이학박사 학위논문

Some problems arising from the dynamics of the Kuramoto oscillators

(쿠라모토 진동자들의 동역학에서 일어나는 문제들에 대한 고찰)

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Some problems arising from the dynamics of the Kuramoto oscillators

A dissertation
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Abstract

In this thesis, we study several problems on the ensemble of Kuramoto oscillators. We present the nonlinear stability of the phase-locked states using a robust ℓ_1-metric as a Lyapunov functional. We show that the phase-locked states are congruent each other in the sense that one phase-locked state is the simply translation of the other and phase-shift is the difference of averaged initial phases. We also show the contration property for measure valued solutions of the kinetic Kuramoto model. We next consider the effect of interaction frustration on the complete synchronization of Kuramoto oscillators. In general, interaction frustration hinders the formation of complete frequency synchronization. For more quantitative estimates, we consider three Kuramoto-type models. Our first model is for an ensemble of Kuramoto oscillators with uniform interaction frustration. Our second model is, as a special case of the first model, a mixture of two identical Kuramoto oscillator groups with distinct natural frequencies. Our third model is like the Kuramoto model for identical oscillators on the bipartite graph. Finally, we investigate the intricate interplay between the inertia and frustration in an ensemble of Kuramoto oscillators. We cannot apply the explicit macro-micro decomposition to reduce the dynamics of intial phases to that of fluctuations. However, we can still derive second-order differential inequalities for the phase of frequency diameters so that the second-order Gronwall inequality method still works well. Moreover, both the analytical and numerical studies demonstrate this fact.

Key words: Kuramoto model, orbital stability, contraction, frustration, sychronization
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Contents

Chapter 1

Introduction

Synchronization is ubiquitous in various disciplines such as physics, biology, chemistry, and the social sciences [59] and recent applications on power system [28, 29]. However, rigorous mathematical treatments of synchronization were initiated only a few decades ago by Winfree [73] and Kuramoto [44, 45], who introduced simple ODE models for limit-cycle oscillators.

Our interest in this thesis lies in the Kuramoto model which is a prototype for synchronization. Kuramoto oscillators can be regarded as point particles rotating on the unit circle. Let $\theta_i = \theta_i(t)$ be the phase of the i-th Kuramoto oscillator. Then their phase are governed by the following first-order ODE system:

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \quad t > 0, \quad i = 1, \dots, N. \tag{1.0.1}$$

where Ω_i, K and N denote a natural frequency of the i-th oscillator, the positive coupling strength, and the number of oscillators, respectively. Each natural frequency is a random variable extracted from some given density function.

For the details of the system (1.0.1), we refer the reader to survey papers and books [1, 4, 44, 62]. Particularly, Ermentrout [30] found a critical coupling at which all oscillators become phase-locked, independent of the number of oscillators. The linear stability of phase-locked state has been studied using tools such as a Lyapunov functional, spectral graph theory, and control theory [3, 7, 24, 43, 52, 53, 54, 68]. Moreover, some complete frequency synchronization estimates have been provided for the Kuramoto model (1.0.1) in [17, 21, 25, 26, 32].

The problems we will consider are:

1. When two initial configurations of Kuramoto oscillators have the same averaged phases, what happens to the ℓ_1-distance between them? If the configurations with different phase averages? How about the kinetic Kuramoto equation? Can we have a similar result?

2. If there is frustration as a phase shift in the Kuramoto model, what is the effect of frustration to the synchronization?

3. If there is the interplay between inertial effect and interaction frustration in an ensemble of Kuramoto oscillators, what happens to the synchronization?

In the first topic, we will deal with two versions of Kuramoto model: *particle* and *kinetic*. First, we will show that the phase-locked states whose existence is guaranteed in [17] are orbitally stable in the ℓ_1-metric. For a fixed K and a given distribution of natural frequencies, the phase-locked states issued from different initial phase configurations have exactly the same structure, which means that one is simply the translation of the other and the phase shift is exactly equal to the difference of averaged phases. Second, we will present a contraction property of the kinetic Kuramoto equation(KKE) in the Wasserstein p-distance for measure valued solutions with the same natural frequency distribution by using a strategy similar to the one described in [12, 48]. We define a cumulative distribution function of a density function f for the KKE, and we derive a new integro-differential equation using its pseudo-inverse function. Then, we use simple techniques for the optimal mass transport in one-dimension, i.e., the equivalence relation between the Wasserstein p-distance and the L^p-distance of the corresponding pseudo-inverse of F in order to obtain the exponential decay estimate of the Wasserstein p-distance between two measure-valued solutions.

In the second topic, we consider three Kuramoto-type models with finite population of oscillators under frustration. First, we study the general case of nonidentical Kuramoto oscillators with natural frequencies that are distributed. Second, as a special case, we deal with an ensemble consisting of two groups of identical oscillators with only two different natural frequency. When two identical Kuramoto oscillator groups are mixed, the whole configuration evolves into the segregated state and then asymptotically toward the phase-locked state. Finally, we consider a special Kuramoto-type model that was recently derived from the Van der Pol equations for two coupled oscillator systems in the work of Lück and Pikovsky [49] where a thermodynamic limit based on the order parameter was studied. The main contribution of this work is to present some explicit sufficient conditions on the parameters and initial configurations to reach the complete synchronization for each of the three models.

In the third topic, we present several analytical conditions leading to complete synchronization after studying the dynamics of phase and frequency diameters through the second-oder Grownwall-type differential inequalities. For identical Kuramoto oscillators, we derive two frameworks depending on the relative amounts of inertia leading to asymptotic complete synchronization. In fact, we show that complete synchronization can be attained exponentially fast in every case for a restricted class of initial configurations. Moreover, we provide a nearly optimal decay exponent for the small-inertia regime, where both the size of the inertia region and the decay exponent depend on the strength of the frustration. For nonidentical oscillators, we present two sufficient conditions with small inertia and large inertia, respectively.

Let us give a brief outline of this thesis. In Chapter 2, we will introduce the Kuramoto model and review relevant results for the Kuramoto model. We will develop the results from these in the following chapters. In Chapter 3, we will discuss the nonlinear orbital stability of phase-locked states arising from the ensemble of non-identical Kuramoto oscillators. Moreover, we will present contractivity estimates for the kinetic Kuramoto model obtained from the Kuramoto phase model in the mean-field limit. In Chapter 4, we will consider three Kuramoto-type models with frustration. First, we will study a general case with nonidentical oscillators. Second, as a special case, we will study an ensemble of two groups of identical oscillators. Third, we will consider a Kuramoto-type model that was recently derived from the

Van der Pol equations for two coupled oscillator systems in the work of Lück and Pikovsky [49]. In Chapter 5, we will discuss how asymptotic synchronization can arise from the competition between synchronization factors such as strong coupling strength and desynchronization factors such as inertia and frustration. In Chapter 6, we will give the conclusion of this thesis and discuss the future works.

All the contents in this thesis stem from a series of papers published(accepted or submitted) during the PhD studies.

Chapter 2

Preliminaries

In this chapter, we briefly review the Kuramoto model and the kinetic Kuramoto equation.

2.1 The Kuramoto model

In this section, we study the relevant results about the Kuramoto model.

The Kuramoto model (1.0.1) and its variants for synchronization have been heuristically derived from the complex Ginzburg-Landau equation [45] and the coupled system of Josephson junction arrays as averaged ones [64, 71]. Kuramoto introduced a simplest nontrivial model for a temporally organized system by a self-sustained oscillator z governed by

$$\dot{z} = (i\Omega + \alpha)z - \beta|z|^2 z, \quad z \in \mathbb{C}.$$

He considered a population of such oscillators $z_1, z_2, \ldots, z_N$ with various frequencies and interactions between every pair:

$$\dot{z}_s = (i\Omega_s + \alpha)z_s - \beta|z_s|^2 z_s + \sum_{r \neq s} K_{rs} z_r, \quad r, s = 1, 2, \ldots, N.$$

He assumed that

- $K_{rs} = K/N$ independently of r and s,
- $\alpha/\beta,\ \Omega_s,\ K$ is finite as $\alpha, \beta \to \infty$.

We put $z_s = \rho_s e^{i\theta_s}$, then we can have $\rho_s = \sqrt{\alpha/\beta}$ due to the second assumption. Hence we only consider the equation

$$\dot{\theta}_s = \Omega_s + \frac{K}{N}\sum_{r\neq s}\sin(\theta_r - \theta_s).$$

It is easy to see from this equation that the average phase rotates on the unit circle with a constant average natural frequency Ω_c:

$$\frac{d\theta_c}{dt} = \Omega_c, \quad \text{i.e.,} \quad \theta_c(t) = \theta_c(0) + t\Omega_c, \quad t \geq 0,$$

where

$$\theta_c := \frac{1}{N}\sum_{i=1}^{N}\theta_i, \quad \omega_c := \frac{1}{N}\sum_{i=1}^{N}\omega_i, \quad \Omega_c := \frac{1}{N}\sum_{i=1}^{N}\Omega_i,$$

and $\omega_i := \dot{\theta}_i$ is the instantaneous frequency of the i-th oscillator.

On the other hand, the fluctuations $(\hat{\theta}_i, \hat{\Omega}_i) := (\theta_i - \theta_c, \Omega_i - \Omega_c)$ satisfy equations of the same form:

$$\dot{\hat{\theta}}_i = \hat{\omega}_i = \hat{\Omega}_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\hat{\theta}_j - \hat{\theta}_i),$$

with the additional algebraic constraints

$$\sum_{i=1}^{N}\hat{\theta}_i = 0, \qquad \sum_{i=1}^{N}\hat{\Omega}_i = 0. \tag{2.1.1}$$

The above conservation laws (2.1.1) for the Kuramoto model without frustration are crucially used in its rigorous study. We next recall the definitions of a few synchronization concepts for Kuramoto-type oscillator models, as these will be used throughout this thesis.

Definition 2.1.1. *Let $\theta(t) = (\theta_1(t), \ldots, \theta_N(t))$ be the ensemble phase of Kuramoto oscillators.*

1. *The Kuramoto ensemble asymptotically exhibits complete phase synchronization if and only if the relative phase differences go to zero asymptotically:*

$$\lim_{t\to\infty} |\theta_i(t) - \theta_j(t)| = 0, \qquad \forall\, i \neq j.$$

2. *The Kuramoto ensemble asymptotically exhibits complete frequency synchronization if and only if the relative frequency differences go to zero asymptotically:*

$$\lim_{t\to\infty} |\omega_i(t) - \omega_j(t)| = 0, \qquad \forall\, i \neq j.$$

3. *The dynamical state $\theta(t) = (\theta_1(t), \dots, \theta_N(t))$ asymptotically approaches the phase-locked state if and only if each relative phase difference goes to a constant as $t \to \infty$; i.e.,*

$$\lim_{t\to\infty} |\theta_i(t) - \theta_j(t)| = \theta_{ij}, \quad \textit{for all} \quad i, j \in \{1, \dots, N\}.$$

We next introduce several notations to be used throughout the thesis. For a given time $t \geq 0$,

$$\begin{aligned}
\theta_m(t) &:= \min_{1\le i\le N} \theta_i(t), \qquad \theta_M(t) := \max_{1\le i\le N} \theta_i(t),\\
\omega_m(t) &:= \min_{1\le i\le N} \omega_i(t), \qquad \omega_M(t) := \max_{1\le i\le N} \omega_i(t),\\
D(\theta(t)) &:= \theta_M(t) - \theta_m(t), \quad D(\omega(t)) := \omega_M(t) - \omega_m(t),\\
D(\Omega) &: = \max_{1\le i,j\le N} |\Omega_i - \Omega_j|, \quad K_e := \frac{D(\Omega)}{\sin D(\theta^0)}, \quad K_{ef} := \frac{D(\Omega)}{1-\sin|\alpha|},
\end{aligned}$$

where $\theta^0 := \theta(0)$. Note that $D(\theta(t))$ is Lipschitz continuous and differentiable except at times of collision between the extremal phases and their neighboring phases.

We recall the following results which is related to this thesis without proof.

Theorem 2.1.1. [17] *Let $\theta = \theta(t)$ be the global smooth the solution to the system* (1.0.1) *satisfying*

$$0 < D(\theta^0) < \pi, \quad D(\Omega) > 0, \quad K > K_e. \tag{2.1.2}$$

Then there exists $t_0 > 0$ such that

$$D(\omega(t_0))e^{-K(t-t_0)} \le D(\omega(t)) \le D(\omega(t_0))e^{-K(\cos D^\infty)(t-t_0)}, \quad t \ge t_0,$$

where D^∞ is the dual angle of initial phase diameter $D(\theta^0)$, i.e.,

$$D^\infty \in \left(0, \frac{\pi}{2}\right), \quad \sin D^\infty = \sin D(\theta^0).$$

Proposition 2.1.1. [17] *Let* $\theta = \theta(t)$ *be the global smooth solution to the system* (1.0.1) *satisfying*

$$0 < D(\theta^0) < \pi, \quad D(\Omega) > 0, \quad K > K_e.$$

Then there exists $t_0 > 0$ *such that*

$$D(\theta(t)) \leq D^\infty \quad \textit{for} \quad t \geq t_0.$$

2.2 The kinetic Kuramoto equation

In this section, we briefly review the kinetic mean-field model for the Kuramoto model. The kinetic Kuramoto equation has been widely used in the literature [1] to analyze the phase transition from a completely disordered state to a partially ordered state as the coupling strength increases from zero. Suppose that $g = g(\Omega)$ is an integrable steady probability density function for natural frequencies with a compact support (see (2.2.5) for details). Let $f = f(\theta, \Omega, t)$ be the probability density function of Kuramoto oscillators in $\theta \in \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$ with a natural frequency Ω at time t as in [46]. The kinetic Kuramoto equation(KKE) is given as follows:

$$\begin{aligned} &\partial_t f + \partial_\theta(\omega[f]f) = 0, \qquad (\theta, \Omega) \in \mathbb{T} \times \mathbb{R},\ t > 0, \\ &\omega[f](\theta, \Omega, t) = \Omega - K \int_{\mathbb{T}} \sin(\theta - \theta_*)\rho(\theta_*, t) d\theta_*, \quad \rho(\theta_*, t) := \int_{\mathbb{R}} f d\Omega_*, \end{aligned} \tag{2.2.3}$$

subject to the initial data:

$$f(\theta, \Omega, 0) = f_0(\theta, \Omega), \quad \int_{\mathbb{T}} f_0 d\theta = g(\Omega). \tag{2.2.4}$$

The natural phase-velocity (frequency) Ω is assumed to be a random variable extracted from the density function $g = g(\Omega)$:

$$\begin{aligned} &g(-\Omega) = g(\Omega), \quad \text{supp}(g) \text{ is bounded}, \\ &\int_{\mathbb{R}} \Omega g(\Omega) d\Omega = 0, \quad \int_{\mathbb{R}} g(\Omega) d\Omega = 1. \end{aligned} \tag{2.2.5}$$

We next study a measure-theoretic formulation of the KKE. Let $\mathcal{M}([0,2\pi)\times\mathbb{R})$ be the set of nonnegative Radon measures on $[0,2\pi)\times\mathbb{R}$, which can be regarded as nonnegative bounded linear functionals on $\mathcal{C}([0,2\pi)\times\mathbb{R})$. For a Radon measure $\nu\in\mathcal{M}([0,2\pi)\times\mathbb{R})$, we use the standard duality relation:

$$\langle \nu, h\rangle := \int_0^{2\pi}\int_{\mathbb{R}} h(\theta,\Omega)\nu(d\theta,d\Omega), \quad h\in\mathcal{C}_0([0,2\pi)\times\mathbb{R}),$$

where $\mathcal{C}_0^k$ denotes the set of functions with k continuous derivatives and vanishing at infinity. Note that since $\theta\in[0,2\pi)$ is a 2π-periodic variable, $h(\theta,\Omega)$ is a 2π-periodic function with respect to θ on $[0,2\pi)\times\mathbb{R}$. The definition of a measure-valued solution to equation (2.2.3) is given as follows. From now on, $\mathcal{C}_w([0,T);\mathcal{M}([0,2\pi)\times\mathbb{R}))$ denotes a space of all weakly continuous time-dependent measures.

Definition 2.2.1. *For $T\in[0,\infty)$, let $\mu\in\mathcal{C}_w([0,T);\mathcal{M}([0,2\pi)\times\mathbb{R}))$ be a measure valued solution to (2.2.3) with an initial Radon measure $\mu_0\in\mathcal{M}([0,2\pi)\times\mathbb{R})$ if and only if μ satisfies the following conditions:*

1. *μ is weakly continuous:*

$$\langle \mu_t, h\rangle \quad \text{is continuous as a function of } t, \ \forall\ h\in\mathcal{C}_0([0,2\pi)\times\mathbb{R}).$$

2. *μ satisfies the integral equation: $\forall\ h\in\mathcal{C}_0^1([0,2\pi)\times\mathbb{R}\times[0,T))$,*

$$\langle \mu_t, h(\cdot,\cdot,t)\rangle - \langle \mu_0, h(\cdot,\cdot,0)\rangle = \int_0^t \langle \mu_s, \partial_s h + \omega[\mu]\partial_\theta h\rangle ds, \qquad (2.2.6)$$

where $\omega[\mu](\theta,\Omega,s)$ is defined by

$$\omega[\mu](\theta,\Omega,s) := \Omega - K(\mu_s * \sin)\theta\,. \qquad (2.2.7)$$

Here $*$ denotes the standard convolution, i.e.,

$$(\mu_s * \sin)\theta = \int_0^{2\pi}\int_{\mathbb{R}} \sin(\theta-\theta_*)\mu_s(d\theta_*,d\Omega).$$

Lemma 2.2.1. [10] *Suppose that the density function $g = g(\Omega)$ has a compact support and the initial measure satisfies*

$$\langle \mu_0, \Omega \rangle = 0,$$

and let $\mu \in \mathcal{C}_w([0,T); \mathcal{M}([0,2\pi) \times \mathbb{R}))$ be a measure valued solution to (2.2.3)*. Then for $t \geq 0$, we have*

$$\langle \mu_t, 1 \rangle = \langle \mu_0, 1 \rangle = 1, \qquad \langle \mu_t, \theta \rangle = \langle \mu_0, \theta \rangle, \qquad t \geq 0.$$

Theorem 2.2.1. [10] *For any $\mu_0 \in \mathcal{M}([0,2\pi) \times \mathbb{R})$, let μ_t be a unique measure valued solution to KKE* (2.2.3) *with initial data μ_0. Then μ_t can be approximated as a sum of Dirac measures of the form:*

$$\mu_t^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\theta_i(t)} \otimes \delta_{\Omega_i(t)}.$$

Furthermore, there holds

$$d(\mu_t, \mu_t^N) \to 0, \quad \text{as } N \to \infty.$$

We now introduce some notations to be used throughout Section 3.2. Let $\mu \in \mathcal{C}([0,T); \mathcal{M}([0,2\pi) \times \mathbb{R}))$ be a measure valued solution to (2.2.3), and let $R(t)$ and $P(t)$ be the orthogonal θ and Ω-projections of $\mathrm{supp}(\mu_t)$ respectively, i.e.,

$$\begin{aligned} R(t) &:= \mathbb{P}_\theta \mathrm{supp}(\mu_t) = \{\theta \in [0,2\pi) : (\theta, \Omega) \in \mathrm{supp}(\mu_t)\}, \\ P(t) &:= \mathbb{P}_\Omega \mathrm{supp}(\mu_t) = \{\Omega \in \mathbb{R} : (\theta, \Omega) \in \mathrm{supp}(\mu_t)\}. \end{aligned}$$

Then it is easy to see that

$$P(t) = P(0), \qquad t \geq 0.$$

We also set

$$\begin{aligned} D_\theta(\mu_t) &:= \mathrm{diam}(R(t)), \quad D_\Omega(\mu_t) := \mathrm{diam}(P(t)), \quad M(t) := \langle \mu_t, 1 \rangle, \\ \theta_c(t) &:= \frac{1}{M(t)} \langle \mu_t, \theta \rangle, \quad \Omega_c(t) := \frac{1}{M(t)} \langle \mu_t, \Omega \rangle, \end{aligned}$$

where $\mathrm{diam}(A) := \sup_{x,y\in A}|x-y|$ for $A \subset \mathbb{R}$. We observe from Lemma 2.2.1 that

$$M(t) = \langle \mu_t, 1\rangle = \langle \mu_0, 1\rangle = M(0) = 1,$$

and since $\Omega_c(0) = 0$, we obtain

$$\Omega_c(t) = 0 \quad \text{and} \quad \theta_c(t) = \theta_c(0), \quad t \geq 0.$$

Remark 2.2.1. *Throughout the thesis, without loss of generality, we assume that $\langle \mu_0, \theta\rangle = \pi$ in order to avoid any possible confusion arising from the periodicity of θ. In fact, if the oscillators satisfy the assumption in Lemma 3.2.1, the orthogonal θ-projection of supp(μ_t), $R(t)$ is confined to the interval $(0, 2\pi)$ for all $t \geq 0$. This property will also be significantly used in Section 3.2 (see Lemma 3.2.2).*

Chapter 3

Nonlinear stability

In this chapter, we will discuss the nonlinear orbital stability of phase-locked states arising from the ensemble of non-identical Kuramoto oscillators. Moreover, we will present contractivity estimates for the kinetic Kuramoto model obtained from the Kuramoto phase model in the mean-field limit. This chapter is based on joint works in [10] and [17].

3.1 Orbital stability of phase-locked states

In this section, we study the nonlinear stability of the phase-locked states whose existence is guaranteed by Theorem 2.1.1.

For any two smooth configurations $\theta = (\theta_1, \cdots, \theta_N), \tilde{\theta} = (\tilde{\theta}_1, \cdots, \tilde{\theta}_N) \in \mathbb{T}^N$, we define ℓ_1-distance by

$$\|\theta - \tilde{\theta}\|_1 := \sum_{i=1}^{N} |\theta_i - \tilde{\theta}_i|.$$

We first state our main theorem of this section. In this theorem, we will present ℓ_1-contraction estimate.

Theorem 3.1.1. *Let θ and $\tilde{\theta}$ be the global smooth solutions to the system* (1.0.1) *with initial data θ^0 and $\tilde{\theta}^0$, respectively satisfying*

$$0 < D(\tilde{\theta}^0) \leq \ D(\theta^0) < \pi \quad \textit{and} \quad K > \max\left\{ \frac{D(\Omega)}{\sin D(\theta^0)}, \frac{D(\Omega)}{\sin D(\tilde{\theta}^0)} \right\}.$$

Then we have the following estimates:

1. *If* $\theta_c(0) \neq \tilde{\theta}_c(0)$, *then there exists* $t_0 > 0$ *such that*

$$\|(\theta - \tilde{\theta})(t)\|_1 + \int_{t_0}^{t} \Lambda_1(s) ds \leq \|(\theta - \tilde{\theta})(t_0)\|_1 \quad \text{for} \quad t \geq t_0,$$

where the nonnegative functional $\Lambda_1(s)$ *is defined by*

$$\begin{aligned} \Lambda_1(s) \quad &:= \quad \frac{K \sin D^\infty}{N D^\infty} \cos D^\infty \Big[\Big(|I^0(s)| + 2|I^-(s)| \Big) \sum_{i \in I^+(s)} |(\theta_i - \tilde{\theta}_i)(s)| \\ &+ \quad \Big(|I^0(s)| + 2|I^+(s)| \Big) \sum_{i \in I^-(s)} |(\theta_i - \tilde{\theta}_i)(s)| \Big]. \end{aligned}$$

2. *If* $\theta_c(0) = \tilde{\theta}_c(0)$, *then we have*

$$||(\theta - \tilde{\theta})(t_0)||_1 e^{-K(t-t_0)} \leq ||(\theta - \tilde{\theta})(t)||_1 \leq ||(\theta - \tilde{\theta})(t_0)||_1 e^{-\frac{K \sin 2D^\infty}{2D^\infty}(t-t_0)},$$

for $t \geq t_0$. *Hence the decay rate lies in the interval* $[K \frac{\sin 2D^\infty}{2D^\infty}, K]$ *in a large-time regime.*

We need some elementary estimates in the following lemma to prove our theorem. Recall the sign function defined by

$$\mathrm{sgn}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

For a given $\alpha = (\alpha_1, \cdots, \alpha_N) \in \mathbb{T}^N$, we decompose the index set $I := \{1, \cdots, N\}$ as the disjoint union of three subsets:

$$I^+ := \{i \mid \alpha_i > 0\}, \quad I^0 := \{i \mid \alpha_i = 0\}, \quad I^- := \{i \mid \alpha_i < 0\}.$$

We set

$$\Delta_{ij} := \big(\mathrm{sgn}(\alpha_i) - \mathrm{sgn}(\alpha_j)\big) \sin\Big(\frac{\alpha_j - \alpha_i}{2}\Big).$$

Lemma 3.1.1. *Let* $\alpha \in \mathbb{T}^N$ *satisfy*

$$\max_{1\le i,j\le N} |\alpha_j - \alpha_i| \le 2d < \pi \quad \textit{for some} \quad d > 0.$$

Then we have

$$\sum_{i,j=1}^{N} \Delta_{ij} \le \begin{cases} -\frac{\sin d}{d}\Big[\Big(|I^0| + 2|I^-|\Big)\sum_{i\in I^+}|\alpha_i| + \Big(|I^0| + 2|I^+|\Big)\sum_{i\in I^-}|\alpha_i|\Big], & \sum_{i=1}^{N}\alpha_i \ne 0, \\ -\frac{N\sin d}{d}\sum_{i=1}^{N}|\alpha_i|, & \sum_{i=1}^{N}\alpha_i = 0, \end{cases}$$

and

$$\sum_{i,j=1}^{N} \Delta_{ij} \ge \begin{cases} -\Big[\Big(|I^0| + 2|I^-|\Big)\sum_{i\in I^+}|\alpha_i| + \Big(|I^0| + 2|I^+|\Big)\sum_{i\in I^-}|\alpha_i|\Big], & \sum_{i=1}^{N}\alpha_i \ne 0, \\ -N\sum_{i=1}^{N}|\alpha_i|, & \sum_{i=1}^{N}\alpha_i = 0. \end{cases}$$

Proof. (i) First, we consider the case of $\sum_{i=1}^{N}\alpha_i \ne 0$. Note that when α_i and α_j have the same sign, we get

$$\Delta_{ij} = 0.$$

The nontrivial cases are summarized in the following table.

