}$$

Definition 3.2.4. The **energy** of a map $w \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^k)$ is defined as

$$E(w) := \int_{-\infty}^{\infty} ||\partial_s w||_m^2 ds.$$

Lemma 3.2.5. Let $w \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^k)$ be a gradient flow line of $\mathcal{A}^{\mathcal{H}}_{F_s}$ with finite energy. Then we have the following estimate.

$$E(w) \le \mathcal{A}^{\mathcal{H}}_{F_-}(w_-) - \mathcal{A}^{\mathcal{H}}_{F_+}(w_+) + \int_{-\infty}^{\infty} ||\partial_s F_s||_- ds \tag{3.2.4}$$

where $w_\pm := \lim_{s \to \pm\infty} w(s) \in \mathrm{Crit}\mathcal{A}^{\mathcal{H}}_{F_s}$. Moreover, equality holds if $\partial_s F_s = 0$.

Proof. The following computation proves the lemma.

$$\begin{aligned}
E(w) &= -\int_{-\infty}^{\infty} d\mathcal{A}^{\mathcal{H}}_{F_s}\big(w(s)\big)[\partial_s w(s)]ds \\
&= -\int_{-\infty}^{\infty} \frac{d}{ds}\Big(\mathcal{A}^{\mathcal{H}}_{F_s}\big(w(s)\big)\Big)ds + \int_{-\infty}^{\infty} \big(\partial_s \mathcal{A}^{\mathcal{H}}_{F_s}\big)\big(w(s)\big)ds \\
&= \mathcal{A}^{\mathcal{H}}_{F_-}(w_-) - \mathcal{A}^{\mathcal{H}}_{F_+}(w_+) - \int_{-\infty}^{\infty}\int_0^1 \partial_s F_s(w)dtds \\
&\leq \mathcal{A}^{\mathcal{H}}_{F_-}(w_-) - \mathcal{A}^{\mathcal{H}}_{F_+}(w_+) + \int_{-\infty}^{\infty} ||\partial_s F_s||_- ds \ .
\end{aligned}$$

□

Remark 3.2.6. We note that $\int_{-\infty}^{\infty} ||\partial_s F_s||_- ds$ has a finite value since $\partial_s F_s$ has a compact support by construction.

Proposition 3.2.7. *$\mathcal{A}^{\mathcal{H}}_{F_s}$ has a uniform bound along gradient flow lines.*

Proof. For any gradient flow line $w \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^k)$ of $\mathcal{A}^{\mathcal{H}}_F$ and $s_1 < s_2 \in \mathbb{R}$, we calculate

$$\begin{aligned}
0 &\leq \int_{s_1}^{s_2} ||\partial_s w||^2_m \, ds \\
&= -\int_{s_1}^{s_2} d\mathcal{A}^{\mathcal{H}}_{F_s}(w(s))(\partial_s w)ds \\
&= \mathcal{A}^{\mathcal{H}}_{F_{s_1}}(w(s_1)) - \mathcal{A}^{\mathcal{H}}_{F_{s_2}}(w(s_2)) - \int_{s_1}^{s_2}\int_0^1 \partial_s F_s(t,v)dtds \\
&\leq \mathcal{A}^{\mathcal{H}}_{F_{s_1}}(w(s_1)) - \mathcal{A}^{\mathcal{H}}_{F_{s_2}}(w(s_2)) + \int_{s_1}^{s_2} ||\partial_s F_s||_- ds.
\end{aligned}$$

From the above inequality we obtain

$$\mathcal{A}^{\mathcal{H}}_{F_{s_2}}(w(s_2)) \leq \mathcal{A}^{\mathcal{H}}_{F_-}(w_-) + \int_{-\infty}^{\infty} ||\partial_s F_s||_- ds,$$
$$\mathcal{A}^{\mathcal{H}}_{F_{s_1}}(w(s_1)) \geq \mathcal{A}^{\mathcal{H}}_{F_+}(w_+) - \int_{-\infty}^{\infty} ||\partial_s F_s||_- ds.$$

This proves the proposition. □

3.2.1 Compactness

In this subsection, we prove Theorem 3.2.8 which is a vital ingredient for all our results. Here, Σ is assumed to be a closed restricted contact coisotropic submanifold. However for a perturbation $F \in \mathfrak{F}$, adapting an idea in [Ka2] we are able to prove the theorem in the contact case as well. We also need the assumptions ii) and iii).

Recall that $\Sigma = \mathcal{G}^{-1}(0)$. For compactness, we cut-off $\mathcal{G}$ to be constant away from Σ. More precisely, $M \setminus G_i^{-1}(0)$ consists of two parts M_i^+ and M_i^- such that $\pm G_i|_{M_i^\pm} > 0$ for $1 \leq i \leq k$. Therefore we are able to modify G_i so that for a small $\epsilon > 0$,

$$G_i = \begin{cases} \textbf{\textit{unchanged}} & \text{on } G_i^{-1}(-\epsilon, \epsilon), \\ \textbf{\textit{constant}} & \text{near infinity.} \end{cases}$$

for all $1 \leq i \leq k$. Note that $\mathcal{G}$ is **still** Poisson-commuting on $\bigcup_{i=1}^k G_i^{-1}(-\epsilon, \epsilon)$ after such a modification and thus Proposition 3.1.1 and Proposition 3.2.2 remain true.

Theorem 3.2.8. *Let $\{w^\nu = (u^\nu, \tau^\nu)\}_{\nu \in \mathbb{N}}$ be a sequence of gradient flow lines of $\mathcal{A}^{\mathcal{H}}_{F_s}$ for which there exist $a \leq b$ such that*

$$a \leq \mathcal{A}^{\mathcal{H}}_{F_s}(w^\nu(s)) \leq b, \quad \textit{for all } \nu \in \mathbb{N},\ s \in \mathbb{R}. \tag{3.2.5}$$

Then for every reparametrization sequence $\sigma_\nu \in \mathbb{R}$ the sequence $w^\nu(\cdot + \sigma_\nu)$ has a convergent subsequence in the C^∞_{loc}-topology. That is, $\{w^\nu\}_{\nu\in\mathbb{N}}$ has a subsequence which converges with all derivatives on every compact subset to a gradient flow line $w \in C^\infty(\mathbb{R}\times S^1, M)\times C^\infty(\mathbb{R},\mathbb{R}^k)$.

Proof. Once we prove Theorem 3.2.11 which is a new feature of Rabinowitz Floer theory, the rest of the proof is established by the following steps which are standard by now in Floer theory.

1. Since (M,ω) is geometrically bounded and we have modified $\mathcal{G}$ so that $\mathcal{G}$ is constant near infinity, we have a uniform bound on images of u^ν, see [AL] (also see [Mc, Lemma 2.4] for the convex at infinity case).

2. Due to Lemma 3.2.5 and Proposition 3.2.7, we have a uniform energy bound on u^ν and this implies a uniform bound on $\partial_s u^\nu$ except finitely many points.

3. On such finitely many points where the derivative $\partial_s u^\nu$ explodes, we can detect nonconstant J-holomorphic spheres, see [McS, Chapter 4.2]. However this so-called bubbling-off phenomenon does not occur due to symplectic asphericity.

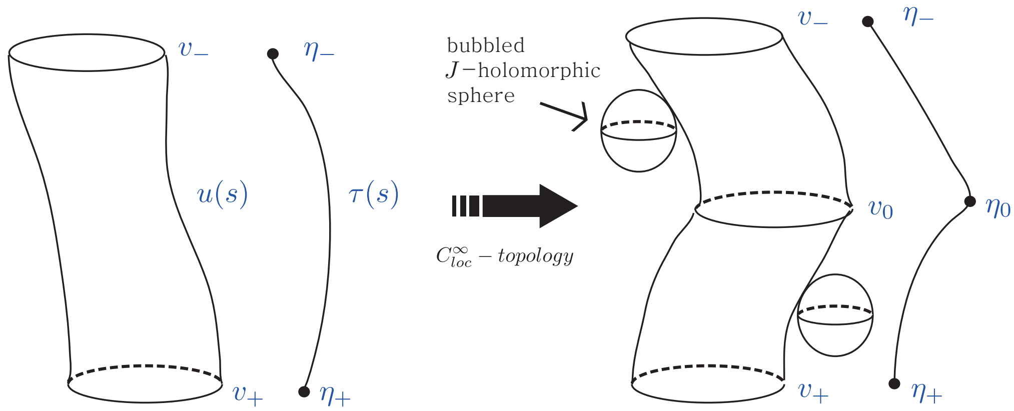
4. By Theorem 3.2.11, we have a uniform bound on $\tau_1^\nu,\dots,\tau_k^\nu$. From the gradient flow equation

$$\partial_s u^\nu + J(t,u^\nu)\Big(\partial_t u^\nu - \sum_{i=1}^k \tau_i^\nu(s) X_{G_i}(u^\nu)\Big) = 0,$$

   we obtain a uniform bound on $\partial_t u^\nu$ as well.

5. Now we can apply the elliptic bootstrapping argument in Floer theory, see [McS, Theorem B.4.2] and hence the assertion follows.

□



We first prove the following fundamental lemma which is a key step in proving Theorem 3.2.11.

Lemma 3.2.9. There exist $\epsilon > 0$ and $C > 0$ such that for $(v, \eta) \in \mathcal{L} \times \mathbb{R}^k$,

$$||\nabla_m \mathcal{A}^{\mathcal{H}}_{F_s}(v, \eta)||_m < \epsilon \quad \text{implies} \quad |\eta_i| \leq C\big(|\mathcal{A}^{\mathcal{H}}_{F_s}(v, \eta)| + 1\big) \ \text{ for all } 1 \leq i \leq k.$$

Proof. The proof proceeds in three steps.

Step 1: There exists a small constant $\delta \in (0, \delta_0)$ satisfying the following. Assume $v(t) \in U_\delta$ for $t \in (1/2, 1)$. Then there exists $C_0 > 0$ such that

$$|\eta_i| \leq C_0\big(|\mathcal{A}^{\mathcal{H}}_{F_s}(v, \eta)| + ||\nabla_m \mathcal{A}^{\mathcal{H}}_{F_s}(v, \eta)||_m + 1\big), \quad i = 1, \ldots, k.$$

Proof of Step 1. Recall that there exists a family of definite matrices

$$\Phi = (\Phi_{i,j}) : \mathcal{G}^{-1}(0) \to \mathrm{Mat}(k \times k)$$

such that

$$X_{G_i} = \Phi R_i, \quad 1 \leq i \leq k$$

and we have assumed

$$\int_{S^1} \chi(t)\Phi(v(t))dt \in \mathrm{Mat}(k \times k)$$

is invertible for any $v \in C^\infty(S^1, \Sigma)$ contractible in M, see Remark 2.1.1. For each $j = 1, \dots, k$,

$$\begin{aligned}
\mathcal{A}^{\mathcal{H}}_{F_s}(v,\eta) &= -\int_0^1 v^*\lambda_j - \sum_{i=1}^k \eta_i \int_0^1 H_i(t,v)dt - \int_0^1 F_s(t,v)dt \\
&= -\int_0^1 \lambda_j\big(\partial_t v - \sum_{i=1}^k \eta_i X_{H_i}(t,v) - X_{F_s}(t,v)\big)dt - \sum_{i=1}^k \int_0^1 \lambda_j\big(\eta_i X_{H_i}(t,v)\big)dt \\
&\quad - \int_0^1 \lambda_j(X_{F_s}(t,v))dt - \sum_{i=1}^k \eta_i \int_0^1 H_i(t,v)dt - \int_0^1 F_s(t,v)dt \\
&= -\int_0^1 \lambda_j\big(\nabla_m \mathcal{A}^{\mathcal{H}}_{F_s}(v,\eta)\big)dt - \sum_{i=1}^k \eta_i \int_0^1 \chi(t)\lambda_j\big(\sum_{\ell=1}^k \Phi_{i,\ell} R_\ell(v)\big)dt \\
&\quad - \int_0^1 \lambda_j(X_{F_s}(t,v))dt - \sum_{i=1}^k \eta_i \int_0^1 H_i(t,v)dt - \int_0^1 F_s(t,v)dt \\
&= -\int_0^1 \lambda_j\big(\nabla_m \mathcal{A}^{\mathcal{H}}_{F_s}(v,\eta)\big)dt - \sum_{i=1}^k \eta_i \int_0^1 \chi(t)\Phi_{i,j}(v)dt \\
&\quad - \int_0^1 \lambda_j(X_{F_s}(t,v))dt - \sum_{i=1}^k \eta_i \int_0^1 \chi(t)G_i(v)dt - \int_0^1 F_s(t,v)dt
\end{aligned}$$

Thus we have

$$-\sum_{i=1}^k \eta_i\chi(t) \int_0^1 \big(\Phi_{i,j} + G_i\big)(v)dt = \mathcal{A}^{\mathcal{H}}_{F_s}(v,\eta) + \int_0^1 \lambda_j\big(\nabla_m \mathcal{A}^{\mathcal{H}}_F(v,\eta) + X_{F_s}(t,v)\big) + F_s(t,v)dt$$

and

$$\Gamma(v)\begin{pmatrix}\eta_1\\ \vdots\\ \eta_k\end{pmatrix}=\begin{pmatrix}\mathcal{A}^{\mathcal{H}}_{F_s}(v,\eta)+\int_0^1\lambda_1\big(\nabla_m\mathcal{A}^{\mathcal{H}}_{F_s}(v,\eta)+X_{F_s}(t,v)\big)+F_s(t,v)dt\\ \vdots\\ \mathcal{A}^{\mathcal{H}}_{F_s}(v,\eta)+\int_0^1\lambda_k\big(\nabla_m\mathcal{A}^{\mathcal{H}}_{F_s}(v,\eta)+X_{F_s}(t,v)\big)+F_s(t,v)dt\end{pmatrix}$$

where $\Gamma(v)$ is a $k\times k$ matrix defined by

$$\Gamma(v):=\Big[-\int_0^1\chi(t)\big(\Phi_{i,j}+G_i\big)(v)dt\Big]_{1\le i,j\le k}$$

We choose small $\delta>0$ so that $\Gamma(v)$ is still invertible for any $v\subset U_\delta:=\mathcal{G}^{-1}(-\delta,\delta)$. Therefore,

$$\begin{pmatrix}\eta_1\\ \vdots\\ \eta_k\end{pmatrix}=\Gamma(v)^{-1}\begin{pmatrix}\mathcal{A}^{\mathcal{H}}_{F_s}(v,\eta)+\int_0^1\lambda_1\big(\nabla_m\mathcal{A}^{\mathcal{H}}_{F_s}(v,\eta)+X_{F_s}(t,v)\big)+F_s(t,v)dt\\ \vdots\\ \mathcal{A}^{\mathcal{H}}_{F_s}(v,\eta)+\int_0^1\lambda_k\big(\nabla_m\mathcal{A}^{\mathcal{H}}_{F_s}(v,\eta)+X_{F_s}(t,v)\big)+F_s(t,v)dt\end{pmatrix}.$$

Since

$$||\lambda_i||_{L^\infty(U_\delta)},\ ||(\Phi_{i,j}+G_i)||_{L^\infty(U_\delta)},\ ||F_s||_{L^\infty(U_\delta)},\ ||X_{F_s}||_{L^\infty(U_\delta)}<\infty,$$

there exists a constant $C_0>0$ such that

$$|\eta_j|\le C_0\big(|\mathcal{A}^{\mathcal{H}}_{F_s}(v,\eta)|+||\nabla_m\mathcal{A}^{\mathcal{H}}_{F_s}(v,\eta)||_m+1\big),\quad \forall j=1,\dots,k.$$

Step 2: If there is $t\in(\frac{1}{2},1)$ such that $v(t)\notin U_\delta$ then $||\nabla_m\mathcal{A}^{\mathcal{H}}_{F_s}(v,\eta)||_m\ge\epsilon$.

Proof of Step 2. The assumption $v(t)\notin U_\delta$ means that there exists $i\in\{1,\dots,k\}$ such that $v(t)\notin U^i_\delta:=G_i^{-1}(-\delta,\delta)$. If in addition, $v(t)\in M-U^i_{\delta/2}$ for all $t\in(\frac{1}{2},1)$ then we easily have

$$||\nabla_m\mathcal{A}^{\mathcal{H}}_{F_s}(v,\eta)||_m\ge\left|\int_0^1 H_i(t,v(t))dt\right|=\left|\int_{1/2}^1\chi(t)G_i(v(t))dt\right|\ge\frac{\delta}{2}.$$

Otherwise there exists $t' \in (\frac{1}{2}, 1)$ such that $v(t') \in U^i_{\delta/2}$. Thus we can find $t_0, t_1 \in (\frac{1}{2}, 1)$ such that

$$v(t_0) \in \partial U^i_{\delta/2}, \;\; v(t_1) \in \partial U^i_{\delta}, \;\; v(t) \in U^i_{\delta} - U^i_{\delta/2}, \quad \forall t \in [t_0, t_1],$$

or

$$v(t_1) \in \partial U^i_{\delta}, \;\; v(t_0) \in \partial U^i_{\delta/2}, \;\; v(t) \in U^i_{\delta} - U^i_{\delta/2}, \quad \forall t \in [t_1, t_0].$$

We treat only the first case. The latter case is analogous. With

$$\mathfrak{P} := \max_{x \in U_\delta} ||\nabla_g G_i(x)||_g < \infty$$

we estimate,

$$\begin{aligned}
\mathfrak{P}||\nabla_m \mathcal{A}^{\mathcal{H}}_{F_s}(v,\eta)||_m &\geq \mathfrak{P}||\partial_t v - \sum_{j=1}^{k} \eta_j X_{H_j}(t,v) - X_{F_s}(t,v)||_{L^2} \\
&\geq \mathfrak{P}||\partial_t v - \sum_{j=1}^{k} \eta_j X_{H_j}(t,v) - X_{F_s}(t,v)||_{L^1} \\
&\geq \int_{t_0}^{t_1} ||\partial_t v - \sum_{j=1}^{k} \eta_j X_{H_j}(t,v) - X_{F_s}(t,v)||_g ||\nabla_g G_i(v(t))||_g dt \\
&\geq \Big| \int_{t_0}^{t_1} \big\langle \nabla_g G_i(v(t)), \partial_t v(t) - \sum_{j=1}^{k} \eta_j X_{H_j}(t,v) - X_{F_s}(t,v) \big\rangle_g dt \Big| \\
&= \Big| \int_{t_0}^{t_1} dG_i(v(t)) \big(\partial_t v(t) - \sum_{j=1}^{k} \eta_j X_{H_j}(t,v) - \underbrace{X_{F_s}(t,v)}_{=0} \big) dt \Big| \\
&= \Big| \int_{t_0}^{t_1} \frac{d}{dt} G_i(v(t)) dt - \underbrace{dG_i(v) \big(\sum_{j=1}^{k} \eta_j X_{H_j}(t,v) \big)}_{=0} \Big| \\
&\geq \big| G_i(v(t_1)) \big| - \big| G_i(v(t_0)) \big| \\
&= \frac{\delta}{2}.
\end{aligned} \tag{3.2.6}$$

Thus Step 2 follows with $\epsilon = \min\{\frac{\delta}{2}, \frac{\delta}{2\mathfrak{P}}\}$.

Step 3: Proof of the lemma.

