이학박사 학위논문

Homotopy cyclic A-infinity algebras, potentials and related cohomology theories

(호모토피 순환 A-무한대수, 잠재함수와 코호몰로지 이론)

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Homotopy cyclic A-infinity algebras, potentials and related cohomology theories

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Abstract

Homotopy cyclic A-infinity algebras, potentials and related cohomology theories

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An A_∞-algebra has "associative up to homotopy" structure. For an A_∞-algebra A, we give a definition of strong homotopy inner products(if exist) which is the homotopy notion of cyclic inner products due to Kontsevich. From strong homotopy inner products we get several invariants which we call "potentials". We study their homotopy natures, gauge invariances etc. Also we find an explicit correspondence between cohomology elements of A and isomorphism classes of strong homotopy inner products on A.

Key words: A-infinity algebra, strong homotopy inner product, potential, negative cyclic cohomology, formal noncommutative manifold
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Contents

Chapter 1

Introduction

In this thesis we study a special kind of inner products on A_∞-algebras. Our first goal is to study invariants(which we call "potentials") defined by them, and the second goal is to find cohomology theories which parametrize such inner products.

A_∞-algebras are "homotopy-transferred algebras" from associative algebras, in the sense that if a chain complex B is homotopy equivalent to a differential graded algebra A, then via homotopy equivalence the associative product on A does not inherit an associative product on B, but if we collect all the failures of associativity, we get the A_∞-algebra structure on B. For the detail see [Va].

There are several motivations to study A_∞-structure. A_∞-algebras were first discovered by Stasheff[St] in his study of H-spaces. For example, if we have a singular cochain group of a topological space X which is an associative algebra by cup product, then its cohomology $H^*(X)$ also has the induced cup product, but some higher product structures which are known as Massey products are also hidden. They give rise to an A_∞-structure on $H^*(X)$.

A_∞-structures also appear in Fukaya-Oh-Ohta-Ono's work [FOOO1], [FOOO2]. They proved that there exist (filtered) A_∞-structures on the Floer cochain complexes of Lagrangian submanifolds on symplectic manifolds, and studied their obstructions to define Floer cohomologies on them. If we consider all Lagrangian submanifolds of a symplectic manifold M and Floer cohomologies between each other, we get an A_∞-category which is called the *Fukaya category* of M, and its derived category is a main ingredient of homological mirror symmetry due to Kontsevich[Ko2].

On an A_∞-algebra, we will define cyclic inner products and potentials. Although we postpone the definition of cyclic inner products to the next chapter, we just fix a notation for the inner product as

$$\langle\ ,\ \rangle : C \otimes C \to \boldsymbol{k}$$

where C is the underlying vector space of given A_∞-algebra.

Cyclic inner products appear in various contexts. Fukaya proved that on the filtered A_∞-algebra structure on a compact Lagrangian submanifold L in a compact(or convex at infinity) symplectic manifold M, a cyclic inner product is given by Poincaré duality[Fu1]. And Costello[Cos] proved that the category of open topological conformal field theory is homotopy equivalent to the category of Calabi-Yau categories which can be considered as categorifications of cyclic A_∞-algebras.

Unlike the homotopy nature of A_∞-algebras, a cyclic inner product is not preserved under homotopy equivalences. Hence it is very natural to search for the definition of "homotopy transferred inner products" as we have homotopy transferred algebra structures from associative products. This procedure is due to [C1] and will be explained in the next chapter. If an A_∞-algebra A is equipped with a strong homotopy inner product, we call A a *homotopy cyclic* A_∞*-algebra*. Cyclic inner products and strong homotopy inner products on A can be described as A_∞-bimodule homomorphisms from A to A^* satisfying several properties. In the perspective of noncommutative geometry, A_∞-algebras correspond to formal noncommutative manifolds(abbreviated formal manifolds from now on). If an A_∞-algebra A has a cyclic inner product, then it gives a constant symplectic form on the corresponding formal manifold. If A just admits a strong homotopy inner product, then it corresponds to a (possibly) nonconstant symplectic form. Note that cyclic inner products are special cases of strong homotopy inner products. If a strong homotopy inner product is given on A, we define potential Φ^A on coordinates of the corresponding formal manifold.

Definition 1.0.1. *Suppose that A is a homotopy cyclic A_∞-algebra with inner product given by a bimodule map $\phi : A \to A^*$. The* potential $\Phi^A(\boldsymbol{x})$ *is defined by*

$$\begin{aligned}\Phi^A(\boldsymbol{x}) &= \sum_{N=1}^{\infty} \Phi^A_N(\boldsymbol{x}) \\ &:= \sum_{N=1}^{\infty} \sum_{p+q+k=N} \frac{1}{N+1} \langle \boldsymbol{x}, \boldsymbol{x}, \cdots, \boldsymbol{x}, \underline{m^A_k(\boldsymbol{x}, \boldsymbol{x}, \cdots, \boldsymbol{x})}, \boldsymbol{x}, \cdots, \boldsymbol{x} | \boldsymbol{x} \rangle_{p,q}\end{aligned}$$

where $\boldsymbol{x} = \sum_i e_i x_i$, $\{e_i\}$ is a basis of vector space C, x_i are formal parameters with $\deg(x_i) = -\deg(e_i)$*.*

$$\langle \boldsymbol{x}, ..., \boldsymbol{x}, \underline{m^A_k(\boldsymbol{x}, ..., \boldsymbol{x})}, \boldsymbol{x}, ..., \boldsymbol{x} | \boldsymbol{x} \rangle_{p,q} := \phi_{p,q}(\boldsymbol{x}, ..., \boldsymbol{x}, \underline{m^A_k(\boldsymbol{x}, ..., \boldsymbol{x})}, \boldsymbol{x}, ..., \boldsymbol{x})(\boldsymbol{x}).$$

In particular, if ϕ is a cyclic inner product, then the potential is

$$\Phi^A(\boldsymbol{x}) = \sum_{k=1}^{\infty} \frac{1}{k+1} \langle m^A_k(\boldsymbol{x}, ..., \boldsymbol{x}), \boldsymbol{x} \rangle.$$

Now we state our first main theorem.

Theorem A. *Let A be an A_∞-algebra with a strong homotopy inner product ϕ is given. Suppose that we have an A_∞-quasi-isomorphism $h : B \to A$ with a commuting diagram*

$$\begin{array}{ccc} A & \xleftarrow{\tilde{h}} & B \\ {\scriptstyle \phi}\big\downarrow & & \big\downarrow{\scriptstyle \psi \,|\, cyc} \\ A^* & \xrightarrow{\tilde{h}^*} & B^* \end{array} \tag{1.0.1}$$

where B is a cyclic A_∞-algebra.

Then

$$\Phi^B = h^* \Phi^A.$$

Potentials of cyclic A_∞-algebras have been of interest among physicists, as called an action of a string field theory. The meaning of Theorem A is that they can be considered as homotopy invariants under adoption of strong homotopy inner products.

The second main theorem is a cohomological interpretation of strong homotopy inner products. It is motivated by the following theorem of Kontsevich-Soibelman[KS1].

Theorem 1.0.2 ([KS1] Theorem 10.2.2)**.** *For weakly unital, compact A_∞-algebra A, cyclic cohomology class which is homologically non-degenerate gives rise to a class of isomorphisms of cyclic inner products on a minimal model of A.*

Our theorem gives an explicit correspondence of this theorem.

Theorem B. *For a weakly unital compact A_∞-algebra A, a homologically non-degenerate negative cyclic cohomology class $[\phi]$ gives rise to an isomorphism class of strong homotopy inner products on A.*

Finally, we propose that there is another kind of potentials from a strong homotopy inner product, which is related to generalized holonomy maps due to [ATZ]. We give its definition and the statement of the final main theorem below.

Theorem C. *Let A be a unital homotopy cyclic A_∞-algebra. Define*

$$\Psi^A(\boldsymbol{x}) := \sum_{p,q\geq 0} \frac{1}{p+q+1} \langle \underbrace{\boldsymbol{x}, \cdots, \boldsymbol{x}}_{p}, \underline{\boldsymbol{x}}, \underbrace{\boldsymbol{x}, \cdots, \boldsymbol{x}}_{q} | I \rangle_{p,q}. \tag{1.0.2}$$

where $\boldsymbol{x} \in \mathcal{MC}(A)$.

Then Ψ^A is invariant under the gauge equivalence.

The thesis will be devoted to give explanations of various definitions, notations and facts which are implicit in the above statements, and to give proofs of Theorem A, B and C.

Chapter 2

A_∞-algebras and homotopy cyclicity

2.1 A_∞-algebra

From now on, all vector spaces are over $\boldsymbol{k}$.

Definition 2.1.1. *A (coassociative)* coalgebra *is a vector space C with an operation $\Delta : C \to C \otimes C$, which is called a* comultiplication*, together with a following commuting diagram:*

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta\otimes 1} \\ C \otimes C & \xrightarrow{1\otimes\Delta} & C \otimes C \otimes C. \end{array}$$

A coderivation *$\delta : C \to C$ is a linear map which satisfies*

$$\Delta \circ \delta = (\delta \otimes 1 + 1 \otimes \delta) \circ \Delta : C \to C \otimes C.$$

Next we recall the basic notions of A_∞-algebras. Let $\boldsymbol{k}$ be the field containing $\mathbb{Q}$ (for example $\mathbb{Q}, \mathbb{R}, \mathbb{C}$) with $\mathrm{char}(\boldsymbol{k}) = 0$. Let $C = \bigoplus_{j\in\mathbb{Z}} C^j$ be a graded vector space over $\boldsymbol{k}$. Consider its suspension $(C[1])^m = C^{m+1}$ and $|x_i|'$

is the shifted grading $|x_i| - 1$. The *tensor-coalgebra* of $C[1]$ over $\boldsymbol{k}$ is given by $BC := \bigoplus_{k\geq 1} T_k(C[1])$, where

$$T_k(C[1]) = \underbrace{C[1] \otimes \cdots \otimes C[1]}_{k}, \tag{2.1.1}$$

with the comultiplication $\Delta : BC \longrightarrow BC \otimes BC$ defined by

$$\Delta(v_1 \otimes \cdots \otimes v_n) := \sum_{i=1}^{n} (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_n). \tag{2.1.2}$$

It is easy to see that Δ is indeed a comultiplication. Now, consider a family of maps of degree one

$$m_k : T_k(C[1]) \to C[1], \quad \text{for } k = 1, 2, \cdots .$$

We can extend m_k uniquely to a coderivation

$$\widehat{m}_k(x_1 \otimes \cdots \otimes x_n) = \sum_{i=1}^{n-k+1} (-1)^{|x_1|' + \cdots + |x_{i-1}|'} x_1 \otimes \cdots \otimes m_k(x_i, \cdots, x_{i+k-1}) \otimes \cdots \otimes x_n \tag{2.1.3}$$

for $k \leq n$ and $\widehat{m}_k(x_1 \otimes \cdots \otimes x_n) = 0$ for $k > n$. Again, it is easy to check that δ is a coderivation.

The coderivation $\widehat{d} = \sum_{k=1}^{\infty} \widehat{m}_k$ is well-defined as a map from BC to BC. The A_∞-equations are equivalent to the equality $\widehat{d} \circ \widehat{d} = 0$, or equivalently,

Definition 2.1.2. *An* A_∞-algebra A *is a* $\mathbb{Z}$*-graded vector space* C *over a field* $\boldsymbol{k}$ *which is equipped with a family of multilinear maps(of degree 1)*

$$m_k : C[1]^{\otimes k} \to C[1], \ k \geq 1$$

which satisfies the following relation:

$$\sum_{k_1+k_2=k+1} \sum_{i=1}^{k_1-1} (-1)^{\epsilon} m_{k_1}(x_1, ..., x_{i-1}, m_{k_2}(x_i, ..., x_{i+k_2-1}), ..., x_k) = 0 \tag{2.1.4}$$

where $\epsilon = |x_1|' + |x_2|' + \cdots + |x_{i-1}|' := |x_1| - 1 + |x_2| - 1 + \cdots + |x_{i-1}| - 1$. *We also call* A *a* strong homotopy associative algebra.