Cases	sgn(α_i)	sgn(α_j)	Δ_{ij}
I	1	0	$-\sin\frac{\lvert\alpha_i\rvert}{2}$
II	-1	0	$-\sin\frac{\lvert\alpha_i\rvert}{2}$
III	0	1	$-\sin\frac{\lvert\alpha_j\rvert}{2}$
IV	0	-1	$-\sin\frac{\lvert\alpha_j\rvert}{2}$
V	1	-1	$-2\sin\left(\frac{\lvert\alpha_i\rvert+\lvert\alpha_j\rvert}{2}\right)$
VI	-1	1	$-2\sin\left(\frac{\lvert\alpha_i\rvert+\lvert\alpha_j\rvert}{2}\right)$

- Case 1: Note that the above table and the simple inequality

$$\sin x \geq \frac{\sin d}{d} x, \ \ x \in [0, d]$$

imply

$$\begin{aligned}
\sum_{(i,j)\in I^+\times I^0} \Delta_{ij} &= -\sum_{(i,j)\in I^+\times I^0} \sin\frac{|\alpha_i|}{2} \\
&\leq -\frac{\sin d}{2d} \sum_{(i,j)\in I^+\times I^0} |\alpha_i| = -\frac{\sin d}{2d} |I^0| \sum_{i\in I^+} |\alpha_i|.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\sum_{(i,j)\in I^0\times I^+} \Delta_{ij} &\leq -\frac{\sin d}{2d} |I^0| \sum_{i\in I^+} |\alpha_i|, \\
\sum_{(i,j)\in I^0\times I^-} \Delta_{ij} &\leq -\frac{\sin d}{2d} |I^0| \sum_{i\in I^-} |\alpha_i|, \\
\sum_{(i,j)\in I^-\times I^0} \Delta_{ij} &\leq -\frac{\sin d}{2d} |I^0| \sum_{i\in I^-} |\alpha_i|.
\end{aligned}$$

- Case 2: Similar to Case 1, we have

$$\begin{aligned}
\sum_{(i,j)\in I^+\times I^-} \Delta_{ij} &\leq -\frac{\sin d}{d} \Big(|I^-| \sum_{i\in I^+} |\alpha_i| + |I^+| \sum_{i\in I^-} |\alpha_i| \Big), \\
\sum_{(i,j)\in I^-\times I^+} \Delta_{ij} &\leq -\frac{\sin d}{d} \Big(|I^+| \sum_{i\in I^-} |\alpha_i| + |I^-| \sum_{i\in I^+} |\alpha_i| \Big).
\end{aligned}$$

Finally, we combine Case 1 and Case 2 to get

$$\sum_{i,j=1}^{N} \Delta_{ij} \leq -\frac{\sin d}{d} \Big[\Big(|I^0| + 2|I^-| \Big) \sum_{i\in I^+} |\alpha_i| + \Big(|I^0| + 2|I^+| \Big) \sum_{i\in I^-} |\alpha_i| \Big].$$

(ii) Next, suppose that $\sum_{i=1}^{N} \alpha_i = 0$. Then we have

$$\sum_{i\in I^-} |\alpha_i| = -\sum_{i\in I^-} \alpha_i = \sum_{i\in I^+} \alpha_i = \sum_{i\in I^+} |\alpha_i|,$$

which imply

$$\sum_{i\in I^-}|\alpha_i| = \sum_{i\in I^+}|\alpha_i| = \frac{1}{2}\sum_{i=1}^{N}|\alpha_i|.$$

Hence we can further simplify

$$\begin{aligned}\sum_{i,j=1}^{N}\Delta_{ij} &\le -\frac{2\sin d}{d}\Big(|I^-|+|I^0|+|I^+|\Big)\sum_{i\in I^+}|\alpha_i| \\ &= -\frac{2N\sin d}{d}\sum_{i\in I^+}|\alpha_i| = -\frac{N\sin d}{d}\sum_{i=1}^{N}|\alpha_i|.\end{aligned}$$

Finally, by using the fact $\sin x \le x$ for $x \ge 0$ and the same argument as the above, we have the lower bound of $\sum_{i,j=1}^{N}\Delta_{ij}$. This completes the proof. $\square$

The proof of Theorem 3.1.1: We use the system (1.0.1) and elementary identity for trigonometric functions to find

$$\begin{aligned}&\frac{d}{dt}(\theta_i - \tilde{\theta}_i) \\ &\quad= \frac{K}{N}\sum_{j=1}^{N}\big(\sin(\theta_j-\theta_i) - \sin(\tilde{\theta}_j - \tilde{\theta}_i)\big) \\ &\quad= \frac{2K}{N}\sum_{j=1}^{N}\cos\left(\frac{\theta_j-\theta_i}{2} + \frac{\tilde{\theta}_j-\tilde{\theta}_i}{2}\right)\sin\left(\frac{\theta_j-\tilde{\theta}_j}{2} - \frac{\theta_i-\tilde{\theta}_i}{2}\right).\end{aligned} \tag{3.1.1}$$

We next multiply (3.1.1) by $\mathrm{sgn}(\theta_i - \tilde{\theta}_i)$ and sum it over i to get

$$\begin{aligned}
&\frac{d}{dt}\sum_{i=1}^{N}|\theta_i - \tilde{\theta}_i| \\
&\quad = \frac{2K}{N}\sum_{i,j=1}^{N}\mathrm{sgn}(\theta_i - \tilde{\theta}_i)\cos\left(\frac{\theta_j - \theta_i}{2} + \frac{\tilde{\theta}_j - \tilde{\theta}_i}{2}\right) \\
&\quad \times \sin\left(\frac{\theta_j - \tilde{\theta}_j}{2} - \frac{\theta_i - \tilde{\theta}_i}{2}\right) \\
&\quad = \frac{K}{N}\sum_{i,j=1}^{N}\Big\{\mathrm{sgn}(\theta_i - \tilde{\theta}_i) - \mathrm{sgn}(\theta_j - \tilde{\theta}_j)\Big\}\cos\left(\frac{\theta_j - \theta_i}{2} + \frac{\tilde{\theta}_j - \tilde{\theta}_i}{2}\right) \\
&\quad \times \sin\left(\frac{\theta_j - \tilde{\theta}_j}{2} - \frac{\theta_i - \tilde{\theta}_i}{2}\right).
\end{aligned} \tag{3.1.2}$$

We now use the following inequality (see Lemma 3.1.1)

$$\big(\mathrm{sgn}(\alpha) - \mathrm{sgn}(\beta)\big)\sin\left(\frac{\beta - \alpha}{2}\right) \le 0.$$

It follows from Proposition 2.1.1 that there exists t_0 such that

$$D(\theta(t)) \le D^\infty, \quad D(\tilde{\theta}(t)) \le \tilde{D}^\infty, \quad t \ge t_0,$$

and eventually we have

$$D(\theta(t)), D(\tilde{\theta}(t)) \le D^\infty, \quad t \ge t_0,$$

by the assumption $K > \max\left\{\frac{D(\Omega)}{\sin D(\theta^0)}, \frac{D(\Omega)}{\sin D(\tilde{\theta}^0)}\right\}$. Then this yields

$$0 < \cos D^\infty \le \cos\left(\frac{\theta_j - \theta_i}{2} + \frac{\tilde{\theta}_j - \tilde{\theta}_i}{2}\right) \le 1.$$

(i) It follows from (3.1.2) and the above estimate that

$$\begin{aligned}
&\frac{d}{dt}\sum_{i=1}^{N}|\theta_i-\tilde{\theta}_i| \\
&\quad\le \frac{K}{N}\cos D^\infty \sum_{i,j=1}^{N}\Big\{\mathrm{sgn}(\theta_i-\tilde{\theta}_i)-\mathrm{sgn}(\theta_j-\tilde{\theta}_j)\Big\}\sin\left(\frac{\theta_j-\tilde{\theta}_j}{2}-\frac{\theta_i-\tilde{\theta}_i}{2}\right) \\
&\quad= -\frac{K\sin D^\infty}{ND^\infty}\cos D^\infty\Big[\Big(|I^0(t)|+2|I^-(t)|\Big)\sum_{i\in I^+(t)}|\theta_i-\tilde{\theta}_i| \\
&\quad+\Big(|I^0(t)|+2|I^+(t)|\Big)\sum_{i\in I^-(t)}|\theta_i-\tilde{\theta}_i|\Big] \\
&\quad=: -\Lambda_1.
\end{aligned} \tag{3.1.3}$$

Hence we have

$$\frac{d}{dt}||\theta-\tilde{\theta}||_1+\Lambda_1\le 0.$$

We integrate the above differential inequality from $s=t_0$ to $s=t$ to get the desired result.

(ii) Suppose the initial phase averages are equal, i.e.

$$\theta_c(0)=\tilde{\theta}_c(0).$$

In (3.1.3), we use the following identity:

$$\sum_{i,j=1}^{N}\big(\mathrm{sgn}(\theta_i-\tilde{\theta}_i)-\mathrm{sgn}(\theta_j-\tilde{\theta}_j)\big)\sin\left(\frac{\theta_j-\tilde{\theta}_j}{2}-\frac{\theta_i-\tilde{\theta}_i}{2}\right)\le -\frac{N\sin D^\infty}{D^\infty}\sum_{i=1}^{N}|\theta_i-\tilde{\theta}_i|$$

to find the Gronwall's inequality:

$$\begin{aligned}
&\frac{d}{dt}\sum_{i=1}^{N}|\theta_i-\tilde{\theta}_i| \\
&\quad\le \frac{K}{N}\cos D^\infty\sum_{i,j=1}^{N}\big(\mathrm{sgn}(\theta_i-\tilde{\theta}_i)-\mathrm{sgn}(\theta_j-\tilde{\theta}_j)\big)\sin\left(\frac{\theta_j-\tilde{\theta}_j}{2}-\frac{\theta_i-\tilde{\theta}_i}{2}\right) \\
&\quad\le -\frac{K\sin D^\infty\cos D^\infty}{D^\infty}\sum_{i=1}^{N}|\theta_i-\tilde{\theta}_i| \quad \text{for} \quad t\ge t_0.
\end{aligned}$$

By the same argument as the above, we also have

$$\frac{d}{dt}\sum_{i=1}^{N}|\theta_i-\tilde{\theta}_i| \geq -K\sum_{i=1}^{N}|\theta_i-\tilde{\theta}_i| \quad \text{for} \quad t \geq t_0.$$

Then the standard Gronwall's lemma yields the desired result.

Remark 3.1.1. *If we choose $\tilde{\theta}^0 = \theta^e$ to be the phase-locked state, then the result of Theorem 4.1 implies the nonlinear stability of θ^e:*

$$\|\theta(t)-\theta^e\|_1 \leq \|\theta(t_0)-\theta^e\|_1 \times \begin{cases} 1, & \theta_c(0) \neq \frac{1}{N}\sum_{i=1}^{N}\theta_i^e, \\ \exp\Big[-\frac{K\sin 2D^\infty}{2D^\infty}(t-t_0)\Big], & \theta_c(0) = \frac{1}{N}\sum_{i=1}^{N}\theta_i^e. \end{cases}$$

as long as θ^0, θ^e and K satisfy the assumptions of Theorem 3.1.1. Hence when the initial data θ^0 have the same average as the given phase-locked state θ^e, the Kuramoto flow θ converges to the given θ^e exponentially fast. Moreover we can see the asymptotic behavior of a perturbation of the phase-locked state in next corollary.

A corollary of Theorem 3.1.1 and translation invariance of the Kuramoto model (1.0.1) yields the orbital stability of the phase-locked states. Let θ^e be any reference phase-locked state guaranteed by Theorem 2.1.1 starting from some initial configurations satisfying the assumptions of Theorem 2.1.1.

Corollary 3.1.1. (Orbital stability of phase-locked states) *Suppose θ^0 and θ^e satisfy the assumptions in Theorem 3.1.1, and let θ be the global smooth solution to the system* (1.0.1) *with initial data θ^0 which is a perturbation of the phase-locked state θ^e. Then the solution $\theta = \theta(t)$ converges to the translated phase-locked state exponentially fast in ℓ_1-metric:*

$$\lim_{t\to\infty} ||\theta(t) - (\theta^e + \beta I_N)||_1 = 0,$$

where β which is the averaged mass of the perturbation and I_N are defined as follows.

$$\beta := \frac{1}{N}\Big(\sum_{i=1}^{N}\theta_i^0 - \sum_{i=1}^{N}\theta_i^e\Big), \quad I_N := (1, \cdots, 1) \in \mathbb{Z}^N.$$

Proof. The case $\beta = 0$ has already been treated in Theorem 3.1.1. Hence we consider the case $\beta \neq 0$. We will use the translation-invariance of the Kuramoto model:

> If θ is the solution to the Kuramoto model with initial data θ^0, then $\theta + \gamma I_N$ is the unique solution of the Kuramoto model corresponding to initial data $\theta^0 + \gamma I_N$ for any constant γ.

Note that assumptions in Theorem 3.1.1 are translation-invariant and the translated phase-locked solution

$$\tilde{\theta}^e := \theta^e + \beta I_N,$$

has the same averaged phase as θ. Hence we apply the second assertion in Theorem 3.1.1 to find

$$||\theta(t_0)-\tilde{\theta}^e||_1 e^{-K(t-t_0)} \le ||\theta(t)-\tilde{\theta}^e||_1 \le ||\theta(t_0)-\tilde{\theta}^e||_1 e^{-\frac{K\sin 2D^\infty}{2D^\infty}(t-t_0)} \quad \text{for} \quad t \ge t_0.$$

This yields that the phase-locked state $\lim_{t\to\infty}\theta(t)$ arising from the initial configuration θ^0 is exactly equal to $\theta^e + \beta I_N$. □

3.2 Stability estimate of the kinetic Kuramoto equation

In this section, we present the strict contractivity of measure valued solutions to the KKE by using the method of optimal mass transport [12, 48, 69]. The strict contractivity result generalizes the ℓ_1-contraction result for the KM in [17].

3.2.1 Alternative formulation of the KKE

In this subsection, we derive an alternative form of the KKE, which is more convenient for deriving estimates in terms of the Wasserstein-distance. First, we study the existence of an invariant set for the KKE.

Lemma 3.2.1. *Suppose that the initial probability measure μ_0 and the coupling strength K satisfy*

$$0 < D_\theta(\mu_0) < \pi, \quad 0 < D_\Omega(\mu_0) < \infty, \quad K > \frac{D_\Omega(\mu_0)}{\sin D_\theta(\mu_0)}.$$

Then, there exist $t_0 > 0$ and $D^\infty \in (0, \frac{\pi}{2})$ such that the measure valued solution μ to (2.2.3) *with initial datum μ_0 satisfies*

$$D_\theta(\mu_t) \le D^\infty, \quad t \ge t_0.$$

Proof. Let $N > 0$ be given. Then we have the following approximation μ_0^N for μ_0:

$$\mu_0^N = \frac{1}{N}\sum_{i=1}^N \delta_{\theta_{i0}} \otimes \delta_{\Omega_{i0}}.$$

We now solve the Cauchy problem for KM:

$$\begin{cases} \dfrac{d\theta_i}{dt} = \Omega_i + \dfrac{K}{N}\displaystyle\sum_{j=1}^N \sin(\theta_j - \theta_i), \quad t > 0, \\ \dfrac{d\Omega_i}{dt} = 0. \end{cases}$$

subject to initial data $(\theta_i(0), \Omega_i(0)) = (\theta_{i0}, \Omega_{i0})$. Theorem 2.2.1 implies that

$$d(\mu_t, \mu_t^N) \to 0 \quad \text{as} \quad N \to \infty\,,$$

and thus, $D_\theta(\mu_t^N) \to D_\theta(\mu_t)$ and $D_\Omega(\mu_t^N) \to D_\Omega(\mu_t)$ as $N \to \infty$. Hence we can take N large enough such that $D_\Omega(\mu_0^N)$ and $D_\theta(\mu_0^N)$ satisfies the conditions of Proposition 2.1.1. Thus, we find that there exist $t_0^N > 0$ and $D^{\infty,N}$ such that

$$D_\theta(\mu_t^N) \le D^{\infty,N}, \quad t \ge t_0^N, \quad \text{for } N \text{ large enough},$$

where

$$t_0^N := \frac{D_\theta(\mu_0^N) - D^{\infty,N}}{K\sin D_\theta(\mu_0^N) - D_\Omega(\mu_0^N)}, \quad D^{\infty,N} := \arcsin\Big[\frac{D_\Omega(\mu_0^N)}{K}\Big] \in \Big(0, \frac{\pi}{2}\Big).$$

We now let $N \to \infty$ to obtain the desired result. $\square$

In the remainder of this section, from Remark 2.2.1, we assume that

$$R(t) \subset \big(0, 2\pi\big) \quad \text{and} \quad t \geq 0. \tag{3.2.4}$$

Under this assumption the solution is given by a smooth particle density function $f(\theta, \Omega, t)$ in L^1 for all $t \geq 0$. For a given Ω, we consider a one-particle density function f as a function of θ. Then we define the pseudo cumulative distribution function of f:

$$F(\theta, \Omega, t) := \int_0^\theta f(\theta_*, \Omega, t) d\tilde{\theta}, \qquad (\theta, \Omega, t) \in [0, 2\pi) \times \mathbb{R} \times \mathbb{R}_+,$$

and a pseudo-inverse ϕ of $F(\cdot, \Omega, t)$ as a function of θ:

$$\phi(\eta, \Omega, t) := \inf\{\theta : F(\theta, \Omega, t) > \eta\}, \quad \eta \in [0, g(\Omega)].$$

As long as there is no confusion, we use the notation $F^{-1}(\eta, \Omega, t) = \phi$ as the pseudo inverse of F as θ-function. Then it is easy to see that

$$F(\phi(\eta, \Omega, t), \Omega, t) = \eta. \tag{3.2.5}$$

Lemma 3.2.2. *Let μ be a measure-valued solution to* (2.2.3)-(2.2.4)*, and let ϕ be the pseudo-inverse function of the cumulative distribution function F. Then we have*

$$(i) \;\; \max\{\theta \mid \theta \in R(t)\} = \max_{\Omega \in supp(g)} \phi(g(\Omega), \Omega, t).$$
$$(ii) \;\; \min\{\theta \mid \theta \in R(t)\} = \min_{\Omega \in supp(g)} \phi(0, \Omega, t).$$
$$(iii) \;\; \max_{\Omega \in supp(g)} \phi(g(\Omega), \Omega, t) - \min_{\Omega \in supp(g)} \phi(0, \Omega, t) \leq D^\infty, \quad t \geq t_0.$$

Proof. Since the estimate for (ii) is similar to that of (i) and the estimate for (iii) follows from the estimates (i) and (ii), we only provide the proof for the estimate (i). For notational simplicity, we set

$$\theta_M := \max\{\theta \mid \theta \in R(t)\}.$$

Then, by definition of μ_t, we have

$$\theta_M = \max\{\theta \mid \theta \in \mathrm{supp}_\theta(f(\theta, \Omega, t)) \text{ and } \; \Omega \in \mathrm{supp}(g)\},$$

where $\text{supp}_\theta(f(\theta,\Omega,t))$ is the θ-projection of $\text{supp}(f(\theta,\Omega,t))$. This yields

$$\theta_M = \max\{\phi(g(\Omega),\Omega,t) \text{ such that } \Omega \in \text{supp}(g)\},$$

by definition of the pseudo-inverse function. This completes the proof. $\square$

Next, we derive an integro-differential equation for the pseudo inverse ϕ. It follows from (3.2.4) that the smooth solution $f(\theta,\Omega,t)$ to (2.2.3)-(2.2.4) satisfies

$$f(0,\Omega,t) = 0, \quad \Omega \in \mathbb{R},\ t \geq 0.$$

We differentiate the relation (3.2.5) in t and use $\partial_\theta F = f$ to get

$$\partial_t F(\theta,\Omega,t)\Big|_{\theta=\phi(\eta,\Omega,t)} + f(\theta,\Omega,t)\Big|_{\theta=\phi(\eta,\Omega,t)}\partial_t\phi(\eta,\Omega,t) = 0.$$

This yields

$$\begin{aligned}
\partial_t\phi(\eta,\Omega,t) &= -\frac{1}{f(\theta,\Omega,t)}\partial_t F(\theta,\Omega,t)\Big|_{\theta=\phi(\eta,\Omega,t)} \\
&= \frac{1}{f(\theta,\Omega,t)}\Big|_{\theta=\phi(\eta,\Omega,t)} \times (\omega[f]f)(\cdot,\Omega,t)\Big|_{\theta=0}^{\theta=\phi(\eta,\Omega,t)} \\
&= \Omega + K\int_{\mathbb{R}}\int_0^{2\pi}\sin(\theta_* - \phi(\eta,\Omega,t))f(\theta_*,\Omega_*,t)d\theta_* d\Omega_* \quad \text{using (3.2.1)} \\
&= \Omega + K\int_{\mathbb{R}}\int_0^{g(\Omega_*)}\sin(\phi(\eta_*,\Omega_*,t) - \phi(\eta,\Omega,t))d\eta_* d\Omega_*,
\end{aligned}$$

where we used $\theta_* = \phi(\eta_*,\Omega_*,t)$ and relation (3.2.5) to see $f(\theta_*,\Omega_*,t)d\theta_* = d\eta_*$. Hence, the pseudo-inverse ϕ satisfies the following integro-differential equation:

$$\partial_t\phi = \Omega + K\int_{\mathbb{R}}\int_0^{g(\Omega_*)}\sin(\phi_* - \phi)d\eta_* d\Omega_*. \tag{3.2.6}$$

where we used abbreviated notations:

$$\phi_* := \phi(\eta_*,\Omega_*,t), \quad \phi := \phi(\eta,\Omega,t).$$

The following results is a simple consequence of the change of variables and Lemma 2.2.1.

Lemma 3.2.3. *Let μ_t be a measure valued solution to* (2.2.3) - (2.2.4) *with an associated pseudo-inverse function ϕ. Then, we have*

$$\int_{\mathbb{R}}\int_0^{g(\Omega)} \phi d\eta d\Omega = \int_{\mathbb{R}}\int_0^{2\pi} \theta\mu_t(d\theta, d\Omega), \quad \frac{d}{dt}\int_{\mathbb{R}}\int_0^{g(\Omega)} \phi d\eta d\Omega = 0.$$

3.2.2 Strict contractivity in the Wasserstein distance

In this subsection, we present the proof of the strict contraction property of the KKE.

For the one-dimensional case, it is well known [12, 69] that the Wasserstein p-distance $W_p(\mu_1, \mu_2)$ between two measures μ_1 and μ_2 is equivalent to the L^p-distance between the corresponding pseudo-inverse functions ϕ_1 and ϕ_2 respectively. Thus, we set

$$W_p(\mu_1, \mu_2)(\Omega, t) := \|\phi_1(\cdot, \Omega, t) - \phi_2(\cdot, \Omega, t)\|_{L^p(0, g(\Omega))}, \quad 1 \le p \le \infty.$$

Since $W_p(\mu_1, \mu_2)$ depends on Ω, we introduce a modified metric on the phase-space (θ, Ω):

$$\widetilde{W}_p(\mu_1, \mu_2)(t) := ||W_p(\mu_1, \mu_2)(\cdot, t)||_{L^p(\mathbb{R})}, \quad 1 \le p \le \infty.$$

Below, we assume that the density function $g(\Omega)$ has compact support. Then, it is easy to see that $\widetilde{W}_p(\mu_1, \mu_2)$ is a metric that satisfies

$$\lim_{p\to\infty} \widetilde{W}_p(\mu_1, \mu_2)(t) = \widetilde{W}_\infty(\mu_1, \mu_2)(t), \qquad t \ge 0. \tag{3.2.7}$$

Recall that the sgn function is defined by

$$\mathrm{sgn}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

Lemma 3.2.4. *Let Φ be a measurable function defined on $[0, g(\Omega)] \times \mathbb{R}$ satisfying*

$$|\Phi(\eta, \Omega)| < \frac{\pi}{2} \quad \textit{and} \quad \int_{\mathbb{R}}\int_0^{g(\Omega)} \Phi(\eta, \Omega) d\eta d\Omega = 0.$$

Then for $1 \le p < \infty$, we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{g(\Omega)}\int_0^{g(\Omega_*)}\Big[|\Phi(\eta,\Omega)|^{p-1}sgn(\Phi(\eta,\Omega)) - |\Phi(\eta_*,\Omega_*)|^{p-1}sgn(\Phi(\eta_*,\Omega_*))\Big]$$
$$\times \sin(\frac{\Phi(\eta_*,\Omega_*)-\Phi(\eta,\Omega)}{2})d\eta_* d\eta d\Omega_* d\Omega \le -\frac{2}{\pi}\int_{\mathbb{R}}\int_0^{g(\Omega)}|\Phi(\eta)|^p d\eta d\Omega.$$

Proof. For notational simplicity, we set

$$\begin{aligned}&\Phi := \Phi(\eta,\Omega),\ \ \Phi_* := \Phi(\eta_*,\Omega_*), \quad \text{and}\\ &\Delta(\eta,\eta_*,\Omega,\Omega_*) := \Big[|\Phi|^{p-1}\mathrm{sgn}(\Phi) - |\Phi_*|^{p-1}\mathrm{sgn}(\Phi_*)\Big]\sin\Big(\frac{\Phi_*-\Phi}{2}\Big),\end{aligned}$$

and we decompose the domain $[0, g(\Omega)] \times \mathbb{R}$ as the disjoint union of three subsets:

$$\begin{aligned}&\mathcal{P} := \{(\eta,\Omega)\ |\ \Phi(\eta,\Omega) > 0\}, \quad \mathcal{Z} := \{(\eta,\Omega)\ |\ \Phi(\eta,\Omega) = 0\},\\ &\mathcal{N} := \{(\eta,\Omega)\ |\ \Phi(\eta,\Omega) < 0\}.\end{aligned}$$

Then it follows from the condition $\int_{\mathbb{R}}\int_0^{g(\Omega)}\Phi d\eta d\Omega = 0$ that

$$\int_{\mathcal{P}}|\Phi| d\eta d\Omega = \int_{\mathcal{N}}|\Phi| d\eta d\Omega. \tag{3.2.8}$$

We use $[0, g(\Omega)] \times \mathbb{R} = \mathcal{P}\cup\mathcal{Z}\cup\mathcal{N}$ to obtain

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{g(\Omega_*)}\int_0^{g(\Omega)}\Delta(\eta,\eta_*,\Omega,\Omega_*)d\eta d\eta_* d\Omega d\Omega_*\\ &\quad = \Big(\underbrace{\int_{\mathcal{P}\times\mathcal{Z}} + \cdots \int_{\mathcal{N}\times\mathcal{P}}}_{\text{distinct signs}} + \underbrace{\int_{\mathcal{P}\times\mathcal{P}} + \int_{\mathcal{N}\times\mathcal{N}} + \int_{\mathcal{Z}\times\mathcal{Z}}}_{\text{same signs}}\Big)\Delta(\eta,\eta_*,\Omega,\Omega_*)d\eta d\eta_* d\Omega d\Omega_*\end{aligned}$$

We now consider the following sub-integrals separately.

$$I(A,B) := \int_{A\times B}\Delta(\eta,\eta_*,\Omega,\Omega_*)d\eta d\eta_* d\Omega d\Omega_*, \quad A, B \in \{\mathcal{P},\mathcal{Z},\mathcal{N}\}.$$

We claim the following:

Case	A	B	$I(A,B) \le$
I	$\mathcal{P}$	$\mathcal{Z}$	$-\frac{\mathcal{L}(\mathcal{Z})}{\pi}\int_{\mathcal{P}} \lvert\Phi_*\rvert^p d\eta_* d\Omega_*$
II	$\mathcal{N}$	$\mathcal{Z}$	$-\frac{\mathcal{L}(\mathcal{Z})}{\pi}\int_{\mathcal{N}} \lvert\Phi_*\rvert^p d\eta_* d\Omega_*$
III	$\mathcal{Z}$	$\mathcal{P}$	$-\frac{\mathcal{L}(\mathcal{Z})}{\pi}\int_{\mathcal{P}} \lvert\Phi\rvert^p d\eta d\Omega$
IV	$\mathcal{Z}$	$\mathcal{N}$	$-\frac{\mathcal{L}(\mathcal{Z})}{\pi}\int_{\mathcal{N}} \lvert\Phi\rvert^p d\eta d\Omega$
V	$\mathcal{P}$	$\mathcal{N}$	$-\frac{1}{\pi}\Big[\mathcal{L}(\mathcal{P})\int_{\mathcal{N}} \lvert\Phi\rvert^p d\eta d\Omega + \mathcal{L}(\mathcal{N})\int_{\mathcal{P}} \lvert\Phi_*\rvert^p d\eta_* d\Omega_*$ $+\int_{\mathcal{N}} \lvert\Phi\rvert^{p-1} d\eta d\Omega \int_{\mathcal{P}} \lvert\Phi_*\rvert d\eta_* d\Omega_*$ $+\int_{\mathcal{P}} \lvert\Phi_*\rvert^{p-1} d\eta_* d\Omega_* \int_{\mathcal{N}} \lvert\Phi\rvert d\eta d\Omega\Big]$
VI	$\mathcal{N}$	$\mathcal{P}$	$-\frac{1}{\pi}\Big[\mathcal{L}(\mathcal{N})\int_{\mathcal{P}} \lvert\Phi\rvert^p d\eta d\Omega + \mathcal{L}(\mathcal{P})\int_{\mathcal{N}} \lvert\Phi_*\rvert^p d\eta_* d\Omega_*$ $+\int_{\mathcal{P}} \lvert\Phi\rvert^{p-1} d\eta d\Omega \int_{\mathcal{N}} \lvert\Phi_*\rvert d\eta_* d\Omega_*$ $+\int_{\mathcal{N}} \lvert\Phi_*\rvert^{p-1} d\eta_* d\Omega_* \int_{\mathcal{P}} \lvert\Phi\rvert d\eta d\Omega\Big]$
VII	$\mathcal{P}$	$\mathcal{P}$	$-\frac{1}{\pi}\Big[\mathcal{L}(\mathcal{P})\int_{\mathcal{P}} \lvert\Phi\rvert^p d\eta d\Omega + \mathcal{L}(\mathcal{P})\int_{\mathcal{P}} \lvert\Phi_*\rvert^p d\eta_* d\Omega_*$ $-\int_{\mathcal{P}} \lvert\Phi\rvert^{p-1} d\eta d\Omega \int_{\mathcal{P}} \lvert\Phi_*\rvert d\eta_* d\Omega_*$ $-\int_{\mathcal{P}} \lvert\Phi_*\rvert^{p-1} d\eta_* d\Omega_* \int_{\mathcal{P}} \lvert\Phi\rvert d\eta d\Omega\Big]$
VIII	$\mathcal{N}$	$\mathcal{N}$	$-\frac{1}{\pi}\Big[\mathcal{L}(\mathcal{N})\int_{\mathcal{N}} \lvert\Phi\rvert^p d\eta d\Omega + \mathcal{L}(\mathcal{N})\int_{\mathcal{N}} \lvert\Phi_*\rvert^p d\eta_* d\Omega_*$ $-\int_{\mathcal{N}} \lvert\Phi\rvert^{p-1} d\eta d\Omega \int_{\mathcal{N}} \lvert\Phi_*\rvert d\eta_* d\Omega_*$ $-\int_{\mathcal{N}} \lvert\Phi_*\rvert^{p-1} d\eta_* d\Omega_* \int_{\mathcal{N}} \lvert\Phi\rvert d\eta d\Omega\Big]$
IX	$\mathcal{Z}$	$\mathcal{Z}$	0

where $\mathcal{L}(A)$ denotes the Lebesgue measure of the set A:

$$\mathcal{L}(A) := \int_A 1 d\eta d\Omega.$$

We also note that

$$\mathcal{L}(\mathcal{P}) + \mathcal{L}(\mathcal{Z}) + \mathcal{L}(\mathcal{N}) = \int_{\mathbb{R}} \int_0^{g(\Omega)} 1 d\eta d\Omega = \int_{\mathbb{R}} g(\Omega) d\Omega = 1.$$

Case I: In this case, we use the definition of $\Delta(\eta, \eta_*, \Omega, \Omega_*)$ and the inequality

$$\sin x \ge \frac{2}{\pi} x, \qquad \text{for } x \in \Big[0, \frac{\pi}{2}\Big],$$

to determine that

$$\Delta(\eta,\eta_*,\Omega,\Omega_*) = -|\Phi_*|^{p-1}\sin\frac{|\Phi_*|}{2} \le -\frac{1}{\pi}|\Phi_*|^p.$$

This yields

$$I(\mathcal{P},\mathcal{Z}) \le -\frac{1}{\pi}\int_{\mathcal{P}\times\mathcal{Z}} |\Phi_*|^p d\eta d\eta_* d\Omega d\Omega_* = -\frac{\mathcal{L}(\mathcal{Z})}{\pi}\int_{\mathcal{P}} |\Phi_*|^p d\eta_* d\Omega_*.$$

Case II - Case IV: The estimates are basically the same as in Case I. Hence, we omit their estimates.