Proof of Step 3. According to Step 2, $v(t) \in U_\delta$ for all $t \in (\frac{1}{2}, 1)$. Then Step 1 completes the proof of the lemma with $C = C_0 + \epsilon + 1$. $\square$

For a given gradient flow line w of $\mathcal{A}^{\mathcal{H}}_{F_s}$ and $\sigma \in \mathbb{R}$, we define

$$\begin{aligned} o(\sigma, w, \epsilon) &:= \inf\big\{\tau \geq 0 \,\big|\, ||\nabla_m \mathcal{A}^{\mathcal{H}}_{F_s}(w(\sigma+\tau))||_m \leq \epsilon\big\}, \\ C_F &:= \int_{-\infty}^{\infty}\int_0^1 \max_{x\in M} ||\partial_s F_s(t,x)||_g dtds \ < \ \infty. \end{aligned} \tag{3.2.7}$$

Lemma 3.2.10. For a gradient flow line w of $\mathcal{A}^{\mathcal{H}}_{F_s}$ with $\lim_{s\to\pm\infty} w(s) = w_\pm$,

$$o(\sigma, w, \epsilon) \leq \frac{\mathcal{A}^{\mathcal{H}}_{F_s}(w_-) - \mathcal{A}^{\mathcal{H}}_{F_s}(w_+) + C_F}{\epsilon^2}.$$

Proof. We compute

$$\begin{aligned} \epsilon^2 o(\sigma, w, \epsilon) &\leq \int_\sigma^{\sigma+o(\sigma,w,\epsilon)} \big|\big|\nabla_m \mathcal{A}^{\mathcal{H}}_{F_s}(w)\big|\big|^2_m ds \\ &\leq \int_{-\infty}^{\infty} -d\mathcal{A}^{\mathcal{H}}_{F_s}(w)(\partial_s w) ds - C_F + C_F \\ &\leq \int_{-\infty}^{\infty} -\frac{d}{ds}\big(\mathcal{A}^{\mathcal{H}}_{F_s}(w(s))\big) ds + C_F \\ &= \mathcal{A}^{\mathcal{H}}_{F_s}(w_-) - \mathcal{A}^{\mathcal{H}}_{F_s}(w_+) + C_F \end{aligned}$$

We obtain a bound on $o(\sigma, w, \epsilon)$ by dividing ϵ^2 in the above inequality. $\square$

Theorem 3.2.11. *Assume that $w = (u, \tau) \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^k)$ is a gradient flow line of $\mathcal{A}^{\mathcal{H}}_{F_s}$ for which there exist $a \leq b$ such that*

$$a \leq \mathcal{A}^{\mathcal{H}}_{F_s}(w(s)) \leq b, \quad \text{for all } s \in \mathbb{R}. \tag{3.2.8}$$

Then the L^∞-norms of τ_i's are uniformly bounded.

As we have mentioned, Theorem 3.2.11 completes the proof of Theorem 3.2.8.

Proof. Using Lemma 3.2.9 and Lemma 3.2.10, we obtain

$$\begin{aligned}|\tau_i(\sigma)| &\leq |\tau_i(\sigma + o(\sigma, w, \epsilon))| + \int_{\sigma}^{\sigma+o(\sigma,w,\epsilon)} |\partial_s \tau_i(s)| ds \\ &\leq C\big(\big|\mathcal{A}^{\mathcal{H}}_{F_s}(w(\sigma + o(\sigma, w, \epsilon)))\big| + 1\big) + o(\sigma, w, \epsilon)||H_i||_{L^\infty} \\ &\leq C(\max\{|a|, |b|\} + 1) + \left(\frac{|b-a| + C_F}{\epsilon^2}\right)||H_i||_{L^\infty}.\end{aligned}$$

□

3.3 Proof of Theorem A

The proof proceeds in two steps. In Step 1, we prove Theorem A under the assumption that Σ is a regular level set of a Poisson commuting Hamiltonian tuple $\mathcal{G}$ satisfying the assumption iii) as before. Then we remove this additional assumption in Step 2.

Step 1. There exists a critical point (v, η) of $\mathcal{A}^{\mathcal{H}}_F$ if $||F|| < \wp(\Sigma)$ and Σ is of restricted contact type with $\Phi : \Sigma \to \mathrm{Mat}_{\mathrm{Def}}(k \times k)$. Moreover the action value of that critical point is uniformly bounded as below:

$$-\,||F|| \leq \mathcal{A}^{\mathcal{H}}_F(v, \eta) \leq ||F||. \tag{3.3.1}$$

Proof of Step 1. We mainly follow the proof of Theorem A in [AF1] which made use of the "stretching the neck" argument. For $0 \leq r$, we choose a smooth family of functions $\varphi_r \in C^\infty(\mathbb{R}, [0, 1])$ satisfying

1. for $r \geq 1$: $\varphi_r'(s) \cdot s \leq 0$ for all $s \in \mathbb{R}$, $\varphi_r(s) = 1$ for $|s| \leq r - 1$, and $\varphi_r(s) = 0$ for $|s| \geq r$,

2. for $r \leq 1$: $\varphi_r(s) \leq r$ for all $s \in \mathbb{R}$ and $\mathrm{Supp}\varphi_r \subset [-1, 1]$,

We note that $\varphi_\infty \equiv 1$ is the limit of φ_r with respect to C^∞_{loc}-topology.

We fix a point $p \in \Sigma$ and consider the moduli space

$$\mathcal{M} := \left\{ (r,w) \in [0,\infty) \times C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^k) \;\middle|\; \begin{array}{c} w \text{ is a gradient flow line of } \mathcal{A}^{\mathcal{H}}_{\varphi_r F} \text{ with} \\ \lim\limits_{s\to-\infty} w(s) = (p,0),\ \lim\limits_{s\to\infty} w(s) \in \Sigma \times \{0\} \end{array} \right\}.$$

Assume on the contrary that there is **no** leafwise coisotropic intersection point of ϕ_F for $||F|| < \wp(\Sigma)$. For $(r,w) \in \mathcal{M}$ with $w_- = (p,0)$ and $w_+ = (q,0)$ in $\Sigma \times \{0\}$, we estimate

$$\begin{aligned} E(w) &= -\int_{-\infty}^{\infty} d\mathcal{A}^{\mathcal{H}}_{\varphi_{r(s)}F}(w(s))(\partial_s w)ds \\ &\le \mathcal{A}^{\mathcal{H}}_0(p,0) - \mathcal{A}^{\mathcal{H}}_0(q,0) + \int_{-\infty}^{\infty} ||\partial_s \varphi_r F||_- ds \\ &= \int_{-\infty}^{\infty} ||\varphi_r'(s) F||_- ds \\ &= \int_{-\infty}^{0} \varphi_r'(s)||F||_- ds - \int_0^{\infty} \varphi_r'(s)||F||_+ ds \\ &= \varphi_r(0)\big(||F||_- + ||F||_+\big) \\ &\le ||F||. \end{aligned}$$

Accordingly we can also estimate,

$$-||F|| \le \mathcal{A}^{\mathcal{H}}_{\varphi_{r_n}F}(w_n(s)) \le ||F||, \qquad (r_n, w_n) \in \mathcal{M}. \tag{3.3.2}$$

Due to the action bound, Theorem 3.2.8 yields that a sequence $\{w_n\}_{n\in\mathbb{N}}$ for $(r_n, w_n) \in \mathcal{M}$ has a convergent subsequence (still denoted w_n) in C^∞_{loc}-topology. We denote by x the limit gradient flow line (which can be a constant gradient flow line). We want to show that $\mathcal{M}$ is compact and so assume by contradiction that $x_+ \notin \Sigma \times \{0\}$ where $x_\pm$ are asymptotic ends of x, i.e. $x_\pm = \lim_{s\to\pm\infty} x(s)$.

Case 1. r_n is bounded.

There is no loss of generality in assuming that $r_n \to r$ as $n \to \infty$. Let $U \in \mathcal{L} \times \mathbb{R}^k$ be an open set containing only the constant critical points of $\mathcal{A}^{\mathcal{H}}_{\varphi_r F}$. Since $x_+ \notin \Sigma \times \{0\}$, we can take for large n, $\sigma_n \in \mathbb{R}$ the last U-entry time of w_n, i.e. $w_n(\sigma_n) \notin U$ and $w_n(s) \in U$ for $s > \sigma_n$. We note that $\sigma_n \to \infty$ as $n \to \infty$ and that the reparametrized sequence $\sigma_n^* w_n$ is a gradient flow line of $\mathcal{A}^{\mathcal{H}}_{\sigma_n^*\varphi_{r_n}F}$ where $\sigma_n^* w_n(\cdot) := w_n(\cdot + \sigma_n)$ and $\sigma_n^*\varphi_{r_n}(\cdot) := \varphi_{r_n}(\cdot + \sigma_n)$. The new sequence $\sigma_n^* w_n$ also has a C^∞_{loc}-convergent subsequence by Theorem 3.2.8 again and we denote by z the limit gradient flow line. Since $r_n \to r$ and $\sigma_n \to \infty$, $\sigma_n^*\varphi_{r_n}$ C^∞_{loc}-converges to the zero function, and thus z is the gradient flow line of $\mathcal{A}^{\mathcal{H}}$. Since $\sigma_n^* w_n \to z$ in C^∞_{loc}-topology, we have

$$E(z) = \int_{-\infty}^{\infty} ||\partial_s z||_m^2 ds = \lim_{T\to\infty} \int_{-T}^{T} ||\partial_s z||_m^2 ds \leq \lim_{T\to\infty} \limsup_{n\in\mathbb{N}} E(w_n) = \limsup_{n\in\mathbb{N}} E(w_n).$$

We observe that $z(0) \notin U$ and the positive asymptotic end $z_+ \in \Sigma \times \{0\}$ since $\Sigma \times \{0\}$ is a Morse-Bott component of $\mathrm{Crit}\mathcal{A}^{\mathcal{H}}$ (see [AF1, Lemma 2.12]) and hence z is a non-constant gradient flow line of $\mathcal{A}^{\mathcal{H}}$. Thus the negative asymptotic end z_- is a critical point of $\mathcal{A}^{\mathcal{H}}$; moreover it is not a constant loop since otherwise z is a non-constant gradient flow line with zero energy $E(z) = 0$. But this case is ruled out by the assumption that $||F|| < \wp(\Sigma)$ as well. To be precise, with $z_- = (v, \eta)$, we can derive the following estimate which contradicts the definition of $\wp(\Sigma)$.

$$0 < |\Omega(v)| = |\mathcal{A}^{\mathcal{H}}_0(z_-)| = E(z) \leq \limsup_{n\in\mathbb{N}} E(w_n) \leq ||F|| < \wp(\Sigma).$$

<u>Case 2</u>. r_n is unbounded.

Without loss of generality, we assume that $r_n \to \infty$ as $n \to \infty$. The limit of $\{w_n\}_{n\in\mathbb{N}}$ is a gradient flow line of $\mathcal{A}^{\mathcal{H}}_F$ since $\beta_\infty \equiv 1$. Then the asymptotic ends of the limit are critical points of $\mathcal{A}^{\mathcal{H}}_F$ which give rise to a leafwise coisotropic intersection point of ϕ_F. It contradicts our assumption and Case 2 is ruled out.

With σ_n the first U-exit time of w_n, the case $x_- \notin \Sigma \times \{0\}$ is analogous. If $x_- = (q, 0) \in \Sigma$ with $q \neq p$, as Case 1, there exists a gradient flow line of $\mathcal{A}^{\mathcal{H}}$ with asymptotic ends $(q, 0)$ and $(p, 0)$. But this cannot occur. Therefore we conclude that the moduli space $\mathcal{M}$ is compact.

Next, we regard the moduli space $\mathcal{M}$ as the zero set of a Fredholm section with index 1 of a Banach bundle over a Banach manifold as in (5.1.1). Moreover, the Fredholm section is already transversal at the $(0, p, 0)$ since Σ is a Morse-Bott component by [AF1, Lemma 2.12]. Therefore we can perturb the Fredholm section away from $(0, p, 0)$ (even if varying J, $(0, p, 0)$ still solves the gradient flow equation) to obtain a transverse Fredholm section whose zero set is a compact one-dimensional smooth manifold with boundary $(0, p, 0)$. But there is no one-dimensional manifold with a single boundary point. This finishes the proof of Claim 1. □

Step 2. End of the proof of Theorem A.

Proof of Step 2. In Step 2, our restricted contact coisotropic submanifold Σ is not necessarily of the form $\Sigma = \mathcal{G}^{-1}(0)$. Recall that on the open neighborhood $U_{\delta_0} \cong \{(q, p_1, \dots, p_k) \in \Sigma \times D_r^k\}$ of Σ, $\omega|_{U_{\delta_0}} = \omega|_\Sigma + \sum_{i=1}^k d(p_i \alpha_i)$ and $X_{p_i} = R_i$ for all $i = 1, \dots, k$.

We consider a family of Hamiltonian tuples $\mathcal{H}_\nu(t, x) = \chi(t)\mathcal{G}_\nu(x)$, $\nu \in \mathbb{N}$ where $\mathcal{H}_\nu = (H_{1,\nu}, \dots, H_{k,\nu})$ and $\mathcal{G}_\nu = (G_{1,\nu}, \dots, G_{k,\nu})$ such that

1. $0 < \epsilon_\nu < \delta$ converges to zero as ν goes to infinity,

2. $G_{i,\nu}|_{U_{\delta_0}} = g_i(p_i)$ for some $g_i \in C^\infty(\mathbb{R})$,

3. for $(x,\mathfrak{p}) \in \Sigma \times (-\delta_0, \delta_0)^k \cong U_{\delta_0}$,

$$G_{i,\nu}|_{U_{2\epsilon_\nu} - U_{\epsilon_\nu/2}}(x,\mathfrak{p}) = \begin{cases} p_i - \epsilon_\nu & \text{if } p_i > 0 \\ -p_i - \epsilon_\nu & \text{if } p_i < 0, \end{cases} \tag{3.3.3}$$

4. $G_{i,\nu}|_{M-U_{\delta_0}} = constant$,

5. $\mathcal{G}_\nu^{-1}(0) = \bigcup_{2^k} \Sigma \times (\pm\epsilon_\nu, \dots, \pm\epsilon_\nu)$.

We note that

$$X_{G_{i,\nu}}|_{\Sigma\times(\pm\epsilon_\nu,\dots,+\epsilon_\nu,\dots,\pm\epsilon_\nu)} = +X_{p_i}, \quad X_{G_{i,\nu}}|_{\Sigma\times(\pm\epsilon_\nu,\dots,-\epsilon_\nu,\dots,\pm\epsilon_\nu)} = -X_{p_i}.$$

By construction, $\mathcal{H}_\nu$ Poisson-commutes and Step 1 guarantees the existence of critical points (v_ν, η_ν) lying on $\mathcal{G}_\nu^{-1}(0)$ for sufficiently large ν because $||F|| < \wp(\Sigma \times \{(\pm\epsilon_\nu, \dots, \pm\epsilon_\nu)\})$ for large $\nu \in \mathbb{N}$. For $(v_\nu, \eta_\nu) \in \mathrm{Crit}\mathcal{A}_F^{\mathcal{H}_\nu}$, v_ν lies on one of the components of $\mathcal{G}_\nu^{-1}(0)$, say $v_\nu \subset \Sigma \times (\epsilon_\nu, \dots, \epsilon_\nu)$. According to Proposition 3.2.2, it holds that

$$\phi_F^1\big(v_\nu(1/2)\big) = v_\nu(0) = \phi_{H_{1,\nu}}^{-\eta_{1,\nu}} \circ \cdots \circ \phi_{H_{k,\nu}}^{-\eta_{k,\nu}}\big(v_\nu(1/2)\big).$$

Then the estimate (3.3.1) in Step 1 implies the following lemma.

Lemma 3.3.1. For $(v_\nu, \eta_\nu) \in \mathrm{Crit}\mathcal{A}_F^{\mathcal{H}_\nu}$, $\eta_{1,\nu}, \dots, \eta_{k,\nu}$ are uniformly bounded in terms of $\lambda_1, \dots, \lambda_k$ and F.

PROOF. We estimate as in (3.3.1): For all $i \in \{1, \dots, k\}$,

$$
\begin{aligned}
||F|| &\geq \big|\mathcal{A}_F^{\mathcal{H}_\nu}(v_\nu, \eta_\nu)\big| \\
&= \Big| \int_0^1 v^*\lambda_i + \int_0^1 \langle \eta, \mathcal{H}_\nu \rangle (t, v_\nu(t))dt + \int_0^1 F(t, v_\nu(t))dt \Big| \\
&= \Big| \int_0^1 \lambda_i(v_\nu)\big(\sum_{j=1}^k \eta_{j,\nu} X_{H_{j,\nu}}(v_\nu) + X_F(t, v_\nu)\big)dt + \int_0^1 F(t, v_\nu(t))dt \Big| \\
&= \frac{3}{4}|\eta_{i,\nu}| - \frac{1}{4(k-1)} \sum_{j \neq i}^k |\eta_{j,\nu}| - \Big| \int_0^1 \lambda_i(v_\nu)\big(X_F(t, v_\nu)\big) + \int_0^1 F(t, v_\nu(t))dt \Big|.
\end{aligned}
$$

Therefore we conclude

$$
\frac{1}{2} \sum_{i=1}^k |\eta_{i,\nu}| \leq k\big(||F|| + \max_{1 \leq i \leq k} ||\lambda_{i|U_{\delta_0/2}}||_{L^\infty} ||X_F||_{L^\infty} + ||F||_{L^\infty}\big).
$$

□

The two sequences of points $\{v_\nu(0)\}_{\nu \in \mathbb{N}}$ and $\{v_\nu(1/2)\}_{\nu \in \mathbb{N}}$ converge up to taking a subsequence (still denoted by $v_\nu(0)$ and $v_\nu(1/2)$) and we denote by

$$
x_0 := \lim_{\nu \to \infty} v_\nu(0), \quad x_{1/2} := \lim_{\nu \to \infty} v_\nu\big(1/2\big).
$$

Obviously x_0 and $x_{1/2}$ are points in Σ. Moreover we know that

$$
x_0 = \lim_{\nu \to \infty} v_\nu(0) = \lim_{\nu \to \infty} \phi_F^1(v_\nu(1/2)) = \phi_F^1(\lim_{\nu \to \infty} v_\nu(1/2)) = \phi_F^1(x_{1/2}). \quad (3.3.4)
$$

Furthermore, due to Lemma 3.3.1, the limit $\{\eta_{i,\nu}\}_{\nu \in \mathbb{N}}$ exists for all i, say

$$
\mathfrak{n}_i := \lim_{\nu \to \infty} \eta_{i,\nu}.
$$

Thus we conclude that x_0 and $x_{1/2}$ lie on the same leaf:

$$x_0 = \lim_{\nu\to\infty} v_\nu(0) = \lim_{\nu\to\infty} \phi_{H_{1,\nu}}^{-\eta_{1,\nu}} \circ \cdots \circ \phi_{H_{k,\nu}}^{-\eta_{k,\nu}}(v_\nu(1/2)) = \phi_{H_1}^{-\mathfrak{n}_1} \circ \cdots \circ \phi_{H_k}^{-\mathfrak{n}_k}(x_{1/2}). \tag{3.3.5}$$

It directly follows

$$\phi_{H_1}^{-\mathfrak{n}_1} \circ \cdots \circ \phi_{H_k}^{-\mathfrak{n}_k}(x_{1/2}) = \phi_F^1(x_{1/2})$$

from (3.3.4) together with (3.3.5). This completes the proof of Theorem A. □

Chapter 4

The existence of a periodic orbit and the leafwise displacement energy

In this chapter, we study the existence of a periodic orbit, i.e. a solution of (1.2.2), together with a relation between its symplectic area and the leafwise displacement energy in the stable case. This proves Theorem D which were proved by Kai Cieliebak, Urs Frauenfelder, and Gabriel Paternain [CFP] for separating stable hypersurfaces. Adapting their idea, we can extend (and slightly improve) their result to stable coisotropic submanifolds.

