이학박사 학위논문

Regularity of fully nonlinear parabolic equations and its applications

(완전 비선형 포물 편미분 방정식의 정칙 이론과 그 응용)

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Regularity of fully nonlinear parabolic equations and its applications

A dissertation
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by

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Abstract

Regularity of fully nonlinear parabolic equations and its applications

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We study the fully nonlinear uniformly parabolic equation

$$F(D^2u) - \partial_t u = f \quad \text{on } \Omega \times (0, T],$$

where Ω is a smooth bounded domain in a complete Riemannian manifold M, and $T \in \mathbb{R}$ is a positive number.

The first part of this thesis is based on joint work with Ki-Ahm Lee [37]. Asymptotic behavior of viscosity solutions to uniformly (or degenerate) parabolic equations has been investigated in the Euclidean space when the operator F is positively homogeneous of order one and $f \equiv 0$. Precisely, the renormalized parabolic solution with positive initial data converges to the related principal eigenfunction as $t \to +\infty$. We also prove that log- concavity (or power concavity) is preserved by the parabolic equation, under the assumption that $\Omega \subset \mathbb{R}^n$ is convex and the operator F is positively homogeneous and concave. Thus the uniform convergence provides such geometric property for the principal eigenfunction.

The second part based on joint work with Seick Kim and Ki-Ahm Lee [36, 38] is devoted to the proof of Krylov-Safonov Harnack inequality for nondivergent uniformly parabolic operators on a complete Riemannian manifold M by obtaining Aleksandrov-Bakelman-Pucci-Krylov-Tso type estimate on M. For linear parabolic operators, we impose certain conditions on the distance function introduced by Kim [35] to establish global Harnack inequality. In the nonlinear parabolic setting, it is required to assume that M has the sectional curvature bounded from below. Lastly, we make use of regularization by sup and inf-convolutions on Riemannian manifolds to prove Harnack inequality for viscosity solutions.

Key words: fully nonlinear parabolic equation, fully nonlinear elliptic eigenvalue problem, porous medium equation, Harnack inequality on Riemannian manifolds, ABP type estimate
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Abstract (in Korean) **i**

Chapter 1

Introduction

Fully nonlinear elliptic and parabolic equations appear in stochastic control theory (for example, the Bellman equation and the Bellman-Isaac equation [42, 13]) and in geometry (for example, mean curvature flows [25]). The theory of fully nonlinear equations in nondivergence form has been developed since the early 1980s due to the breakthrough estimate by Krylov and Safonov [45, 46], and the concept of weak solution called viscosity solution. Krylov-Safonov Harnack inequality based on Aleksandrov-Bakelman-Pucci (ABP) estimate [1, 6, 52] is the analogue of the De Giorgi-Nash-Moser theory for divergent operators (see [26, 29]), and the notion of viscosity solutions introduced by Crandall and Lions [19] and Evans [22, 23] suits fully nonlinear equations in nondivergence form. Existence, Uniqueness of viscosity solutions and regularity theory of fully nonlinear uniformly elliptic and parabolic equations are well understood and we refer to [18, 12, 43, 44, 24, 13, 60, 61].

In this thesis, we are concerned with the fully nonlinear parabolic equation

$$F(D^2u) - \partial_t u = f \quad \text{in } \Omega \times (0, T], \tag{1.0.1}$$

under the assumption that the operator F is uniformly elliptic, and Ω is a smooth bounded domain in the Euclidean space $\mathbb{R}^n$ or in a complete Riemannian manifold M, and $T \in \mathbb{R}$ is a positive number. Assuming F to be positively homogeneous of order one, we investigate long- time behavior of viscosity solutions to (1.0.1) with $f \equiv 0$ and its relation to the principal eigenvalue problem. Preservation of some geometric property is also proved by the parabolic flow when $\Omega \subset \mathbb{R}^n$ is convex and F is also assumed to be concave. On the other hand, we establish Krylov-Safonov Harnack inequality for viscosity solutions to (1.0.1) on a Rimannian manifold M imposing certain conditions on the distance function or the sectional curvature condition on M.

1.1 Long-time asymptotics for parabolic equations

In this section, we consider the following fully nonlinear uniformly or degenerate parabolic equation

$$\begin{cases} F(D^2u^m) - \partial_t u = 0 & \text{in } \Omega \times (0, +\infty), \\ u(\cdot, 0) > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \times (0, +\infty), \end{cases} \tag{1.1.1}$$

in the range of the exponents $m \geq 1$, where F is uniformly elliptic and positively homogeneous of order one, and $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain. When F is the Laplace operator, (1.1.1) is the well-known heat equation for $m = 1$, porous medium equation for $m > 1$, or fast diffusion equation for $0 < m < 1$, respectively, that models linear or nonlinear diffusion of material (for example, heat and gas flows) in various media. For the Laplace operator, the asymptotic behavior of solutions to the uniform, degenerate or singular diffusion equations has been studied by many authors and we refer to [47, 58] and references therein.

In Chapter 3, we shall show that a renormalized limit of $u(\cdot, t)$ as $t \to +\infty$ is the function φ, which solves the following elliptic eigenvalue problem

$$\begin{cases} F(D^2\varphi) + \mu\varphi^p = 0, & \text{in } \Omega, \\ \varphi > 0 & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1.2}$$

where $0 < p := \frac{1}{m} \leq 1$ and $\mu > 0$ is the corresponding eigenvalue depending on m, F and Ω. More precisely, for uniformly parabolic case when $m = 1$, it is proved that the renormalized solution $e^{\mu t}u(\cdot, t)$ converges uniformly to $\gamma^*\varphi$ as $t \to +\infty$, by using the regularity theory and the maximum principle, where $\mu > 0$ is the corresponding eigenvalue of the problem (1.1.2), and the constant $\gamma^* > 0$ is uniquely determined depending only on the initial data. For porous medium type case ($m > 1$), it follows that the unique limit of $t^{\frac{m}{m-1}} u^m(\cdot, t)$ is the positive eigenfunction associated with the eigenvalue $\mu = \frac{1}{m-1}$ from the barrier argument with separable solutions. The existence and uniqueness up to a multiplicative constant of the principal eigenfunction for an elliptic, positively homogeneous operator F of order one were proven by Ishii and Yoshimura [31], and they also showed that the principal eigenvalue $\mu > 0$ is unique. The simplified proof can be found in [2], where they investigated the principal eigenvalues of fully nonlinear operators via maximum principle method (see also [8, 9, 53]). For the sub-linear case $0 < p < 1$, the unique positive eigenfunction of (1.1.2) can be established by using a barrier argument.

Secondly, we study some geometric property of viscosity solutions to (1.1.1). Under the additional assumptions that Ω is convex and F is concave, $\log(u)$ for $m = 1$, and $u^{\frac{m-1}{2}}$ for $m > 1$ will turn out to be geometric quantities that preserve the concavity for all time. The argument for the Laplace operator in [47] seems to hold for fully nonlinear case, but it is not straight forward due to the nonlinearity of the operator. Thus sophisticated geometric computations using approximation of the nonlinear elliptic operator are employed to investigate geometric quantities, which will satisfy maximum principle. For the porous medium type equations, we impose an extra assumption on initial data due to lack of global regularity. In general, we need to prove a weighted $C^{2,\alpha}$ estimate up to the boundary. As a consequence, the principal eigenfunction has such geometric property from the uniform convergence, that is, $\log(\varphi)$ is concave in the case $p = 1$, and $\varphi^{\frac{1-p}{2}}$ is concave for $0 < p < 1$. This implies the convexity of the super-level sets of the positive eigenfunction of (1.1.2). Since 1950s, the convexity of the level sets of positive eigenfunctions for the Laplace operator has been investigated by many authors; see [10, 40, 34, 47, 50, 27]. For the simplest case when a domain is a ball and the operator is Laplacian, there is a unique rotationally symmetric solution by the Alexandrov reflection argument which is decreasing and has convex super-level sets. Therefore, our results give that the geometric property is preserved under some nonlinear, concave perturbation of the operator.

Notation 1.1.1.

- $D^2u \geq 0$, and $D^2u \leq 0$ are understood in the usual sense of quadratic forms. For instance, $D^2u \geq 0$ means that D^2u is positive semidefinite.

- In order to avoid confusion between coordinates and partial derivatives, we will use the standard subindex notation to denote the former, while partial derivatives will be denoted in the form $f_{,\alpha}$ for $\frac{\partial f}{\partial x_\alpha}$. The second order partial derivatives will be denoted in the form $f_{,\alpha\beta}$ for $\frac{\partial^2 f}{\partial x_\alpha \partial x_\beta}$. This notation is usual in some parts of the physics literature. However, we denote by f_ν and f_τ the normal and tangential derivatives, respectively, since no confusion is expected.

1.2 Parabolic Harnack inequality on Riemannian manifolds

Krylov-Safonov Harnack inequality on a complete Riemannian manifold M for uniformly parabolic operators in nondivergence form is established in Chapter 4. The Harnack inequality is well understood for the divergent operators on Riemannian manifolds, where the volume doubling property and the weak Poincaré inequality hold. Indeed, it is shown that the two conditions above imply the Harnack inequality for divergent parabolic operators; see [28, 54].

In the setting of elliptic equations in nondivergence form on M, Cabré [11] obtained Krylov-Safonov type Harnack inequality of classical solutions to linear, uniformly elliptic equations when M has nonnegative sectional curvature. ABP estimate is essential to develop the regularity theory for fully nonlinear equations such as Krylov-Safonov Harnack inequality, which is proved using affine functions in the Euclidean space. Since affine functions can not be generalized into an intrinsic notion on Riemannian manifolds, Cabré considered the functions of the squared distance as appropriate replacements for the affine functions. Later, Kim [35] improved Cabré's result removing the sectional curvature assumption and imposing the certain conditions on the distance function; see (1.2.2) below. Recently, Wang and Zhang [62] obtained a version of ABP estimate on M with a lower bound of Ricci curvature, and Harnack inequality of classical solutions for nonlinear uniformly elliptic operators on M with the sectional curvature bounded from below.

We first consider linear, uniformly parabolic equations of nondivergence type defined by

$$\mathscr{L}u := \text{trace}\,(A_{x,t} \circ D^2 u) - \partial_t u = f, \tag{1.2.1}$$

where $A_{x,t}$ is a positive definite symmetric endomorphism of $T_x M$ for any $x \in M$ with the assumption that

$$\lambda |X|^2 \le \langle A_{x,t} X, X \rangle \le \Lambda |X|^2, \quad \forall x \in M, \ \ \forall X \in T_x M.$$

We assume essentially the same conditions introduced by Kim [35] that for the distance function $d_p := d(\cdot, p)$ on M, there is a positive constant $a_{\mathscr{L}}$ such that

$$\begin{aligned} \triangle d_p(x) &\le \frac{n-1}{d_p(x)} \quad \text{for} \quad x \notin \text{ cut locus of } p \cup \{p\}, \quad \forall p \in M, \\ \mathscr{L} d_p(x) &\le \frac{a_{\mathscr{L}}}{d_p(x)} \quad \text{for} \quad x \notin \text{ cut locus of } p \cup \{p\}, \quad \forall p \in M. \end{aligned} \tag{1.2.2}$$

The first condition of (1.2.2) implies Bishop's volume comparison theorem (see [48]), in particular, the underlying manifold M has a global volume doubling property. Under the assumption (1.2.2), Krylov-Safonov Harnack inequality is proved for classical solutions to (1.2.1) in Section 4.1, which gives in particular a new, nondivergent proof for Li-Yau Harnack inequality for the heat operator on M with nonnegative Ricci curvature. ABP-Krylov-Tso estimate discovered by Krylov [41] in the Euclidean case (see also [57, 60]) is a parabolic analogue of the ABP estimate, and a key ingredient in proving parabolic Harnack inequality. In order to prove ABP-Krylov-Tso type estimate (Lemma 4.1.3) on Riemannian manifolds, an intrinsically geometric version of Krylov-Tso normal map, namely,

$$\Phi(x,t) := \left(\exp_x \nabla_x u(x,t), -\frac{1}{2} d^2 \left(x, \exp_x \nabla u(x,t) \right) - u(x,t) \right)$$

is introduced. The map Φ is called the parabolic normal map related to $u(x,t)$ and its Jacobian determinant is explicitly computed in Lemma 4.1.2.

Influenced by Wang and Zhang [62], we prove Harnack inequality for the following fully nonlinear uniformly parabolic equation

$$F(D^2 u) - \partial_t u = f \tag{1.2.3}$$

assuming that M has the sectional curvature bounded from below by $-\kappa$ for $\kappa \ge 0$ in Section 4.2. We introduce the parabolic contact set $\mathcal{A}_{a,b}$ for $a, b > 0$, which consists of a point $(\bar{x}, \bar{t}) \in M \times \mathbb{R}$, where a concave paraboloid

$$-\frac{a}{2} d_y^2(x) + bt + C \quad \text{(for some } C\text{)}$$

touches u from below at $(\bar{x}, \bar{t})$ in a parabolic neighborhood of $(\bar{x}, \bar{t})$, i.e, $B_r(\bar{x}) \times (\bar{t} - r^2, \bar{t}]$ for some $r > 0$. Under the assumption that the Ricci curvature of M is bounded from below, the Jacobian determinant of the parabolic normal map on the contact set $\mathcal{A}_{a,b}$ is estimated using the theory of Jacobi fields with the help of [62], which is essential to prove a priori Harnack estimate. When dealing with the fully nonlinear operators, the sectional curvature condition is required, and then we obtain local Harnack inequality due to the local uniform doubling property of negatively curved manifolds; see Bishop-Gromov Theorem 2.2.4.

Recently, the notion of viscosity solutions introduced by Ishii [30] has been extended on Riemannian manifolds in [5, 51, 64], where they proved comparison, uniqueness and existence results for the viscosity solutions to fully nonlinear elliptic and parabolic equations on Riemannian manifolds. In Section 4.3, we obtain Krylov-Safonov Harnack inequality for viscosity solutions to (1.2.3) from a priori estimates by using regularization by sup- and inf-convolutions, proposed by Jensen [32] in the Euclidean space. For $\varepsilon > 0$, the inf-convolution of u is defined as

$$u_\varepsilon(x_0) := \inf_{y \in \Omega} \left\{ u(y) + \frac{1}{2\varepsilon} d^2(y, x_0) \right\} \quad \text{for } x_0 \in \Omega \subset M.$$

Then, the inf-convolution is semi-concave and hence admits the Hessian almost everywhere thanks to Aleksandrov theorem, [1, 7]. We shall prove in Proposition 4.3.3 that a class of all viscosity solutions for uniformly parabolic operators is invariant under the regularization processes of sup- and inf-convolutions, where the sectional curvature is bounded from below. Therefore, the application of a priori estimate to sup- and inf-convolutions of viscosity solutions gives Harnack inequality for viscosity solutions.

Notation 1.2.1.

- Let $r > 0, \rho > 0$, $x_0 \in M$ and $t_0 \in \mathbb{R}$. We denote

  $$K_{r,\rho}(x_0, t_0) := B_r(x_0) \times (t_0 - \rho, t_0],$$

  where $B_r(x_0)$ is a geodesic ball of radius r centered at x_0. In particular, we denote

  $$K_r(x_0, t_0) := K_{r,r^2}(x_0, t_0).$$

- We denote
$$\fint_Q f := \frac{1}{|Q|}\int_Q f,$$
where $|Q|$ stands for the volume of a set Q of M or $M \times \mathbb{R}$.

We conclude the introduction with a short summary of each chapter of this thesis. In Chapter 2, we briefly recall the theory of viscosity solutions to fully nonlinear elliptic and parabolic equations. We also give some results on Riemannian geometry that are used in the thesis. In Chapter 3, we study long time asymptotics and geometric property for fully nonlinear parabolic equations and the related elliptic eigenvalue problem in the Euclidean space. In Chapter 4, we establish Krylov-Safonov Harnack inequality for viscosity solutions on Riemannian manifolds.

Chapter 2

Preliminaries

2.1 Viscosity solutions

In this section, we give an overview of the theory of viscosity solutions to fully nonlinear equations. The concept of viscosity solutions gives us a way to understand a nonsmooth function as a solution of equations in nondivergence form using maximum principle. The existence of viscosity solutions is obtained most often through the Perron method and uniqueness results. Viscosity solutions provide a general existence and uniqueness theory and stability and compatibility with classical solutions . We refer to [18, 13, 60, 61] and references therein for the results of existence, uniqueness, nice properties and regularity of viscosity solutions.

2.1.1 Uniformly elliptic operator

We introduce the uniformly elliptic operator which is a generalization of the Laplace operator, and give some properties and examples of the uniformly elliptic operators.

Definition 2.1.1. *Let* $\mathrm{Sym}(n)$ *denote the set of* $n \times n$ *symmetric matrices. An operator* $F : \mathrm{Sym}(n) \to \mathbb{R}$ *is said to be uniformly elliptic with the so-called ellipticity constants* $0 < \lambda \leq \Lambda$, *if for any* $S \in \mathrm{Sym}(n)$, *and for any positive semidefinite* $P \in \mathrm{Sym}(n)$, *we have*

$$\lambda \,\mathrm{trace}(P) \leq F(S + P) - F(S) \leq \Lambda \,\mathrm{trace}(P).$$

We state the basic hypotheses on the operator $F : \mathrm{Sym}(n) \to \mathbb{R}$ which will be commonly assumed:

(F1) F is uniformly elliptic and $F(0) = 0$.

(F2) F is positively homogeneous of order one; for all $t \geq 0$ and $S \in \mathrm{Sym}(n)$,

$$F(tS) = tF(S).$$

(F3) F is concave.

We may extend F on $\mathbb{R}^{n^2}$ by defining $F(A) := F\left(\frac{A+A^T}{2}\right)$ for a nonsymmetric matrix A. Examples of the operator satisfying (F1), (F2), and (F3) are Bellman operator, and the Pucci's extremal operator $\mathcal{M}^-$ defined as follows.

Definition 2.1.2 (Pucci's extremal operators)**.** *For $0 < \lambda \leq \Lambda$ (called ellipticity constants), the Pucci's extremal operators are defined as follows: for any $S \in$* $\mathrm{Sym}(n)$,

$$\mathcal{M}^+_{\lambda,\Lambda}(S) := \mathcal{M}^+(S) = \lambda \sum_{e_i<0} e_i + \Lambda \sum_{e_i>0} e_i,$$
$$\mathcal{M}^-_{\lambda,\Lambda}(S) := \mathcal{M}^-(S) = \Lambda \sum_{e_i<0} e_i + \lambda \sum_{e_i>0} e_i,$$

where $e_i = e_i(S)$ are the eigenvalues of S.

In the special case when $\lambda = \Lambda = 1$, the Pucci's extremal operators $\mathcal{M}^\pm$ simply coincide with the trace operator. We notice that the hypothesis (F1) is equivalent to the following: for any $S, P \in \mathrm{Sym}(n)$,

(F1') $$\mathcal{M}^-(P) \leq F(S + P) - F(S) \leq \mathcal{M}^+(P), \quad \text{and} \quad F(0) = 0.$$

We state some properties of the Pucci's operators as a following lemma and refer to [13] for the proof.

Lemma 2.1.1. *Let* $\mathrm{Sym}(n)$ *denote the set of $n \times n$ symmetric matrices. For $S, P \in$* $\mathrm{Sym}(n)$, *the followings hold:*

(a)

$$\mathcal{M}^+(S) = \sup_{A \in \mathcal{S}_{\lambda,\Lambda}} \operatorname{trace}(AS), \quad \textit{and} \quad \mathcal{M}^-(S) = \inf_{A \in \mathcal{S}_{\lambda,\Lambda}} \operatorname{trace}(AS),$$

where $\mathcal{S}_{\lambda,\Lambda}$ *consists of positive definite symmetric matrices in* Sym(n), *whose eigenvalues lie in* $[\lambda, \Lambda]$.

(b) $\mathcal{M}^-(-S) = -\mathcal{M}^+(S)$.

(c) $\mathcal{M}^-(S + P) \leq \mathcal{M}^-(S) + \mathcal{M}^+(P) \leq \mathcal{M}^+(S + P) \leq \mathcal{M}^+(S) + \mathcal{M}^+(P)$.

2.1.2 Viscosity solutions

We recall viscosity solutions, which are the proper notion of the weak solutions for the fully nonlinear elliptic and parabolic equations in nondivergence form.

Definition 2.1.3. *Let* $F : \mathrm{Sym}(n) \to \mathbb{R}$ *be a uniformly elliptic operator, and let* f *and* u *be continuous functions defined on a bounded domain* $\Omega \subset \mathbb{R}^n$. *A function* u *is said to be a* viscosity subsolution *(respectively,* viscosity supersolution*) of*

$$F(D^2u) = f \quad \textit{in} \quad \Omega \tag{2.1.1}$$

when the following holds: if $u - \phi$ *has a local maximum at* x_0 *for any* $x_0 \in \Omega$ *and* $\phi \in C^2(\Omega)$, *then we have*

$$F(D^2\phi(x_0)) \geq f(x_0)$$

(respectively, if $u-\phi$ *has a local minimum at* x_0, *then we have* $F(D^2\phi(x_0)) \leq f(x_0)$*). We say that* u *is a viscosity solution of* $F(D^2u) = f$ *in* Ω *when it is both a viscosity subsolution and supersolution.*

Viscosity solution for the parabolic equation is defined analogously; see Subsection 2.2.3. The notion of viscosity solutions is compatible with the classical notion of solutions as the following lemma.

Lemma 2.1.2. *Assume that* $u \in C^2(\Omega)$. *Then,* u *is a viscosity subsolution of* (2.1.1) *in* Ω *if and only if* $F(D^2u(x)) \geq f(x)$ *for any* $x \in \Omega$.

We end this subsection by mentioning basic facts of viscosity sub and supersolutions of uniformly elliptic equations. The following is the comparison principle which gives the uniqueness of the viscosity solution.

Proposition 2.1.3 (Comparison principle)**.** *Let F be uniformly elliptic, and let $f \in C(\overline{\Omega})$, where $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain. Let $u, v \in C(\overline{\Omega})$ be viscosity sub and super solutions to* (2.1.1)*, respectively. If $u \leq v$ on $\partial\Omega$, then we have $u \leq v \quad on\ \Omega$.*

Proposition 2.1.4 (Hopf's Lemma)**.** *Let F be uniformly elliptic and let $u \in C(\overline{\Omega})$ be a viscosity subsolution of* (2.1.1) *with $f \equiv 0$ satisfying $u \not\equiv 0$ and $u \leq 0$ in Ω, where $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain. If $u(x_0) = 0$ for $x_0 \in \partial\Omega$, then we have*

$$\liminf_{x \in \Omega \to x_0} \frac{u(x_0) - u(x)}{|x_0 - x|} > 0.$$

2.1.3 Regularity for uniformly elliptic and parabolic equations

In this subsection, we summarize regularity estimates for the following fully nonlinear uniformly elliptic equation

$$F(D^2u) = f \quad \text{in } \Omega, \tag{2.1.2}$$

where we assume that F satisfies (F1), and $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain. We refer to [13, 26] and references therein for the proofs.

(i) (Harnack inequality [13, Theorem 4.3]) Let $u \in C(\Omega)$ be a nonnegative viscosity solution of (2.1.2) for $f \in C(\Omega) \cap L^n(\Omega)$. Then for any compact subset $K \subset \Omega$, we have

$$\sup_K u \leq C \left\{ \inf_K u + \|f\|_{L^n(\Omega)} \right\}$$

where $C > 0$ depends only on n, λ, Λ, K and Ω.

(ii) (Local regularity) Let $u \in C(\Omega)$ be a viscosity solution of (2.1.2). The followings hold for an appropriate function f in Ω.

  (a) [13, Proposition 4.10] Hölder regularity for $f \in C(\Omega)$.

  (b) [13, Corollary 5.7 and Theorem 8.3] $C^{1,\alpha}$-regularity ($0 < \alpha < 1$) for a Hölder continuous function f in Ω.

We also assume that F is concave (or convex).

(c) [13, Theorem 6.6] $C^{1,1}$-regularity for $f \equiv 0$.

(d) [13, Theorem 6.1 and Theorem 8.1] $C^{2,\alpha}$-regularity ($0 < \alpha < 1$) for a Hölder continuous function f on Ω.

(e) [13, Theorem 7.1] $W^{2,p}$- regularity for $f \in L^p(\Omega)$ and $n < p < \infty$.

(iii) (Global Regularity) Let $u \in C\left(\overline{\Omega}\right)$ be a viscosity solution of

$$\begin{cases} F(D^2u) = f & \text{in} \quad \Omega, \\ \quad u = g & \text{on} \quad \partial\Omega. \end{cases}$$

(a) [13, Proposition 4.14] Hölder regularity for $f \in C(\Omega) \cap L^n(\Omega)$ and $g \in C^\beta(\partial\Omega)$ $(0 < \beta \le 1)$.

(b) [56] $C^{1,\alpha}$-regularity ($0 < \alpha < 1$) for $f \equiv 0$ and $g \in C^{1,\beta}(\partial\Omega)$ ($0 < \beta \le 1$).

We also assume that F is concave (or convex).

(c) [13, Proposition 9.8] $C^{2,\alpha}$-regularity ($0 < \alpha < 1$) for $f \equiv 0$ and $g \in C^3(\partial\Omega)$.

(d) [63] $W^{2,p}$-regularity for $f \in L^p(\Omega)$ and $g \in W^{2,p}(\Omega)$ ($n < p < \infty$).

For viscosity solutions to the fully nonlinear uniformly parabolic equation

$$F(D^2u) - \partial_t u = f \quad \text{in} \quad \Omega \times (0, T],$$

similar regularity results can be found in [49, 60, 61] and references therein.

2.2 Riemannian geometry

In this thesis, let (M, g) be a smooth, complete Riemannian manifold of dimension n, where g is the Riemannian metric and $\mathrm{Vol} := \mathrm{Vol}_g$ is the Riemannian measure on M. We denote $\langle X, Y\rangle := g(X, Y)$ and $|X|^2 := \langle X, X\rangle$ for $X, Y \in T_xM$, where T_xM is the tangent space at $x \in M$. Let $d(\cdot, \cdot)$ be the distance function on M. For a given point $y \in M$, $d_y(x)$ denotes the distance function to y, i.e., $d_y(x) := d(x, y)$.

We recall the exponential map $\exp : TM \to M$. If $\gamma_{x,X} : \mathbb{R} \to M$ is the geodesic starting at $x \in M$ with velocity $X \in T_xM$, then the exponential map is defined by

$$\exp_x(X) := \gamma_{x,X}(1).$$

We observe that the geodesic $\gamma_{x,X}$ is defined for all time since M is complete. For $X \in T_xM$ with $|X| = 1$, we define the cut time $t_c(X)$ as

$$t_c(X) := \sup\{t > 0 : \exp_x(sX) \text{ is minimizing between } x \text{ and } \exp_x(tX)\}.$$

The cut locus of $x \in M$, denoted by $\mathrm{Cut}(x)$, is defined by

$$\mathrm{Cut}(x) := \{\exp_x(t_c(X)X) : X \in T_xM \text{ with } |X| = 1,\ t_c(X) < +\infty\}.$$

If we define

$$E_x := \{tX \in T_xM : 0 \le t < t_c(X),\ X \in T_xM \text{ with } |X| = 1\} \subset T_xM,$$

it can be proved that $\mathrm{Cut}(x) = \exp_x(\partial E_x)$, $M = \exp_x(E_x) \cup \mathrm{Cut}(x)$, and $\exp_x : E_x \to \exp_x(E_x)$ is a diffeomorphism. We note that $\mathrm{Cut}(x)$ is closed and has measure zero. Given two points x and $y \notin \mathrm{Cut}(x)$, there exists a unique minimizing geodesic $\exp_x(tX)$ (for $X \in E_x$) joining x to y with $y = \exp_x(X)$, and we will write $X = \exp_x^{-1}(y)$. For any $x \notin \mathrm{Cut}(y) \cup \{y\}$, the distance function d_y is smooth at x, and the Gauss lemma implies that

$$\nabla d_y(x) = -\frac{\exp_x^{-1}(y)}{|\exp_x^{-1}(y)|},$$

and

$$\nabla(d_y^2/2)(x) = -\exp_x^{-1}(y).$$

The injectivity radius at x of M is defined as

$$i_M(x) := \sup\{r > 0 : \exp_x \text{ is a diffeomorphism from } B_r(0) \text{ onto } B_r(x)\}.$$

We note that $i_M(x) > 0$ for any $x \in M$ and the map $x \mapsto i_M(x)$ is continuous.

We recall the Hessian of a C^2- function u on M defined as

$$D^2u\,(X, Y) := \langle \nabla_X \nabla u, Y\rangle,$$

for any vector fields X, Y on M, where ∇ denotes the Riemannian connection of M, and ∇u is the gradient of u. The Hessian D^2u is a symmetric 2-tensor in $\mathrm{Sym}\,TM$, whose value at $x \in M$ depends only on u and the values X, Y at x. By a canonical identification of the space of symmetric bilinear forms on T_xM with the space of symmetric endomorphisms of T_xM, the Hessian of u at $x \in M$ can be also viewed as a symmetric endomorphism of T_xM:

$$D^2u(x) \cdot X = \nabla_X \nabla u, \quad \forall X \in T_xM.$$

We will write $D^2u(x)\,(X, X) = \left\langle D^2u(x) \cdot X,\ X \right\rangle$ for $X \in T_xM$.

Let ξ be a vector field along a differentiable curve $\gamma : [0, a] \to M$. We denote by $\frac{D\xi}{dt}(t) = \nabla_{\dot{\gamma}(t)}\xi(t)$, the covariant derivative of ξ along γ. A vector field ξ along γ is said to be parallel along γ when

$$\frac{D\xi}{dt}(t) \equiv 0 \quad \text{on } [0, a].$$

If $\gamma : [0, 1] \to M$ is a unique minimizing geodesic joining x to y, then for any $\zeta \in T_xM$, there exists a unique parallel vector field, denoted by $L_{x,y}\zeta(t)$, along γ such that $L_{x,y}\zeta(0) = \zeta$. The parallel transport of ζ from x to y , denoted by $L_{x,y}\zeta$, is defined as

$$L_{x,y}\zeta := L_{x,y}\zeta(1) \in T_yM,$$

which will induce a linear isometry $L_{x,y} : T_xM \to T_yM$. We note that $L_{y,x} = L_{x,y}^{-1}$ and

$$\left\langle L_{x,y}\zeta, \nu \right\rangle_y = \left\langle \zeta, L_{y,x}\nu \right\rangle_x, \quad \forall \zeta \in T_xM,\ \ \nu \in T_yM. \tag{2.2.1}$$

We also define the parallel transport of a symmetric bilinear form along the unique minimizing geodesic; see [5, p. 311].

Definition 2.2.1. *Let $x, y \in M$, and let $\gamma : [0, 1] \to M$ be a unique minimizing geodesic joining x to y. For $S \in \mathrm{Sym}\,TM_x$, the parallel transport of S from x to y, denoted by $L_{x,y} \circ S$, is a symmetric bilinear form on T_yM satisfying*

$$\left\langle \left(L_{x,y} \circ S\right) \cdot \nu, \nu \right\rangle_y := \left\langle S \cdot \left(L_{y,x}\nu\right), L_{y,x}\nu \right\rangle_x, \qquad \forall \nu \in T_yM.$$

Identifying the space of symmetric bilinear forms on T_yM with the space of symmetric endomorphisms of T_yM, $L_{x,y} \circ S$ can be considered as a symmetric endomorphism of T_yM such that

$$\left(L_{x,y} \circ S\right) \cdot \nu = L_{x,y}\left(S \cdot \left(L_{y,x}\nu\right)\right), \qquad \forall \nu \in T_yM.$$

Then it is not difficult to check that S and $L_{x,y} \circ S$ have the same eigenvalues.

Let the Riemannain curvature tensor be defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

where ∇ denotes the Riemannian connection of M. For two linearly independent vectors $X, Y \in T_x M$, we define the sectional curvature of the plane determined by X and Y as

$$\mathrm{Sec}(X,Y) := \frac{\langle R(X,Y)X, Y\rangle}{|X|^2|Y|^2 - \langle X, Y\rangle^2}.$$

Let Ric denote the Ricci curvature tensor defined as follows: for a unit vector $X \in T_x M$ and an orthonormal basis $\{X, e_2, \cdots, e_n\}$ of $T_x M$,

$$\mathrm{Ric}(X,X) = \sum_{j=2}^{n} \mathrm{Sec}(X, e_j).$$

As usual, $\mathrm{Ric} \geq \kappa$ on M $(\kappa \in \mathbb{R})$ stands for $\mathrm{Ric}_x \geq \kappa g_x$ for all $x \in M$.

Let M and N be Riemannian manifolds of dimension n and $\phi : M \to N$ be smooth. The Jacobian of ϕ is the absolute value of determinant of the differential $d\phi$, i.e.,

$$\mathrm{Jac}\,\phi(x) := |\det d\phi(x)| \quad \text{for } x \in M.$$

The following is the area formula, which follows easily from the area formula in Euclidean space and a partition of unity.

Lemma 2.2.1 (Area formula)**.** *For any smooth function $\phi : M \times \mathbb{R} \to M \times \mathbb{R}$ and any measurable set $E \subset M \times \mathbb{R}$, we have*

$$\int_E \mathrm{Jac}\,\phi(x,t) dV(x,t) = \int_{M\times\mathbb{R}} \mathcal{H}^0[E \cap \phi^{-1}(y,s)] dV(y,s),$$

where $\mathcal{H}^0$ is the counting measure.

2.2.1 Variation formulas and Volume comparison

First, we recall the first and second variations of the energy function (see for instance, [21]).

Lemma 2.2.2 (First and second variations of energy)**.** *Let $\gamma : [0,1] \to M$ be a minimizing geodesic, and ξ be a vector field along γ. For small $\epsilon > 0$, let $h : (-\epsilon, \epsilon) \times [0,1] \to M$ be a variation of γ defined as*

$$h(r,t) := \exp_{\gamma(t)} r\xi(t).$$

Define the energy function of the variation

$$E(r) := \int_0^1 \left| \frac{\partial h}{\partial t}(r,t) \right|^2 dt, \qquad \textit{for } \; r \in (-\epsilon, \epsilon).$$

Then, we have

(a)

$$E(0) = d^2\left(\gamma(0), \gamma(1)\right),$$

(b)

$$\frac{1}{2}E'(0) = \langle \xi(1), \dot\gamma(1) \rangle - \langle \xi(0), \dot\gamma(0) \rangle,$$

(c)

$$\frac{1}{2}E''(0) = \int_0^1 \left\{ \left\langle \frac{D\xi}{dt}, \frac{D\xi}{dt} \right\rangle - \langle R\left(\dot\gamma(t), \xi(t)\right)\dot\gamma(t),\; \xi(t) \rangle \right\} dt.$$

In particular, if a vector field ξ is parallel along γ, then we have $\frac{D\xi}{dt} \equiv 0$ and $\langle \xi, \dot\gamma \rangle \equiv C$ (for $C \in \mathbb{R}$) on $[0,1]$. In this case, we have the following estimate:

$$E(r) = E(0) - r^2 \int_0^1 \langle R\left(\dot\gamma(t), \xi(t)\right)\dot\gamma(t),\; \xi(t) \rangle \, dt + o\left(r^2\right). \tag{2.2.2}$$

Now, we state some known results on Riemannian manifolds. Under a certain condition on the distance function, we have the estimate for Jacobian of the exponential map and Bishop's volume comparison theorem as follows The proof can be found in [35, p. 286] (see also [48]).

Lemma 2.2.3. *Suppose that M satisfies*

$$\triangle d_p(x) \le \frac{n-1}{d_p(x)} \quad \textit{for} \quad x \notin \mathrm{Cut}(p) \cup \{p\}.$$

(i) For any $x \in M$ *and* $X \in E_x$,

$$\mathrm{Jac}\exp_x(X) = |\det d\exp_x(X)| \leq 1.$$

(ii) (Bishop) For any $x \in M$, $\mathrm{Vol}(B_R(x))/R^n$ *is nonincreasing with respect to R, where* $B_R(x)$ *is a geodesic ball of radius R centered at x. Namely,*

$$\frac{\mathrm{Vol}(B_R(x))}{\mathrm{Vol}(B_r(x))} \leq \frac{R^n}{r^n} \quad \text{if } 0 < r < R.$$

In particular, M satisfies the volume doubling property; i.e., $\mathrm{Vol}(B_{2R}(x)) \leq 2^n \mathrm{Vol}(B_R(x))$.

Assuming Ricci curvature to be bounded from below (see [59] for instance), we have the following volume doubling property in general.

Theorem 2.2.4 (Bishop-Gromov)**.** *Assume that* $\mathrm{Ric} \geq -(n-1)\kappa$ *on M for* $\kappa \geq 0$. *For any* $0 < r < R$, *we have*

$$\frac{\mathrm{Vol}(B_{2r}(z))}{\mathrm{Vol}(B_r(z))} \leq 2^n \cosh^{n-1}\left(2\sqrt{\kappa}R\right). \tag{2.2.3}$$

We observe that the doubling property (2.2.3) implies that for any $0 < r < R < R_0$,

$$\frac{\mathrm{Vol}(B_R(z))}{\mathrm{Vol}(B_r(z))} \leq \mathcal{D}\left(\frac{R}{r}\right)^{\log_2 \mathcal{D}},$$

where $\mathcal{D} := 2^n \cosh^{n-1}\left(2\sqrt{\kappa}R_0\right)$ is the so-called doubling constant. Using the volume doubling property, it is easy to prove the following lemma.

Lemma 2.2.5. *Assume that for any* $z \in M$ *and* $0 < r < 2R_0$, *there exists a doubling constant* $\mathcal{D} > 0$ *such that*

$$\mathrm{Vol}(B_{2r}(z)) \leq \mathcal{D}\,\mathrm{Vol}(B_r(z)).$$

Then we have that for any $B_r(y) \subset B_R(z)$ *with* $0 < r < R < R_0$,

$$\left\{\fint_{B_r(y)} \left|r^2 f\right|^{n\theta}\right\}^{\frac{1}{n\theta}} \leq 2\left\{\fint_{B_R(z)} \left|R^2 f\right|^{n\theta}\right\}^{\frac{1}{n\theta}}; \qquad \theta := \frac{1}{n}\log_2 \mathcal{D}. \tag{2.2.4}$$

In particular, if the sectional curvature of M is bounded from below by $-\kappa$ *(*$\kappa \geq 0$*), then* (2.2.4) *holds with* $\theta := 1 + \log_2 \cosh(4\sqrt{\kappa}R_0)$.

In the parabolic setting, it follows from Lemma 2.2.5 that for any $0 < r < R < R_0$,

$$\left\{ \fint_{K_{r,\alpha r^2}(y,s)} \left| r^2 f \right|^{n\theta+1} \right\}^{\frac{1}{n\theta+1}} \leq 2\alpha^{-\frac{1}{n\theta+1}} \left\{ \fint_{K_R(z,t)} \left| R^2 f \right|^{n\theta+1} \right\}^{\frac{1}{n\theta+1}}, \tag{2.2.5}$$

where $K_{r,\alpha r^2}(y,s) := B_r(y) \times (s - \alpha r^2, s] \subset K_R(z,t) = B_R(z) \times (t - R^2, t]$ for $\alpha > 0$.

2.2.2 Semi-concavity

Semi-concavity of functions on Riemannian manifolds is a natural generalization of concavity. The work of Bangert [7] concerning semi-concave functions enables us to deal with functions that are not twice differentiable in the usual sense.

Definition 2.2.2. *Let Ω be an open set of M. A function $\phi : \Omega \to \mathbb{R}$ is said to be semi-concave at $x_0 \in \Omega$ if there exist a geodesically convex ball $B_r(x_0)$ with $0 < r < i_M(x_0)$, and a smooth function $\Psi : B_r(x_0) \to \mathbb{R}$ such that $\phi + \Psi$ is geodesically concave on $B_r(x_0)$. A function ϕ is semi-concave on Ω if it is semi-concave at each point in Ω.*

The following local characterization of semi-concavity is quoted from [17, Lemma 3.11].

Lemma 2.2.6. *Let $\phi : \Omega \to \mathbb{R}$ be a continuous function and let $x_0 \in \Omega$, where $\Omega \subset M$ is open. Assume that there exist a neighborhood U of x_0, and a constant $C > 0$ such that for any $x \in U$ and $X \in T_xM$ with $|X| = 1$,*

$$\limsup_{r\to 0} \frac{\phi(\exp_x rX) + \phi(\exp_x -rX) - 2\phi(x)}{r^2} \leq C.$$

Then ϕ is semi-concave at x_0.

Hessian bound for the squared distance function is the following lemma which is proved in [17, Lemma 3.12] using the formula for the second variation of energy. According to the local characterization of semi-concavity combined with Lemma 2.2.7, d_y^2 is semi-concave on a bounded open set $\Omega \subset M$ for any $y \in M$, provided that the sectional curvature of M is bounded from below.

Lemma 2.2.7. *Let $x, y \in M$. If* Sec $\geq -\kappa$ $(\kappa \geq 0)$ *along a minimizing geodesic joining x to y, then for any $X \in T_xM$ with $|X| = 1$,*

$$\limsup_{r\to 0} \frac{d_y^2(\exp_x rX) + d_y^2(\exp_x -rX) - 2d_y^2(x)}{r^2} \leq 2\sqrt{\kappa}d_y(x)\coth\left(\sqrt{\kappa}d_y(x)\right).$$

The following result from Bangert is an extension of Aleksandrov's second differentiability theorem that a convex function has second derivatives almost everywhere in the Euclidean space [2] (see also [59, Chapter 14]) .

Theorem 2.2.8 (Aleksandrov-Bangert, [7])**.** *Let $\Omega \subset M$ be an open set and let $\phi : \Omega \to \mathbb{R}$ be semi-concave. Then for almost every $x \in \Omega$, ϕ is differentiable at x, and there exists a symmetric operator $A(x) : T_xM \to T_xM$ characterized by any one of the two equivalent properties:*

(a) for $\xi \in T_xM$, $\quad A(x)\cdot\xi = \nabla_\xi\nabla\phi(x)$,

(b) $\phi(\exp_x \xi) = \phi(x) + \langle\nabla\phi(x), \xi\rangle + \frac{1}{2}\langle A(x)\cdot\xi, \xi\rangle + o\left(|\xi|^2\right) \quad$ as $\xi \to 0$.

The operator $A(x)$ and its associated symmetric bilinear from on T_xM are denoted by $D^2\phi(x)$ and called the Hessian of ϕ at x when no confusion is possible.

2.2.3 Viscosity solutions on Riemannian manifolds

In this subsection, we consider a refined definition of viscosity solutions to parabolic equations slightly different from the usual definition in [64]; see [60] for the Euclidean case.

Definition 2.2.3. *Let $\Omega \subset M$ be open and $T > 0$. Let $u : \Omega\times(0, T] \to \mathbb{R}$ be a lower semi-continuous function. We say that u has a local minimum at $(x_0, t_0) \in \Omega\times(0, T]$ in the parabolic sense if there exists $r > 0$ such that*

$$u(x, t) \geq u(x_0, t_0) \quad \textit{for all } (x, t) \in K_r(x_0, t_0) := B_r(x_0) \times (t_0 - r^2, t_0].$$

Similarly, we can define a local maximum in the parabolic sense.

Definition 2.2.4 (Viscosity sub and super- differentials)**.** *Let $\Omega \subset M$ be open and $T > 0$. Let $u : \Omega \times (0, T] \to \mathbb{R}$ be a lower semi-continuous function. We define the second order parabolic subjet of u at $(x, t) \in \Omega \times (0, T]$ by*

$$\mathcal{P}^{2,-}u(x,t) := \Big\{ \Big(\partial_t\varphi(x,t), \nabla\varphi(x,t), D^2\varphi(x,t)\Big) \in \mathbb{R} \times T_xM \times \operatorname{Sym} TM_x : \varphi \in C^{2,1}\left(\Omega \times (0,T]\right), \\ u - \varphi \ \text{ has a local minimum at } (x,t) \text{ in the parabolic sense}\Big\}.$$

If $(p, \zeta, A) \in \mathcal{P}^{2,-}u(x, t)$, then (p, ζ) and A are called a first order subdifferential (with respect to (t, x)), and a second order subdifferential (with respect to x) of u at (x, t), respectively.

In a similar way, for an upper semi-continuous function $u : \Omega \times (0, T] \to \mathbb{R}$, we define the second order parabolic superjet of u at $(x, t) \in \Omega \times (0, T]$ by

$$\mathcal{P}^{2,+}u(x,t) := \Big\{ \Big(\partial_t\varphi(x,t), \nabla\varphi(x,t), D^2\varphi(x,t)\Big) \in \mathbb{R} \times T_xM \times \operatorname{Sym} TM_x : \varphi \in C^{2,1}\left(\Omega \times (0,T]\right), \\ u - \varphi \ \text{ has a local maximum at } (x,t) \text{ in the parabolic sense}\Big\}.$$

The following characterization of the parabolic subjet $\mathcal{P}^{2,-}u$ can be obtained by a simple modification of [5, Proposition 2.2], [64, Proposition 2.2].

Lemma 2.2.9. *Let $u : \Omega \times (0, T] \to \mathbb{R}$ be a lower semi-continuous function and let $(x, t) \in \Omega \times (0, T]$. The following statements are equivalent:*

(a) $(p, \zeta, A) \in \mathcal{P}^{2,-}u(x, t)$,

(b) for $\xi \in T_xM$ and $\sigma \leq 0$,

$$u\left(\exp_x \xi, t + \sigma\right) \geq u(x,t) + \langle \zeta, \xi \rangle + \sigma p + \frac{1}{2}\langle A \cdot \xi, \xi \rangle + o\left(|\xi|^2 + |\sigma|\right) \ \text{ as } (\xi, \sigma) \to (0,0).$$

Definition 2.2.5 (Viscosity solution)**.** *Let $F : M \times \mathbb{R} \times \mathbb{R} \times TM \times \operatorname{Sym} TM \to \mathbb{R}$, and let $\Omega \subset M$ be open and $T > 0$. We say that an upper semi-continuous function $u : \Omega \times (0, T] \to \mathbb{R}$ is a parabolic viscosity subsolution of the equation $\partial_t u = F(x, t, u, \nabla u, D^2u)$ in $\Omega \times (0, T]$ if*

$$p - F\left(x, t, u(x,t), \zeta, A\right) \leq 0$$

for any $(x, t) \in \Omega \times (0, T]$ and $(p, \zeta, A) \in \mathcal{P}^{2,+}u(x, t)$. Similarly, a lower semi-continuous function $u : \Omega \times (0, T] \to \mathbb{R}$ is said to be a parabolic viscosity supersolution of the equation $\partial_t u = F(x, t, u, \nabla u, D^2u)$ in $\Omega \times (0, T]$ if

$$p - F\left(x, t, u(x,t), \zeta, A\right) \geq 0$$

for any $(x,t) \in \Omega \times (0,T]$ *and* $(p,\zeta,A) \in \mathcal{P}^{2,-}u(x,t)$. *We say that u is a parabolic viscosity solution if u is both a parabolic viscosity subsolution and a parabolic viscosity supersolution.*

We remark that parabolic viscosity solutions at the present time will not be influenced by what is to happen in the future. In the Euclidean space, Juutinen [33] showed that a refined definition of parabolic viscosity solutions is equivalent to the usual one if comparison principle holds. Whenever we refer to a "viscosity (sub or super) solution" to parabolic equations in this thesis, we always mean a "parabolic viscosity (sub or super) solution" for simplicity.

Chapter 3

Asymptotic behavior for fully nonlinear parabolic equations and eigenvalue problems

In this chapter, we study large time behavior of solutions to the following fully nonlinear parabolic equation

$$\begin{cases} F(D^2u^m) - \partial_t u = 0 & \text{in } \Omega \times (0, +\infty), \\ u(\cdot, 0) > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \times (0, +\infty), \end{cases} \tag{3.0.1}$$

in the range of the exponents $m \geq 1$. In this chapter, F is always assumed to be uniformly elliptic with $F(0) = 0$, and $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain. We will impose the common hypothesis (F2) or (F3) on the nonlinear operator F.

3.1 Uniformly parabolic equations

3.1.1 Elliptic eigenvalue problem

For a nonlinear operator F satisfying (F1) and (F2), the existence and uniqueness of principal half-eigenvalues have been explored in [31], and the simplified proof can be found in [2, Theorem 3.4].

Theorem 3.1.1 ([31])**.** *Suppose that F satisfies $(F1)$ and $(F2)$. Then there exist $\varphi \in C^{1,\alpha}(\overline{\Omega}), (0 < \alpha < 1),$ and $\mu > 0$ such that $\varphi > 0$ in Ω and φ satisfies*

$$\begin{cases} -F(D^2\varphi) = \mu\varphi & \text{in } \Omega, \\ \varphi(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{\textbf{EV}}$$

Moreover, μ is unique in the sense that if ρ is another eigenvalue of F in Ω associated with a nonnegative eigenfunction, then $\mu = \rho$; and is simple in the sense that if ψ in $C^0(\overline{\Omega})$ is a solution of (**EV**) *with ψ in place of φ, then ψ is a constant multiple of φ.*

3.1.2 Long-time asymptotics for uniformly parabolic equations

In this subsection, we study fully nonlinear uniformly parabolic equation

$$\begin{cases} F(D^2u) - \partial_t u = 0 & \text{in } \Omega \times (0, +\infty), \\ u(\cdot, 0) = u_0 \in C^0(\overline{\Omega}), & \\ u = 0 & \text{on } \partial\Omega \times (0, +\infty). \end{cases} \tag{3.1.1}$$

In the entire subsection, we assume that Ω is a smooth bounded domain of $\mathbb{R}^n$ and F satisfies $(F1)$. We also assume that initial data $u_0 \in C^0(\overline{\Omega})$ is nonnegative in Ω. In particular, we analyze the asymptotic behavior of the solution u of (3.1.1) when time goes infinity. First, we find the exact decay rate of u comparing with barriers which are separable solutions of the form $\varphi(x)e^{-\mu t}$, where $\varphi(x)$ is the positive eigenfunction and $\mu > 0$ is the principal eigenvalue in Theorem 3.1.1.

Lemma 3.1.2. *Suppose that F satisfies (F1) and (F2). Let u be the solution of* (3.1.1) *and let φ be the positive eigenfunction associated with the eigenvalue $\mu > 0$ in Theorem 3.1.1. For a nonnegative and nonzero $u_0 \in C^0(\overline{\Omega})$, there exist $T_o > 0$ and $0 < C_1 < C_2 < +\infty$ such that*

$$C_1\varphi(x)e^{-\mu T_o} < u(x, T_o) < C_2\varphi(x)e^{-\mu T_o}, \quad \forall x \in \Omega.$$

Moreover, we have

$$C_1\varphi(x)e^{-\mu t} < u(x, t) < C_2\varphi(x)e^{-\mu t}, \quad \forall (x, t) \in \Omega \times [T_o, +\infty). \tag{3.1.2}$$

PROOF. (i) First, we construct a subsolution of $F(D^2w) - w_t = 0$, whose support expands in time. Define $g(x,t) = \frac{1}{t^\beta}\exp\left(-\alpha\frac{r^2}{t}\right)$ for $\alpha = \frac{1}{4\lambda}$, $\beta = \frac{\Lambda n}{2\lambda}$ and $r = |x|$. Then it is easy to see that at the point $(r, 0, \cdots, 0)$,

$$\begin{aligned} \partial_{ij}g &= 0 \qquad \text{if } i \neq j, \\ \partial_{11}g &= 2\alpha\frac{g}{t^2}(2\alpha r^2 - t), \\ \text{and } \partial_{ii}g &= -2\alpha\frac{g}{t} \qquad \text{if } i > 1. \end{aligned}$$

Since $\mathcal{M}^-$ is rotationally symmetric, we can check for $r^2 < \frac{t}{2\alpha}$

$$\mathcal{M}^-(D^2g) - g_t = \Lambda\partial_{11}g + (n-1)\Lambda\partial_{22}g - g_t = \frac{g}{t^2}\left\{t(\beta - 2\alpha\Lambda n) + \alpha r^2(4\Lambda\alpha - 1)\right\} \geq 0$$

and for $r^2 \geq \frac{t}{2\alpha}$

$$\mathcal{M}^-(D^2g) - g_t = \frac{g}{t^2}\left\{t[\beta - 2\alpha(\lambda + (n-1)\Lambda] + \alpha r^2(4\lambda\alpha - 1)\right\} \geq 0.$$

This implies that g satisfies $F(D^2u) - u_t \geq \mathcal{M}^-(D^2u) - u_t \geq 0$. Now we define for any $x_o \in \Omega$,

$$h(x,t) := \max\left\{c_o\frac{1}{(t+\tau_o)^\beta}\exp\left(-\alpha\frac{|x-x_0|^2}{t+\tau_o}\right) - \delta_o,\ 0\right\},$$

where positive constants c_o, τ_o, and δ_o will be chosen later. Then h is also a subsolution of $F(D^2w) - w_t = 0$ as long as $\operatorname{supp} h(\cdot,t) \subset \Omega$.

Select a point x_o and $\eta > 0$ such that $u_0(x_o) > 0, u_0 > 0$ on $\overline{B_\eta}(x_o)$ and $B_{2\eta}(x_o) \subset \Omega$. We recall that $u_0 \in C^0(\overline{\Omega})$ is nonzero and nonnegative. We assume that $\eta > 0$ satisfies $\bigcup_{x\in\Omega_{(-2\eta)}} B_{2\eta}(x) \subset \Omega$ for $\Omega_{(-2\eta)} := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > 2\eta\}$.