Case V: In this case, we have

$$\begin{aligned}\Delta(\eta,\eta_*,\Omega,\Omega_*) &= -(|\Phi|^{p-1}+|\Phi_*|^{p-1})\sin\left(\frac{|\Phi_*|+|\Phi|}{2}\right)\\ &\le -\frac{1}{\pi}\Big(|\Phi|^p+|\Phi_*|^p+|\Phi|^{p-1}|\Phi_*|+|\Phi_*|^{p-1}|\Phi|\Big).\end{aligned}$$

This yields the desired result.
Case VI: Once we interchange $\mathcal{P}\longleftrightarrow\mathcal{N}$, the same estimate holds.

Case VII: In this case, we need to consider two subcases:

$$\text{Either } \Phi > \Phi_* > 0 \quad \text{or} \quad \Phi_* \ge \Phi > 0.$$

By considering each case, we have

$$\begin{aligned}\Delta(\eta,\eta_*,\Omega,\Omega_*) &= \big(|\Phi|^{p-1}-|\Phi_*|^{p-1}\big)\sin\left(\frac{\Phi_*-\Phi}{2}\right)\\ &\le \frac{1}{\pi}\left(|\Phi|^{p-1}-|\Phi_*|^{p-1}\right)\left(|\Phi_*|-|\Phi|\right)\\ &= -\frac{1}{\pi}\Big(|\Phi|^p+|\Phi_*|^p-|\Phi|^{p-1}|\Phi_*|-|\Phi_*|^{p-1}|\Phi|\Big).\end{aligned}$$

This yields the desired result.

Case VIII:: The estimate is exactly the same as in Case VII. Hence we omit its estimate.

Case IX:: The estimate is trivial.

We now add all cases and use (3.2.8) to find

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{g(\Omega_*)}\int_0^{g(\Omega)} &\Delta(\eta,\eta_*,\Omega,\Omega_*)d\eta d\eta_* d\Omega d\Omega_* \\ &\leq -\frac{2}{\pi}\Big(\mathcal{L}(\mathcal{P})+\mathcal{L}(\mathcal{Z})+\mathcal{L}(\mathcal{N})\Big)\int_{\mathbb{R}}\int_0^{g(\Omega)}|\Phi|^p d\eta d\Omega \\ &= -\frac{2}{\pi}\int_{\mathbb{R}}\int_0^{g(\Omega)}|\Phi|^p d\eta d\Omega.\end{aligned}$$

□

Theorem 3.2.1. *Suppose that two initial measures* $\mu_0,\ \nu_0 \in \mathcal{M}([0,2\pi)\times\mathbb{R})$ *and* K *satisfy*

$$\begin{aligned}&(i)\ 0 < D_\theta(\nu_0) \leq D_\theta(\mu_0) < \pi, \\ &\qquad \int_{[0,2\pi]\times\mathbb{R}} \theta\mu_0(d\theta,d\Omega) = \int_{[0,2\pi]\times\mathbb{R}} \theta\nu_0(d\theta,d\Omega) = \pi. \\ &(ii)\ K > D_\Omega(\mu_0)\max\Big\{\frac{1}{\sin D_\theta(\mu_0)},\frac{1}{\sin D_\theta(\nu_0)}\Big\},\end{aligned}$$

and let μ_t *and* ν_t *be two measure valued solutions to* (2.2.3) - (2.2.4) *corresponding to initial data* μ_0 *and* ν_0*, respectively. Then, there exists* $t_0 > 0$ *such that*

$$\widetilde{W}_p(\mu_t,\nu_t) \leq \exp\Big[-\frac{2K\cos D^\infty}{\pi}(t-t_0)\Big]\widetilde{W}_p(\mu_{t_0},\nu_{t_0}), \quad t > t_0,\ 1\leq p\leq\infty.$$

Proof. First, we consider the case where $p \in [1,\infty)$. Note that the Wasserstein distance in one-space dimension is equivalent to the L^p-distance of its corresponding pseudo inverse distribution function. Remember that we are assuming that the solutions are smooth, hence it is more convenient to obtain the L^p-estimate from equation (3.2.6). Denoting by $\phi_i,\ i=1,2$ the pseudo inverse functions associated to μ_t and ν_t respectively, we get

$$\partial_t\phi_i = \Omega + K\int_{\mathbb{R}}\int_0^{g(\Omega_*)}\sin(\phi_{i*}-\phi_i)d\eta_* d\Omega_*\,,$$

for $i = 1, 2$. Then the above equations imply that

$$\begin{aligned}\partial_t(\phi_1 - \phi_2) &= K \int_{\mathbb{R}} \int_0^{g(\Omega_*)} \Big(\sin(\phi_{1*} - \phi_1) - \sin(\phi_{2*} - \phi_2) \Big) d\eta_* d\Omega_*, \\ &= 2K \int_{\mathbb{R}} \int_0^{g(\Omega_*)} \cos\left(\frac{\phi_{1*} - \phi_1}{2} + \frac{\phi_{2*} - \phi_2}{2} \right) \sin\left(\frac{\phi_{1*} - \phi_1}{2} - \frac{\phi_{2*} - \phi_2}{2} \right) d\eta_* d\Omega_*.\end{aligned} \tag{3.2.9}$$

We multiply (3.2.9) by $p\mathrm{sgn}(\phi_1 - \phi_2)|\phi_1 - \phi_2|^{p-1}$ and integrate over $[0, g(\Omega)] \times \mathbb{R}$ using the symmetry $(\eta, \Omega) \Longleftrightarrow (\eta_*, \Omega_*)$ to obtain

$$\begin{aligned}&\frac{d}{dt}||\phi_1 - \phi_2||_{L^p}^p \\ &= 2pK \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^{g(\Omega)} \int_0^{g(\Omega_*)} \Big[\cos\left(\frac{\phi_{1*} - \phi_1}{2} + \frac{\phi_{2*} - \phi_2}{2} \right) \sin\left(\frac{\phi_{1*} - \phi_1}{2} - \frac{\phi_{2*} - \phi_2}{2} \right) \\ &\times \big[|\phi_1 - \phi_2|^{p-1} \mathrm{sgn}\,(\phi_1 - \phi_2) - |\phi_{1*} - \phi_{2*}|^{p-1} \mathrm{sgn}\,(\phi_{1*} - \phi_{2*}) \big] \Big] d\eta_* d\eta d\Omega_* d\Omega.\end{aligned}$$

It follows from the proof of Lemma 3.2.4 that for all $a,\ b \in \mathbb{R}$,

$$\big(|a|^{p-1} \mathrm{sgn}(a) - |b|^{p-1} \mathrm{sgn}(b) \big) \sin\left(\frac{b - a}{2} \right) \leq 0.$$

On the other hand, Lemma 3.2.1 implies that there exists t_0 such that

$$D_\theta(\mu_t) \leq D^\infty, \quad D_\theta(\nu_t) \leq D^\infty, \quad t \geq t_0,$$

and we use Lemma 3.2.2 to obtain

$$\begin{aligned}&\max_{\Omega \in \mathrm{supp}(g)} \phi_1(g(\Omega), \Omega, t) - \min_{\Omega \in \mathrm{supp}(g)} \phi_1(0, \Omega, t) \leq D^\infty, \\ &\max_{\Omega \in \mathrm{supp}(g)} \phi_2(g(\Omega), \Omega, t) - \min_{\Omega \in \mathrm{supp}(g)} \phi_2(0, \Omega, t) \leq D^\infty, \quad t \geq t_0.\end{aligned}$$

Then, this yields

$$0 < \cos D^\infty \leq \cos\left(\frac{\phi_{1*} - \phi_1}{2} + \frac{\phi_{2*} - \phi_2}{2} \right).$$

Hence, we obtain

$$\frac{d}{dt}||\phi_1 - \phi_2||_{L^p}^p \leq 2pK \cos D^\infty \mathcal{J}$$

where

$$\mathcal{J} := \int_{\mathbb{R}\times\mathbb{R}} \int_0^{g(\Omega)} \int_0^{g(\Omega_*)} \sin\left(\frac{\phi_{1*}-\phi_{2*}}{2} - \frac{\phi_1-\phi_2}{2}\right)$$
$$\times \left[|\phi_1-\phi_2|^{p-1}\mathrm{sgn}\,(\phi_1-\phi_2) - |\phi_{1*}-\phi_{2*}|^{p-1}\mathrm{sgn}\,(\phi_{1*}-\phi_{2*})\right] d\eta_* d\eta d\Omega_* d\Omega.$$

If we set $\Phi := \phi_1 - \phi_2$, then

$$|\Phi_* - \Phi| \le |\phi_{1*}-\phi_1| + |\phi_{2*}-\phi_2| \le 2D^\infty < \pi, \quad t > t_0.$$

Since μ_0, ν_0 have the same center of mass, it follows from Lemma 3.2.3 that

$$\int_{\mathbb{R}} \int_0^{g(\Omega)} \Phi d\eta d\Omega = 0, \quad t > t_0.$$

Thus, we can apply Lemma 3.2.4 with $\Phi = \phi_1 - \phi_2$ to obtain

$$\frac{d}{dt}\widetilde{W}_p^p(\mu_t,\nu_t) \le -\frac{2pK\cos D^\infty}{\pi}\widetilde{W}_p^p(\mu_t,\nu_t), \quad t \ge t_0.$$

This yields

$$\widetilde{W}_p(\mu_t,\nu_t) \le \exp\Big(-\frac{2K\cos D^\infty}{\pi}(t-t_0)\Big)\widetilde{W}_p(\mu_{t_0},\nu_{t_0}). \tag{3.2.10}$$

In the case of $p=\infty$, we use (3.2.7) and (3.2.10) to obtain

$$\widetilde{W}_\infty(\mu_t,\nu_t) \le \exp\Big(-\frac{2K\cos D^\infty}{\pi}(t-t_0)\Big)\widetilde{W}_\infty(\mu_{t_0},\nu_{t_0}).$$

This completes the proof for smooth solutions. As mentioned above a simple approximation argument as in Subsection 3.3 in [10] finishes the proof for measure valued solutions. □

Remark 3.2.1. *The assumption in Theorem 3.2.1 on the initial measures to have equal mean in θ is not restricted. Due to Lemma 3.2.3, the mean in θ is preserved in time. Thus, we can always restrict to the equal mean in θ case by translational invariance of* (2.2.3).

Chapter 4

Kuramoto type models with frustration

In this chapter, we will study the dynamic interplay between distinct natural frequencies (*intrinsic frustration*) and phase shift in interactions (*interaction frustration*) among Kuramoto oscillators. This chapter is based on joint works in [34].

4.1 Kuramoto model with frustration

Kuramoto and Sakaguchi [61] proposed a variant of the Kuramoto model in which the coupling function incorporated frustration (phase shift) so that richer dynamical phenomena would be observed than that with no frustration. Let α_{ji} be the frustration between the j-th and i-th oscillators, which is assumed to be symmetric in i and j. In this situation, the dynamics of Kuramoto oscillators is governed by the following ODE system:

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{i=1}^{N} \sin(\theta_j - \theta_i + \alpha_{ji}), \qquad -\frac{\pi}{2} < \alpha_{ji} < \frac{\pi}{2}. \tag{4.1.1}$$

Note that the R.H.S. of (4.1.1) is Lipschitz continuous, so the well-posedness of the system (4.1.1) is well known from the Cauchy-Lipschitz theory. Thus, what matters about the solutions is the dynamic behavior such as the relaxation process, the shape of phase-locked states, and the existence of global attractors, etc. In general, the frustration hinders synchronization, so coupling strength greater than that of the original Kuramoto model without frustration is needed to guarantee global synchronization. The use of frustration is needed for modeling real physical and biological systems, but it causes numerous mathematical difficulties in analyzing the synchronization. For this reason many studies about frustration have been mostly based on the numerical approach.[23, 47, 55, 57, 58, 66, 75] Furthermore, one of the most important reasons for the interest in frustration is that the Kuramoto model with frustration has no conservation law, even for identical oscillators (see Example 4.1.1). Thus, the standard energy method [17, 19, 32, 33, 36, 37] based on a conservation law cannot be applied in our setting. Hence, analyzing the large-time behavior of physical systems without conservation laws is challenging itself, and to the best knowledge of the authors, there are so far no general tools for dealing with such systems without conservation laws.

In this chapter, we describe three Kuramoto type models with frustrations, which are investigated in later sections. The first two models are concerned with nonidentical oscillators with (interaction) frustrations. Model A corresponds to the system (4.1.1) with a uniform frustration $\alpha_{ij} = \alpha$, and Model B corresponds to the special case of Model A in which there are only two distinct natural frequencies. Model C deals with identical oscillators on a bipartite graph; i.e., interactions arise only between members from different groups.

- Model A: (Uniform frustration)

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i + \alpha), \quad i = 1, \dots, N.$$

- Model B: (Interaction of two identical oscillator groups under frustration)

$$\dot{\theta}_i = \Omega_1 + \frac{K}{2N}\Big[\sum_{j=1}^{N}\sin(\theta_j - \theta_i + \alpha) + \sum_{j=1}^{N}\sin(\phi_j - \theta_i + \alpha)\Big], \quad i = 1, \dots, N,$$

$$\dot{\phi}_i = \Omega_2 + \frac{K}{2N}\Big[\sum_{j=1}^{N}\sin(\phi_j - \phi_i + \alpha) + \sum_{j=1}^{N}\sin(\theta_j - \phi_i + \alpha)\Big], \quad i = 1, \dots, N.$$

- Model C: (Identical oscillators on a bipartite graph)

$$\dot{\theta}_i = \frac{\mu}{N_2}\sum_{j=1}^{N_2}\sin(\phi_j - 2\theta_i), \qquad i = 1, \ldots, N_1,$$
$$\dot{\phi}_i = \frac{1-\mu}{N_1}\sum_{j=1}^{N_1}\sin(2\theta_j - \phi_i + \alpha), \qquad i = 1, \ldots, N_2,$$

where $\mu \in (0,1)$ is a positive constant. This model was recently derived from the Van der Pol equations for two coupled oscillator systems in the work of Lück and Pikovsky [49] .

As examples, we consider some simple situations to illustrate how the frustration causes the system to deviate from the original Kuramoto model (1.0.1).

Example 4.1.1. *Consider the system of two oscillators with frustration and identical natural frequencies $\Omega_1 = \Omega_2 = 0$:*

$$\begin{aligned}\frac{d\theta_1}{dt} &= \frac{K}{2}\sin(\theta_2 - \theta_1 + \alpha), \quad t > 0,\\ \frac{d\theta_2}{dt} &= \frac{K}{2}\sin(\theta_1 - \theta_2 + \alpha).\end{aligned}$$

We introduce the mean values of the phase and instantaneous frequency:

$$\theta_c := \frac{\theta_1 + \theta_2}{2}, \quad \omega_c := \frac{\omega_1 + \omega_2}{2}.$$

Then, the mean phase satisfies

$$\begin{aligned}\frac{d\theta_c}{dt} &= \frac{K}{2}\Big[\sin(\theta_2 - \theta_1 + \alpha) + \sin(\theta_1 - \theta_2 + \alpha)\Big]\\ &= K\cos(\theta_1 - \theta_2)\sin\alpha.\end{aligned}$$

Thus, we do not have conservation of total phase.

Before we close this section, we present an elementary estimate for a system of differential inequalities to be used in Section 4.3.

Lemma 4.1.1. *Let X, Y, and Z be differentiable functions satisfying the following system of differential inequalities:*

$$\begin{aligned} &\dot{X} \le -cX + (\beta + \gamma \sin(Y + Z + |\alpha|))X^2, \quad t \ge t_0, \\ &\dot{Y} \le -cY + (\beta + \gamma \sin(X + Z + |\alpha|))Y^2, \\ &\dot{Z} \le -2cZ + \delta + \frac{K}{2}(\sin(X + \alpha) + \sin(Y - \alpha)), \end{aligned} \tag{4.1.2}$$

where c, β, γ, and δ are positive constants. Then, there exist $t_1 \ge t_0 > 0$ such that

$$\begin{aligned} &X(t) \le \frac{c}{\beta + \gamma + C(X_0)e^{c(t-t_0)}}, \quad t \ge t_0, \\ &Y(t) \le \frac{c}{\beta + \gamma + C(Y_0)e^{c(t-t_0)}}, \\ &Z(t) \le Z(t_0)e^{-2c(t-t_0)} + \frac{\delta + K}{2c}\Big(1 - e^{-2c(t-t_0)}\Big), \end{aligned}$$

where $X_0 = X(t_0)$ and $Y_0 = Y(t_0)$.

Proof. • (Estimate of X): We use $\sin(Y + Z + |\alpha|) \le 1$ to obtain

$$\dot{X} \le X(-c + (\beta + \gamma)X).$$

This yields

$$X(t) \le \frac{c}{\beta + \gamma + C(X(t_0))e^{c(t-t_0)}}, \quad t \ge t_0, \tag{4.1.3}$$

where $C(X(t_0))$ is given as follows:

$$C(X(t_0)) := \Big|\frac{c - (\beta + \gamma)X(t_0)}{X(t_0)}\Big|.$$

• (Estimate of Y): Similarly, we have

$$Y(t) \le \frac{c}{\beta + \gamma + C(Y_0)e^{c(t-t_0)}}, \quad t \ge t_0. \tag{4.1.4}$$

• (Estimate of Z): We use the elementary relation

$$\begin{aligned} |\sin(X + \alpha) + \sin(Y - \alpha)| &= 2\Big|\sin\Big(\frac{X + Y}{2}\Big)\cos\Big(\frac{X - Y}{2} + \alpha\Big)\Big| \\ &\le |X| + |Y|, \end{aligned}$$

to derive

$$\dot{Z} \leq -2cZ + \delta + \frac{K}{2}(|X| + |Y|).$$

On the other hand, it follows from (4.1.3) and (4.1.4) that there exists $t_1 \geq t_0$ such that

$$|X(t)| + |Y(t)| \leq \frac{2\delta}{K}, \quad t \geq t_1.$$

Thus, we have

$$\dot{Z} \leq -2cZ + 2\delta, \quad t \geq t_1.$$

This yields

$$Z(t) \leq Z(t_1)e^{-2c(t-t_1)} + \frac{\delta}{c}\Big(1 - e^{-2c(t-t_t)}\Big), \quad t \geq t_1.$$

□

Remark 4.1.1. *From the result of Lemma 2.1, we see that*

$$X(t) = \mathcal{O}(1)e^{-ct}, \quad Y(t) = \mathcal{O}(1)e^{-ct}, \quad \text{as } t \to \infty \quad \text{and} \quad \lim_{t\to\infty} Z(t) \leq \frac{\delta}{c}.$$

In the following three sections, we derive sufficient conditions leading to complete synchronization of the aforementioned three models.

4.2 Synchronization estimate for Model A

In this section, we study the synchronizability of the Model A.

Recall that the Kuramoto model with a uniform interaction frustration is given by

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i + \alpha), \quad t \geq 0, \quad i = 1, \dots, N, \quad |\alpha| < \frac{\pi}{2}. \tag{4.2.5}$$

So far, the sufficient conditions for the Kuramoto model ($\alpha = 0$) have been extensively studied in literature [17, 19, 21, 25, 26, 32, 33, 36, 37]. Our synchronization analysis relies on two steps. First, we show the existence of a positively invariant region so that within a finite time the initial configurations are confined to the interval with length $\pi/2$. Second, we derive a Gronwall-type inequality for the frequency diameter that is subsequently defined to conclude the complete frequency synchronization.

4.2.1 Existence of a trapping region

In this subsection, we study the existence of a positively invariant region for the Kuramoto model. For this, we follow the approach of Choi *et al.* [17].

Lemma 4.2.1. (Existence of a positively invariant set) *Suppose that the natural frequencies, coupling strength, and initial data satisfy*

$$D(\Omega) > 0, \quad K \geq K_{ef} := \frac{D(\Omega)}{1 - \sin|\alpha|}, \quad 0 < D(\theta^0) < D_*^\infty - |\alpha|,$$

where D_*^∞ is the unique solution of the following equation:

$$\sin x = \frac{D(\Omega) + K \sin|\alpha|}{K}, \quad x \in \left(\frac{\pi}{2}, \pi\right).$$

Then, for the global solution to (4.2.5)*, we have*

$$\sup_{t \geq 0} D(\theta(t)) \leq D_*^\infty - |\alpha|.$$

Proof. We define a set $\mathcal{T}$ and its supremum $T^* \in [0, \infty]$:

$$\mathcal{T} := \left\{ T \geq 0 \mid D(\theta(t)) < D_*^\infty - |\alpha|, \ \forall\, t \in [0, T) \right\}, \quad T^* := \sup \mathcal{T}.$$

Since $D(\theta^0) < D_*^\infty - |\alpha|$ and $D(\theta(t))$ is continuous, there exists $T > 0$ such that

$$D(\theta(t)) < D_*^\infty - |\alpha|, \quad \forall\, t \in [0, T).$$

Hence, the set $\mathcal{T}$ is not empty. We now claim that

$$T^* = +\infty.$$

Suppose not; i.e., $T^* < \infty$. Then, we have

$$D(\theta(T^*)) \geq D_*^\infty - |\alpha|, \quad \text{and} \quad D(\theta(t)) < D_*^\infty - |\alpha|, \quad \forall\, t \in [0, T^*).$$

On the other hand, we have for $t \in [0, T^*)$,

$$\begin{aligned}
\frac{d}{dt} D(\theta(t)) &\leq \Omega_M - \Omega_m + \frac{K}{N} \sum_{j=1}^N \Big[\sin(\theta_j - \theta_M + \alpha) - \sin(\theta_j - \theta_m + \alpha) \Big] \\
&\leq D(\Omega) + \frac{K \cos\alpha}{N} \sum_{j=1}^N \Big[\sin(\theta_j - \theta_M) - \sin(\theta_j - \theta_m) \Big] \\
&\quad + \frac{K \sin\alpha}{N} \sum_{j=1}^N \Big[\cos(\theta_j - \theta_M) - \cos(\theta_j - \theta_m) \Big].
\end{aligned}$$

We consider two cases according to the sign of α.

- Case 1: $\alpha \in [0, \frac{\pi}{2})$. In this case, we have

$$\begin{aligned}
&\frac{d}{dt} D(\theta(t)) \\
&\qquad \le D(\Omega) + \frac{K \cos\alpha \sin D(\theta)}{N D(\theta)} \sum_{j=1}^{N} \Big[(\theta_j - \theta_M) - (\theta_j - \theta_m) \Big] \\
&+ \frac{K \sin\alpha}{N} \sum_{j=1}^{N} \Big[1 - \cos D(\theta) \Big] \\
&\qquad = D(\Omega) - K \Big[\sin\left(D(\theta) + \alpha \right) - \sin\alpha \Big] \\
&\qquad = D(\Omega) - K \Big[\sin\left(D(\theta) + |\alpha| \right) - \sin|\alpha| \Big].
\end{aligned}$$

- Case 2: $\alpha \in (-\frac{\pi}{2}, 0)$. In this case, we have

$$\begin{aligned}
&\frac{d}{dt} D(\theta(t)) \\
&\qquad \le D(\Omega) + \frac{K \cos\alpha \sin D(\theta)}{N D(\theta)} \sum_{j=1}^{N} \Big[(\theta_j - \theta_M) - (\theta_j - \theta_m) \Big] \\
&+ \frac{K \sin\alpha}{N} \sum_{j=1}^{N} \Big[\cos D(\theta) - 1 \Big] \\
&\qquad = D(\Omega) - K \Big[\sin\left(D(\theta) - \alpha \right) + \sin\alpha \Big] \\
&\qquad = D(\Omega) - K \Big[\sin\left(D(\theta) + |\alpha| \right) - \sin|\alpha| \Big].
\end{aligned}$$

Here we used the condition:

$$\sin(\theta_j - \theta_M) \le \frac{\sin D(\theta)}{D(\theta)} (\theta_j - \theta_M), \quad \sin(\theta_j - \theta_m) \ge \frac{\sin D(\theta)}{D(\theta)} (\theta_j - \theta_m),$$

and

$$\cos(\theta_j - \theta_M), \cos(\theta_j - \theta_m) \le 1, \quad \cos(\theta_j - \theta_m), \cos(\theta_j - \theta_M) \ge \cos D(\theta).$$

Then we have

$$\frac{d}{dt} D(\theta(t)) \le D(\Omega) + K \sin|\alpha| - K \sin(D(\theta(t)) + |\alpha|). \tag{4.2.6}$$

Since $D(\theta(t)) + |\alpha| < D_*^\infty < \pi$ for $t \in [0, T^*)$, we obtain

$$\dot{D}(\theta(t)) \le D(\Omega) + K \sin|\alpha| - \frac{K \sin D_*^\infty}{D_*^\infty}(D(\theta(t)) + |\alpha|).$$

We now use the above inequality to observe that

$$D(\theta(t)) \le \Big(D(\theta^0) - D_*^\infty + |\alpha|\Big) e^{-\frac{K \sin D_*^\infty}{D_*^\infty} t} + D_*^\infty - |\alpha|, \quad t \in [0, T^*).$$

This implies

$$\begin{aligned} D_*^\infty - |\alpha| \le D(\theta(T^*)) &\le \Big(D(\theta^0) - D_*^\infty + |\alpha|\Big) e^{-\frac{K \sin D_*^\infty}{D_*^\infty} T^*} + D_*^\infty - |\alpha| \\ &< D_*^\infty - |\alpha|, \end{aligned}$$

which is a contradiction. Hence, $T^* = \infty$. □

Remark 4.2.1. *(i) Note that the lower bound on the coupling strength K is improved even for the zero frustration case $\alpha = 0$. In [17], the lower bound K_e is defined to be dependent on the diameter of initial phase diameter $D(\theta_0)$, more precisely*

$$K > \frac{D(\Omega)}{\sin D(\theta^0)}.$$

However, in our case, the lower bound for K does not depend on the initial phase diameter.

(ii) The arguments used in the proof of Lemma 4.2.1 also imply that $D(\theta(t))$ satisfies

$$\dot{D}(\theta) \le D(\Omega) + K \sin|\alpha| - K \sin\big(D(\theta) + |\alpha|\big), \ \textit{a.e. } t, \tag{4.2.7}$$

where $\dot{D}(\theta(t)) := \frac{d}{dt} D(\theta(t))$.

4.2.2 Entrance to the exponential stability regime

In this subsection, we study the transition of the phase ensemble to the exponential stability regime within a finite time. We introduce a reference angle D^∞:

$$D^\infty \in \left(0, \frac{\pi}{2}\right), \qquad \sin D^\infty = \frac{D(\Omega) + K \sin|\alpha|}{K}. \tag{4.2.8}$$

Remark 4.2.2. *1. Note that D^∞ is the dual angle of D^∞_* in Lemma 4.2.1 satisfying*

$$\sin D^\infty = \sin D^\infty_* = \frac{D(\Omega) + K \sin|\alpha|}{K}.$$

2. Since

$$\sin D^\infty = \frac{D(\Omega)}{K} + \sin|\alpha| > \sin|\alpha|,$$

we have $D^\infty > |\alpha|$.

3. The inequality (4.2.7) *implies that $D(\theta(t))$ is decreasing if*

$$D(\theta(t)) \in (D^\infty - |\alpha|, D^\infty_* - |\alpha|).$$

Proposition 4.2.1. *Suppose that the natural frequencies, coupling strength and initial data satisfy*

$$D(\Omega) > 0, \quad K > K_{ef}, \quad 0 < D(\theta^0) < D^\infty_* - |\alpha|.$$

Let $\theta(t)$ be the global solution to (4.2.5)*. Then for any $0 < \epsilon \ll 1$, there exists $t_0 = t_0(\epsilon) > 0$ such that*

$$D(\theta(t)) \le D^\infty - |\alpha| + \epsilon, \quad \text{for} \quad t \ge t_0.$$

In particular, we can choose ϵ so small that $D(\theta(t)) + |\alpha| \le D^\infty + \epsilon < \frac{\pi}{2}$ for $t \ge t_0$.