Let Σ be a closed stable coisotropic submanifold in a symplectically aspherical symplectic manifold (M, ω) which is geometrically bounded. As in Theorem A we first assume that $\Sigma = \mathcal{G}^{-1}(0)$ for some Poisson commuting Hamiltonian tuple $\mathcal{G} \in C^\infty(M, \mathbb{R}^k)$, but this additional assumption will be removed in the second step. Suppose that Σ is **displaced by** $F \in C_c^\infty(S^1 \times M)$, i.e. $\phi_F(\Sigma) \cap \Sigma = \emptyset$. We consider again the smooth family of functions $\varphi_r \in C^\infty(\mathbb{R}, [0,1])$ defined in the proof of Theorem A. As before, we fix a point $p \in \Sigma$ and consider the moduli space $\mathcal{M}$ defined by

$$\mathcal{M} = \left\{ (r,w) \in [0,\infty) \times C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^k) \;\middle|\; \begin{array}{c} w \text{ is a gradient flow line of } \mathcal{A}^{\mathcal{H}}_{\varphi_r F} \text{ with} \\ \lim\limits_{s \to -\infty} w(s) = (p,0), \; \lim\limits_{s \to \infty} w(s) \in \Sigma \times \{0\} \end{array} \right\}.$$

Theorem 4.0.2. *For $(r,w) \in \mathcal{M}$ where $w = (u,\tau)$, τ and r are uniformly bounded.*

In the previous sections we showed how Rabinowitz Floer theory for hypersurfaces can be generalized to our set-up. Since the proof of Theorem 4.0.2 needs several technical lemmas and auxiliary action functionals as in the contact case [Ka2],we refer the reader to [CFP, Section 4.3] or [Ka3] instead of giving a proof .

4.1 Proof of Theorem D

Step 1. We know that a sequence $\{(r_n, w_n)\}_{n \in \mathbb{N}}$ in $\mathcal{M}$ has a C^∞_{loc}-convergent subsequence due to Theorem 4.0.2 together with the argument in the proof of Theorem 3.2.8. We denote by (r,w) the limit which is a gradient flow line of $\mathcal{A}^{\mathcal{H}}_{\varphi_r F}$. Again by compactness, w asymptotically converges to $w_\pm = (v_\pm, \eta_\pm) \in \mathrm{Crit}\mathcal{A}^{\mathcal{H}}$ since $\varphi_r(\pm\infty) = 0$. If $(r,w) \in \mathcal{M}$, the moduli space $\mathcal{M}$ is a one dimensional compact manifold with a single boundary point $\{(0,p,0)\}$ (after perturbing a Fredholm section as in the proof of Theorem A). However such a manifold does not exist and therefore one of the asymptotic ends $w_\pm$ of w is a nontrivial solution of (1.2.2). For simplicity, let us assume $w_+ \notin \Sigma \times \{0\}$. Following the notation from the proof of Theorem A, we consider $\sigma_n \in \mathbb{R}$ the last U-entry time. Then $\sigma_n^* w_n$ is a gradient flow line of $\mathcal{A}^{\mathcal{H}}_{\sigma_n^* \varphi_{r_n} F}$ and C^∞_{loc}-converges to a non-constant gradient flow line z of $\mathcal{A}^{\mathcal{H}}$ with $z(0) \notin$

U and $z_+ \in \Sigma\times\{0\}$.[1] By compactness and the energy estimate, $z_- = (v,\eta) \in \mathrm{Crit}\mathcal{A}^{\mathcal{H}}$ and z_- is a nontrivial solution of (1.2.2). Moreover, by (3.3.2), we have

$$-||F|| \le \mathcal{A}^{\mathcal{H}}_{\sigma_n^*\varphi_{r_n}F}(\sigma_n^* w_n(s)) \le ||F||, \quad \forall s \in \mathbb{R}.$$

As n goes to infinity, it holds that

$$-\,||F|| \le \Omega(v) = \mathcal{A}^{\mathcal{H}}(z_-) \le ||F|| \tag{4.1.1}$$

for every Hamiltonian function $F \in C^\infty_c(S^1\times M)$ displacing Σ. Since $\mathcal{A}^{\mathcal{H}}(z_+) = 0$ and the action value of $\mathcal{A}^{\mathcal{H}}$ decreases along z,

$$\big|\Omega(v)\big| = \big|\mathcal{A}^{\mathcal{H}}(z_-)\big| > 0. \tag{4.1.2}$$

(4.1.1) and (4.1.2) prove Theorem E provided that Σ is a level set of some Poisson-commuting Hamiltonian tuple.

Step 2. Now we consider the situation that Σ is not necessarily a level set of some Poisson-commuting Hamiltonian tuple. We choose a family of Hamiltonian tuples $\mathcal{H}_\nu(t,x) = \chi(t)\mathcal{G}_\nu(x)$, $\nu \in \mathbb{N}$ where $\mathcal{H}_\nu = (H_{1,\nu},\dots,H_{k,\nu})$ and $\mathcal{G}_\nu = (G_{1,\nu},\dots,G_{k,\nu})$ such that

1. $0 < \epsilon_\nu < \min\{1/4k, \delta_0/2, \delta_1\}$ converges to zero as ν goes to infinity,

2. $G_{i,\nu}|_{U_{\delta_0}} = g_i(p_i)$ for some $g_i \in C^\infty(\mathbb{R})$,

3. for $(x,\mathfrak{p}) \in \Sigma \times (-\delta_0,\delta_0)^k \cong U_{\delta_0}$,

$$G_{i,\nu}|_{U_{2\epsilon_\nu}-U_{\epsilon_\nu/2}}(x,\mathfrak{p}) = \begin{cases} p_i - \epsilon_\nu & \text{if } p_i > 0 \\ -p_i - \epsilon_\nu & \text{if } p_i < 0, \end{cases}$$

[1] Honestly speaking, we did not prove C^∞_{loc}-convergence of $(r_n, \sigma_n^* w_n)$; but it follows from the proof of Theorem 4.0.2.

4. $G_{i,\nu}|_{M-U_{\delta_0}} = constant$,

5. $\mathcal{G}_\nu^{-1}(0) = \bigcup_{2^k} \Sigma \times (\pm\epsilon_\nu, \dots, \pm\epsilon_\nu)$.

With this defining Hamiltonian tuple $\mathcal{H}_\nu$, the argument in Step 1 still works and thus there exists $v_\epsilon \in \mathcal{G}_\nu^{-1}(0)$ a solution of (1.2.2) satisfying $0 < \Omega(v_\epsilon) \leq e(\mathcal{G}_\nu^{-1}(0))$. Since $\mathcal{G}_\nu^{-1}(0)$ is disconnected, v_ϵ lies in one of its connected components, say $v_\epsilon \subset \Sigma_\epsilon$. Since there is a diffeomorphism ψ_ϵ between Σ_ϵ and Σ, $\psi_\epsilon(v_\epsilon)$ is a loop solving (1.2.2), contractible in M with $\Omega(\psi_\epsilon(v_\epsilon)) = \Omega(v_\epsilon) > 0$. Moreover if we have chosen sufficiently large ν, $e(\Sigma) = e(\mathcal{G}_\nu^{-1}(0))$. For simplicity, let us assume that $e(\Sigma) + \varepsilon < e(\mathcal{G}_\nu^{-1}(0))$ for some small $\varepsilon > 0$ and for all $\nu \in \mathbb{N}$; it means that there is $F \in C_c^\infty(S^1 \times M)$ such that $||F|| \in (e(\Sigma), e(\Sigma) + \varepsilon)$ such that $\phi_F(\Sigma) \cap \Sigma = \emptyset$; but if ν is big enough, ϕ_F also displaces $\mathcal{G}_\nu^{-1}(0)$ and it contradicts $||F|| < e(\mathcal{G}_\nu^{-1}(0))$. Hence, we have proved that

$$0 < \Omega(\psi_\epsilon(v_\epsilon)) = \Omega(v_\epsilon) \leq e(\mathcal{G}_\nu^{-1}(0)) = e(\Sigma).$$

$\square$

Remark 4.1.1. If one succeeds in proving compactness of gradient flow lines of the perturbed Rabinowitz action functional in the stable case, Theorem D is an immediate consequence of the invariance property of Rabinowitz Floer homology.

Chapter 5

Rabinowitz Floer homology

In the hypersurface case, [CFP, AF1] proved that the (perturbed) Rabinowitz action functional is generically Morse-Bott (Morse). Their argument undeniably continues to hold in our set-up. That is, $\mathcal{A}^{\mathcal{G}}$ is Morse-Bott and $\mathcal{A}^{\mathcal{H}}_F$ is Morse for a generic perturbation $F \in C^\infty_c(S^1 \times M)$. Furthermore, we know that gradient flow lines of the Rabinowitz action functional are compact modulo breaking (see (F1) and (F2) below) for restricted contact coisotropic submanifolds due to Theorem 3.2.8. Therefore we can define Floer homologies of $\mathcal{A}^{\mathcal{G}}$ and $\mathcal{A}^{\mathcal{H}}_F$ as usual.[1] As one expects, these two Floer homologies are isomorphic by the standard continuation method in Floer theory. Here we only treat the restricted contact case and refer to Remark 2.5.1 for other cases. As before, (M,ω) is an exact symplectic manifold being geometrically bounded with a family of ω-compatible almost complex structures $J = J(s,t)$.

[1] $\mathcal{A}^{\mathcal{G}}$ is never Morse since there is a S^1-symmetry coming from time-shift on the critical points set. However $\mathcal{A}^{\mathcal{G}}$ is Morse-Bott for a generic coisotropic submanifold, thus we can define Morse-Bott homology of $\mathcal{A}^{\mathcal{G}}$ by counting gradient flow lines with cascades, see [Fr]. Since Rabinowitz Floer homology is invariant under homotopies there is no loss of generality in assuming $\mathcal{A}^{\mathcal{H}}$ is Morse-Bott, see [CFP].

5.1 Boundary Operator

We can assign some index to critical points of $\mathcal{A}_F^{\mathcal{H}}$, namely the transverse Conley-Zehnder index.[2] But we omit the definition, referring the reader to [BO2, CF, MP]. We denote the index by

$$\mu : \mathrm{Crit}\mathcal{A}_F^{\mathcal{H}} \longrightarrow \mathbb{Z}.$$

Here we assumed that the first Chern class c_1 vanishes over $\pi_2(M)$ for simplicity; otherwise the index μ is well defined modulo $2N$ where N is the minimal Chern number of (M, ω).

Let $\mathcal{M}_J(w_-, w_+)$ be the moduli space of gradient flow lines of $\mathcal{A}_F^{\mathcal{H}}$ with asymptotic ends $w_\pm \in \mathrm{Crit}\mathcal{A}_F^{\mathcal{H}}$.

$$\mathcal{M}_J(w_-, w_+) := \left\{ (u,\tau) \in C^\infty(\mathbb{R} \times S^1, M) \times C^\infty(\mathbb{R}, \mathbb{R}^k) \;\middle|\; \begin{array}{r} (u,\tau) \text{ solves (3.2.3)}, \\ \lim\limits_{s\to\pm\infty}(u,\tau) = w_\pm \end{array} \right\}.$$

In order to show that $\mathcal{M}_J(w_-, w_+)$ is a finite dimensional smooth manifold, we interpret it as the zero set of a Fredholm section of a Banach bundle over a Banach space. Let $\mathcal{P}(w_-, w_+)$ be the Banach manifold given by

$$\mathcal{P}(w_-, w_+) := \left\{ (u,\tau) \in W^{1,2}(\mathbb{R} \times S^1, M) \times W^{1,2}(\mathbb{R}, \mathbb{R}^k) \;\middle|\; \lim_{s\to\pm\infty}(u,\tau) = w_\pm \right\}$$

and $\mathcal{E}$ be the Banach bundle over $\mathcal{P}(w_-, w_+)$ whose fibre at $(u,\tau) \in \mathcal{P}(w_-, w_+)$ is

$$\mathcal{E}_{(u,\tau)} := L^2(\mathbb{R} \times S^1, u^*TM \times \tau^*T\mathbb{R}^k).$$

Then the moduli space $\mathcal{M}(w_-, w_+)$ is the zero set of the section

$$s_J : \mathcal{P}(w_-, w_+) \longrightarrow \mathcal{E}, \quad s_J(u,\tau) = \big(\bar\partial_{\mathcal{H},F,J}(u), \bar\partial_1(\tau_1), \cdots, \bar\partial_k(\tau_k)\big) \tag{5.1.1}$$

[2] We can define Floer homology of $\mathcal{A}_F^{\mathcal{H}}$ without this index.

defined by

$$\left.\begin{aligned}\bar{\partial}_{\mathcal{H},F,J}(u) &= \partial_s u + J(s,t,u)\Big(\partial_t u - \sum_{i=1}^{k} \eta_i X_{H_i}(t,u) - X_{F_s}(t,u)\Big)\\ \bar{\partial}_i(\tau_i) &= \partial_s \tau_i - \int_0^1 H_i(t,u)dt, \qquad 1 \le i \le k\end{aligned}\right\}$$

where $\tau = (\tau_1, \dots, \tau_k)$. It turns out that this section is Fredholm. Then we regard the moduli space as the zero set of this section, $\mathcal{M}_J(w_-, w_+) = s_J^{-1}(0)$. Let

$$Ds_J(u,\tau) : T_{(u,\tau)}\mathcal{P}(w_-, w_+) \longrightarrow \mathcal{E}_{(u,\tau)}$$

be the vertical differential of s_J at (u,τ). It is known that $Ds_J(u,\tau)$ is surjective for a generic ω-compatible almost complex structure J and for any $(u,\tau) \in s_J^{-1}(0)$, see [FHS, Section 5] and [BO1]. This transversality issues (surjectivity of $Ds_J(u,\tau)$) can now also be settled using the framework of polyfolds developed by Hofer-Wysocki-Zehnder [HWZ1, HWZ2, HWZ3]. Thus we perturb the section s_J (varying J slightly) so that $Ds_J(u,\tau)$ is surjective and the implicit function theorem yields that $s_J^{-1}(0) = \mathcal{M}_J(w_-, w_+)$ is a smooth finite dimensional manifold. Moreover the dimension of the moduli space $\mathcal{M}_J(w_-, w_+)$ coincides with the dimension of the kernel of Ds_J which in turn is the same as the Fredholm index of s_J since it is surjective; besides, the Fredholm index of s_J can be computed in terms of the indices of $\mu(w_-)$ and $\mu(w_+)$ using the spectral flow [RS, BO2, CF]. In conclusion, we have the identity

$$\dim \mathcal{M}_J(w_-, w_+) = \mu(w_-) - \mu(w_+), \quad w_\pm \in \mathrm{Crit}\mathcal{A}_F^{\mathcal{H}}.$$

We suppress the subindex J in $\mathcal{M}_J(w_-, w_+)$ for notational convenience. We divide out the $\mathbb{R}$-action on $\mathcal{M}(w_-, w_+)$ defined by shifting the gradient flow lines in the s-variable. Then we obtain the moduli space of unparametrized gradient flow lines which we denote by

$$\widehat{\mathcal{M}}(w_-, w_+) := \mathcal{M}(w_-, w_+)/\mathbb{R}.$$

For the compactification of the moduli space $\mathcal{M}(w_-, w_+)$, we recall the **Floer-Gromov convergence**. A sequence $\{(u^\nu, \tau^\nu)\}_{\nu\in\mathbb{N}}$ in $\mathcal{M}(w_-, w_+)$ is said to Floer-Gromov converge to a broken gradient flow lines $\{(u_j, \tau_j)\}_{j=1}^m$ where $z_0, \dots, z_m \in \mathrm{Crit}\mathcal{A}^{\mathcal{H}}_{F_s}$ with $z_0 = w_-$ and $z_m = w_+$, and

$$(u_j, \tau_j) \in \mathcal{M}(z_{j-1}, z_j), \quad j \in \{1, \dots, m\}$$

if there exist $\sigma_j^\nu \in \mathbb{R}$ such that reparametrized sequences $(u^\nu, \tau^\nu)(\sigma_j^\nu + \cdot)$ converge to (u_j, τ_j) for all $j \in \{1, \dots, m\}$ in the C^∞_{loc}-topology. The following statements are the key ingredients for boundary operators of various Floer homologies, including Rabinowitz Floer homology.

(F1) The moduli space $\mathcal{M}(w_-, w_+)$ is a one dimensional compact smooth manifold with respect to the topology of Floer-Gromov convergence when $\mu(w_-) - \mu(w_+) = 1$.[3] Accordingly, $\widehat{\mathcal{M}}(w_-, w_+)$ is a finite set.

(F2) Let $\widehat{\mathcal{M}}_c(w_-, w_+)$ be the compactification of $\widehat{\mathcal{M}}(w_-, w_+)$ with respect to the topology of Floer-Gromov convergence. If $\mu(w_-) - \mu(w_+) = 2$, $\widehat{\mathcal{M}}_c(w_-, w_+)$ is a compact one-dimensional manifold whose boundary is

$$\partial\widehat{\mathcal{M}}_c(w_-, w_+) = \bigcup_z \widehat{\mathcal{M}}(w_-, z) \times \widehat{\mathcal{M}}(z, w_+) \tag{5.1.2}$$

where the union runs over $z \in \mathrm{Crit}\mathcal{A}^{\mathcal{H}}_F$ with $\mu(w_-) - \mu(z) = 1$.

(F1) follows from the elliptic bootstrapping argument as discussed in Theorem 3.2.8, see also Floer's beautiful paper [Fl2]. (F2) is proved by Floer's gluing theorem [Fl1].

[3] Without help of the Conley-Zehnder index, we can rephrase that the one dimensional component of $\mathcal{M}(w_-, w_+)$ is a compact smooth manifold.