We set $m_0 := \min_{B_\eta(x_o)} u_0 > 0$. By choosing c_o, τ_o and δ_o such that

$$\eta^2 = 4\Lambda n\tau_o(> 2\Lambda n\tau_o), \quad \frac{c_o}{\tau_o{}^\beta}\exp\left(-\alpha\frac{\eta^2}{\tau_o}\right) = \delta_o \quad \text{and} \quad \frac{c_o}{\tau_o{}^\beta} - \delta_o = m_0,$$

we can show that $\operatorname{supp} h(\cdot,0) \subset B_\eta(x_o)$ and $h(x,0) \leq m_o$ in $B_\eta(x_o)$, and that the support of $h(x,t)$ is increasing for $0 < t \leq t_0 := \frac{1}{e}\left(\frac{c_o}{\delta_o}\right)^{1/\beta} - \tau_o = \frac{e-1}{4\Lambda\lambda}\eta^2$. We also have that

$$\operatorname{supp} h(\cdot, t_0) = B_{\sqrt{\frac{e}{2}}\eta}(x_o) \quad \text{at } t_0 = \frac{e-1}{4\Lambda\lambda}\eta^2.$$

Comparison principle implies that $h(x,t) \le u(x,t)$ in $\Omega\times(0,t_0]$ and hence $u(x,t_0) \ge h(x,t_0) > 0$ in $B_{\sqrt{\frac{e}{2}}\eta}(x_o)$ at $t_0 = \dfrac{e-1}{4\Lambda\lambda}\eta^2 > 0$. So far, we have proved that if $u(\cdot,0) > 0$ on $\overline{B_\eta}(x_o)$, then

$$u(\cdot,t_0) > 0 \quad \text{on} \quad B_{\sqrt{\frac{e}{2}}\eta}(x_o) \quad \text{at } t_0 = \frac{e-1}{4\Lambda\lambda}\eta^2 > 0.$$

By setting $1+2\epsilon := \sqrt{\frac{e}{2}}$, we also have that $u(\cdot,t_0) > 0$ in $\overline{B_{(1+\epsilon)\eta}}(x_o)$.

(ii) Now, we apply the above argument repeatedly to show that $u(\cdot,T_o) > 0$ in Ω for some $T_o > 0$. For any $y \in \Omega_{(-(1+2\epsilon)\eta)} := \{x\in\Omega : \text{dist}\,(x,\partial\Omega) > (1+2\epsilon)\eta)\}$, we have a chain of uniform number of balls $\left\{B_\eta(x^k)\right\}_{k=1}^N$ such that $x^1 = x_o$, $\partial B_{\epsilon\eta}(x^N) = y$, and $|x^k - x^{k+1}| \le \epsilon\eta$ for $k = 1,\cdots,N-1$. The number $N\in\mathbb{N}$ of balls is bounded by a uniform constant depending on ϵ,η,n and Ω. Applying (i) N- times with barriers h (after suitable translation in space variables at each step) , we deduce that there exist a time $t_1 > 0$ such that $u(\cdot,t_1) > 0$ on $\overline{\Omega_{(-2\epsilon)}}$. Indeed, once $u(x,t^k) > 0$ in some ball $\overline{B_{(1+\epsilon)\eta}}(x^k)$ at $t = t^k$, then $u > 0$ in $\overline{B_{(1+\epsilon)\eta}}(x^k)\times[t^k,+\infty)$, by comparison with a separable solution $\delta_k\varphi_1(x)e^{-\mu_1 t}$ for small $\delta_k > 0$, where φ_1 is the positive eigenfunction in $B_{(1+\epsilon+\epsilon_k)\eta}(x^k)$ (for some small $\epsilon_k > 0$) associated with $\mu_1 > 0$. Therefore, we conclude that there is a time $T_o > 0$ such that

$$u(x,T_o) > 0 \quad \text{in}\,\Omega \quad \text{and} \quad |\nabla u(y,T_o)| > 0,\;\; \forall y\in\partial\Omega$$

by applying (i) again. The second inequality comes from the nontrivial gradient property of the barrier h.

(iii) We choose $C_1 > 0$ small such that $C_1\varphi(x)e^{-\mu T_o} < u(x,T_o)$ in Ω since $|\nabla u(y,T_o)| > 0$ for any $y\in\partial\Omega$. Since u is $C^{1+\gamma}\left(\overline{\Omega}\times[T_o,T_o+1]\right)$, there is $C_2 > 0$ such that

$$C_1\varphi(x)e^{-\mu T_o} < u(x,T_o) < C_2\varphi(x)e^{-\mu T_o} \quad \text{in }\,\Omega.$$

Therefore, the comparison principle implies that (3.1.2). □

Under the assumption that F satisfies (F1) and (F2), we refine the asymptotic behavior of solutions to (3.1.1). Let u be the solution of (3.1.1) and $\mu > 0$ be the principal eigenvalue in Theorem 3.1.1. Define the renormalized function

$$v(x,t) := e^{\mu t}u(x,t). \tag{3.1.3}$$

Then, $v(x,t)$ satisfies

$$v_t = F(D^2v) + \mu v \quad \text{in} \quad \Omega\times(0,+\infty).$$

From Lemma 3.1.2, we deduce the following corollary.

Corollary 3.1.3. *Under the same assumption of Lemma 3.1.2,* $v(x,t) = e^{\mu t}u(x,t)$ *has the following estimates:*

$$C_1\varphi(x) < v(x,t) < C_2\varphi(x) \quad \textit{in} \quad \Omega \times [T_o, +\infty),$$
$$\|v(x,t)\|_{L^\infty(\Omega\times[T_o,+\infty))} \le \frac{C_2}{C_1}\|v(x,T_o)\|_{L^\infty(\Omega)},$$

where $0 < C_1 < C_2 < +\infty$ *and* $T_o > 0$ *are in Lemma 3.1.2.*

Proof. From (3.1.2), we have

$$C_1\varphi(x) < v(x,t) < C_2\varphi(x) \le \frac{C_2}{C_1}e^{\mu T_o}u(x,T_o) = \frac{C_2}{C_1}v(x,T_o) \quad \text{in} \quad \Omega \times [T_o, +\infty),$$

and hence the results follows. □

Now, we shall show that the renormalized parabolic flow $v(x,t) = e^{\mu t}u(x,t)$ converges uniformly to the unique limit as $t \to +\infty$, which is the positive eigenfunction in Theorem 3.1.1. In order to obtain the uniform convergence to the positive eigenfunction, we use the approach presented by Armstrong and Trokhimtchouk [3], who studied the long-time behavior of solutions to the uniformly parabolic equations in $\mathbb{R}^n \times (0, +\infty)$.

Proposition 3.1.4. *Suppose F satisfies (F1) and (F2). Let u be the solution of* (3.1.1) *with a nonzero nonnegative initial data* $u_0 \in C^0(\overline{\Omega})$ *and let* φ *be the positive eigenfunction associated with the principal eigenvalue* $\mu > 0$ *in Theorem 3.1.1. Define* $v(x,t) := e^{\mu t}u(x,t)$*. Then, there exists a unique constant* $\gamma^* > 0$ *depending on* u_0 *such that*

$$\|v(x,t) - \gamma^*\varphi(x)\|_{C^0_x(\overline{\Omega})} \to 0 \quad \textit{as } t \to +\infty.$$

Proof. We recall that v is bounded from Corollary 3.1.3 and then

$$\sup_{s\ge1} \|v(\cdot, \cdot + s)\|_{C^\alpha(\overline{\Omega}\times[0,+\infty))} < +\infty \quad \text{for } 0 < \alpha < 1,$$

from the uniform Hölder regularity (see Theorem 4.23, [60]). For a given sequence $\{s_n\}$ such that $s_n \to +\infty$ as $n \to +\infty$, we find a subsequence $\{s_{n_k}\}$ and a function $w \in C^\alpha\left(\overline{\Omega} \times [0, +\infty)\right)$ such that

$$v(x, t + s_{n_k}) \to w(x,t) \quad \text{locally uniformly in } \overline{\Omega} \times [0, +\infty) \quad \text{as } n_k \to +\infty,$$

according to Arzela-Ascoli Theorem. Then a limit w satisfies $F(D^2w)+\mu w-w_t = 0$ in $\Omega \times (0, +\infty)$. Now, let $\mathcal{A}$ be the set of all sequential limits of $\{v(\cdot, \cdot + s)\}_{s\geq T_o}$, where $T_o > 0$ is given in Corollary 3.1.3. Then any $w \in \mathcal{A}$ satisfies that

$$F(D^2w) + \mu w - w_t = 0 \quad \text{ in } \Omega \times (0, +\infty)$$

and

$$C_1\varphi(x) \leq w(x, t) \leq C_2\varphi(x) \quad \text{ in } \Omega \times (0, +\infty)$$

for some constant $0 < C_1 < C_2 < +\infty$ from Corollary 3.1.3. We define

$$\gamma^* := \inf \{\gamma > 0 : \exists\, w \in \mathcal{A} \text{ such that } \ w \leq \gamma\varphi \ \text{ in } \Omega \times (0, +\infty)\}\,.$$

We note that $0 < C_1 \leq \gamma^* \leq C_2 < +\infty$. We are going to prove that $\mathcal{A} = \{\gamma^*\varphi\}$.

First, we show that $w \leq \gamma^*\varphi$ for any $w \in \mathcal{A}$. Fix $\epsilon > 0$. There exists $\tilde{w} \in \mathcal{A}$ such that $\tilde{w} \leq (\gamma^* + \epsilon)\varphi$ by the definition of γ^* and then we have a sequence of functions, $\{v_n := v(\cdot, \cdot + s_n)\}$, converging to $\tilde{w}$ locally uniformly as $s_n \to +\infty$. This implies that there is $N > 0$ such that $\|v_n(x, 1) - \tilde{w}(x, 1)\|_{L^\infty(\Omega)} \leq \epsilon$ for all $n \geq N$. Maximum principle for $e^{-\mu(t-1)}(v_n - \tilde{w})$ gives us that

$$|v_n(x, t) - \tilde{w}(x, t)| \leq \epsilon e^{\mu(t-1)}, \quad \forall (x, t) \in \Omega \times [1, +\infty)$$

since $e^{-\mu(t-1)}(v_n - \tilde{w})$ satisfies

$$\mathcal{M}^-(D^2z) - z_t \leq 0 \leq \mathcal{M}^+(D^2z) - z_t \quad \text{ in } \Omega \times (1, +\infty)$$

and $e^{-\mu(t-1)}(v_n - \tilde{w}) = 0$ on $\partial\Omega \times [1, +\infty)$. We apply the Regularity Theory to $e^{-\mu(t-1)}(v_n - \tilde{w})$ to estimate

$$\|\nabla_x (v_n(x, 2) - \tilde{w}(x, 2))\|_{L^\infty(\Omega)} \leq C_o e^\mu \epsilon,$$

where the uniform constant $C_o > 0$ depends only on λ, Λ, n and Ω. Since $v_n(x, 2) - \tilde{w}(x, 2) = 0$ for $x \in \partial\Omega$, $\|v_n(x, 2) - \tilde{w}(x, 2)\|_{L^\infty(\Omega)} \leq e^\mu\epsilon$, and $\|\nabla_x (v_n(x, 2) - \tilde{w}(x, 2))\|_{L^\infty(\Omega)} \leq C_o e^\mu \epsilon$, we deduce that

$$|v_n(x, 2) - \tilde{w}(x, 2)| \leq \tilde{C}\epsilon\varphi(x) \quad \text{ in } \ \Omega$$

for some uniform constant $\tilde{C} > 0$ depending only on C_o, Ω, and φ. Therefore we have

$$v_n(x, 2) = v(x, 2 + s_n) \leq \tilde{w}(x, 2) + \tilde{C}\epsilon\varphi(x) \leq (\gamma^* + \epsilon + \tilde{C}\epsilon)\varphi(x) \quad \text{ in } \Omega,$$

for some large $s_n > 0$. By using maximum principle for $u(x,t) - e^{-\mu t}(\gamma^* + \epsilon + \tilde{C}\epsilon)\varphi(x)$, we conclude that

$$v(x,t) = e^{\mu t}u(x,t) \leq (\gamma^* + \epsilon + \tilde{C}\epsilon)\varphi(x) \quad \text{for} \quad t \geq 2 + s_n.$$

and it follows that

$$w \leq (\gamma^* + \epsilon + \tilde{C}\epsilon)\varphi \quad \text{for all} \quad w \in \mathcal{A}.$$

Since ϵ is arbitrary and $\tilde{C}$ is uniform, we have that $w \leq \gamma^*\varphi$ for all $w \in \mathcal{A}$.

Second, we show that $\mathcal{A}$ has only one element. Assume that $\tilde{w} \not\equiv \gamma^*\varphi$ for some $\tilde{w} \in \mathcal{A}$. By the definition of $\mathcal{A}$, we can find a sequence of functions $\{v(\cdot, \cdot + s_n)\}$ whose limit is $\tilde{w}$. Now we set $u_1(x,t) := e^{-\mu t}\tilde{w}(x,t)$ and $u_2(x,t) := \gamma^*\varphi(x)e^{-\mu t}$. Then $u_1 - u_2$ satisfies

$$\begin{cases} \mathcal{M}^-(D^2 z) - z_t \leq 0 \leq \mathcal{M}^+(D^2 z) - z_t & \text{in } \Omega \times (0, +\infty), \\ z = 0 & \text{on } \partial\Omega \times (0, +\infty). \end{cases}$$

It is easy to check that $\tilde{w}(\cdot, 0) \lneqq \gamma^*\varphi$. Indeed, if not, the uniqueness says that $\tilde{w} \equiv \gamma^*\varphi$ in $\Omega \times (0, +\infty)$, which is a contradiction. Thus the strong maximum principle and Hopf's Lemma imply that $u_2(x,1) - u_1(x,1) > 0$ in Ω and

$$u_2(x,1) - u_1(x,1) \geq \delta\varphi(x) \quad \text{in} \quad \Omega$$

for small $\delta > 0$, namely, $\tilde{w}(x,1) \leq (\gamma^* - \delta e^{\mu})\varphi(x)$ in Ω. Therefore we have that

$$e^{\mu(t+1)}u_1(x,t+1) = \tilde{w}(x,t+1) \leq (\gamma^* - \delta e^{\mu})\varphi(x) \quad \text{in } \Omega \times (0, +\infty)$$

from the comparison principle. Setting $t_n := s_n + 1$, we get

$$v(x, t + t_n) \to \tilde{w}(x, t+1) \quad \text{locally uniformly in } \overline{\Omega} \times [0, +\infty),$$

as $n \to +\infty$, which is a contradiction to the definition of γ^* since $\tilde{w}(x,t+1) \leq (\gamma^* - \delta e^{\mu})\varphi(x)$. Therefore we conclude that $\mathcal{A} = \{\gamma^*\varphi\}$.

Now we take $s_n = n$ for $n \in \mathbb{N}$. Then for $\epsilon > 0$, we can choose $N \in \mathbb{N}$ such that if $n \geq N$, then $\|v(\cdot, \cdot + n) - \gamma^*\varphi\|_{C^0(\overline{\Omega}\times[0,1])} \leq \epsilon$, which means

$$\|v(x,t) - \gamma^*\varphi(x)\|_{L^\infty(\Omega\times[N,+\infty))} \leq \epsilon.$$

This finishes the proof. □

Under the additional assumption that F is concave, we obtain the following corollary.

Corollary 3.1.5. *Suppose that F satisfies (F1), (F2) and (F3). Let u, v and φ be the functions in Proposition 3.1.4. Then we have*

$$\|v(x,t) - \gamma^*\varphi(x)\|_{C^k_x(\overline{\Omega})} \to 0 \quad \text{as } t \to +\infty$$

for $k = 1, 2$, where $\gamma^* > 0$ is the constant in Proposition 3.1.4.

Proof. Since F is concave, the eigenfunction φ is of $C^{2,\alpha}(\overline{\Omega})$ and u and v also belong to $C^{2,\alpha}\left(\overline{\Omega} \times (0, +\infty)\right)$. Moreover, we have the uniform $C^{2,\alpha}$ estimate:

$$\sup_{s \geq 1} \|v(\cdot, \cdot + s)\|_{C^{2,\alpha}\left(\overline{\Omega}\times[0,+\infty)\right)} < +\infty \tag{3.1.4}$$

since v is bounded from Corollary 3.1.3. Arzela-Ascoli Theorem says that for any $\{s_n\}$ such that $s_n \to +\infty$, there is a subsequence $\{s_{n_k}\}$ satisfying

$$\begin{cases} v_{n_k} := v(\cdot, \cdot + s_{n_k}) & \to & \gamma^*\varphi \\ \nabla_x v_{n_k} & \to & \gamma^*\nabla\varphi \\ D^2_x v_{n_k} & \to & \gamma^* D^2\varphi \end{cases}$$

locally uniformly in $\overline{\Omega} \times [0, +\infty)$ as $n_k \to +\infty$, since $v(\cdot, t)$ converges uniformly to the unique limit $\gamma^*\varphi$ from Proposition 3.1.4. Therefore, as $t \to +\infty$, $\nabla_x v(\cdot, t)$ and $D^2_x v(\cdot, t)$ converge to $\gamma^*\nabla\phi$ and $\gamma^* D^2\phi$ in $\overline{\Omega}$, respectively, and then the uniform convergence follows from the uniform $C^{2,\alpha}$ estimate, (3.1.4). □

3.1.3 Log- concavity

In this subsection, we study log-concavity of solutions of (3.1.1) and (**EV**) provided that a smooth bounded domain Ω is convex, and the operator F satisfies (F1), (F2) and (F3). First, let us approximate the operator with smooth operators as follows.

Lemma 3.1.6. *Suppose that F satisfies (F1), (F2) and (F3). Then there are smooth operators $F^\epsilon : \mathbb{S}^{n\times n} \to \mathbb{R}$, which converges to F uniformly and satisfies (F1), (F3) and*

$$\left|F^\epsilon_{ij}(M)M_{ij} - F^\epsilon(M)\right| \leq \sqrt{n}\Lambda\epsilon \quad \text{for } M = (M_{ij}) \in \mathbb{S}^{n\times n}, \tag{3.1.5}$$

where $F^\epsilon_{ij}(M) := \dfrac{\partial F^\epsilon}{\partial p_{ij}}(M)$.

Proof. We extend F to $\mathbb{R}^{n^2}$ by $F(M) = F\left(\frac{M+M^t}{2}\right)$. Then we can show that F is Lipschitz continuous in $\mathbb{R}^{n^2}$ with a Lipschitz constant $\sqrt{n}\Lambda$ by using the uniform ellipticity of F, $(F1)$, and the fact that $\mathcal{M}^+(N) = \Lambda tr(N) \leq \sqrt{n}\Lambda\left(\sum_{i,j=1}^{n} N_{ij}^2\right)^{\frac{1}{2}}$ for $0 \leq N = (N_{ij}) \in \mathbb{S}^{n\times n}$.

Let $\psi \in C_0^\infty(\mathbb{R}^{n^2})$ be a standard mollifier with $\int_{\mathbb{R}^{n^2}} \psi(Z)dZ = 1$ and $\operatorname{supp}(\psi) \subset B_1(0)$ and let $\psi_\epsilon(Z) = \dfrac{1}{\epsilon^{n^2}}\psi\left(\dfrac{Z}{\epsilon}\right)$. Define F^ϵ as

$$F^\epsilon(Z) := F * \psi_\epsilon(Z) = \int_{\mathbb{R}^{n^2}} F(Z-Y)\psi_\epsilon(Y)dY.$$

Then it is easy to show that F^ϵ is smooth, uniformly elliptic (with the same ellipticity constants λ, Λ) and concave. We can also show that F^ϵ satisfies $F^\epsilon(M) = F^\epsilon(M^t)$ and

$$|F^\epsilon(M) - F(M)| \leq \sqrt{n}\Lambda\epsilon$$

since F is Lipschitz continuous in $\mathbb{R}^{n^2}$ with a Lipschitz constant $\sqrt{n}\Lambda$. Thus F^ϵ converges uniformly to F.

Now, it remains to show that for any $M = (M_{ij}) \in \mathbb{S}^{n\times n}$,

$$\left|F_{ij}^\epsilon(M)M_{ij} - F^\epsilon(M)\right| = |DF^\epsilon(M)\cdot M - F^\epsilon(M)| \leq \sqrt{n}\Lambda\epsilon.$$

Since F is Lipschitz continuous, F is differentiable almost everywhere from Rademacher's Theorem. Moreover, we have $\|DF\|_{L^\infty(\mathbb{R}^{n^2})} \leq \sqrt{n}\Lambda$. The condition $(F2)$ implies that $\dfrac{F((1+t)Z) - F(Z)}{t} = F(Z)$ for $Z \in \mathbb{R}^{n^2}$ and $t > 0$ and hence

$$DF(Z)\cdot Z = F(Z) \quad \text{a.e. } Z \in \mathbb{R}^{n^2}.$$

Thus we have

$$\begin{aligned} DF^\epsilon(Z)\cdot Z - F^\epsilon(Z) &= \int_{\mathbb{R}^{n^2}} \left(DF(Y)\cdot Z - F(Y)\right)\psi_\epsilon(Z-Y)\,dY \\ &= \int_{\mathbb{R}^{n^2}} DF(Y)\cdot(Z-Y)\,\psi_\epsilon(Z-Y)\,dY \end{aligned}$$

and then $|DF^\epsilon(Z)\cdot Z - F^\epsilon(Z)| \leq \sqrt{n}\Lambda\epsilon$ for $Z \in \mathbb{R}^{n^2}$. Therefore we conclude that $\left|F_{ij}^\epsilon(M)M_{ij} - F^\epsilon(M)\right| \leq \sqrt{n}\Lambda\epsilon$ for $M \in \mathbb{S}^{n\times n}$. □

Remark 3.1.7.

i) If a differentiable operator F satisfies (F2), F should be linear. If F also satisfies $(F1)$, then F becomes Laplacian after a suitable transformation.

ii) Let $\sigma_k(D^2u) := \sum_{i_1<\cdots<i_k} \lambda_{i_1}\cdots\lambda_{i_k}$ for the eigenvalues $\lambda_1 \le \cdots \le \lambda_n$ of D^2u. Then the operator $F(D^2u) := \sigma_k(D^2u)^{\frac{1}{k}}$ satisfies the conditions $(F2)$ and $(F3)$.

Now, we prove that log-concavity is preserved under the parabolic flow of (3.1.1) when initial data has such geometric property.

Lemma 3.1.8. *Suppose that F satisfies $(F1), (F2)$ and $(F3)$ and that Ω is strictly convex. Let u be the solution of (3.1.1) with initial data $u_0 \in C_b(\overline{\Omega})$. If $\log u_0$ is concave, then the solution $u(x,t)$ is log-concave in the x variables for all $t \in (0,+\infty)$, i.e.,*

$$D_x^2 \log u \le 0 \quad \text{in } \ \Omega \times (0,+\infty).$$

Proof. (i) First, we prove the preservation of log-concavity assuming that $u_0 \in C^{2,\gamma}(\overline{\Omega}) \cap C_b(\overline{\Omega})$ $(0 < \gamma < 1)$, $D_x^2 \log u_0 \le 0$ in Ω, and $F(D^2u_0) = 0$ on $\partial\Omega$. We approximate F by F^ϵ as Lemma 3.1.6 and we may assume that $F^\epsilon(0) = 0$ by subtracting $F^\epsilon(0)$ to F^ϵ. We also approximate u_0 by $u_0^\epsilon \in C^{2,\gamma}(\overline{\Omega}) \cap C_b(\overline{\Omega})$ for small $\epsilon > 0$ such that

$$u_0^\epsilon \to u_0 \quad \text{in } C^{2,\gamma}(\overline{\Omega}), \ \ D_x^2 \log u_0^\epsilon \le \epsilon \mathbf{I} \ \text{ in } \ \Omega, \quad \text{and} \quad F^\epsilon(D^2u_0^\epsilon) = 0 \quad \text{on} \quad \partial\Omega.$$

In fact, let u_0^ϵ be the solution of

$$\begin{cases} F^\epsilon(D^2u_0^\epsilon) = F(D^2u_0) & \text{in } \Omega, \\ u_0^\epsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

Then we have the uniform global $C^{2,\gamma}$ estimate for u_0^ϵ $(0 < \gamma < 1)$ from [13] so we construct such initial data u_0^ϵ from Arzela-Ascoli Theorem. Indeed, we first note that $u_0^\epsilon \ge \delta_o u_0$ uniformly in Ω for some $\delta_o > 0$ since $u_0 \in C_b(\overline{\Omega})$. Then the argument below, (3.1.8) says that there is a uniform $\eta > 0$ such that $D^2 \log u_0^\epsilon \le 0$ on $\Omega \backslash \Omega_{(-\eta)}$ and then uniform C^2 convergence in $\Omega_{(-\eta)}$, up to a subsequence, implies that $D^2 \log u_0^\epsilon \le \epsilon \mathbf{I}$ on $\Omega\backslash\Omega_{(-\eta)}$ for small $\epsilon > 0$. Thus we deduce that $D_x^2 \log u_0^\epsilon \le \epsilon\mathbf{I}$ in Ω for small $\epsilon > 0$.

Let u^ϵ be the solution of (3.1.1) with the operator F^ϵ and initial data u_0^ϵ.Then we have the uniform global $C^{2,\gamma}$ estimate for u^ϵ $(0 < \gamma < 1)$ from Theorem 3.2 in [61] ; for a fixed $T > 0$,

$$\|u^\epsilon\|_{C^{2,\gamma}(\overline{\Omega}\times[0,T])} < C \quad \text{uniformly.} \tag{3.1.6}$$

The uniform $C^{2,\gamma}$ estimate gives the uniform convergence of u^ϵ to u in $C^2\left(\overline{\Omega}\times[0,T]\right)$ by Arzela-Ascoli Theorem since the family of viscosity solutions is closed in the topology of local uniform convergence, where we recall that u is the solution of (3.1.1) with the operator F and initial data u_0. Since $F^\epsilon \geq \mathcal{M}^-$, we have the uniform lower bound from the comparison principle;

$$u^\epsilon \geq \tilde{u} > 0 \quad \text{in } \ \Omega\times(0,+\infty),$$

where $\tilde{u}$ is the solution of (3.1.1) with the Pucci's operator $\mathcal{M}^-$ and initial data $\delta_o u_0$. We also observe that $|\nabla_x u^\epsilon(x,t)| > c_o > 0$ uniformly for $(x,t)\in\partial\Omega\times[0,T]$ for small $c_o > 0$.

To show log-concavity of u, we consider an approximating solution u^ϵ and we set

$$g^\epsilon := \log u^\epsilon,$$

which is finite and smooth in Ω and takes the value $g^\epsilon = -\infty$ on $\partial\Omega\times[0,+\infty)$. The function g^ϵ satisfies the equation

$$\partial_t g = e^{-g} F^\epsilon\left(e^g\left(D^2 g + DgDg^t\right)\right) \quad \text{in } \ \Omega\times(0,+\infty).$$

To estimate the maximum of its second derivatives, we define, for a given $0 < \delta < 1$,

$$Z(t) := \sup_{y\in\Omega}\sup_{|e_\beta|=1} \ g^\epsilon_{,\beta\beta}(y,t) + \psi(t),$$

where e_β is a unit vector in $\mathbb{R}^n$ and $\psi(t) := -\delta - \delta\tan\left(3K\sqrt{\delta}\,t\right)$. The constant $K > 0$, independent of $\epsilon > 0$ and $\delta > 0$, will be chosen later.

Now, fix $T > 0$ and suppose that there exists $t_o \in \left[0, \min\left(\frac{\pi}{12K\sqrt{\delta}}, T\right)\right] \subset [0,T]$ such that $Z(t_o) = 0$. We may assume that t_o is the first time for Z to vanish and

$$Z(t_o) = g^\epsilon_{,\alpha\alpha}(x_o,t_o) + \psi(t_o) = 0$$

for some direction $e_\alpha \in \mathbb{R}^n$ with $|e_\alpha| = 1$ and some point $x_o \in \overline{\Omega}$. Then the unit vectior e_α is an eigen-direction of the Hessian matrix $D_x^2 g^\epsilon(x_o, t_o)$, which implies that

$$g^\epsilon_{,\alpha\beta}(x_o, t_o) = 0 \quad \text{for} \quad \beta \neq \alpha,$$

using orthonormal coordinates in which e_α is taken as one of the coordinate axes. We notice that $Z(0) \leq \epsilon - \delta < 0$ if $0 < \epsilon < \delta$ from the assumption on the initial data u_0, which implies that $t_o > 0$.

We claim that x_o is an interior point of Ω by proving that for any $\tilde{x} \in \partial\Omega, t > 0$

$$g^\epsilon_{,\alpha\alpha}(x, t) = \frac{u^\epsilon\, u^\epsilon_{,\alpha\alpha} - (u^\epsilon_{,\alpha})^2}{(u^\epsilon)^2} \to -\infty \quad \text{as } \ \Omega \ni x \to \tilde{x} \in \partial\Omega. \tag{3.1.7}$$

The above inequality holds when e_α is not a tangential direction to $\partial\Omega$ at $\tilde{x}$, since $|D^2 u^\epsilon|$ is uniformly bounded and $u^\epsilon = 0$ on $\partial\Omega$, $|\nabla u^\epsilon| > 0$ on $\partial\Omega$ by Hopf's lemma. If e_α is a tangential direction to $\partial\Omega$ at $\tilde{x}$, we take a coordinate system such that $\tilde{x} = 0$ and the tangent plane to $\partial\Omega$ at $\tilde{x} = 0 \in \partial\Omega$ is $x_n = 0$ with e_n being the inner normal vector. Let the boundary be given locally by the equation $x_n = f(x')$ for $x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}$. We introduce the change of variables;

$$y_i = x_i \ (i = 1, \cdots, n-1), \quad y_n = x_n - f(x'), \quad v(y, t) = u^\epsilon(x, t).$$

Then along any tangent direction e_τ to $\partial\Omega$ at $\tilde{x}$, we have

$$u^\epsilon_{,\tau\tau}(x, t) = v_{,\tau\tau}(y, t) - 2v_{,n\tau}(y, t)\, f_\tau(x') + v_{,nn}(y, t)\, f_\tau^2(x') - v_{,n}(y, t)\, f_{,\tau\tau}(x').$$

Using the fact that $v_{,ii}(0, t) = 0$ from the zero boundary condition and $f_{,i}(0) = 0$ for $i = 1, \cdots, n-1$, we obtain

$$u^\epsilon_{,\tau\tau}(\tilde{x}, t) = u^\epsilon_{,\tau\tau}(0, t) = -v_{,n}(0, t)\, f_{,\tau\tau}(0) = -u^\epsilon_{,n}(\tilde{x}, t) f_{,\tau\tau}(0) < 0$$

from Hopf's lemma since $f_{,\tau\tau}(0) > 0$. We note that $f_{,\tau\tau}(0) > 0$ for a tangent vector e_τ since Ω is strictly convex and that $u^\epsilon_\tau(\tilde{x}, t) = 0$ from the zero boundary condition. Thus $g^\epsilon_{,\tau\tau}(x, t)$ tends to $-\infty$ when $x \in \Omega$ goes to $\tilde{x} \in \partial\Omega$ for any tangential vector e_τ to $\partial\Omega$ at $\tilde{x} \in \partial\Omega$, so this is true for the tangential vector e_α. Therefore we have proved (3.1.7).

Moreover, from the uniform $C^{2,\gamma}$ estimate of u^ϵ, (3.1.6), we can find a small $\eta > 0$, independent of $\epsilon, \delta > 0$, such that

$$g^\epsilon_{,\alpha\alpha}(x, t) \leq 0 \quad \text{for } (x, t) \in \left(\Omega \backslash \Omega_{(-\eta)}\right) \times [0, T],$$

where $\Omega_{(-\eta)} = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \eta\}$. Indeed, we first observe that for any $(x,t) \in \partial\Omega \times [0,T]$,

$$u^\epsilon_{,\tau\tau}(x,t) < -c_o\kappa_o < 0 \quad \text{for any tangential direction } \tau \text{ to } \partial\Omega \text{ at } x,$$

with $\kappa_o > 0$, the lower bound of the curvature of $\partial\Omega$, from the above argument, since $|\nabla_x u^\epsilon| > c_o > 0$ uniformly on $\partial\Omega \times [0,T]$. If $\eta > 0$ is small enough, then for $x \in \Omega \setminus \Omega_{(-\eta)}$, there exists a unique $\tilde{x} \in \partial\Omega$ such that $|x - \tilde{x}| = \operatorname{dist}(x, \partial\Omega)$. For each $\tilde{x} \in \partial\Omega$, let $\nu(\tilde{x})$ be the outer normal vector to $\partial\Omega$ at $\tilde{x} \in \partial\Omega$ and let $\tau(\tilde{x})$ be the unit vector such that $\tau(\tilde{x}) \perp \nu(\tilde{x})$ and

$$e_\alpha := \beta_1(\tilde{x})\tau(\tilde{x}) + \beta_2(\tilde{x})\nu(\tilde{x})$$

with $\beta_1^2 + \beta_2^2 = 1$. We define $\nu(x) := \nu(\tilde{x})$ and $\tau(x) := \tau(\tilde{x})$ for each $x \in \Omega\backslash\Omega_{(-\eta)}$, where $\tilde{x}$ is the unique boundary point of Ω such that $|x - \tilde{x}| = \operatorname{dist}(x, \partial\Omega)$. Then the uniform $C^{2,\gamma}$ estimate implies that there is a uniform small $\eta > 0$ such that

$$|u^\epsilon_\nu(x,t)| > \frac{c_o}{2}, \quad u^\epsilon_{,\tau\tau}(x,t) < -\frac{c_o\kappa_o}{2} \quad \text{on } \left(\Omega\backslash\Omega_{(-\eta)}\right) \times [0,T].$$

For each $x \in \Omega\backslash\Omega_{(-\eta)}$ and $t \in [0,T]$ we have

$$\begin{aligned} u^\epsilon_{,\tau}(x,t) = u^\epsilon_{,\tau(\tilde{x})}(x,t) &= u^\epsilon_{,\tau(\tilde{x})}(\tilde{x},t) + \nabla u^\epsilon_{,\tau(\tilde{x})}(x^*,t)\cdot(x-\tilde{x}) \\ &\le 0 + C_o|x-\tilde{x}| = C_o\operatorname{dist}(x,\partial\Omega) \le \tilde{C}_o u^\epsilon(x,t) \end{aligned}$$

for some $x^* \in \Omega$ and for uniform constants $0 < C_o < \tilde{C}_o$ from the uniform $C^{2,\gamma}$ estimates since $|\nabla_x u^\epsilon| > c_o > 0$ uniformly on $\partial\Omega \times [0,T]$. Thus for $x \in \Omega\backslash\Omega_{(-\eta)}$, we have

$$\begin{aligned} (u^\epsilon)^2 g^\epsilon_{,\alpha\alpha}(x,t) &= u^\epsilon u^\epsilon_{,\alpha\alpha}(x,t) - (u^\epsilon_{,\alpha}(x,t))^2 \\ &= u^\epsilon\left\{\beta_1^2(\tilde{x})u^\epsilon_{,\tau\tau}(x,t) + 2\beta_1(\tilde{x})\beta_2(\tilde{x})u^\epsilon_{,\tau\nu}(x,t) + \beta_2^2(\tilde{x})u^\epsilon_{,\nu\nu}(x,t)\right\} \\ &\quad - \beta_2^2(\tilde{x})\left(u^\epsilon_\nu(x,t)\right)^2 - \beta_1^2(\tilde{x})(u^\epsilon_\tau(x,t))^2 - 2\beta_1(\tilde{x})\beta_2(\tilde{x})u^\epsilon_\tau(x,t)u^\epsilon_\nu(x,t) \\ &\le u^\epsilon\left\{\beta_1^2(\tilde{x})u^\epsilon_{,\tau\tau}(x,t) + 2\beta_1(\tilde{x})\beta_2(\tilde{x})u^\epsilon_{,\tau\nu}(x,t) + \beta_2^2(\tilde{x})u^\epsilon_{,\nu\nu}(x,t)\right\} \\ &\quad - \frac{\beta_2^2}{2}(\tilde{x})\left(u^\epsilon_\nu(x,t)\right)^2 + \beta_1^2(\tilde{x})(u^\epsilon_\tau(x,t))^2 \\ &\le u^\epsilon(x,t)\left(-\frac{c_o\kappa_o}{2}\beta_1^2 + \frac{c_o\kappa_o}{4}\beta_1^2 + C_o\beta_2^2\right) - \frac{c_o^2}{8}\beta_2^2 + \tilde{C}_o^2(u^\epsilon(x,t))^2\beta_1^2 \\ &= -\tilde{C}_o^2 u^\epsilon(x,t)\left\{\frac{c_o\kappa_o}{4\tilde{C}_o^2} - u^\epsilon(x,t)\right\}\beta_1^2 - \left\{\frac{c_o^2}{8} - C_o u^\epsilon(x,t)\right\}\beta_2^2, \end{aligned} \tag{3.1.8}$$

for uniform $\tilde{C}_o > C_o > 0$, where we use Young's inequality and the uniform $C^{2,\gamma}$ estimate. By using the uniform $C^{2,\gamma}$ estimate again, we choose $\eta > 0$ sufficiently small to deduce that $g^\epsilon_{,\alpha\alpha} \leq 0$ in $\left(\Omega \backslash \Omega_{(-\eta)}\right) \times [0,T]$. Therefore, we conclude that the maximum point x_o belongs to $\overline{\Omega}_{(-\eta)}$ since $g^\epsilon_{,\alpha\alpha}(x_o,t_o) = -\psi(t_o) > \delta > 0$.

Next, we look at the evolution equation of $g^\epsilon_{,\alpha\alpha}$, which is given by the equation as below;

$$\begin{aligned}\partial_t g_{,\alpha\alpha} &= F^\epsilon_{ij} \cdot \left(D_{ij} g_{,\alpha\alpha} + D_i g_{,\alpha\alpha} D_j g + D_i g D_j g_{,\alpha\alpha} + 2 D_i g_{,\alpha} D_j g_{,\alpha}\right) \\ &\quad + (g^2_{,\alpha} - g_{,\alpha\alpha}) \left\{ e^{-g} F^\epsilon \left(e^g \left(D_{ij} g + D_i g D_j g \right)\right) - F^\epsilon_{ij} \cdot \left(D_{ij} g + D_i g D_j g \right)\right\} \\ &\quad + e^{-g} F^\epsilon_{ij,kl} \cdot \left\{ e^g \left(D_{ij} g + D_i g D_j g \right)\right\}_\alpha \left\{ e^g \left(D_{kl} g + D_k g D_l g \right)\right\}_\alpha ,\end{aligned}$$

where $F^\epsilon_{ij} = \dfrac{\partial F^\epsilon}{\partial p_{ij}} \left(e^{g^\epsilon} \left(D^2 g^\epsilon + D g^\epsilon (D g^\epsilon)^t \right)\right)$ and $F^\epsilon_{ij,kl} = \dfrac{\partial^2 F^\epsilon}{\partial p_{ij} \partial p_{kl}} \left(e^{g^\epsilon} \left(D^2 g^\epsilon + D g^\epsilon (D g^\epsilon)^t \right)\right)$. Since F^ϵ satisfies $(F3)$ and (3.1.5) with the constant $2\sqrt{n}\Lambda\epsilon$ instead of $\sqrt{n}\Lambda\epsilon$, it follows that

$$\partial_t g^\epsilon_{\alpha\alpha} \leq F^\epsilon_{ij} \cdot \left(D_{ij} g^\epsilon_{,\alpha\alpha} + D_i g^\epsilon_{,\alpha\alpha} D_j g^\epsilon + D_i g^\epsilon D_j g^\epsilon_{,\alpha\alpha} + 2 D_i g^\epsilon_{,\alpha} D_j g^\epsilon_{,\alpha}\right) + 2\sqrt{n}\Lambda e^{-g^\epsilon} \left| (g^\epsilon_{,\alpha})^2 - g^\epsilon_{,\alpha\alpha}\right| \epsilon.$$

At the point of maximum (x_o, t_o), we see that

$$g^\epsilon_{,\alpha\alpha} = -\psi > \delta > 0, \ \nabla_x g^\epsilon_{,\alpha\alpha} = 0, \ D^2_x g^\epsilon_{,\alpha\alpha} \leq 0 \quad \text{and} \ \ g^\epsilon_{,\alpha\beta} = 0, \ \forall \beta \neq \alpha.$$

Thus using the ellipticity condition of F^ϵ, we have at the point of maximum (x_o, t_o),

$$\begin{aligned}\partial_t g^\epsilon_{,\alpha\alpha} &\leq 2 F^\epsilon_{\alpha\alpha} \left(g^\epsilon_{,\alpha\alpha}\right)^2 + 2\sqrt{n}\Lambda e^{-g^\epsilon} \left((g^\epsilon_{,\alpha})^2 + g^\epsilon_{,\alpha\alpha}\right) \epsilon \\ &\leq 2\Lambda \left(g^\epsilon_{,\alpha\alpha}\right)^2 + 2\sqrt{n}\Lambda \frac{\left| u^\epsilon_{,\alpha\alpha}\right|}{(u^\epsilon)^2} \epsilon \\ &\leq 2\Lambda \left(g^\epsilon_{,\alpha\alpha}\right)^2 + K\epsilon,\end{aligned}$$

for a uniform constant $K > 0$, where we choose a uniform $K > 0$ bigger than $2\sqrt{n}\Lambda \left(1 + \sup\limits_{\overline{\Omega}_{(-\eta)} \times [0,T]} \dfrac{|D^2_x u^\epsilon|}{(u^\epsilon)^2}\right)$ from the uniform $C^{2,\gamma}$ estimates for u^ϵ.

On the other hand, when the supremum of $Z(t) - \psi(t) = \sup\limits_{y \in \Omega} \sup\limits_{|e_\beta| = 1} g_{,\beta\beta}(y,t)$ is achieved at a point $x(t) \in \Omega$ with a unit vector $e_{\beta(t)}$ at each time $t > 0$, we can check that $\nabla_x g_{,\beta(t)\beta(t)} = 0$ and $g_{,\beta(t)\beta'(t)} = 0$ at the point $(x(t), t)$ using $|e_\beta(t)|^2 = 1$. Thus, we have that at the first time $t_o > 0$,

$$\begin{aligned} 0 \le Z'(t_o) &= \partial_t g^\epsilon_{,\alpha\alpha}(x_o, t_o) + \psi_t(t_o) \\ &\le \psi_t + 2\Lambda\psi^2 + K\epsilon \le \psi_t + K(\psi^2 + \epsilon). \end{aligned} \tag{3.1.9}$$

But, it is easy to check that if $0 < \epsilon < \delta^2$,

$$\psi_t + K(\psi^2 + \epsilon) < \frac{3K(-\delta^{3/2} + \delta^2)}{\cos(3K\sqrt{\delta}t)} < 0 \quad \text{for } 3K\sqrt{\delta}\,t < \frac{\pi}{2},$$

which is a contradiction to (3.1.9). Therefore, the function Z never reaches 0 for $t \in \left[0, \min\left(\frac{\pi}{12K\sqrt{\delta}}, T\right)\right]$ and hence we obtain that

$$\sup_{y\in\Omega}\sup_{|e_\alpha|=1} \partial_{\alpha\alpha}\log u^\epsilon(y,t) < -\psi(t) \le 2\delta \quad \text{for } t \in \left[0, \min\left(\frac{\pi}{12K\sqrt{\delta}}, T\right)\right]$$

for $0 < \epsilon < \delta^2$. Using the uniform $C^{2,\gamma}$-estimates of u^ϵ,we let ϵ go to 0 and then let δ go to 0 to obtain

$$D_x^2 \log u \le 0 \quad \text{in} \quad \Omega \times [0, T].$$

Therefore, $u(x,t)$ is log-concave in the x variables in $\Omega \times (0, +\infty)$ since T is arbitrary.

(ii) For general initial data $u_0 \in C_b(\overline{\Omega})$, we will show that there is a sequence of log-concave functions $u_{0j} \in C^{2,\gamma}(\overline{\Omega}) \cap C_b(\overline{\Omega})$ satisfying that $F(D^2 u_{0j}) = 0$ on $\partial\Omega$, which converge to u_0 uniformly in $\overline{\Omega}$.

First, we may assume that $u_0 \in C^\infty(\overline{\Omega}) \cap C_b(\overline{\Omega})$ is strictly log-concave. Indeed, we perform a mollification to obtain an approximating sequence $u_{0j} \in C^\infty(\overline{\Omega})$ of log-concave functions, which converges to u_0 uniformly in $\overline{\Omega}$. We modify u_{0j} to make it strictly log-concave;

$$\tilde{u}_{0j}(x) := u_{0j}(x)\exp(-c_j|x|^2),$$

where $c_j > 0$ converges to 0 as $j \to +\infty$.

For a strictly log-concave $u_0 \in C^\infty(\overline{\Omega}) \cap C_b(\overline{\Omega})$, we consider

$$\begin{cases} F(D^2 u_{0j}) = \xi_j \cdot F(D^2 u_0) & \text{in } \Omega, \\ u_{0j} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 \le \xi_j \le 1$ satisfies that $\xi_j \in C_0^\infty(\Omega)$, and $\xi_j \equiv 1$ in $\Omega_{(-1/j)}$ for $\Omega_{(-\delta)} := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\}$. Since we have the global uniform $C^{1,\alpha}$ estimate ($0 < \alpha < 1$) from [13], u_{0j} converges to u_0 in $C^{1,\alpha}(\overline{\Omega})$ as $j \to +\infty$, up to a subsequence. Since $u_{0j} \in C^{2,\gamma}(\overline{\Omega})$, and $F(D^2 u_{0j}) = 0$ on $\partial\Omega$, it remains to show that u_{0j} is log-concave for large j in order to obtain the desired initial data u_{oj} converging to u_0 uniformly.

We use the uniform global $C^{1,\alpha}$ estimate and the scaling property with a similar argument as in the proof of Theorem 3.2.2 to show that there is a uniform constant $\eta_o > 0$ such that

$$|D^2 u_{0j}(x)| \le C \operatorname{dist}(x, \partial\Omega)^{-1+\alpha} \quad \text{for } x \in \Omega \backslash \Omega_{(-\eta_o)},$$

where η_o and $C > 0$ are uniform with respect to j. By choosing a uniform small $0 < \eta < \eta_o$, we have

$$u_{0j}(x) D^2 u_{0j}(x) - \nabla u_{0j}(x) \nabla u_{0j}^t(x) \le C \operatorname{dist}(x, \partial\Omega)^{1+(-1+\alpha)} \mathbf{I} - \delta_o^2 \mathbf{I} \le 0 \quad \text{for} \quad x \in \Omega \backslash \Omega_{(-\eta)}$$

for uniform $C > 0$ and $\delta_o > 0$ since $u_0, u_{oj} \in C_b(\overline{\Omega})$. For $\Omega_{(-\eta)}$, we have the uniform interior $C^{2,\gamma}$ estimate of u_{0j} if j is large enough. So we have that $D^2 \log u_{0j} \le 0$ in $\Omega_{(-\eta)}$ since $\log u_0$ is strictly concave. Therefore, we deduce that u_{0j} is log-concave. Thus we have proved that for a given log-concave function $u_0 \in C_b(\overline{\Omega})$, there is a sequence of log-concave functions $u_{0j} \in C^{2,\gamma}(\overline{\Omega}) \cap C_b(\overline{\Omega})$ satisfying that $F(D^2 u_{0j}) = 0$ on $\partial\Omega$, which converges to u_0 uniformly in $\overline{\Omega}$.

Let u_{0j} converge to u_0 uniformly in $\overline{\Omega}$ and let u_j and u be the solution of (3.1.1) with the operator F and initial data u_{0j} and u_0, respectively. From the maximum principle for $u_j - u$, we have

$$\|u_j - u\|_{L^\infty(\Omega \times [0,+\infty))} \le \|u_{0j} - u_0\|_{L^\infty(\Omega)} \to 0 \quad \text{as} \quad j \to +\infty$$

since $u_j - u$ satisfies $\mathcal{M}^-(D^2 v) - v_t \le 0 \le \mathcal{M}^+(D^2 v) - v_t$ in $\Omega \times (0, +\infty)$. Then log-concavity of u_j from (i);

$$\frac{1}{2}\Big(\log u_j(x,t) + \log u_j(y,t)\Big) - \log u_j\left(\frac{x+y}{2}, t\right) \le 0 \quad \text{for } x, y \in \Omega, \ t \in (0, +\infty),$$

is preserved under the uniform convergence. Therefore, we conclude that $u(x,t)$ is log-concave in the x variables for any $t > 0$.

□

Corollary 3.1.9. *Suppose that F satisfies (F1), (F2) and (F3) and that Ω is convex. Let u be the solution of* (3.1.1) *with initial data $u_0 \in C_b(\overline{\Omega})$. If u_0 is log-concave, then the solution $u(x,t)$ is log-concave in the x variables for $t > 0$.*

Proof. We approximate Ω by Ω_j which are strictly convex, smooth and bounded domains in $\mathbb{R}^n$ and approximate initial data u_0 by $u_{0j} \in C_b(\overline{\Omega}_j)$. Let u_j be the solution of (3.1.1) with the operator F and initial data u_{0j} in $\Omega_j \times (0,+\infty)$. Then we have uniform local Hölder estimates for u_j in $\Omega \times (0,+\infty)$, namely, $\|u_j\|_{C^\alpha(K\times[t_0,t_1])} < C$ uniformly for each compact subset $K \times [t_0,t_1]$ of $\Omega \times (0,+\infty)$. We can also check that u_j converges to u pointwise in $\Omega \times [0,+\infty)$ by using the maximum principle. So we have uniform convergence of u_j to u locally in $\Omega \times (0,+\infty)$, up to a subsequence, from Arzela-Ascoli Theorem. Therefore for any $K \times [t_0,t_1] \subset \Omega \times (0,+\infty)$, we let j go to $+\infty$ to obtain

$$\frac{1}{2}\left(\log u(x,t) + \log u(y,t)\right) - \log u\left(\frac{x+y}{2}, t\right) \le 0 \quad \text{for } x,y \in K,\ t \in [t_0,t_1],$$

which completes the proof. □

Remark 3.1.10.

i) *We note that any concave function in a convex domain Ω is log-concave.*

ii) *It is well-known that the distance function $dist(x,\partial\Omega)$ is concave for a convex domain Ω, so Lemma 3.1.8 and Corollary 3.1.9 are not void.*

Corollary 3.1.11 (Log-concavity)**.** *Suppose that F satisfies (F1), (F2) and (F3), and that Ω is convex. Then, the positive eigenfunction $\varphi(x)$ in Theorem 3.1.1 is log-concave, i.e., $D^2\log\varphi(x) \le 0$ for $x \in \Omega$.*

Proof. We take the distance function as an initial data of the uniformly parabolic equation (3.1.1). Then Corollary 3.1.9 yields that for $x,y \in \Omega$ and $t > 0$,

$$\frac{1}{2}\left(\log u(x,t) + \log u(y,t)\right) - \log u\left(\frac{x+y}{2}, t\right) \le 0.$$

From the uniform convergence of $e^{\mu t}u(\cdot,t)$ to $\gamma^*\varphi$ in Proposition 3.1.4, we conclude that

$$\frac{1}{2}\left(\log\varphi(x) + \log\varphi(y)\right) - \log\varphi\left(\frac{x+y}{2}\right) \le 0 \quad \text{for } x,y \in \Omega.$$

□

3.2 Degenerate parabolic equations

3.2.1 Sub-linear elliptic eigenvalue problems

In the range of the exponents $m > 1$, we consider the following elliptic equation

$$\begin{cases} -F(D^2 f^m) = \frac{1}{m-1} f & \text{in } \Omega, \\ f = 0 & \text{on } \partial\Omega, \\ f > 0 & \text{in } \Omega, \end{cases} \tag{3.2.1}$$

which is the asymptotic profile, after appropriate normalization (Proposition 3.2.4), of solutions to the following parabolic equation

$$\begin{cases} F(D^2 u^m) - \partial_t u = 0 & \text{in } \Omega \times (0, +\infty), \quad m > 1, \\ u(\cdot, 0) = u_0, & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \times (0, +\infty). \end{cases} \tag{3.2.2}$$

For (3.2.2), we require initial data u_0 to have nontrivial bounded gradient of u_0^m on $\partial\Omega$, i.e.,

$$u_0^m \in C_b(\overline{\Omega}),$$

where

$$C_b(\overline{\Omega}) := \left\{ h \in C^0(\overline{\Omega}) \,|\, c_o \operatorname{dist}(x, \partial\Omega) \le h(x) \le C_o \operatorname{dist}(x, \partial\Omega) \text{ for } 0 < c_o \le C_o < +\infty \right\}.$$

We note that if f is a solution of (3.2.1), then $\phi := f^m$ is a solution of (**NLEV**) associated with the exponent $p = \frac{1}{m}$ and the eigenvalue $\mu = \frac{1}{m-1}$.

For the sublinear case, $0 < p = \frac{1}{m} < 1$, we can generalize the comparison principle as follows and we refer to Section 2, [4] for Laplace operator. We also found [2, Theorem 3.3] which dealt with comparison between viscosity sub- and super-solutions of general elliptic equations.

Lemma 3.2.1 (Comparison principle)**.** *Suppose that F satisfies (F1), and either (F2) or (F3). Let v and w be in $C^2(\Omega) \cap C^1(\overline{\Omega})$ such that $v \not\equiv 0$, $v, w \ge 0$ in Ω and*

$$F(D^2 v) + \frac{1}{m-1} v^{\frac{1}{m}} \le 0 \le F(D^2 w) + \frac{1}{m-1} w^{\frac{1}{m}} \quad \text{in } \Omega.$$

If $v \ge 0 \ge w$ on $\partial\Omega$, then $v \ge w$ in Ω.

Proof. (i) First, we assume that v is a strict supersolution, i.e., $F(D^2v)+\frac{1}{m-1}v^{\frac{1}{m}} < 0$ in Ω. Then we will show that $v > w$ in Ω. On the contrary, suppose that $v \le w$ for some point in Ω. Since a nonzero and nonnegative function v satisfies

$$\mathcal{M}^-(D^2v) \le F(D^2v) < -\frac{1}{m-1}v^{\frac{1}{m}} \le 0 \quad \text{in} \quad \Omega,$$

we have that $v > 0$ in Ω and $|\nabla v| > \delta_0 > 0$ on $\partial\Omega \cap \{v = 0\}$ for some $\delta_0 > 0$ from the strong minimum principle and Hopf's lemma. This implies that there is a small $\epsilon > 0$ such that $v \ge \epsilon w$ since $w \in C^1(\overline{\Omega})$ and $w = 0$ on $\partial\Omega$. Define

$$t^* := \sup\{t > 0 \,|\, v \ge tw \;\text{ in } \Omega\}.$$

Then $0 < \epsilon \le t^* \le 1$ from the assumption that $v \le w$ for some point in Ω since v is positive in Ω. Now we set $z := v - t^*w$ and then a nonnegative function z vanishes at some point in $\overline{\Omega}$ and satisfies

$$\begin{aligned}\mathcal{M}^-(D^2z) \le F(D^2v) - F(D^2t^*w) &< \frac{1}{m-1}\left(t^*w^{\frac{1}{m}} - v^{\frac{1}{m}}\right)\\ &\le \frac{1}{m-1}\left((t^*w)^{\frac{1}{m}} - v^{\frac{1}{m}}\right) \le 0 \quad \text{in } \Omega,\end{aligned}$$

where we use the hypotheses on the operator F in the first two inequalities. Since $\mathcal{M}^-(D^2z) < 0$ in Ω, we have that $z \not\equiv 0$, so the strong minimum principle and Hopf's lemma imply that $z > 0$ in Ω and $|\nabla z| > \delta_1 > 0$ on $\partial\Omega \cap \{z = 0\}$ for some $\delta_1 > 0$. Then we can choose $\epsilon_0 > 0$ small so that $z \ge \epsilon_0 w$ in Ω, since $w = 0$ on $\partial\Omega$ and $w \in C^1(\overline{\Omega})$. It is a contradiction to the definition of t^*. Therefore, we conclude that $v > w$ in Ω.