Convention 2.1.3. *In the shifted environment, the signs obey Koszul convention, so from now on we write all signs as* $(-1)^{Kos}$*, and it is easy to recover them. Also we fix a ground field* $\boldsymbol{k}$ *of all* A_∞*-algebras. Finally, when we say* $A = (C, \{m_k\})$ *is an* A_∞*-algebra, it means that the underlying vector space of the* A_∞*-algebra* A *is* C *and* $m_1, m_2, \cdots$ *are defining multilinear maps. But later we will abuse notations as if* A *itself is again the underlying vector space, not taking new symbol* C*.*

Definition 2.1.4. *An element* $I \in C^0 = C^{-1}[1]$ *is called a* unit *if*

$$m_{k+1}(x_1, ..., I, ..., x_k) = 0 \text{ for } k \neq 1,$$

$$m_2(I, x) = (-1)^{|x|} m_2(x, I) = x.$$

If an A_∞*-algebra* A *has a unit, then we call* A *a* unital A_∞-algebra.

We give the first three relations of (2.1.4) explicitly.

- For $k = 1$, $m_1^2(a) = 0$ for all a, so m_1 is a differential.
- For $k = 2$,

$$m_1(m_2(a, b)) - m_2(m_1(a), b) - (-1)^{|a|'} m_2(a, m_1(b)) = 0,$$

  i.e. m_1 is a derivation with respect to m_2.
- For $k = 3$,

$$\begin{aligned} & m_2(m_2(a, b), c) \pm m_2(a, m_2(b, c)) \\ = \ & \pm m_1(m_3(a, b, c)) \pm m_3(m_1(a), b, c) \pm m_3(a, m_1(b), c) \pm m_3(a, b, m_1(c)), \end{aligned}$$

  i.e. m_2 may NOT be associative, but only associative *up to homotopy.*

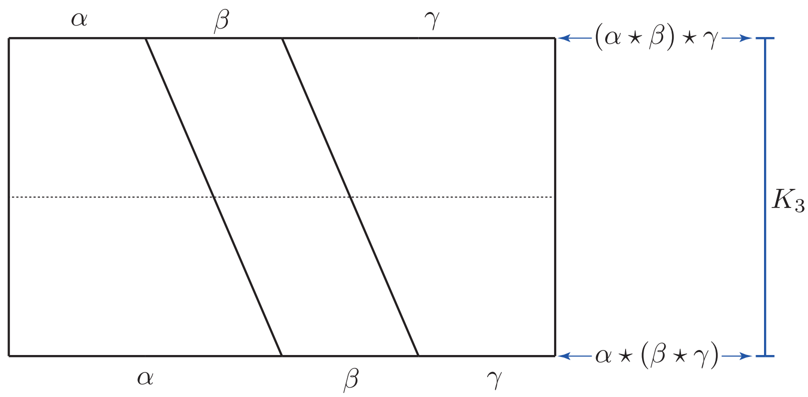


Figure 2.1: Homotopy between two loop products

The reason why we said "up to homotopy" is encoded in the Figure 2.1, which clearly describes a homotopy between loop products $(\alpha \star \beta) \star \gamma$ and $\alpha \star (\beta \star \gamma)$. They are not same, but are same up to homotopy.

Now we define an A_∞-homomorphism between two A_∞-algebras. Given two coalgebras C and D, a *map of coalgebras* is a linear map $\hat{f} : C \to D$ such that $\Delta_D \circ \hat{f} = \hat{f} \circ \Delta_C$, where Δ_C and Δ_D are comultiplications of C and D, respectively.

In the cotensor coalgebra case, the family of maps of degree 0

$$f_k : B_k C_1 \to C_2[1] \ \text{ for } k = 1, 2, \cdots$$

clearly induce the coalgebra map $\hat{f} : BC_1 \to BC_2$, which for $x_1 \otimes \cdots \otimes x_k \in B_k C_1$ is defined by the formula

$$\hat{f}(x_1 \otimes \cdots \otimes x_k) = \sum_{1 \le k_1 \le \cdots \le k_n \le k} f_{k_1}(x_1, \cdots, x_{k_1}) \otimes \cdots \otimes f_{k-k_n}(x_{k_n+1}, \cdots, x_k).$$

The map $\hat{f}$ is called an A_∞-*homomorphism* if

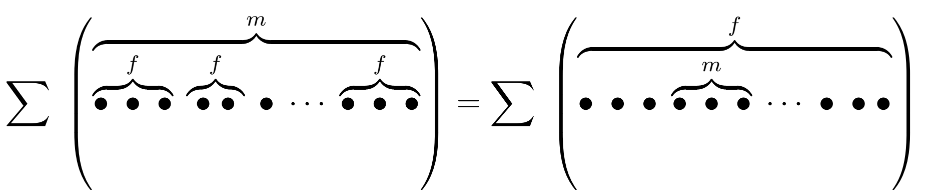
$$\hat{d} \circ \hat{f} = \hat{f} \circ \hat{d},$$

or equivalently,

$$d \circ \hat{f} = f \circ \hat{d}$$

where $d := pr \circ \hat{d}$, $f := pr \circ \hat{f}$ and pr is the projection $BC \to C[1]$.

The below is a pictorial description of an A_∞-homomorphism $\hat{f}$.



An A_∞-algebra $(C, \{m_i\})$ is called *compact* if $H^\bullet(C, m_1)$ is finite dimensional and is called *minimal* if $m_1 \equiv 0$.

2.2 Homotopy equivalence of A_∞-algebras

In this section we collect the definition and properties of homotopy equivalences of A_∞-algebras without proofs and details involved. For the proofs and the explicit constructions of models we refer to [FOOO1]. The idea for the definition of a homotopy between two A_∞-homomorphisms is to consider algebraic analogue of a homotopy between maps of topological spaces. Recall that a homotopy between two maps f and g between topological spaces X to Y is a map $H : [0,1] \times X \to Y$ such that $H|_{\{0\}\times X} = f$ and $H|_{\{1\}\times X} = g$. It leads us to the following definitions.

Definition 2.2.1. *Let $A = (C, \{m_k\})$ be an A_∞-algebra. An A_∞-algebra $\mathcal{A} = (\mathcal{C}, \{m_k\})$ is a* model *of $[0,1] \times A$ if there are A_∞-homomorphisms*

$$inc : A \to \mathcal{A}, \; ev_0 : \mathcal{A} \to A, \; ev_1 : \mathcal{A} \to A$$

such that

- *$inc_k : B_kC \to \mathcal{C}[1]$ is zero if $k \neq 1$, and the same holds for ev_0 and ev_1.*
- *$ev_0 \circ inc = ev_1 \circ inc = identity.$*

- *$inc_1 : (C, m_1) \to (\mathcal{C}, m_1)$ is a cochain homotopy equivalence, and $(ev_0)_1, (ev_1)_1 : (\mathcal{C}, m_1) \to (C, m_1)$ are cochain homotopy equivalences.*
- *The cochain homomorphism $(ev_0)_1 \oplus (ev_1)_1 : \mathcal{C} \to C \oplus C$ is surjective.*

Definition 2.2.2. *(Homotopy between two A_∞-homomorphisms) Let A, B be A_∞-algebras, and $f, g : A \to B$ be A_∞-homomorphisms. Let $\mathcal{B}$ be a model of $[0,1] \times B$.*

We say f is homotopic *to g via $\mathcal{B}$ and write $f \sim_{\mathcal{B}} g$, if there is an A_∞-homomorphism $F : A \to \mathcal{B}$ such that $ev_0 \circ F = f$, $ev_1 \circ F = g$. We call F the homotopy between f and g.*

Then [FOOO1] proves that the relation $\sim_{\mathcal{B}}$ does not depend on the choice of $\mathcal{B}$, and that it is indeed an equivalence relation. So we have a definition for homotopies of A_∞-homomorphisms. For A_∞-algebras A and B we say that an A_∞-homomorphism $f : A \to B$ is a *homotopy equivalence* if there is another A_∞-homomorphism $g : B \to A$ such that $f \circ g$ is homotopic to id_B and $g \circ f$ is homotopic to id_A(the identity map id_A of an A_∞-algebra $A = (C, \{m_k\})$ is given by $(id_A)_1 = id_C : C \to C$, and $(id_A)_k = 0$ for $k \neq 1$).

Remark 2.2.3. *Let A and B be A_∞-algebras, with underlying vector spaces C and D respectively. By taking an explicit model of $[0,1] \times B$, [FOOO1] also shows that an A_∞-homomorphisms $f, g : A \to B$ are homotopic if and only if there exists a family of maps $h_k : B_k C \to D[1]$ of degree -1 such that*

$$\begin{aligned} & \sum (-1)^{Kos} m(f(a_1, ..., a_i), h(a_{i+1}, ..., a_j), g(a_{j+1}, ..., a_n)) \\ = \quad & f(a_1, ..., a_n) - g(a_1, ..., a_n) - \sum (-1)^{Kos} h(a_1, ..., a_{i'}, m(a_{i'+1}, ..., a_{j'}), a_{j'+1}, ..., a_n). \end{aligned}$$

Observe that it is very similiar to the form of cochain homotopies.

Let f be an A_∞-homomorphism. By definitions of A_∞-algebras and homomorphisms, m_1 defines a cochain complex, and f_1 defines a cochain map between m_1-complexes. We say f is a *weak homotopy equivalence* if f_1 induces an m_1-cochain homotopy equivalence.

Now, we state the following result, which is very useful and important for the remaining parts.

Theorem 2.2.4. *([FOOO1], Whitehead theorem for A_∞-algebras) A weak homotopy equivalence of A_∞-algebras is a homotopy equivalence.*

In other words, the homotopy equivalence of A_∞-algebras is just the homotopy equivalence of the underlying m_1-cochain complexes. The proof involves a careful examination of obstruction theory of A_∞-algebras.

Furthermore, if we only consider the ground ring as $\mathbb{Z}$ or a field, then it is not hard to show that quasi-isomorphisms of cochain complexes over the ground ring are in fact cochain homotopy equivalences. Such a restriction was assumed at first. So if $f_1 : C \to D$ is a quasi-isomorphism of m_1-complexes and is a part of an A_∞-homomorphism $f : A \to B$, then we call such f an A_∞-quasi-isomorphism, and consider it as a homotopy equivalence between A and B.

2.3 A_∞-bimodules and inner products

Now we recall the definition of A_∞-bimodules and homomorphisms between them. For oridinary bimodules over associative algebras, there is a compatibility axiom relating algebra multiplications and scalar actions. In the A_∞-case, we need to generalize such compatibility condition up to homotopy.

Definition 2.3.1. *Let $A = (C, \{m_k^A\})$ and $B = (D, \{m_k^B\})$ be A_∞-algebras and let M be a $\mathbb{Z}$-graded vector space over $\boldsymbol{k}$. Suppose that we have a family of maps*

$$d_{k,l} : C[1]^{\otimes k} \otimes \underline{M[1]} \otimes D[1]^{\otimes l} \to M[1]$$

of degree 1 *for all* $k, l \geq 0$. *Then* M *is an* A-B-bimodule *if* $\{d_{k,l}\}$ *satisfies*

$$\sum_{i=1}^{k}\sum_{j=1}^{k-i+1}(-1)^{Kos}d_{k-i+1,l}(a_1,...,m_i^A(a_j,...,a_{i+j-1}),...,\underline{m},b_1,...,b_l)$$
$$+\sum_{i=0}^{k}\sum_{j=0}^{l}(-1)^{Kos}d_{k-i,j}(a_1,...,\underline{d_{i,j}(a_{k-i+1},...,m,b_1,...,b_j)},...,b_l)$$
$$+\sum_{i=1}^{l}\sum_{j=1}^{l-i+1}(-1)^{Kos}d_{k,l-i+1}(a_1,...,\underline{m},...,m_i^B(b_j,...,b_{i+j-1}),...,b_l)=0$$

for all $(a_1,...,a_k,\underline{m},b_1,...,b_l) \in C[1]^{\otimes k}\otimes \underline{M[1]}\otimes D[1]^{\otimes l}$.