We denote by $\mathrm{Crit}_q\mathcal{A}_F^{\mathcal{H}}$ the set of critical point of $\mathcal{A}_F^{\mathcal{H}}$ of index $q \in \mathbb{Z}$, i.e. $\mu((v,\eta)) = q$ for $(v,\eta) \in \mathrm{Crit}_q\mathcal{A}_F^{\mathcal{H}}$. We define a $\mathbb{Z}/2$-vector space

$$\mathrm{CF}_q(\mathcal{A}_F^{\mathcal{H}}) := \Big\{\xi = \sum_{(v,\eta)\in\mathrm{Crit}_q\mathcal{A}_F^{\mathcal{H}}} \xi_{(v,\eta)}(v,\eta) \,\Big|\, \xi_{(v,\eta)} \in \mathbb{Z}/2\Big\}$$

where $\xi_{(v,\eta)}$ satisfies the finiteness condition:

$$\#\big\{(v,\eta) \in \mathrm{Crit}_q\mathcal{A}_F^{\mathcal{H}} \,\big|\, \xi_{(v,\eta)} \neq 0,\ \mathcal{A}_F^{\mathcal{H}}(v,\eta) \geq \kappa\big\} < \infty, \quad \forall\kappa \in \mathbb{R}.$$

We denote by $n(w_-, w_+)$ be the parity of elements of the finite set $\widehat{\mathcal{M}}(w_-, w_+)$ when $\mu(w_-) - \mu(w_+) = 1$, see (F1) above. Then the boundary operators $\{\partial_q\}_{\{q\in\mathbb{Z}\}}$ are defined by

$$\begin{aligned}\partial_q : \mathrm{CF}_q(\mathcal{A}_F^{\mathcal{H}}) &\longrightarrow \mathrm{CF}_{q-1}(\mathcal{A}_F^{\mathcal{H}})\\ w_- \in \mathrm{Crit}_q\mathcal{A}_F^{\mathcal{H}} &\longmapsto \sum_{w_+\in\mathrm{Crit}_{q-1}\mathcal{A}_F^{\mathcal{H}}} n(w_-, w_+)\cdot w_+.\end{aligned}$$

Due to (F2), we know $\partial_{q-1}\circ\partial_q = 0$ (in $\mathbb{Z}/2$) so that $(\mathrm{CF}_\bullet(\mathcal{A}_F^{\mathcal{H}}), \partial_\bullet)$ is a chain complex indeed. We define **Rabinowitz Floer homology** by

$$\mathrm{HF}_q(\mathcal{A}_F^{\mathcal{H}}) := \mathrm{H}_q(\mathrm{CF}_\bullet(\mathcal{A}_F^{\mathcal{H}}), \partial_\bullet), \quad \mathrm{RFH}_q(\Sigma, M) := \mathrm{HF}_q(\mathcal{A}^{\mathcal{G}}).$$

To be exact, since $\mathcal{A}^{\mathcal{G}}$ is Morse-Bott, $\mathrm{HF}(\mathcal{A}^{\mathcal{G}})$ is defined by Frauenfelder's Morse-Bott homology [Fr, Appendix A]. We note that $\mathrm{Crit}\mathcal{A}^{\mathcal{G}}$ consists of Σ and circles. We pick a Morse function f on $\mathrm{Crit}\mathcal{A}^{\mathcal{G}}$ and then the boundary operator for $\mathrm{HF}(\mathcal{A}^{\mathcal{G}})$ is defined by counting gradient flow lines of $\mathcal{A}^{\mathcal{G}}$ (called cascades) together with gradient flow lines of f. Note that if there is no nonconstant solution of (1.2.2), $\mathrm{Crit}\mathcal{A}^{\mathcal{G}} \cong \Sigma$ and thus there are no cascades since the energy of each cascade is positive. Thus if this is the case, $\mathrm{HF}(\mathcal{A}^{\mathcal{G}}) \cong \mathrm{H}(\Sigma; \mathbb{Z}/2)$.

5.2 Continuation Homomorphism

Given any two Hamiltonian functions F and K in $C_c^\infty(S^1 \times M)$, we consider the homotopies $D_s^\pm \in C^\infty(S^1 \times M)$, $s \in \mathbb{R}$,

$$D_s^+(t,x) := K(t,x) + \varphi_+(s)\big(F(t,x) - K(t,x)\big)$$

and

$$D_s^-(t,x) := K(t,x) + \varphi_-(s)\big(F(t,x) - K(t,x)\big)$$

where $\varphi_\pm \in C^\infty(\mathbb{R}, [0,1])$ are cut-off functions defined by

$$\varphi_+(s) = \begin{cases} 0 & s \leq -1 \\ 1 & s \geq 1 \end{cases} \qquad \varphi_-(s) = \begin{cases} 1 & s \leq -1 \\ 0 & s \geq 1. \end{cases}$$

We consider the time-dependent version of the gradient flow equation:

$$\left.\begin{aligned} &\partial_s u + J_s(t,u)\big(\partial_t u - \sum_{i=1}^k \tau_i X_{H_i}(t,u) - X_{D_s^+}(t,u)\big) = 0 \\ &\partial_s \tau_i - \int_0^1 H_i(t,u)dt = 0, \qquad 1 \leq i \leq k. \end{aligned}\right\} \tag{5.2.1}$$

The solutions of (5.2.1) with an asymptotic condition form the following moduli space:

$$\mathcal{M}(w_K, w_F) := \left\{ w \in C^\infty(\mathbb{R} \times S^1, M) \times C^\infty(\mathbb{R}, \mathbb{R}^k) \;\middle|\; \begin{array}{l} w = (u,\tau) \text{ solves (5.2.1) with} \\ \lim\limits_{s \to \pm\infty} w(s) = w_{F/K} \in \mathrm{Crit}\mathcal{A}^{\mathcal{H}}_{F/K} \end{array} \right\}.$$

As we discussed in the previous subsection, it is also a well-known fact in Floer theory that the moduli space $\mathcal{M}(w_K, w_F)$ is a smooth manifold of dimension $\mu(w_K) - \mu(w_F)$ for a generic homotopy. In particular, it is known that $\mathcal{M}(w_K, w_F)$ is a finite set when w_K and w_F have the same index and thus we denote the parity of $\mathcal{M}(w_K, w_F)$ by $n(w_K, w_F)$ if this is the case. Then we define the continuation homomorphism as follows.

$$\begin{aligned}\Phi_K^F : \mathrm{CF}_q(\mathcal{A}_K^{\mathcal{H}}) &\longrightarrow \mathrm{CF}_q(\mathcal{A}_F^{\mathcal{H}}) \\ w_K \in \mathrm{Crit}_q \mathcal{A}_K^{\mathcal{H}} &\longmapsto \sum_{w_F \in \mathrm{Crit}_q \mathcal{A}_F^{\mathcal{H}}} n(w_K, w_F) \cdot w_F.\end{aligned}$$

In the same way, we also define

$$\Phi_F^K : \mathrm{CF}_q(\mathcal{A}_F^{\mathcal{H}}) \longrightarrow \mathrm{CF}_q(\mathcal{A}_K^{\mathcal{H}})$$

using the other homotopy D_s^-. Then we obtain the invariance property of Rabinowitz Floer homology via the continuation homomorphisms using a homotopy of homotopies $D_s^r(t,x) := K(t,s) + \varphi_r(s)(F(t,x) - K(t,x))$ where $\varphi_r : \mathbb{R} \to [0,1]$, $r \in \mathbb{R}$ and $\varphi_r = \varphi_\pm$ if $\pm r \geq 1$, see [Sa, Section 3.4] [4]:

Theorem 5.2.1. *Rabinowitz Floer homology is independent of the choice of perturbations up to canonical isomorphism. In particular, it holds that*

$$\mathrm{RFH}(\Sigma, M) \cong \mathrm{HF}(\mathcal{A}_F^{\mathcal{H}}), \quad F \in C_c^\infty(S^1 \times M).$$

For the later purpose, we compare the action values of $\mathcal{A}_K^{\mathcal{H}}$ and $\mathcal{A}_F^{\mathcal{H}}$:

Proposition 5.2.2. *If the moduli space $\mathcal{M}(w_K, w_F)$ is not empty,*

$$\mathcal{A}_F^{\mathcal{H}}(w_F) \leq \mathcal{A}_K^{\mathcal{H}}(w_K) + ||F - K||_-.$$

[4] Here we again make use of Floer-Gromov compactness and Floer's gluing theorem.

PROOF. We pick $w \in \mathcal{M}(w_K, w_F)$ and estimate its energy:

$$\begin{aligned}
0 &\leq E(w) \\
&= -\int_{-\infty}^{\infty} d\mathcal{A}^{\mathcal{H}}_{D_s^+}(w(s))[\partial_s w] ds \\
&= -\int_{-\infty}^{\infty} \frac{d}{ds}\big(\mathcal{A}^{\mathcal{H}}_{D_s^+}(w(s))\big) ds - \int_{-\infty}^{\infty}\int_0^1 {\varphi_+}'(s)\big(F(t,w(s)) - K(t,w(s))\big) dt ds \\
&\leq \mathcal{A}^{\mathcal{H}}_{D^+_{-\infty}}(w_K) - \mathcal{A}^{\mathcal{H}}_{D^+_{\infty}}(w_F) - \int_{-\infty}^{\infty} {\varphi_+}'(s) \int_0^1 \big(F(t,w(s)) - K(t,w(s))\big) dt ds \\
&\leq \mathcal{A}^{\mathcal{H}}_K(w_K) - \mathcal{A}^{\mathcal{H}}_F(w_F) + ||F-K||_-.
\end{aligned}$$

□

5.3 Proof of Theorem E

Suppose that there are no leafwise coisotropic intersection points for some $\phi_F \in \mathrm{Ham}_c(M,\omega)$. Then the set $\mathrm{Crit}\mathcal{A}^{\mathcal{H}}_F$ is empty since otherwise a critical point of $\mathcal{A}^{\mathcal{H}}_F$ gives rise to a leafwise coisotropic intersection point. Thus $\mathrm{HF}(\mathcal{A}^{\mathcal{H}}_F) = 0$ and Theorem 5.2.1 proves (i).

If there are only constant solutions of (1.2.2), no cascades appear in the boundary operator of Morse-Bott homology. Thus the Rabinowitz Floer homology of (Σ, M) is isomorphic to the Morse homology of Σ and hence to the singular homology of Σ. This proves (iii).

Suppose there are only constant solutions of (1.2.2). Due to (iii), we know that the Rabinowitz Floer homology of (Σ, M) is isomorphic to the singular homology of Σ. While the singular homology of Σ never vanishes, the Rabinowitz Floer homology of (Σ, M) vanishes by (i) since Σ is displaceable. This contradiction proves (ii). □

5.4 Filtered Rabinowitz Floer Homology

For $a < b \in \mathbb{R}$ which are not critical values of $\mathcal{A}_F^{\mathcal{H}}$, we define the $\mathbb{Z}/2$-vector space

$$\mathrm{CF}_q^{(a,b)}(\mathcal{A}_F^{\mathcal{H}}) := \mathrm{Crit}_q^{(a,b)}(\mathcal{A}_F^{\mathcal{H}}) \otimes \mathbb{Z}/2$$

where

$$\mathrm{Crit}_q^{(a,b)}(\mathcal{A}_F^{\mathcal{H}}) := \big\{(v,\eta) \in \mathrm{Crit}_q \mathcal{A}_F^{\mathcal{H}} \,\big|\, \mathcal{A}_F^{\mathcal{H}}(v,\eta) \in (a,b)\big\}.$$

Then $\big(\mathrm{CF}_*^{(-\infty,b)}(\mathcal{A}_F^{\mathcal{H}}), \partial_*^b\big)$ is a sub-complex of $\big(\mathrm{CF}_*(\mathcal{A}_F^{\mathcal{H}}), \partial_*\big)$ since (negative) gradient flow lines of $\mathcal{A}_F^{\mathcal{H}}$ flow downhill. Here $\partial_*^b := \partial_*|_{\mathrm{CF}_*^{(-\infty,b)}}$. There are canonical homomorphisms

$$i_a^{b,c} : \mathrm{CF}_q^{(a,b)}(\mathcal{A}_F^{\mathcal{H}}) \longrightarrow \mathrm{CF}_q^{(a,c)}(\mathcal{A}_F^{\mathcal{H}}), \qquad a \le b \le c$$

and

$$\pi_{a,b}^c : \mathrm{CF}_q^{(a,c)}(\mathcal{A}_F^{\mathcal{H}}) \longrightarrow \mathrm{CF}_q^{(b,c)}(\mathcal{A}_F^{\mathcal{H}}), \qquad a \le b \le c.$$

$i_a^{b,c}$ is a natural inclusion and $\pi_{a,b}^c$ is a projection along $\mathrm{CF}_q^{(a,b)}(\mathcal{A}_F^{\mathcal{H}})$. We note that

$$\mathrm{CF}_q^{(a,c)}(\mathcal{A}_F^{\mathcal{H}}) = \mathrm{CF}_q^{(a,b)}(\mathcal{A}_F^{\mathcal{H}}) \oplus \mathrm{CF}_q^{(b,c)}(\mathcal{A}_F^{\mathcal{H}}),$$

We suppress the indices a, b, and c if there is no confusion. The short exact sequence

$$0 \longrightarrow \mathrm{CF}_q^{(-\infty,a)}(\mathcal{A}_F^{\mathcal{H}}) \xrightarrow{\ i\ } \mathrm{CF}_q^{(-\infty,b)}(\mathcal{A}_F^{\mathcal{H}}) \xrightarrow{\ \pi\ } \mathrm{CF}_q^{(a,b)}(\mathcal{A}_F^{\mathcal{H}}) \longrightarrow 0$$

gives rise to a boundary operator ∂_{a*}^b on $\mathrm{CF}_*^{(a,b)}(\mathcal{A}_F^{\mathcal{H}})$ and this induces a homology group, namely the **filtered Rabinowitz Floer homology**:

$$\mathrm{HF}_q^{(a,b)}(\mathcal{A}_F^{\mathcal{H}}) = \mathrm{H}_q(\mathrm{CF}_\bullet^{(a,b)}(\mathcal{A}_F^{\mathcal{H}}), \partial_{a\bullet}^b).$$

More generally for $a \leq b \leq c$, we have

$$0 \longrightarrow \mathrm{CF}_q^{(a,b)}(\mathcal{A}_F^{\mathcal{H}}) \xrightarrow{i} \mathrm{CF}_q^{(a,c)}(\mathcal{A}_F^{\mathcal{H}}) \xrightarrow{\pi} \mathrm{CF}_q^{(b,c)}(\mathcal{A}_F^{\mathcal{H}}) \longrightarrow 0.$$

The canonical homomorphisms i, π, and the boundary map ∂ are compatible with each other so that they induce canonical homomorphisms on the homology level. Thus we have

$$\cdots \xrightarrow{\delta} \mathrm{HF}_q^{(a,b)}(\mathcal{A}_F^{\mathcal{H}}) \xrightarrow{i_*} \mathrm{HF}_q^{(a,c)}(\mathcal{A}_F^{\mathcal{H}}) \xrightarrow{\pi_*} \mathrm{HF}_q^{(b,c)}(\mathcal{A}_F^{\mathcal{H}}) \xrightarrow{\delta} \mathrm{HF}_{q-1}^{(a,b)}(\mathcal{A}_F^{\mathcal{H}}) \xrightarrow{i_*} \cdots.$$

where δ is the connecting homomorphism.

Corollary 5.4.1. *In the filtered case, the canonical homomorphism is given by*

$$(\Phi_K^F)_* : \mathrm{HF}_q^{(a,b)}(\mathcal{A}_K^{\mathcal{H}}) \longrightarrow \mathrm{HF}_q^{(a-||F-K||_-,b+||F-K||_-)}(\mathcal{A}_F^{\mathcal{H}}).$$

Proof. This is a well-known fact in Floer theory; it follows from the comparison of the action values of $\mathcal{A}_K^{\mathcal{H}}$ and $\mathcal{A}_F^{\mathcal{H}}$, see Proposition 5.2.2. □

5.5 Proof of Theorem B

All of the lemmas and the propositions in this subsection were established for hypersurfaces in [AF1]. Without doubt, their arguments continue to hold in our situation, but we outline the arguments for the sake of completeness.

For $||F|| < \wp(\Sigma)$, we define

$$\mathrm{Crit}_{\mathrm{loc}}(\mathcal{A}_F^{\mathcal{H}}) := \left\{ (v,\eta) \in \mathrm{Crit}\mathcal{A}_F^{\mathcal{H}} \,\middle|\, -||F||_+ \leq \mathcal{A}_F^{\mathcal{H}}(v,\eta) \leq ||F||_- \right\}.$$

We note that the set $\mathrm{Crit}_{\mathrm{loc}}(\mathcal{A}_F^{\mathcal{H}})$ is finite. This follows from the Arzela-Ascoli theorem since the Lagrange multipliers η_i's are uniformly bounded according to Theorem 3.2.11. We define the finite dimensional $\mathbb{Z}/2$ vector space

$$\mathrm{CF}_{\mathrm{loc}}(\mathcal{A}_F^{\mathcal{H}}) := \mathrm{Crit}_{\mathrm{loc}}(\mathcal{A}_F^{\mathcal{H}}) \otimes \mathbb{Z}/2 \,.$$

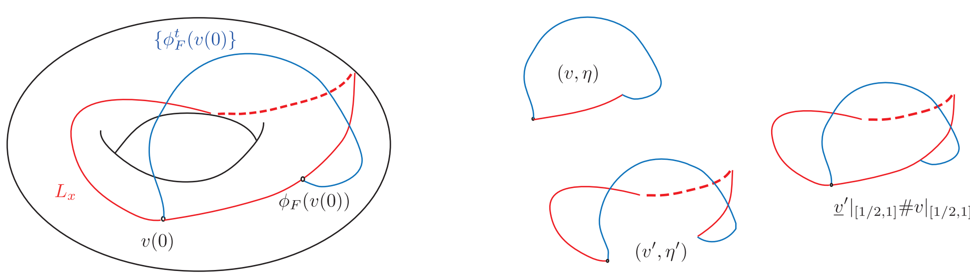
$(\mathrm{CF}_{\mathrm{loc}}(\mathcal{A}_F^{\mathcal{H}}), \partial_{\mathrm{loc}})$ is a chain complex and the **local Rabinowitz Floer homology** is defined by

$$\mathrm{HF}_{\mathrm{loc}}(\mathcal{A}_F^{\mathcal{H}}) := \mathrm{H}(\mathrm{CF}_{\mathrm{loc}}(\mathcal{A}_F^{\mathcal{H}}), \partial_{\mathrm{loc}}).$$

Proposition 5.5.1. *For $F \in C_c^\infty(S^1, \mathcal{M})$ with $||F|| < \wp(\Sigma)$, the following inequalities hold.*

$$\#\left\{\begin{array}{c}\textit{Leafwise coisotropic}\\ \textit{intersection points of } \phi_F\end{array}\right\} \geq \dim \mathrm{CF}_{\mathrm{loc}}(\mathcal{A}_F^{\mathcal{H}}) \geq \dim \mathrm{HF}_{\mathrm{loc}}(\mathcal{A}_F^{\mathcal{H}}) \,.$$

Proof. We briefly sketch the proof and refer to [AF1, Lemma 2.19] for details. The last inequality is obvious. For the first inequality, it suffices to show that different critical points of $\mathcal{A}_F^{\mathcal{H}}$ give rise to different leafwise coisotropic intersection points. If two distinct critical points (v, η), $(v', \eta') \in \mathrm{Crit}_{\mathrm{loc}}\mathcal{A}_F^{\mathcal{H}}$ give rise to the same leafwise coisotropic intersection point, then $\gamma := \underline{v}'|_{[1/2,1]} \# v|_{[1/2,1]}$, where $\underline{v}(t) = v(1-t)$ and $\#$ is the path catenation operator, is a periodic orbit solving (1.2.2), see pictures below. Moreover a close look at γ reveals that $\Omega(\gamma) \leq ||F|| < \wp(\Sigma)$ which contradicts the definition of $\wp(\Sigma)$. $\square$



Proposition 5.5.2. *The local Rabinowitz Floer homology of $\mathcal{A}^{\mathcal{H}}$ is isomorphic to the singular homology of Σ, i.e.*

$$\mathrm{H}(\Sigma;\mathbb{Z}/2) \overset{\Theta}{\cong} \mathrm{HF}_{\mathrm{loc}}(\mathcal{A}^{\mathcal{H}}) \ .$$

Proof. The set $\mathrm{Crit}_{\mathrm{loc}}\mathcal{A}^{\mathcal{H}}$ consists of critical points of $\mathcal{A}^{\mathcal{H}}$ whose action values are zero which in turn implies $\mathrm{Crit}_{\mathrm{loc}}\mathcal{A}^{\mathcal{H}} \cong \Sigma$. Therefore no cascades appear in the boundary operator and $\mathrm{HF}_{\mathrm{loc}}(\mathcal{A}^{\mathcal{H}})$ is isomorphic to Morse homology of Σ. $\square$

The lemma below directly follows from the definition of $\wp(\Sigma)$.