(ii) Now we assume that v is a supersolution, i.e., $F(D^2v) + \frac{1}{m-1}v^{\frac{1}{m}} \le 0$ in Ω and we approximate v by strict supersolutions. We note that $v > 0$ in Ω from the strong minimum principle. Let $v^\epsilon := (1+\epsilon)v$ for $0 < \epsilon < 1$. Then v^ϵ satisfies $v^\epsilon > v$ and

$$\begin{aligned}F(D^2v^\epsilon) + \frac{1}{m-1}(v^\epsilon)^{\frac{1}{m}} &\le (1+\epsilon)F(D^2v) + \frac{(1+\epsilon)^{\frac{1}{m}}}{m-1}v^{\frac{1}{m}}\\ &\le \frac{1}{m-1}v^{\frac{1}{m}}\left\{(1+\epsilon)^{\frac{1}{m}} - (1+\epsilon)\right\} < 0 \quad \text{in } \Omega.\end{aligned}$$

From (i), we get $v^\epsilon > w$ in Ω. Letting $\epsilon \to 0$, it follows that $v \ge w$ in Ω. □

Theorem 3.2.2 (Existence and Uniqueness)**.** *Suppose F satisfies (F1) and (F2). Let $0 < p := \frac{1}{m} < 1$. The nonlinear eigenvalue problem* (**NLEV**) *has a unique solution $\phi \in C^{0,1}(\overline{\Omega}) \cap C^{1,\alpha}(\Omega)$, ($0 < \alpha < 1$), namely, ϕ satisfies*

$$\begin{cases} -F(D^2\phi) = \frac{1}{m-1}\phi^p & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \\ \phi > 0 & \text{in } \Omega. \end{cases} \tag{NLEV}$$

Proof. (i) First, the uniqueness of the solution follows from the comparison principle. To prove the existence, we use Perron's method via comparison principle so we establish positive super and sub-solutions with zero boundary data. Let h be the solution of

$$\begin{cases} F(D^2h) = -1 & \text{in } \Omega, \\ h = 0 & \text{on } \partial\Omega, \\ h > 0 & \text{in } \Omega. \end{cases}$$

If we select $t > 0$ satisfying $t^{1-\frac{1}{m}}\|h\|_{L^\infty(\Omega)}^{-\frac{1}{m}} = \frac{1}{m-1}$, then we have

$$F(D^2(th)) = -t = -t^{1-\frac{1}{m}}h^{-\frac{1}{m}}(th)^{\frac{1}{m}} \leq -\frac{1}{m-1}(th)^{\frac{1}{m}} \quad \text{in } \Omega,$$

i.e., $h^+ := th$ is a supersolution. Now, let φ be the positive eigenfunction of (**EV**) at Theorem 3.1.1. By choosing $s > 0$ such that $\mu\left(s\|\varphi\|_{L^\infty(\Omega)}\right)^{1-\frac{1}{m}} = \frac{1}{m-1}$, we have

$$F(D^2(s\varphi)) \geq -\frac{1}{m-1}(s\varphi)^{\frac{1}{m}} \quad \text{in } \Omega.$$

Thus, $h^- := s\varphi$ is a subsolution. So we have constructed a supersolution h^+ and a subsolution h^-. Comparison principle implies that $h^- \leq h^+$ and then there is a viscosity solution ϕ to (**NLEV**) such that $h^- \leq \phi \leq h^+$ from Perron's method, [18].

(ii) Now we show that $\phi \in C^{0,1}(\overline{\Omega}) \cap C^{1,\alpha}(\Omega)$. First, ϕ belongs to $L^\infty(\Omega)$ from comparison since $h^+ \in L^\infty(\Omega)$ by Aleksandrov-Bakelman-Pucci estimates, [13]. Then ϕ is of $C^{1,\alpha}(\Omega)$ from the regularity theory of uniformly elliptic equations, [13] since $\phi^p \in L^\infty(\Omega)$.

To show Lipschitz regularity of ϕ up to the boundary, we recall that

$$c_o \operatorname{dist}(x, \partial\Omega) \leq h^- \leq \phi \leq h^+ \leq C_o \operatorname{dist}(x, \partial\Omega) \tag{3.2.3}$$

for some $0 < c_o \le C_o < +\infty$ from Hopf's Lemma for h^- and $C^{0,1}(\overline{\Omega})$ - regularity of h^+, [13]. Let $\delta > 0$ be a fixed constant such that $\cup_{x\in\Omega_{(-\epsilon)}} B_\epsilon(x) \subset \Omega$ for any $0 < \epsilon \le \delta$, where $\Omega_{(-\epsilon)} := \{x \in \Omega : \text{dist }(x, \partial\Omega) > \epsilon\}$. For $x_o \in \Omega \setminus \Omega_{(-\delta)}$, set

$$2\epsilon := \text{dist}\,(x_o, \partial\Omega) \le \delta.$$

Now we scale the function ϕ in the following way;

$$\phi_\epsilon(x) := \frac{1}{\epsilon}\phi(x_o + \epsilon x) \quad \text{in } \; B_1(0).$$

Then (3.2.3) implies that $0 < c_o \le \phi_\epsilon(x) \le 3C_o$ in $B_1(0)$ and ϕ_ϵ satisfies

$$F(D^2\phi_\epsilon) = -\frac{\epsilon^{1+p}}{m-1}\phi_\epsilon^p \quad \text{in } \; \Omega$$

with $\|\phi_\epsilon^p\|_{L^\infty(B_1(0))} \le (3C_o)^p$. From the regularity theory,[13], we have

$$|D\phi(x_o)| = |D\phi_\epsilon(0)| \le \tilde{C} \quad \text{for a uniform constant } \; \tilde{C} > 0,$$

where $\tilde{C}$ depends only on C_o, p, λ, Λ and n. Therefore, we have $|D\phi(x_o)| \le \tilde{C}$ for any $x_o \in \Omega \setminus \Omega_{(-\delta)}$ and then we conclude that

$$\|D\phi\|_{L^\infty(\Omega)} \le \tilde{C} + \|D\phi\|_{L^\infty\left(\overline{\Omega}_{(-\delta)}\right)} < +\infty$$

from the interior $C^{1,\alpha}$ estimates. □

Remark 3.2.3. *From Hopf's Lemma, the eigenfunction ϕ in Theorem 3.2.2 has nontrivial bounded gradient on the boundary, that is, $\inf_{\partial\Omega} |\nabla\phi| > \delta_o > 0$ for some $\delta_o > 0$. So, ϕ belongs to $C_b(\overline{\Omega})$ from Lipschitz regularity.*

3.2.2 Long-time asymptotics for degenerate parabolic equations

In this subsection, we study the asymptotic behavior of the solution of (3.2.2) in the range of exponents $m > 1$ when $t \to +\infty$.

Proposition 3.2.4. *Suppose that F satisfies (F1) and (F2). Let u be the solution of* (3.2.2) *with $u_0^m \in C_b(\overline{\Omega})$ and let ϕ is the solution of* (**NLEV**) *in Theorem 3.2.2. Set $W(x,t) := \frac{\phi(x)}{(1+t)^{\frac{m}{m-1}}}$. Then, we have*

(i)

$$t^{\frac{m}{m-1}}|u^m(x,t) - W(x,t)| \to 0 \quad \textit{uniformly for } x \in \Omega, \quad \textit{as } t \to +\infty,$$

(ii) for $(x,t) \in \Omega \times [1,+\infty)$,

$$c_o\, dist\,(x,\partial\Omega) < t^{\frac{m}{m-1}} u^m(x,t) < C_o\, dist\,(x,\partial\Omega)$$

(iii) for $(x,t) \in \Omega \times [2,+\infty)$,

$$t^{\frac{m}{m-1}}|\nabla u^m(x,t)| < C_o,$$

where the uniform constants $0 < c_o < C_o$ *depend on* u_0.

Proof. (i) We recall that $\phi \in C^{0,1}(\overline{\Omega})$ and $\inf_{\partial\Omega}|\nabla\phi| > \delta > 0$ for some $\delta > 0$. We can choose $\tau_1 > \tau_2 > 0$ such that

$$\phi(x)\tau_1^{-\frac{m}{m-1}} \le u_0^m(x) \le \phi(x)\tau_2^{-\frac{m}{m-1}} \quad \text{for } x \in \Omega$$

since $u_0^m \in C_b(\overline{\Omega})$. The comparison principle implies that for $(x,t) \in \Omega \times (0,+\infty)$,

$$\phi(x)(\tau_1 + t)^{-\frac{m}{m-1}} \le u^m(x,t) \le \phi(x)(\tau_2 + t)^{-\frac{m}{m-1}}$$

since $\phi(x)(\kappa + t)^{-\frac{m}{m-1}}$ is a separable solution of (3.2.2) for any $\kappa > 0$. Thus, we have for $(x,t) \in \Omega \times (0,+\infty)$,

$$\begin{aligned} t^{\frac{m}{m-1}}|u^m(x,t) - W(x,t)| &\le t^{\frac{m}{m-1}}\phi(x) \cdot \max\left\{\left|(\tau_i + t)^{-\frac{m}{m-1}} - (1+t)^{-\frac{m}{m-1}}\right|, i = 1,2\right\} \\ &\le \max_{x\in\Omega}\phi(x) \cdot \max\left\{\left|\frac{t^{\frac{m}{m-1}}}{(\tau_i + t)^{\frac{m}{m-1}}} - \frac{t^{\frac{m}{m-1}}}{(1+t)^{\frac{m}{m-1}}}\right|, i = 1,2\right\} \end{aligned}$$

and hence we deduce that $\lim_{t\to+\infty} t^{\frac{m}{m-1}}\|u^m(\cdot,t) - W(\cdot,t)\|_{L^\infty(\Omega)} = 0$.

(ii) Let $w := u^m$. In the proof of (i), we have for $(x,t) \in \Omega \times (0,+\infty)$,

$$\phi(x)(\tau_1 + t)^{-\frac{m}{m-1}} \le w(x,t) = u^m(x,t) \le \phi(x)(\tau_2 + t)^{-\frac{m}{m-1}}.$$

Since $\phi \in C^{0,1}(\overline{\Omega})$ and $\inf_{\partial\Omega}|\nabla\phi| > \delta > 0$ for some $\delta > 0$, we can find positive constants $0 < c_1 < C_1 < +\infty$ such that for $(x,t) \in \Omega \times (0,+\infty)$,

$$\frac{c_1}{(1+t)^{\frac{m}{m-1}}}\,\mathrm{dist}\,(x,\partial\Omega) \le w(x,t) \le \frac{C_1}{(1+t)^{\frac{m}{m-1}}}\,\mathrm{dist}\,(x,\partial\Omega)$$

and then for $(x,t) \in \Omega \times [1,+\infty)$,

$$\frac{c_1}{2^{\frac{m}{m-1}} t^{\frac{m}{m-1}}} \operatorname{dist}(x,\partial\Omega) \ \leq w(x,t) = u^m(x,t) \leq \frac{C_1}{t^{\frac{m}{m-1}}} \operatorname{dist}(x,\partial\Omega). \tag{3.2.4}$$

Therefore, (ii) follows.

(iii) Let $0 < \delta_o < 1$ be a constant such that $B_{\delta_o}(x) \subset \Omega$ for $x \in \Omega_{(-\delta_o)} := \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) > \delta_o\}$. Let $(x_o,t_o) \in (\Omega\backslash\Omega_{(-\delta_o)}) \times [2,+\infty)$. For $x_o \in \Omega\backslash\Omega_{(-\delta_o)}$, we set $\operatorname{dist}(x_o,\partial\Omega) = 2\sigma$. According to (3.2.4), it follows that for $(x,t) \in B_\sigma(x_o) \times [t_o/2, t_o]$,

$$c_2\sigma \leq t_o^{\frac{m}{m-1}} w(x,t) \leq C_2\sigma, \tag{3.2.5}$$

where $c_2 := c_1 2^{-\frac{m}{m-1}}$ and $C_2 := 3C_1 2^{\frac{m}{m-1}}$. We define for $(\tilde{x},\tilde{t}) \in B_1(0) \times [\sigma^{-1-1/m} - 1, \sigma^{-1-1/m}]$,

$$\tilde{w}(\tilde{x},\tilde{t}) := \frac{t_o^{\frac{m}{m-1}}}{\sigma} w\left(x_o + \sigma\tilde{x}, \sigma^{1+1/m} t_o \tilde{t}\right),$$

where $x_o + \sigma\tilde{x} \in B_\sigma(x_o)$ and $\sigma^{1+1/m} t_o \tilde{t} \in [t_o - \sigma^{1+1/m} t_o, t_o] \subset [t_o/2, t_o]$. From the scaling property, $\tilde{w}$ satisfies

$$m\tilde{w}^{1-\frac{1}{m}} F(D^2\tilde{w}) - \tilde{w}_t = 0 \quad \text{in} \quad B_1(0) \times [\sigma^{-1-1/m} - 1, \sigma^{-1-1/m}].$$

From (3.2.5), we have

$$c_2 \leq \tilde{w} \leq C_2 \quad \text{in} \quad B_1(0) \times [\sigma^{-1-1/m} - 1, \sigma^{-1-1/m}],$$

which implies that $\tilde{w}$ solves a uniformly parabolic equation in $B_1(0) \times [\sigma^{-1-1/m} - 1, \sigma^{-1-1/m}]$ with the ellipticity constants depending only on m, c_2, C_2, λ and Λ. From uniform gradient estimates for uniformly parabolic equations, Theorem 1.3 and Theorem 4.8 in [61], we obtain that

$$|\nabla\tilde{w}(0,\sigma^{-1-1/m})| < C,$$

where $C > 0$ depends only on m, c_2, C_2, λ and Λ. Therefore we deduce that

$$t_o^{\frac{m}{m-1}} |\nabla w(x_o,t_o)| = |\nabla\tilde{w}(0,\sigma^{-1-1/m})| < C$$

and hence

$$t^{\frac{m}{m-1}} |\nabla w(x,t)| < C \quad \text{uniformly for } (x,t) \in (\Omega\backslash\Omega_{(-\delta_o)}) \times [2,+\infty).$$

For $t_o \in [2, +\infty)$, (3.2.4) gives that

$$\tilde{c} \le t_o^{\frac{m}{m-1}} w(x,t) \le \tilde{C} \quad \text{for} \quad (x,t) \in \Omega_{(-\delta_o/2)} \times [t_o/2, t_o],$$

where $\tilde{c} := c_1 2^{-\frac{m}{m-1}} \delta_o/2$ and $\tilde{C} := C_1 2^{\frac{m}{m-1}} \operatorname{diam}(\Omega)$. We define for $(\tilde{x}, \tilde{t}) \in \Omega_{(-\delta_o/2)} \times [1/2, 1]$,

$$\tilde{w}(\tilde{x}, \tilde{t}) := t_o^{\frac{m}{m-1}} w(\tilde{x}, t_o \tilde{t}),$$

where $t_o \tilde{t} \in [t_o/2, t_o]$ and $\tilde{c} \le \tilde{w} \le \tilde{C}$ in $\Omega_{(-\delta_o/2)} \times [1/2, 1]$. From the scaling property, $\tilde{w}$ satisfies

$$m\tilde{w}^{1-\frac{1}{m}} F(D^2 \tilde{w}) - \tilde{w}_t = 0 \quad \text{in} \quad \Omega_{(-\delta_o/2)} \times [1/2, 1],$$

which is uniformly parabolic in $\Omega_{(-\delta_o/2)} \times [1/2, 1]$ with the ellipticity constants depending only on $m, \tilde{c}, \tilde{C}, \lambda$ and Λ. Thus we have

$$t_o^{\frac{m}{m-1}} |\nabla w(x, t_o)| = |\nabla \tilde{w}(x, 1)| < C \quad \text{uniformly for } x \in \Omega_{(-\frac{3}{4}\delta_0)}$$

from uniform gradient estimates, Theorem 1.3 and Theorem 4.8 in [61], where $C > 0$ depends only on $m, \tilde{c}, \tilde{C}, \lambda$ and Λ. Therefore, we conclude that $t^{\frac{m}{m-1}} |\nabla w(x,t)|$ is uniformly bounded in $\Omega_{(-\frac{3}{4}\delta_0)} \times [2, +\infty)$ and hence

$$t^{\frac{m}{m-1}} |\nabla w(x,t)| < C \quad \text{uniformly in } \Omega \times [2, +\infty)$$

by putting together with the above argument. □

Corollary 3.2.5. *Under the same assumption of Proposition 3.2.4, we also assume that F is concave. For each compact subset K of Ω, we have*

$$\left\| t^{\frac{m}{m-1}} u^m(\cdot, t) - \phi \right\|_{C_x^k(K)} \to 0 \quad \text{as } t \to +\infty$$

for $k = 0, 1, 2$.

Proof. Since

$$\begin{aligned} |t^{\frac{m}{m-1}} u^m(x,t) - \phi(x)| &\le t^{\frac{m}{m-1}} |u^m(x,t) - W(x,t)| + |t^{\frac{m}{m-1}} W(x,t) - \phi(x)| \\ &\le t^{\frac{m}{m-1}} |u^m(x,t) - W(x,t)| + \phi(x) \left| \left(\frac{t}{1+t} \right)^{\frac{m}{m-1}} - 1 \right|, \end{aligned}$$

we have the uniform convergence of $t^{\frac{m}{m-1}}u^m(\cdot,t)$ to the limit ϕ as $t \to +\infty$ from Proposition 3.2.4. For $K \Subset \Omega$, we can find a compact set K' such that $K \Subset K' \Subset \Omega$. Then there is a time $T > 0$ such that for $(x,t) \in K' \times [T, +\infty)$,

$$-\frac{1}{2}\inf_{K'}\phi \leq t^{\frac{m}{m-1}}u^m(x,t) - \phi(x) \leq \sup_{K'}\phi.$$

Let $w := u^m$. Then we have

$$\frac{1}{2}\inf_{K'}\phi \leq t^{\frac{m}{m-1}}w(x,t) \leq 2\sup_{K'}\phi.$$

For $t_o \in [2T, +\infty)$, we have

$$\tilde{c} \leq t_o^{\frac{m}{m-1}}w(x,t) = t_o^{\frac{m}{m-1}}u^m(x,t) \leq \tilde{C} \quad \text{for } (x,t) \in K' \times [t_o/2, t_o],$$

where $\tilde{c} := \frac{1}{2}\inf_{K'}\phi$ and $\tilde{C} := 2^{1+\frac{m}{m-1}}\sup_{K'}\phi$. From (iii) in Proposition 3.2.4, we also have

$$t_o^{\frac{m}{m-1}}|\nabla w(x,t)| < 2^{\frac{m}{m-1}}C_o \quad \text{uniformly for } (x,t) \in \Omega \times [t_o/2, t_o],$$

and then for $(x,t) \in K' \times [t_o/2, t_o]$,

$$\left|\nabla\left(t_o^{\frac{m}{m-1}}w\right)^{1-\frac{1}{m}}\right| = \frac{m-1}{m}\left(t_o^{\frac{m}{m-1}}w\right)^{-\frac{1}{m}}|t_o^{\frac{m}{m-1}}\nabla w(x,t)| \leq \frac{m-1}{m}(\tilde{c})^{-\frac{1}{m}}2^{\frac{m}{m-1}}C_o,$$

where a uniform constant $C_o > 0$ is in Proposition 3.2.4. By using a similar argument as in the proof of (iii), Proposition 3.2.4 and the concavity of F, we can apply Theorem 1.1 and Theorem 4.13 in [61], to deduce

$$\left\|t_o^{\frac{m}{m-1}}u^m(\cdot,t_o)\right\|_{C^{2,\alpha}(K)} = \left\|t_o^{\frac{m}{m-1}}w(\cdot,t_o)\right\|_{C^{2,\alpha}(K)} < C,$$

where $0 < \alpha < 1$ and a uniform constant $C > 0$ depends only on $m, \tilde{c}, \tilde{C}, C_o, \lambda, \Lambda, K$ and K'. So we have proved

$$\|t^{\frac{m}{m-1}}u^m(\cdot,t)\|_{C^{2,\alpha}(K)} < C \quad \text{uniformly for } t \in [2T, +\infty).$$

Now we use Arzela-Ascoli Theorem and the uniform convergence of $t^{\frac{m}{m-1}}u^m(\cdot,t)$ to the unique limit ϕ to conclude that

$$\|t^{\frac{m}{m-1}}u^m(\cdot,t) - \phi\|_{C_x^k(K)} \to 0 \quad \text{as } t \to +\infty$$

for $k = 1, 2$. For details of the proof, we refer to Corollary 3.1.5. □

Corollary 3.2.6. *Suppose that F satisfies (F1) and (F2). Let u be the solution of* (3.2.2) *with $u_0^m \in C_b(\overline{\Omega})$. Set $U(x,t) := \frac{f(x)}{(1+t)^{\frac{1}{m-1}}}$, where f solves*

$$\begin{cases} -F(D^2 f^m) = \frac{1}{m-1} f & \text{in } \Omega, \\ f = 0 & \text{on } \partial\Omega, \\ f > 0 & \text{in } \Omega. \end{cases}$$

Then, we have

$$t^{\frac{1}{m-1}} |u(x,t) - U(x,t)| \to 0 \quad \text{uniformly for} \quad x \in \Omega, \quad \text{as } t \to +\infty,$$

and hence

$$\lim_{t\to+\infty} \|t^{\frac{1}{m-1}} u(\cdot,t) - f\|_{L^\infty(\Omega)} = 0.$$

Proof. Let ϕ be the solution of (**NLEV**) in Theorem 3.2.2. Since $\phi = f^m$ and $(1-r)^m \leq 1 - r^m$ for $0 \leq r \leq 1$, the result follows from Proposition 3.2.4. □

3.2.3 Square-root concavity of the pressure

Let u be the solution of the problem (3.2.2) with $u_0^m \in C_b(\overline{\Omega})$. Let $v := u^{m-1}$ be the pressure and let $v =: w^2$. We prove the concavity of w in spatial variables for any $t > 0$ if the initial data u_0 has the concavity of $u_0^{\frac{m-1}{2}}$, under some assumption. The function $w = \sqrt{v}$ is a suitable quantity to perform geometrical investigation, which was demonstrated by Daskalopoulos, Hamilton and Lee in [20] for the Laplace operator.

First, let us approximate the problem (3.2.2) as follows; for $0 < \eta < 1$,

$$\begin{cases} F(D^2 u_\eta^m) - \partial_t u_\eta = 0 & \text{in } \Omega \times (0,\infty), \\ u_\eta = \eta & \text{on } \partial\Omega \times (0,\infty), \\ u_\eta(\cdot,0) = u_{\eta,0} \geq \eta & \text{in } \Omega. \end{cases} \tag{3.2.6}$$

Let $g_\eta := u_\eta^m$. Then g_η satisfies the following problem:

$$\begin{cases} m g_\eta^{1-1/m} F(D^2 g_\eta) - \partial_t g_\eta = 0 & \text{in } \Omega \times (0,\infty), \\ g_\eta = \eta^m & \text{on } \partial\Omega \times (0,\infty), \\ g_\eta(\cdot,0) = g_{\eta,0} = u_{\eta,0}^m \geq \eta^m & \text{in } \Omega. \end{cases} \tag{3.2.7}$$

We note that g_η satisfies a uniformly parabolic equation in $\Omega \times (0, +\infty)$ for a fixed $\eta > 0$ since $g_\eta \geq \eta^m$ from the comparison principle. We assume that $g_{\eta,0}$ satisfies

$$\frac{1}{2} g_0 \leq g_{\eta,0} - \eta^m \leq 2g_0 \quad \text{in} \quad \Omega. \tag{3.2.8}$$

Lemma 3.2.7. *Suppose that F satisfies $(F1)$. Let g_η be the solution of (3.2.7) with the initial data $g_{\eta,0} \in C^0(\overline{\Omega})$ satisfying (3.2.8) for some $g_0 \in C_b(\overline{\Omega})$. There are uniform positive constants c_0, c_1 and K with respect to $0 < \eta < 1$ and F such that*

$$c_0\, dist\,(x, \partial\Omega)\, e^{-Kt} < g_\eta(x,t) - \eta^m < c_1\, dist\,(x, \partial\Omega), \quad \forall (x,t) \in \Omega \times [0, +\infty),$$

and

$$0 < c_0 e^{-Kt} < |\nabla_x g_\eta(x,t)| < c_1, \quad \forall (x,t) \in \partial\Omega \times [0, +\infty).$$

Proof. We establish a subsolution and a supersolution of (3.2.7). From Theorem 3.1.1, let φ^- solve

$$\begin{cases} -\mathcal{M}^-(D^2\varphi^-) = \mu^-\varphi^- & \text{in } \Omega, \\ \varphi^- = 0 & \text{on } \partial\Omega, \end{cases} \tag{\textbf{EV}}$$

associated with the eigenvalue $\mu^- > 0$. We may assume that $g_{\eta,0} \geq \eta^m + \varphi^-$ by multiplying a positive constant to φ^- since $g_{\eta,0} - \eta^m \geq \frac{1}{2} g_0$ and $g_0 \in C_b(\overline{\Omega})$. We define

$$h(x,t) := \eta^m + \varphi^-(x)e^{-Kt} \quad \text{ for } K := \mu^- m \left(1 + \|\varphi^-\|_{L^\infty(\Omega)}\right)^{1-1/m} > 0.$$

Then we have

$$\begin{aligned} mh^{1-1/m}F(D^2h) - h_t &\geq mh^{1-1/m}\mathcal{M}^-(D^2h) - h_t \\ &= mh^{1-1/m}e^{-Kt}\left\{\mathcal{M}^-(D^2\varphi^-) + \frac{K\varphi^-}{mh^{1-1/m}}\right\} \\ &= mh^{1-1/m}e^{-Kt}\varphi^-\left\{-\mu^- + \mu^- \frac{m\left(1 + \|\varphi^-\|_{L^\infty(\Omega)}\right)^{1-1/m}}{m(\eta^m + \varphi^- e^{-Kt})^{1-1/m}}\right\} \\ &\geq 0 \qquad \text{in} \quad \Omega \times (0, +\infty), \end{aligned}$$

$h = \eta^m$ on $\partial\Omega \times [0, +\infty)$ and $h(\cdot, 0) = \eta^m + \varphi^- \leq g_{\eta,0}$ in Ω. Thus the comparison principle gives that

$$g_\eta(x,t) \geq h(x,t) = \eta^m + \varphi^-(x)e^{-Kt} \quad \text{for } (x,t) \in \Omega \times [0, +\infty),$$

where $K > 0$ depends only on the initial data g_0. Thus, we find $c_0 > 0$ such that

$$g_\eta(x,t) > \eta^m + c_0 \,\mathrm{dist}\,(x, \partial\Omega)\, e^{-Kt}, \quad \forall (x,t) \in \Omega \times [0, +\infty)$$

and $|\nabla g_\eta(x,t)| > c_0 e^{-Kt}$ for $(x,t) \in \partial\Omega \times (0, +\infty)$ since $\inf_{\partial\Omega} |\nabla \varphi^-| > 0$.

On the other hand, let φ^+ be the positive eigenfunction of

$$\begin{cases} -\mathcal{M}^+(D^2\varphi^+) = \frac{1}{m-1}(\varphi^+)^{\frac{1}{m}} & \text{in } \Omega, \\ \varphi^+ = 0 & \text{on } \partial\Omega. \end{cases}$$

from Theorem 3.2.2. Multiplying a positive constant to φ^+, we assume that $g_{\eta,0} \leq \eta^m + \varphi^+$ and that φ^+ is the positive eigenfunction associated with the eigenvalue $\mu^+ > 0$ since $g_{\eta,0} - \eta^m \leq 2g_0$ and $g_0 \in C_b(\overline{\Omega})$. If we define $h := \eta^m + \varphi^+$, then h satisfies

$$mh^{1-1/m}F(D^2h) - h_t \leq mh^{1-1/m}\mathcal{M}^+(D^2h) = mh^{1-1/m}\left\{-\mu^+(\varphi^+)^{\frac{1}{m}}\right\} < 0 \quad \text{in } \Omega \times (0, +\infty).$$

From the comparison principle, we obtain that

$$g_\eta \leq \eta^m + \varphi^+ \quad \text{in} \quad \Omega \times [0, +\infty).$$

Thus there is a uniform constant $c_1 > 0$, depending only on g_0, such that $g_\eta(x,t) < \eta^m + c_1 \,\mathrm{dist}\,(x, \partial\Omega)$ for $(x,t) \in \Omega \times (0, +\infty)$ and $|\nabla g_\eta| < c_1$ on $\partial\Omega \times (0, +\infty)$ since $\varphi^+ \in C^{0,1}(\overline{\Omega})$. □

Lemma 3.2.8. *Under the same condition as Lemma 3.2.7, we also assume that $g_{\eta,0}$ converges to g_0 in $C^\gamma(\overline{\Omega})$ $(0 < \gamma \leq 1)$ when η tends to* 0. *Let u be the solution of* (3.2.2) *with the initial data $u_0 := g_0^{\frac{1}{m}}$ and let $g := u^m$. Then for any $T > 0$, g_η converges to g uniformly in $\overline{\Omega} \times [0, T]$, up to a subsequence, when η tends to* 0.

Proof. Let $0 < \epsilon < 1$. From Lemma 3.2.7, we have

$$0 < \delta \leq g_\eta \leq M \quad \text{in} \quad \Omega_{(-\epsilon)} \times [0, T]$$

for $\delta := c_0 \epsilon e^{-KT}$ and $M := c_1 \,\mathrm{diam}\,(\Omega)$, where $\Omega_{(-\epsilon)} := \{x \in \Omega : \,\mathrm{dist}\,(x, \Omega) > \epsilon\}$. Thus g_η satisfies

$$\pm\left\{\mathcal{M}^{\pm}_{\tilde{\delta}\lambda, \tilde{M}\Lambda}(D^2 g_\eta) - \partial_t g_\eta\right\} \geq 0 \quad \text{in} \quad \Omega_{(-\epsilon)} \times (0, T]$$

for $\tilde{\delta} := m\delta^{1-1/m}$ and $\tilde{M} := mM^{1-1/m}$. Then $\{g_\eta\}$ are equicontinuous in $\Omega_{(-2\epsilon)}\times[0,T]$ from [61] since $g_{\eta,0} \in C^\gamma(\overline{\Omega})$ for $0 < \gamma \leq 1$. We use Arzela-Ascoli Theorem to deduce that g_η converges to a continuous function locally uniformly in $\Omega \times [0,T]$, as η tends to 0, up to a subsequence.

We recall that the family of viscosity solutions is closed in the topology of local uniform convergence, and that

$$0 < g_\eta \leq \eta^m + c_1 \operatorname{dist}(x, \partial\Omega) \quad \text{in} \quad \Omega \times [0, +\infty)$$

from Lemma 3.2.7. Therefore, we deduce that g_η converges to g uniformly in $\overline{\Omega} \times [0,T]$, up to a subsequence, as η tends to 0, since $g_{\eta,0}$ converges to g_0 in $C^\gamma(\overline{\Omega})$ and the solution to (3.2.2) is unique. □

From the above lemma, it suffices to show the concavity of $g_\eta^{\frac{m-1}{2m}} = u_\eta^{\frac{m-1}{2}}$ for the square-root concavity of the pressure of u.

Lemma 3.2.9 (Aronson-Bénilan inequality)**.** *Suppose that F satisfies (F1) and (F3). Let g_η be the solution of* (3.2.7) *with the initial data $g_{\eta,0} \in C^0(\overline{\Omega})$ satisfying* (3.2.8) *for some $g_0 \in C_b(\overline{\Omega})$, and $u_\eta := g_\eta^{1/m}$. Then we have*

$$\partial_t g_\eta \geq -\frac{m}{m-1} \cdot \frac{g_\eta}{t} \quad \textit{and} \quad \partial_t u_\eta \geq -\frac{1}{m-1} \cdot \frac{u_\eta}{t}, \quad \forall (x,t) \in \Omega \times (0, +\infty). \tag{3.2.9}$$

Proof. (i) First, we assume that F is smooth. Let $\delta > 0$ and let C be any positive constant bigger than $\frac{m}{m-1}$. We can select $\tau_\delta \in (-\delta, 0)$ such that

$$\partial_t g_\eta(x,\delta) + C\frac{g_\eta(x,\delta)}{\delta + \tau_\delta} \geq \eta^m, \quad \forall x \in \Omega,$$

since $\partial_t g_\eta(\cdot,\delta)$ is bounded and $g_\eta(\cdot,\delta) \geq \eta^m$ in Ω. We define

$$Z(t) := \inf_{x\in\Omega} \left(\partial_t g_\eta(x,t) + C\frac{g_\eta(x,t)}{t + \tau_\delta} \right).$$

We note that $Z(\delta) \geq \eta^m > 0$.

Suppose that there is $t_o \in (\delta, +\infty)$ such that $Z(t_o) = 0$. We may assume that t_o is the first time for Z to vanish and hence $Z_t(t_o) \leq 0$. Since

$$\left(\partial_t g_\eta + C\frac{g_\eta}{t + \tau_\delta} \right) = 0 + C\frac{g_\eta}{t + \tau_\delta} > 0 \quad \text{on} \quad \partial\Omega \times [\delta, +\infty),$$

the infimum of Z at time $t = t_o$ is achieved at an interior point x_o of Ω. Then we have that $\partial_t g_\eta(x_o, t_o) < 0$, and $\left(\partial_t g_\eta\right)^2 = C^2 \dfrac{g_\eta^2}{(t_o + \tau_\delta)^2} > 0$ at the minimum point $(x_o, t_o) \in \Omega \times (\delta, +\infty)$.

We consider the function

$$\Psi(s) := m g_\eta^{1-1/m}(x,t) F\left((1-s) D^2 g_\eta(x,t)\right) - (1-s)\partial_t g_\eta(x,t)$$

for any $(x,t) \in \Omega \times (0, +\infty)$. We note that $\Psi(0) = \Psi(1) = 0$. We use the concavity of F to obtain that g_η satisfies

$$m g_\eta^{1-1/m} F_{ij}(D^2 g_\eta) D_{ij} g_\eta - \partial_t g_\eta \le 0 \quad \text{in } \Omega \times (0, +\infty).$$

At the minimum point (x_o, t_o), we have

$$\begin{aligned}
Z_t &= \partial_t \left(\partial_t g_\eta + C \frac{g_\eta}{t + \tau_\delta} \right)(x_o, t_o) \\
&= \left(1 - \frac{1}{m}\right) m g_\eta^{-1/m} F(D^2 g_\eta) \partial_t g_\eta + m g_\eta^{1-1/m} F_{ij}(D^2 g_\eta) \cdot D_{ij} \partial_t g_\eta + C \frac{\partial_t g_\eta}{t + \tau_\delta} - C \frac{g_\eta}{(t+\tau_\delta)^2} \\
&= \left(1 - \frac{1}{m}\right) \frac{(\partial_t g_\eta)^2}{g_\eta} + m g_\eta^{1-1/m} F_{ij}(D^2 g_\eta) \cdot D_{ij} \left(\partial_t g_\eta + C \frac{g_\eta}{t + \tau_\delta} \right) \\
&\quad - m g_\eta^{1-1/m} F_{ij}(D^2 g_\eta) \cdot D_{ij} \left(C \frac{g_\eta}{t+\tau_\delta} \right) + C \frac{\partial_t g_\eta}{t + \tau_\delta} - C \frac{g_\eta}{(t+\tau_\delta)^2} \\
&\ge \left(1 - \frac{1}{m}\right) \frac{(\partial_t g_\eta)^2}{g_\eta} - C \frac{\partial_t g_\eta}{t + \tau_\delta} + C \frac{\partial_t g_\eta}{t + \tau_\delta} - C \frac{g_\eta}{(t+\tau_\delta)^2} \\
&= \left(1 - \frac{1}{m}\right) C^2 \frac{g_\eta}{(t+\tau_\delta)^2} - C \frac{g_\eta}{(t+\tau_\delta)^2} = C \frac{g_\eta}{(t+\tau_\delta)^2} \left(\frac{m-1}{m} C - 1 \right) > 0,
\end{aligned}$$

which is a contradiction since $Z_t(t_o) \le 0$. Therefore we deduce that $Z(t) > 0$ for any $t > \delta$, i.e.,

$$\partial_t g_\eta > -C \frac{g_\eta}{t + \tau_\delta} \ge -C \frac{g_\eta}{t - \delta}, \quad \forall t > \delta.$$

Since $\delta > 0$ and $C > \frac{m}{m-1}$ are arbitrary, we conclude that

$$\partial_t g_\eta \ge -\frac{m}{m-1} \cdot \frac{g_\eta}{t}, \quad \forall (x,t) \in \Omega \times (0, +\infty)$$

and hence the result follows.

(ii) In general, we approximate F by smooth operators F^ϵ using the mollification as in Lemma 3.1.6. Let g_η^ϵ be the solution of (3.2.7) with the operator F^ϵ and the same initial data $g_{\eta,0}$. Let $g_\eta^\pm$ be the solution of (3.2.7) with the operator $\mathcal{M}^\pm$ and the initial data $g_{\eta,0}$, respectively. From the comparison principle, we have that

$$0 < \eta^m \le g_\eta^- \le g_\eta^\epsilon \le g_\eta^+ < +\infty \quad \text{in } \Omega \times (0, +\infty),$$

and hence g_η^ϵ solves the uniformly parabolic equation in $\Omega \times (0, +\infty)$, where $0 < \eta < 1$ is fixed. Then, g_η^ϵ and $\partial_t g_\eta^\epsilon$ converge to g_η and $\partial_t g_\eta$, locally uniformly in $\overline{\Omega} \times (0, +\infty)$, respectively, when ϵ goes to 0, up to a subsequence. In fact, g_η^ϵ converges to g_η in $C^{1,\alpha}(\overline{\Omega} \times (0, +\infty))$ since g_η^ϵ and g_η are uniformly bounded, where $0 < \eta < 1$ is fixed. From the uniform interior $C^{2,\alpha}$- estimate of the uniformly parabolic equation [61], we have the convergence of $\partial_t g_\eta^\epsilon$ to $\partial_t g_\eta$ locally uniformly in $\overline{\Omega} \times (0, +\infty)$ when ϵ tends to 0. Therefore, we use (i) for g_η^ϵ, to conclude that

$$\partial_t g_\eta \ge -\frac{m}{m-1} \cdot \frac{g_\eta}{t} \quad \text{for} \quad t > 0,$$

completing the proof. □

Corollary 3.2.10 (Aronson-Bénilan inequality)**.** *Suppose that F satisfies (F1) and (F3). Let u be the solution of* (3.2.2) *with* $u_0^m \in C_b(\overline{\Omega}) \cap C^\gamma(\overline{\Omega})$ *for some* $0 < \gamma \le 1$. *Then we have*

$$\partial_t u \ge -\frac{1}{m-1} \cdot \frac{u}{t} \quad \textit{in } \Omega \times (0, +\infty).$$

Proof. For $0 < \eta < 1$, we find $g_{\eta,0} \in C^\gamma(\overline{\Omega})$ satisfying (3.2.8) and converging to $g_0 := u_0^m$ in $C^\gamma(\overline{\Omega})$ as η goes to 0. Let g_η be the solution of (3.2.7) with the initial data $g_{\eta,0}$, and let $g := u^m$. From Lemma 3.2.8, g_η converges to g uniformly in each compact subset of $\overline{\Omega} \times [0, +\infty)$ when η tends to 0, up to a subsequence. As in the proof of Lemma 3.2.8, we have the local uniform Hölder estimate of g_η in $\Omega \times (0, +\infty)$ and then we use Theorem 1.3 and Theorem 4.8 in [61] to obtain the uniform interior $C^{2,\tilde{\gamma}}$- estimate of g_η $(0 < \tilde{\gamma} < 1)$ in each compact subset of $\Omega \times (0, +\infty)$. Thus, $\partial_t g_\eta$ converges to $\partial_t g$ locally uniformly in $\Omega \times (0, +\infty)$, when η goes to 0, up to a subsequence. Therefore, Lemma 3.2.9 and the convergence of g_η to g imply that

$$\partial_t g \ge -\frac{m}{m-1} \cdot \frac{g}{t} \quad \text{in } \Omega \times (0, +\infty),$$

which finishes the proof. □

Lemma 3.2.11. *Suppose that F satisfies (F1) and (F3). Let g_η be the solution of* (3.2.7) *with the initial data $g_{\eta,0} \in C^{2,\gamma}(\overline{\Omega})$, $(0 < \gamma < 1)$ satisfying* (3.2.8) *for some $g_0 \in C_b(\overline{\Omega})$, and*

$$F(D^2 g_{\eta,0}) \leq 0 \quad in \quad \Omega.$$

Then g_η is nonincreasing in time.

Proof. Fix $0 < \eta < 1$ and $T > 0$. We show that

$$\partial_t g_\eta \leq 0 \quad \text{in} \quad \Omega \times (0, T].$$

We approximate the operator F by smooth operators F^ϵ using mollification as in Lemma 3.1.6 and we may assume that $F^\epsilon(0) = 0$ by subtraction of $F^\epsilon(0)$ to F^ϵ. We also approximate the initial data $g_{\eta,0}$ by $g^\epsilon_{\eta,0}$ satisfying $F^\epsilon(D^2 g^\epsilon_{\eta,0}) = 0$ on $\partial\Omega$ and $F^\epsilon(D^2 g^\epsilon_{\eta,0}) \leq 0$ in Ω. Indeed, let $g^\epsilon_{\eta,0}$ be the solution of the following elliptic problem

$$\begin{cases} F^\epsilon(D^2 h) = \xi_\epsilon \cdot \left\{ F^\epsilon(D^2 g_{\eta,0}) - 2\sqrt{n}\Lambda\epsilon \right\} & \text{in } \Omega, \\ h = \eta^m & \text{on } \partial\Omega, \end{cases}$$

where $0 \leq \xi_\epsilon \leq 1$ satisfies that $\xi_\epsilon \in C_0^\infty(\Omega)$, and $\xi_\epsilon \equiv 1$ in $\Omega_{(-\epsilon)}$ for $\Omega_{(-\epsilon)} := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \epsilon\}$. Then the solution $g^\epsilon_{\eta,0}$ is such initial data since $F^\epsilon(D^2 g_{\eta,0}) \leq F(D^2 g_{\eta,0}) + 2\sqrt{n}\Lambda\epsilon \leq 2\sqrt{n}\Lambda\epsilon$ as in the proof of Lemma 3.1.6. We notice that $g^\epsilon_{\eta,0}$ and $g_{\eta,0}$ have a uniform $C^{1,\gamma}$- estimate in $\overline{\Omega}$ for $0 < \gamma < 1$ from [13] since $F(D^2 g_{\eta,0})$ and $\xi_\epsilon \cdot \left\{ F^\epsilon(D^2 g_{\eta,0}) - 2\sqrt{n}\Lambda\epsilon \right\}$ are uniformly bounded. Then Arzela-Ascoli Theorem gives that $g^\epsilon_{\eta,0}$ converges to $g_{\eta,0}$ uniformly in $\overline{\Omega}$, as ϵ tends to 0, up to a subsequence, where $0 < \eta < 1$ is fixed.

Let g^ϵ_η be the solution of (3.2.7) with F^ϵ and $g^\epsilon_{\eta,0}$ in place of F and g_0, respectively. From the global Hölder regularity [61] and Arzela-Ascoli Theorem, we have that g^ϵ_η converges uniformly to g_η in $\overline{\Omega} \times [0, T]$, as ϵ tends to 0, up to a subsequence. We recall that $g^\epsilon_\eta \in C^{2,\gamma}(\overline{\Omega} \times [0, T])$ (see [61]) solves a uniformly parabolic equation (3.2.7) in $\Omega \times (0, T]$, where ϵ and η are fixed.

For a fixed $\epsilon > 0$, we will show that $\partial_t g^\epsilon_\eta \leq 0$ in $\Omega \times (0, T]$. Define

$$h := \partial_t g^\epsilon_\eta - \delta t - \delta$$

for small $0 < \delta < 1$. Then h is negative on the parabolic boundary of $\Omega \times (0, T]$. Indeed, for $(x, t) \in \partial\Omega \times [0, +\infty)$, we have that $h = 0 - \delta t - \delta \leq -\delta < 0$, and for

$(x, 0) \in \Omega \times \{t = 0\}$, we have

$$h = \partial_t g_\eta^\epsilon - \delta \leq -\delta < 0 \quad \text{in} \quad \Omega$$

since $\partial_t g_\eta^\epsilon(\cdot, 0) = m(g_{\eta,0}^\epsilon)^{1-1/m} F^\epsilon(D^2 g_{\eta,0}^\epsilon) \leq 0$ in Ω.

Suppose that there is $t_o \in (0, T]$ such that $h(x_o, t_o) = 0$ at some point $x_o \in \Omega$ for the first time. Then the point (x_o, t_o) is a maximum point of h in $\Omega \times (0, t_o]$, and hence at the maximum point (x_o, t_o), we have

$$\begin{aligned} 0 \geq m(g_\eta^\epsilon)^{1-1/m} F_{ij}^\epsilon(D^2 g_\eta^\epsilon) D_{ij} h - h_t &= -\left(1 - \frac{1}{m}\right) \frac{\left(\partial_t g_\eta^\epsilon\right)^2}{g_\eta^\epsilon} + \delta \\ = -\left(1 - \frac{1}{m}\right) \frac{\delta^2 (t_o + 1)^2}{g_\eta^\epsilon} + \delta &\geq -\left(1 - \frac{1}{m}\right) \frac{\delta^2 (T + 1)^2}{\eta^m} + \delta. \end{aligned}$$

However, it is a contradiction if we select δ small enough. Thus, for given $0 < \epsilon, \eta < 1$, and $T > 0$, we find a small $\delta(\eta, T) > 0$ such that if $0 < \delta < \delta(\eta, T)$, then $h < 0$ in $\Omega \times [0, T]$, i.e.,

$$\partial_t g_\eta^\epsilon < \delta t + \delta \qquad \text{in} \ \ \Omega \times [0, T] \ .$$

Letting δ go to 0, we have $\partial_t g_\eta^\epsilon \leq 0$ in $\Omega \times [0, T]$, i.e., for $(x, t) \in \Omega \times [0, T]$,

$$g_\eta^\epsilon(x, t + s) - g_\eta^\epsilon(x, t) \leq 0, \quad \forall s > 0.$$

Using the uniform convergence of g_η^ϵ to g_η, we let ϵ go to 0 to deduce

$$\partial_t g_\eta \leq 0 \quad \text{in} \quad \Omega \times [0, T],$$

completing the proof. □

Lemma 3.2.12. *Suppose that a smooth operator F satisfies (F1) and (F3). Let g_η be the solution of* (3.2.7) *with the initial data $g_{\eta,0} \in C^{2,\gamma}(\overline{\Omega})$ $(0 < \gamma < 1)$ satisfying* (3.2.8) *for some $g_0 \in C_b(\overline{\Omega})$. We assume that $F(D^2 g_{\eta,0}) = 0$ on $\partial\Omega$ and*

$$-\tilde{C} g_{\eta,0}^{\frac{1}{m}} \leq F(D^2 g_{\eta,0}) \leq 0 \quad \textit{in} \quad \Omega$$

for some $\tilde{C} > 0$. Then we have

$$-m\tilde{C} g_\eta \leq \partial_t g_\eta \leq 0 \quad \textit{in} \ \ \Omega \times (0, +\infty).$$

Proof. According to Lemma 3.2.11, $\partial_t g_\eta$ is nonpositive in $\Omega \times (0, +\infty)$. Define a linearized operator

$$H[h] := m g_\eta^{1-1/m} F_{ij} \cdot D_{ij} h - \partial_t h + \left(1 - \frac{1}{m}\right) \frac{\partial_t g_\eta}{g_\eta} h$$

for $F_{ij} := \frac{\partial F}{\partial p_{ij}}(D^2 g_\eta)$. Then we have that $H[\partial_t g_\eta] = 0$ in $\Omega \times (0, +\infty)$, and $-g_\eta$ satisfies

$$H[-g_\eta] \geq 0 + \left(1 - \frac{1}{m}\right) \frac{\partial_t g_\eta}{g_\eta}(-g_\eta) \geq 0 \quad \text{in } \Omega \times (0, +\infty)$$

since F is concave (see the proof of Lemma 3.2.9), and $\partial_t g_\eta \leq 0$ in $\Omega \times (0, +\infty)$. We note that $-m\tilde{C} g_{\eta,0} \leq \partial_t g_{\eta,0} = m g_{\eta,0}^{1-\frac{1}{m}} F(D^2 g_{\eta,0})$ in Ω and that $-m\tilde{C} g_\eta < 0 = \partial_t g_\eta$ on $\partial\Omega \times [0, +\infty)$. Therefore the result follows from the comparison principle. □

Lemma 3.2.13. *Suppose that a smooth operator F satisfies (F1) and (F3). and that Ω is strictly convex. Let g_η be the solution of* (3.2.7) *with the initial data $g_{\eta,0} \in C^{2,\gamma}(\overline{\Omega})$, $(0 < \gamma < 1)$ satisfying* (3.2.8) *for some $g_0 \in C_b(\overline{\Omega})$. We assume that $F(D^2 g_{\eta,0}) = 0$ on $\partial\Omega$, and*

$$-\tilde{C} g_{\eta,0}^{1/m} \leq F(D^2 g_{\eta,0}) \leq 0 \quad \text{in} \quad \Omega$$

for a uniform constant $\tilde{C} > 0$ with respect to $0 < \eta < 1$. We also assume that $g_{\eta,o}$ has a uniform C^2- estimate in $\overline{\Omega}$ with respect to $0 < \eta < 1$. Then for $T > 0$, we have

$$|D_x^2 g_\eta| < C(T) \quad \text{uniformly on } \partial\Omega \times [0, T], \tag{3.2.10}$$

where $C(T) > 0$ depends only on $m, n, \lambda, \Lambda, T, \tilde{C}$, the boundary gradient estimate of g_0, and the uniform C^2- estimate of $g_{\eta,0}$.

Proof. (i) Since $g_{\eta,o}$ has a uniform Lipschitz estimate with respect to $0 < \eta < 1$, Lemma 3.2.7 and Lemma 3.2.11 imply that

$$|\nabla g_\eta| < C \quad \text{uniformly in } \overline{\Omega} \times [0, +\infty), \tag{3.2.11}$$

where $C > 0$ is uniform with respect to $0 < \eta < 1$. In fact, for any unit vector $e_\alpha \in \mathbb{R}^n$, $\partial_\alpha g_\eta$ satisfies

$$m g_\eta^{1-1/m} F_{ij} \cdot D_{ij}(\partial_\alpha g_\eta) - \partial_t(\partial_\alpha g_\eta) + \left(1 - \frac{1}{m}\right) \frac{\partial_t g_\eta}{g_\eta} \partial_\alpha g_\eta = 0 \quad \text{in } \Omega \times (0, +\infty)$$

for $F_{ij} := \frac{\partial F}{\partial p_{ij}}(D^2 g_\eta)$. Since $\partial_t g_\eta \leq 0$ from Lemma 3.2.11, the comparison principle implies that $\partial_\alpha g_\eta$ is uniformly bounded by a constant depending only on the uniform Lipschitz estimate of $g_{\eta,0}$, and the uniform boundary gradient estimate in Lemma 3.2.7.

(ii) We fix $\eta > 0$, and denote g_η by g for simplicity. We fix a boundary point $x_o \in \partial\Omega$ and denote x_o by the origin. Now we introduce the coordinate system such that the tangent plane to $\partial\Omega$ at 0 is $x_n = 0$ with e_n being the inner normal vector. When $e_\tau = e_i$, $(i = 1, \cdots, n-1)$ is tangential to $\partial\Omega$ at 0, we have $g_\tau = 0$ and $g_{\tau\tau} = -g_{,n}\kappa_\tau < 0$ at 0 as in the proof of Lemma 3.1.8, where κ_τ is the curvature of $\partial\Omega$ at 0 in the direction e_τ. Using the uniform boundary estimates in Lemma 3.2.7, and the strict convexity of $\partial\Omega$, we obtain

$$0 < c(T) < -g_{\tau\tau}(0,t) < C, \quad \forall t \in [0,T] \tag{3.2.12}$$

for any tangential unit vector e_τ to $\partial\Omega$ at 0, where $0 < c(T) < C$ are independent of $0 < \eta < 1$. Thus, there is $C > 0$, independent of $0 < \eta < 1$, such that

$$|\partial_{e_i,e_j} g(0,t)| < C, \quad \forall t \in [0,T], \ (1 \leq i, j \leq n-1).$$

(iii) Near the origin, $\partial\Omega$ is represented by $x_n = \psi(x') = \frac{1}{2}A_{ij}x_i x_j + O(|x'|^3)$. Since Ω is strictly convex, the eigenvalues of $\left(A_{ij}\right)$ lie in $[\kappa_0, \kappa_1]$ for some $0 < \kappa_0 < \kappa_1$. After a change of coordinate of $\mathbb{R}^{n-1}$, the boundary of Ω near 0 becomes $x_n = \tilde{\psi}(x') = \frac{1}{2}|x'|^2 + O(|x'|^3)$ and the operator F will be transformed to a new operator $\tilde{F}$ with new elliptic coefficients $\tilde{\lambda} = \tilde{\lambda}(\lambda, \Lambda, \kappa_0, \kappa_1)$ and $\tilde{\Lambda} = \tilde{\Lambda}(\lambda, \Lambda, \kappa_0, \kappa_1)$ that are uniformly bounded and positive. So $\partial\Omega$ is close to a unit ball with an error $O(|x'|^3)$ near the origin. For simplicity, we assume that $\Omega = B_1(e_n)$ and denote $B_1(e_n)$ by B_1. The general domain can be considered with a simple modification as [14].

