이학박사 학위논문

Categorification and Supercategorification of Highest Weight Modules over Quantum Generalized Kac-Moody Algebras

(일반화된 양자 캐츠-무디 대수의 최고치 모듈의 카테고리화 그리고 슈퍼카테고리화)

2013년 2월

서울대학교 대학원

수리과학부

오 세 진

Categorification and Supercategorification of Highest Weight Modules over Quantum Generalized Kac-Moody Algebras

(일반화된 양자 캐츠-무디 대수의 최고치 모듈의 카테고리화 그리고 슈퍼카테고리화)

지도교수 강 석 진

이 논문을 이학박사 학위논문으로 제출함

2012년 12월

서울대학교 대학원

수리과학부

오 세 진

오 세 진의 이학박사 학위논문을 인준함

2012년 12월

위 원 장	______________________________	(인)
부 위 원 장	______________________________	(인)
위 원	______________________________	(인)
위 원	______________________________	(인)
위 원	______________________________	(인)

Categorification and Supercategorification of Highest Weight Modules over Quantum Generalized Kac-Moody Algebras

A dissertation
submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
to the faculty of the Graduate School of
Seoul National University

by

Se-jin Oh

Dissertation Director : Professor Seok-Jin Kang

Department of Mathematical Sciences
Seoul National University

February 2013

Abstract

Let $U_q(\mathfrak{g})$ be one of the quantum generalized Kac-Moody algebras and let $V(\Lambda)$ be integrable highest weight $U_q(\mathfrak{g})$-module with highest weight Λ. We prove that $V(\Lambda)$ can be categorified from the cyclotomic quiver Hecke algebra R^Λ and supercategorified from the cyclotomic quiver Hecke superalgebras $\mathcal{R}^\Lambda$. Moreover, since $U_q^-(\mathfrak{g})$ is the projective limit of $V(\Lambda)$, $U_q^-(\mathfrak{g})$ can also be categorified via the quiver Hecke algebra R and supercategorified via the quiver Hecke superalgebras $\mathcal{R}$.

Key words: categorification, perfect basis, quantum generalize Kac-Moody algebras, quiver Hecke algebras, quiver Hecke superalgebras, supercategorification
Student Number: 2007-20281

Contents

Chapter 1

Introduction

The Grothendieck group $[\mathscr{C}]$ of abelian category $\mathscr{C}$ is the abelian group generated by $[X]$ (X is an object of $\mathscr{C}$) with the defining relations:

$$\text{if } 0 \to X' \to X \to X'' \to 0 \text{ is an exact sequence, then } [X] = [X'] + [X''].$$

We say that an algebra A *categorifies* an algebra (or a module) B if the Grothendieck group of *some* A-module category $\mathscr{C}$ is isomorphic to B ([6, 9]). More precisely, elements of B replaces by modules in $\mathscr{C}$, functions on B by functors on $\mathscr{C}$ and relations between functions on B by natural isomorphisms between functors on $\mathscr{C}$:

$$\begin{tikzcd} B \arrow[r, "\simeq"] \arrow[rr, bend left, "\text{categorification}"] & {[\mathscr{C}]} & \boxed{\begin{array}{c}\text{some } A\text{-module}\\ \text{category } \mathscr{C}\end{array}} \arrow[l, "\textit{taking}", "\textit{Grothendieck group}"'] \end{tikzcd}.$$

The process of taking Grothendieck group can be understood as a "extracting information", since we do *not* consider the module itself, *but* the composition series of the module. In this situation, we say that B is *embedded properly* in $\mathscr{C}$. Thus, the category $\mathscr{C}$ has rich structure comparing with B.

With a similar picture, we have understood the quantization of Kac-Moody algebras:

$$\begin{tikzcd} \boxed{\begin{array}{c}\text{Kac-Moody}\\ \text{algebra } U(\mathfrak{g})\end{array}} \arrow[r, "\simeq"] \arrow[rr, bend left, "\text{quantization}"] & U^1(\mathfrak{g}) & \boxed{\begin{array}{c}\text{quantum Kac-Moody}\\ \text{algebra } U_q(\mathfrak{g})\end{array}} \arrow[l, "1\leftarrow q"'] \end{tikzcd}.$$

In [27], Lascoux-Leclerc-Thibon conjectured that the coefficients of Kashiwara's lower global basis (=Lusztig's canonical basis) of type $A^{(1)}_{\ell-1}$ tells us the composition multiplicities of simple modules inside Specht modules over Hecke algebras. In other word, they conjectured that the Grothendieck group of modules over Hecke algebras might encode the information of the module $V(\Lambda_0)$ over $U(A^{(1)}_{\ell-1})$.

Soon after, Ariki ([1]) stated and proved a generalization of the conjecture by using the method of geometric representation theory and Specht module theory. More precisely, he showed that

(Ar1) the Grothendieck groups of the categories of finitely generated projective modules over affine and cyclotomic Hecke algebra $\mathcal{H}$ and $\mathcal{H}^{\Lambda}$ are isomorphic to $U^-_{\mathbb{Z}}(A^{(1)}_{\ell-1})$ and $V(\Lambda)$ for all integral dominant weight Λ, respectively,

(Ar2) the set of isomorphism classes of projective indecomposable modules corresponds to Kashiwara's lower global basis (= Lusztig's canonical basis) under the isomorphism, which implies the Lascoux-Leclerc-Thibon conjecture.

We have referred this to as the Lascoux-Leclerc-Thibon-Ariki theory.

After that, many mathematicians tried to extend the Lascoux-Leclerc-Thibon-Ariki theory to general settings, such as Kac-Moody algebras for other types, quantum Kac-Moody algebras and variants of (quantum) Kac-Moody algebras. In [4], Brundan and Kleshchev proved that, when the defining parameter is primitive $(2\ell+1)$-th root of unity, the crystals consisting of simple modules over affine and cyclotomic Hecke-Clifford superalgebras are isomorphic to the crystal $B(\infty)$ and $B(\Lambda)$ of type $A^{(2)}_{2\ell}$. In [32], Tsuchioka proved a similar results for type $D_{\ell+1}^{(2)}$ by considering affine Hecke-Clifford superalgebras when the defining parameter is a primitive $2(\ell+1)$-th root of unity. However, they only gave isomorphisms in the level of crystals.

In 2008, Khovanov and Lauda ([24, 25]) and Rouquier ([33]) independently introduced certain graded algebras which depend on the root system $R = \bigoplus_{\beta \in Q^+} R(\beta)$, called the *Khovanov-Lauda-Rouquier algebras* or *quiver Hecke algebras*, which is a breakthrough in many aspects. Comparing with previous results on the categorification theory, the quiver Hecke algebras provide categorifications of *quantum* Kac-Moody algebras corresponding to *arbitrary* symmetrizable Cartan datum. More precisely, for each Cartan datum, we can define the quiver Hecke algebra R such that

$$U_{\mathbb{A}}^-(\mathfrak{g}) \simeq [\mathrm{Proj}(R)] = \bigoplus_{\beta \in Q^+} [\mathrm{Proj}(R(\beta))],$$

where $U_{\mathbb{A}}^-(\mathfrak{g})$ is the integral form of the negative part $U_q^-(\mathfrak{g})$ of $U_q(\mathfrak{g})$ with $\mathbb{A} = \mathbb{Z}[q, q^{-1}]$ and $[\mathrm{Proj}(R)]$ is the Grothendieck group of the category of finitely generated graded projective R-modules (cf. (Ar1)). When the generalized Cartan matrix is symmetric, Varangnolo and Vasserot ([34]) proved that the set of isomorphism classes of projective indecomposable modules in $[\mathrm{Proj}(R)]$ corresponds to Kashiwara's lower global basis (= Lusztig's canonical basis) under this isomorphism (cf. (Ar2)). The quiver Hecke algebra R has a special quotient, the *cyclotomic quiver Hecke algebra* $R^\Lambda = \bigoplus_{\beta \in Q^+} R^\Lambda(\beta)$ for each dominant integral weight Λ. In [24], Khovanov and Lauda stated a conjecture, which is now referred to as the *cyclotomic conjecture*, that

$$V_{\mathbb{A}}^-(\Lambda) \simeq [\mathrm{Proj}(R^\Lambda)] = \bigoplus_{\beta \in Q^+} [\mathrm{Proj}(R^\Lambda(\beta))],$$

where $V_{\mathbb{A}}^-(\Lambda)$ is the $U_{\mathbb{A}}(\mathfrak{g})$-module generated by v_Λ. At the level of crystal, Lauda and Vazirani ([28]) proved this conjecture for all types; i.e., the set of all isomorphism classes of irreducible modules has a crystal structure which is isomorphic to $B(\Lambda)$. For $A_{\ell-1}^{(1)}$ and A_∞ cases, Brundan and Kleshchev proved this conjecture by constructing an isomorphism between R^Λ and $\mathcal{H}^\Lambda$ as *graded algebras* ([5]). Finally, Kang and Kashiwara ([14]) proved this conjecture for all types by investigating the properties of R^Λ itself. In their proof, the main difficult steps were showing the following:

(KK1) The functors E_i^Λ and F_i^Λ corresponding to the Chevalley generators e_i and f_i are well-defined on $\mathsf{Proj}(R^\Lambda)$,

(KK2) The commutation relation derived from the natural isomorphisms between the functors E_i^Λ and F_i^Λ satisfies the axiom of $\mathfrak{sl}_2$-categorification developed by Chuang and Rouquier [7].

In [19], Kang, Oh and Park extended the study of the quiver Hecke algebras to the case of generalized quantum Kac-Moody algebras. In that paper, introducing the polynomial P_i for each index i, they defined the *generalized* Khovanov-Lauda-Rouquier algebras or *generalized* quiver Hecke algebras R which categorify the integral form of the negative half of generalized quantum Kac-Moody algebras corresponding to the Borcherds-Cartan data. In [15], Kang, Kashiwara and Oh proved the cyclotomic theorem in this case.

In [17], Kang, Kashiwara and Tsuchioka introduced a new family of graded superalgebras $\mathcal{R}$, called *the quiver Hecke superalgebras*, which can be understood as the super-version of the quiver Hecke algebras. Moreover, they proved that quiver Hecke superalgebras are weakly Morita superequivalent to affine Hecke-Clifford superalgebras after suitable completions. Since the quiver Hecke superalgebras has a natural $(\mathbb{Z}\times\mathbb{Z}_2)$-grading, the $\mathbb{Z}$-graded module category over $\mathcal{R}$ has a natural *supercategory structure* with endofunctor Π induced by parity involution $\phi_{\mathcal{R}}$ on superalgebras. As in non-super case, we say that a superalgebra $\mathcal{A}$ *supercategorifies* an algebra (or a module) $\mathcal{B}$ if the Grothendieck group of some $\mathcal{A}$-module supercategory is isomorphic to $\mathcal{B}$. Recently, Kang, Kashiwara and Oh ([16]) proved that quiver Hecke superalgebras and their cyclotomic quotients supercategorify $U_{\mathbb{A}}^-(\mathfrak{g})$ and $V_{\mathbb{A}}(\Lambda)$, respectively.

In this thesis, we will show that

- $U_{\mathbb{A}}^-(\mathfrak{g}) \hookrightarrow [\mathsf{Proj}(\mathsf{R})]$ and hence $[\mathsf{Rep}(\mathsf{R})] \twoheadrightarrow U_{\mathbb{A}}^-(\mathfrak{g})^\vee$,
  $U_{\mathbb{A}}^-(\mathfrak{g}) \simeq [\mathsf{Proj}(\mathsf{R})]$ and $[\mathsf{Rep}(\mathsf{R})] \simeq U_{\mathbb{A}}^-(\mathfrak{g})^\vee$ if $a_{ii}\neq 0$ for all $i\in I$,
- $V_{\mathbb{A}}(\Lambda) \simeq [\mathsf{Proj}(\mathsf{R}^\Lambda)]$ and $V_{\mathbb{A}}(\Lambda)^\vee \simeq [\mathsf{Proj}(\mathsf{R}^\Lambda)]$ if $a_{ii}\neq 0$ for all $i\in I$,

where

(a) $U_{\mathbb{A}}^{-}(\mathfrak{g})^{\vee}$ and $V_{\mathbb{A}}(\Lambda)^{\vee}$ are dual of $U_{\mathbb{A}}^{-}(\mathfrak{g})$ and $V_{\mathbb{A}}(\Lambda)$, respectively,

(b) R is the quiver Hecke algebra associated with the Borcherds-Cartan datum and a_{ii} is the (i,i)-entry of the Borcherds-Cartan matrix,

(c) $[\mathrm{Rep}(\mathsf{R})]$ and $[\mathrm{Rep}(\mathsf{R}^{\Lambda})]$ are the Grothendieck group of the category of finite dimensional graded R-modules and R^{Λ}-modules, respectively.

To accomplish these goals, we will employ the framework given in [14, 15]. However, unlike those papers, we will use *the perfect bases* introduced by Berenstein and Kazhdan ([2]). More precisely, introducing the notion of *strong* perfect bases, we show that $[\mathrm{Rep}(\mathsf{R})]$ is isomorphic to $U_{\mathbb{A}}^{-}(\mathfrak{g})^{\vee}$ and hence $[\mathsf{Proj}(\mathsf{R})]$ is isomorphic to $U_{\mathbb{A}}^{-}(\mathfrak{g})$ by duality.

After that we will prove similar results for quiver Hecke superalgebras $\mathcal{R}$. Notice that, in this case, the Borcherds-Cartan datum is indeed the Cartan datum and the Cartan matrix A is colored by I_{odd}.

This thesis is organized as follows. In Section 2.1 and 2.2, we recall the definition of a generalized Kac-Moody algebras $U(\mathfrak{g})$, a quantum generalized Kac-Moody algebras $U_q(\mathfrak{g})$ and some of their properties, which were proved in [3, 13]. In Section 2.3, we recall the lower crystal basis theory for $U_q(\mathfrak{g})$ developed in [12]. In Section 2.4, 2.5 and 2.6, we will develop the upper crystal and upper global basis theory for $U_q(\mathfrak{g})$ and its modules, which implies that existence of a perfect basis. In particular, we will give a characterization of $V_{\mathbb{A}}(\Lambda)^{\vee}$ in terms of strong perfect bases (See Proposition 2.5.3).

In Section 3.1, we will give the definition of the quiver Hecke algebras R for quantum generalized Kac-Moody algebras. In Section 3.2 and 3.3, we will prove the Poincaré-Birkhoff-Witt-type basis for R by constructing faithful representations of R. In Section 3.4, we will show that the categorical Serre relations hold for $[\mathsf{Proj}(\mathsf{R})]$ and $U_{\mathbb{A}}^{-}(\mathfrak{g})^{\vee}$ is *embedded properly* in $[\mathsf{Proj}(\mathsf{R})]$ if there exists an index $i \in I$ such that $a_{ii} = 0$. In Section 3.5, we will show that one can choose the set of isomorphism classes of irreducible R-modules satisfying the axioms of strong perfect bases. In Section 3.6, we define the simple root functors E_i, F_i and $\overline{F}_i$ and prove the existence of natural isomorphisms and short exact sequences, which play a crucial role in categorification of R. In Section 3.7, we prove that the functors E_i^{Λ} and F_i^{Λ}

are well-defined on $\mathsf{Proj}(\mathsf{R}^\Lambda)$ and they satisfy the axiom of $\mathfrak{sl}_2$-categorification. In Section 3.8, we conclude that R and R^Λ categorify the quantum generalized Kac-Moody algebras and their integrable highest weight modules.

In Chapter 4, we will prove super-versions of results given in Chapter 3 via quiver Hecke superalgebras $\mathcal{R}$. We will follow the same framework as in Chapter 3. Thus we will sometimes omit the proof or only give a sketch of the proof. However, working in the super case entails doing the following: (i) in Section 4.1, we recall the notion of supercategories and superbimodules. (ii) since all the categories dealt with in this chapter are supercategories, we need to determine the effect of the endofunctor Π on the supercategories (See Theorem 4.3.1). (iii) since $\mathcal{R}$ is a superalgebra, the $\mathbb{Z}_2$-grading must be considered in each computation. Thus, in super case, computations are generally much more complicated.

This thesis is based on the series of papers [15, 16, 18, 19]. The first two papers are in collaboration with Seok-Jin Kang and Masaki Kashiwara. The last two papers are jointly written with Seok-Jin Kang and Euiyong Park.

Chapter 2

Quantum generalized Kac-Moody algebras

2.1 Generalized Kac-Moody algebras

In this section, we briefly recall the definition of generalized Kac-Moody algebra associated with a Borcherds-Cartan datum and review its properties.

Let I be a countable (possibly infinite) index set. An integral square matrix $A = (a_{ij})_{i,j \in I}$ is called a *Borcherds-Cartan matrix* if it satisfies

$$\text{(i) } a_{ii} \in \{2,\ 0,\ -2, \ldots\}, \quad \text{(ii) } a_{ij} \le 0 \text{ for } i \ne j, \quad \text{(iii) } a_{ij} = 0 \text{ if } a_{ji} = 0.$$

For $i \in I$, i is said to be *real* if $a_{ii} = 2$ and i is said to be *imaginary* if $a_{ii} \le 0$. Set $I^{\mathrm{re}} = \{i \in I \mid a_{ii} = 2\}$ and $I^{\mathrm{im}} = \{i \in I \mid a_{ii} \le 0\}$. In this paper, we assume that A is *symmetrizable*; i.e., there is a diagonal matrix $D = \mathrm{diag}(d_i \in \mathbb{Z}_{>0} \mid i \in I)$ such that DA is symmetric.

A *Borcherds-Cartan datum* $(A, P, \Pi, \Pi^\vee)$ consists of

(i) a Borcherds-Cartan matrix A,

(ii) a free abelian group P, called the *weight lattice*,

(iii) $\Pi = \{\alpha_i \in P \mid \ i \in I\}$, called the set of *simple roots*,

(iv) $\Pi^\vee = \{h_i \mid i \in I\} \subset P^\vee := \mathrm{Hom}(P, \mathbb{Z})$, called the set of *simple coroots*, satisfying the following conditions:

(a) $\langle h_i, \alpha_j \rangle = a_{ij}$ for all $i, j \in I$,

(b) Π is linearly independent.

The weight lattice P has a symmetric bilinear pairing $(\ |\ )$ satisfying

$$(\alpha_i|\lambda) = d_i\langle h_i, \lambda\rangle \text{ for all } \lambda \in P.$$

Therefore, we have $(\alpha_i|\alpha_j) = d_i a_{ij}$. We set $P^+ := \{\Lambda \in P \mid \langle h_i, \Lambda\rangle \in \mathbb{Z}_{\geq 0}, \text{ for all } i \in I\}$, which is called the set of *dominant integral weights*. The free abelian group $Q := \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ is called the *root lattice*. Set $Q^+ = \sum_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$. For $\beta = \sum k_i\alpha_i \in Q^+$, $|\beta| := \sum_{i\in I} k_i$ is called the *height* of β.

Definition 2.1.1. [3] *The* generalized Kac-Moody algebra $\mathfrak{g}$ *associated with a Borcherds-Cartan datum* $(A, P, \Pi, \Pi^\vee)$ *is the Lie algebra over* $\mathbb{Q}$ *generated by* e_i, f_i $(i \in I)$ *and* $h \in P^\vee$ *satisfying the following relations:*

(i) $[h, h'] = 0$, *for all* $h, h' \in P^\vee$,

(ii) $[h, e_i] = \langle h, \alpha_i\rangle e_i$, $[h, f_i] = -\langle h, \alpha_i\rangle f_i$ *for all* $h \in P^\vee$,

(iii) $[e_i, f_j] = \delta_{ij} h_i$ *for* $i, j \in I$,

(iv) $(\mathrm{ad} e_i)^{1-a_{ij}}(e_j) = (\mathrm{ad} f_i)^{1-a_{ij}}(f_j) = 0$ *if* $i \in I^{\mathrm{re}}$ *and* $i \neq j$,

(v) $[e_i, e_j] = [f_i, f_j] = 0$ *if* $a_{ij} = 0$.

We denote by $U(\mathfrak{g})$ *the universal enveloping algebra* of $\mathfrak{g}$ and $U^+(\mathfrak{g})$(resp. $U^-(\mathfrak{g})$) the subalgebra of $U(\mathfrak{g})$ generated by e_i $(i \in I)$ (resp. f_i $(i \in I)$).

Definition 2.1.2. *We define* $\mathcal{O}_{int}$ *to be the category consisting of* $U(\mathfrak{g})$-*modules* V *satisfying the following properties:*

(i) V *has a* weight space decomposition *with finite-dimensional weight spaces; i.e.,*
$$V = \bigoplus_{\mu\in P} V_\mu \quad \textit{with} \quad \dim_{\mathbb{Q}} V_\mu < \infty,$$
where $V_\mu = \{v \in V \mid h\, v = \langle h, \mu\rangle v \textit{ for all } h \in P^\vee\}$,

(ii) *there are finitely many* $\lambda_1, \ldots, \lambda_s \in P$ *such that*

$$\mathrm{wt}(V) := \{\mu \in P \mid V_\mu \neq 0\} \subset \bigcup_{i=1}^{s} (\lambda_i - Q_+),$$

(iii) *the action of* f_i *on* V *is locally nilpotent for* $i \in I^{\mathrm{re}}$,

(iv) *if* $i \in I^{\mathrm{im}}$, *then* $\langle h_i, \mu\rangle \in \mathbb{Z}_{\geq 0}$ *for all* $\mu \in \mathrm{wt}(V)$,

(v) *if* $i \in I^{\mathrm{im}}$ *and* $\langle h_i, \mu\rangle = 0$, *then* $f_i V_\mu = 0$,

(vi) *if* $i \in I^{\mathrm{im}}$ *and* $\langle h_i, \mu\rangle = -a_{ii}$, *then* $e_i V_\mu = 0$.

For $\lambda \in P$, a $U(\mathfrak{g})$-module V which admits a weight space decomposition is called a *highest weight module* with *highest weight* λ and *highest weight vector* v_λ if there exists a nonzero element $v_\lambda \in V_\lambda$ such that

$$(1)\ V = U(\mathfrak{g})v_\lambda, \quad (2)\ hv_\lambda = \langle h, \lambda\rangle v_\lambda, \quad (3)\ e_i v_\lambda = 0 \ \text{ for all } \ i \in I.$$

Let $J(\lambda)$ be the left ideal of $U(\mathfrak{g})$ generated by e_i, $h - \langle h, \lambda\rangle 1$ for all $i \in I, h \in P^\vee$. Set $M(\lambda) := U(\mathfrak{g})/J(\lambda)$, which is called the *Verma module*.

Proposition 2.1.1. [3]

(a) $M(\lambda)$ *is a highest weight module with highest weight* λ *and highest weight vector* $v_\lambda := 1 + J(\lambda)$.

(b) $M(\lambda)$ *is a free* $U^-(\mathfrak{g})$*-module of rank 1 generated by* v_λ.

(c) *Every highest weight* $U(\mathfrak{g})$*-module with highest weight* λ *is isomorphic to a quotient of* $M(\lambda)$.

(d) $M(\lambda)$ *has a unique maximal proper submodule* $R(\lambda)$.

If we set $V(\lambda) := M(\lambda)/R(\lambda)$, then $V(\lambda)$ is the unique up to isomorphism irreducible highest weight $U(\mathfrak{g})$-module with highest weight λ.

2.2 Quantum deformation

Throughout this section, we will deal with the quantum generalized Kac-Moody algebra $U_q(\mathfrak{g})$. By taking the classical limit, the algebra $U_q(\mathfrak{g})$ can be considered as a quantum deformation of $U(\mathfrak{g})$. Hence $U(\mathfrak{g})$ *embedded properly* $U_q(\mathfrak{g})$ and $U_q(\mathfrak{g})$ has more algebraic information which reflect the properties of $U(\mathfrak{g})$ and its modules. We mainly follow [12, 13].

Let q be an indeterminate and $m, n \in \mathbb{Z}_{\geq 0}$. Set

$$q_i = q^{d_i} \text{ for } i \in I.$$

If $i \in I^{\text{re}}$, define

$$[n]_i := \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! := \prod_{k=1}^{n} [k]_i, \quad \begin{bmatrix} m \\ n \end{bmatrix}_i = \frac{[m]_i!}{[m-n]_i![n]_i!}.$$

If $a_{ii} < 0$, set $c_i = -\frac{1}{2} a_{ii} \in \mathbb{Z}_{>0}$ and define

$$\{n\}_i := \frac{q_i^{c_i \cdot n} - q_i^{-c_i \cdot n}}{q_i^{c_i} - q_i^{-c_i}}, \quad \{n\}_i! := \prod_{k=1}^{n} \{k\}_i, \quad \begin{Bmatrix} m \\ n \end{Bmatrix}_i = \frac{\{m\}_i!}{\{m-n\}_i!\{n\}_i!}.$$

If $a_{ii} = 0$, set $c_{ii} = 0$ and define

$$\{n\}_i := n, \quad \{n\}_i! := n!, \quad \begin{Bmatrix} m \\ n \end{Bmatrix}_i = \binom{m}{n}.$$

Definition 2.2.1. [13] *The* quantum generalized Kac-Moody algebra $U_q(\mathfrak{g})$ *associated with a Borcherds-Cartan datum* $(A, P, \Pi, \Pi^\vee)$ *is the associative algebra over* $\mathbb{Q}(q)$ *with 1 generated by* e_i, f_i $(i \in I)$ *and* q^h $(h \in P^\vee)$ *satisfying following relations:*

(i) $q^0 = 1, q^h q^{h'} = q^{h+h'}$ *for* $h, h' \in P^\vee$,

(ii) $q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i,\ q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} f_i$ *for* $h \in P^\vee, i \in I$,

(iii) $e_i f_j - f_j e_i = \delta_{ij} \dfrac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$, *where* $K_i = q^{d_i \cdot h_i}$,

(iv) $\displaystyle\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_i e_i^{1-a_{ij}-r} e_j e_i^r = 0 \quad \textit{if } i \in I^{\mathrm{re}} \textit{ and } i \neq j,$

(v) $\displaystyle\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_i f_i^{1-a_{ij}-r} f_j f_i^r = 0 \quad \textit{if } i \in I^{\mathrm{re}} \textit{ and } i \neq j,$

(vi) $e_i e_j - e_j e_i = 0,\ f_i f_j - f_j f_i = 0 \quad \textit{if } a_{ij} = 0.$

Let $U_q^+(\mathfrak{g})$ (resp. $U_q^-(\mathfrak{g})$) be the subalgebra of $U_q(\mathfrak{g})$ generated by the elements e_i (resp. f_i), and let $U_q^0(\mathfrak{g})$ be the subalgebra of $U_q(\mathfrak{g})$ generated by q^h ($h \in P^\vee$). Then we have the *triangular decomposition*

$$U_q(\mathfrak{g}) \cong U_q^-(\mathfrak{g}) \otimes U_q^0(\mathfrak{g}) \otimes U_q^+(\mathfrak{g}),$$

and the *root space decomposition*

$$U_q(\mathfrak{g}) = \bigoplus_{\alpha \in Q} U_q(\mathfrak{g})_\alpha,$$

where $U_q(\mathfrak{g})_\alpha := \{x \in U_q(\mathfrak{g}) \mid q^h x q^{-h} = q^{\langle h, \alpha \rangle} x \text{ for any } h \in P^\vee\}$. Define a $\mathbb{Q}$-algebra automorphism ${}^-\colon U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ by

$$e_i \mapsto e_i, \quad f_i \mapsto f_i, \quad q^h \mapsto q^{-h}, \quad q \mapsto q^{-1}. \tag{2.1}$$

Let $\mathbb{A} = \mathbb{Z}[q, q^{-1}]$. For $n \in \mathbb{Z}_{>0}$, set

$$e_i^{(n)} = \begin{cases} \dfrac{e_i^n}{[n]_i!} & \text{if } i \in I^{\mathrm{re}}, \\ e_i^n & \text{if } i \in I^{\mathrm{im}}, \end{cases} \qquad f_i^{(n)} = \begin{cases} \dfrac{f_i^n}{[n]_i!} & \text{if } i \in I^{\mathrm{re}}, \\ f_i^n & \text{if } i \in I^{\mathrm{im}}, \end{cases}.$$

Denote by $U_{\mathbb{A}}^-(\mathfrak{g})$ (resp. $U_{\mathbb{A}}^+(\mathfrak{g})$) the $\mathbb{A}$-subalgebra of $U_q^-(\mathfrak{g})$ generated by $f_i^{(n)}$ (resp. $e_i^{(n)}$) and denote by $U_{\mathbb{A}}^0(\mathfrak{g})$ the $\mathbb{A}$-subalgebra generated by q^h and $\prod_{k=1}^m \dfrac{1-q^k q^h}{1-q^k}$ for all $m \in \mathbb{Z}_{>0}$ and $h \in P^\vee$. Let $U_{\mathbb{A}}(\mathfrak{g})$ be the $\mathbb{A}$-subalgebra generated by $U_{\mathbb{A}}^0(\mathfrak{g})$, $U_{\mathbb{A}}^+(\mathfrak{g})$ and $U_{\mathbb{A}}^-(\mathfrak{g})$. Then $U_{\mathbb{A}}(\mathfrak{g})$ also has the triangular decomposition

$$U_{\mathbb{A}}(\mathfrak{g}) \simeq U_{\mathbb{A}}^-(\mathfrak{g}) \otimes U_{\mathbb{A}}^0(\mathfrak{g}) \otimes U_{\mathbb{A}}^+(\mathfrak{g}).$$

Define a twisted algebra structure on $U_q^-(\mathfrak{g}) \otimes U_q^-(\mathfrak{g})$ as follows:

$$(x_1 \otimes x_2)(y_1 \otimes y_2) = q^{-(\beta_2|\gamma_1)}(x_1 y_1 \otimes x_2 y_2),$$

where $x_i \in U_q^-(\mathfrak{g})_{\beta_i}$ and $y_i \in U_q^-(\mathfrak{g})_{\gamma_i}$ $(i = 1, 2)$. Then there is an algebra homomorphism $\Delta_0 : U_q^-(\mathfrak{g}) \to U_q^-(\mathfrak{g}) \otimes U_q^-(\mathfrak{g})$ satisfying

$$\Delta_0(f_i) := f_i \otimes 1 + 1 \otimes f_i \ (i \in I). \tag{2.2}$$

Definition 2.2.2. *We define $\mathcal{O}^q_{int}$ to be the category consisting of $U_q(\mathfrak{g})$-modules V satisfying the following properties:*

(i) *V has a* weight space decomposition *with finite-dimensional weight spaces; i.e.,*

$$V = \bigoplus_{\mu \in P} V_\mu \quad \textit{with} \quad \dim V_\mu < \infty,$$

where $V_\mu = \{v \in V \mid q^h\, v = q^{\langle h, \mu\rangle} v,\ \text{for all } h \in P^\vee\}$,

(ii) *there are finitely many $\lambda_1, \ldots, \lambda_s \in P$ such that*

$$\mathrm{wt}(V) := \{\mu \in P \mid V_\mu \neq 0\} \subset \bigcup_{i=1}^{s} (\lambda_i - Q_+),$$

(iii) *the action of f_i on V is locally nilpotent for $i \in I^{\mathrm{re}}$,*

(iv) *if $i \in I^{\mathrm{im}}$, then $\langle h_i, \mu\rangle \in \mathbb{Z}_{\geq 0}$ for all $\mu \in \mathrm{wt}(V)$,*

(v) *if $i \in I^{\mathrm{im}}$ and $\langle h_i, \mu\rangle = 0$, then $f_i V_\mu = 0$,*

(vi) *if $i \in I^{\mathrm{im}}$ and $\langle h_i, \mu\rangle = -a_{ii}$, then $e_i V_\mu = 0$.*

For $\lambda \in P$, a $U_q(\mathfrak{g})$-module V which admits a weight space decomposition is called a *highest weight module* with *highest weight* λ and *highest weight vector* v_λ if there exists a nonzero vector $v_\lambda \in V_\lambda$ such that

$$(1)\ V = U_q(\mathfrak{g}) v_\lambda, \quad (2)\ q^h v_\lambda = q^{\langle h, \lambda\rangle} v_\lambda, \quad (3)\ e_i v_\lambda = 0 \quad \text{for all } i \in I. \tag{2.3}$$

Let $J_q(\lambda)$ be the left ideal of $U_q(\mathfrak{g})$ generated by $e_i, q^h - q^{\langle h, \lambda\rangle} 1$ for all $iinI, h \in P^\vee$. Set $M_q(\lambda) := U_q(\mathfrak{g})/J_q(\lambda)$, which is called the *Verma module*.

Proposition 2.2.1. [13]

(i) *$M_q(\lambda)$ is a highest weight module with highest weight λ and highest weight vector $v_\lambda := 1 + J_q(\lambda)$.*

(ii) *$M_q(\lambda)$ is a free $U_q^-(\mathfrak{g})$-module generated by v_λ.*

(iii) *Every highest weight $U_q(\mathfrak{g})$-module with highest weight λ is a quotient of $M_q(\lambda)$.*

(iv) *$M_q(\lambda)$ has the unique maximal proper submodule $R_q(\lambda)$.*

The quotient $V_q(\lambda) := M_q(\lambda)/R_q(\lambda)$ is an irreducible highest weight module. The following theorem shows that there exists a 1-1 correspondence between P^+ and the set of irreducible objects in $\mathcal{O}^q_{int}$.

Theorem 2.2.1. [12, Theorem 3.7] *Every $U_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{int}$ is isomorphic to a direct sum of irreducible highest weight modules $V_q(\Lambda)$ with $\Lambda \in P^+$.*

Let $\mathbb{A}_1 = \{f/g \in \mathbb{Q}(q) \mid f, g \in \mathbb{Q}[q], g(1) \neq 0\}$ and $\mathbb{J}$ the ideal of $\mathbb{A}_1$ generated by the element $q - 1 \in \mathbb{A}_1$. Note that $\mathbb{A}_1/\mathbb{J} \simeq \mathbb{Q}$. Let $U^0_{\mathbb{A}_1}(\mathfrak{g})$ be the $\mathbb{A}_1$-subalgebra of $U_q(\mathfrak{g})$ generated by q^h and $\dfrac{q^h - 1}{q - 1}$ for $h \in P^\vee$. We denote by $U^+_{\mathbb{A}_1}(\mathfrak{g})$ (reps. $U^-_{\mathbb{A}_1}(\mathfrak{g})$) the subalgebra of $U_q(\mathfrak{g})$ generated by $e_i^{(n)}$ (resp. $f_i^{(n)}$) for $i \in I$ and $n \in \mathbb{Z}_{\geq 0}$. Let $U_{\mathbb{A}_1}(\mathfrak{g})$ be the $\mathbb{A}_1$-subalgebra of $U_q(\mathfrak{g})$ generated by $U^0_{\mathbb{A}_1}(\mathfrak{g}), U^+_{\mathbb{A}_1}(\mathfrak{g})$ and $U^-_{\mathbb{A}_1}(\mathfrak{g})$. Then we have

$$U_{\mathbb{A}_1}(\mathfrak{g}) \cong U^-_{\mathbb{A}_1}(\mathfrak{g}) \otimes U^0_{\mathbb{A}_1}(\mathfrak{g}) \otimes U^+_{\mathbb{A}_1}(\mathfrak{g}).$$

For an irreducible highest weight $U_q(\mathfrak{g})$-module $V_q(\lambda) \in \mathcal{O}^q_{int}$ with highest weight λ and highest weight vector v_λ, we define

$$V_{\mathbb{A}_1}(\lambda) := U_{\mathbb{A}_1}(\mathfrak{g}) v_\lambda = U^-_{\mathbb{A}_1}(\mathfrak{g}) v_\lambda,$$

and

$$U^1 := U_{\mathbb{A}_1}(\mathfrak{g})/\mathbb{J}U_{\mathbb{A}_1}(\mathfrak{g}) \simeq \mathbb{Q} \otimes_{\mathbb{A}_1} U_{\mathbb{A}_1}(\mathfrak{g}), \;\; V^1(\lambda) := V_{\mathbb{A}_1}(\lambda)/\mathbb{J}V_{\mathbb{A}_1}(\lambda) \simeq \mathbb{Q} \otimes_{\mathbb{A}_1} V_{\mathbb{A}_1}(\lambda).$$

Then we have

$$V^1(\lambda) = \bigoplus_{\mu \in P} V^1_\mu \quad \text{and} \quad \dim_{\mathbb{Q}} V^1_\mu = \operatorname{rank}_{\mathbb{A}_1}(V_{\mathbb{A}_1})_\mu = \dim_{\mathbb{Q}(q)} V_\mu,$$

where $V^1_\mu = \mathbb{Q} \otimes_{\mathbb{A}_1} V_{\mathbb{A}_1}(\lambda)_\mu$ for $\mu \in P$.

Consider the following natural projection maps

$$U_{\mathbb{A}_1}(\mathfrak{g}) \longrightarrow U^1 \quad \text{and} \quad V_{\mathbb{A}_1}(\lambda) \longrightarrow V^1(\lambda).$$

The process obtaining $V^1(\lambda)$ from $V_q(\lambda)$ via the above projection maps is referred to as taking the *classical limit*.

Theorem 2.2.2. [13]

(a) *There is an algebra isomorphism between U^1 and $U(\mathfrak{g})$.*

(b) *For an irreducible highest weight $U_q(\mathfrak{g})$-module $V_q(\Lambda) \in \mathcal{O}^q_{int}$, $V^1(\Lambda)$ becomes a $U(\mathfrak{g})$-module, and is isomorphic to $V(\Lambda) \in \mathcal{O}_{int}$.*

Corollary 2.2.1. *Every $U(\mathfrak{g})$-module in $\mathcal{O}_{int}$ is isomorphic to a direct sum of irreducible highest weight modules $V(\Lambda)$ with $\Lambda \in P^+$.*

For a fixed $i \in I$, let

(2.4) U_i be the subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i, q^h for all $h \in P^\vee$.

Then U_i can be considered as a quantum generalized Kac-Moody algebra associated with $A = (a_{ii})$.

Proposition 2.2.2. [12] *Suppose $a_{ii} \leq 0$ and let V be the irreducible highest weight U_i-module of highest weight λ and highest wight vector v_λ. Then we have:*

(a) *if $\langle h_i, \lambda \rangle = 0$, then $V = \mathbb{Q}(q)v_\lambda$,*

(b) *if $\langle h_i, \lambda \rangle > 0$, then V has a basis $\{f_i^n v_\lambda\}_{n \geq 0}$.*

2.3 Lower crystal bases

In this section, we briefly review the lower crystal basis theory of quantum generalized Kac-Moody algebra and its integrable modules which was developed in [12].

Fix $i \in I$. For any $P \in U_q^-(\mathfrak{g})$, there exist unique elements $Q, R \in U_q^-(\mathfrak{g})$ such that

$$e_i P - P e_i = \frac{K_i Q - K_i^{-1} R}{q_i - q_i^{-1}}.$$

We define the endomorphisms $e_i', e_i'' : U_q^-(\mathfrak{g}) \to U_q^-(\mathfrak{g})$ by

$$e_i'(P) = R, \quad \mathrm{e_i''}(P) = Q.$$

Consider f_i as the endomorphism of $U_q^-(\mathfrak{g})$ defined by left multiplication by f_i. Then we have

$$e_i' f_j = \delta_{ij} + q_i^{-a_{ij}} f_j e_i'. \tag{2.5}$$

Let

$$e_i'^{(n)} = \begin{cases} (e_i')^n & \text{if } i \in I^{\mathrm{re}}, \\ \dfrac{(e_i')^n}{\{n\}_i!} & \text{if } i \in I^{\mathrm{im}}, \end{cases}$$

Then we obtain the following commutation relations:
(2.6)

$$e_i'^{(n)} f_j^{(m)} = \begin{cases} \displaystyle\sum_{k=0}^{n} q_i^{-2nm+(n+m)k-k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_i f_i^{(m-k)} e_i'^{(n-k)} & \text{if } i = j \text{ and } i \in I^{\mathrm{re}}, \\ \displaystyle\sum_{k=0}^{m} q_i^{-c_i(-2nm+(n+m)k-k(k-1)/2)} \begin{Bmatrix} m \\ k \end{Bmatrix}_i f_i^{(m-k)} e_i'^{(n-k)} & \text{if } i = j \text{ and } i \in I^{\mathrm{im}}, \\ q_i^{-nma_{ij}} f_j^{(m)} e_i'^{(n)} & \text{if } i \neq j. \end{cases}$$

Definition 2.3.1. *The quantum boson algebra $B_q(\mathfrak{g})$ associated with a Borcherds-Cartan matrix A is the associative algebra over $\mathbb{Q}(q)$ generated by e_i', f_i $(i \in I)$ satisfying the following relations:*

(i) $e_i' f_j = q_i^{-a_{ij}} f_j e_i' + \delta_{ij}$,

(ii) $\displaystyle\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_i {e'_i}^{1-a_{ij}-r} e'_j {e'_i}^r = 0 \qquad \text{if } i \in I^{\mathrm{re}},\ i \neq j,$

(iii) $\displaystyle\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_i {f_i}^{1-a_{ij}-r} f_j {f_i}^r = 0 \qquad \text{if } i \in I^{\mathrm{re}},\ i \neq j,$

(iv) $e'_i e'_j - e'_j e'_i = 0,\ f_i f_j - f_j f_i = 0 \qquad \text{if } a_{ij} = 0.$

We denote by $B_{\mathbb{A}}^{\mathrm{low}}(\mathfrak{g})$ (resp. $B_{\mathbb{A}}^{\mathrm{up}}(\mathfrak{g})$) the $\mathbb{A}$-subalgebra of $B_q(\mathfrak{g})$ generated by e'_i and $f_i^{(n)}$ (resp. by ${e'_i}^n/[n]_i!$ and f_i) for all $i \in I$ and $n \in \mathbb{Z}_{>0}$.

Proposition 2.3.1. [12, 21, 19]

(a) *If $x \in U_q^-(\mathfrak{g})$ and $e'_i x = 0$ for all $i \in I$, then x is a constant multiple of $\mathbf{1}$, the identity element of $U_q^-(\mathfrak{g})$.*

(b) *$U_q^-(\mathfrak{g})$ is a simple $B_q(\mathfrak{g})$-module.*

For any homogeneous element $u \in U_q^-(\mathfrak{g})$, u can be expressed uniquely as

$$u = \sum_{l \geq 0} f_i^{(l)} u_l, \tag{2.7}$$

where $e'_i u_l = 0$ for every $l \geq 0$ and $u_l = 0$ for $l \gg 0$. We call this the *i-string decomposition* of u in $U_q^-(\mathfrak{g})$. We define the *lower Kashiwara operators* $\tilde{e}_i$, $\tilde{f}_i$ $(i \in I)$ of $U_q^-(\mathfrak{g})$ by

$$\tilde{e}_i u = \sum_{k \geq 1} f_i^{(k-1)} u_k, \quad \tilde{f}_i u = \sum_{k \geq 0} f_i^{(k+1)} u_k.$$

Let $\mathbb{A}_0 = \{f/g \in \mathbb{Q}(q) \mid f, g \in \mathbb{Q}[q], g(0) \neq 0\}$.

Definition 2.3.2. *A lower crystal basis of $U_q^-(\mathfrak{g})$ is a pair (L, B) satisfying the following conditions:*

(i) *L is a free $\mathbb{A}_0$-module of $U_q^-(\mathfrak{g})$ such that $U_q^-(\mathfrak{g}) = \mathbb{Q}(q) \otimes_{\mathbb{A}_0} L$ and $L = \bigoplus_{\alpha \in Q^+} L_{-\alpha}$, where $L_{-\alpha} := L \cap U_q^-(\mathfrak{g})_{-\alpha}$,*

(ii) *B is a $\mathbb{Q}$-basis of L/qL such that $B = \bigsqcup_{\alpha \in Q^+} B_{-\alpha}$, where $B_{-\alpha} := B \cap (L_{-\alpha}/qL_{-\alpha})$,*

(iii) $\tilde{e}_i B \subset B \sqcup \{0\}$, $\tilde{f}_i B \subset B$ *for all* $i \in I$,

(iv) *For* $b, b' \in B$ *and* $i \in I$, $b' = \tilde{f}_i b$ *if and only if* $b = \tilde{e}_i b'$.

Proposition 2.3.2. [12, Theorem 7.1 (b)] *Let* $L(\infty)$ *be the free* $\mathbb{A}_0$*-module of* $U_q^-(\mathfrak{g})$ *generated by* $\{\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} \mathbf{1} \mid r \geq 0, i_k \in I\}$ *and let*

$$B(\infty) = \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} \mathbf{1} + qL(\infty) \mid r \geq 0, i_k \in I\} \setminus \{0\}.$$

Then the pair $(L(\infty), B(\infty))$ *is the unique lower crystal basis of* $U_q^-(\mathfrak{g})$.

Let M be a $U_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{int}$. For any $i \in I$ and $u \in M_\mu$, the element u can be expressed uniquely as

$$u = \sum_{k \geq 0} f_i^{(k)} u_k, \tag{2.8}$$

where $u_k \in M_{\mu + k\alpha_i}$ and $e_i u_k = 0$. We call this the *i-string decomposition* of u. We define the *lower Kashiwara operators* $\tilde{e}_i, \tilde{f}_i$ $(i \in I)$ by

$$\tilde{e}_i u = \sum_{k \geq 1} f_i^{(k-1)} u_k, \quad \tilde{f}_i u = \sum_{k \geq 0} f_i^{(k+1)} u_k.$$

Definition 2.3.3. *A lower crystal basis of* $U_q(\mathfrak{g})$*-module* M *is a pair* (L, B) *satisfying the following conditions:*

(i) L *is a free* $\mathbb{A}_0$*-module of* M *such that* $M = \mathbb{Q}(q) \otimes_{\mathbb{A}_0} L$ *and* $L = \bigoplus_{\lambda \in P} L_\lambda$, *where* $L_\lambda := L \cap M_\lambda$,

(ii) B *is* $\mathbb{Q}$*-basis of* L/qL *such that* $B = \bigsqcup_{\lambda \in P} B_\lambda$, *where* $B_\lambda := B \cap L_\lambda / qL_\lambda$,

(iii) $\tilde{e}_i B \subset B \sqcup \{0\}$, $\tilde{f}_i B \subset B \sqcup \{0\}$ *for all* $i \in I$,

(iv) *For* $b, b' \in B$ *and* $i \in I$, $b' = \tilde{f}_i b$ *if and only if* $b = \tilde{e}_i b'$.

Proposition 2.3.3. [12, Theorem 7.1 (a)] *For* $\lambda \in P^+$, *let* $L(\lambda)$ *be the free* $\mathbb{A}_0$*-module of* $V(\lambda)$ *generated by* $\{\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} v_\lambda \mid r \geq 0, i_k \in I\}$ *and let*

$$B(\lambda) = \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} v_\lambda + qL(\lambda) \mid r \geq 0, i_k \in I\} \setminus \{0\}.$$

Then the pair $(L(\lambda), B(\lambda))$ *is the unique lower crystal basis of* $V(\lambda)$.

2.4 Upper crystal bases

In this section, we will develop the upper crystal bases of $U_q^-(\mathfrak{g})$ and $V_q(\Lambda)$ which are *dual* to the lower crystal bases. From the upper crystal bases, we can derived the global bases which will provide the existence of perfect bases of $U^-(\mathfrak{g})$ and $V(\Lambda)$ in the succeeding section .

Consider the anti-automorphism φ on $B_q(\mathfrak{g})$ defined by

$$e_i' \mapsto f_i \quad \text{and} \quad f_i \mapsto e_i'.$$

We define the symmetric bilinear forms $(\ ,\ )_K$ and $(\ ,\ )_L$ on $U_q^-(\mathfrak{g})$ as follows (cf. [21, Proposition 3.4.4], [29, Chapter 1]):

$$\begin{aligned} &(\mathbf{1},\mathbf{1})_K = 1, \quad (e_i'x, y)_K = (x, f_i y)_K, \\ &(\mathbf{1},\mathbf{1})_L = 1, \quad (f_i, f_j)_L = \delta_{ij}(1-q_i^2)^{-1}, \quad (x, yz)_L = (\Delta_0(x), y\otimes z)_L \end{aligned} \tag{2.9}$$

for $x, y, z \in U_q^-(\mathfrak{g})$.

Lemma 2.4.1.

1. *The bilinear form $(\ ,\ )_K$ on $U_q^-(\mathfrak{g})$ is nondegenerate.*
2. *For homogeneous elements $x \in U_q^-(\mathfrak{g})_{-\alpha}$ and $y \in U_q^-(\mathfrak{g})_{-\beta}$, we have*
$$(x,y)_L = \prod_{i\in I} \frac{1}{(1-q_i^2)^{k_i}}(x,y)_K,$$
where $\alpha = \sum_{i\in I} k_i\alpha_i \in Q^+$. Hence $(\ ,\ )_L$ is nondegenerate.
3. *For any $x, y \in U_q^-(\mathfrak{g})$, we have*
$$(e_i'x, y)_L = (1-q_i^2)(x, f_i y)_L.$$

Proof. The assertion (1) is proved in [12].

It was shown in [31, (2.4)] that the bilinear form $(\ ,\ )_K$ satisfies

$$(x, yz)_K = \sum_n (x_n^{(1)}, y)_K (x_n^{(2)}, z)_K,$$

where $\Delta_0(x) = \sum_n x_n^{(1)} \otimes x_n^{(2)}$. Then assertion (2) can be proved by induction on $|\alpha|$.

To prove assertion (3), without loss of generality, we may assume that $x \in U_q^-(\mathfrak{g})_{-\alpha}$, where $\alpha = -\sum_i k_i \alpha_i \in -Q^+$. Then by (2) and the definition of $(\ ,\ )_K$, we have

$$\begin{aligned}(e_i' x, y)_L &= \frac{1}{(1-q_i^2)^{k_i - 1}} \prod_{j \neq i} \frac{1}{(1-q_j^2)^{k_j}} (e_i' x, y)_K \\ &= \frac{1-q_i^2}{(1-q_i^2)^{k_i}} \prod_{j \neq i} \frac{1}{(1-q_j^2)^{k_j}} (x, f_i y)_K \\ &= (1-q_i^2)(x, f_i y)_L,\end{aligned}$$

which proves assertion (3). □

Now, we define the *upper Kashiwara operators* for the $B_q(\mathfrak{g})$-module $U_q^-(\mathfrak{g})$. Let $u \in U_q^-(\mathfrak{g})$ such that $e_i' u = 0$. Then, for $n \in \mathbb{Z}_{\geq 0}$, we define the upper Kashiwara operators $\tilde{E}_i, \tilde{F}_i$ by

$$\tilde{E}_i(f_i^{(n)} u) = \begin{cases} \dfrac{q_i^{-(n-1)}}{[n]_i} f_i^{(n-1)} u & \text{if } i \in I^{\mathrm{re}}, \\ \{n\}_i q_i^{c_i(n-1)} f_i^{(n-1)} u & \text{if } i \in I^{\mathrm{im}}, \end{cases}$$

$$\tilde{F}_i(f_i^{(n)} u) = \begin{cases} q_i^n [n+1]_i f_i^{(n+1)} u & \text{if } i \in I^{\mathrm{re}}, \\ \dfrac{1}{\{n+1\}_i q_i^{c_i n}} f_i^{(n+1)} u & \text{if } i \in I^{\mathrm{im}}. \end{cases}$$

From the i-string decomposition (2.7), the upper Kashiwara operators $\tilde{E}_i$ and $\tilde{F}_i$ can be extended to the whole space $U_q^-(\mathfrak{g})$ by linearity.

Definition 2.4.1. *An upper crystal basis of $U_q^-(\mathfrak{g})$ is a pair $(L^\vee, B^\vee)$ satisfying the following conditions:*

(i) *$L^\vee$ is a free $\mathbb{A}_0$-module of $U_q^-(\mathfrak{g})$ such that $U_q^-(\mathfrak{g}) = \mathbb{Q}(q) \otimes_{\mathbb{A}_0} L^\vee$ and $L^\vee = \bigoplus_{\alpha \in Q^+} L^\vee_{-\alpha}$, where $L^\vee_{-\alpha} := L^\vee \cap U_q^-(\mathfrak{g})_{-\alpha}$,*

(ii) *$B^\vee$ is a $\mathbb{Q}$-basis of $L^\vee / q L^\vee$ such that $B^\vee = \bigsqcup_{\alpha \in Q^+} B^\vee_{-\alpha}$, where $B^\vee_{-\alpha} := B^\vee \cap (L^\vee_{-\alpha} / q L^\vee_{-\alpha})$,*

(iii) $\tilde{E}_i B^\vee \subset B^\vee \sqcup \{0\}$, $\tilde{F}_i B^\vee \subset B^\vee$ *for all* $i \in I$,

(iv) *For* $b, b' \in B^\vee$ *and* $i \in I$, $b' = \tilde{F}_i b$ *if and only if* $b = \tilde{E}_i b'$.