Lemma 5.5.3. For any $(a,b) \subset (-\wp(\Sigma), \wp(\Sigma))$, we have an isomorphism

$$\mathrm{HF}^{(a,b)}(\mathcal{A}^{\mathcal{H}}) \cong \mathrm{HF}_{\mathrm{loc}}(\mathcal{A}^{\mathcal{H}}).$$

Proposition 5.5.4. *If $||F|| < \wp(\Sigma)$, there exists an injective homomorphism*

$$\iota : \mathrm{H}(\Sigma;\mathbb{Z}/2) \longrightarrow \mathrm{HF}_{\mathrm{loc}}(\mathcal{A}^{\mathcal{H}}_F) \ .$$

In particular, $\dim \mathrm{HF}_{\mathrm{loc}}(\mathcal{A}^{\mathcal{H}}_F) \geq \dim \mathrm{H}(\Sigma;\mathbb{Z}/2)$.

Proof. We pick $a \in \mathbb{R}$ with $0 < a < ||F|| < \wp(\Sigma)$ then using the continuation homomorphism in Corollary 5.4.1, we obtain

$$(\Phi_0^F)_* : \mathrm{HF}_{\mathrm{loc}}(\mathcal{A}^{\mathcal{H}}) \cong \mathrm{HF}^{(-a,0)}(\mathcal{A}^{\mathcal{H}}) \longrightarrow \mathrm{HF}^{(-a+||F||_-,|F||_-)}(\mathcal{A}^{\mathcal{H}}_F) \cong \mathrm{HF}_{\mathrm{loc}}(\mathcal{A}^{\mathcal{H}}_F).$$

On the other hand, we also have

$$(\Phi_F^0)_* : \mathrm{HF}^{(-a+||F||_-,|F||_-)}(\mathcal{A}^{\mathcal{H}}_F) \longrightarrow \mathrm{HF}^{(-a+||F||,||F||)}(\mathcal{A}^{\mathcal{H}}) \cong \mathrm{RFH}_{\mathrm{loc}}(\Sigma, M).$$

Using a homotopy of homotopies $D_s^r(t,x) = \varphi_r(s)F(t,x)$, we deduce

$$(\Phi_F^0)_* \circ (\Phi_0^F)_* = \mathrm{id}_{\mathrm{HF}_{\mathrm{loc}}(\mathcal{A}^{\mathcal{H}})}.$$

Therefore $(\Phi_0^F)_*$ is injective and the proposition follows with

$$\iota := (\Phi_0^F)_* \circ \Theta.$$

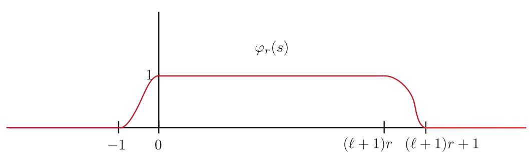
$\square$

Proof of Theorem B. It directly follows from Proposition 5.5.1 and Proposition 5.5.4. $\square$

5.6 Proof of Theorem C

We give a sketch of the proof here and refer to [AMo] for details.[5]

As before, $F \in C_c^\infty(S^1 \times M)$ with $||F|| < \wp(\Sigma)$. Let $\ell \in \mathbb{N}$. For $r \geq 0$, we choose a smooth family of functions $\varphi_r \in C^\infty(\mathbb{R}, [0,1])$.



We consider the following moduli space.

$$\mathcal{M}(r) := \left\{ w \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^k) \;\middle|\; \begin{array}{c} w \text{ is a gradient flow line of} \\ \mathcal{A}^{\mathcal{H}}_{\varphi_r F} \text{ with } \lim\limits_{s \to \pm\infty} w(s) \in \Sigma \times \{0\} \end{array} \right\}.$$

Note that $\mathcal{M}(0) \cong \Sigma$. Moreover one can show that $\mathcal{M}(r)$ is compact in the sense of Theorem 3.2.8.[6]

[5]We tacitly assume all transversality conditions of evaluation maps and Fredholm sections involved (or hidden) in the proof. These conditions are true up to small perturbations, as a matter of fact.

[6]The proof is similar to the corresponding part of the proof of Theorem A.

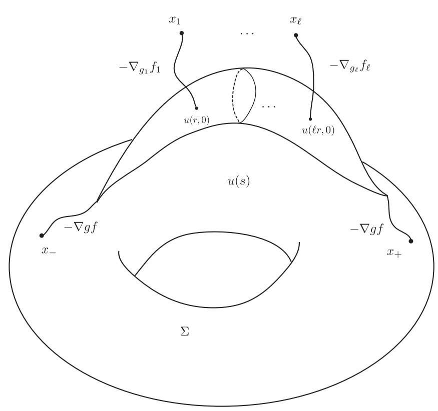
Now we consider the evaluation map

$$\begin{aligned} ev_r : \mathcal{M}(r) &\longrightarrow M^{\times \ell} \\ w = (u, \tau) &\longmapsto \big(u(r,0), \dots u(\ell r, 0)\big). \end{aligned}$$

For generic Morse functions f_i and Riemannian metrics g_i on M and f, g on Σ and for any $x = (x_1, \dots, x_\ell, x_-, x_+) \in \mathrm{Crit} f_1 \times \cdots \times \mathrm{Crit} f_\ell \times \mathrm{Crit} f \times \mathrm{Crit} f$,

$$\mathcal{M}(r, x) := \left\{ w = (u,\tau) \in \mathcal{M}(r) \,\middle|\, \begin{array}{c} \lim\limits_{s \to \pm\infty} u(s) \in W^{u/s}(x_\pm, f) \\ ev_r(u) \in W^s(x_1, f_1) \times \cdots \times W^s(x_\ell, f_\ell) \end{array} \right\}$$

is a smooth manifold. The map defined by



$(u,\tau) \in \mathcal{M}(r,x)$

$$\begin{aligned} \theta_r : CM^*(f_1) \otimes \cdots \otimes CM^*(f_\ell) \otimes CM_*(f) &\longrightarrow CM_*(f) \\ (x_1 \otimes \cdots \otimes x_\ell) \otimes x_- &\longmapsto \sum_{x_+ \in \mathrm{Crit} f} \#_2 \mathcal{M}(r,x) \cdot x_+. \end{aligned}$$

is a chain map. Since $\mathcal{M}(r,x)$ is chain homotopy equivalent to $\mathcal{M}(0,x)$ via the moduli space $\mathcal{M}[0,r] := \{(e,w) \,|\, e \in [0,r],\, w \in \mathcal{M}(r)\}$, θ_r is chain ho-motopic to θ_0. The map θ_0 induces the cohomology operation

$$\begin{aligned}\Theta : H^*(M)^{\otimes \ell} \otimes H_*(\Sigma) &\longrightarrow H_*(\Sigma),\\ (a_1 \otimes \cdots \otimes a_\ell) \otimes b &\longmapsto (a_1 \cup \cdots \cup a_\ell)|_\Sigma \cap b.\end{aligned}$$

Let $\ell = \mathrm{cl}(\Sigma, M)$ so that the cohomology operation Θ is nonzero, and hence $\mathcal{M}(r,x) \neq \emptyset$ for some $x \in \mathrm{Crit} f_1 \times \cdots \times \mathrm{Crit} f_\ell \times \mathrm{Crit} f \times \mathrm{Crit} f$ and for all $r \in \mathbb{R}$. We may assume that Morse functions $f, f_1, \ldots, f_\ell$ and Riemannian metrics $g, g_1, \ldots g_\ell$ satisfy the following generic condition.

- $W^s(x_i, f_i)$ does not intersect with the set of leafwise coisotropic intersection points for $x_i \in \mathrm{Crit} f_i$ with nonzero Morse index.

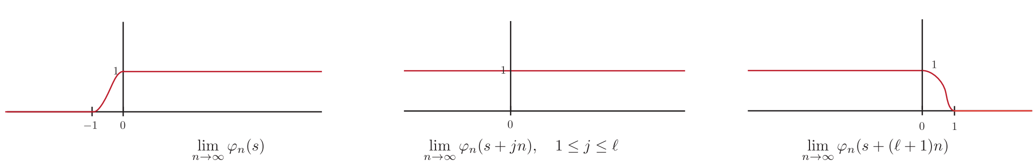
We choose a sequence $w^n = (u^n, \tau^n) \in \mathcal{M}(n, x)$, $n \in \mathbb{N}$. That is,

$$\begin{cases} \partial_s u^n(s,t) + J(s,t,u^n)\big(\partial_t u^n - \displaystyle\sum_{i=1}^k \tau_i^n(s) X_{H_i}(t,u^n) - \varphi_n(s) X_F(t,u^n)\big) = 0, & \\ \partial_s \tau_i^n - \displaystyle\int_0^1 \mathcal{H}(t,u^n)dt = 0, & 1 \leq \forall i \leq k. \end{cases}$$

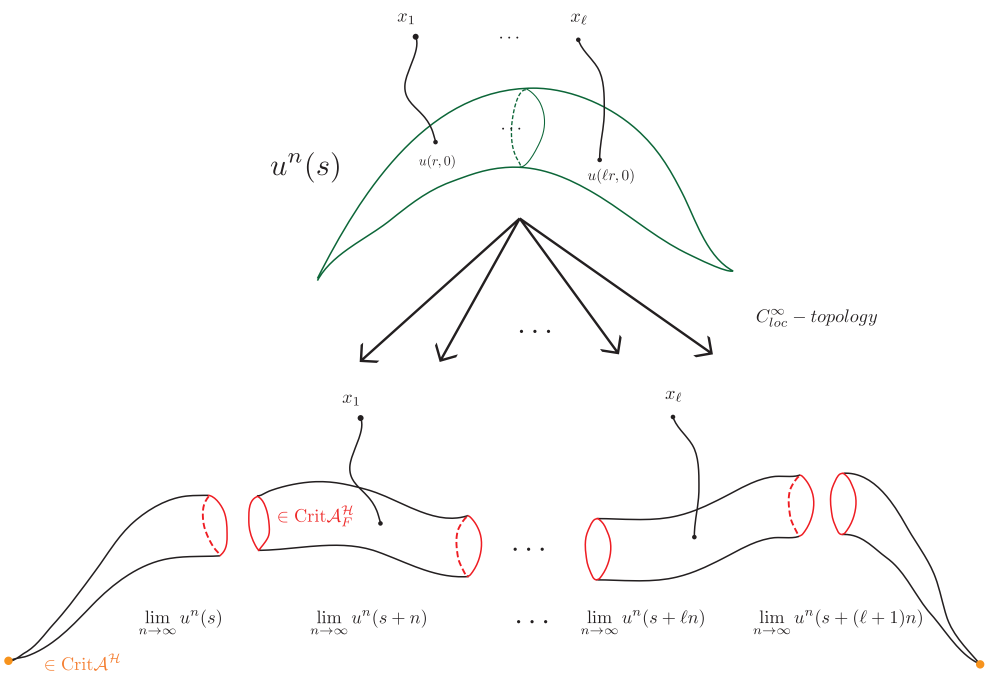
Consider the following $\ell + 2$ sequences of maps:

$$w^n(s + jn), \quad j \in \{0, \ldots \ell + 1\}.$$

The limits of $\varphi_n(s + jn)$, $0 \leq j \leq \ell + 1$ in the C^∞_{loc}-topology look like as pictures below and in particular $\varphi_n(s+jn)F$ converges to F for $1 \leq j \leq \ell$. By applying Theorem 3.2.8, $w^n(s + jn)$ converges (up to subsequence) to



some map $\widehat{w}_j$ in the C^∞_{loc}-topology for $0 \leq j \leq \ell+1$. Note that $\widehat{w}_j$ is a gradient flow line of $\mathcal{A}^{\mathcal{H}}_F$ for $1 \leq j \leq \ell$ and in particular $\widehat{w}_j(\pm\infty) \in \mathrm{Crit}\mathcal{A}^{\mathcal{H}}_F$ for $1 \leq j \leq \ell$. Since we have assumed that $W^s(x_i, f_i)$ does not intersect with



the set of leafwise coisotropic intersection points for $x_i \in \mathrm{Crit} f_i$ with nonzero Morse index, $\widehat{w}_j$, $1 \leq j \leq \ell$ are not constant gradient flow lines. Therefore $\ell+1$ critical points

$$\widehat{w}_1(-\infty),\ \widehat{w}_2(-\infty), \cdots, \widehat{w}_\ell(-\infty),\ \widehat{w}_\ell(\infty)$$

of $\mathcal{A}^{\mathcal{H}}_F$ are distinct. Moreover as in the proof of Theorem A, the assumption $||F|| < \wp(\Sigma)$ guarantees that they give rise to distinct leafwise coisotropic intersection points. This shows the existence of $\mathrm{cl}(\Sigma, M)+1$ leafwise coisotropic intersection points. $\square$

Chapter 6

Künneth formula in Rabinowitz Floer homology

In this chapter, we analyze the Rabinowitz Floer action functional for a product of restricted contact hypersurfaces in a product of symplectic manifolds and derive a Künneth formula for Rabinowitz Floer homology. Consider restricted contact hypersurfaces (Σ_1, λ_1) resp. (Σ_2, λ_2) in exact symplectic manifolds $(M_1, \omega_1 = d\lambda_1)$ resp. $(M_2, \omega_2 = d\lambda_2)$. Moreover we assume that Σ_1 resp. Σ_2 bounds a compact region in M_1 resp. M_2 and that those M_1 and M_2 are geometrically bounded. We introduce projection maps $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$; then $(M_1 \times M_2, \omega_1 \oplus \omega_2)$ admits the symplectic structure $\omega_1 \oplus \omega_2 = \pi_1^* \omega_1 + \pi_2^* \omega_2$.

6.1 Rabinowitz action functional for product manifolds

Since Σ_1 and Σ_2 are restricted contact hypersurfaces, there exist associated Liouville vector fields Y_1 resp. Y_2 on M_1 resp. M_2 such that $\mathcal{L}_{Y_i} \omega_i = \omega_i$ and $Y_i \pitchfork \Sigma_i$ for $i = 1, 2$. We denote by $\phi^t_{Y_i}$ the flow of Y_i and fix $\delta > 0$ such that $\phi^t_{Y_i}|_{\Sigma_i}$ is defined for $|t| < \delta$. Since Σ_1 resp. Σ_2 bounds a compact region in M_1 resp. M_2, we are able to define Hamiltonian functions $G_1 \in C^\infty(M_1)$ and $G_2 \in C^\infty(M_2)$ so that

1. $G_1^{-1}(0) = \Sigma_1$ and $G_2^{-1}(0) = \Sigma_2$ are regular level sets;
2. dG_1 and dG_2 have compact supports;
3. $G_i(\phi^t_{Y_i}(x_i)) = t$ for all $x_i \in \Sigma_i$, $i = 1, 2$, and $|t| < \delta$;

We extend G_1, G_2 to be defined on the whole of $M_1 \times M_2$:

$$\begin{aligned} \widetilde{G}_i := \pi_i^* G_i : M_1 \times M_2 &\longrightarrow \mathbb{R}, \qquad i = 1, 2 \\ (x_1, x_2) &\longmapsto G_i(x_i). \end{aligned}$$

We denote by $\mathcal{L} = \mathcal{L}_{M_1 \times M_2} \subset C^\infty(S^1, M_1 \times M_2)$ the space of contractible loops in $M_1 \times M_2$. The perturbed Rabinowitz action functional $\mathcal{A}_F^{\widetilde{G}_1, \widetilde{G}_2}(v, \eta_1, \eta_2) : \mathcal{L} \times \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$\mathcal{A}_F^{\widetilde{G}_1, \widetilde{G}_2}(v, \eta_1, \eta_2) = -\int_0^1 v^*(\lambda_1 \oplus \lambda_2) - \eta_1 \int_0^1 \widetilde{G}_1(v) dt - \eta_2 \int_0^1 \widetilde{G}_2(v) dt$$

where $\lambda_1 \oplus \lambda_2 := \pi_1^* \lambda_1 + \pi_2^* \lambda_2$. The real numbers η_1 and η_2 can be thought of as Lagrange multipliers as before. A critical point $(v, \eta_1, \eta_2) \in \mathrm{Crit} \mathcal{A}_F^{\widetilde{G}_1, \widetilde{G}_2}$ satisfies

$$\left.\begin{aligned} &\partial_t v = \eta_1 X_{\widetilde{G}_1}(v) + \eta_2 X_{\widetilde{G}_2}(v), \\ &\int_0^1 \widetilde{G}_1(v) dt = 0, \\ &\int_0^1 \widetilde{G}_2(v) dt = 0. \end{aligned}\right\} \tag{6.1.1}$$

We choose a compatible almost complex structure J_1 on M_1 and define the metric on (M_1, ω_1) by $g_1(\cdot, \cdot) = \omega_1(\cdot, J_1 \cdot)$. Analogously we also define the metric $g_2(\cdot, \cdot) = \omega_2(\cdot, J_2 \cdot)$ on (M_2, ω_2). Then $g = g_1 \oplus g_2$ which is the metric on $(M_1 \times M_2, \omega_1 \oplus \omega_2)$ induces a metric m on the tangent space $T_{(v,\eta_1,\eta_2)}(\mathcal{L} \times \mathbb{R}^2) \cong T_v\mathcal{L} \times \mathbb{R}^2$ as follows:

$$m_{(v,\eta_1,\eta_2)}\big((\hat{v}^1, \hat{\eta}_1^1, \hat{\eta}_2^1), (\hat{v}^2, \hat{\eta}_1^2, \hat{\eta}_2^2)\big) := \int_0^1 g_v(\hat{v}^1, \hat{v}^2)dt + \hat{\eta}_1^1\hat{\eta}_1^2 + \hat{\eta}_2^1\hat{\eta}_2^2 \, .$$

In this set-up, the gradient flow equation

$$\partial_s w(s) + \nabla_m \mathcal{A}_F^{\widetilde{G}_1,\widetilde{G}_2}(w(s)) = 0, \quad w \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^2)$$

can be interpreted as maps $u(s,t) : \mathbb{R} \times S^1 \to M_1 \times M_2$ and $\tau_1(s)$, $\tau_2(s) : \mathbb{R} \to \mathbb{R}$ solving

$$\left.\begin{aligned} &\partial_s u + J(t,u)\big(\partial_t v - \tau_1 X_{\widetilde{G}_1}(u) - \tau_2 X_{\widetilde{G}_2}(u)\big) = 0, \\ &\partial_s \tau_1 - \int_0^1 \widetilde{G}_1(u)dt = 0, \\ &\partial_s \tau_2 - \int_0^1 \widetilde{G}_2(u)dt = 0. \end{aligned}\right\} \tag{6.1.2}$$

6.1.1 Compactness

In order to define Rabinowitz Floer homology, we prove the compactness theorem for gradient flow lines of the Rabinowitz action functional in this subsection.