Proof. We consider the ordinary differential equation

$$\dot{y} = D(\Omega) + K \sin|\alpha| - K \sin y. \tag{4.2.9}$$

Our assumption on K implies that $y_* = D^\infty$ is an equilibrium point for the equation (4.2.9). y_* is locally stable because in the neighborhood of y_*, $\frac{dy}{dt} < 0$ for $y > y_*$ and $\frac{dy}{dt} > 0$ for $y < y_*$. Moreover, for any initial value $y(0)$ with $0 < y(0) < D^\infty_*$, the trajectory $y(t)$ monotonically approaches y_*. Therefore, for any $\epsilon > 0$, there exists a time t_0 such that

$$|y(t) - y_*| < \epsilon, \quad \forall\, t \ge t_0.$$

We now apply this analysis on (4.2.9) and the principle of comparison with (4.2.7) to find

$$D(\theta(t)) + |\alpha| < D^\infty + \epsilon, \quad \forall\, t \geq t_0.$$

That is,

$$D(\theta(t)) < D^\infty - |\alpha| + \epsilon, \quad \forall\, t \geq t_0.$$

□

4.2.3 Relaxation estimate

In this subsection, we provide an estimate of the relaxation toward the phase-locked state. We next derive the Gronwall's differential inequality for the frequency diameter $D(\omega(t))$. We differentiate the system (4.2.5) with respect to time t to obtain

$$\frac{d\omega_i}{dt} = \frac{K}{N}\sum_{j=1}^{N}\cos(\theta_j - \theta_i + \alpha)(\omega_j - \omega_i).$$

This yields

$$\begin{aligned}\frac{d}{dt}D(\omega(t)) &= \frac{K}{N}\sum_{j=1}^{N}\Big[\cos(\theta_j - \theta_M^\omega + \alpha)(\omega_j - \omega_M) - \cos(\theta_j - \theta_m^\omega + \alpha)(\omega_j - \omega_m)\Big] \\ &\leq \frac{K}{N}\cos\left(D^\infty + \epsilon\right)\sum_{j=1}^{N}\Big[(\omega_j - \omega_M) - (\omega_j - \omega_m)\Big] \\ &= -K\cos\left(D^\infty + \epsilon\right)D(\omega(t)), \qquad t \geq t_0,\end{aligned} \tag{4.2.10}$$

where θ_M^ω and θ_m^ω denote the phases of the oscillators, which have the maximum and minimum instantaneous frequencies, respectively. In summary, we have

$$\frac{d}{dt}D(\omega(t)) \leq -K\cos\left(D^\infty + \epsilon\right)D(\omega(t)), \qquad t \geq t_0. \tag{4.2.11}$$

We are now ready to make the relaxation estimate for Model A.

Theorem 4.2.1. *Suppose that the natural frequencies, coupling strength, and initial data satisfy*

$$D(\Omega) > 0, \quad K > K_{ef}, \quad 0 < D(\theta^0) < D_*^\infty - |\alpha|.$$

Then, for any $0 < \epsilon \ll 1$ *with* $D^\infty + \epsilon < \frac{\pi}{2}$, *there exists* $t_0 > 0$ *such that*

$$D(\omega(t_0))e^{-K(t-t_0)} \le D(\omega(t)) \le D(\omega(t_0))e^{-K\cos(D^\infty+\epsilon)(t-t_0)}, \quad t \ge t_0.$$

Proof. By Proposition 4.2.1, there exists some $t_0 > 0$ such that

$$|\theta_j(t) - \theta_i(t)| \le D(\theta(t)) \le D^\infty - |\alpha| + \epsilon, \qquad \forall\, t \ge t_0.$$

• (Upper bound estimate): We combine the above result with (4.2.11) to find

$$D(\omega(t)) \le D(\omega(t_0)) \exp\Big[- K \cos\big(D^\infty + \epsilon\big)(t - t_0)\Big], \quad t \ge t_0.$$

Hence, the frequency diameter decays exponentially to zero.

• (Lower bound estimate): We use $\cos\theta \le 1$ in (4.2.10) to find

$$D(\omega(t)) \ge D(\omega(t_0)) \exp[-K(t-t_0)], \quad t \ge t_0.$$

□

4.3 Synchronization estimates for Model B

In this section, we consider the interaction of two identical oscillator groups with different natural frequencies. In the absence of frustration, this case has already been considered by Ha and Kang [33], who found that the resulting phase-locked states are configurations of two-point clusters and the relaxation speed can be exponential or algebraic depending on the size of K compared to the size of the intrinsic frustration $D(\Omega)$. More precisely, we consider the ensemble of mixed Kuramoto oscillators with two distinct natural frequencies. In this case, the plausible scenario has two stages. First the ensemble will evolve from the mixed phase into the segregated phase; i.e., two identical oscillator groups will separate into two groups that will each have the same natural frequency. Then, these two groups will evolve to become the point clusters, respectively.

Let $\{\theta_i\}_{i=1}^N$ and $\{\phi_i\}_{i=1}^N$ be the identical oscillator groups with natural frequencies Ω_1 and Ω_2, respectively. In this situation, the dynamics of the oscillator groups is governed by the following coupled system:

$$\begin{aligned}
\dot{\theta}_i &= \Omega_1 + \frac{K}{2N}\Big[\sum_{j=1}^N \sin(\theta_j - \theta_i + \alpha) + \sum_{j=1}^N \sin(\phi_j - \theta_i + \alpha)\Big], \quad i = 1, \dots, N, \\
\dot{\phi}_i &= \Omega_2 + \frac{K}{2N}\Big[\sum_{j=1}^N \sin(\phi_j - \phi_i + \alpha) + \sum_{j=1}^N \sin(\theta_j - \phi_i + \alpha)\Big], \quad i = 1, \dots, N,
\end{aligned} \tag{4.3.12}$$

subject to initial data and frustration,

$$(\theta_i, \phi_i)(0) = (\theta_i^0, \phi_j^0), \qquad |\alpha| < \frac{\pi}{2}, \tag{4.3.13}$$

where $K > 0$ denotes the coupling strength and N denotes the number of oscillators in each group. Without loss of generality, we assume that

$$\Omega_1 > \Omega_2.$$

As an extreme case, we consider the situation where the identical oscillators with the same natural frequencies are collapsed to a single phase at $t = 0$; i.e.,

$$\theta_i^0 = \theta_1^0, \qquad \text{and} \qquad \phi_i^0 = \phi_1^0, \quad i = 2, \dots, N.$$

Then, by the uniqueness of the solution to the system (4.3.12), we have

$$\theta_1(t) = \cdots = \theta_N(t), \quad \phi_1(t) = \cdots = \phi_N(t), \quad t \geq 0.$$

We set

$$D(\Omega) = \Omega_1 - \Omega_2, \quad \text{and} \quad \Delta(t) := \theta_1(t) - \phi_1(t), \quad t \geq 0.$$

From (4.3.12), we have

$$
\begin{aligned}
\dot{\theta}_1 &= \Omega_1 + \frac{K}{2}\sin\alpha + \frac{K}{2N}\sum_{j=1}^{N}\sin(-\Delta+\alpha),\\
\dot{\phi}_1 &= \Omega_2 + \frac{K}{2}\sin\alpha + \frac{K}{2N}\sum_{j=1}^{N}\sin(\Delta+\alpha).
\end{aligned} \tag{4.3.14}
$$

We subtract the second equation from the first equation to obtain

$$
\dot{\Delta} = D(\Omega) - K\cos\alpha\sin\Delta, \tag{4.3.15}
$$

which is a standard Adler's equation.

4.3.1 Existence of a trapping region

In this subsection, we will find a trapping region for the system (4.3.12). We set

$$
\begin{aligned}
\theta_m(t) &:= \min_{1\le i\le N}\theta_i(t), \qquad \theta_M(t) := \max_{1\le i\le N}\theta_i(t),\\
\phi_m(t) &:= \min_{1\le i\le N}\phi_i(t), \qquad \phi_M(t) := \max_{1\le i\le N}\phi_i(t),\\
D(\theta(t)) &:= \theta_M(t)-\theta_m(t), \quad D(\phi(t)) := \phi_M(t)-\phi_m(t),\\
D(\theta,\phi)(t) &:= \max_{1\le i\le N}\{\theta_i,\phi_i\} - \min_{1\le i\le N}\{\theta_i,\phi_i\}.
\end{aligned} \tag{4.3.16}
$$

We also set

$$
\Delta(t) := \theta_m(t) - \phi_M(t).
$$

It is easy to see that if $\Delta(t)\ge 0$, then

$$
D(\theta,\phi)(t) = D(\theta(t)) + D(\phi(t)) + \Delta(t).
$$

That is, the sum of the three partial diameters is exactly the total diameter of the phases $\{\theta(t),\phi(t)\}$. The next lemma demonstrates a kind of monotonicity property among Kuramoto oscillators with the same natural frequency.

Lemma 4.3.1. *Let* $\{\theta_i, \phi_i\}$ *be the global solution to the system* (4.3.12)-(4.3.13)*, with*

$$\phi_1^0 \le \phi_2^0 \le \cdots \le \phi_N^0.$$

Then for all $t > 0$*, we have*

$$\phi_1(t) \le \phi_2(t) \le \cdots \le \phi_N(t).$$

Proof. It is sufficient to show that

$$\phi_i^0 \le \phi_j^0 \quad \Longrightarrow \quad \phi_i(t) \le \phi_j(t),\ t > 0.$$

Suppose that there exists some time $t_0 > 0$ such that $\phi_i(t_0) = \phi_j(t_0)$. Then, by the uniqueness of the solution, we observe that $\phi_i(t) = \phi_j(t)$, $\forall\, t \ge t_0$. Moreover, owing to the analyticity of $\{\theta_i, \phi_i\}$, we can derive $\phi_i(t) = \phi_j(t)$, $\forall\, t \ge 0$. Hence, we have

$$\phi_i^0 < \phi_j^0 \quad \Longrightarrow \quad \phi_i(t) < \phi_j(t),\ t > 0.$$

□

Remark 4.3.1. *The same result holds for the group* $\{\theta_i\}$ *with natural frequency* Ω_1*. Lemma 4.3.1 means that no collision occurs in either group during the evolution process, unless the oscillators initially have the same phase. Hence, the oscillators that take the maximum and minimum phases in each group are fixed forever. This implies that the extremal phases* $\theta_M(t)$*,* $\theta_m(t)$*,* $\phi_m(t)$*,* $\phi_M(t)$ *and partial diameters* $D(\theta(t))$*,* $D(\phi(t))$ *are analytic functions.*

Lemma 4.3.2. *Let* $(\theta, \phi) = (\theta(t), \phi(t))$ *be the smooth solution to* (4.3.12)-(4.3.13) *satisfying*

$$D(\theta, \phi)(0) \le R, \quad \textit{for some constant } R < \frac{\pi}{2} - |\alpha|.$$

Then, we have

$$\frac{d}{dt} D(\theta(t))\Big|_{t=0} \le 0, \qquad \frac{d}{dt} D(\phi(t))\Big|_{t=0} \le 0.$$

Proof. (1) We use (4.3.12) and the definition of $D(\theta(t))$ to obtain

$$\begin{aligned}
\frac{d}{dt}D(\theta(t)) &= \dot{\theta}_M(t) - \dot{\theta}_m(t) \\
&= \frac{K}{2N}\sum_{j=1}^{N}\Big\{\sin(\theta_j - \theta_M + \alpha) - \sin(\theta_j - \theta_m + \alpha)\Big\} \\
&\quad + \frac{K}{2N}\sum_{j=1}^{N}\Big\{\sin(\phi_j - \theta_M + \alpha) - \sin(\phi_j - \theta_m + \alpha)\Big\} \\
&= -\frac{K}{N}\sin\frac{D(\theta)}{2}\sum_{j=1}^{N}\cos\Big(\frac{\theta_j - \theta_M}{2} + \frac{\theta_j - \theta_m}{2} + \alpha\Big) \\
&\quad - \frac{K}{N}\sin\frac{D(\theta)}{2}\sum_{j=1}^{N}\cos\Big(\frac{\phi_j - \theta_M}{2} + \frac{\phi_j - \theta_m}{2} + \alpha\Big).
\end{aligned}$$

Note that at $t = 0$ we have

$$\begin{aligned}
-\frac{R}{2} \le -\frac{D(\theta)}{2} \le& \frac{\theta_j - \theta_M}{2} + \frac{\theta_j - \theta_m}{2} \le \frac{D(\theta)}{2} \le \frac{R}{2}, \\
-R \le \phi_j - \theta_M \le& \frac{\phi_j - \theta_M}{2} + \frac{\phi_j - \theta_m}{2} \le \phi_j - \theta_m \le R.
\end{aligned}$$

This yields

$$\frac{d}{dt}D(\theta(t))\Big|_{t=0} \le 0.$$

(2) We also have the simialr result for $D(\phi(t))$. Note that

$$\begin{aligned}
\frac{d}{dt}D(\phi(t)) =& -\frac{K}{N}\sin\frac{D(\phi)}{2}\sum_{j=1}^{N}\cos\Big(\frac{\phi_j - \phi_M}{2} + \frac{\phi_j - \phi_m}{2} + \alpha\Big) \\
& - \frac{K}{N}\sin\frac{D(\phi)}{2}\sum_{j=1}^{N}\cos\Big(\frac{\theta_j - \phi_M}{2} + \frac{\theta_j - \phi_m}{2} + \alpha\Big).
\end{aligned}$$

Since

$$\begin{aligned}
-\frac{R}{2} \le -\frac{D(\phi)}{2} \le& \frac{\phi_j - \phi_M}{2} + \frac{\phi_j - \phi_m}{2} \le \frac{D(\phi)}{2} \le \frac{R}{2}, \quad t = 0, \\
-R \le \theta_j - \phi_M \le& \frac{\theta_j - \phi_M}{2} + \frac{\theta_j - \phi_m}{2} \le \theta_j - \phi_m \le R,
\end{aligned}$$

we have

$$\frac{d}{dt} D(\phi(t))\big|_{t=0} \leq 0.$$

$\square$

Proposition 4.3.1. *Suppose that the coupling strength K satisfies*

$$K > \frac{D(\Omega)}{1 - \sin|\alpha|},$$

and let $(\theta, \phi) = (\theta(t), \phi(t))$ be the smooth solution to (4.3.12)-(4.3.13) *satisfying*

$$D(\theta, \phi)(0) \leq R, \quad \text{for some constant } R < \frac{\pi}{2} - |\alpha|.$$

Then we have the following two assertions:

(1) $D(\theta, \phi)(t) \leq \begin{cases} R, & \text{if } R \geq D^\infty - |\alpha|, \\ D^\infty - |\alpha|, & \text{if } R < D^\infty - |\alpha|, \end{cases} \quad \forall\, t \geq 0;$

(2) $D(\theta(t)) \leq D(\theta^0)$, *and* $D(\phi(t)) \leq D(\phi^0), \quad \forall\, t \geq 0.$

Here, D^∞ is given by (4.2.8).

Proof. (i) Since our system (4.3.12) is a special case for the system (4.2.5), we can use the analysis in Section 4.2, in particular, Remark 4.2.2 (3) and the analysis in Proposition 4.2.1 to see the first assertion.

(ii) For the non-increasing property of the phase diameters $D(\theta(t))$ and $D(\phi(t))$, we use Lemma 4.3.2 and standard continuation arguments. $\square$

Remark 4.3.2. *The assertion (1) is equivalent to $D(\theta, \phi)(t) \leq \max\{R,\, D^\infty - |\alpha|\}$, $\forall\, t \geq 0$. Thus, with loss of generality we may simply say $D(\theta, \phi)(t) \leq R$, since we can choose $R = D^\infty - |\alpha|$ if $D(\theta, \phi)(0) \leq D^\infty - |\alpha|$.*

4.3.2 From mixed stage to segregated stage

In this subsection, we present the emergence from the mixture of oscillators to the segregation phase consisting of two identical Kuramoto oscillator groups without overlapping.

Lemma 4.3.3. *Let $(\theta, \phi) = (\theta(t), \phi(t))$ be the smooth solution to* (4.3.12)-(4.3.13) *satisfying*

$$D(\theta, \phi)(0) \le R, \quad \textit{for some constant } R < \frac{\pi}{2} - |\alpha|.$$

Then there exists a time t_0 such that

$$\Delta(t_0) > 0.$$

Proof. We consider three cases depending on the initial phases:

$$\Delta(0) > 0, \quad \Delta(0) = 0, \quad \text{and} \quad \Delta(0) < 0.$$

• Case 1 ($\Delta(0) > 0$): In this case, we have nothing to prove.

• Case 2 ($\Delta(0) = 0$): By definition of Δ, we have

$$\begin{aligned}
\frac{d}{dt}\Delta(t) &= D(\Omega) + \frac{K}{2N}\sum_{j=1}^{N} \sin\left(\theta_j - \theta_m + \alpha\right) + \frac{K}{2N}\sum_{j=1}^{N} \sin\left(\phi_j - \theta_m + \alpha\right) \\
&\quad - \frac{K}{2N}\sum_{j=1}^{N} \sin\left(\phi_j - \phi_M + \alpha\right) - \frac{K}{2N}\sum_{j=1}^{N} \sin\left(\theta_j - \phi_M + \alpha\right) \\
&= D(\Omega) + \frac{K}{2N}\sum_{j=1}^{N} \Big[\sin(\theta_j - \theta_m + \alpha) - \sin(\theta_j - \phi_M + \alpha) \Big] \\
&\quad + \frac{K}{2N}\sum_{j=1}^{N} \Big[\sin(\phi_j - \theta_m + \alpha) - \sin(\phi_j - \phi_M + \alpha) \Big] \\
&= D(\Omega) + \frac{K}{N}\sum_{j=1}^{N} \cos\Big(\frac{\theta_j - \theta_m}{2} + \frac{\theta_j - \phi_M}{2} + \alpha \Big) \sin\Big(\frac{\phi_M - \theta_m}{2} \Big) \\
&\quad + \frac{K}{N}\sum_{j=1}^{N} \cos\Big(\frac{\phi_j - \theta_m}{2} + \frac{\phi_j - \phi_M}{2} + \alpha \Big) \sin\Big(\frac{\phi_M - \theta_m}{2} \Big) \\
&= D(\Omega) \\
&\quad - \frac{2K}{N} \sin\Big(\frac{\Delta}{2}\Big) \sum_{j=1}^{N} \cos\Big(\frac{\theta_j - \theta_m}{2} + \frac{\phi_j - \phi_M}{2} + \alpha \Big) \cos\Big(\frac{\theta_j - \phi_j}{2} \Big).
\end{aligned} \tag{4.3.17}$$

It is easy to see from $\Delta(0) = 0$ that

$$\dot{\Delta}(0) = D(\Omega) > 0.$$

Hence, there exists a small $t_0 > 0$ such that $\Delta(t_0) > 0$.

- Case 3 ($\Delta(0) < 0$): Suppose there is no such t, i.e.,

$$\Delta(t) < 0, \quad \forall\, t \leq 0. \tag{4.3.18}$$

Note that

$$\begin{gathered} -\frac{D(\phi)}{2} + \alpha \leq \frac{\theta_j - \theta_m}{2} + \frac{\phi_j - \phi_M}{2} + \alpha \leq \frac{D(\theta)}{2} + \alpha, \\ \Delta \leq \theta_j - \phi_j \leq D(\theta, \phi). \end{gathered}$$

Since $D(\theta), D(\phi) \leq D(\theta, \phi) \leq R$ and $|\Delta| \leq D(\theta, \phi)$, in (4.3.17) we obtain

$$\begin{aligned} \dot{\Delta} &\geq D(\Omega) - 2K \sin\frac{\Delta}{2} \cos\left(\frac{R}{2} + |\alpha|\right) \cos\frac{R}{2} \\ &\geq D(\Omega) - \frac{2K}{\pi} \cos\frac{R}{2} \cos\left(\frac{R}{2} + |\alpha|\right)\Delta, \end{aligned}$$

where we used $\sin x \leq \frac{2}{\pi}x,\ x \in \left(-\frac{\pi}{2}, 0\right)$. Since $R < \frac{\pi}{2} - |\alpha|$ and $|\alpha| < \frac{\pi}{2}$, we note that $\frac{R}{2} + |\alpha| < \frac{\pi}{2}$. Therefore we have

$$\begin{aligned} \Delta(t) \geq\ & \frac{\pi D(\Omega)}{2K} \sec\frac{R}{2} \sec\left(\frac{R}{2} + |\alpha|\right) \\ & + \left(\Delta(0) - \frac{\pi D(\Omega)}{2K} \sec\frac{R}{2} \sec\left(\frac{R}{2} + |\alpha|\right)\right) \\ & \times \exp\left\{ -\frac{\pi D(\Omega) t}{2K} \sec\frac{R}{2} \sec\left(\frac{R}{2} + |\alpha|\right) \right\}. \end{aligned}$$

This yields

$$\lim_{t\to\infty} \Delta(t) \geq \frac{\pi D(\Omega)}{2K} \sec\frac{R}{2} \sec\left(\frac{R}{2} + |\alpha|\right) > 0.$$

Hence there exists a sufficiently large t_0 such that

$$\Delta(t_0) > \frac{\pi D(\Omega)}{4K} \sec\frac{R}{2} \sec\left(\frac{R}{2} + |\alpha|\right) > 0.$$

This is contradictory to our assumption (4.3.18). $\square$

Proposition 4.3.2. *Suppose that the coupling strength K satisfies*

$$K > \frac{D(\Omega)}{1 - \sin|\alpha|},$$

and let $(\theta, \phi) = (\theta(t), \phi(t))$ be the smooth solution to (4.3.12)-(4.3.13) satisfying

$$D(\theta, \phi)(0) \leq R, \quad \textit{for some constant } R < \frac{\pi}{2} - |\alpha|.$$

Then, there exists a finite-time t_0 such that group difference $\Delta(t)$ is uniformly bounded below for $t \geq t_0$:

$$\Delta(t) := \theta_m(t) - \phi_M(t) \geq \min\left\{\Delta(t_0), \frac{D(\Omega)}{K}\right\}, \quad \forall\, t \geq t_0.$$

Proof. By Lemma 4.3.3, there exists a time t_0 such that $\Delta(t_0) > 0$. Define a set $\mathcal{T}$ and its supremum $T^* \in [t_0, \infty]$:

$$\mathcal{T} := \left\{T \geq t_0 \ : \ \Delta(t) > 0, \ \forall\, t \in [t_0, T)\right\}, \qquad T^* := \sup \mathcal{T}.$$

Since $\Delta(t_0) > 0$, by the continuity of $\Delta(t)$, there exists $T > 0$ such that

$$\Delta(t) > 0, \quad \forall\, t \in [t_0, T).$$

Hence $T \in \mathcal{T}$, that is, the set $\mathcal{T}$ is not empty. We now claim:

$$T^* = +\infty.$$

Suppose not, i.e., $T^* < \infty$. Then we have

$$\lim_{t \to T^*-} \Delta(t) = 0, \tag{4.3.19}$$

and

$$\Delta(t) > 0, \quad \forall\, t \in [t_0, T^*). \tag{4.3.20}$$

We now use (4.3.17) again to see that

$$\begin{aligned}
\dot{\Delta}(t) &\geq D(\Omega) - \frac{2K}{N} \sin\left(\frac{\Delta}{2}\right) \sum_{j=1}^{N} \cos\left(\frac{\Delta}{2}\right) \\
&= D(\Omega) - K \sin\Delta, \quad \text{for} \quad t \in [t_0, T^*),
\end{aligned}$$

where we used

$$\cos\Big(\frac{\theta_j-\theta_m}{2}+\frac{\phi_j-\phi_M}{2}+\alpha\Big)\le 1,\quad \cos\Big(\frac{\theta_j-\phi_j}{2}\Big)\le\cos\Big(\frac{\Delta}{2}\Big).$$

Then we have

$$\Delta(t)\ge\Delta(t_0)e^{-K(t-t_0)}+\frac{D(\Omega)}{K}\left(1-e^{-K(t-t_0)}\right),\quad \forall\, t\in[t_0,T^*). \tag{4.3.21}$$

This implies

$$0=\lim_{t\to T^*-}\Delta(t)\ge\Delta(t_0)e^{-K(T^*-t_0)}+\frac{D(\Omega)}{K}\left(1-e^{-K(T^*-t_0)}\right)>0,$$

which gives a contradiction. Hence $T^*=\infty$. Then we recall (4.3.21) to see that

$$\Delta(t)\ge C:=\min\Big\{\Delta(t_0),\frac{D(\Omega)}{K}\Big\},\quad \forall\, t\ge t_0.$$

□

Remark 4.3.3. *By Lemma 4.3.3 and Proposition 4.3.1, if* $\Delta(0)\le 0$*, after a finite-time* t_0*,* $\Delta(t_0)$ *becomes positive and the trapping condition on the phase diameter is still valid. Then we can regard* $(\theta,\phi)(t_0)$ *as a new initial configuration. In the next subsection, without loss of generality we will assume initially* $\Delta(0)>0$.

4.3.3 Formation of the two-point cluster configuration

In this subsection, we study the asymptotic formation of two-point cluster configurations from initial configurations satisfying some conditions. For this, we derive nonlinear Gronwall-type inequalities for $D(\theta)$, $D(\phi)$, and Δ.

Lemma 4.3.4. *Let* $(\theta,\phi)=(\theta(t),\phi(t))$ *be the smooth solution to* (4.3.12)–(4.3.13) *satisfying*

$$\Delta(0)>0,\quad D(\theta,\phi)(0)\le R,\quad \textit{for some constant } R<\tfrac{\pi}{2}-|\alpha|.$$

Then, $D(\theta(t)), D(\phi(t))$, and $\Delta(t)$ satisfy the following system of differential inequalities:

$$\begin{aligned}
\dot{D}(\theta(t)) &\leq -\frac{K\cos\alpha}{\pi}D(\theta(t)) + \frac{K}{2}\Big[\sin|\alpha| + \sin\big(\Delta(t) + D(\phi(t)) + |\alpha|\big)\Big]D(\theta(t))^2,\\
\dot{D}(\phi(t)) &\leq -\frac{K\cos\alpha}{\pi}D(\phi(t)) + \frac{K}{2}\Big[\sin|\alpha| + \sin\big(\Delta(t) + D(\theta(t)) + |\alpha|\big)\Big]D(\phi(t))^2,\\
\dot{\Delta}(t) &\leq D(\Omega) - \frac{2K\cos\alpha}{\pi}\Delta(t) + \frac{K}{2}\Big[\sin(D(\theta(t)) + \alpha) + \sin(D(\phi(t)) - \alpha)\Big].
\end{aligned}$$

Proof. (i) From the proof of Lemma 4.1, we already know that

$$\begin{aligned}
\frac{d}{dt}D(\theta(t)) = &-\frac{K}{N}\sin\frac{D(\theta)}{2}\sum_{j=1}^{N}\cos\Big(\frac{\theta_j - \theta_M}{2} + \frac{\theta_j - \theta_m}{2} + \alpha\Big)\\
&-\frac{K}{N}\sin\frac{D(\theta)}{2}\sum_{j=1}^{N}\cos\Big(\frac{\phi_j - \theta_M}{2} + \frac{\phi_j - \theta_m}{2} + \alpha\Big).
\end{aligned}$$

Note that

$$\begin{aligned}
-\frac{D(\theta)}{2} &\leq \frac{\theta_j - \theta_M}{2} + \frac{\theta_j - \theta_m}{2} \leq \frac{D(\theta)}{2},\\
-\Delta - D(\phi) - \frac{D(\theta)}{2} &\leq \frac{\phi_j - \theta_M}{2} + \frac{\phi_j - \theta_m}{2} \leq -\Delta - \frac{D(\theta)}{2}, \quad \frac{D(\theta)}{2} + |\alpha| < \frac{\pi}{2}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\frac{d}{dt}D(\theta(t)) &\leq -K\sin\frac{D(\theta)}{2}\cos\Big(\frac{D(\theta)}{2} + |\alpha|\Big)\\
&\quad - K\sin\frac{D(\theta)}{2}\cos\Big(\Delta + D(\phi) + \frac{D(\theta)}{2} + |\alpha|\Big)\\
&= -\frac{K}{2}\Big[\cos\alpha + \cos\big(\Delta + D(\phi) + |\alpha|\big)\Big]\sin D(\theta)\\
&\quad + K\Big[\sin|\alpha| + \sin\big(\Delta + D(\phi) + |\alpha|\big)\Big]\sin^2\frac{D(\theta)}{2}\\
&\leq -\frac{K}{\pi}\Big[\cos\alpha + \cos\big(\Delta + D(\phi) + |\alpha|\big)\Big]D(\theta)\\
&\quad + \frac{K}{2}\Big[\sin|\alpha| + \sin\big(\Delta + D(\phi) + |\alpha|\big)\Big]D(\theta)^2,
\end{aligned}$$

where we used the relations

$$\sin x \geq \frac{2}{\pi}x, \quad 2\sin^2\frac{x}{2} = 1 - \cos x \leq x^2 \quad \text{for} \quad x \in \left[0, \frac{\pi}{2}\right].$$

To obtain the upper bound, we used the inequality $\cos(\Delta + D(\phi) + |\alpha|) > 0$.