(iv) We claim that $|\partial_{e_k,e_n} g(0,t)| \leq C$ for $t \in [0,T]$, where $C > 0$ is independent of η, and $k = 1, \cdots, n-1$. For positive constants A_1, A_2, and A_3, which will be fixed later, we define

$$\begin{aligned} w_\pm(x,t) &:= \partial_{T_k} g \pm A_1 \sum_{l=1}^{n-1} g_{,l}^2 \pm A_2 x_n^2 \pm A_3 \left(1 - |x - e_n|^2\right)^{2-\rho} \\ &:= (1 - x_n) g_{,k} + x_k g_{,n} \pm A_1 \sum_{l=1}^{n-1} g_{,l}^2 \pm A_2 x_n^2 \pm A_3 \left(1 - |x - e_n|^2\right)^{2-\rho}, \end{aligned}$$

where $\rho := 1 - \frac{1}{m}$, and $\partial_{T_k} g := (1 - x_n)g_{,k} + x_k g_{,n}$ is a directional derivative and coincides with a tangential derivative on ∂B_1. Define

$$v := A_4 x_n$$

for a uniform constant $A_4 > 0$, which will be chosen large later.

Now, we consider a linearized operator

$$H[h] := mg^{1-\frac{1}{m}} F_{ij} \cdot D_{ij}h - \partial_t h$$

with $F_{ij} := \frac{\partial F}{\partial p_{ij}}(D^2 g)$. We will show that

$$H[w_+] \geq 0 = H[v] \quad \text{and} \quad H[w_-] \leq 0 = H[-v] \quad \text{in} \quad B_1 \times (0, T] \tag{3.2.13}$$

for sufficiently large constants A_1, A_2, and A_3.

We can select $A_4 > 0$ large so that

$$-v \leq w_- \leq w_+ \leq v \quad \text{on} \quad B_1 \times \{0\}$$

since $g_{\eta,o}$ has a uniform C^2- estimate in $\overline{\Omega}$, and satisfies that $\partial_{T_k} g_{\eta,o} = 0$ on ∂B_1, $\partial_l g_{\eta,o}(0) = 0$ for $l = 1, \cdots, n-1$, and $|x|^2 \leq 2x_n$ for $x \in B_1$.

We recall that $|\nabla_x g| < c_1$ on $\partial B_1 \times [0, +\infty)$ from Lemma 3.2.7. Since $g = \eta^m$ on ∂B_1 and

$$\begin{aligned} g_{,l}^2 &= [(1 - x_n)g_{,l} + x_l g_{,n} + x_n g_{,l} - x_l g_{,n}]^2 \leq 2[(1 - x_n)g_{,l} + x_l g_{,n}]^2 + 2(x_n g_{,l} - x_l g_{,n})^2 \\ &\leq 2[(1 - x_n)g_{,l} + x_l g_{,n}]^2 + 8c_1^2|x|^2 = 8c_1^2|x|^2 \quad \text{on } \partial B_1 \times [0, +\infty), \end{aligned}$$

we see that for $(x, t) \in \partial B_1 \times [0, +\infty)$,

$$-\left\{8(n-1)c_1^2 A_1 + A_2\right\}|x|^2 \leq w_-(x, t) \leq w_+(x, t) \leq \left\{8(n-1)c_1^2 A_1 + A_2\right\}|x|^2.$$

Since $|x|^2 = 2x_n$ for $x \in \partial B_1$, we obtain that

$$-2\left\{8(n-1)c_1^2 A_1 + A_2\right\}x_n \leq w_- \leq w_+ \leq 2\left\{8(n-1)c_1^2 A_1 + A_2\right\}x_n, \quad \forall (x, t) \in \partial B_1 \times [0, T].$$

Thus for a large $A_4 > 0$, we have that

$$-v \leq w_- \leq w_+ \leq v \quad \text{on} \ \partial_p (B_1 \times (0, T]).$$

If we prove (3.2.13), then the comparison principle gives

$$-v \le w_- \le w_+ \le v \ \text{ in } \ B_1 \times [0, T].$$

Therefore, we deduce that, for $1 \le k \le n-1$,

$$|\partial_{kn} g(0,t)| = |\partial_n w_\pm(0,t)| \le |\partial_n v(0)| = A_4 \quad \text{for } t \in [0,T].$$

So, it remains to show (3.2.13) by choosing suitable constants A_1, A_2, and A_3. Note that $\partial_t g_\eta / g_\eta$ is uniformly bounded in $\Omega \times (0, +\infty)$ from Lemma 3.2.12. We use (3.2.11), Lemma 3.2.12, and the ellipticity of F to have

$$\begin{aligned}
H[w_+] \ge & -\left(1 - \frac{1}{m}\right) \frac{\partial_t g}{g} \left\{ (1-x_n) g_{,k} + x_k g_{,n} + 2A_1 \sum_{l=1}^{n-1} g_{,l}^2 \right\} \\
& + 2mg^{1-\frac{1}{m}} \left(-\sum_{i=1}^{n} F_{ni} g_{,ki} + \sum_{i=1}^{n} F_{ki} g_{,ni} + A_1 \sum_{l=1}^{n-1} F_{ij} g_{,li} g_{,lj} + A_2 F_{nn} \right) \\
& + A_3 (2-\rho)(1-\rho) m g^{1-\frac{1}{m}} \left(1 - |x - e_n|^2\right)^{-\rho} 4\lambda |x - e_n|^2 \\
& - A_3 (2-\rho) m g^{1-\frac{1}{m}} \left(1 - |x-e_n|^2\right)^{1-\rho} n\Lambda \\
\ge & -C(1 + A_1) + 2mg^{1-\frac{1}{m}} \left\{ -C \left(\sum_{l,i=1}^{n} |g_{,li}|^2 \right)^{\frac{1}{2}} + A_1 \lambda \sum_{l=1}^{n-1} \sum_{i=1}^{n} g_{,li}^2 + A_2 \lambda / 2 \right\} \\
& + mg^{1-\frac{1}{m}} A_2 \lambda + A_3 (2-\rho)(1-\rho) m g^{1-\frac{1}{m}} \left(1 - |x-e_n|^2\right)^{-\rho} 4\lambda |x-e_n|^2 \\
& - A_3 (2-\rho) m g^{1-\frac{1}{m}} \left(1 - |x - e_n|^2\right)^{1-\rho} n\Lambda
\end{aligned}$$

for a uniform $C > 0$ with respect to $0 < \eta < 1$. Using the equation $mg^{1-\frac{1}{m}} F(D^2 g) - \partial_t g = 0 = F(0)$ and the ellipticity of F, it follows that (see the proof of Theorem 9.5 in [13])

$$g_{,nn}^2 \le C \sum_{(i,j) \ne (n,n)} |g_{,ij}|^2 + C \left| \frac{\partial_t g}{g} \cdot g^{1/m} \right|^2 \quad \text{in } \ B_1 \times (0, +\infty). \tag{3.2.14}$$

Using (3.2.14) and Lemma 3.2.12, we have

$$\begin{aligned}H[w_+] \geq& -C(1+A_1) + 2mg^{1-\frac{1}{m}}\left\{-C\left(\sum_{l,i=1}^{n}|g_{,li}|^2\right)^{\frac{1}{2}} + \frac{A_1}{C}\sum_{l,i=1}^{n} g_{,li}^2 + A_2\lambda/2\right\}\\&+ m\lambda A_2 g^{1-\frac{1}{m}} + A_3(2-\rho)(1-\rho)mg^{1-\frac{1}{m}}\left(1-|x-e_n|^2\right)^{-\rho}4\lambda|x-e_n|^2\\&- A_3(2-\rho)mg^{1-\frac{1}{m}}\left(1-|x-e_n|^2\right)^{1-\rho} n\Lambda\end{aligned}$$

for a large $C > 0$, independent of $0 < \eta < 1$. Selecting $A_2 \geq A_1$ and $A_1^2 \geq \frac{C^3}{2\lambda}$ (see [13, Theorem 9.5]), we obtain

$$\begin{aligned}H[w_+] \geq& -C(1+A_1) + m\lambda A_2 g^{1-\frac{1}{m}} + A_3(2-\rho)(1-\rho)mg^{1-\frac{1}{m}}\left(1-|x-e_n|^2\right)^{-\rho}4\lambda|x-e_n|^2\\&- A_3(2-\rho)mg^{1-\frac{1}{m}}\left(1-|x-e_n|^2\right)^{1/m} n\Lambda.\end{aligned}$$

Since $g_\eta \geq \delta\left(1-|x-e_n|^2\right)$ in $B_1 \times (0,T]$ for a uniform $\delta = \delta(T) > 0$ with respect to $0 < \eta < 1$ from Lemma 3.2.7, we have

$$\begin{aligned}H[w_+] \geq& -C(1+A_1) + m\lambda A_2\delta^{1-\frac{1}{m}}\left(1-|x-e_n|^2\right)^{1-\frac{1}{m}}\\&+ A_3\left\{m(2-\rho)(1-\rho)\delta^{1-\frac{1}{m}}4\lambda|x-e_n|^2 - C\left(1-|x-e_n|^2\right)^{1/m}\right\}\end{aligned}$$

for a large $C > 0$ independent of $0 < \eta < 1$. Choosing A_3 and A_2 large, we deduce $H[w_+] \geq 0$ in $B_1 \times (0,T]$, and hence

$$H[w_+] \geq 0 = H[v] \quad \text{in } B_1 \times (0,T].$$

Similarly, we have $H[w_-] \leq 0 = H[-v]$ in $B_1 \times (0,T]$. Therefore, we have proved that

$$|\partial_{kn} g(0,t)| \leq A_4 \quad \text{for all } t \in [0,T],$$

where A_4 is uniform with respect to $0 < \eta < 1$.

(v) Lastly, since $g_{,nn}^2(0,t) \leq C \sum_{(i,j)\neq(n,n)} |g_{,ij}(0,t)|^2$ for $(0,t) \in \partial\Omega \times (0,+\infty)$ from (3.2.14), we have

$$|\partial_{nn} g_\eta(0,t)| \leq C, \quad \forall t \in (0,T],$$

where $C > 0$ is independent of $0 < \eta < 1$. Therefore, we have

$$|D_x^2 g_\eta(0,t)| \le C, \quad \forall t \in [0,T],$$

and hence (3.2.10) follows since $x_o = 0$ is an arbitrary point of $\partial\Omega$. □

Lemma 3.2.14. *Suppose that F satisfies (F1) and (F3). and that Ω is strictly convex. Let g_η be the solution of* (3.2.7) *with the initial data $g_{\eta,0} \in C^{2,\gamma}(\overline{\Omega})$, $(0 < \gamma < 1)$ satisfying* (3.2.8) *for some $g_0 \in C_b(\overline{\Omega})$. We also assume that for $T > 0$,*

$$|D_x^2 g_\eta| < C(T) \quad \textit{uniformly on } \partial\Omega \times (0,T], \tag{3.2.15}$$

where $C(T) > 0$ is independent of $0 < \eta < 1$. Let $w_\eta := g_\eta^{\frac{m-1}{2m}}$. Then, there exist $\eta(T) > 0$ and $c(T) > 0$ such that if $0 < \eta < \eta(T)$, then

$$w_{\eta,\alpha\alpha}(x,t) = \frac{m-1}{2m g_\eta^{2-\frac{m-1}{2m}}}\left(g_\eta g_{\eta,\alpha\alpha} - \frac{m+1}{2m} g_{\eta,\alpha}^2\right) \le -\frac{m-1}{2m}\frac{c(T)}{\eta^{\frac{m+1}{2}}}, \quad \forall (x,t) \in \partial\Omega \times (0,T], \tag{3.2.16}$$

for any unit vector $e_\alpha \in \mathbb{R}^n$, where $\eta(T) > 0$ and $c(T) > 0$ depend only on $C(T), c_0e^{-KT}$ and the lower bound of the curvature of $\partial\Omega$, and the uniform constant $c_0e^{-KT} > 0$ is as in Lemma 3.2.7.

Proof. We fix $\eta > 0$. For simplicity, we denote g_η by g. Fix a $(x_o,t_o) \in \partial\Omega \times (0,T]$. We may assume $x_o = 0$ and introduce the coordinate system such that $x_o = 0$ and that the tangent plane at 0 is $x_n = 0$ with e_n being the inner normal vector at the origin. When $e_\tau = e_i$, $(i = 1, \cdots, n-1)$ is tangential to $\partial\Omega$ at $x_o = 0$, we have $g_\tau = 0$ and $g_{\tau\tau} = -g_{,n}\kappa_\tau < 0$ at 0 as in the proof of Lemma 3.1.8, where $\kappa_\tau > 0$ is the curvature of $\partial\Omega$ at 0 in the direction e_τ.

According to the uniform boundary gradient estimates in Lemma 3.2.7 and the strict convexity of $\partial\Omega$, we have

$$0 < c_1(T) < -g_{\tau\tau} = |\nabla g| \cdot \kappa_\tau < C_1 \quad \text{at } (0,t_o) \in \partial\Omega \times (0,T] \tag{3.2.17}$$

for any tangential unit vector e_τ to $\partial\Omega$ at 0, where $c_1(T) > 0$ and $C_1 > 0$ are uniform with respect to $0 < \eta < 1$. Then we have

$$g(0,t_o) \cdot g_{\tau\tau}(0,t_o) - \frac{m+1}{2m} g_\tau^2(0,t_o) \le -c_1(T)\eta^m - 0 = -c_1(T)\eta^m$$

for any tangent unit vector e_τ to $\partial\Omega$ at $x_o = 0 \in \partial\Omega$, where $c_1(T) > 0$ depends only on $c_0 e^{-KT}$ and the lower bound of the curvature of $\partial\Omega$.

Let e_α be any unit vector in $\mathbb{R}^n$. We decompose $e_\alpha := \beta_1 e_\tau + \beta_2 e_\nu$ with $\beta_1^2 + \beta_2^2 = 1$, where unit vectors e_ν and e_τ are normal and tangent to $\partial\Omega$ at 0, respectively. We use (3.2.17), (3.2.15) and Lemma 3.2.7 to have at $(0, t_o) \in \partial\Omega \times (0, T]$,

$$\begin{aligned} gg_{,\alpha\alpha}(0,t_o) - \frac{m+1}{2m} g_{,\alpha}^2(0,t_o) &= g\left(\beta_1^2 g_{\tau\tau} + 2\beta_1\beta_2 g_{\tau\nu} + \beta_2^2 g_{\nu\nu}\right) - \frac{m+1}{2m}\beta_2^2 g_\nu^2 \\ &\le g\left\{-c_1(T)\beta_1^2 + C(T)\left(2\beta_1\beta_2 + \beta_2^2\right)\right\} - \beta_2^2 \delta_o(T) \end{aligned}$$

for a uniform $\delta_o(T) > 0$ depending on $m, c_0 e^{-KT}$. We use Young's inequality to deduce that at $(0, t_o) \in \partial\Omega \times (0, T]$,

$$\begin{aligned} gg_{,\alpha\alpha}(0,t_o) - \frac{m+1}{2m} g_{,\alpha}^2(0,t_o) &\le g\left\{-\frac{c_1(T)}{2}\beta_1^2 + \widetilde{C}(T)\beta_2^2\right\} - \beta_2^2\delta_o(T) \\ &= -\eta^m \frac{c_1(T)}{2}\beta_1^2 + \left\{\eta^m \widetilde{C}(T) - \delta_o(T)\right\}\beta_2^2 \\ &\le -\frac{\min\{c_1(T), \delta_o(T)\}}{2}\eta^m(\beta_1^2 + \beta_2^2) =: -c(T)\eta^m \end{aligned}$$

with $\widetilde{C}(T) := C(T) + \frac{2C(T)^2}{c_1(T)}$, for small $0 < \eta^m < \{\eta(T)\}^m := \frac{\delta_o(T)}{2\widetilde{C}(T)}$. Thus we conclude

$$g_\eta(x_o,t_o) g_{\eta,\alpha\alpha}(x_o,t_o) - \frac{m+1}{2m} g_{\eta,\alpha}^2(x_o,t_o) \le -c(T)\eta^m$$

for any direction e_α. Since (x_o, t_o) is an arbitrary point of $\partial\Omega \times (0, T]$, we deduce

$$w_{\eta,\alpha\alpha} = \frac{m-1}{2m}\frac{1}{\eta^{2m-\frac{m-1}{2}}}\left(g_\eta g_{\eta,\alpha\alpha} - \frac{m+1}{2m} g_{\eta,\alpha}^2\right) \le -\frac{m-1}{2m}\frac{c(T)}{\eta^{\frac{m+1}{2}}} \quad \text{on} \quad \partial\Omega \times (0, T],$$

completing the proof. □

Lemma 3.2.15. *Suppose that F satisfies (F1) and (F3), and that Ω is strictly convex. Then there exist $g_0 \in C_b(\overline{\Omega}) \cap C^{2,\gamma}(\overline{\Omega})$, $(0 < \gamma < 1)$, and $0 < \eta_o < 1$ satisfying the following properties :*

(i) g_0 *and* $g_{\eta,0} := g_0 + \eta^m$ *satisfy*

$$F(D^2\psi) = 0 \quad \text{on } \partial\Omega, \quad \text{and} \quad -\tilde{C}\psi^{1/m} \le F(D^2\psi) \le 0 \quad \text{in} \quad \Omega$$

for a uniform constant $\tilde{C} > 0$ with respect to $0 < \eta < \eta_o$ and F, *(ii)* g_0 *and* $g_{\eta,0}$ *have a uniform* $C^{2,\gamma}$*- estimate in* $\overline{\Omega}$ *with respect to* $0 < \eta < \eta_o$ *and* F,

(iii) $g_0^{\frac{m-1}{2m}}$ *and* $g_{\eta,0}^{\frac{m-1}{2m}}$ *are concave in* Ω *for* $0 < \eta < \eta_o$,

(iv) $g_{\eta,0}$ *converges to* g_0 *in* $C^2(\overline{\Omega})$ *as* η *tends to* 0, *where* $\eta_o > 0$ *is a uniform constant.*

Proof. Let d be the distance function to $\partial\Omega$, which is concave in Ω, and let h be the solution to

$$\begin{cases} F(D^2h) = -d^{\frac{1}{m}} & \text{in } \Omega, \\ h = 0 & \text{on } \partial\Omega. \end{cases}$$

Note that h has a uniform $C^{2,\gamma}$- estimate $(0 < \gamma < 1)$ in $\overline{\Omega}$ with respect to F. Then there exist uniform numbers $\epsilon_o, \eta_o > 0$, and a convex domain $\Omega_o \Subset \Omega$ such that $|\nabla h| > \epsilon_o$ in $\Omega \setminus \Omega_o$ from Hopf's Lemma, the level sets of h are strictly convex in $\Omega \setminus \Omega_o$, and that $D^2 (h + \eta^m)^{\frac{m-1}{2m}} \leq 0$ in $\Omega \setminus \Omega_o$ for $0 \leq \eta < \eta_o$ since $h \in C^{2,\gamma}(\overline{\Omega})$, and

$$D^2 (h + \eta^m)^{\frac{m-1}{2m}} = \frac{m-1}{2m}(h + \eta^m)^{\frac{m-1}{2m}-2} \left\{(h + \eta^m)D^2h - \frac{m+1}{2m}\nabla h \nabla h^t\right\}$$

(we refer to (3.1.8), and the proof of Lemma 3.2.14).

Let $\delta_0 < \delta_1 < \delta_2$ be small positive numbers to be fixed later satisfying $\Omega \setminus \{h > \delta_2\} \subset \Omega \setminus \Omega_o$. Define

$$\tilde{h} := h - C_o(h - \delta_0)^3 \quad \text{in } \{h > \delta_0\} \setminus \Omega_o$$

for some $C_o > 0$, which will be chosen later. Then we can find $0 < \delta_0 < \delta_1 < \delta_2 < 1$, and $C_o > 0$ such that $\nabla\tilde{h} \parallel \nabla h$, $\frac{1}{2} < \nabla\tilde{h} \cdot \nabla h < 1$ in $\{h > \delta_0\} \setminus \{h > \delta_2\}$, $D^2\tilde{h} \leq 0$ in $\{h > \delta_1\} \setminus \{h > \delta_2\}$, and that $F(D^2\tilde{h}) \leq 0$, $D^2(\tilde{h} + \eta^m)^{\frac{m-1}{2m}} \leq 0$ in $\{h > \delta_0\} \setminus \{h > \delta_1\}$ for $0 \leq \eta < \eta_o$.

Now we set

$$g_0 := h \quad \text{in } \Omega \setminus \{h > \delta_0\}, \quad g_0 := \tilde{h} \quad \text{in } \{h > \delta_0\} \setminus \{h > \delta_2\},$$

and

$$g_0 := \delta_2 - C_o(\delta_2 - \delta_0)^3 \quad \text{on } \{h > \delta_2\}.$$

Then g_o is concave in $\{h > \delta_1\}$. By regularizing g_0 in $\{h > (\delta_1 + \delta_2)/2\}$, we obtain $g_0 \in C^{2,\gamma}(\overline{\Omega})$ such that g_0 is concave in $\{h > \delta_1\}$, and that

$$\begin{cases} -\tilde{C} g_0^{\frac{1}{m}} \leq F(D^2 g_0) \leq 0 & \text{in } \Omega \setminus \{h > \delta_1\}, \\ g_0 = 0 & \text{on } \partial\Omega \end{cases}$$

for some $\tilde{C} > 0$ since $h \in C_b(\overline{\Omega}) \cap C^{2,\gamma}(\overline{\Omega})$, and $F(D^2\tilde{h}) \leq 0$ in $\{h > \delta_0\} \setminus \{h > \delta_1\}$. We notice that $g_0^{\frac{m-1}{2m}}$ is concave in $\{h > \delta_1\}$ since g_0 is concave in $\{h > \delta_1\}$. Therefore, we set $g_{\eta,0} := g_0 + \eta^m$, and hence g_0 and $g_{\eta,0}$ satisfy $(i) - (iv)$, where we recall $D^2(g_0 + \eta^m)^{\frac{m-1}{2m}} \leq 0$ in $\Omega \setminus \{h > \delta_1\}$ for $0 \leq \eta < \eta_o$. □

Lemma 3.2.16. *Suppose that F satisfies (F1), (F2) and (F3) and Ω is a strictly convex bounded domain. Let g_0 be an initial data in Lemma 3.2.15, u be the solution of* (3.2.2) *with initial data $u_0 := g_0^{\frac{1}{m}}$, and $g := u^m$. Then $u^{\frac{m-1}{2}}$ is concave in the spatial variables for any $t > 0$, i.e.,*

$$D_x^2 u^{\frac{m-1}{2}} \leq 0 \quad \textit{in} \quad \Omega \times (0, +\infty).$$

Proof. Let g_η be the solution of (3.2.7) with the initial data $g_{\eta,0}$, where $g_{\eta,0}$ is as in Lemma 3.2.15. For a fixed $T > 0$, it suffices to show the concavity of $w_\eta := u_\eta^{\frac{m-1}{2}} = g_\eta^{\frac{m-1}{2m}}$ for small $0 < \eta \ll 1$;

$$\frac{1}{2}\left(g_\eta^{\frac{m-1}{2m}}(x,t) + g_\eta^{\frac{m-1}{2m}}(y,t)\right) - g_\eta^{\frac{m-1}{2m}}\left(\frac{x+y}{2}, t\right) \leq 0, \quad \forall x, y \in \Omega, \ \ \forall t \in [0, T]$$

since the uniform convergence of g_η to g in Lemma 3.2.8 preserves the concavity.

Now, we approximate F by a smooth operator F^ϵ as in Lemma 3.1.6, and we may assume that $F^\epsilon(0) = 0$. For any $\epsilon > 0$, let g_0^ϵ and $g_{\eta,0}^\epsilon$ be the initial data as in Lemma 3.2.15 with F^ϵ in place of F, and let g_η^ϵ be the solution of (3.2.7) with F^ϵ and $g_{\eta,0}^\epsilon$. We note that Lemmas 3.2.13 and 3.2.14 hold for g_η^ϵ for $0 < \epsilon < 1$ and $0 < \eta < \min\{\eta_o, \eta(T)\}$, where $\eta(T)$ and η_o are the uniform constants as in Lemma 3.2.14 and Lemma 3.2.15, respectively.

Fix $0 < \eta < \min\{\eta(T), \eta_o\}$. Since $g_{\eta,0}^\epsilon$ is uniformly bounded with respect to $0 < \epsilon < 1$, where η is fixed, the comparison principle implies

$$0 < \eta^m \leq g_\eta^\epsilon \leq C < +\infty \quad \text{in} \quad \Omega \times (0, +\infty).$$

Since g_η^ϵ is uniformly bounded with respect to $0 < \epsilon < 1$, we have the uniform global $C^{1,\gamma}$- estimate ($0 < \gamma < 1$) for g_η^ϵ in $\overline{\Omega} \times [0,T]$ with respect to $0 < \epsilon < 1$ from [61] and hence the uniform $C^{2,\gamma}$ estimate for g_η^ϵ in $\overline{\Omega} \times [0,T]$ using Theorem 1.1 in [61], where $0 < \eta < \min\{\eta(T), \eta_o\}$ is fixed. According to Arzela-Ascoli Theorem, g_η^ϵ converges uniformly to g_η in $C^2(\overline{\Omega} \times [0,T])$, up to a subsequence, since $g_{\eta,0}^\epsilon$ converges to $g_{\eta,0}$ uniformly in $\overline{\Omega}$ as ϵ tends to 0, up to a subsequence. Thus we consider the concavity of $w_\eta^\epsilon := (u_\eta^\epsilon)^{\frac{m-1}{2}}$ for small $0 < \epsilon < 1$.

We note that the function g_η^ϵ solves

$$mg^{1-1/m} F^\epsilon(D^2 g) = \partial_t g \quad \text{in} \quad \Omega \times (0,T],$$

which is uniformly parabolic for a given $\eta > 0$.

The geometric quantity w_η^ϵ satisfies

$$\partial_t w = \frac{m-1}{2} w^{\frac{m-3}{m-1}} F^\epsilon \left(\frac{2m}{m-1} w^{\frac{3-m}{m-1}} \left(w^2 D^2 w + \frac{m+1}{m-1} w Dw Dw^t \right) \right) \quad \text{in} \quad \Omega \times (0,T].$$

After the change of the time $t \mapsto mt$, the above equation will be transformed to

$$\partial_t w = \frac{m-1}{2m} w^{\frac{m-3}{m-1}} F^\epsilon \left(\frac{2m}{m-1} w^{\frac{3-m}{m-1}} \left(w^2 D^2 w + rw Dw Dw^t \right) \right), \tag{3.2.18}$$

with $r = \frac{m+1}{m-1}$. By taking differentiation twice, the function w_η^ϵ satisfies

$$\begin{aligned}
\partial_t w_{,\alpha\beta} = {} & \frac{m-1}{2m} w^{\frac{m-3}{m-1}} F^\epsilon_{ij,kl} \cdot \left(\frac{2m}{m-1} w^{\frac{3-m}{m-1}} \left(w^2 D_{ij} w + \frac{m+1}{m-1} w D_i w D_j w \right) \right)_{,\alpha} \\
& \cdot \left(\frac{2m}{m-1} w^{\frac{3-m}{m-1}} \left(w^2 D_{kl} w + \frac{m+1}{m-1} w D_k w D_l w \right) \right)_{,\beta} \\
& + F^\epsilon_{ij} \cdot (2 w_{,\alpha} w_{,\beta} D_{ij} w + 2 w w_{,\alpha\beta} D_{ij} w + 2 w w_{,\alpha} D_{ij} w_{,\beta} + 2 w w_{,\beta} D_{ij} w_{,\alpha} + w^2 D_{ij} w_{,\alpha\beta} \\
& + r w_{,\alpha\beta} D_i w D_j w + 2 r w_{,\alpha} D_i w_{,\beta} D_j w + 2 r w_{,\beta} D_i w_{,\alpha} D_j w \\
& + 2 r w D_i w_{,\alpha} D_j w_{,\beta} + 2 r w D_i w_{,\alpha\beta} D_j w) \\
& + \frac{m-1}{2m} \frac{m-3}{m-1} w^{\frac{m-3}{m-1}-1} w_{,\alpha\beta} F^\epsilon \left(\frac{2m}{m-1} w^{\frac{3-m}{m-1}} \left(w^2 D^2 w + rw Dw Dw^T \right) \right) \\
& - \frac{m-3}{m-1} w^{-1} w_{,\alpha\beta} F^\epsilon_{ij} \cdot \left(w^2 D_{ij} w + rw D_i w D_j w \right) \\
& - \frac{m-1}{2m} \frac{m-3}{m-1} \frac{2}{m-1} w^{\frac{m-3}{m-1}-2} w_{,\alpha} w_{,\beta} F^\epsilon \left(\frac{2m}{m-1} w^{\frac{3-m}{m-1}} \left(w^2 D^2 w + rw Dw Dw^T \right) \right) \\
& + \frac{m-3}{m-1} \frac{2}{m-1} w^{-2} w_{,\alpha} w_{,\beta} F^\epsilon_{ij} \cdot \left(w^2 D_{ij} w + rw D_i w D_j w \right),
\end{aligned}$$

with $F^{\epsilon}_{ij} = \frac{\partial F^{\epsilon}}{\partial p_{ij}} \left(\frac{2m}{m-1}(w^{\epsilon}_{\eta})^{\frac{3-m}{m-2}} \left((w^{\epsilon}_{\eta})^2 D^2 w^{\epsilon}_{\eta} + r w^{\epsilon}_{\eta} D w^{\epsilon}_{\eta} (D w^{\epsilon}_{\eta})^t\right)\right)$ and

$$F^{\epsilon}_{ij,kl} = \frac{\partial^2 F^{\epsilon}}{\partial p_{ij} \partial p_{kl}} \left(\frac{2m}{m-1}(w^{\epsilon}_{\eta})^{\frac{3-m}{m-2}} \left((w^{\epsilon}_{\eta})^2 D^2 w^{\epsilon}_{\eta} + r w^{\epsilon}_{\eta} D w^{\epsilon}_{\eta} (D w^{\epsilon}_{\eta})^t\right)\right).$$

In order to study the concavity of w^{ϵ}_{η}, we consider for given $0 < \epsilon < \delta < 1$,

$$\sup_{y\in\Omega} \sup_{|e_\beta|=1} \partial_{\beta\beta} w^{\epsilon}_{\eta}(y,t) + \psi(t),$$

where $e_\beta \in \mathbb{R}^n$ is a unit vector, and $\psi(t) := -\epsilon - e^{-1/\delta} e^{2Kt} \tan(K\sqrt{\delta}t)$ for some uniform constant $K > 0$, which will be chosen later and independent of $0 < \epsilon < \delta < 1$.

Now suppose that there is a time $t_o \in \left[0, \min\left(\frac{\pi}{4K\sqrt{\delta}}, T\right)\right] \subset [0,T]$ such that

$$\sup_{y\in\Omega} \sup_{|e_\beta|=1} \partial_{\beta\beta} w^{\epsilon}_{\eta}(y,t_o) + \psi(t_o) = 0.$$

We may assume that t_o is the first time for it to be zero. From the assumption on the initial data $g^{\epsilon}_{\eta,0}$ that $D^2 (g^{\epsilon}_{\eta,0})^{\frac{m-1}{2m}} \le 0$ in Ω, we have

$$\sup_{y\in\Omega} \sup_{|e_\beta|=1} \partial_{\beta\beta} w^{\epsilon}_{\eta}(y,0) + \psi(0) \le \psi(0) = -\epsilon < 0,$$

and hence $t_o > 0$.

We assume that the supremum

$$\sup_{y\in\Omega} \sup_{|e_\beta|=1} \partial_{\beta\beta} w^{\epsilon}_{\eta}(x,t_o) = \partial_{\overline{\alpha\alpha}} w^{\epsilon}_{\eta}(x_o,t_o) = -\psi(t_o) > 0$$

is achieved at $(x_o,t_o) \in \overline{\Omega} \times \left(0, \min\left(\frac{\pi}{4K\sqrt{\delta}}, T\right)\right]$ with some direction $e_{\overline{\alpha}}$. From the boundary estimate of the second derivatives, (3.2.16) in Lemma 3.2.14, the maximum point x_o should be an interior point of Ω since $\partial_{\overline{\alpha\alpha}} w^{\epsilon}_{\eta}(x_o,t_o) = -\psi(t_o) > 0$.

Without losing of generality, we assume that $x_o = 0$ and introduce orthonormal coordinates in which $e_{\overline{\alpha}}$ is taken as one of the coordinate axes so that

$$\partial_{\overline{\alpha}\beta} w^{\epsilon}_{\eta}(0,t_o) = 0 \quad \text{for } \beta \ne \overline{\alpha}.$$

In order to create extra terms, we perturb second derivatives of w^{ϵ}_{η} and we use the function

$$Z(x,t) := \partial_{\alpha\beta} w^{\epsilon}_{\eta}(x,t) \xi^{\alpha}(x) \xi^{\beta}(x)$$

where $\xi^{\beta}(x) := \delta_{\overline{\alpha}\beta} + c_{\overline{\alpha}}x^{\beta} + \frac{1}{2}c_{\overline{\alpha}}c_{\alpha}x^{\alpha}x^{\beta}$ and we denote $\vec{\xi}^{t} := (\xi^1, \cdots, \xi^n)$ (see [47]). We choose $c_{\beta} \in \mathbb{R}$ so that

$$-4(w^{\epsilon}_{\eta})^2 c_{\beta} + 4w^{\epsilon}_{\eta}\,\partial_{\beta}w^{\epsilon}_{\eta} = 0 \quad \text{ at the maximum point } (0, t_o).$$

We notice that $Z(0, t_o) = \partial_{\overline{\alpha\alpha}}w^{\epsilon}_{\eta}(0, t_o) = -\psi(t_o) > 0$.

Now we define

$$Y(x, t) := Z(x, t) + \psi(t)|\vec{\xi}(x)|^2 = \vec{\xi}^{t}(x)\left\{D^2_x w^{\epsilon}_{\eta}(x, t) + \psi(t)\mathbf{I}\right\}\vec{\xi}(x).$$

We have that

$$\partial_t Y \geq 0, \quad D^2_x Y \leq 0 \;\text{ and }\; \nabla_x Y = 0 \quad \text{at} \quad (0, t_o), \tag{3.2.19}$$

since

$$\begin{aligned} &D^2_x w^{\epsilon}_{\eta}(x, t) + \psi(t)\mathbf{I} \leq 0 \quad \text{for } (x, t) \in \Omega \times (0, t_o], \\ \text{and}\quad &Y(0, t_o) = \vec{\xi}^{t}(0)\left\{D^2_x w^{\epsilon}_{\eta}(0, t_o) + \psi(t_o)\mathbf{I}\right\}\vec{\xi}(0) = \partial_{\overline{\alpha\alpha}}w^{\epsilon}_{\eta}(0, t_o) + \psi(t_o) = 0. \end{aligned}$$

A simple computation gives that at $(0, t_o)$,

$$\begin{aligned} Z_{,i} &= \partial_{\alpha\beta i}w^{\epsilon}_{\eta}\xi^{\alpha}\xi^{\beta} + 2\partial_{\beta i}w^{\epsilon}_{\eta}c_{\alpha}\xi^{\alpha}\xi^{\beta} \\ Z_{,ij} &= \partial_{\alpha\beta ij}w^{\epsilon}_{\eta}\xi^{\alpha}\xi^{\beta} + 4\partial_{\beta ij}w^{\epsilon}_{\eta}c_{\alpha}\xi^{\alpha}\xi^{\beta} + 2\partial_{\beta i}w^{\epsilon}_{\eta}c_j c_{\alpha}\xi^{\alpha}\xi^{\beta} + 2\partial_{ij}w^{\epsilon}_{\eta}c_{\alpha}c_{\beta}\xi^{\alpha}\xi^{\beta}. \end{aligned}$$

Thus at the maximum point $(0, t_o)$ of all second derivatives of w^{ϵ}_{η}, we have

$$\begin{aligned} Z_t &= \partial_{\alpha\beta t}w^{\epsilon}_{\eta}\xi^{\alpha}\xi^{\beta} \\ &\leq w^2 F^{\epsilon}_{ij}\cdot Z_{,ij} + F^{\epsilon}_{ij}\cdot w_{,\beta ij}\xi^{\alpha}\xi^{\beta}\left(-4w^2 c_{\alpha} + 4ww_{,\alpha}\right) + 2F^{\epsilon}_{ij}\cdot w_{,ij}\left(-w^2 c^2_{\overline{\alpha}} + w^2_{,\overline{\alpha}}\right) \\ &\quad + \left(2wF^{\epsilon}_{ij}\cdot D_{ij}w + rF^{\epsilon}_{ij}\cdot D_i w D_j w\right)w_{,\overline{\alpha\alpha}} + 4rF^{\epsilon}_{\overline{\alpha}j}\cdot D_j w\, w_{,\overline{\alpha}}w_{,\overline{\alpha\alpha}} - 2w^2 F^{\epsilon}_{\overline{\alpha}j}\cdot c_j c_{\overline{\alpha}}\, w_{,\overline{\alpha\alpha}} \\ &\quad + 2rwF^{\epsilon}_{ij}\cdot D_j w D_i w_{,\overline{\alpha\alpha}} + 2rwF^{\epsilon}_{\overline{\alpha\alpha}}\cdot w^2_{,\overline{\alpha\alpha}} \\ &\quad + \frac{|m-3|}{2m}w^{-\frac{m+1}{m-1}}\left\{ww_{,\overline{\alpha\alpha}} + \frac{2}{m-1}w^2_{,\overline{\alpha}}\right\}2\sqrt{n}\Lambda\epsilon, \\ &\leq w^2 F^{\epsilon}_{ij}\cdot\partial_{ij}\left(|\vec{\xi}|^2\right)(-\psi(t_o)) + \left(2w\cdot n\Lambda\partial_{\overline{\alpha\alpha}}w + r\Lambda|\nabla w|^2\right)Z \\ &\quad + 4r\Lambda|\nabla w|^2 Z + 2w^2\Lambda\left(\sum_j c^2_j\right)Z \\ &\quad + 2rw\Lambda Z^2 + \frac{|m-3|}{2m}w^{-\frac{m+1}{m-1}}\left\{ww_{,\overline{\alpha\alpha}} + \frac{2}{m-1}w^2_{,\overline{\alpha}}\right\}2\sqrt{n}\Lambda\epsilon, \end{aligned}$$

where we use the concavity of F^ϵ, (3.1.5) in Lemma 3.1.6, (3.2.19), $(F1)$, and the choice of c_α.

Since $F^\epsilon_{ij}\cdot\partial_{ij}\left(|\vec{\xi}|^2\right) = 2c_{\overline{\alpha}}\sum_{j=1}^n\left(F_{\overline{\alpha}j}c_j + F_{jj}c_{\overline{\alpha}}\right) \le 2(n+1)\Lambda|c_{\overline{\alpha}}|\left(\sum_j c_j^2\right)^{1/2}$ from $(F1)$, $w := w^\epsilon_\eta$ satisfies that at $(0,t_o)$,

$$\begin{aligned}
&Z_t(0,t_o) = \partial_{\alpha\beta t}w^\epsilon_\eta\xi^\alpha\xi^\beta\\
&\le\left\{2w^\epsilon_\eta n\Lambda + 2rw^\epsilon_\eta\Lambda\right\}Z^2 + \left\{(w^\epsilon_\eta)^2\cdot 2(n+1)\Lambda\left(\sum_j c_j^2\right) + 5r\Lambda|\nabla w^\epsilon_\eta|^2 + 2(w^\epsilon_\eta)^2\Lambda\left(\sum_j c_j^2\right)\right.\\
&\quad\left.+\frac{|m-3|}{2m}(w^\epsilon_\eta)^{-\frac{2}{m-1}}2\sqrt{n}\Lambda\right\}Z + \left\{\frac{|m-3|}{m(m-1)}(w^\epsilon_\eta)^{-\frac{m+1}{m-1}}(\partial_{\overline{\alpha}}w^\epsilon_\eta)^2\cdot 2\sqrt{n}\Lambda\right\}\epsilon.
\end{aligned}$$

Since $c_\beta = \frac{\partial_\beta w^\epsilon_\eta(0,t_o)}{w^\epsilon_\eta(0,t_o)}$, we use the uniform global $C^{1,\gamma}$ estimate for g^ϵ_η (with respect to $0<\epsilon<1$) to find a large constant $K = K_\eta > 0$ independent of $0<\epsilon<\delta<1$ such that

$$Z_t(0,t_o) \le K\left(Z^2 + Z + \epsilon\right),$$

where η is fixed. Thus we obtain that

$$\begin{aligned}
0 \le \partial_t Y(0,t_o) &= Z_t(0,t_o) + \psi_t(t_o)\\
&\le K\left(Z^2+Z+\epsilon\right) + \psi_t(t_o) = \psi_t(t_o) + K\left(\psi^2(t_o) - \psi(t_o) + \epsilon\right).
\end{aligned}$$

On the other hand, we can check that $\psi(t) := -\epsilon - e^{-1/\delta}e^{2Kt}\tan(K\sqrt{\delta}t)$ satisfies

$$\psi_t + K(\psi^2 - \psi + \epsilon) < 0, \quad \forall 0 < t \le \frac{\pi}{4K\sqrt{\delta}},$$

for small $0<\epsilon\ll\delta\ll 1$, which are uniform numbers with respect to T,η, and K. This implies a contradiction to the fact that $\partial_t Y(0,t_o) \ge 0$ if $t_o \in \left(0, \min\left(\frac{\pi}{4K\sqrt{\delta}}, T\right)\right]$. Therefore, we deduce that for small $0<\epsilon\ll\delta\ll 1$,

$$\sup_{y\in\Omega}\sup_{|e_\beta|=1}\partial_{\beta\beta}w^\epsilon_\eta(y,t) < -\psi(t), \quad \forall t\in\left[0,\min\left(\frac{\pi}{4K\sqrt{\delta}},T\right)\right],$$

i.e.,

$$\sup_{y\in\Omega}\sup_{|e_\beta|=1}\partial_{\beta\beta}(u^\epsilon_\eta)^{\frac{m-1}{2}}(y,t) < \epsilon + e^{-1/\delta+\pi/(2\sqrt{\delta})}, \quad \forall t\in\left[0,\min\left(\frac{\pi}{4K\sqrt{\delta}},T\right)\right].$$

Using the uniform $C^{2,\gamma}$- estimate of g_η^ϵ, we let δ and ϵ go to 0 to conclude that

$$D_x^2 u_\eta^{\frac{m-1}{2}} \leq 0 \qquad \text{in } \ \Omega \times (0, T],$$

which means

$$\frac{1}{2}\left\{u_\eta^{\frac{m-1}{2}}(x,t) + u_\eta^{\frac{m-1}{2}}(y,t)\right\} - u_\eta^{\frac{m-1}{2}}\left(\frac{x+y}{2}, t\right) \leq 0 \quad \forall x, y \in \Omega, \ \ \forall t \in [0, T].$$

This finishes the proof. □

Corollary 3.2.17 (Square-root Concavity). *Let F satisfy (F1), (F2) and (F3). If Ω is strictly convex, then $\phi^{\frac{1-p}{2}}$ is concave, where ϕ is the positive eigenfunction Theorem 3.2.2 and $p := \frac{1}{m}$.*

Proof. We choose the initial data g_0 as in Lemma 3.2.15. Let u be the solution of (3.2.2) with initial data $u_0 := g_0^{\frac{1}{m}}$. Lemma 3.2.16 implies

$$D_x^2 u^{\frac{m-1}{2}} \leq 0 \quad \text{in } \Omega \times (0, \infty).$$

The uniform convergence in Corollary 3.2.5, namely,

$$t^{\frac{m}{m-1}} u^m(x,t) \to \phi(x) \quad \text{uniformly for } \ x \in \Omega \quad \text{as } t \to +\infty,$$

preserves the concavity. Therefore, it follows that

$$\frac{1}{2}\left\{\phi^{\frac{m-1}{2m}}(x) + \phi^{\frac{m-1}{2m}}(y)\right\} - \phi^{\frac{m-1}{2m}}\left(\frac{x+y}{2}\right) \leq 0 \quad \text{for } \ x, y \in \Omega.$$

□

Corollary 3.2.18. *Let F satisfy (F1), (F2) and (F3). If Ω is convex, then $\phi^{\frac{1-p}{2}}$ is concave, where ϕ is the positive eigenfunction Theorem 3.2.2 and $p := \frac{1}{m}$.*

Chapter 4

Parabolic Harnack inequality on Riemannian manifolds

In this chapter, we establish Harnack inequality for uniformly parabolic operators in nondivergence form on a smooth, complete Riemannian manifold (M, g) of dimension n. We first prove Harnack inequality of smooth solutions to uniformly parabolic equations by using Aleksandrov-Bakelman-Pucci-Krylov-Tso type estimate (Lemmas 4.1.3, 4.2.3) and Christ's Theorem 4.1.10 (see also Lemma 4.1.11) on M. By applying a priori Harnack estimates in Section 4.1 and Section 4.2 to the sup- and inf-convolutions u_ε of the viscosity solution u, we shall show Harnack inequality for viscosity solutions.

4.1 Harnack inequality for linear parabolic operators

In this section, we consider linear uniformly parabolic equation

$$\mathscr{L}u := \text{trace}\,(A_{x,t} \circ D^2u) - \partial_t u = f \quad \text{in } M \times (0, +\infty), \tag{4.1.1}$$

where $A_{x,t}$ is a positive definite symmetric endomorphism of T_xM for any $x \in M$ with the assumption that

$$\lambda|X|^2 \leq \langle A_{x,t}X, X\rangle \leq \Lambda|X|^2 \quad \forall x \in M, \ \ \forall X \in T_xM.$$

Assuming in this section that there exists $a_{\mathscr{L}} > 0$ such that for all $p \in M$,

$$\triangle d_p(x) \le \frac{n-1}{d_p(x)} \quad \text{for} \quad x \notin \mathrm{Cut}(p) \cup \{p\}, \tag{4.1.2}$$

$$\mathscr{L} d_p(x) \le \frac{a_{\mathscr{L}}}{d_p(x)} \quad \text{for} \quad x \notin \mathrm{Cut}(p) \cup \{p\}, \quad t \in \mathbb{R}, \tag{4.1.3}$$

instead of the curvature condition on M, we prove global Harnack inequality of smooth solutions to (4.1.1). Here, (4.1.2) and (4.1.3) are essentially the same condition introduced by Kim [35] in the elliptic case.

4.1.1 ABP-Krylov-Tso type estimate

In this section, we obtain Aleksandrov-Bakelman-Pucci-Krylov-Tso type estimate (Lemma 4.1.3) which is a crucial ingredient in proving Krylov-Safonov Harnack inequality. In the simplified proof of classical ABP-Krylov-Tso estimate [57], the normal map

$$(x,t) \mapsto (\nabla u(x,t), u(x,t) - \nabla u(x,t) \cdot x)$$

plays a role to bound the maximum of u by estimating the measure of the image of the normal map, where the second term is considered (up to a sign) as the Legendre transform of u. As Cabré used paraboloids instead in [11], we introduce an intrinsically geometric version of Krylov-Tso normal map, namely,

$$\Phi(x,t) := \left(\exp_x \nabla_x u(x,t), -\frac{1}{2} d\left(x, \exp_x \nabla u(x,t)\right)^2 - u(x,t)\right).$$

which is called the parabolic normal map related to $u(x,t)$.

First, we quote the following lemma from Lemma 3.2 in [11], in which the Jacobian of the map $x \mapsto \exp_x(\nabla v(x))$ is computed explicitly.

Lemma 4.1.1 (Cabré)**.** *Let v be a smooth function in an open set Ω of M. Define the map $\phi : \Omega \to M$ by*

$$\phi(x) := \exp_x \nabla v(x).$$

Let $x \in \Omega$ and suppose that $\nabla v(x) \in E_x$. Set $y := \phi(x)$. Then we have

$$\mathrm{Jac}\, \phi(x) = \mathrm{Jac} \exp_x(\nabla v(x)) \cdot \left| \det D^2 \left(v + d_y^2/2\right)(x) \right|,$$

where $\mathrm{Jac} \exp_x(\nabla v(x))$ *denotes the Jacobian of* $\exp_x$*, a map from T_xM to M, at the point $\nabla v(x) \in T_xM$.*

As a parabolic analogue of Lemma 4.1.1, we have direct computation of the Jacobian of the parabolic normal map Φ below.

Lemma 4.1.2. *Let v be a smooth function in an open set K of $M \times \mathbb{R}$. Define the map $\phi : K \to M$ by*

$$\phi(x,t) := \exp_x \nabla_x v(x,t)$$

and the map $\Phi : K \to M \times \mathbb{R}$ by

$$\Phi(x,t) := \left(\phi(x,t), -\frac{1}{2} d\left(x, \phi(x,t)\right)^2 - v(x,t)\right).$$

Let $(x,t) \in K$ and assume that $\nabla_x v(x,t) \in E_x$. Set $y := \phi(x,t)$. Then

$$\operatorname{Jac}\Phi(x,t) = \operatorname{Jac}\exp_x(\nabla_x v(x,t)) \cdot \left|(-v_t) \det\left(D_x^2\left(v + d_y^2/2\right)\right)\right|,$$

where $\operatorname{Jac}\exp_x(\nabla_x v(x,t))$ *denotes the Jacobian of* $\exp_x$ *at the point* $\nabla_x v(x,t) \in T_xM$.

Proof. We may assume that $\nabla_x v(x,t) \neq 0$, which is equivalent to $x \neq y$. Let $(\xi,\sigma) \in T_xM \times \mathbb{R}\backslash\{(0,0)\}$ and let $\gamma = (\gamma_1,\gamma_2)$ be the geodesic with $\gamma(0) = (x,t)$ and $\gamma'(0) = (\xi,\sigma)$. We note that $\gamma_1(\tau) = \exp_x \tau\xi$ and $\gamma_2(\tau) = t + \sigma\tau$. Set

$$Y(s,\tau) := \exp_{\gamma_1(\tau)}\left[s\nabla_x v(\gamma(\tau))\right].$$

Consider the family of geodesics (in the parameter s)

$$\Pi(s,\tau) := \left(Y(s,\tau), \gamma_2(\tau) - s\left\{\frac{1}{2} d\left(\gamma_1(\tau), \phi(\gamma(\tau))\right)^2 + v(\gamma(\tau)) + \gamma_2(\tau)\right\}\right)$$

that joins $\Pi(0,\tau) = \gamma(\tau)$ to $\Pi(1,\tau) = \Phi(\gamma(\tau))$. Then we define

$$J(s) := \frac{\partial}{\partial\tau}\Big|_{\tau=0} \Pi(s,\tau),$$

which is a Jacobi field along

$$X(s) := \left(\exp_x(s\nabla_x v(x,t)), t - s\left\{\frac{1}{2} d\left(x, \phi(x,t)\right)^2 + v(x,t) + t\right\}\right).$$

Simple computation says that

$$J(0) = (\xi,\sigma) \quad \text{and} \quad J(1) = \frac{\partial}{\partial\tau}\Big|_{\tau=0} \Phi(\gamma(\tau)) = d\Phi(x,t)\cdot(\xi,\sigma).$$

We also have

$$D_sJ(0) = \Big(D_x^2 v(x,t)\,\xi + \sigma\nabla_x v_t(x,t)\,, \quad -\sigma v_t(x,t) - \sigma \\ - \Big\langle \nabla_x \left(d_x^2/2\right)(y),\ d\exp_x(\nabla_x v(x,t))\cdot \left(D_x^2\left(v + d_y^2/2\right)(x,t)\cdot\xi + \sigma\nabla_x v_t(x,t)\right)\Big\rangle\Big).$$

In fact, we have

$$D_sJ(0) = D_s\frac{\partial\Pi}{\partial\tau}\Big|_{s=0,\tau=0} = D_\tau\frac{\partial\Pi}{\partial s}\Big|_{s=0,\tau=0}$$
$$= D_\tau\Big|_{\tau=0}\left(\nabla_x v(\gamma(\tau)), -\frac{1}{2}d(\gamma_1(\tau),\phi(\gamma(\tau)))^2 - v(\gamma(\tau)) - \gamma_2(\tau)\right)$$
$$= \Big(D_x^2 v(x,t)\cdot\xi + \sigma\nabla_x v_t(x,t),$$
$$-\left\langle\nabla_x(d_y^2/2)(x),\ \xi\right\rangle - \left\langle\nabla_x(d_x^2/2)(\phi(x,t)),\ \frac{\partial}{\partial\tau}\phi(\gamma(\tau))\Big|_{\tau=0}\right\rangle - \langle\nabla_x v(x,t),\ \xi\rangle - \sigma v_t - \sigma\Big)$$
$$= \left(D_x^2 v\cdot\xi + \sigma\nabla_x v_t, -\left\langle\nabla_x(d_x^2/2)(\phi(x,t)),\ \frac{\partial}{\partial\tau}\phi(\gamma(\tau))\Big|_{\tau=0}\right\rangle - \sigma v_t - \sigma\right),$$

since $\nabla_x(d_y^2/2)(x) = -\exp_x^{-1}(y) = -\nabla_x v(x,t)$. Then we use Lemma 4.1.1 to obtain

$$\frac{\partial}{\partial\tau}\phi(\gamma(\tau))\Big|_{\tau=0} = d\exp_x(\nabla_x v(x,t))\cdot\left(D_x^2\left(v + d_y^2/2\right)(x,t)\cdot\xi + \sigma\nabla_x v_t(x,t)\right).$$

On the other hand, consider the Jacobi field $J_{\xi,\sigma}$ along $X(s)$ satisfying

$$J_{\xi,\sigma}(0) = (\xi,\sigma) \quad \text{and} \quad J_{\xi,\sigma}(1) = (0,0).$$

Then we can check that

$$J_{\xi,\sigma}(s) = \frac{\partial}{\partial\tau}\Psi\Big|_{\tau=0} \quad \text{and} \quad D_sJ_{\xi,\sigma}(0) = \left(-D_x^2\left(d_y^2/2\right)(x)\cdot\xi, -\sigma\right),$$

where

$$\Psi(s,\tau) := \left(\exp_{\gamma_1(\tau)} s\exp^{-1}_{\gamma_1(\tau)}\phi(x,t), \gamma_2(\tau) - s\left\{\frac{1}{2}d(x,\phi(x,t))^2 + v(x,t) + \gamma_2(\tau)\right\}\right).$$

(We refer [11, Lemma 3.2] for the proof.)