We specified module elements(and the module itself) by underlines to avoid confusion. If $A = B$, then we call M an A_∞-*bimodule over* A. Since we will only be concerned with such cases, we just give a definition of homomorphisms between A_∞-bimodules over a fixed A_∞-algebra A.

Definition 2.3.2. *Let* $(M, \{d^M_{k,l}\})$ *and* $(N, \{d^N_{k,l}\})$ *be* A_∞*-bimodules over* $A = (C, \{m_k\})$. *An* A_∞-bimodule homomorphism *between* M *and* N *is a family of maps*

$$f_{k,l} : C[1]^{\otimes k}\otimes \underline{M[1]}\otimes C[1]^{\otimes l}\to N[1]$$

of degree 0 *for all* $k, l \geq 0$ *such that*

$$\sum_{i=1}^{k}\sum_{j=1}^{k-i+1}(-1)^{Kos}f_{k-i+1,l}(a_1,...,m_i(a_j,...,a_{i+j-1}),...,\underline{m},...,a_{k+l+1})$$
$$+\sum_{j=1}^{k}\sum_{i=k-j+2}^{k+l-j+2}(-1)^{Kos}f_{j,k+l-i-j+3}(a_1,...,\underline{d^M_{k-j+1,i+j-k-2}(a_j,...,m,....,a_{i+j-1})},...,a_{k+l+1})$$
$$+\sum_{i=1}^{l}\sum_{j=k+2}^{k+l-i+2}(-1)^{Kos}f_{k,l-i+1}(a_1,...,\underline{m},...,m_i(a_j,...,a_{i+j-1}),...,a_{k+l+1})$$
$$=\sum_{j=1}^{k+1}\sum_{i=k-j+2}^{k+l-j+2}(-1)^{Kos}d^N_{j,k+l-i-j+3}(a_1,...,f_{k-j+1,i+j-k-2}(a_j,...,\underline{m},...,a_{i+j-1}),...,a_{k+l+1}).$$

For the case of $M = A$, we may set $d_{k,l} := m_{k+l+1}$. For the case of the dual $M = A^*$, we define the bimodule structure $d^*_{k,l}$ as

$$d^*_{k,l}(x_1, ..., x_k, \underline{v}^*, x_{k+1}, ..., x_{k+l})(w) := (-1)^{Kos} v^*(m_{k+l+1}(x_{k+1}, ..., x_{k+l}, w, x_1, ..., x_k)). \tag{2.3.1}$$

Now we are able to define a cyclic inner product on an A_∞-algebra.

Definition 2.3.3. *An A_∞-algebra $A = (C, \{m_k\})$ is said to have a* cyclic inner product *if there exists a skew-symmetric nondegenerate bilinear map*

$$\langle , \rangle : C \otimes C \to \boldsymbol{k} \tag{2.3.2}$$

such that for all integers $k \geq 1$,

$$\langle m_k(x_1, ..., x_k), x_{k+1} \rangle = (-1)^{Kos} \langle m_k(x_2, ..., x_{k+1}), x_1 \rangle.$$

When A has a cyclic inner product, we call A a cyclic A_∞-algebra.

Cyclic inner products are expressed as A_∞-bimodule maps with certain properties.

Lemma 2.3.4 ([C1] Lemma 3.1)**.** *Let ψ be an A_∞-bimodule homomorphism $\psi : A \to A^*$. Define*

$$\langle a, b \rangle := \psi_{0,0}(a)(b),$$

and suppose that $\langle , \rangle$ is nondegenerate. Then, it defines a cyclic inner product on A if

1. *$\psi_{k,l} \equiv 0$ for $(k, l) \neq (0, 0)$*
2. *$\psi_{0,0}(a)(b) = -(-1)^{|a|'|b|'} \psi_{0,0}(b)(a)$.*

Conversely, any cyclic symmetric inner product $\langle , \rangle$ on A give rise to an A_∞-bimodule map $\psi : A \to A^*$ with (1) and (2).

Remark 2.3.5. *If an A_∞-algebra A has $m_2 \neq 0$ and $m_1 = m_3 = m_4 = \cdots = 0$, then it is in fact an associative algebra. In this case A is a cyclic A_∞-algebra* *if and only if A is a Frobenius algebra, which is an associative algebra with an inner product*

$$\langle a \cdot b, c\rangle = \langle a, b \cdot c\rangle = \pm\langle b \cdot c, a\rangle.$$

If $a \mapsto \langle a, \ \rangle$ is a map $A \to A^*$, then the cyclicity is equivalent to that this map is an A-bimodule homomorphism between A and A^*.

There is a notion of cyclic A_∞-homomorphism due to Kajiura[Kaj].

Definition 2.3.6. *An A_∞-homomorphism $\{h_k\}_{k\geq 1}$ between two cyclic A_∞-algebras is called a* cyclic A_∞-homomorphism *if*

1. *h_1 preserves inner product $\langle a, b\rangle = \langle h_1(a), h_1(b)\rangle$.*
2. $$\sum_{i+j=k} \langle h_i(x_1, \cdots, x_i), h_j(x_{i+1}, \cdots, x_k)\rangle = 0.$$

We also recall that A_∞-homomorphism $f : A \to B$ can be also understood as an A_∞-bimodule homomorphism $\widetilde{f} : A \to B$ over (f, f). In this case,

$$\widetilde{f}_{k,l} : C[1]^{\otimes k} \otimes \underline{C[1]} \otimes C[1]^{\otimes l} \to D$$

is defined by $\widetilde{f}_{k,l} = f_{k+l+1}$. One can check that $\widetilde{f}$ satisfies $\widetilde{f} \circ \widehat{m}^A = m^B \circ \widehat{\widetilde{f}}$.

Now, we define strong homotopy inner products whose definition is modified from [C1].

Definition 2.3.7. *Let A be an A_∞-algebra. We call an A_∞-bimodule map $\phi : A \to A^*$ a* strong homotopy inner product *if it satisfies the following properties.*

1. *(Skew-symmetry) $\phi_{k,l}(\vec{a}, \underline{v}, \vec{b})(w) = -(-1)^{Kos}\phi_{l,k}(\vec{b}, \underline{w}, \vec{a})(v)$.*
2. *(Closedness) for any choice of a family $(a_1, ..., a_{l+1})$ and any choice of indices $1 \leq i < j < k \leq l+1$, we have*

$$\phi(..., \underline{a_i}, ...)(a_j) + (-1)^{Kos}\phi(..., \underline{a_j}, ...)(a_k) + (-1)^{Kos}\phi(..., \underline{a_k}, ...)(a_i) = 0.$$

3. *(Homological non-degeneracy) for any non-zero $[a] \in H^\bullet(A)$ with $a \in A$, there exists an element $[b] \in H^\bullet(A)$ with $b \in A$, such that $\phi_{0,0}(a)(b) \neq 0$.*

Remark 2.3.8. *The reason for the name of the second condition is that it is equivalent to the closedness of the corresponding noncommutative 2-form on the formal manifold which corresponds to A.*

Before we proceed, we introduce a few diagrams which helps to understand the axioms of strong homotopy inner products. A value of $\phi_{k,l}$ is expressed as in the Figure 2.3. Then the skew-symmetry is given by the Figure 2.3, and the closedness is given by the Figure 2.3.

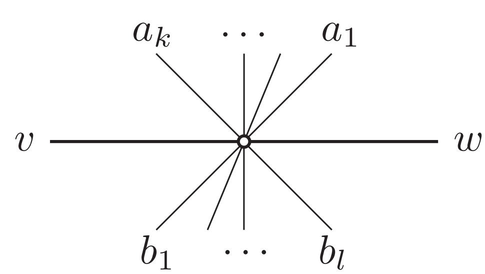


Figure 2.2: $\phi_{k,l}(a_1, ..., a_k, \underline{v}, b_1, ..., b_l)(w)$

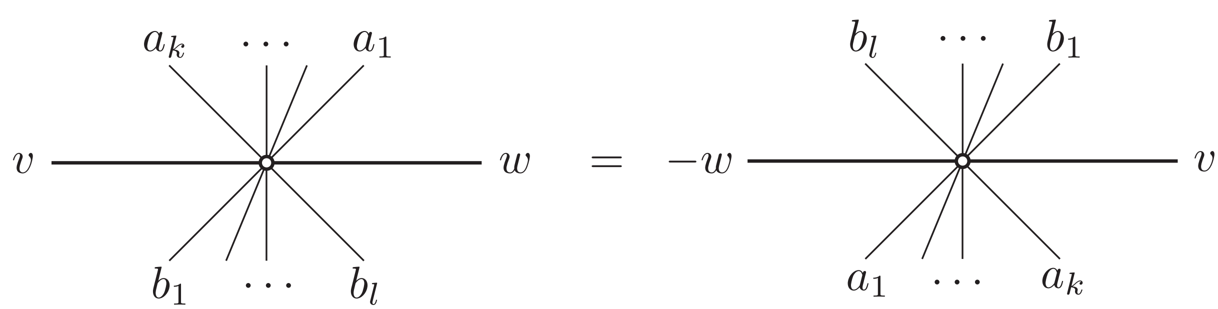


Figure 2.3: Skew-symmetry

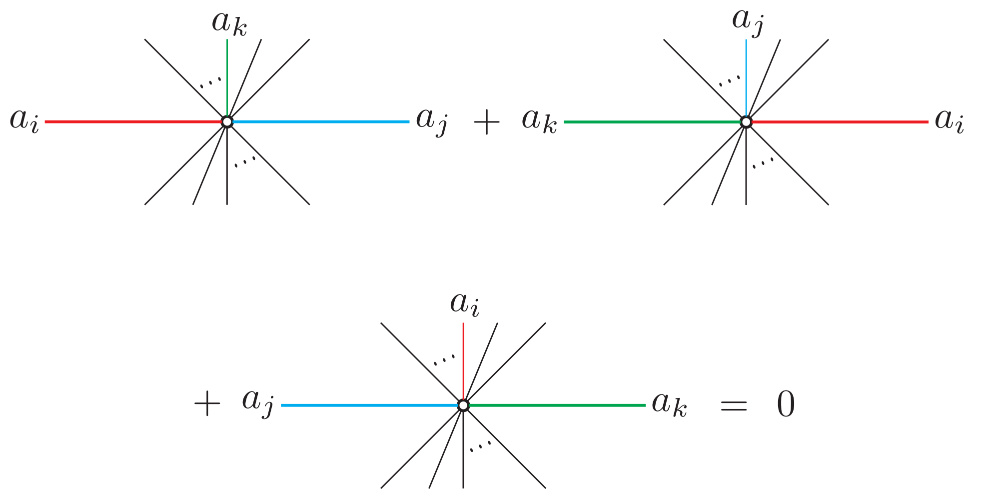


Figure 2.4: Closedness

Originally, in [C1], a strong homotopy inner product was defined by an A_∞-bimodule map $\phi : A \to A^*$ such that the following commutative diagram exists:

$$\begin{array}{ccc} A & \xrightarrow{\;g=\widetilde{f}\;} & B \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\psi}\ {\scriptstyle cyc} \\ A^* & \xleftarrow[\;g^*\;]{} & B^* \end{array} \tag{2.3.3}$$

where f is an A_∞-quasi-isomorphism and $\widetilde{f}$ is an A_∞-bimodule map induced by f as we discussed above. Then Cho's main result in [C1] is as following:

Theorem 2.3.9. *If such ϕ exists, then there exists an A_∞-bimodule map ϕ' on A which satisfies the above three conditions of definition 2.3.7.*

But the original definition does not fit well to the context of noncommutative geometry. Namely, suppose that we have ϕ satisfying definition 2.3.7, which corresponds exactly to a noncommutative symplectic form. Then ϕ does not always become exactly a strong homotopy inner product if we follow its original definition in [C1], but is only *equivalent* to one which satisfies the original definition(the notion of equivalence will be given later). That is why we modify the definition of strong homotopy inner products and introduce the notion of their equivalence.