We introduce two auxiliary action functionals $\mathcal{A}_1, \mathcal{A}_2 : \mathcal{L}_{M_1 \times M_2} \times \mathbb{R}^2 \to \mathbb{R}$:

$$\begin{aligned} \mathcal{A}_1(v, \eta_1, \eta_2) &:= \int_0^1 v^*\pi_1^*\lambda_1 - \eta_1 \int_0^1 G_1(v)dt \\ \mathcal{A}_2(v, \eta_1, \eta_2) &:= \int_0^1 v^*\pi_2^*\lambda_2 - \eta_2 \int_0^1 G_2(v)dt. \end{aligned}$$

Lemma 6.1.1. Let $w = (v, \eta_1, \eta_2) \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^2)$ be a gradient flow line of $\mathcal{A}_F^{\widetilde{G}_1,\widetilde{G}_2}$ with asymptotic ends $w_- = (v_-, \eta_{1-}, \eta_{2-})$ and $w_+ = (v_+, \eta_{1+}, \eta_{2+})$.

Then the action values of $\mathcal{A}_1$ and $\mathcal{A}_2$ are bounded along w in terms of the asymptotic data:

$$\begin{aligned} &\text{(i)} \quad \mathcal{A}_1(w(s)) \leq 2|\mathcal{A}_1(w_-)| + |\mathcal{A}_1(w_+)|, \quad \forall s \in \mathbb{R}; \\ &\text{(ii)} \quad \mathcal{A}_2(w(s)) \leq 2|\mathcal{A}_2(w_-)| + |\mathcal{A}_2(w_+)|, \quad \forall s \in \mathbb{R}. \end{aligned}$$

Proof. We only show the first inequality, the latter one is proved in a similar way. Since it holds that $i_{X_{\widetilde{G}_2}}\pi_1^*\omega_1 = 0$, we compute

$$\begin{aligned} \frac{d}{ds}\mathcal{A}_1(w(s)) &= d\mathcal{A}_1(w(s))[\partial_s w(s)] \\ &= \int_0^1 \pi_1^*\omega_1\big(\partial_t v, \partial_s v\big) - \int_0^1 \omega_1 \oplus \omega_2\big(\eta_1 X_{\widetilde{G}_1}(v), \partial_s v\big) - \Big(\int_0^1 \widetilde{G}_1(v)dt\Big)^2 \\ &= \int_0^1 \pi_1^*\omega_1\big(\partial_t v - \eta_1 X_{\widetilde{G}_1}(v), \partial_s v\big)dt - \Big(\int_0^1 \widetilde{G}_1(v)dt\Big)^2 \\ &= -\int_0^1 \pi_1^*\omega_1(\partial_s v, J\partial_s v)dt - \Big(\int_0^1 \widetilde{G}_1(v)dt\Big)^2. \end{aligned}$$

Integrating the above equality from $-\infty$ to any $s_0 \in \mathbb{R}$, we have

$$\begin{aligned} \mathcal{A}_1(w(s_0)) - \mathcal{A}_1(w_-) &= \int_{-\infty}^{s_0} \frac{d}{ds}\mathcal{A}_1(w(s))ds \\ &= -\int_{-\infty}^{s_0}\int_0^1 \pi_1^*\omega_1(\partial_s v, J\partial_s v)dtds - \int_{-\infty}^{s_0}\Big(\int_0^1 \widetilde{G}_1(v)dt\Big)^2 ds. \end{aligned}$$

We set

$$\mathbf{B}(s) := \int_0^1 \pi_1^*\omega_1(\partial_s v, J\partial_s v)dt + \Big(\int_0^1 \widetilde{G}_1(v)dt\Big)^2.$$

Therefore the following estimate can be derived for any $s_0 \in \mathbb{R}$

$$|\mathcal{A}_1(w(s_0))| \leq |\mathcal{A}_1(w_+)| + \Big|\int_{-\infty}^{s_0} \mathbf{B}(s)ds\Big|,$$

and it remains to find a bound for $|\int_{-\infty}^{s_0} \mathbf{B}(s)ds|$. Since $\mathbf{B}(s)$ is nonnegative, we are able to estimate as the following. By setting $s_0 = \infty$, we have

$$\mathcal{A}_1(w_+) - \mathcal{A}_1(w_-) = -\int_{-\infty}^{\infty} \mathbf{B}(s)ds$$

Using the above formula, we obtain

$$\Big|\int_{-\infty}^{s_0} \mathbf{B}(s)ds\Big| \le \Big|\int_{-\infty}^{\infty} \mathbf{B}(s)ds\Big| \le |\mathcal{A}_1(w_+)| + |\mathcal{A}_1(w_-)|.$$

Thus we finally deduce

$$|\mathcal{A}_1(w(s_0))| \le |\mathcal{A}_1(w_+)| + 2|\mathcal{A}_1(w_-)|, \quad \forall s_0 \in \mathbb{R}.$$

□

Lemma 6.1.2. Assume that $v \subset U_\delta := \widetilde{G}_1^{-1}(-\delta,\delta) \cap \widetilde{G}_2^{-1}(-\delta,\delta)$ with $0 < 2\delta < \min\{1, \delta_0\}$. Then there exists $C_i > 0$ satisfying

$$|\eta_i| \le C_i\Big(|\mathcal{A}_i(v,\eta)| + ||\nabla_m \mathcal{A}^{\widetilde{G}_1,\widetilde{G}_2}||_m + 1\Big), \qquad i = 1, 2.$$

Proof. We estimate

$$\begin{aligned} |\mathcal{A}_i(v,\eta_1,\eta_2)| &= \Big|\int_0^1 v^*\pi_i^*\lambda_i + \eta_i \int_0^1 \widetilde{G}_i(v)dt\Big| \\ &\ge \Big|\eta_i \int_0^1 \pi_i^*\lambda_i(v)\big(X_{\widetilde{G}_i}(v)\big)dt\Big| - \Big|\eta_i \int_0^1 \widetilde{G}_i(v)dt\Big|| \\ &\quad - \Big|\int_0^1 \pi_i^*\lambda_i(v)\big(\partial_t v - \eta_1 X_{\widetilde{G}_1}(v) - \eta_2 X_{\widetilde{G}_2}(v)\big)dt\Big| \\ &\ge |\eta_i| - \delta|\eta_i| - C_{i,\delta}||\partial_t v - \eta_1 X_{\widetilde{G}_1}(v) - \eta_2 X_{\widetilde{G}_2}(v))||_{L^1} \\ &\ge |\eta_i| - \delta|\eta_i| - C_{i,\delta}||\nabla_m \mathcal{A}^{\widetilde{G}_1,\widetilde{G}_2}||_m \end{aligned}$$

where $C_{i,\delta} := ||\pi_i^*\lambda_i|_{U_\delta}||_{L^\infty}$. The second inequality holds since $\pi_i^*\lambda_i(X_{\widetilde{G}_j}) = 0$

if $i \neq j$. This estimate finishes the lemma with

$$C_i := \max\left\{\frac{1}{1-\delta}, \frac{C_{i,\delta}}{1-\delta}, \right\}, \qquad i = 1,\, 2.$$

$\square$

Along arguments in Chapter 3, one can easily show the following fundamental lemma using previous two lemmas.

Lemma 6.1.3. For a gradient flow line $w = (u, \tau_1, \tau_2) \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^2)$ of $\mathcal{A}^{\widetilde{G}_1, \widetilde{G}_2}$, the following assertion holds for $i = 1,\, 2$ with some $C, \epsilon > 0$.

$$|\tau_i| \leq C\big(|\mathcal{A}_i(w_-)| + |\mathcal{A}_i(w_+)| + 1\big) \quad \text{ if } \quad ||\nabla_m \mathcal{A}^{\widetilde{G}_1, \widetilde{G}_2}(u, \tau_1, \tau_2)||_m < \epsilon.$$

The following compactness theorem immediately follows from the fundamental lemma as before, see Chapter 3.

Theorem 6.1.4. *Let $\{w_n\}_{n \in \mathbb{N}}$ be a sequence of gradient flow lines of $\mathcal{A}^{\widetilde{G}_1, \widetilde{G}_2}$ for which there exist $a < b$ such that*

$$a \leq \mathcal{A}^{\widetilde{G}_1, \widetilde{G}_2}(w_n(s)) \leq b, \qquad \textit{for all } s \in \mathbb{R}.$$

Then for every reparametrization sequence $\sigma_n \in \mathbb{R}$, the sequence $w_n(\cdot + \sigma_n)$ has a subsequence which is converges in $C^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{L} \times \mathbb{R}^2)$.

This theorem enables us to define the Rabinowitz Floer homology

$$\mathrm{RFH}(\Sigma_1 \times \Sigma_2, M_1 \times M_2) = \mathrm{H}\big(\mathrm{CF}(\mathcal{A}^{\widetilde{G}_1, \widetilde{G}_2}), \partial^{1,2}\big).$$

6.2 Proof of Theorem F

Thanks to the previous section, we are ready to define Rabinowitz Floer homology of $(\Sigma_1 \times \Sigma_2, M_1 \times M_2)$ and to prove Theorem F. Consider the Rabinowitz action functionals $\mathcal{A}^{G_1} : \mathcal{L}_{M_1} \times \mathbb{R} \to \mathbb{R}$ and $\mathcal{A}^{G_2} : \mathcal{L}_{M_2} \times \mathbb{R} \to \mathbb{R}$:

- $\mathcal{A}^{G_1}(v_1, \eta_1) = -\int_0^1 v_1^* \lambda_1 - \eta_1 \int_0^1 G_1(v_1) dt,$
- $\mathcal{A}^{G_2}(v_2, \eta_2) = -\int_0^1 v_2^* \lambda_2 - \eta_2 \int_0^1 G_2(v_2) dt.$

Recall for $i = 1, 2$ that $(v_i, \eta_i) \in \mathrm{Crit}\mathcal{A}^{G_i}$ if and only if

$$\partial_t v_i = \eta_i X_{G_i}(v_i), \quad \int_0^1 G_1(v_i) dt = 0, \tag{6.2.1}$$

and $w_i(s,t) = \big(u_i(s,t), \tau_i(s)\big) : \mathbb{R} \times S^1 \to M_i \times \mathbb{R}$ is a gradient flow line of $\mathcal{A}^{G_i}$ if and only if

$$\partial_s u_i + J_i(t, u_i)\big(\partial_t u_i - \eta_i X_{G_i}(u_i)\big) = 0, \quad \partial_s \tau_i - \int_0^1 G_i(u_i) dt = 0. \tag{6.2.2}$$

Then we define chain complexes $\mathrm{CF}(\mathcal{A}^{G_1})$, $\mathrm{CF}(\mathcal{A}^{G_2})$ and their boundary operators ∂^1, ∂^2 analogously as before and denote their Floer homologies by

$$\mathrm{RFH}(\Sigma_1, M_1) = \mathrm{H}\big(\mathrm{CF}(\mathcal{A}^{G_1}), \partial^1\big), \quad \mathrm{RFH}(\Sigma_2, M_2) = \mathrm{H}\big(\mathrm{CF}(\mathcal{A}^{G_2}), \partial^2\big).$$

Next, for a Künneth formula, we define the tensor product of chain complexes by

$$\big(\mathrm{CF}_*(\mathcal{A}^{G_1}) \otimes \mathrm{CF}_*(\mathcal{A}^{G_2})\big)_n := \bigoplus_{i=0}^{n} \mathrm{CF}_i(\mathcal{A}^{G_1}) \otimes \mathrm{CF}_{n-i}(\mathcal{A}^{G_2}).$$

together with the boundary operator $\partial_n^{\otimes}$ given by

$$\partial_n^{\otimes}\big((v_1, \eta_1)_i \otimes (v_2, \eta_2)_{n-i}\big) = \partial_i^1 (v_1, \eta_1)_i \otimes (v_2, \eta_2)_{n-i} + (v_1, \eta_1)_i \otimes \partial_{n-i}^2 (v_2, \eta_2)_{n-i}.$$

Comparing the critical point equations (6.1.1) and (6.2.1), we easily notice that $\big((v_1, v_2), \eta_1, \eta_2\big) = (v, \eta_1, \eta_2) \in \mathrm{Crit}\mathcal{A}^{G_1,G_2}$ if and only if $(v_1, \eta_1) \in \mathrm{Crit}\mathcal{A}^{G_1}$ and $(v_2, \eta_2) \in \mathrm{Crit}\mathcal{A}^{G_2}$ where $v_1 = \pi_1 \circ v : S^1 \to M_1$ and $v_2 = \pi_2 \circ v : S^1 \to M_2$ for the projections π_1, π_2. Here, $(v_1, v_2) \in C^\infty(S^1, M_1 \times M_2)$ is defined by

$$\begin{aligned}(v_1, v_2) : S^1 &\longrightarrow M_1 \times M_2,\\ t &\longmapsto (v_1(t), v_2(t)).\end{aligned}$$

Moreover since the Conley-Zehnder index behaves additively, we have

$$\mathrm{Crit}_n(\mathcal{A}^{\widetilde{G}_1,\widetilde{G}_2}) = \bigcup_{i+j=n} \mathrm{Crit}_i(\mathcal{A}^{G_1}) \times \mathrm{Crit}_j(\mathcal{A}^{G_2}),$$

and we are able to define a chain homomorphism:

$$\begin{aligned}P_n : \big(\mathrm{CF}_*(\mathcal{A}^{G_1}) \otimes \mathrm{CF}_*(\mathcal{A}^{G_2})\big)_n &\longrightarrow \mathrm{CF}_n(\mathcal{A}^{\widetilde{G}_1,\widetilde{G}_2}),\\ (v_1, \eta_1) \otimes (v_2, \eta_2) &\longmapsto \big((v_1, v_2), \eta_1, \eta_2\big).\end{aligned}$$

To verify that P_n is a chain homomorphism, we need to show that

$$\partial_n^{1,2} \circ P_n = P_{n-1} \circ \partial_n^{\otimes}.$$

For $w_{1-} = (v_{1-}, \eta_{1-}) \in \mathrm{Crit}\mathcal{A}^{G_1}$ and $w_{2-} = (v_{2-}, \eta_{2-}) \in \mathrm{Crit}\mathcal{A}^{G_2}$, we compute

$$\begin{aligned}
\partial_n^{1,2} \circ P_n(w_{1-} \otimes w_{2-}) &= \partial_n^{1,2} \underbrace{\big((v_{1-}, v_{2-}), \eta_{1-}, \eta_{2-}\big)}_{=:w_-} \\
&= \sum_{\substack{w_+ \in \mathrm{Crit}\mathcal{A}^{\widetilde{G}_1, \widetilde{G}_2};\\ \mu(w_+)=\mu(w_-)-1}} \#_2 \mathcal{M}\{w_-, w_+\} w_+ \\
&= \sum_{\substack{(v_{1+},\eta_{1+}) \in \mathrm{Crit}\mathcal{A}^{G_1};\\ \mu(w_{1+})=\mu(w_{1-})-1}} \#_2 \mathcal{M}\big\{w_-, ((v_{1+}, v_{2-}), \eta_{1+}, \eta_{2-})\big\}\big((v_{1+}, v_{2-}), \eta_{1+}, \eta_{2-}\big) \\
&\quad + \sum_{\substack{(v_{2+},\eta_{2+}) \in \mathrm{Crit}\mathcal{A}^{G_2};\\ \mu(w_{2+})=\mu(w_{2-})-1}} \#_2 \mathcal{M}\big\{w_-, ((v_{1-}, v_{2+}), \eta_{1-}, \eta_{2+})\big\}\big((v_{1-}, v_{2+}), \eta_{1-}, \eta_{2+}\big) \\
&= \sum_{\substack{(v_{1+},\eta_{1+}) \in \mathrm{Crit}\mathcal{A}^{G_1};\\ \mu(w_{1+})=\mu(w_{1-})-1}} \#_2 \mathcal{M}\big\{w_{1-}, w_{1+}\big\} P_{n-1}(w_{1+} \otimes w_{2-}) \\
&\quad + \sum_{\substack{(v_{2+},\eta_{2+}) \in \mathrm{Crit}\mathcal{A}^{G_2};\\ \mu(w_{2+})=\mu(w_{2-})-1}} \#_2 \mathcal{M}\big\{w_{2-}, w_{2+}\big\} P_{n-1}(w_{1-} \otimes w_{2+}) \\
&= P_{n-1}(\partial_i^1 w_{1-} \otimes w_{2-}) + P_{n-1}(w_{1-} \otimes \partial_{n-i}^2 w_{2-}) \\
&= P_{n-1} \circ \partial_n^{\otimes}(w_{1-} \otimes w_{2-}).
\end{aligned}$$

where $\mathcal{M}\big\{w_{1-}, w_{1+}\big\}$ resp. $\mathcal{M}\big\{w_{2-}, w_{2+}\big\}$ is the moduli space which consists of gradient flow lines with cascades of $\mathcal{A}^{G_1}$ resp. $\mathcal{A}^{G_2}$. The fourth equality follows by comparing (6.1.2) together with (6.2.2). Therefore we have an isomorphism

$$(P_\bullet)_* : \mathrm{H}_\bullet\big(\mathrm{CF}(\mathcal{A}^{G_1}) \otimes \mathrm{CF}(\mathcal{A}^{G_2})\big) \xrightarrow{\cong} \mathrm{H}_\bullet(\mathrm{CF}(\mathcal{A}^{\widetilde{G}_1, \widetilde{G}_2})) = \mathrm{RFH}_\bullet(\Sigma_1 \times \Sigma_2, M_1 \times M_2).$$

Finally, the algebraic Künneth formula enable us to derive the desired (topological) Künneth formula in Rabinowitz Floer homology.

$$\mathrm{RFH}_n(\Sigma_1 \times \Sigma_2, M_1 \times M_2) \cong \bigoplus_{p=0}^{n} \mathrm{RFH}_p(\Sigma_1, M_1) \otimes \mathrm{RFH}_{n-p}(\Sigma_2, M_2).$$

6.3 Proof of Theorem G

In this section, we do not consider Σ_2 and let (M_2, ω_2) be closed and symplectically aspherical, i.e. $\omega_2|_{\pi_2(M_2)} = 0$. To prove Statement (G1) in Theorem G, we need a compactness theorem for gradient flow lines of the perturbed Rabinowitz action functional on $(\Sigma_1 \times M_2, M_1 \times M_2)$ with an arbitrary perturbation $F \in C_c^\infty(S^1 \times M_1 \times M_2)$. For that reason, we analyze the Rabinowitz action functional again. Once we establish the fundamental lemma, then the remaining steps are exactly same as before. We assume that $\Sigma_1 \times M_2$ bounds a compact region in $M_1 \times M_2$ for Statement (G2). As before, we choose a defining Hamiltonian function $G \in C^\infty(M_1)$ so that

1. $G^{-1}(0) = \Sigma_1$ is a regular level set and dG has a compact support.