Define $\tilde{J}_{\xi,\sigma} := J - J_{\xi,\sigma}$. The Jacobi field $\tilde{J}_{\xi,\sigma}$ along $X(s)$ satisfying

$$\tilde{J}_{\xi,\sigma}(0) = (0,0) \quad \text{and} \quad D_s\tilde{J}_{\xi,\sigma}(0) = D_sJ(0) - D_sJ_{\xi,\sigma}(0)$$

is written by

$$d\exp_{(x,t)}(sX'(0)) \cdot \left(sD_s\tilde{J}_{\xi,\sigma}(0)\right).$$

Therefore, we have

$$J(1) = \tilde{J}_{\xi,\sigma}(1) = d\exp_{(x,t)}\left(\nabla_x v(x,t), -\frac{1}{2}d(x,y)^2 - v(x,t) - t\right) \cdot \left(D_s J(0) - D_s J_{\xi,\sigma}(0)\right),$$

which means

$$\begin{aligned}
&d\Phi(x,t)\cdot(\xi,\sigma) = d\exp_{(x,t)}\left(\nabla_x v(x,t), -\frac{1}{2}d(x,y)^2 - v(x,t) - t\right)\cdot \\
&\left(D_x^2\left(v + d_y^2/2\right)(x,t)\cdot\xi + \sigma\nabla_x v_t(x,t)\,,\right. \\
&\left.-\sigma v_t - \left\langle \nabla_x\left(d_x^2/2\right)(y),\ d\exp_x(\nabla_x v(x,t))\cdot\left(D_x^2\left(v + d_y^2/2\right)(x,t)\cdot\xi + \sigma\nabla_x v_t(x,t)\right)\right\rangle\right) \\
&= \left(d\exp_x\left(\nabla_x v(x,t)\right)\cdot\left(D_x^2\left(v + d_y^2/2\right)(x,t)\cdot\xi + \sigma\nabla_x v_t(x,t)\right)\,,\right. \\
&\left.-\sigma v_t - \left\langle \nabla_x\left(d_x^2/2\right)(y),\ d\exp_x(\nabla_x v(x,t))\cdot\left(D_x^2\left(v + d_y^2/2\right)(x,t)\cdot\xi + \sigma\nabla_x v_t(x,t)\right)\right\rangle\right).
\end{aligned}$$

To calculate the Jacobian of Φ, we introduce an orthonormal basis $\{e_1, \cdots, e_n\}$ of T_xM and an orthonormal basis $\{\overline{e}_1, \cdots, \overline{e}_n\}$ of $T_yM = T_{\exp_x \nabla v(x,t)}M$. By setting for $i, j = 1, \cdots, n$,

$$\begin{aligned}
A_{ij} &:= \left\langle \overline{e}_i,\ d\exp_x\left(\nabla_x v(x,t)\right)\cdot\left(D_x^2\left(v + d_y^2/2\right)(x,t)e_j\right)\right\rangle, \\
b_i &:= \left\langle \overline{e}_i,\ d\exp_x\left(\nabla_x v(x,t)\right)\cdot\nabla v_t(x,t)\right\rangle, \\
c_i &:= \left\langle \overline{e}_i,\ \nabla_x\left(d_x^2/2\right)(y)\right\rangle,
\end{aligned}$$

the Jacobian matrix of Φ at (x,t) is

$$\begin{pmatrix} A_{ij} & b_i \\ -c_k A_{kj} & -v_t - b_k c_k \end{pmatrix}.$$

Lastly, we use the row operations to deduce that

$$\mathrm{Jac}\,\Phi(x,t) = \left|\det\begin{pmatrix} A_{ij} & b_i \\ 0 & -v_t \end{pmatrix}\right| = \left|(-v_t)\det(A_{ij})\right|.$$

This completes the proof. □

The following lemma will play a key role to estimate sublevel sets of u in Lemma 4.1.6 and then to prove a decay estimate of the distribution function of u in Lemma 4.1.14. This ABP-type lemma corresponds to [11, Lemma 4.1].

Lemma 4.1.3. *Suppose that M satisfies the condition* (4.1.3). *Let $z_o \in M$, $R > 0$, and $0 < \eta < 1$. Let u be a smooth function in $K_{\alpha_1 R, \alpha_2 R^2}(z_o, 0) \subset M \times \mathbb{R}$ satisfying*

$$u \geq 0 \quad in \quad K_{\alpha_1 R, \alpha_2 R^2}(z_o, 0) \backslash K_{\beta_1 R, \beta_2 R^2}(z_o, 0) \quad and \quad \inf_{K_{2R}(z_o,0)} u \leq 1, \tag{4.1.4}$$

where $\alpha_1 := \frac{11}{\eta}$, $\alpha_2 := 4 + \eta^2 + \frac{\eta^4}{4}$, $\beta_1 := \frac{9}{\eta}$, and $\beta_2 := 4 + \eta^2$. Then we have

$$|B_R(z_o)| \cdot R^2 \leq C(\eta, n, \lambda) \int_{\{u \leq M_\eta\} \cap K_{\beta_1 R, \beta_2 R^2}(z_o,0)} \left\{ \left(R^2 \mathscr{L} u + a_{\mathscr{L}} + \Lambda + 1 \right)^+ \right\}^{n+1} \tag{4.1.5}$$

where the constant $M_\eta > 0$ depends only on $\eta > 0$ and $C(\eta, n, \lambda) > 0$ depends only on η, n and λ.

Proof. For any $\bar{y} \in B_R(z_o)$, we define

$$w_{\bar{y}}(x,t) := \frac{1}{2} R^2 u(x,t) + \frac{1}{2} d_{\bar{y}}^2(x) - C_\eta t, \quad C_\eta := \frac{6}{\eta^2}.$$

From the assumption (4.1.4), it is easy to check that

$$\inf_{K_{2R}(z_o,0)} w_{\bar{y}} \leq \left(5 + \frac{24}{\eta^2} \right) R^2 =: A_\eta R^2,$$

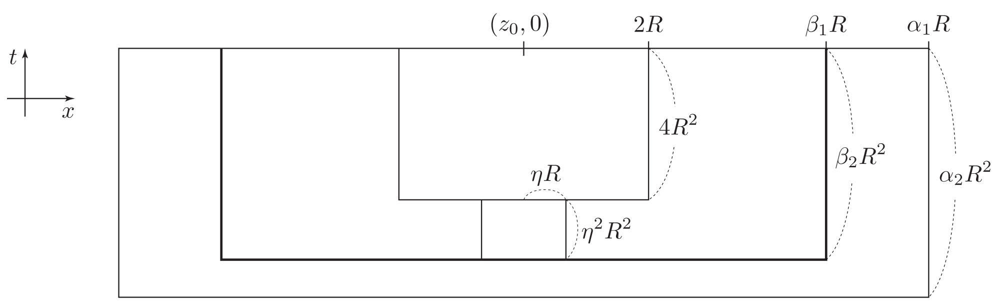

Figure 4.1: $\alpha_1 := \frac{11}{\eta}, \alpha_2 := 4 + \eta^2 + \frac{\eta^4}{4}, \beta_1 := \frac{9}{\eta}, \beta_2 := 4 + \eta^2$.

and

$$w_{\bar{y}} \geq \left(6 + \frac{24}{\eta^2}\right) R^2 = (A_\eta + 1) R^2 \quad \text{on} \ \ K_{\alpha_1 R, \alpha_2 R^2}(z_o, 0) \backslash K_{\beta_1 R, \beta_2 R^2}(z_o, 0).$$

From the above observation, for any $(\bar{y}, \overline{h}) \in B_R(z_o) \times \left(A_\eta R^2, (A_\eta + 1) R^2\right)$, we can find a time $\bar{t} \in \left(-\beta_2 R^2, 0\right)$ such that

$$\overline{h} = \inf_{B_{\beta_1 R}(z_o) \times (-\beta_2 R^2, \bar{t}]} w_{\bar{y}}(z, \tau) = w_{\bar{y}}\left(\overline{x}, \bar{t}\right),$$

where the infimum is achieved at an interior point $\overline{x}$ of $B_{\beta_1 R}(z_o)$. By the same argument as in [11, pp. 637-638], we have the following relation:

$$\overline{y} = \exp_x \nabla_x \left(\frac{1}{2} R^2 u\right) \left(\overline{x}, \bar{t}\right).$$

Now, we consider the map $\Phi : K_{\alpha_1 R, \alpha_2 R^2}(z_o, 0) \to M \times \mathbb{R}$ (with $v(x,t) = \frac{1}{2} R^2 u(x,t) - C_\eta t$ in Lemma 4.1.2) defined as

$$\Phi(x,t) := \left(\exp_x \nabla_x \left(\frac{1}{2} R^2 u\right)(x,t), -\frac{1}{2} d\left(x, \exp_x \nabla_x \left(\frac{1}{2} R^2 u\right)(x,t)\right)^2 - \frac{1}{2} R^2 u(x,t) + C_\eta t\right).$$

Define a set

$$E := \left\{(x,t) \in K_{\beta_1 R, \beta_2 R^2}(z_o, 0) \ : \ \exists y \in B_R(z_o) \quad \text{s.t.} \quad w_y(x,t) = \inf_{B_{\beta_1 R}(z_o) \times (-\beta_2 R^2, t]} w_y \leq (A_\eta + 1) R^2 \right\}.$$

The set E is a subset of the contact set in $K_{\beta_1 R, \beta_2 R^2}(z_o, 0)$ that contains a point (x,t), where a concave paraboloid $-\frac{1}{2} d_y^2(x) + C_\eta t + C$ (for some C) touches $\frac{1}{2} R^2 u$ from below. Thus we have proved that for any $(y,s) \in B_R(z_o) \times (-(A_\eta + 1) R^2, -A_\eta R^2)$, there is at least one $(x,t) \in E$ such that $(y,s) = \Phi(x,t)$, namely,

$$B_R(z_o) \times \left(-(A_\eta + 1) R^2, -A_\eta R^2\right) \subset \Phi(E).$$

So Area formula gives

$$|B_R(z_o)| \cdot R^2 \leq \int_{M \times \mathbb{R}} \mathcal{H}^0 \left[E \cap \Phi^{-1}(y,s)\right] dV(y,s) = \int_E \text{Jac}\, \Phi(x,t) dV(x,t). \quad (4.1.6)$$

We notice that for $(x,t) \in E$ and $y \in B_R(z_o)$, $w_y(x,t) = \frac{1}{2}R^2u(x,t) + \frac{1}{2}d_y^2(x) - C_\eta t \leq (A_\eta + 1)R^2$ and hence $u(x,t) \leq 2(A_\eta + 1) =: M_\eta$ for $(x,t) \in E$.

Lastly, we claim that for $(x,t) \in E$,

$$\operatorname{Jac}\Phi(x,t) \leq \frac{1}{(n+1)^{n+1}\lambda^n}\left\{\left(\frac{1}{2}R^2\mathscr{L}u(x,t) + a_{\mathscr{L}} + \Lambda + C_\eta\right)^+\right\}^{n+1}. \tag{4.1.7}$$

Fix $(x,t) \in E$ and $y \in B_R(z_o)$ to satisfy

$$w_y(x,t) = \inf_{B_{\beta_1 R}(z_o)\times(-\beta_2 R^2, t]} w_y.$$

We recall that $y = \exp_x \nabla_x\left(\frac{1}{2}R^2u\right)(x,t)$ (see [11, pp. 637-638]).

If x is not a cut point of y, then Lemma 4.1.2 (with $v(x,t) = \frac{1}{2}R^2u(x,t) - C_\eta t$) and Lemma 2.2.3 (i) imply that

$$\operatorname{Jac}\Phi(x,t) \leq \left|\left(-\frac{1}{2}R^2u_t + C_\eta\right)\det\left(D_x^2\left(\frac{1}{2}R^2u + \frac{1}{2}d_y^2\right)\right)(x,t)\right|.$$

Since the minimum of w_y in $B_{\beta_1 R}(z_o) \times (-\beta_2 R^2, t]$ is achieved at (x,t), we have

$$0 \leq D_x^2 w_y(x,t) = D_x^2\left(\frac{1}{2}R^2u + \frac{1}{2}d_y^2\right) \quad\text{and}\quad 0 \geq \partial_t w_y(x,t) = \frac{1}{2}R^2u_t - C_\eta,$$

where $D_x^2 w_y(x,t) \geq 0$ means that the Hessian of w_y at (x,t) is positive semidefinite. Therefore, by using the geometric and arithmetic means inequality, we get

$$\begin{aligned}
\operatorname{Jac}\Phi(x,t) &\leq \left(-\frac{1}{2}R^2u_t + C_\eta\right)\det\left(D_x^2\left(\frac{1}{2}R^2u + \frac{1}{2}d_y^2\right)\right)(x,t)\\
&\leq \frac{1}{\lambda^n}\left(-\frac{1}{2}R^2u_t + C_\eta\right)\det A_{x,t}\det\left(D_x^2\left(\frac{1}{2}R^2u + \frac{1}{2}d_y^2\right)\right)\\
&\leq \frac{1}{(n+1)^{n+1}\lambda^n}\left\{\operatorname{tr}\left(A_{x,t}\circ D_x^2\left(\frac{1}{2}R^2u + \frac{1}{2}d_y^2\right)\right) - \frac{1}{2}R^2u_t + C_\eta\right\}^{n+1}\\
&= \frac{1}{(n+1)^{n+1}\lambda^n}\left\{\frac{1}{2}R^2\mathscr{L}u(x,t) + \mathscr{L}\left[\frac{1}{2}d_y^2\right] + C_\eta\right\}^{n+1}\\
&\leq \frac{1}{(n+1)^{n+1}\lambda^n}\left\{\frac{1}{2}R^2\mathscr{L}u + a_{\mathscr{L}} + \Lambda + C_\eta\right\}^{n+1}\\
&= \frac{1}{(n+1)^{n+1}\lambda^n}\left\{\left(\frac{1}{2}R^2\mathscr{L}u + a_{\mathscr{L}} + \Lambda + C_\eta\right)^+\right\}^{n+1},
\end{aligned}$$

where we used

$$\mathscr{L}\left[d_y^2/2\right] = d_y\mathscr{L}d_y + \left\langle A_{x,t}\nabla d_y, \nabla d_y\right\rangle \le a_{\mathscr{L}} + \Lambda|\nabla d_y|^2.$$

When x is a cut point of y, we make use of upper barrier technique due to Calabi [15]. Since $y = \exp_x \nabla_x\left(\frac{1}{2}R^2u\right)(x,t)$, x is not a cut point of $y_\sigma := \phi_\sigma(x,t) := \exp_x \nabla_x\left(\frac{\sigma}{2}R^2u\right)(x,t)$ for $0 \le \sigma < 1$. Now we consider

$$\Phi_\sigma(z,\tau) := \left(\phi_\sigma(z,\tau), -\frac{\sigma}{2}R^2u(z,\tau) - \frac{1}{2}d\left(z,\phi_\sigma(z,\tau)\right)^2 + C_\eta\tau\right)$$

instead of Φ since $\mathrm{Jac}\,\Phi(x,t) = \lim_{\sigma\uparrow 1}\mathrm{Jac}\,\Phi_\sigma(x,t)$. As before, we have

$$\mathrm{Jac}\,\Phi_\sigma(x,t) \le \left|\left(-\frac{\sigma}{2}R^2u_t + C_\eta\right)\det\left(D_x^2\left(\frac{\sigma}{2}R^2u + \frac{1}{2}d_{y_\sigma}^2\right)\right)(x,t)\right|.$$

We note that

$$\begin{aligned}&\liminf_{\sigma\uparrow 1}\left|\left(-\frac{\sigma}{2}R^2u_t + C_\eta\right)\det\left(D_x^2\left(\frac{\sigma}{2}R^2u + \frac{1}{2}d_{y_\sigma}^2\right)\right)(x,t)\right|\\ &= \liminf_{\sigma\uparrow 1}\left|\left(-\partial_t w_{y_\sigma}(x,t)\right)\det\left(D_x^2 w_{y_\sigma}\right)(x,t)\right|\end{aligned}$$

for $w_{y_\sigma}(z,\tau) := \frac{1}{2}R^2u(z,\tau) + \frac{1}{2}d_{y_\sigma}^2(z) - C_\eta\tau$. According to the triangle inequality, we have

$$\begin{aligned}w_y(z,\tau) &\le \frac{1}{2}R^2u(z,\tau) + \frac{1}{2}\left(d_{y_\sigma}(z) + d(y_\sigma,y)\right)^2 - C_\eta\tau\\ &= w_{y_\sigma}(z,\tau) + d(y_\sigma,y)d_{y_\sigma}(z) + \frac{1}{2}d(y_\sigma,y)^2,\end{aligned}$$

where the equality holds at $(z,\tau) = (x,t)$. Since w_y has the minimum at (x,t) in $B_{\beta_1 R}(z_o) \times (-\beta_2R^2, t]$, the minimum of $w_{y_\sigma}(z,\tau) + d(y_\sigma,y)d_{y_\sigma}(z)$ (in $B_{\beta_1 R}(z_o) \times (-\beta_2R^2, t]$) is also achieved at (x,t), that implies that

$$D_x^2\left(w_{y_\sigma} + d(y_\sigma,y)d_{y_\sigma}\right)(x,t) \ge 0, \quad \partial_t w_{y_\sigma}(x,t) \le 0.$$

To bound $D^2y_\sigma(x)$ uniformly in $\sigma \in [1/2, 1)$, we recall the Hessian comparison theorem (see [54],[55]): Let $-k^2$ ($k > 0$) be a lower bound of sectional curvature along the minimal geodesic joining x and y. Then for $0 < \sigma < 1$,

$$D^2 d_{y_\sigma}(x) \le k\coth(k d_{y_\sigma}(x))\mathrm{Id}$$

and hence we find a constant $N > 0$ independent of σ such that

$$D^2 d_{y_\sigma}(x) \le N\mathrm{Id} \quad \text{for } \frac{1}{2} \le \sigma < 1.$$

Following the above argument, for $\frac{1}{2} \le \sigma < 1$, we obtain

$$\begin{aligned}
0 &\le \liminf_{\sigma\uparrow 1} \left(-\partial_t w_{y_\sigma}(x,t)\right) \det\left(D_x^2 w_{y_\sigma} + d(y_\sigma, y) D^2 d_{y_\sigma}\right)(x,t) \\
&\le \liminf_{\sigma\uparrow 1} \left(-\partial_t w_{y_\sigma}(x,t)\right) \det\left(D_x^2 w_{y_\sigma} + d(y_\sigma, y) N\mathrm{Id}\right)(x,t) \\
&\le \liminf_{\sigma\uparrow 1} \frac{1}{(n+1)^{n+1}\lambda^n} \left\{\frac{1}{2} R^2 \mathscr{L} u + a_{\mathscr{L}} + \Lambda + C_\eta + d(y_\sigma, y) n\Lambda N\right\}^{n+1} \\
&\le \frac{1}{(n+1)^{n+1}\lambda^n} \left\{\left(\frac{1}{2} R^2 \mathscr{L} u + a_{\mathscr{L}} + \Lambda + C_\eta\right)^+\right\}^{n+1}.
\end{aligned}$$

Then we deduce that

$$\mathrm{Jac}\,\Phi(x,t) \le \frac{1}{(n+1)^{n+1}\lambda^n} \left\{\left(\frac{1}{2} R^2 \mathscr{L} u(x,t) + a_{\mathscr{L}} + \Lambda + C_\eta\right)^+\right\}^{n+1}$$

since

$$\begin{aligned}
\liminf_{\sigma\uparrow 1} \left|\det\left(D_x^2 w_{y_\sigma}\right)(x,t)\right| &= \liminf_{\sigma\uparrow 1} \left|\det\left(D_x^2 w_{y_\sigma} + d(y_\sigma, y) N\mathrm{Id}\right)(x,t)\right| \\
&= \liminf_{\sigma\uparrow 1} \det\left(D_x^2 w_{y_\sigma} + d(y_\sigma, y) N\mathrm{Id}\right)(x,t).
\end{aligned}$$

We conclude that (4.1.7) is true for $(x,t) \in E$. Therefore the estimate (4.1.5) follows from (4.1.6) since $E \subset \left\{u \le M_\eta\right\} \cap K_{\beta_1 R, \beta_2 R^2}(z_o, 0)$. □

4.1.2 Barrier functions

We modify the barrier function of [60] to construct a barrier function in the Riemannian case. First, we fix some constants that will be used frequently (see Figure 4.1); for a given $0 < \eta < 1$,

$$\alpha_1 := \frac{11}{\eta},\ \alpha_2 := 4 + \eta^2 + \frac{\eta^4}{4},\ \beta_1 := \frac{9}{\eta} \ \text{ and } \ \beta_2 := 4 + \eta^2.$$

Lemma 4.1.4. *Suppose that M satisfies the condition* (4.1.3). *Let $z_o \in M$, $R > 0$ and $0 < \eta < 1$. There exists a continuous function $v_\eta(x,t)$ in $K_{\alpha_1 R, \alpha_2 R^2}(z_o, \beta_2 R^2)$, which is smooth in $(M\setminus \mathrm{Cut}(z_o)) \cap K_{\alpha_1 R, \alpha_2 R^2}(z_o, \beta_2 R^2)$ such that*

(i) $v_\eta(x,t) \geq 0$ *in* $K_{\alpha_1 R, \alpha_2 R^2}(z_o, \beta_2 R^2) \setminus K_{\beta_1 R, \beta_2 R^2}(z_o, \beta_2 R^2)$,

(ii) $v_\eta(x,t) \leq 0$ *in* $K_{2R}(z_o, \beta_2 R^2)$,

(iii) $R^2 \mathscr{L} v_\eta + a_{\mathscr{L}} + \Lambda + 1 \leq 0$ *a.e. in* $K_{\beta_1 R, \beta_2 R^2}(z_o, \beta_2 R^2) \setminus K_{\frac{\eta}{2}R}(z_o, \frac{\eta^2}{4} R^2)$,

(iv) $R^2 \mathscr{L} v_\eta \leq C_\eta$ *a.e. in* $K_{\beta_1 R, \beta_2 R^2}(z_o, \beta_2 R^2)$,

(v) $v_\eta(x,t) \geq -C_\eta$ *in* $K_{\alpha_1 R, \alpha_2 R^2}(z_o, \beta_2 R^2)$.

Here, the constant $C_\eta > 0$ depends only on $\eta, n, \lambda, \Lambda, a_{\mathscr{L}}$ (independent of R and z_o).

Proof. Fix $0 < \eta < 1$. Consider

$$h(s,t) := -Ae^{-mt}\left(1 - \frac{s}{\beta_1^2}\right)^l \frac{1}{(4\pi t)^{n/2}} \exp\left(-\alpha \frac{s}{t}\right) \quad \text{for } t > 0,$$

as in Lemma 3.22 of [60] and define

$$\psi(s,t) := h(s,t) + (a_{\mathscr{L}} + \Lambda + 1)t \quad \text{in } [0, \beta_1^2] \times [0, \beta_2] \setminus [0, \tfrac{\eta^2}{4}] \times [0, \tfrac{\eta^2}{4}],$$

where the positive constants A, m, l, α (depending only on $\eta, n, \lambda, \Lambda, a_{\mathscr{L}}$) will be chosen later. In particular, l will be an odd number in $\mathbb{N}$. We extend ψ smoothly in $[0, \alpha_1^2] \times [-\frac{\eta^4}{4}, \beta_2]$ to satisfy

$$\begin{aligned} \psi &\geq 0 \quad \text{on } [0, \alpha_1^2] \times [-\tfrac{\eta^4}{4}, \beta_2] \setminus [0, \beta_1^2] \times [0, \beta_2], \\ \psi &\geq -C_\eta \quad \text{on } [0, \alpha_1^2] \times [-\tfrac{\eta^4}{4}, \beta_2], \end{aligned}$$

and

$$\sup_{[0,\beta_1^2]\times[0,\beta_2]} \{2a_{\mathscr{L}} |\partial_s \psi| + \Lambda (2|\partial_s \psi| + 4s|\partial_{ss}\psi|) + |\partial_t \psi|\}(s,t) < C_\eta$$

for some $C_\eta > 0$. We also assume that $\psi(s,t)$ is nondecreasing with respect to s in $[0, \alpha_1^2] \times [-\frac{\eta^4}{4}, \beta_2]$. We define

$$v_\eta(x,t) = v(x,t) := \psi\left(\frac{d_{z_o}^2(x)}{R^2}, \frac{t}{R^2}\right) \quad \text{for} \quad (x,t) \in K_{\alpha_1 R, \alpha_2 R^2}(z_o, \beta_2 R^2),$$

where d_{z_o} is the distance function to z_o. Properties (i) and (v) are trivial.

We denote $d_{z_o}(x)$ and $h\left(\frac{d_{z_o}^2(x)}{R^2}, \frac{t}{R^2}\right)$ by $d(x)$ and $\phi(x,t)$ for simplicity and we notice that for $(x,t) \in K_{\beta_1 R, \beta_2 R^2}(z_o, \beta_2 R^2) \backslash K_{\frac{\eta}{2}R}(z_o, \frac{\eta^2}{4}R^2)$,

$$v(x,t) = h\left(\frac{d^2(x)}{R^2}, \frac{t}{R^2}\right) + (a_{\mathscr{L}} + \Lambda + 1)\frac{t}{R^2} = \phi(x,t) + (a_{\mathscr{L}} + \Lambda + 1)\frac{t}{R^2}$$

and $\phi(x,t)$ is negative in $K_{\beta_1 R, \beta_2 R^2}(z_o, \beta_2 R^2)$.

Now, we claim that

$$\mathscr{L}\phi \leq 0 \quad \text{a.e. in } K_{\beta_1 R, \beta_2 R^2}(z_o, \beta_2 R^2) \backslash K_{\frac{\eta}{2}R}(z_o, \tfrac{\eta^2}{4}R^2). \tag{4.1.8}$$

Once (4.1.8) is proved, then property (iii) follows from the simple calculation that $R^2\mathscr{L}\left[(a_{\mathscr{L}} + \Lambda + 1)\frac{t}{R^2}\right] = -(a_{\mathscr{L}} + \Lambda + 1)$ in $K_{\beta_1 R, \beta_2 R^2}(z_o, \beta_2 R^2)$. Now we use the identity

$$\mathscr{L}[\varphi(u(x),t)] = \partial_u \varphi(u,t)\mathscr{L}u + \partial_{uu}\varphi(u,t)\langle A_{x,t}\nabla u, \nabla u\rangle - \partial_t \varphi(u,t)$$

to obtain

$$\begin{aligned}\mathscr{L}\phi = {} & \frac{2d}{R^2}\partial_s h\left(\frac{d^2}{R^2}, \frac{t}{R^2}\right) Ld \\ & + \left\{\frac{2}{R^2}\partial_s h + \frac{4d^2}{R^4}\partial_{ss} h\right\}\left(\frac{d^2}{R^2}, \frac{t}{R^2}\right)\langle A_{x,t}\nabla d, \nabla d\rangle - \frac{1}{R^2}\partial_t h\left(\frac{d^2}{R^2}, \frac{t}{R^2}\right).\end{aligned}$$

Since $d \cdot \mathscr{L} d \leq a_{\mathscr{L}}$ and $\lambda \leq \langle A_{x,t} \nabla d, \nabla d \rangle \leq \Lambda$ in $M \backslash \operatorname{Cut}(z_o)$, we have that

$$
\begin{aligned}
&\frac{(\beta_1^2 R^2 - d^2)^2}{(-\phi)} \mathscr{L} \phi \\
&= (\beta_1^2 R^2 - d^2) \left\{ 2l + (\beta_1^2 R^2 - d^2) \frac{2\alpha}{t} \right\} d \mathscr{L} d \\
&- \left\{ l(l-1) 4d^2 + 2l(\beta_1^2 R^2 - d^2) \frac{4\alpha d^2}{t} + (\beta_1^2 R^2 - d^2)^2 \frac{4\alpha^2 d^2}{t^2} \right\} \langle A_{x,t} \nabla d, \nabla d \rangle \\
&+ (\beta_1^2 R^2 - d^2) \left\{ 2l + (\beta_1^2 R^2 - d^2) \frac{2\alpha}{t} \right\} \langle A_{x,t} \nabla d, \nabla d \rangle \\
&+ (\beta_1^2 R^2 - d^2)^2 \frac{\alpha d^2}{t^2} - (\beta_1^2 R^2 - d^2)^2 \left(\frac{n}{2t} + \frac{m}{R^2} \right) \\
&\leq 2l(\beta_1^2 R^2 - d^2)(a_{\mathscr{L}} + \Lambda) + (\beta_1^2 R^2 - d^2)^2 \left\{ \frac{2\alpha}{t}(a_{\mathscr{L}} + \Lambda) + \frac{\alpha d^2}{t^2} \right\} \\
&- l(l-1) 4d^2 \lambda - (\beta_1^2 R^2 - d^2)^2 \left(\frac{4\alpha^2 d^2}{t^2} \lambda + \frac{n}{2t} \right) \\
&- 2l(\beta_1^2 R^2 - d^2) \frac{4\alpha d^2}{t} \lambda - \frac{m}{R^2} (\beta_1^2 R^2 - d^2)^2 \quad \text{a.e. in } K_{\beta_1 R, \beta_2 R^2}(z_o, \beta_2 R^2).
\end{aligned}
$$

By choosing

$$
\begin{aligned}
&\alpha := \frac{1}{4\lambda}, \quad \frac{2\beta_1^2}{\eta^2 \lambda}(a_{\mathscr{L}} + \Lambda) + 1 \leq l := 2l' + 1 \ \ (\text{for some } l' \in \mathbb{N}), \\
&m := 2 \cdot \max \left\{ \frac{8\alpha}{\eta^2}(a_{\mathscr{L}} + \Lambda), \ \frac{2l(a_{\mathscr{L}} + \Lambda)}{\beta_1^2 - \frac{\eta^2}{4}} \right\},
\end{aligned}
\tag{4.1.9}
$$

we deduce

$$
\frac{(\beta_1^2 R^2 - d^2)^2}{(-\phi)} \mathscr{L} \phi \leq 0 \quad \text{a.e. in } K_{\beta_1 R, \beta_2 R^2}(z_o, \beta_2 R^2) \backslash K_{\frac{\eta}{2} R}(z_o, \tfrac{\eta^2}{4} R^2).
$$

Indeed, we divide the domain $K_{\beta_1 R, \beta_2 R^2}(z_o, \beta_2 R^2) \backslash K_{\frac{\eta}{2} R}(z_o, \frac{\eta^2}{4} R^2)$ into three regions such that

$$
K_{\beta_1 R, \beta_2 R^2}(z_o, \beta_2 R^2) \backslash K_{\frac{\eta}{2} R}(z_o, \frac{\eta^2}{4} R^2) =: A_1 \cup A_2 \cup A_3,
$$

where $A_1 := \{0 \leq \frac{t}{R^2} \leq \frac{\eta^2}{4}, \frac{\eta}{2} \leq \frac{d}{R} \leq \beta_1\}$, $A_2 := \{\frac{\eta^2}{4} \leq \frac{t}{R^2} \leq \beta_2, \frac{\eta}{2} \leq \frac{d}{R} \leq \beta_1\}$ and

$A_3 := \{\frac{\eta^2}{4} \le \frac{t}{R^2} \le \beta_2, 0 \le \frac{d}{R} \le \frac{\eta}{2}\}$. We can check that

$$\frac{(\beta_1^2 R^2 - d^2)^2}{(-\phi)} \mathscr{L}\phi \le 0 \quad \text{a.e. in } K_{\beta_1 R, \beta_2 R^2}(z_o, \beta_2 R^2) \backslash K_{\frac{\eta}{2}R}(z_o, \tfrac{\eta^2}{4}R^2)$$

by choosing α and l large in A_1, m large in A_2 and A_3 as in (4.1.9). Therefore, we have proved (4.1.8).

From the assumption on ψ, we have that for a.e. $(x,t) \in K_{\beta_1 R, \beta_2 R^2}(z_o, \beta_2 R^2)$,

$$\begin{aligned} R^2 \mathscr{L} v(x,t) &= 2\partial_s \psi \left(\frac{d^2}{R^2}, \frac{t}{R^2} \right) d \cdot \mathscr{L} d + \left\{ 2\partial_s \psi + \frac{4d^2}{R^2} \partial_{ss} \psi \right\} \left(\frac{d^2}{R^2}, \frac{t}{R^2} \right) \langle A_{x,t} \nabla d, \nabla d \rangle \\ &\quad - \partial_t \psi \left(\frac{d^2}{R^2}, \frac{t}{R^2} \right) \\ &\le \sup_{[0,\beta_1^2]\times[0,\beta_2]} \{ 2a_{\mathscr{L}} \partial_s \psi + \Lambda \left(2\partial_s \psi + 4s |\partial_{ss}\psi| \right) + |\partial_t \psi| \} (s,t) < C_\eta. \end{aligned}$$

This proves property (iv).

In order to show (ii), we take $A > 0$ large enough so that for $(x,t) \in K_{2R}(z_o, \beta_2 R^2)$,

$$v(x,t) \le -Ae^{-\beta_2 m} \left(1 - \frac{4}{\beta_1^2} \right)^l \frac{1}{(4\pi\beta_2)^{n/2}} e^{-4\alpha/\eta^2} + (a_{\mathscr{L}} + \Lambda + 1)\beta_2 \le 0.$$

This finishes the proof of the lemma. □

Now we apply Lemma 4.1.3 to $u + v_\eta$ with v_η constructed in Lemma 4.1.4 and translated in time. Since the barrier function $v_\eta(x,t) = \psi_\eta \left(\frac{d_{z_o}^2(x)}{R^2}, \frac{t}{R^2} \right)$ is not smooth on $\mathrm{Cut}(z_o)$, we need to approximate v_η by a sequence of smooth functions as Cabré's approach at [11]. We recall that the cut locus of z_o is closed and has measure zero. It is not hard to verify the following lemma and we just refer to [11] Lemmas 5.3, 5.4.

Lemma 4.1.5. *Let $z_o \in M$, $R > 0$ and let $\psi : \mathbb{R}^+ \times [0,T] \to \mathbb{R}$ be a smooth function such that $\psi(s,t)$ is nondecreasing with respect to s for any $t \in [0,T]$. Let $v(x,t) := \psi\left(d_{z_o}^2(x), t\right)$. Then there exist a smooth function $0 \le \varphi(x) \le 1$ on M satisfying*

$$\varphi \equiv 1 \quad \textit{in } B_{\beta_1 R}(z_o) \textit{ and } \quad \operatorname{supp} \varphi \subset B_{\frac{10}{\eta}R}(z_o)$$

and a sequence $\{w_k\}_{k=1}^\infty$ *of smooth functions in* $M \times [0,T]$ *such that*

$$\begin{cases} w_k \to \varphi v & \text{uniformly in } M \times [0,T], \\ \partial_t w_k \to \varphi \partial_t v & \text{uniformly in } M \times [0,T], \\ D_x^2 w_k \leq C\mathrm{Id} & \text{in } M \times [0,T], \\ D_x^2 w_k \to D_x^2 v & \text{a.e. in } B_{\beta_1 R}(z_o) \times [0,T], \end{cases}$$

where the constant $C > 0$ *is independent of* k.

Lemma 4.1.6. *Suppose that* M *satisfies the conditions* (4.1.2),(4.1.3). *Let* $z_o \in M, R > 0$, *and* $0 < \eta < 1$. *Let* u *be a smooth function such that* $\mathscr{L}u \leq f$ *in* $K_{\alpha_1 R,\, \alpha_2 R^2}(z_o, 4R^2)$ *such that*

$$u \geq 0 \quad in \quad K_{\alpha_1 R,\, \alpha_2 R^2}(z_o, 4R^2) \backslash K_{\beta_1 R, \beta_2 R^2}(z_o, 4R^2)$$

and

$$\inf_{K_{2R}(z_o, 4R^2)} u \leq 1.$$

Then, there exist uniform constants $M_\eta > 1, 0 < \mu_\eta < 1$, *and* $0 < \varepsilon_\eta < 1$ *such that*

$$\frac{\left|\left\{u \leq M_\eta\right\} \cap K_{\eta R}(z_o, 0)\right|}{\left|K_{\alpha_1 R,\, \alpha_2 R^2}(z_o, 4R^2)\right|} \geq \mu_\eta, \tag{4.1.10}$$

provided

$$R^2 \left(\fint_{K_{\alpha_1 R,\, \alpha_2 R^2}(z_o, 4R^2)} |f^+|^{n+1} \right)^{\frac{1}{n+1}} \leq \varepsilon_\eta, \tag{4.1.11}$$

where $M_\eta > 0$, $0 < \varepsilon_\eta$, $\mu_\eta < 1$ *depend only on* $\eta, n, \lambda, \Lambda$ *and* $a_{\mathscr{L}}$.

Proof. Let v_η be the barrier function in Lemma 4.1.4 after translation in time (by $-\eta^2 R^2$) and let $\{w_k\}_{k=1}^\infty$ be a sequence of smooth functions approximating v_η as in Lemma 4.1.5. We notice that $u + v_\eta \geq 0$ in $K_{\alpha_1 R,\, \alpha_2 R^2}(z_o, 4R^2) \backslash K_{\beta_1 R, \beta_2 R^2}(z_o, 4R^2)$ and $\inf_{K_{2R}(z_o, 4R^2)} (u + v_\eta) \leq 1$. Thanks to the uniform convergence of w_k to φv_η, we consider a sequence $\{\varepsilon_k\}_{k=1}^\infty$ converging to 0 such that $\sup_{K_{2R}(z_o, 4R^2)} w_k \leq \varepsilon_k$ and

$$w_k \geq -\varepsilon_k \;\text{ in }\; K_{\alpha_1 R,\, \alpha_2 R^2}(z_o, 4R^2) \backslash K_{\beta_1 R, \beta_2 R^2}(z_o, 4R^2),$$

and define
$$\overline{w}_k := \frac{u + w_k + \varepsilon_k}{1 + 2\varepsilon_k}.$$
Then $\overline{w}_k$ satisfies the hypotheses of Lemma 4.1.3 (after translation in time by $4R^2$). Now we replace u by $\overline{w}_k$ in (4.1.5) and then the uniform convergence implies that for a given $0 < \delta < 1$, we have

$$|B_R(z_o)|R^2 \le C(\eta, n, \lambda) \int_{\{u+v_\eta \le M_\eta + \delta\} \cap K_{\beta_1 R, \beta_2 R^2}(z_o, 4R^2)} \left\{ \left(R^2 \mathscr{L} \overline{w}_k + a_{\mathscr{L}} + \Lambda + 1 \right)^+ \right\}^{n+1}$$

if k is sufficiently large. Since $D_x^2 w_k \le C\mathrm{Id}$ and $|\partial_t w_k| < C$ uniformly in k on $K_{\beta_1 R, \beta_2 R^2}(z_o, 4R^2)$, we use the dominated convergence theorem to let k go to $+\infty$. Letting δ go to 0, we obtain

$$\begin{aligned} |B_R(z_o)| \cdot R^2 &\le C(\eta, n, \lambda) \int_{\{u+v_\eta \le M_\eta\} \cap K_{\beta_1 R, \beta_2 R^2}(z_o, 4R^2)} \left\{ \left(R^2 \mathscr{L}[u + v_\eta] + a_{\mathscr{L}} + \Lambda + 1 \right)^+ \right\}^{n+1} \\ &= C(\eta, n, \lambda) \int_{E_1 \cup E_2} \left\{ \left(R^2 \mathscr{L}[u + v_\eta] + a_{\mathscr{L}} + \Lambda + 1 \right)^+ \right\}^{n+1}, \end{aligned}$$

where $E_1 := \{u + v_\eta \le M_\eta\} \cap \left(K_{\beta_1 R, \beta_2 R^2}(z_o, 4R^2) \setminus K_{\eta R}(z_o, 0) \right)$ and $E_2 := \{u + v_\eta \le M_\eta\} \cap K_{\eta R}(z_o, 0)$. From properties (iii) and (iv) of v_η in Lemma 4.1.4 and Bishop's volume comparison theorem, we deduce that

$$\begin{aligned} |K_{\alpha_1 R, \alpha_2 R^2}(z_o, 4R^2)|^{\frac{1}{n+1}} &\le C_\eta \left\| \left(R^2 \mathscr{L} u \right)^+ \right\|_{L^{n+1}\left(K_{\beta_1 R, \beta_2 R^2}(z_o, 4R^2)\right)} + C_\eta \left\| \chi_{E_2} \right\|_{L^{n+1}\left(K_{\beta_1 R, \beta_2 R^2}(z_o, 4R^2)\right)} \\ &\le C_\eta \| R^2 f^+ \|_{L^{n+1}\left(K_{\alpha_1 R, \alpha_2 R^2}(z_o, 4R^2)\right)} + C_\eta \left| \left\{ u + v_\eta \le M_\eta \right\} \cap K_{\eta R}(z_o, 0) \right|^{\frac{1}{n+1}}, \end{aligned}$$

where $C_\eta > 0$ depends only on n, λ and $\eta > 0$. We note that $\{u \le M_\eta - v_\eta\} \subset \{u \le M_\eta + C_\eta\}$ from (v) in Lemma 4.1.4. Therefore, by taking

$$\varepsilon_\eta = \frac{1}{2C_\eta}, \quad M'_\eta = M_\eta + C_\eta \quad \text{and} \quad \mu_\eta^{\frac{1}{n+1}} = \frac{1}{2C_\eta},$$

we conclude that $\dfrac{\left| \left\{ u \le M'_\eta \right\} \cap K_{\eta R}(z_o, 0) \right|}{\left| K_{\alpha_1 R, \alpha_2 R^2}(z_o, 4R^2) \right|} \ge \mu_\eta > 0.$ □

Using iteration of Lemma 4.1.6, we have the following corollaries.

Corollary 4.1.7. *Suppose that M satisfies the conditions* (4.1.2),(4.1.3). *Let $z_o \in M$ and $0 < \eta < 1$. For $i \in \mathbb{N}$, let $\overline{R}_i := \left(\frac{2}{\eta}\right)^{i-1} R$ and $\bar{t}_i := \sum_{j=1}^{i} 4\overline{R}_j^2$. Let u be a nonnegative smooth function such that $\mathscr{L}u \le f$ in $\bigcup_{i=1}^{k} K_{\alpha_1 \overline{R}_i, \alpha_2 \overline{R}_i^2}(z_o, \bar{t}_i)$ for some $k \in \mathbb{N}$. We assume that for $h > 0$, $\inf_{\bigcup_{i=1}^{k} K_{2\overline{R}_i}(z_o,\bar{t}_i)} u \le h$ and*

$$\overline{R}_i^2 \left(\fint_{K_{\alpha_1 \overline{R}_i, \alpha_2 \overline{R}_i^2}(z_o,\bar{t}_i)} |f^+|^{n+1} \right)^{\frac{1}{n+1}} \le \varepsilon_\eta h M_\eta^{k-i}, \quad \forall 1 \le i \le k.$$

Then we have

$$\frac{\left| \left\{ u \le h M_\eta^k \right\} \cap K_{\eta R}(z_o, 0) \right|}{\left| K_{\alpha_1 R, \alpha_2 R^2}(z_o, 4R^2) \right|} \ge \mu_\eta, \tag{4.1.12}$$

where $M_\eta, \varepsilon_\eta, \mu_\eta$ are the same uniform constants as in Lemma 4.1.6.

Proof. We may assume $h = 1$ since $v := \frac{u}{h}$ satisfies $\mathscr{L}v = \frac{1}{h}\mathscr{L}u \le \frac{f}{h}$. We use the induction on k to show the lemma. When $k = 1$, it is immediate from Lemma 4.2.5.

Now suppose that (4.1.12) is true for $k-1$. By assumption, we find a $j_o \in \mathbb{N}$ such that $1 \le j_o \le k$ and $\inf_{K_{2\overline{R}_{j_o}}(z_o,\bar{t}_{j_o})} u = \inf_{\bigcup_{i=1}^{k} K_{2\overline{R}_i}(z_o,\bar{t}_i)} u \le 1$. Define $v := u/M_\eta^{k-j_o}$. Then v satisfies that $\mathscr{L}v \le f/M_\eta^{k-j_o}$, $\inf_{K_{2\overline{R}_{j_o}}(z_o,\bar{t}_{j_o})} v \le 1$ and

$$\overline{R}_{j_o}^2 \left(\fint_{K_{\alpha_1 \overline{R}_{j_o}, \alpha_2 \overline{R}_{j_o}^2}(z_o,\bar{t}_{j_o})} \left| f^+/M_\eta^{k-j_o} \right|^{n+1} \right)^{\frac{1}{n+1}} \le \varepsilon_\eta.$$

Applying Lemma 4.1.6 to v in $K_{\alpha_1 \overline{R}_{j_o}, \alpha_2 \overline{R}_{j_o}^2}(z_o, \bar{t}_{j_o})$, we deduce

$$\frac{\left| \left\{ v \le M_\eta \right\} \cap K_{\eta \overline{R}_{j_o}}(z_o, \bar{t}_{j_o} - 4R_{j_o}^2) \right|}{\left| K_{\alpha_1 \overline{R}_{j_o}, \alpha_2 \overline{R}_{j_o}^2}(z_o, \bar{t}_{j_o}) \right|} = \frac{\left| \left\{ v \le M_\eta \right\} \cap K_{2\overline{R}_{j_o-1}}(z_o, \bar{t}_{j_o-1}) \right|}{\left| K_{\alpha_1 \overline{R}_{j_o}, \alpha_2 \overline{R}_{j_o}^2}(z_o, \bar{t}_{j_o}) \right|} \ge \mu_\eta > 0$$

which implies that $\inf_{\bigcup_{i=1}^{j_o-1} K_{2\overline{R}_i}(z_o,\bar{t}_i)} u \le \inf_{K_{2\overline{R}_{j_o-1}}(z_o,\bar{t}_{j_o-1})} u \le M_\eta^{k-j_o+1}$. Therefore, we use the induction hypothesis for $j_o - 1 (\le k-1)$ to conclude

$$\frac{\left| \left\{ u/M_\eta^{k-j_o+1} \le M_\eta^{j_o-1} \right\} \cap K_{\eta R}(z_o, 0) \right|}{\left| K_{\alpha_1 R, \alpha_2 R^2}(z_o, 4R^2) \right|} \ge \mu_\eta > 0,$$

which implies (4.1.12). □

We remark that Lemma 4.1.6 and Corollary 4.1.7 hold for any $M'_\eta \geq M_\eta$. The following is a simple technical lemma that will be used in the proof of Proposition 4.1.9.

Lemma 4.1.8. *Let $A, D > 0$ and $\varepsilon > 0$. Let u be a nonnegative smooth function such that $\mathscr{L}u \leq f$ in $\overline{B_R(z_o)} \times (-AR^2, 0]$ with*

$$R^2 \left(\fint_{B_R(z_o)\times(-AR^2,0]} |f^+|^{n+1} \right)^{\frac{1}{n+1}} \leq \varepsilon.$$

Then, there exists a sequence u_k of nonnegative smooth functions in $B_R(z_o) \times (-AR^2, DR^2]$ such that u_k converges to u locally uniformly in $B_R(z_o) \times (-AR^2, 0]$ and $\mathscr{L}u_k \leq g_k$ in $B_R(z_o) \times (-AR^2, DR^2]$ with

$$R^2 \left(\fint_{B_R(z_o)\times(-AR^2,DR^2]} |g_k^+|^{n+1} \right)^{\frac{1}{n+1}} \leq \varepsilon.$$

Proof. First, we define for $(x, t) \in B_R(z_o) \times (-\infty, DR^2]$,

$$\overline{u}(x,t) := \begin{cases} 0 & \text{for } \ t \in (-\infty, -AR^2], \\ u(x,t) & \text{for } \ t \in (-AR^2, 0], \\ u(x,0) + St & \text{for } \ t \in (0, DR^2], \end{cases}$$

where $S := \sup_{B_R(z_o)} \{(\mathscr{L}u)^+(x,0) + |u_t(x,0)|\}$. Then $\overline{u}$ is Lipschitz continuous with respect to time in $B_R(z_o) \times (-AR^2, DR^2]$ and satisfies

$$\mathscr{L}\overline{u}(x,t) \leq \overline{f}(x,t) := \begin{cases} 0 & \text{for } t \in (-\infty, -AR^2), \\ f(x,t) & \text{for } t \in (-AR^2, 0), \\ \mathscr{L}u(x,0) + u_t(x,0) - S \ \leq 0 & \text{for } t \in (0, DR^2]. \end{cases}$$

Let $\varepsilon_k > 0$ converge to 0 as $k \to +\infty$, and let φ be a nonnegative smooth function such that $\varphi(t) = 0$ for $t \notin (0,1)$ and $\int_{\mathbb{R}} \varphi(t)dt = 1$. We define $\varphi_k(t) := \dfrac{1}{\varepsilon_k}\varphi\left(\dfrac{t}{\varepsilon_k}\right)$ and

$$u_k(x,t) := \int_{\mathbb{R}} \overline{u}(x,s)\varphi_k(t-s)ds, \quad \forall (x,t) \in B_R(z_o) \times (-\infty, DR^2],$$

where we notice that the above integral is calculated over $(t-\varepsilon_k, t) \subset \mathbb{R}$. Then, a smooth function u_k satisfies

$$\mathscr{L}u_k(x,t) = \int_{\mathbb{R}} \mathscr{L}\overline{u}(x,s)\varphi_k(t-s)ds \le g_k(x,t), \quad \forall (x,t) \in B_R(z_o) \times (-\infty, DR^2],$$

where $g_k(x,t) := \int_{\mathbb{R}} \overline{f}^+(x,s)\varphi_k(t-s)ds \ge 0$. We also have

$$\begin{aligned} R^2\left(\fint_{B_R(z_o)\times(-AR^2,DR^2]} |g_k^+|^{n+1}\right)^{\frac{1}{n+1}} &\le \frac{R^2}{\{|B_R(z_o)|\cdot(A+D)R^2\}^{\frac{1}{n+1}}}\|\overline{f}^+\|_{L^{n+1}\left(B_R(z_o)\times(-AR^2-\varepsilon_k, DR^2-\varepsilon_k]\right)} \\ &\le \frac{R^2}{\{|B_R(z_o)|\cdot(A+D)R^2\}^{\frac{1}{n+1}}}\|\overline{f}^+\|_{L^{n+1}\left(B_R(z_o)\times(-AR^2,0]\right)} \\ &\le \left(\frac{A}{A+D}\right)^{\frac{1}{n+1}}\varepsilon < \varepsilon, \end{aligned}$$

which finishes the proof. □

Proposition 4.1.9. *Suppose that M satisfies the conditions* (4.1.2),(4.1.3). *Let $z_o \in M, R > 0, 0 < \eta < \frac{1}{2}$ and $\tau \in [3,16]$. Let u be a nonnegative smooth function such that $\mathscr{L}u \le f$ in $B_{\frac{49}{\eta^3}R}(z_o) \times \left(-3R^2, \frac{\tau R^2}{\eta^2}\right]$. Assume that $\inf_{B_R(z_o)\times\left[\frac{2R^2}{\eta^2},\frac{\tau R^2}{\eta^2}\right]} u \le 1$ and*

$$R^2\left(\fint_{B_{\frac{49}{\eta^3}R}(z_o)\times\left(-3R^2,\frac{\tau R^2}{\eta^2}\right]} |f^+|^{n+1}\right)^{\frac{1}{n+1}} \le \varepsilon'_\eta$$

for a uniform constant $0 < \varepsilon'_\eta < 1$. Let $r > 0$ satisfy $\left(\frac{\eta}{2}\right)^N R \le r < \left(\frac{\eta}{2}\right)^{N-1} R$ for some $N \in \mathbb{N}$ and let (z_1, t_1) be a point such that $d(z_o, z_1) < R$ and $|t_1| < R^2$. Then there exists a uniform constant $M'_\eta > 1$ (independent of r, N, z_1 and t_1) such that

$$\frac{\left|\left\{u \le {M'_\eta}^{N+2}\right\} \cap K_{\eta r}(z_1,t_1)\right|}{\left|K_{\alpha_1 r, \alpha_2 r^2}(z_1, t_1+4r^2)\right|} \ge \mu_\eta > 0,$$

where $0 < \mu_\eta < 1$ is the constant in Lemma 4.1.6.

Proof. (i) From Lemma 4.1.8, we approximate u by nonnegative smooth functions u_k, which are defined on $B_{\frac{48}{\eta^3}R}(z_o)\times\left(-3R^2, \frac{64R^2}{(4-\eta^2)\eta^6}+R^2\right]$. We find functions u_k and g_k such that u_k converges locally uniformly to u in $B_{\frac{48}{\eta^3}R}(z_o)\times\left(-3R^2, \frac{\tau R^2}{\eta^2}\right]$, and satisfies

$$\mathscr{L}u_k \le g_k \quad \text{in } B_{\frac{48}{\eta^3}R}(z_o)\times\left(-3R^2, \frac{64R^2}{(4-\eta^2)\eta^6}+R^2\right],$$

and

$$R^2\left(\fint_{B_{\frac{48}{\eta^3}R}(z_o)\times\left(-3R^2,\frac{64R^2}{(4-\eta^2)\eta^6}+R^2\right]}|g_k^+|^{n+1}\right)^{\frac{1}{n+1}} \le \frac{49}{48}\varepsilon'_\eta < 2\varepsilon'_\eta$$

by using the volume comparison theorem and Lemma 4.1.8. For a small $\delta > 0$, we consider $w_k := \dfrac{u_k}{1+\delta}$ and then for large k, w_k satisfies $\inf_{B_R(z_o)\times\left[\frac{2R^2}{\eta^2},\frac{\tau R^2}{\eta^2}\right]} w_k \le 1$, $\mathscr{L}w_k \le g_k$ in $B_{\frac{48}{\eta^3}R}(z_o)\times\left(-3R^2, \frac{64R^2}{(4-\eta^2)\eta^6}+R^2\right]$, and

$$R^2\left(\fint_{B_{\frac{48}{\eta^3}R}(z_o)\times\left(-3R^2,\frac{64R^2}{(4-\eta^2)\eta^6}+R^2\right]}|g_k^+|^{n+1}\right)^{\frac{1}{n+1}} < 2\varepsilon'_\eta,$$

according to the local uniform convergence of u_k to u in Lemma 4.1.8. So if we show the proposition for w_k, the local uniform convergence will imply that the result holds for u by letting $k \to +\infty$ and $\delta \to 0$. Now we assume that u is a nonnegative smooth function in $B_{\frac{48}{\eta^3}R}(z_o)\times\left(-3R^2, \frac{64R^2}{(4-\eta^2)\eta^6}+R^2\right]$ satisfying the same hypotheses as w_k.