Theorem 2.3.9 can be rephrased as the following theorem.

Theorem 2.3.10. *[C1] Let $\phi : A \to A^*$ be an A_∞-bimodule map.*

1. *If ϕ is a strong homotopy inner product(in the modified sense), then there exists an A_∞-algebra B with a cyclic inner product $\psi : B \to B^*$ and an A_∞-quasi-isomorphism $\iota : B \to A$ satisfying the following commutative diagram of A_∞-bimodule homomorphisms*

$$\begin{array}{ccc} A & \xleftarrow{\ \tilde{\iota}\ } & B \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\psi}\ {\scriptstyle cyc} \\ A^* & \xrightarrow{\ \tilde{\iota}^*\ } & B^* \end{array} \tag{2.3.4}$$

2. *If there exists a cyclic A_∞-algebra B with $\psi : B \to B^*$ and an A_∞-quasi-isomorphism $f : A \to B$ such that the following diagram of A_∞-bimodules over A commutes:*

$$\begin{array}{ccc} A & \xrightarrow[\ g=\tilde{f}\ ]{} & B \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\psi}\ {\scriptstyle cyc} \\ A^* & \xleftarrow[\ g^*\ ]{} & B^* \end{array} \tag{2.3.5}$$

   then ϕ is a strong homotopy inner product.

If $\phi_{0,0}$ is nondegenerate in the chain level, then one can find B such that both diagrams (2.3.4), (2.3.5) holds.

Proof. If $\phi_{0,0}$ is nondegenerate on the chain level, one can find B with an A_∞-isomorphism $f : A \to B$ from the proof of Theorem 2.3.9 making commuting diagram (2.3.3). Hence one can find exact inverse of f to make the commuting diagram (2.3.4).

Also, the statement (2) can be checked without much difficulty from the commuting diagram, so we only consider the statement (1). We explain that the proof of the theorem 2.3.9 given in [C1] is enough to prove the existence of the diagram (2.3.4): We recall from [C1] that the first step of the construction of the cyclic A_∞-algebra B when A is only homologically non-degenerate was considering the minimal model $\iota : H^\bullet(A) \to A$ and consider the pull back $\iota^*\phi$.

$$\begin{CD} A @<{\widetilde{\iota}}<< H^\bullet(A) @>{\widetilde{f}}>> H^\bullet(A) \\ @V{\phi}VV @V{\iota^*\phi}VV @V{cyc}VV \\ A^* @>{\widetilde{\iota}^*}>> (H^\bullet(A))^* @<{\widetilde{f}^*}<< (H^\bullet(A))^* \end{CD} \tag{2.3.6}$$

Then $\iota^*\phi$ is non-degenerate and skew symmetric and closed, and one proves the theorem for $\iota^*\phi$ to find $f : H^\bullet(A) \to H^\bullet(A)$ with the above commutative diagram. As the quasi-isomorphism f on $H^\bullet(A)$ is in fact an isomorphism, hence there exists explicit inverse f^{-1} and we obtain the diagram (2.3.4). $\square$

We can also prove the following corollary.

Corollary 2.3.11. *Let $\phi : A \to A^*$ be a strong homotopy inner product. Suppose we have an A_∞-quasi-isomorphism $f : A \to H^\bullet(A)$ with the commuting diagram*

$$\begin{CD} A @>{g=\widetilde{f}}>> H^\bullet(A) \\ @V{\phi}VV @V{\psi}V{cyc}V \\ A^* @<{\widetilde{g}^*}<< H^\bullet(A)^* \end{CD} \tag{2.3.7}$$

then, there exists an A_∞-quasi-isomorphism $h : H^\bullet(A) \to A$ with the commuting diagram (with the same ψ as the above)

$$\begin{CD} A @<{\tilde{h}}<< H^\bullet(A) \\ @V{\phi}VV @V{\psi}V{cyc}V \\ A^* @>{\tilde{h}^*}>> (H^\bullet(A))^* \end{CD} \tag{2.3.8}$$

Proof. By the decomposition theorem of A_∞-algebras(see [Kaj]), the map f has a right inverse A_∞-quasi-homomorphism, say $h : H^\bullet(A) \to A$ such that

$f \circ h = id$. To see this, consider an A_∞-isomorphism η

$$\eta : A \to A^{dc} := A^H \oplus A^{lc}$$

to the direct sum of the minimal A_∞-algebra A^H and the linear contractible A^{lc}.

Let $\pi : A^{dc} \to A^H$ be the projection and $i : A^H \to A^{dc}$ be the inclusion where the both are A_∞-quasi-isomorphisms with $\pi \circ i = id$. As f is an A_∞-quasi-isomorphism, $f \circ \eta^{-1} \circ i : A^H \to H^\bullet(A)$ is an A_∞-isomorphism, hence has an A_∞-inverse say ξ. Then, we define the right A_∞ inverse $h = \eta^{-1} \circ i \circ \xi$. The property $f \circ h = id$ can be checked immediately. The second diagram then follows from the first commuting diagram. □

Now we define the equivalence of strong homotopy inner products.

Definition 2.3.12. *Two strong homotopy inner products $\phi : A \to A^*$ and $\psi : B \to B^*$ are said to be* equivalent *if there exists a cyclic minimal A_∞-algebra H with a quasi-isomorphism to A and B, with the following commutative diagram:*

$$\begin{array}{ccccc} A & \xleftarrow{qis} & H & \xrightarrow{qis} & B \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle cyc} & & \downarrow{\scriptstyle \psi} \\ A^* & \longrightarrow & H^* & \longleftarrow & B^* \end{array}$$

One can actually choose H to be a minimal(or canonical) model.

Given a strong homotopy inner product $\phi : B \to B^*$, and an A_∞-quasi-isomorphism $f : A \to B$, we may define a pullback $f^*\phi : A \to A^*$

$$\begin{array}{ccc} A & \xrightarrow[\widetilde{f}]{} & B \\ {\scriptstyle f^*\phi}\downarrow & & {\scriptstyle \phi}\downarrow \\ A^* & \xleftarrow[\widetilde{f^*}]{} & B^* \end{array}$$

as a composition : $f^*\phi = \widetilde{f^*} \circ \widehat{\phi} \circ \widehat{\widetilde{f}}$ where $\widehat{\phi}$ and $\widehat{\widetilde{f}}$ denote the extensions to higher tensor powers, see section 3 of [C1].

Proposition 2.3.13. *$f^*\phi$ defines a strong homotopy inner product on A which is equivalent to ϕ.*

Proof. Since $\phi : B \to B^*$ is skew-symmetric and closed, so is $f^*\phi$ by lemma 5.6 of [C1]. It is not hard to check that $f^*\phi$ is also homologically non-degenerate as f is a quasi-isomorphism. Hence, $f^*\phi$, by Definition 2.3.7, is a strong homotopy inner product. Hence there exist an A_∞-algebra C which is cyclic symmetric ($\psi : C \to C^*$), and A_∞-quasi-homomorphism $h : C \to A$ with the following commutative diagrams.

$$\begin{array}{ccccc} C & \xrightarrow[\widetilde{h}]{} & A & \xrightarrow[\widetilde{f}]{} & B \\ \psi\downarrow & & f^*\phi\downarrow & & \phi\downarrow \\ C^* & \xleftarrow[\widetilde{h}^*]{} & A^* & \xleftarrow[\widetilde{f}^*]{} & (B)^* \end{array} .$$

From the diagram, it is easy to see that ϕ and $f^*\phi$ are equivalent in the sense of definition 2.3.12. □

Remark 2.3.14. *In general, A_∞-quasi-isomorphisms do not preserve cyclic property of A_∞-algebra, i.e. a pullback of a cyclic inner product may not be again cyclic. That is why we need to consider strong homotopy inner products, which is given via any A_∞-quasi-isomorphism from a cyclic inner product. The notion of a cyclic A_∞-homomorphism, which preserves cyclic property of A_∞-algebra, was first considered by Kajiura[Kaj] from the condition $f^*\omega = \omega'$ so that both ω and ω' are constant coefficient symplectic forms.*

Chapter 3

Formal noncommutative geometry

We recall the language of formal noncommutative geometry mainly from [KS1], which provides more geometric point of view of the related homotopy theories. For a more systematic exposition, we refer readers to Kontsevich and Soibelman[KS1], Kajiura[Kaj] or Hamilton and Lazarev[HL2].

3.1 Noncommutative function rings and vector fields

We restrict ourselves to A_∞-algebras on a finite dimensional vector space C over a field $\boldsymbol{k}$(to prove the main theorem for compact A_∞-algebras, we will pullback all the related notions to $H^\bullet(C, m_1)$ which is finite dimensional). We choose a basis of C and denote it as $\{e_1, \cdots, e_n\}$. Consider the dual space $C^* = \mathrm{Hom}(C, \boldsymbol{k})$ and denote the dual basis as $x_1, \cdots, x_n$ whose degrees are given as $|x_i|' = -|e_i|'$.

As an A_∞-algebra is given by coalgebra with codifferential, its dual becomes a (noncommutative) differential graded algebra (DGA) or a formal manifold in the language of Kontsevich and Soibelman [KS1]. Namely, the dual of the coalgebra $(BC)^*$ is a DGA. To have a unit, actually, one should work with an augmented bar complex $BC^+ := BC \oplus \boldsymbol{k}$ with the comultiplication $\Delta : BC^+ \to BC^+ \otimes BC^+$ defined as in (2.1.2) with the sum from $i = 0$ to $i = n$.

Then, the dual space $(BC^+)^*$ may be considered as a function ring $\mathcal{O}(X)$ such that

$$\mathcal{O}(X) = \boldsymbol{k}\langle\langle x_1, ..., x_n\rangle\rangle.$$

Namely, it is a *noncommutative* formal power series ring, regarded as the ring of regular functions on the formal noncommutative manifold X.

Dual of the codifferential $\hat{d}$ becomes a differential of the DGA, which may be considered as a vector field on X. The vector field Q on X corresponding $\hat{d}$ may be defined by

$$Q := \sum_{k\rangle 0} a_i^{j_1,\cdots,j_k} x_{j_1} x_{j_2} \cdots x_{j_k} \frac{\partial}{\partial x_i}$$

where the coefficients a's are defined by the A_∞-operations

$$m_k(e_{j_1}, \cdots, e_{j_k}) = \sum a_i^{j_1,\cdots,j_k} e_i.$$

Note that Q may be an infinite sum and is regarded as a formal vector field.

The A_∞-equation $\hat{d} \circ \hat{d} = \frac{1}{2}[\hat{d}, \hat{d}] = 0$ implies the relation $[Q, Q] = 0$ or the following identities between the coefficients of Q for each s

$$0 = \sum_{\substack{k_1+k_2=k+1 \\ j,l}} (-1)^{\epsilon_1} a_s^{i_1,\cdots,i_{j-1},l,i_{j+k_2},\cdots,i_k} \cdot a_l^{i_j,\cdots,i_{j+k_2-1}}$$

Here $\frac{\partial}{\partial x^i}$ acts on $\mathcal{O}(X)$ in a natural way: for example(assume each of $k(x), f(x)$ and $g(x)$ has homogeneous degree),

$$k(x)\frac{\partial}{\partial x^1}(f(x)g(x)) = k(x)\frac{\partial}{\partial x^1}(f(x))\cdot g(x) + (-1)^{(|k(x)|'-|x_1|')(|f(x)|')} f(x)\cdot k(x)\frac{\partial}{\partial x^1}(g(x)).$$

Here we set $|\frac{\partial}{\partial x_i}|' = -|x_i|'$. Then one may check that A_∞-equation corresponds to $[Q, Q] = 0$ or more precisely, for any $f(x)$, we have

$$[Q, Q](f) = Q(Q(f)) - (-1)^{(1\cdot 1)} Q(Q(f)) = 2Q(Q(f)) = 0.$$

In the non-graded situation, $[Q, Q]$ is zero for any vector field Q, but now we need to consider gradings when we take the commutator. In particular, if $|Q|$ is odd, then $[Q, Q]$ is not automatically zero any more, and if it is zero, Q is a very special vector field. In particular, the structure of an A_∞-algebra on C is equivalent to the noncommutative (pointed) formal manifold X equipped with a vector field Q with $[Q, Q] = 0$. Here pointed means that we consider the case of $m_0 = 0$, or in another way, we consider formal series with no constant term.