Lemma 2.4.2. *For any* $u, v \in U_q^-(\mathfrak{g})$, *we have*

$$(\tilde{E}_i u, v) = (u, \tilde{f}_i v) \quad \text{and} \quad (\tilde{F}_i u, v) = (u, \tilde{e}_i v).$$

Proof. First, we consider the case $\tilde{E}_i$. Fixing $i \in I$, it suffices to consider $I = \{i\}$. Moreover, by the i-string decomposition (2.7), we may assume u, v are of the form $f_i^{(k)} u_k$ and $f_i^{(k-1)} v_{k-1}$ where $u_k, v_{k-1} \in \mathrm{Ker}(e_i')$. For $i \in I^{\mathrm{re}}$, we have

$$(f_i^{(k)} u_k, \frac{f_i}{[k]_i} f_i^{(k-1)} v_{k-1}) = \frac{1}{[k]_i} (e_i' f_i^{(k)} u_k, f_i^{(k-1)} v_{k-1}).$$

Since $e_i' f_i^{(k)} u_k = q_i^{-k+1} f_i^{(k-1)} u_k$, we have

$$(f_i^{(k)} u_k, \tilde{f}_i (f_i^{(k-1)} v_{k-1})) = \frac{q_i^{-k+1}}{[k]_i} (f_i^{(k-1)} u_k, f_i^{(k-1)} v_{k-1}) = (\tilde{E}_i (f_i^{(k)} u_k), f_i^{(k-1)} v_{k-1}).$$

For $i \in I^{\mathrm{im}}$,

$$(f_i^k u_k, f_i (f_i^{k-1} v_{k-1})) = (e_i' (f_i^k u_k), f_i^{k-1} v_{k-1}).$$

By (2.6), $e_i'(f_i^k u_k) = \{k\}_i q_i^{c_i(k-1)} f_i^{k-1} u_k$. Thus

$$(f_i^k u_k, \tilde{f}_i (f_i^{k-1} v_{k-1})) = \{k\}_i q_i^{c_i(k-1)} (f_i^{k-1} u_k, f_i^{k-1} v_{k-1}) = (\tilde{E}_i (f_i^k u_k), f_i^{k-1} v_{k-1}).$$

In a similar way, we can prove the desired formula for the $\tilde{F}_i$ case. □

Lemma 2.4.3. *Let* $u \in U_q^-(\mathfrak{g})$, *and* n *be the smallest integer such that* $e_i'^{n+1} u = 0$. *Then we have*

$$e_i'^n u = \begin{cases} [n]_i! \tilde{E}_i^n u & \text{if } i \in I^{\mathrm{re}}, \\ \tilde{E}_i^n u & \text{if } i \in I^{\mathrm{im}}. \end{cases}$$

Proof. For $u \in U_q^-(\mathfrak{g})$ and $i \in I$, consider the i-string decomposition: $u = \sum_{l=0}^n f_i^{(l)} u_l$, where $e_i' u_l = 0$. If $i \in I^{\mathrm{re}}$, then by (2.6) and the definition of $\tilde{E}_i$, we have

$$e_i'^n u = q_i^{-n(n-1)/2} u_n, \quad \tilde{E}_i^n u = \frac{q_i^{-n(n-1)/2}}{[n]_i!} u_n.$$

Similarly, if $i \in I^{\rm im}$, we obtain

$$e_i'^{(n)} u = q_i^{c_i n(n-1)/2} u_n, \quad \tilde{E}_i^n u = \{n\}_i! q_i^{c_i n(n-1)/2} u_n,$$

which proves our assertion. $\square$

Let $(L(\infty), B(\infty))$ be the lower crystal basis of $U_q^-(\mathfrak{g})$. Set

$$L(\infty)^\vee = \{u \in U_q^-(\mathfrak{g}) \mid (u, L(\infty))_K \subset \mathbb{A}_0\}.$$

We also denote by $(\ ,\ )_K : L(\infty)^\vee / qL(\infty)^\vee \times L(\infty)/qL(\infty) \to \mathbb{Q}$ the nondegenerate bilinear form induced by the bilinear form $(\ ,\ )_K$ on $U_q^-(\mathfrak{g})$. Let

$$B(\infty)^\vee = \{b \mid b \in B(\infty)\}$$

be the $\mathbb{Q}$-basis of $L(\infty)^\vee / qL(\infty)^\vee$ which is dual to $B(\infty)$ with respect to $(\ ,\ )_K$.

Proposition 2.4.1. *The pair $(L(\infty)^\vee, B(\infty)^\vee)$ is an upper crystal basis of $U_q^-(\mathfrak{g})$.*

Proof. The proof is almost the same as in [22]. $\square$

Let V be a $U_q(\mathfrak{g})$-module in $\mathcal{O}^q_{int}$, and take a weight vector $u \in V_\lambda$ with $e_i u = 0$. Then, for $n \in \mathbb{Z}_{\geq 0}$, we define *the upper Kashiwara operators* $\tilde{E}_i, \tilde{F}_i$ by

$$\tilde{E}_i(f_i^{(n)} u) = \begin{cases} \dfrac{[\langle h_i, \lambda\rangle - n + 1]_i}{[n]_i} f_i^{(n-1)} u & i \in I^{\rm re}, \\ \{n\}_i [\langle h_i, \lambda\rangle + c_i(n-1)]_i f_i^{(n-1)} u & i \in I^{\rm im}, \end{cases}$$

$$\tilde{F}_i(f_i^{(n)} u) = \begin{cases} \dfrac{[n+1]_i}{[\langle h_i, \lambda\rangle - n]_i} f_i^{(n+1)} u & i \in I^{\rm re}, \\ \dfrac{1}{\{n+1\}_i [\langle h_i, \lambda\rangle + c_i(n)]_i} f_i^{(n+1)} u & i \in I^{\rm im}. \end{cases}$$

From the i-string decomposition (2.8), the upper Kashiwara operators $\tilde{E}_i$ and $\tilde{F}_i$ can be extended to the whole space V by linearity.

Remark 2.4.1. *Recall the definition U_i in (2.4). Using the U_i-module structure on $V_q(\lambda)$ for $i \in I^{\rm im}$, we have*

$$e_i f_i^n v_\lambda = \{n\}_i [\lambda(h_i) + c_i(n-1)]_i f_i^{n-1} v_\lambda. \tag{2.10}$$

Thus we can see $\tilde{E}_i v = e_i v$ for all $v \in V$ and $i \in I$.

Now, we define the upper crystal bases for integrable modules over quantum generalized Kac-Moody algebras in a similar manner to $U_q^-(\mathfrak{g})$.

Definition 2.4.2. *An upper crystal basis of V is a pair $(L^\vee, B^\vee)$ satisfying the following conditions:*

(i) *$L^\vee$ is a free $\mathbb{A}_0$-module of $U_q^-(\mathfrak{g})$ such that $V = \mathbb{Q}(q) \otimes_{\mathbb{A}_0} L^\vee$ and $L^\vee = \bigoplus_{\alpha \in P} L^\vee_\alpha$, where $L^\vee_\alpha := L^\vee \cap V_\alpha$,*

(ii) *$B^\vee$ is a $\mathbb{Q}$-basis of $L^\vee/qL^\vee$ such that $B^\vee = \bigsqcup_{\alpha \in P} B^\vee_\alpha$, where $B^\vee_\alpha := B^\vee \cap (L^\vee_\alpha / qL^\vee_\alpha)$,*

(iii) *$\tilde{E}_i B^\vee \subset B^\vee \sqcup \{0\}$, $\quad \tilde{F}_i B^\vee \subset B^\vee \sqcup \{0\}$ for all $i \in I$,*

(iv) *For $b, b' \in B^\vee$ and $i \in I$, $b' = \tilde{F}_i b$ if and only if $b = \tilde{E}_i b'$.*

Consider the anti-automorphism of $U_q(\mathfrak{g})$ given by

$$e_i \mapsto f_i, \quad f_i \mapsto e_i \ \text{ and } \ q^h \mapsto q^h.$$

As the case of $B_q(\mathfrak{g})$-module $U_q^-(\mathfrak{g})$, one can show that there exists a unique non-degenerate symmetric bilinear form $(\ ,\ )_K$ on $V_q(\Lambda)$ $(\Lambda \in P^+)$ with highest weight vector v_Λ satisfying

$$(v_\Lambda, v_\Lambda)_K = 1, \quad (e_i u, v)_K = (u, f_i v)_K \text{ for } u,\ v \in V_q(\Lambda). \tag{2.11}$$

Lemma 2.4.4. *We have*

$$(\tilde{E}_i u, v)_K = (u, \tilde{f}_i v)_K \quad \textit{and} \quad (\tilde{F}_i u, v)_K = (u, \tilde{e}_i v)_K \ \textit{ for all } u, v \in V_q(\Lambda).$$

Proof. For $i \in I^{\mathrm{re}}$, our assertion is proved in [22]. For $i \in I^{\mathrm{im}}$, it suffices to consider the module $V_q(\Lambda) \in \mathcal{O}^q_{int}$ for the case $|I| = 1$, $I = I^{\mathrm{im}}$. Moreover, by the i-string decomposition and (2.11), we may assume u, v are of the form $f_i^k v_\Lambda$. Then we have

$$\begin{aligned}(f_i^k v_\lambda, f_i(f_i^{k-1} v_\lambda))_K &= (e_i(f_i^k v_\lambda), f_i^{k-1} v_\lambda)_K \\ &= \{k\}_i [\langle h_i, \lambda\rangle + c_i(k-1)]_i (f_i^{k-1} v_\lambda, f_i^{k-1} v_\lambda)_K.\end{aligned}$$

Thus we obtain

$$\begin{aligned}(f_i^k v_\lambda, \tilde{f}_i(f_i^{k-1} v_\lambda))_K &= \{k\}_i[\langle h_i, \lambda\rangle + c_i(k-1)]_i (f_i^{k-1} v_\lambda, f_i^{k-1} v_\lambda)_K \\ &= (\tilde{E}_i(f_i^k v_\lambda), f_i^{k-1} v_\lambda)_K.\end{aligned}$$

In a similar way, we can prove the desired formula for the $\tilde{F}_i$ case. □

Let (L, B) be a lower crystal basis of a $U_q(\mathfrak{g})$-module V in $\mathcal{O}^q_{int}$. Set

$$L^\vee = \{u \in V \mid (u, L)_K \subset \mathbb{A}_0\}.$$

We will also denote by $(\ ,\ )_K : L^\vee/qL^\vee \times L\ qL \to \mathbb{Q}$ the nondegenerate bilinear pairing induced by the bilinear form $(\ ,\ )_K$ on V. Let $b = \{b \mid b \in B\}$ be the $\mathbb{Q}$-basis of $L^\vee/qL^\vee$ which is dual to B with respect to $(\ ,\ )_K$.

Proposition 2.4.2. *The pair $(L^\vee, B^\vee)$ is an upper crystal basis of V.*

Proof. By a standard argument on nondegenerate bilinear forms, it is straightforward to verify all the conditions for upper crystal basis. We will only check the last condition. For $b_1^\vee, b_2^\vee \in b$ and $i \in I$, we have

$$\begin{aligned}\tilde{F}_i b_1^\vee = b_2^\vee &\iff 1 = (\tilde{F}_i b_1^\vee, b_2)_K = (b_1^\vee, \tilde{e}_i b_2)_K \iff \tilde{e}_i b_2 = b_1 \iff \tilde{f}_i b_1 = b_2 \\ &\iff 1 = (b_2^\vee, \tilde{f}_i b_1)_K = (\tilde{E}_i b_2^\vee, b_1)_K \iff \tilde{E}_i b_2^\vee = b_1^\vee\end{aligned}$$

as desired. □

2.5 Upper global bases

Crystal bases treated in the preceding sections can be understood as bases at $q = 0$. Globalizing these, we can get $\mathbb{Q}(q)$-bases, the *global bases*. In particular, we will study the upper global bases in more detail.

Let $\mathbb{A}_\infty$ be the subring of $\mathbb{Q}(q)$ consisting of regular functions at ∞. Let V be a $\mathbb{Q}(q)$-vector space. Let $V_\mathbb{A}$ (resp. L_0 and L_∞) be an $\mathbb{A}$-submodule (resp. $\mathbb{A}_0$-submodule and $\mathbb{A}_\infty$-submodule) of V.

Definition 2.5.1. *We say that $(V_\mathbb{A}, L_0, L_\infty)$ is a balanced triple if*

(i) $V \cong \mathbb{Q}(q) \otimes_\mathbb{A} V_\mathbb{A} \cong \mathbb{Q}(q) \otimes_{\mathbb{A}_0} L_0 \cong \mathbb{Q}(q) \otimes_{\mathbb{A}_\infty} L_\infty$ *as $\mathbb{Q}$-vector spaces,*

(ii) $V_{\mathbb{A}} \cong \mathbb{A} \otimes_{\mathbb{Q}} E, \quad L_0 \cong \mathbb{A}_0 \otimes_{\mathbb{Q}} E, \quad L_\infty \cong \mathbb{A}_\infty \otimes_{\mathbb{Q}} E$, *where* $E = V_{\mathbb{A}} \cap L_0 \cap L_\infty$.

Note that if $(V_{\mathbb{A}}, L_0, L_\infty)$ is a balanced triple, there exists a natural $\mathbb{Q}$-vector space isomorphism $E \xrightarrow{\sim} L_0/qL_0$, and vice versa. Equivalently, there exists a natural $\mathbb{Q}$-vector space isomorphism $E \xrightarrow{\sim} L_\infty/q^{-1}L_\infty$.

Recall the $\mathbb{Q}$-algebra automorphism $^-$: $U_q^-(\mathfrak{g}) \to U_q^-(\mathfrak{g})$ given in (2.1). For any given $\mathbb{A}_0$-submodule L of $U_q^-(\mathfrak{g})$, we denote by $\overline{L}$ the image of L under the involution $^-$.

Proposition 2.5.1. [12] $(U_{\mathbb{A}}^-(\mathfrak{g}), L(\infty), \overline{L(\infty)})$ *is a balanced triple for* $U_q^-(\mathfrak{g})$.

Define

$$\begin{aligned} U_{\mathbb{A}}^-(\mathfrak{g})^\vee &= \{u \in U_q^-(\mathfrak{g}) \mid (u, U_{\mathbb{A}}^-(\mathfrak{g}))_K \subset \mathbb{A}\}, \\ L(\infty)^\vee &= \{u \in U_q^-(\mathfrak{g}) \mid (u, L(\infty))_K \subset \mathbb{A}_0\}, \\ \overline{L(\infty)}^\vee &= \{u \in U_q^-(\mathfrak{g}) \mid (u, \overline{L(\infty)})_K \subset \mathbb{A}_\infty\}. \end{aligned}$$

By the same argument as in [22], one can verify that $(U_{\mathbb{A}}^-(\mathfrak{g})^\vee, L(\infty)^\vee, \overline{L(\infty)}^\vee)$ is a balanced triple for $U_q^-(\mathfrak{g})$. Hence there is a natural isomorphism

$$E^\vee := U_{\mathbb{A}}^-(\mathfrak{g})^\vee \cap L(\infty)^\vee \cap \overline{L(\infty)}^\vee \xrightarrow{\sim} L(\infty)^\vee / qL(\infty)^\vee.$$

Let $G^\vee$ denote the inverse of this isomorphism and set

$$\mathbb{B}(\infty) = \{G^\vee(b) \mid b \in B(\infty)^\vee\}.$$

Lemma 2.5.1. *Let* $b \in L(\infty)^\vee / qL(\infty)^\vee$ *and* $n \in \mathbb{Z}_{\geq 0}$.

(a) *If* $\tilde{E}_i^{n+1} b = 0$, *then* $e_i'^n G^\vee(b) = \begin{cases} [n]_i! G^\vee(\tilde{E}_i^n b) & \text{if } i \in I^{\mathrm{re}}, \\ G^\vee(\tilde{E}_i^n b) & \text{if } i \in I^{\mathrm{im}}. \end{cases}$

(b) $e_i'^{n+1} G^\vee(b) = 0$ *if and only if* $\tilde{E}_i^{n+1} b = 0$.

Proof. We first prove assertion (1). Let $i \in I^{\mathrm{re}}$. Since $\varphi(\frac{1}{[n]_i!} e_i'^n) = f_i^{(n)}$, by Lemma 2.4.3, we obtain

$$\frac{1}{[n]_i!} e_i'^n G^\vee(b) = \tilde{E}_i^n G^\vee(b) \in U_{\mathbb{A}}^-(\mathfrak{g})^\vee \cap L(\infty)^\vee \cap \overline{L(\infty)}^\vee,$$

which yields $\dfrac{1}{[n]_i!}e_i'^{n}G^\vee(b) = G^\vee(\tilde{E}_i^n b)$.

Similarly, for $i \in I^{\mathrm{im}}$, it follows from $\varphi(e_i'^{n}) = f_i^{(n)}$ that

$$e_i'^{n}G^\vee(b) = \tilde{E}_i^n G^\vee(b) \in U_{\mathbb{A}}^-(\mathfrak{g})^\vee \cap L(\infty)^\vee \cap \overline{L(\infty)}^\vee.$$

Thus we have $e_i'^{n}G^\vee(b) = G^\vee(\tilde{E}_i^n b)$.

For assertion (2), it is obvious that $e_i'^{n+1}G^\vee(b) = 0$ implies $\tilde{E}_i^{n+1}b = 0$. To prove the converse, suppose $e_i'^{n+1}G^\vee(b) \neq 0$ and take the smallest $m > n$ such that $e_i'^{m+1}G^\vee(b) = 0$. By (1), we have

$$e_i'^{m}G^\vee(b) = \begin{cases} [m]_i! G^\vee(\tilde{E}_i^m b) = 0, & \text{if } i \in I^{\mathrm{re}}, \\ G^\vee(\tilde{E}_i^m b) = 0, & \text{if } i \in I^{\mathrm{im}}, \end{cases}$$

which is a contradiction to the choice of m. Hence we conclude $e_i'^{n+1}G^\vee(b) = 0$. □

For $v \in U_q^-(\mathfrak{g})$, we define

$$\varepsilon_i^{\mathrm{or}}(v) = \min\{n \in \mathbb{Z}_{\geq 0} \mid e_i'^{n+1}v = 0\}.$$

Proposition 2.5.2. *For $b \in B(\infty)^\vee$, we have*

$$e_i'G^\vee(b) = \begin{cases} [\varepsilon_i^{\mathrm{or}}(b)]_i G^\vee(\tilde{E}_i b) + \displaystyle\sum_{\varepsilon_i^{\mathrm{or}}(b') < \varepsilon_i^{\mathrm{or}}(b)-1} E_{b,b'}^i G^\vee(b') & \text{if } i \in I^{\mathrm{re}}, \\ G^\vee(\tilde{E}_i b) + \displaystyle\sum_{\varepsilon_i^{\mathrm{or}}(b') < \varepsilon_i^{\mathrm{or}}(b)-1} E_{b,b'}^i G^\vee(b') & \text{if } i \in I^{\mathrm{im}}, \end{cases}$$

$$(2.12)\qquad f_iG^\vee(b) = \begin{cases} q_i^{-\varepsilon_i^{\mathrm{or}}(b)} G^\vee(\tilde{F}_i b) + \displaystyle\sum_{\varepsilon_i^{\mathrm{or}}(b') \leq \varepsilon_i^{\mathrm{or}}(b)} F_{b,b'}^i G^\vee(b') & \text{if } i \in I^{\mathrm{re}}, \\ \{\varepsilon_i^{\mathrm{or}}(b)+1\}_i q_i^{c_i(\varepsilon_i^{\mathrm{or}}(b)+1)} G^\vee(\tilde{F}_i b) & \\ \qquad + \displaystyle\sum_{\varepsilon_i^{\mathrm{or}}(b') \leq \varepsilon_i^{\mathrm{or}}(b)} F_{b,b'}^i G^\vee(b') & \text{if } i \in I^{\mathrm{im}}. \end{cases}$$

for some $E_{b,b'}^i, F_{b,b'}^i \in \mathbb{Q}(q)$.

Proof. If $i \in I^{\mathrm{re}}$, our assertions were proved in [22]. We will prove the case when $i \in I^{\mathrm{im}}$. Set $n = \varepsilon_i^{\mathrm{or}}(b)$. By Lemma 2.5.1 and Definition 2.4.1 (iv), we have

$$e_i'^{n}G^\vee(b) = G^\vee(\tilde{E}_i^n b) = G^\vee(\tilde{E}_i^{n-1}\tilde{E}_i b) = e_i'^{n-1}G^\vee(\tilde{E}_i b),$$

which implies

$$e_i' G^\vee(b) - G^\vee(\tilde{E}_i b) \in \mathrm{Ker}(e_i'^{\,n-1}).$$

Using equation (2.6), we get

$$e_i'^{(n+1)} f_i G^\vee(b) = (q_i^{2c_i(n+1)} f_i e_i'^{(n+1)} + q_i^{c_i(n+1)} e_i'^{(n)}) G^\vee(b).$$

Hence Lemma 2.5.1 yields

$$e_i'^{(n+1)} f_i G^\vee(b) = \frac{1}{\{n\}_i!} q_i^{c_i(n+1)} e_i'^{\,n} G^\vee(b) = \frac{1}{\{n\}_i!} q_i^{c_i(n+1)} G^\vee(\tilde{E}_i^n b).$$

Using Lemma 2.5.1 again, we obtain

$$\begin{aligned}\frac{1}{\{n\}_i!} q_i^{c_i(n+1)} G^\vee(\tilde{E}_i^{n+1} \tilde{F}_i b) &= \frac{1}{\{n\}_i!} q_i^{c_i(n+1)} e_i'^{\,n+1} G^\vee(\tilde{F}_i b) \\ &= \{n+1\}_i q_i^{c_i(n+1)} e_i'^{(n+1)} G^\vee(\tilde{F}_i b).\end{aligned}$$

Thus we have

$$f_i G^\vee(b) - \{n+1\}_i q_i^{c_i(n+1)} G^\vee(\tilde{F}_i b) \in \mathrm{Ker}(e_i'^{\,n+1})$$

as desired. □

Now we construct the *upper global bases* for $U_q(\mathfrak{g})$-modules in $\mathcal{O}_{int}^q$. From the involution of $U_q(\mathfrak{g})$ given in (2.1), we can induce a $\mathbb{Q}$-linear automorphism on $V_q(\lambda)$ given by

$$uv_\lambda \mapsto \bar{u} v_\lambda \quad \text{for } u \in U_q(\mathfrak{g}), \tag{2.13}$$

where v_λ is a highest weight vector of $V_q(\lambda)$. Let $\overline{L(\lambda)}$ be the image of $L(\lambda)$ under the automorphism in (2.13).

Set

$$V_{\mathbb{A}}(\lambda) = U_{\mathbb{A}}(\mathfrak{g}) v_\lambda = U_{\mathbb{A}}^-(\mathfrak{g}) v_\lambda.$$

Define

$$\begin{aligned} V_{\mathbb{A}}(\lambda)^\vee &= \{u \in V_q(\lambda) \mid (u, V_{\mathbb{A}}(\lambda))_K \subset \mathbb{A}\}, \\ L(\lambda)^\vee &= \{u \in V_q(\lambda) \mid (u, L(\lambda))_K \subset \mathbb{A}_0\}, \\ \overline{L(\lambda)}^\vee &= \{u \in V_q(\lambda) \mid (u, \overline{L(\lambda)})_K \subset \mathbb{A}_\infty\}. \end{aligned}$$

As in the case of $U_q^-(\mathfrak{g})$, we can conclude that $(V_{\mathbb{A}}(\lambda)^\vee, L(\lambda)^\vee, \overline{L(\lambda)}^\vee)$ is a balanced triple for $V_q(\lambda)$. Hence there is a natural isomorphism

$$E = V_{\mathbb{A}}(\lambda)^\vee \cap L(\lambda)^\vee \cap \overline{L(\lambda)}^\vee \xrightarrow{\sim} L(\lambda)^\vee / qL(\lambda)^\vee.$$

We also denote by $G^\vee$ the inverse of this isomorphism and set

$$\mathbb{B}(\lambda) := \{G^\vee(b) \mid b \in B(\lambda)^\vee\}.$$

Then $\mathbb{B}(\lambda)$ is an $\mathbb{A}$-basis of $V_{\mathbb{A}}(\lambda)^\vee$ satisfying the following properties.

Proposition 2.5.3.

(a) *For $b \in B(\lambda)^\vee$, $G^\vee(b)$ is a unique element of $V_{\mathbb{A}}(\lambda)^\vee \cap L(\lambda)^\vee$ such that*

$$G^\vee(b) \equiv b \mod qL(\lambda)^\vee, \qquad \overline{G^\vee(b)} = G^\vee(b).$$

(b) *For $i \in I^{im}$, $b \in B(\lambda)^\vee$ and $n \in \mathbb{Z}_{\geq 0}$, we have*

$$G^\vee(\tilde{E}_i^n b) = e_i^n G^\vee(b).$$

Proof. Assertion (1) can be proved in the same manner as in [12]. To prove (2), note that we have

$$\overline{e_i G^\vee(b)} = e_i G^\vee(b), \quad \tilde{E}_i u = e_i u \text{ for all } b \in B(\lambda)^\vee, u \in V_q(\lambda).$$

It follows that

$$\overline{e_i^n G^\vee(b)} = e_i^n G^\vee(b), \quad e_i^n G^\vee(b) \equiv \tilde{E}_i^n b \mod qL(\lambda)^\vee.$$

By (1), we conclude $G^\vee(\tilde{E}_i^n b) = e_i^n G(b)$. □

For any $V \in \mathcal{O}_{int}^q$ and $v \in V$, define

$$\begin{aligned} \varepsilon_i^{\mathrm{or}}(v) &:= \min\{n \in \mathbb{Z}_{\geq 0} \mid e_i'^{n+1} v = 0\}, \\ \varphi_i(v) &:= \min\{n \in \mathbb{Z}_{\geq 0} \mid f_i^{n+1} v = 0\}. \end{aligned} \tag{2.14}$$

Then, using Proposition 2.5.3 and Proposition 2.5.2, we have the following formulas:

For $b \in B(\Lambda)^\vee$,
(2.15)

$$e_i G^\vee(b) = \begin{cases} [\varepsilon_i^{\mathrm{or}}(b)]_i G^\vee(\tilde{E}_i b) + \displaystyle\sum_{\varepsilon_i^{\mathrm{or}}(b') < \varepsilon_i^{\mathrm{or}}(b)-1} E^i_{b,b'} G^\vee(b') & \text{if } i \in I^{\mathrm{re}}, \\ G^\vee(\tilde{E}_i b) & \text{if } i \in I^{\mathrm{im}}, \end{cases}$$

$$f_i G^\vee(b) = \begin{cases} [\varphi_i(b)]_i G^\vee(\tilde{F}_i b) + \displaystyle\sum_{\varepsilon_i^{\mathrm{or}}(b') \le \varepsilon_i^{\mathrm{or}}(b)} F^i_{b,b'} G^\vee(b') & \text{if } i \in I^{re}, \\ \{\varepsilon_i^{\mathrm{or}}(b)+1\}_i [\langle h_i, \lambda\rangle + c_i(\varepsilon_i^{\mathrm{or}}(b))]_i G^\vee(\tilde{E}_i b) & \\ \qquad + \displaystyle\sum_{\varepsilon_i^{\mathrm{or}}(b') \le \varepsilon_i^{\mathrm{or}}(b)} F^i_{b,b'} G^\vee(b') & \text{if } i \in I^{im} \end{cases}$$

for some $E^i_{b,b'}, F^i_{b,b'} \in \mathbb{Q}[q, q^{-1}]$.

2.6 Perfect bases

By abstracting the property of the upper global bases, we can define the notion of a perfect basis. In the end of this section, we will give a characterization of $V_{\mathbb{A}}(\Lambda)^\vee$ with respect to the strong perfect basis. This will play a crucial role in proving the categorification theory.

Let $\mathbf{k}$ be a commutative ring. Let $V = \bigoplus_{\lambda \in P} V_\lambda$ be a P-graded $\mathbf{k}$-vector space. We assume that there are finitely many $\lambda_1, \ldots, \lambda_s \in P$ such that

$$\mathrm{wt}(V) := \{\mu \in P \mid V_\mu \neq 0\} \subset \bigcup_{i=1}^{s} (\lambda_i - Q^+).$$

Furthermore, assume that e_i $(i \in I)$ acts on V in such a way that $e_i V_\lambda \subset V_{\lambda+\alpha_i}$. Recall the definition of $\varepsilon_i^{\mathrm{or}}$ in (2.14). For any $v \in V \setminus \{0\}$ and $i \in I$, we define

$$\varepsilon_i^*(v) := \begin{cases} \varepsilon_i^{\mathrm{or}}(v) & \text{if } i \in I^{\mathrm{re}} \text{ or } \varepsilon_i^{\mathrm{or}}(v) = 0, \\ 1 & \text{if } i \in I^{\mathrm{im}} \text{ and } \varepsilon_i^{\mathrm{or}}(v) > 0. \end{cases}$$

If $v = 0$, then we will use the convention $\varepsilon_i^*(0) = -\infty$. One can easily check that, for $k \in \mathbb{Z}_{\ge 0}$,

$$V_i^{<k} := \{v \in V \mid \varepsilon_i^{\mathrm{or}}(v) < k\} = \mathrm{Ker}\, e_i^k.$$

Definition 2.6.1. *[2, 18]*

(i) *A* $\mathbf{k}$*-basis* B *of* V *is called a* perfect basis *if*

(a) $B = \bigsqcup_{\mu \in \mathrm{wt}(V)} B_\mu$, *where* $B_\mu := B \cap V_\mu$,

(b) *for any* $b \in B$ *and* $i \in I$ *with* $e_i(b) \neq 0$, *there exists a unique element* $\tilde{\mathsf{e}}_i(b) \in B$ *such that*

$$e_i b - \mathsf{c}_i(b)\, \tilde{\mathsf{e}}_i(b) \in V_i^{<\varepsilon_i^{\mathrm{or}}(b)-1} \text{ for some } \mathsf{c}_i(b) \in \mathbf{k}^\times,$$

(c) *if* $b, b' \in B$ *and* $i \in I$ *satisfy* $\varepsilon_i^{\mathrm{or}}(b) = \varepsilon_i^{\mathrm{or}}(b') > 0$ *and* $\tilde{\mathsf{e}}_i(b) = \tilde{\mathsf{e}}_i(b')$, *then* $b = b'$.

(ii) *Assume that* $\mathbf{k}$ *contains* $\mathbb{Q}(q)$. *We say that a perfect basis is* strong *if* $\mathsf{c}_i(b) = [\varepsilon_i^*(b)]_i$ *for any* $b \in B$ *and* $i \in I$; *i.e.,*

$$e_i b - [\varepsilon_i^*(b)]_i \tilde{\mathsf{e}}_i(b) \in V_i^{<\varepsilon_i^{\mathrm{or}}(b)-1}. \tag{2.16}$$

Theorem 2.6.1. *For* $\Lambda \in P^+$,

(a) $U_q^-(\mathfrak{g})$ *and* $V_q(\Lambda)$ *have strong perfect bases.*

(b) $U(\mathfrak{g})$ *and* $V(\Lambda)$ *have strong perfect bases.*

Proof. The first assertion comes from (2.12) and (2.15). By taking the classical limit to the upper global bases of $U_q^-(\mathfrak{g})$ and $V_q(\Lambda)$, the second assertion follows. □

For a perfect basis B, we set $\tilde{\mathsf{e}}_i(b) = 0$ if $e_i b = 0$. We can easily see that for a perfect basis B

$$V_i^{<k} = \bigoplus_{b \in B,\, \varepsilon_i^{\mathrm{or}}(b) < k} \mathbf{k} b. \tag{2.17}$$

For any sequence $\mathbf{i} = (i_1, \ldots, i_m) \in I^m$ $(m \geq 1)$, we inductively define a binary relation $\preceq_{\mathbf{i}}$ on $V \setminus \{0\}$ as follows:

$$\text{if } \mathbf{i} = (i),\ v \preceq_{\mathbf{i}} v' \Leftrightarrow \varepsilon_i^{\mathrm{or}}(v) \leq \varepsilon_i^{\mathrm{or}}(v'),$$

$$\text{if } \mathbf{i} = (i; \mathbf{i}'),\ v \preceq_{\mathbf{i}} v' \Leftrightarrow \begin{cases} \varepsilon_i^{\mathrm{or}}(v) < \varepsilon_i^{\mathrm{or}}(v') \text{ or} \\ \varepsilon_i^{\mathrm{or}}(v) = \varepsilon_i^{\mathrm{or}}(v'),\ e_i^{\varepsilon_i^{\mathrm{or}}(v)}(v) \preceq_{\mathbf{i}'} e_i^{\varepsilon_i^{\mathrm{or}}(v')}(v'). \end{cases}$$

We write

$$\text{(i) } v \equiv_{\mathbf{i}} v' \text{ if } v \preceq_{\mathbf{i}} v' \text{ and } v' \preceq_{\mathbf{i}} v, \quad \text{(ii)} v' \prec_{\mathbf{i}} v \text{ if } v' \preceq_{\mathbf{i}} v \text{ and } v \not\equiv_{\mathbf{i}} v'.$$

We can easily verify:

(1) for all $v \in V \setminus \{0\}$, $V^{\prec_{\mathbf{i}} v} := \{0\} \bigsqcup \{v' \in V \setminus \{0\} \mid v' \prec_{\mathbf{i}} v\}$ forms a $\mathbf{k}$-linear subspace of V.

(2) if $v \not\equiv_{\mathbf{i}} v'$, then $v + v' \equiv_{\mathbf{i}} \begin{cases} v & \text{if } v' \prec_{\mathbf{i}} v, \\ v' & \text{if } v \prec_{\mathbf{i}} v'. \end{cases}$

For i and $v \in V \setminus \{0\}$, we set

$$e_i{}^{\mathrm{top}}(v) := e_i^{\varepsilon_i^{\mathrm{or}}(v)} v.$$

For each $\mathbf{i} = (i_1, \ldots, i_m) \in I^m$, define a map $e_{\mathbf{i}}{}^{\mathrm{top}} \colon V \setminus \{0\} \to V \setminus \{0\}$ by

$$e_{\mathbf{i}}{}^{\mathrm{top}} := e_{i_m}^{\mathrm{top}} \circ \cdots \circ e_{i_1}^{\mathrm{top}}. \tag{2.18}$$

Then $e_{\mathbf{i}}{}^{\mathrm{top}} B \subset \mathbf{k}^{\times} B$.

Let $V^H := \{v \in V \mid e_i v = 0 \text{ for all } i \in I\}$ be the space of highest weight vectors in V and let $B^H = V^H \cap B$ be the set of highest weight vectors in B. Then, (2.17) implies that

$$V^H = \bigoplus_{b \in B^H} \mathbf{k} b.$$

In [2], Berenstein and Kazhdan proved the following version of the uniqueness theorem for perfect bases.

Theorem 2.6.2. [2] *Let B and B' be perfect bases of V such that $B^H = (B')^H$. Then there exist a map $\psi \colon B \xrightarrow{\sim} B'$ and a map $\xi \colon B \to \mathbf{k}^{\times}$ such that $\psi(b) - \xi(b) b \in V^{\prec_{\mathbf{i}} b}$ holds for any $b \in B$ and any $\mathbf{i} = (i_1, \ldots, i_m)$ satisfying $e_{\mathbf{i}}^{\mathrm{top}}(b) \in V^H$. Moreover, such ψ and ξ are unique and ψ commutes with $\tilde{\mathsf{e}}_i$ and $\varepsilon_i^{\mathrm{or}}$ $(i \in I)$.*

From now on, we set $\mathbf{k} = \mathbb{Q}(q)$. We define

$$e_i{}^{(\mathrm{top})}(v) := e_i^{(\varepsilon_i^{\mathrm{or}}(v))} v.$$

Then we can easily see the following:

(2.19) If B is a strong perfect basis, then $e_i{}^{(\mathrm{top})}(b) = \tilde{\mathsf{e}}_i^{\varepsilon_i^{\mathrm{or}}(b)} b$ for any $b \in B$.

As in the way of (2.18), for each $\mathbf{i} = (i_1, \ldots, i_m) \in I^m$, we can define $e_{\mathbf{i}}{}^{(\mathrm{top})}$ as

$$e_{\mathbf{i}}{}^{(\mathrm{top})} := e_{i_m}^{(\mathrm{top})} \circ \cdots \circ e_{i_1}^{(\mathrm{top})}.$$

Thus, if B is a strong perfect basis, then we have $e_{\mathbf{i}}{}^{(\mathrm{top})} B \subset B$.

Lemma 2.6.1. *Let B be a strong perfect basis of V.*

(a) *For any finite subset S of B, there exists a finite sequence $\mathbf{i} = (i_1, \ldots, i_m)$ of I such that $e_{\mathbf{i}}^{(\mathrm{top})}(b) \in B^H$ for any $b \in S$.*

(b) *Let $b_0 \in B^H$ and let $\mathbf{i} = (i_1, \ldots, i_m)$ be a finite sequence in I. Then $S := \left\{ b \in B \mid e_{\mathbf{i}}^{(\mathrm{top})}(b) = b_0 \right\}$*
is linearly ordered by $\preceq_{\mathbf{i}}$.

Proof. (a) is evident. In order to see (b), it is enough to show that if $b, b' \in S$ satisfy $b \equiv_{\mathbf{i}} b'$, then $b = b'$. If we set $v_0 = b$, $\ell_k = \varepsilon_{i_k}^{\mathrm{or}}(v_{k-1})$ and $v_k = e_{i_k}^{(\ell_k)} v_{k-1}$ $(1 \le k \le m)$, then $v_m = b_0$. Similarly, if we set $v'_0 = b'$, $\ell'_k = \varepsilon_{i_k}^{\mathrm{or}}(v'_{k-1})$ and $v'_k = e_{i_k}^{(\ell'_k)} v'_{k-1}$ $(1 \le k \le m)$, then $\ell'_k = \ell_k$ and $v'_m = b_0$. Thus we have $v_k = \tilde{\mathsf{e}}_{i_k}^{\ell_k} v_{k-1}$ and $v'_k = \tilde{\mathsf{e}}_{i_k}^{\ell_k} v'_{k-1}$. Hence Definition 2.6.1 (i) (c) shows that $v'_k = v_k$ for all k. $\square$

The following proposition gives a characterization of $V_{\mathbb{A}}(\Lambda)^{\vee}$ by using strong perfect bases.

Proposition 2.6.1. *Let M be a $U_q(\mathfrak{g})$-module in $\mathcal{O}_{int}^q$ such that* $\mathrm{wt}(M) \subset \Lambda - Q^+$. *Let $M_{\mathbb{A}}$ be an $\mathbb{A}$-submodule of M. Assume that*

- $e_i^{(n)} M_{\mathbb{A}} \subset M_{\mathbb{A}}$ *for all $i \in I$,*
- $(M_{\mathbb{A}})_{\Lambda} = \mathbb{A} v_{\Lambda}$ *and M has a strong perfect basis $B \subset M_{\mathbb{A}}$ such that $B^H = \{v_{\Lambda}\}$.*

Then we have

(a) $M_{\mathbb{A}} \simeq V_{\mathbb{A}}(\Lambda)^{\vee}$,

(b) B *is an* $\mathbb{A}$*-basis of* $M_{\mathbb{A}}$.

Proof. Since M has only one highest weight vector v_Λ, M is isomorphic to the irreducible module $V_q(\Lambda) \in \mathcal{O}^q_{int}$. Since $(M_{\mathbb{A}})_\Lambda = \mathbb{A}v_\Lambda$ and

$$V_{\mathbb{A}}(\Lambda)^{\vee}_{\lambda} = \left\{ u \in V(\Lambda)_\lambda \;\middle|\; \begin{array}{c} e_{i_1}^{(a_1)} \cdots e_{i_\ell}^{(a_\ell)} u \in \mathbb{A}v_\Lambda \text{ for all } (i_1, \cdots, i_\ell) \\ \text{such that } \sum_{k=1}^{\ell} a_k \alpha_{i_k} + \lambda = \Lambda \end{array} \right\},$$

it is obvious that $M_{\mathbb{A}} \subset V_{\mathbb{A}}(\Lambda)^{\vee}$.

Thus it suffices to show that $V_{\mathbb{A}}(\Lambda)^{\vee} \subset M_{\mathbb{A}}$. For any $u \in V_{\mathbb{A}}(\Lambda)^{\vee}$, we have $u = \sum_{b \in B} c_b b$ with $c_b \in \mathbb{Q}(q)$. By Lemma 2.6.1(a), we have a sequence $\mathbf{i} = (i_1, \ldots, i_m)$ for the set $B(u) := \{b \in B | c_b \neq 0\}$ satisfying $e_{\mathbf{i}}^{(\mathrm{top})}(b) = v_\Lambda$ for all $b \in B(u)$. Then Lemma 2.6.1(b) tells us that $B(u)$ is the linearly ordered set with respect to $\prec_{\mathbf{i}}$. By using descending induction with respect to the order, we shall show that $c_b \in \mathbb{A}$. For the maximal element $\mathbf{b} \in B(u)$, $e_{\mathbf{i}}^{\mathrm{top}}(\mathbf{b}) = e_{\mathbf{i}}^{\mathrm{top}}(u) = c_{\mathbf{b}}\mathbf{b}$. Thus we can start an induction. Assume that $c_{b'} \in \mathbb{A}$ for any $b' \in B$ such that $b \prec_{\mathbf{i}} b'$. Then setting $v_0 = b$, $\ell_k = \varepsilon_{i_k}(v_{k-1})$ and $v_k = e_{i_k}^{(\ell_k)} v_{k-1}$ $(1 \leq k \leq m)$, we have $e_{i_m}^{(\ell_m)} \cdots e_{i_1}^{(\ell_1)} u = c_b v_\Lambda + \sum\limits_{b \prec_{\mathbf{i}} b'} c_{b'} e_{i_m}^{(\ell_m)} \cdots e_{i_1}^{(\ell_1)} b' \in V_{\mathbb{A}}(\Lambda)^{\vee}$. Hence we obtain $c_b \in \mathbb{A}$. $\square$

Chapter 3

Categorification

3.1 The quiver Hecke algebra R

In this section, we construct the quiver Hecke algebra R associated with a Borcherds-Cartan matrix A.

We take a graded commutative ring $\mathbf{k} = \oplus_{n \in \mathbb{Z}_{\geq 0}} \mathbf{k}_n$ as the base ring. For $i, j \in I$ and $r, s \in \mathbb{Z}_{\geq 0}$, we take $t_{i,j;(r,s)} \in \mathbf{k}_{-2((\alpha_i|\alpha_j)+rd_i+sd_j)}$ and $u_{i;(r',s')} \in \mathbf{k}_{-2d_i(1-r'-s'-\frac{a_{ii}}{2})}$ such that

$$
\begin{aligned}
&t_{i,j;(-a_{ij},0)} \in \mathbf{k}_0^{\times}, \quad t_{i,j;(r,s)} = t_{j,i;(s,r)}, \\
&u_{i;(1-\frac{a_{ii}}{2},0)}, \; u_{i;(0,1-\frac{a_{ii}}{2})} \in \mathbf{k}_0^{\times}.
\end{aligned} \tag{3.1}
$$

In particular, $t_{i,j;(r,s)} = 0$ if $i = j$.

Remark 3.1.1. *We sometimes assume that*

$$
\mathbf{k} = \mathbf{k}_0. \tag{3.2}
$$

Under this assumption, $t_{i,j;(r,s)} = u_{i;(r',s')} = 0$ *for* (r,s), $(r',s') \in \mathbb{Z}_{\geq 0}^2$ *such that* $(\alpha_i|\alpha_j) + d_i p + d_j q \neq 0$ *and* $1 - \frac{a_{ii}}{2} - r' - s' \neq 0$.

Let $\mathsf{Q}_{i,j}$ and P_i be elements in $\mathbf{k}[w,z]$ which are of the form

$$
\begin{aligned}
&\mathsf{Q}_{i,j}(w,z) = \delta_{i,j} \sum_{r,s \in \mathbb{Z}_{\geq 0}} t_{i,j;(r,s)} w^r z^s, \\
&\mathsf{P}_i(w,z) = \sum_{r+s \leq 1-\frac{a_{ii}}{2}} u_{i;(r,s)} w^r z^s.
\end{aligned} \tag{3.3}
$$

Let $s_{a,b} = (a, b)$ be the transposition on $\mathbf{k}[x_1, \ldots, x_n]$ which interchanges x_a and x_b. Define the operator ∂_a on $\mathbf{k}[x_1, \ldots, x_n]$ by

$$\partial_{a,b} f = \frac{s_{a,b} f - f}{x_a - x_b}$$

and let $\partial_a := \partial_{a,a+1}$.

For the sake of simplicity, we assume that I is a finite set.

Definition 3.1.1 ([19]). *The quiver Hecke algebra $\mathsf{R}(n)$ of degree n associated with the data $(A, P, \Pi, \Pi^\vee)$, $(\mathsf{Q}_{i,j})_{i,j\in I}$ and $(\mathsf{P}_i)_{i\in I}$ is the associative algebra over $\mathbf{k}$ generated by $e(\nu)$ $(\nu \in I^n)$, x_k $(1 \le k \le n)$, τ_a $(1 \le a \le n-1)$ with the following defining relations:*

(R1) $e(\mu)e(\nu) = \delta_{\mu,\nu} e(\nu)$ *for all* μ, $\nu \in I^n$, *and* $1 = \sum_{\nu\in I^n} e(\nu)$,

(R2) $x_p x_q = x_q x_p$,

(R3) $x_p e(\nu) = e(\nu) x_p$ *and* $\tau_a e(\nu) = e(s_a\nu)\tau_a$, *where* $s_a = (a, a+1)$ *is the transposition on the set of sequences,*

(R4) $\tau_a x_p e(\nu) = x_{s_a(p)} \tau_a e(\nu)$ *if* $p \ne a$, $a+1$,

(R5) $(\tau_a x_{a+1} - x_a \tau_a) e(\nu) = (x_{a+1}\tau_a - \tau_a x_a) e(\nu) = \delta_{\nu_a,\nu_{a+1}} \mathsf{P}_{\nu_a}(x_a, x_{a+1}) e(\nu)$,

(R6) $\tau_a^2 e(\nu) = \begin{cases} (\partial_a \mathsf{P}_{\nu_a}(x_a, x_{a+1}))\tau_a e(\nu) & \text{if } \nu_a = \nu_{a+1}, \\ \mathsf{Q}_{\nu_a,\nu_{a+1}}(x_a, x_{a+1}) e(\nu) & \text{if } \nu_a \ne \nu_{a+1}, \end{cases}$

(R7) $\tau_a \tau_b = \tau_b \tau_a$ *if* $|a - b| > 1$,

(R8) $(\tau_{a+1}\tau_a\tau_{a+1} - \tau_a\tau_{a+1}\tau_a) e(\nu)$

$$= \begin{cases} \mathsf{P}_{\nu_a}(x_a, x_{a+2}) \overline{\mathsf{Q}}_{\nu_a,\nu_{a+1}}(x_a, x_{a+1}, x_{a+2}) e(\nu) & \text{if } \nu_a = \nu_{a+2} \ne \nu_{a+1}, \\ \overline{\mathsf{P}}'_{\nu_a}(x_a, x_{a+1}, x_{a+2})\tau_a e(\nu) & \\ \qquad\qquad + \overline{\mathsf{P}}''_{\nu_a}(x_a, x_{a+1}, x_{a+2})\tau_{a+1} e(\nu) & \text{if } \nu_a = \nu_{a+1} = \nu_{a+2}, \\ \qquad\qquad 0 & \text{otherwise,} \end{cases}$$

where

$$\overline{\mathsf{P}}'_i(u,v,w)=\overline{\mathsf{P}}'_i(v,u,w):=+\frac{\mathsf{P}_i(v,u)\mathsf{P}_i(u,w)}{(u-v)(u-w)}+\frac{\mathsf{P}_i(u,w)\mathsf{P}_i(v,w)}{(u-w)(v-w)}-\frac{\mathsf{P}_i(u,v)\mathsf{P}_i(v,w)}{(u-v)(v-w)},$$
$$\overline{\mathsf{P}}''_i(u,v,w)=\overline{\mathsf{P}}''_i(u,w,v):=-\frac{\mathsf{P}_i(u,v)\mathsf{P}_i(u,w)}{(u-v)(u-w)}-\frac{\mathsf{P}_i(u,w)\mathsf{P}_i(w,v)}{(u-w)(v-w)}+\frac{\mathsf{P}_i(u,v)\mathsf{P}_i(v,w)}{(u-v)(v-w)},$$
$$\overline{\mathsf{Q}}_{i,j}(u,v,w):=\frac{\mathsf{Q}_{i,j}(u,v)-\mathsf{Q}_{i,j}(w,v)}{u-w}.$$

Let us assign a $\mathbb{Z}$-grading on the generators as follows: for all $\nu\in I^n$, $1\le k\le n$ and $1\le \ell<n$

(3.4) $\quad \deg(e(\nu))=0,\quad \deg(x_ke(\nu))=2d_{\nu_k},\quad \deg(\tau_\ell e(\nu))=-(\alpha_{\nu_\ell}|\alpha_{\nu_{\ell+1}}).$

Then one can check that all relations in Definition 3.1.1 are homogeneous. Hence $\mathsf{R}(n)$ has a natural $\mathbb{Z}$-grading induced by (3.4).

We understand that $\mathsf{R}(0)\simeq \mathbf{k}$, and $\mathsf{R}(1)$ is isomorphic to $\mathbf{k}^I[x_1]$ where $\mathbf{k}^I=\bigoplus_{i\in I}\mathbf{k}e(i)$ is the direct sum of the copies $\mathbf{k}e(i)$ of the algebra $\mathbf{k}$.

Remark 3.1.2. *For each w in the symmetric group S_n, we choose a reduced expression $s_{i_1}\cdots s_{i_\ell}$ of w and write $\tau_w=\tau_{i_1}\cdots\tau_{i_\ell}$. Then, from the relations given in Definition 3.1.1, we can see that the set*

$$\{\tau_w x_1^{a_1}\cdots x_n^{a_n}e(\nu)\mid a_1,\dots,a_n\in\mathbb{Z}_{\ge 0},\ w\in S_n,\ \nu\in I^n\}$$

spans the $\mathbf{k}$*-module* $\mathsf{R}(n)$.

For $\nu=(\nu_1,\dots,\nu_n)\in I^n$ and $1\le m\le n$, we define

$$\nu_{<m}=(\nu_1,\dots,\nu_{m-1}),\qquad \nu_{\le m}=(\nu_1,\dots,\nu_m),$$
$$\nu_{>m}=(\nu_{m+1},\dots,\nu_n),\qquad \nu_{\ge m}=(\nu_m,\dots,\nu_n).$$

For pairwise distinct $a,b,c\in\{1,\dots n\}$, let us define

$$e_{a,b}=\sum_{\substack{\nu\in I^n,\\ \nu_a=\nu_b}}e(\nu),\quad \mathsf{P}_{a,b}=\sum_{\substack{\nu\in I^n,\\ \nu_a=\nu_b}}\mathsf{P}_{\nu_a}(x_a,x_b)e(\nu),$$
$$\overline{\mathsf{Q}}_{a,b,c}=\sum_{\substack{\nu\in I^n,\\ \nu_a=\nu_c\neq\nu_b}}\frac{\mathsf{Q}_{\nu_a,\nu_b}(x_a,x_b)-\mathsf{Q}_{\nu_a,\nu_b}(x_c,x_b)}{x_a-x_c}e(\nu),\quad \overline{\mathsf{Q}}_a:=\overline{\mathsf{Q}}_{a,a+1,a+2},$$

$$\overline{\mathsf{P}}'_{a,b,c} = \sum_{\substack{\nu \in I^n, \\ \nu_a = \nu_b = \nu_c}} \overline{\mathsf{P}}'_{\nu_a}(x_a, x_b, x_c) e(\nu), \quad \overline{\mathsf{P}}'_a := \overline{\mathsf{P}}'_{a,a+1,a+2},$$
$$\overline{\mathsf{P}}''_{a,b,c} = \sum_{\substack{\nu \in I^n, \\ \nu_a = \nu_b = \nu_c}} \overline{\mathsf{P}}''_{\nu_a}(x_a, x_b, x_c) e(\nu), \quad \overline{\mathsf{P}}''_a := \overline{\mathsf{P}}''_{a,a+1,a+2}.$$

Then we have

$$\tau_{a+1}\tau_a\tau_{a+1} - \tau_a\tau_{a+1}\tau_a = \overline{\mathsf{Q}}_a \mathsf{P}_{a,a+2} + \overline{\mathsf{P}}'_a \tau_a + \overline{\mathsf{P}}''_a \tau_{a+1}.$$

We define the operator, also denoted by $\partial_{a,b}$, on $\bigoplus_{\nu \in I^n} \mathbf{k}[x_1, \ldots, x_n] e(\nu)$, by

$$\partial_{a,b} f = \frac{s_{a,b} f - f}{x_a - x_b} e_{a,b}, \quad \partial_a := \partial_{a,a+1}.$$

Then we obtain

$$\tau_a f - (s_a f)\tau_a = f\tau_a - \tau_a(s_a f) = (\partial_a f)\mathsf{P}_{a,a+1}. \tag{3.5}$$

Using the formula (3.5), we have

$$\overline{\mathsf{P}}'_a \tau_a = \tau_a \overline{\mathsf{P}}'_a \ \text{ and } \ \overline{\mathsf{P}}''_a \tau_{a+1} = \tau_{a+1} \overline{\mathsf{P}}''_a.$$

For $\beta \in Q^+$ with $|\beta| = n$, we set

$$I^\beta = \{\nu = (\nu_1, \ldots, \nu_n) \in I^n \mid \alpha_{\nu_1} + \cdots + \alpha_{\nu_n} = \beta\},$$
$$I^{(\beta)} = \{\nu = (\nu_1^{a_1}, \ldots, \nu_k^{a_k}) \in I^n \mid a_1\alpha_{\nu_1} + \cdots + a_k\alpha_{\nu_k} = \beta\}.$$

We define

$$\mathsf{R}(m,n) = \mathsf{R}(m) \otimes_{\mathbf{k}} \mathsf{R}(n) \subset \mathsf{R}(m+n),$$
$$e(n) = \sum_{\nu \in I^n} e(\nu), \quad e(\beta) = \sum_{\nu \in I^\beta} e(\nu), \quad e(\alpha,\beta) = \sum_{\mu \in I^\alpha, \ \nu \in I^\beta} e(\mu,\nu),$$
$$\mathsf{R}(\beta) = e(\beta)\mathsf{R}(n), \quad \mathsf{R}(\alpha,\beta) = \mathsf{R}(\alpha) \otimes_{\mathbf{k}} \mathsf{R}(\beta) \subset \mathsf{R}(\alpha+\beta),$$
$$e(n, i^k) = \sum_{\substack{\nu \in I^{n+k}, \\ \nu_{n+1} = \cdots = \nu_{n+k} = i}} e(\nu), \quad e(i^k, n) = \sum_{\substack{\nu \in I^{n+k}, \\ \nu_1 = \cdots = \nu_k = i}} e(\nu),$$
$$e(\beta, i^k) = e(\beta, k\alpha_i) = e(\beta + k\alpha_i) e(n, i^k),$$
$$e(i^k, \beta) = e(k\alpha_i, \beta) = e(\beta + k\alpha_i) e(i^k, n),$$

for $\alpha, \beta \in Q^+$.