(ii) We use Corollary 4.1.7 so we need to check the two hypotheses with $k = N+2$ and $h = 1$. As in the corollary, we define for $i \in \mathbb{N}$,

$$\bar{r}_i := \left(\frac{2}{\eta}\right)^{i-1} r \quad \text{and} \quad \bar{t}_i := t_1 + \sum_{j=1}^{i} 4\bar{r}_j^2.$$

Using the conditions on r, z_1, and t_1, simple computation says that for $0 < \eta < 1/2$,

$$\begin{aligned} &B_{2\bar{r}_{N+1}}(z_1) \supset B_{2R}(z_1) \supset B_R(z_o),\\ &\bar{t}_N < R^2 + \frac{16R^2}{4-\eta^2} < \frac{2R^2}{\eta^2} < \frac{16R^2}{\eta^2} < -R^2 + \frac{4(4+\eta^2)R^2}{\eta^2} < \bar{t}_{N+2}. \end{aligned}$$

Thus we have $B_{2\overline{r}_{N+1}}(z_1)\times(\bar{t}_N,\bar{t}_{N+2})\supset B_R(z_o)\times\left[\frac{2R^2}{\eta^2},\frac{16R^2}{\eta^2}\right]\supset B_R(z_o)\times\left[\frac{2R^2}{\eta^2},\frac{\tau R^2}{\eta^2}\right]$ for $0<\eta<\frac{1}{2}$ and hence $\inf\limits_{\bigcup_{i=1}^{N+2}K_{2\bar{r}_i}(z_1,\bar{t}_i)} u\le\inf\limits_{\bigcup_{i=N+1}^{N+2}K_{2\bar{r}_i}(z_1,\bar{t}_i)} u\le 1$. We remark that $\overline{r}_{N+2}$ is comparable to R.

Now, it suffices to show for some large $M'_\eta\ge M_\eta$, and small $0<\varepsilon'_\eta<\varepsilon_\eta$, we have

$$\bar{r}_i^2\left(\fint_{K_{\alpha_1\bar{r}_i,\alpha_2\bar{r}_i^2}(z_1,\bar{t}_i)}|f^+|^{n+1}\right)^{\frac{1}{n+1}}\le\varepsilon_\eta {M'_\eta}^{N+2-i},\quad\forall 1\le i\le N+2,\tag{4.1.13}$$

where M_η and ε_η are the constants in Corollary 4.1.7. We notice that $B_{\beta_1\overline{r}_{N+2}}(z_o)\subset B_{\alpha_1\overline{r}_{N+2}}(z_1)\subset B_{\frac{12}{\eta}\cdot\frac{4}{\eta^2}R}(z_o)$ and

$$\bigcup_{i=1}^{N+2}K_{\alpha_1\bar{r}_i,\alpha_2\bar{r}_i^2}(z_1,\bar{t}_i)\subset B_{\frac{48}{\eta^3}R}(z_o)\times\left(-3R^2,\frac{64R^2}{(4-\eta^2)\eta^6}+R^2\right]$$

since $d(z_o,z_1)<R,|t_1|<R^2$ and $\frac{2}{\eta}R\le\overline{r}_{N+2}<\frac{4}{\eta^2}R$. Then for $i=1,2,\cdots,N+2$, we have

$$\begin{aligned}\bar{r}_i^{2(n+1)}\fint_{K_{\alpha_1\bar{r}_i,\alpha_2\bar{r}_i^2}(z_1,\bar{t}_i)}|f^+|^{n+1}&\le\left(\frac{4}{\eta^2}\right)^{2(n+1)}\frac{R^{2(n+1)}}{|K_{\alpha_1\bar{r}_i,\alpha_2\bar{r}_i^2}(z_1,\bar{t}_i)|}\|f^+\|^{n+1}_{L^{n+1}\left(B_{\frac{48}{\eta^3}R}(z_o)\times\left(-3R^2,\frac{64R^2}{(4-\eta^2)\eta^6}+R^2\right]\right)}\\&\le\left(\frac{4}{\eta^2}\right)^{2(n+1)}(2\varepsilon'_\eta)^{n+1}\frac{\left|B_{\frac{48}{\eta^3}R}(z_o)\times\left(-3R^2,\frac{64R^2}{(4-\eta^2)\eta^6}+R^2\right]\right|}{|K_{\alpha_1\bar{r}_i,\alpha_2\bar{r}_i^2}(z_1,\bar{t}_i)|}\\&\le C(n,\eta){\varepsilon'_\eta}^{n+1}\frac{|B_{\frac{48}{\eta^3}R}(z_o)|R^2}{|B_{\alpha_1\bar{r}_i}(z_1)|\bar{r}_i^2}\le C(n,\eta){\varepsilon'_\eta}^{n+1}\frac{\left|B_{\beta_1\bar{r}_{N+2}}(z_o)\right|\bar{r}_{N+2}^2}{|B_{\alpha_1\bar{r}_i}(z_1)|\bar{r}_i^2},\end{aligned}$$

where we use that $\frac{2}{\eta}R\le\overline{r}_{N+2}<\frac{4}{\eta^2}R$ and the volume comparison theorem in the last inequality and the constant $C(n,\eta)>0$ depending only on n and η, may change from line to line. Since $d(z_o,z_1)<R$, we use the volume comparison theorem

again to obtain

$$\begin{aligned}\bar{r}_i^{2(n+1)} \fint_{K_{\alpha_1\bar{r}_i,\alpha_2\bar{r}_i^2}(z_1,\bar{t}_i)} |f^+|^{n+1} &\leq C(n,\eta){\varepsilon'_\eta}^{n+1} \frac{\left|B_{\beta_1\bar{r}_{N+2}}(z_o)\right| \bar{r}_{N+2}^2}{|B_{\alpha_1\bar{r}_i}(z_1)|\bar{r}_i^2} \\ &\leq C(n,\eta){\varepsilon'_\eta}^{n+1} \frac{\left|B_{\alpha_1\bar{r}_{N+2}}(z_1)\right| \bar{r}_{N+2}^2}{|B_{\alpha_1\bar{r}_i}(z_1)|\bar{r}_i^2} \\ &\leq C(n,\eta){\varepsilon'_\eta}^{n+1} \left(\frac{\bar{r}_{N+2}}{\bar{r}_i}\right)^{n+2} \leq C(n,\eta){\varepsilon'_\eta}^{n+1} \left(\frac{2}{\eta}\right)^{(n+2)(N+2-i)}.\end{aligned}$$

We select $M'_\eta > M_\eta$ large and $0 < \varepsilon'_\eta < \varepsilon_\eta$ small enough to satisfy

$$C(n,\eta){\varepsilon'_\eta}^{n+1} \left(\frac{2}{\eta}\right)^{(n+2)(N+2-i)} \leq \varepsilon_\eta^{n+1} {M'_\eta}^{(n+1)(N+2-i)}, \quad \forall 1 \leq i \leq N+2,$$

which proves (4.1.13). Therefore, Corollary 4.1.7 (after translation in time by t_1) gives

$$\frac{\left|\left\{u \leq {M'_\eta}^{N+2}\right\} \cap K_{\eta r}(z_1, t_1)\right|}{\left|K_{\alpha_1 r,\, \alpha_2 r^2}(z_1, t_1 + 4r^2)]\right|} \geq \mu_\eta > 0.$$

□

4.1.3 Parabolic version of the Calderón-Zygmund decomposition

Throughout this subsection, we assume that a complete Riemannian manifold M satisfies the condition (4.1.2). We introduce a parabolic version of the Calderón-Zygmund lemma (Lemma 4.1.13) to prove power decay of super-level sets in Lemma 4.1.14 (see [60, 11, 13]). Christ [16] proved that the following theorem holds for so-called "spaces of homogeneous type", which is a generalization of Euclidean dyadic decomposition. In harmonic analysis, a metric space X is called a space of homogeneous type when X equips a nonnegative Borel measure ν satisfying the doubling property

$$\nu(B_{2R}(x)) \leq A_1 \nu(B_R(x)) < +\infty, \quad \forall x \in X,\ R > 0,$$

for some constant A_1 independent of x and R. From Bishop's volume comparison, a complete Riemannian manifold M satisfying the condition (4.1.2) is a space of homogeneous type with $A_1 = 2^n$.

Theorem 4.1.10 (Christ)**.** *There exist a countable collection* $\{Q^{k,\alpha} \subset M : k \in \mathbb{Z}, \alpha \in I_k\}$ *of open subsets of* M *and positive constants* $0 < \delta_0 < 1$, c_1 *and* c_2 *(with* $2c_1 \le c_2$ *) that depend only on n, such that*

(i) $\left|M \setminus \bigcup_\alpha Q^{k,\alpha}\right| = 0$ *for* $k \in \mathbb{Z}$*,*

(ii) if $l \le k$, $\alpha \in I_k$*, and* $\beta \in I_l$*, then either* $Q^{k,\alpha} \subset Q^{l,\beta}$ *or* $Q^{k,\alpha} \cap Q^{l,\beta} = \emptyset$*,*

(iii) for any (k, α) *and any* $l < k$*, there is a unique* β *such that* $Q^{k,\alpha} \subset Q^{l,\beta}$*,*

(iv) $diam(Q^{k,\alpha}) \le c_2\delta_0^k$*,*

(v) any $Q^{k,\alpha}$ *contains some ball* $B_{c_1\delta_0^k}(z^{k,\alpha})$*.*

For convenience, we will use the following notation.

Definition 4.1.1 (Dyadic cubes on M)**.**

(i) The open set $Q = Q^{k,\alpha}$ *in Theorem 4.1.10 is called a dyadic cube of generation k on M. From the property (iii) in Theorem 4.1.10, for any* (k, α)*, there is a unique* β *such that* $Q^{k,\alpha} \subset Q^{k-1,\beta}$*. We call* $Q^{k-1,\beta}$ *the predecessor of* $Q^{k,\alpha}$*. When* $Q := Q^{k,\alpha}$*, we denote the predecessor* $Q^{k-1,\beta}$ *by* $\widetilde{Q}$ *for simplicity.*

(ii) For a given $R > 0$*, we define* $k_R \in \mathbb{N}$ *to satisfy*

$$c_2\delta_0^{k_R-1} < R \le c_2\delta_0^{k_R-2}.$$

The number k_R means that a dyadic cube of generation k_R is comparable to a ball of radius R.

For the rest of this section, we fix some small numbers;

$$\delta := \frac{2c_1}{c_2}\delta_0 \in (0, \delta_0), \;\; \delta_1 := \frac{\delta_0(1-\delta_0)}{2} \in \left(0, \frac{\delta_0}{2}\right),$$

$$\eta := \min(\delta, \delta_1) \in \left(0, \frac{1}{2}\right) \quad \text{and} \quad \kappa := \frac{\eta}{2}\sqrt{1-\delta_0^2}.$$

By using the dyadic decomposition of a manifold M, we have the following decomposition of $M \times (T_1, T_2]$ in space and time. For time variable, we take the standard euclidean dyadic decomposition.

Lemma 4.1.11. *There exists a countable collection* $\{K^{k,\alpha} \subset M \times (T_1, T_2] : k \in \mathbb{Z}, \alpha \in J_k\}$ *of subsets of* $M \times (T_1, T_2] \subset M \times \mathbb{R}$ *and positive constants* $0 < \delta_0 < 1$, c_1 *and* c_2 *(with* $2c_1 \le c_2$*) that depend only on* n, *such that*

(i) $\left|M \times (T_1, T_2] \setminus \bigcup_\alpha K^{k,\alpha}\right| = 0$ *for* $k \in \mathbb{Z}$,

(ii) if $l \le k$, $\alpha \in J_k$, *and* $\beta \in J_l$, *then either* $K^{k,\alpha} \subset K^{l,\beta}$ *or* $K^{k,\alpha} \cap K^{l,\beta} = \emptyset$,

(iii) for any (k, α) *and any* $l < k$, *there is a unique* β *such that* $K^{k,\alpha} \subset K^{l,\beta}$,

(iv) $diam(K^{k,\alpha}) \le c_2\delta_0^k \times c_2^2\delta_0^{2k}$,

(v) any $K^{k,\alpha}$ *contains some cylinder* $B_{c_1\delta_0^k}(z^{k,\alpha}) \times (t^{k,\alpha} - c_1^2\delta_0^{2k}, t^{k,\alpha}]$.

Proof. To decompose in time variable, for each $k \in \mathbb{Z}$, we select the largest integer $N_k \in \mathbb{Z}$ to satisfy

$$\frac{1}{4}c_2^2\delta_0^{2k} \le \frac{T_2 - T_1}{2^{2N_k}} < c_2^2\delta_0^{2k}.$$

For k-th generation, we split the interval $(T_1, T_2]$ into 2^{2N_k} disjoint subintervals which have the same length. Then we obtain $|J_k| = |I_k| \cdot 2^{2N_k}$ disjoint subsets on $M \times (T_1, T_2]$ satisfying properties (i)-(v). □

For the rest of this section, let $\{K^{k,\alpha} \subset M \times (T_1, T_2] : k \in \mathbb{Z}, \alpha \in J_k\}$ be the parabolic dyadic decomposition of $M \times (T_1, T_2]$ as in Lemma 4.1.11.

Definition 4.1.2 (Parabolic dyadic cubes).

(i) $K = K^{k,\alpha}$ *is called a parabolic dyadic cube of generation* k. *If* $K := K^{k,\alpha} \subset K^{k-1,\beta} =: \widetilde{K}$, *we say* $\widetilde{K}$ *is the predecessor of* K.

(ii) For a parabolic dyadic cube K *of generation* k, *we define* $l(k)$ *to be the length of* K *in time variable, namely,* $l(k) = \frac{T_2 - T_1}{2^{2N_k}}$ *for* $M \times (T_1, T_2]$ *in Lemma 4.1.11.*

We quote the following technical lemma proven by Cabré [11, Lemma 6.5].

Lemma 4.1.12 (Cabré). *Let* $z_o \in M$ *and* $R > 0$. *Then we have the following.*

(i) If Q *is a dyadic cube of generation* k *such that*

$$k \ge k_R \quad \textit{and} \quad Q \subset B_R(z_o),$$

then there exist $z_1 \in Q$ *and* $r_k \in (0, R/2)$ *such that*

$$B_{\delta r_k}(z_1) \subset Q \subset \widetilde{Q} \subset \overline{B_{2r_k}(z_1)} \subset B_{\frac{11}{\eta} r_k}(z_1) \subset B_{\frac{11}{\eta} R}(z_o) \tag{4.1.14}$$

and

$$B_{\frac{9}{\eta} R}(z_o) \subset B_{\frac{11}{\eta} R}(z_1). \tag{4.1.15}$$

In fact, for $k \geq k_R$*, the above radius* r_k *is defined by*

$$r_k := \frac{1}{2} c_2 \delta_0^{k-1} = \frac{c_1}{\delta} \delta_0^k.$$

(ii) If Q *is a dyadic cube of generation* k_R *and* $d(z_o, Q) \leq \delta_1 R$*, then* $Q \subset B_R(z_o)$ *and hence* (4.1.14) *and* (4.1.15) *hold for some* $z_1 \in Q$ *and* $r_{k_R} \in \left[\frac{\delta_0 R}{2}, \frac{R}{2}\right)$*. Moreover,*

$$B_{\delta_1 R}(z_o) \subset B_{2r_{k_R}}(z_1).$$

(iii) There exists at least one dyadic cube Q *of generation* k_R *such that* $d(z_o, Q) \leq \delta_1 R$*.*

We remark that for $k \geq k_R$,

$$\eta^2 r_k^2 \leq \delta_0^2 r_k^2 = \frac{1}{4} c_2^2 \delta_0^{2k} \leq l(k) < c_2^2 \delta_0^{2k} = 4 r_{k+1}^2$$

and (4.1.14) gives that for any $a \in \mathbb{R}$,

$$K_{\eta r_k}(z_1, a) \subset Q \times (a - l(k), a] \subset \overline{K_{2r_k}(z_1, a)} \tag{4.1.16}$$

Definition 4.1.3. *Let* $m \in \mathbb{N}$*. For any parabolic dyadic cube* $K := Q \times (a - l(k), a]$ *of generation* k*, the elongation of* K *along time in* m *steps (see [39]) , denoted by* $\overline{K}^m$*, is defined by*

$$\overline{K}^m := \widetilde{Q} \times (a, a + m \cdot l(k-1)],$$

where $l(k)$ *is the length of a parabolic dyadic cube of generation* k *in time and* $\widetilde{Q}$ *is the predecessor of* Q *in space. The elongation* $\overline{K}^m$ *is the union of the stacks of parabolic dyadic cubes congruent to the predecessor of* K*.*

Now we have a parabolic version of Calderón-Zygmund lemma. The proof of lemma is the same as Euclidean case so we refer to [60] for the proof.

Lemma 4.1.13 (Lemma 3.23, [60])**.** *Let* $K_1 = Q_1 \times (a - l(k_0), a]$ *be a parabolic dyadic cube of generation* k_0 *in* $M \times (T_1, T_2]$, *and let* $0 < \alpha < 1$ *and* $m \in \mathbb{N}$. *Let* $\mathcal{A} \subset K_1$ *be a measurable set such that* $|\mathcal{A} \cap K_1| \leq \alpha |K_1|$ *and let*

$$\mathcal{A}_\alpha^m := \cup \left\{ \overline{K}^m : |K \cap \mathcal{A}| > \alpha |K|,\ K, \text{ a parabolic dyadic cube in } K_1 \right\} \cap (Q_1 \times \mathbb{R}).$$

Then, we have

$$|\mathcal{A}_\alpha^m| \geq \frac{m}{(m+1)\alpha} |\mathcal{A}|.$$

4.1.4 Proof of parabolic Harnack inequality

In order to prove the parabolic Harnack inequality, we take the approach presented in [60] and iterate Lemma 4.1.6 with Christ decomposition (Theorem 4.1.10) and Calderón-Zygmund type lemma (Lemma 4.1.13). We begin this subsection with recalling that $\eta \in \left(0, \frac{1}{2}\right)$ is fixed as in the previous subsection. So the uniform constants $\mu_\eta, \varepsilon'_\eta$ and M'_η in Proposition 4.1.9 are also fixed and we denote them by μ, ε_0 and M_0 for simplicity.

We select an integer $m > 1$ large enough to satisfy

$$\frac{m}{(m+1)(1-\mu)} > \frac{1}{1 - \frac{\mu}{2}},$$

where $0 < \mu < 1$ is the constant in Lemma 4.1.6. For $T_1 := -3R^2$ and $T_2 := \left(\frac{16}{\eta^2} + 1 + m\right) R^2$, we consider a parabolic dyadic decomposition of $M \times (T_1, T_2]$ in Lemma 4.1.11 and fix the decomposition for this subsection.

Lemma 4.1.14. *Suppose that* M *satisfies the conditions* (4.1.2),(4.1.3). *Let* $z_o \in M, R > 0$ *and* $\tau \in [3, 16]$. *Let* u *be a nonnegative smooth function such that* $\mathscr{L} u \leq f$ *in* $B_{\frac{50}{\eta^3} R}(z_o) \times \left(-3R^2, \frac{\tau R^2}{\eta^2}\right]$. *Assume that*

$$\inf_{B_R(z_o) \times \left[\frac{2R^2}{\eta^2}, \frac{\tau R^2}{\eta^2}\right]} u \leq 1$$

and

$$R^2 \left(\fint_{B_{\frac{50}{\eta^3} R}(z_o) \times \left(-3R^2, \frac{\tau R^2}{\eta^2}\right]} |f^+|^{n+1} \right)^{\frac{1}{n+1}} \leq \varepsilon_1$$

for a uniform constant $0 < \varepsilon_1 < \varepsilon_0$. *Let* K_1 *be a parabolic dyadic cube of generation* k_R *such that*

$$K_1 := Q_1 \times (t_1 - l(k_R), t_1] \subset Q_1 \times (-R^2, R^2),$$

where Q_1 *is a dyadic cube of generation* k_R *such that* $d(z_o, Q_1) \leq \delta_1 R$. *Then for* $i = 1, 2, \cdots$, *we have*

$$\frac{\left|\{u > M_1^i\} \cap K_1\right|}{|K_1|} < \left(1 - \frac{\mu}{2}\right)^i, \tag{4.1.17}$$

where $0 < \varepsilon_1 < \varepsilon_0$ *and* $M_1 > 0$ *depend only on* n, λ, Λ, *and* $a_{\mathscr{L}}$.

Proof. (i) As Proposition 4.1.9, we use Lemma 4.1.8 to assume that a nonnegative smooth function u defined on $B_{\frac{49}{\eta^3}R}(z_o) \times (T_1, T_2]$ satisfies that $\inf_{B_R(z_o)\times\left[\frac{2R^2}{\eta^2}, \frac{\tau R^2}{\eta^2}\right]} u \leq 1$ and $\mathscr{L}u \leq f$ in $B_{\frac{49}{\eta^3}R}(z_o) \times (T_1, T_2]$ for some f with

$$R^2 \left(\fint_{B_{\frac{49}{\eta^3}R}(z_o)\times(T_1,T_2]} |f^+|^{n+1} \right)^{\frac{1}{n+1}} \leq \frac{50}{49}\varepsilon_1 < 2\varepsilon_1.$$

(ii) According to Lemma 4.1.12, there exists a dyadic cube $Q_1 \subset B_R(z_o)$ of generation k_R such that $d(z_o, Q_1) \leq \delta_1 R$. We find $z_1 \in Q_1$ and $r_{k_R} \in [\frac{\delta_0}{2}R, \frac{1}{2}R)$ satisfying (4.1.14),(4.1.15) and $B_{\delta_1 R}(z_o) \subset B_{2r_{k_R}}(z_1)$. Since $\eta^2 r_{k_R}^2 \leq l(k_R) < 4r_{k_R+1}^2 = 4\delta_0^2 r_{k_R}^2 < \delta_0^2 R^2$, we find $t_1 \in (-R^2 + l(k_R), R^2)$ such that $K_1 := Q_1 \times (t_1 - l(k_R), t_1]$ is a parabolic dyadic cube of generation k_R of $M \times (T_1, T_2]$. From (4.1.16), we also have that

$$K_{\eta r_{k_R}}(z_1, t_1) \subset K_1 \subset \overline{K_{2r_{k_R}}(z_1, t_1)}.$$

We use the induction to prove (4.1.17) so we first check the case $i = 1$. We notice that $d(z_o, z_1) < R$, $r_{k_R} \in [\frac{\delta_0}{2}R, \frac{1}{2}R) \subset (\frac{\eta}{2}R, R)$ and $|t_1| < R^2$. We set $\varepsilon_1 := \left(\frac{3/\eta^2+3}{16/\eta^2+m+4}\right)^{\frac{1}{n+1}} \frac{\varepsilon_0}{2}$. Then, u satisfies the hypotheses of Proposition 4.1.9 with $r = r_{k_R}$ and $N = 1$, so we deduce that

$$0 < \mu \leq \frac{\left|\{u \leq M_0^3\} \cap K_{\eta r_{k_R}}(z_1, t_1)\right|}{|K_{\alpha_1 r_{k_R}, \alpha_2 r_{k_R}^2}(z_1, t_1 + 4r_{k_R}^2)|} = \frac{\left|\{u \leq M_0^3\} \cap K_{\eta r_{k_R}}(z_1, t_1)\right|}{|K_{\alpha_1 r_{k_R}, \alpha_2 r_{k_R}^2}(z_1, t_1)|} < \frac{\left|\{u \leq M_0^3\} \cap K_1\right|}{|K_1|}.$$

Thus, we have for $M_1 \geq M_0^3$,

$$\frac{|\{u > M_1\} \cap K_1|}{|K_1|} \leq 1 - \mu < 1 - \frac{\mu}{2}.$$

(iii) Now, suppose that (4.1.17) is true for i, that is,

$$\frac{\left|\{u > M_1^i\} \cap K_1\right|}{|K_1|} < \left(1 - \frac{\mu}{2}\right)^i.$$

To show the (i+1)-th step, define for $h > 0$,

$$\mathcal{B}_h := \{u > h\} \cap B_{\frac{49}{\eta^3}R}(z_o) \times (T_1, T_2].$$

We know $\dfrac{\left|\mathcal{B}_{M_1^i} \cap K_1\right|}{|K_1|} < \left(1 - \dfrac{\mu}{2}\right)^i$. If $h > 0$ is a constant such that

$$\frac{|\mathcal{A}|}{|K_1|} \geq \left(1 - \frac{\mu}{2}\right)^{i+1} \quad \text{for} \quad \mathcal{A} := \mathcal{B}_{hM_1^i} \cap K_1,$$

then we will show that $h < M_1$ for a uniform constant $M_1 > M_0 > 1$, that will be fixed later.

Suppose on the contrary that $h \geq M_1$. From (ii), we have $\dfrac{|\mathcal{A}|}{|K_1|} \leq \dfrac{|\mathcal{B}_{M_0^3} \cap K_1|}{|K_1|} \leq 1 - \mu$ for $M_1 \geq M_0^3$ and $h \geq 1$. Applying Lemma 4.1.13 to $\mathcal{A}$ with $\alpha = 1 - \mu$, it follows that

$$|\mathcal{A}_{1-\mu}^m| \geq \frac{m}{(m+1)(1-\mu)}|\mathcal{A}| > \frac{1}{1 - \frac{\mu}{2}}|\mathcal{A}|.$$

We claim that

$$\mathcal{A}_{1-\mu}^m \subset \mathcal{B}_{\frac{hM_1^i}{M_0^m}} \tag{4.1.18}$$

for $h \geq C_1 M_0^m > 1$, where a uniform constant $C_1 > 0$ will be chosen. If not, there is a point $(x_1, s_1) \in \mathcal{A}_{1-\mu}^m \backslash \mathcal{B}_{\frac{hM_1^i}{M_0^m}}$ and we find a parabolic dyadic cube $K := Q \times (a - l(k), a] \subset K_1$ of generation $k (> k_R)$ such that

$$|\mathcal{A} \cap K| > (1 - \mu)|K| \quad \text{and} \quad (x_1, s_1) \in \overline{K}^m$$

from the definition of $\mathcal{A}^m_{1-\mu}$. According to Lemma 4.1.12, there exist $z_1 \in Q \subset \underline{Q_1} \subset B_R(z_o)$ and $r_k \in (0, R/2)$ satisfying (4.1.14), (4.1.15), $K_{\eta r_k}(z_1, a) \subset K \subset \overline{K_{2r_k}(z_1, a)}$ and

$$(x_1, s_1) \in \overline{K}^m = \widetilde{Q} \times (a, a + m \cdot l(k-1)] \subset \overline{B}_{2r_k}(z_1) \times (a, a + m \cdot 4r_k^2].$$

We note that

$$\inf_{B_{2r_k}(z_1)\times(a,a+m\cdot 4r_k^2]} u \le u(x_1, s_1) \le \frac{hM_1^i}{M_0^m}$$

and

$$B_{\alpha_1 r_k}(z_1) \times \left(a - (\eta^2 + \eta^4/4) r_k^2, a + m \cdot 4r_k^2\right] \subset B_{\alpha_1 R}(z_o) \times \left(-3R^2, (1+m)R^2\right],$$

since $r_k < R/2$ and $a \in (t_1 - l(k_R), t_1] \subset (-R^2, R^2)$. We also have that for $j = 1, \cdots, m$,

$$r_k^2 \left(\fint_{K_{\alpha_1 r_k, \alpha_2 r_k^2}(z_1, a+(m-j+1)\cdot 4r_k^2)} |f^+|^{n+1} \right)^{\frac{1}{n+1}} \le \varepsilon_0 \frac{hM_1^i}{M_0^{m-j+1}}. \tag{4.1.19}$$

Indeed, the volume comparison theorem and the property (4.1.15) will give that

$$\begin{aligned}
r_k^2 \left(\fint_{K_{\alpha_1 r_k, \alpha_2 r_k^2}(z_1, a+(m-j+1)\cdot 4r_k^2)} |f^+|^{n+1} \right)^{\frac{1}{n+1}} &= \frac{r_k^{\frac{n}{n+1}} \cdot r_k^{\frac{n+2}{n+1}}}{\left|B_{\alpha_1 r_k}(z_1)\right|^{\frac{1}{n+1}} \left(\alpha_2 r_k^2\right)^{\frac{1}{n+1}}} \|f^+\|_{L^{n+1}} \\
\le \frac{R^{\frac{n}{n+1}} \cdot R^{\frac{n+2}{n+1}}}{\left|B_{\alpha_1 R}(z_1)\right|^{\frac{1}{n+1}} (\alpha_2 R^2)^{\frac{1}{n+1}}} \|f^+\|_{L^{n+1}\left(B_{\frac{12}{\eta}R}(z_o)\times(T_1,T_2]\right)} \\
\le \frac{R^2}{\left|B_{\beta_1 R}(z_o)\right|^{\frac{1}{n+1}} (\alpha_2 R^2)^{\frac{1}{n+1}}} \|f^+\|_{L^{n+1}\left(B_{\frac{49}{\eta^3}}(z_o)\times(T_1,T_2]\right)} \\
\le \frac{C_1 R^2/2}{\left|B_{\frac{49}{\eta^3}}(z_o) \times (T_1, T_2]\right|^{\frac{1}{n+1}}} \|f^+\|_{L^{n+1}\left(B_{\frac{49}{\eta^3}}(z_o)\times(T_1,T_2]\right)} < C_1 \varepsilon_1,
\end{aligned}$$

where a uniform constant $C_1 > 1$ depends only on η, n and m. For $h \ge C_1 M_0^m$ and $M_1 > 1$, we have that

$$r_k^2 \left(\fint_{K_{\alpha_1 r_k, \alpha_2 r_k^2}(z_1,\, a+(m-j+1)\cdot 4r_k^2)} |f^+|^{n+1} \right)^{\frac{1}{n+1}} < C_1 \varepsilon_1 < \varepsilon_0 \frac{hM_1^i}{M_0^m} \le \varepsilon_0 \frac{hM_1^i}{M_0^{m-j+1}},$$

which proves (4.1.19). Thus, we can apply Lemma 4.1.6 iteratively to $\tilde{u}_j := \frac{M_0^{m-j+1}}{hM_1^i}u$, for $1 \le j \le m$, to deduce

$$\mu \le \frac{\left|\left\{u \le hM_1^i\right\} \cap K_{\eta r_k}(z_1, a)\right|}{|K_{\alpha_1 r_k, \alpha_2 r_k^2}(z_1, a + 4r_k^2)|} < \frac{\left|\left\{u \le hM_1^i\right\} \cap K\right|}{|K|}.$$

However, this contradicts to the fact that $|\mathcal{A} \cap K| > (1-\mu)|K|$. Therefore, we have proved that $\mathcal{A}^m_{1-\mu} \subset \mathcal{B}_{\frac{hM_1^i}{M_0^m}}$ for $h \ge C_1 M_0^m$.

(iv) Since $|\mathcal{B}_{M_1^i} \cap K_1| < \left(1 - \frac{\mu}{2}\right)^i |K_1|$, we have that $\left|\mathcal{B}_{\frac{hM_1^i}{M_0^m}} \cap K_1\right| \le |\mathcal{B}_{M_1^i} \cap K_1| < \left(1 - \frac{\mu}{2}\right)^i |K_1| \le \frac{1}{1-\frac{\mu}{2}}|\mathcal{A}|$ for $h \ge C_1 M_0^m$. Then, by using (4.1.18), we obtain

$$\begin{aligned} |\mathcal{A}^m_{1-\mu} \backslash K_1| &= |\mathcal{A}^m_{1-\mu}| - |\mathcal{A}^m_{1-\mu} \cap K_1| \\ &\ge \frac{m}{(m+1)(1-\mu)}|\mathcal{A}| - \left|\mathcal{B}_{\frac{hM_1^i}{M_0^m}} \cap K_1\right| \\ &> \left(\frac{m}{(m+1)(1-\mu)} - \frac{1}{1-\frac{\mu}{2}}\right)|\mathcal{A}| =: \alpha|\mathcal{A}| \ge \alpha\left(1-\frac{\mu}{2}\right)^{i+1}|K_1| \end{aligned}$$

with $\alpha := \frac{m}{(m+1)(1-\mu)} - \frac{1}{1-\frac{\mu}{2}} > 0$. We find a point $(x_1, s_1) \in \mathcal{A}^m_{1-\mu} \backslash K_1$ and a parabolic dyadic cube $K := Q \times (a - l(k), a] \subset K_1$ of generation $k(> k_R)$ such that $(x_1, s_1) \in \overline{K}^m$, and $|\mathcal{A} \cap K| > (1-\mu)|K|$. We may assume that

$$s_1 > t_1 + \frac{\alpha}{2}\left(1 - \frac{\mu}{2}\right)^{i+1} l(k_R)$$

since $\mathcal{A}^m_{1-\mu} \subset Q_1 \times (t_1 - l(k_R), +\infty)$ and $\frac{|\mathcal{A}^m_{1-\mu} \backslash K_1|}{|Q_1|} > \alpha\left(1 - \frac{\mu}{2}\right)^{i+1} l(k_R)$. Using Lemma 4.1.12 again, there exist $z_1 \in Q \subset Q_1 \subset B_R(z_o)$ and $r_k \in (0, R/2)$ satisfying (4.1.14),(4.1.15), and $K_{\eta r_k}(z_1, a) \subset K \subset \overline{K_{2r_k}(z_1, a)}$. Then we have

$$s_1 \le a + m \cdot l(k-1) < t_1 + m \cdot 4r_k^2$$

and hence

$$r_k \ge \frac{\sqrt{\alpha}}{\sqrt{8m}}\left(1 - \frac{\mu}{2}\right)^{\frac{i+1}{2}} \sqrt{l(k_R)} \ge \frac{\sqrt{\alpha}\delta_0^2}{4\sqrt{2m}}\left(1 - \frac{\mu}{2}\right)^{\frac{i+1}{2}} R \ge \left(\frac{\eta}{2}\right)^{Ni} R$$

for a uniform integer $N > 0$ independent of $i \in \mathbb{N}$. We apply Proposition 4.1.9 to u in order to get

$$\mu \leq \frac{\left|\left\{u \leq M_0^{Ni+2}\right\} \cap K_{\eta r_k}(z_1, a)\right|}{\left|K_{\alpha_1 r_k, \alpha_2 r_k^2}(z_1, a + 4r_k^2)\right|} \leq \frac{\left|\left\{u \leq M_0^{(N+2)i}\right\} \cap K\right|}{|K|},$$

since $r_k \geq \left(\frac{\eta}{2}\right)^{Ni} R$, and $(z_1, a) \in K \subset K_1 \subset B_R(z_o) \times (-R^2, R^2)$. If $h \geq M_1 := \max\{C_1 M_0^m, M_0^{N+2}\}$, this implies

$$1 - \mu > \frac{\left|\left\{u > M_0^{(N+2)i}\right\} \cap K\right|}{|K|} \geq \frac{\left|\left\{u > hM_1^i\right\} \cap K\right|}{|K|} = \frac{|\mathcal{A} \cap K|}{|K|},$$

which is a contradiction to the fact that $|\mathcal{A} \cap K| > (1 - \mu)|K|$. Thus, we have $h < M_1$ for a uniform constant $M_1 := \max\{C_1 M_0^m, M_0^{N+2}\}$. Therefore, we conclude that $\frac{|\{u > M_1^{i+1}\} \cap K_1|}{|K_1|} < \left(1 - \frac{\mu}{2}\right)^{i+1}$, completing the proof. □

The following corollary is a direct consequence of Lemma 4.1.14, which estimates the distribution function of u.

Corollary 4.1.15. *Under the same assumption as Lemma 4.1.14, we have*

$$\frac{|\{u \geq h\} \cap K_1|}{|K_1|} \leq dh^{-\epsilon} \quad \forall h > 0, \tag{4.1.20}$$

where $d > 0$ and $0 < \epsilon < 1$ depend only on n, λ, Λ, and $a_{\mathscr{L}}$.

Another consequence of Lemma 4.1.14 is a weak Harnack inequality for nonnegative supersolutions to $\mathscr{L}u = f$.

Corollary 4.1.16. *Under the same assumption as Lemma 4.1.14, we have for $p_o := \frac{\epsilon}{2}$,*

$$\left(\frac{1}{|K_{\kappa R}(z_o, 0)|} \int_{K_{\kappa R}(z_o, 0)} u^{p_o}\right)^{\frac{1}{p_o}} \leq C, \tag{4.1.21}$$

where $C > 0$ depends only on n, λ, Λ, and $a_{\mathscr{L}}$.

Proof. Let $k = k_R$ and let $\left\{K^{k,\alpha} := Q^{k,\alpha} \times (t^{k,\alpha} - l(k), t^{k,\alpha}]\right\}_{\alpha \in J'_k}$ be a family of parabolic dyadic cubes intersecting $K_{\kappa R}(z_o, 0)$. For $\alpha \in J'_k$, we have that $K^{k,\alpha} \subset B_{(\kappa+\delta_0)R}(z_o) \times (-R^2, R^2]$ since $d(z_o, Q^{k,\alpha}) \leq \kappa R (< \delta_1 R)$, $diam(Q^{k,\alpha}) \leq c_2 \delta_0^k \leq \delta_0 R$, and $-R^2 + l(k) < -\kappa^2 R^2 \leq t^{k,\alpha} \leq l(k) < \delta_0^2 R^2$. Since

$$
\begin{aligned}
|K_{\kappa R}(z_o, 0)| &\geq \left(\frac{\kappa}{\kappa + \delta_0}\right)^n |B_{(\kappa+\delta_0)R}(z_o)| \cdot \kappa^2 R^2 \geq \left(\frac{\kappa}{\kappa + \delta_0}\right)^n \sum_{\alpha \in J'_k} |Q^{k,\alpha}| \cdot \kappa^2 R^2 \\
&\geq \left(\frac{\kappa}{\kappa + \delta_0}\right)^n \sum_{\alpha \in J'_k} |B_{c_1 \delta_0^k}(z^{k,\alpha})| \cdot \kappa^2 R^2 \geq \left(\frac{\kappa}{\kappa + \delta_0}\right)^n \sum_{\alpha \in J'_k} |B_{\frac{\delta\delta_0}{2} R}(z^{k,\alpha})| \cdot \kappa^2 R^2 \\
&\geq \left(\frac{\kappa}{\kappa + \delta_0} \cdot \frac{\delta\delta_0}{2(\delta_0 + 2\kappa)}\right)^n \sum_{\alpha \in J'_k} |B_{(\delta_0 + 2\kappa)R}(z^{k,\alpha})| \cdot \kappa^2 R^2 \\
&\geq \left(\frac{\kappa}{\kappa + \delta_0} \cdot \frac{\delta\delta_0}{2(\delta_0 + 2\kappa)}\right)^n \sum_{\alpha \in J'_k} |B_{\kappa R}(z_o)| \cdot \kappa^2 R^2,
\end{aligned}
$$

the number $|J'_k|$ of parabolic dyadic cubes intersecting $K_{\kappa R}(z_o, 0)$ is uniformly bounded. Thus for some $K^{k,\alpha}$ with $\alpha \in J'_k$, we have

$$
\begin{aligned}
\int_{K_{\kappa R}(z_o)} u^{p_o} &\leq |J'_k| \cdot \int_{K^{k,\alpha}} u^{p_o} \\
&\leq |J'_k| \cdot \left\{ |K^{k,\alpha}| + p_o \int_1^\infty h^{p_o - 1} |\{u \geq h\} \cap K^{k,\alpha}| dh \right\} \\
&\leq |J'_k| \cdot \left\{ |K^{k,\alpha}| + p_o d |K^{k,\alpha}| \int_1^\infty h^{p_o - 1 - \epsilon} dh \right\}.
\end{aligned}
$$

from Corollary 4.1.15, where d and ϵ are the constants in Corollary 4.1.15.

By using the volume comparison theorem, we conclude that

$$
\frac{1}{|K_{\kappa R}(z_o, 0)|} \int_{K_{\kappa R}(z_o, 0)} u^{p_o} \leq C_0 \frac{|K^{k,\alpha}|}{|K_{\kappa R}(z_o, 0)|} \leq C_0 \left(\frac{\kappa + \delta_0}{\kappa}\right)^n \cdot \frac{\delta_0^2}{\kappa^2}
$$

for $C_0 := |J'_k| \cdot \left\{1 + p_o d \int_1^\infty h^{-1-\epsilon/2} dh\right\}$ since $K^{k,\alpha} \subset B_{(\kappa+\delta_0)R}(z_o) \times (t^{k,\alpha} - \delta_0^2 R^2, t^{k,\alpha}]$. □

So far, we have dealt with nonnegative supersolutions. Now, we consider a nonnegative solution u of $\mathscr{L} u = f$. We apply Corollary 4.1.15 as in [11] (see also [60]) to solutions $C_1 - C_2 u$ for some constants C_1 and C_2.

Lemma 4.1.17. *Suppose that M satisfies the conditions* (4.1.2),(4.1.3). *Let $z_o \in M, R > 0$ and $\tau \in [3,16]$. Let u be a nonnegative smooth function such that $\mathscr{L}u = f$ in $B_{\frac{50}{\eta^3}R}(z_o) \times \left(-3R^2, \frac{\tau R^2}{\eta^2}\right]$. Assume that $\inf_{B_R(z_o)\times\left[\frac{2R^2}{\eta^2}, \frac{\tau R^2}{\eta^2}\right]} u \le 1$ and*

$$R^2 \left(\fint_{B_{\frac{50}{\eta^3}R}(z_o)\times\left(-3R^2, \frac{\tau R^2}{\eta^2}\right]} |f|^{n+1} \right)^{\frac{1}{n+1}} \le \frac{\varepsilon_1}{4} =: \varepsilon$$

for a uniform constant $0 < \varepsilon_1 < 1$ as in Lemma 4.1.14.

Then there exist constants $\sigma > 0$ and $\tilde{M}_0 > 1$ depending on n, λ, Λ and $a_{\mathscr{L}}$ such that for $\nu := \frac{\tilde{M}_0}{\tilde{M}_0 - 1/2} > 1$, the following holds:

If $j \ge 1$ is an integer and $z_1 \in M$ and $t_1 \in \mathbb{R}$ satisfy

$$d(z_o, z_1) \le \kappa R, \quad |t_1| \le \kappa^2 R^2$$

and

$$u(z_1, t_1) \ge \nu^{j-1}\tilde{M}_0,$$

then

(i) $K_{\frac{50}{\eta^3}L_j, \left(3+\frac{\tau}{\eta^2}\right)L_j^2}(z_1, t_1) \subset B_{\frac{50}{\eta^3}R}(z_o) \times \left(-3R^2, \frac{\tau R^2}{\eta^2}\right],$

(ii) $\sup_{K_{\frac{50}{\eta^3}L_j, \left(3+\frac{\tau}{\eta^2}\right)L_j^2}(z_1,t_1)} u \ge \nu^j \tilde{M}_0,$

where $L_j := \sigma \tilde{M}_0^{-\frac{\epsilon}{n+2}} \nu^{-\frac{j\epsilon}{n+2}} R$ and $0 < \epsilon < 1$ as in Corollary 4.1.15.

Proof. We select $\sigma > 0$ and $\tilde{M}_0 > 1$ large so that

$$\sigma > \frac{c_2}{c_1\delta_0} (2d2^{\epsilon})^{\frac{1}{n+2}}$$

and

$$\sigma \tilde{M}_0^{-\frac{\epsilon}{n+2}} + d\tilde{M}_0^{-\epsilon} \le \frac{\kappa}{4},$$

where d, ϵ, c_1, c_2 and δ_0 are the constants in Corollary 4.1.15 and Theorem 4.1.10. Since $L_j \le \frac{\kappa R}{4} < \frac{\eta R}{8}$, $d(z_o, z_1) \le \kappa R < R$ and $|t_1| \le \kappa^2 R^2 < \frac{\eta^2 R^2}{4}$, we have

$$B_{\frac{50}{\eta^3}L_j}(z_1) \times \left(t_1 - \left(3 + \frac{\tau}{\eta^2}\right) L_j^2, t_1\right] \subset B_{\frac{50}{\eta^3}R}(z_o) \times \left(-3R^2, \frac{\tau R^2}{\eta^2}\right],$$

so (i) is true.

Now, suppose on the contrary that

$$\sup_{K_{\frac{50}{\eta^3}L_j,\left(3+\frac{\tau}{\eta^2}\right)L_j^2}(z_1,t_1)} u < \nu^j \tilde{M}_0.$$

Let $k_j := k_{L_j} \geq k_R$ with L_j in Definition 4.1.1. From Lemma 4.1.12, there exists a dyadic cube Q_{L_j} of generation k_j such that $d(z_1, Q_{L_j}) \leq \delta_1 L_j$. We also find a parabolic dyadic cube K_{L_j} of generation k_j such that

$$K_{L_j} \subset Q_{L_j} \times \left(t_1 - \frac{\tau L_j^2}{\eta^2} - L_j^2, t_1 - \frac{\tau L_j^2}{\eta^2} + L_j^2\right)$$

since $l(k_j) < \delta_0^2 L_j^2$. Let K_1 be the unique predecessor of K_{L_j} of generation k_R, that is,

$$K_{L_j} \subset K_1 := Q_1 \times (a - l(k_R), a].$$

Then we have

$$d(z_o, Q_1) \leq d(z_o, Q_{L_j}) \leq d(z_o, z_1) + d(z_1, Q_{L_j}) \leq \kappa R + \delta_1 L_j < \delta_1 R$$
$$\text{and } (a - l(k_R), a] \subset (-R^2, R^2)$$

since

$$l(k_R) + |t_1| + \frac{\tau L_j^2}{\eta^2} + L_j^2 \leq l(k_R) + |t_1| + \frac{16 L_j^2}{\eta^2} + L_j^2$$
$$\leq \delta_0^2 R^2 + \kappa^2 R^2 + \left(\frac{16}{\eta^2} + 1\right)\frac{\kappa^2}{16} R^2 = \left\{\delta_0^2 + \left(\frac{16}{\eta^2} + 17\right)\frac{\eta^4(1-\delta_0^2)}{64}\right\} R^2 < R^2.$$

Now, we apply Corollary 4.1.15 to u with K_1 to obtain

$$\left|\left\{u \geq \nu^j \frac{\tilde{M}_0}{2}\right\} \cap K_{L_j}\right| \leq \left|\left\{u \geq \nu^j \frac{\tilde{M}_0}{2}\right\} \cap K_1\right| \leq d\left(\nu^j \frac{\tilde{M}_0}{2}\right)^{-\epsilon} |K_1|. \tag{4.1.22}$$

On the other hand, we consider the function

$$w := \frac{\nu \tilde{M}_0 - u/\nu^{j-1}}{(\nu - 1)\tilde{M}_0},$$

which is nonnegative and satisfies

$$\mathscr{L}w = -\frac{f}{\nu^{j-1}(\nu-1)\tilde{M}_0} \quad \text{in} \quad K_{\frac{50}{\eta^3}L_j, \left(3+\frac{\tau}{\eta^2}\right)L_j^2}(z_1, t_1)$$

from the assumption. We also have $w(z_1, t_1) \le 1$ and

$$\frac{|f|}{\nu^{j-1}(\nu-1)\tilde{M}_0} \le \frac{|f|}{(\nu-1)\tilde{M}_0} = \frac{2(\tilde{M}_0 - 1/2)|f|}{\tilde{M}_0} \le 2|f|.$$

By using the volume comparison theorem with $L_j \le \frac{\kappa}{4}R < \frac{\eta R}{8}$ and $B_{\frac{11}{\eta}\frac{4}{\eta^2}\frac{\eta R}{8}}(z_o) \subset B_{\frac{50}{\eta^3}\frac{\eta R}{8}}(z_1)$, we get

$$\begin{aligned}
L_j^2\left(\fint_{K_{\frac{50}{\eta^3}L_j,\left(3+\frac{\tau}{\eta^2}\right)L_j^2}(z_1,t_1)} |2f|^{n+1}\right)^{\frac{1}{n+1}} &= \frac{2L_j^2}{|B_{\frac{50}{\eta^3}L_j}(z_1)|^{\frac{1}{n+1}}\left\{\left(3+\frac{\tau}{\eta^2}\right)L_j^2\right\}^{\frac{1}{n+1}}}\|f\|_{L^{n+1}} \\
&\le \frac{2(\eta R/8)^2}{|B_{\frac{50}{\eta^3}\cdot\frac{\eta R}{8}}(z_1)|^{\frac{1}{n+1}}\left\{\left(3+\frac{\tau}{\eta^2}\right)(\eta R/8)^2\right\}^{\frac{1}{n+1}}}\|f\|_{L^{n+1}} \\
&\le \frac{2(\eta R/8)^2}{|B_{\frac{11}{\eta}\frac{4}{\eta^2}\frac{\eta R}{8}}(z_o)|^{\frac{1}{n+1}}\left\{\left(3+\frac{\tau}{\eta^2}\right)(\eta R/8)^2\right\}^{\frac{1}{n+1}}}\|f\|_{L^{n+1}} \\
&\le \frac{2R^2}{|B_{\frac{11}{\eta}\frac{4}{\eta^2}R}(z_o)|^{\frac{1}{n+1}}\left\{\left(3+\frac{\tau}{\eta^2}\right)R^2\right\}^{\frac{1}{n+1}}}\|f\|_{L^{n+1}\left(B_{\frac{50}{\eta^3}R}(z_o)\times\left(-3R^2,\frac{\tau R^2}{\eta^2}\right]\right)} \\
&\le 2\left(\frac{50}{44}\right)^{\frac{n}{n+1}}\frac{\varepsilon_1}{4} \le \varepsilon_1.
\end{aligned}$$

Applying Corollary 4.1.15 to w in K_{L_j}, we deduce that $|\{w \ge \tilde{M}_0\} \cap K_{L_j}| \le d\tilde{M}_0^{-\epsilon}|K_{L_j}|$, i.e.,

$$\left|\left\{u \le \nu^j\frac{\tilde{M}_0}{2}\right\} \cap K_{L_j}\right| \le d\tilde{M}_0^{-\epsilon}|K_{L_j}|.$$

Putting together with (4.1.22), we obtain

$$|K_{L_j}| \le 2d2^{\epsilon}\nu^{-j\epsilon}\tilde{M}_0^{-\epsilon}|K_1|$$

since $d\tilde{M}_0^{-\epsilon} \le \frac{\kappa}{2} < 1/2$. From Theorem 4.1.10, there is a point $z_* \in Q_{L_j}$ such that $B_{c_1\delta_0^{k_j}}(z_*) \subset Q_{L_j} \subset Q_1 \subset \overline{B}_{c_2\delta_0^{k_R}}(z_*)$. Then we have

$$\begin{aligned}\left|B_{c_1\delta_0^{k_j}}(z_*)\right| \cdot c_1^2\delta_0^{2k_j} \le \left|B_{c_1\delta_0^{k_j}}(z_*)\right| \cdot l(k_j) &\le |K_{L_j}| \\ &\le 2d2^\epsilon \nu^{-j\epsilon}\tilde{M}_0^{-\epsilon}|K_1| = 2d2^\epsilon\nu^{-j\epsilon}\tilde{M}_0^{-\epsilon}|Q_1|\cdot l(k_R) \\ &< 2d2^\epsilon\nu^{-j\epsilon}\tilde{M}_0^{-\epsilon}|\overline{B}_{c_2\delta_0^{k_R}}(z_*)| \cdot c_2^2\delta_0^{2k_R} \\ &\le 2d2^\epsilon\nu^{-j\epsilon}\tilde{M}_0^{-\epsilon}\left(\frac{c_2\delta_0^{k_R}}{c_1\delta_0^{k_j}}\right)^n \left|B_{c_1\delta_0^{k_j}}(z_*)\right| c_2^2\delta_0^{2k_R}\end{aligned}$$

from the volume comparison theorem. This means

$$\delta_0^{k_j} < (2d2^\epsilon)^{\frac{1}{n+2}}\, \tilde{M}_0^{-\frac{\epsilon}{n+2}}\nu^{-\frac{j\epsilon}{n+2}}\frac{c_2}{c_1}\delta_0^{k_R}.$$

Since $c_2\delta_0^{k_R-1} < R \le c_2\delta_0^{k_R-2}$, we deduce that

$$\begin{aligned} L_j \le c_2\delta_0^{k_j-2} &\le \frac{c_2^2}{c_1\delta_0^2}(2d2^\epsilon)^{\frac{1}{n+2}}\, \tilde{M}_0^{-\frac{\epsilon}{n+2}}\nu^{-\frac{j\epsilon}{n+2}}\delta_0^{k_R} \\ &< \frac{c_2}{c_1\delta_0}(2d2^\epsilon)^{\frac{1}{n+2}}\, \tilde{M}_0^{-\frac{\epsilon}{n+2}}\nu^{-\frac{j\epsilon}{n+2}}R < \sigma\tilde{M}_0^{-\frac{\epsilon}{n+2}}\nu^{-\frac{j\epsilon}{n+2}}R = L_j,\end{aligned}$$

in contradiction to the definition of L_j. Therefore, (ii) is true. □

Thus we deduce the following lemma from Lemma 4.1.17.

Lemma 4.1.18. *Suppose that M satisfies the conditions* (4.1.2),(4.1.3). *Let $z_o \in M, R > 0$ and $\tau \in [3, 16]$. Let u be a nonnegative smooth function such that $\mathscr{L}u = f$ in $B_{\frac{50}{\eta^3}R}(z_o) \times \left(-3R^2, \frac{\tau R^2}{\eta^2}\right]$. Assume that*

$$\inf_{B_R(z_o)\times\left[\frac{2R^2}{\eta^2},\frac{\tau R^2}{\eta^2}\right]} u \le 1$$

and

$$R^2\left(\fint_{B_{\frac{50}{\eta^3}R}(z_o)\times\left(-3R^2,\frac{\tau R^2}{\eta^2}\right]} |f|^{n+1}\right)^{\frac{1}{n+1}} \le \varepsilon$$

for a uniform constant $0 < \varepsilon < 1$ *in Lemma 4.1.17. Then*

$$\sup_{B_{\frac{\kappa R}{2}}(z_o)\times\left(-\frac{\kappa^2R^2}{4},\frac{\kappa^2R^2}{4}\right)} u \le C,$$

where $C > 0$ *depends only on* n, λ, Λ *and* $a_{\mathscr{L}}$.