A cohomomorphism between coalgebras corresponding to A_∞-algebras naturally corresponds to the algebra homomorphism compatible with derivations. Namely for two A_∞-algebras $(A, m_*^A), (B, m_*^B)$ and an A_∞-homomorphism $h : B \to A$, the formal change of coordinates of the dual variables are given as follows. We assume B is finite dimensional as a vector space, and denote by $\{f_*\}$ its basis, and introduce corresponding formal variables y_* as before. Suppose

$$h_k(f_{j_1}, \cdots, f_{j_k}) = h^i_{j_1,\cdots,j_k} e_i, \quad h^i_{j_1,\cdots,j_k} \in R.$$

Then, algebra homomorphism is defined by changing each variable as

$$x_i \mapsto h^i_{j_{11}} y_{j_{11}} + h^i_{j_{21},j_{22}} y_{j_{21}} y_{j_{22}} + \cdots + h^i_{j_{l1},\cdots,j_{lk}} y_{j_{l1}} \cdots y_{j_{lk}} + \cdots. \tag{3.1.1}$$

We refer readers to [Kaj] for detailed explanation on this point.

3.2 Noncommutative de Rham theory

There is a noncommutative version of de Rham(or Karoubi) theory(see for example [KS1]). The main difference from the commutative case is that the space X where the differential forms should live does not really exist but is only considered hypothetical, and the right de Rham complex in the noncommutative case is the cyclic de Rham complex.

First, one may introduce the de Rham forms as follows. Consider $\mathcal{O}(T[1]X) := \boldsymbol{k}\langle\langle x_i, dx_i\rangle\rangle$, where dx_i are another formal variables such that $|dx_i| = |x_i|$. There are additional signs when dealing with these forms or vector fields, which we follow the definition in [Kaj]. First denote

$$\sharp(dx_i) = 1, \sharp(x_i) = 0, \sharp(\frac{\partial}{\partial x^i}) = -1$$

and in general, by denoting x_i or dx_i by ϕ, one defines

$$\sharp(\phi^1 \cdots \phi^k) = \sum_{j=1}^{k} \sharp(\phi^j).$$

And the Koszul sign rule in this case is given by considering the sign $\sharp$ and $|\cdot|'$ separate. For example, graded commutator is defined (for homogeneous elements) by

$$[f(\phi), g(\phi)] = f(\phi)g(\phi) - (-1)^{(|f(\phi)|'|g(\phi)|' + \sharp(f(\phi))\sharp(g(\phi)))} g(\phi)f(\phi).$$

Cyclic functions are defined by the quotient

$$\Omega^0_{cyc}(X) = \mathcal{O}(X)/[\mathcal{O}(X), \mathcal{O}(X)]_{top}.$$

Here one takes the closure of algebraic commutator in the adic topology. The space of *cyclic differential forms* on X is defined similarly by

$$\Omega_{cyc}(X) = \mathcal{O}(T[1]X)/[\mathcal{O}(T[1]X), \mathcal{O}(T[1]X)]_{top}.$$

Cyclic noncommutative one forms on X , $\Omega^1_{cyc}(X)$ is generated by expressions as $x_{i_1} \cdots x_{i_k} dx_{i_{k+1}}$, where by cyclic rotation, dx_* may be regarded as being in the last slot. But in general, cyclic 2-form is generated by equivalence classes of elements like

$$x_{i_1} \cdots x_{i_p} dx_a x_{j_1} \cdots x_{j_q} dx_b x_{k_1} \cdots x_{k_r}.$$

Hence, unlike in the ordinary commutative case, $\Omega^s_{cyc}(X)$ does not vanish for $s > \dim(V)$. The usual de Rham differential d descends to the quotient $\Omega_{cyc}(X)$ and we denote it as d_{cycl} as in [KS1]. $(\Omega_{cyc}(X), d_{cycl})$ is the noncommutative de Rham complex.

Furthermore, it is well-known that the *contraction map*(or *interior product*)

$$i_\xi : \mathcal{O}(T[1]X) \to \mathcal{O}(T[1]X)$$

can be defined by $i_\xi(f) = 0$, $i_\xi(df) = \xi(f)$ for all $f \in \mathcal{O}(T[1]X)$. Now, one defines the Lie derivative

$$\mathcal{L}_\xi = [d, i_\xi] = d \circ i_\xi + i_\xi \circ d.$$

As $\mathcal{L}_\xi$ is also a derivation, we have for any $f(\phi), g(\phi) \in \mathcal{O}(T[1]X)$,

$$\mathcal{L}_\xi([f(\phi), g(\phi)]) \subset [\mathcal{O}(T[1]X), \mathcal{O}(T[1]X)].$$

Hence, $\mathcal{L}_\xi$ is well-defined on cyclic forms $\Omega_{cyc}(T[1]X)$. Like the standard differential calculus, the following holds true also for the noncommutative de Rham complex. We remark that these identities are also called "geometric identities" in some literatures.

$$[d, d] = 0, [d, \mathcal{L}_\xi] = 0,$$

$$[\mathcal{L}_\xi, i_\eta] = i_{[\xi,\eta]}, [\mathcal{L}_\xi, \mathcal{L}_\eta] = \mathcal{L}_{[\xi,\eta]}, [i_\xi, i_\eta] = 0.$$

We mention two of the well-known theorems. The first one is

Theorem 3.2.1 (Poincaré lemma)**.** *The cohomology of* $(\Omega_{cyc}(X), d)$ *is trivial.*

Proof. We follow Lemma 4.8 of [Kaj]. One can define the explicit contracting homotopy H satisfying $dH + Hd = Id$ as follows. Denote an element of $\Omega_{cyc}(X)$ as $a = \frac{1}{k} a_{i_1 \cdots i_k} \phi_{i_1} \cdots \phi_{i_k}$ where $\phi_{i_j} = x_{i_j}$ or $\phi_{i_j} = dx_{i_j}$. Then, H is defined by

$$H(a) = \sum_i (-1)^{\sharp_{i_1} + \cdots \sharp_{i_{j-1}}} \frac{1}{k} a_{i_1 \cdots i_k} \phi_{i_1} \cdots (H(\phi_{i_j})) \cdot \phi_{i_k}.$$

where $H(x_{i_j}) = 0$ and $H(dx_{i_j}) = x_{i_j}$. □

Theorem 3.2.2. *(Darboux theorem) Any symplectic form on a formal noncommutative manifold can be transformed to the constant (coefficient) symplectic form by a coordinate transformation.*

We refer readers to [Gi], [Kaj] or Theorem 6.5.3 of this paper for its proof in the filtered case.

3.3 Kontsevich-Soibelman's theorem

We give a brief sketch of the proof of the Kontsevich-Soibelman's theorem for readers' convenience, and refer readers to [KS1] for more details.

Proof. Consider a symplectic form ω, satisfying $d_{cycl}\omega = 0, \mathcal{L}_Q(\omega) = 0$, which is a cycle of the complex $(\Omega^{2,cl}_{cyc}(X), \mathcal{L}_Q)$, where cl means d_{cycl}-closed elements.

By the Poincaré lemma, there exists an element $\alpha \in \Omega^1_{cyc}(X)/d_{cyc}\Omega^0_{cyc}(X)$ such that $d_{cycl}\alpha = \omega$. This provides an isomorphism of complexes:

$$d_{cycl} : \left(\frac{\Omega^1_{cyc}(X)}{d_{cycl}\Omega^0_{cyc}(X)}, \mathcal{L}_Q \right) \to (\Omega^{2,cl}_{cyc}(X), \mathcal{L}_Q).$$

We remark that as it is an isomorphism, there exists an inverse, but we do not know any map from $\Omega^{2,cl}_{cyc}(X) \to \Omega^1_{cyc}(X)$ which is a chain map with respect to $\mathcal{L}_Q$ which is a source of some complications. For example, the contracting homotopy in the proof of the Poincaré lemma does not commute with the differential $\mathcal{L}_Q$.

Kontsevich and Soibelman has proved that the following map via $adb \to [a, b]$

$$\left(\frac{\Omega^1_{cyc}(X)}{d_{cycl}\Omega^0_{cyc}(X)}, \mathcal{L}_Q \right) \to ([\mathcal{O}(X), \mathcal{O}(X)]_{top}, \mathcal{L}_Q),$$

is a quasi-isomorphism. From the definition $\Omega^0_{cyc}(X) = \mathcal{O}(X)/[\mathcal{O}(X), \mathcal{O}(X)]_{top}$, we have a short exact sequence of $\mathcal{L}_Q$-complexes,

$$0 \to [\mathcal{O}(X), \mathcal{O}(X)]_{top} \to \mathcal{O}(X)/\boldsymbol{k} \to \Omega^0_{cyc}(X)/\boldsymbol{k} \to 0.$$

Note that $(\mathcal{O}(X)/\boldsymbol{k}, \mathcal{L}_Q)$ is acyclic ([KS2] Prop. 8.4.1), hence $(\Omega^{2,cl}_{cyc}(X), \mathcal{L}_Q)$ is quasi-isomorphic to $(\Omega^0_{cyc}(X)/\boldsymbol{k}, \mathcal{L}_Q)$ which is the cyclic cohomology of A (see Lemma 6.1.2). □

To show that the resulting cyclic structure really depends only on the $\mathcal{L}_Q$-cohomology class of ω, they prove

Lemma 3.3.1 ([KS2] Lemma 11.2.6)**.** *Let $\omega_1 = \omega + \mathcal{L}_Q(d\alpha)$. Then there exists a vector field v such that $v(x_0) = 0$, $[v, Q] = 0$, and $\mathcal{L}_v(\omega) = \mathcal{L}_Q(d\alpha)$.*

Proof. As in the proof of Darboux lemma, we need to find a vector field v, satisfying the condition $\mathcal{L}_v\omega = \mathcal{L}_Q(d\alpha)$. Let $\beta = \mathcal{L}_Q(\alpha)$. Then, $d\beta = \mathcal{L}_Q(d\alpha)$.

Hence, the desired equation $\mathcal{L}_v\omega = \mathcal{L}_Q(d\alpha)$, is equivalent to

$$di_v\omega = d\beta = d\mathcal{L}_Q(\alpha).$$

Hence, we solve

$$i_v\omega = \beta,$$

which is possible by the non-degeneracy of ω.

We also claim that any such solution v automatically satisfies $[Q, v] = 0$. To see this, note that

$$\mathcal{L}_Q \circ i_v\omega = \mathcal{L}_Q \circ \mathcal{L}_Q(\alpha) = \mathcal{L}_{[Q,Q]}(\alpha) = 0.$$

But the first term equals

$$\mathcal{L}_Q \circ i_v\omega = i_v\mathcal{L}_Q\omega + i_{[Q,v]}\omega = i_{[Q,v]}\omega.$$

Hence, $i_{[Q,v]}\omega = 0$ and this implies the claim as ω is non-degenerate. □

The above lemma suggests that there exist an A_∞-automorphism (preserving A_∞-structure) which transforms the symplectic form $\omega + \mathcal{L}_Q(d\alpha)$ to ω, thus proving that the cyclic structure depends only on the $\mathcal{L}_Q$-cohomology class. But we found that the construction of such an automorphism is rather involved which occupies the whole section 6.3.