(ii) The result is derived in the same manner as that of (i) above. We use

$$\begin{aligned} -\frac{D(\phi)}{2} &\leq \frac{\phi_j - \phi_M}{2} + \frac{\phi_j - \phi_m}{2} \leq \frac{D(\phi)}{2}, \\ \Delta + \frac{D(\phi)}{2} &\leq \frac{\theta_j - \phi_M}{2} + \frac{\theta_j - \phi_m}{2} \leq \Delta + D(\theta) + \frac{D(\phi)}{2}, \end{aligned}$$

to obtain

$$\begin{aligned} \frac{d}{dt}D(\phi(t)) &\leq -K\sin\frac{D(\phi)}{2}\cos\left(\frac{D(\phi)}{2} + |\alpha|\right) \\ &\quad - K\sin\frac{D(\phi)}{2}\cos\left(\Delta + D(\theta) + \frac{D(\phi)}{2} + |\alpha|\right) \\ &= -\frac{K}{2}\Big[\cos\alpha + \cos\big(\Delta + D(\theta) + |\alpha|\big)\Big]\sin D(\phi) \\ &\quad + K\Big[\sin|\alpha| + \sin\big(\Delta + D(\theta) + |\alpha|\big)\Big]\sin^2\frac{D(\phi)}{2} \\ &\leq -\frac{K}{\pi}\Big[\cos\alpha + \cos\big(\Delta + D(\theta) + |\alpha|\big)\Big]D(\phi) \\ &\quad + \frac{K}{2}\Big[\sin|\alpha| + \sin\big(\Delta + D(\theta) + |\alpha|\big)\Big]D(\phi)^2. \end{aligned}$$

(iii) Note that

$$-\Delta - D(\phi) \leq \phi_j - \theta_m \leq -\Delta, \quad \Delta \leq \theta_j - \phi_M \leq \Delta + D(\theta). \qquad (4.3.22)$$

By the definition of $\Delta(t)$ and the relations (4.3.22), we obtain

$$
\begin{aligned}
\frac{d}{dt}\Delta(t) &= \dot{\theta}_m(t) - \dot{\phi}_M(t) \\
&= D(\Omega) + \frac{K}{2N}\sum_{j=1}^{N}\sin(\theta_j - \theta_m + \alpha) + \frac{K}{2N}\sum_{j=1}^{N}\sin(\phi_j - \theta_m + \alpha) \\
&\quad - \frac{K}{2N}\sum_{j=1}^{N}\sin(\phi_j - \phi_M + \alpha) - \frac{K}{2N}\sum_{j=1}^{N}\sin(\theta_j - \phi_M + \alpha) \\
&\le D(\Omega) + \frac{K}{2}\sin(D(\theta) + \alpha) + \frac{K}{2}\sin(-\Delta + \alpha) \\
&\quad - \frac{K}{2}\sin(-D(\phi) + \alpha) - \frac{K}{2}\sin(\Delta + \alpha),
\end{aligned}
$$

where we used the monotonicity of $\sin x,\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. We then have the result

$$
\begin{aligned}
\frac{d}{dt}\Delta(t) &\le D(\Omega) - K\cos\alpha\sin\Delta + \frac{K}{2}\sin(D(\theta) + \alpha) + \frac{K}{2}\sin(D(\phi) - \alpha) \\
&\le D(\Omega) - \frac{2K\cos\alpha}{\pi}\Delta + \frac{K}{2}\Big[\sin(D(\theta) + \alpha) + \sin(D(\phi) - \alpha)\Big].
\end{aligned}
$$

□

Theorem 4.3.1. *Let $(\theta, \phi) = (\theta(t), \phi(t))$ be the smooth solution to* (4.3.12)–(4.3.13) *satisfying*

$$\Delta(0) > 0, \quad D(\theta, \phi)(0) \le R, \quad \text{for some constant } R < \tfrac{\pi}{2} - |\alpha|.$$

Then, $D(\theta)$, $D(\phi)$, and Δ satisfy the following estimates:

$$
\begin{aligned}
&(i)\ D(\theta(t)) = \mathcal{O}(1)\exp\Big(-\frac{K\cos\alpha}{\pi}t\Big), \quad \text{as } t \to \infty. \\
&(ii)\ D(\phi(t)) = \mathcal{O}(1)\exp\Big(-\frac{K\cos\alpha}{\pi}t\Big). \\
&(iii)\ \lim_{t\to\infty}\Delta(t) \le \frac{\pi D(\Omega)}{K\cos\alpha}.
\end{aligned}
$$

Proof. We use Lemma 4.1.1 and Lemma 4.3.2 to obtain the desired result, i.e., we set

$$
\begin{aligned}
&X(t) = D(\theta(t)), \quad Y(t) = D(\phi(t)), \quad Z(t) = \Delta(t), \\
&c = \frac{K\cos\alpha}{\pi}, \quad \beta = \frac{K\sin|\alpha|}{2}, \quad \gamma = \frac{K}{2}, \quad \delta = D(\Omega).
\end{aligned}
$$

□

4.4 Synchronization estimates for Model C

In this section, we present synchronization estimates for Model C, which is

$$\begin{aligned} \dot{\theta}_i &= \frac{\mu}{N_2}\sum_{j=1}^{N_2}\sin(\phi_j - 2\theta_i), \qquad i = 1, \dots, N_1. \\ \dot{\phi}_i &= \frac{1-\mu}{N_1}\sum_{j=1}^{N_1}\sin(2\theta_j - \phi_i + \alpha), \qquad i = 1, \dots, N_2. \end{aligned} \tag{4.4.23}$$

Owing to the symmetry $(\theta, \phi, \alpha) \to (-\theta, -\phi, -\alpha)$, without loss of generality, we can assume that $\alpha \in (0, \frac{\pi}{2})$. To transform the system (4.4.23) into the familiar form, we introduce a new variable

$$\tilde{\theta}_i := 2\theta_i.$$

Then, the system (4.4.23) can be rewritten as

$$\begin{aligned} \dot{\tilde{\theta}}_i &= \frac{2\mu}{N_2}\sum_{j=1}^{N_2}\sin(\phi_j - \tilde{\theta}_i), \qquad i = 1, \dots, N_1, \\ \dot{\phi}_i &= \frac{1-\mu}{N_1}\sum_{j=1}^{N_1}\sin(\tilde{\theta}_j - \phi_i + \alpha), \qquad i = 1, \dots, N_2, \end{aligned}$$

Throughout this section, we still use θ_i to denote $\tilde{\theta}_i$; i.e., we have

$$\begin{aligned} \dot{\theta}_i &= \frac{2\mu}{N_2}\sum_{j=1}^{N_2}\sin(\phi_j - \theta_i), \qquad i = 1, \dots, N_1, \\ \dot{\phi}_i &= \frac{1-\mu}{N_1}\sum_{j=1}^{N_1}\sin(\theta_j - \phi_i + \alpha), \qquad i = 1, \dots, N_2, \end{aligned} \tag{4.4.24}$$

Note that there are no interactions between duplicate groups, so the interactions occur only for different groups. Thus, the system (4.4.24) can be regarded as the Kuramoto model on a bipartite graph, which can be applied in political science.

Example 4.4.1. *As a simple example, we consider a two-oscillator system consisting of θ_1 and ϕ_1; i.e.,*

$$\begin{aligned}\frac{d\theta_1}{dt} &= 2\mu\sin(\phi_1-\theta_1),\\ \frac{d\phi_1}{dt} &= (1-\mu)\cos\alpha\sin(\theta_1-\phi_1)+(1-\mu)\sin\alpha\cos(\theta_1-\phi_1).\end{aligned}$$

We introduce the difference $\psi := \phi_1-\theta_1$. Then, the system above becomes a single equation for ψ:

$$\begin{aligned}\frac{d\psi}{dt} &= -(2\mu+(1-\mu)\cos\alpha)\sin\psi+(1-\mu)\sin\alpha\cos\psi\\ &= -\sqrt{4\mu^2+4\mu(1-\mu)\cos\alpha+(1-\mu)^2}\sin(\psi-\Phi),\end{aligned}$$

where

$$\Phi = \arctan\left(\frac{(1-\mu)\sin\alpha}{2\mu+(1-\mu)\cos\alpha}\right) \in \left(0,\frac{\pi}{2}\right). \tag{4.4.25}$$

Note that, due to the frustration, complete phase synchronization does not occur, even though the oscillators are identical.

4.4.1 Existence of a trapping region

In this subsection, we study the existence of a positive invariant region. We define the extremal phases of each group, the partial diameters, and the total diameter in ways similar to those in (4.3.16). We set

$$\Delta(t) := \phi_m(t)-\theta_M(t).$$

Note that the arguments in Lemma 4.3.1 can be applied to Model C. Thus, the results in Lemma 4.3.1 and the statement in Remark 4.3.1 still hold. That is, there are no collisions between the oscillators in each group.

Lemma 4.4.1. *Suppose that the initial configuration satisfies*

$$D(\theta,\phi)(0) < R, \qquad \textit{for some constant } R < \tfrac{\pi}{2}-\alpha.$$

Then, we have

$$\frac{d^+}{dt}D(\theta(t))\Big|_{t=0} \le 0, \quad \frac{d^+}{dt}D(\phi(t))\Big|_{t=0} \le 0.$$

Proof. By direct calculation, we have

$$
\begin{aligned}
\frac{d^+}{dt} D(\theta(t))\Big|_{t=0} &= \frac{2\mu}{N_2} \sum_{j=1}^{N_2} \Big[\sin(\phi_j(0) - \theta_M(0)) - \sin(\phi_j(0) - \theta_m(0)) \Big] \\
&= -\frac{4\mu}{N_2} \sin \frac{D(\theta(0))}{2} \sum_{j=1}^{N_2} \cos\left(\frac{\phi_j(0) - \theta_M(0)}{2} + \frac{\phi_j(0) - \theta_m(0)}{2} \right) \\
&\le -4\mu \cos R \sin \frac{D(\theta(0))}{2} \\
&\le 0,
\end{aligned}
$$

where we used

$$
-R \le \phi_j - \theta_M \le \frac{\phi_j - \theta_M}{2} + \frac{\phi_j - \theta_m}{2} \le \phi_j - \theta_m < R.
$$

We obtain the estimate for $D(\phi(t))$ similarly:

$$
\begin{aligned}
&\frac{d^+}{dt} D(\phi(t))\Big|_{t=0} \\
&\quad = \frac{1-\mu}{N_1} \sum_{j=1}^{N_1} \Big[\sin(\theta_j(0) - \phi_M(0) + \alpha) - \sin(\theta_j(0) - \phi_m(0) + \alpha) \Big] \\
&\quad = -\frac{2(1-\mu)}{N_1} \sin \frac{D(\phi(0))}{2} \sum_{j=1}^{N_2} \cos\left(\frac{\theta_j(0) - \phi_M(0)}{2} + \frac{\theta_j(0) - \phi_m(0)}{2} + \alpha \right) \\
&\quad \le -2(1-\mu) \cos(R+\alpha) \sin \frac{D(\phi(0))}{2} \\
&\quad \le 0.
\end{aligned}
$$

□

Lemma 4.4.2. *Let* $(\theta, \phi) = (\theta(t), \phi(t))$ *be the smooth solution to* (4.4.24) *satisfying*

$$
\max\big\{ |\Delta(0)|, \Phi \big\} + D(\theta(0)) + D(\phi(0)) < R, \quad \textit{for some constant } R < \tfrac{\pi}{2} - \alpha. \tag{4.4.26}
$$

Then, we have the following:

(i) $D(\theta, \phi)(t) < R, \quad \forall\, t \ge 0;$

(ii) $D(\theta(t)) \le D(\theta(0))$, *and* $D(\phi(t)) \le D(\phi(0))$, $\quad \forall\, t \ge 0$.

Proof. We define a set $\mathcal{T}_1$ and its supremum $T_1^* \in [0, \infty]$:

$$\mathcal{T}_1 := \Big\{ T \in \mathbb{R}_+ : D(\theta,\phi)(t) < R \text{ and } \Delta(t) < \max\{\Delta(0), \Phi\} + \alpha,\ \forall\, t \in [0, T) \Big\},$$
$$T_1^* := \sup \mathcal{T}_1.$$

where Φ is the positive constant defined by (4.4.25).

It follows from (4.4.26) that we have

$$\begin{aligned} & D(\theta,\phi)(0) = \Delta(0) + D(\theta(0)) + D(\phi(0)) < R, && \text{if } \Delta(0) > 0, \\ & D(\theta,\phi)(0) \le D(\theta(0)) + D(\phi(0)) < R, && \text{if } \Delta(0) < 0, \end{aligned}$$

and

$$\Delta(0) < \max\{\Delta(0), \Phi\} + \alpha.$$

By the continuity of $D(\theta,\phi)(t)$ and $\Delta(t)$, there exists $T > 0$ such that $T \in \mathcal{T}_1$; i.e., the set $\mathcal{T}_1$ is not empty and $T_1^* > 0$. We now claim that

$$T_1^* = +\infty.$$

Proof of claim: Suppose not; i.e., $T_1^* < \infty$. Then, at $t = T_1^*$ we have

$$\text{either} \quad D(\theta,\phi)(T_1^*) = R \quad \text{or} \quad \Delta(t) = \max\{\Delta(0), \Phi\} + \alpha, \tag{4.4.27}$$

and

$$D(\theta,\phi)(t) < R \quad \text{and} \quad \Delta(t) < \max\{\Delta(0), \Phi\} + \alpha \quad \forall\, t \in [0, T_1^*). \tag{4.4.28}$$

Then, by Lemma 4.4.1, we deduce that for all $t \in [0, T_1^*)$ the partial diameters $D(\theta(t))$ and $D(\phi(t))$ are nonexpanding. Hence, we have

$$D(\theta(t)) \le D(\theta(0)) \quad \text{and} \quad D(\phi(t)) \le D(\phi(0)), \qquad \forall\, t \in [0, T_1^*). \tag{4.4.29}$$

Next, we estimate the lower and upper bounds for $\Delta(t)$ in $[0, T_1^*)$. Obviously, we have

$$\frac{d}{dt}\Delta(t) = -\frac{2\mu}{N_2}\sum_{j=1}^{N_2} \sin(\phi_j - \theta_M) + \frac{1-\mu}{N_1}\sum_{j=1}^{N_1} \sin(\theta_j - \phi_m + \alpha).$$

Note that

$$\Delta \le \phi_j - \theta_M \le \Delta + D(\phi), \quad -\Delta - D(\theta) \le \theta_j - \phi_m \le -\Delta, \quad \forall\, t \in [0, T_1^*).$$

We use (4.4.28)–(4.4.29) to obtain the upper bound

$$\begin{aligned}
\dot{\Delta} &\le -2\mu \sin\Delta - (1-\mu)\sin(\Delta - \alpha)\\
&= -(2\mu + (1-\mu)\cos\alpha)\sin\Delta + (1-\mu)\sin\alpha\cos\Delta\\
&= -A\sin\Delta + B\cos\Delta\\
&= -\sqrt{A^2+B^2}\sin(\Delta - \Phi).
\end{aligned}$$

We note that the solution of the ODE

$$\dot{x} = -\sqrt{A^2+B^2}\sin(x-\Phi), \qquad x(0) = \Delta(0) \in (-\frac{\pi}{2}, \frac{\pi}{2}),$$

monotonically approaches the equilibrium $x_* = \Phi$; therefore, by the principle of comparison we have an upper bound

$$\Delta(t) \le \max\{\Delta(0), \Phi\}, \qquad t \in [0, T_1^*). \tag{4.4.30}$$

Similarly, we use (4.4.28)–(4.4.29) to derive

$$\begin{aligned}
\dot{\Delta} &\ge -2\mu\sin(\Delta + D(\phi)) - (1-\mu)\sin\big(\Delta + D(\theta) - \alpha\big)\\
&\ge -2\mu\sin(\Delta + D(\phi(0))) - (1-\mu)\sin\big(\Delta + D(\theta(0)) - \alpha\big)\\
&= -\sin\Delta\Big(2\mu\cos(D(\phi(0))) + (1-\mu)\cos(D(\theta(0)) - \alpha)\Big)\\
&\quad - \cos\Delta\Big(2\mu\sin(D(\phi(0))) + (1-\mu)\sin(D(\theta(0)) - \alpha)\Big)\\
&= -C\sin(\Delta + \hat{\Phi}),
\end{aligned} \tag{4.4.31}$$

where $C = C(D(\theta(0)), D(\phi(0)), \alpha, \mu)$ is a positive constant and $\hat{\Phi}$ is given by

$$\hat{\Phi} = \arctan\Big[\frac{2\mu\sin(D(\phi(0))) + (1-\mu)\sin(D(\theta(0)) - \alpha)}{2\mu\cos(D(\phi(0))) + (1-\mu)\cos(D(\theta(0)) - \alpha)}\Big] \in \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big).$$

We use the elementary inequality

$$\frac{x_1}{y_1} \le \frac{x_1 + x_2}{y_1 + y_2} \le \frac{x_2}{y_2}, \quad \text{if } \frac{x_1}{y_1} \le \frac{x_2}{y_2} \text{ and } y_1,\, y_2 > 0,$$

as well as the monotonicity of $\tan x$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, to observe that $\hat{\Phi}$ is located within the interval bounded by $D(\theta(0)) - \alpha$ and $D(\phi(0))$.

Consider the following ODE:

$$\dot{y} = -C\sin(y + \hat{\Phi}), \qquad y(0) = \Delta(0).$$

Its solution $y(t)$ monotonically approaches the equilibrium $y_* = -\hat{\Phi}$; thus, the trajectory $y(t)$ is confined within the interval bounded by $-\hat{\Phi}$ and $\Delta(0)$. We now recall (4.4.31) and use the principle of comparison to derive

$$\Delta(t) \geq y(t).$$

Therefore, we have

$$\Delta(t) \geq \min\{-\hat{\Phi},\, \Delta(0)\} \geq \min\left\{\alpha - D(\theta(0)),\, -D(\phi(0)),\, \Delta(0)\right\}. \tag{4.4.32}$$

We combine (4.4.30) and (4.4.32), the estimates of the upper and lower bounds, to obtain

$$-\max\{D(\theta(0)), D(\phi(0)), \Delta(0)\} \leq \Delta(t) \leq \max\{\Delta(0), \Phi\}, \qquad \forall\, t \in [0, T_1^*). \tag{4.4.33}$$

Then, from the continuity we know that the estimate (4.4.33) is still valid for $t = T_1^*$; thus,

$$-\max\{D(\theta(0)), D(\phi(0)), \Delta(0)\} \leq \Delta(T_1^*) < \max\{\Delta(0), \Phi\} + \alpha. \tag{4.4.34}$$

We now combine (4.4.29) and (4.4.33) to estimate $D(\theta, \phi)(t)$ on $[0, T_1^*)$.

• Case 1 $(\Delta(t) \geq 0)$: In this case, we have

$$D(\theta, \phi)(t) = \Delta(t) + D(\theta(t)) + D(\phi(t)) \leq \max\{\Delta(0), \Phi\} + D(\theta(0)) + D(\phi(0)).$$

• Case 2 $(\Delta(t) < 0)$: In this case, we have

$$D(\theta, \phi)(t) \leq D(\theta(t)) + D(\phi(t)) \leq D(\theta(0)) + D(\phi(0)).$$

We use the continuity of $D(\theta, \phi)(t)$ and combine Cases 1 and 2 at $t = T_1^*$ to observe that

$$D(\theta, \phi)(T_1^*) \leq \max\{\Delta(0), \Phi\} + D(\theta(0)) + D(\phi(0)) < R. \tag{4.4.35}$$

Note that (4.4.34)–(4.4.35) contradict (4.4.27). Hence, $T_1^* = \infty$, which implies that

$$D(\theta,\phi)(t) < R, \qquad \forall\, t \geq 0.$$

Thus, we have the result(i) desired. To derive the result(ii), we use the result(i) and Lemma 4.4.1. □

4.4.2 Relaxation estimate

In this subsection, by means of the following theorem, we present an estimate of the relaxation toward the phase-locked states.

Theorem 4.4.1. *Let* $(\theta,\phi) = (\theta(t),\phi(t))$ *be the smooth solution to* (4.4.24) *with initial data satisfying*

$$\max\big\{|\Delta(0)|, \Phi\big\} + D(\theta(0)) + D(\phi(0)) < R, \quad \textit{for some constant } R < \tfrac{\pi}{2} - \alpha.$$

Then each group exhibits asymptotic phase synchronization and the two groups will asymptotically become separated. More precisely, $D(\theta(t))$*,* $D(\phi(t))$ *and* $\Delta(t)$ *satisfy the following estimates:*

$$(i)\ D(\theta(t)) \leq D(\theta(0)) \exp\Big\{-\Big(\frac{4\mu\cos R}{\pi}\Big)t\Big\}, \quad t \geq 0.$$
$$(ii)\ D(\phi(t)) \leq D(\phi(0)) \exp\Big\{-\Big(\frac{2(1-\mu)\cos(R+\alpha)}{\pi}\Big)t\Big\}.$$
$$(iii)\ \Delta(t) \to \Phi, \quad \textit{as } t \to \infty.$$

Here, Φ *is given by* (4.4.25).

Proof. First, by Lemma 4.4.2, we have

$$D(\theta,\phi)(t) < R, \quad D(\theta(t)) \leq D(\theta(0)), \quad D(\phi(t)) \leq D(\phi(0)), \quad \forall\, t \geq 0.$$

(i) In the proof of Lemma 4.4.2, we used

$$\dot{D}(\theta) \leq -4\mu\cos R \sin\frac{D(\theta)}{2} \leq -\frac{4\mu}{\pi}\cos R D(\theta),$$

where we used $\sin x \geq \frac{2}{\pi}x$, $x \in \left(0, \frac{\pi}{2}\right)$. Then, the standard Gronwall estimate yields the desired result.

(ii) Similarly, we have

$$\dot{D}(\phi) \leq -2(1-\mu)\cos(R+\alpha)\sin\frac{D(\phi)}{2} \leq -\frac{2(1-\mu)}{\pi}\cos(R+\alpha)D(\phi).$$

(iii) This assertion follows from the phase synchronization of each group and the analysis in Example 4.4.1. □

4.5 Numerical examples

In this section, we present several numerical examples and compare the simulation results with the analytical results in this chapter.

4.5.1 Model A

For the simulation, we used the fourth-order Runge-Kutta method and employed the parameters

$$N = 100, \quad K = 20.$$

The natural frequencies Ω_i were randomly chosen from the interval $(-1, 1)$ so that

$$D(\Omega) = 1.9867, \quad \Omega_c = 0.$$

The initial configuration θ_i^0 was randomly chosen from the interval $(0, \pi - 2 \times 0.7)$ satisfying

$$D(\theta^0) = 1.7130.$$

For a fixed initial configuration, we will compare the case $\alpha = 0$ with other cases (i.e., $\alpha = 0.3, 0.7$) to observe the effects of the frustrations. Note that $D(\theta^0)$ satisfies our assumptions in Theorem 4.2.1. The initial and limiting phase configurations are displayed in Figure 4.1(a) and (b).

Although we do not have the optimal decay rate for $D(\omega(t))$ in Theorem 4.2.1, the upper bound estimate for $D(\omega(t))$ suggests that the decay rate is proportional to $\cos(D^\infty + \epsilon) \approx \cos D^\infty$. Recall that the reference angle

D^∞ is defined by (4.2.8). We observe that as α increases, $\cos D^\infty$ decrease, i.e., the decay rate may decrease, which can be seen in Figure 4.1(c).

4.5.2 Model B

Figure 4.2 shows the effects of α on the dynamic behavior of θ and ϕ. We consider effects such as preservation of segregated states and relaxation to the two-point clusters. For this simulation, we employed

$$N = 40, \quad K = 10.$$

The initial configurations of θ and ϕ were randomly chosen from $(0, \pi - 2 \times 0.8)$, and Ω was chosen from $(0, 1)$ so that $\Omega_1 > \Omega_2$. From the numerical simulations, we observed that the segregated states are robust, as expected from Proposition 4.3.1. We can see the effect of α on the asymptotic phase of θ and ϕ in Figure 4.2(c).

Figure 4.3 shows the effect of K for fixed N and α. In this simulation, we used

$$N = 40, \quad \alpha = 0.8,$$

and the initial configurations θ^0, ϕ^0 and Ω was the same as for Figure 4.2.

We see in Figure 4.3 that the value of Δ is affected by that of K. Based on these two simulations, we conclude that the behavior of θ and ϕ is asymptotically like that for the two-oscillators system.

4.5.3 Model C

To produce Figure 4.4, we used

$$N_1 = 30, \quad N_2 = 70, \quad \alpha = 0.3.$$

The initial configuration of θ^0 and ϕ^0 was chosen randomly from $[0, \pi]$ and $[0, \pi - 2 \times 0.3]$, respectively. We compare the case $\mu = 0.2$ with the case $\mu = 0.8$ with the same initial configuration. In Figure 4.4(b), we observe that θ and ϕ are synchronized to two distinct point clusters. In Figures 4.4(c) and 4.4(d), we observe that if μ is small, then the sub-configuration of ϕ becomes synchronized faster than the sub-configuration of θ, whereas if μ is large, then θ synchronizes faster than ϕ. Note that the group ϕ is under the effect of frustration. Hence, when μ is large, the group ϕ becomes synchronized much slower than the group θ, because of the low coupling strength and the effect of the frustration on the dynamics of ϕ. This is why the difference in relaxation rates in Figure 4.4(d) is much greater than the difference in relaxation rates in Figure 4.4(c). Recall that the group distance Δ depends on the frustration α and the coupling strength μ. Thus, in the next two sets of simulations, we investigated the dynamic behavior of Δ with respect to α and μ separately.

Figure 4.5 shows the relation between α and the group distance Δ. For this simulation, we used

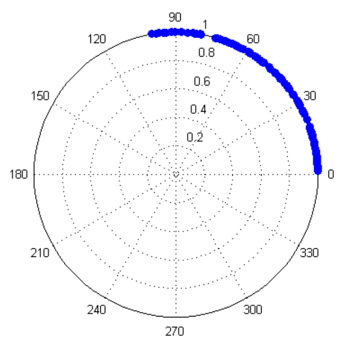
$$N_1 = 70, \quad N_2 = 30, \quad \mu = 0.6,$$

and the initial configurations θ^0 and ϕ^0 were chosen randomly from $[0, \pi - 2 \times 0.9]$. Note that if $\alpha = 0$, then θ and ϕ are completely synchronized, even though there is no coupling strength within the group. The same phenomenon exists in the Kuramoto model for identical oscillators. We see that Δ is monotonically increasing as α increases.

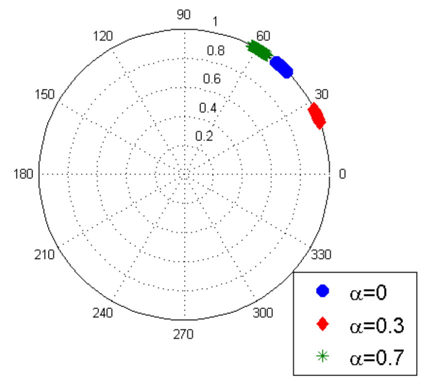
Figure 4.6 shows the relation between μ and Δ. For this simulation, we used

$$N_1 = 60, \quad N_2 = 40, \quad \alpha = 0.5,$$

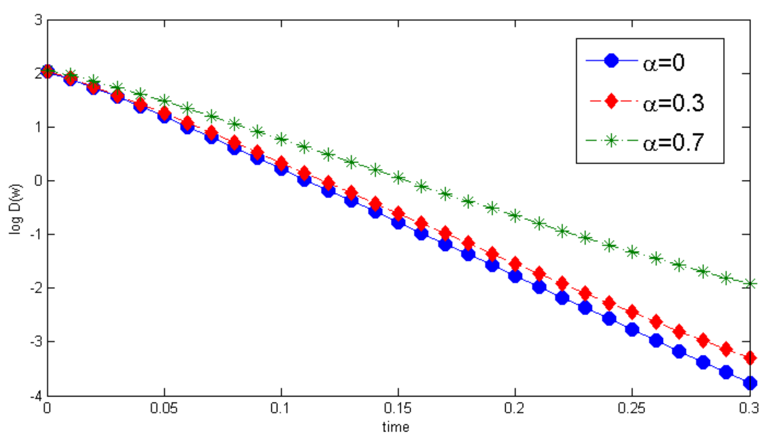
and the initial configurations θ^0 and ϕ^0 were chosen ramdomly from$[0, \pi - 2 \times 0.5]$. From the numerical simulations, it is easy to see that the group distance Δ decreases as μ increases.