Let $\mathsf{Mod}(\mathsf{R}(\beta))$ be a category of arbitrary $\mathbb{Z}$-graded $\mathsf{R}(\beta)$-modules. The morphisms in the category are $\mathsf{R}(\beta)$-homomorphisms which are homogeneous. Let $\mathsf{Proj}(\mathsf{R}(\beta))$ (resp. $\mathsf{Rep}(\mathsf{R}(\beta))$) be the full subcategory consisting of finitely generated projective (resp. finite dimensional over $\mathbf{k}_0$) $\mathbb{Z}$-graded $(\mathsf{R}(\beta))$-modules. For a $\mathbb{Z}$-graded $\mathsf{R}(\beta)$-module $M = \bigoplus_{t\in\mathbb{Z}} M_t$, let $M\langle k\rangle$ denote the graded $\mathsf{R}(\beta)$-module whose $\mathbb{Z}$-grading is shifted by k from one of M; i.e., $M\langle k\rangle_t := \bigoplus_{t\in\mathbb{Z}} M_{t+k}$. We also denote by q the grading shift functor

$$(q \cdot M)_t = M_{t-1}. \tag{3.6}$$

We denote by $[\mathsf{Proj}(\mathsf{R}(\beta))]$ and $[\mathsf{Rep}(\mathsf{R}(\beta))]$ the Grothendieck group of $\mathsf{Proj}(\mathsf{R}(\beta))$ and $\mathsf{Rep}(\mathsf{R}(\beta))$, respectively. These group have the $\mathbb{A}$-module structures induced by the grading shift functor q; i.e., $q[M] = [q \cdot M]$ where $[M]$ is the isomorphism class of an $\mathsf{R}(\beta)$-module M.

Let $\psi : \mathsf{R}(\beta) \to \mathsf{R}(\beta)$ be the anti-involution given by

$$\psi(ab) = \psi(b)\psi(a), \quad \psi(e(\nu)) = e(\nu), \quad \psi(x_k) = x_k, \quad \psi(\tau_l) = \tau_l \tag{3.7}$$

for all a, $b \in \mathsf{R}(\beta)$ and generators of $\mathsf{R}(\beta)$.

For any $M \in \mathsf{Mod}(\mathsf{R}(\beta))$, we denote by M^ψ the graded right $\mathsf{R}(\beta)$-modules whose right action is induced by the involution ψ. Namely,

$$v \cdot r = \psi(r)v \quad \text{for } v \in M^\psi,\ r \in \mathsf{R}(\beta).$$

For $\alpha, \beta \in Q^+$, consider the natural embedding

$$\iota_{\alpha,\beta} \colon \mathsf{R}(\alpha, \beta) \hookrightarrow \mathsf{R}(\alpha + \beta).$$

For $M \in \mathsf{Mod}(\mathsf{R}(\alpha, \beta))$ and $N \in \mathsf{Mod}(\mathsf{R}(\alpha + \beta))$, we define

$$\begin{aligned}
&\mathrm{Ind}_{\alpha,\beta}(M) = \mathsf{R}(\alpha + \beta)e(\alpha, \beta) \otimes_{\mathsf{R}(\alpha,\beta)} M \in \mathsf{Mod}(\mathsf{R}(\alpha + \beta)),\\
&\mathrm{Res}_{\alpha,\beta}(N) = e(\alpha, \beta)N \in \mathsf{Mod}(\mathsf{R}(\alpha, \beta)).
\end{aligned}$$

Then Frobenius reciprocity holds in this setting:

$$\mathrm{Hom}_{\mathsf{R}(\alpha+\beta)}(\mathrm{Ind}_{\alpha,\beta}(M), N) \simeq \mathrm{Hom}_{\mathsf{R}(\alpha,\beta)}(M, \mathrm{Res}_{\alpha,\beta}N). \tag{3.8}$$

3.2 The algebra $\mathsf{R}(n\alpha_i)$.

In this section, we will study the algebra $\mathsf{R}(n\alpha_i)$ which is a special case where $\beta = n\alpha_i$. However, the properties of $\mathsf{R}(n\alpha_i)$ depend on the value of a_{ii}. For $i \in I^{\mathrm{re}}$, this algebra was treated in [24, 33].

Throughout this section, we assume that

(3.9) $\mathbf{k}_0$ is a field and the components $\mathbf{k}_t$ are finite-dimensional over $\mathbf{k}_0$.

Under condition (3.9), the $\mathbb{Z}$-graded algebra $\mathsf{R}(\beta)$ satisfies the conditions:

(3.10)
(a) its $\mathbb{Z}$-grading is bounded below and
(b) each homogeneous subspace $\mathsf{R}(\beta)_t$ is finite-dimensional over $\mathbf{k}_0$ $(t \in \mathbb{Z})$.

Hence we have

(3.11)
(i) $\mathsf{R}(\beta)$ has the Krull-Schmidt direct sum property for finitely generated modules,
(ii) any simple object in $\mathsf{Mod}(\mathsf{R}(\beta))$ is finite-dimensional over $\mathbf{k}_0$ and has an indecomposable finitely generated projective cover (unique up to isomorphism),
(iii) there are finitely many simple modules in $\mathsf{Rep}(\mathsf{R}(\beta))$ up to grade shifts and isomorphisms.

Thus $\mathsf{Rep}(\mathsf{R}(\beta))$ contains all irreducible $\mathsf{R}(\beta)$-modules and

(3.12) the set of all the isomorphism classes of $\mathsf{R}(\beta)$-modules, $\mathrm{Irr}_q(\mathsf{R}(\beta))$,

forms a $\mathbb{Z}$-*basis* of $[\mathsf{Rep}(\mathsf{R}(\beta))]$.

Now, we will study the representation theory of $\mathsf{R}(n\alpha_i)$. To do this, we need to consider its primitive idempotents first.

The case when $a_{ii} = 2$ (See [24, 33] for more detail). For $1 \le k < n$, let $\mathbf{b}_k := \tau_k x_{k+1} \in \mathsf{R}(n\alpha_i)$. Then, by a direct computation, we have

$$\begin{aligned} &\mathbf{b}_r \mathbf{b}_s = \mathbf{b}_s \mathbf{b}_r \quad \text{if } |r-s| > 1, \\ &\mathbf{b}_r \mathbf{b}_{r+1} \mathbf{b}_r = \mathbf{b}_{r+1} \mathbf{b}_r \mathbf{b}_{r+1}, \quad \text{for } 1 \le r < n-1. \end{aligned} \tag{3.13}$$

Thus, for $w \in S_n$, $\mathbf{b}_w$ is well-defined.

We denote by $w[1,n]$ the longest element of the symmetric group S_n, and set

$$\mathbf{b}(i^n) := \mathbf{b}_{w[1,n]}. \tag{3.14}$$

Then (3.13) implies

$$\mathbf{b}(i^n)^2 = \mathbf{b}(i^n) \quad \text{and} \quad \mathbf{b}_k \mathbf{b}(i^n) = \mathbf{b}(i^n) \mathbf{b}_k = \mathbf{b}(i^n) \quad \text{for } 1 \le k < n.$$

The algebra $\mathsf{R}(n\alpha_i)$ decomposes into the direct sum of indecomposable projective graded modules over $\mathsf{R}(n\alpha_i)$ as follows:

$$\mathsf{R}(n\alpha_i) \simeq [n]_i! \mathsf{P}(i^n), \tag{3.15}$$

where

$$\mathsf{P}(i^n) := \mathsf{R}(n\alpha_i) \mathbf{b}(i^n) \left\langle \frac{d_i n(n-1)}{2} \right\rangle.$$

Note that $\mathsf{P}(i^n)$ is an indecomposable projective graded module unique up to isomorphism and grading shift. Note also that

$$\mathsf{R}(n\alpha_i) \mathbf{b}(i^n) \simeq \mathsf{R}(n\alpha_i) / \sum_{k=1}^{n-1} \mathsf{R}(n\alpha_i) \tau_k.$$

On the other hand, there exists an irreducible graded $\mathsf{R}(n\alpha_i)$-module $\mathsf{L}(i^n)$ which is unique up to isomorphism and grading shift:

$$\mathsf{L}(i^n) := \mathrm{Ind}_{\mathbf{k}[x_1] \otimes \cdots \otimes \mathbf{k}[x_n]}^{\mathsf{R}(n\alpha_i)} \mathbf{1}, \tag{3.16}$$

where $\mathbf{1}$ is the trivial $\mathbf{k}[x_1] \otimes \cdots \otimes \mathbf{k}[x_n]$-module which is isomorphic to $\mathbf{k}_0$. Hence $\mathsf{L}(i^n)$ is the $\mathsf{R}(n\alpha_i)$-module generated by the element $u(i^n)$ of degree 0 such that

$$x_k u(i^n) = 0 \ (1 \le k \le n), \quad \mathbf{k}_s u(i^n) = 0 \ (s > 0).$$

The case when $a_{ii} < 0$. Recall the $\mathbb{Z}$-grading on $\mathsf{R}(n)$ which is defined in (3.4). In this case, all the generators have positive grading. Thus, the algebra $\mathsf{R}(n\alpha_i)$ has a unique idempotent $\mathbf{b}(i^n) := e(i^n)$; i.e.,

(3.17) $\quad \mathsf{R}(n\alpha_i)$ is the unique projective indecomposable $\mathsf{R}(n\alpha_i)$-module.

Therefore, by (3.11), we can conclude that $\mathsf{R}(n\alpha_i)$ has the unique 1-dimensional irreducible graded module $\mathsf{L}(i^n) = \mathbf{k}_0 u(i^n)$ up to isomorphism and grading shift; i.e.,

$$e(i^n)u(i^n) = u(i^n),\ x_k u(i^n) = 0,\ \tau_a u(i^n) = 0,\ \mathbf{k}_s u(i^n) = 0\ (s > 0), \tag{3.18}$$

for all $1 \le k \le m$ and $1 \le a \le m-1$.

The case when $a_{ii} = 0$ If we take $\mathsf{P}_i(w,z) = w+z$, then $\mathsf{R}(3\alpha_i)$ has $-\tau_1\tau_2$, $-\tau_2\tau_1$ and $1+\tau_1\tau_2+\tau_2\tau_1$ as orthogonal primitive idempotents. In general, $\mathsf{R}(n\alpha_i)$ has many primitive idempotents. Hence the algebra $\mathsf{R}(n\alpha_i)$ is not a principal indecomposable $\mathsf{R}(n\alpha_i)$-module and $\mathsf{R}(n\alpha_i)$ has many irreducible modules. In this case, we also set $\mathbf{b}(i^n) := e(i^n)$ although $e(i^n)$ is not the unique idempotent.

Hereafter, we construct a graded faithful representation of $\mathsf{R}(n\alpha_i)$. For a polynomial ring $\mathbf{k}[X_1,\dots,X_n]$, we assign the degree of X_k $(1 \le k \le n)$ as $2d_i$. We define the action of x_k $(1 \le k \le n)$ and τ_a $(1 \le a \le n-1)$ on $\mathbf{k}[X_1,\dots,X_n]$ as follows:

$$\begin{aligned} x_k \cdot f(X_1,\dots,X_n) &:= X_k f(X_1,\dots,X_n),\\ \tau_a \cdot f(X_1,\dots,X_n) &:= \mathsf{P}_i(X_a,X_{a+1})\partial_a(f(X_1,\dots,X_n)) \end{aligned} \tag{3.19}$$

for $f(X_1,\dots,X_n) \in \mathbf{k}[X_1,\dots,X_n]$.

Proposition 3.2.1. *The polynomial ring* $\mathbf{k}[X_1,\dots,X_n]$ *is a faithful representation of* $\mathsf{R}(n\alpha_i)$.

Proof. If $i \in I^{\mathrm{re}}$, our assertion was shown in [24, Example 2.2]. We may assume that $i \in I^{\mathrm{im}}$. Note that for $w \in S_n$,

(3.20) ∂_w is well-defined and $\{\partial_w x_1^{a_1}\cdots x_n^{a_n} \mid a_1,\dots a_n \in \mathbb{Z}_{\ge 0},\ w \in S_n\}$ is a linearly independent subset of $\mathrm{End}(\mathbf{k}[X_1,\dots,X_n])$.

To prove our assertion, we need to prove that the map $\gamma_i : \mathsf{R}(n\alpha_i) \to \mathrm{End}(\mathbf{k}[X_1, \dots, X_n])$ induced by (3.19) is indeed a homomorphism with kernel equal to $\{0\}$. The relations given in Definition 3.1.1 except for ($\mathsf{R}5$), ($\mathsf{R}6$) and ($\mathsf{R}8$) are obvious. For the relation ($\mathsf{R}5$), we have

$$\tau_a x_{a+1} \cdot f = \mathsf{P}_i(X_a, X_{a+1}) \frac{X_a s_a(f) - X_{a+1} f}{X_a - X_{a+1}},$$
$$x_a \tau_a \cdot f = \mathsf{P}_i(X_a, X_{a+1}) \frac{X_a s_a(f) - X_a f}{X_a - X_{a+1}},$$

for any $f \in \mathbf{k}[X_1, \dots, X_n]$. Thus we have

$$\tau_a x_{a+1} - x_a f \tau_a = \mathsf{P}_i(x_a, x_{a+1}), \quad \text{in } \mathrm{End}(\mathbf{k}[X_1, \dots, X_n]).$$

In a similar way, we can show that $x_{a+1}\tau_a - \tau_a x_a = \mathsf{P}_i(x_a, x_{a+1})$ in $\mathrm{End}(\mathbf{k}[X_1, \dots, X_n])$. By a direct computation, we have

$$\tau_a^2 \cdot f = \frac{\mathsf{P}_i(X_{a+1}, X_a) - \mathsf{P}_i(X_a, X_{a+1})}{X_a - X_{a+1}} \mathsf{P}_i(X_a, X_{a+1}) \frac{s_a(f) - f}{X_a - X_{a+1}}.$$

Since $\gamma_i(\tau_a) = \mathsf{P}_i(X_a, X_{a+1})\partial_a$, ($\mathsf{R}6$) holds. To check relation ($\mathsf{R}8$), it suffices to show that the assertion hold for $n = 3$. Set

$$A = \frac{\mathsf{P}_i(x_1, x_2)}{x_1 - x_2},\ A' = \frac{\mathsf{P}_i(x_2, x_1)}{x_1 - x_2},\ B = \frac{\mathsf{P}_i(x_2, x_3)}{x_2 - x_3},\ B' = \frac{\mathsf{P}_i(x_3, x_2)}{x_2 - x_3},\ C = \frac{\mathsf{P}_i(x_1, x_3)}{x_1 - x_3}.$$

Then we have

$$\gamma_i(\tau_2\tau_1\tau_2) = ABC(s_2s_1s_2 - s_2s_1 - s_1s_2 + s_1) - BB'C(1 - s_2) + AB^2(s_2 - 1),$$
$$\gamma_i(\tau_1\tau_2\tau_1) = ABC(s_1s_2s_1 - s_1s_2 - s_2s_1 + s_2) - AA'C(1 - s_1) + A^2B(s_1 - 1).$$

Thus

$$\begin{aligned}\gamma_i(\tau_2\tau_1\tau_2 - \tau_1\tau_2\tau_1) &= (A'C + BC - AB)\,(A(s_1 - 1)) \\ &\quad - (AB + B'C - AC)\,(B(s_2 - 1))\,.\end{aligned}$$

Since $\tau_1 = A(s_1 - 1)$ and $\tau_2 = B(s_2 - 1)$ in $\mathrm{End}(\mathbf{k}[X_1, X_2, X_3])$, ($\mathsf{R}6$) also holds. Hence γ_i is a homomorphism.

In remains to show that γ_i is injective. Recall the notation τ_w for $w \in S_n$ in Remark 3.1.2. We take a nonzero element

$$y = \tau_{w_1} f_1 + \cdots + \tau_{w_r} f_r \text{ for some } f_k \in \mathbf{k}[x_1, \cdots, x_n] \ (1 \le k \le r) \text{ in } \mathsf{R}(n\alpha_i)$$

such that $w_a \neq w_b$ in S_n $(1 \le a \neq b \le r)$. Then, $\gamma_i(y)$ can be written as

$$\gamma_i(y) = \partial_{w_1} f_1' + \cdots + \partial_{w_r} f_r'$$

for some nonzero f_k'. By (3.20), $\gamma_i(y)$ is nonzero which yields our assertion. $\square$

Corollary 3.2.1. *The set*

$$\{\tau_w x_1^{a_1} \cdots x_n^{a_n} \ | a_1, \ldots, a_n \in \mathbb{Z}_{\ge 0}, \ w \in S_n\}$$

forms a basis of $\mathbf{k}$*-module* $\mathsf{R}(n\alpha_i)$*.*

Proof. It is an immediate consequence of Remark 3.1.2 and Proposition 3.2.1. $\square$

By Corollary 3.2.1, for $i \in I^{\mathrm{re}}$, the $\mathsf{R}(n\alpha_i)$-module $\mathsf{L}(i^n)$ has a $\mathbf{k}_0$-basis

$$\{\tau_w \cdot u(i^n) \mid \ w \in S_n\}.$$

Set

$$L_k := \{v \in \mathsf{L}(i^n) \mid x_n^k \cdot v = 0\} \quad (k \ge 0).$$

Since x_n commutes with all x_i $(i = 1, \ldots, n-1)$ and τ_j $(j = 1, \ldots, n-2)$, L_k has an $\mathsf{R}((n-1)\alpha_i)$-module structure. Moreover L_k has a $\mathbf{k}_0$-basis

$$\{\tau_w \tau_{n-1} \cdots \tau_s u(i^n) \mid w \in S_{n-1}, n-k+1 \le s \le n\},$$

and $L_n = \mathsf{L}(i^n)$. Thus we have a module isomorphism

$$L_k/L_{k-1} \simeq \mathsf{L}(i^{n-1}) \langle 2d_i(1-k) \rangle \quad \text{for } 1 \le k \le n.6 \tag{3.21}$$

Here the grading shift is caused by the degree of $\tau_{n-1} \cdots \tau_{n-k+1} u(i^n)$. Note that $L_k = x_n^{n-k} \mathsf{L}(i^n)$ for $0 \le k \le n$.

3.3 The Poincaré-Birkhoff-Witt-type basis.

In this section, we will prove that the set given in Remark 3.1.2 is indeed a basis of $\mathbf{k}$-module $\mathsf{R}(n)$. From this fact, we can define functor between categories of projective $\mathsf{R}(n)$-modules which play a crucial role in the later sections.

Take a total order $\prec$ on I. For $\beta \in Q^+$ with $|\beta| = n$, let

$$\mathfrak{Pol}_\beta = \bigoplus_{\nu \in I^\beta} \mathbf{k}[X_1, \dots, X_n]E(\nu) \ \text{ and } \ \mathfrak{Pol}_n = \bigoplus_{\beta \in Q^+,\ |\beta|=n} \mathfrak{Pol}_\beta.$$

We define the action of the generators $e(\nu)$ ($\nu \in I^n$), x_k ($1 \le k \le n$) and τ_a ($1 \le a \le n-1$) on $\mathfrak{Pol}_n$ as follows:

$$\begin{aligned} &e(\mu) \cdot fE(\nu) = \delta_{\mu,\nu} fE(\nu),\ x_k \cdot fE(\nu) = X_k fE(\nu), \\ &\tau_a \cdot fE(\nu) = \begin{cases} \mathsf{P}_{\nu_a}(X_a, X_{a+1})\partial_a(f)E(\nu) & \text{if } \nu_a = \nu_{a+1}, \\ \mathsf{Q}_{\nu_a,\nu_{a+1}}(X_a, X_{a+1}) s_a(f) E(s_a \cdot \nu) & \text{if } \nu_a \succ \nu_{a+1}, \\ s_a(f)E(s_a \cdot \nu) & \text{if } \nu_a \prec \nu_{a+1}, \end{cases} \end{aligned} \tag{3.22}$$

for $f \in \mathbf{k}[X_1, \dots, X_n]$.

Lemma 3.3.1. *$\mathfrak{Pol}_n$ is a well-defined $\mathsf{R}(n)$-module.*

Proof. We have to verify that the defining relations given in 3.1.1 hold in $\mathrm{End}(\mathfrak{Pol}_n)$. It suffices to assume that $n = 3$ and $|I| \le 3$. If $|I| = 1$, we already proved this in Proposition 3.2.1. Thus, we may assume that $|I| = 2$ or $|I| = 3$. By a simple computation, one can check that the relations except for ($\mathsf{R}8$) hold trivially. Hence we will consider only relation ($\mathsf{R}8$) under the following cases:
Case (i): Let $\nu = (i, j, i)$ with $i \ne j$. Without loss of generality, we may assume $i \prec j$. For $a, b, c \in \mathbb{Z}_{\geq>0}$, Set $X^{a,b,c} := X_1^a X_2^b X_3^c$. By a direct computation, we have

$$\begin{aligned} \tau_1\tau_2\tau_1 \cdot X^{a,b,c}E(\nu) &= \mathsf{P}_i(X_1, X_3)\mathsf{Q}_{ij}(X_1, X_2)\frac{X^{a,b,c} - X^{c,b,a}}{X_1 - X_3}E(\nu), \\ \tau_2\tau_1\tau_2 \cdot X^{a,b,c}E(\nu) &= \mathsf{P}_i(X_1, X_3)\frac{\mathsf{Q}_{ij}(X_3, X_2)X^{a,b,c} - \mathsf{Q}_{ij}(X_1, X_2)X^{c,b,a}}{X_1 - X_3}E(\nu), \end{aligned}$$

which yield

$$(\tau_1\tau_2\tau_1 - \tau_2\tau_1\tau_2)e(\nu) = \mathsf{P}_i(x_1,x_3)\frac{\mathsf{Q}_{ij}(x_3,x_2) - \mathsf{Q}_{ij}(x_1,x_2)}{x_3 - x_1}e(\nu) \text{ in } \mathrm{End}(\mathfrak{Pol}_n).$$

Case (ii): Let $\nu = (i,j,k)$ such that i, j, k are distinct. Since the other cases are similar, we will only prove our assertion when $i \succ j \succ k$. Then we have

$$\begin{aligned}
\tau_1\tau_2\tau_1 \cdot X^{a,b,c}E(\nu) &= \mathsf{Q}_{ij}(X_2,X_3)\mathsf{Q}_{jk}(X_1,X_2)\mathsf{Q}_{ik}(X_1,X_3)X^{c,b,a}E(\nu),\\
\tau_2\tau_1\tau_2 \cdot X^{a,b,c}E(\nu) &= \mathsf{Q}_{ij}(X_2,X_3)\mathsf{Q}_{jk}(X_1,X_2)\mathsf{Q}_{ik}(X_1,X_3)X^{c,b,a}E(\nu),
\end{aligned}$$

which implies that $(\tau_1\tau_2\tau_1 - \tau_2\tau_1\tau_2)e(\nu) = 0$ in $\mathrm{End}(\mathfrak{Pol}_n)$.
Case (iii): Similarly as above, we consider $\mathbf{i} = (i,i,j)$ with $i \succ j$ only. Then

$$\begin{aligned}
\tau_1\tau_2\tau_1 \cdot X^{a,b,c}E(\nu) &= \mathsf{Q}_{ij}(X_1,X_2)\mathsf{Q}_{ij}(X_1,X_3)\mathsf{P}_i(X_2,X_3)\frac{X^{c,a,b} - X^{c,b,a}}{X_2 - X_3}E(\nu),\\
\tau_2\tau_1\tau_2 \cdot X^{a,b,c}E(\nu) &= \mathsf{Q}_{ij}(X_1,X_2)\mathsf{Q}_{ij}(X_1,X_3)\mathsf{P}_i(X_2,X_3)\frac{X^{c,a,b} - X^{c,b,a}}{X_2 - X_3}E(\nu).
\end{aligned}$$

Hence we have $(\tau_1\tau_2\tau_1 - \tau_2\tau_1\tau_2)e(\nu) = 0$ in $\mathrm{End}(\mathfrak{Pol}_n)$, which completes the proof. □

By the first relation in (3.22), we can naturally deduce the $\mathsf{R}(\beta)$-module on $\mathfrak{Pol}_\beta$. Note that $\mathsf{R}(\beta) = \bigoplus_{\mu,\nu\in I^\beta} e(\mu)\mathsf{R}(\beta)e(\nu)$. For μ, $\nu \in I^\beta$, let

$${}_\mu S_\nu = \{w \mid w \in S_{|}\beta|,\ w(\mu) = \nu\}.$$

By Remark 3.1.2, for $\beta \in Q^+$ with $|\beta| = n$,

$${}_\mu B_\nu := \{\tau_w x_1^{a_1}\cdots x_n^{a_n}e(\nu) | a_1,\ldots,a_n \in \mathbb{Z}_{\geq 0},\ w \in {}_\mu S_\nu\}$$

spans $e(\mu)\mathsf{R}(\alpha)e(\nu)$.

Theorem 3.3.1. *Fix $\nu \in I^\beta$. Then the set ${}_\mu B_\nu$ is a basis of the* $\mathbf{k}$*-module $e(\mu)\mathsf{R}(\beta)e(\nu)$ for any $\mu \in I^\beta$.*

Proof. Let $<$ be the lexicographic order of I^β arising from the order $\prec$ of I, and let ${}_\eta w_\mu$ be the one of the elements in ${}_\eta S_\mu$ for η, $\mu \in I^\beta$. Let

$$\gamma_\beta : \mathsf{R}(\beta) \longrightarrow \mathrm{End}(\mathfrak{Pol}_\beta)$$

be the algebra homomorphism given in (3.22). We will show that $\gamma_\beta({}_\mu B_\nu)$ is linearly independent for any $\mu \in I^\beta$, which would imply the set ${}_\mu B_\nu$ is linearly independent. We prove our claim using induction on the lexicographic order $<$ on I^β.

Let $\nu \in I^\beta$, and let

$$\mu = (\underbrace{j_1 \dots j_1}_{k_1} \underbrace{j_2 \dots j_2}_{k_2} \cdots \underbrace{j_r \dots j_r}_{k_r}) \in I^\beta$$

such that $j_1 \prec j_2 \prec \cdots \prec j_r$. Note that μ is a minimal element in I^β. Let m be a linear combination of ${}_\mu B_\nu$ such that $\gamma_\beta(m) = 0$. Note that m can be expressed as

$$m = \sum_s \tau_{w_s} \tau_{{}_\mu w_\nu} x^{\mathbf{a}_s} e(\nu)$$

for some $\mathbf{a}_s \in \mathbb{Z}^m_{\geq 0}$, $x^{\mathbf{a}_s} := x_1^{a_{s_1}} \cdots x_n^{a_{s_n}}$ and some $w_s \in S_{k_1} \times \cdots \times S_{k_r}$. It follows from Lemma 3.3.1 that $\gamma_\beta\left(\tau_{{}_\mu w_\nu} e(\nu)\right)$ is a linear map $\mathbf{k}[X_1, \dots, X_n]E(\nu)$ to $\mathbf{k}[X_1, \dots, X_n]E(\mu)$ sending $E(\nu)$ to $E(\mu)$. Hence,

$$\gamma_\beta(m) = 0 \text{ if and only if } \gamma_\beta\left(\sum_s \tau_{w_s} x^{{}_\mu w_\nu \cdot (\mathbf{a}_s)} e(\nu)\right) = 0.$$

Since $\gamma_\beta\left(\sum_s \tau_{w_s} x^{{}_\mu w_\nu \cdot (\mathbf{a}_s)} e(\nu)\right)$ is in $\mathrm{End}(\mathfrak{Pol}_\beta)$ sending $E(\nu)$ to $E(\mu)$, by Proposition 3.2.1, we have

$$\gamma_\beta\left(\sum_s \tau_{w_s} x^{{}_\mu w_\nu \cdot (\mathbf{a}_s)} e(\mu)\right) = 0 \text{ if and only if } \sum_s \tau_{w_s} x^{{}_\mu w_\nu \cdot (\mathbf{a}_s)} e(\mu) = 0,$$

which implies $m = 0$. Therefore, $\gamma_\beta({}_\mu B_\nu)$ is linearly independent.

We now consider the case when η is an arbitrary sequence in I^β such that $\eta > \mu$. This step can be proved by a similar induction argument as in [24, Theorem 2.5], which completes the proof. □

For any α, $\beta \in Q^+$, $e(\alpha, \beta)\mathsf{R}(\alpha + \beta)$ has a natural graded $\mathsf{R}(\alpha, \beta)$-module structure.

Corollary 3.3.1. *$e(\alpha, \beta)\mathsf{R}(\alpha + \beta)$ is a free graded left $\mathsf{R}(\alpha, \beta)$-module.*

Proof. Let $n := |\alpha|$, $m := |\beta|$ and $S_n \times S_n \backslash S_{n+m}$ be the set of the minimal right $S_n \times S_n$-coset representatives of S_{n+m}. For $w \in S_n \times S_n \backslash S_{n+m}$, set

$$\widehat{\tau}_w = \sum_{\substack{\nu \in I^\alpha, \\ \mu \in I^\beta}} e(\nu, \mu)\tau_w e(w^{-1} \cdot (\nu, \mu)).$$

Then, Theorem 3.3.1 tells that

$$\{\widehat{\tau}_w | w \in S_n \times S_n \backslash S_{n+m}\}$$

is a basis of the left graded $\mathsf{R}(\alpha, \beta)$-module $e(\alpha, \beta)\mathsf{R}(\alpha + \beta)$. □

By the above corollary, we can conclude that

(3.23) $\mathrm{Ind}_{\alpha,\beta}$ and $\mathrm{Res}_{\alpha,\beta}$ take projective modules to projective modules.

Hence we have the linear maps

$$\begin{aligned} &[\mathrm{Ind}_{\alpha,\beta}] : [\mathrm{Proj}(\mathsf{R}(\alpha))] \times [\mathrm{Proj}(\mathsf{R}(\beta))] \to [\mathrm{Proj}(\mathsf{R}(\alpha + \beta))], \\ &[\mathrm{Res}_{\alpha,\beta}] : [\mathrm{Proj}(\mathsf{R}(\alpha + \beta))] \to [\mathrm{Proj}(\mathsf{R}(\alpha))] \times [\mathrm{Proj}(\mathsf{R}(\beta))]. \end{aligned} \tag{3.24}$$

Given $\alpha, \alpha', \beta, \beta' \in Q^+$ with $\alpha + \beta = \alpha' + \beta'$, let

$${}_{\alpha,\beta}\mathsf{R}_{\alpha',\beta'} := e(\alpha, \beta)\mathsf{R}(\alpha + \beta)e(\alpha', \beta'). \tag{3.25}$$

We also write ${}_{\alpha+\beta}\mathsf{R}_{\alpha,\beta} := \mathsf{R}(\alpha+\beta)e(\alpha, \beta)$ and ${}_{\alpha',\beta'}\mathsf{R}_{\alpha'+\beta'} := e(\alpha', \beta')\mathsf{R}(\alpha'+\beta')$. Note that ${}_{\alpha,\beta}\mathsf{R}_{\alpha',\beta'}$ is a $(\mathsf{R}(\alpha, \beta), \mathsf{R}(\alpha', \beta'))$-bimodule. Now we obtain Mackey's Theorem for the quiver Hecke algebras.

Proposition 3.3.1. *The $(\mathsf{R}(\alpha, \beta), \mathsf{R}(\alpha', \beta'))$-bimodule ${}_{\alpha,\beta}\mathsf{R}_{\alpha',\beta'}$ has a graded filtration with graded subquotients isomorphic to*

$$({}_{\alpha}\mathsf{R}_{\alpha-\gamma,\gamma}) \otimes ({}_{\beta}\mathsf{R}_{\beta+\gamma-\beta',\beta'-\gamma}) \otimes_{\mathsf{R}'} ({}_{\alpha-\gamma,\alpha'+\gamma-\alpha}\mathsf{R}_{\alpha'}) \otimes ({}_{\gamma,\beta'-\gamma}\mathsf{R}_{\beta'})\langle -(\gamma|\beta+\gamma-\beta')\rangle,$$

where $\mathsf{R}' = \mathsf{R}(\alpha - \gamma) \otimes \mathsf{R}(\gamma) \otimes \mathsf{R}(\beta + \gamma - \beta') \otimes \mathsf{R}(\beta' - \gamma)$ and γ ranges over the set of $\gamma \in Q^+$ such that $\alpha - \gamma$, $\beta' - \gamma$ and $\beta + \gamma - \beta' = \alpha' + \gamma - \alpha$ belong to Q^+.

3.4 Quantum Serre relations

In this section, we will show that the Grothendieck group $[\mathsf{Proj}(\mathsf{R}(\beta))]$ satisfies the categorical Serre relations.

Recall the definition of the idempotent $\mathbf{b}(i^n)$ in Section 3.2. For $\nu = (\nu_1^{a_1}, \ldots, \nu_k^{a_k}) \in I^{(\beta)}$, we define

$$\mathbf{b}(\nu) := \mathbf{b}(\nu_1^{a_1}) \otimes \cdots \otimes \mathbf{b}(\nu_k^{a_k}) \in \mathsf{R}(\beta).$$

Since each $\mathbf{b}(\nu_r^{a_r})$ $(1 \le r \le k)$ is an idempotent, so is $\mathbf{b}(\nu)$ and

$$\mathsf{P}(\nu) := \mathsf{R}(\beta)\mathbf{b}(\nu) \left\langle \sum_{\substack{r=1,\ldots,k,\\ i_r \in I^{\mathrm{re}}}} \frac{a_k(a_k-1)(\alpha_{i_r}|\alpha_{i_r})}{4} \right\rangle$$

is a projective graded $\mathsf{R}(\beta)$-module. Recall that, for $i \in I^{\mathrm{im}}$,

$$\mathsf{P}(i^n) = \mathsf{R}(n\alpha_i).$$

Now, we will prove that the quantum Serre relations hold on $[\mathsf{Proj}(\mathsf{R}(\beta))]$. Suppose that we have $i \in I^{\mathrm{re}}$ and $j \in I$ such that $i \neq j$ and $a_{ij} \neq 0$. Let $b = 1 - a_{ij}$ and take nonnegative integers $m, n \ge 0$ with $m + n = b$. Note that

$$\begin{aligned} e(i^m, j, i^n)\tau_b \cdots \tau_{m+1} &= \tau_b \cdots \tau_{m+1} e(i^{m+1}, j, i^{n-1}), \\ e(i^m, j, i^n)\tau_1 \cdots \tau_{m+1} &= \tau_1 \cdots \tau_m e(i^{m-1}, j, i^{n+1}). \end{aligned}$$

Define the homogeneous elements

$$\begin{aligned} \alpha^+_{m,n} &:= e(i^m, j, i^n)\tau_b \cdots \tau_{m+1}\mathbf{b}((i^{m+1}, j, i^{n-1})), \\ \alpha^-_{m,n} &:= e(i^m, j, i^n)\tau_1 \cdots \tau_m\mathbf{b}((i^{m-1}, j, i^{n+1})). \end{aligned}$$

Choose a pair of sequences ν and μ such that $(\nu, i^m, j, i^n, \mu) \in I^{(\alpha)}$. Then these elements give rise to homomorphisms of graded projective modules

$$\begin{aligned} d^+_{m,n} : \mathsf{P}(\mu, i^m, j, i^n, \nu) &\longrightarrow \mathsf{P}(\mu, i^{m+1}, j, i^{n-1}, \nu), \\ v &\longmapsto v \cdot \mathbf{b}(\mu) \otimes \alpha^+_{a,b} \otimes \mathbf{b}(\nu), \\ d^-_{m,n} : \mathsf{P}(\mu, i^m, j, i^n, \nu) &\longrightarrow \mathsf{P}(\mu, i^{m-1}, j, i^{n+1}, \nu), \\ v &\longmapsto v \cdot \mathbf{b}(\mu) \otimes \alpha^-_{a,b} \otimes \mathbf{b}(\nu). \end{aligned} \tag{3.26}$$

Set $d^+_{b,0} = 0$ and $d^-_{0,b} = 0$. Then we have

$$0 \rightleftarrows \mathsf{P}(\mu, j, i^b, \nu) \underset{d^-_{1,b-1}}{\overset{d^+_{0,b}}{\rightleftarrows}} \cdots \underset{d^-_{m,n}}{\overset{d^+_{m-1,n+1}}{\rightleftarrows}} \mathsf{P}(\mu, i^m, j, i^n, \nu) \underset{d^-_{m+1,n-1}}{\overset{d^+_{m,n}}{\rightleftarrows}}$$

$$\mathsf{P}(\mu, i^{m+1}, j, i^{n-1}, \nu) \underset{d^-_{m+2,n-2}}{\overset{d^+_{m+1,n-1}}{\rightleftarrows}} \cdots \underset{d^-_{b,0}}{\overset{d^+_{b-1,1}}{\rightleftarrows}} \mathsf{P}(\mu, i^b, j, \nu) \rightleftarrows 0.$$

Lemma 3.4.1.

(a) $d^+_{m,n} \circ d^+_{m-1,n+1} = 0, \quad d^-_{m,n} \circ d^-_{m+1,n-1} = 0$ *for* $m\ ,n > 0$.

(b) $d^+_{b-1,1} \circ d^-_{b,0} = (-1)^{b-1} t_{i,j;(-a_{ij},0)} \mathrm{id}, \quad d^-_{1,b-1} \circ d^+_{0,b} = t_{i,j;(-a_{ij},0)} \mathrm{id}$.

(c) *For* $1 < m, n < b$, *we have*

$$d^+_{m-1,n+1} \circ d^-_{m,n} - d^-_{m+1,n-1} \circ d^+_{m,n} = (-1)^{m-1} t_{i,j;(-a_{ij},0)} \mathrm{id}.$$

Proof. If $j \in I^{\mathrm{re}}$, this lemma was proved in [25, 33]. We will prove our lemma when $j \in I^{\mathrm{im}}$.

Let $\mathbf{b}_{m,n} = \mathbf{b}((i^m, j, i^n))$ for $m, n \geq 0$. Since $i \in I^{\mathrm{re}}$, it follows from [25, 33] that

$$\begin{aligned}
\alpha^+_{m,n} &= \tau_b \cdots \tau_{b-n} \mathbf{b}_{m+1,n-1} = \mathbf{b}_{m,n} \tau_b \cdots \tau_{b-n} \mathbf{b}_{m+1,n-1}, \\
\alpha^-_{m,n} &= \tau_1 \cdots \tau_m \mathbf{b}_{m-1,n+1} = \mathbf{b}_{m,n} \tau_1 \cdots \tau_a \mathbf{b}_{m-1,n+1}.
\end{aligned}$$

By a direct computation, we have

$$\begin{aligned}
\alpha^+_{m-1,n+1} \alpha^+_{m,n} &= \mathbf{b}_{m-1,n+1} \tau_b \cdots \tau_m \mathbf{b}_{m,n} \mathbf{b}_{m,n} \tau_b \cdots \tau_{m+1} \mathbf{b}_{m+1,n-1} \\
&= \mathbf{b}_{m-1,n+1} \tau_b \cdots \tau_m \tau_b \cdots \tau_{m+1} \mathbf{b}_{m+1,n-1} \\
&= 0.
\end{aligned}$$

In the same manner, we get $\alpha^-_{m+1,n-1} \alpha^-_{m,n} = 0$.

On the other hand, for $a, b > 0$, we obtain

$$\begin{aligned}
\alpha^+_{m,n} \alpha^-_{m+1,n-1} &= \mathbf{b}_{m,n} \tau_b \cdots \tau_{m+1} \mathbf{b}_{m+1,n-1} \mathbf{b}_{m+1,n-1} \tau_1 \cdots \tau_{m+1} \mathbf{b}_{m,n} \\
&= \tau_1 \cdots \tau_{m-1} \tau_b \cdots \tau_{m+1} \tau_m \tau_{m+1} \mathbf{b}_{a,b}, \\
\alpha^-_{m,n} \alpha^+_{m-1,n+1} &= \mathbf{b}_{m,n} \tau_1 \cdots \tau_m \mathbf{b}_{m-1,n+1} \mathbf{b}_{m-1,n+1} \tau_b \cdots \tau_m \mathbf{b}_{m,n} \\
&= \tau_1 \cdots \tau_{m-1} \tau_b \cdots \tau_m \tau_{m+1} \tau_m \mathbf{b}_{m,n},
\end{aligned}$$

which implies

$$\alpha^-_{b,0}\alpha^+_{b-1,1} = (-1)^{b-1}t_{i,j;(-a_{ij},0)}\mathbf{b}_{b,0}, \quad \alpha^+_{0,b}\alpha^-_{1,b-1} = t_{i,j;(-a_{ij},0)}\mathbf{b}_{0,b},$$

and

$$\begin{aligned}\alpha^-_{m,n}\alpha^+_{m-1,n+1} - \alpha^+_{m,n}\alpha^-_{m+1,n-1} &= \tau_1\cdots\tau_{m-1}\tau_b\cdots\tau_{m+2}(\tau_m\tau_{m+1}\tau_m - \tau_{m+1}\tau_m\tau_{m+1})\mathbf{b}_{m,n}\\ &= \tau_1\cdots\tau_{m-1}\tau_b\cdots\tau_{m+2}(\overline{Q}_{m,m+1}(x_m, x_{m+1}, x_{m+2})\mathbf{b}_{a,b}\\ &= (-1)^{m-1}t_{i,j;(-a_{ij},0)}\mathbf{b}_{a,b}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}&\alpha^+_{m-1,n+1}\alpha^+_{m,n} = 0, \quad \alpha^-_{m+1,n-1}\alpha^-_{m,n} = 0,\\ &\alpha^-_{b,0}\alpha^+_{b-1,1} = (-1)^{b-1}t_{i,j;(-a_{ij},0)}\mathbf{b}_{b,0}, \quad \alpha^+_{0,b}\alpha^-_{1,b-1} = t_{i,j;(-a_{ij},0)}\mathbf{b}_{0,b},\\ &\alpha^-_{m,n}\alpha^+_{m-1,n+1} - \alpha^+_{m,n}\alpha^-_{m+1,n-1} = (-1)^{m-1}t_{i,j;(-a_{ij},0)}\mathbf{b}_{m,n},\end{aligned}$$

as desired. □

Theorem 3.4.1. *For any pair of sequences μ and ν,*

(a) *if $a_{ij} = 0$ for $i \neq j$, then $[\mathsf{P}(\mu, i, j, \nu)] = [\mathsf{P}(\mu, j, i, \nu)]$*

(b) *If $i \in I^{\mathrm{re}}$ and $j \in I$ with $i \neq j$, then*

$$\sum_{k=0}^{1-a_{ij}} (-1)^k [\mathsf{P}(\mu, i^k, j, i^{1-a_{ij}-k}, \nu)] = 0.$$

Proof. Without loss of generality, we assume that $i < j$. If $a_{ij} = 0$, let τ^- (resp. τ^+) be the element in R changing (ij) to (ji) (resp. (ji) to (ij)) and define

$$d^- : \mathsf{P}(\mu, i, j, \nu) \to \mathsf{P}(\mu, j, i, \nu) \text{ (resp. } d^+ : \mathsf{P}(\mu, j, i, \nu) \to \mathsf{P}(\mu, i, j, \nu))$$

to be the map given by right multiplication by $t_{i,j;(-a_{ij},0)}\tau^-$ (resp. $t_{ji}\tau^+$). From the defining relation, we see that d^+ and d^- are inverses to each other. Hence

$$[\mathsf{P}(\mu, i, j, \nu)] = [\mathsf{P}(\mu, j, i, \nu)].$$

Suppose that $a_{ij} \neq 0$ and $i \in I^{\mathrm{re}}$. By Lemma 3.4.1, the complex $\big(\mathsf{P}(\mu, i^m, j, i^n, \nu), d^+_{m,n}\big)$ becomes an exact sequence with the splitting maps $(-1)^{m-1} t_{i,j;(-a_{ij},0)} d^-_{m,n}$. Therefore, our assertion follows from the Euler-Poincaré principle. $\square$

For a module $N = \bigoplus_{t\in\mathbb{Z}} N_t \in \mathsf{Mod}(\mathsf{R}(\beta))$ such that $\dim_{\mathbf{k}_0} N_t < \infty$, define a formal power series

$$\dim_q(N) = \sum_{t\in\mathbb{Z}} (\dim_{\mathbf{k}_0} N_t) q^t.$$

Note that for $N \in \mathsf{Proj}(\mathsf{R}(\beta))$, $\dim_q(N)$ is well-defined.

By similar arguments to the one in [24, Section 2.5], we have the lemma below:

Lemma 3.4.2. *Assume that* (3.2) *holds; i.e.,*$\mathbf{k} = \mathbf{k}_0$. *The* $\mathbb{A}$*-linear pairing* $(\,,\,) : [\mathsf{Proj}(\mathsf{R}(\beta))] \times [\mathsf{Proj}(\mathsf{R}(\beta))] \to \mathbb{Z}[[q, q^{-1}]]$ *defined by*

$$([P], [Q]) = \dim_q(P^\psi \otimes_{\mathsf{R}(\beta)} Q)$$

is a nondegenerate symmetric bilinear form on $\mathsf{Proj}(\mathsf{R}(\beta))$.

Given $\mu \in I^\alpha$ and $\nu \in I^\beta$, a sequence $\eta \in I^{\alpha+\beta}$ is called a *shuffle* of μ and ν if η is a permutation of (μ, ν) such that μ and ν are subsequences of η. For a shuffle η of $\mu \in I^\alpha$ and $\nu \in I^\beta$, let

$$\deg(\mu, \nu, \eta) = \deg(\tau_w e(\mu, \nu)),$$

where w is the element in $S_{|\alpha|+|\beta|}/S_{|\alpha|} \times S_{|\beta|}$ corresponding to η.

For $M \in \mathsf{Mod}(\mathsf{R}(\alpha))$ and $N \in \mathsf{Mod}(\mathsf{R}(\beta))$, $M \boxtimes N$ will denote the outer tensor product of M and N. Then, from Proposition 3.3.1, one can check that

$$\begin{aligned} &\mathrm{Ind}_{\alpha,\beta}(\mathsf{P}(\mu) \boxtimes \mathsf{P}(\nu)) \simeq \mathsf{P}((\mu, \nu)) && \text{for } \mu \in I^\alpha,\ \nu \in I^\beta, \\ &\mathrm{Res}_{\alpha,\beta}\mathsf{P}(\eta) \simeq \bigoplus_{\mu,\nu} \mathsf{P}(\mu) \boxtimes \mathsf{P}(\nu)\langle -\deg(\mu, \nu, \eta)\rangle && \text{for } \eta \in I^{\alpha+\beta}, \end{aligned} \tag{3.27}$$

where the sum is taken over all $\mu \in I^\alpha$, $\nu \in I^\beta$ such that η can be expressed as a shuffle of μ and ν. Set

$$\mathsf{Proj}(\mathsf{R}) = \bigoplus_{\beta\in Q^+} \mathsf{Proj}(\mathsf{R}(\beta)) \text{ and } [\mathsf{Proj}(\mathsf{R})] = \bigoplus_{\beta\in Q^+} [\mathsf{Proj}(\mathsf{R}(\beta))].$$

Then we can extend the linear maps $[\mathrm{Ind}_{\alpha,\beta}]$ and $[\mathrm{Res}_{\alpha,\beta}]$ in (3.24) to $[\mathrm{Ind}]$ and $[\mathrm{Res}]$ on $[\mathsf{Proj}(\mathsf{R})]$ by the following:

$$\begin{aligned}
[\mathrm{Ind}] :& [\mathsf{Proj}(\mathsf{R})] \otimes [\mathsf{Proj}(\mathsf{R})] \longrightarrow [\mathsf{Proj}(\mathsf{R})] \quad \text{given by} \quad ([M],[N]) \mapsto [\mathrm{Ind}_{\alpha,\beta} M \boxtimes N], \\
[\mathrm{Res}] :& [\mathsf{Proj}(\mathsf{R})] \longrightarrow [\mathsf{Proj}(\mathsf{R})] \otimes [\mathsf{Proj}(\mathsf{R})] \quad \text{given by} \quad [L] \mapsto \sum_{\alpha',\beta' \in Q^+} [\mathrm{Res}_{\alpha',\beta'} L].
\end{aligned}$$

where

- $M \in \mathsf{Proj}(\mathsf{R}(\alpha))$ and $N \in \mathsf{Proj}(\mathsf{R}(\beta))$,
- the sum is taken over all α',$\beta' \in Q^+$ for $\mathsf{R}(\alpha' + \beta')$-module L.

We denote by $[M][N]$ the product $[\mathrm{Ind}]([M],[N])$ of $[M]$ and $[N]$ in $[\mathsf{Proj}(\mathsf{R})]$.

Proposition 3.4.1.

(a) *The pair* $([\mathsf{Proj}(\mathsf{R})], \mathrm{Ind})$ *becomes an associative unital* $\mathbb{A}$*-algebra.*

(b) *The pair* $([\mathsf{Proj}(\mathsf{R})], \mathrm{Res})$ *becomes a coassociative counital* $\mathbb{A}$*-coalgebra.*

Proof. Our assertions on associativity and coassociativity follow from the transitivity of induction and restriction. Define

$$\begin{aligned}
\iota : \mathbb{A} \longrightarrow [\mathsf{Proj}(\mathsf{R})] \quad & \text{by} \quad \iota(\sum_k a_k q^k) = \sum_k a_k q^k \mathbf{1}, \\
\epsilon : [\mathsf{Proj}(\mathsf{R})] \longrightarrow \mathbb{A} \quad & \text{by} \quad \epsilon([M]) = c([M^0]),
\end{aligned}$$

where $\mathbf{1}$ is the trivial module over $\mathsf{R}(0)$-module which is isomorphic to $\mathbf{k}$, $[M^0]$ is the image of $[M]$ under the natural projection

$$[\mathsf{Proj}(\mathsf{R})] = \bigoplus_{\beta \in Q^+} [\mathsf{Proj}(\mathsf{R}(\beta))] \longrightarrow [\mathsf{Proj}(\mathsf{R}(0))]$$

and $c([M^0])$ is the coefficient of $[M^0]$ with respect to the canonical basis of $[\mathsf{Proj}(\mathsf{R}(0))]$. Then one can verify that ι (resp. ϵ) is the unit (resp. counit) of $[\mathsf{Proj}(\mathsf{R})]$. $\square$

We define the algebra structure on $[\mathsf{Proj}(\mathsf{R})] \otimes [\mathsf{Proj}(\mathsf{R})]$ by

$$([M_1] \otimes [M_2]) \cdot ([N_1] \otimes [N_2]) = q^{-(\beta_2|\gamma_1)}[M_1][N_1] \otimes [M_2][N_2]$$

for $M_i \in [\mathsf{Proj}(\mathsf{R}(\beta_i))]$, $N_i \in [\mathsf{Proj}(\mathsf{R}(\gamma_i))]$ $(i = 1, 2)$. Using Proposition 3.3.1, we prove:

Proposition 3.4.2. $[\mathrm{Res}] : [\mathsf{Proj}(\mathsf{R})] \longrightarrow [\mathsf{Proj}(\mathsf{R})] \otimes [\mathsf{Proj}(\mathsf{R})]$ *is an algebra homomorphism.*

Proposition 3.4.3. *Under assumption* (3.2)*, the bilinear pairing* $(\ ,\ ) : [\mathsf{Proj}(\mathsf{R})] \otimes [\mathsf{Proj}(\mathsf{R})] \to \mathbb{Q}(q)$ *satisfies the following properties:*

(a) $(\mathbf{1}, \mathbf{1}) = 1$,

(b) $([\mathsf{P}(i)], [\mathsf{P}(j)]) = \delta_{ij}(1 - q_i^2)^{-1}$ *for* $i, j \in I$,

(c) $([L], [M][N]) = (\mathrm{Res}[L], [M] \otimes [N])$ *for* $[L], [M], [N] \in [\mathsf{Proj}(\mathsf{R})]$,

(d) $([L][M], [N]) = ([L] \otimes [M], \mathrm{Res}[N])$ *for* $[L], [M], [N] \in [\mathsf{Proj}(\mathsf{R})]$.

Proof. Assertions (a) and (b) follow from the $\mathbb{Z}$-grading (3.4) on $\mathsf{R}(\alpha)$. Suppose that $L \in \mathsf{Proj}(\mathsf{R}(\alpha+\beta))$, $M \in \mathsf{Proj}(\mathsf{R}(\alpha))$ and $N \in \mathsf{Proj}(\mathsf{R}(\beta))$. Then we have

$$\begin{aligned}([L], [M][N]) &= \dim_q(L^{\psi} \otimes_{\mathsf{R}(\alpha+\beta)} \mathrm{Ind}_{\alpha,\beta}(M \boxtimes N)) \\ &= \dim_q((\mathrm{Res}_{\alpha,\beta}L)^{\psi} \otimes_{\mathsf{R}(\alpha,\beta)} M \boxtimes N) = (\mathrm{Res}_{\alpha,\beta}L, M \boxtimes N),\end{aligned}$$

which yields that $([L], [M][N]) = (\mathrm{Res}[L], [M] \otimes [N])$.