2. $G_i(\phi_Y^t(x)) = t$ for all $x \in \Sigma_i$, and $|t| < \delta$;

where Y is the Liouville vector field for $\Sigma_1 \subset M_1$. We define $\widetilde{G} \in C^\infty(M_1 \times M_2)$ by $\widetilde{G}(x_1, x_2) = G(x_1)$ so that $\widetilde{G}$ is a defining Hamiltonian function for $\Sigma_1 \times M_2$. We let $\widetilde{H}(t,x) = \chi(t)\widetilde{G}(x) \in C^\infty(S^1 \times M_1 \times M_2)$ for $\chi \in C^\infty(S^1, \mathbb{R}_{\geq 0})$ with $\int_0^1 \chi(t)dt = 1$ and $\mathrm{Supp}\chi \subset (1/2, 1)$. With a perturbation $F \in C_c^\infty(S^1 \times M_1 \times M_2)$ satisfying $F(t, \cdot) = 0$ for $t \in (1/2, 1)$, the perturbed Rabinowitz action functional $\mathcal{A}_F^{\widetilde{H}} : \mathcal{L} \times \mathbb{R} \to \mathbb{R}$ is given by

$$\mathcal{A}_F^{\widetilde{H}}(v, \eta) = -\int_{D^2} \bar{v}^* \omega_1 \oplus \omega_2 - \eta \int_0^1 \widetilde{H}(t, v)dt - \int_0^1 F(t, v)dt$$

where $\mathcal{L} = \mathcal{L}_{M_1 \times M_2} \subset C^\infty(S^1, M_1 \times M_2)$ is the space of contractible loops in $M_1 \times M_2$ and $\bar{v} : D^2 \to M_1 \times M_2$ is a filling disk of v.

We prove the following key lemma using a kind of isoperimetric inequality.

Lemma 6.3.1. Let $w(s,t) = (v(s,t), \eta(s)) \in C^\infty(\mathbb{R}\times S^1, M_1 \times M_2) \times C^\infty(\mathbb{R}, \mathbb{R})$ be a gradient flow line of $\mathcal{A}_F^{\widetilde{H}}$. We set $\gamma(t) = v(s_0, t) \in C^\infty(S^1, M_1 \times M_2)$ for some fixed $s_0 \in \mathbb{R}$. Then $\int_{D^2} \bar{\gamma}^* \pi_2^* \omega_2$ is uniformly bounded provided

$$||\nabla_m \mathcal{A}_F^{\widetilde{H}}(v(s_0, \cdot), \eta(s_0))||_m < \epsilon$$

for some $\epsilon > 0$:

$$\Big| \int_{D^2} \bar{\gamma}^* \pi_2^* \omega_2 \Big| \leq \max_{x \in \widetilde{M_2}} \big\{ ||\lambda_{\widetilde{M_2}}(x)||_{\tilde{g}_2} \,\big|\, d_{\tilde{g}_2}(x, \widetilde{M}_\star) < \epsilon + ||X_F||_{L^\infty} \big\} \big(\epsilon + ||X_F||_{L^\infty}\big). \tag{6.3.1}$$

where $\widetilde{M_2}$ is the universal covering of M_2; $\tilde{g}_2$ is the lifting of the metric $g_2(\cdot,\cdot) = \omega_2(\cdot, J_2 \cdot)$ on M_2; $\widetilde{M}_\star$ is a fundamental domain in $\widetilde{M_2}$; $d_{\tilde{g}_2}(x, \widetilde{M}_\star)$ is the distance between x and $\widetilde{M}_\star$; the value on the right hand side of (6.3.1) is finite since $\widetilde{M}_\star \cong M_2$ is compact.

Proof. We write $v(s,t)$ as $v(s,t) = (v_1, v_2)(s,t)$ where $v_1 : \mathbb{R} \times S^1 \to M_1$ and $v_2 : \mathbb{R} \times S^1 \to M_2$. Let $\gamma \in C^\infty(S^1, M_1 \times M_2)$ be defined by $\gamma(t) = v(s_0, t)$ for some $s_0 \in \mathbb{R}$. Since γ is contractible and M_2 is symplectically aspherical, the value of $\int_{D^2} \bar{\gamma}^* \pi_2^* \omega_2$ is well-defined. Let $\gamma_2 := \pi_2 \circ \gamma$. We also consider $(\widetilde{M_2}, \widetilde{\omega_2})$ the universal cover of M_2 where $\widetilde{\omega_2}$ is the lift of ω_2 and we also lift the metric g_2 on M_2 which we write as $\tilde{g}_2$. Since we have assumed the symplectically asphericity of (M_2, ω_2), there exists a primitive one form $\lambda_{\widetilde{M_2}}$ of $\widetilde{\omega_2}$. Let $\widetilde{M}_\star (\cong M_2)$ be one of the fundamental domains in $\widetilde{M_2}$ and $\tilde{v}(s,t) : \mathbb{R} \times S^1 \to M_1 \times \widetilde{M_2}$ be the lift of v such that $\tilde{v}(s_0, t) = \tilde{\gamma}(t)$ intersects $M_1 \times \widetilde{M}_\star$. Now, we can show the following kind of isoperimetric inequality.

This inequality concludes the proof.

$$\begin{aligned}
\Big|\int_{D^2}\bar{\gamma}^*\pi_2^*\omega_2\Big| = \Big|\int_{D^2}(\tilde{\bar{\gamma}}_2)^*\widetilde{\omega_2}\Big| &= \Big|\int_0^1 \tilde{\gamma}_2^*\lambda_{\widetilde{M}_2}\Big| \\
&\leq ||\lambda_{\widetilde{M}_2}|_{\gamma_2(S^1)}||_{L^\infty}\int_0^1 ||\partial_t\tilde{\gamma}_2||_{\tilde{g}_2}dt \\
&= ||\lambda_{\widetilde{M}_2}|_{\gamma_2(S^1)}||_{L^\infty}\int_0^1 ||\partial_t\gamma_2||_{g_2}dt \\
&= ||\lambda_{\widetilde{M}_2}|_{\gamma_2(S^1)}||_{L^\infty}\int_0^1 ||J\partial_s\gamma_2 + \pi_{2*}X_F(t,\gamma_2)||_{g_2}dt \\
&\leq \lambda_{\mathrm{Max}}\big(||\nabla_m\mathcal{A}_F^{\widetilde{H}}(v(s_0,\cdot),\eta(s_0))||_m + ||X_F||_{L^\infty}\big).
\end{aligned}$$

where

$$\begin{aligned}
\lambda_{\mathrm{Max}} &:= \max_{x\in\widetilde{M}_2}\Big\{||\lambda_{\widetilde{M}_2}(x)||_{\tilde{g}_2}\,\Big|\, d_{\tilde{g}_2}(x,\widetilde{M}_\star) < \int_0^1 ||\partial_t\gamma_2||_{g_2}dt\Big\} \\
&\leq \max_{x\in\widetilde{M}_2}\big\{||\lambda_{\widetilde{M}_2}(x)||_{\tilde{g}_2}\,\big|\, d_{\tilde{g}_2}(x,\widetilde{M}_\star) < ||\nabla_m\mathcal{A}_F^{\widetilde{H}}(v(s_0,\cdot),\eta(s_0))||_m + ||X_F||_{L^\infty}\big\}.
\end{aligned}$$

□

The following two lemmas can be proved similarly as before.

Lemma 6.3.2. We assume that for $(v,\eta)\in C^\infty(S^1, M_1\times M_2)\times\mathbb{R}$, $v(t)\in U_\delta := \widetilde{G}^{-1}(-\delta,\delta)$ for all $t\in(\frac{1}{2},1)$ with $0<2\delta<\min\{1,\delta_0\}$. Then there exists $C>0$ satisfying

$$|\eta| \leq C\Big(|\mathcal{A}_F^{\widetilde{H}}(v,\eta)| + ||\nabla_m\mathcal{A}_F^{\widetilde{H}}(v,\eta)||_m + \Big|\int_{D^2}\bar{v}^*\pi_2^*\omega_2\Big| + 1\Big).$$

Lemma 6.3.3. For $(v,\eta)\in C^\infty(S^1, M_1\times M_2)\times\mathbb{R}$ if there exists $t\in[\frac{1}{2},1]$ such that $v(t)\notin U_\delta$, then $||\nabla_m\mathcal{A}_F^{\widetilde{H}}(v,\eta)||_m > \epsilon$ for some $\epsilon=\epsilon_\delta$.

Due to the three previous lemmas, we are able to deduce the fundamental lemma in the situation of Theorem G, and thus we obtain a uniform L^∞-

bound on the Lagrange multiplier η.

Lemma 6.3.4. For a gradient flow line $w(s) = (v,\eta)(s) \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R})$, the following assertions holds with some $C, \epsilon > 0$. If $||\nabla_m \mathcal{A}_F^{\widetilde{H}}(v,\eta)||_m < \epsilon$,

$|\eta| \leq C\big(|\mathcal{A}_F^{\widetilde{H}}(w_-)|+|\mathcal{A}_F^{\widetilde{H}}(w_+)|+\epsilon+\Xi_\epsilon+1\big)$ provided that $||\nabla_m \mathcal{A}_F^{\widetilde{H}}(v,\eta)||_m < \epsilon$

where $\Xi_\epsilon = \max\big\{||\lambda_{\widetilde{M_2}}(x)||_{\tilde{g}_2} \,|\, d_{\tilde{g}_2}(x, \widetilde{M}_\star) < \epsilon + ||X_F||_{L^\infty}\big\}\big(\epsilon + ||X_F||_{L^\infty}\big) < \infty$.

Proof. The proof is almost same as the proof of Lemma 6.1.3. Since

$$||\nabla_m \mathcal{A}_F^{\widetilde{H}}(v,\eta)||_m < \epsilon,$$

$v(t) \subset U_\delta$ for $t \in (\frac{1}{2}, 1)$ by Lemma 6.3.3. Thus Lemma 6.3.1 and Lemma 6.3.2 prove the lemma. □

This fundamental lemma proves compactness of gradient flow lines and enables us to find a leafwise intersection points. Let $\phi \in \mathrm{Ham}_c(M_1 \times M_2, \omega_1 \oplus \omega_2)$ be a Hamiltonian diffeomorphism with the Hofer norm less than $\wp(\Sigma_1, \lambda_1)$. Then there exists a leafwise coisotropic intersection point even if $\Sigma_1 \times M_2$ does not bound a compact region in $M_1 \times M_2$, see the proof of Theorem A.

Next, we define the Rabinowitz Floer homology for $(\Sigma_1 \times M_2, M_1 \times M_2)$ in the same way as before and derive the Künneth formula in this situation. We consider another two action functionals $\mathcal{A}^H : \mathcal{L}_{M_1} \times \mathbb{R} \to \mathbb{R}$ and $\mathcal{A} : \mathcal{L}_{M_2} \to \mathbb{R}$ defined by

$$\mathcal{A}^H(v_1,\eta) := -\int_0^1 v_1^*\lambda_1 - \eta\int_0^1 H(t,v)dt, \quad \mathcal{A}(v_2) := -\int_{D^2} \bar{v}_2^*\omega_2.$$

where $H(t,x) = \chi(t)G(x) \in C^\infty(S^1 \times M_1)$. As in the proof of Theorem F, we compare critical points of $\mathcal{A}^{\widetilde{H}}$ and critical points of $\mathcal{A}^H$ as follows.

$$\mathrm{Crit}_n(\mathcal{A}^{\widetilde{H}}) = \bigcup_{i+j=n} \mathrm{Crit}_i(\mathcal{A}^H) \times \mathrm{Crit}_j(\mathcal{A}).$$

Since Crit$\mathcal{A}$ consists of one component M_2, any gradient flow line with cascades of $\mathcal{A}$ necessarily has zero cascades, and hence is simply a gradient flow line of an additional Morse function $f \in C^\infty(M_2)$. Thus the chain group for the Morse-Bott homology of $\mathcal{A}$ is given by $\mathrm{CF}(\mathcal{A}, f) = \mathrm{CM}(f)$. Here CM stands for the Morse complex. The following map is a chain isomorphism, which can be verified using the methods of the previous subsection.

$$\begin{aligned} P_n : \big(\mathrm{CF}_*(\mathcal{A}^H) \otimes \mathrm{CM}_*(f)\big)_n &\longrightarrow \mathrm{CF}_n(\mathcal{A}^{\widetilde{H}}), \\ (v_1, \eta) \otimes v_2 &\longmapsto \big((v_1, v_2), \eta\big). \end{aligned}$$

Therefore it induces an isomorphism on the homology level

$$(P_\bullet)_* : \mathrm{H}_\bullet\big(\mathrm{CF}(\mathcal{A}^H) \otimes \mathrm{CM}(f)\big) \xrightarrow{\cong} \mathrm{H}_\bullet\big(\mathrm{CF}(\mathcal{A}^{\widetilde{H}})\big) = \mathrm{RFH}_\bullet(\Sigma_1 \times M_2, M_1 \times M_2)$$

and the Künneth formula for $(\Sigma_1 \times M_2, M_1 \times M_2)$ directly follows:

$$\mathrm{RFH}_n(\Sigma_1 \times M_2, M_1 \times M_2) \cong \bigoplus_{p=0}^{n} \mathrm{RFH}_p(\Sigma_1, M_1) \otimes \mathrm{H}_{n-p}(M_2).$$

Chapter 7

Infinitely many leafwise coisotropic intersection points

As we have mentioned, we do not have a compactness theorem for the perturbed Rabinowitz action functional on product manifolds in general. For that reason, the existence problem of leafwise coisotropic intersection points for a product of restricted contact hypersurfaces is still open. However if a product of restricted contact hypersurfaces is of restricted contact type again, we have proved the compactness theorem in Chapter 3. Therefore we are able to find leafwise coisotropic intersection points using the Künneth formula derived in the previous chapter on restricted contact coisotropic submanifolds of product type. In particular, we find a class of restricted contact coisotropic submanifolds which have infinitely many leafwise coisotropic intersection points for a generic perturbations using the Künneth formula.

7.1 Proofs of Corollary F and Corollary G

Since the Rabinowitz action functional can be defined for each homotopy class of loops, we can define the Rabinowitz Floer homology $\mathrm{RFH}(\Sigma, M, \gamma)$

for $\gamma \in [S^1, M]$. Note that $\mathrm{RFH}(\Sigma, M)$ considered so far, equals $\mathrm{RFH}(\Sigma, M, x)$, $x \in M$. We also can define Rabinowitz Floer homology on the full loop space $\Lambda N := C^\infty(S^1, M)$ and denote it by $\mathbf{RFH}(\Sigma, M)$. Then we have

$$\mathbf{RFH}_*(\Sigma, M) = \bigoplus_{\gamma \in [S^1, M]} \mathrm{RFH}_*(\Sigma, M, \gamma).$$

Theorem 7.1.1. *[CFO, AS] For a unit cotangent bundle S^*N over a closed Riemannian manifold N,*

$$\mathbf{RFH}_*(S^*N, T^*N) \cong \begin{cases} \mathrm{H}_*(\Lambda N), & * > 1, \\ \mathrm{H}^{-*+1}(\Lambda N), & * < 0. \end{cases}$$

Since the Künneth formula obviously holds for **RFH** as well, the following corollary directly follows.

Corollary 7.1.2. *Let Σ_1 be a restricted contact hypersurface in (M_1, ω_1) bounding a compact region. If $\mathbf{RFH}_*(\Sigma_1, M_1) \neq 0$, and $\dim \mathrm{H}_*(\Lambda N) = \infty$ then*

$$\dim \mathbf{RFH}_*(\Sigma_1 \times S^*N, M_1 \times T^*N) = \infty.$$

Accordingly, if $\Sigma_1 \times S^*N$ is of contact type again, $\Sigma_1 \times S^*N$ has infinitely many leafwise coisotropic intersection points or a periodic leafwise coisotropic intersection point for a generic perturbation $\phi_F \in \mathrm{Ham}_c(M_1 \times M_2)$.

From now on, we investigate leafwise coisotropic intersection points on $(S^*S^1 \times S^*N, T^*S^1 \times T^*N)$.

Lemma 7.1.3. $S^*S^1 \times S^*N$ is a contact submanifold of codimension two in $T^*S^1 \times T^*N$.

Proof. $(T^*S^1, \omega_{\mathrm{S}^1,\mathrm{can}}) \cong (S^1 \times \mathbb{R}, d\theta \wedge dr)$ where θ is the angular coordinate on S^1 and r is the coordinate on $\mathbb{R}$. Then $d\theta \wedge dr$ has two global primitives $-rd\theta$ and $-rd\theta + d\theta$. We can easily check that $S^*S^1 \times S^*N$ carries a contact structure with $-rd\theta \oplus \lambda_{\mathrm{N,can}}$ and $(-rd\theta + d\theta) \oplus \lambda_{\mathrm{N,can}}$ where $\lambda_{\mathrm{N,can}}$ is the canonical one form on T^*N. $\square$

To exclude periodic leafwise coisotropic intersection points, we consider the loop space Ω defined by

$$\Omega := \big\{ v = (v_1, v_2) \in C^\infty(S^1, T^*S^1 \times T^*N) \,\big|\, v_1 \text{ is contractible in } T^*S^1 \big\}.$$

Then we consider the Rabinowitz action functional on this loop space, $\mathcal{A}^{\widetilde{G}_1, \widetilde{G}_2} : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ which defines the Rabinowitz Floer homology $\mathrm{RFH}(S^*S^1 \times S^*N, T^*S^1 \times T^*N, \Omega)$. Moreover the following type of the Künneth formula holds.

$$\mathrm{RFH}_n(S^*S^1 \times S^*N, T^*S^1 \times T^*N, \Omega) \cong \bigoplus_{p=0}^{n} \mathrm{RFH}_p(S^*S^1, T^*S^1) \otimes \mathbf{RFH}_{n-p}(S^*N, T^*N).$$

Therefore $\mathrm{RFH}(S^*S^1 \times S^*N, T^*S^1 \times T^*N, \Omega)$ is of infinite dimensional whenever $\dim \mathrm{H}_*(\Lambda N) = \infty$ and Lemma 7.1.4 below yields that there are infinitely many leafwise coisotropic intersection points for a generic perturbation $\phi_F \in \mathrm{Ham}_c(T^*S^1 \times T^*N)$ if $\dim N \geq 2$. This proves Corollary F.