(a) Initial phase configuration

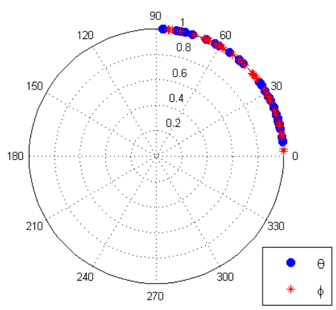


(b) Final phase configuration

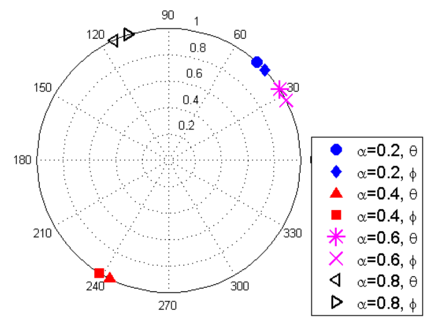


(c) $\log D(\omega)$ for $\alpha = 0, 0.3, 0.7$

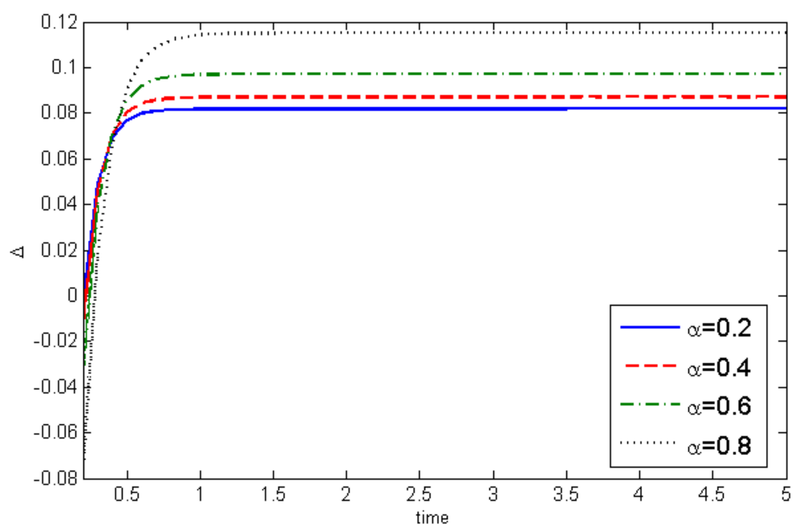
Figure 4.1: Model A: The initial and limiting phase configurations are displayed in (a) and (b). We observe that as α increases, $\cos D^{\infty}$ decrease, i.e., the decay rate may decrease, which can be seen in (c).



(a) Initial phase configuration



(b) Final phase configuration



(c) Dynamics of Δ for $\alpha = 0.2, 0.4, 0.6, 0.8$

Figure 4.2: Model B: Figure shows the effects of α on the dynamic behavior of θ and ϕ. The initial configurations of θ and ϕ were randomly chosen from $(0, \pi - 2 \times 0.8)$, and Ω was chosen from $(0, 1)$ so that $\Omega_1 > \Omega_2$. From the numerical simulations, we observed that the segregated states are robust. We can see the effect of α on the asymptotic phase of θ and ϕ in (c)

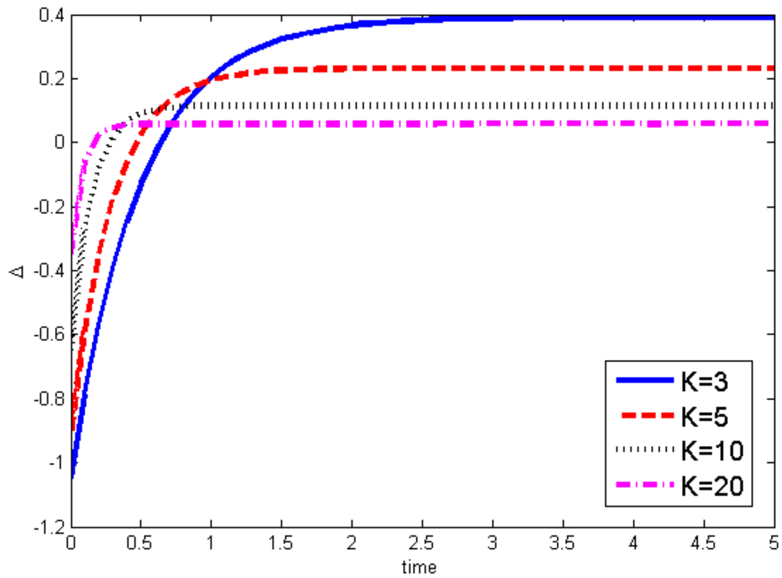
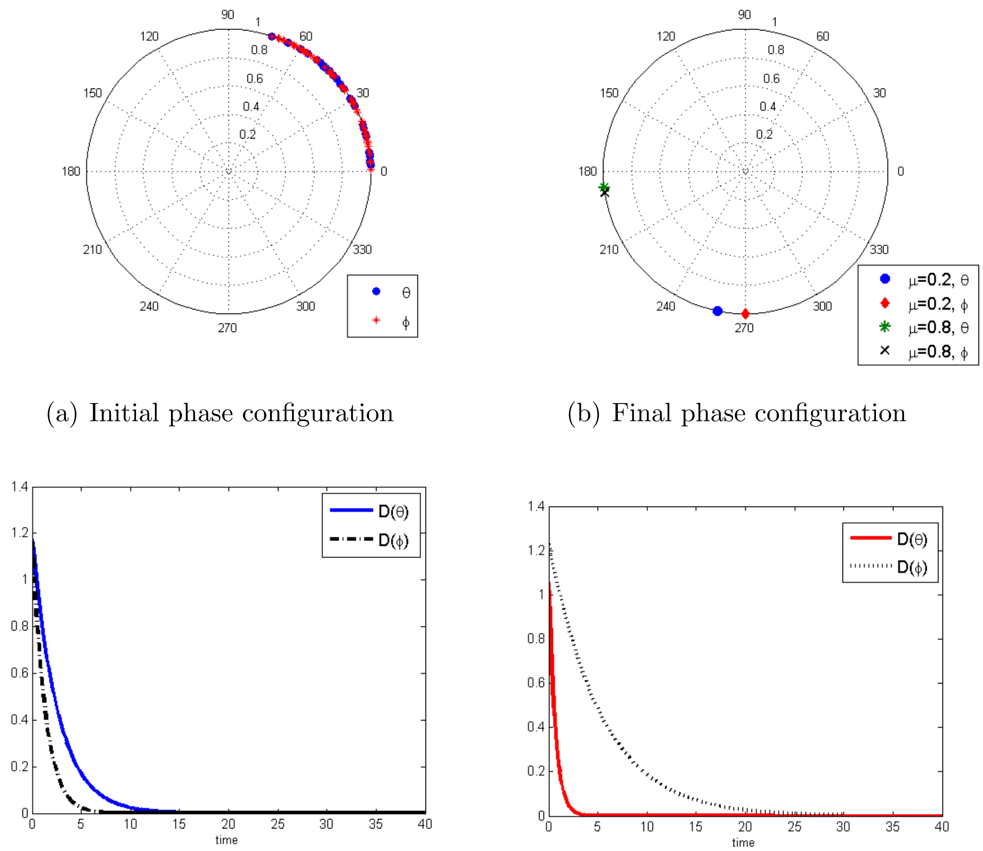


Figure 4.3: Model B: dynamics of Δ for $K = 3, 5, 10, 20$. The initial configurations θ^0, ϕ^0 and Ω were the same as for Figure 4.2. We can see that the value of Δ is affected by that of K. Based on these two simulations, we conclude that the behavior of θ and ϕ is asymptotically like that for the two-oscillators system.



(a) Initial phase configuration

(b) Final phase configuration

(c) $\mu = 0.2$

(d) $\mu = 0.8$

Figure 4.4: Model C: dynamics of phases for $\mu = 0.2, 0.8$. The initial configuration of θ^0 and ϕ^0 was chosen randomly from $[0, \pi]$ and $[0, \pi - 2 \times 0.3]$, respectively. We compare the case $\mu = 0.2$ with the case $\mu = 0.8$ with the same initial configuration. In Figure 4(b), we observe that θ and ϕ are synchronized to two distinct point clusters. In (c) and (d), we observe that if μ is small, then the sub-configuration of ϕ becomes synchronized faster than the sub-configuration of θ, whereas if μ is large, then θ synchronizes faster than ϕ. Note that the group ϕ is under the effect of frustration.

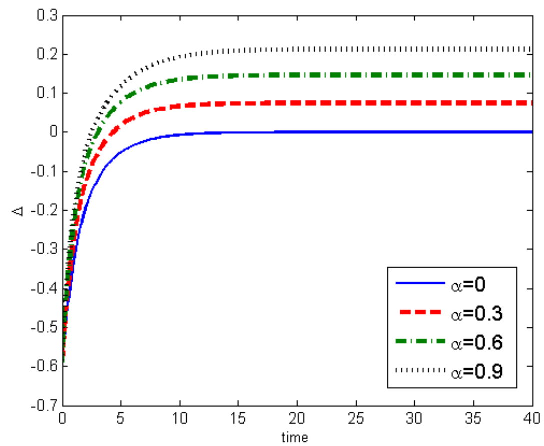


Figure 4.5: Model C: dynamics of Δ for $\alpha = 0, 0.3, 0.6, 0.9$. The initial configurations θ^0 and ϕ^0 were chosen randomly from $[0, \pi - 2 \times 0.9]$. Note that if $\alpha = 0$, then θ and ϕ are completely synchronized, even though there is no coupling strength within the group. We see that Δ is monotonically increasing as α increases.

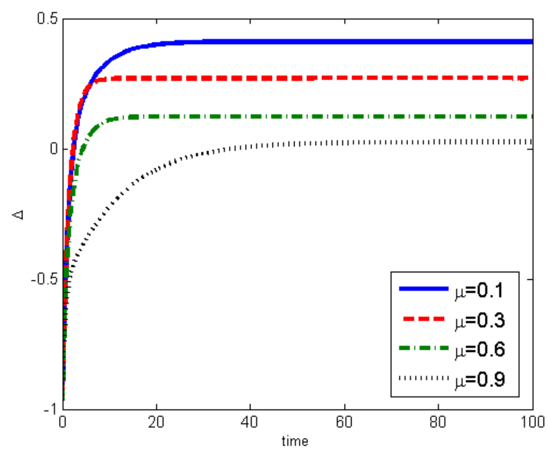


Figure 4.6: Model C: dynamics of Δ for $\mu = 0.1, 0.3, 0.6, 0.9$. The initial configurations θ^0 and ϕ^0 were chosen ramdomly from$[0, \pi - 2 \times 0.5]$. We observe that the group distance Δ decreases as μ increases.

Chapter 5

Kuramoto model with inertia and frustration

In this chapter, we will study the intricate interplay between inertial effect and interaction frustration in as ensemble of Kuramoto oscillators. This chapter is based on joint works in [35].

The inertial effect on the Kuramoto model was first conceived by Ermentrout [31] to explain the slow synchronization of certain biological systems, e.g. fireflies of the Pteroptyx malaccae, but the inertia Kuramoto model appears in the modeling of superconducting Josephson junction arrays [70, 71]. Mathematically, incorporating the inertial effect to the Kuramoto model is simply to add the second order term $m\ddot{\theta}_i$, and the inertia causes rich phenomena from the dynamical view point [2, 19, 41, 42, 67].

5.1 The Kuramoto model with inertia

In this section, we briefly review the second-order Kuramoto model.

Let $\theta_i = \theta_i(t) \in \mathbb{R}$ be the phase of i-th oscillator with a natural frequency Ω_i. Then, the Kuramoto model with inertia effect can be written as

$$m\ddot{\theta}_i + \dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \quad t > 0,\ i = 1, 2, \dots, N, \tag{5.1.1}$$

We rewrite the system (5.1.1) as a system of first-order ODEs: for $i = 1, \cdots, N$,

$$\begin{aligned} &\dot{\theta}_i = \omega_i, \quad t > 0, \\ &\dot{\omega}_i = \frac{1}{m}\Big[-\omega_i + \Omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i)\Big]. \end{aligned} \tag{5.1.2}$$

We introduce macro-variables (center-of-mass frame) and micro-variables (fluctuations around macro-variables) as follows:

$$\begin{aligned} &\Omega_c := \frac{1}{N}\sum_{i=1}^{N}\Omega_i, \quad \theta_c := \frac{1}{N}\sum_{i=1}^{N}\theta_i, \quad \omega_c := \frac{1}{N}\sum_{i=1}^{N}\omega_i, \\ &\hat{\Omega}_i := \Omega_i - \Omega_c, \qquad \hat{\theta}_i := \theta_i - \theta_c, \qquad \hat{\omega}_i := \omega_i - \omega_c. \end{aligned}$$

Then, due to $\displaystyle\sum_{i,j=1}^{N} \sin(\theta_j - \theta_i) = 0$, we can easily see that the macro-variables θ_c and ω_c satisfy

$$\frac{d\theta_c}{dt} = \omega_c, \qquad m\frac{d\omega_c}{dt} = -\omega_c + \Omega_c. \tag{5.1.3}$$

By direct calculation, θ_c and ω_c are given by the following explicit analytic forms:

$$\begin{aligned} &\theta_c(t) = \theta_c(0) + t\Omega_c + m(\omega_c(0) - \Omega_c)(1 - e^{-\frac{t}{m}}), \\ &\omega_c(t) = \Omega_c + (\omega_c(0) - \Omega_c)e^{-\frac{t}{m}}. \end{aligned} \tag{5.1.4}$$

Note that ω_c is uniformly bounded, whereas θ_c tends to the traveling profile with a constant speed Ω_c :

$$\lim_{t\to\infty} \Big|\theta_c(t) - \big(\theta_c(0) + m(\omega_c(0) - \Omega_c) + t\Omega_c\big)\Big| = 0, \qquad \lim_{t\to\infty} |\omega_c(t) - \Omega_c| = 0. \tag{5.1.5}$$

On the other hand, the micro-variable $\hat{\theta}_i$ satisfies the same dynamics as the original one (5.1.1):

$$m\ddot{\hat{\theta}}_i + \dot{\hat{\theta}}_i = \hat{\Omega}_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\hat{\theta}_j - \hat{\theta}_i), \qquad i = 1, 2, \dots, N, \tag{5.1.6}$$

and the total sums of micro variables are zero:

$$\sum_{i=1}^{N} \hat{\Omega}_i = 0, \qquad \sum_{i=1}^{N} \hat{\theta}_i(t) = 0, \qquad \sum_{i=1}^{N} \hat{\omega}_i(t) = 0 \qquad t \geq 0.$$

The above macro-micro decomposition for the second-order Kuramoto model without frustration are crucially used in its rigorous study. However, for the Kuramoto model with frustration and inertia, in general (5.1.3) is not true; thus we cannot derive an macro-micro variables systems (5.1.3)-(5.1.6) in which the macro-variable system can be easily solved.

Before we close this section, we present basic a priori estimates related to a second-order differential inequality, which will be crucially used in this chapter. Consider a nonnegative function satisfying the following second-order differential inequality:

$$\begin{aligned} &a\ddot{y} + b\dot{y} + cy + d \leq 0, \quad t > 0, \\ &y(0) = y_0, \quad \dot{y}(0) = y_1, \end{aligned} \tag{5.1.7}$$

where a, b, c and d are constants with $a > 0$ and $c \neq 0$. We set ν_1 and ν_2 as follows:

$$\nu_1 := \frac{b + \sqrt{b^2 - 4ac}}{2a}, \quad \nu_2 := \frac{b - \sqrt{b^2 - 4ac}}{2a}.$$

Lemma 5.1.1. [19] *Let $y = y(t)$ be a nonnegative C^2-function satisfying the differential inequality* (5.1.7).

(i) If $b^2 - 4ac > 0$, then

$$y(t) \leq \Big(y_0 + \frac{d}{c}\Big)e^{-\nu_1 t} + a\frac{e^{-\nu_2 t} - e^{-\nu_1 t}}{\sqrt{b^2 - 4ac}}\Big(y_1 + \nu_1 y_0 + \frac{2d}{b - \sqrt{b^2 - 4ac}}\Big) - \frac{d}{c};$$

(ii) If $b^2 - 4ac \leq 0$, then

$$y(t) \leq e^{-\frac{b}{2a}t}\Big[y_0 + \frac{4ad}{b^2} + \Big(\frac{b}{2a}y_0 + y_1 + \frac{2d}{b}\Big)t\Big] - \frac{4ad}{b^2}.$$

Remark 5.1.1. *If $y(t)$ satisfies*

$$a\ddot{y} + b\dot{y} + cy + d \geq 0, \qquad b^2 - 4ac > 0,$$

then we have

$$y(t) \geq \Big(y_0 + \frac{d}{c}\Big)e^{-\nu_1 t} + a\frac{e^{-\nu_2 t} - e^{-\nu_1 t}}{\sqrt{b^2-4ac}}\Big(y_1 + \nu_1 y_0 + \frac{2d}{b - \sqrt{b^2-4ac}}\Big) - \frac{d}{c}.$$

5.2 Synchronization estimate: identical oscillators

In this section, we present complete phase synchronization estimates to the Kuramoto model with a uniform inertia and interaction frustration for identical oscillators: for $i = 1, \cdots, N$,

$$m\ddot{\theta}_i + \dot{\theta}_i = \Omega + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i + \alpha), \quad t \geq 0, \quad |\alpha| < \frac{\pi}{2}. \tag{5.2.8}$$

5.2.1 Notations

In this subsection, we present several notations to be used later. Note that the non-differentiable points for the extremal functions are the subset of collision times in phase and frequency. Hence, those points are countable and isolated. Therefore, extremal functions are piecewise C^2 with respect to time t, and Lipschitz continuous, and the asymptotic complete phase-frequency synchronization in Defintion 2.1.1 can be rephrased as the zero convergence of phase-frequency diameters as $t \to \infty$:

$$\lim_{t\to\infty} D(\theta(t)) = 0, \qquad \lim_{t\to\infty} D(\omega(t)) = 0.$$

For $m > 0$ and configuration $(\theta(t), \omega(t))$ we set

$$\begin{aligned} &\mathcal{C}_l(t) := \max\Big\{D(\theta(t)) + |\alpha|,\, D(\theta(t)) + |\alpha| + lm\dot{D}(\theta(t))\Big\}, \\ &R^l(t) := \frac{\sin \mathcal{C}_l(t)}{\mathcal{C}_l(t)}, \qquad t \geq 0,\ \ l = 1, 2, \ldots. \end{aligned} \tag{5.2.9}$$

Here, $\dot{D}(\theta(t))$ is given by

$$\dot{D}(\theta(t)) := \dot{\theta}_M(t) - \dot{\theta}_m(t) \quad \Big(\leq D(\omega(t)) \Big).$$

Note that (5.2.9) implies

$$D(\theta(t)) \leq \mathcal{C}_l(t) - |\alpha|.$$

5.2.2 Complete synchronization

For the complete phase synchronization, we employ a kind of boot strapping argument. First we show that the phase diameter is uniformly bounded, and then we improve our rough estimate to get the zero convergence of phase diameter.

Lemma 5.2.1. *Suppose that* m, K *and initial configuration* (θ_0, ω_0) *satisfy*

$$m > 0, \quad K > 0, \quad 0 < \mathcal{C}_1(0) < \pi - |\alpha|.$$

Then for any solution to (5.1.1)*, we have*

$$D(\theta(t)) \leq \mathcal{C}_1(0) - |\alpha|, \quad t \geq 0.$$

Proof. We define

$$\Gamma := \{T \geq 0 \ : D(\theta(t)) < \pi - 2|\alpha|, \quad \forall\, t \in [0, T)\}, \qquad T_* := \sup \Gamma.$$

Since $\mathcal{C}_1(0) < \pi - |\alpha|$, we have $D(\theta(0)) < \pi - 2|\alpha|$. Due to the continuity of $D(\theta(t))$, there exists $\varepsilon > 0$ such that

$$D(\theta(t)) < \pi - 2|\alpha|, \quad \forall\, t \in [0, \varepsilon).$$

This means that the set Γ is nonempty and T_* is well-defined. We now claim:

$$T_* = +\infty.$$

Suppose not, i.e., $T_* < +\infty$. Then, by the continuity of $D(\theta(t))$, we have

$$D(\theta(T_*)) = \pi - 2|\alpha|. \tag{5.2.10}$$

On the other hand, it follows from the system (5.2.8) that we have

$$\begin{aligned} & m\ddot{D}(\theta(t)) + \dot{D}(\theta(t)) \\ & \quad = -\frac{2K}{N}\sin\frac{D(\theta(t))}{2}\sum_{j=1}^{N}\cos\left(\frac{\theta_j(t)-\theta_M(t)}{2}+\frac{\theta_j(t)-\theta_m(t)}{2}+\alpha\right) \end{aligned} \tag{5.2.11}$$

Note that for all $t \in [0, T_*)$,

$$\left|\frac{\theta_j(t)-\theta_M(t)}{2}+\frac{\theta_j(t)-\theta_m(t)}{2}+\alpha\right| \le \frac{D(\theta(t))}{2}+|\alpha| < \frac{\pi}{2}, \quad \frac{D(\theta(t))}{2} \in \left[0, \frac{\pi}{2}\right).$$

Thus, (5.2.11) implies

$$m\ddot{D}(\theta(t)) + \dot{D}(\theta(t)) \le 0. \tag{5.2.12}$$

Then we have

$$D(\theta(t)) \le D(\theta(0)) + m(1-e^{-\frac{t}{m}})\dot{D}(\theta(0)), \quad \forall\, t \in [0, T_*). \tag{5.2.13}$$

By the continuity of $D(\theta(t))$, we can see that

$$\begin{aligned} D(\theta(T_*)) &= \lim_{t\to T_*-} D(\theta(t)) \\ &\le D(\theta(0)) + m(1-e^{-\frac{T_*}{m}})\dot{D}(\theta(0)) \le \mathcal{C}_1(0) - |\alpha| < \pi - 2|\alpha|. \end{aligned}$$

This is a contradiction to (5.2.10). Thus, $T_* = +\infty$. We now turn back to (5.2.13) with $T_* = +\infty$ to find the desired estimate. □

Theorem 5.2.1. *Suppose that m, K and initial configuration (θ_0, ω_0) satisfy*

$$m > 0, \quad K > 0, \quad 0 < \mathcal{C}_1(0) < \pi - |\alpha|.$$

Then, we have complete phase-frequency synchronization asymptotically.

Proof. (i) (*Estimate on $D(\theta(t))$*): First of all, we recall (5.2.11) in the proof of Proposition 5.2.1

$$\begin{aligned} & m\ddot{D}(\theta(t)) + \dot{D}(\theta(t)) \\ & \quad = -\frac{2K}{N}\sin\frac{D(\theta(t))}{2}\sum_{j=1}^{N}\cos\left(\frac{\theta_j(t)-\theta_M(t)}{2}+\frac{\theta_j(t)-\theta_m(t)}{2}+\alpha\right). \end{aligned}$$

From Proposition 5.2.1, for all $t \geq 0$, we have

$$\left|\frac{\theta_j(t)-\theta_M(t)}{2}+\frac{\theta_j(t)-\theta_m(t)}{2}+\alpha\right| \leq \frac{D(\theta(t))}{2}+|\alpha| \leq \frac{\mathcal{C}_1(0)+|\alpha|}{2},$$

$$\text{and} \qquad \frac{D(\theta(t))}{2} \in \left[0, \frac{\pi}{2}\right).$$

Hence, the differential inequality (5.2.11) implies

$$\begin{aligned} m\ddot{D}(\theta(t))+\dot{D}(\theta(t)) &\leq -2K\cos\left(\frac{\mathcal{C}_1(0)+|\alpha|}{2}\right)\sin\frac{D(\theta(t))}{2} \\ &\leq -\frac{2K}{\pi}\cos\left(\frac{\mathcal{C}_1(0)+|\alpha|}{2}\right)D(\theta(t)), \end{aligned}$$

i.e.,

$$m\ddot{D}(\theta(t))+\dot{D}(\theta(t))+\frac{2K}{\pi}\cos\left(\frac{\mathcal{C}_1(0)+|\alpha|}{2}\right)D(\theta(t)) \leq 0, \quad a.e.\ t. \tag{5.2.14}$$

Here we used the inequality

$$\sin x \geq \frac{2}{\pi}x, \quad x \in \left[0, \frac{\pi}{2}\right).$$

We now consider two cases.

- Case 1: $mK < \frac{\pi}{8\cos\left(\frac{\mathcal{C}_1(0)+|\alpha|}{2}\right)}$. In this case, we have

$$1-4m\frac{2K}{\pi}\cos\left(\frac{\mathcal{C}_1(0)+|\alpha|}{2}\right) > 0. \tag{5.2.15}$$

Hence, we can use (5.2.14) and Lemma 5.1.1 (i) with

$$a=m, \quad b=1, \quad c=\frac{2K}{\pi}\cos\left(\frac{\mathcal{C}_1(0)+|\alpha|}{2}\right), \quad d=0,$$

to obtain an upper bound estimate:

$$D(\theta(t)) \leq Ce^{-\lambda_1 t}, \qquad \lambda_1=\frac{1-\sqrt{1-\frac{8mK}{\pi}\cos\left(\frac{\mathcal{C}_1(0)+|\alpha|}{2}\right)}}{2m}, \tag{5.2.16}$$

where $C > 0$ is some constant.

- Case 2: $mK \geq \frac{\pi}{8\cos\left(\frac{\mathcal{C}_1(0)+|\alpha|}{2}\right)}$. In this case, we have

$$1 - 4m\frac{2K}{\pi}\cos\left(\frac{\mathcal{C}_1(0)+|\alpha|}{2}\right) \leq 0.$$

Then we can use (5.2.14) and Lemma 5.1.1 (ii) with

$$a = m, \quad b = 1, \quad c = \frac{2K}{\pi}\cos\left(\frac{\mathcal{C}_1(0)+|\alpha|}{2}\right), \quad d = 0,$$

to obtain an upper bound estimate:

$$D(\theta(t)) \leq e^{-\frac{t}{2m}}\Big[D(\theta(0)) + \Big(\frac{D(\theta(0))}{2m} + \dot{D}(\theta(0))\Big)t\Big] = |\mathcal{O}(1)|e^{-(\frac{1}{2m}-\eta)t}. \tag{5.2.17}$$

Here η is any constant with $0 < \eta \ll \frac{1}{2m}$. We combine the relations (5.2.16) and (5.2.17) to see that $D(\theta(t))$ exponentially decays to 0 for all $m, K \geq 0$.

(ii) [*Estimate on* $D(\omega(t))$] We obtain the following equation from (5.2.8):

$$m\ddot{\omega}_i + \dot{\omega}_i = \frac{K}{N}\sum_{j=1}^{N}\cos(\theta_j - \theta_i + \alpha)(\omega_j - \omega_i).$$

By the definition of ω_M and ω_m, we have

$$m\ddot{\omega}_M + \dot{\omega}_M \leq K\cos\mathcal{C}_1(0)\sum_{j=1}^{N}(\omega_j - \omega_M),$$

$$m\ddot{\omega}_m + \dot{\omega}_m \geq K\cos\mathcal{C}_1(0)\sum_{j=1}^{N}(\omega_j - \omega_m),$$

where we used

$$\cos(\theta_j - \theta_M + \alpha),\ \cos(\theta_j - \theta_m + \alpha) \geq \cos(D(\theta) + |\alpha|) \geq \cos\mathcal{C}_1(0), \quad \forall\, t.$$

This yields

$$m\ddot{D}(\omega(t)) + \dot{D}(\omega(t)) + K\cos\mathcal{C}_1(0)D(\omega(t)) \leq 0, \quad a.e.\ t. \tag{5.2.18}$$

We again consider two cases here.

- Case 1: $mK < \frac{1}{4\cos \mathcal{C}_1(0)}$. In this case, we have

$$1 - 4mK\cos \mathcal{C}_1 > 0.$$

Hence, we can use (5.2.18) and Lemma 5.1.1 (i) with

$$a = m, \quad b = 1, \quad c = K\cos \mathcal{C}_1(0), \quad d = 0,$$

to obtain an upper bound estimate:

$$D(\omega(t)) \le Ce^{-\lambda_2 t}, \qquad \lambda_2 = \frac{1 - \sqrt{1 - 4mK\cos \mathcal{C}_1(0)}}{2m},$$

where $C > 0$ is some constant.

- Case 2: $mK \ge \frac{1}{4\cos \mathcal{C}_1(0)}$. In this case, we have

$$1 - 4mK\cos \mathcal{C}_1(0) \le 0.$$

Then we use (5.2.18) and Lemma 5.1.1 (ii) with

$$a = m, \quad b = 1, \quad c = K\cos \mathcal{C}_1(0), \quad d = 0,$$

to obtain an upper bound estimate:

$$D(\omega(t)) \le e^{-\frac{t}{2m}}\Big[D(\omega(0)) + \Big(\frac{D(\omega(0))}{2m} + \dot{D}(\omega(0))\Big)t\Big] = |\mathcal{O}(1)|e^{-(\frac{1}{2m} - \eta)t}.$$

Here η is any constant with $0 < \eta \ll \frac{1}{2m}$. $\square$

We next derive a refined estimate for the asymptotic decay exponent for small inertia regime.