Chapter 4

Review of cohomology theories on A_∞-algebras

We need another main ingredients, namely cohomology theories of A_∞-algebras, to understand and prove Theorem B and Theorem C. From now on, all A_∞-algebras are assumed to be unital.

4.1 Hochschild (co)homology for A_∞-algebras

Hochschild homology of an A_∞-algebra $A = (A, \{m_k\})$ as an A_∞-module over itself is defined as follows. Denote

$$C^k(A, A) = A \otimes A[1]^{\otimes k}, \tag{4.1.1}$$

and its degree $\bullet$ part by $C^k_\bullet(A, A)$. We define the Hochschild chain complex

$$(C_\bullet(A, A), b) = \big(\oplus_{k\geq 0} C^k_\bullet(A, A), b\big), \tag{4.1.2}$$

where the degree one differential b is defined as follows: for $v \in A$ and $x_i \in A$,

$$b(\underline{v}\otimes x_1\otimes\cdots\otimes x_k) = \sum_{\substack{0\leq j\leq k+1-i\\ 1\leq i}} (-1)^{\epsilon_1}\underline{v}\otimes\cdots\otimes x_{i-1}\otimes m_j(x_i,\cdots,x_{i+j-1})\otimes\cdots\otimes x_k$$

$$+\sum_{\substack{0\le i,j\le k\\ i+j\le k}}(-1)^{\epsilon_2}\underline{m_{i+j+1}\big(x_{k-i+1},\cdots,x_k,v,x_1,\cdots,x_j\big)}\otimes x_{j+1}\otimes\cdots\otimes x_{k-i}. \quad (4.1.3)$$

We underline module elements to avoid confusion. We note again that the signs follow the Koszul sign convention:

$$\epsilon_1=|v|'+|x_1|'+\cdots+|x_{i-1}|',\ \epsilon_2=\Big(\sum_{s=1}^{i}|x_{k-i+s}|'\Big)\Big(|v|'+\sum_{t=1}^{j}|x_t|'\Big).$$

Combining the Koszul sign rules and A_∞-relation (2.1.4), we have $b^2=0$.

We introduce Figure 4.1 to understand the Hochschild differential better.

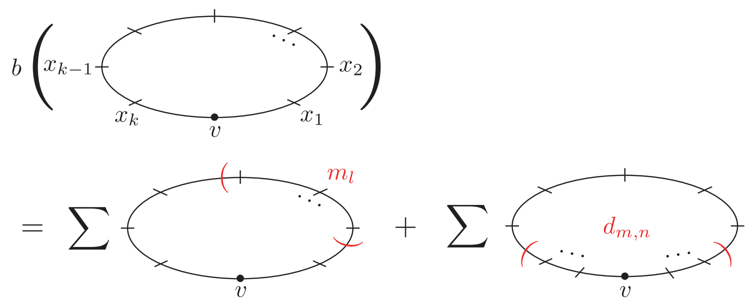


Figure 4.1: The Hochschild boundary map b

Similarly, one can define the reduced Hochschild homology by considering $A^{red}:=A/R\cdot I$ where I is the unit of A, and set $C^k_{red}(A,A)=A\otimes(A^{red}[1])^{\otimes k}$ instead, and the resulting homology is isomorphic to the standard one.

The cochain complex obtained by taking a dual of the reduced Hochschild chain complex, $\big((A^{red}_\bullet(A,A))^*,b^*\big)$, defines the Hochschild cohomology $H^\bullet_{red}(A,A^*)$. Here, cochain elements are given by the maps $\{f_n:(A^{red}[1])^{\otimes n}\to C^*\}$ and the degree one differential b^* is given by

$$b^*f(a_1,...,a_n)=\sum(-1)^{Kos}f(a_1,...,m_k(...),...,a_n)+\sum(-1)^{Kos}d^*(a_1,...,\underline{f(...)},...,a_n).$$

We recall that d^* was defined in (2.3.1).

4.2 (Negative) cyclic cohomology for A_∞-algebras

In this section, we briefly review the definition of the cyclic and negative cyclic cohomology of a unital A_∞-algebra. For weakly unital case, see for example, [HL2] or [C2]. Given an A_∞-algebra, there exists a Tsygan's bicomplex. Consider the Hochschild chain complex $C_\bullet(A,A)$ defined in (4.1.2). For the cyclic generator $t_{n+1} \in \mathbb{Z}/(n+1)\mathbb{Z}$, we define its action on $A^{\otimes(n+1)}$ as follows:

$$t_{n+1} \cdot (x_0, ..., x_n) = (-1)^{|x_n|'(|x_0|'+\cdots+|x_{n-1}|'}(x_n, x_0, ..., x_{n-1}).$$

Here, we set t_1 to be identity on A and write the identity map as 1. Consider

$$N_{n+1} := 1 + t_{n+1} + t_{n+1}^2 + \cdots t_{n+1}^n.$$

As in the classical case, we have the natural augmented exact sequence:

$$A^{\otimes(n+1)} \xleftarrow{1-t_{n+1}} A^{\otimes(n+1)} \xleftarrow{N_{n+1}} A^{\otimes(n+1)} \xleftarrow{1-t_{n+1}} A^{\otimes(n+1)} \xleftarrow{N_{n+1}} \cdots.$$

We consider $\bigoplus_{n=1}^\infty N_n$ action on $\bigoplus_{n=1}^\infty A^{\otimes n}$ and denote it as

$$N : C_\bullet(A,A) \to C_\bullet(A,A).$$

We can also similarly define $(1-t) : C_\bullet(A,A) \to C_\bullet(A,A)$.

Recall that in the classical case, cyclic bicomplex has even columns which are the copies of the Hochschild complex, and odd columns which are the copies of the bar complex. Bicomplex for A_∞ case is constructed in a similar way. Even columns will be given by $(C_\bullet(A,A), b)$. Consider $\hat{d}$ operation on $C_\bullet(A,A)$ considered as a subspace of BC. The homology of the chain complex $(C_\bullet(A,A), \hat{d})$ vanishes, and this will be the odd columns. We set $b' := \hat{d}$ to follow the standard notation. The following lemma is a standard fact.

Lemma 4.2.1. *We have the following identities on $C_\bullet(A,A)$:*

$$b(1-t) = (1-t)b', b'N = Nb. \tag{4.2.1}$$

We thus obtain the cyclic bicomplex(analogous to Tsygan's) defined as follows.

Definition 4.2.2. *Define*

$$CC_{pq}(A) := C_q(A, A) \text{ for all } p \geq 0,\ q \in \mathbb{Z}.$$

We define differentials on the double complex as

$$b : CC_{pq}(A) \to CC_{p(q+1)}(A) \text{ for } p \text{ even},$$

$$-b' : CC_{pq}(A) \to CC_{p(q+1)}(A) \text{ for } p \text{ odd},$$

$$1 - t : CC_{pq}(A) \to CC_{(p-1)q}(A) \text{ for } p \text{ odd},$$

$$N : CC_{pq}(A) \to CC_{(p-1)q}(A) \text{ for } p \text{ even}.$$

As a diagram, we have

$$\begin{array}{ccccccccc}
\uparrow b & & \uparrow -b' & & \uparrow b & & \uparrow -b' & & \\
C_1(A,A) & \xleftarrow{1-t} & C_1(A,A) & \xleftarrow{N} & C_1(A,A) & \xleftarrow{1-t} & C_1(A,A) & \xleftarrow{N} & \\
\uparrow b & & \uparrow -b' & & \uparrow b & & \uparrow -b' & & \\
C_0(A,A) & \xleftarrow{1-t} & C_0(A,A) & \xleftarrow{N} & C_0(A,A) & \xleftarrow{1-t} & C_0(A,A) & \xleftarrow{N} & \\
\uparrow b & & \uparrow -b' & & \uparrow b & & \uparrow -b' & & \\
C_{-1}(A,A) & \xleftarrow{1-t} & C_{-1}(A,A) & \xleftarrow{N} & C_{-1}(A,A) & \xleftarrow{1-t} & C_{-1}(A,A) & \xleftarrow{N} & \\
\uparrow b & & \uparrow -b' & & \uparrow b & & \uparrow -b' & &
\end{array} \tag{4.2.2}$$

Denote $C^\lambda_\bullet(A) := \operatorname{coker}(1-t) = C_\bullet(A, A)/\operatorname{im}(1-t)$. Then as in the classical case, we can prove that its homology with differentials inherited from Hochschild boundary b is the same as that of the above bicomplex.

We have another bicomplex named by (b, B)-complex, which can be also used to define cyclic homology. The *Connes' B-operator* is defined by

$$B = (1-t)sN,$$

where

$$s(a_1 \otimes \cdots \otimes a_n) := I \otimes a_1 \otimes \cdots \otimes a_n.$$

Cyclic and negative cyclic cochain complex can be obtained by taking the dual of the (b, B)-complex in the following way.

Consider the following unbounded (b^*, B^*) complex.

$$\begin{array}{ccccccccc}
 & & \uparrow b^* & & \uparrow b^* & & \uparrow b^* & & \\
\xrightarrow{B^*} & & C^2_{red}(A, A^*) & \xrightarrow{B^*} & C^1_{red}(A, A^*) & \xrightarrow{B^*} & C^0_{red}(A, A^*) & \xrightarrow{B^*} & \\
 & & \uparrow b^* & & \uparrow b^* & & \uparrow b^* & & \\
\xrightarrow{B^*} & & C^1_{red}(A, A^*) & \xrightarrow{B^*} & C^0_{red}(A, A^*) & \xrightarrow{B^*} & C^{-1}_{red}(A, A^*) & \xrightarrow{B^*} & \\
 & & \uparrow b^* & & \uparrow b^* & & \uparrow b^* & & \\
\xrightarrow{B^*} & & C^0_{red}(A, A^*) & \xrightarrow{B^*} & C^{-1}_{red}(A, A^*) & \xrightarrow{B^*} & C^{-2}_{red}(A, A^*) & \xrightarrow{B^*} & \\
 & & \uparrow b^* & & \uparrow b^* & & \uparrow b^* & &
\end{array} \tag{4.2.3}$$

Here $(C^\bullet_{red}(A, A^*), b^*)$ is the dual of reduced Hochschild cochain complex in the previous subsection, and the dual of the Connes-Tsygan's operator is given by

$$B^* f(a_1, ..., a_n) = \sum_i f(a_i, a_{i+1}, ..., a_{i-1})(I).$$

The double complex $CC^\bullet_-(A, A^*)$ which consists of only negative columns of (4.2.3) defines the *negative cyclic cohomology* of an A_∞-algebra A, which we denote by $HC^\bullet_-(A)$. Here, elements in the line parallel to $y = -x$ line has the same total degree of the double complex, and we consider the direct sums instead of the direct products so that its dualization would give the negative cyclic homology which is given by the direct products (with suitable finiteness assumption).

The double complex $CC^\bullet(A, A^*)$ which consists of only nonnegative columns of (4.2.3) defines the cyclic cohomology of the A_∞-algebra A, which we denote by $HC^\bullet(A)$. Here we use the direct products instead of direct sums so that the dualization of the cyclic homology would give cyclic cohomology.

The double complex $CP^\bullet(A, A^*)$ which consists of all columns of (4.2.3) defines the periodic cyclic cohomology of the A_∞-algebra A, which we denote by $HP^\bullet(A)$. Here we use the direct products.

As usual, one obtains the following spectral sequence of these three homology theories given by the inclusion of $CC^\bullet(A, A^*)$ to $CP^\bullet(A, A^*)$:

$$\cdots \to HC^n(A) \to HP^n(A) \to HC^n_-(A) \to HC^{n+1}(A) \to \cdots \qquad (4.2.4)$$

Here the map $HC^n_-(A) \to HC^{n+1}(A)$ is induced by B^*.