Assertion (4) can be proved in the same manner. □

Define a map $\Phi : U_{\mathbb{A}}^{-}(\mathfrak{g}) \longrightarrow [\mathsf{Proj}(\mathsf{R})]$ by

$$f_{i_1}^{(k_1)} \cdots f_{i_r}^{(k_r)} \longmapsto [\mathsf{P}(i_1^{k_1}, \ldots, i_r^{k_r})]. \tag{3.28}$$

Theorem 3.4.2. *Under assumption* (3.2)*, the map* Φ *is an injective algebra homomorphism.*

Proof. By Theorem 3.4.1, Φ is an algebra homomorphism. Since both of the maps Δ_0 and Res are algebra homomorphisms and

$$\Delta_0(f_i) = f_i \otimes \mathbf{1} + \mathbf{1} \otimes f_i,\ \mathrm{Res}(\mathsf{P}(i)) = \mathsf{P}(i) \otimes \mathbf{1} + \mathbf{1} \otimes \mathsf{P}(i)\ (i \in I),$$

by (2.9) and Proposition 3.4.3, we have

$$(x, y)_L = (\Phi(x), \Phi(y)) \text{ for all } x, y \in U_{\mathbb{A}}^-(\mathfrak{g}).$$

Hence KerΦ is contained in the radical of the bilinear form $(\ ,\ )_L$, which is nondegenerate. Now our assertion follows immediately. □

3.5 Crystal structure and strong perfect bases

In this section, we investigate the structure of $\mathsf{RepR}(\beta)$. From this, we can choose a set of irreducible $\mathsf{R}(\beta)$-modules which give a strong perfect basis of $[\mathsf{RepR}(\beta)]$. Throughout this section, we assume that

- $\mathbf{k}_0$ is a field and the components $\mathbf{k}_t$'s are finite-dimensional over $\mathbf{k}_0$ and
- $a_{ii} \neq 0$ for all $i \in I$.

For $i \in I$, define

$$\begin{aligned}
&\Delta_{i^k} M = e(\beta - k\alpha_i, i^k) M \in \mathsf{Rep}(\mathsf{R}(\beta - k\alpha_i, k\alpha_i)),\\
&\varepsilon_i^{\mathrm{or}}(M) = \max\{k \geq 0 \mid \Delta_{i^k} M \neq 0\},\\
&E_i(M) = e(\beta - \alpha_i, i) M \in \mathsf{Rep}(\mathsf{R}(\beta - \alpha_i)),\\
&F_i'(M) = \mathrm{Ind}_{\beta, \alpha_i}(M \boxtimes \mathsf{L}(i)) \in \mathsf{Rep}(\mathsf{R}(\beta + \alpha_i)),\\
&\tilde{e}_i(M) = \mathrm{soc}((E_i(M))) \in \mathsf{Rep}(\mathsf{R}(\beta - \alpha_i)),\\
&\tilde{f}_i(M) = \mathrm{hd}((F_i' M)) \in \mathsf{Rep}(\mathsf{R}(\beta + \alpha_i)).
\end{aligned} \tag{3.29}$$

Here, $\mathrm{soc}(M)$ means the *socle* of M, the largest semisimple subobject of M and $\mathrm{hd}(M)$ means the *head* of M, the largest semisimple quotient of M. We set $\varepsilon_i(M) = -\infty$ for $M = 0$.
For $M \in \mathsf{Rep}(\mathsf{R}(\beta))$, $M_k \in \mathsf{Rep}(\mathsf{R}(\beta_k))$ $(k = 1, \ldots, m)$ and $d \in \mathbb{Z}_{>0}$, set

$$M^{\boxtimes d} = \underbrace{M \boxtimes \cdots \boxtimes M}_{d}, \qquad \mathop{\boxtimes}_{k=1}^{m} M_k = M_1 \boxtimes \cdots \boxtimes M_m.$$

We also define the *character* $\mathrm{ch}_q(M)$ of M to be

$$\mathrm{ch}_q(M) = \sum_{\nu \in I^\beta} (\dim_q(e(\nu)M))\nu.$$

By Frobenius reciprocity, we have

$$\begin{aligned}\mathrm{Hom}_{\mathsf{R}(\beta)}(\mathrm{Ind}_{\beta-m\alpha_i,m\alpha_i}(N \boxtimes \mathsf{L}(i^m)), M) \\ \simeq \mathrm{Hom}_{\mathsf{R}(\beta-m\alpha_i,m\alpha_i)}(N \boxtimes \mathsf{L}(i^m), \Delta_{i^m} M)\end{aligned} \tag{3.30}$$

for $N \in \mathsf{Mod}(\mathsf{R}(\beta - m\alpha_i))$ and $M \in \mathsf{Mod}(\mathsf{R}(\beta))$.

Lemma 3.5.1. *For $i \in I^{\mathrm{im}}$, take $m_1, \ldots, m_k \in \mathbb{Z}_{>0}$ and set $m = m_1 + \cdots + m_k$. Then the following statements hold.*

(a) $[\mathrm{Res}_{m_1\alpha_i,\ldots,m_k\alpha_i}\mathsf{L}(i^m)] = [\mathop{\boxtimes}\limits_{\ell=1}^{k} \mathsf{L}(i^{m_\ell})]$.

(b) $\mathrm{hd}(\mathrm{Ind}_{m_1\alpha_i,\ldots,m_k\alpha_i}(\mathop{\boxtimes}\limits_{\ell=1}^{k} \mathsf{L}(i^{m_\ell}))) \simeq \mathsf{L}(i^m)$.

Proof. Assertion (a) follows from definition (3.18). To prove (2), for simplicity, we assume $k = 2$. Let $L = \mathrm{Ind}(L_1 \boxtimes L_2)$, where $L_j := \mathsf{L}(i^{m_j})$ $(j = 1, 2)$. Set

$$L' = \{x \in L \mid \deg(x) > 0\}.$$

Since all generators of $\mathsf{R}((m_1+m_2)\alpha_i)$ have non-negative $\mathbb{Z}$-degree, it becomes an $\mathsf{R}((m_1+m_2)\alpha_i)$-module. Then, since $(L_1 \boxtimes L_2) \cap L' = \{0\}$, L' is the unique maximal submodule of L; i.e., $L/L' \simeq \mathsf{L}(i^m)$ as a graded module. We will show that $\mathrm{hd}(L)$ is irreducible. By a direct computation,

$$\mathrm{ch}_q(L) = \sum_{w \in S_{m_1+m_2}/S_{m_1}\times S_{m_2}} q^{-\ell(w)(\alpha_i|\alpha_i)} i^n = (i^n) + a(q)(i^n),$$

where $a(q)$ is a polynomial $Z[q]$ without constant term. Note that $\mathrm{ch}_q(L_1 \boxtimes L_2) = (i^n)$. For any quotient Q of L, by Frobenius reciprocity (3.8), we have an injective homomorphism

$$L_1 \boxtimes L_2 \hookrightarrow \mathrm{Res}_{m_1\alpha_i, m_2\alpha_i} Q,$$

which yields

$$\mathrm{ch}_q(Q) = (i^n) + a'(q)(i^n),$$

where $a'(q) \in \mathbb{Z}[q]$ without constant term. Therefore, $\mathrm{hd}(L)$ has only one summand, and hence it is irreducible. $\square$

Recall the natural $\mathbb{Z}$-basis $\mathrm{Irr}_q(\mathsf{R}(\beta))$ of $[\mathsf{Rep}(\mathsf{R}(\beta))]$ (see (3.12)).

Lemma 3.5.2. *Let $[M] \in \mathrm{Irr}_q(\mathsf{R}(\beta))$ and let $N \boxtimes \mathsf{L}(i^m)$ be an irreducible submodule of the $\mathsf{R}(\beta - m\alpha_i, m\alpha_i)$-module $\Delta_{i^m} M$. Then $\varepsilon_i^{\mathrm{or}}(N) = \varepsilon_i^{\mathrm{or}}(M) - m$.*

Proof. If $i \in I^{\mathrm{re}}$, then the proof is the same as that of [24, Lemma 3.6]. If $i \in I^{\mathrm{im}}$, by the definition, we have $\varepsilon_i^{\mathrm{or}}(N) \le \varepsilon_i^{\mathrm{or}}(M) - m$. From equation (3.30), we obtain

$$0 \to K \to \mathrm{Ind}(N \boxtimes \mathsf{L}(i^m)) \to M \to 0$$

for some submodule K of $\mathrm{Ind}(N \boxtimes \mathsf{L}(i^m))$. By Proposition 3.3.1 and the exactness of the functor Δ_{i^k}, we can conclude that $\varepsilon_i^{\mathrm{or}}(N) \ge \varepsilon_i^{\mathrm{or}}(M) - m$. Thus our assertion follows. $\square$

Lemma 3.5.3. *Let $[N] \in \mathrm{Irr}_q(\mathsf{R}(\beta))$ with $[E_i][N] = 0$ and let $M = \mathrm{Ind}(N \boxtimes \mathsf{L}(i^m))$. Here $[E_i^k]$ is the map from $[\mathsf{Rep}(\mathsf{R}(\beta))]$ to $[\mathsf{Rep}(\mathsf{R}(\beta - k\alpha_i))]$ induced by the exact functor E_i^k for $k \in \mathbb{Z}_{\ge 0}$. Then we have*

(a) $[\Delta_{i^m} M] = [N \boxtimes \mathsf{L}(i^m)] \in \mathrm{Irr}_q(\mathsf{R}(\beta, m\alpha_i))$,

(b) $[\mathrm{hd}(M)] \in \mathrm{Irr}_q(\mathsf{R}(\beta + m\alpha_i))$ *with* $\varepsilon_i^{\mathrm{or}}(\mathrm{hd}(M)) = m$.

Proof. Our assertion can be proved in the same manner as in [24, Lemma 3.7]. $\square$

Lemma 3.5.4. *For $[M] \in \mathrm{Irr}_q(\mathsf{R}(\beta))$ with $\varepsilon = \varepsilon_i^{\mathrm{or}}(M)$,*

$$[\Delta_{i^\varepsilon} M] = [N \boxtimes \mathsf{L}(i^\varepsilon)]$$

for some $[N] \in \mathrm{Irr}_q(\mathsf{R}(\beta - \varepsilon\alpha_i))$ with $\varepsilon_i^{\mathrm{or}}(N) = 0$.

Proof. Our assertion can be proved in the same manner as in [26, Lemma 5.1.4] (cf. [24, Lemma 3.8]). $\square$

Lemma 3.5.5. *Suppose that $i \in I^{\mathrm{im}}$ and $[N] \in \mathrm{Irr}_q(\mathsf{R}(\beta))$ with $[E_i][N] = 0$. Let $[M] = [\mathrm{Ind}\left(N \boxtimes (\mathop{\boxtimes}\limits_{\ell=1}^{k} \mathsf{L}(i^{m_\ell}))\right)]$ for some positive integers $m_1, \dots m_k \in \mathbb{Z}_{>0}$ and set $m = m_1 + \cdots + m_k$. Then*

(a) $[\mathrm{hd}(M)] \in \mathrm{Irr}_q(\mathsf{R}(\beta + \alpha_i))$,

(b) $\varepsilon_i^{\mathrm{or}}(\mathrm{hd}(M)) = m$.

Proof. By the definition, we have

$$[\Delta_{i^m} M] = [N \boxtimes \mathrm{Ind}(\mathop{\boxtimes}\limits_{\ell=1}^{k} \mathsf{L}(i^{m_\ell}))].$$

Then we have

$$[\Delta_{i^m} M] = \sum_{w} q^{-\ell(w)(\alpha_i|\alpha_i)}[N \boxtimes \mathsf{L}(i^m)] = [N \boxtimes \mathsf{L}(i^m)] + a(q)[N \boxtimes \mathsf{L}(i^m)],$$

where w runs over all the elements in $S_m/S_{m_1} \times \cdots \times S_{m_k}$ and $a(q) \in \mathbb{Z}[q]$ without constant term. By Frobenius reciprocity (3.30), for any quotient Q of M, there is a nontrivial homomorphism of degree 0

$$\Delta_{i^m} M \simeq N \boxtimes \mathrm{Ind}(\mathop{\boxtimes}\limits_{\ell=1}^{k} \mathsf{L}(i^{m_\ell})) \to \Delta_{i^m} Q.$$

By Lemma 3.5.1 (2), we have

$$[\Delta_{i^m} Q] = [N \boxtimes \mathsf{L}(i^m)] + a(q)[N \boxtimes \mathsf{L}(i^m)],$$

for some $a(q)$ with $a(q) \in \mathbb{Z}[q]$ without constant term. Therefore, by the same argument as in Lemma 3.5.1, $\mathrm{hd}(M)$ is irreducible and $\varepsilon_i^{\mathrm{or}}(\mathrm{hd}(M)) = m$. $\square$

Lemma 3.5.6. *Let $[N] \in \mathrm{Irr}_q(\mathsf{R}(\beta))$ and let $[M] = [\mathrm{Ind}(N \boxtimes \mathsf{L}(i^m))]$. Then*

$$[\mathrm{hd}(M)] \in \mathrm{Irr}_q(\mathsf{R}(\beta + m\alpha_i)) \quad \textit{with} \quad \varepsilon_i^{\mathrm{or}}(\mathrm{hd}(M)) = \varepsilon_i^{\mathrm{or}}(N) + m.$$

Proof. If $i \in I^{\mathrm{re}}$, then the proof is identical to that of [26, Lemma 5.1.5] (cf. [24, Lemma 3.9]). Suppose that $i \in I^{\mathrm{im}}$. Let $\varepsilon = \varepsilon_i^{\mathrm{or}}(N)$. By Lemma 3.5.4, we have

$$[\Delta_{i^\varepsilon} N] = [K \boxtimes \mathsf{L}(i^\varepsilon)]$$

for $[K] \in \mathrm{Irr}_q(\mathsf{R}(\beta - m\alpha_i))$ with $\varepsilon_i^{\mathrm{or}}(K) = 0$. By Frobenius reciprocity (3.30), there is a surjective homomorphism

$$\mathrm{Ind}(K \boxtimes \mathsf{L}(i^{\varepsilon})) \twoheadrightarrow N,$$

which yields

$$\mathrm{Ind}(K \boxtimes \mathsf{L}(i^{\varepsilon}) \boxtimes \mathsf{L}(i^m)) \twoheadrightarrow \mathrm{Ind}(N \boxtimes \mathsf{L}(i^m)).$$

Therefore, our assertion follows from Lemma 3.5.5. □

Lemma 3.5.7. *For $[M] \in \mathrm{Irr}_q(\mathsf{R}(\beta))$and $0 \leq m \leq \varepsilon_i^{\mathrm{or}}(M)$, the submodule $\mathrm{soc}\Delta_{i^m} M$ of M is an irreducible module of the form $L \boxtimes \mathsf{L}(i^m)$ with $\varepsilon_i^{\mathrm{or}}(L) = \varepsilon_i^{\mathrm{or}}(M) - m$ for some $L \in \mathrm{Irr}_q(\mathsf{R}(\beta - m\alpha_i))$.*

Proof. If $i \in I^{\mathrm{re}}$, then the proof is the same as that of [26, Lemma 5.1.6] (cf. [24, Lemma 3.10]). If $i \in I^{\mathrm{im}}$, let $\varepsilon = \varepsilon_i^{\mathrm{or}}(M)$. Note that every summand of $\mathrm{soc}\Delta_{i^m} M$ has the form $L \boxtimes \mathsf{L}(i^m)$ for $L \in \mathrm{Irr}_q(\mathsf{R}(\beta - m\alpha_i))$. It follows from Lemma 3.5.2 that

$$\varepsilon_i^{\mathrm{or}}(L) = \varepsilon - m,$$

so that $\Delta_{i^{\varepsilon-m}}(L) \boxtimes \mathsf{L}(i^m) \neq 0$. It is clear that $\mathrm{Res}^{\beta-\varepsilon\alpha_i, \varepsilon\alpha_i}_{\alpha-\varepsilon\alpha_i, (\varepsilon-m)\alpha_i, m\alpha_i} \Delta_{i^{\varepsilon}} M$ has $\Delta_{i^{\varepsilon-m}}(L) \boxtimes \mathsf{L}(i^m)$ as a submodule. On the other hand, by Lemma 3.5.1 and Lemma 3.5.4, there exists an irreducible $N \in \mathrm{Irr}_q(\mathsf{R}(\beta - \varepsilon\alpha_i))$ such that

$$[\mathrm{Res}^{\beta-\varepsilon\alpha_i, \varepsilon\alpha_i}_{\beta-\varepsilon\alpha_i, (\varepsilon-m)\alpha_i, m\alpha_i} \Delta_{i^{\varepsilon}} M] = [N \boxtimes \mathsf{L}(i^{\varepsilon-m}) \boxtimes \mathsf{L}(i^m)],$$

which is irreducible. Hence $\mathrm{soc}\Delta_{i^m} M$ is irreducible and isomorphic to $L \boxtimes \mathsf{L}(i^m)$. □

Lemma 3.5.8. *For $[M] \in \mathrm{Irr}_q(\mathsf{R}(\beta))$ with $[E_i^k][M] = 0$, $[M]$ is a linear combination of modules $[N]$, where $[N] \in \mathrm{Irr}_q(\mathsf{R}(\beta))$ with $\varepsilon_i^{\mathrm{or}}(N) < k$.*

Proof. Write $[M] = \sum a_N [N]$, where $a_N \in \mathbb{Z}$ and N ranges over the set of isomorphic classes of irreducible $R(\beta)$-modules. Let ℓ be the largest $\varepsilon_i^{\mathrm{or}}(N)$ with $a_N \neq 0$. Then by Lemma 3.5.7, $[E_i^{\ell}][M] = \dim_{\mathbf{k}_0} \mathsf{L}(i^{\ell}) \sum\limits_{\varepsilon_i^{\mathrm{or}}(M)=\ell} a_M [\tilde{e}_i^{\ell} M]$. Hence if $\ell \geq k$, then $[E_i^{\ell}][M] = 0$, which is a contradiction. Hence we obtain the desired result. □

By Lemma 3.5.6 and Lemma 3.5.7, the operators $\tilde{e}_i$ and $\tilde{f}_i$ take irreducible modules to irreducible modules or 0, and

$$\varepsilon_i^{\mathrm{or}}(M) = \max\{k \geq 0 \mid \tilde{e}_i^k M \neq 0\}, \quad \varepsilon_i^{\mathrm{or}}(\tilde{f}_i M) = \varepsilon_i^{\mathrm{or}}(M) + 1.$$

Lemma 3.5.9. *For* $[M] \in \mathrm{Irr}_q(\mathsf{R}(\beta))$, *we have*

$$[\mathrm{soc} E_i^m M] = [\tilde{e}_i^m M], \quad [\mathrm{hd}(F_i'^m M)] = [\tilde{f}_i^m M].$$

Proof. If $i \in I^{\mathrm{re}}$, then the proof is the same as in [26, Lemma 5.2.1]. Suppose that $i \in I^{\mathrm{im}}$. Now, we focus on the first assertion. Since the case $m > \varepsilon_i^{\mathrm{or}}(M)$ is trivial, we may assume that $m \leq \varepsilon_i^{\mathrm{or}}(M)$. Since $\mathsf{L}(i) \boxtimes \tilde{e}_i M \hookrightarrow \Delta_i M$, we have

$$\tilde{e}_i^m M \boxtimes \ \mathsf{L}(i)^{\boxtimes m} \hookrightarrow \mathrm{Res}^{\beta - m\alpha_i, m\alpha_i}_{\beta - m\alpha_i, \alpha_i, \ldots \alpha_i} \Delta_{i^m} M,$$

which implies there is a nontrivial homomorphism

$$\tilde{e}_i^m M \boxtimes \ \mathrm{Ind}(\mathsf{L}(i)^{\boxtimes m}) \longrightarrow \Delta_{i^m} M.$$

Since any quotient of $\mathrm{Ind}(\mathsf{L}(i)^{\boxtimes m})$ has a 1-dimensional submodule, $\Delta_{i^m} M$ has a submodule which is isomorphic to $\tilde{e}_i^m M \boxtimes \mathsf{L}(i^m)$. Hence the first assertion follows from Lemma 3.5.7.

For the second assertion, by the definition of $\tilde{f}_i$, there is a nontrivial homomorphism $\mathrm{Ind}(M \boxtimes \mathrm{Ind}(\mathsf{L}(i)^{\boxtimes m}) \twoheadrightarrow \tilde{f}_i^m M$. By Lemma 3.5.6, we have

$$[\mathrm{hd}(\mathrm{Ind}(M \boxtimes \mathrm{Ind}(\mathsf{L}(i)^{\boxtimes m}))] = [\tilde{f}_i^m M].$$

On the other hand, the nontrivial homomorphism

$$\mathrm{Ind}(\mathsf{L}(i)^{\boxtimes m}) \longrightarrow \mathsf{L}(i^m)$$

induces a nontrivial homomorphism

$$\mathrm{Ind}(M \boxtimes \mathrm{Ind}(\mathsf{L}(i)^{\boxtimes m})) \longrightarrow \mathrm{Ind}(M \boxtimes \mathsf{L}(i^m)).$$

Therefore, we conclude $[\mathrm{hd}(F_i'^m M)] = [\tilde{f}_i^m M]$. □

Lemma 3.5.10. *For* $[M] \in \mathrm{Irr}_q(\mathsf{R}(\beta))$ *and* $[N] \in \mathrm{Irr}_q(\mathsf{R}(\beta + \alpha_i))$, *we have*

$$[\tilde{f}_i M] = [N] \ \textit{ if and only if } \ [M] = [\tilde{e}_i N].$$

Proof. Using Lemma 3.5.9, our assertion can be proved in the same manner as in [26, Lemma 5.2.3] □

For $M \in \mathsf{Rep}(\mathsf{R}(\beta))$, set

$$\varepsilon_i^*(M) = \begin{cases} \varepsilon_i^{\mathrm{or}}(M) & \text{if } i \in I^{\mathrm{re}} \text{ or } \varepsilon_i^{\mathrm{or}}(v) = 0, \\ 1 & \text{if } i \in I^{\mathrm{im}} \text{ and } \varepsilon_i^{\mathrm{or}}(v) > 0. \end{cases}$$

Proposition 3.5.1. *For $[M] \in \mathrm{Irr}_q(\mathsf{R}(\beta))$, assume that $\varepsilon := \varepsilon_i^{\mathrm{or}}(M) > 0$ and set $\varepsilon^* := \varepsilon_i^*(M)$. Then we have*

$$[E_i][M] = q_i^{1-\varepsilon^*}[\varepsilon^*]_i[\tilde{e}_i M] + \sum_k [N_k], \tag{3.31}$$

where the N_k are irreducible modules with $\varepsilon_i^{\mathrm{or}}(N_k) < \varepsilon_i^{\mathrm{or}}(\tilde{e}_i M) = \varepsilon - 1$.

Proof. By Lemma 3.5.7 , we have

$$[\Delta_{i^\varepsilon} M] = [\tilde{e}_i^\varepsilon M \boxtimes \mathsf{L}(i^\varepsilon)] \ \text{ and } \ [\Delta_{i^{\varepsilon-1}} \tilde{e}_i M] = [\tilde{e}_i^\varepsilon M \boxtimes \mathsf{L}(i^{\varepsilon-1})].$$

On the other hand, (3.21) and Lemma 3.5.1 imply that

$$[\mathsf{L}(i^\varepsilon)] = q_i^{1-\varepsilon^*}[\varepsilon^*]_i[\mathsf{L}(i^{\varepsilon-1})]$$

as an element of $[\mathsf{Rep}(\mathsf{R}(0))]$. Thus we obtain

$$[E_i^{\varepsilon-1}]\left([E_i M] - q_i^{1-\varepsilon^*}[\varepsilon^*]_i[\tilde{e}_i M]\right) = 0.$$

Hence the desired result follows from Lemma 3.5.8. □

For any $M \in \mathsf{Mod}(\mathsf{R}(\beta))$, we denote by $M^* = \mathrm{Hom}_{\mathbf{k}_0}(M, \mathbf{k}_0)$ the $\mathbf{k}_0$-dual of M whose left $R(\beta)$-module structure is induced by the anti-involution ψ given in (3.7): namely, $(af)(s) = f(\psi(a)s)$ for $f \in \mathrm{Hom}_{\mathbf{k}_0}(M, \mathbf{k}_0)$, $a \in R(\beta)$ and $s \in M$. We say that M is *self-dual* if $M^* \simeq M$ as modules over $\mathsf{R}(\beta)$.

Lemma 3.5.11. *For $M \in \mathrm{Irr}_q(\mathsf{R}(\beta))$with $\varepsilon_i^{\mathrm{or}}(M) > 0$, we have*

$$[(q_i^{1-\varepsilon_i^*(M)}\tilde{e}_i M)^*] = [q_i^{1-\varepsilon_i^*(M)}\tilde{e}_i (M^*)]. \tag{3.32}$$

Proof. Set $\varepsilon^* = \varepsilon_i^*(M)$. By (3.31), we have

$$[E_i][M] = [\varepsilon^*]_i[q_i^{1-\varepsilon^*}\tilde{e}_i M] + \sum_k [N_k].$$

Here N_k's are irreducible modules with $\varepsilon_i^{\mathrm{or}}(N_k) < \varepsilon_i^{\mathrm{or}}(M) - 1$. Since E_i commutes with the duality functor, we have

$$[E_i][M^*] = [\varepsilon^*]_i[\big(q_i^{1-\varepsilon^*}\tilde{e}_i M\big)^*] + \sum_k [(N_k)^*].$$

On the other hand, applying (3.31) to M^*, we obtain

$$[E_i][M^*] = [\varepsilon^*]_i[q_i^{1-\varepsilon^*}\tilde{e}_i(M^*)] + \sum_{k'} [N'_{k'}]$$

with $\varepsilon_i^{\mathrm{or}}(N'_{k'}) < \varepsilon_i^{\mathrm{or}}(M) - 1$. Hence we obtain the desired result. □

By a similar argument to the one in [24, Corollary 3.19], we have the following lemma:

Lemma 3.5.12. *For $[M] \in \mathrm{Irr}_q(\mathsf{R}(\beta))$, we have*

$$\mathbf{k}_0 \simeq \mathrm{End}_{R(\beta)}(M).$$

Proposition 3.5.2. *For $[M] \in \mathrm{Irr}_q(\mathsf{R}(\beta))$, there exists $r \in \mathbb{Z}$ such that $q^r M$ is self-dual, that is,*

$$[(q^r M)^*] = [q^r M].$$

Proof. Using induction on $|\beta|$, we shall show that there exists $r \in \mathbb{Z}$ such that

$$q^r M \quad \text{is self-dual.}$$

Assume $|\beta| > 0$ and take $i \in I$ such that $\varepsilon_i^{\mathrm{or}}(M) > 0$. Set $\varepsilon^* = \varepsilon_i^*(M)$. Then, by the induction hypothesis, there exists $r \in \mathbb{Z}$ such that $q^r q_i^{1-\varepsilon^*}\tilde{e}_i M$ is self-dual. Then the preceding lemma implies

$$[q_i^{1-\varepsilon^*}\tilde{e}_i(q^r M)] = [\big(q_i^{1-\varepsilon^*}\tilde{e}_i(q^r M)\big)^*] = [q_i^{1-\varepsilon^*}\tilde{e}_i\,((q^r M)^*)].$$

Hence by Lemma 3.5.10, we get $[q^r M] = [(q^r M)^*]$. □

Finally we obtain the following theorem which shows the existence of strong perfect basis of $[\mathsf{Rep}(\mathsf{R}(\beta))]$.

Theorem 3.5.1. *For $\beta \in Q^+$, let $\mathrm{Irr}_0(\mathsf{R}(\beta))$ be the set of isomorphism classes of self-dual irreducible $\mathsf{R}(\beta)$-modules. Then*

$$\{ [M] \mid M \in \mathrm{Irr}_0(\mathsf{R}(\beta))\}$$

is an $\mathbb{A}$-basis of $[\mathsf{Rep}(\mathsf{R}(\beta))]$. Moreover, it is a strong perfect basis; i.e., it satisfies the property (3.31)*.*

Proof. The proof is an immediate consequence of Proposition 3.5.2 and (3.31). ☐

Set

$$\mathsf{Rep}(\mathsf{R}) = \bigoplus_{\beta \in Q^+} \mathsf{Rep}(\mathsf{R}(\beta)) \text{ and } [\mathsf{Rep}(\mathsf{R})] = \bigoplus_{\beta \in Q^+} [\mathsf{Rep}(\mathsf{R}(\beta))].$$

The following lemma is a categorification of the q-boson relation.

Lemma 3.5.13.

$$[E_i][F'_j] = q^{-(\alpha_i|\alpha_j)}[F'_j][E_i] + \delta_{ij}\mathrm{Id} \in \mathrm{End}_{\mathbb{A}}([\mathsf{Rep}(\mathsf{R})]). \tag{3.33}$$

Here $[F'_j]$ is the endomorphism on $[\mathsf{Rep}(\mathsf{R})]$ induced by the functor F'_j.

Proof. Choose any $[M] \in \mathsf{Rep}(\mathsf{R}(\beta))$. Then, from Proposition 3.3.1, we have

$$\begin{aligned}
[E_i][F'_j][M] &= [E'_i]([\mathrm{Ind}_{\beta,\alpha_j}(M \boxtimes \mathsf{L}(j))]) \\
&= [M][E_i\mathsf{L}(j)] + [\mathrm{Ind}_{\beta-\alpha_i,\alpha_j}(E_i(M) \boxtimes \mathsf{L}(j))\langle(\alpha_j|\alpha_i)\rangle] \\
&= \delta_{ij}[M] + q^{-(\alpha_i|\alpha_j)}[F'_j][E_i][M],
\end{aligned}$$

which yield our assertion. ☐

3.6 The functors E_i, F_i and $\overline{F}_i$

In this section, we define the functors E_i, F_i and $\overline{F}_i$ on $\mathsf{Mod}(\mathsf{R}(\beta))$. The functorial relations among them will be ingredients of the categorification theorem for cyclotomic quiver Hecke algebras in the later section.

Recall the notion of $\mathbf{k}^I = \bigoplus_{i\in I} \mathbf{k}e(i)$. By Corollary 3.3.1, we have a decomposition
(3.34)

$$\mathsf{R}(n+1) = \bigoplus_{a=1}^{n+1} \mathsf{R}(n) \otimes_{\mathbf{k}} \mathbf{k}^I[x_{n+1}]\tau_n \dots \tau_a = \bigoplus_{a=1}^{n+1} \tau_a \dots \tau_n \mathbf{k}^I[x_{n+1}] \otimes_{\mathbf{k}} \mathsf{R}(n)$$

as left-$\mathsf{R}(n,1)$-modules (resp. right $\mathsf{R}(n,1)$-modules). Here, when $a = n+1$, we understand $\tau_n \dots \tau_a = \tau_a \dots \tau_n = 1$.

Let $\xi_n \colon \mathsf{R}(n) \to \mathsf{R}(n+1)$ be the algebra homomorphism given by

$$\xi_n(x_k) = x_{k+1}, \quad \xi_n(\tau_\ell) = \tau_{\ell+1}, \quad \xi_n(e(\nu)) = \sum_{i\in I} e(i,\nu) \tag{3.35}$$

for all $1 \le k \le n$, $1 \le \ell < n$ and $\nu \in I^n$. We denote by $\mathsf{R}^1(n)$ the image of ξ_n.

For each $i \in I$ and $\beta \in Q^+$, let $\overline{\mathcal{F}}_{i,\beta} := \mathsf{R}(\beta+\alpha_i)v(i,\beta)$ be the $\mathsf{R}(\beta+\alpha_i)$-module generated by $v(i,\beta)$ of degree 0 with the defining relation $e(i,\beta)v(i,\beta) = v(i,\beta)$. The module $\overline{\mathcal{F}}_{i,\beta}$ has an $(\mathsf{R}(\beta+\alpha_i), \mathsf{R}(\beta))$-bimodule structure whose right $\mathsf{R}(\beta)$-action is given by

$$av(i,\beta) \cdot b = a\xi_n(b)v(i,\beta) \quad \text{for } a \in \mathsf{R}(\beta+\alpha_i) \text{ and } b \in \mathsf{R}(\beta).$$

In a similar way, we define the $(\mathsf{R}(n+1), \mathsf{R}(n))$-bimodule structure on $\mathsf{R}(n+1)v(1,n)$ by

$$av(1,n) \cdot b = a\xi_n(b)v(1,n) \quad \text{for } a \in \mathsf{R}(n+1) \text{ and } b \in \mathsf{R}(n).$$

Hence

$$\mathsf{R}(n+1)v(1,n) \simeq \bigoplus_{i\in I,\ |\beta|=n} \mathsf{R}(\beta+\alpha_i)v(i,\beta).$$

Let

$$\begin{aligned}&E_i\colon \mathsf{Mod}(\mathsf{R}(\beta+\alpha_i)) \to \mathsf{Mod}(\mathsf{R}(\beta)),\\ &F_i,\ \overline{F}_i\colon \mathsf{Mod}(\mathsf{R}(\beta)) \to \mathsf{Mod}(\mathsf{R}(\beta+\alpha_i))\end{aligned}$$

be the functors given by

$$\begin{aligned}E_i(N) &= e(\beta,i)N \simeq e(\beta,i)\mathsf{R}(\beta+\alpha_i)\otimes_{\mathsf{R}(\beta+\alpha_i)} N\\ &\qquad\qquad\simeq \mathrm{Hom}_{\mathsf{R}(\beta+\alpha_i)}(\mathsf{R}(\beta+\alpha_i)e(\beta,i),N),\\ F_i(M) &= \mathsf{R}(\beta+\alpha_i)e(\beta,i)\otimes_{\mathsf{R}(\beta)} M,\\ \overline{F}_i(M) &= \overline{\mathcal{F}}_{i,\beta}\otimes_{\mathsf{R}(\beta)} M\end{aligned}$$

for $N \in \mathsf{Mod}(\mathsf{R}(\beta+\alpha_i))$ and $M \in \mathsf{Mod}(\mathsf{R}(\beta))$.

From now on, We shall investigate the commutation relations for the functors E_i, F_i and $\overline{F}_i$ ($i \in I$).

Proposition 3.6.1. *The homomorphism of* $(\mathsf{R}(n),\mathsf{R}(n-1))$*-bimodules*

$$\tilde{\rho}\colon \mathsf{R}(n)e(n-1,j)\underset{\mathsf{R}(n-1)}{\otimes} q^{-(\alpha_i|\alpha_j)}e(n-1,i)\mathsf{R}(n) \longrightarrow e(n,i)\mathsf{R}(n+1)e(n,j)$$

given by

$$x\otimes y \longmapsto x\tau_n y, \qquad x\in \mathsf{R}(n)e(n-1,j),\ y\in e(n-1,i)\mathsf{R}(n)$$

induces an isomorphism of $(\mathsf{R}(n),\mathsf{R}(n))$*-bimodules*

$$\begin{aligned}\rho\colon \mathsf{R}(n)e(n-1,j)\underset{\mathsf{R}(n-1)}{\otimes} q^{-(\alpha_i|\alpha_j)}e(n-1,i)\mathsf{R}(n)\oplus e(n,i)\mathsf{R}(n,1)e(n,j)\\ \xrightarrow{\ \sim\ } e(n,i)\mathsf{R}(n+1)e(n,j).\end{aligned}\tag{3.36}$$

Proof. The homomorphism $\tilde{\rho}$ is well-defined since we have

$$a\ e(n-1,j)\tau_n e(n-1,i) = e(n-1,j)\tau_n e(n-1,i)\ a \text{ for any } a\in \mathsf{R}(n-1).$$

Thus it induces a homomorphism

$$\rho'\colon \mathsf{R}(n)e(n-1,j)\underset{\mathsf{R}(n-1)}{\otimes} q^{-(\alpha_i|\alpha_j)}e(n-1,i)\mathsf{R}(n) \to \frac{e(n,i)\mathsf{R}(n+1)e(n,j)}{e(n,i)\mathsf{R}(n,1)e(n,j)}.$$

Thus it is enough to show that ρ' is an isomorphism. Since

$$\mathsf{R}(n) = \bigoplus_{a=1}^{n} \tau_a \cdots \tau_{n-1} \mathbf{k}^I[x_n] \otimes_{\mathbf{k}} \mathsf{R}(n-1),$$

we have

$$\begin{aligned}
&\mathsf{R}(n)e(n-1,j) \otimes_{\mathsf{R}(n-1)} e(n-1,i)\mathsf{R}(n) \\
&\qquad = \left(\bigoplus_{a=1}^{n} \tau_a \cdots \tau_{n-1} \mathbf{k}[x_n e(j)] \otimes_{\mathbf{k}} \mathsf{R}(n-1) \right) \otimes_{\mathsf{R}(n-1)} e(n-1,i)\mathsf{R}(n) \\
&\qquad \simeq \bigoplus_{a=1}^{n} \tau_a \cdots \tau_{n-1} \mathbf{k}[x_n e(j)] \otimes_{\mathbf{k}} e(n-1,i)\mathsf{R}(n).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{e(n,i)\mathsf{R}(n+1)e(n,j)}{e(n,i)\mathsf{R}(n,1)e(n,j)} &\simeq \frac{\bigoplus_{a=1}^{n+1} e(n,i)\tau_a \cdots \tau_n \mathbf{k}[x_{n+1}e(j)] \otimes_{\mathbf{k}} e(n-1,i)\mathsf{R}(n)}{e(n,i)\mathbf{k}[x_{n+1}e(j)] \otimes_{\mathbf{k}} \mathsf{R}(n)} \\
&\simeq \bigoplus_{a=1}^{n} e(n,i)\tau_a \cdots \tau_n \mathbf{k}[x_{n+1}e(j)] \otimes_{\mathbf{k}} e(n-1,i)\mathsf{R}(n).
\end{aligned}$$

By (3.5), for $f \in \mathbf{k}[x_n e(j)]$, $y \in e(n-1,i)\mathsf{R}(n)$ and $1 \le a \le n$, we have

$$\begin{aligned}
\tau_a \cdots \tau_{n-1} f \tau_n y &= \tau_a \cdots \tau_{n-1} \left(\tau_n (s_n f) + (\partial_n f)\mathsf{P}_{n,n+1} \right) y \\
&\equiv \tau_a \cdots \tau_n (s_n f) y \qquad \mod e(n,i)\mathsf{R}(n,1)e(n,j).
\end{aligned}$$

Hence ρ' is right $\mathsf{R}(n)$-linear and $\rho'(\tau_a \cdots \tau_{n-1} f) = \tau_a \cdots \tau_n (s_n f)$. Since $f \mapsto s_n f$ induces an isomorphism $\mathbf{k}[x_n e(j)] \simeq \mathbf{k}[x_{n+1} e(j)]$, our assertion follows. $\square$

Theorem 3.6.1. *There exist natural isomorphism*

$$E_i F_j \xrightarrow{\sim} q^{-(\alpha_i|\alpha_j)} F_j E_i \oplus \delta_{i,j} \mathbf{k}[t_i] \otimes \mathrm{Id},$$

where t_i is an indeterminate of degree $(\alpha_i|\alpha_i)$ and

$$\mathbf{k}[t_i] \otimes \mathrm{Id} \colon \mathsf{Mod}(\mathsf{R}(\beta)) \to \mathsf{Mod}(\mathsf{R}(\beta))$$

is the functor defined by $M \mapsto \mathbf{k}[t_i] \otimes M$.

Proof. Note that the kernels of F_jE_i and E_iF_j on $\mathsf{Mod}(\mathsf{R}(\beta))$ are given by

$$\mathsf{R}(\beta-\alpha_i+\alpha_j)e(\beta-\alpha_i,j)\underset{\mathsf{R}(\beta-\alpha_i)}{\otimes}e(\beta-\alpha_i,\alpha_i)\mathsf{R}(\alpha_i) \quad \text{and}$$
$$e(\beta+\alpha_j-\alpha_i,i)\mathsf{R}(\beta+\alpha_j)e(\beta,j),$$

respectively. Since

$$\begin{aligned}&\mathsf{R}(n)e(n-1,j)\otimes_{\mathsf{R}(n-1)}e(n-1,i)\mathsf{R}(n)e(\beta)\\ &\qquad\simeq \mathsf{R}(\beta-\alpha_i+\alpha_j)e(\beta-\alpha_i,j)\underset{\mathsf{R}(\beta-\alpha_i)}{\otimes}e(\beta-\alpha_i,\alpha_i)\mathsf{R}(\beta) \quad \text{and}\\ &e(n,i)\mathsf{R}(n+1)e(n,j)e(\beta,j)=e(\beta+\alpha_j-\alpha_i,i)\mathsf{R}(\beta+\alpha_j)e(\beta,j),\end{aligned}$$

our assertion is obtained by applying the exact functor $\bullet\, e(\beta,j)$ on (3.36). ∎

By a similar argument to that given in [14, Proposition 3.7], we have the proposition below:

Proposition 3.6.2. *There exists an injective* $(\mathsf{R}(n),\mathsf{R}(n))$*-bimodule homomorphism*

$$\Phi\colon \mathsf{R}(n)v(1,n-1)\otimes_{\mathsf{R}(n-1)}\mathsf{R}(n)\to\mathsf{R}(n+1)v(1,n)$$

given by

$$x\,v(1,n-1)\otimes y\longmapsto x\,\xi_n(y)v(1,n) \quad \textit{for all } x,y\in\mathsf{R}(n).$$

Moreover, its image $\mathsf{R}(n)\mathsf{R}^1(n)$ *has a decomposition*

$$\mathsf{R}(n)\mathsf{R}^1(n)=\bigoplus_{a=2}^{n+1}\mathsf{R}(n,1)\tau_n\cdots\tau_a=\bigoplus_{a=0}^{n-1}\tau_a\cdots\tau_1\mathsf{R}(1,n).$$

Lemma 3.6.1. *For all* $1\le k\le n$ *and* $1\le \ell\le n-1$,

(a) $x_k\tau_n\cdots\tau_1\equiv\tau_n\cdots\tau_1x_{k+1}$,

(b) $\tau_\ell\tau_n\cdots\tau_1\equiv\tau_n\cdots\tau_1\tau_{\ell+1}$,

(c) $x_{n+1}\tau_n\cdots\tau_1\equiv\tau_n\cdots\tau_1x_1 \bmod \mathsf{R}(n)\mathsf{R}^1(n)$.

Thus

$$\begin{aligned} a\tau_n \cdots \tau_1 e(i,\beta) &\equiv \tau_n \cdots \tau_1 e(i,\beta)\xi_n(a), \\ x_{n+1}e(\beta,i)\tau_n \cdots \tau_1 e(i,\beta) &\equiv \tau_n \cdots \tau_1 x_1 e(i,\beta) \\ &\qquad \bmod \mathsf{R}(n)\mathsf{R}^1(n) \text{ for any } a \in \mathsf{R}(\beta). \end{aligned} \tag{3.37}$$

Proof. We will verify that for $f \in \mathbf{k}[x_1, \cdots, x_{n+1}]$

$$\tau_n\tau_{n-1}\cdots\tau_k\ f\ \tau_\ell\cdots\tau_1 \equiv 0 \quad \bmod \mathsf{R}(n)\mathsf{R}^1(n) \text{ if } \ell+2 \le k \le n+1. \tag{3.38}$$

We shall prove this by using downward induction on k. If $k = n+1$, it is trivial.

Assume that $k \le n$ and our assertion is true for $k+1$. Then we have

$$\begin{aligned} \tau_n \cdots \tau_k f \tau_\ell \cdots \tau_1 &= \tau_n \cdots \tau_{k+1}(s_k(f)\tau_k + f')\tau_\ell \cdots \tau_1 \\ &= \tau_n \cdots \tau_{k+1} s_k(f)\tau_\ell \cdots \tau_1\tau_k + \tau_n \cdots \tau_{k+1} f'\tau_\ell \cdots \tau_1 \end{aligned} \tag{3.39}$$

for some $f' \in \mathbf{k}[x_1, \cdots, x_{n+1}]$. Since $\tau_k \in \mathsf{R}^1(n)$, all the terms in the right-hand side of (3.39) are 0 mod $\mathsf{R}(n)\mathsf{R}^1(n)$ by the induction hypothesis. Hence our assertion holds.

(a) For $1 \le k \le n$, we have

$$\begin{aligned} x_k\tau_n \cdots \tau_1 &= \tau_n \cdots \tau_{k+1}x_k\tau_k \cdots \tau_1 \\ &= \tau_n \cdots \tau_{k+1}\tau_k x_{k+1}\tau_{k-1} \cdots \tau_1 - \tau_n \cdots \tau_{k+1}\mathsf{P}_{k,k+1}\tau_{k-1} \cdots \tau_1. \end{aligned}$$

Then the second term is 0 mod $\mathsf{R}(n)\mathsf{R}^1(n)$ by (3.38), and the first term is equal to

$$(\tau_n \cdots \tau_{k+1}\tau_k)(\tau_{k-1} \cdots \tau_1)x_{k+1},$$

which implies our first assertion.

(b) For $1 \le \ell \le n-1$, we have

$$\begin{aligned} &\tau_\ell\tau_n \cdots \tau_1 = \tau_n \cdots \tau_{\ell+2}\tau_\ell\tau_{\ell+1}\tau_\ell \cdots \tau_1 \\ &= \tau_n \cdots \tau_{\ell+2}(\tau_{\ell+1}\tau_\ell\tau_{\ell+1} - \overline{\mathsf{Q}}_\ell\mathsf{P}_{\ell,\ell+2} - \overline{\mathsf{P}}'_\ell\tau_\ell - \tau_{\ell+1}\overline{\mathsf{P}}''_\ell)\tau_{\ell-1} \cdots \tau_1 \\ &= \tau_n \cdots \tau_1\tau_{\ell+1} - \tau_n \cdots \tau_{\ell+2}(\overline{\mathsf{Q}}_\ell\mathsf{P}_{\ell,\ell+2})\tau_{\ell-1} \cdots \tau_1 \\ &\quad - \tau_n \cdots \tau_{\ell+2}(\overline{\mathsf{P}}'_\ell)\tau_\ell \cdots \tau_1 - \tau_n \cdots \tau_{\ell+1}(\overline{\mathsf{P}}''_\ell)\tau_{\ell-1} \cdots \tau_1. \end{aligned}$$

By (3.38), the terms except the first one are 0 mod $\mathsf{R}(n)\mathsf{R}^1(n)$.
(c) If $k = n+1$, we have

$$\begin{aligned} x_{n+1}\tau_n\cdots\tau_1 &= (\tau_n x_n + \mathsf{P}_{n,n+1})\tau_{n-1}\cdots\tau_1 \\ &= \tau_n x_n \tau_{n-1}\cdots\tau_1 + \mathsf{P}_{n,n+1}\tau_{n-1}\cdots\tau_1 \\ &\equiv \tau_n x_n \tau_{n-1}\cdots\tau_1 \\ &\quad\vdots \\ &\equiv \tau_n\cdots\tau_1 x_1 \quad \text{mod } \mathsf{R}(n)\mathsf{R}^1(n). \end{aligned}$$

$\square$

By Proposition 3.6.2, there exists a right $\mathsf{R}(n)$-linear map

$$\varphi_1\colon \mathsf{R}(n+1)v(1,n) \to \mathsf{R}(n)\otimes \mathbf{k}^I[x_{n+1}]$$

given by

$$\begin{aligned} \mathsf{R}(n+1)v(1,n) \to &\mathrm{Coker}(\Phi) \cong \frac{\bigoplus_{a=1}^{n+1}\mathsf{R}(n,1)\tau_n\cdots\tau_a}{\bigoplus_{a=2}^{n+1}\mathsf{R}(n,1)\tau_n\cdots\tau_a} \\ &\xleftarrow{\sim} \mathsf{R}(n,1)\tau_n\cdots\tau_1 \xleftarrow{\sim} \mathsf{R}(n,1) \cong \mathsf{R}(n)\otimes\mathbf{k}^I[x_{n+1}] \\ &\cong \mathsf{R}(n)\otimes\mathbf{k}^I[t]. \end{aligned} \tag{3.40}$$

Similarly, there is another map $\varphi_2\colon \mathsf{R}(n+1)v(1,n) \to \mathbf{k}^I[x_1]\otimes\mathsf{R}(n)$ given by

$$\begin{aligned} \mathsf{R}(n+1)v(1,n) \to &\mathrm{Coker}(\Phi) \cong \frac{\bigoplus_{a=0}^{n}\tau_a\cdots\tau_1\mathsf{R}(1,n)}{\bigoplus_{a=0}^{n-1}\tau_a\cdots\tau_1\mathsf{R}(1,n)} \\ &\xleftarrow{\sim} \tau_n\cdots\tau_1\mathsf{R}(1,n) \xleftarrow{\sim} \mathsf{R}(1,n) \cong \mathbf{k}^I[x_1]\otimes\mathsf{R}(n) \\ &\cong \mathbf{k}^I[t]\otimes\mathsf{R}(n). \end{aligned} \tag{3.41}$$

By restricting Φ to

$$\mathsf{R}(\beta+\alpha_j-\alpha_i)v(j,\beta-\alpha_i)\otimes_{\mathsf{R}(\beta-\alpha_i)} e(\beta-\alpha_i,i)\mathsf{R}(\beta),$$

which is the kernel of $\overline{F}_j E_i$ on $\mathsf{Mod}(\mathsf{R}(\beta))$, (3.40) and (3.41) can be rewritten as

$$\begin{aligned}
&e(\beta+\alpha_j-\alpha_i,i)\mathsf{R}(\beta+\alpha_j)v(j,\beta)\\
&\qquad\qquad \xrightarrow{\varphi_1} \operatorname{Coker}(\Phi) \cong \frac{\bigoplus_{a=1}^{n+1}\mathsf{R}(\beta+\alpha_j-\alpha_i,i)\tau_n\cdots\tau_a e(j,\beta)}{\bigoplus_{a=2}^{n+1}\mathsf{R}(\beta+\alpha_j-\alpha_i,i)\tau_n\cdots\tau_a e(j,\beta)}\\
&\qquad\qquad \xleftarrow{\sim} \delta_{i,j}\mathsf{R}(\beta,i)\tau_n\cdots\tau_1 \xleftarrow{\sim} \delta_{i,j}\mathsf{R}(\beta,i)\\
&\qquad\qquad \simeq \delta_{i,j}\mathsf{R}(\beta)\otimes\mathbf{k}[x_{n+1}e(i)] \simeq \delta_{i,j}\mathbf{k}[t_i]\otimes\mathsf{R}(\beta)
\end{aligned}$$

and

$$\begin{aligned}
&e(\beta+\alpha_j-\alpha_i,i)\mathsf{R}(\beta+\alpha_j)v(j,\beta)\\
&\qquad\qquad \xrightarrow{\varphi_2} \operatorname{Coker}(\Phi) \cong \frac{\bigoplus_{a=0}^{n} e(\beta+\alpha_j-\alpha_i,i)\tau_a\cdots\tau_1\mathsf{R}(j,\beta)}{\bigoplus_{a=0}^{n-1} e(\beta+\alpha_j-\alpha_i,i)\tau_a\cdots\tau_1\mathsf{R}(j,\beta)}\\
&\qquad\qquad \xleftarrow{\sim} \delta_{i,j}\tau_n\cdots\tau_1\mathsf{R}(i,\beta) \xleftarrow{\sim} \delta_{i,j}\mathsf{R}(i,\beta)\\
&\qquad\qquad \simeq \delta_{i,j}\mathbf{k}[x_1e(i)]\otimes\mathsf{R}(\beta) \simeq \delta_{i,j}\mathbf{k}[t_i]\otimes\mathsf{R}(\beta).
\end{aligned}$$

Therefore by (3.37), φ_1 and φ_2 coincide and we obtain:

Theorem 3.6.2.

(i) *There is a natural isomorphism*

$$\overline{F}_j E_i \xrightarrow{\sim} E_i\overline{F}_j \quad \text{for } i \neq j.$$

(ii) *There is an exact sequence in* $\mathsf{Mod}(\mathsf{R}(\beta))$

$$0 \to \overline{F}_i E_i M \to E_i \overline{F}_i M \to q^{-(\alpha_i|\beta)}\mathbf{k}[t_i]\otimes M \to 0,$$

which is functorial in $M \in \mathsf{Mod}(\mathsf{R}(\beta))$. *Here* t_i *is an indeterminate of degree* $(\alpha_i|\alpha_i)$.

3.7 Cyclotomic quotient

Let $\Lambda \in P^+$ be a dominant integral weight. In this section, we investigate the structure of $\mathsf{R}^\Lambda(\beta)$. Then we observe that how the functors E_i^Λ and F_i^Λ act on $\mathsf{Mod}(\mathsf{R}^\Lambda(\beta))$ and the commutation relation between them.

For $\Lambda \in P^+$ and $i \in I$, we choose a monic polynomial of degree $\langle h_i, \Lambda\rangle$

$$\mathfrak{a}_i^\Lambda(u) = \sum_{k=0}^{\langle h_i,\Lambda\rangle} c_{i;k} u^{\langle h_i,\Lambda\rangle - k} \tag{3.42}$$

with $c_{i;k} \in \mathbf{k}_{2kd_i}$ and $c_{i,0} = 1$.