Theorem 5.2.2. *(Nearly-optimal decay exponent) Suppose that m, K and initial configuration satisfy*

$$m > 0, \quad mK < \frac{1}{4\cos\alpha}, \quad 0 < \mathcal{C}_1(0) < \pi - |\alpha|.$$

Then for any sufficiently small $\varepsilon > 0$, there exists time $t_* > 0$ such that

$$C_1 e^{-\lambda_0 (t - t_*)} \le D(\theta(t)) \le C_2 e^{-\lambda_0' (t - t_*)}, \quad \forall\, t > t_*,$$

where $\lambda_0 = \frac{1 - \sqrt{1 - 4mK\cos(|\alpha| - \varepsilon)}}{2m}$, $\quad \lambda_0' = \frac{1 - \sqrt{1 - 4mK\cos(|\alpha| + \varepsilon)}}{2m}$.

Proof. • *(Lower bound estimate)*: First of all, due to $mK < \frac{1}{4\cos\alpha}$, we can find a positive $\varepsilon_0 \ll 1$, such that

$$1 - 4mK\cos(|\alpha| - \varepsilon) > 0, \quad \text{for all } \varepsilon < \varepsilon_0. \tag{5.2.19}$$

We now recall the relation (5.2.11), i.e.,

$$\begin{aligned} &m\ddot{D}(\theta(t)) + \dot{D}(\theta(t)) \\ &\quad = -\frac{2K}{N}\sin\frac{D(\theta(t))}{2}\sum_{j=1}^{N}\cos\left(\frac{\theta_j(t) - \theta_M(t)}{2} + \frac{\theta_j(t) - \theta_m(t)}{2} + \alpha\right). \end{aligned}$$

By Theorem 5.2.1, we see that $D(\theta(t))$ decays to 0 exponentially fast for any choice of positive parameters m and K. Therefore, for all $\varepsilon < \varepsilon_0$, there exists some $t_* > 0$ such that

$$D(\theta(t)) < \varepsilon, \quad \forall\, t \geq t_*.$$

This implies that

$$\cos\left(\frac{\theta_j(t) - \theta_M(t)}{2} + \frac{\theta_j(t) - \theta_m(t)}{2} + \alpha\right) < \cos(|\alpha| - \varepsilon), \quad \forall\, t \geq t_*.$$

Hence, we can use the inequality (5.2.11) and $\sin\frac{D(\theta(t))}{2} \leq \frac{D(\theta(t))}{2}$ to find that

$$m\ddot{D}(\theta(t)) + \dot{D}(\theta(t)) + KD(\theta(t))\cos(|\alpha| - \varepsilon) \geq 0, \quad \textit{a.e. } t \geq t_*.$$

Thanks to (5.2.19), we can apply Lemma 5.1.1 to derive a lower bound estimate:

$$D(\theta(t)) \geq C_1 e^{-\lambda_0 t}, \qquad \forall\, t \geq t_*.$$

• *(Upper bound estimate)*: Obviously, for the same ε, we have

$$1 - 4mK\cos(|\alpha| + \varepsilon) > 0.$$

Note that for the same t_*, we have

$$\cos\left(\frac{\theta_j(t) - \theta_M(t)}{2} + \frac{\theta_j(t) - \theta_m(t)}{2} + \alpha\right) > \cos(|\alpha| + \varepsilon), \quad \forall\, t \geq t_*.$$

Note that $\sin x = x + \mathcal{O}(x^3)$ for $x \ll 1$. Then it follows from (5.2.11) that

$$m\ddot{D}(\theta(t)) + \dot{D}(\theta(t)) + KD(\theta(t))\cos(|\alpha| + \varepsilon) \leq 0, \quad \textit{a.e. } t \geq t_*.$$

Then we apply Lemma 5.1.1 (i) to derive the desired upper bound estimate:

$$D(\theta(t)) \leq C_2 e^{-\lambda_0' t}, \qquad \forall\, t \geq t_*.$$

□

Remark 5.2.1. *Theorem 5.2.2 gives a nearly-optimal decay rate estimate for $D(\theta(t))$ under the small inertia regime. Since ε can be sufficiently small, we have*

$$\lambda_0 \approx \lambda_0' \approx \frac{1 - \sqrt{1 - 4mK\cos\alpha}}{2m}.$$

5.3 Synchronization estimate: non-identical oscillators

In this section, we consider non-identical oscillators with distributed natural frequencies. Recall the Kuramoto model with a uniform inertia and interaction frustration: for $i = 1, \cdots, N$,

$$m\ddot{\theta}_i + \dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i + \alpha), \quad t \geq 0, \quad |\alpha| < \frac{\pi}{2}. \tag{5.3.20}$$

This is equivalent to the following first order system:

$$\begin{aligned} \dot{\theta}_i &= \omega_i\,, \\ m\dot{\omega}_i &= -\omega_i + \Omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i + \alpha). \end{aligned} \tag{5.3.21}$$

We will derive two frameworks for the asymptotic synchronization depending on the strength of inertia.

5.3.1 A small inertia regime

In this part we consider the small inertia regime. We first introduce several structural conditions on the parameters of the system (5.3.20). For a fixed $K > 0$,

- ($\mathcal{H}_s1$) The oscillators are non-identical and the diameter of natural frequencies satisfy
$$K > \frac{D(\Omega)}{1-\sin|\alpha|}.$$
- ($\mathcal{H}_s2$) The strength of inertia m satisfies
$$m < \frac{\Delta^\infty}{4K\sin\Delta^\infty},$$

where Δ^∞ is the (unique) root of the following trigonometric equation:
$$\sin x = \frac{D(\Omega) + K\sin|\alpha|}{K}, \quad x \in \Big(0, \frac{\pi}{2}\Big).$$
Here we note that the condition (H_s1) guarantees the above trigonometric equation has a solution, since the right-hand side is a positive number less than 1.

We set
$$\mu_1 := \frac{1+\sqrt{1-4mK\bar{R}_1}}{2m}, \qquad \mu_2 := \frac{1-\sqrt{1-4mK\bar{R}_1}}{2m},$$
where $\bar{R}_1$ is a constant given by
$$\bar{R}_1 := \frac{\sin\Delta^\infty}{\Delta^\infty}.$$

Lemma 5.3.1. *Suppose that ($\mathcal{H}_s1$) and ($\mathcal{H}_s2$) hold, and the initial configuration satisfies*
$$\mathcal{C}_2(0) < \Delta^\infty.$$
Then, we have
$$\dot{D}(\theta(0)) + \mu_1(D(\theta(0)) + |\alpha|) - \frac{2(D(\Omega) + K\sin|\alpha|)}{1-\sqrt{1-4mK\bar{R}_1}} \le 0.$$

Proof. By direct calculation, we have
$$\dot{D}(\theta(0)) + \mu_1(D(\theta(0)) + |\alpha|) - \frac{2(D(\Omega) + K\sin|\alpha|)}{1-\sqrt{1-4mK\bar{R}_1}}$$

$$
\begin{aligned}
&= \dot{D}(\theta(0)) + \frac{1+\sqrt{1-4mK\bar{R}_1}}{2m}(D(\theta(0)) + |\alpha|) \\
&- \frac{(D(\Omega) + K\sin|\alpha|)(1+\sqrt{1-4mK\bar{R}_1})}{2mK\bar{R}_1} \\
&= \frac{1}{2m}\Big[2m\dot{D}(\theta(0)) + D(\theta(0)) + |\alpha| - \Delta^\infty \\
&+ \sqrt{1-4mK\bar{R}_1}\Big(D(\theta(0)) + |\alpha| - \Delta^\infty\Big)\Big] \\
&\le \frac{1}{2m}\Big[\mathcal{C}_2(0) - \Delta^\infty + \sqrt{1-4mK\bar{R}_1}\big(C_2(0) - \Delta^\infty\big)\Big] \\
&\le 0.
\end{aligned}
$$

□

We can derive a trapping region for the phases based on Lemma 5.3.1.

Proposition 5.3.1. *Suppose that* $(\mathcal{H}_s1)$ *and* $(\mathcal{H}_s2)$ *hold, and the initial configuration satisfies*

$$\mathcal{C}_2(0) < \Delta^\infty.$$

Then, for any global solution to the system (5.3.20)*, we have*

$$D(\theta(t)) \le \Delta^\infty - |\alpha|, \qquad t \ge 0.$$

Proof. We use the proof by contradiction. We set

$$\mathcal{T} := \{T \in [0,\infty) : D(\theta(t)) < \Delta^\infty - |\alpha|, \ \forall\, t \in [0,T)\}, \quad T_* := \sup \mathcal{T}.$$

Since $\mathcal{C}_2(0) < \Delta^\infty$, we see that $D(\theta_0) < \Delta^\infty - |\alpha|$. By continuity of $D(\theta(t))$, the set $\mathcal{T}$ contains some small interval $[0,\varepsilon)$, which means that T_* is well-defined. We claim that

$$T_* = +\infty.$$

Suppose not, i.e., $T_* < +\infty$. Then by continuity we should have

$$D(\theta(T_*)) = \Delta^\infty - |\alpha|. \tag{5.3.22}$$

We now make an estimate for the diameter $D(\theta(t))$. By system equation

(5.3.20) we have

$$
\begin{aligned}
& m\ddot{D}(\theta(t)) + \dot{D}(\theta(t)) \\
& \quad \le \Omega_M - \Omega_m + \frac{K}{N}\sum_{j=1}^{N}\Big[\sin(\theta_j - \theta_M + \alpha) - \sin(\theta_j - \theta_m + \alpha)\Big] \\
& \quad \le D(\Omega) + \frac{K\cos\alpha}{N}\sum_{j=1}^{N}\Big[\sin(\theta_j - \theta_M) - \sin(\theta_j - \theta_m)\Big] \\
& \quad + \frac{K\sin\alpha}{N}\sum_{j=1}^{N}\Big[\cos(\theta_j - \theta_M) - \cos(\theta_j - \theta_m)\Big], \quad t \in [0, T_*).
\end{aligned}
$$

We consider two cases according to the sign of α.

- Case 1: $\alpha \in [0, \frac{\pi}{2})$. In this case, we have

$$
\begin{aligned}
& m\ddot{D}(\theta(t)) + \dot{D}(\theta(t)) \\
& \quad \le D(\Omega) + \frac{K\cos\alpha\sin D(\theta(t))}{ND(\theta(t))}\sum_{j=1}^{N}\Big[(\theta_j(t) - \theta_M(t)) - (\theta_j(t) - \theta_m(t))\Big] \\
& \quad + \frac{K\sin\alpha}{N}\sum_{j=1}^{N}\Big[1 - \cos D(\theta(t))\Big] \\
& \quad = D(\Omega) - K\Big[\sin\left(D(\theta(t)) + \alpha\right) - \sin\alpha\Big] \\
& \quad = D(\Omega) - K\Big[\sin\left(D(\theta(t)) + |\alpha|\right) - \sin|\alpha|\Big].
\end{aligned}
$$

• Case 2: $\alpha \in (-\frac{\pi}{2}, 0)$. In this case, we have

$$\begin{aligned}
&m\ddot{D}(\theta(t)) + \dot{D}(\theta(t)) \\
&\quad \le D(\Omega) + \frac{K\cos\alpha \sin D(\theta(t))}{ND(\theta(t))} \sum_{j=1}^{N} \Big[(\theta_j(t) - \theta_M(t)) - (\theta_j(t) - \theta_m(t))\Big] \\
&\qquad + \frac{K\sin\alpha}{N} \sum_{j=1}^{N} \Big[\cos D(\theta(t)) - 1\Big] \\
&\quad = D(\Omega) - K\Big[\sin\big(D(\theta(t)) - \alpha\big) + \sin\alpha\Big] \\
&\quad = D(\Omega) - K\Big[\sin\big(D(\theta(t)) + |\alpha|\big) - \sin|\alpha|\Big],
\end{aligned}$$

where we used

$$\cos(\theta_j(t) - \theta_M(t)), \quad \cos(\theta_j(t) - \theta_m(t)) \le 1,$$

and $D(\theta(t)) < \Delta^\infty - |\alpha| < \frac{\pi}{2}, \ \forall\, t \in [0, T_*)$, which means

$$\cos(\theta_j(t) - \theta_m(t)), \cos(\theta_j(t) - \theta_M(t)) \ge \cos D(\theta(t)), \quad \forall\, t \in [0, T_*).$$

We combine Case 1 and Case 2 to get a differential inequality as follows:

$$m\ddot{D}(\theta(t)) + \dot{D}(\theta(t)) \le D(\Omega) - K\big[\sin\big(D(\theta(t)) + |\alpha|\big) - \sin|\alpha|\big], \quad a.e.\ t \in [0, T_*). \tag{5.3.23}$$

We now set

$$\Delta(t) := D(\theta(t)) + |\alpha|,$$

then the inequality (5.3.23) implies

$$m\ddot{\Delta}(t) + \dot{\Delta}(t) + K\sin\Delta(t) - D(\Omega) - K\sin|\alpha| \le 0, \ \ a.e.\ t \in [0, T_*).$$

Since $\Delta(t) < \Delta^\infty, \ t \in [0, T_*)$, we can use the inequality

$$\frac{\sin\Delta(t)}{\Delta(t)} \ge \frac{\sin\Delta^\infty}{\Delta^\infty} \ (0 \le t < T_*)$$

to obtain

$$m\ddot{\Delta}(t) + \dot{\Delta}(t) + K\bar{R}_1\Delta(t) - D(\Omega) - K\sin|\alpha| \le 0, \quad a.e.\ t \in [0, T_*). \tag{5.3.24}$$

Since $1 - 4mK\bar{R}_1 > 0$, we can apply Lemma 5.1.1 (i) with

$$a = m, \quad b = 1, \quad c = K\bar{R}_1, \quad d = -D(\Omega) - K\sin|\alpha|,$$

to obtain

$$\begin{aligned}
\Delta(t) &\le \left(\Delta(0) - \frac{D(\Omega) + K\sin|\alpha|}{K\bar{R}_1}\right) e^{-\mu_1 t} + m\frac{e^{-\mu_2 t} - e^{-\mu_1 t}}{\sqrt{1 - 4mK\bar{R}_1}} \\
&\quad \times \left(\dot{\Delta}(0) + \mu_1\Delta(0) - \frac{2(D(\Omega) + K\sin|\alpha|)}{1 - \sqrt{1 - 4mK\bar{R}_1}}\right) + \frac{D(\Omega) + K\sin|\alpha|}{K\bar{R}_1} \\
&= \Delta(0)e^{-\mu_1 t} + \frac{D(\Omega) + K\sin|\alpha|}{K\bar{R}_1}\left(1 - e^{-\mu_1 t}\right) \\
&\quad + m\frac{e^{-\mu_2 t} - e^{-\mu_1 t}}{\sqrt{1 - 4mK\bar{R}_1}}\left(\dot{\Delta}(0) + \mu_1\Delta(0) - \frac{2(D(\Omega) + K\sin|\alpha|)}{1 - \sqrt{1 - 4mK\bar{R}_1}}\right).
\end{aligned}$$

By assumption $(\mathcal{H}_s 2)$ we have

$$\frac{D(\Omega) + K\sin|\alpha|}{K\bar{R}_1} = \frac{\sin\Delta^\infty}{\bar{R}_1} = \Delta^\infty.$$

Thus, we have

$$\begin{aligned}
\Delta(t) &\le \Delta^\infty + (\Delta(0) - \Delta^\infty)e^{-\mu_1 t} \\
&\quad + m\frac{e^{-\mu_2 t} - e^{-\mu_1 t}}{\sqrt{1 - 4mK\bar{R}_1}}\left(\dot{\Delta}(0) + \mu_1\Delta(0) - \frac{2(D(\Omega) + K\sin|\alpha|)}{1 - \sqrt{1 - 4mK\bar{R}_1}}\right), \quad t \in [0, T_*).
\end{aligned}$$

We now employ Lemma 5.3.1 and $0 < \mu_2 < \mu_1$ to find that

$$\Delta(t) \le \Delta^\infty + (\Delta(0) - \Delta^\infty)e^{-\mu_1 t}, \quad t \in [0, T_*).$$

From the continuity of $\Delta(t)$ we have

$$\Delta(T_*) < \Delta^\infty, \quad \text{i.e.,} \quad D(\theta(T_*)) < \Delta^\infty - |\alpha|.$$

This contradicts to (5.3.22); thus $T_* = +\infty$ and the desired estimate is established. $\square$

We now give the main result for the complete synchronization under small inertia regime.

Theorem 5.3.1. *Suppose that* $(\mathcal{H}_s1)$ *and* $(\mathcal{H}_s2)$ *hold, and the initial configuration satisfies*

$$\mathcal{C}_2(0) < \Delta^\infty.$$

Then for any global solutions to the system (5.3.20)*, we have asymptotic complete frequency synchronization:*

$$D(\omega(t)) \le |\mathcal{O}(1)|e^{-\mu_3 t}, \quad \mu_3 := \frac{1-\sqrt{1-4mK\cos\Delta^\infty}}{2m}.$$

Proof. First of all, the frequency diameter $D(\omega(t))$ is piecewise C^2 and continuous every where. Its non-differentiable points are a subset of the collision time in frequency, which are isolated. More precisely, there exist at most a countable number of times $0 := t_0 < t_1 < \cdots < t_\infty \le \infty$ such that $D(\omega(t))$ is differentiable in each time interval (t_k, t_{k+1}), $k = 1, 2, \ldots$. For a given time interval (t_k, t_{k+1}), we choose two indices i_1, i_2 such that

$$\omega_{i_1}(t) = \omega_M(t) \quad \text{and} \quad \omega_{i_2}(t) = \omega_m(t), \quad t \in (t_k, t_{k+1}).$$

By Proposition 5.3.1, we see that for all $i, j = 1, 2, \cdots, N$,

$$|\theta_i(t) - \theta_j(t)| \le D(\theta(t)) < \Delta^\infty - |\alpha|, \quad t \in (t_k, t_{k+1}),$$

which implies

$$|\theta_i(t) - \theta_j(t) + \alpha| < \Delta^\infty < \frac{\pi}{2}, \quad t \in (t_k, t_{k+1}),$$

and

$$\cos\big(\theta_i(t) - \theta_j(t) + \alpha\big) > \cos\Delta^\infty, \quad t \in (t_k, t_{k+1}).$$

On the other hand, we use the system (5.3.21) to find

$$\begin{aligned} m\ddot{\omega}_{i_1} + \dot{\omega}_{i_1} &= \frac{K}{N}\sum_{j=1}^N \cos(\theta_j - \theta_{i_1} + \alpha)(\omega_j - \omega_{i_1}) \\ &\le \frac{K\cos\Delta^\infty}{N}\sum_{j=1}^N(\omega_j - \omega_{i_1}), \quad t \in (t_k, t_{k+1}). \end{aligned} \tag{5.3.25}$$

Here we used $\Delta^\infty < \frac{\pi}{2}$ and $\omega_j - \omega_{i_1} \le 0$. Similarly, we can derive an inequality for ω_{i_2} as follows:

$$m\ddot{\omega}_{i_2} + \dot{\omega}_{i_2} \ge \frac{K\cos\Delta^\infty}{N}\sum_{j=1}^{N}(\omega_j - \omega_{i_2}), \quad t \in (t_k, t_{k+1}). \tag{5.3.26}$$

We combine the estimates (5.3.25)-(5.3.26) for each time interval to obtain

$$m\ddot{D}(\omega) + \dot{D}(\omega) + K(\cos\Delta^\infty)D(\omega) \le 0 \quad a.e.\ t. \tag{5.3.27}$$

By the assumption $(\mathcal{H}_s 2)$ and $\frac{\sin\Delta^\infty}{\Delta^\infty} \ge \cos\Delta^\infty$, we see that

$$1 - 4mK\cos\Delta^\infty \ge 1 - 4mK\frac{\sin\Delta^\infty}{\Delta^\infty} > 0.$$

Then we can apply Lemma 5.1.1 (i) with

$$a = m, \quad b = 1, \quad c = K\cos\Delta^\infty, \quad d = 0,$$

to obtain the desired estimate. □

5.3.2 A large inertia regime

In this part we consider a large inertia case. We first impose several structural conditions on the parameters of the system (5.3.20). For a fixed $K > 0$,

- $(\mathcal{H}_\ell 1)$ The inertia satisfies

$$mK \ge \frac{\pi}{8}.$$

- $(\mathcal{H}_\ell 2)$ The natural frequencies and frustration satisfy

$$4m(D(\Omega) + K\sin|\alpha|) < \frac{\pi}{2}.$$

For convenience we denote

$$\Delta_\ell^\infty := 4m(D(\Omega) + K\sin|\alpha|).$$

Again we derive a trapping region estimate and then give the main result on complete synchronization.

Proposition 5.3.2. *Suppose that* $(\mathcal{H}_\ell 1)$ *and* $(\mathcal{H}_\ell 2)$ *hold, and the initial configuration satisfies*

$$\mathcal{C}_2(0) < \Delta_\ell^\infty.$$

Then for the smooth global solution to system (5.3.20)*, we have*

$$D(\theta(t)) < \Delta_\ell^\infty - |\alpha|, \quad t \geq 0.$$

Proof. We use the proof by contradiction. We set

$$\mathcal{T}_\ell := \{T \in [0, \infty) : D(\theta(t)) < \Delta_\ell^\infty - |\alpha|, \ \forall\, t \in [0, T)\}, \quad T_\ell^* := \sup \mathcal{T}_\ell.$$

Since $\mathcal{C}_2(0) < \Delta_\ell^\infty$, we see that $D(\theta_0) < \Delta_\ell^\infty - |\alpha|$. By continuity of $D(\theta(t))$, the set $\mathcal{T}_\ell$ contains some small interval $[0, \varepsilon)$, which means that $\mathcal{T}_\ell$ is nonempty and T_ℓ^* is well-defined. We claim that

$$T_\ell^* = +\infty.$$

Suppose not, i.e., $T_\ell^* < +\infty$. Then by continuity we should have

$$D(\theta(T_\ell^*)) = \Delta_\ell^\infty - |\alpha|. \tag{5.3.28}$$

By the same argument as in Proposition 5.3.1 we have

$$m\ddot{\Delta}(t) + \dot{\Delta}(t) + K\bar{R}_2\Delta(t) - D(\Omega) - K \sin|\alpha| \leq 0, \quad a.e.\ t \in [0, T_\ell^*), \tag{5.3.29}$$

where $\Delta(t) := D(\theta(t)) + |\alpha|$ and

$$\bar{R}_2 := \frac{\sin \Delta_\ell^\infty}{\Delta_\ell^\infty}.$$

Thanks to the assumptions $(\mathcal{H}_\ell 1)$ and $(\mathcal{H}_\ell 2)$, we can use the elementary inequality

$$\sin x \geq \frac{2}{\pi}x, \quad x \in \left(0, \frac{\pi}{2}\right),$$

to derive

$$1 - 4mK\bar{R}_2 \leq 0.$$

Then, we can apply Lemma 5.1.1 (ii) with

$$a = m, \quad b = 1, \quad c = K\bar{R}_2, \quad d = -D(\Omega) - K\sin|\alpha|,$$

to see

$$\begin{aligned}
\Delta(t) &\le e^{-\frac{1}{2m}t}\Bigg[\Delta(0) - 4m(D(\Omega) + K\sin|\alpha|) \\
&\qquad + \left(\frac{1}{2m}\Delta(0) + \dot{\Delta}(0) - 2(D(\Omega) + K\sin|\alpha|)\right)t\Bigg] \\
&\quad + 4m(D(\Omega) + K\sin|\alpha|) \\
&\le \frac{e^{-\frac{1}{2m}t}}{2m}\Big(\mathcal{C}_2(0) - \Delta_\ell^\infty\Big)t + e^{-\frac{1}{2m}t}\big(\mathcal{C}_2(0) - \Delta_\ell^\infty\big) + \Delta_\ell^\infty, \qquad t \in [0, T_*).
\end{aligned}$$

Hence, we have

$$\Delta(T_\ell^*) < \Delta_\ell^\infty, \quad \text{i.e.,} \quad D(\theta(T_\ell^*)) < \Delta(t) - |\alpha|.$$

This contradicts to (5.3.28); thus $T_\ell^* = +\infty$ and the desired estimate is established. $\square$

We set

$$\bar{\mu}_1 := \frac{1 + \sqrt{1 - 4mK\cos\Delta_\ell^\infty}}{2m}, \quad \bar{\mu}_2 := \frac{1 - \sqrt{1 - 4mK\cos\Delta_\ell^\infty}}{2m},$$

Theorem 5.3.2. *Suppose that* $(\mathcal{H}_\ell 1)$ *and* $(\mathcal{H}_\ell 2)$ *hold, and the initial configuration satisfies*

$$\mathcal{C}_2(0) < \Delta_\ell^\infty.$$

Then for any global solution to the system (5.3.20) *with initial data* $(\theta_0, \dot{\theta}_0)$*, we have complete frequency synchronization, more precisely,*

$$D(\omega(t)) \le \max\{U_1(t), U_2(t)\}, \quad t \ge 0,$$

where U_1 *and* U_2 *are given by the following relations:*

$$\begin{aligned}
U_1(t) &:= D(\omega(0))e^{-\bar{\mu}_1 t} + m\frac{e^{-\bar{\mu}_2 t} - e^{-\bar{\mu}_1 t}}{\sqrt{1 - 4mK\cos\Delta_\ell^\infty}}\Big(\dot{D}(\omega(0)) + \bar{\mu}_1 D(\omega(0))\Big), \\
U_2(t) &:= e^{-\frac{t}{2m}}\Big[D(\omega(0)) + \Big(\frac{D(\omega(0))}{2m} + \dot{D}(\omega(0))\Big)t\Big].
\end{aligned}$$

Proof. By the similar argument as in Theorem 5.3.1 we can derive an differential inequality as follows:

$$m\ddot{D}(\omega) + \dot{D}(\omega) + K(\cos\Delta_\ell^\infty)D(\omega) \le 0 \quad a.e.\ t. \tag{5.3.30}$$

Next, we apply Lemma 5.1.1 with

$$a = m, \quad b = 1, \quad c = K\cos\Delta_\ell^\infty, \quad d = 0,$$

in two cases depending on different regions of the discriminant $b^2 - 4ac$.

• Case 1$(1 - 4mK\cos\Delta_\ell^\infty > 0)$: In this case, we apply Lemma 5.1.1 (i) to find

$$D(\omega(t)) \le D(\omega(0))e^{-\bar{\mu}_1 t} + m\frac{e^{-\bar{\mu}_2 t} - e^{-\bar{\mu}_1 t}}{\sqrt{1 - 4mK\cos\Delta_\ell^\infty}}\Big(\dot{D}(\omega(0)) + \bar{\mu}_1 D(\omega(0))\Big).$$

• Case 2 $(1 - 4mK\cos\Delta_\ell^\infty \le 0)$: In this case, we apply Lemma 5.1.1 (ii) to find

$$D(\omega(t)) \le e^{-\frac{t}{2m}}\Big[D(\omega(0)) + \Big(\frac{D(\omega(0))}{2m} + \dot{D}(\omega(0))\Big)t\Big].$$

Then we combine the estimates in two cases to obtain the desired result. □

5.4 Numerical simulations

In this section, we present several numerical examples to display the complex interplay between inertia and frustration, and we compare simulation results with analytical results in previous sections. For the simulations, we used the fourth-order Runge-Kutta method.

5.4.1 Identical oscillators

In this subsection, we discuss several numerical simulations for identical oscillators and compare these with the analytical results in Section 5.2. To see the interplay between inertia m and frustration α, we study the effect of each while the other is fixed. In all simulations, we used $K = 1$ and $N = 100$.

Fixed frustration and varying inertia

To see the effect of inertia at fixed frustration as in Theorem 5.2.2, we performed two sets of simulations with $\alpha = 0.3$ and $\alpha = 0.9$, respectively. First of all, Theorem 5.2.1 predicts that for all positive values of m and K, the diameters $D(\theta(t)$ and $D(\omega(t))$ decay exponentially fast. As noted in Theorem 5.2.2 and Remark 5.2.1, if $m < m_c(K, \alpha) := \dfrac{1}{4K\cos\alpha}$, then we have a nearly optimal exponential decay which means that $\log D(\theta(t))$ decays linearly. Moreover, for either case, as we increase the strength of the inertia from $m = 0.1$, the decay of $D(\theta(t))$ and $D(\omega(t))$ becomes more rapid until m reaches the threshold m_c.

In the case of Figure 5.1, with $\alpha = 0.3$, the initial configuration was randomly chosen from the interval $(0, \pi - 2 \times 0.3)$ satisfying

$$D(\theta_0) = 1.9867.$$

Note that in Figure 5.1 oscillatory motions emerge in the regime $m \geq 0.27$, whereas no oscillatory motions appear in the regime $m < 0.26$. This is exactly as predicted by Theorem 5.2.2: if $m < \frac{1}{4\cos 0.3} \approx 0.2617$, then $\log D(\theta(t))$ decays linearly.