These homology theories for arbitrary A_∞-algebras are in general difficult to deal with, and we are mainly interested in the case that the A_∞-algebra $A = (C, \{m_k\})$ satisfies either $C^{>0} \equiv 0$ or $C^{<0} \equiv 0$ before shifting degrees. We remark that the usual (non-graded) algebras may be considered as A_∞-algebras and after degree shifting, all elements have degree -1. In this case the Hochschild complex $C_{\geq 0}(A, A) \equiv 0$ for degree reasons. The examples from geometry, for example the usual de Rham complex, has degree from 0 to N. In particular, if we assume that C^0 is generated by the unit (in cohomology), then it is easy to show that the Hochschild cochain complex satisfies $C^{>1}_{red}(A, A^*) \equiv 0$. We also remark that by the standard spectral sequence arguments, homotopy equivalent A_∞-algebras have isomorphic Hochschild (co)homology classes. As we have used direct sums to define negative cyclic cohomology, the usual usual invariant-coinvariant relation gives rise to the following lemma:

Lemma 4.2.3 ([HL1] Lemma 3.6)**.** *Let (A, m) be a weakly unital A_∞-algebra for which there exists an integer N such that $H^k(V, V^*) = 0$ for $k > N$. Then, for any integer n, we have*

$$HC^n_-(A) \cong HC^{n+1}(A).$$

The equivalence is given by the map in the long exact sequence (4.2.4), and this is also the relation between the cohomology classes used in Theorem 1.0.2 and Theorem C.

Chapter 5

Potentials of homotopy cyclic A_∞-algebras and proof of theorem A

5.1 Potentials

Let an A_∞-algebra $(A, \{m_k^A\})$ be given a strong homotopy inner product $\phi : A \to A^*$ which consists of a family of maps

$$\phi_{p,q} : A^{\otimes p} \otimes \underline{A} \otimes A^{\otimes q} \to A^*.$$

We denote by

$$\langle x_1, \cdots, x_p, \underline{v}, y_1, \cdots, y_q \mid w \rangle_{p,q} := \phi_{p,q}(x_1, \cdots, x_p, \underline{v}, y_1, \cdots, y_q)(w). \quad (5.1.1)$$

As in the cyclic case, let $\{e_i\}$ be a basis of A as a vector space, which is assumed to be finite dimensional(one may use the pullback defined in the previous section using the inclusion $\iota : H^\bullet(A) \to A$ in the case that $H^\bullet(A)$ is finite dimensional). Define $\boldsymbol{x} = \sum_i e_i x_i$ where x_i are formal parameters with $\deg(x_i) = -\deg(e_i)$.

Now we give a definition of the potentials for strong homotopy inner products.

Definition 5.1.1. *The* potential *of an A_∞-algebra $(A, \{m_k^A\})$ with a strong homotopy inner product $\phi : A \to A^*$ is defined as*

$$\begin{aligned}\Phi^A(\boldsymbol{x}) &= \sum_{N=1}^{\infty} \Phi_N^A(\boldsymbol{x}) \\ &:= \sum_{N=1}^{\infty} \sum_{p+q+k=N} \frac{1}{N+1} \langle \boldsymbol{x}, \boldsymbol{x}, \cdots, \boldsymbol{x}, \underline{m_k^A(\boldsymbol{x}, \boldsymbol{x}, \cdots, \boldsymbol{x})}, \boldsymbol{x}, \cdots, \boldsymbol{x} \mid \boldsymbol{x} \rangle_{p,q}\end{aligned} \tag{5.1.2}$$

The definition itself is somewhat similar to that of cyclic case (1.0.1). But in (1.0.1), the fraction $\dfrac{1}{k+1}$ was to cancel out by repetitive contribution to the potential due to cyclic symmetry (2.3.2), whereas in the strong homotopy case, such cyclic symmetry of the rotation of arguments do not exist. Namely, in general

$$\langle e_1, \cdots, \underline{m_i(e_j, \cdots, e_{j+i-1})}, \cdots, e_k \mid e_{k+1} \rangle \neq \pm \langle e_2, \cdots, \underline{m_i(e_{j+1}, \cdots, e_{j+i})}, \cdots, e_{k+1} \mid e_1 \rangle.$$

We later show that the combination of A_∞-bimodule equation, skew-symmetry and closed condition will compensate the absence of the strict cyclic symmetry.

We explain how the potential behaves under pullbacks, and this will show the relation between potentials of equivalent strong homotopy inner products. For an A_∞-quasi-isomorphism $h : B \to A$, the *pullback* of a potential is defined as follows: We assume B is finite dimensional as a vector space, and denote by $\{f_i\}$ its basis, and introduce corresponding formal variables y_i as before. Suppose

$$h_k(f_{j_1}, \cdots, f_{j_k}) = h^i_{j_1,\cdots,j_k} e_i, \quad h^i_{j_1,\cdots,j_k} \in \boldsymbol{k}.$$

Then, we set

$$x_i \mapsto h^i_{j_{11}} y_{j_{11}} + h^i_{j_{21},j_{22}} y_{j_{21}} y_{j_{22}} + \cdots + h^i_{j_{l1},\cdots,j_{lk}} y_{j_{l1}} \cdots y_{j_{lk}} + \cdots. \tag{5.1.3}$$

Then, one defines the pullback $h^*\Phi^A$ by using the above change of coordinate formula. Namely, $h^*\Phi^A$ is given by the replacement of $\boldsymbol{x}$ by $\sum_{k\geq 1} h_k(\boldsymbol{y}^{\otimes k})$ in the formula of Φ^A, where $\boldsymbol{y} := \sum f_i y_i$. Here y_i are formal variables corresponding to f_i as above.

5.2 Theorem A

Now, we are ready to state and prove our first main theorem.

Theorem 5.2.1. *(Theorem A) Let $\phi : A \to A^*$ be a strong homotopy inner product. Let B be a cyclic A_∞-algebra with a quasi-isomorphism $h : B \to A$ providing the commutative diagram (2.3.4). Then, we have*

$$\Phi^B = h^*\Phi^A$$

Proof. The overall scheme of the proof, which is first to differentiate and then to compare, follows that of [C1] (idea due to Kajiura [Kaj] in the unfiltered case). The main difficulty, and the essential part of the proof is the first step where we take (formal) partial derivatives on each side. The following lemma shows that after partial differentiation, the fraction on each summand disappears.

Lemma 5.2.2.

$$\begin{aligned}\frac{\partial}{\partial x_i}\Phi^A_N(\boldsymbol{x}) &= \frac{\partial}{\partial x_i}\sum_{p+q+k=N}\frac{1}{N+1}\langle \boldsymbol{x},\boldsymbol{x},\cdots,\boldsymbol{x},\underline{m^A_k(\boldsymbol{x},\boldsymbol{x},\cdots,\boldsymbol{x})},\boldsymbol{x},\cdots,\boldsymbol{x}\mid \boldsymbol{x}\rangle_{p,q}\\ &= \sum_{p+q+k=N}\langle \boldsymbol{x},\boldsymbol{x},\cdots,\boldsymbol{x},\underline{m^A_k(\boldsymbol{x},\boldsymbol{x},\cdots,\boldsymbol{x})},\boldsymbol{x},\cdots,\boldsymbol{x}\mid e_i\rangle_{p,q}.\end{aligned}$$

We assume the lemma for a moment and show the proof of the theorem using the lemma. Let $\{f_i\}$ be a basis of $H^\bullet(A)$, and let $\{y_i\}$ be corresponding formal variables for $\{f_i\}$, namely $\boldsymbol{y} := \sum_i y_i f_i$.

We let $h^{sum}(\boldsymbol{y}) := \sum_{k\geq 1} h_k(\boldsymbol{y}^{\otimes k})$. Then

$$\frac{\partial}{\partial y_i}\Phi^{H^\bullet(A)} = \sum_{k\geq 1}\langle m_k^{H^\bullet(A)}(\boldsymbol{y},\cdots,\boldsymbol{y}), f_i\rangle$$

by cyclic symmetry, and

$$\begin{aligned}\frac{\partial}{\partial y_i}h^*\Phi^A &= \frac{\partial}{\partial y_i}\sum_{\substack{k\geq 1\\ p+q+k=N}}\frac{1}{N+1}\langle h^{sum}(\boldsymbol{y})^{\otimes p}, \underline{m_k^A(h^{sum}(\boldsymbol{y}),\cdots,h^{sum}(\boldsymbol{y}))}, h^{sum}(\boldsymbol{y})^{\otimes q} \mid h^{sum}(\boldsymbol{y})\rangle\\ &= \sum_{\substack{N\geq 1\\ p+q+k=N}}\langle h^{sum}(\boldsymbol{y})^{\otimes p}, \underline{m_k^A(h^{sum}(\boldsymbol{y}),\cdots,h^{sum}(\boldsymbol{y}))}, h^{sum}(\boldsymbol{y})^{\otimes q} \mid \frac{\partial}{\partial y_i}h^{sum}(\boldsymbol{y})\rangle\end{aligned}$$

by the above lemma. From the diagram (2.3.4), we have $\psi = \widetilde{h}^*\circ\widehat{\phi}\circ\widehat{\widetilde{h}}$, where all maps are $H^\bullet(A)$-bimodule homomorphisms, consider following:

$$\sum_{\substack{p,q\geq 0\\ k\geq 1}}\psi(\boldsymbol{y}^{\otimes p}, \underline{m_k^{H^\bullet(A)}(\overrightarrow{\boldsymbol{y}})}, \boldsymbol{y}^{\otimes q})(f_i) = \sum_{\substack{p,q\geq 0\\ k\geq 1}}(\widetilde{h}^*\circ\widehat{\phi}\circ\widehat{\widetilde{h}})(\boldsymbol{y}^{\otimes p}, \underline{m_k^{H^\bullet(A)}(\overrightarrow{\boldsymbol{y}})}, \boldsymbol{y}^{\otimes q})(f_i) \tag{5.2.1}$$

$$\begin{aligned}&= \sum_{\substack{p,q\geq 0\\ k\geq 1}}\sum_{\substack{p_1+p_2+p_3=p\\ q_1+q_2+q_3=q}}\widetilde{h}^*(\boldsymbol{y}^{\otimes p_3}, \phi(\widehat{h}(\boldsymbol{y}^{\otimes p_2}), \underline{h_{p_1+q_1+1}(\boldsymbol{y}^{\otimes p_1}, \underline{m_k^{H^\bullet(A)}(\overrightarrow{\boldsymbol{y}})}, \boldsymbol{y}^{\otimes q_1})}, \widehat{h}(\boldsymbol{y}^{\otimes q_2})), \boldsymbol{y}^{\otimes q_3})(f_i)\\ &= \sum_{\substack{p,q\geq 0\\ k\geq 1}}\sum_{\substack{p_1+p_2+p_3=p\\ q_1+q_2+q_3=q}}\phi(\widehat{h}(\boldsymbol{y}^{\otimes p_2}), \underline{h_{p_1+q_1+1}(\boldsymbol{y}^{\otimes p_1}, \underline{m_k^{H^\bullet(A)}(\overrightarrow{\boldsymbol{y}})}, \boldsymbol{y}^{\otimes q_1})}, \widehat{h}(\boldsymbol{y}^{\otimes q_2}))(h_{p_3+q_3+1}(\boldsymbol{y}^{\otimes q_3}, \underline{f_i}, \boldsymbol{y}^{\otimes p_3}))\\ &= \sum_{\substack{p,q\geq 0\\ k\geq 1}}\sum_{\substack{p_1+p_2+p_3=p\\ q_1+q_2+q_3=q}}\langle\widehat{h}(\boldsymbol{y}^{\otimes p_2}), \underline{h_{p_1+q_1+1}(\boldsymbol{y}^{\otimes p_1}, \underline{m_k^{H^\bullet(A)}(\overrightarrow{\boldsymbol{y}})}, \boldsymbol{y}^{\otimes q_1})}, \widehat{h}(\boldsymbol{y}^{\otimes q_2}) \mid h_{p_3+q_3+1}(\boldsymbol{y}^{\otimes q_3}, \underline{f_i}, \boldsymbol{y}^{\otimes p_3})\rangle\\ &= \sum_{\substack{N\geq 1\\ p+q+k=N}}\langle h^{sum}(\boldsymbol{y})^{\otimes p}, \underline{m_k^A(h^{sum}(\boldsymbol{y}),\cdots,h^{sum}(\boldsymbol{y}))}, h^{sum}(\boldsymbol{y})^{\otimes q} \mid \frac{\partial}{\partial y_i}h^{sum}(\boldsymbol{y})\rangle\\ &= \frac{\partial}{\partial y_i}h^*\Phi^A.\end{aligned}$$