In order to prove that there is generically no periodic leafwise coisotropic intersection points, we use an argument in [AF2]. Consider $\mathcal{A}_F^{\widetilde{H}_1, \widetilde{H}_2} : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ where $\widetilde{H}_i(t,x) = \chi(t) G_i(x) \in C^\infty(S^1 \times M_1 \times M_2)$, $i = 1, 2$ and where $F \in C_c^\infty(S^1 \times M_1 \times M_2)$ with $F(t, \cdot) = 0$ for $t \in (1/2, 1)$. We denote by $\mathcal{R}$ the set of periodic Reeb orbits in T^*N which has dimension one. It is convenient to introduce the following sets:

$$\mathcal{F}^j := \big\{ F \in C_c^j(S^1 \times T^*S^1 \times T^*N) \,\big|\, F(t, \cdot) = 0,\ \forall t \in \big[\tfrac{1}{2}, 1\big] \big\}, \quad \mathcal{F} := \bigcap_{j=1}^{\infty} \mathcal{F}^j.$$

Lemma 7.1.4. If $\dim N \geq 2$, the following set is dense in $\mathcal{F}$.

$$\mathcal{F}_{S^*S^1\times S^*N} := \left\{ F \in \mathcal{F} \;\middle|\; \begin{array}{c} \mathcal{A}_F^{\widetilde{H}_1,\widetilde{H}_2} \text{ is Morse, } v(0)\cap(S^*S^1\times R)=\emptyset \\ \text{for all } (v,\eta_1,\eta_2)\in \mathrm{Crit}\mathcal{A}_F^{\widetilde{H}_1,\widetilde{H}_2},\ R\in\mathcal{R}. \end{array} \right\}.$$

Proof. We denote by

$$\Omega^{1,2} := \big\{ v=(v_1,v_2)\in W^{1,2}(S^1,T^*S^1\times T^*N)\,\big|\, v_1 \text{ is contractible in } T^*S^1 \big\}.$$

the loop space which is indeed a Hilbert manifold. Let $\mathcal{E}$ be the L^2-bundle over $\Omega^{1,2}$ with $\mathcal{E}_v = L^2(S^1, v^*T(S^*S^1\times S^*N))$. We consider the section

$$S:\Omega^{1,2}\times\mathbb{R}^2\times\mathcal{F}^j \longrightarrow \mathcal{E}^\vee\times\mathbb{R}^2 \quad \text{defined by} \quad S(v,\eta_1,\eta_2,F) := d\mathcal{A}_F^{\widetilde{H}_1,\widetilde{H}_2}(v,\eta_1,\eta_2).$$

Here the symbol $\vee$ represents the dual space. At $(v,\eta_1,\eta_2,F)\in S^{-1}(0)$, the vertical differential

$$DS: T_{(v,\eta_1,\eta_2,F)}\Omega^{1,2}\times\mathbb{R}^2\times\mathcal{F}^j \longrightarrow \mathcal{E}_v^\vee\times\mathbb{R}^2$$

is given by the pairing

$$\big\langle DS_{(v,\eta_1,\eta_2,F)}[\hat{v}^1,\hat{\eta}_1^1,\hat{\eta}_2^1,\hat{F}],[\hat{v}^2,\hat{\eta}_2^1,\hat{\eta}_2^2]\big\rangle = \mathcal{H}_{\mathcal{A}_F^{\widetilde{H}_1,\widetilde{H}_2}}[(\hat{v}^1,\hat{\eta}_1^1,\hat{\eta}_2^1),(\hat{v}^2,\hat{\eta}_1^2,\hat{\eta}_2^2)] + \int_0^1 \hat{F}(t,v)dt.$$

where $\mathcal{H}_{\mathcal{A}_F^{\widetilde{H}_1,\widetilde{H}_2}}$ is the Hessian of $\mathcal{A}_F^{\widetilde{H}_1,\widetilde{H}_2}$. As shown in [AF1], we know that for $(v,\eta_1,\eta_2,F)\in S^{-1}(0)$, $DS_{(v,\eta_1,\eta_2,F)}$ is surjective on the space

$$\mathcal{V} := \big\{ (\hat{v},\hat{\eta}_1,\hat{\eta}_2,\hat{F}) \in T_{(v,\eta_1,\eta_2,F)}(\Omega^{1,2}\times\mathbb{R}^2\times\mathcal{F}^j)\,\big|\,\hat{v}(0)=0 \big\}.$$

Next, we consider the evaluation map

$$\begin{aligned}\mathrm{ev}: \mathcal{M} &\longrightarrow S^*S^1 \times S^*N,\\ (v,\eta_1,\eta_2,F) &\longmapsto v(0).\end{aligned}$$

The surjectivity of $DS_{(v,\eta_1,\eta_2,F)}|_{\mathcal{V}}$ implies that ev is a submersion, see a lemma due to Salamon [AF2, Lemma 3.5]. Then $\mathcal{M}_{\mathcal{R}} := \mathrm{ev}^{-1}(S^*S^1 \times \mathcal{R})$ is a submanifold in $\mathcal{M}$ of

$$\mathrm{codim}(\mathcal{M}_{\mathcal{R}}/\mathcal{M}) = \mathrm{codim}(S^*S^1 \times \mathcal{R}/S^*S^1 \times S^*N).$$

We consider the projections $\Pi : \mathcal{M} \to \mathcal{F}^j$ and $\Pi_{\mathcal{R}} := \Pi_{|\mathcal{M}_{\mathcal{R}}}$. Then $\mathcal{A}_F^{\widetilde{H}_1,\widetilde{H}_2}$ is Morse if and only if F is a regular value of Π, which is a generic property by Sard-Smale theorem (for j large enough). The set $\Pi^{-1}(F)$ of leafwise coisotropic intersection points for F is manifold of required dimension zero since it is a critical set of $\mathcal{A}_F^{\widetilde{H}_1,\widetilde{H}_2}$. On the other hand, $\Pi_{\mathcal{R}}^{-1}(F)$ is a manifold of dimension

$$0 + \dim \mathcal{M}_{\mathcal{R}} - \dim \mathcal{M} = -\mathrm{codim}(\mathcal{M}_{\mathcal{R}}/\mathcal{M}) < 0$$

since we have assumed $\dim N \geq 2$. Therefore ev does not intersect $S^*S^1 \times \mathcal{R}$, so the set

$$\mathcal{F}^j_{S^*S^1\times S^*N} := \mathcal{F}_{S^*S^1\times S^*N} \cap \mathcal{F}^j$$

is dense in $\mathcal{F}$ for all $j \in \mathbb{N}$. Since $\mathcal{F}_{S^*S^1\times S^*N}$ is the countable intersection of $\mathcal{F}^j_{S^*S^1\times S^*N}$ for $j \in \mathbb{N}$, it is dense again in $\mathcal{F}$ and the lemma is proved. $\square$

In the case of Theorem G, we consider the Rabinowitz action functional $\mathcal{A}_F^{\widetilde{H}} : \Omega_{M_2} \times \mathbb{R} \to \mathbb{R}$ by where

$$\Omega_{M_2} : \big\{ v = (v_1, v_2) \in C^\infty(S^1, M_1 \times M_2) \,\big|\, v_2 \text{ is contractible in } M_2 \big\}.$$

In a similar vein as above, we are able to prove Corollary G.

Corollary 7.1.5. *Let* (M_2, ω_2) *be a closed symplectically aspherical symplectic manifold. If a closed manifold* N *has* $\dim \mathrm{H}_*(\Lambda N) = \infty$*,*

$$\dim \mathrm{RFH}_*(S^*N \times M_2, T^*N \times M_2, \Omega_{M_2}) = \infty.$$

Therefore, if $\dim N \geq 2$*,* $S^*N \times M_2$ *has infinitely many leafwise coisotropic intersection points for a generic perturbation.*

Remark 7.1.6. Corollary F and Corollary G still holds when we deal with a fiber-wise star shaped hypersurface in T^*N instead of S^*N, see [AF2].

Bibliography

[AF1] P. Albers, U. Frauenfelder, *Leaf-wise intersections and Rabinowitz Floer homology,* Journal of Topology and Analysis **2** (2010), no. 1, 77–98.

[AF2] P. Albers, U. Frauenfelder, *Infinitely many leaf-wise intersection points on cotangent bundles,* Expositiones Mathematicae **30** (2012), no. 2, 168–182.

[AF3] P. Albers, U. Frauenfelder, *Spectral invariants in Rabinowitz Floer homology and global Hamiltonian perturbation*, Journal of Modern dynamics **4** (2010), no. 2, 329–357.

[AF4] P. Albers, U. Frauenfelder, *A remark on a theorem by Ekeland-Hofer,* Israel Journal of Mathematics **187** (2012), no. 1, 485–491.

[AF5] P. Albers, U. Frauenfelder, *Rabinowitz Floer homology : A survey*, Global Differential Geometry, Springer Proceedings in Mathematics (2012), 437–461.

[AF6] P. Albers, U. Frauenfelder, *A variational approach to Givental's nonlinear Maslov index*, Geometric and Functional Analysis (GAFA), (2012), 1033–1050.

[AL] M. Audin and J. Lafontaine, editors. "Holomorphic curves in symplectic geometry", Progress Math. **117**, Birkhäuser Verlag , 1994.

BIBLIOGRAPHY

[AMo] P. Albers, A. Momin, *Cup-length estimates for leaf-wise intersections*, Mathematical Proceedings of the Cambridge Philosophical Society **149** (2010), 539–551.

[AMe1] P. Albers, W. Merry, *Translated points and Rabinowitz Floer homology*, Journal of Fixed Point Theory and Applications, (2013), 201–214.

[AMe2] P. Albers, W. Merry, *Orderability, contact non-squeezing, and Rabinowitz Floer homology*, (2013) arXiv:1302.6576.

[AFMe] P. Albers, U. Fuchs, W. Merry, *Orderability and the Weinstein Conjecture*, (2013) arXiv:1310.0786.

[AMc] P. Albers, M. McLean, Non-displaceable contact embeddings and infinitely many leaf-wise intersections, Journal of Symplectic Geometry **9** (2011), no. 3, 271–284.

[AS] A. Abbondandolo, M. Schwarz, *Estimates and computations in Rabinowitz-Floer homology,* Journal of Topology and Analysis **1** (2009), no. 4, 307–405.

[Ba] A. Banyaga, *On fixed points of symplectic maps,* Inventiones Mathematicae, **50** (1980), no. 3, 215–229.

[Bo1] P. Bolle, *Une condition de contact pour les sous-variétés coïsotropes d'une variété symplectique,* Comptes Rendus de l'Académie des Sciences - Series I **322** (1996), 83–86.

[Bo2] P. Bolle, *A contact condition for p-dimensional submanifolds of a symplectic manifold ($2 \leq p \leq n$),* Mathematische Zeitschrift **227** (1998), 211–230.

[BF] Y. Bae, U. Frauenfelder, *Continuation homomorphism in Rabinowitz Floer homology for symplectic deformations*, Math. Proc. Camb. Phil. Soc. (2011), **151** 471–502.

BIBLIOGRAPHY

[BO1] F. Bourgeois, A. oancea, *Fredholm theory and transversality for the parametrized and for the S^1-invariant symplectic action,* Journal of the European Mathematical Society **12** (2010), no. 5, 1181–1229.

[BO2] F. Bourgeois, A. oancea, *The index of Floer moduli problems for parametrized action functionals,* Geometriae Dedicata **165** (2013), no. 1, 5–24.

[CF] K. Cieliebak, U. Frauenfelder, *A Floer homology for exact contact embeddings*, Pacific Journal of Mathematics **249** (2009), no. 2, 251–316.

[CFO] K. Cieliebak, U. Frauenfelder, A. Oancea, *Rabinowitz Floer homology and symplectic homology,* Annales scientifiques de l'ENS **43** (2010), fasc. 6, 957–1015.

[CFP] K. Cieliebak, U. Frauenfelder, G.P. Paternain, *Symplectic topology of Mañé's critical values,* Geometry & Topology **14** (2010) 1765–1870.

[Dr] D. L. Dragnev, Symplectic rigidity, symplectic fixed points and global perturbations of Hamiltonian systems. Comm. Pure Appl. Math, **(**61) (2008), 346–370.

[EH] I. Ekeland, H. Hofer, *Two symplectic fixed-point theorems with applications to Hamiltonian dynamics,* J. Math. Pures Appl. **68** (1989), no. 4, 467–489.

[FHS] A. Floer, H. Hofer, D. Salamon, *Transversality in elliptic Morse theory for the symplectic action*, Duke Math. J. **80** (1995), no.1, 251–292.

[Fl1] A. Floer, *Morse theory for Lagrangian intersections*, J. Diff. Geom. **28** (1988), 513–547.

[Fl2] A. Floer, *The unregularizaed gradient flow of the symplectic action*, Comm. Pure Appl. Math. **41** (1988), 775–813.

BIBLIOGRAPHY

[Fr] U. Frauenfelder, *The Arnold-Givental conjecture and moment Floer homology,* Int. Math. Res. Not. **42** (2004), 2179–2269.

[FS] U. Frauenfelder, F. Schlenk, *S^1-equivariant Rabinowitz-Floer homology*, (2013) arXiv:1312.5201.

[Gi] V. L. Ginzburg, *Coisotropic intersections,* Duke Math. J. **140** (2007), no. 1, 111–163.

[Gü] B. Gürel, *Leafwise coisotropic intersections,* Int. Math. Res. Not. (2010), no. 5, 914–931.

[GG] V. L. Ginzburg, B. Gürel, *A C^2-smooth counterexample to the Hamiltonian Seifert conjecture in R^4*, Ann. of Math. **158** (2003), 953–976.

[Ho] H. Hofer, *On the topological properties of symplectic maps,* Proc. Roy. Soc. Edinburgh Sect. A **115** (1990), no. 1-2, 25–38.

[HWZ1] H. Hofer, K.Wysocki, E. Zehnder, *A general Fredholm theory I: A splicing-based differential geometry*, J. Eur. Math. Soc. **9** (2007), 841–876.

[HWZ2] H. Hofer, K.Wysocki, E. Zehnder, *A general Fredholm theory II: Implicit function theorems*, GAFA. **19** (2009), 206–293.

[HWZ3] H. Hofer, K.Wysocki, E. Zehnder, *A general Fredholm theory III: Fredholm functors and Polyfolds*, Geometry and Topology **13** (2009), 2279–2387.

[Ka1] J. Kang, *Survival of infinitely many critical points for the Rabinowitz action functional,* J. Modern Dynamics **4** (2010), no. 4, 733–739.

[Ka2] J. Kang, *Existence of leafwise intersection points in the unrestricted case,* Israel Journal of Mathematics, (2012) **190**, no. 1, 111–134.

[Ka3] J. Kang, *Generalized Rabinowitz Floer homology and coisotropic intersections*, Int. Math. Res. Notices, (2013), no. 10, 2271–2322.

[Ka4] J. Kang, *Kuňneth formula in Rabinowitz Floer homology,* Cal. Var. Partial Differential Equations, (2013) **47**, No. 1, 763–787.

[Ke] E. Kerman, *Displacement energy of coisotropic submanifolds and Hofer's geometry,* J. Modern dynamics, (2008) **2** no. 3, 471–497.

[McS] D. McDuff, D. Salamon, "J-holomorphic curves and symplectic topology", AMS Colloquium Publication, **52**, 2004.

[MS] C.C. Moore, C. Schochet, "Global analysis on foliated spaces", Mathematical Sciences Research Institute Publications, **9**, 1988.

[MP] W. Merry, G. Paternain, *Index computations in Rabinowitz Floer homology*, J. Fixed Point Theory Appl. **10** (2011), No. 1, 88–111.

[MMP] L. Macarini, W. Merry, G. Paternain, *On the growth rate of leaf-wise intersections*, J. Symplectic Geom. **12** (2012), No. 4, 601–653.

[Mc] D. McDuff, *Symplectic manifolds with contact type boundaries*, Invent. Math. **103** (1991), no.3, 651–671.

[Me1] W. Merry, *On the Rabinowitz Floer homology of twisted cotangent bundles*, Calc. Var. Partial Differential Equations, **42** (2011), 355–404.

[Me2] W. Merry, *Lagrangian Rabinowitz Floer homology and twisted cotangent bundles*, Geom. Ded., (2013) DOI 10.1007/s10711-013-9903-9.

[Mo] J. Moser, *A fixed point theorem in symplectic geometry,* Acta Math. **141** (1978), no. 1-2, 17–34.

[Oa] A. Oancea, The Künneth formula in Floer homology for manifolds with restricted contact type boundary, Math. Ann.**334** (2006), 51–91.

BIBLIOGRAPHY

[Po] L. Polterovich, "The geometry of the group of symplectic diffeomorphisms", Lectures in Mathematics–Birkhäuser, (2001).

[Ra] P.H. Rabinowitz, *Periodic solutions of Hamiltonian systems,* Comm. Pure. Appl. Math **31** (1978). 157–184.

[RS] J.W. Robbin, D.A. Salamon *The spectral flow and the Maslov index*, Bulletin of the LMS **27** (1995), 1–33.

[Sa] D.A. Salamon, *Lectures on Floer homology,* in "Symplectic Geometry and Topology", Eds: Y. Eliashberg and L. Traynor, IAS/Park City Mathematics series **7** (1999), 143–230.

[Sc] M. Schwarz, "Morse homology", Birkhäuser, (1993)

[SV] D. Sullivan, M. Vigué-Poirrier, *The homology theory of the closed geodesic problem*, J. Differential Geometry **11** (1976), no. 4, 633–644.

[Us] M. Usher, *Boundary depth in Floer theory and its applications to Hamiltonian dynamics and coisotropic submanifolds,* Israel Journal of Mathematics **184** (2011), no. 1, 1–57.

[Zi1] F. Ziltener, *Coisotropic Submanifolds, leafwise fixed points, and presymplectic embeddings,* Journal of Symplectic Geom. **8** (2010), no. 1, 1–24.

[Zi2] F. Ziltener, *A Maslov map for coisotropic submanifolds, leaf-wise fixed points and presymplectic non-embeddings,* (2009), arXiv:0911.1460.

국문초록

Urs Frauenfelder와 Kai Cieliebak은 Paul Rabinowitz가 자율적 해밀턴 시스템에서 주기궤도들 찾기 위해 제안한 라그랑즈 승수 함수를 사용하여 Rabinowitz Floer homology 이론을 개발하였다.

이 논문에서는 우리는 임의의 여차원을 가지는 여등방성 부분다양체 위의 역학구조를 분석하는데 적합한 여러개의 Lagrange 상수들을 가지는 일반화된 Rabinowitz 함수를 연구할 것이다. 우리는 일반화된 Rabinowitz 함수를 사용하여 여등방성 궤적 교차점, 여등방성 부분 다양체의 전치가능성, 그리고 여등방성 부분다양체의 Rabinowitz Floer homology 등에 관해 연구할 것이다. 우리는 또한 Rabinowitz Floer homology의 Künneth 공식을 유도하여 무한개의 여등방 궤적 교차점을 가지는 여등방성 부분다양체들을 찾을 것이다. 이 연구는 여러 개의 운동 상수 (보존량) 를 가지는 운동 시스템을 연구하는데 중요한 역할을 할 것이다.

주요어휘: 라비노위츠 플로어 호몰로지, 해밀턴 역학, 보존량, 여등방성 부분다양체, 여등방 궤적교차점.
학번: 2008-20276

감사의 글

5 년이란 긴 시간동안 저를 아낌없이 지도해주신 Urs Frauenfelder 교수님께 감사드립니다. 어떻게 수학이란 학문을 공부하고 연구하여야하는지 배울 수 있었던 소중한 시간이었습니다. 저를 사교기하의 길로 인도해주시고 아낌없는 가르침과 조언을 주시는 조철현 교수님께 진심으로 감사의 말씀을 올립니다. Otto van Koert 교수님과의 토론 역시 폭 넓은 지식의 밑거름이 되었음에 감사드립니다. 1 년동안 Münster 대학에서 연구할 수 있도록 해주시며 많은 도움을 주신 Peter Albers 교수님께도 다시 한번 감사의 인사를 올립니다. 이 학위논문 심사를 비롯하여 많은 도움과 가르침을 주신 박종일 교수님과 이재혁 교수님께도 감사드립니다.

그리고 같이 공부하고 토론하며 즐거운 대학원 생활을 보내도록 해준 저의 모든 선후배, 친구들에게도 고맙습니다. 앞으로도 같은 길을 걸어가며 서로에게 든든한 동료가 되기를 바랍니다.

마지막으로 저의 든든한 후원자인 가족들에게 고마움과 사랑을 전합니다. 항상 제 옆을 지켜주는 혜선에게는 특별한 ♡ 을 보냅니다.