Proof. We take $j_o \in \mathbb{N}$ such that

$$\sum_{j=j_o}^{\infty} \frac{50}{\eta^3} L_j < \frac{\kappa R}{2} \quad \text{and} \quad \sum_{j=j_o}^{\infty}\left(3+\frac{16}{\eta^2}\right)L_j^2 < \frac{\kappa^2R^2}{4}.$$

We claim that $\sup_{B_{\frac{\kappa R}{2}}(z_o)\times\left(-\frac{\kappa^2R^2}{4},\frac{\kappa^2R^2}{4}\right)} u \le \nu^{j_o-1}\tilde{M}_0$ with $\tilde{M}_0 > 1$ as in Lemma 4.1.17. If it does not hold, then there is a point $(z_{j_o}, t_{j_o}) \in B_{\frac{\kappa R}{2}}(z_o) \times \left(-\frac{\kappa^2R^2}{4}, \frac{\kappa^2R^2}{4}\right)$ such that $u(z_{j_o}, t_{j_o}) > \nu^{j_o-1}\tilde{M}_0$. Applying Lemma 4.1.17 with $(z_1, t_1) = (z_{j_o}, t_{j_o})$, we can find a point $(z_{j_o+1}, t_{j_o+1}) \in K_{\frac{50}{\eta^3}L_j,\left(3+\frac{\tau}{\eta^2}\right)L_j^2}(z_{j_o}, t_{j_o})$ such that

$$u(z_{j_o+1}, t_{j_o+1}) \ge \nu^{j_o}\tilde{M}_0.$$

According to the choice of j_o, we have

$$d(z_o, z_{j_o+1}) \le d(z_o, z_{j_o}) + d(z_{j_o}, z_{j_o+1}) < \frac{\kappa R}{2} + \frac{\kappa R}{2} = \kappa R$$

and

$$|t_{j_o+1}| \le |t_{j_o}| + |t_{j_o} - t_{j_o+1}| < \frac{\kappa^2R^2}{4} + \frac{\kappa^2R^2}{4} < \kappa^2R^2.$$

Thus we iterate this argument to obtain a sequence of points (z_j, t_j) for $j \ge j_o$ satisfying

$$d(z_o, z_j) \le \kappa R, \; |t_j| \le \kappa^2R^2 \quad \text{and} \quad u(z_j, t_j) \ge \nu^{j-1}\tilde{M}_0,$$

since $d(z_o, z_j) \le d(z_o, z_{j_o}) + \sum_{i=j_o}^{\infty} d(z_i, z_{i+1}) \le \frac{\kappa R}{2} + \sum_{i=j_o}^{\infty}\frac{50}{\eta^3}L_i < \kappa R$ and $|t_i| \le |t_{j_o}| + \sum_{i=j_o}^{\infty}|t_i - t_{i+1}| \le \frac{\kappa^2R^2}{4} + \sum_{i=j_o}^{\infty}\left(3+\frac{\tau}{\eta^2}\right)L_i^2 < \kappa^2R^2$ for $j \ge j_o$. This contradicts to the continuity of u and therefore we conclude that

$$\sup_{B_{\frac{\kappa R}{2}}(z_o)\times\left(-\frac{\kappa^2R^2}{4},\frac{\kappa^2R^2}{4}\right)} u \le \nu^{j_o-1}\tilde{M}_0.$$

□

Now the Harnack inequality follows easily from Lemma 4.1.18 by using a standard covering argument and the volume comparison theorem.

Theorem 4.1.19 (Harnack Inequality)**.** *Suppose that M satisfies the conditions* (4.1.2),(4.1.3). *Let $z_o \in M$, and $R > 0$. Let u be a nonnegative smooth function in $K_{2R}(0, 4R^2) \subset M \times \mathbb{R}$. Then*

$$\sup_{K_R(z_o,2R^2)} u \le C\left\{ \inf_{K_R(z_o,4R^2)} u + R^2 \left(\fint_{K_{2R}(z_o,4R^2)} |\mathscr{L}u|^{n+1} \right)^{\frac{1}{n+1}} \right\}, \tag{4.1.23}$$

where $C > 0$ is a constant depending only on n, λ, Λ and $a_{\mathscr{L}}$.

Proof. According to Lemma 4.1.18, for $\tau \in [3, 16]$, a nonnegative smooth function v in $K_{\frac{50}{\eta^3}r, \left(3+\frac{\tau}{\eta^2}\right)r^2}\left(\bar{x}, \bar{t} + \frac{\tau r^2}{\eta^2}\right)$ satisfies

$$\sup_{K_{\frac{\kappa r}{2}}(\bar{x},\bar{t})} v \le C\left\{ \inf_{K_{\frac{\kappa r}{2}}(\bar{x},\bar{t}+\frac{\tau r^2}{\eta^2})} v + r^2 \left(\fint_{K_{\frac{50}{\eta^3}r, \left(3+\frac{\tau}{\eta^2}\right)r^2}\left(\bar{x},\bar{t}+\frac{\tau r^2}{\eta^2}\right)} |\mathscr{L}v|^{n+1} \right)^{\frac{1}{n+1}} \right\} \tag{4.1.24}$$

since $\frac{\kappa}{2} < 1$.

Now, let $(x, t) \in K_R(z_o, 2R^2) = B_R(z_o) \times (R^2, 2R^2]$ and $(y, s) \in K_R(z_o, 4R^2) = B_R(z_o) \times (3R^2, 4R^2]$. We show that

$$u(x, t) \le C\left\{ u(y, s) + \frac{R^2}{|K_{2R}(z_o, 4R^2)|^{\frac{1}{n+1}}} \|\mathscr{L}u\|_{L^{n+1}(K_{2R}(z_o,4R^2))} \right\}$$

for a uniform constant $C > 0$ depending only on n, λ, Λ and $a_{\mathscr{L}}$. We consider a piecewise C^1 path $\gamma : [0, l] \to M$, $\gamma(0) = x, \gamma(l) = y$, $l < 2R$, consisting of a minimal geodesic parametrized by arc length joining x and z_o, followed by a minimal geodesic parametrized by arc length joining z_o and y. We notice that $\gamma([0, l]) \subset B_R(z_o)$ and $d(\gamma(s_1), \gamma(s_2)) \le |s_1 - s_2|$.

We can select uniform constants $A > 0$ and $N \in \mathbb{N}$ such that

$$A := \max\left\{ \frac{64}{3\kappa\eta^2}, \frac{50}{\eta^3} \right\} \quad \text{and} \quad \frac{3}{16}\eta^2 A^2 \le N \le \frac{1}{3}\eta^2 A^2$$

since $\frac{16-9}{16\cdot 3}\eta^2 A^2 \ge \frac{7\cdot 4}{3^2\kappa} A > 1$. For $i = 0, 1, \cdots, N$, we define

$$(x_i, t_i) := \left(\gamma\left(i\frac{l}{N} \right), i\,\frac{s-t}{N} + t \right) \in B_R(z_o) \times [R^2, 4R^2].$$

Then we have $(x_0, t_0) = (x, t), (x_N, t_N) = (y, s)$ and for $i = 0, \cdots, N-1$,

$$d(x_{i+1}, x_i) \le \frac{l}{N} < \frac{2R}{N} \le \frac{64}{3\kappa\eta^2 A} \cdot \frac{\kappa R}{2A} \le \frac{\kappa}{2}\frac{R}{A},$$
$$\frac{3R^2}{\eta^2 A^2} \le \frac{R^2}{N} \le t_{i+1} - t_i = \frac{s-t}{N} \le \frac{3R^2}{N} \le \frac{16R^2}{\eta^2 A^2}.$$

We also have that $K_{\frac{50}{\eta^3}\frac{R}{A}, 3\frac{R^2}{A^2}+t_i-t_{i-1}}(x_i, t_i) \subset K_{2R}(z_o, 4R^2)$ for $i = 1, \cdots, N$ since $\frac{50}{\eta^3}\frac{R}{A} \le R$. We apply the estimate (4.1.24) with $r = \frac{R}{A}, \tau = (t_{i+1} - t_i)\frac{\eta^2 A^2}{R^2}$ and $(\bar{x}, \bar{t}) = (x_{i+1}, t_{i+1})$ for $i = 0, 1, \cdots, N-1$ and use the volume comparison theorem to have

$$\begin{aligned}
u(x_i, t_i) &\le C\left\{u(x_{i+1}, t_{i+1}) + \frac{(R/A)^2}{|K_{\frac{50}{\eta^3}\frac{R}{A}, 3\frac{R^2}{A^2}+t_i-t_{i-1}}(x_{i+1}, t_{i+1})|^{\frac{1}{n+1}}}\|\mathscr{L}u\|_{L^{n+1}(K_{2R}(z_o, 4R^2))}\right\} \\
&\le C\left\{u(x_{i+1}, t_{i+1}) + \frac{(R/A)^2}{|B_{\frac{50}{\eta^3}\frac{R}{A}}(x_{i+1}) \cdot \left(3 + \frac{3}{\eta^2}\right)\frac{R^2}{A^2}|^{\frac{1}{n+1}}}\|\mathscr{L}u\|_{L^{n+1}(K_{2R}(z_o, 4R^2))}\right\} \\
&\le C\left\{u(x_{i+1}, t_{i+1}) + \frac{R^2}{|B_{3R}(x_{i+1}) \cdot 4R^2|^{\frac{1}{n+1}}}\|\mathscr{L}u\|_{L^{n+1}(K_{2R}(z_o, 4R^2))}\right\},
\end{aligned}$$

where a uniform constant $C > 0$ may change from line to line. Since $B_{3R}(x_{i+1}) \supset B_{2R}(z_o)$, we deduce that

$$u(x_i, t_i) \le C\left\{u(x_{i+1}, t_{i+1}) + \frac{R^2}{|B_{2R}(z_o) \cdot 4R^2|^{\frac{1}{n+1}}}\|\mathscr{L}u\|_{L^{n+1}(K_{2R}(z_o, 4R^2))}\right\}.$$

Therefore, we conclude that

$$u(x, t) \le C\left\{u(y, s) + \frac{R^2}{|K_{2R}(z_o, 4R^2)|^{\frac{1}{n+1}}}\|\mathscr{L}u\|_{L^{n+1}(K_{2R}(z_o, 4R^2))}\right\}$$

for a uniform constant $C > 0$ since $N \in \mathbb{N}$ is uniform. □

4.1.5 Weak Harnack inequality

Arguing in a similar way as Theorem 4.1.19, Corollary 4.1.16 gives the following weak Harnack inequality.

Theorem 4.1.20 (Weak Harnack Inequality)**.** *Suppose that M satisfies the conditions* (4.1.2),(4.1.3)*. Let $z_o \in M$, and $R > 0$. Let u be a nonnegative smooth function such that $\mathscr{L}u \leq f$ in $K_{2R}(z_o, 4R^2)$. Then*

$$\left(\fint_{K_R(z_o,2R^2)} u^{p_o}\right)^{\frac{1}{p_o}} \leq C\left\{\inf_{K_R(z_o,4R^2)} u + R^2\left(\fint_{K_{2R}(z_o,4R^2)} |f^+|^{n+1}\right)^{\frac{1}{n+1}}\right\},$$

where $0 < p_o < 1$ and $C > 0$ depend only on n, λ, Λ and $a_{\mathscr{L}}$.

Proof. Let $\epsilon > 0$ be the constant in Corollary 4.1.15 and let $p_o := \frac{\epsilon}{2}$. We consider a parabolic decomposition of $M\times(0, 4R^2]$ according to Lemma 4.1.11. Let $k := k_{\frac{\kappa R}{A}}$ for the constant $A > 0$ in the proof of Theorem 4.1.19. Let $\left\{K^{k,\alpha} := Q^{k,\alpha} \times (t^{k,\alpha} - l(k), t^{k,\alpha}]\right\}_{\alpha\in J'_k}$ be a family of parabolic dyadic cubes intersecting $K_R(z_o, 2R^2)$. We note that $diam(Q^{k,\alpha}) \leq c_2\delta_0^k \leq \delta_0 \cdot \frac{\kappa R}{A}$ and $l(k) \leq \delta_0^2 \cdot \frac{\kappa^2 R^2}{A^2}$. Following the same argument as Corollary 4.1.16, we deduce that $|J'_k|$ is uniformly bounded and

$$\int_{K_R(z_o,2R^2)} u^{p_o} \leq |J'_k| \int_{K^{k,\alpha}} u^{p_o}$$

for some $K^{k,\alpha}$ with $\alpha \in J'_k$. Then we find $(x, t) \in K^{k,\alpha} \cap B_R(z_o) \times [R^2, (2 + \delta_0^2\frac{\kappa^2}{A^2})R^2]$ such that $K^{k,\alpha} \subset K_{\frac{\kappa R}{A}}(x, t)$ since $diam(Q^{k,\alpha}) \leq \delta_0 \cdot \frac{\kappa R}{A}$ and $l(k) \leq \delta_0^2 \cdot \frac{\kappa^2 R^2}{A^2}$. Since $d(z_o, x) \leq R$ and $B_{\frac{\kappa R}{A}}(x) \subset B_{(1+\frac{\kappa}{A})R}(z_o)$, we have

$$\frac{1}{|K_R(z_o, 2R^2)|}\int_{K_R(z_o,2R^2)} u^{p_o} \leq \frac{C_0}{|K_{\frac{\kappa R}{A}}(x, t)|}\int_{K_{\frac{\kappa R}{A}}(x,t)} u^{p_o} \tag{4.1.25}$$

for $C_0 := |J'_k|\left(1 + \frac{\kappa}{A}\right)^n \cdot \frac{\kappa^2}{A^2}$ by using the volume comparison theorem.

We set

$$\inf_{K_R(z_o,4R^2)} u =: u(y, s)$$

for some $(y, s) \in \overline{K_R(z_o, 4R^2)}$. As in the proof of Theorem 4.1.19 we take a piecewise geodesic path γ connecting x to y. Let $N \in \mathbb{N}$ be the constant in Theorem 4.1.19. For $i = 0, 1, \cdots, N$, we define

$$(x_i, t_i) := \left(\gamma\left(i\frac{l}{N}\right), i\frac{s - t}{N} + t\right) \in B_R(z_o) \times [R^2, 4R^2].$$

Then we have $(x_0, t_0) = (x, t), (x_N, t_N) = (y, s)$ and for $i = 0, \cdots, N-1$,

$$d(x_{i+1}, x_i) < \frac{\kappa}{2} \cdot \frac{R}{A} \quad \text{and} \quad \frac{3R^2}{\eta^2 A^2} \le t_{i+1} - t_i \le \frac{16R^2}{\eta^2 A^2}.$$

It is easy to check that for any $i = 0, 1, \cdots, N-1$, $B_{\frac{\kappa R}{A}}(x_i) \cap B_{\frac{\kappa R}{A}}(x_{i+1}) \supset B_{\frac{\kappa R}{2A}}(x_{i+1})$ and hence

$$\begin{aligned} \inf_{K_{\frac{\kappa R}{A}}(x_i, t_{i+1})} u \le \inf_{K_{\frac{\kappa R}{2A}}(x_{i+1}, t_{i+1})} u &\le \left\{ \frac{1}{|K_{\frac{\kappa R}{2A}}(x_{i+1}, t_{i+1})|} \int_{K_{\frac{\kappa R}{2A}}(x_{i+1}, t_{i+1})} u^{p_o} \right\}^{\frac{1}{p_o}} \\ &\le 2^{\frac{n+2}{p_o}} \left\{ \frac{1}{|K_{\frac{\kappa R}{A}}(x_{i+1}, t_{i+1})|} \int_{K_{\frac{\kappa R}{A}}(x_{i+1}, t_{i+1})} u^{p_o} \right\}^{\frac{1}{p_o}}. \end{aligned} \tag{4.1.26}$$

On the other hand, Corollary 4.1.16 says that for $i = 0, 1, \cdots, N-1$

$$\begin{aligned} &\left\{ \frac{1}{|K_{\frac{\kappa R}{A}}(x_i, t_i)|} \int_{K_{\frac{\kappa R}{A}}(x_i, t_i)} u^{p_o} \right\}^{1/p_o} \\ &\le C \left\{ \inf_{K_{\frac{\kappa R}{A}}(x_i, t_{i+1})} u + \frac{(R/A)^2}{\left| K_{\frac{50}{\eta^3} \cdot \frac{R}{A}, 3\frac{R^2}{A^2} + t_{i+1} - t_i}(x_i, t_{i+1}) \right|^{\frac{1}{n+1}}} \left\| f^+ \right\|_{L^{n+1}(K_{2R}(z_o, 4R^2))} \right\} \\ &\le C \left\{ \inf_{K_{\frac{\kappa R}{A}}(x_i, t_{i+1})} u + \frac{R^2}{\left| K_{2R}(z_o, 4R^2) \right|^{\frac{1}{n+1}}} \left\| f^+ \right\|_{L^{n+1}(K_{2R}(z_o, 4R^2))} \right\} \end{aligned}$$

by using the same argument as Theorem 4.1.19 with the volume comparison theorem. Combining with (4.1.26), we deduce

$$
\begin{aligned}
&\left\{\frac{1}{|K_{\frac{\kappa R}{A}}(x,t)|}\int_{K_{\frac{\kappa R}{A}}(x,t)} u^{p_o}\right\}^{1/p_o} \\
&\leq C\left\{\inf_{K_{\frac{\kappa R}{A}}(x_{N-1},t_N)} u + \frac{R^2}{\left|K_{2R}\left(z_o,4R^2\right)\right|^{\frac{1}{n+1}}}\left\|f^+\right\|_{L^{n+1}(K_{2R}(z_o,4R^2))}\right\} \\
&\leq C\left\{\inf_{K_{\frac{\kappa R}{2A}}(x_N,t_N)} u + \frac{R^2}{\left|K_{2R}\left(z_o,4R^2\right)\right|^{\frac{1}{n+1}}}\left\|f^+\right\|_{L^{n+1}(K_{2R}(z_o,4R^2))}\right\} \\
&\leq C\left\{u(y,s) + \frac{R^2}{\left|K_{2R}\left(z_o,4R^2\right)\right|^{\frac{1}{n+1}}}\left\|f^+\right\|_{L^{n+1}(K_{2R}(z_o,4R^2))}\right\},
\end{aligned}
$$

for a uniform constant $C > 0$ since $N \in \mathbb{N}$ is uniform. Therefore, the result follows from (4.1.25). □

4.2 Harnack inequality for nonlinear parabolic operators

The aim of this section is to prove Proposition 4.2.5, which is a main ingredient of a priori Harnack estimate. We begin with the definition of the contact set for the elliptic case from [62].

Definition 4.2.1. *Let Ω be a bounded open set in M and let $u \in C(\Omega)$. For a given $a > 0$ and a compact set $E \subset M$, the contact set associated with u of opening a with vertex set E is defined by*

$$
\mathcal{A}_a(E;\Omega;u) := \left\{x \in \Omega : \exists y \in E \ \ s.t.\ \inf_{\Omega}\left(u + \frac{a}{2}d_y^2\right) = u(x) + \frac{a}{2}d_y^2(x)\right\}.
$$

The following lemma is quoted from [62, Proof of Theorem 1.2] and [11, Proof of Lemma 4.1] (see also [17, Proposition 2.5] and [59, Chapter 14]).

Lemma 4.2.1. *Assume that*

$$
\mathrm{Ric} \geq -\kappa \quad \text{on } M, \quad \text{for } \kappa \geq 0.
$$

Let Ω be a bounded open set in M and E be a compact set in M. For $a > 0$ and a smooth function u on Ω, we define the map $\tilde{\phi} : \Omega \to M$ as

$$\tilde{\phi}(x) := \exp_x a^{-1}\nabla u(x).$$

Then, we have the following : Let $x \in \mathcal{A}_a(E; \Omega; u)$.

(a) If $y \in E$ satisfies

$$\inf_{\Omega}\left(u + \frac{a}{2}d_y^2\right) = u(x) + \frac{a}{2}d_y^2(x),$$

then $y = \tilde{\phi}(x) = \exp_x a^{-1}\nabla u(x), \quad x \notin \mathrm{Cut}(y), \quad$ *and* $\quad \frac{1}{a}\nabla u(x) = -d_y(x)\nabla d_y(x).$

(b)

$$\mathrm{Jac}\,\tilde{\phi}(x) \le \mathcal{S}^n\left(\sqrt{\frac{\kappa}{n}}\frac{|\nabla u|}{a}\right)\left\{\mathcal{H}\left(\sqrt{\frac{\kappa}{n}}\frac{|\nabla u|}{a}\right) + \frac{\triangle u}{na}\right\}^n, \tag{4.2.1}$$

where

$$\mathcal{H}(\tau) = \tau\coth(\tau), \qquad \mathcal{S}(\tau) = \sinh(\tau)/\tau, \qquad \tau \ge 0.$$

Now we define a parabolic version of the contact set which contains a point $(\bar{x}, \bar{t}) \in M \times \mathbb{R}$, where a concave paraboloid $-\frac{a}{2}d_y^2(x) + bt + C$ (for some $a, b > 0$ and C) touches u from below at $(\bar{x}, \bar{t})$ in a parabolic neighborhood of $(\bar{x}, \bar{t})$, i.e., in $K_r(\bar{x}, \bar{t})$ for some $r > 0$.

Definition 4.2.2. *Let Ω be a bounded open set in M and let $u \in C(\Omega \times (0, T])$ for $T > 0$. For given $a, b > 0$ and a compact set $E \subset M$, the parabolic contact set associated with u is defined by*

$$\mathcal{A}_{a,b}(E; \Omega \times (0, T]; u)$$
$$:= \left\{(x, t) \in \Omega \times (0, T] : \exists y \in E \ s.t. \inf_{\Omega\times(0,t]}\left(u(z,\tau) + \frac{a}{2}d_y^2(z) - b\tau\right) = u(x,t) + \frac{a}{2}d_y^2(x) - bt\right\}.$$

As in Section 4.1, for $u \in C^{2,1}(\Omega \times (0, T])$, we define the map $\phi : \Omega \times (0, T] \to M$ by

$$\phi(x, t) := \exp_x a^{-1}\nabla u(x, t),$$

and define the parabolic normal map $\Phi : \Omega \times (0, T] \to M \times \mathbb{R}$ by

$$\Phi(x, t) := \left(\phi(x, t), -\frac{1}{2}d^2\left(x, \phi(x, t)\right) - a^{-1}\left\{u(x, t) - bt\right\}\right).$$

Lemma 4.2.2. *Assume that*

$$\mathrm{Ric} \geq -\kappa \quad \textit{on } M, \quad \textit{for } \kappa \geq 0.$$

Let Ω *be a bounded open set in* M, *and let* u *be a smooth function on* $\Omega \times (0, T]$ *for* $T > 0$. *For any compact set* $E \subset M$, $a, b > 0$, *and* $0 < \tilde{\lambda} \leq 1$, *we have that if* $(x,t) \in \mathcal{A}_{a,b}(E; \Omega \times (0,T]; u)$, *then*

$$\mathrm{Jac}\,\Phi(x,t) \leq \frac{1}{(n+1)^{n+1}} \mathscr{S}^{n+1}\left(\sqrt{\frac{\kappa}{n}}\frac{|\nabla u|}{a}\right)\left[\left\{\frac{\tilde{\lambda}\triangle u - \partial_t u}{\tilde{\lambda} a} + \frac{b}{\tilde{\lambda} a} + n\mathscr{H}\left(\sqrt{\frac{\kappa}{n}}\frac{|\nabla u|}{a}\right)\right\}^+\right]^{n+1}, \tag{4.2.2}$$

where

$$\mathscr{H}(\tau) = \tau \coth(\tau), \qquad \mathscr{S}(\tau) = \sinh(\tau)/\tau, \qquad \tau \geq 0.$$

Proof. Let $(x,t) \in \mathcal{A}_{a,b} := \mathcal{A}_{a,b}(E; \Omega \times (0,T]; u) \subset \Omega \times (0,T]$. From the definition of the parabolic contact set, there exists a vertex $y \in E$ such that

$$\inf_{\Omega \times (0,t]} \left(u(z,\tau) + \frac{a}{2} d_y^2(z) - b\tau\right) = u(x,t) + \frac{a}{2} d_y^2(x) - bt.$$

According to Lemma 4.2.1, we have that

$$y = \phi(x,t) = \exp_x a^{-1}\nabla u(x,t), \quad x \notin \mathrm{Cut}(y), \quad \text{and} \quad \frac{1}{a}\nabla u(x,t) = -d_y(x)\nabla d_y(x)$$

since

$$\inf_{\Omega}\left(u(z,t) + \frac{a}{2} d_y^2(z)\right) = u(x,t) + \frac{a}{2} d_y^2(x).$$

We notice that $D^2\left(u + \frac{a}{2} d_y^2\right)(x,t) \geq 0$ and $\partial_t u(x,t) - b \leq 0$. Now we set

$$\tilde{\phi} := \phi(\cdot, t) \; : \Omega \ni z \;\mapsto\; \exp_z a^{-1}\nabla u(z,t) \in M$$

to obtain from Lemma 4.2.1 that

$$\mathrm{Jac}\,\tilde{\phi}(x) \leq \mathscr{S}^n\left(\sqrt{\frac{\kappa}{n}}\frac{|\nabla u|}{a}\right)\left\{\mathscr{H}\left(\sqrt{\frac{\kappa}{n}}\frac{|\nabla u|}{a}\right) + \frac{\triangle u}{na}\right\}^n (x,t). \tag{4.2.3}$$

By a simple calculation, we have that for $(\xi, \sigma) \in T_x M \times \mathbb{R}\backslash\{(0,0)\}$,

$$d\Phi(x,t)\cdot(\xi,\sigma) = \left(d\tilde{\phi}\cdot\xi + \sigma\frac{\partial\phi}{\partial t}, \;\; -\left\langle \nabla\left(d_x^2/2\right)(y),\, d\tilde{\phi}\cdot\xi + \sigma\frac{\partial\phi}{\partial t}\right\rangle_y - a^{-1}\sigma\left(\partial_t u - b\right)\right),$$

where $\frac{\partial\phi}{\partial t}(x,t) = \frac{d}{d\tau}\Big|_{\tau=0}\phi(x,t+\tau) \in T_yM$ and we used $\nabla\left(d_y^2/2\right)(x) = -a^{-1}\nabla u(x,t)$. To compute the Jacobian of Φ, we introduce an orthonormal basis $\{e_1,\cdots,e_n\}$ of T_xM and an orthonormal basis $\{\overline{e}_1,\cdots,\overline{e}_n\}$ of $T_yM = T_{\phi(x,t)}M$. By setting for $i,j = 1,\cdots,n$,

$$A_{ij} := \left\langle \overline{e}_i,\, d\tilde{\phi}\cdot e_j\right\rangle,\ b_i := \left\langle \overline{e}_i,\, \frac{\partial\phi}{\partial t}\right\rangle,\ \text{and}\ c_i := \left\langle \overline{e}_i,\, \nabla\left(d_x^2/2\right)(y)\right\rangle,$$

the Jacobian matrix of Φ at (x,t) is

$$\begin{pmatrix} A_{ij} & b_i \\ -c_kA_{kj} & -c_kb_k + a^{-1}(b-\partial_t u) \end{pmatrix}.$$

Using the row operations and (4.2.3), we deduce that

$$\begin{aligned}
\operatorname{Jac}\Phi(x,t) &= \left|\det\begin{pmatrix} A_{ij} & b_i \\ 0 & a^{-1}(b-\partial_t u)\end{pmatrix}\right| = a^{-1}(b-\partial_t u)\operatorname{Jac}\tilde{\phi}(x) \\
&\le a^{-1}(b-\partial_t u)\,\mathscr{S}^n\left(\sqrt{\frac{\kappa}{n}}\frac{|\nabla u|}{a}\right)\left\{\mathscr{H}\left(\sqrt{\frac{\kappa}{n}}\frac{|\nabla u|}{a}\right) + \frac{\triangle u}{na}\right\}^n(x,t),
\end{aligned}$$

where we note that $(b-\partial_t u)(x,t) \ge 0$ and $\operatorname{Jac}\tilde{\phi}(x) \ge 0$. According to the geometric and arithmetic means inequality, we conclude that

$$\begin{aligned}
\operatorname{Jac}\Phi(x,t) &\le \frac{1}{(n+1)^{n+1}}\left[n\mathscr{S}\left(\sqrt{\frac{\kappa}{n}}\frac{|\nabla u|}{a}\right)\left\{\mathscr{H}\left(\sqrt{\frac{\kappa}{n}}\frac{|\nabla u|}{a}\right) + \frac{\triangle u}{na}\right\} + \frac{b-\partial_t u}{a}\right]^{n+1} \\
&= \frac{1}{(n+1)^{n+1}}\left[\mathscr{S}\left(\sqrt{\frac{\kappa}{n}}\frac{|\nabla u|}{a}\right)\left\{\frac{\tilde{\lambda}\triangle u - \partial_t u + b}{\tilde{\lambda}a} + n\mathscr{H}\left(\sqrt{\frac{\kappa}{n}}\frac{|\nabla u|}{a}\right)\right\}\right. \\
&\qquad\left. + \left\{1 - \frac{1}{\tilde{\lambda}}\mathscr{S}\left(\sqrt{\frac{\kappa}{n}}\frac{|\nabla u|}{a}\right)\right\}\frac{b-\partial_t u}{a}\right]^{n+1} \\
&\le \frac{1}{(n+1)^{n+1}}\left[\mathscr{S}\left(\sqrt{\frac{\kappa}{n}}\frac{|\nabla u|}{a}\right)\left\{\frac{\tilde{\lambda}\triangle u - \partial_t u + b}{\tilde{\lambda}a} + n\mathscr{H}\left(\sqrt{\frac{\kappa}{n}}\frac{|\nabla u|}{a}\right)\right\}\right]^{n+1}
\end{aligned}$$

since $(b-\partial_t u)(x,t) \ge 0$ and $\mathscr{S}(\tau) = \sinh(\tau)/\tau \ge 1 \ge \tilde{\lambda}$ for all $\tau \ge 0$. □

Assuming the sectional curvature of M to be bounded from below, we have ABP-Krylov-Tso type estimate in the following lemma, which will play a key role to estimate sublevel sets of u in Proposition 4.2.5; see also Lemma 4.1.3 and Figure 4.1.

Lemma 4.2.3. *Assume that*

$$\mathrm{Sec} \geq -\kappa \quad \text{on } M, \quad \text{for } \kappa \geq 0.$$

Let $R_0 > 0$ *and* $0 < \eta < 1$. *For* $z_0 \in M$, *and* $0 < R \leq R_0$, *let* u *be a smooth function in* $K_{\alpha_1 R, \alpha_2 R^2}(z_0, 0) \subset M \times \mathbb{R}$ *such that*

$$u \geq 0 \quad \text{in} \quad K_{\alpha_1 R, \alpha_2 R^2}(z_0, 0) \backslash K_{\beta_1 R, \beta_2 R^2}(z_0, 0) \quad \text{and} \quad \inf_{K_{2R}(z_0,0)} u \leq 1, \tag{4.2.4}$$

where $\alpha_1 := \frac{11}{\eta}$, $\alpha_2 := 4 + \eta^2 + \frac{\eta^4}{4}$, $\beta_1 := \frac{9}{\eta}$, *and* $\beta_2 := 4 + \eta^2$. *Then we have*

$$|B_R(z_0)| \cdot R^2 \leq \int_{\{u \leq M_\eta\} \cap K_{\beta_1 R, \beta_2 R^2}(z_0,0)} \mathscr{S}^{n+1} \cdot \left[\left\{ \frac{R^2}{2\lambda} \left\{ \mathcal{M}^-(D^2 u) - \partial_t u \right\} + \frac{6}{\lambda \eta^2} + (n+1) \frac{\Lambda}{\lambda} \mathscr{H} \right\}^+ \right]^{n+1}, \tag{4.2.5}$$

where the constant $M_\eta > 0$ *depends only on* $\eta > 0$, *and*

$$\mathscr{S} := \mathscr{S}\left(2\alpha_1 \sqrt{\kappa} R_0\right), \quad \mathscr{H} := \mathscr{H}\left(2\alpha_1 \sqrt{\kappa} R_0\right)$$

for $\mathscr{S}(\tau) = \sinh(\tau)/\tau$, *and* $\mathscr{H}(\tau) = \tau \coth(\tau)$.

Proof. We consider the parabolic contact set

$$\mathcal{A}_{a,b}\left(\overline{B}_R(z_0); K_{\alpha_1 R, \alpha_2 R^2}(z_0, 0); u\right) \quad \text{for } a := \tfrac{2}{R^2} \text{ and } b := \tfrac{12}{\eta^2 R^2},$$

which will be denoted by $\mathcal{A}$ for simplicity. As in the proof of Lemma 4.1.3, for any $\overline{y} \in B_R(z_0)$, we define

$$w_{\overline{y}}(x,t) := \frac{1}{2} R^2 u(x,t) + \frac{1}{2} d_{\overline{y}}^2(x) - C_\eta t, \quad C_\eta := \frac{b}{a} = \frac{6}{\eta^2}.$$

From the assumption (4.2.4), we see that

$$\inf_{K_{2R}(z_0,0)} w_{\overline{y}} \leq \left(5 + \frac{24}{\eta^2}\right) R^2 =: A_\eta R^2,$$

and

$$w_{\overline{y}} \geq \left(6 + \frac{24}{\eta^2}\right) R^2 = (A_\eta + 1) R^2 \quad \text{on } K_{\alpha_1 R, \alpha_2 R^2}(z_0, 0) \backslash K_{\beta_1 R, \beta_2 R^2}(z_0, 0).$$

Then we deduce that for any $\left(\overline{y}, \overline{h}\right) \in B_R(z_0) \times \left(A_\eta R^2, (A_\eta + 1)R^2\right)$, there exists a time $\bar{t} \in \left(-\beta_2 R^2, 0\right)$ such that

$$\overline{h} = \inf_{B_{\alpha_1 R}(z_0) \times \left(-\alpha_2 R^2, \bar{t}\right]} w_{\overline{y}} = w_{\overline{y}}\left(\overline{x}, \bar{t}\right),$$

where the infimum is achieved at an interior point $\overline{x} \in B_{\beta_1 R}(z_0)$. This means that $(\overline{x}, \bar{t})$ is a parabolic contact point, i.e., $(\overline{x}, \bar{t}) \in \mathcal{A}$. According to Lemma 4.2.1, we observe that

$$\overline{y} = \exp_{\overline{x}}\left(\frac{1}{2} R^2 \, \nabla u\left(\overline{x}, \bar{t}\right)\right), \quad \text{and} \quad \overline{x} \notin \mathrm{Cut}(\overline{y}).$$

Now, we define the map $\phi : K_{\alpha_1 R, \alpha_2 R^2}(z_0, 0) \to M$ as

$$\phi(x,t) := \exp_x\left(\frac{1}{2} R^2 \nabla u(x,t)\right),$$

and the map $\Phi : K_{\alpha_1 R, \alpha_2 R^2}(z_0, 0) \to M \times \mathbb{R}$ as

$$\Phi(x,t) := \left(\phi(x,t), -\frac{1}{2} d^2\left(x, \phi(x,t)\right) - \frac{1}{2} R^2 u(x,t) + C_\eta t\right).$$

We also define

$$\tilde{\mathcal{A}} := \left\{ (x,t) \in K_{\beta_1 R, \beta_2 R^2}(z_0, 0) \; : \; \exists y \in B_R(z_0) \text{ s.t. } w_y(x,t) = \inf_{B_{\alpha_1 R}(z_0) \times \left(-\alpha_2 R^2, t\right]} w_y \le (A_\eta + 1) R^2 \right\}.$$

According to the argument above, we have proved that for any $(y, s) \in B_R(z_0) \times \left(-(A_\eta + 1)R^2, -A_\eta R^2\right)$, there exists a point $(x,t) \in \tilde{\mathcal{A}}$ such that $(y,s) = \Phi(x,t)$, that is,

$$B_R(z_0) \times \left(-(A_\eta + 1)R^2, -A_\eta R^2\right) \subset \Phi(\tilde{\mathcal{A}}).$$

Thus, the area formula provides

$$|B_R(z_0)| \cdot R^2 \le \int_{M \times \mathbb{R}} \mathcal{H}^0\left[\tilde{\mathcal{A}} \cap \Phi^{-1}(y,s)\right] dV(y,s) = \int_{\tilde{\mathcal{A}}} \mathrm{Jac}\, \Phi(x,t) dV(x,t). \tag{4.2.6}$$

We note that

$$\tilde{\mathcal{A}} \subset \mathcal{A} \cap K_{\beta_1 R, \beta_2 R^2}(z_0, 0) \cap \{u \le M_\eta\} \tag{4.2.7}$$

for $M_\eta := 2(A_\eta + 1)$ since $\frac{1}{2}R^2u(x,t) \le w_y(x,t) \le (A_\eta+1)R^2$ for $(x,t) \in \tilde{\mathcal{A}}$.

Next, we claim that for $(x,t) \in \mathcal{A}$,

$$\operatorname{Jac}\Phi(x,t) \le \frac{1}{(n+1)^{n+1}}\mathscr{S}^{n+1}\left(2\alpha_1\sqrt{\kappa}R_0\right)\left[\left\{\frac{nR^2}{2\lambda}\left(\frac{\lambda}{n}\triangle u - \partial_t u\right) + \frac{6n}{\lambda\eta^2} + n\mathscr{H}\left(2\alpha_1\sqrt{\kappa}R_0\right)\right\}^+\right]^{n+1}. \tag{4.2.8}$$

From Lemma 4.2.1, if $(x,t) \in \mathcal{A}$, then we have

$$\frac{R^2}{2}\nabla u(x,t) = -d_y(x)\nabla d_y(x) \quad \text{for} \quad y := \phi(x,t) \in \overline{B}_R(z_0); \quad x \notin \operatorname{Cut}(y),$$

and hence

$$\frac{R^2}{2}|\nabla u(x,t)| = d_y(x) \le d(y,z_0) + d(z_0,x) \le R + \alpha_1 R \le 2\alpha_1 R_0. \tag{4.2.9}$$

Using Lemma 4.2.2 (with $\tilde{\lambda} = \lambda/n$) and (4.2.9), we deduce that for $(x,t) \in \mathcal{A}$,

$$\begin{aligned}
&(n+1)\operatorname{Jac}\Phi(x,t)^{\frac{1}{n+1}}\\
&\quad\le \mathscr{S}\left(\sqrt{\frac{(n-1)\kappa}{n}}\frac{R^2|\nabla u|}{2}\right)\left\{\frac{nR^2}{2\lambda}\left(\frac{\lambda}{n}\triangle u - \partial_t u\right) + \frac{6n}{\lambda\eta^2} + n\mathscr{H}\left(\sqrt{\frac{(n-1)\kappa}{n}}\frac{R^2|\nabla u|}{2}\right)\right\}^+\\
&\quad\le \mathscr{S}\left(2\alpha_1\sqrt{\kappa}R_0\right)\left\{\frac{nR^2}{2\lambda}\left(\frac{\lambda}{n}\triangle u - \partial_t u\right) + \frac{6n}{\lambda\eta^2} + n\mathscr{H}\left(2\alpha_1\sqrt{\kappa}R_0\right)\right\}^+,
\end{aligned}$$

since $\mathscr{H}(\tau)$ and $\mathscr{S}(\tau)$ are nondecreasing for $\tau \ge 0$. This proves (4.2.8).

Lastly, we shall show that for $(x,t) \in \mathcal{A}$,

$$\frac{\lambda}{n}\triangle u \le \mathcal{M}^-(D^2u) + \frac{2n\Lambda}{R^2}\mathscr{H}\left(2\alpha_1\sqrt{\kappa}R_0\right). \tag{4.2.10}$$

Indeed, for $(x,t) \in \mathcal{A}$, we recall Lemma 4.2.1 again to see

$$D^2\left(u + \frac{1}{R^2}d_y^2\right)(x,t) \ge 0 \quad \text{for } y := \phi(x,t); \quad x \notin \operatorname{Cut}(y),$$

i.e., the Hessian of $R^2u + d_y^2$ at (x,t) is positive semidefinite. From Lemma 2.2.7 and (4.2.9), it follows that

$$D^2u(x,t) \ge -\frac{2}{R^2}D^2\left(\frac{1}{2}d_y^2\right)(x) \ge -\frac{2}{R^2}\mathscr{H}\left(\sqrt{\kappa}\,d_y(x)\right)g_x \ge -\frac{2}{R^2}\mathscr{H}\left(2\alpha_1\sqrt{\kappa}R_0\right)g_x.$$

Let μ_1 be the largest eigenvalue of $D^2u(x,t)$. If $\mu_1 \geq 0$, then we have

$$\begin{aligned}\mathcal{M}^-(D^2u(x,t)) &\geq \lambda\mu_1 - (n-1)\Lambda\frac{2}{R^2}\mathscr{H}\left(2\alpha_1\sqrt{\kappa}R_0\right)\\ &\geq \frac{\lambda}{n}\triangle u - n\Lambda\frac{2}{R^2}\mathscr{H}\left(2\alpha_1\sqrt{\kappa}R_0\right).\end{aligned}$$

If $\mu_1 < 0$, then we have

$$\mathcal{M}^-(D^2u(x,t)) = \Lambda\triangle u \geq -n\Lambda\frac{2}{R^2}\mathscr{H}\left(2\alpha_1\sqrt{\kappa}R_0\right) \geq \frac{\lambda}{n}\triangle u - n\Lambda\frac{2}{R^2}\mathscr{H}\left(2\alpha_1\sqrt{\kappa}R_0\right),$$

which proves (4.2.10) for $(x,t) \in \mathcal{A}$. Therefore, the ABP-Krylov-Tso type estimate (4.2.5) follows from (4.2.6), (4.2.7) (4.2.8) and (4.2.10). □

As Subsection 4.1.2, we construct the barrier as below. First, we fix some constants that will be used frequently (see Figure 4.1); for a given $0 < \eta < 1$,

$$\alpha_1 := \frac{11}{\eta},\ \alpha_2 := 4 + \eta^2 + \frac{\eta^4}{4},\ \beta_1 := \frac{9}{\eta}\quad\text{and}\quad \beta_2 := 4 + \eta^2.$$

Lemma 4.2.4. *Assume that*

$$\text{Sec} \geq -\kappa \quad \textit{on } M, \quad \textit{for } \kappa \geq 0.$$

Let $R_0 > 0$ and $0 < \eta < 1$. For $z_0 \in M$, and $0 < R \leq R_0$, there exists a continuous function $v_\eta(x,t)$ in $K_{\alpha_1 R,\alpha_2 R^2}(z_0,\beta_2 R^2)$, which is smooth in $(M\setminus \text{Cut}(z_0)) \cap K_{\alpha_1 R,\alpha_2 R^2}(z_0,\beta_2 R^2)$, such that

(a) $v_\eta(x,t) \geq 0$ in $K_{\alpha_1 R,\alpha_2 R^2}(z_0,\beta_2 R^2) \setminus K_{\beta_1 R,\beta_2 R^2}(z_0,\beta_2 R^2)$,

(b) $v_\eta(x,t) \leq 0$ in $K_{2R}(z_0,\beta_2 R^2)$,

(c) $R^2\left\{\mathcal{M}^+(D^2 v_\eta) - \partial_t v_\eta\right\} + \frac{12}{\eta^2} + 2(n+1)\Lambda\mathscr{H}\left(2\alpha_1\sqrt{\kappa}R_0\right) \leq 0$ a.e. in $K_{\beta_1 R,\beta_2 R^2}(z_0,\beta_2 R^2)\setminus K_{\frac{\eta}{2}R}(z_0,\frac{\eta^2}{4}R^2)$,

(d) $R^2\left\{\mathcal{M}^+(D^2 v_\eta) - \partial_t v_\eta\right\} \leq C_\eta$ a.e. in $K_{\beta_1 R,\beta_2 R^2}(z_0,\beta_2 R^2)$,

(e) $v_\eta(x,t) \geq -C_\eta$ in $K_{\alpha_1 R,\alpha_2 R^2}(z_0,\beta_2 R^2)$,

where $\mathscr{H}(\tau) = \tau\coth(\tau)$, and the constant $C_\eta > 0$ depends only on $\eta, n, \lambda, \Lambda, \sqrt{\kappa}R_0$ (independent of R and z_0).

Proof. As in Lemma 4.1.4, we consider

$$h(s,t) := -Ae^{-mt}\left(1-\frac{s}{\beta_1^2}\right)^l \frac{1}{(4\pi t)^{n/2}}\exp\left(-\alpha\frac{s}{t}\right) \quad \text{for } t>0,$$

and define

$$\psi_\eta(s,t) := h(s,t) + \widetilde{C}t \quad \text{in } [0,\beta_1^2]\times[0,\beta_2]\backslash[0,\tfrac{\eta^2}{4}]\times[0,\tfrac{\eta^2}{4}],$$

where $\widetilde{C} := 12/\eta^2 + 2(n+1)\Lambda\mathscr{H}\left(2\alpha_1\sqrt{\kappa}R_0\right)$, and the positive constants A, m, l, α (depending only on $\eta, n, \lambda, \Lambda, \sqrt{\kappa}R_0$) will be chosen later. Extending ψ_η smoothly in $[0,\alpha_1^2]\times[-\frac{\eta^4}{4},\beta_2]$ to satisfy (a) and (e), we define

$$v_\eta(x,t) := \psi_\eta\left(\frac{d_{z_0}^2(x)}{R^2}, \frac{t}{R^2}\right) \quad \text{for} \quad (x,t)\in K_{\alpha_1 R,\,\alpha_2 R^2}(z_0,\beta_2 R^2),$$

where d_{z_0} is the distance function to z_0. We may assume that $\psi_\eta(s,t)$ is nondecreasing with respect to s in $[0,\alpha_1^2]\times[-\frac{\eta^4}{4},\beta_2]$.

We recall that

$$\left\langle D^2\left(d_{z_0}^2/2\right)(x)\cdot\xi,\xi\right\rangle = \left\langle d_{z_0}D^2 d_{z_0}(x)\cdot\xi,\xi\right\rangle + \langle\nabla d_{z_0}(x),\xi\rangle^2, \quad \forall\xi\in T_xM, \quad x\notin \mathrm{Cut}(z_0),$$

and

$$\mathcal{M}^+\left(D^2\left(d_{z_0}^2/2\right)(x)\right) \le n\Lambda\mathscr{H}\left(\sqrt{\kappa}d_{z_0}(x)\right) \le n\Lambda\mathscr{H}\left(\alpha_1\sqrt{\kappa}R_0\right), \quad \forall x\in B_{\beta_1 R}(z_0)\backslash\,\mathrm{Cut}(z_0),$$

from Lemma 2.2.7. Following the proof of Lemma 4.1.4, and using Lemma 2.1.1 (a), we can select positive constants A, m, l, α, depending only on $\eta, n, \lambda, \Lambda, \sqrt{\kappa}R_0$, such that (b), (c), and (d) hold. □

The following proposition is obtained by applying Lemma 4.2.3 to $u+v_\eta$ with v_η, constructed in Lemma 4.2.4 and translated in time, due to Lemma 4.1.5 .

Proposition 4.2.5. *Assume that*

$$\mathrm{Sec} \ge -\kappa \quad \textit{on } M, \quad \textit{for } \kappa\ge 0,$$

and that F satisfies (F1). Let $0<\eta<1$ and $K_{\alpha_1 R,\,\alpha_2 R^2}(z_0,4R^2)\subset K_{R_0}(x_0,t_0)\subset M\times\mathbb{R}$. Let u be a smooth function on $K_{\alpha_1 R,\,\alpha_2 R^2}(z_0,4R^2)$ such that

$$F(D^2u)-\partial_t u \le f \quad \textit{in} \quad K_{\beta_1 R,\,\beta_2 R^2}(z_0,4R^2),$$

$$u \geq 0 \quad in \quad K_{\alpha_1 R, \alpha_2 R^2}(z_0, 4R^2) \backslash K_{\beta_1 R, \beta_2 R^2}(z_0, 4R^2),$$

and

$$\inf_{K_{2R}(z_0, 4R^2)} u \leq 1.$$

Then, there exist uniform constants $M_\eta > 1, 0 < \mu_\eta < 1$*, and* $0 < \epsilon_\eta < 1$ *such that*

$$\frac{\left|\left\{u \leq M_\eta\right\} \cap K_{\eta R}(z_0, 0)\right|}{\left|K_{\beta_1 R, \beta_2 R^2}(z_0, 4R^2)\right|} \geq \mu_\eta,$$

provided

$$\left(\fint_{K_{\beta_1 R, \beta_2 R^2}(z_0, 4R^2)} \left|\beta_1^2 R^2 f^+\right|^{n\theta+1} \right)^{\frac{1}{n\theta+1}} \leq \epsilon_\eta; \quad f^+ := \max(f, 0),$$

where $\theta := 1 + \log_2 \cosh(4\sqrt{\kappa} R_0)$*, and* $M_\eta > 0$*,* $0 < \mu_\eta, \epsilon_\eta < 1$ *depend only on* $\eta, n, \lambda, \Lambda$ *and* $\sqrt{\kappa} R_0$*.*

Proof. Let v_η be the barrier function as in Lemma 4.2.4 after translation in time (by $-\eta^2 R^2$) and let $\{w_k\}_{k=1}^\infty$ be a sequence of smooth functions approximating v_η from Lemma 4.1.5. We notice that $u + v_\eta \geq 0$ in $K_{\alpha_1 R, \alpha_2 R^2}(z_0, 4R^2) \backslash K_{\beta_1 R, \beta_2 R^2}(z_0, 4R^2)$ and $\inf_{K_{2R}(z_0, 4R^2)} (u + v_\eta) \leq 1$. We can apply Lemma 4.2.3 to $u + w_k$ after a slight modification as in the proof of Lemma 4.1.6, and use the dominated convergence theorem to let k go to $+\infty$ due to Lemma 4.1.5. Thus we obtain

$$|B_R(z_0)| \cdot R^2 \leq C_1 \int_{\{u+v_\eta \leq M_\eta\} \cap K_{\beta_1 R, \beta_2 R^2}(z_0, 4R^2)} \left[\left\{ R^2 \left\{ \mathcal{M}^-(D^2 u + D^2 v_\eta) - \partial_t (u + v_\eta) \right\} + C_2 \right\}^+ \right]^{n+1},$$

where $C_1 := \mathscr{S}^{n+1}\left(2\sqrt{\kappa} R_0\right) / (2\lambda)^{n+1}$, and $C_2 := 12/\eta^2 + 2(n+1)\Lambda \mathscr{H}\left(2\sqrt{\kappa} R_0\right)$. Using Lemma 2.1.1, (F1) and the properties $(c), (d)$ of v_η in Lemma 4.2.4, we have

$$\begin{aligned}
|B_R(z_0)| \cdot R^2 &\leq C_1 \int_{E_1 \cup E_2} \left[\left\{ R^2 \left\{ \mathcal{M}^-(D^2 u) - \partial_t u \right\} + R^2 \left\{ \mathcal{M}^+(D^2 v_\eta) - \partial_t v_\eta \right\} + C_2 \right\}^+ \right]^{n+1} \\
&\leq C_1 \int_{E_1 \cup E_2} \left[\left\{ R^2 \left\{ F(D^2 u) - \partial_t u \right\} + R^2 \left\{ \mathcal{M}^+(D^2 v_\eta) - \partial_t v_\eta \right\} + C_2 \right\}^+ \right]^{n+1} \\
&\leq C_1 \int_{K_{\beta_1 R, \beta_2 R^2}(z_0, 4R^2)} \left| R^2 f^+ + (C_\eta + C_2) \chi_{E_2} \right|^{n+1},
\end{aligned}$$

where $E_1 := \{u + v_\eta \leq M_\eta\} \cap \left(K_{\beta_1 R,\beta_2 R^2}(z_0, 4R^2)\backslash K_{\eta R}(z_0, 0)\right)$ and $E_2 := \{u + v_\eta \leq M_\eta\} \cap K_{\eta R}(z_0, 0)$. Then, it follows that

$$\begin{aligned}\frac{|B_R(z_0)| \cdot R^2}{\left|K_{\beta_1 R,\beta_2 R^2}(z_0, 4R^2)\right|} &\leq C_3 \fint_{K_{\beta_1 R,\beta_2 R^2}(z_0,4R^2)} \left|\beta_1^2 R^2 f^+ + \chi_{E_2}\right|^{n+1} \\ &\leq C_3 \left(\fint_{K_{\beta_1 R,\beta_2 R^2}(z_0,4R^2)} \left|\beta_1^2 R^2 f^+\right|^{n\theta+1}\right)^{\frac{n+1}{n\theta+1}} + C_3 \frac{|E_2|^{\frac{n+1}{n\theta+1}}}{\left|K_{\beta_1 R,\beta_2 R^2}(z_0, 4R^2)\right|^{\frac{n+1}{n\theta+1}}}\end{aligned}$$

for $\theta := 1 + \log_2 \cosh(4\sqrt{\kappa}R_0) \geq 1$, where a uniform constant $C_3 > 0$ depending only on $\eta, n, \lambda, \Lambda$ and $\sqrt{\kappa}R_0$ may change from line to line. Therefore, Bishop-Gromov's Theorem 2.2.4 implies that

$$\begin{aligned}\frac{|E_2|^{\frac{n+1}{n\theta+1}}}{\left|K_{\beta_1 R,\beta_2 R^2}(z_0, 4R^2)\right|^{\frac{n+1}{n\theta+1}}} + \left(\fint_{K_{\beta_1 R,\beta_2 R^2}(z_0,4R^2)} \left|\beta_1^2 R^2 f^+\right|^{n\theta+1}\right)^{\frac{n+1}{n\theta+1}} &\geq \frac{1}{C_3} \frac{|B_R(z_0)| \cdot R^2}{\left|K_{\beta_1 R,\beta_2 R^2}(z_0, 4R^2)\right|} \\ &\geq \frac{1}{C_3} \frac{1}{\mathcal{D}} \left(\frac{1}{\beta_1}\right)^{\log_2 \mathcal{D}} \frac{1}{\beta_2} =: 2\mu_\eta^{\frac{n+1}{n\theta+1}}\end{aligned}$$

for $\mathcal{D} := 2^n \cosh^{n-1}(2\sqrt{\kappa}R_0)$. By selecting $\epsilon_\eta := \mu_\eta^{\frac{1}{n\theta+1}}$, we conclude that

$$\mu_\eta \leq \frac{\left|\left\{u + v_\eta \leq M_\eta\right\} \cap K_{\eta R}(z_0, 0)\right|}{\left|K_{\beta_1 R,\beta_2 R^2}(z_0, 4R^2)\right|} \leq \frac{\left|\left\{u \leq \tilde{M}_\eta\right\} \cap K_{\eta R}(z_0, 0)\right|}{\left|K_{\beta_1 R,\beta_2 R^2}(z_0, 4R^2)\right|}$$

for $\tilde{M}_\eta := M_\eta + C_\eta$ depending only on $\eta, n, \lambda, \Lambda$ and $\sqrt{\kappa}R_0$ since $v_\eta \geq -C_\eta$ in $K_{\beta_1 R,\beta_2 R^2}(z_0, 4R^2)$ from Lemma 4.2.4. □

Therefore, we have the following Harnack inequality.