In the case of Figure 5.2, with $\alpha = 0.9$, the initial configuration was extracted from $(0, \pi - 2 \times 0.9)$. Note that in Figure 5.2 oscillatory motions appear in the relaxation process for $m \geq 0.41$, which coincides with the result predicted by Theorem 5.2.2:

$$m_c = \frac{1}{4\cos 0.9} \approx 0.4022.$$

Another observation in these simulations is, as m is increased beyond m_c, the decay rate becomes smaller as m increases (see Figure 5.1 (c)-(d) and Figure 5.2 (c)-(d)). This is opposite to regime $m < m_c$.

Fixed inertia and varying frustration

In this subsection, we consider the situation in which the frustration α is varied for a fixed m to concentrate on the effect of α. Again, it follows from

Theorem 5.2.2 that if $\alpha > \alpha_c(m, K) := \arccos\left(\frac{1}{4mK}\right)$, then $D(\theta(t))$ and $D(\omega(t))$ will decay exponentially fast without any oscillatory motion. To confirm this theoretical expectation, we employed the fixed value $m = 0.3$, and the initial configuration was chosen from $(0, \pi)$. With these settings, the critical frustration α_c theoretically satisfies

$$\alpha_c(0.3, 1) = \arccos\frac{5}{6} \approx 0.5857.$$

Figure 5.3 shows the qualitative difference between the regimes $\alpha \leq 0.5$ and $\alpha \geq 0.6$. In particular, $\log D(\theta(t))$ decays linearly when $\alpha \geq 0.6$ and the asymptotic decay rate (the slope of the line) decreases as α increases, which is expected from Theorem 5.2.2. Note that for small $\alpha < \alpha_c$, the oscillatory phase appears, and the oscillation period increases as α is increased toward α_c from some value below α_c. We conjecture that the period of oscillation diverges to ∞ as $\alpha \to \alpha_c$.

Relation between oscillation period and m

In this subsection, we observe the relation between the oscillation period and the parameters m and α through numerical simulations. In particular, we focus on the effect of the frustration α on the oscillation period. For this, we first consider the case with no frustration; i.e., $\alpha = 0$. In this case, as shown by Figure 5.4, the oscillation period decreases as m increases from 0.3 to 0.5. However, for $m \geq 0.5$, Table 5.1 shows that the period begins to increase as m increases. Here, Table 5.1 was obtained by examining the time of the local minimum of $\log(D(\theta(t)))$ for each value of m. Of course, in the presence of frustration, similar phenomena occur. However, some other things occur as well. For the simulations with fixed $\alpha = 0.2$ and $\alpha = 0.8$, we refer to Tables 5.2–5.3 and Figures 5.5–5.6. It follows from Tables 5.2–5.3 that as m is increased the periods begin to decrease but then increase at some critical inertia $m_c(\alpha)$. Although we have conducted only two simulations, we conjecture that $m_c(\alpha)$ is an increasing function with respect to α.

m	Starting and ending times for each period						Average period
0.3	4.59	8.80	13.01	17.23			4.21
0.4	3.27	6.51	9.75	13.00	16.24	19.49	3.24
0.5	2.90	6.05	9.19	12.33	15.47	18.61	3.14
0.6	2.73	5.92	9.11	12.29	15.48	18.66	3.19
0.7	2.63	5.91	9.19	12.47	15.75	19.03	3.28
0.8	2.56	5.96	9.35	12.74	16.13	19.52	3.39
0.9	2.52	6.04	9.55	13.05	16.56		3.51
1.0	2.48	6.13	9.76	13.39	17.01		3.63
1.1	2.46	6.23	9.98	13.73	17.48		3.75
1.2	2.44	6.34	10.21	14.08	17.95		3.87
1.3	2.42	6.45	10.44	14.42	18.41		3.99

Table 5.1: Evolution of the oscillation period for $(K, \alpha) = (1, 0)$

m	Starting and ending times for each period						Average period
0.3	4.61	9.11	13.60	18.09			4.49
0.4	3.07	6.40	9.74	13.07	16.40	19.74	3.30
0.5	2.65	5.85	9.06	12.27	15.48	18.68	3.21
0.6	2.45	5.69	8.94	12.18	15.42	18.66	3.24
0.7	2.33	5.66	8.99	12.32	15.65	18.98	3.33
0.8	2.25	5.69	9.13	12.56	16.01	19.44	3.44
0.9	2.19	5.75	9.31	12.86	16.42	19.98	3.56
1.0	2.14	5.83	9.51	13.19	16.87		3.68
1.1	2.11	5.92	9.72	13.52	17.32		3.80
1.2	2.08	6.02	9.94	13.86	17.78		3.92
1.3	2.06	6.13	10.17	14.20	18.24		4.04

Table 5.2: Evolution of the oscillation period for $(K, \alpha) = (1, 0.2)$

m	Starting and ending times for each period						Average period
0.4	7.92	15.34	22.72				7.40
0.5	4.50	9.51	14.52	19.51	24.54	29.47	4.99
0.6	3.53	8.13	12.72	17.33	21.92	26.52	4.60
0.7	3.05	7.58	12.07	16.59	21.10	25.61	4.51
0.8	2.78	7.34	11.87	16.40	20.93	25.47	4.54
0.9	2.59	7.24	11.84	16.44	21.05	25.65	4.61
1.0	2.47	7.23	11.91	16.62	21.31	26.02	4.71
1.1	2.37	7.26	12.05	16.84	21.68	26.49	4.82
1.2	2.30	7.32	12.25	17.16	22.09	27.01	4.94
1.3	2.24	7.40	12.44	17.48	22.52	27.57	5.06

Table 5.3: Evolution of the oscillation period for $(K, \alpha) = (1, 0.8)$

5.4.2 Nonidentical oscillators

In this subsection, we present numerical simulations for nonidentical oscillators. For these simulations, the natural frequencies were randomly chosen from the interval $(-0.5, 0.5)$ to satisfy

$$D(\Omega) = 0.9593, \quad \Omega_c = 0.$$

To prepare Figure 5.7, we took the frustration and the coupling strength as

$$\alpha = 0.1, \quad K = 2.$$

It is easy to see that the above choices satisfy the condition $(\mathcal{H}_s 1)$ in Section 5.3; i.e.,

$$\frac{D(\Omega)}{1 - \sin|\alpha|} \approx 0.9994.$$

In this case, since $\Delta^\infty = \arcsin\left(\frac{D(\Omega)}{K} + \sin|\alpha|\right) \approx 0.5819$, the condition $(\mathcal{H}_s 2)$ requires that the inertia satisfy

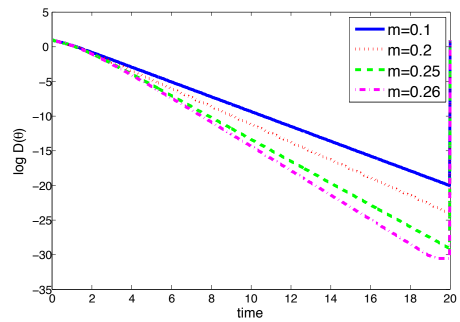
$$m < \frac{\Delta^\infty}{4K \sin \Delta^\infty} \approx 0.13235.$$

We observe that if $m \leq 0.13$, then $\log D(\omega(t))$ decays linearly. Although the inertia $m > 0.13$ in Figures 5.7(c) and 5.7(d) does not satisfy $(\mathcal{H}_\ell 1)$, we see that the oscillations also occur in this case. Hence, the assumptions in Section 5.3 are not necessary for the complete frequency synchronization.

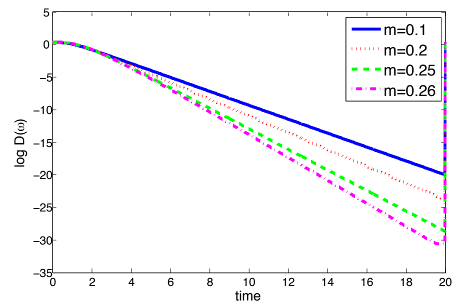
Figure 5.8 compares simulations with different α but otherwise the same. We took $\alpha = 0.8$ and $K = 4$ satisfying

$$K > \frac{D(\Omega)}{1 - \sin|\alpha|} \approx 3.1828.$$

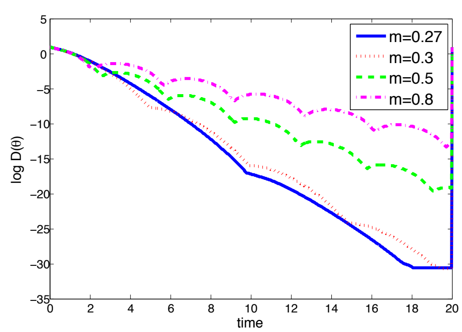
In this case the inertia m has to satisfy $m < 0.0815$ for the small-inertia results and $0.0982 \leq m < 0.1042$ for the large-inertia results. However, Figure 5.8 shows that these assumptions are not necessary. Although m does not satisfy our assumptions, we observe that $\log D(\omega(t))$ decays linearly for small inertia and undergoes oscillations for large inertia. Moreover, the asymptotic behavior of $\log D(\theta(t))$ does not depend on the value of m.



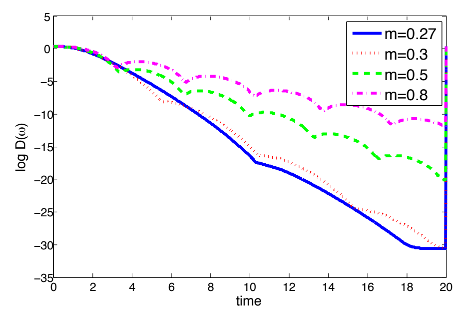
(a) $\log D(\theta(t))$ for small inertia



(b) $\log D(\omega(t))$ for small inertia

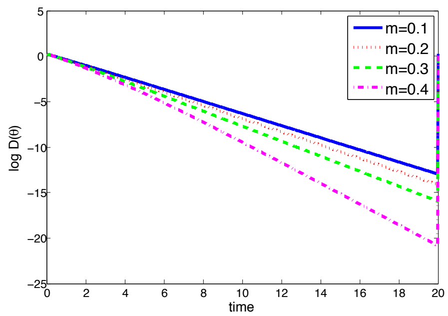


(c) $\log D(\theta(t))$ for large inertia

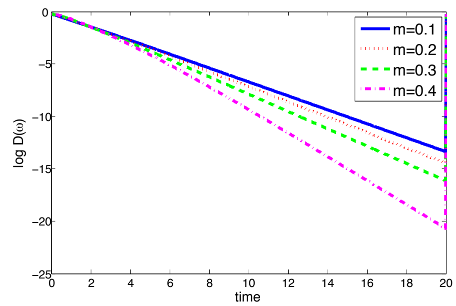


(d) $\log D(\omega(t))$ for large inertia

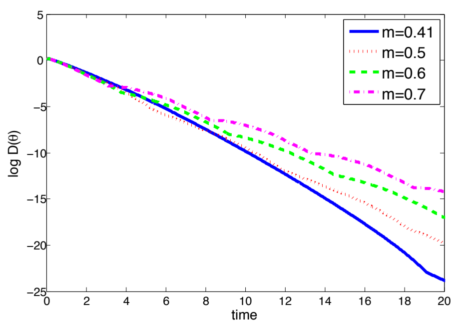
Figure 5.1: The relaxation of phase and frequency diameters for $(K, \alpha) = (1, 0.3)$.



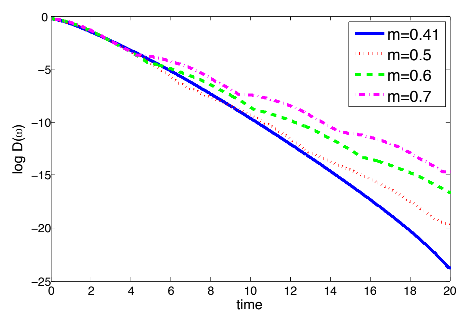
(a) $\log D(\theta(t))$ for small inertia



(b) $\log D(\omega(t))$ for small inertia

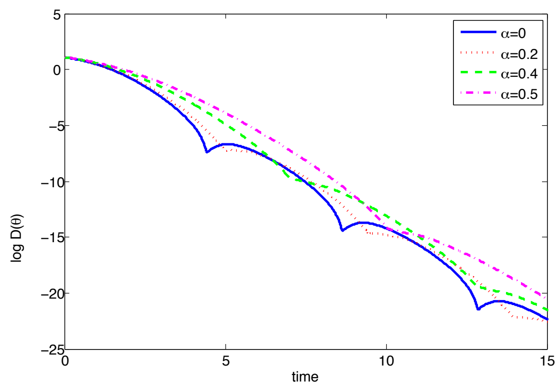


(c) $\log D(\theta(t))$ for large inertia

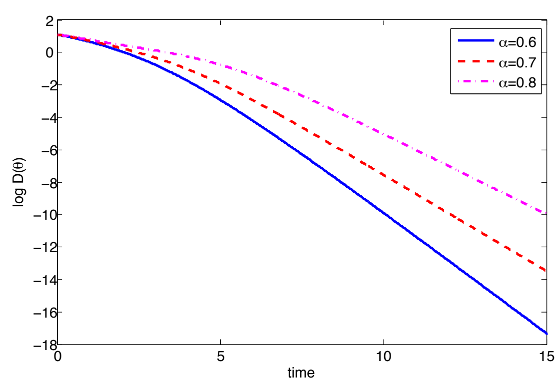


(d) $\log D(\omega(t))$ for large inertia

Figure 5.2: The relaxation of phase and frequency diameter for $(K, \alpha) = (1, 0.9)$.



(a) $\log D(\theta(t))$ for small α



(b) $\log D(\theta(t))$ for large α

Figure 5.3: The behaviors of $D(\theta)$ for $(K, m) = (1, 0.3)$

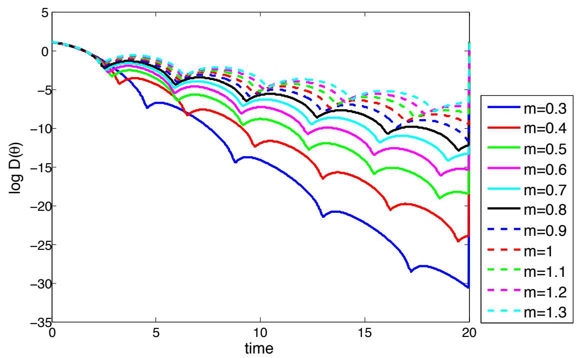


Figure 5.4: Dynamics of $\log D(\theta(t))$ for $\alpha = 0$

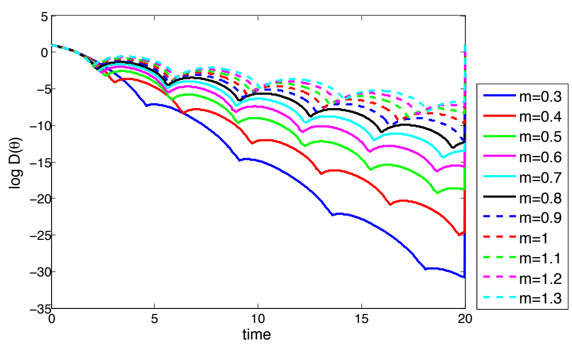


Figure 5.5: Dynamics of $\log D(\theta(t))$ for $\alpha = 0.2$

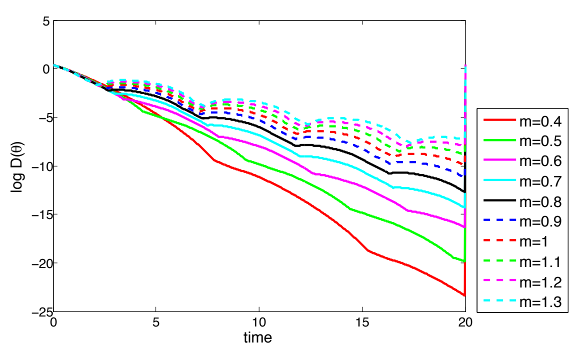
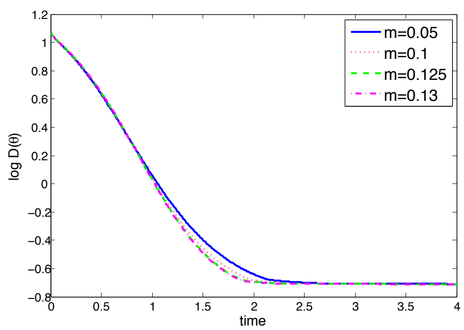
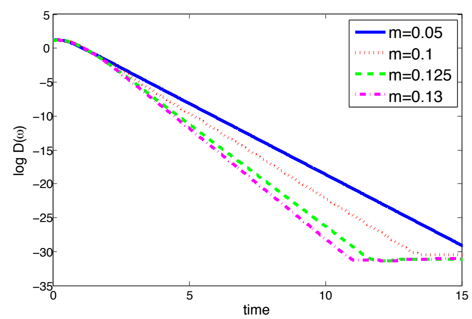


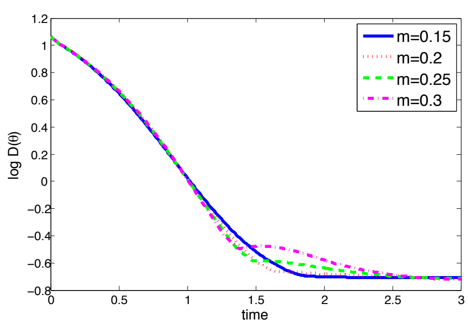
Figure 5.6: Dynamics of $\log D(\theta(t))$ for $\alpha = 0.8$



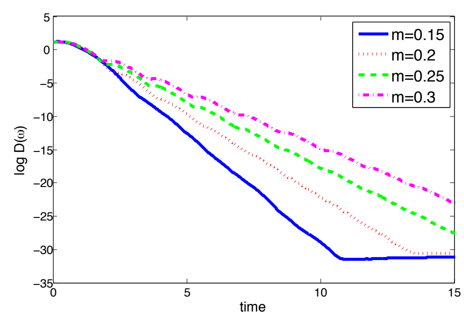
(a) $\log D(\theta(t))$ for $m \leq 0.13$



(b) $\log D(\omega(t))$ for $m \leq 0.13$

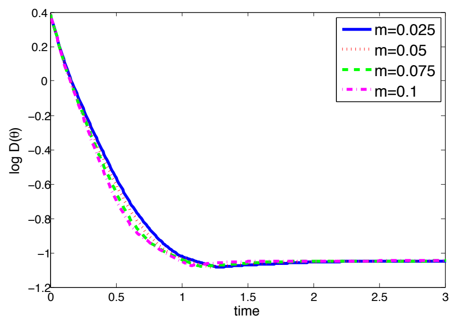


(c) $\log D(\theta(t))$ for $m \geq 0.15$

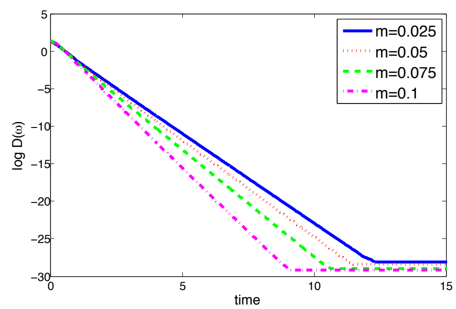


(d) $\log D(\omega(t))$ for $m \geq 0.15$

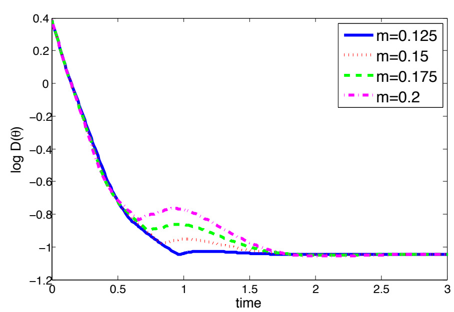
Figure 5.7: The behaviors of phase and frequency diameters for $(K, \alpha) = (2, 0.1)$



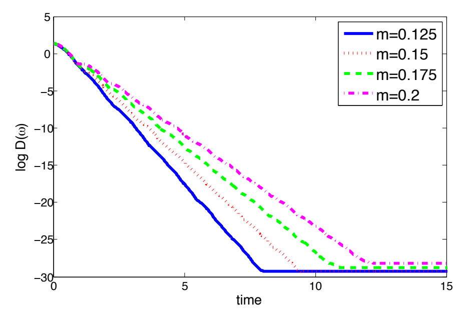
(a) $\log D(\theta(t))$ for small inertia



(b) $\log D(\omega(t))$ for small inertia



(c) $\log D(\theta(t))$ for large inertia



(d) $\log D(\omega(t))$ for large inertia

Figure 5.8: The behaviors of phase and frequency diameters for $(K, \alpha) = (4, 0.8)$

Chapter 6

Conclusion and future works

6.1 Conclusion

In this thesis, we studied several problems in the ensemble of Kuramoto oscillators. We show the nonlinear stability of the phase-locked states using a robust ℓ_1-metric as a Lyapunov functional. The main result of Section 3.1 says that the phase-locked states are congruent each other in the sense that one phase-locked state is the simply translation of the other and phase-shift is the difference of averaged initial phases. Our stability approach is very elementary and nonlinear in the sense that we do not use any linearization arguments of the Kuramoto model at the phase-locked states and do not require a prior spectral information. We also show the contraction property for measure valued solutions of the KKE. If two initial Radon measures have the same natural frequency density function and strength of coupling, we show that the Wasserstein p-distance between corresponding measure valued solutions is exponentially decreasing in time. This contraction principle is more general than previous ℓ_1-contraction properties of Section 3.1.

In Chapter 4, we studied the effect of interaction frustration on the complete synchronization of Kuramoto oscillators. In general, interaction frustration hinders the formation of complete (frequency) synchronization. Hence, even for the same initial configuration, we need a larger coupling strength to ensure the synchronization in the presence of nonzero interaction frustration. For more quantitative estimates, we considered three Kuramoto-type models. Our first model is for an ensemble of Kuramoto oscillators with uniform interaction frustration. In this case, we derived the explicit sufficient conditions of Theorem 4.2.1 for the initial configurations, coupling strength, and frustration that lead to complete frequency synchronization. Although we do not have an optimal rate of decay in the frequency diameter, the upper bound estimate of Theorem 4.2.1 suggests how the frustration slows the decay, and this coincides with Figure 1(b). Our second model is a special case of the first model; i.e., the ensemble is simply a mixture of two identical Kuramoto oscillator groups with distinct natural frequencies. In this case, we showed that the mixed configuration evolves toward two-point cluster configurations exponentially fast. We also estimated the lower and upper bounds on the distance between two point clusters in terms of the system parameter K, the frustration α, and $D(\Omega)$. Our third model is like the Kuramoto model for identical oscillators on the bipartite graph. In this case, like in the second model, the configuration evolves toward the two-point cluster configuration and, furthermore, we obtained the exact asymptotic diameter of the two-point cluster configuration.

In Chapter 5, we investigated the intricate interplay between the inertia and frustration in an ensemble of Kuramoto oscillators. As shown in Section 5.1, we cannot apply the explicit macro-micro decomposition to reduce the dynamics of initial phases to that of fluctuations. However, we can still derive second-order differential inequalities for the phase or frequency diameters so that the second-order Gronwall inequality method still works well. We presented several sufficient conditions on the parameters and initial configurations to guarantee asymptotic complete synchronization of phase or frequency. The results exhibit the interplay between the inertia and interaction frustration in the relaxation process. For identical oscillators, a nearly optimal decay rate for the phase diameter was presented, with the strength of the inertia less than some critical value depending on the strength of the frustration. Moreover, both the analytical and numerical studies demonstrated this fact.

6.2 Future works

We obtained some results for the dynamics of Kuramoto oscillators. However, we always have constraint on initial phase configurations: $D(\theta^0) < \pi$. This condition is improved in comparison with [32] and [21]. But we can find many studies without initial condition which is based on the numerical approach. Because This stems from mathematical technics, it is possible to eliminate this condition. Moreover, the transition and relaxation stages have been studied in [17] for initial configurations with a diameter greater than $\pi/2$. Using this results, we may shrink the diameter for any initial condition in a finite time. Then we can deal with the problem in a similar way as before. Hopefully, we will show all the results for the synchronization without initial phase configurations.

We next consider a system of two coupled Kuramoto oscillators with frustration:

$$\begin{aligned} \frac{d\theta_1}{dt} &= \Omega_1 + \frac{K}{2}\sin(\theta_2 - \theta_1 + \alpha), \quad t > 0, \\ \frac{d\theta_2}{dt} &= \Omega_2 + \frac{K}{2}\sin(\theta_1 - \theta_2 + \alpha), \end{aligned} \tag{6.2.1}$$

where the natural frequencies Ω_i and frustration α are assumed to satisfy

$$\Omega_1 > \Omega_2, \qquad |\alpha| < \frac{\pi}{2}.$$

To reduce the number of equations, we introduce the following differences:

$$\theta := \theta_1 - \theta_2, \quad \Omega := \Omega_1 - \Omega_2.$$

Then, the system (6.2.1) becomes a single equation for the differences:

$$\frac{d\theta}{dt} = \Omega - K(\cos\alpha)\sin\theta. \tag{6.2.2}$$

We easily see that to obtain complete frequency synchronization we need to satisfy the condition

$$K \geq \frac{\Omega}{\cos\alpha}.$$

However, this critical value is less than our condition for the coupling strength $K_{ef} = \frac{D(\Omega)}{1-\sin|\alpha|}$. As mentioned before, this condition K_{ef} is improved on the previous papers [17, 32] in some sense. Hence we try to obtain more optimal condition for the coupling strength K.

Finally, we need to improve our results for more general case:

$$\dot{\theta}_i = \Omega_i + \sum_{j=1}^{N} \frac{K_{ij}}{N} \sin(\theta_j - \theta_i + \alpha_{ij}).$$

We only dealt with uniform frustration in Chapter 4 and 5, though frustration is an interaction between oscillators. So we can consider the non-uniform constant frustration, and furthermore it is a natural problem that frustration is a function of t, changing with time. As a first step to generalize the results, we can consider two identical oscillators groups with different frustration:

$$\begin{aligned}
\dot{\theta}_i &= \Omega_1 + \frac{K_{11}}{N_1} \sum_{k=1}^{N_1} \sin(\theta_k - \theta_i + \alpha_{11}) + \frac{K_{12}}{N_2} \sum_{k=1}^{N_2} \sin(\phi_k - \theta_i + \alpha_{12}), \\
\dot{\phi}_j &= \Omega_2 + \frac{K_{21}}{N_2} \sum_{k=1}^{N_2} \sin(\phi_k - \phi_j + \alpha_{21}) + \frac{K_{22}}{N_1} \sum_{k=1}^{N_1} \sin(\theta_k - \phi_j + \alpha_{22}),
\end{aligned}$$

for $i = 1, \dots, N_1,\ j = 1, \dots, N_2$. In this case, it is meaningful that we compare the interaction frustration(α_{12} and α_{22}) between two identical oscillator groups(θ's and ϕ's) with the intra-frustration(α_{11} and α_{21}) of each group.

Moreover we do not need to assume that the interaction occurs between all mutual oscillators. When a very large group of birds fly together, for example, each bird is affected by only few birds close to itself. Additionally, the network problem is natural, in which there is an interaction between only connected objects. Hence, we can improve Model C in Chapter 4 for the network:

$$\dot{\theta}_i = \Omega_i + \sum_{\theta_j \sim \theta_i} K_{ij} \sin(\theta_j - \theta_i + \alpha_{ij}).$$

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국문초록

이 논문에서 우리는 쿠라모토 진동자들에서 일어나는 문제들에 대해 알아보았다. 우리는 ℓ_1 측도를 이용하여 위상 동기 상태의 비선형 안정성을 제시하였다. 어떤 위상 동기 상태는 다른 위상동기의 초기 평균상태의 차이만큼 위상 변위되었다는 의미에서 서로 동치라고 할 수 있다. 또한 우리는 쿠라모토 운동방정식의 해에 대한 축소 성질을 증명하였다. 우리는 쿠라모토 진동자들의 동기화에 관한 상호 방해의 효과에 대해 알아보았다. 일반적으로 상호 방해는 동기화의 형성을 저지한다. 우리는 세가지 상황을 고려하였는데, 첫번째는 모든 입자 사이의 상호 방해가 같은 경우이다. 두번째로는 첫번째 모델의 특별한 경우로, 서로 다른 빈도를 갖는 동질 진동자 집단이 섞여 있는 경우이다. 세번째 모델은 양분 그래프 상에 있는 진동자들에 관한 것이다. 마지막으로 우리는 관성과 방해의 상호작용에 대해 알아보았다. 여기에서 우리는 위상과 빈도의 직경에 대한 2차 그론월 부등식을 유도하였다. 이를 보여 줄 해석적 연구와 수치적 연구를 함께 하였다.

주요어휘: 쿠라모토 모델, 궤도 안정성, 축소, 방해, 동기화
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