Given $\beta \in Q^+$ with $|\beta| = n$, a dominant integral weight $\Lambda \in P^+$ and k ($1 \le k \le n$), set

$$\mathfrak{a}^\Lambda(x_k) = \sum_{\nu \in I^\beta} \mathfrak{a}^\Lambda_{\nu_k}(x_k) e(\nu) \in \mathsf{R}(\beta).$$

Definition 3.7.1. *For $\beta \in Q^+$ and $\Lambda \in P^+$, the cyclotomic quiver Hecke algebra $\mathsf{R}^\Lambda(\beta)$ at β (resp. $\mathsf{R}^\Lambda(n)$ of degree n) is the quotient algebra*

$$\mathsf{R}^\Lambda(\beta) = \frac{\mathsf{R}(\beta)}{\mathsf{R}(\beta)\mathfrak{a}^\Lambda(x_1)\mathsf{R}(\beta)} \quad \textit{and} \quad \mathsf{R}^\Lambda(n) = \bigoplus_{\beta \in Q^+,\ |\beta| = n} \mathsf{R}^\Lambda(\beta).$$

Lemma 3.7.1. *Let $\nu \in I^n$ be such that $\nu_a = \nu_{a+1}$ for some $1 \le a < n$. Then, for an $\mathsf{R}(n)$-module M and $f \in \mathbf{k}[x_1, \ldots, x_n]$, $fe(\nu)M = 0$ implies*

$$(\partial_a f)\mathsf{P}_{\nu_a}(x_a, x_{a+1})\mathsf{P}_{\nu_a}(x_{a+1}, x_a)e(\nu)M = 0,$$
$$(s_a f)\mathsf{P}_{\nu_a}(x_a, x_{a+1})\mathsf{P}_{\nu_a}(x_{a+1}, x_a)e(\nu)M = 0.$$

Proof. Note that $\tau_a e(\nu) = e(\nu)\tau_a$ and $\tau_a^2 e(\nu) = (\partial_a \mathsf{P}_{\nu_a}(x_a, x_{a+1}))\tau_a e(\nu)$. Thus we have

$$\begin{aligned}
&(x_a - x_{a+1})\tau_a f \tau_a e(\nu) \\
&= (x_a - x_{a+1})((s_a f)\tau_a + (\partial_a f)\mathsf{P}_{\nu_a}(x_a, x_{a+1}))\tau_a e(\nu) \\
&= (x_a - x_{a+1})\Big((\partial_a \mathsf{P}_{\nu_a}(x_a, x_{a+1}))(s_a f) + (\partial_a f)\mathsf{P}_{\nu_a}(x_a, x_{a+1})\Big)\tau_a e(\nu) \\
&= (\mathsf{P}_{\nu_a}(x_{a+1}, x_a) - \mathsf{P}_{\nu_a}(x_a, x_{a+1}))\,(s_a f)\tau_a e(\nu) + \mathsf{P}_{\nu_a}(x_a, x_{a+1})(s_a(f) - f)\tau_a e(\nu) \\
&= \mathsf{P}_{\nu_a}(x_{a+1}, x_a)(s_a f)\tau_a e(\nu) - \mathsf{P}_{\nu_a}(x_a, x_{a+1}) f \tau_a e(\nu) \\
&= \mathsf{P}_{\nu_a}(x_{a+1}, x_a)\,(\tau_a f - (\partial_a f)\mathsf{P}_{\nu_a}(x_a, x_{a+1}))\,e(\nu) - \mathsf{P}_{\nu_a}(x_a, x_{a+1}) f \tau_a e(\nu).
\end{aligned}$$

Thus

$$(\partial_a f)\mathsf{P}_{\nu_a}(x_a, x_{a+1})\mathsf{P}_{\nu_a}(x_{a+1}, x_a)e(\nu)M = 0.$$

Since $(x_a - x_{a+1})(\partial_a f) = s_a f - f$, we have

$$(s_a f)\mathsf{P}_{\nu_a}(x_a, x_{a+1})\mathsf{P}_{\nu_a}(x_{a+1}, x_a)e(\nu)M = 0.$$

$\square$

Lemma 3.7.2. *Let $\beta \in Q^+$ with $|\beta| = n$.*

(i) *There exists a monic polynomial $g(u) \in \mathbf{k}[u]$ such that $g(x_a) = 0$ in $\mathsf{R}^\Lambda(\beta)$ for any a $(1 \le a \le n)$.*

(ii) *If $i \in I^{\mathrm{re}}$, then there exists $m \in \mathbb{Z}_{\ge 0}$ such that $\mathsf{R}^\Lambda(\beta + k\alpha_i) = 0$ for any $k \ge m$.*

Proof. (i) By induction on a, it is enough to show that

For any monic polynomial $g(u)$, there exists a monic polynomial $h(u)$ such that $h(x_{a+1})M = 0$ for any $\mathsf{R}(\beta)$-module M with $g(x_a)M = 0$.

If $\nu_a = \nu_{a+1}$, then Lemma 3.7.1 implies that

$$g(x_{a+1})\mathsf{P}_{\nu_a}(x_a, x_{a+1})\mathsf{P}_{\nu_a}(x_{a+1}, x_a)e(\nu)M = 0.$$

By the definition of $\mathsf{P}_i(u, v)$ given in (3.1), $g(x_{a+1})\mathsf{P}_{\nu_a}(x_a, x_{a+1})\mathsf{P}_{\nu_a}(x_{a+1}, x_a)$ is a monic polynomial in x_{a+1} with coefficients in $\mathbf{k}[x_a]$. Hence we can choose a monic polynomial $h(x_{a+1})$ in the ideal generated by $g(x_a)$ and $g(x_{a+1})\mathsf{P}_{\nu_a}(x_a, x_{a+1})\mathsf{P}_{\nu_a}(x_{a+1}, x_a)$ in $\mathbf{k}[x_a, x_{a+1}]$. Thus

$$h(x_{a+1})e(\nu)M = 0.$$

If $\nu_a \ne \nu_{a+1}$, then

$$g(x_{a+1})\mathsf{Q}_{\nu_a,\nu_{a+1}}(x_a, x_{a+1})e(\nu)M = g(x_{a+1})\tau_a^2 e(\nu)M = \tau_a g(x_a)e(s_a\nu)\tau_a M = 0.$$

Since $g(x_{a+1})\mathsf{Q}_{\nu_a,\nu_{a+1}}(x_a, x_{a+1})$ is a monic polynomial in x_{a+1} with coefficients in $\mathbf{k}[x_a]$, we can choose a monic polynomial $h(x_{a+1})$ as in the case of $\nu_a = \nu_{a+1}$.

(ii) For $\nu \in I^n$, set $\mathrm{Supp}_i(\nu) = \#\{k \mid 1 \le k \le n \text{ and } \nu_k = i\}$. Our assertion is equivalent to:

(3.43) For all n, there exists $k_n \in \mathbb{Z}_{\ge 0}$ such that $e(\nu)\mathsf{R}^\Lambda(n + k_n) = 0$ for any $\nu \in I^{n+k_n}$ with $\mathrm{Supp}_i(\nu) \ge k_n$.

If $e(\nu)\mathsf{R}^\Lambda(n+k)=0$ for any $\nu\in I^{n+k}$ such that $\mathrm{Supp}_i(\nu)\ge k$, then one can easily see that

(3.44)
$$e(\nu')\mathsf{R}^\Lambda(n+k')=0 \text{ for any } k'\ge k \text{ and } \nu'\in I^{n+k'} \text{ with } \mathrm{Supp}_i(\nu'_{\le n+k})\ge k.$$

In order to prove (3.43), we will use induction on n. Assume that there exists $k=k_{n-1}$ such that

$$e(\nu)\mathsf{R}^\Lambda(n-1+k)=0 \quad \text{if } \mathrm{Supp}_i(\nu)\ge k.$$

By (i), there exists a monic polynomial $g(u)$ of degree $m\ge 0$ such that $g(x_{n+k})\mathsf{R}^\Lambda(n+k)=0$. It suffices to show

$$e(\nu)\mathsf{R}^\Lambda(n+k+m)=0 \text{ for } \mathrm{Supp}_i(\nu)\ge k+m.$$

If $\mathrm{Supp}_i(\nu_{\le n+k-1})\ge k$, then by (3.44) $e(\nu)\mathsf{R}^\Lambda(n+k+m)=0$. Thus we may assume that $\mathrm{Supp}_i(\nu_{\le n+k-1})\le k-1$. Hence we have $\nu_{\ge n+k}=(i,\dots,i)$. Then the repeated application of Lemma 3.7.1 imply

$$(\partial_{n+k+m-1}\cdots\partial_{n+k}g(x_{n+k}))e(\nu)\mathsf{R}^\Lambda(n+k+m)=0.$$

Since $\partial_{n+k+m-1}\cdots\partial_{n+k}g(x_{n+k})=\pm 1$, we can choose $k_n=k+m$. □

Lemma 3.7.3. *If $i\in I^{\mathrm{im}}$ and $\langle h_i,\Lambda-\beta\rangle=0$, then*

$$\mathsf{R}^\Lambda(\beta+\alpha_i)=0.$$

Proof. Since $\langle h_i,\Lambda\rangle$, $\langle h_i,-\beta\rangle\ge 0$, the hypothesis $\langle h_i,\Lambda-\beta\rangle=0$ implies $\langle h_i,\Lambda\rangle=0$ and $\langle h_i,\beta\rangle=0$. Thus for all $j\in\mathrm{Supp}(\beta)\setminus\{i\}$, we have $a_{ij}=0$. In particular, we have $\mathsf{Q}_{j,i}\in\mathbf{k}_0^\times$. Since $\langle h_i,\Lambda\rangle=0$, we have $e(i,\beta)\mathsf{R}^\Lambda(\beta+\alpha_i)=0$. For $\nu\in I^{\beta+\alpha_i}$, let k be the smallest integer such that $\nu_k=i$. We shall show $e(\nu)\mathsf{R}^\Lambda(\beta+\alpha_i)=0$ by induction on k. If k=1, it is obvious. Assume $k>1$. Hence $\mathsf{Q}_{\nu_{k-1},\nu_k}e(\nu)\mathsf{R}^\Lambda(\beta+\alpha_i)=\tau_{k-1}e(s_{k-1}\nu)\tau_{k-1}\mathsf{R}^\Lambda(\beta+\alpha_i)$ vanishes since $(s_{k-1}\nu)_{k-1}=i$. Since $\mathsf{Q}_{\nu_{k-1},\nu_k}\in\mathbf{k}_0^\times$, we obtain the desired result $e(\nu)\mathsf{R}^\Lambda(\beta+\alpha_i)=0$. □

For each $i \in I$, we define the functors

$$E_i^\Lambda \colon \mathsf{Mod}(\mathsf{R}^\Lambda(\beta+\alpha_i)) \to \mathsf{Mod}(\mathsf{R}^\Lambda(\beta)),$$
$$F_i^\Lambda \colon \mathsf{Mod}(\mathsf{R}^\Lambda(\beta)) \to \mathsf{Mod}(\mathsf{R}^\Lambda(\beta+\alpha_i)),$$

by

$$E_i^\Lambda(N) = e(\beta,i)N = e(\beta,i)\mathsf{R}^\Lambda(\beta+\alpha_i) \otimes_{\mathsf{R}^\Lambda(\beta+\alpha_i)} N,$$
$$F_i^\Lambda(M) = \mathsf{R}^\Lambda(\beta+\alpha_i)e(\beta,i) \otimes_{\mathsf{R}^\Lambda(\beta)} M,$$

where $M \in \mathsf{Mod}(\mathsf{R}^\Lambda(\beta+\alpha_i))$ and $N \in \mathsf{Mod}(\mathsf{R}^\Lambda(\beta))$.

We introduce $(\mathsf{R}(\beta+\alpha_i), \mathsf{R}^\Lambda(\beta))$-bimodules

$$\begin{aligned} F^\Lambda &:= \mathsf{R}^\Lambda(\beta+\alpha_i)e(\beta,i), \\ K_0 &:= \mathsf{R}(\beta+\alpha_i)e(\beta,i) \otimes_{\mathsf{R}(\beta)} \mathsf{R}^\Lambda(\beta), \\ K_1 &:= \mathsf{R}(\beta+\alpha_i)v(i,\beta) \otimes_{\mathsf{R}(\beta)} \mathsf{R}^\Lambda(\beta). \end{aligned} \tag{3.45}$$

The bimodules F^Λ, K_0 and K_1 are the kernels of the functors F_i^Λ, F_i and $\overline{F}_i$ from $\mathsf{Mod}(\mathsf{R}^\Lambda(\beta))$ to $\mathsf{Mod}(\mathsf{R}(\beta+\alpha_i))$, respectively.

Let t_i be an indeterminate of degree $2d_i$. Then $\mathbf{k}[t_i]$ acts from the right on $\mathsf{R}(\beta+\alpha_i)e(i,\beta)$ and K_1 by multiplying x_1. Similarly, $\mathbf{k}[t_i]$ acts from the right on $\mathsf{R}(\beta+\alpha_i)e(\beta,i)$, F^Λ and K_1 by multiplying x_{n+1}. Thus K_0, F^Λ and K_1 have an $(\mathsf{R}(\beta+\alpha_i), \mathbf{k}[t_i] \otimes \mathsf{R}^\Lambda(\beta))$-bimodule structure.

By a similar argument to the one given in [14, Lemma 4.8, Lemma 4.16], we obtain the following lemmas.

Lemma 3.7.4.

(i) *Both K_1 and K_0 are finitely generated projective right $\mathbf{k}[t_i] \otimes \mathsf{R}^\Lambda(\beta)$-modules.*

(ii) *In particular, for any $f(x_1, \ldots, x_{n+1}) \in \mathbf{k}[x_1, \ldots, x_{n+1}]$ which is a monic polynomial in x_1, the right multiplication by f on K_1 induces an injective endomorphism of K_1.*

Lemma 3.7.5. *For $i \in I$ and $\beta \in Q^+$ with $|\beta| = n$, we have*

(i) $\mathsf{R}(\beta+\alpha_i)\mathfrak{a}^\Lambda(x_1)\mathsf{R}(\beta+\alpha_i) = \sum_{a=0}^{n} \mathsf{R}(\beta+\alpha_i)\mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_a$,

(ii) $\mathsf{R}(\beta+\alpha_i)\mathfrak{a}^\Lambda(x_1)\mathsf{R}(\beta+\alpha_i)e(\beta,i)$
$= \mathsf{R}(\beta+\alpha_i)\mathfrak{a}^\Lambda(x_1)R(\beta)e(\beta,i) + \mathsf{R}(\beta+\alpha_i)\mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_n e(\beta,i)$.

Let $\mathrm{pr}\colon K_0 \to F^\Lambda$ be the canonical projection and $\widetilde{P}\colon \mathsf{R}(\beta+\alpha_i)e(i,\beta) \to K_0$ be the right multiplication by $\mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_n$ whose degree is

$$2d_i\langle h_i,\Lambda\rangle + (\alpha_i|-\beta) = (\alpha_i|2\Lambda-\beta).$$

Then, using Lemma 3.7.5, one can see that

$$\mathrm{Im}(\widetilde{P}) = \mathrm{Ker}(\mathrm{pr}) = \frac{\mathsf{R}(\beta+\alpha_i)\mathfrak{a}^\Lambda(x_1)\mathsf{R}(\beta+\alpha_i)e(\beta,i)}{\mathsf{R}(\beta+\alpha_i)\mathfrak{a}^\Lambda(x_1)\mathsf{R}(\beta)e(\beta,i)} \subset K_0. \tag{3.46}$$

Lemma 3.7.6. *The map $\widetilde{P}\colon \mathsf{R}(\beta+\alpha_i)e(i,\beta) \to K_0$ is a right $\mathbf{k}[t_i]\otimes\mathsf{R}(\beta)$-linear homomorphism; i.e., for all $S \in \mathsf{R}(\beta+\alpha_i)$, $1 \le a \le n$ and $1 \le b \le n-1$,*

$$\widetilde{P}(Sx_{a+1}) = \widetilde{P}(S)x_a, \quad \widetilde{P}(Sx_1) = \widetilde{P}(S)x_{n+1}, \quad \widetilde{P}(S\tau_{b+1}) = \widetilde{P}(S)\tau_b.$$

Proof. First, we will verify that for $f \in \mathbf{k}[x_1,\dots,x_{n+1}]$ and $\ell+2 \le k \le n+1$,

(3.47)
$$\mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_\ell f\tau_k\cdots\tau_n e(\beta,i) \equiv 0 \mod \mathsf{R}(\beta+\alpha_i)\mathfrak{a}^\Lambda(x_1)\mathsf{R}(\beta)e(\beta,i).$$

We will prove this by using downward induction on k. It is trivial for $k = n+1$. Assume that $k \le n$ and our assertion is true for $k+1$. Then we have

(3.48)
$$\begin{aligned}\mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_\ell f\tau_k\cdots\tau_n e(\beta,i) &= \tau_k\mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_\ell s_k(f)\tau_{k+1}\cdots\tau_n e(\beta,i)\\ &\quad + \mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_\ell f'\tau_{k+1}\cdots\tau_n e(\beta,i)\end{aligned}$$

for some $f' \in \mathbf{k}[x_1,\dots,x_{n+1}]$, and both the terms in the right-hand side of (3.48) are 0 mod $\mathsf{R}(\beta+\alpha_i)\mathfrak{a}^\Lambda(x_1)\mathsf{R}(\beta)e(\beta,i)$ by the induction hypothesis. Thus we obtain (3.47).

For $1 \le a \le n$, we have

$$\begin{aligned}&x_{a+1}(\mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_n e(\beta,i))\\ &= \mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_{a-1}(x_{a+1}\tau_a)\tau_{a+1}\cdots\tau_n e(\beta,i),\\ &= \mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_{a-1}(\tau_a x_a + \mathsf{P}_{a,a+1})\tau_{a+1}\cdots\tau_n e(\beta,i),\\ &= \mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_n x_a e(\beta,i) + \mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_{a-1}\mathsf{P}_{a,a+1}\tau_{a+1}\cdots\tau_n e(\beta,i),\\ &\equiv \mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_n x_a e(\beta,i) \quad \text{(by (3.47))}.\end{aligned}$$

For the second assertion, we have

$$
\begin{aligned}
&x_1(\mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_n e(\beta,i))\\
&=\mathfrak{a}^\Lambda(x_1)(\tau_1x_2-\mathsf{P}_{1,2})\tau_2\cdots\tau_n e(\beta,i)\\
&=\mathfrak{a}^\Lambda(x_1)\tau_1x_2\tau_2\cdots\tau_n e(\beta,i)-\mathsf{P}_{1,2}\tau_2\cdots\tau_n\mathfrak{a}^\Lambda(x_1)e(\beta,i)\\
&\equiv\mathfrak{a}^\Lambda(x_1)\tau_1x_2\tau_2\cdots\tau_n e(\beta,i)\\
&=\mathfrak{a}^\Lambda(x_1)\tau_1\tau_2x_3\tau_3\cdots\tau_n e(\beta,i)-\mathfrak{a}^\Lambda(x_1)\tau_1\mathsf{P}_{2,3}\tau_3\cdots\tau_n e(\beta,i)\\
&\equiv\mathfrak{a}^\Lambda(x_1)\tau_1\tau_2x_3\tau_3\cdots\tau_n e(\beta,i)\quad\text{(by (3.47))}\\
&\qquad\vdots\\
&\equiv\mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_n x_{n+1}e(\beta,i)\quad\operatorname{mod}\ \mathsf{R}(\beta+\alpha_i)\mathfrak{a}^\Lambda(x_1)\mathsf{R}(\beta)e(\beta,i).
\end{aligned}
$$

For $1\le b\le n-1$, we have

$$
\begin{aligned}
&\tau_{b+1}(\mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_n e(\beta,i))\\
&=\mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_{b-1}(\tau_{b+1}\tau_b\tau_{b+1})\tau_{b+2}\cdots\tau_n e(\beta,i)\\
&=\mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_{b-1}(\tau_b\tau_{b+1}\tau_b+\overline{\mathsf{Q}}_b\mathsf{P}_{b,b+2}+\tau_b\overline{\mathsf{P}}'_b+\overline{\mathsf{P}}''_b\tau_{b+1})\tau_{b+2}\cdots\tau_n e(\beta,i)\\
&=\mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_n\tau_b e(\beta,i)+\mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_{b-1}(\overline{\mathsf{Q}}_b\mathsf{P}_{b,b+2})\tau_{b+2}\cdots\tau_n e(\beta,i)\\
&\quad+\mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_b(\overline{\mathsf{P}}'_b)\tau_{b+2}\cdots\tau_n e(\beta,i)+\mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_{b-1}(\overline{\mathsf{P}}''_b)\tau_{b+1}\cdots\tau_n e(\beta,i).
\end{aligned}
$$

By (3.47), all the terms except the first one are 0 mod $\mathsf{R}(\beta+\alpha_i)\mathfrak{a}^\Lambda(x_1)\mathsf{R}(\beta)e(\beta,i)$. Thus we obtain

$$\tau_{b+1}\mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_n e(\beta,i)\equiv\mathfrak{a}^\Lambda(x_1)\cdots\tau_n\tau_b e(\beta,i)\quad\operatorname{mod}\ \mathsf{R}(\beta+\alpha_i)\mathfrak{a}^\Lambda(x_1)\mathsf{R}(\beta)e(\beta,i).$$

□

Since $\widetilde{P}$ is right $\mathbf{k}[t_i]\otimes\mathsf{R}(\beta)$-linear and maps $\mathsf{R}(\beta+\alpha_i)\mathfrak{a}^\Lambda(x_2)\mathsf{R}^1(\beta)e(i,\beta)$ to $\mathsf{R}(\beta+\alpha_i)\mathfrak{a}^\Lambda(x_1)\mathsf{R}(\beta)e(\beta,i)$, it induces a map

$$P\colon K_1\to K_0,$$

which is an $(\mathsf{R}(\beta+\alpha_i),\mathbf{k}[t_i]\otimes\mathsf{R}(\beta))$-bilinear homomorphism. By (3.46), we get an exact sequence of $(\mathsf{R}(\beta+\alpha_i),\mathbf{k}[t_i]\otimes\mathsf{R}(\beta))$-bimodules

$$K_1\xrightarrow{P}K_0\xrightarrow{\mathrm{pr}}F^\Lambda\longrightarrow 0.$$

We will show that P is actually injective by constructing an $(\mathsf{R}(\beta+\alpha_i), \mathsf{R}(\beta)\otimes \mathbf{k}[t_i])$-bilinear homomorphism Q such that $Q \circ P$ is injective.

For $1 \le a \le n$, we define the elements φ_a and g_a of $\mathsf{R}(\beta+\alpha_i)$ by

$$\varphi_a = \sum_{\substack{\nu \in I^{\beta+\alpha_i},\\ \nu_a \neq \nu_{a+1}}} \tau_a e(\nu) + \sum_{\substack{\nu \in I^{\beta+\alpha_i},\\ \nu_a = \nu_{a+1}}} (\mathsf{P}_{\nu_a}(x_a, x_{a+1}) - (x_{a+1} - x_a)\tau_a) e(\nu) \tag{3.49}$$

and

$$g_a = \sum_{\substack{\nu \in I^{\beta+\alpha_i},\\ \nu_a \neq \nu_{a+1}}} \tau_a e(\nu) + \sum_{\substack{\nu \in I^{\beta+\alpha_i},\\ \nu_a = \nu_{a+1}}} ((x_{a+1} - x_a)\mathsf{P}_{\nu_a}(x_a, x_{a+1}) - (x_{a+1} - x_a)\tau_a) e(\nu) \tag{3.50}$$

The elements φ_a's are called *intertwiners*, and the elements g_a are their variants of them.

Lemma 3.7.7. *For $1 \le a \le n$ and $\nu \in I^{n+1}$, we have*

$$\begin{aligned} &\varphi_a e(\nu) = e(s_a\nu)\varphi_a, \quad x_{s_a(b)}\varphi_a e(\nu) = \varphi_a x_b e(\nu) \ (1 \le b \le n+1), \\ &\tau_b \varphi_a e(\nu) = \varphi_a \tau_b e(\nu) \ \ \textit{if } |b-a| > 1, \quad \tau_a \varphi_{a+1}\varphi_a = \varphi_{a+1}\varphi_a\tau_{a+1}, \end{aligned} \tag{3.51}$$

and

$$\begin{aligned} &g_a e(\nu) = e(s_a\nu) g_a, \quad x_{s_a(b)} g_a e(\nu) = g_a x_b e(\nu) \ (1 \le b \le n+1), \\ &\tau_b g_a e(\nu) = g_a \tau_b e(\nu) \ \ \textit{if } |b-a| > 1, \quad \tau_a g_{a+1} g_a = g_{a+1} g_a \tau_{a+1}. \end{aligned} \tag{3.52}$$

Proof. By the defining relations in Definition 3.1.1, the first and the third equalities can be verified immediately. We will prove the second equality in (3.51) when $\nu_a = \nu_{a+1} \in I$. Let $b = a$. Then

$$\begin{aligned} x_{a+1}\varphi_a e(\nu) &= x_{a+1}\mathsf{P}_{\nu_a}(x_a, x_{a+1}) - (x_{a+1} - x_a)(x_{a+1}\tau_a) e(\nu) \\ &= x_{a+1}\mathsf{P}_{\nu_a}(x_a, x_{a+1}) - (x_{a+1} - x_a)(-\tau_a x_a + \mathsf{P}_{\nu_a}(x_a, x_{a+1})) e(\nu), \end{aligned}$$

and

$$\varphi_a x_a e(\nu) = x_a \mathsf{P}_{\nu_a}(x_a, x_{a+1}) - (x_{a+1} - x_a)(-\tau_a x_a) e(\nu).$$

Therefore we have

$$x_{a+1}\varphi_a e(\nu) - \varphi_a x_a e(\nu) = 0.$$

Similarly, we can prove the equality when $b = a + 1$.

By relation ($\mathsf{R}8$) in Definition 3.1.1, $S = \tau_a\varphi_{a+1}\varphi_a - \varphi_{a+1}\varphi_a\tau_{a+1}$ does not contain the term $\tau_{a+1}\tau_a\tau_{a+1}$ and $\tau_a\tau_{a+1}\tau_a$ and is contained in the $\mathbf{k}[x_a, x_{a+1}, x_{a+2}]$-module generated by 1, τ_a, τ_{a+1}, $\tau_a\tau_{a+1}$, $\tau_{a+1}\tau_a$. That is, S can be expressed as

$$S = \mathsf{T}_1 + \mathsf{T}_2\tau_a + \mathsf{T}_3\tau_{a+1} + \mathsf{T}_4\tau_a\tau_{a+1} + \mathsf{T}_5\tau_{a+1}\tau_a$$

for some $\mathsf{T}_i \in \mathbf{k}[x_a, x_{a+1}, x_{a+2}]$ $(1 \le i \le 5)$. By a similar argument given in [14, Lemma 4.12], we have

$$Sx_b = x_{s_{a,a+2}(b)}S \quad \text{for all } b.$$

Then one can show that all T_i must be zero. Thus our second assertion holds. □

Proposition 3.7.1.

(i) *Let* $\widetilde{Q}\colon \mathsf{R}(\beta+\alpha_i)e(\beta, i) \to K_1$ *be the left* $\mathsf{R}(\beta+\alpha_i)$*-linear homomorphism given by the multiplication of* $g_n \cdots g_1$ *from the right. Then* $\widetilde{Q}$ *is a right* $(\mathsf{R}(\beta) \otimes k[t_i])$*-linear homomorphism. That is,*

$$\widetilde{Q}(Sx_a) = \widetilde{Q}(S)x_{a+1} \ (1 \le a \le n), \quad \widetilde{Q}(Sx_{n+1}) = \widetilde{Q}(S)x_1$$
$$\widetilde{Q}(S\tau_b) = \widetilde{Q}(S)\tau_{b+1} \ (1 \le b \le n-1)$$

for any $S \in \mathsf{R}(\beta + \alpha_i)e(\beta, i)$.

(ii) *The map* $\widetilde{Q}$ *induces a well-defined* $(\mathsf{R}(\beta + \alpha_i), \mathsf{R}(\beta) \otimes k[t_i])$*-bilinear homomorphism*

$$Q\colon K_0 \to K_1.$$

Proof. The proof follows immediately from the preceding lemma. □

Theorem 3.7.1. *For each* $\nu \in I^\beta$, *set*

$$\mathsf{A}_\nu(t_i) = \mathfrak{a}_i^\Lambda(t_i) \prod_{\substack{1 \le a \le n, \\ \nu_a \ne i}} \mathsf{Q}_{i,\nu_a}(t_i, x_a) \prod_{\substack{1 \le a \le n, \\ \nu_a = i \in I}} \mathsf{P}(t_i, x_a)\mathsf{P}(x_a, t_i)e(\nu),$$

and define

$$\mathsf{A}(t_i) := \sum_{\nu \in I^\beta} \mathsf{A}_\nu(t_i) \in \mathbf{k}[t_i] \otimes \mathsf{R}^\Lambda(\beta).$$

Then the composition

$$Q \circ P \colon K_1 \to K_1$$

coincides with the right multiplication by $\mathsf{A}(t_i)$*; i.e.,*

$$\begin{tikzcd} K_1 \arrow[r, "P=\mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_n"] \arrow[d, "\mathsf{A}(t_i)"'] & K_0 \arrow[dl, "Q=g_n\cdots g_1"] \\ K_1. & \end{tikzcd}$$

Proof. it suffices to show that

$$\begin{aligned} \mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_n g_n\cdots g_1 e(i,\nu) &= \mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_n e(\nu,i) g_n\cdots g_1 \\ &\equiv \mathsf{A}'_\nu \mod \mathsf{R}(\beta+\alpha_i)\mathfrak{a}^\Lambda(x_2)\mathsf{R}^1(\beta), \end{aligned} \tag{3.53}$$

where

$$\mathsf{A}'_\nu = \mathfrak{a}_i^\Lambda(x_1) \prod_{\substack{1\le a\le n,\\ \nu_a\ne i}} \mathsf{Q}_{i,\nu_a}(x_1,x_{a+1}) \prod_{\substack{1\le a\le n,\\ \nu_a=i\in I}} \mathsf{P}(x_1,x_{a+1})\mathsf{P}(x_{a+1},x_1)e(\nu).$$

We will prove (3.53) by induction on $|\beta| = n$. If $n=0$, the assertion is obvious. Thus we may assume that $n \ge 1$.

Note that we have

(3.54)

$$\tau_n e(\nu,i) g_n = \begin{cases} \tau_n e(\nu,i)\tau_n = \mathsf{Q}_{i,\nu_n}(x_n,x_{n+1})e(\nu_{<n},i,\nu_n) & \text{if } \nu_n \ne i, \\ \tau_n(x_{n+1}-x_n)\mathsf{P}_i(x_{n+1},x_n)e(\nu,i) & \text{if } \nu_n = i. \end{cases}$$

(i) We first assume that $\nu_n \ne i$. Then we have

$$\begin{aligned} &\mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_n g_n\cdots g_1 e(i,\nu) \\ &= \mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_{n-1}\mathsf{Q}_{i,\nu_n}(x_n,x_{n+1})g_{n-1}\cdots g_1 e(i,\nu) \\ &= \mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_{n-1}g_{n-1}\cdots g_1 e(i,\nu)\mathsf{Q}_{i,\nu_n}(x_1,x_{n+1}) \\ &\equiv \mathsf{A}'_{\nu_{<n}}\mathsf{Q}_{i,\nu_n}(x_1,x_{n+1}) = \mathsf{A}'_\nu \qquad \mod \mathsf{R}(\beta+\alpha_i)\mathfrak{a}^\Lambda(x_2)\mathsf{R}^1(\beta)e(i,\beta). \end{aligned}$$

(ii) If $\nu_n = i$, then we have

$$\begin{aligned}
&\mathfrak{a}^{\Lambda}(x_1)\tau_1\cdots\tau_n g_n\cdots g_1 e(i,\nu)\\
&= \mathfrak{a}^{\Lambda}(x_1)\tau_1\cdots\tau_n(x_{n+1}-x_n)\mathsf{P}_i(x_{n+1},x_n)g_{n-1}\cdots g_1 e(i,\nu)\\
&= \mathfrak{a}^{\Lambda}(x_1)\tau_1\cdots\tau_n g_{n-1}\cdots g_1(x_{n+1}-x_1)\mathsf{P}_i(x_{n+1},x_1)e(i,\nu).
\end{aligned} \tag{3.55}$$

Note that

$$\begin{aligned}
\mathfrak{a}^{\Lambda}(x_1)\tau_1\cdots\tau_{n-1}g_n\cdots g_1 e(i,\nu) = g_n\cdots g_1\mathfrak{a}^{\Lambda}(x_2)\tau_2\cdots\tau_n \equiv 0\\
\bmod \mathsf{R}(\beta+\alpha_i)\mathfrak{a}^{\Lambda}(x_2)\mathsf{R}^1(\beta)e(i,\beta).
\end{aligned} \tag{3.56}$$

By the definition of g_n, formula (3.56) can be written as

$$\mathfrak{a}^{\Lambda}(x_1)\tau_1\cdots\tau_{n-1}\left(\tau_n(x_{n+1}-x_n)^2-(x_{n+1}-x_n)\mathsf{P}_{\nu_a}(x_a,x_{a+1})\right)g_{n-1}\cdots g_1 e(i,\nu)\equiv 0.$$

Thus

$$\begin{aligned}
&\mathfrak{a}^{\Lambda}(x_1)\tau_1\cdots\tau_{n-1}\tau_n g_{n-1}\cdots g_1(x_{n+1}-x_1)^2 e(i,\nu)\\
&\equiv \mathfrak{a}^{\Lambda}(x_1)\tau_1\cdots\tau_{n-1}g_{n-1}\cdots g_1(x_{n+1}-x_1)\mathsf{P}_{\nu_a}(x_1,x_{a+1})e(i,\nu)\\
&\equiv \mathsf{A}'_{\nu_{<n}}(x_{n+1}-x_1)\mathsf{P}_{\nu_a}(x_1,x_{a+1}).
\end{aligned}$$

Since right multiplication by $(x_{n+1}-x_1)$ on K_1 is injective by Lemma 3.7.4, we conclude that

$$\mathfrak{a}^{\Lambda}(x_1)\tau_1\cdots\tau_{n-1}\tau_n g_{n-1}\cdots g_1(x_{n+1}-x_1)e(i,\nu)\equiv \mathsf{A}'_{\nu_{<n}}\mathsf{P}_{\nu_a}(x_1,x_{a+1})$$

which implies

$$\begin{aligned}
&\mathfrak{a}^{\Lambda}(x_1)\tau_1\cdots\tau_n g_{n-1}\cdots g_1(x_{n+1}-x_1)\mathsf{P}_i(x_{n+1},x_1)e(i,\nu)\\
&\qquad\equiv \mathsf{A}'_{\nu_{<n}}\mathsf{P}_{\nu_a}(x_1,x_{a+1})\mathsf{P}_{\nu_a}(x_{a+1},x_1).
\end{aligned}$$

Then, (3.55), together with $\mathsf{A}'_{\nu} = \mathsf{A}'_{\nu_{<n}}\mathsf{P}_{\nu_a}(x_1,x_{a+1})\mathsf{P}_{\nu_a}(x_{a+1},x_1)$, implies the desired result. □

By applying the same argument as in [14, Lemma 4.19], we have the following lemma.

Corollary 3.7.1. *The following diagram commutes*

$$\begin{tikzcd} K_1 \arrow[r, "P"] \arrow[d, "\mathsf{A}(t_i)"'] & K_0 \arrow[d, "\mathsf{A}(t_i)"] \arrow[dl, "Q"'] \\ K_1 \arrow[r, "P"'] & K_0. \end{tikzcd}$$

Since K_1 is a projective $\mathsf{R}^\Lambda(\beta) \otimes \mathbf{k}[t_i]$-module by Lemma 3.7.4 and $\mathsf{A}(t_i)$ is a monic polynomial in t_i (up to a multiple of an invertible element), by a similar argument to the one in [14, Lemma 4.17, Lemma 4.18], we conclude:

Theorem 3.7.2. *The module F^Λ is a projective right $\mathsf{R}^\Lambda(\beta)$-module and we have a short exact sequence consisting of right projective $\mathsf{R}^\Lambda(\beta)$-modules:*

$$0 \to K_1 \xrightarrow{P} K_0 \to F^\Lambda \to 0. \tag{3.57}$$

Since K_1, K_0 and F^Λ are the kernels of the functors $\overline{F}_i$, F_i and F_i^Λ, respectively, we have:

Corollary 3.7.2. *For any $i \in I$ and $\beta \in Q^+$, there exists an exact sequence of $\mathsf{R}(\beta + \alpha_i)$-modules*

$$0 \to q^{(\alpha_i|2\Lambda-\beta)}\overline{F}_i M \to F_i M \to F_i^\Lambda M \to 0, \tag{3.58}$$

which is functorial in $M \in \mathsf{Mod}(\mathsf{R}^\Lambda(\beta))$.

For $\alpha \in Q^+$, let $\mathsf{Proj}(\mathsf{R}^\Lambda(\alpha))$ denote the category of finitely generated projective $\mathbb{Z}$-graded $\mathsf{R}^\Lambda(\alpha)$-modules, and let $\mathsf{Rep}(\mathsf{R}^\Lambda(\alpha))$ be the category of $\mathbb{Z}$-graded $\mathsf{R}^\Lambda(\alpha)$-modules which are finite dimensional over $\mathbf{k}_0$. Then we conclude that the functors E_i^Λ and F_i^Λ are well-defined on $\bigoplus_{\alpha \in Q^+} \mathsf{Proj}(\mathsf{R}^\Lambda(\alpha))$ and $\bigoplus_{\alpha \in Q^+} \mathsf{Rep}(\mathsf{R}^\Lambda(\alpha))$:

Theorem 3.7.3. *Set*

$$\mathsf{Proj}(\mathsf{R}^\Lambda) = \bigoplus_{\alpha \in Q^+} \mathsf{Proj}(\mathsf{R}^\Lambda(\alpha)), \quad \mathsf{Rep}(\mathsf{R}^\Lambda) = \bigoplus_{\alpha \in Q^+} \mathsf{Rep}(\mathsf{R}^\Lambda(\alpha)).$$

Then the functors E_i^Λ and F_i^Λ are well-defined exact functors on $\mathsf{Proj}(\mathsf{R}^\Lambda)$ and $\mathsf{Rep}(\mathsf{R}^\Lambda)$, and they induce endomorphisms of the Grothendieck groups $[\mathsf{Proj}(\mathsf{R}^\Lambda)]$ and $[\mathsf{Rep}(\mathsf{R}^\Lambda)]$.

Proof. By Theorem 3.7.2, F^Λ is a finitely generated projective module as a right $\mathsf{R}^\Lambda(\beta)$-module and as a left $\mathsf{R}^\Lambda(\beta+\alpha_i)$-module . Similarly, $e(\beta,i)\mathsf{R}^\Lambda(\beta+\alpha_i)$ is a finitely generated projective module as a left $\mathsf{R}^\Lambda(\beta)$-module and as a right $\mathsf{R}^\Lambda(\beta+\alpha_i)$-module. Now our assertions follow from these facts immediately. □

Theorem 3.7.4. *For $i \neq j \in I$, there exists a natural isomorphism*

$$F_j^\Lambda E_i^\Lambda \simeq q_i^{-a_{ij}} E_i^\Lambda F_j^\Lambda. \tag{3.59}$$

Proof. By Proposition 3.6.1, we already know

$$e(n,i)\mathsf{R}(n+1)e(n,j) \simeq q_i^{-a_{ij}}\mathsf{R}(n)e(n-1,j)\otimes_{\mathsf{R}(n-1)} e(n-1,j)\mathsf{R}(n). \tag{3.60}$$

Applying the functor $\mathsf{R}^\Lambda(n)\otimes_{\mathsf{R}(n)} \bullet \otimes_{\mathsf{R}(n)} \mathsf{R}^\Lambda(n)e(\beta)$ on (3.60), we obtain

$$\begin{aligned}&\frac{e(n,i)\mathsf{R}(n+1)e(\beta,j)}{e(n,i)\mathsf{R}(n)\mathfrak{a}^\Lambda(x_1)\mathsf{R}(n+1)e(\beta,j)+e(n,i)\mathsf{R}(n+1)\mathfrak{a}^\Lambda(x_1)\mathsf{R}(n)e(\beta,j)}\\ &\qquad\simeq \mathsf{R}^\Lambda(n)e(n-1,j)\otimes_{\mathsf{R}^\Lambda(n-1)} e(n-1,i)\mathsf{R}^\Lambda(n)e(\beta) = F_j^\Lambda E_i^\Lambda \mathsf{R}^\Lambda(\beta).\end{aligned}$$

Note that

$$E_i^\Lambda F_j^\Lambda \mathsf{R}^\Lambda(\beta) = \left(\frac{e(n,i)\mathsf{R}(n+1)e(n,j)}{e(n,i)\mathsf{R}(n+1)\mathfrak{a}^\Lambda(x_1)\mathsf{R}(n+1)e(n,j)}\right)e(\beta).$$

Thus it suffices to show that

$$\begin{aligned}&e(n,i)\mathsf{R}(n+1)\mathfrak{a}^\Lambda(x_1)\mathsf{R}(n+1)e(n,j)\\ &\quad= e(n,i)\mathsf{R}(n)\mathfrak{a}^\Lambda(x_1)\mathsf{R}(n+1)e(n,j)+e(n,i)\mathsf{R}(n+1)\mathfrak{a}^\Lambda(x_1)\mathsf{R}(n)e(n,j).\end{aligned} \tag{3.61}$$

Since $\mathfrak{a}^\Lambda(x_1)\tau_k = \tau_k\mathfrak{a}^\Lambda(x_1)$ for all $k \geq 2$, we have

$$\begin{aligned}\mathsf{R}(n+1)\mathfrak{a}^\Lambda(x_1)\mathsf{R}(n+1) &= \sum_{a=1}^{n+1}\mathsf{R}(n+1)\mathfrak{a}^\Lambda(x_1)\tau_a\cdots\tau_n\mathsf{R}(n,1)\\ &= \mathsf{R}(n+1)\mathfrak{a}^\Lambda(x_1)\mathsf{R}(n,1)+\mathsf{R}(n+1)\mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_n\mathsf{R}(n,1)\\ &= \mathsf{R}(n+1)\mathfrak{a}^\Lambda(x_1)\mathsf{R}(n,1)+\sum_{a=1}^{n+1}\mathsf{R}(n,1)\tau_n\cdots\tau_a\mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_n\mathsf{R}(n,1)\\ &= \mathsf{R}(n+1)\mathfrak{a}^\Lambda(x_1)\mathsf{R}(n,1)+\mathsf{R}(n,1)\mathfrak{a}^\Lambda(x_1)\mathsf{R}(n+1)\\ &\qquad+\mathsf{R}(n,1)\tau_n\cdots\tau_1\mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_n\mathsf{R}(n,1).\end{aligned}$$

For $i \neq j$, we get

$$e(n,i)\mathsf{R}(n,1)\tau_n \cdots \tau_1 \mathfrak{a}^{\Lambda}(x_1)\tau_1 \cdots \tau_n \mathsf{R}(n,1)e(n,j) = 0,$$

and our assertion (3.61) follows. □

Consider the following commutative diagram with exact rows and columns derived from Theorem 3.6.1, Theorem 3.6.2 and Corollary 3.7.2:
(3.62)

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & q_i^{(\alpha_i|2\Lambda-\beta)}\overline{F}_i E_i M \arrow[r] \arrow[d] & q_i^{-a_{ii}} F_i E_i M \arrow[r] \arrow[d] & q_i^{-a_{ii}} F_i^{\Lambda} E_i^{\Lambda} M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & q_i^{(\alpha_i|2\Lambda-\beta)} E_i \overline{F}_i M \arrow[r] \arrow[d] & E_i F_i M \arrow[r] \arrow[d] & E_i^{\Lambda} F_i^{\Lambda} M \arrow[r] & 0 \\
 & q_i^{(\alpha_i|2\Lambda-2\beta)}\mathbf{k}[t_i] \otimes M \arrow[r] \arrow[d] & \mathbf{k}[t_i] \otimes M \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

By taking the kernel modules, we obtain the following commutative diagram of $(\mathsf{R}(\beta), \mathsf{R}^{\Lambda}(\beta))$-modules:

(3.63)

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & q_i^{(\alpha_i|2\Lambda-\beta)} K_1' \arrow[r, "P'"] \arrow[d] & q_i^{-a_{ii}} K_0' \arrow[r, "G"] \arrow[d, "F"] & q_i^{-a_{ii}} F_i^\Lambda E_i^\Lambda \mathsf{R}^\Lambda(\beta) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & q_i^{(\alpha_i|2\Lambda-\beta)} E_i K_1 \arrow[r, "P"] \arrow[d, "B"'] & E_i K_0 \arrow[r] \arrow[d, "C"] & E_i^\Lambda F_i^\Lambda \mathsf{R}^\Lambda(\beta) \arrow[r] & 0 \\
 & q_i^{(\alpha_i|2\Lambda-2\beta)} \mathbf{k}[t_i] \otimes \mathsf{R}^\Lambda(\beta) \arrow[r, "A"] \arrow[d] & \mathbf{k}[t_i] \otimes \mathsf{R}^\Lambda(\beta) \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

where

$$\begin{aligned}
K_0' &= F_i E_i \mathsf{R}^\Lambda(\beta) = \mathsf{R}(\beta) e(\beta-\alpha_i, i) \otimes_{\mathsf{R}(\beta-\alpha_i)} e(\beta-\alpha_i)\mathsf{R}^\Lambda(\beta) \\
K_1' &= \overline{F}_i E_i \mathsf{R}^\Lambda(\beta) = \mathsf{R}(\beta) e(i, \beta-\alpha_i) \otimes_{\mathsf{R}(\beta-\alpha_i)} e(\beta-\alpha_i)\mathsf{R}^1(\beta) \otimes_{\mathsf{R}(\beta)} \mathsf{R}^\Lambda(\beta) \\
&= \mathsf{R}(\beta) e(i, \beta-\alpha_i) \otimes_{\mathsf{R}(\beta-\alpha_i)} \mathsf{R}^\Lambda(\beta).
\end{aligned}$$

The homomorphisms in the diagram (3.63) can be described as follows:

- P' is given by the right multiplication by $\mathfrak{a}^\Lambda(x_1)\tau_1 \cdots \tau_{n-1}$ on $\mathsf{R}(\beta)e(i, \beta-\alpha_i)$.
- A is $\mathsf{R}^\Lambda(\beta)$-linear but *not* $\mathbf{k}[t_i]$-linear.
- B is given by taking the coefficient of $\tau_n \cdots \tau_1$ and is $(\mathsf{R}(\beta)\otimes\mathbf{k}[x_{n+1}], \mathbf{k}[x_1]\otimes \mathsf{R}^1(\beta))$-bilinear.
- C is $(\mathsf{R}(\beta), \mathsf{R}^\Lambda(\beta))$-bilinear but does *not* commute with t_i.
- P is the right multiplication by $\mathfrak{a}^\Lambda(x_1)\tau_1 \cdots \tau_n$ and is $(\mathsf{R}(\beta), \mathsf{R}^\Lambda(\beta) \otimes \mathbf{k}[t_i])$-bilinear.
- F is the multiplication by τ_n (See Proposition 3.6.1).

Let p be the number of α_i appearing in β. Note that the degree of t_i in

$$t_i^{-\langle h_i,\Lambda\rangle}\mathsf{A}_\nu = \prod_{\substack{1\le a\le n,\\ \nu_a\neq i}} \mathsf{Q}_{i,\nu_a}(t_i,x_a) \prod_{\substack{1\le a\le n,\\ \nu_a=i}} \mathsf{P}_i(t_i,x_a)\mathsf{P}_i(x_a,t_i),$$

denoted by $\deg_{t_i}(t_i^{-\langle h_i,\Lambda\rangle}\mathsf{A}_\nu)$, is given by

$$-\langle h_i,\beta-p\alpha_i\rangle+2p(1-\frac{a_{ii}}{2}) = -\langle h_i,\beta\rangle+pa_{ii}+2p-pa_{ii} = -\langle h_i,\beta\rangle+2p.$$

Define an invertible element $\gamma\in\mathbf{k}^\times$ by

$$\begin{aligned}(-1)^p &\prod_{\substack{1\le a\le n,\\ \nu_a\neq i}} \mathsf{Q}_{i,\nu_a}(t_i,x_a) \prod_{\substack{1\le a\le n,\\ \nu_a=i}} \mathsf{P}_i(t_i,x_{a+1})\mathsf{P}_i(x_{a+1},t_i)\\ &= \gamma^{-1}t_i^{-\langle h_i,\beta\rangle+2p} + (\text{ terms of degree } < -\langle h_i,\beta\rangle+2p \text{ in } t_i)\,.\end{aligned} \tag{3.64}$$

Set $\lambda=\Lambda-\beta$ and

$$\varphi_k = A(t_i^k)\in\mathbf{k}[t_i]\otimes\mathsf{R}^\Lambda(\beta), \tag{3.65}$$

which is of degree $2(\alpha_i|\lambda)+2d_ik = 2d_i(\langle h_i,\lambda\rangle+k)$.

Proposition 3.7.2. *If $\langle h_i,\lambda\rangle+k\ge 0$, then $\gamma\varphi_k$ is a monic polynomial in t_i of degree $\langle h_i,\lambda\rangle+k$.*

Note that for $m<0$, we say that a polynomial φ is a monic polynomial of degree m if $\varphi=0$.

To prove Proposition 3.7.2, we need some preparation. Let

$$z=\sum_{k\in\mathbb{Z}_{>0}} a_k\otimes b_k \in \mathsf{R}(\beta)e(\beta-\alpha_i,i)\otimes_{\mathsf{R}(\beta-\alpha_i)} e(\beta-\alpha_i,i)\mathsf{R}^\Lambda(\beta),$$

where $a_k\in\mathsf{R}(\beta)e(\beta-\alpha_i,i)$ and $b_k\in e(\beta-\alpha_i,i)\mathsf{R}^\Lambda(\beta)$. Define a map $E\colon K_0'\to E_iK_0$ by

$$z\mapsto \sum_{k\in\mathbb{Z}_{>0}} a_k\mathsf{P}_i(x_n,x_{n+1})b_k. \tag{3.66}$$

Lemma 3.7.8. *For $z\in\mathsf{R}(\beta)e(\beta-\alpha_i)\otimes_{\mathsf{R}(\beta-\alpha_i)} e(\beta-\alpha_i,i)\mathsf{R}^\Lambda(\beta)$, we have*

$$F(z)x_{n+1} = F(z(x_n\otimes 1))+E(z). \tag{3.67}$$

Proof. Let $z = a \otimes b \in \mathsf{R}(\beta)e(\beta-\alpha_i, i) \otimes_{\mathsf{R}(\beta-\alpha_i)} e(\beta-\alpha_i, i)\mathsf{R}^\Lambda(\beta)$, where $a \in \mathsf{R}(\beta)e(\beta-\alpha_i, i)$ and $b \in e(\beta-\alpha_i, i)\mathsf{R}^\Lambda(\beta)$. Then

$$F(z) = a\tau_n b, \qquad E(z) = a\mathsf{P}_i(x_n, x_{n+1})b.$$

Thus

$$\begin{aligned} F(z)x_{n+1} &= a\tau_n b x_{n+1} = a\tau_n x_{n+1} b = a(x_n\tau_n + \mathsf{P}_i(x_n, x_{n+1}))b \\ &= a x_n \tau_n b + a\mathsf{P}_i(x_n, x_{n+1})b \\ &= F(a x_n \otimes b) + E(z) = F(z(x_n \otimes 1)) + E(z). \end{aligned}$$

□

By Proposition 3.6.1, we have

$$\begin{aligned} &e(\beta, i)\mathsf{R}(\beta+\alpha_i)e(\beta, i) \otimes_{\mathsf{R}(\beta)} \mathsf{R}^\Lambda(\beta) \\ &\quad = F\big(\mathsf{R}(\beta)e(\beta-\alpha_i, i) \otimes_{\mathsf{R}(\beta-\alpha_i)} e(\beta-\alpha_i, i)\mathsf{R}^\Lambda(\beta)\big) \oplus \mathbf{k}[t_i] \otimes \mathsf{R}^\Lambda(\beta), \end{aligned} \tag{3.68}$$

where $t_i = x_{n+1}$. Using the decomposition (3.68), we write

$$P(\tau_n \cdots \tau_1 x_1^k) = F(\psi_k) + \varphi_k \tag{3.69}$$

for uniquely determined $\psi_k \in K_0'$ and $\varphi_k \in \mathbf{k}[t_i] \otimes \mathsf{R}^\Lambda(\beta)$.

Using (3.65), we have

$$A(t_i^k) = AB(\tau_n \cdots \tau_1 x_1^k) = CP(\tau_n \cdots \tau_1 x_1^k) = \varphi_k.$$

Thus one can verify that the definition of φ_k coincides with the definition given in (3.65).

Since

$$\begin{aligned} F(\psi_{k+1}) + \varphi_{k+1} &= P(\tau_n \cdots \tau_1 x_1^{k+1}) = P(\tau_n \cdots \tau_1 x_1^k)x_{n+1} \\ &= (F(\psi_k) + \varphi_k)x_{n+1} = F(\psi_k(x_n \otimes 1)) + E(\psi_k) + \varphi_k t_i, \end{aligned}$$

we have

$$\psi_{k+1} = \psi_k(x_n \otimes 1), \qquad \varphi_{k+1} = E(\psi_k) + \varphi_k t_i. \tag{3.70}$$

Now we will prove Proposition 3.7.2. By Lemma 3.7.1, we have

$$g_n \cdots g_1 x_1^k \tau_1 \cdots \tau_n = x_{n+1}^k \mathfrak{a}_i^\Lambda(x_{n+1}) \prod_{\substack{1\le a\le n,\\ \nu_a\ne i}} \mathsf{Q}_{i,\nu_a}(x_{n+1},x_a) \prod_{\substack{1\le a\le n,\\ \nu_a = i}} \mathsf{P}_i(x_{n+1},x_a)\mathsf{P}_i(x_a,x_{n+1})$$

in $e(\beta,i)\mathsf{R}(\beta+\alpha_i)e(\beta,i)\otimes \mathsf{R}^\Lambda(\beta)$, which implies

$$\begin{aligned} AB(g_n\cdots g_1 x_1^k) &= C\Big(x_{n+1}^k \mathfrak{a}_i^\Lambda(x_{n+1}) \prod_{\substack{1\le a\le n,\\ \nu_a\ne i}} \mathsf{Q}_{i,\nu_a}(x_{n+1},x_a) \prod_{\substack{1\le a\le n,\\ \nu_a = i}} \mathsf{P}_i(x_{n+1},x_a)\mathsf{P}_i(x_a,x_{n+1})\Big) \\ &= t_i^k \mathfrak{a}_i^\Lambda(t_i) \prod_{\substack{1\le a\le n,\\ \nu_a\ne i}} \mathsf{Q}_{i,\nu_a}(t_i,x_a) \prod_{\substack{1\le a\le n,\\ \nu_a = i}} \mathsf{P}_i(t_i,x_a)\mathsf{P}_i(x_a,t_i). \end{aligned}$$

On the other hand, since B is the map taking the coefficient of $\tau_n\cdots\tau_1$, we have

$$\begin{aligned} B(g_n\cdots g_1x_1^k) &= B\left(\prod_{\nu_a=i}(-(x_{n+1}-x_a)^2)x_{n+1}^k\tau_n\cdots\tau_1\right) \\ &= t_i^k\prod_{\nu_a=i}(-(t_i-x_a)^2). \end{aligned}$$

Thus we have

$$A(t_i^k\prod_{\nu_a=i}(t_i-x_a)^2) = (-1)^p t_i^k \mathfrak{a}_i^\Lambda(t_i) \prod_{\substack{1\le a\le n,\\ \nu_a\ne i}} \mathsf{Q}_{i,\nu_a}(t_i,x_a) \prod_{\substack{1\le a\le n,\\ \nu_a = i}} \mathsf{P}_i(t_i,x_a)\mathsf{P}_i(x_a,t_i). \tag{3.71}$$

Set

$$\begin{aligned} \mathsf{S}_i &= \prod_{\nu_a=i}(t_i-x_a)^2, \\ \mathsf{F}_i &= \gamma(-1)^p\mathfrak{a}_i^\Lambda(t_i) \prod_{\substack{1\le a\le n,\\ \nu_a\ne i}} \mathsf{Q}_{i,\nu_a}(t_i,x_a) \prod_{\substack{1\le a\le n,\\ \nu_a = i}} \mathsf{P}_i(t_i,x_a)\mathsf{P}_i(x_a,t_i) \in \mathbf{k}[t_i]\otimes\mathsf{R}^\Lambda(\beta). \end{aligned}$$

Then S_i and F_i are monic polynomials in t_i of degree $2p$ and $\langle h_i,\lambda\rangle + 2p$, respectively. Note that they are contained in the center of $\mathbf{k}[t_i]\otimes\mathsf{R}^\Lambda(\beta)$. Then (3.71) can be expressed as the following form:

$$\gamma A(t_i^k\mathsf{S}_i) = t_i^k\mathsf{F}_i.$$

Note that

- if $p = 0$, then $K'_0 = 0$ and
- if $i \in I^{\mathrm{im}}$ such that $\langle h_i, \lambda \rangle = 0$ and $p > 0$, then $\mathsf{R}^{\Lambda}(\beta) = 0$.