Theorem 4.2.6 (Harnack inequality)**.** *Assume that M has sectional curvature bounded from below by $-\kappa$ for $\kappa \geq 0$, i.e.,* Sec $\geq -\kappa$ *on M, and F satisfies (F1). Let u be a nonnegative smooth function in $K_{2R}(x_0, 4R^2) \subset M \times \mathbb{R}$. Then we have*

$$\sup_{K_R(x_0,2R^2)} u \leq C \left\{\inf_{K_R(x_0,4R^2)} u + R^2 \left(\fint_{K_{2R}(x_0,4R^2)} \left|F(D^2 u) - \partial_t u\right|^{n\theta+1}\right)^{\frac{1}{n\theta+1}}\right\},$$

where $\theta := 1 + \log_2 \cosh(8\sqrt{\kappa}R)$ and $C > 0$ is a uniform constant depending only on n, λ, Λ and $\sqrt{\kappa}R$.

Theorem 4.2.7 (Weak Harnack inequality)**.** *Assume that* Sec $\geq -\kappa$ *on* M *for* $\kappa \geq 0$, *and* F *satisfies (F1). Let* u *be a nonnegative function such that* $F(D^2u)-\partial_t u \leq f$ *in* $K_{2R}(x_0, 4R^2)$. *Then we have*

$$\left(\fint_{K_R(x_0,2R^2)} u^p\right)^{\frac{1}{p}} \leq C\left\{\inf_{K_R(x_0,4R^2)} u + R^2\left(\fint_{K_{2R}(x_0,4R^2)} |f^+|^{n\theta+1}\right)^{\frac{1}{n\theta+1}}\right\}; \quad f^+ := \max(f, 0),$$

where $\theta := 1 + \log_2 \cosh(8\sqrt{\kappa}R)$, *and the positive constants* $p \in (0, 1)$ *and* C *are uniform depending only on* n, λ, Λ, *and* $\sqrt{\kappa}R$.

Sketch of proof of Theorems 4.2.6 and 4.2.7

Theorems 4.2.6 and 4.2.7 follow from Proposition 4.2.5 and a standard covering argument using Bishop and Gromov's Theorem 2.2.4. In fact, we follow Section 4.1 to prove a decay estimate for the distribution function of a supersolution u to $F(D^2u) - \partial_t u = f$ in $K_{2R}(x_0, 4R^2)$ by using Proposition 4.2.5 and a parabolic version of the Calderón-Zygmund decomposition Lemma 4.1.11. We note that M has local doubling property Bishop and Gromov's Theorem 2.2.4 since Sec $\geq -\kappa$. Then, the weak Harnack inequality in Theorem 4.2.7 easily follows. To complete the proof of Theorem 4.2.6, we apply Proposition 4.2.5, and obtain the same decay estimate for $w := C_1 - C_2 u$ (for $C_1, C_2 > 0$), which satisfies

$$\mathcal{M}^-(D^2w) - \partial_t w = -C_2\left\{\mathcal{M}^+(D^2u) - \partial_t u\right\} \leq -C_1\left\{F(D^2u) - \partial_t u\right\} = -C_1 f.$$

□

4.3 Harnack inequality for viscosity solutions

4.3.1 Sup- and inf-convolution

In this subsection, we study the sup- and inf- convolutions introduced by Jensen[32] (see also [13, Chapter 5]) to regularize continuous viscosity solutions. Let $\Omega \subset M$ be a bounded open set, and let u be a continuous function on $\overline{\Omega} \times [T_0, T_2]$ for $T_2 > T_0$. For $\varepsilon > 0$, we define the inf-convolution of u (with respect to $\Omega\times(T_0, T_2]$), denoted by u_ε, as follows: for $(x_0, t_0) \in \overline{\Omega} \times [T_0, T_2]$,

$$u_\varepsilon(x_0, t_0) := \inf_{(y,s)\in\overline{\Omega}\times[T_0,T_2]} \left\{u(y, s) + \frac{1}{2\varepsilon}\left\{d^2(y, x_0) + |s - t_0|^2\right\}\right\}. \tag{4.3.1}$$

Lemma 4.3.1. *For $u \in C\left(\overline{\Omega} \times [T_0, T_2]\right)$, let u_ε be the inf-convolution of u with respect to $\Omega \times (T_0, T_2]$. Let $(x_0, t_0) \in \overline{\Omega} \times [T_0, T_2]$.*

(a) If $0 < \varepsilon < \varepsilon'$, then $u_{\varepsilon'}(x_0, t_0) \le u_\varepsilon(x_0, t_0) \le u(x_0, t_0)$.

(b) There exists $(y_0, s_0) \in \overline{\Omega} \times [T_0, T_2]$ such that $u_\varepsilon(x_0, t_0) = u(y_0, s_0) + \frac{1}{2\varepsilon}\left\{d^2(y_0, x_0) + |s_0 - t_0|^2\right\}$.

(c) $d^2(y_0, x_0) + |s_0 - t_0|^2 \le 2\varepsilon|u(x_0, t_0) - u(y_0, s_0)| \le 4\varepsilon\|u\|_{L^\infty(\Omega \times (T_0, T_2])}$.

(d) $u_\varepsilon \uparrow u$ uniformly in $\overline{\Omega} \times [T_0, T_2]$.

(e) u_ε is Lipschitz continuous in $\overline{\Omega} \times [T_0, T_2]$: for $(x_0, t_0), (x_1, t_1) \in \overline{\Omega} \times [T_0, T_2]$,

$$|u_\varepsilon(x_0, t_0) - u_\varepsilon(x_1, t_1)| \le \frac{3}{2\varepsilon} \operatorname{diam}(\Omega) d(x_0, x_1) + \frac{3}{2\varepsilon}(T_2 - T_0)|t_0 - t_1|. \quad (4.3.2)$$

Proof. From the definition of u_ε, (a) and (b) are obvious. From (a) and (b), it follows that

$$\frac{1}{2\varepsilon}\left\{d^2(y_0, x_0) + |s_0 - t_0|^2\right\} = u_\varepsilon(x_0, t_0) - u(y_0, s_0) \le u(x_0, t_0) - u(y_0, s_0),$$

proving (c). To show (d), we observe that

$$0 \le u(x_0, t_0) - u_\varepsilon(x_0, t_0) \le u(x_0, t_0) - u(y_0, s_0).$$

We use (c) and the uniform continuity of u on $\overline{\Omega} \times [T_0, T_2]$ to deduce that u_ε converges to u uniformly on $\overline{\Omega} \times [T_0, T_2]$.

Now we prove (e). For $(y, s) \in \overline{\Omega} \times [T_0, T_2]$, we have

$$\begin{aligned}
u_\varepsilon(x_0, t_0) &\le u(y, s) + \frac{1}{2\varepsilon}\left\{d^2(y, x_0) + |s - t_0|^2\right\} \\
&\le u(y, s) + \frac{1}{2\varepsilon}\left\{(d(y, x_1) + d(x_1, x_0))^2 + (|s - t_1| + |t_1 - t_0|)^2\right\} \\
&= u(y, s) + \frac{1}{2\varepsilon}\left\{d^2(y, x_1) + d^2(x_0, x_1) + 2d(y, x_1)d(x_0, x_1) + (|s - t_1| + |t_0 - t_1|)^2\right\} \\
&\le u(y, s) + \frac{1}{2\varepsilon}\left\{d^2(y, x_1) + |s - t_1|^2\right\} + \frac{3}{2\varepsilon} \operatorname{diam}(\Omega) d(x_0, x_1) + \frac{3}{2\varepsilon}(T_2 - T_0)|t_0 - t_1|.
\end{aligned}$$

Taking the infimum of the right hand side, we conclude (4.3.2), that is, u_ε is Lipschitz continuous on $\overline{\Omega} \times [T_0, T_2]$. □

Now, we show the semi-concavity of the inf-convolution, and hence the inf-convolution is twice differentiable almost everywhere in the sense of Aleksandrov and Bangert's Theorem 2.2.8.

Lemma 4.3.2. *Assume that*

$$\mathrm{Sec} \geq -\kappa \quad \text{on } M, \quad \text{for } \kappa \geq 0$$

For $u \in C\left(\overline{\Omega} \times [T_0, T_2]\right)$, let u_ε be the inf-convolution of u with respect to $\Omega \times (T_0, T_2]$, where $\Omega \subset M$ is a bounded open set, and $T_0 < T_2$.

(a) u_ε is semi-concave in $\Omega \times (T_0, T_2)$. Moreover, for almost every $(x, t) \in \Omega \times (T_0, T_2)$, u_ε is differentiable at (x, t), and there exists the Hessian $D^2 u_\varepsilon(x, t)$ (in the sense of Aleksandrov-Bangert's Theorem 2.2.8) such that

$$u_\varepsilon(\exp_x \xi, t + \sigma) = u_\varepsilon(x, t) + \langle \nabla u_\varepsilon(x, t), \xi \rangle + \sigma \partial_t u_\varepsilon(x, t) + \frac{1}{2}\left\langle D^2 u_\varepsilon(x, t) \cdot \xi, \xi \right\rangle + o\left(|\xi|^2 + |\sigma|\right) \tag{4.3.3}$$

as $(\xi, \sigma) \in T_x M \times \mathbb{R} \to (0, 0)$.

(b) $D^2 u_\varepsilon(x, t) \leq \frac{1}{\varepsilon} \sqrt{\kappa}\, \mathrm{diam}(\Omega) \coth\left(\sqrt{\kappa}\, \mathrm{diam}(\Omega)\right) g_x$ a.e. in $\Omega \times (T_0, T_2)$.

(c) Let $H \times (T_1, T_2]$ be a subset such that $\overline{H} \times [T_1, T_2] \subset \Omega \times (T_0, T_2]$, where H is open, and $T_0 < T_1 < T_2$. Then, there exist a smooth function φ on $M \times (-\infty, T_2]$ satisfying

$$0 \leq \varphi \leq 1 \text{ on } M \times (-\infty, T_2], \quad \varphi \equiv 1 \text{ in } \overline{H} \times [T_1, T_2] \quad \text{and} \quad \mathrm{supp}\, \varphi \subset \Omega \times (T_0, T_2],$$

and a sequence $\{w_k\}_{k=1}^\infty$ of smooth functions on $M \times (-\infty, T_2]$ such that

$$\begin{cases} w_k \to \varphi u_\varepsilon & \text{uniformly in } M \times (-\infty, T_2] \text{ as } k \to +\infty, \\ |\nabla w_k| + |\partial_t w_k| \leq C & \text{in } M \times (-\infty, T_2], \\ \partial_t w_k \to \partial_t u_\varepsilon & \text{a.e. in } H \times (T_1, T_2) \text{ as } k \to +\infty, \\ D^2 w_k \leq C g & \text{in } M \times (-\infty, T_2], \\ D^2 w_k \to D^2 u_\varepsilon & \text{a.e. in } H \times (T_1, T_2) \text{ as } k \to +\infty, \end{cases}$$

where the constant $C > 0$ is independent of k.

Proof. To prove semi-concavity of u_ε in $\Omega\times(T_0,T_2)$, we fix $(x_0,t_0)\in\Omega\times(T_0,T_2)$, and find $(y_0,s_0)\in\overline{\Omega}\times[T_0,T_2]$ satisfying

$$u_\varepsilon(x_0,t_0)=u(y_0,s_0)+\frac{1}{2\varepsilon}\left\{d^2(y_0,x_0)+|s_0-t_0|^2\right\}.$$

For any $\xi\in T_{x_0}M$ with $|\xi|=1$, and for small $r\in\mathbb{R}$, it follows from the definition of the inf-convolution u_ε that

$$\begin{aligned}
&u_\varepsilon\left(\exp_{x_0} r\xi,\ t_0+r\right)+u_\varepsilon\left(\exp_{x_0}-r\xi,\ t_0-r\right)-2u_\varepsilon(x_0,t_0)\\
&\quad\le u(y_0,s_0)+\frac{1}{2\varepsilon}\left\{d^2\left(y_0,\exp_{x_0} r\xi\right)+|s_0-(t_0+r)|^2\right\}\\
&\qquad+u(y_0,s_0)+\frac{1}{2\varepsilon}\left\{d^2\left(y_0,\exp_{x_0}-r\xi\right)+|s_0-(t_0-r)|^2\right\}-2u_\varepsilon(x_0,t_0)\\
&\quad\le\frac{1}{2\varepsilon}\left\{d_{y_0}^2\left(\exp_{x_0} r\xi\right)+d_{y_0}^2\left(\exp_{x_0}-r\xi\right)-2d_{y_0}^2(x_0)\right\}+\frac{1}{\varepsilon}r^2.
\end{aligned}$$

Then, we use Lemma 2.2.7 to obtain that for any $\xi\in T_{x_0}M$ with $|\xi|=1$,

$$\begin{aligned}
&\limsup_{r\to0}\frac{u_\varepsilon\left(\exp_{x_0} r\xi,\ t_0+r\right)+u_\varepsilon\left(\exp_{x_0}-r\xi,\ t_0-r\right)-2u_\varepsilon(x_0,t_0)}{r^2}\\
&\le\limsup_{r\to0}\frac{1}{2\varepsilon}\frac{d_{y_0}^2\left(\exp_{x_0} r\xi\right)+d_{y_0}^2\left(\exp_{x_0}-r\xi\right)-2d_{y_0}^2(x_0)}{r^2}+\frac{1}{\varepsilon}\\
&\le\frac{1}{\varepsilon}\sqrt{\kappa}d_{y_0}(x_0)\coth\left(\sqrt{\kappa}d_{y_0}(x_0)\right)+\frac{1}{\varepsilon}\\
&\le\frac{1}{\varepsilon}\sqrt{\kappa}\operatorname{diam}(\Omega)\coth\left(\sqrt{\kappa}\operatorname{diam}(\Omega)\right)+\frac{1}{\varepsilon},
\end{aligned}\tag{4.3.4}$$

where we note that $\tau\coth(\tau)$ is nondecreasing with respect to $\tau\ge0$. We recall that u_ε is Lipschitz continuous on $\overline{\Omega}\times[T_0,T_2]$ according to Lemma 4.3.1. Since $(x_0,t_0)\in\Omega\times(T_0,T_2)$ is arbitrary, (4.3.4) and Lemma 2.2.6 imply that u_ε is semi-concave on $\Omega\times(T_0,T_2)$. Thus, u_ε admits the Hessian almost everywhere in $\Omega\times(T_0,T_2)$ satisfying (4.3.3) from Aleksandrov and Bangert's Theorem 2.2.8. The upper bound of the Hessian in (b) follows from (4.3.3) and (4.3.4).

We use a standard mollification and a partition of unity to approximate φu_ε by a sequence $\{w_k\}_{k=1}^\infty$ of smooth functions in (c), where a mollifier is supported in $(-\delta,0]$ with respect to time (for small $\delta>0$), not in $(-\delta,\delta)$. By using Lipschitz continuity of u_ε on $\overline{\Omega}\times[T_0,T_2]$ and semi-concavity on $\Omega\times(T_0,T_2)$, it is not difficult to prove the properties of w_k. For the details, we refer to the proof of Lemma 5.3 in [11]. □

Next, we shall prove that if u is a viscosity supersolution of the equation

$$F(D^2u) - \partial_t u = f \quad \text{in } \Omega \times (T_0, T_2] \subset M \times \mathbb{R},$$

then the inf-convolution u_ϵ is still a viscosity supersolution; see [18, Lemma A.5] for the Euclidean case.

Proposition 4.3.3. *Assume that*

$$\text{Sec} \geq -\kappa \quad \text{on } M, \quad \text{for } \kappa \geq 0.$$

Let H and Ω be bounded open sets in M such that $\overline{H} \subset \Omega$, and $T_0 < T_1 < T_2$. Let $u \in C\left(\overline{\Omega} \times [T_0, T_2]\right)$, and let ω denote a modulus of continuity of u on $\overline{\Omega} \times [T_0, T_2]$, which is nondecreasing on $(0, +\infty)$ with $\omega(0+) = 0$. For $\varepsilon > 0$, let u_ε be the inf-convolution of u with respect to $\Omega \times (T_0, T_2]$. Then, there exists $\varepsilon_0 > 0$ depending only on $\|u\|_{L^\infty(\overline{\Omega} \times [T_0, T_2])}$, H, Ω, T_0, and T_1, such that if $0 < \varepsilon < \varepsilon_0$, then the following statements hold: Let $(x_0, t_0) \in \overline{H} \times [T_1, T_2]$, and let $(y_0, s_0) \in \overline{\Omega} \times [T_0, T_2]$ satisfy

$$u_\varepsilon(x_0, t_0) = u(y_0, s_0) + \frac{1}{2\varepsilon}\left\{d^2(y_0, x_0) + |s_0 - t_0|^2\right\}.$$

(a) We have that

$$(y_0, s_0) \in \Omega \times (T_0, T_2],$$

and there is a unique minimizing geodesic joining x_0 to y_0.

(b) If $(p, \zeta, A) \in \mathcal{P}^{2,-}u_\varepsilon(x_0, t_0)$, then we have

$$y_0 = \exp_{x_0}(-\varepsilon\zeta), \quad \text{and} \quad s_0 \in [t_0 - \varepsilon p, T_2].$$

(c) If $(p, \zeta, A) \in \mathcal{P}^{2,-}u_\varepsilon(x_0, t_0)$, then we have

$$\left(p,\, L_{x_0,y_0}\zeta,\, L_{x_0,y_0} \circ A - 2\kappa\, \omega\left(2\sqrt{\varepsilon\, \|u\|_{L^\infty\left(\overline{\Omega}\times[T_0,T_2]\right)}}\right) g_{y_0}\right) \in \mathcal{P}^{2,-}u(y_0, s_0),$$

where L_{x_0,y_0} stands for the parallel transport along the unique minimizing geodesic joining x_0 to $y_0 = \exp_{x_0}(-\varepsilon\zeta)$.

Proof. By recalling Lemma 4.3.1, (c), we see that

$$(y_0, s_0) \in \overline{B}_{2\sqrt{m\varepsilon}}(x_0) \times \left(\left[t_0 - 2\sqrt{m\varepsilon},\ t_0 + 2\sqrt{m\varepsilon} \right] \cap [T_0, T_2] \right),$$

for $m := \|u\|_{L^\infty(\overline{\Omega}\times[T_0,T_2])}$. Since the distance between $\overline{H}$ and $\partial\Omega$ is positive, we select $\varepsilon_0 > 0$ so small that

$$2\sqrt{m\varepsilon_0} < \min\left\{ d(\overline{H}, \partial\Omega),\ T_1 - T_0 \right\} =: \delta_0,$$

where $d(\overline{H}, \partial\Omega)$ means the distance between $\overline{H}$ and $\partial\Omega$. For $0 < \varepsilon < \varepsilon_0$, we have that

$$(y_0, s_0) \in \Omega \times (T_0, T_2]$$

since $(x_0, t_0) \in \overline{H} \times [T_1, T_2]$. We observe that

$$i_{\overline{\Omega}} := \inf\left\{ i_M(x) : x \in \overline{\Omega} \right\} > 0$$

since $\overline{\Omega}$ is compact from Hopf- Rinow Theorem and the map $x \mapsto i_M(x)$ is continuous. Now, we select

$$\varepsilon_0 := \frac{1}{8\|u\|_{L^\infty(\overline{\Omega}\times[T_0,T_2])}} \min\left\{ \delta_0^2,\ i_{\overline{\Omega}}^2 \right\}.$$

Then we have that for $0 < \varepsilon < \varepsilon_0$,

$$d^2(x_0, y_0) \le 4\varepsilon\|u\|_{L^\infty(\overline{\Omega}\times[T_0,T_2])} < 4\varepsilon_0\|u\|_{L^\infty(\overline{\Omega}\times[T_0,T_2])} < i_{\overline{\Omega}}^2,$$

and hence $d(x_0, y_0) < i_{\overline{\Omega}} \le \min\{i_M(x_0),\ i_M(y_0)\}$, which implies the uniqueness of a minimizing geodesic joining x_0 to y_0. This finishes the proof of (a).

From (a), there exists a unique vector $X \in T_{x_0}M$ such that

$$y_0 = \exp_{x_0} X, \quad \text{and} \quad |X| = d(x_0, y_0).$$

First, we claim that if $(p, \zeta, A) \in \mathcal{P}^{2,-}u_\varepsilon(x_0, t_0)$, then $y_0 = \exp_{x_0}(-\varepsilon\zeta)$, namely, $X = -\varepsilon\zeta$. Since $(p, \zeta, A) \in \mathcal{P}^{2,-}u_\varepsilon(x_0, t_0)$, we have that for any $\xi \in T_{x_0}M$ with

$|\xi| = 1$, small $r \in \mathbb{R}, \sigma \le 0$ and for any $(y, s) \in \overline{\Omega} \times [T_0, T_2]$,

$$\begin{aligned}
u(y, s) &+ \frac{1}{2\varepsilon}\left\{d^2\left(y, \exp_{x_0} r\xi\right) + |s - (t_0 + \sigma)|^2\right\}\\
&\ge u_\varepsilon\left(\exp_{x_0} r\xi, t_0 + \sigma\right)\\
&\ge u_\varepsilon(x_0, t_0) + r\langle\zeta, \xi\rangle + \sigma p + \frac{r^2\langle A\cdot\xi, \xi\rangle}{2} + o\left(r^2 + |\sigma|\right)\\
&= u(y_0, s_0) + \frac{1}{2\varepsilon}\left\{d^2(y_0, x_0) + |s_0 - t_0|^2\right\} + r\langle\zeta, \xi\rangle + \sigma p + \frac{r^2\langle A\cdot\xi, \xi\rangle}{2} + o\left(r^2 + |\sigma|\right).
\end{aligned} \tag{4.3.5}$$

When $(y, s) = (y_0, s_0)$ and $\sigma = 0$ in (4.3.5), we see that for small $r \ge 0$,

$$\begin{aligned}
\frac{1}{2\varepsilon}\{d(y_0, x_0) + r\}^2 &\ge \frac{1}{2\varepsilon}d^2\left(y_0, \exp_{x_0} r\xi\right)\\
&\ge \frac{1}{2\varepsilon}d^2(x_0, y_0) + r\langle\zeta, \xi\rangle + \frac{r^2\langle A\cdot\xi, \xi\rangle}{2} + o\left(r^2\right),
\end{aligned}$$

and hence for small $r \ge 0$,

$$rd(x_0, y_0) \ge r\langle\varepsilon\zeta, \xi\rangle + O\left(r^2\right), \qquad \forall \xi \in T_{x_0}M \text{ with } |\xi| = 1. \tag{4.3.6}$$

If $X = 0$, (4.3.6) implies that $\langle\varepsilon\zeta, \xi\rangle = 0$ for all $\xi \in T_{x_0}M$. Thus we deduce that $\zeta = 0$ and $y_0 = \exp_{x_0} 0 = \exp_{x_0}(-\varepsilon\zeta)$.

Now, we assume that $X \ne 0$. If $(y, s) = (y_0, s_0)$, $\sigma = 0$, and $\xi = X/|X| = X/d(x_0, y_0)$ in (4.3.5), then we have that for small $r \ge 0$,

$$\frac{1}{2\varepsilon}\{d(x_0, y_0) - r\}^2 \ge \frac{1}{2\varepsilon}d^2(x_0, y_0) + r\langle\zeta, \xi\rangle + \frac{r^2\langle A\cdot\xi, \xi\rangle}{2} + o\left(r^2\right)$$

and hence for small $r \ge 0$,

$$-rd(x_0, y_0) \ge r\langle\varepsilon\zeta, X/|X|\rangle + O\left(r^2\right). \tag{4.3.7}$$

For small $r \ge 0$, (4.3.6) and (4.3.7) imply that

$$\langle-\varepsilon\zeta, \xi\rangle \le |X| = d(x_0, y_0), \qquad \forall \xi \in T_{x_0}M \quad \text{with } |\xi| = 1,$$

and

$$\langle-\varepsilon\zeta, X/|X|\rangle = |X| = d(x_0, y_0).$$

Then, it follows that $-\varepsilon\zeta = X$ and hence $y_0 = \exp_{x_0} X = \exp_{x_0}(-\varepsilon\zeta)$ for $X \neq 0$. Thus we have proved that $y_0 = \exp_{x_0}(-\varepsilon\zeta)$.

When $(y, s) = (y_0, s_0)$ and $r = 0$ in (4.3.5), we have that for small $\sigma \leq 0$,

$$\frac{1}{2\varepsilon}|s_0 - t_0 - \sigma|^2 \geq \frac{1}{2\varepsilon}|s_0 - t_0|^2 + \sigma p + o(|\sigma|),$$

which implies that $s_0 \geq t_0 - \varepsilon p$. This proves (b).

To show (c), we recall that there is a unique minimizing geodesic joining x_0 to y_0, and $(y_0, s_0) \in \Omega \times (T_0, T_2]$ according to (a). Using the parallel transport, we rewrite (4.3.5) as follows: for any $\nu \in T_{y_0}M$ with $|\nu| = 1$, and small $r \in \mathbb{R}, \sigma \leq 0$, and for $(y, s) \in \overline{\Omega} \times [T_0, T_2]$,

$$\begin{aligned} u(y, s) \geq\ & u(y_0, s_0) + r\left\langle \zeta, L_{y_0,x_0}\nu \right\rangle_{x_0} + \sigma p + \frac{r^2}{2}\left\langle A \cdot \left(L_{y_0,x_0}\nu\right), L_{y_0,x_0}\nu \right\rangle_{x_0} \\ & + \frac{1}{2\varepsilon}\left\{d^2(y_0, x_0) - d^2\left(y, \exp_{x_0} rL_{y_0,x_0}\nu\right)\right\} + \frac{1}{2\varepsilon}\left\{|s_0 - t_0|^2 - |s - t_0 - \sigma|^2\right\} + o(r^2 + |\sigma|). \end{aligned}$$

By setting $(y, s) := \left(\exp_{y_0} r\nu, s_0 + \sigma\right)$ for small $r \in \mathbb{R}$, $\sigma \leq 0$, we claim that

$$\begin{aligned} u\left(\exp_{y_0} r\nu, s_0 + \sigma\right) \geq\ & u(y_0, s_0) + r\left\langle L_{x_0,y_0}\zeta, \nu \right\rangle_{y_0} + \sigma p + \frac{r^2}{2}\left\langle \left(L_{x_0,y_0} \circ A\right) \cdot \nu, \nu \right\rangle_{y_0} \\ & + \frac{1}{2\varepsilon}\left\{d^2(y_0, x_0) - d^2\left(\exp_{y_0} r\nu,\ \exp_{x_0} L_{y_0,x_0} r\nu\right)\right\} + o\left(r^2 + |\sigma|\right) \\ \geq\ & u(y_0, s_0) + r\left\langle L_{x_0,y_0}\zeta, \nu \right\rangle_{y_0} + \sigma p + \frac{r^2}{2}\left\langle \left(L_{x_0,y_0} \circ A\right) \cdot \nu, \nu \right\rangle_{y_0} \\ & - \frac{1}{2\varepsilon} r^2 \kappa\, d^2(x_0, y_0) + o\left(r^2 + |\sigma|\right). \end{aligned} \tag{4.3.8}$$

The first inequality is immediate from (2.2.1) and Definition 2.2.1. To prove the second inequality in (4.3.8), we consider a unique minimizing geodesic

$$\gamma(t) := \exp_{x_0}(-t\varepsilon\zeta)$$

joining $\gamma(0) = x_0$ to $\gamma(1) = y_0 = \exp_{x_0}(-\varepsilon\zeta)$. For a given $\nu \in T_{y_0}M$ with $|\nu| = 1$, define a variational field

$$\nu(t) := L_{y_0,\gamma(t)}\nu \in T_{\gamma(t)}M$$

along γ, where $v(0) = L_{y_0,x_0}v$, and $v(1) = v$. For small $\epsilon > 0$, we define a variation $h : (-\epsilon, \epsilon) \times [0, 1] \to M$, of γ,

$$h(r, t) := \exp_{\gamma(t)} rv(t).$$

The energy is defined as

$$E(r) := \int_0^1 \left| \frac{\partial h}{\partial t}(r, t) \right|^2 dt.$$

We use the second variation of energy formula (2.2.2) to obtain

$$E(r) = E(0) - r^2 \int_0^1 \langle R(\dot{\gamma}(t), v(t))\, \dot{\gamma}(t), v(t) \rangle \, dt + o\left(r^2\right)$$

since γ is a unique minimizing geodesic, and $v(t)$ is parallel transported along γ. Since $|v(t)| = |v| = 1$, and $|\dot{\gamma}(t)| = |\dot{\gamma}(0)| = d(x_0, y_0)$ for $t \in [0, 1]$, we have that

$$\begin{aligned}
E(0) - E(r) &= r^2 \int_0^1 \langle R(\dot{\gamma}(t), v(t))\, \dot{\gamma}(t), v(t) \rangle \, dt + o\left(r^2\right) \\
&= r^2 \int_0^1 \operatorname{Sec}(\dot{\gamma}(t), v(t)) \cdot \left(|\dot{\gamma}(t)|^2 - \langle \dot{\gamma}(t), v(t) \rangle^2 \right) dt + o\left(r^2\right) \\
&\geq -r^2 \int_0^1 \kappa \left(|\dot{\gamma}(t)|^2 - \langle \dot{\gamma}(t), v(t) \rangle^2 \right) dt + o\left(r^2\right) \\
&\geq -r^2 \kappa \, |\dot{\gamma}(0)|^2 + o\left(r^2\right) \; = -r^2 \kappa \, d^2(x_0, y_0) + o\left(r^2\right).
\end{aligned}$$

Recalling that $E(0) = d^2(x_0, y_0)$, and

$$E(r) \geq d^2\left(\exp_{\gamma(0)} rv(0), \exp_{\gamma(1)} rv(1)\right) = d^2\left(\exp_{x_0} L_{y_0,x_0} rv, \, \exp_{y_0} rv\right),$$

we obtain

$$\begin{aligned}
d^2(x_0, y_0) - d^2\left(\exp_{x_0} L_{y_0,x_0} rv, \exp_{y_0} rv\right) &\geq E(0) - E(r) \\
&\geq -r^2 \kappa \, d^2(x_0, y_0) + o\left(r^2\right),
\end{aligned}$$

which proves the second inequality of (4.3.8).

Since $d^2(x_0, y_0) + |t_0 - s_0|^2 \leq 4\varepsilon \|u\|_{L^\infty\left(\overline{\Omega} \times [T_0, T_2]\right)}$ from Lemma 4.3.1, (c), it follows that

$$d^2(x_0, y_0) \leq 2\varepsilon \, |u(x_0, t_0) - u(y_0, s_0)| \leq 2\varepsilon \, \omega\left(2 \sqrt{\varepsilon \|u\|_{L^\infty\left(\overline{\Omega} \times [T_0, T_2]\right)}}\right), \tag{4.3.9}$$

where ω is a modulus of continuity of u on $\overline{\Omega} \times [T_0, T_2]$. Therefore, we use (4.3.8) and (4.3.9) to conclude that for any $v \in T_{y_0}M$ with $|v| = 1$, and for small $r \in \mathbb{R}, \sigma \leq 0$,

$$u\left(\exp_{y_0} rv, s_0 + \sigma\right) \geq u(y_0, s_0) + r\left\langle L_{x_0,y_0}\zeta, v\right\rangle_{y_0} + \sigma p + \frac{r^2}{2}\left\langle\left(L_{x_0,y_0} \circ A\right) \cdot v, v\right\rangle_{y_0}$$
$$- r^2\kappa\,\omega\left(2\sqrt{\varepsilon\|u\|_{L^\infty\left(\overline{\Omega}\times[T_0,T_2]\right)}}\right) + o\left(r^2 + |\sigma|\right).$$

Therefore, Lemma 2.2.9 implies

$$\left(p,\ L_{x_0,y_0}\zeta,\ L_{x_0,y_0} \circ A - 2\kappa\,\omega\left(2\sqrt{\varepsilon\|u\|_{L^\infty\left(\overline{\Omega}\times[T_0,T_2]\right)}}\right) g_{y_0}\right) \in \mathcal{P}^{2,-}u(y_0, s_0).$$

□

Now, we recall the intrinsic uniform continuity of the operator with respect to x from [5], which is a natural extension of the Euclidean notion of uniform continuity of the operator with respect to x.

Definition 4.3.1. *The operator F : $\operatorname{Sym} TM \to \mathbb{R}$ is said to be intrinsically uniformly continuous with respect to x if there exists a modulus of continuity $\omega_F : [0, +\infty) \to [0, +\infty)$ with $\omega_F(0+) = 0$ such that*

$$F(S) - F\left(L_{x,y} \circ S\right) \leq \omega_F\left(d(x, y)\right) \tag{F2}$$

for any $S \in \operatorname{Sym} TM_x$, and $x, y \in M$ with $d(x, y) < \min\{i_M(x),\ i_M(y)\}$.

We may assume that ω_F is nondecreasing on $(0, +\infty)$. Recall some examples of the intrinsically uniformly continuous operator from [5].

Remark 4.3.4. *(a) When $M = \mathbb{R}^n$, we have $L_{x,y} \circ S \equiv S$ so* (F2) *holds.*

(b) In general, we consider the operator F, which depends only on the eigenvalues of $S \in \operatorname{Sym} TM$, of the form :

$$F(S) = G(\text{ eigenvalues of } S\ ) \quad \textit{for some } G. \tag{4.3.10}$$

Since S and $L_{x,y} \circ S$ have the same eigenvalues, the operator F satisfies intrinsic uniform continuity with respect to x (with $\omega_F \equiv 0$). The trace and determinant of S are typical examples of the operator satisfying (4.3.10).

(c) Pucci's extremal operators $\mathcal{M}^{\pm}$ *satisfy* (4.3.10), (F2) *and (F1).*

Lemma 4.3.5. *Under the same assumption as Proposition 4.3.3, we also assume that* F *satisfies (F1) and* (F2). *For* $f \in C(\Omega \times (T_0, T_2])$, *let* $u \in C\left(\overline{\Omega} \times [T_0, T_2]\right)$ *be a viscosity supersolution of*

$$F(D^2 u) - \partial_t u = f \quad \text{in } \Omega \times (T_0, T_2].$$

If $0 < \varepsilon < \varepsilon_0$, *then the inf-convolution* u_ε *(with respect to* $\Omega \times (T_0, T_2]$*) is a viscosity supersolution of*

$$F(D^2 u_\varepsilon) - \partial_t u_\varepsilon = f_\varepsilon \quad \text{on } H \times (T_1, T_2],$$

where $\varepsilon_0 > 0$ *is the constant as in Proposition 4.3.3, and*

$$f_\varepsilon(x,t) := \sup_{\overline{B}_{2\sqrt{m\varepsilon}}(x) \times \left[t - 2\sqrt{m\varepsilon},\, \min\left\{t + 2\sqrt{m\varepsilon},\, T_2\right\}\right]} f + \omega_F\left(2\sqrt{m\varepsilon}\right) + 2n\Lambda\kappa\,\omega\left(2\sqrt{m\varepsilon}\right)$$

for $m := \|u\|_{L^\infty\left(\overline{\Omega} \times [T_0, T_2]\right)}$. *Moreover, we have*

$$F(D^2 u_\varepsilon) - \partial_t u_\varepsilon \le f_\varepsilon \quad \text{a.e. in } H \times (T_1, T_2).$$

Proof. Fix $0 < \varepsilon < \varepsilon_0$. Let $\varphi \in C^{2,1}(H \times (T_1, T_2])$ be a function such that $u_\varepsilon - \varphi$ has a local minimum at $(x_0, t_0) \in H \times (T_1, T_2]$ in the parabolic sense. Then we have

$$\left(\partial_t \varphi(x_0, t_0), \nabla \varphi(x_0, t_0), D^2 \varphi(x_0, t_0)\right) \in \mathcal{P}^{2,-} u_\varepsilon(x_0, t_0).$$

We apply Proposition 4.3.3 to have that

$$\left(\partial_t \varphi(x_0, t_0), L_{x_0, y_0} \nabla \varphi(x_0, t_0), L_{x_0, y_0} \circ D^2 \varphi(x_0, t_0) - 2\kappa\omega\left(2\sqrt{m\varepsilon}\right) g_{y_0}\right) \in \mathcal{P}^{2,-} u(y_0, s_0)$$

for

$$y_0 := \exp_{x_0}(-\varepsilon \nabla \varphi(x_0, t_0)) \in \overline{B}_{2\sqrt{m\varepsilon}}(x_0) \subset \Omega,$$

and some $s_0 \in \left[t_0 - 2\sqrt{m\varepsilon}, \min\left\{t_0 + 2\sqrt{m\varepsilon}, T_2\right\}\right] \subset (T_0, T_2]$. Since u is a viscosity supersolution in $\Omega \times (T_0, T_2]$, we see that

$$\begin{aligned} f(y_0, s_0) &\ge F\left(L_{x_0, y_0} \circ D^2 \varphi(x_0, t_0) - 2\kappa\,\omega\left(2\sqrt{m\varepsilon}\right) g_{y_0}\right) - \partial_t \varphi(x_0, t_0) \\ &\ge F\left(L_{x_0, y_0} \circ D^2 \varphi(x_0, t_0)\right) - n\Lambda \cdot 2\kappa\,\omega\left(2\sqrt{m\varepsilon}\right) - \partial_t \varphi(x_0, t_0) \\ &\ge F\left(D^2 \varphi(x_0, t_0)\right) - \omega_F\left(d(x_0, y_0)\right) - 2n\Lambda\kappa\,\omega\left(2\sqrt{m\varepsilon}\right) - \partial_t \varphi(x_0, t_0) \end{aligned}$$

using the uniform ellipticity and intrinsic uniform continuity of F. Thus, we deduce that

$$\begin{aligned} F\left(D^2\varphi(x_0,t_0)\right) - \partial_t\varphi(x_0,t_0) &\le f(y_0,s_0) + \omega_F\left(d(x_0,y_0)\right) + 2n\Lambda\kappa\,\omega\left(2\sqrt{m\varepsilon}\right) \\ &\le f_\varepsilon(x_0,t_0). \end{aligned}$$

Therefore, u_ε is a viscosity supersolution of $F(D^2u_\varepsilon) - \partial_t u_\varepsilon = f_\varepsilon$ in $H \times (T_1, T_2]$.

According to Lemma 4.3.2, u_ε admits the Hessian almost everywhere in $\Omega \times (T_0, T_2)$ satisfying (4.3.3). For almost every $(x,t) \in \Omega \times (T_0, T_2)$, we use (4.3.3) and Lemma 2.2.9 to deduce

$$\left(\partial_t u_\varepsilon(x,t),\ \nabla u_\varepsilon(x,t),\ D^2u_\varepsilon(x,t)\right) \in \mathcal{P}^{2,-}u_\varepsilon(x,t) \cap \mathcal{P}^{2,+}u_\varepsilon(x,t).$$

Therefore, we conclude that

$$F(D^2u_\varepsilon) - \partial_t u_\varepsilon \le f_\varepsilon \quad \text{a.e. in} \quad H \times (T_1, T_2),$$

since u_ε is a viscosity supersolution in $H \times (T_1, T_2]$. □

For a viscosity subsolution, we can obtain similar results to Lemmas 4.3.1,4.3.2, 4.3.5, and Proposition 4.3.3 using the sup-convolution:

$$u^\varepsilon(x_0,t_0) := \sup_{(y,s)\in\overline{\Omega}\times[T_0,T_2]} \left\{ u(y,s) - \frac{1}{2\varepsilon}\left\{d^2(y,x_0) + |s-t_0|^2\right\}\right\} \quad \text{for } (x_0,t_0) \in \overline{\Omega} \times [T_0, T_2].$$

4.3.2 Proof of parabolic Harnack inequality

Now we shall prove Proposition 4.3.6 from a priori estimate in Section 4.2.

Proposition 4.3.6. *Assume that*

$$\mathrm{Sec} \ge -\kappa \quad \text{on } M, \quad \text{for } \kappa \ge 0,$$

and that F satisfies (F1). Let $0 < \eta < 1$ and $K_{\alpha_1 R, \alpha_2 R^2}(z_0, 4R^2) \subset K_{R_0}(x_0,t_0) \subset M \times \mathbb{R}$. For $f \in C\left(K_{R_0}(x_0,t_0)\right)$, let $u \in C\left(K_{R_0}(x_0,t_0)\right)$ be a viscosity supersolution of

$$F(D^2u) - \partial_t u = f \quad \text{in} \quad K_{\alpha_1 R, \alpha_2 R^2}(z_0, 4R^2),$$

such that

$$u \geq 0 \quad in \quad K_{\alpha_1 R,\, \alpha_2 R^2}(z_0, 4R^2) \backslash K_{\beta_1 R, \beta_2 R^2}(z_0, 4R^2),$$

and

$$\inf_{K_{2R}(z_0, 4R^2)} u \leq 1.$$

Then, there exist uniform constants $M_\eta > 1, 0 < \mu_\eta < 1,$ *and* $0 < \epsilon_\eta < 1$ *such that*

$$\frac{\left|\left\{u \leq M_\eta\right\} \cap K_{\eta R}(z_0, 0)\right|}{\left|K_{\alpha_1 R,\, \alpha_2 R^2}(z_0, 4R^2)\right|} \geq \mu_\eta,$$

provided

$$\left(\fint_{K_{R_0}(x_0, t_0)} \left|R_0^2 f^+\right|^{n\theta+1} \right)^{\frac{1}{n\theta+1}} \leq \epsilon_\eta, \tag{4.3.11}$$

where $\theta := 1 + \log_2 \cosh(4\sqrt{\kappa} R_0)$, *and* $M_\eta > 0$, $0 < \mu_\eta, \epsilon_\eta < 1$ *depend only on* $\eta, n, \lambda, \Lambda$ *and* $\sqrt{\kappa} R_0$.

Proof. It suffices to prove the proposition for $F = \mathcal{M}^-$ from (F1) (or (F1')). Setting $\tilde{\alpha}_1 := (\alpha_1 + \beta_1)/2$, and $\tilde{\alpha}_2 := (\alpha_2 + \beta_2)/2$, we define

$$\Omega \times (T_0, T_2] := K_{\tilde{\alpha}_1 R, \tilde{\alpha}_2 R^2}(z_0, 4R^2), \quad \text{and} \quad H \times (T_1, T_2] := K_{\beta_1 R, \beta_2 R^2}(z_0, 4R^2).$$

We note that u and f belong to $C\left(\overline{\Omega} \times [T_0, T_2]\right)$, and we denote by ω the modulus of continuity of u on $\overline{\Omega} \times [T_0, T_2]$, which is nondecreasing with $\omega(0+) = 0$.

For $\varepsilon > 0$, let u_ε be the inf-convolution of u with respect to $\Omega \times (T_0, T_2]$ as in (4.3.1). According to Lemma 4.3.5, there exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then u_ε satisfies

$$\mathcal{M}^-(D^2 u_\varepsilon) - \partial_t u_\varepsilon \leq f_\varepsilon \qquad \text{a.e. in} \;\; K_{\beta_1 R, \beta_2 R^2}(z_0, 4R^2),$$

where f_ε is defined as follows: for $(x, t) \in K_{\beta_1 R, \beta_2 R^2}(z_0, 4R^2)$,

$$f_\varepsilon(x,t) := \sup_{\overline{B}_{2\sqrt{m\varepsilon}}(x) \times \left[t - 2\sqrt{m\varepsilon}, \min\left\{t + 2\sqrt{m\varepsilon}, T_2\right\}\right]} f + 2n\Lambda\kappa\, \omega\left(2\sqrt{m\varepsilon}\right); \;\; m := \|u\|_{L^\infty\left(\overline{K}_{\tilde{\alpha}_1 R, \tilde{\alpha}_2 R^2}(z_0, 4R^2)\right)},$$

and we recall that $\mathcal{M}^-$ is intrinsically uniformly continuous with respect to x with $\omega_{\mathcal{M}^-} \equiv 0$. Using (2.2.5) and (4.3.11), we have that

$$\begin{aligned} \left(\fint_{K_{\beta_1 R, \beta_2 R^2}(z_0, 4R^2)} \left|\beta_1^2 R^2 f^+\right|^{n\theta+1} \right)^{\frac{1}{n\theta+1}} &\leq 2 \left(\frac{\beta_2}{\beta_1^2}\right)^{-\frac{1}{n\theta+1}} \left(\fint_{K_{R_0}(x_0, t_0)} |R_0^2 f^+|^{n\theta+1} \right)^{\frac{1}{n\theta+1}} \\ &\leq 2 \left(\frac{\beta_2}{\beta_1^2}\right)^{-\frac{1}{n\theta+1}} \epsilon_\eta =: \tilde{\epsilon}_\eta, \end{aligned}$$

and hence for small $\varepsilon > 0$,

$$\left(\fint_{K_{\beta_1 R, \beta_2 R^2}(z_0, 4R^2)} \left| \beta_1^2 R^2 \left\{ \mathcal{M}^-(D^2 u_\varepsilon) - \partial_t u_\varepsilon \right\}^+ \right|^{n\theta+1} \right)^{\frac{1}{n\theta+1}} \le \left(\fint_{K_{\beta_1 R, \beta_2 R^2}(z_0, 4R^2)} \left| \beta_1^2 R^2 f_\varepsilon^+ \right|^{n\theta+1} \right)^{\frac{1}{n\theta+1}} \le 2\tilde{\epsilon}_\eta, \tag{4.3.12}$$

since f_ε converges uniformly to f in $K_{\beta_1 R, \beta_2 R^2}(z_0, 4R^2)$. For a fixed $\delta > 0$, we may assume that for small $\varepsilon > 0$,

$$u_\varepsilon \ge -\delta \quad \text{in} \quad K_{\tilde{\alpha}_1 R, \tilde{\alpha}_2 R^2}(z_0, 4R^2) \backslash K_{\beta_1 R, \beta_2 R^2}(z_0, 4R^2),$$

and

$$\inf_{K_{2R}(z_0, 4R^2)} u_\varepsilon \le 1 + \delta$$

since u_ε converges uniformly to u in $K_{\tilde{\alpha}_1 R, \tilde{\alpha}_2 R^2}(z_0, 4R^2)$ from Lemma 4.3.1.

Now, we fix a small $\varepsilon > 0$. According to Lemma 4.3.2, (c), there is a smooth function φ on $M \times (-\infty, T_2]$ satisfying $0 \le \varphi \le 1$ on $M \times (-\infty, T_2]$,

$$\varphi \equiv 1 \quad \text{in } \overline{K}_{\beta_1 R, \beta_2 R^2}(z_0, 4R^2), \quad \text{and} \quad \operatorname{supp} \varphi \subset K_{\tilde{\alpha}_1 R, \tilde{\alpha}_2 R^2}(z_0, 4R^2),$$

and we find a sequence $\{w_k\}_{k=1}^\infty$ of smooth functions on $M \times (-\infty, T_2]$ satisfying

$$\begin{cases} w_k \to \varphi u_\varepsilon & \text{uniformly in } M \times (-\infty, T_2] \text{ as } k \to +\infty, \\ |\nabla w_k| + |\partial_t w_k| \le C & \text{in } M \times (-\infty, T_2], \\ \partial_t w_k \to \partial_t u_\varepsilon & \text{a.e. in } K_{\beta_1 R, \beta_2 R^2}(z_0, 4R^2) \text{ as } k \to +\infty, \\ D^2 w_k \le C g & \text{in } M \times (-\infty, T_2], \\ D^2 w_k \to D^2 u_\varepsilon & \text{a.e. in } K_{\beta_1 R, \beta_2 R^2}(z_0, 4R^2) \text{ as } k \to +\infty, \end{cases}$$

where the constant $C > 0$ is independent of k. For large k, we may assume that

$$w_k \ge -2\delta \ \text{ in } K_{\alpha_1 R, \alpha_2 R^2}(z_0, 4R^2) \backslash K_{\beta_1 R, \beta_2 R^2}(z_0, 4R^2), \qquad \inf_{K_{2R}(z_0, 4R^2)} \frac{w_k + 2\delta}{1 + 4\delta} \le 1,$$

and

$$\left(\fint_{K_{\beta_1 R, \beta_2 R^2}(z_0, 4R^2)} \left| \beta_1^2 R^2 \left\{ \mathcal{M}^-(D^2 w_k) - \partial_t w_k \right\}^+ \right|^{n\theta+1} \right)^{\frac{1}{n\theta+1}} \le 4\tilde{\epsilon}_\eta,$$

where we used the dominated convergence theorem to obtain the last estimate from (4.3.12).

Selecting $\epsilon_\eta > 0$ small enough, we apply Proposition 4.2.5 to $\dfrac{w_k + 2\delta}{1 + 4\delta}$ (for large k) to obtain

$$\frac{\left|\left\{w_k + 2\delta \le (1 + 4\delta)M_\eta\right\} \cap K_{\eta R}(z_0, 0)\right|}{\left|K_{\beta_1 R, \beta_2 R^2}(z_0, 4R^2)\right|} \ge \mu_\eta.$$

By letting $k \to +\infty$, we have

$$\frac{\left|\left\{u_\varepsilon + \delta \le (1 + 4\delta)M_\eta\right\} \cap K_{\eta R}(z_0, 0)\right|}{\left|K_{\beta_1 R, \beta_2 R^2}(z_0, 4R^2)\right|} \ge \mu_\eta.$$

Since u_ε converges uniformly to u in $K_{\tilde{\alpha}_1 R, \tilde{\alpha}_2 R^2}(z_0, 4R^2)$, we let $\varepsilon \to 0$ and $\delta \to 0$, and use Bishop-Gromov's Theorem 2.2.4 to deduce that

$$\frac{\left|\left\{u \le M_\eta\right\} \cap K_{\eta R}(z_0, 0)\right|}{\left|K_{\alpha_1 R, \alpha_2 R^2}(z_0, 4R^2)\right|} \ge \frac{1}{\mathcal{D}}\left(\frac{\beta_1}{\alpha_1}\right)^{\log_2 \mathcal{D}} \frac{\beta_2}{\alpha_2} \frac{\left|\left\{u \le M_\eta\right\} \cap K_{\eta R}(z_0, 0)\right|}{\left|K_{\beta_1 R, \beta_2 R^2}(z_0, 4R^2)\right|} \ge \frac{1}{\mathcal{D}}\left(\frac{\beta_1}{\alpha_1}\right)^{\log_2 \mathcal{D}} \frac{\beta_2}{\alpha_2}\mu_\eta > 0$$

for $\mathcal{D} := 2^n \cosh^{n-1}(4\sqrt{\kappa}R_0)$, which finishes the proof. □

Therefore, Harnack inequality is obtained according to Proposition 4.3.6.

Theorem 4.3.7 (Harnack inequality). *Assume that M has sectional curvature bounded from below by $-\kappa$ for $\kappa \ge 0$, i.e.,* Sec $\ge -\kappa$ *on M, and F satisfies (F1). Let $f \in C\left(K_{2R}(x_0, 4R^2)\right)$. If $u \in C\left(K_{2R}(x_0, 4R^2)\right)$ is a nonnegative viscosity solution of the equation $F(D^2u) - \partial_t u = f$ in $K_{2R}(x_0, 4R^2)$, then we have*

$$\sup_{K_R(x_0, 2R^2)} u \le C\left\{\inf_{K_R(x_0, 4R^2)} u + R^2\left(\fint_{K_{2R}(x_0, 4R^2)} |f|^{n\theta+1}\right)^{\frac{1}{n\theta+1}}\right\},$$

where $\theta := 1 + \log_2 \cosh(8\sqrt{\kappa}R)$ and $C > 0$ is a uniform constant depending only on n, λ, Λ and $\sqrt{\kappa}R$.

Theorem 4.3.8 (Weak Harnack inequality). *Assume that* Sec $\ge -\kappa$ *on M for $\kappa \ge 0$, and F satisfies (F1). Let $f \in C\left(K_{2R}(x_0, 4R^2)\right)$. If $u \in C\left(K_{2R}(x_0, 4R^2)\right)$ is a nonnegative viscosity supersolution of the equation $F(D^2u) - \partial_t u = f$ in $K_{2R}(x_0, 4R^2)$, then we have*

$$\left(\fint_{K_R(x_0, 2R^2)} u^p\right)^{\frac{1}{p}} \le C\left\{\inf_{K_R(x_0, 4R^2)} u + R^2\left(\fint_{K_{2R}(x_0, 4R^2)} |f^+|^{n\theta+1}\right)^{\frac{1}{n\theta+1}}\right\}; \quad f^+ := \max(f, 0),$$

where $\theta := 1 + \log_2 \cosh(8\sqrt{\kappa}R)$, *and the positive constants* $p \in (0,1)$ *and* C *are uniform depending only on* n, λ, Λ, *and* $\sqrt{\kappa}R$.

Sketch of proof of Theorems 4.3.7 and 4.3.8

Proposition 4.3.6 and Bishop and Gromov's Theorem 2.2.4 imply Theorems 4.3.7 and 4.3.8 following the proofs of Theorems 4.2.6 and 4.2.7. The main difference is the fact that u solves the parabolic equation in the viscosity sense so it is necessary to mention that $w := C_1 - C_2 u$ (for $C_1, C_2 > 0$) satisfies

$$\mathcal{M}^-(D^2 w) - \partial_t w = -C_2 \left\{ \mathcal{M}^+(D^2 u) - \partial_t u \right\} \leq -C_1 \left\{ F(D^2 u) - \partial_t u \right\} = -C_1 f.$$

in the viscosity sense. □

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국문초록

이 학위 논문에서는 비발산 구조를 갖는 완전 비선형 포물 방정식의 해의 정칙 이론과 그 응용에 대하여 연구하였다.

첫번째 장은 완전 비선형 고른 포물형 및 퇴화된 포물형 방정식의 해의 점근적 행동 양상에 대한 연구이다. 먼저, 포물 방정식의 정규화 된 해가 시간이 흐름에 따라 방정식과 관련된 완전 비선형 타원 작용소의 제 1 고유 함수로 수렴함을 증명하였다. 또한 볼록한 영역에서 오목한 완전 비선형 제차 작용소가 주어졌을때, 포물형 해의 초기 기하적 구조-특정한 오목성 (log-concavity, power concavity)-가 보존되는 것을 보였다. 위의 수렴성을 이용하면 제 1 고유 함수 또한 같은 기하적 구조를 가짐을 알 수 있다.

두번째 장에서는 완전 리만 다양체 위에서 완전 비선형 포물 방정식의 해를 다루었는데, 특히 정칙 이론의 초석이 되는 포물형 Harnack 부등식을 증명하였다. 선형 작용소에 대해서는 거리 함수로 정의된 특정한 조건을 가정하고 정칙인 해의 대역적 Harnack 부등식을 얻었다. 또 단면 곡률의 하한을 가지는 다양체에 대해 비선형 작용소의 국소적 Harnack 부등식을 보였다. 마지막으로 Jensen의 sup- and inf-convolution을 이용하여, 연속 해인 viscosity 해에 대한 Harnack 부등식을 증명하였다.

주요어휘: 완전 비선형 포물 방정식, 비선형 타원형 고유치 문제, 퇴화된 포물 방정식, Harnack 부등식, ABP 추정
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