Thus, to prove Proposition 3.7.2, we may assume that

$$p > 0, \text{ and if } i \in I^{\mathrm{im}}, \text{ then } \langle h_i, \lambda \rangle > 0. \tag{3.72}$$

Lemma 3.7.9. *For any $k \geq 0$, we have*

$$t_i^k \mathsf{F}_i = (\gamma\varphi_k)\mathsf{S}_i + \mathsf{h}_k, \tag{3.73}$$

for some polynomial h_k in t_i with $\deg_{t_i}(\mathsf{h}_k) \leq 2p - \frac{a_{ii}}{2}$. Moreover, if $i \in I^{\mathrm{re}}$, then $\gamma\varphi_k$ coincides with the quotient of $t_i^k \mathsf{F}_i$ by S_i, and if $i \in I^{\mathrm{im}}$, then $\deg_{t_i}(\mathsf{h}_k) < \deg_{t_i}(t_i^k \mathsf{F}_i)$.

Proof. By (3.70), we have

$$\varphi_{k+1} = \varphi_k t_i + E(\psi_k), \qquad \psi_k = \psi_0(x_n^k \otimes 1).$$

Since x_n is a nilpotent element in $\mathsf{R}^{\Lambda}(\beta)$,

$$E(\psi_k) = E(\psi_0(x_n^k \otimes 1)) = 0 \quad \text{ for } k \gg 0.$$

In particular,

$$A(\mathsf{S}_i t_i^k) = \mathsf{S}_i A(t_i^k) \quad \text{ for } k \gg 0,$$

which yields

$$\mathsf{S}_i \gamma\varphi_k = \gamma \mathsf{S}_i A(t_i^k) = \gamma A(\mathsf{S}_i t_i^k) = t_i^k \mathsf{F}_i.$$

We will use downward induction. Assume that our assertion is true for $k+1$. Since

$$t_i^{k+1}\mathsf{F}_i = (\gamma\varphi_{k+1})\mathsf{S}_i + \mathsf{h}_{k+1} = \gamma(t_i\varphi_k + E(\psi_k))\mathsf{S}_i + \mathsf{h}_{k+1},$$

we have

$$t_i \mathsf{h}_k = t_i(t_i^k \mathsf{F}_i - (\gamma\varphi_k)\mathsf{S}_i) = \gamma E(\psi_k)\mathsf{S}_i + \mathsf{h}_{k+1}.$$

Since $\deg_{t_i}(\gamma E(\psi_k)) \leq 1 - \frac{a_{ii}}{2}$ and $\deg_{t_i}(\mathsf{S}_i) = 2p$, we get

$$\deg_{t_i}(\gamma E(\psi_k)\mathsf{S}_i) \leq 2p + 1 - \frac{a_{ii}}{2}.$$

Hence

$$\deg_{t_i}(\mathsf{h}_k) \le 2p - \frac{a_{ii}}{2}.$$

For $i \in I^{\mathrm{re}}$, our second assertion holds since $\deg_{t_i}(\mathsf{h}_k) < \deg_{t_i}(\mathsf{S}_i)$.

Assume that $i \in I^{\mathrm{im}}$. Then we have $\langle h_i, \lambda\rangle = \langle h_i, \Lambda\rangle - \langle h_i, \beta\rangle \ge -\langle h_i, \beta\rangle \ge -a_{ii}$. Hence if $a_{ii} < 0$, then $\langle h_i, \lambda\rangle > -a_{ii}/2$, and if $a_{ii} = 0$, then $\langle h_i, \lambda\rangle > 0 = -a_{ii}/2$ by (3.72). In both cases, we have

$$\deg_{t_i}(t_i^k \mathsf{F}_i) = \langle h_i, \lambda\rangle + 2p + k > 2p - \frac{a_{ii}}{2} \ge \deg_{t_i}(\mathsf{h}_k).$$

Hence the last assertion holds. $\square$

Thus by Lemma 3.7.9, we can conclude that $\gamma\varphi_k$ is a monic polynomial in t_i of degree $\langle h_i, \lambda\rangle + k$, which completes the proof of Proposition 3.7.2.

Theorem 3.7.5. *Let $\lambda = \Lambda - \beta$. Then there exist natural isomorphisms of endofunctors on $\mathsf{Mod}(\mathsf{R}^\Lambda(\beta))$ given below.*

(i) *If $\langle h_i, \lambda\rangle \ge 0$, then we have*

$$q_i^{-a_{ii}} F_i^\Lambda E_i^\Lambda \oplus \bigoplus_{k=0}^{\langle h_i,\lambda\rangle - 1} q_i^{2k} \xrightarrow{\sim} E_i^\Lambda F_i^\Lambda. \tag{3.74}$$

(ii) *If $\langle h_i, \lambda\rangle < 0$, then we have*

$$q_i^{-a_{ii}} F_i^\Lambda E_i^\Lambda \xrightarrow{\sim} E_i^\Lambda F_i^\Lambda \oplus \bigoplus_{k=0}^{-\langle h_i,\lambda\rangle - 1} q_i^{-2k-2}. \tag{3.75}$$

Proof. Due to Proposition 3.7.2 and (3.70), we can apply the arguments in [14, Theorem 5.2] with a slight modification. Hence we will give only a sketch of the proof.

From the Snake Lemma, we get an exact sequences of $\mathsf{R}^\Lambda(\beta)$-bimodules:

$$0 \to \mathrm{Ker}A \to q_i^{-a_{ii}} F_i^\Lambda E_i^\Lambda \mathsf{R}^\Lambda(\beta) \to E_i^\Lambda F_i^\Lambda \mathsf{R}^\Lambda(\beta) \to \mathrm{Coker}A \to 0.$$

If $a := \langle h_i, \lambda\rangle \ge 0$, then Proposition 3.7.2 yields

$$\mathrm{KerA} = 0, \quad \bigoplus_{k=0}^{a-1} \mathbf{k} t_i^k \otimes \mathsf{R}^\Lambda(\beta) \simeq \mathrm{Coker}A$$

and our first assertion follows.

If $a := \langle h_i, \lambda\rangle < 0$, then Proposition 3.7.2 implies CokerA $= 0$. By (3.70), we can prove that there is an isomorphism

$$\mathrm{Ker}A \simeq \bigoplus_{k=0}^{-a-1} \mathbf{k}t_i^k \otimes \mathsf{R}^\Lambda(\beta),$$

which completes the proof. $\square$

3.8 Categorification

In this section, based on the natural isomorphisms of functors in previous sections, we will show that the quiver Hecke algebras $\mathsf{R}(\beta)$ and their cyclotomic quotients provide categorifications of $V_{\mathbb{A}}(\Lambda)$ and $U_{\mathbb{A}}^-(\mathfrak{g})$, respectively.

From now on, we assume that $\mathbf{k}_0$ is a field and the $\mathbf{k}_t$'s are finite-dimensional over $\mathbf{k}_0$.

Recall the anti-involution ψ: $\mathsf{R}^\Lambda(\beta) \to \mathsf{R}^\Lambda(\beta)$ given by (3.7). For $N \in \mathsf{Mod}(\mathsf{R}^\Lambda(\beta))$, let N^ψ be the right $\mathsf{R}^\Lambda(\beta)$-module obtained from N by the anti-involution ψ of $\mathsf{R}^\Lambda(\beta)$. By (3.11) and Lemma 3.5.12, we have a non-degenerate pairing

$$[\mathsf{Proj}(\mathsf{R}^\Lambda)] \times [\mathsf{Rep}(\mathsf{R}^\Lambda)] \to \mathbb{A} \tag{3.76}$$

given by

$$([P], [M]) \mapsto \sum_{n\in\mathbb{Z}} q^n \dim_{\mathbf{k}_0}(P^\psi \otimes_{\mathsf{R}^\Lambda} M)_n.$$

From Theorem 3.7.3, we can define endomorphisms E_i and F_i, induced by E_i^Λ and F_i^Λ, on the Grothendieck groups $[\mathsf{Proj}(\mathsf{R}^\Lambda)]$ and $[\mathsf{Rep}(\mathsf{R}^\Lambda)]$ as follows:

$$\begin{aligned} &[\mathsf{Proj}(\mathsf{R}^\Lambda(\beta))] \underset{\mathsf{E}_i := [q_i^{1-\langle h_i, \Lambda-\beta\rangle} E_i^\Lambda]}{\overset{\mathsf{F}_i := [F_i^\Lambda]}{\rightleftarrows}} [\mathsf{Proj}(\mathsf{R}^\Lambda(\beta+\alpha_i))]\ , \\ &[\mathsf{Rep}(\mathsf{R}^\Lambda(\beta))] \underset{\mathsf{E}_i := [E_i^\Lambda]}{\overset{\mathsf{F}_i := [q_i^{1-\langle h_i, \Lambda-\beta\rangle} F_i^\Lambda]}{\rightleftarrows}} [\mathsf{Rep}(\mathsf{R}^\Lambda(\beta+\alpha_i))]. \end{aligned} \tag{3.77}$$

Then, from the isomorphisms (3.59), (3.74) and (3.75), we obtain the following identities in $[\mathsf{Proj}(\mathsf{R}^\Lambda(\beta))]$ and $[\mathsf{Rep}(\mathsf{R}^\Lambda)(\beta)]$:

$$\begin{aligned}
&\mathsf{E}_i\mathsf{F}_j = \mathsf{F}_j\mathsf{E}_i \quad \text{if } i \neq j,\\
&\mathsf{E}_i\mathsf{F}_i = \mathsf{F}_i\mathsf{E}_i + \frac{q_i^{\langle h_i,\Lambda-\beta\rangle} - q_i^{-\langle h_i,\Lambda-\beta\rangle}}{q_i - q_i^{-1}} \quad \text{if } \langle h_i,\Lambda-\beta\rangle \geq 0,\\
&\mathsf{E}_i\mathsf{F}_i + \frac{q_i^{-\langle h_i,\Lambda-\beta\rangle} - q_i^{\langle h_i,\Lambda-\beta\rangle}}{q_i - q_i^{-1}} = \mathsf{F}_i\mathsf{E}_i \quad \text{if } \langle h_i,\Lambda-\beta\rangle \leq 0.
\end{aligned} \tag{3.78}$$

Let K_i be an endomorphism on $[\mathsf{Proj}(\mathsf{R}^\Lambda(\beta))]$ and $[\mathsf{Rep}(\mathsf{R}^\Lambda(\beta))]$ given by

$$\mathsf{K}_i|_{[\mathsf{Proj}(\mathsf{R}^\Lambda(\beta))]} := q_i^{\langle h_i,\Lambda-\beta\rangle}, \quad \mathsf{K}_i|_{[\mathsf{Rep}(\mathsf{R}^\Lambda(\beta))]} := q_i^{\langle h_i,\Lambda-\beta\rangle}. \tag{3.79}$$

Then (3.78) can be rewritten as the third relation in Definition 2.2.1:

$$[\mathsf{E}_i, \mathsf{F}_j] = \delta_{i,j}\frac{\mathsf{K}_i - \mathsf{K}_i^{-1}}{q_i - q_i^{-1}}. \tag{3.80}$$

Now we assume that $a_{ii} \neq 0$ for all $i \in I$. Define the functors $E_i^{\Lambda^{(n)}}$ and $F_i^{\Lambda^{(n)}}$

$$\begin{aligned}
F_i^{\Lambda^{(n)}} &: \mathsf{Mod}(\mathsf{R}^\Lambda(\beta)) \to \mathsf{Mod}(\mathsf{R}^\Lambda(\beta + n\alpha_i)),\\
E_i^{\Lambda^{(n)}} &: \mathsf{Mod}(\mathsf{R}^\Lambda(\beta + n\alpha_i)) \to \mathsf{Mod}(\mathsf{R}^\Lambda(\beta)),
\end{aligned}$$

by

$$\begin{aligned}
E_i^{\Lambda^{(n)}} &\colon N \longmapsto (\mathsf{R}^\Lambda(\beta) \otimes \mathsf{P}(i^n))^\psi \otimes_{\mathsf{R}^\Lambda(\beta)\otimes\mathsf{R}(n\alpha_i)} e(\beta, i^n)N,\\
F_i^{\Lambda^{(n)}} &\colon M \longmapsto \mathsf{R}^\Lambda(\beta + n\alpha_i)e(\beta, i^n) \otimes_{\mathsf{R}^\Lambda(\beta)\otimes\mathsf{R}(n\alpha_i)} (M \boxtimes \mathsf{P}(i^n))
\end{aligned}$$

for $M \in \mathsf{Mod}(\mathsf{R}^\Lambda(\beta))$ and $N \in \mathsf{Mod}(\mathsf{R}^\Lambda(\beta + n\alpha_i))$. Then they induce the endomorphism

$$\mathsf{E}_i^{(n)} := \begin{cases} \mathsf{E}_i^n/[n]_i! & \text{if } i \in I^{\mathrm{re}},\\ \mathsf{E}_i^n & \text{if } i \in I^{\mathrm{im}}, \end{cases} \quad \text{and} \quad \mathsf{F}_i^{(n)} := \begin{cases} \mathsf{F}_i^n/[n]_i! & \text{if } i \in I^{\mathrm{re}},\\ \mathsf{F}_i^n & \text{if } i \in I^{\mathrm{im}}, \end{cases}$$

by (3.15) and (3.17).

Note that

(3.81)
(i) the action of E_i on $[\mathsf{Proj}(\mathsf{R}^\Lambda)]$ and $[\mathsf{Rep}(\mathsf{R}^\Lambda)]$ is locally nilpotent,
(ii) if the module $[M]$ in $[\mathsf{Rep}(\mathsf{R}^\Lambda(\beta))]$ satisfies $\mathsf{E}_i[M] = 0$ for all $i \in I$, then $\beta = 0$.

By Lemma 3.7.2 and Lemma 3.7.3, we see that

(a) for $i \in I^{\mathrm{re}}$, the action F_i on $[\mathsf{Proj}(\mathsf{R}^\Lambda)]$ and $[\mathsf{Rep}(\mathsf{R}^\Lambda)]$ are locally nilpotent.

(b) for $i \in I^{\mathrm{im}}$ and $\beta \in Q^+$ with $\langle h_i, \Lambda - \beta \rangle = 0$, $\mathsf{F}_i[\mathsf{Proj}(\mathsf{R}^\Lambda(\beta)] = 0$.

(cf. See (iii), (iv) in Definition 2.2.2.)

Therefore, by (3.80) and [23, Proposition B.1], the functors F_i and E_i satisfy the quantum Serre relations in Definition 3.1.1. Hence $[\mathsf{Proj}(\mathsf{R}^\Lambda)]$ and $[\mathsf{Rep}\mathsf{R}^\Lambda]$ are endowed with a $U_{\mathbb{A}}(\mathfrak{g})$-module structure.

Note that $[\mathsf{Proj}(\mathsf{R})]:=\bigoplus_{\beta \in Q^+}[\mathsf{Proj}(\mathsf{R}(\beta))]$ and $[\mathsf{Rep}(\mathsf{R})]:=\bigoplus_{\beta \in Q^+}[\mathsf{Rep}(\mathsf{R}(\beta))]$ are also $\mathbb{A}$-dual to each other. The exact functors $E_i \colon \mathsf{Rep}(\mathsf{R}(\beta + \alpha_i)) \to \mathsf{Rep}(\mathsf{R}(\beta))$ and $F_i' \colon \mathsf{Rep}(\mathsf{R}(\beta)) \to \mathsf{Rep}(\mathsf{R}(\beta + \alpha_i))$ defined in (3.29) induce endomorphisms E_i' and F_i' on $[\mathsf{Rep}(\mathsf{R})]$, respectively. Hence, (3.33) implies the following commutation relation in $[\mathsf{Rep}(\mathsf{R})]$:

$$\mathsf{E}_i'\mathsf{F}_j' = q^{-(\alpha_i|\alpha_j)}\mathsf{F}_j'\mathsf{E}_i' + \delta_{i,j}. \tag{3.82}$$

Similarly, we define

$$\begin{tikzcd} \mathsf{Proj}(\mathsf{R}(\beta)) \arrow[r, shift left, "F_i"] & \mathsf{Proj}(\mathsf{R}(\beta + \alpha_i)) \arrow[l, shift left, "E_i'"] \end{tikzcd},$$

by

$$F_i P := \mathsf{R}(\beta + \alpha_i)e(\beta, i) \otimes_{\mathsf{R}(\beta)} P \quad \text{and} \quad E_i' Q := \frac{e(\beta, i)\mathsf{R}(\beta + \alpha_i)}{e(\beta, i)x_{n+1}\mathsf{R}(\beta + \alpha_i)} \otimes_{\mathsf{R}(\beta+\alpha_i)} Q,$$

where $|\beta| = n$. Then they are well-defined on $\mathsf{Proj}(\mathsf{R})$, and we obtain an exact sequence

$$0 \longrightarrow \delta_{i,j}\ \mathrm{id} \longrightarrow E_i' F_j \longrightarrow q^{-(\alpha_i|\alpha_j)} F_j E_i' \longrightarrow 0.$$

Thus the exact functors induce the endomorphisms E_i' and F_i' on $[\mathsf{Proj}(\mathsf{R})]$ and satisfy the same equation in (3.82) (See [19, Lemma 5.1], for more details).

Let $\mathrm{Irr}_0(\mathsf{R}^\Lambda(\beta))$ be the set of isomorphism classes of self-dual irreducible $\mathsf{R}^\Lambda(\beta)$-modules, and $\mathrm{Irr}_0(\mathsf{R}^\Lambda) := \bigsqcup_{\beta \in Q^+} \mathrm{Irr}_0(\mathsf{R}^\Lambda(\beta))$. Then $\{[S] \mid S \in \mathrm{Irr}_0(\mathsf{R}^\Lambda)\}$ is a strong perfect basis of $[\mathsf{Rep}(\mathsf{R}^\Lambda)]$ by Theorem 3.5.1. By Proposition 2.6.1, (3.81)(ii) and (3.76), we conclude:

Theorem 3.8.1. *Let $U_q(\mathfrak{g})$ be the quantum generalized Kac-Moody algebra associated with the Cartan matrix A with $a_{ii} \neq 0$ for all $i \in I$. For $\Lambda \in P^+$, we have*

$$V_{\mathbb{A}}(\Lambda)^\vee \simeq [\mathsf{Rep}(\mathsf{R}^\Lambda)] \quad \textit{and} \quad V_{\mathbb{A}}(\Lambda) \simeq [\mathsf{Proj}(\mathsf{R}^\Lambda)] \tag{3.83}$$

as $U_{\mathbb{A}}(\mathfrak{g})$-modules.

The fully faithful exact functor $\mathsf{Rep}(\mathsf{R}^\Lambda(\beta)) \to \mathsf{Rep}(\mathsf{R}(\beta))$ induces an $\mathbb{A}$-linear homomorphism $[\mathsf{Rep}(\mathsf{R}^\Lambda)] \to [\mathsf{Rep}(\mathsf{R})]$. Hence $[\mathsf{Rep}(\mathsf{R}^\Lambda)] \to [\mathsf{Rep}(\mathsf{R})]$ is injective and its cokernel is a free $\mathbb{A}$-module. By the duality, the homomorphism $[\mathsf{Proj}(\mathsf{R})] \to [\mathsf{Proj}(\mathsf{R}^\Lambda)]$ is surjective.

As a $U_{\mathbb{A}}^-(\mathfrak{g})$-module, $U_{\mathbb{A}}^-(\mathfrak{g})$ is the projective limit of $V_{\mathbb{A}}(\Lambda)$. Hence, Theorem 3.8.1 implies the following corollary:

Corollary 3.8.1. *There exist isomorphisms:*

$$U_{\mathbb{A}}^-(\mathfrak{g})^\vee \simeq [\mathsf{Rep}(\mathsf{R})] \quad \textit{as a } B_{\mathbb{A}}^{\mathrm{up}}(\mathfrak{g})\textit{-module},$$
$$U_{\mathbb{A}}^-(\mathfrak{g}) \simeq [\mathsf{Proj}(\mathsf{R})] \quad \textit{as a } B_{\mathbb{A}}^{\mathrm{low}}(\mathfrak{g})\textit{-module}.$$

Chapter 4

Supercategorification

Throughout this chapter, we assume that

$$I^{\mathrm{im}} = \emptyset; \text{ i.e., for all } i \in I,\ a_{ii} = 2.$$

4.1 Supercategories and superbimodules

In this section, we recall the notion of supercategory, superfunctor and superbimodule and their basic properties (See [17, Section 2] for more details).

Definition 4.1.1.

(i) *A* supercategory *is a category* $\mathscr{C}$ *equipped with an endofunctor* $\Pi_{\mathscr{C}}$ *of* $\mathscr{C}$ *and an isomorphism* $\xi_{\mathscr{C}} \colon \Pi_{\mathscr{C}}^2 \xrightarrow{\sim} \mathrm{id}_{\mathscr{C}}$ *such that* $\xi_{\mathscr{C}} \circ \Pi_{\mathscr{C}} = \Pi_{\mathscr{C}} \circ \xi_{\mathscr{C}} \in \mathrm{Hom}(\Pi_{\mathscr{C}}^3, \Pi_{\mathscr{C}})$.

(ii) *For a pair of supercategories* $(\mathscr{C}, \Pi, \xi)$ *and* $(\mathscr{C}', \Pi', \xi')$*, a* superfunctor *from* $(\mathscr{C}, \Pi, \xi)$ *to* $(\mathscr{C}', \Pi', \xi')$ *is a pair consisting of a functor* $F \colon \mathscr{C} \to \mathscr{C}'$ *and an isomorphism* $\alpha_F \colon F \circ \Pi \xrightarrow{\sim} \Pi' \circ F$ *such that the following diagram commutes:*

(4.1)
$$\begin{tikzcd}
F \circ \Pi^2 \arrow[r, "\alpha_F \circ \Pi"] \arrow[d, "F \circ \xi"] & \Pi' \circ F \circ \Pi \arrow[r, "\Pi' \circ \alpha_F"] & \Pi'^2 \circ F \arrow[d, "\xi' \circ F"] \\
F \arrow[rr, "\mathrm{id}_F"] & & F
\end{tikzcd}$$

If F is an equivalence of categories, we say that (F, α_F) is an equivalence of supercategories.

(iii) *Let (F, α_F) and $(F', \alpha_{F'})$ be superfunctors from a supercategory $(\mathscr{C}, \Pi, \xi)$ to $(\mathscr{C}', \Pi', \xi')$. A morphism from (F, α_F) to $(F', \alpha_{F'})$ is a morphism of functors $\varphi \colon F \to F'$ such that*

$$\begin{tikzcd} F \circ \Pi \arrow[r, "\varphi\circ\Pi"] \arrow[d, "\alpha_F"'] & F' \circ \Pi \arrow[d, "\alpha_{F'}"'] \\ \Pi' \circ F \arrow[r, "\Pi'\circ\varphi"] & \Pi' \circ F' \end{tikzcd}$$

commutes.

In this paper, a supercategory is assumed to be an additive category.

A superalgebra is a $\mathbb{Z}_2$-graded algebra. Let $A = A_0 \oplus A_1$ be a superalgebra. We denote by ϕ_A the involution of A given by $\phi_A(a) = (-1)^\epsilon a$ for $a \in A_\epsilon$ with $\epsilon = 0, 1$. We call ϕ_A the *parity involution* of the superalgebra A. An A-supermodule is an A-module with a decomposition $M = M_0 \oplus M_1$ such that $A_\epsilon M_{\epsilon'} \subset M_{\epsilon+\epsilon'}$ $(\epsilon, \epsilon' \in \mathbb{Z}_2)$. For an A-supermodule M, we denote by $\phi_M \colon M \to M$ the involution of M given by $\phi_M|_{M_\epsilon} = (-1)^\epsilon \operatorname{id}_{M_\epsilon}$. We call ϕ_M the *parity involution* of the A-supermodule M. Then we have $\phi_M(ax) = \phi_A(a)\phi_M(x)$ for any $a \in A$ and $x \in M$.

Example 4.1.1.

(a) *The category of A-modules $\mathcal{M}\mathrm{od}(A)$ has a natural supercategory structure induced by the parity involution ϕ_A; i.e., for $M \in \mathcal{M}\mathrm{od}(A)$,*

$$\Pi M := \{\pi(x) \mid x \in M\} \text{ with } \pi(x) + \pi(x') = \pi(x + x') \text{ and}$$
$$a \cdot \pi(x) := \pi(\phi_A(a) \cdot x) \text{ for } a \in A.$$

The isomorphism $\xi \colon \Pi^2 \to \operatorname{id}$ is given by $\pi(\pi(x)) \mapsto x$.

(b) *Let $\mathscr{M}\mathrm{od}(A)$ be the category of A-supermodules. The morphisms in this category are A-linear homomorphisms which preserve the $\mathbb{Z}_2$-grading. Then $\mathscr{M}\mathrm{od}(A)$ has a natural supercategory structure induced by the parity* *shift; i.e.,*

$$(\Pi M)_\epsilon := \{\pi(x) \mid x \in M_{1-\epsilon}\} \ (\epsilon = 0, 1) \text{ and}$$
$$a \cdot \pi(x) := \pi(\phi_A(a) \cdot x) \text{ for } a \in A \text{ and } x \in M.$$

The isomorphism $\xi_M \colon \Pi^2 M \to M$ *is given by* $\pi(\pi(x)) \mapsto x$ $(x \in M)$.

Definition 4.1.2. *Let* $(\mathscr{C}, \Pi, \xi)$ *be a supercategory. The Grothendieck group* $[\mathscr{C}]$ *of* $\mathscr{C}$ *is the abelian group generated by* $[X]$ *(*X *is an object of* $\mathscr{C}$*) with the defining relations:*

$$\text{if } 0 \to X' \to X \to X'' \to 0 \text{ is an exact sequence, then } [X] = [X'] + [X''].$$

Let A and B be superalgebras. An (A, B)-*superbimodule* is an (A, B)-bimodule with a $\mathbb{Z}_2$-grading compatible with the left action of A and the right action of B. For an (A, B)-superbimodule L, we have a functor $F_L \colon \mathcal{M}\mathrm{od}(B) \to \mathcal{M}\mathrm{od}(A)$ given by $N \mapsto L \otimes_B N$ for $N \in \mathcal{M}\mathrm{od}(B)$. Then F_L is indeed a superfunctor with an isomorphism

$$\alpha_{F_L} \colon F_L \Pi N = L \otimes_B \Pi N \to \Pi F_L N = \Pi(L \otimes_B N)$$

which is given by $s \otimes \pi(x) \mapsto \pi(\phi_L(s) \otimes x)$ $(s \in L,\ x \in N)$.

For an (A, B)-superbimodule L, the superbimodule structure of ΠL is given as follows:

$$a \cdot \pi(s) \cdot b = \pi(\phi_A \cdot (a) s \cdot b) \quad \text{for all } s \in L, a \in A \text{ and } b \in B.$$

Then there exists a natural isomorphism between superfunctors $\eta \colon F_{\Pi L} \xrightarrow{\sim} \Pi \circ F_L$. The isomorphism $\eta_N \colon (\Pi L) \otimes_B N \xrightarrow{\sim} \Pi(L \otimes_B N)$ is given by $\pi(s) \otimes x \mapsto \pi(s \otimes x)$. It is an isomorphism of superfunctors since one can easily check the commutativity of the following diagram:

$$\begin{tikzcd}
F_{\Pi L} \circ \Pi \arrow[r, "\eta \circ \Pi"] \arrow[dd, "\alpha_{F_{\Pi L}}"'] & \Pi \circ F_L \circ \Pi \arrow[dr, "\Pi \circ \alpha_{F_L}"] \arrow[dd, "\alpha_{(\Pi \circ F_L)}"] & \\
& & \Pi \circ \Pi \circ F_L \arrow[dl, "\alpha_\Pi \circ F_L = -\mathrm{id}_{\Pi \circ \Pi \circ F_L}"] \\
\Pi \circ F_{\Pi L} \arrow[r, "\Pi \circ \eta"] & \Pi \circ \Pi \circ F_L. &
\end{tikzcd}$$

by using the fact $\phi_{\Pi L}(\pi(s)) = -\pi(\phi_L(s))$.

4.2 The quiver Hecke superalgebra $\mathcal{R}$

In this section, we recall the definition of quiver Hecke superalgebras and their properties which were proved in [17].

We assume that a decomposition $I = I_{\mathrm{even}} \bigsqcup I_{\mathrm{odd}}$ is given. We say that a Borcherds-Cartan matrix $A = (a_{ij})_{i,j \in I}$ is *colored* by I_{odd} if

$$a_{ij} \in 2\mathbb{Z} \quad \text{for all } i \in I_{\mathrm{odd}} \text{ and } \ j \in I.$$

From now on, we assume that A is colored by I_{odd}. We use the same graded ring $\mathbf{k}$ in Chapter 3.

We define the *parity* function $\mathrm{p}\colon I \to \{0,1\}$ by

$$\mathrm{p}(i) = 1 \quad \text{if } i \in I_{\mathrm{odd}} \quad \text{and} \quad \mathrm{p}(i) = 0 \quad \text{if } i \in I_{\mathrm{even}}.$$

Then we naturally extend the parity function on I^n and Q^+ as follows:

$$\mathrm{p}(\nu) := \sum_{k=1}^{n} \mathrm{p}(\nu_k), \ \mathrm{p}(\beta) := \sum_{k=1}^{r} \mathrm{p}(i_k) \ \text{ for all } \nu \in I^n \text{ and } \beta = \sum_{k=1}^{r} \alpha_{i_k} \in Q^+.$$

For $i \neq j \in I$ and $r, s \in \mathbb{Z}_{\geq 0}$, we take $\mathtt{t}_{i,j;(r,s)} \in \mathbf{k}_{-2(\alpha_i|\alpha_j) - r(\alpha_i|\alpha_i) - s(\alpha_j|\alpha_j)}$ such that

$$\mathtt{t}_{i,j;(-a_{ij},0)} \in \mathbf{k}_0^{\times}, \quad \mathtt{t}_{i,j;(r,s)} = \mathtt{t}_{j,i;(s,r)},$$
$$\mathtt{t}_{i,j;(r,s)} = 0 \quad \text{if } i \in I_{\mathrm{odd}} \text{ and } r \text{ is an odd integer.}$$

For any $\nu \in I^n$ $(n \geq 2)$, let

$$\mathcal{P}_\nu := \mathbf{k}\langle x_1, \ldots, x_n \rangle / \langle x_a x_b - (-1)^{\mathrm{p}(\nu_a)\mathrm{p}(\nu_b)} x_b x_a \rangle_{1 \leq a < b \leq n}$$

be the superalgebra generated by x_k $(1 \leq k \leq n)$ with $\mathbb{Z} \times \mathbb{Z}_2$-degree $((\alpha_{\nu_k}|\alpha_{\nu_k}), \mathrm{p}(\nu_k))$ $(k = 1, \ldots, \mathrm{n})$.
For $i, j \in I$, we choose an element $\mathcal{Q}_{i,j}$ in $\mathcal{P}_{(ij)}$ which is of the form

$$\mathcal{Q}_{i,j}(w,z) = \delta_{i,j} \sum_{r,s \in \mathbb{Z}_{\geq 0}} \mathtt{t}_{i,j;(r,s)} w^r z^s. \tag{4.2}$$

Then $\mathcal{Q}_{i,j}(w,z)$ is an even element and $\mathcal{Q} = (\mathcal{Q}_{i,j})_{i,j \in I}$ satisfies
(4.3)

$$\mathcal{Q}_{i,j}(w,z) = \mathcal{Q}_{j,i}(z,w) \ \text{ and } \mathcal{Q}_{i,j}(w,z) = \mathcal{Q}_{i,j}((-1)^{\mathrm{p}(i)}w, z) \quad \text{for } i, j \in I.$$

Definition 4.2.1 ([17]). *The* quiver Hecke superalgebra $\mathcal{R}(n)$ *of degree* n *associated with the Cartan datum* $(A, P, \Pi, \Pi^\vee)$ *and* $(\mathcal{Q}_{i,j})_{i,j\in I}$ *is the associative superalgebra over* $\mathbf{k}$ *generated by* $e(\nu)$ $(\nu \in I^n)$, x_k $(1 \le k \le n)$, τ_a $(1 \le a \le n-1)$ *with the parity*

$$\mathrm{p}(e(\nu)) = 0, \quad \mathrm{p}(x_k e(\nu)) = \mathrm{p}(\nu_k), \quad \mathrm{p}(\tau_a e(\nu)) = \mathrm{p}(\nu_a)\mathrm{p}(\nu_{a+1}) \tag{4.4}$$

subject to the following defining relations:

($\mathcal{R}$1) $e(\mu)e(\nu) = \delta_{\mu,\nu} e(\nu)$ *for all* $\mu,\ \nu \in I^n$, *and* $1 = \sum_{\nu\in I^n} e(\nu)$,

($\mathcal{R}$2) $x_p x_q e(\nu) = (-1)^{\mathrm{p}(\nu_p)\mathrm{p}(\nu_q)} x_q x_p e(\nu)$ *if* $p \neq q$,

($\mathcal{R}$3) $x_p e(\nu) = e(\nu) x_p$ *and* $\tau_a e(\nu) = e(s_a \nu)\tau_a$, *where* $s_a = (a, a+1)$ *is the transposition on the set of sequences,*

($\mathcal{R}$4) $\tau_a x_p e(\nu) = (-1)^{\mathrm{p}(\nu_p)\mathrm{p}(\nu_a)\mathrm{p}(\nu_{a+1})} x_p \tau_a e(\nu)$, *if* $p \neq a,\ a+1$,

($\mathcal{R}$5)
$$(\tau_a x_{a+1} - (-1)^{\mathrm{p}(\nu_a)\mathrm{p}(\nu_{a+1})} x_a \tau_a) e(\nu) = (x_{a+1}\tau_a - (-1)^{\mathrm{p}(\nu_a)\mathrm{p}(\nu_{a+1})} \tau_a x_a) e(\nu) = \delta_{\nu_a,\nu_{a+1}} e(\nu),$$

($\mathcal{R}$6) $\tau_a^2 e(\nu) = \mathcal{Q}_{\nu_a,\nu_{a+1}}(x_a, x_{a+1}) e(\nu)$,

($\mathcal{R}$7) $\tau_a \tau_b e(\nu) = (-1)^{\mathrm{p}(\nu_a)\mathrm{p}(\nu_{a+1})\mathrm{p}(\nu_b)\mathrm{p}(\nu_{b+1})} \tau_b \tau_a e(\nu)$ *if* $|a-b| > 1$,

($\mathcal{R}$8)
$$(\tau_{a+1}\tau_a\tau_{a+1} - \tau_a\tau_{a+1}\tau_a) e(\nu) = \begin{cases} \dfrac{\mathcal{Q}_{\nu_a,\nu_{a+1}}(x_{a+2}, x_{a+1}) - \mathcal{Q}_{\nu_a,\nu_{a+1}}(x_a, x_{a+1})}{x_{a+2} - x_a} e(\nu) & \text{if } \nu_a = \nu_{a+2} \in I_{\mathrm{even}}, \\ (-1)^{\mathrm{p}(\nu_{a+1})}(x_{a+2} - x_a) \dfrac{\mathcal{Q}_{\nu_a,\nu_{a+1}}(x_{a+2}, x_{a+1}) - \mathcal{Q}_{\nu_a,\nu_{a+1}}(x_a, x_{a+1})}{x_{a+2}^2 - x_a^2} e(\nu) & \\ & \text{if } \nu_a = \nu_{a+2} \in I_{\mathrm{odd}}, \\ 0 & \text{otherwise}. \end{cases}$$

Remark 4.2.1. *If* $I_{\mathrm{odd}} = \emptyset$, *the quiver Hecke superalgebra* $\mathcal{R}(n)$ *is the same as the quiver Hecke algebra* $\mathsf{R}(n)$.

The algebra $\mathcal{R}(n)$ is also $\mathbb{Z}$-graded via the following assignment:

$$\deg_{\mathbb{Z}}(e(\nu)) = 0, \quad \deg_{\mathbb{Z}}(x_k e(\nu)) = (\alpha_{\nu_k}|\alpha_{\nu_k}), \quad \deg_{\mathbb{Z}}(\tau_a e(\nu)) = -(\alpha_{\nu_a}|\alpha_{\nu_{a+1}}).$$

For $a, b \in \{1, \dots, n\}$ with $a \neq b$, we define the elements of $\mathcal{R}(n)$ by

$$e^{\mathrm{ev}}_{a,b} = \sum_{\substack{\nu \in I^n,\\ \nu_a = \nu_b \in I_{\mathrm{even}}}} e(\nu), \quad e^{\mathrm{od}}_{a,b} = \sum_{\substack{\nu \in I^n,\\ \nu_a = \nu_b \in I_{\mathrm{odd}}}} e(\nu) \quad \text{and} \quad e_{a,b} = e^{\mathrm{ev}}_{a,b} + e^{\mathrm{od}}_{a,b}. \tag{4.5}$$

Let $\mathcal{P}^{\mathrm{ev}}_\nu$ be the subalgebra of $\mathcal{P}_\nu$ generated by $x_k^{1+\mathrm{p}(\nu_k)}$ $(1 \leq k \leq n)$. Then $\mathcal{P}^{\mathrm{ev}}_\nu$ is isomorphic to the polynomial ring $\mathbf{k}[x_1^{1+\mathrm{p}(\nu_1)}, \cdots, x_n^{1+\mathrm{p}(\nu_n)}]$. Set

$$\mathcal{P}_n = \bigoplus_{\nu \in I^n} \mathcal{P}_\nu e(\nu) \quad \text{and} \quad \mathcal{P}^{\mathrm{ev}}_n = \bigoplus_{\nu \in I^n} \mathcal{P}^{\mathrm{ev}}_\nu e(\nu).$$

Then $\mathcal{P}^{\mathrm{ev}}_n$ is contained in the center of $\mathcal{P}_n$.

By (4.3),

$$\mathcal{Q}_{\nu_a,\nu_{a+1}}(x_a, x_{a+1})e(\nu) \text{ belongs to } \mathcal{P}^{\mathrm{ev}}_n \text{ for all } \nu \in I^n \text{ and } 1 \leq a < n. \tag{4.6}$$

For $1 \leq k < n$, we define the algebra endomorphism $\overline{s}_k$ of $\mathcal{P}_n$ by

$$\overline{s}_k(x_p e(\nu)) = (-1)^{\mathrm{p}(\nu_k)\mathrm{p}(\nu_{k+1})\mathrm{p}(\nu_p)} x_{s_k(p)} e(s_k \nu) \quad \text{for } 1 \leq p \leq n, \tag{4.7}$$

where $s_k = (k, k+1) \in S_n$ is the transposition which acts on I^n in a natural way. For $f \in \mathcal{P}_n$ and $1 \leq k < n$, define

$$\begin{aligned} \overline{\partial}_k f &= \frac{f - \overline{s}_k f}{x_{k+1} - x_k} e^{\mathrm{ev}}_{k,k+1} + \frac{(x_{k+1} - x_k) f - \overline{s}_k f (x_{k+1} - x_k)}{x_{k+1}^2 - x_k^2} e^{\mathrm{od}}_{k,k+1}, \\ f^{\overline{\partial}_k} &= \frac{f - \overline{s}_k f}{x_{k+1} - x_k} e^{\mathrm{ev}}_{k,k+1} + \frac{f(x_{k+1} - x_k) - (x_{k+1} - x_k)(\overline{s}_k f)}{x_{k+1}^2 - x_k^2} e^{\mathrm{od}}_{k,k+1}. \end{aligned} \tag{4.8}$$

Then one can easily show that

$$\overline{\partial}_k f,\ f^{\overline{\partial}_k} \in \mathcal{P}_n, \quad \tau_k f = (\overline{s}_k f)\tau_k + \overline{\partial}_k f, \quad f\tau_k = \tau_k(\overline{s}_k f) + f^{\overline{\partial}_k} \tag{4.9}$$

and

$$\begin{aligned} &\overline{\partial}_k(x_j) = (x_j)^{\overline{\partial}_k} = \delta_{j,k+1} e^{\mathrm{ev}}_{k,k+1} - \delta_{j,k} e^{\mathrm{ev}}_{k,k+1} + \delta_{j,k+1} e^{\mathrm{od}}_{k,k+1} + \delta_{j,k} e^{\mathrm{od}}_{k,k+1}, \\ &\overline{\partial}_k(fg) = (\overline{\partial}_k f)g + (\overline{s}_k f)\overline{\partial}_k g, \quad (fg)^{\overline{\partial}_k} = f(g^{\overline{\partial}_k}) + (f^{\overline{\partial}_k})\overline{s}_k g. \end{aligned}$$

As in the case of the quiver Hecke algebras, we define

$$\begin{aligned}
&\mathcal{R}(m,n) = \mathcal{R}(m) \otimes_{\mathbf{k}} \mathcal{R}(n) \subset \mathcal{R}(m+n),\\
&e(n) = \sum_{\nu \in I^n} e(\nu), \quad e(\beta) = \sum_{\nu \in I^\beta} e(\nu), \quad e(\alpha,\beta) = \sum_{\mu \in I^\alpha,\ \nu \in I^\beta} e(\mu,\nu),\\
&\mathcal{R}(\beta) = e(\beta)\mathcal{R}(n), \quad \mathcal{R}(\alpha,\beta) = \mathcal{R}(\alpha) \otimes_{\mathbf{k}} \mathcal{R}(\beta) \subset \mathcal{R}(\alpha+\beta),\\
&e(n,i^k) = \sum_{\substack{\nu \in I^{n+k},\\ \nu_{n+1}=\cdots=\nu_{n+k}=i}} e(\nu), \quad e(i^k,n) = \sum_{\substack{\nu \in I^{n+k},\\ \nu_1=\cdots=\nu_k=i}} e(\nu),\\
&e(\beta,i^k) = e(\beta,k\alpha_i) = e(\beta+k\alpha_i)e(n,i^k),\\
&e(i^k,\beta) = e(k\alpha_i,\beta) = e(\beta+k\alpha_i)e(i^k,n)
\end{aligned}$$

for $\alpha, \beta \in Q^+$.

Proposition 4.2.1 ([17, Corollary 3.15])**.** *For each $w \in S_n$, we choose a reduced expression $s_{i_1} \cdots s_{i_\ell}$ of w and write $\tau_w = \tau_{i_1} \cdots \tau_{i_\ell}$. Then*

$$\{x_1^{a_1} \cdots x_n^{a_n} \tau_w e(\nu) \mid a = (a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n,\ w \in S_n,\ \nu \in I^n\}$$

forms a basis of the $\mathbf{k}$-module $\mathcal{R}(n)$.

By the proposition above we have:

Lemma 4.2.1. *The algebra $\mathcal{R}(n+1)$ has a direct sum decomposition*

$$\mathcal{R}(n+1) = \bigoplus_{a=1}^{n+1} \mathcal{R}(n,1)\tau_n \cdots \tau_a = \bigoplus_{a=1}^{n+1} \mathcal{R}(n) \otimes \mathbf{k}^I[x_{n+1}]\tau_n \cdots \tau_a. \tag{4.10}$$

In particular, $\mathcal{R}(n+1)$ is a free $\mathcal{R}(n,1)$-module of rank $n+1$.

Let $\mathcal{M}\mathrm{od}(\mathcal{R}(\beta))$ (resp. $\mathcal{P}\mathrm{roj}(\mathcal{R}(\beta))$, $\mathcal{R}\mathrm{ep}(\mathcal{R}(\beta))$) be the category of arbitrary (resp. finitely generated projective, finite dimensional over $\mathbf{k}_0$) $\mathbb{Z}$-graded $\mathcal{R}(\beta)$-modules. The morphisms in these categories are $\mathcal{R}(\beta)$-homomorphisms which are homogeneous with respect to the $\mathbb{Z}$-grading. By the observation in Example 4.1.1 (a), these categories have a supercategory structure induced by the parity involution $\phi := \phi_{\mathcal{R}(\beta)}$.

Let π be an *odd* element with the defining equation $\pi^2 = 1$. For any superring $\mathcal{K}$, we define

$$\mathcal{K}^\pi := \mathcal{K} \otimes_{\mathbb{Z}} \mathbb{Z}[\pi] \simeq \mathcal{K} \oplus \mathcal{K}\pi.$$

Thus $\mathbb{A}^\pi = \mathbb{Z}[q, q^{-1}, \pi]$ with $\pi^2 = 1$. We denote by $[\mathcal{P}\mathrm{roj}(\mathcal{R}(\beta))]$ and $[\mathcal{R}\mathrm{ep}(\mathcal{R}(\beta))]$ the Grothendieck group of $\mathcal{P}\mathrm{roj}(\mathcal{R}(\beta))$ and $\mathcal{R}\mathrm{ep}(\mathcal{R}(\beta))$, respectively (See Definition 4.1.2). Then $[\mathcal{P}\mathrm{roj}(\mathcal{R}(\beta))]$ and $[\mathcal{R}\mathrm{ep}(\mathcal{R}(\beta))]$ have the $\mathbb{A}^\pi$-module structure given by $q[M] = [qM]$ and $\pi[M] = [\Pi M]$, where $[M]$ is the isomorphism classes of an $\mathcal{R}(\beta)$-module M.

Hereafter, *an $\mathcal{R}(n)$-module always means a $\mathbb{Z}$-graded $\mathcal{R}(n)$-module.*

In a manner similar to (3.25), we can define

$${}_{\alpha,\beta}\mathcal{R}_{\alpha',\beta'} := e(\alpha,\beta)\mathcal{R}(\alpha+\beta)e(\alpha',\beta')$$

and we have Mackey's Theorem for quiver Hecke superalgebras as follows:

Proposition 4.2.2. *The $\mathbb{Z}\times\mathbb{Z}_2$-graded $(\mathcal{R}(\alpha,\beta),\mathcal{R}(\alpha',\beta'))$-superbimodule ${}_{\alpha,\beta}\mathcal{R}_{\alpha',\beta'}$ has a graded filtration with graded subquotients isomorphic to*

$$\Pi^{\mathrm{p}(\gamma)\mathrm{p}(\beta+\gamma-\beta')}({}_{\alpha}\mathcal{R}_{\alpha-\gamma,\gamma})\otimes({}_{\beta}\mathcal{R}_{\beta+\gamma-\beta',\beta'-\gamma})\otimes_{\mathcal{R}'}({}_{\alpha-\gamma,\alpha'+\gamma-\alpha}\mathcal{R}_{\alpha'})\otimes({}_{\gamma,\beta'-\gamma}\mathcal{R}_{\beta'})\langle-(\gamma|\beta+\gamma-\beta')\rangle,$$

where $\mathcal{R}' = \mathcal{R}(\alpha-\gamma)\otimes\mathcal{R}(\gamma)\otimes\mathcal{R}(\beta+\gamma-\beta')\otimes\mathcal{R}(\beta'-\gamma)$ and γ ranges over the set of $\gamma\in Q^+$ such that $\alpha-\gamma$, $\beta'-\gamma$ and $\beta+\gamma-\beta' = \alpha'+\gamma-\alpha$ belong to Q^+.

4.3 Strong perfect basis of $[\mathcal{R}\mathrm{ep}(\mathcal{R}(\beta))]$

In this section, we study the structure of $\mathcal{R}(n\alpha_i)$ as an $\mathcal{R}(n\alpha_i)$-supermodule and choose a set of irreducible $\mathcal{R}(\beta)$-modules which provides a strong perfect basis of $[\mathcal{R}\mathrm{ep}(\mathcal{R}(\beta))]$ as in Section 3.2 and Section 3.5, respectively. Although, the quiver Hecke superalgebras are different from the quiver Hecke algebras, we can apply many results in those sections. However, since the supercategory $[\mathcal{M}\mathrm{od}(\mathcal{R}(\beta))]$ has the endofunctor Π, we need to investigate how the endofunctor Π affects the structure of $\mathcal{R}(n\alpha_i)$ and $[\mathcal{R}\mathrm{ep}(\mathcal{R}(\beta))]$.

Throughout this section, we assume that (3.9) holds; i.e.,

$\mathbf{k}_0$ is a field and the $\mathbf{k}_t$'s are finite-dimensional over $\mathbf{k}_0$.

Under this assumption, the quiver Hecke superalgebra $\mathcal{R}(\beta)$ satisfies the properties in (3.10); i.e., as a $\mathbb{Z}$-graded algebra, the $\mathbb{Z}$-grading of $\mathcal{R}(\beta)$ is bounded below and each $\mathbb{Z}$-homogeneous subspace is finite dimensional over $\mathbf{k}_0$. Hence $\mathcal{R}(\beta)$ holds the properties in (3.11). In other words,

(4.11)

(i) there exists a $1-1$ correspondence between the projective indecomposable modules in $\mathcal{P}\mathrm{roj}(\mathcal{R}(\beta))$ and the simple modules in $\mathcal{R}\mathrm{ep}(\mathcal{R}(\beta))$,

(ii) $[\mathcal{R}\mathrm{ep}(\mathcal{R}(\beta))]$ is a finite dimensional $\mathbb{A}^\pi$-vector space and has a natural $\mathbb{Z}$-basis $\mathcal{I}\mathrm{rr}_q(\mathcal{R}(\beta))$, consisting of the isomorphism classes of simple $\mathcal{R}(\beta)$-modules.

We now consider the algebra $\mathcal{R}(n\alpha_i)$. By Remark 4.2.1, it suffices to assume that $i \in I_{\mathrm{odd}}$.

For $1 \le k < n$, we can check that the element $\mathbf{b}(i^n)$ defined in the same way as (3.14) also is an idempotent.

Set

$$\Pi_i = \Pi^{\mathrm{p}(i)}, \qquad \pi_i := \pi^{\mathrm{p}(i)} \tag{4.12}$$

and define

$$[n]_i^\pi = \frac{(\pi_i q_i)^n - q_i^{-n}}{\pi_i q_i - q_i^{-1}}, \quad [n]_i^\pi! = \prod_{k=1}^n [k]_i^\pi. \tag{4.13}$$

Proposition 4.3.1. *[8] The algebra $R(n\alpha_i)$ decomposes into the direct sum of projective indecomposable $\mathbb{Z}\times\mathbb{Z}_2$-modules as follows:*

$$\mathcal{R}(n\alpha_i) \simeq [n]_i^\pi! \mathcal{P}(i^n), \tag{4.14}$$

where

$$\mathcal{P}(i^n) := \Pi_i^{\frac{n(n-1)}{2}} \mathcal{R}(n\alpha_i)\mathbf{b}(i^n) \left\langle \frac{n(n-1)}{4}(\alpha_i|\alpha_i) \right\rangle. \tag{4.15}$$

Similar to Section 3.2 we have

- $\mathcal{P}(i^n)$ is an indecomposable projective $\mathbb{Z}\times\mathbb{Z}_2$-graded module unique up to isomorphism and $\mathbb{Z}\times\mathbb{Z}_2$-grading shift.
- there exists an irreducible $\mathbb{Z}\times\mathbb{Z}_2$-graded $R(n\alpha_i)$-module $\mathcal{L}(i^n)$ which is unique up to isomorphism and $\mathbb{Z}\times\mathbb{Z}_2$-grading shift:

$$\mathcal{L}(i^n) := \mathrm{Ind}_{\mathbf{k}[x_1]\otimes\cdots\otimes\mathbf{k}[x_n]}^{\mathcal{R}(n\alpha_i)}\mathbf{1}, \tag{4.16}$$

  which is isomorphic to $\mathbf{k}_0$.

By Proposition 4.2.1, the $\mathcal{R}(n\alpha_i)$-module $\mathcal{L}(i^n)$ has a $\mathbf{k}_0$-basis

$$\{\tau_w \cdot u(i^n) \mid \ w \in S_n\}.$$

As in the Section 3.2, set

$$\mathcal{L}_k := \{v \in \mathcal{L}(i^n) \mid x_n^k \cdot v = 0\} \quad (k \geq 0).$$

Then we have a supermodule isomorphism

$$\mathcal{L}_k/\mathcal{L}_{k-1} \simeq \Pi_i^{k-1}\mathcal{L}(i^{n-1})\,\langle(1-k)(\alpha_i|\alpha_i)\rangle \qquad \text{for } 1 \leq k \leq n. \tag{4.17}$$

Here the $\mathbb{Z}\times\mathbb{Z}_2$-grading shift is caused by the $(\mathbb{Z}\times\mathbb{Z}_2)$-degree of $\tau_{n-1}\cdots\tau_{n-k+1}u(i^n)$.

Remark 4.3.1. *In general, τ_w depends on the choice of reduced expressions of w. However, we still write τ_w after choosing a reduced expression of w. In $I=\{i\}$ case, by the axioms in Definition 4.2.1, $\pm\tau_w$ does not depend on the choice of reduced expressions of $w\in S_n$; i.e., for any two reduced expressions $w = s_{i_1}\cdots s_{i_r} = s_{j_1}\cdots s_{j_r}$, we have*

$$\tau_{i_1}\cdots\tau_{i_r} = \pm\tau_{j_1}\cdots\tau_{j_r}.$$

Lemma 4.3.1. *Let $w[1,n]$ be the longest element of S_n. Then we have*

$$\mathcal{R}(n\alpha_i)\tau_{w[1,n]}\mathcal{R}(n\alpha_i) = \mathcal{R}(n\alpha_i).$$

Proof. We have

$$\tau_{w[1,n]} = \tau_{w[1,n-1]}\tau_{n-1}\tau_{n-2}\cdots\tau_1.$$

Hence by induction, it is enough to show that for $1 \le a \le n-1$

$$\tau_{w[1,n-1]}\tau_{n-1}\tau_{n-2}\cdots\tau_{a+1} \in \mathcal{R}(n\alpha_i)\tau_{w[1,n-1]}\tau_{n-1}\tau_{n-2}\cdots\tau_a\mathcal{R}(n\alpha_i). \tag{4.18}$$

Note that

$$\begin{aligned}
x_n\tau_{w[1,n-1]}\tau_{n-1}\tau_{n-2}\cdots\tau_a &= \pm\tau_{w[1,n-1]}x_n\tau_{n-1}\tau_{n-2}\cdots\tau_a \\
&= \pm\tau_{w[1,n-1]}(\pm\tau_{n-1}x_{n-1}+1)\tau_{n-2}\cdots\tau_a \\
&= \pm\tau_{w[1,n-1]}\tau_{n-1}x_{n-1}\tau_{n-2}\cdots\tau_a \\
&\quad\vdots \\
&= \pm\tau_{w[1,n-1]}\tau_{n-1}\tau_{n-2}\cdots\tau_{a+1}x_{a+1}\tau_a \\
&= \pm\tau_{w[1,n-1]}\tau_{n-1}\tau_{n-2}\cdots\tau_{a+1}(\pm\tau_a x_a+1).
\end{aligned}$$

Thus we have (4.18) and our assertion follows. □

Now, we will choose the strong perfect basis of $[\mathcal{R}\mathrm{ep}(\mathcal{R}(\beta))]$. To do this, we need to employ many results in Section 3.5. However, in the quiver Hecke superalgebra case, we can apply the arguments with slight modification. Thus we will focus on the perfect basis property arising from the set of irreducible $\mathcal{R}(\beta)$-modules ($\beta \in Q^+$) by using the results in Section 3.5 as ingredients.

For $M \in \mathcal{R}\mathrm{ep}(\mathcal{R}(\beta))$ and $i \in I$, we define,

$$\begin{aligned}
&\Delta_{i^k}M = e(\beta-k\alpha_i, i^k)M \in \mathcal{R}\mathrm{ep}(\mathcal{R}(\beta-k\alpha_i, k\alpha_i)),\\
&\varepsilon_i(M) = \max\{k \ge 0 \mid \Delta_{i^k}M \ne 0\},\\
&E_i(M) = e(\beta-\alpha_i, i)M \in \mathcal{R}\mathrm{ep}(\mathcal{R}(\beta-\alpha_i)),\\
&F_i'(M) = \mathrm{Ind}_{\beta,\alpha_i}(M \boxtimes \mathcal{L}(i)) \in \mathcal{R}\mathrm{ep}(\mathcal{R}(\beta+\alpha_i)),\\
&\tilde{e}_i(M) = \mathrm{soc}(E_i(M)) \in \mathcal{R}\mathrm{ep}(\mathcal{R}(\beta-\alpha_i)),\\
&\tilde{f}_i(M) = \mathrm{hd}(F_i'M) \in \mathcal{R}\mathrm{ep}(\mathcal{R}(\beta+\alpha_i)).
\end{aligned} \tag{4.19}$$

These definitions are almost the same as (3.29). But we drop the superscript $^{\mathrm{or}}$ for ε_i, since $I = I^{\mathrm{re}}$ in this chapter.

Then we have the following statements: (cf. Lemma 3.5.4, Lemma 3.5.6, Lemma 3.5.7, Lemma 3.5.8, Lemma 3.5.10, Lemma 3.5.12)

(I1) For $[M] \in \mathrm{Irr}_q(\mathcal{R}\mathrm{ep}(\mathcal{R}(\beta)))$ with $\varepsilon = \varepsilon_i(M) > 0$,

$\Delta_{i^\varepsilon} M \simeq N \boxtimes \mathcal{L}(i^\varepsilon)$ for some $[N] \in \mathrm{Irr}_q(\mathcal{R}\mathrm{ep}(\mathcal{R}(\beta - \varepsilon\alpha_i)))$ with $\varepsilon_i(N) = 0$.

Moreover, $[N] = [\tilde{e}_i^\varepsilon(M)]$.

(I2) For $[M] \in \mathrm{Irr}_q(\mathcal{R}\mathrm{ep}(\mathcal{R}(\beta)))$,

$[\tilde{f}_i M] \in \mathrm{Irr}_q(\mathcal{R}\mathrm{ep}(\mathcal{R}(\beta + \alpha_i)))$ and $[\tilde{e}_i M] \in \mathrm{Irr}_q(\mathcal{R}\mathrm{ep}(\mathcal{R}(\beta - \alpha_i)))$ if $\varepsilon_i(M) \geq 0$.

(I3) For $[M] \in [\mathcal{R}\mathrm{ep}(R(\beta))]$ with $[E_i^k][M] = 0$,

$$[M] = \sum_k [N_k] \quad \text{for } [N_k] \in \mathrm{Irr}_q(\mathcal{R}\mathrm{ep}(\mathcal{R}(\beta))) \text{ with } \varepsilon_i(M) < k.$$

(I4) For $[M] \in \mathrm{Irr}_q(\mathcal{R}\mathrm{ep}(\mathcal{R}(\beta)))$,

$$[\tilde{e}_i \tilde{f}_i M] = [M] \quad \text{and} \quad [\tilde{f}_i \tilde{e}_i M] = [M] \quad \text{if } \varepsilon_i(M) \geq 0.$$

(I5) For $[M] \in \mathrm{Irr}_q(\mathcal{R}\mathrm{ep}(\mathcal{R}(\beta)))$,

$$\mathbf{k}_0 \simeq \mathrm{End}_{\mathcal{R}(\beta)}(M).$$

Now we are ready to prove the following fundamental result on irreducible modules over quiver Hecke superalgebras.

Theorem 4.3.1. Π *acts as the identity on* $[\mathcal{R}\mathrm{ep}(R(\beta))]$ *and* $[\mathcal{P}\mathrm{roj}(R(\beta))]$. *Hence* $[\mathcal{R}\mathrm{ep}(R(\beta))]$ *and* $[\mathcal{P}\mathrm{roj}(R(\beta))]$ *are indeed* $\mathbb{A}$*-modules.*

Proof. We shall prove

$$\Pi M \simeq M \quad \text{for } M \in \mathcal{I}\mathrm{rr}_q(R(\beta))$$

by induction on $|\beta|$. If $|\beta| > 0$, there exists $i \in I$ such that $\varepsilon_i(M) > 0$. Since the endofunctor Π commutes with the functor E_i

$$\tilde{e}_i(\Pi M) \simeq (\Pi \tilde{e}_i M).$$

By induction hypothesis, $\Pi \tilde{e}_i(M) \simeq \tilde{e}_i M$. Hence we obtain

$$\tilde{e}_i M \simeq \tilde{e}_i \Pi M.$$

Then $\Pi M \simeq M$ follows from (I4). The second assertion follows from (4.11)(i). $\square$

Now, we have the same result as in Proposition 3.5.1 for the quiver Hecke superalgebras.

Proposition 4.3.2. *For $[M] \in \mathrm{Irr}_q(\mathcal{R}(\beta))$ with $\varepsilon := \varepsilon_i(M) > 0$. Then we have*

$$[E_i][M] = q_i^{1-\varepsilon}[\varepsilon]_i[\tilde{e}_i M] + \sum_k [N_k],$$

where $N_k \in \mathrm{Irr}_q(\mathcal{R}(\beta - \alpha_i))$ with $\varepsilon_i(N_k) < \varepsilon_i(\tilde{e}_i M) = \varepsilon - 1$.

Proof. By applying (I1) to the irreducible modules M and $\tilde{e}_i M$, we have

$$\Delta_{i^\varepsilon} M \simeq \tilde{e}_i^\varepsilon M \boxtimes \mathcal{L}(i^\varepsilon) \quad \text{and} \quad \Delta_{i^{\varepsilon-1}} \tilde{e}_i M \simeq \tilde{e}_i^\varepsilon M \boxtimes \mathcal{L}(i^{\varepsilon-1}).$$

On the other hand, (4.17),

$$[E_i][\mathcal{L}(i^\varepsilon)] = \pi_i^{1-\varepsilon} q_i^{1-\varepsilon}[\varepsilon]_i^\pi [\mathcal{L}(i^{\varepsilon-1})] \in [\mathcal{R}\mathrm{ep}(\mathcal{R}((\varepsilon-1)\alpha_i))].$$

Hence we have

$$[E_i^{\varepsilon-1}]\big([E_i M] - \pi_i^{1-\varepsilon} q_i^{1-\varepsilon}[\varepsilon]_i^\pi [\tilde{e}_i M]\big) = 0.$$

Then, from (I1) and Theorem 4.3.1, we can conclude that

$$[E_i][M] = q_i^{1-\varepsilon}[\varepsilon]_i[\tilde{e}_i M] + \sum_k [N_k].$$

□

Let $\psi \colon \mathcal{R}(\beta) \to \mathcal{R}(\beta)$ be the involution given by

$$\psi(ab) = \psi(b)\psi(a), \quad \psi(e(\nu)) = e(\nu), \quad \psi(x_k) = x_k, \quad \psi(\tau_l) = \tau_l, \tag{4.20}$$

for all $a, b \in \mathcal{R}(\beta)$.

Set

$$\mathcal{R}\mathrm{ep}(\mathcal{R}) = \bigoplus_{\beta \in Q^+} \mathcal{R}\mathrm{ep}(\mathcal{R}(\beta)) \text{ and } [\mathcal{R}\mathrm{ep}(\mathcal{R})] = \bigoplus_{\beta \in Q^+} [\mathcal{R}\mathrm{ep}(\mathcal{R}(\beta))].$$

Analogous to the definitions in Section 3.5, we can define

$$M^* = \mathrm{Hom}_{\mathbf{k}_0}(M, \mathbf{k}_0) \text{ and } M \text{ a self-dual } \mathcal{R}(\beta)\text{-module if } M^* \simeq M.$$

Moreover, we have the following theorem:

Theorem 4.3.2. *For $\beta \in Q^+$, we let $\mathcal{I}\mathrm{rr}_0(\mathcal{R}(\beta))$ be the set of isomorphism classes of self-dual irreducible $\mathcal{R}(\beta)$-modules. Moreover,*

$$\mathcal{I}\mathrm{rr}_0(\mathcal{R}) := \bigsqcup_{\beta \in Q^+} \mathcal{I}\mathrm{rr}_0(\mathcal{R}(\beta))$$

is a strong perfect basis of $[\mathcal{R}\mathrm{ep}(\mathcal{R})]$; i.e., for $[M] \in \mathcal{I}\mathrm{rr}_0(\mathcal{R}(\beta))$, there exists a unique $[\tilde{\mathsf{e}}_i M] \in \mathcal{I}\mathrm{rr}_0(\mathcal{R}(\beta - \alpha_i))$ such that

$$[E_i][M] = [\varepsilon_i(M)]_i[\tilde{\mathsf{e}}_i M] + \sum_k [N_k],$$

where N_k are irreducible modules in $\mathcal{R}\mathrm{ep}(\mathcal{R}(\beta - \alpha_i))$ with $\varepsilon_i(N_k) < \varepsilon_i(M) - 1$.

Proof. We can apply the same arguments given in Lemma 3.5.11 and Proposition 3.5.2. □

By Theorem 4.3.1 and the argument in Lemma 3.5.13, we have

$$[E_i][F'_j] = q^{-(\alpha_i|\alpha_j)}\Pi^{\mathrm{p}(i)\mathrm{p}(j)}[F'_j][E_i] + \delta_{ij}\mathrm{Id} \in \mathrm{End}_{\mathbb{A}}([\mathcal{R}\mathrm{ep}(\mathcal{R})]). \tag{4.21}$$

4.4 The superfunctors E_i, F_i and $\overline{F}_i$

In this section, we will construct super-analogues of the functors in Section 3.6. The ideas of all proofs in this section originated from Section 3.6. But we need to apply the arguments given in the section very carefully, since $\mathcal{R}(\beta)$ is a superalgebra and hence the $\mathbb{Z}_2$-grading should be considered in each computation. To avoid repetition, we will give only the ingredients of the proofs.

Let

$$\begin{aligned} E_i &: \mathcal{M}\mathrm{od}(\mathcal{R}(\beta + \alpha_i)) \to \mathcal{M}\mathrm{od}(\mathcal{R}(\beta)), \\ F_i &: \mathcal{M}\mathrm{od}(\mathcal{R}(\beta)) \to \mathcal{M}\mathrm{od}(\mathcal{R}(\beta + \alpha_i)) \end{aligned}$$

be the superfunctors given by

$$\begin{aligned} E_i(N) &= e(\beta, i)N \simeq e(\beta, i)R(\beta + \alpha_i) \otimes_{R(\beta+\alpha_i)} N \\ &\simeq \mathrm{Hom}_{R(\beta+\alpha_i)}(R(\beta + \alpha_i)e(\beta, i), N), \\ F_i(M) &= R(\beta + \alpha_i)e(\beta, i) \otimes_{R(\beta)} M \end{aligned}$$

for $N \in \mathcal{M}\mathrm{od}(\mathcal{R}(\beta+\alpha_i))$ and $M \in \mathcal{M}\mathrm{od}(\mathcal{R}(\beta))$.

Note that the definitions of E_i and F_i are essentially the same as in the Section 3.6. But, in this case, the kernels of E_i (resp. F_i) have natural $(\mathcal{R}(\beta), \mathcal{R}(\beta+\alpha_i))$-superbimodule (resp. $(\mathcal{R}(\beta+\alpha_i), \mathcal{R}(\beta))$-superbimodule) structure. Hence by Section 4.1, they become superfunctors.

Moreover, (F_i, E_i) is an adjoint pair; i.e.,

$$\mathrm{Hom}_{\mathcal{R}(\beta+\alpha_i)}(F_iM, N) \simeq \mathrm{Hom}_{\mathcal{R}(\beta)}(M, E_iN).$$

Let $n = |\beta|$. There are natural transformations:

$$\begin{aligned} &x_{E_i}\colon E_i \to \Pi_i q_i^{-2} E_i, && x_{F_i}\colon F_i \to \Pi_i q_i^{-2} F_i, \\ &\tau_{E_{ij}}\colon E_iE_j \to \Pi^{\mathrm{p}(i)\mathrm{p}(j)} q^{(\alpha_i|\alpha_j)} E_jE_i, && \tau_{F_{ij}}\colon F_iF_j \to \Pi^{\mathrm{p}(i)\mathrm{p}(j)} q^{(\alpha_i|\alpha_j)} F_jF_i \end{aligned}$$

induced by

(a) the left multiplication by x_{n+1} on $e(\beta, i)N$ for $N \in \mathcal{M}\mathrm{od}(\mathcal{R}(\beta+\alpha_i))$,

(b) the right multiplication by x_{n+1} on the kernel $\mathcal{R}(\beta+\alpha_i)e(\beta, i)$ of the functor F_i,

(c) the left multiplication by τ_{n+1} on $e(\beta, i, j)N$ for $N \in \mathcal{M}\mathrm{od}(\mathcal{R}(\beta+\alpha_i+\alpha_j))$,

(d) the right multiplication by τ_{n+1} on the kernel $\mathcal{R}(\beta+\alpha_i+\alpha_j)e(\beta, j, i)$ of the functor F_iF_j.

By the adjoint property, $\tau_{E_{ij}}$ induces a natural transformation

$$F_jE_i \to \Pi^{\mathrm{p}(i)\mathrm{p}(j)} q^{(\alpha_i|\alpha_j)} E_iF_j.$$

Theorem 4.4.1. *The homomorphism of* $(\mathcal{R}(n), \mathcal{R}(n-1))$*-superbimodules*

$$\mathcal{R}(n)e(n-1, j) \underset{\mathcal{R}(n-1)}{\otimes} q^{-(\alpha_i|\alpha_j)} \Pi^{\mathrm{p}(i)\mathrm{p}(j)} e(n-1, i)\mathcal{R}(n) \longrightarrow e(n, i)\mathcal{R}(n+1)e(n, j)$$

given by

$$x \otimes \pi^{\mathrm{p}(i)\mathrm{p}(j)} y \longmapsto x\tau_n y, \qquad x \in \mathcal{R}(n)e(n-1, j),\ y \in e(n-1, i)\mathcal{R}(n) \tag{4.22}$$

induces a natural isomorphism between superfunctors

$$E_iF_j \xrightarrow{\sim} q^{-(\alpha_i|\alpha_j)}F_j\Pi^{\mathrm{p}(i)\mathrm{p}(j)}E_i \oplus \delta_{i,j}\mathbf{k}[t_i]\otimes \mathrm{Id}.$$

Here, t_i is an indeterminate of $(\mathbb{Z}\times\mathbb{Z}_2)$-degree $\big((\alpha_i|\alpha_i),\mathrm{p}(i)\big)$ and

$$\mathbf{k}[t_i]\otimes \mathrm{Id}\colon \mathcal{M}\mathrm{od}(\mathcal{R}(\beta)) \to \mathcal{M}\mathrm{od}(\mathcal{R}(\beta))$$

is the superfunctor defined by $M \mapsto \mathbf{k}[t_i]\otimes M$.

Proof. We can apply the same arguments given in Proposition 3.6.1 and Theorem 4.4.1. Note that, in this case, the endofunctor $\Pi^{\mathrm{p}(i)\mathrm{p}(j)}$ arises from the $\mathbb{Z}_2$-grading of $\tau_n e(n-1,i,j)$. □

Remark 4.4.1. *For an $R(\beta)$-module M, the $R(\beta)$-module structure on $\mathbf{k}[t_i]\otimes M$ is given by*

$$a(t_i^k\otimes s) = t_i^k\otimes \phi^{k\mathrm{p}(i)}(a)s \quad \text{for } a\in R(\beta),\ s\in M,$$

where $\phi := \phi_{\mathcal{R}(\beta)}$ in Example 4.1.1(a). Thus we have an isomorphism of functors

$$\mathbf{k}[t_i]\otimes \mathrm{Id} \simeq \bigoplus_{k\geq 0}(q^{(\alpha_i|\alpha_i)}\Pi_i)^k.$$

Recall the map ξ_n in (3.35). We can define a map, also denoted by ξ_n, from $\mathcal{R}(n)$ to $\mathcal{R}(n+1)$ in a similar way and denoted by $\mathcal{R}^1(n)$ the image of ξ_n.

For each $i\in I$ and $\beta\in Q^+$, let $\overline{\mathcal{F}}_{i,\beta} := \mathcal{R}(\beta+\alpha_i)v(i,\beta)$ be the $\mathcal{R}(\beta+\alpha_i)$-supermodule generated by $v(i,\beta)$ of $\mathbb{Z}\times\mathbb{Z}_2$-degree $(0,0)$ with the defining relation $e(i,\beta)v(i,\beta) = v(i,\beta)$. The supermodule $\overline{\mathcal{F}}_{i,\beta}$ has an $(\mathcal{R}(\beta+\alpha_i),\mathcal{R}(\beta))$-superbimodule structure whose right $\mathcal{R}(\beta)$-action is given by

$$av(i,\beta)\cdot b = a\xi_n(b)v(i,\beta) \quad \text{for } a\in\mathcal{R}(\beta+\alpha_i) \text{ and } b\in R(\beta).$$

Then following Section 4.1, we can define the $(\mathcal{R}(n+1),\mathcal{R}(n))$-superbimodule $\mathcal{R}(n+1)v(1,n)$ such that

$$\mathcal{R}(n+1)v(1,n) \simeq \bigoplus_{i\in I,\ |\beta|=n}\mathcal{R}(\beta+\alpha_i)v(i,\beta).$$

Now, for each $i \in I$, we define the superfunctor

$$\overline{F}_i\colon \mathcal{M}\mathrm{od}(R(\beta)) \to \mathcal{M}\mathrm{od}(R(\beta+\alpha_i)) \text{ by } N \mapsto \overline{\mathcal{F}}_{i,\beta} \otimes_{R(\beta)} N.$$

By a direct calculation, for $1 \le k \le n$, $1 \le \ell \le n-1$ and $\nu \in I^\beta$, we can easily see that

$$\begin{aligned}
&x_k e(\nu,i)\tau_n\cdots\tau_1 e(i,\nu) \equiv (-1)^{\mathrm{p}(i)\mathrm{p}(\beta)\mathrm{p}(\nu_k)}\tau_n\cdots\tau_1 x_{k+1}e(i,\nu),\\
&\tau_\ell e(\nu,i)\tau_n\cdots\tau_1 e(i,\nu) \equiv (-1)^{\mathrm{p}(i)\mathrm{p}(\beta)\mathrm{p}(\nu_\ell)\mathrm{p}(\nu_{\ell+1})}\tau_n\cdots\tau_1 \tau_{\ell+1}e(i,\nu),\\
&x_{n+1} e(\nu,i)\tau_n\cdots\tau_1 e(i,\nu) \equiv (-1)^{\mathrm{p}(i)\mathrm{p}(\beta)}\tau_n\cdots\tau_1 x_1 e(i,\nu) \mod \mathcal{R}(n)\mathcal{R}^1(n).
\end{aligned}$$

Note that

$$\begin{aligned}
&\mathrm{p}(\tau_n\cdots\tau_1 e(i,\nu)) = \mathrm{p}(i)\mathrm{p}(\beta), \quad \mathrm{p}(x_k e(\nu,i)) = \mathrm{p}(\nu_k),\\
&\mathrm{p}(\tau_\ell e(\nu,i)) = \mathrm{p}(\nu_\ell)\mathrm{p}(\nu_{\ell+1}), \quad \mathrm{p}(x_{n+1}e(\nu,i)) = \mathrm{p}(i).
\end{aligned}$$

Hence

$$\begin{aligned}
a\tau_n\cdots\tau_1 e(i,\beta) \equiv&\tau_n\cdots\tau_1 e(i,\beta)\phi^{\mathrm{p}(i)\mathrm{p}(\beta)}(\xi_n(a)),\\
x_{n+1}e(\beta,i)\tau_n\cdots\tau_1 e(i,\beta) \equiv&(-1)^{\mathrm{p}(i)\mathrm{p}(\beta)}\tau_n\cdots\tau_1 x_1 e(i,\beta)\\
&\mod \mathcal{R}(n)\mathcal{R}^1(n) \text{ for any } a \in \mathcal{R}(\beta).
\end{aligned} \tag{4.23}$$

Theorem 4.4.2. *There is an exact sequence in $\mathcal{M}\mathrm{od}(\mathcal{R}(\beta))$*

$$0 \to \overline{F}_j E_i M \to E_i \overline{F}_j M \to \delta_{i,j}\Pi^{\mathrm{p}(i)\mathrm{p}(\beta)} q^{-(\alpha_i|\beta)}\mathbf{k}[t_i] \otimes M \to 0,$$

which is functorial in $M \in \mathcal{M}\mathrm{od}(\mathcal{R}(\beta))$. Here t_i is an indeterminate of $(\mathbb{Z}\times\mathbb{Z}_2)$-degree $((\alpha_i|\alpha_i), \mathrm{p}(i))$.

Proof. The proof can be obtained by applying the arguments given in Proposition 3.6.2 and Theorem 3.6.2 with $(\mathbb{Z}\times\mathbb{Z}_2)$-grading consideration (4.23). □

4.5 Cyclotomic quotient

In this section, we define the cyclotomic quotient of the quiver Hecke superalgebra and investigate its structure.

For $\Lambda \in P^+$ and $i \in I$, we choose a monic polynomial of degree $\langle h_i, \Lambda\rangle$

$$\mathfrak{a}_i^\Lambda(u) = \sum_{k=0}^{\langle h_i,\Lambda\rangle} c_{i;k} u^{\langle h_i,\Lambda\rangle - k} \tag{4.24}$$

with $c_{i;k} \in \mathbf{k}_{k(\alpha_i|\alpha_i)}$ such that $c_{i,0} = 1$ and $c_{i;k} = 0$ if $i \in I_{\mathrm{odd}}$ and k is odd.

Hence $\mathfrak{a}_i^\Lambda(x_1)e(i)$ has the $\mathbb{Z} \times \mathbb{Z}_2$-degree

$$(\langle h_i, \Lambda\rangle(\alpha_i|\alpha_i), \mathrm{p}(i)\langle h_i, \Lambda\rangle).$$

For $1 \le k \le n$, define

$$\mathfrak{a}^\Lambda(x_k) = \sum_{\nu \in I^n} \mathfrak{a}^\Lambda_{\nu_k}(x_k)e(\nu) \in \mathcal{R}(n).$$

Definition 4.5.1. *Let $\beta \in Q^+$ and $\Lambda \in \mathtt{P}^+$. The cyclotomic quiver Hecke superalgebra $\mathcal{R}^\Lambda(\beta)$ at β is the quotient algebra*

$$\mathcal{R}^\Lambda(\beta) = \frac{\mathcal{R}(\beta)}{\mathcal{R}(\beta)\mathfrak{a}^\Lambda(x_1)\mathcal{R}(\beta)}.$$

We shall prove that the cyclotomic quotients are finitely generated over $\mathbf{k}$. For the definition of $\overline{s}_a$ and $\overline{\partial}_a$, see (4.7) and (4.8).

Lemma 4.5.1. *Assume that $fe(\nu)M = 0$ for $M \in \mathcal{M}\mathrm{od}(\mathcal{R}(n))$, $f \in \mathcal{P}_n$, $\nu \in I^n$ and $1 \le a < n$ such that $\nu_a = \nu_{a+1} = i$. Then we have*

$$(\overline{\partial}_a f)(x_a - x_{a+1})^{\mathrm{p}(i)} e(\nu)M = 0 \quad \textit{and}$$
$$(x_a^2 + x_{a+1}^2)^{\mathrm{p}(i)}(\overline{s}_a f)e(\nu)M = (\overline{s}_a f)(x_a^2 + x_{a+1}^2)^{\mathrm{p}(i)} e(\nu)M = 0.$$

Proof. By (4.8), we have

$$\begin{aligned}
&(x_{a+1}^{1+\mathrm{p}(i)} - x_a^{1+\mathrm{p}(i)})\tau_a f \tau_a e(\nu) = \left((x_{a+1} - x_a)^{\mathrm{p}(i)} f - \overline{s}_a f (x_{a+1} - x_a)^{\mathrm{p}(i)}\right) \tau_a e(\nu)\\
&= \left((x_{a+1} - x_a)^{\mathrm{p}(i)} f \tau_a - (-1)^{\mathrm{p}(i)} \overline{s}_a f \tau_a (x_{a+1} - x_a)^{\mathrm{p}(i)}\right) e(\nu)\\
&= \left((x_{a+1} - x_a)^{\mathrm{p}(i)} f \tau_a - (-1)^{\mathrm{p}(i)} \tau_a f (x_{a+1} - x_a)^{\mathrm{p}(i)} + (-1)^{\mathrm{p}(i)} \overline{\partial}_a f (x_{a+1} - x_a)^{\mathrm{p}(i)}\right) e(\nu).
\end{aligned}$$

Hence we have $(\overline{\partial}_a f)(x_a - x_{a+1})^{\mathrm{p}(i)} e(\nu)M = 0$. It follows that

$$\begin{aligned}
0 &= (x_a^{1+\mathrm{p}(i)} - x_{a+1}^{1+\mathrm{p}(i)})(\overline{\partial}_a f)(x_a - x_{a+1})^{\mathrm{p}(i)} e(\nu)M\\
&= \left((x_a - x_{a+1})^{\mathrm{p}(i)} f (x_a - x_{a+1})^{\mathrm{p}(i)} - \overline{s}_a f (x_a - x_{a+1})^{2\mathrm{p}(i)}\right) e(\nu)M.
\end{aligned}$$

Thus $(x_a - x_{a+1})^{2\mathrm{p}(i)}(\overline{s}_a f)e(\nu)M = (x_a^2 + x_{a+1}^2)^{\mathrm{p}(i)}(\overline{s}_a f)e(\nu)M = 0.$ $\square$

Lemma 4.5.2. *There exists a monic polynomial $g(u)$ with coefficients in* $\mathbf{k}$ *such that $g(x_a^2) = 0$ in $\mathcal{R}^\Lambda(\beta)$ $(1 \le a \le n)$.*

Proof. If $a = 1$, $g(x_1^2) = \prod_{i \in I} \mathfrak{a}_i^\Lambda(-x_1)\mathfrak{a}_i^\Lambda(x_1)$ satisfies the condition. Hence, by induction on a, it is enough to show the following statement:

For any monic polynomial $g(u) \in \mathbf{k}[u]$ and $\nu \in I^n$, we can find a monic polynomial $h(u) \in \mathbf{k}[u]$ such that $h(x_{a+1}^2)e(\nu)M = 0$ for any $\mathcal{R}(\beta)$-module M with $g(x_a^2)M = 0$.

(i) Suppose $\nu_a \neq \nu_{a+1}$. In this case, we have

$$g(x_{a+1}^2)\mathcal{Q}_{\nu_a,\nu_{a+1}}(x_a, x_{a+1})e(\nu)M = g(x_{a+1}^2)\tau_a^2 e(\nu)M = \tau_a g(x_a^2)\tau_a e(\nu)M = 0.$$

Since $\mathcal{Q}_{\nu_a,\nu_{a+1}}(x_a, x_{a+1})$ is a monic polynomial in $x_{a+1}^{1+p(\nu_{a+1})}$ with coefficients in $\mathbf{k}[x_a^{1+\mathrm{p}(\nu_a)}]$, there exists a monic polynomial $h(u)$ such that

$$\begin{aligned} h(x_{a+1}^2) \in &\mathbf{k}[x_a^{1+\mathrm{p}(\nu_a)}e(\nu), x_{a+1}^{1+\mathrm{p}(\nu_{a+1})}e(\nu)]g(x_a^2) + \\ &\qquad \mathbf{k}[x_a^{1+\mathrm{p}(\nu_a)}e(\nu), x_{a+1}^{1+\mathrm{p}(\nu_{a+1})}e(\nu)]g(x_{a+1}^2)\mathcal{Q}_{\nu_a,\nu_{a+1}}(x_a, x_{a+1}). \end{aligned}$$

Then $h(x_{a+1}^2)e(\nu)M = 0$.

(ii) Suppose $\nu_a = \nu_{a+1}$. Then Lemma 4.5.1 implies

$$g(x_{a+1}^2)(x_a^2 + x_{a+1}^2)^{\mathrm{p}(\nu_a)}e(\nu)M = 0.$$

Then we can apply the same argument as (i). □

Lemma 4.5.3. *Let $f \in \mathcal{P}_{n+1}$ be a monic polynomial of degree m in x_{n+1} whose coefficients are contained in $\mathcal{P}_n \otimes \mathbf{k}^I$. Set $\overline{\mathcal{R}} = e(n, i^{m+1})\mathcal{R}(n+m+1)e(n, i^{m+1})$. Then we have*

$$\overline{\mathcal{R}} f \overline{\mathcal{R}} = \overline{\mathcal{R}}.$$

Proof. We will prove the following statement by induction on k:

$$\overline{\partial}_{k-1} \cdots \overline{\partial}_{n+1} f e(n, i^{m+1})\tau_{w[n+1,k]} \in \overline{\mathcal{R}} f \overline{\mathcal{R}} \tag{4.25}$$

for $n+1 \le k \le n+m+1$. Here $w[n+1,k]$ is the longest element of the subgroup $S_{[n+1,k]}$ generated by s_a $(n+1 \le a < k)$ (See Remark 4.3.1). Assuming (4.25), by multiplying $\tau_{w[n+1,k]^{-1}w[n+1,k+1]}$ from the right, we have

$$\overline{\partial}_{k-1} \cdots \overline{\partial}_{n+1} f e(n, i^{m+1})\tau_{w[n+1,k+1]} \in \overline{\mathcal{R}} f \overline{\mathcal{R}}.$$

By multiplying τ_k from the left, we have

$$\begin{aligned}
&\tau_k(\overline{\partial}_{k-1}\cdots\overline{\partial}_{n+1}f)e(n,i^{m+1})\tau_{w[n+1,k+1]} \\
&\qquad = \big(\overline{s}_k(\overline{\partial}_{k-1}\cdots\overline{\partial}_{n+1}f)\tau_k+\overline{\partial}_k\cdots\overline{\partial}_{n+1}f\big)e(n,i^{m+1})\tau_{w[n+1,k+1]} \\
&\qquad = \overline{\partial}_k\cdots\overline{\partial}_{n+1}fe(n,i^{m+1})\tau_{w[n+1,k+1]}\in\overline{\mathcal{R}}f\overline{\mathcal{R}}.
\end{aligned}$$

Here we have used the fact that $\tau_k\tau_{w[n+1,k+1]}=0$.

Thus the induction proceeds and we obtain (4.25) for any k. Since $\overline{\partial}_{n+m}\cdots\overline{\partial}_{n+1}f=1$, our assertion follows from $\overline{\mathcal{R}}\tau_{w[n+1,n+m+1]}\overline{\mathcal{R}}=\overline{\mathcal{R}}$ in Lemma 4.3.1. □

Corollary 4.5.1. *For $\beta\in Q^+$ with $|\beta|=n$ and $i\in I$, there exists m such that*

$$\mathcal{R}^\Lambda(\beta+k\alpha_i)=0 \text{ for any } k\geq m.$$

Proof. By Lemma 4.5.2, there exists a monic polynomial $g(u)$ of degree m such that

$$g(x_n)\mathcal{R}^\Lambda(\beta)=0.$$

Lemma 4.5.3 implies $e(n,i^k)\mathcal{R}^\Lambda(\beta+k\alpha_i)=0$ for $k>m$. Now our assertion follows from arguments similar to those for Lemma 3.7.2. □

4.6 The superfunctors E_i^Λ and F_i^Λ

In this section, we define the superfunctors E_i^Λ and F_i^Λ on $\mathcal{M}\mathrm{od}(\mathcal{R}^\Lambda(\beta))$ and investigate their action on $\mathcal{P}\mathrm{roj}(\mathcal{R}^\Lambda(\beta))$ and $\mathcal{R}\mathrm{ep}(\mathcal{R}^\Lambda(\beta))$. After that we study their commutation relations as functors, which will give categorical $U_{\mathbb{A}}(\mathfrak{g})$-module structure on $[\mathcal{P}\mathrm{roj}(\mathcal{R}^\Lambda(\beta))]$ and $[\mathcal{R}\mathrm{ep}(\mathcal{R}^\Lambda(\beta))]$ in the last section.

For each $i\in I$, we define the superfunctors

$$\begin{aligned}
&E_i^\Lambda\colon \mathcal{M}\mathrm{od}(\mathcal{R}^\Lambda(\beta+\alpha_i))\to\mathcal{M}\mathrm{od}(\mathcal{R}^\Lambda(\beta)),\\
&F_i^\Lambda\colon \mathcal{M}\mathrm{od}(\mathcal{R}^\Lambda(\beta))\to\mathcal{M}\mathrm{od}(\mathcal{R}^\Lambda(\beta+\alpha_i))
\end{aligned}$$

by

$$\begin{aligned}
&E_i^\Lambda(N)=e(\beta,i)N=e(\beta,i)\mathcal{R}^\Lambda(\beta+\alpha_i)\otimes_{\mathcal{R}^\Lambda(\beta+\alpha_i)}N,\\
&F_i^\Lambda(M)=\mathcal{R}^\Lambda(\beta+\alpha_i)e(\beta,i)\otimes_{\mathcal{R}^\Lambda(\beta)}M
\end{aligned}$$

for $M \in \mathcal{M}\mathrm{od}(\mathcal{R}^\Lambda(\beta))$ and $N \in \mathcal{M}\mathrm{od}(\mathcal{R}^\Lambda(\beta+\alpha_i))$.

For each $i \in I$, $\beta \in Q^+$ and $m \in \mathbb{Z}$, let

$$\mathcal{K}_{i,\beta}^m := \mathcal{R}(\beta+\alpha_i)v(i,\beta)T_i^m$$

be the $\mathcal{R}(\beta+\alpha_i)$-supermodule generated by $v(i,\beta)T_i^m$ with the defining relation

$$e(i,\beta)v(i,\beta)T_i^m = v(i,\beta)T_i^m.$$

We assign to $v(i,\beta)T_i^m$ the $(\mathbb{Z}\times\mathbb{Z}_2)$-degree $(0,0)$. The supermodule $\mathcal{K}_{i,\beta}^m$ has an $(\mathcal{R}(\beta+\alpha_i), \mathbf{k}[t_i]\otimes\mathcal{R}(\beta))$-superbimodule structure whose right $\mathbf{k}[t_i]\otimes\mathcal{R}(\beta)$-action is given by

$$\begin{aligned} &av(i,\beta)T_i^m \cdot b = a\xi_n(b)v(i,\beta)T_i^m, \\ &av(i,\beta)T_i^m \cdot t_i = a\phi_i^m(x_1)v(i,\beta)T_i^m = (-1)^{\mathrm{p}(i)}ax_1v(i,\beta)T_i^m \end{aligned} \tag{4.26}$$

for $a \in \mathcal{R}(\beta+\alpha_i)$ and $b \in \mathcal{R}(\beta)$. Here, $\phi_i{:=}\phi^{\mathrm{p}(i)}$ and ϕ is the parity involution (see Example 4.1.1(a)).

In the sequel, we sometimes omit the $\mathbb{Z}$-grading shift functor q when the $\mathbb{Z}$-grading can be neglected.

Set $\Lambda_i := \langle h_i, \Lambda\rangle$. We introduce $(\mathcal{R}(\beta+\alpha_i), \mathcal{R}^\Lambda(\beta))$-superbimodules

$$\begin{aligned} F^\Lambda &:= \mathcal{R}^\Lambda(\beta+\alpha_i)e(\beta,i), \\ K_0 &:= \mathcal{R}(\beta+\alpha_i)e(\beta,i)\otimes_{\mathcal{R}(\beta)}\mathcal{R}^\Lambda(\beta), \\ K_1 &:= \mathcal{K}_{i,\beta}^{\Lambda_i}\otimes_{\mathcal{R}(\beta)}\Pi_i^{\Lambda_i+\mathrm{p}(\beta)}\mathcal{R}^\Lambda(\beta) \\ &\;= \mathcal{R}(\beta+\alpha_i)v(i,\beta)T_i^{\Lambda_i}\otimes_{\mathcal{R}(\beta)}\Pi_i^{\Lambda_i+\mathrm{p}(\beta)}\mathcal{R}^\Lambda(\beta). \end{aligned} \tag{4.27}$$

For $i \in I$, let t_i be an indeterminate of $\mathbb{Z}\times\mathbb{Z}_2$-degree $((\alpha_i|\alpha_i), \mathrm{p}(i))$. Then $\mathbf{k}[t_i]$ is a superalgebra. The superalgebra $\mathbf{k}[t_i]$ acts on K_1 from the right by the formula given in (4.26). Namely,

$$\Big(av(i,\beta)T_i^{\Lambda_i}\otimes\pi_i^{\Lambda_i+\mathrm{p}(\beta)}b\Big)t_i = a\phi_i^{\mathrm{p}(\beta)}(x_1)v(i,\beta)T_i^{\Lambda_i}\otimes\pi_i^{\Lambda_i+\mathrm{p}(\beta)}\phi_i(b)$$

for $a \in \mathcal{R}(\beta+\alpha_i)$ and $b \in \mathcal{R}^\Lambda(\beta)$. Here $\pi_i{:=}\pi^{\mathrm{p}(i)}$ and $\phi_i = \phi^{\mathrm{p}(i)}$. On the other hand, $\mathbf{k}[t_i]$ acts on $\mathcal{R}(\beta+\alpha_i)e(\beta,i)$, F^Λ and K_0 by multiplying by x_{n+1} from the right. Thus K_0, F^Λ and K_1 have a graded $\big(\mathcal{R}(\beta+\alpha_i), \mathbf{k}[t_i]\otimes\mathcal{R}^\Lambda(\beta)\big)$-superbimodule structure.

By a similar argument to Lemma 3.7.4 and 3.7.5, we have quiver Hecke superalgebra versions of those lemmas; i.e.; for $i \in I$ and $\beta \in Q^+$ with $|\beta| = n$, we have the following statements:

(i) Both K_1 and K_0 are finitely generated projective right $\mathbf{k}[t_i] \otimes \mathcal{R}^\Lambda(\beta)$-supermodules.

(ii) For any monic polynomial $f(t_i) \in \mathcal{P}_n[t_i]$, right multiplication by $f(t_i)$ on K_1 induces an injective endomorphism of K_1.

(iii) $\mathcal{R}(\beta+\alpha_i)\mathfrak{a}^\Lambda(x_1)\mathcal{R}(\beta+\alpha_i) = \sum_{a=0}^{n} \mathcal{R}(\beta+\alpha_i)\mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_a$,

(iv) $\mathcal{R}(\beta+\alpha_i)\mathfrak{a}^\Lambda(x_1)\mathcal{R}(\beta+\alpha_i)e(\beta,i)$
$= \mathcal{R}(\beta+\alpha_i)\mathfrak{a}^\Lambda(x_1)\mathcal{R}(\beta)e(\beta,i) + \mathcal{R}(\beta+\alpha_i)\mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_n e(\beta,i)$.

Applying an argument to the one for Lemma 3.7.6, we have

$$\begin{aligned} x_1\mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_n e(\nu,i) &\equiv (-1)^{\mathrm{p}(\nu)\mathrm{p}(i)}\mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_n x_{n+1}e(\nu,i), \\ \mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_n e(\beta,i)c &\equiv \phi_i^{\Lambda_i+\mathrm{p}(\beta)}(\xi_n(c))\mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_n e(\beta,i) \\ &\qquad \mod \mathcal{R}(n+1)\mathfrak{a}^\Lambda(x_1)\mathcal{R}(\beta)e(\beta,i) \end{aligned} \tag{4.28}$$

for any $\beta \in Q^+$ with $|\beta| = n$, $\nu \in I^\beta$ and $c \in \mathcal{R}(n)$.

Let $P\colon K_1 \to K_0$ be the homomorphism defined by

$$xv(i,\beta)T_i^{\Lambda_i} \otimes \pi_i^{\Lambda_i+\mathrm{p}(\beta)}y \longmapsto x\mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_n e(\beta,i)\otimes y \tag{4.29}$$

for $x \in \mathcal{R}(\beta+\alpha_i)$ and $y \in \mathcal{R}^\Lambda(\beta)$. Then, by (4.28), P becomes an $(\mathcal{R}(\beta+\alpha_i), \mathbf{k}[t_i]\otimes\mathcal{R}(\beta))$-superbimodule homomorphism.

Let $\mathrm{pr}\colon K_0 \to F^\Lambda$ be the canonical projection. Then by a similar reason as (3.46), we have an exact sequence of $\big(\mathcal{R}(\beta+\alpha_i), \mathbf{k}[t_i]\otimes\mathcal{R}(\beta)\big)$-superbimodules

$$K_1 \xrightarrow{P} K_0 \xrightarrow{\mathrm{pr}} F^\Lambda \longrightarrow 0.$$

As in the quiver Hecke algebra case, we will construct a map $(\mathcal{R}(\beta+\alpha_i), \mathbf{k}[t_i]\otimes\mathcal{R}(\beta))$-bilinear homomorphism Q such that $Q\circ P$ is injective. This then implies that P is indeed injective.

For $1 \le a \le n$, we define the elements φ_a and g_a of $\mathcal{R}(\beta+\alpha_i)$ which can be understood to be super-extensions of (3.49) and (3.50):

$$\varphi_a = \sum_{\substack{\nu \in I^{\beta+\alpha_i},\\ \nu_a \neq \nu_{a+1}}} \tau_a e(\nu) + \sum_{\substack{\nu \in I^{\beta+\alpha_i},\\ \nu_a = \nu_{a+1}}} ((x_{a+1}-x_a)^{\mathrm{p}(\nu_a)} - (x_{a+1}^{1+\mathrm{p}(\nu_a)} - x_a^{1+\mathrm{p}(\nu_a)})\tau_a)e(\nu)$$

and

$$\begin{aligned} g_a = &\sum_{\substack{\nu \in I^{\beta+\alpha_i},\\ \nu_a \neq \nu_{a+1}}} \tau_a e(\nu) \\ &+ \sum_{\substack{\nu \in I^{\beta+\alpha_i},\\ \nu_a = \nu_{a+1}}} (x_{a+1}^{1+\mathrm{p}(\nu_a)} - x_a^{1+\mathrm{p}(\nu_a)})\big((x_{a+1}-x_a)^{\mathrm{p}(\nu_a)} - (x_{a+1}^{1+\mathrm{p}(\nu_a)} - x_a^{1+\mathrm{p}(\nu_a)})\tau_a\big)e(\nu). \end{aligned}$$

Note that if $\nu_a = \nu_{a+1}$,

(4.30)
$$\begin{aligned} \varphi_a e(\nu) &= (x_a^{1+\mathrm{p}(\nu_a)}\tau_a - \tau_a x_a^{1+\mathrm{p}(\nu_a)})e(\nu) = (\tau_a x_{a+1}^{1+\mathrm{p}(\nu_a)} - x_{a+1}^{1+\mathrm{p}(\nu_a)}\tau_a)e(\nu) \\ &= ((x_a - x_{a+1})^{\mathrm{p}(\nu_a)} - \tau_a(x_a^{1+\mathrm{p}(\nu_a)} - x_{a+1}^{1+\mathrm{p}(\nu_a)}))e(\nu), \end{aligned}$$

and

$$g_a e(\nu) = (x_{a+1}^{1+\mathrm{p}(\nu_a)} - x_a^{1+\mathrm{p}(\nu_a)})^{\delta_{\nu_a,\nu_{a+1}}} \varphi_a e(\nu), \tag{4.31}$$

where $(x_{a+1}^{1+\mathrm{p}(\nu_a)} - x_a^{1+\mathrm{p}(\nu_a)})^{\delta_{\nu_a,\nu_{a+1}}} e(\nu)$ is an even element.

Lemma 4.6.1. *For $1 \le a \le n$ and $\nu \in I^{n+1}$, we have*

$$\begin{aligned} &\varphi_a e(\nu) = e(s_a\nu)\varphi_a, \\ &x_{s_a(b)}\varphi_a e(\nu) = (-1)^{\mathrm{p}(\nu_a)\mathrm{p}(\nu_{a+1})\mathrm{p}(\nu_b)}\varphi_a x_b e(\nu) \quad (1 \le b \le n+1), \\ &\tau_b \varphi_a e(\nu) = (-1)^{\mathrm{p}(\nu_a)\mathrm{p}(\nu_{a+1})\mathrm{p}(\nu_b)\mathrm{p}(\nu_{b+1})}\varphi_a \tau_b e(\nu) \quad \textit{if } |b-a|>1, \\ &\tau_a \varphi_{a+1}\varphi_a = \varphi_{a+1}\varphi_a\tau_{a+1}, \end{aligned} \tag{4.32}$$

and

$$\begin{aligned} &g_a e(\nu) = e(s_a\nu)g_a, \\ &x_{s_a(b)} g_a e(\nu) = (-1)^{\mathrm{p}(\nu_a)\mathrm{p}(\nu_{a+1})\mathrm{p}(\nu_b)} g_a x_b e(\nu) \quad (1 \le b \le n+1), \\ &\tau_b g_a e(\nu) = (-1)^{\mathrm{p}(\nu_a)\mathrm{p}(\nu_{a+1})\mathrm{p}(\nu_b)\mathrm{p}(\nu_{b+1})} g_a \tau_b e(\nu) \quad \textit{if } |b-a|>1, \\ &\tau_a g_{a+1} g_a = g_{a+1} g_a \tau_{a+1}. \end{aligned} \tag{4.33}$$

Proof. By the defining relations of quiver Hecke superalgebras, the third equality can be verified immediately. If $\nu_a \neq \nu_{a+1}$ or $\nu_a = \nu_{a+1} \in I_{\text{even}}$, the first and second equalities were covered by Lemma 3.7.7. We will prove the second equality in (4.32) when $\nu_a = \nu_{a+1} \in I_{\text{odd}}$. Let $b = a$. Then

$$\begin{aligned} x_{a+1}\varphi_a e(\nu) &= x_{a+1}^2 - x_{a+1}x_a - (x_{a+1}^2 - x_a^2)(x_{a+1}\tau_a)e(\nu) \\ &= x_{a+1}^2 - x_{a+1}x_a - (x_{a+1}^2 - x_a^2)(-\tau_a x_a + 1)e(\nu), \end{aligned}$$

and

$$\varphi_a x_a e(\nu) = x_{a+1}x_a - x_a^2 - (x_{a+1}^2 - x_a^2)(\tau_a x_a)e(\nu).$$

Therefore we have

$$x_{a+1}\varphi_a e(\nu) + \varphi_a x_a e(\nu) = 0.$$

Similarly, we can prove the equality when $b = a + 1$.

Let $S = \tau_a\varphi_{a+1}\varphi_a - \varphi_{a+1}\varphi_a\tau_{a+1}$. Using the second equality, we have

$$\begin{aligned}
(\tau_a\varphi_{a+1}\varphi_a)x_a e(\nu) &= (-1)^{\mathrm{p}(\nu_a)(\mathrm{p}(\nu_a)\mathrm{p}(\nu_{a+1})+\mathrm{p}(\nu_a)\mathrm{p}(\nu_{a+2})+\mathrm{p}(\nu_{a+1})\mathrm{p}(\nu_{a+2}))} \\
&\qquad x_{a+2}(\tau_a\varphi_{a+1}\varphi_a)e(\nu), \\
(\varphi_{a+1}\varphi_a\tau_{a+1})x_a e(\nu) &= (-1)^{\mathrm{p}(\nu_a)(\mathrm{p}(\nu_{a+1})\mathrm{p}(\nu_{a+2})+\mathrm{p}(\nu_a)\mathrm{p}(\nu_{a+2})+\mathrm{p}(\nu_a)\mathrm{p}(\nu_{a+1}))} \\
&\qquad x_{a+2}(\varphi_{a+1}\varphi_a\tau_{a+1})e(\nu), \\
(\tau_a\varphi_{a+1}\varphi_a)x_{a+1} e(\nu) &= (-1)^{\mathrm{p}(\nu_{a+1})(\mathrm{p}(\nu_a)\mathrm{p}(\nu_{a+1})+\mathrm{p}(\nu_a)\mathrm{p}(\nu_{a+2}))}\tau_a x_a\varphi_{a+1}\varphi_a e(\nu) \\
&= (-1)^{\mathrm{p}(\nu_{a+1})(\mathrm{p}(\nu_{a+1})\mathrm{p}(\nu_a)+\mathrm{p}(\nu_a)\mathrm{p}(\nu_{a+2})+\mathrm{p}(\nu_{a+1})\mathrm{p}(\nu_{a+2}))} \\
&\qquad (x_{a+1}\tau_a - e_{a,a+1})\varphi_{a+1}\varphi_a e(\nu), \\
(\varphi_{a+1}\varphi_a\tau_{a+1})x_{a+1} e(\nu) &= (-1)^{\mathrm{p}(\nu_{a+1})\mathrm{p}(\nu_{a+2})}\varphi_{a+1}\varphi_a(x_{a+2}\tau_{a+1} - e_{a+1,a+2})e(\nu) \\
&= (-1)^{\mathrm{p}(\nu_{a+1})(\mathrm{p}(\nu_{a+1})\mathrm{p}(\nu_{a+2})+\mathrm{p}(\nu_a)\mathrm{p}(\nu_{a+2})+\mathrm{p}(\nu_a)\mathrm{p}(\nu_{a+1}))}x_{a+1}\varphi_{a+1}\varphi_a\tau_{a+1}e(\nu) \\
&\qquad - (-1)^{\mathrm{p}(\nu_{a+1})\mathrm{p}(\nu_{a+2})}\varphi_{a+1}\varphi_a e_{a+1,a+2}e(\nu), \\
(\tau_a\varphi_{a+1}\varphi_a)x_{a+2} e(\nu) &= (-1)^{\mathrm{p}(\nu_{a+2})(\mathrm{p}(\nu_a)\mathrm{p}(\nu_{a+1})+\mathrm{p}(\nu_{a+2})\mathrm{p}(\nu_a))}\tau_a x_{a+1}\varphi_{a+1}\varphi_a e(\nu) \\
&= (-1)^{\mathrm{p}(\nu_{a+2})(\mathrm{p}(\nu_a)\mathrm{p}(\nu_{a+1})+\mathrm{p}(\nu_{a+2})\mathrm{p}(\nu_a))} \\
&\qquad ((-1)^{\mathrm{p}(\nu_{a+1})\mathrm{p}(\nu_{a+2})}x_a\tau_a + e_{a,a+1})\varphi_{a+1}\varphi_a e(\nu) \\
&= (-1)^{\mathrm{p}(\nu_{a+2})(\mathrm{p}(\nu_a)\mathrm{p}(\nu_{a+1})+\mathrm{p}(\nu_{a+2})\mathrm{p}(\nu_a)+\mathrm{p}(\nu_{a+1})\mathrm{p}(\nu_{a+2}))}x_a\tau_a\varphi_{a+1}\varphi_a e(\nu) \\
&\qquad + (-1)^{\mathrm{p}(\nu_{a+2})(\mathrm{p}(\nu_a)\mathrm{p}(\nu_{a+1})+\mathrm{p}(\nu_{a+2})\mathrm{p}(\nu_a))}\varphi_{a+1}\varphi_a e_{a+1,a+2}e(\nu), \\
(\varphi_{a+1}\varphi_a\tau_{a+1})x_{a+2} e(\nu) &= \varphi_{a+1}\varphi_a((-1)^{\mathrm{p}(\nu_{a+1})\mathrm{p}(\nu_{a+2})}x_{a+1}\tau_{a+1} + e_{a+1,a+2})e(\nu) \\
&= (-1)^{\mathrm{p}(\nu_{a+2})(\mathrm{p}(\nu_a)\mathrm{p}(\nu_{a+1})+\mathrm{p}(\nu_{a+2})\mathrm{p}(\nu_a)+\mathrm{p}(\nu_{a+1})\mathrm{p}(\nu_{a+2}))}x_a\tau_a\varphi_{a+1}\varphi_a e(\nu) \\
&\qquad + \varphi_{a+1}\varphi_a e_{a+1,a+2}e(\nu).
\end{aligned}$$

Hence we have $Sx_b = \pm x_{s_{a,a+2}b}S$ for all b. Using the argument in Lemma 3.7.7 we conclude that $S = 0$.

The equalities in (4.33) follow from (4.31). □

By the preceding lemma, one can see that

$$(4.34)\qquad \begin{aligned}
a g_n\cdots g_1 e(i,\beta) &= g_n\cdots g_1 e(i,\beta)\phi_i^{\mathrm{p}(\beta)}(\xi_n(a)), \\
x_{n+1}g_n\cdots g_1 e(i,\beta) &= (-1)^{\mathrm{p}(\beta)\mathrm{p}(i)}g_n\cdots g_1 e(i,\beta)x_1.
\end{aligned}$$

for any $a \in \mathcal{R}(\beta)$.

Using a similar method to the construction of P, we obtain the following proposition:

Proposition 4.6.1. *There is an $(\mathcal{R}(\beta+\alpha_i), \mathbf{k}[t_i] \otimes \mathcal{R}^\Lambda(\beta))$-bilinear homomorphism*

$$Q\colon K_0 \to K_1' := \mathcal{R}(\beta+\alpha_i)v(i,\beta) \otimes_{\mathcal{R}(\beta)} \Pi_i^{\mathrm{p}(\beta)}\mathcal{R}^\Lambda(\beta)$$

defined by

$$ae(\beta,i) \otimes b \longmapsto ag_n \cdots g_1 v(i,\beta) \otimes \pi_i^{\mathrm{p}(\beta)} b$$

for $a \in \mathcal{R}(\beta+\alpha_i)$ and $b \in \mathcal{R}^\Lambda(\beta)$. Here, the right action of t_i on K_1' is given by

$$av(i,\beta) \otimes \pi_i^{\mathrm{p}(\beta)} b \longmapsto (-1)^{\mathrm{p}(i)\mathrm{p}(\beta)} a x_1 v(i,\beta) \otimes \pi_i^{\mathrm{p}(\beta)} \phi_i(b).$$

Theorem 4.6.1. *For each $\nu \in I^\beta$, set*

$$\mathcal{A}_\nu(t_i) = \mathfrak{a}_i^\Lambda(t_i) \prod_{\substack{1 \le a \le n,\\ \nu_a \ne i}} \mathcal{Q}_{i,\nu_a}(t_i, x_a) \prod_{\substack{1 \le a \le n,\\ \nu_a = i \in I_{\mathrm{odd}}}} (x_a - t_i)^2 e(\nu),$$

and define

$$\mathcal{A}(t_i) := \sum_{\nu \in I^\beta} \mathcal{A}_\nu(t_i) \in \mathbf{k}[t_i] \otimes \mathcal{R}^\Lambda(\beta).$$

Then the composition

$$Q \circ P\colon K_1 \to K_1'$$

coincides with the right multiplication by $(-1)^{p(i)\Lambda_i \mathrm{p}(\beta)}\mathcal{A}(t_i)$; i.e.,

$$\begin{aligned} av(i,\beta)T_i^{\Lambda_i} \otimes \pi_i^{\Lambda_i+\mathrm{p}(\beta)} b &\longmapsto \big(av(i,\beta) \otimes \pi_i^{\mathrm{p}(\beta)} \phi_i^{\Lambda_i}(b)\big)(-1)^{p(i)\Lambda_i\mathrm{p}(\beta)}\mathcal{A}(t_i) \\ &= av(i,\beta)\mathcal{A}(t_i) \otimes \pi_i^{\mathrm{p}(\beta)} b. \end{aligned}$$

Proof. If $i \in I_{\mathrm{even}}$, we already proved in Theorem 3.7.1. If $i \in I_{\mathrm{odd}}$, then it suffices to show that

$$\begin{aligned} \mathfrak{a}^\Lambda(x_1)\tau_1 \cdots \tau_n g_n \cdots g_1 e(i,\nu) &= \mathfrak{a}^\Lambda(x_1)\tau_1 \cdots \tau_n e(\nu,i) g_n \cdots g_1 \\ &\equiv \mathcal{A}_\nu' \mod \mathcal{R}(\beta+\alpha_i)\mathfrak{a}^\Lambda(x_2)\mathcal{R}^1(\beta), \end{aligned} \tag{4.35}$$

where

$$\mathcal{A}_\nu' = \mathfrak{a}_i^\Lambda(x_1) \prod_{\substack{1 \le a \le n,\\ \nu_a \ne i}} \mathcal{Q}_{i,\nu_a}(x_1, x_{a+1}) \prod_{\substack{1 \le a \le n,\\ \nu_a = i \in I_{\mathrm{odd}}}} (x_{a+1} - x_1)^2 e(i,\nu).$$

As in Theorem 3.7.1, we will use induction on $|\beta| = n$ to prove (4.35). If $n = 0$, it is obvious. Thus we may assume that $n \geq 1$.

Note that, by (4.30), we have

$$\tau_n e(\nu, i) g_n = \begin{cases} \tau_n e(\nu, i)\tau_n = \mathcal{Q}_{i,\nu_n}(x_n, x_{n+1}) e(\nu_{<n}, i, \nu_n) & \text{if } \nu_n \neq i, \\ \tau_n (x_{n+1}^2 - x_n^2)(x_{n+1} - x_n) e(\nu, i) & \text{if } \nu_n = i. \end{cases}$$

(i) We first assume that $\nu_n \neq i$. Then, by (4.6), we have

$$\begin{aligned} &\mathfrak{a}^{\Lambda}(x_1)\tau_1 \cdots \tau_n g_n \cdots g_1 e(i, \nu) \\ &= \mathfrak{a}^{\Lambda}(x_1)\tau_1 \cdots \tau_{n-1} \mathcal{Q}_{i,\nu_n}(x_n, x_{n+1}) g_{n-1} \cdots g_1 e(i, \nu) \\ &= \mathfrak{a}^{\Lambda}(x_1)\tau_1 \cdots \tau_{n-1} g_{n-1} \cdots g_1 e(i, \nu) \mathcal{Q}_{i,\nu_n}(x_1, x_{n+1}) \\ &\equiv \mathcal{A}'_{\nu_{<n}} \mathcal{Q}_{i,\nu_n}(x_1, x_{n+1}) = \mathcal{A}'_{\nu} \qquad \bmod \mathcal{R}(\beta + \alpha_i)\mathfrak{a}^{\Lambda}(x_2)\mathcal{R}^1(\beta) e(i, \beta). \end{aligned}$$

(ii) Assume that $\nu_n = i$. Then we have

$$\begin{aligned} &\mathfrak{a}^{\Lambda}(x_1)\tau_1 \cdots \tau_n g_n \cdots g_1 e(i, \nu) \\ &= \mathfrak{a}^{\Lambda}(x_1)\tau_1 \cdots \tau_n (x_{n+1} - x_n)(x_{n+1}^2 - x_n^2) g_{n-1} \cdots g_1 e(i, \nu). \end{aligned} \tag{4.36}$$

Note that

$$\begin{aligned} \mathfrak{a}^{\Lambda}(x_1)\tau_1 \cdots \tau_{n-1} g_n \cdots g_1 e(i, \nu) = \pm g_n \cdots g_1 \mathfrak{a}^{\Lambda}(x_2)\tau_2 \cdots \tau_n \equiv 0 \\ \bmod \mathcal{R}(\beta + \alpha_i)\mathfrak{a}^{\Lambda}(x_2)\mathcal{R}^1(\beta) e(i, \beta). \end{aligned} \tag{4.37}$$

By (4.30), formula (4.37) can be written as

$$\mathfrak{a}^{\Lambda}(x_1)\tau_1 \cdots \tau_{n-1}(\tau_n(x_{n+1}^2 - x_n^2) - (x_{n+1} - x_n))(x_{n+1}^2 - x_n^2) g_{n-1} \cdots g_1 e(i, \nu) \equiv 0.$$

Thus

$$\begin{aligned} &\mathfrak{a}^{\Lambda}(x_1)\tau_1 \cdots \tau_{n-1}\tau_n (x_{n+1}^2 - x_n^2)^2 g_{n-1} \cdots g_1 e(i, \nu) \\ &\equiv \mathfrak{a}^{\Lambda}(x_1)\tau_1 \cdots \tau_{n-1}(x_{n+1} - x_n)(x_{n+1}^2 - x_n^2) g_{n-1} \cdots g_1 e(i, \nu) \\ &\equiv (-1)^{\mathrm{p}(\nu_{<n})} \mathcal{A}'_{\nu_{<n}} (x_{n+1} - x_1)(x_{n+1}^2 - x_1^2). \end{aligned}$$

Since the right multiplication by $(x_{n+1}^2 - x_1^2)$ and $(x_{n+1} - x_1)$ on K_1 are injective, we conclude that

$$\begin{aligned} &\mathfrak{a}^{\Lambda}(x_1)\tau_1 \cdots \tau_{n-1}\tau_n (x_{n+1}^2 - x_n^2) g_{n-1} \cdots g_1 e(i, \nu) \\ &\qquad \equiv (-1)^{\mathrm{p}(\nu_{<n})} \mathcal{A}'_{\nu_{<n}} (x_{n+1} - x_1), \end{aligned}$$

which implies

$$(-1)^{\mathrm{p}(\nu_{<n})}\mathfrak{a}^{\Lambda}(x_1)\tau_1\cdots\tau_{n-1}\tau_n(x_{n+1}^2-x_n^2)(x_{n+1}-x_n)g_{n-1}\cdots g_1 e(i,\nu) \\ \equiv (-1)^{\mathrm{p}(\nu_{<n})}\mathcal{A}'_{\nu_{<n}}(x_{n+1}-x_1)^2.$$

Then, (4.36), together with $\mathcal{A}'_{\nu}=\mathcal{A}'_{\nu_{<n}}(x_{n+1}-x_1)^2$, implies the desired result. $\square$

By applying the same argument given in Corollary 3.7.1, we have the following lemma.

Corollary 4.6.1. *Set*

$$K'_0 := \mathcal{R}(\beta+\alpha_i)e(\beta,i)T_i^{\Lambda_i}\otimes_{\mathcal{R}(\beta)}\Pi_i^{\Lambda_i}\mathcal{R}^{\Lambda}(\beta).$$

Then the following diagram commutes

$$\begin{tikzcd} K_1 \arrow[r, "P"] \arrow[d, "(-1)^{\mathrm{p}(i)\Lambda_i\mathrm{p}(\beta)}\mathcal{A}(t_i)"'] & K_0 \arrow[d, "(-1)^{\mathrm{p}(i)\Lambda_i\mathrm{p}(\beta)}\mathcal{A}(t_i)"] \arrow[dl, "Q"'] \\ K'_1 \arrow[r, "P'"'] & K'_0. \end{tikzcd}$$

Here, $\Lambda_i=\langle\Lambda,h_i\rangle$ *and* $P'\colon K'_1\to K'_0$ *is given by*

$$av(i,\beta)\otimes\pi_i^{\mathrm{p}(\beta)}b\longmapsto a\mathfrak{a}^{\Lambda}(x_1)\tau_1\cdots\tau_n e(\beta,i)T_i^{\Lambda_i}\otimes\pi_i^{\Lambda_i}b,$$

and $(-1)^{\mathrm{p}(i)\Lambda_i\mathrm{p}(\beta)}\mathcal{A}(t_i)\colon K_0\to K'_0$ *is given by*

$$a\otimes b\longmapsto \big(aT_i^{\Lambda_i}\otimes\pi_i^{\Lambda_i}\phi_i^{\Lambda_i}(b)\big)(-1)^{\mathrm{p}(i)\Lambda_i\mathrm{p}(\beta)}\mathcal{A}(t_i)=(-1)^{\mathrm{p}(i)\Lambda_i\mathrm{p}(\beta)}a\mathcal{A}(t_i)T_i^{\Lambda_i}\otimes\pi_i^{\Lambda_i}b.$$

In particular, for any $\nu\in I^{\beta}$, *we have*

$$g_n\cdots g_1\mathfrak{a}^{\Lambda}(x_1)\tau_1\cdots\tau_n e(\nu,i)\otimes e(\beta) \\ =(-1)^{\mathrm{p}(i)\Lambda_i\mathrm{p}(\beta)}\mathfrak{a}_i^{\Lambda}(x_{n+1})\prod_{\substack{1\le a\le n,\\ \nu_a\ne i}}\mathcal{Q}_{\nu_a,i}(x_a,x_{n+1})\prod_{\substack{1\le a\le n,\\ \nu_a=i\in I_{\mathrm{odd}}}}(x_a-x_{n+1})^2 e(\nu,i)\otimes e(\beta)$$

in $\mathcal{R}(\beta+\alpha_i)e(\beta,i)\otimes_{\mathcal{R}(\beta)}\mathcal{R}^{\Lambda}(\beta)$.

Set

$$\mathcal{P}\mathrm{roj}(\mathcal{R}^\Lambda) = \bigoplus_{\alpha \in Q^+} \mathcal{P}\mathrm{roj}(\mathcal{R}^\Lambda(\alpha)), \quad \mathcal{R}\mathrm{ep}(\mathcal{R}^\Lambda) = \bigoplus_{\alpha \in Q^+} \mathcal{R}\mathrm{ep}(\mathcal{R}^\Lambda(\alpha)).$$

Since K_1 is a projective $\mathcal{R}^\Lambda(\beta) \otimes \mathbf{k}[t_i]$-supermodule and $\mathcal{A}(t_i)$ is a monic (skew)-polynomial in t_i (up to a multiple of an invertible element of $\mathbf{k}$) by applying arguments similar to those in Section 3.7, we have the following theorem:

Theorem 4.6.2. *The module F^Λ is a projective right $\mathcal{R}^\Lambda(\beta)$-supermodule and we have a short exact sequence consisting of right projective $\mathcal{R}^\Lambda(\beta)$-supermodules:*

$$0 \to K_1 \xrightarrow{P} K_0 \to F^\Lambda \to 0. \tag{4.38}$$

Hence the functors E_i^Λ and F_i^Λ are well-defined exact functors on $\mathcal{P}\mathrm{roj}(\mathcal{R}^\Lambda)$ and $\mathcal{R}\mathrm{ep}(\mathcal{R}^\Lambda)$, and they induce endomorphisms on the Grothendieck groups $[\mathcal{P}\mathrm{roj}(\mathcal{R}^\Lambda)]$ and $[\mathcal{R}\mathrm{ep}(\mathcal{R}^\Lambda)]$.

Now, we will show that the superfunctors E_i^Λ and F_i^Λ satisfy certain commutation relations, from which we obtain a supercategorification of $V(\Lambda)$.

By taking the kernels of exact sequences given in Theorem 4.4.1, Theorem 4.4.2 and the exact sequence of superbimodules (4.38), we have the following commutative diagram of $(R(\beta), R^\Lambda(\beta))$-superbimodules:

(4.39)

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & L_1' \arrow[r] \arrow[d] & L_0' \arrow[r] \arrow[d] & q_i^{-2} F_i^\Lambda \Pi_i E_i^\Lambda R^\Lambda(\beta) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L_1 \arrow[r] \arrow[d] & L_0 \arrow[r] \arrow[d] & E_i^\Lambda F_i^\Lambda R^\Lambda(\beta) \arrow[r] & 0 \\
 & \mathbf{k}[t_i] \mathbf{T}_i^{\Lambda_i} \otimes \Pi_i^{\Lambda_i} R^\Lambda(\beta) \arrow[r, "A"] \arrow[d] & \mathbf{k}[t_i] \otimes R^\Lambda(\beta) \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

where

$$\begin{aligned}
L_0' &= q_i^{-2} R(\beta)e(\beta-\alpha_i,i) \otimes_{R(\beta-\alpha_i)} \Pi_i e(\beta-\alpha_i,i)R^\Lambda(\beta),\\
L_1' &= q_i^{(\alpha_i|2\Lambda-\beta)} R(\beta)v(i,\beta-\alpha_i)\mathbf{T}_i^{\Lambda_i} \otimes_{R(\beta-\alpha_i)} \Pi_i^{\Lambda_i+\mathrm{p}(\beta)} e(\beta-\alpha_i,i)R^\Lambda(\beta),\\
L_0 &= e(\beta,i)R(\beta+\alpha_i)e(\beta,i) \otimes_{R(\beta)} R^\Lambda(\beta),\\
L_1 &= q_i^{(\alpha_i|2\Lambda-\beta)} e(\beta,i)R(\beta+\alpha_i)v(i,\beta)\mathbf{T}_i^{\Lambda_i} \otimes_{R(\beta)} \Pi_i^{\Lambda_i+\mathrm{p}(\beta)} R^\Lambda(\beta),
\end{aligned}$$

The homomorphisms in the diagram (4.39) can be described as follows (cf. (3.63)):

- P is given by (4.29). It is $(\mathcal{R}(\beta+\alpha_i), \mathbf{k}[t_i] \otimes \mathcal{R}^\Lambda(\beta))$-bilinear.
- A is defined by chasing the diagram. Note that it is $\mathcal{R}^\Lambda(\beta)$-linear but *not* $\mathbf{k}[t_i]$-linear.
- B is given by taking the coefficient of $\tau_n \cdots \tau_1$. It is $(\mathcal{R}(\beta), \mathbf{k}[t_i] \otimes \mathcal{R}(\beta))$-linear (see the remark below).
- F is given by $a \otimes \pi_i b \longmapsto a\tau_n \otimes b$ for $a \in \mathcal{R}(\beta)e(\beta-\alpha_i,i)$ and $b \in e(\beta-\alpha_i,i)\mathcal{R}^\Lambda(\beta)$ (See Theorem 4.4.1).
- C is the cokernel map of F. It is $(\mathcal{R}(\beta), \mathcal{R}^\Lambda(\beta))$-bilinear but does *not* commute with t_i.

Remark 4.6.1. *The map B can be described as*

$$B\big(x_{n+1}^l a\tau_n \cdots \tau_k v(i,\beta)T_i^{\Lambda_i} \otimes \pi_i^{\Lambda_i+\mathrm{p}(\beta)} b\big) = \delta_{k,1} t_i^l T_i^{\Lambda_i} \otimes \pi_i^{\Lambda_i} \phi_i^{\Lambda_i}(a) b$$

for $a \in \mathcal{R}(\beta)$ and $b \in \mathcal{R}^\Lambda(\beta)$. Then

$$\begin{aligned}
&B\left((x_{n+1}^l a\tau_n \cdots \tau_k v(i,\beta)T_i^{\Lambda_i} \otimes \pi_i^{\Lambda_i+\mathrm{p}(\beta)} b)t_i\right)\\
&= B\left(\delta_{k,1}(-1)^{\mathrm{p}(i)(\Lambda_i+\mathrm{p}(\beta))}(x_{n+1}^l a\tau_n \cdots \tau_k v(i,\beta)T_i^{\Lambda_i} t_i \otimes \pi_i^{\Lambda_i+\mathrm{p}(\beta)} \phi_i(b)\right)\\
&= B\left(\delta_{k,1}(x_{n+1}^l a x_{n+1}\tau_n \cdots \tau_k v(i,\beta)T_i^{\Lambda_i} \otimes \pi_i^{\Lambda_i+\mathrm{p}(\beta)} \phi_i(b)\right)\\
&= B\left(\delta_{k,1}(x_{n+1}^{l+1} \phi_i(a)\tau_n \cdots \tau_k v(i,\beta)T_i^{\Lambda_i} \otimes \pi_i^{\Lambda_i+\mathrm{p}(\beta)} \phi_i(b)\right)\\
&= \delta_{k,1} t_i^{l+1} T_i^{\Lambda_i} \otimes \pi_i^{\Lambda_i} \phi_i^{\Lambda_i+1}(a)\phi_i(b).
\end{aligned}$$

On the other hand,

$$B\left((x_{n+1}^{l}a\tau_n\cdots\tau_k v(i,\beta)T_i^{\Lambda_i}\otimes\pi_i^{\Lambda_i+\mathrm{p}(\beta)}b)\right)t_i=\delta_{k,1}(t_i^lT_i^{\Lambda_i}\otimes\pi_i^{\Lambda_i}\phi_i^{\Lambda_i}(a)b)t_i$$
$$=\delta_{k,1}t_i^{l+1}T_i^{\Lambda_i}\otimes\pi_i^{\Lambda_i}\phi_i^{\Lambda_i+1}(a)\phi_i(b).$$

Thus B is right $(\mathbf{k}[t_i]\otimes\mathcal{R}(\beta))$-linear.

Define

$$\mathbf{T}=T_i^{\Lambda_i}\otimes\pi_i^{\Lambda_i}1\in\mathbf{k}[t_i]T_i^{\Lambda_i}\otimes\Pi_i^{\Lambda_i}\mathcal{R}^{\Lambda}(\beta),\quad \mathbf{T}_1=v(i,\beta)T_i^{\Lambda_i}\otimes\pi_i^{\Lambda_i+\mathrm{p}(\beta)}1\in L_1.$$

The element $\mathbf{T}$ has $\mathbb{Z}_2$-degree $\mathrm{p}(i)\Lambda_i$ and $\mathbf{T}_1$ has $\mathbb{Z}_2$-degree $\mathrm{p}(i)(\Lambda_i+\mathrm{p}(\beta))$. Note that

$$\mathbf{T}t_i=t_i\mathbf{T}\quad\text{ and }\quad\mathbf{T}_1t_i=(-1)^{\mathrm{p}(i)\mathrm{p}(\beta)}t_i\mathbf{T}_1.$$

Let p be the number of α_i appearing in β. Define an invertible element $\gamma\in\mathbf{k}^{\times}$ by

$$(-1)^{\mathrm{p}(i)\Lambda_i\mathrm{p}(\beta)+p}\prod_{\substack{1\le a\le n,\\ \nu_a\neq i}}\mathcal{Q}_{i,\nu_a}(t_i,x_a)\prod_{\substack{1\le a\le n,\\ \nu_a=i\in I_{\mathrm{odd}}}}(x_a-t_i)^2$$
$$=\gamma^{-1}t_i^{-\langle h_i,\beta\rangle+2(1+\mathrm{p}(i))p}+\big(\text{terms of degree}<-\langle h_i,\beta\rangle+2(1+\mathrm{p}(i))p\text{ in }t_i\big).$$

Note that γ does not depend on $\nu\in I^{\beta}$.

Set $\lambda=\Lambda-\beta$ and

$$\varphi_k=A(\mathbf{T}t_i^k)\in\mathbf{k}[t_i]\otimes\mathcal{R}^{\Lambda}(\beta).\tag{4.40}$$

From now on, we investigate the kernel and cokernel of the map A which are the key ingredients of the proof of Theorem 4.6.3 below. For this purpose, the following proposition is crucial.

Proposition 4.6.2. *The element $\gamma\varphi_k$ is a monic (skew)-polynomial in t_i of degree $\langle h_i,\lambda\rangle+k$.*

Here and in the sequel, for $m < 0$, we say that a (skew)-polynomial φ is a monic polynomial of degree m if $\varphi = 0$.

Define a map $E\colon L_0' \to \mathcal{R}^\Lambda(\beta)$ by

$$a \otimes \pi_i b \mapsto a\phi_i(b) \quad \text{for } a \in \mathcal{R}(\beta)e(\beta-\alpha_i, i) \text{ and } b \in e(\beta-\alpha_i, i)\mathcal{R}^\Lambda(\beta).$$

We define the endomorphism $(x_n \otimes 1)$ of L_0' by

$$(a \otimes \pi_i b)(x_n \otimes 1) = (-1)^{\mathrm{p}(i)} a x_n \otimes \pi_i \phi_i(b).$$

Lemma 4.6.2. *Let*

$$L_0' := \mathcal{R}(\beta)e(\beta-\alpha_i, i) \otimes_{\mathcal{R}(\beta-\alpha_i)} \Pi_i e(\beta-\alpha_i, i)\mathcal{R}^\Lambda(\beta).$$

Then for any $z \in L_0'$, we have

$$F(z)t_i = F(z(x_n \otimes 1)) + e(\beta, i) \otimes E(z). \tag{4.41}$$

Proof. We may assume $z = a \otimes \pi_i b$. Note that

$$F(z) = a\tau_n e(\beta-\alpha_i, i^2) \otimes b, \qquad E(z) = a\phi_i(b).$$

Thus

$$\begin{aligned} F(z)t_i &= a\tau_n e(\beta-\alpha_i, i^2)x_{n+1} \otimes \phi_i(b) \\ &= a((-1)^{\mathrm{p}(i)} x_n\tau_n + 1)e(\beta-\alpha_i, i^2) \otimes \phi_i(b) \\ &= (-1)^{\mathrm{p}(i)} a x_n \tau_n e(\beta-\alpha_i, i^2) \otimes \phi_i(b) + ae(\beta-\alpha_i, i^2) \otimes \phi_i(b) \\ &= (-1)^{\mathrm{p}(i)} F(a x_n \otimes \pi_i\phi_i(b)) + e(\beta, i) \otimes E(z) \\ &= F(z(x_n \otimes 1)) + e(\beta, i) \otimes E(z). \end{aligned}$$

□

By Theorem 4.4.1, we have

$$\begin{aligned} &e(\beta, i)\mathcal{R}(\beta+\alpha_i)e(\beta, i) \otimes_{\mathcal{R}(\beta)} \mathcal{R}^\Lambda(\beta) \\ &\quad = F\big((\mathcal{R}(\beta)e(\beta-\alpha_i, i) \otimes_{\mathcal{R}(\beta-\alpha_i)} e(\beta-\alpha_i, i)\mathcal{R}^\Lambda(\beta))\big) \\ &\qquad \oplus e(\beta, i)(\mathbf{k}[t_i] \otimes \mathcal{R}^\Lambda(\beta)). \end{aligned} \tag{4.42}$$

Then reasoning as in 3.69, we may write

$$P(e(\beta,i)\tau_n\cdots\tau_1\mathbf{T}_1t_i^k) = F(\psi_k) + e(\beta,i)\varphi_k$$

for uniquely determined $\psi_k \in L_0'$ and $\varphi_k \in \mathbf{k}[t_i]\otimes\mathcal{R}^\Lambda(\beta)$. On the other hand, we have

$$\begin{aligned}A(\mathbf{T}t_i^k) &= AB(e(\beta,i)\tau_n\cdots\tau_1\mathbf{T}_1t_i^k)\\ &= CP(e(\beta,i)\tau_n\cdots\tau_1\mathbf{T}_1t_i^k) = \varphi_k.\end{aligned}$$

Hence the definition of φ_k coincides with the definition given in (4.40). Note that

$$\begin{aligned}F(\psi_{k+1}) + e(\beta,i)\varphi_{k+1} &= P(e(\beta,i)\tau_n\cdots\tau_1\mathbf{T}_1t_i^{k+1})\\ &= P(e(\beta,i)\tau_n\cdots\tau_1\mathbf{T}_1t_i^k)t_i\\ &= (F(\psi_k)+e(\beta,i)\varphi_k)t_i\\ &= F(\psi_k(x_n\otimes 1)) + e(\beta,i)E(\psi_k) + e(\beta,i)\varphi_kt_i\,,\end{aligned}$$

which yields

$$\psi_{k+1} = \psi_k(x_n\otimes 1), \quad \varphi_{k+1} = E(\psi_k)+\varphi_kt_i. \tag{4.43}$$

Now we will prove Proposition 4.6.2. By Corollary 3.7.1, the equality

$$\begin{aligned}&g_n\cdots g_1x_1^ke(i,\nu)\mathfrak{a}^\Lambda(x_1)\tau_1\cdots\tau_n\\ &= (-1)^{(k+\Lambda_i)\mathrm{p}(i)\mathrm{p}(\beta)}x_{n+1}^k\mathfrak{a}_i^\Lambda(x_{n+1})\prod_{\substack{1\le a\le n,\\ \nu_a\ne i}}\mathcal{Q}_{\nu_a,i}(x_a,x_{n+1})\prod_{\substack{1\le a\le n,\\ \nu_a=i\in I_{\mathrm{odd}}}}(x_a-x_{n+1})^2e(\nu,i)\end{aligned}$$

holds in $\mathcal{R}(\beta+\alpha_i)e(\beta,i)\otimes_{\mathcal{R}(\beta)}\mathcal{R}^\Lambda(\beta)$, which implies

$$\begin{aligned}&AB(g_n\cdots g_1x_1^ke(i,\nu)\mathbf{T}_1)\\ &= C\Big((-1)^{(k+\Lambda_i)\mathrm{p}(i)\mathrm{p}(\beta)}x_{n+1}^k\mathfrak{a}_i^\Lambda(x_{n+1})\prod_{\substack{1\le a\le n,\\ \nu_a\ne i}}\mathcal{Q}_{\nu_a,i}(x_a,x_{n+1})\prod_{\substack{1\le a\le n,\\ \nu_a=i\in I_{\mathrm{odd}}}}(x_a-x_{n+1})^2e(\nu,i)\Big)\\ &= (-1)^{(k+\Lambda_i)\mathrm{p}(i)\mathrm{p}(\beta)}t_i^k\mathfrak{a}_i^\Lambda(t_i)\prod_{\substack{1\le a\le n,\\ \nu_a\ne i}}\mathcal{Q}_{\nu_a,i}(x_a,t_i)\prod_{\substack{1\le a\le n,\\ \nu_a=i\in I_{\mathrm{odd}}}}(x_a-t_i)^2e(\nu).\end{aligned}$$

On the other hand, since B is the map taking the coefficient of $\tau_n \cdots \tau_1$, we have

$$\begin{aligned}
&B(g_n \cdots g_1 x_1^k e(i,\nu)\mathbf{T}_1) \\
&= B\left((-1)^{k\mathrm{p}(i)\mathrm{p}(\beta)} x_{n+1}^k \prod_{\nu_a=i} -(x_{n+1}^{1+\mathrm{p}(i)} - x_a^{1+\mathrm{p}(i)})^2 e(\nu,i)\tau_n \cdots \tau_1 \mathbf{T}_1\right) \\
&= (-1)^{k\mathrm{p}(i)\mathrm{p}(\beta)+p}\, t_i^k \prod_{\nu_a=i} (t_i^{1+\mathrm{p}(i)} - x_a^{1+\mathrm{p}(i)})^2 \mathbf{T} e(\nu).
\end{aligned}$$

Thus we have

$$\begin{aligned}
&A(t_i^k \prod_{\nu_a=i} (t_i^{1+\mathrm{p}(i)} - x_a^{1+\mathrm{p}(i)})^2 \mathbf{T} e(\nu)) \\
&= (-1)^{\Lambda_i \mathrm{p}(i)\mathrm{p}(\beta)+p}\, t_i^k \mathfrak{a}_i^\Lambda(t_i) \prod_{\substack{1\le a\le n,\\ \nu_a\neq i}} \mathcal{Q}_{\nu_a,i}(x_a,t_i) \prod_{\substack{1\le a\le n,\\ \nu_a=i\in I_{\mathrm{odd}}}} (x_a - t_i)^2 e(\nu).
\end{aligned} \tag{4.44}$$

Set

$$\mathsf{S}_i = \sum_{\nu\in I^\beta} \prod_{\nu_a=i} (t_i^{1+\mathrm{p}(i)} - x_a^{1+\mathrm{p}(i)})^2 e(\nu) \in \mathbf{k}[t_i] \otimes \mathcal{R}^\Lambda(\beta),$$

$$\mathsf{F}_i = \gamma(-1)^{\Lambda_i \mathrm{p}(i)\mathrm{p}(\beta)+p} \mathfrak{a}_i^\Lambda(t_i) \sum_{\nu\in I^\beta} \Big(\prod_{\substack{1\le a\le n,\\ \nu_a\neq i}} \mathcal{Q}_{i,\nu_a}(t_i,x_a) \prod_{\substack{1\le a\le n,\\ \nu_a=i\in I_{\mathrm{odd}}}} (x_a - t_i)^2 e(\nu) \Big)$$
$$\in \mathbf{k}[t_i] \otimes \mathcal{R}^\Lambda(\beta).$$

Then they are monic (skew)-polynomials in t_i of degree $2(1+\mathrm{p}(i))p$ and $\langle h_i, \lambda\rangle + 2(1+\mathrm{p}(i))p$, respectively. Note that S_i is contained in the center of $\mathbf{k}[t_i] \otimes \mathcal{R}^\Lambda(\beta)$ and F_i commutes with t_i. Hence (4.44) can be expressed in the following form:

$$\gamma A(t_i^k \mathsf{S}_i \mathbf{T}) = t_i^k \mathsf{F}_i. \tag{4.45}$$

Lemma 4.6.3. *For any $k \ge 0$, we have*

$$t_i^k \mathsf{F}_i = (\gamma\varphi_k)\mathsf{S}_i + \mathsf{h}_k,$$

where $\mathsf{h}_k \in \mathbf{k}[t_i] \otimes \mathcal{R}^\Lambda(\beta)$ is a polynomial in t_i of degree $< 2(1+\mathrm{p}(i))p$. In particular, $\gamma\varphi_k$ coincides with the quotient of $t_i^k \mathsf{F}_i$ by S_i.

Proof. By (4.43),

$$A(at_i) - A(a)t_i \in \mathbf{k}[t_i] \otimes \mathcal{R}^\Lambda(\beta) \text{ is of degree } \le 0 \text{ in } t_i, \tag{4.46}$$

for any $a \in \mathbf{k}[t_i]T_i^\Lambda \otimes \Pi_i^{\Lambda_i}\mathcal{R}^\Lambda(\beta)$. We will show

(4.47)
for any polynomial f in the center of $\mathbf{k}[t_i] \otimes \mathcal{R}^\Lambda(\beta)$ in t_i of degree $m \in \mathbb{Z}_{\ge 0}$ and $a \in \mathbf{k}[t_i]T_i^{\Lambda_i} \otimes \mathcal{R}^\Lambda(\beta)$, $A(af) - A(a)f$ is of degree $< m$.

We will use induction on m. Since A is right $\mathcal{R}^\Lambda(\beta)$-linear, (4.47) holds for $m = 0$. Thus it suffices to show (4.47) when $f = t_i g$. By the induction hypothesis, (4.47) is true for g. Then we have

$$A(af) - A(a)f = (A(at_i g) - A(at_i)g) + (A(at_i) - A(a)t_i)g.$$

It follows that the first term is of degree $< \deg(g)$ in t_i and the second term is of degree $< \deg(g) + 1$, which proves (4.47). Thus we have

$$t_i^k \gamma^{-1} \mathsf{F}_i - \varphi_k \mathsf{S}_i = t_i^k \gamma^{-1} \mathsf{F}_i - A(t_i^k \mathbf{T})\mathsf{S}_i = A(t_i^k \mathsf{S}_i \mathbf{T}) - A(t_i^k \mathbf{T})\mathsf{S}_i$$

by (4.45) and it is of degree $< 2(1 + \mathrm{p}(i))p$ by applying (4.47) for $f = \mathsf{S}_i$. $\square$

Therefore, by Lemma 4.6.3, we conclude $\gamma\varphi_k$ is a monic (skew)-polynomial in t_i of degree $\langle h_i, \lambda\rangle + k$, which completes the proof of Proposition 4.6.2.

Applying the arguments given in Theorem 3.7.4 and Theorem 3.7.5, we have the following theorem.

Theorem 4.6.3. *Let $\lambda = \Lambda - \beta$. Then there exist natural isomorphisms of endofunctors on $\mathcal{M}\mathrm{od}(\mathcal{R}^\Lambda(\beta))$ given below.*

(i) *If $i \ne j$, then we have*

$$E_i^\Lambda F_j^\Lambda \xrightarrow{\sim} q^{-(\alpha_i|\alpha_j)} \Pi^{\mathrm{p}(i)\mathrm{p}(j)} F_j^\Lambda E_i^\Lambda.$$

(ii) *If $\langle h_i, \lambda\rangle \ge 0$, then we have*

$$\Pi_i q_i^{-2} F_i^\Lambda E_i^\Lambda \oplus \bigoplus_{k=0}^{\langle h_i,\lambda\rangle - 1} \Pi_i^k q_i^{2k} \xrightarrow{\sim} E_i^\Lambda F_i^\Lambda.$$

(iii) *If $\langle h_i, \lambda\rangle < 0$, then we have*

$$\Pi_i q_i^{-2} F_i^\Lambda E_i^\Lambda \xrightarrow{\sim} E_i^\Lambda F_i^\Lambda \oplus \bigoplus_{k=0}^{-\langle h_i,\lambda\rangle - 1} \Pi_i^{k+1} q_i^{-2k-2}.$$

4.7 Supercategorification

In this section, we will show that the supercategories consisting of $\mathcal{R}^\Lambda(\beta)$-modules and $\mathcal{R}(\beta)$-modules give supercategorifications of $U_{\mathbb{A}}^-(\mathfrak{g})$ and $V_{\mathbb{A}}(\Lambda)$. Here, we need to recall the Cartan matrix is colored by I_{odd}. We prove these by using the same arguments given in Section 3.8.

We assume that (3.9); i.e.,

$$\mathbf{k}_0 \text{ is a field and the components } \mathbf{k}_t \text{ are finite-dimensional over } \mathbf{k}_0$$

Recall the result of Theorem 4.3.1; i.e.,

$$\Pi \text{ acts as the identity on } [\mathcal{R}\mathrm{ep}(R(\beta))] \text{ and } [\mathcal{P}\mathrm{roj}(R(\beta))].$$

Thus, although the natural isomorphisms in Theorem 4.6.3 are different from the ones of Theorem 3.7.5, we obtain the following identities in $[\mathcal{P}\mathrm{roj}\mathcal{R}^\Lambda(\beta)]$ and $[\mathcal{R}\mathrm{ep}\mathcal{R}^\Lambda(\beta)]$ as in Theorem 3.7.5:

$$\begin{aligned}
&\mathsf{E}_i\mathsf{F}_j = \mathsf{F}_j\mathsf{E}_i \quad \text{if } i \neq j,\\
&\mathsf{E}_i\mathsf{F}_i = \mathsf{F}_i\mathsf{E}_i + \frac{q_i^{\langle h_i,\Lambda-\beta\rangle} - q_i^{-\langle h_i,\Lambda-\beta\rangle}}{q_i - q_i^{-1}} \quad \text{if } \langle h_i, \Lambda-\beta\rangle \geq 0,\\
&\mathsf{E}_i\mathsf{F}_i + \frac{q_i^{-\langle h_i,\Lambda-\beta\rangle} - q_i^{\langle h_i,\Lambda-\beta\rangle}}{q_i - q_i^{-1}} = \mathsf{F}_i\mathsf{E}_i \quad \text{if } \langle h_i, \Lambda-\beta\rangle \leq 0.
\end{aligned} \tag{4.48}$$

Hence they are summarized as

$$[\mathsf{E}_i, \mathsf{F}_j] = \delta_{i,j}\frac{\mathsf{K}_i - \mathsf{K}_i^{-1}}{q_i - q_i^{-1}}$$

(See (3.77) and (3.79) for the definition of E_i, F_i and K_i, respectively.)

Using $\mathcal{P}(i^n)$ and (4.14), we can also define the endomorphisms $\mathsf{E}_i/[n]_i!$ and $\mathsf{F}_i/[n]_i!$ on $[\mathcal{R}\mathrm{ep}(\mathcal{R}(\beta))]$ and $[\mathcal{P}\mathrm{roj}(\mathcal{R}(\beta))]$. Moreover, Lemma 4.5.1 implies that

$$\text{the action } \mathsf{F}_i \text{ on } [\mathcal{P}\mathrm{roj}(\mathcal{R}(\beta))] \text{ and } [\mathcal{R}\mathrm{ep}(\mathcal{R}(\beta))] \text{ is locally nilpotent.}$$

Thus, by [23, Proposition B.1],

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \mathsf{E}_i^{(1-a_{ij}-r)}\mathsf{E}_j\mathsf{E}_i^{(r)} = \sum_{r=0}^{1-a_{ij}} (-1)^r \mathsf{F}_i^{(1-a_{ij}-r)}\mathsf{F}_j\mathsf{F}_i^{r} = 0.$$

Moreover, by Theorem 4.3.1

(i) $[\mathcal{P}\mathrm{roj}(\mathcal{R})] := \bigoplus_{\beta \in Q^+} [\mathcal{P}\mathrm{roj}(\mathcal{R}(\beta))]$ and $[\mathcal{R}\mathrm{ep}(\mathcal{R})] := \bigoplus_{\beta \in Q^+} [\mathcal{R}\mathrm{ep}(\mathcal{R}(\beta))]$ are indeed $\mathbb{A}$-dual to each other,

(ii) (4.21) can be expressed as

$$\mathsf{E}'_i \mathsf{F}'_j = q^{-(\alpha_i|\alpha_j)} \mathsf{F}'_j \mathsf{E}'_i + \delta_{i,j},$$

as an endomorphism of $[\mathcal{P}\mathrm{roj}(\mathcal{R})]$ and $[\mathcal{R}\mathrm{ep}(\mathcal{R})]$ (cf. (3.82)).

Let $\mathcal{I}\mathrm{rr}_0(\mathcal{R}^\Lambda(\beta))$ be the set of isomorphism classes of self-dual irreducible $\mathcal{R}^\Lambda(\beta)$-modules, and $\mathcal{I}\mathrm{rr}_0(\mathcal{R}^\Lambda) := \bigsqcup_{\beta \in Q^+} \mathcal{I}\mathrm{rr}_0(\mathcal{R}^\Lambda(\beta))$. Then $\{[S] \mid S \in \mathcal{I}\mathrm{rr}_0(\mathcal{R}^\Lambda)\}$ is a strong perfect basis of $[\mathcal{R}\mathrm{ep}(\mathcal{R}^\Lambda)]$ by Theorem 4.3.2. Applying the arguments given in Theorem 3.8.1 and Corollary 3.8.1, we have

Theorem 4.7.1. *Let $U_q(\mathfrak{g})$ be the quantum Kac-Moody algebra associated with the Cartan matrix colored by I_{odd}. For $\Lambda \in P^+$, we have the following isomorphisms:*

(a) $V_{\mathbb{A}}(\Lambda)^\vee \simeq [\mathcal{R}\mathrm{ep}(\mathcal{R}^\Lambda)]$ *and* $V_{\mathbb{A}}(\Lambda) \simeq [\mathcal{P}\mathrm{roj}(\mathcal{R}^\Lambda)]$ *as* $U_{\mathbb{A}}(\mathfrak{g})$*-modules,*

(b) $U^-_{\mathbb{A}}(\mathfrak{g})^\vee \simeq [\mathsf{Rep}(\mathcal{R})]$ *as a* $B^{\mathrm{up}}_{\mathbb{A}}(\mathfrak{g})$*-module,*

(c) $U^-_{\mathbb{A}}(\mathfrak{g}) \simeq [\mathsf{Proj}(\mathcal{R})]$ *as a* $B^{\mathrm{low}}_{\mathbb{A}}(\mathfrak{g})$*-module.*

Bibliography

[1] S. Ariki, *On the decomposition numbers of the Hecke algebra of $G(m,1,n)$,* J. Math. Kyoto Univ. **36** (1996), 789–808.

[2] A. Berenstein, D. Kazhdan, *Geometric and unipotent crystals. II. From unipotent bicrystals to crystal bases*, Quantum groups, 13–88, Contemp. Math., 433, Amer. Math. Soc., Providence, RI, 2007.

[3] R. E. Borcherds, Generalized Kac-Moody algebras, J. Algebra **115** (1988), 501–512.

[4] J. Brundan, A. Kleshchev, *Hecke-Clifford superalgebras, crystals of type $A_{2l}^{(2)}$ and modular branching rules for $\hat{S}_n$*, Represent. Theory **5** (2001), 317–403.

[5] J. Brundan and A. Kleshchev, *Blocks of cyclotomic Hecke algebras and Khovanov-Lauda algebras*, Invent. Math. **178** (2009), no. 3, 451–484.

[6] L. Crane, I. B. Frenkel, *Four-dimensional topological quantum field theory, Hopf categories, and the canonical bases*, J. Math. Phys. **35** (1994), 5136-5154.

[7] J. Chuang, R. Rouquier, *Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification*, Ann. Math. **167** (2008), 245–298.

[8] A. Ellis, M. Khovanov, A. Lauda, *The odd nilHecke algebra and its diagrammatics*, arXiv:1111.1320.

[9] I. B. Frenkel, M. Khovanov, *Canonical bases in tensor products and graphical calculus for* $U_q(\mathfrak{sl}_2)$, Duke Math. J. **87** (1997), 409-480.

[10] J. Hong and S.-J. Kang, *Introduction to Quantum Groups and Crystal Bases*, Grad. Stud. Math. 42, Amer. Math. Soc., Providence, RI, 2002.

[11] J. Hu and A. Mathas, *Graded cellular bases for the cyclotomic Khovanov-Lauda- Rouquier algebras of type A*, Adv. Math. **225** (2) (2010), 598-642.

[12] K. Jeong, S.-J. Kang and M. Kashiwara , Crystal bases for quantum Generalized Kac–Moody Algebras, Proc. London Math. Soc. **(3) 90** (2005), 395–438.

[13] S.-J. Kang, Quantum deformations of generalized Kac–Moody algebras and their modules, J. Algebra **175** (1995), 1041–1066.

[14] S.-J. Kang, M. Kashiwara, *Categorification of highest weight modules via Khovanov-Lauda-Rouquier algebras*, arXiv:1102.4677 (2011), to appear in Invent. Math.

[15] S.-J. Kang, M. Kashiwara, S.-j. Oh, *Categorification of highest weight modules over quantum generalized Kac-Moody algebras*, arXiv:1106.2635 (2011), to appear in Mosc. Math. J.

[16] S.-J. Kang, M. Kashiwara, S.-j. Oh, *Supercategorification of quantum Kac-Moody algebras*, arXiv:1206.5933

[17] S.-J. Kang, M. Kashiwara and S. Tsuchioka, *Quiver Hecke superalgebras*, arXiv :1107.1039.

[18] S.-J. Kang, S.-j. Oh, E. Park, *Perfect bases for integrable modules over generalized Kac-Moody algebras*, Algebr. Represent. Theory **14** (2011) no. 3, 571–587.

[19] S.-J. Kang, S.-j. Oh, E. Park, *Categorification of quantum generalized Kac-Moody algebras and crystal bases*, arXiv:1102.5165 (2011), to appear in Int. J. Math.

[20] S.-J. Kang and E. Park, *Irreducible modules over Khovanov-Lauda-Rouquier algebras of type A_n and semistandard tableaux*, J. Algebra **339** (2011), 223-251.

[21] M. Kashiwara, *On crystal bases of the q-analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), no. 2, 465–516.

[22] M. Kashiwara, *Global crystal bases of quantum groups*, Duke Math. J. **69** (1993), no. 2, 455–485.

[23] M. Kashiwara, T. Miwa, J. Petersen, C.-M. Yung, *Perfect crystals and q-deformed Fock spaces*, Selecta Math. (N.S.) **2** (1996), 415–499.

[24] M. Khovanov, A. Lauda, *A diagrammatic approach to categorification of quantum groups I*, Represent. Theory **13** (2009), 309–347.

[25] M. Khovanov and A. Lauda, *A diagramatic approach to categorification of quantum groups. II.*, Trans. Amer. Math. Soc. **363** (2011), 2685-2700.

[26] A. Kleshchev, *Linear and Projective Representations of Symmetric Groups*, Cambridge Tracts in Math., 163, Cambridge University Press, Cambridge, 2005.

[27] A. Lascoux, B. Leclerc, J.-Y. Thibon, *Hecke algebras at roots of unity and crystal bases of quantum affine algebras*, Comm. Math. Phys. **181** (1996), 205–263.

[28] A. Lauda and M. Vazirani, *Crystals from categorified quantum groups*, Adv. Math **228** (2) (2011), 803-861.

[29] G. Lusztig, *Introduction to Quantum Groups*, Birkhäuser, Boston, 1993.

[30] K. C. Misra, T. Miwa, *Crystal base for the basic representation of $U_q(\widehat{\mathfrak{sl}}(n))$*, Commun. Math. Phys. **134** (1990), 79–88.

[31] B. Sevenhant and M. Van Den Bergh, *A relation between a conjecture of Kac and the structure of the Hall algebra*, J. Pure Appl. Algebra **160** (2001), no. 2-3, 319–332.

[32] S. Tsuchioka, *Hecke-Clifford superalgebras and crystals of type* $D_l^{(2)}$, Publ. Res. Inst. Math. Sci. **46** (2010), 423–471.

[33] R. Rouquier, *2-Kac-Moody algebras*, (2008), arXiv:08125023.

[34] M. Varagnolo, E. Vasserot, *Canonical bases and KLR algebras*, J. Reine Angew. Math. **659** (2011), 67-100.

국문초록

우리는 일반화된 양자 캐츠-무디 대수 $U_q(\mathfrak{g})$의 최고치 모듈 $V(\Lambda)$가 퀴버 헥케 대수의 cyclotomic quotient R^{Λ}를 이용하여 카테고리화 됨을 보이고 또한 퀴버 헥케 슈퍼대수의 cyclotomic quotient $\mathcal{R}^{\Lambda}$를 이용하여 슈퍼카테고리화 됨을 증명 하였다. 뿐만 아니라 일반화된 양자 캐츠-무디 대수의 음의 부분대수 $U_q(\mathfrak{g})^-$가 최고치 모듈들의 projective 극한임을 이용하여, 퀴버 헥케 대수 R이 $U_q(\mathfrak{g})^-$을 카테고리화하고 퀴버 헥케 슈퍼대수 $\mathcal{R}$이 $U_q(\mathfrak{g})^-$를 슈퍼카테고리화함을 증명하였다.

주요어휘: 카테고리화, 완전 기저, 일반화된 양자 캐츠-무디 대수, 퀴버 헥케 대수, 퀴버 헥케 슈퍼대수, 슈퍼카태고리화
학번: 2007-20281

감사의 글

가장 먼저 6년이라는 시간동안, 아니 학부를 포함해서 13년이라는 긴 시간동안 부족한 저를 지도해 주시고 보살펴 주신 강석진 교수님께 깊은 감사의 마음을 전하려 합니다. 선생님께서 지난 시간 저에게 보여주신 학문에 대한 열정과 자세, 그리고 끊임없는 노력을 조금이나마 닮아가는 것이 제가 제자된 최소한의 도리가 아닐까 생각해봅니다. 선생님께서 손수 보여주신 모습을 본받아 좋은 수학자가 되도록 더 열심히 노력하겠습니다.

부족한 시간을 할애하여 적지 않은 양의 제 졸업 논문을 읽고 평가해주신 Georgia Benkart 선생님, 조영현 선생님, 변동호 선생님, 오병권 선생님, 권재훈 선생님께도 깊은 감사의 말씀을 전하고 싶습니다. 심사위원 선생님들의 애정어린 조언과 지적이 없었다면 이 논문을 완성하기가 많이 힘들었을 것입니다. 학부와 대학원을 걸쳐 저에게 수학의 아름다움과 매력을 알려주신 김명환 선생님, 이인석 선생님, 김혁 선생님, 이우영 선생님, 김영훈 선생님께 감사드립니다. 부족하고 서툰 저를 지도해 주신 모든 선생님들께 정말 감사드립니다. 특히, 두편의 논문을 저와 함께 해주시고 어리석고 부족한 저의 질문에 너그러히 대답해주신 Masaki Kashiwara 선생님께 감사의 말씀을 전합니다.

졸업하시고 교수님이 되셨음에도 후배들을 잘 챙겨주신 오영탁 선생님, 이규환 선생님, 이동일 선생님 감사합니다. 그리고 모든 선배님들 감사합니다. 저와 함께 세편의 논문을 함께 해주신 의용이형 감사합니다. 저와 함께 공부한 명호, 지혜, 대홍, 한솔, 미란이에게도 지면을 빌어 감사하다는 말 전합니다. 대학원 입학부터 같이 고민하고 공부한 동기들에게도 고맙다는 말 전합니다. 특히 저의 고민과 질문들을 잘 받아주는 경석이, 현호에게 고맙다는 말 전합니다. 그리고 저의 학교 생활을 아름답고 활기차게 해준 모든 자연대 축구부원들에게 깊은 감사의 말을 전하고 싶습니다. 제가 언급하지 못했지만 저를 아껴주는 모든 분들께 감사드립니다.

많이 부족한 저를 평생의 반려자로 허락해 준 저의 반쪽 혜연이에게 평소에 잘 하지 못하는 사랑한다는 말을 전하고 싶습니다. 제가 여기까지 오는데 있어 절대적으로 저를 지지해주고 믿어주신 부모님과 누나들에게 감사의 마음을 전하고 싶습니다. 아들로서, 동생으로서 그리고 남편으로서 부끄럽지 않도록 더 열심히 살겠습니다. 사랑합니다.