Here, we denote by $m_k(\overrightarrow{\boldsymbol{y}})$ the expression $m_k(\boldsymbol{y},\cdots,\boldsymbol{y})$ for simplicity. The last identity holds because the sum is over all $p_1+p_2+p_3 = p$ and $q_1+q_2+q_3 = q$ where p and q run over all nonnegative integers, and there is the A_∞-bimodule relation $\widehat{m^A}\circ\widehat{h} = \widehat{h}\circ\widehat{m^{H^\bullet(A)}}$. We also used the fact that

$$\frac{\partial}{\partial y_i}h_k(\boldsymbol{y}^{\otimes k}) = \sum_{p_3+q_3+1=k}h_{p_3+q_3+1}(\boldsymbol{y}^{\otimes p_3}, f_i, \boldsymbol{y}^{\otimes q_3}).$$

The summands of (5.2.1) are all zero except for $(p,q)=(0,0)$ because ψ is a cyclic inner product. Hence,

$$\frac{\partial}{\partial y_i} h^*\Phi^A = \sum_{k\geq 1} \psi(m_k^{H^\bullet(A)}(\overrightarrow{\boldsymbol{y}}))(f_i) = \sum_{k\geq 1} \langle m_k^{H^\bullet(A)}(\boldsymbol{y},\cdots,\boldsymbol{y}), f_i\rangle = \frac{\partial}{\partial y_i}\Phi^{H^\bullet(A)}.$$

This proves the theorem. □

Proof of lemma 5.2.2. Before we proceed, we give some remarks on the signs. The sign convention used in this paper and in [C1] is the Koszul convention after the degree one shift. For simplicity, we omit the Koszul sign factor and the expressions will appear with $+$ if it agrees with the Koszul sign rule, $-$ if it is the negative of the Koszul sign. We illustrate this for two examples, from which the general convention can be easily understood. The first example is the A_∞-equation with two inputs. We write

$$m_1 m_2(x_1,x_2) + m_2(m_1(x_1),x_2) + m_2(x_1, m_1(x_2)) = 0 \tag{5.2.2}$$

whereas the actual equation is

$$m_1 m_2(x_1,x_2) + m_2(m_1(x_1),x_2) + (-1)^{|x_1|'} m_2(x_1, m_1(x_2)) = 0.$$

The equation (5.2.2) will also be written as

$$m_1 m_2(x_1,x_2) = -m_2(m_1(x_1),x_2) - m_2(x_1, m_1(x_2)).$$

The second example is the equation for $\langle m_2(\underline{x_1},x_2) \mid x_3\rangle$. Note that ϕ being an A_∞-bimodule map $\phi : A \to A^*$ with the induced A_∞-bimodule structure on A^* (see expression (3.3) [C1] for the precise definition) implies the following actual equation.

$$\begin{aligned}&\langle m_2(\underline{x_1},x_2) \mid x_3\rangle + \langle m_1(\underline{x_1}), x_2 \mid x_3\rangle + (-1)^{|x_1|'}\langle \underline{x_1}, m_1(x_2) \mid x_3\rangle \\ &+(-1)^{|x_1|'+|x_2|'}\langle \underline{x_1}, x_2 \mid m_1(x_3)\rangle + (-1)^{|x_1|'}\langle \underline{x_1} \mid m_2(x_2,x_3)\rangle = 0.\end{aligned}$$

In this paper, the above equation will be written simply as

$$\langle m_2(\underline{x_1}, x_2) \mid x_3\rangle + \langle m_1(\underline{x_1}), x_2 \mid x_3\rangle + \langle \underline{x_1}, m_1(x_2) \mid x_3\rangle \\ +\langle \underline{x_1}, x_2 \mid m_1(x_3)\rangle + \langle \underline{x_1} \mid m_2(x_2, x_3)\rangle = 0.$$

Now, we begin the proof of the lemma. From now on, we replace m_k^A by m_k if there is no confusion. By taking a derivative, the expression becomes as follows. For

$$\frac{\partial}{\partial x_i} \sum_{p+q+k=N} \langle \boldsymbol{x}, \boldsymbol{x}, \cdots, \boldsymbol{x}, \underline{m_k(\boldsymbol{x}, \boldsymbol{x}, \cdots, \boldsymbol{x})}, \boldsymbol{x}, \cdots, \boldsymbol{x} \mid \boldsymbol{x}\rangle_{p,q}, \tag{5.2.3}$$

$$\sum_{\substack{p+q+k=N \\ r+s=k-1}} \langle \boldsymbol{x}, \cdots, \boldsymbol{x}, \underline{m_k(\overbrace{\boldsymbol{x}, \cdots, \boldsymbol{x}}^{r}, e_i, \overbrace{\boldsymbol{x}, \cdots, \boldsymbol{x}}^{s})}, \boldsymbol{x}, \cdots, \boldsymbol{x} \mid \boldsymbol{x}\rangle_{p,q}, \tag{5.2.4}$$

$$\sum_{\substack{p+q+k=N \\ r+s=p-1}} \langle \underbrace{\boldsymbol{x}, \cdots, \boldsymbol{x}}_{r}, e_i, \underbrace{\boldsymbol{x}, \cdots, \boldsymbol{x}}_{s}, \underline{m_k(\boldsymbol{x}, \cdots, \boldsymbol{x})}, \boldsymbol{x}, \cdots, \boldsymbol{x} \mid \boldsymbol{x}\rangle_{p,q}, \tag{5.2.5}$$

$$\sum_{\substack{p+q+k=N \\ r+s=q-1}} \langle \boldsymbol{x}, \cdots, \boldsymbol{x}, \underline{m_k(\boldsymbol{x}, \cdots, \boldsymbol{x})}, \underbrace{\boldsymbol{x}, \cdots, \boldsymbol{x}}_{r}, e_i, \underbrace{\boldsymbol{x}, \cdots, \boldsymbol{x}}_{s} \mid \boldsymbol{x}\rangle_{p,q}, \tag{5.2.6}$$

$$\sum_{p+q+k=N} \langle \boldsymbol{x}, \cdots, \boldsymbol{x}, \underline{m_k(\boldsymbol{x}, \cdots, \boldsymbol{x})}, \boldsymbol{x}, \cdots, \boldsymbol{x} \mid e_i\rangle_{p,q}, \tag{5.2.7}$$

we have

$$(5.2.3) = (5.2.4) + (5.2.5) + (5.2.6) + (5.2.7).$$

Now, the lemma can be proved by the following lemma. $\square$

Lemma 5.2.3. *The sum of the terms in (5.2.4),(5.2.5) and (5.2.6) equals to N times of the expression (5.2.7).*

Proof. To prove the lemma, we recall the A_∞-bimodule equation. The equation for A_∞-bimodule homomorphism $A \to A^*$ is

$$\phi \circ \widehat{b_A} = b_{A^*} \circ \widehat{\phi} \tag{5.2.8}$$

with $b_A = m^A$ when A is considered to be an A_∞-bimodule, and b_{A^*} is defined by canonical construction of the dual of the A_∞-bimodule A. Here $\widehat{\phi}$ is the coalgebra homomorphism induced from ϕ (We refer readers to [C1],[T] or [GJ] for details). Let us restrict the equation (5.2.8) to the case $(\boldsymbol{x},\cdots,\boldsymbol{x},\underline{e_i},\boldsymbol{x},\cdots,\boldsymbol{x}) \in A^{\otimes n}\otimes \underline{A}\otimes A^{\otimes m}$ where $n+m+1=N$. For

$$\sum_{\substack{p+j_1=n\\ j_2+q=m}} \langle \boldsymbol{x},\cdots,\boldsymbol{x},\underline{m_{j_1+j_2+1}(\overbrace{\boldsymbol{x},\cdots,\boldsymbol{x}}^{j_1},\underline{e_i},\overbrace{\boldsymbol{x},\cdots,\boldsymbol{x}}^{j_2})},\boldsymbol{x},\cdots,\boldsymbol{x}\mid \boldsymbol{x}\rangle_{p,q}, \tag{5.2.9}$$

$$\sum_{\substack{k_1+k_2+j=n\\ p=k_1+k_2+1}} \langle \underbrace{\boldsymbol{x},\cdots,\boldsymbol{x}}_{k_1},m_j(\boldsymbol{x},\cdots,\boldsymbol{x}),\underbrace{\boldsymbol{x},\cdots,\boldsymbol{x}}_{k_2},\underline{e_i},\boldsymbol{x},\cdots,\boldsymbol{x}\mid \boldsymbol{x}\rangle^{dum}_{p,m}, \tag{5.2.10}$$

$$\sum_{\substack{l_1+l_2+h=m\\ q=l_1+l_2+1}} \langle \boldsymbol{x},\cdots,\boldsymbol{x},\underline{e_i},\underbrace{\boldsymbol{x},\cdots,\boldsymbol{x}}_{l_1},m_h(\boldsymbol{x},\cdots,\boldsymbol{x}),\underbrace{\boldsymbol{x},\cdots,\boldsymbol{x}}_{l_2}\mid \boldsymbol{x}\rangle^{dum}_{n,q}, \tag{5.2.11}$$

$$\sum_{\substack{p+k_1=m\\ k_2+q=n}} \langle \boldsymbol{x},\cdots,\boldsymbol{x},\underline{m_{k_1+k_2+1}(\overbrace{\boldsymbol{x},\cdots,\boldsymbol{x}}^{k_1},\underline{\boldsymbol{x}},\overbrace{\boldsymbol{x}\cdots\boldsymbol{x}}^{k_2})},\boldsymbol{x},\cdots,\boldsymbol{x}\mid e_i\rangle_{p,q}, \tag{5.2.12}$$

we have

$$(5.2.9)+(5.2.10)+(5.2.11)=(5.2.12).$$

It is important to note that the expression in the summand (5.2.12) is obtained in $k:=k_1+k_2+1$ different ways according to the position of the (underlined) bimodule element $\underline{x}$. Namely, different choices of a bimodule element still give rise to equivalent expressions. We also observe that (5.2.9)=(5.2.4) after summing over $n+m+1=N$.

We apply skew-symmetry to (5.2.10) and (5.2.11), namely we have

$$-(5.2.10) = \sum_{p+j+k_1+k_2+1=N} \langle \boldsymbol{x}^{\otimes p},\underline{\boldsymbol{x}},\boldsymbol{x}^{\otimes k_1},m_j(\vec{\boldsymbol{x}}),\boldsymbol{x}^{\otimes k_2}\mid e_i\rangle, \tag{5.2.13}$$

$$-(5.2.11) = \sum_{p+j+k_1+k_2+1=N} \langle \boldsymbol{x}^{\otimes p},m_j(\vec{\boldsymbol{x}}),\boldsymbol{x}^{\otimes k_1},\underline{\boldsymbol{x}},\boldsymbol{x}^{\otimes k_2}\mid e_i\rangle. \tag{5.2.14}$$

Here we set $m_j(\vec{\boldsymbol{x}}) := m_j(\boldsymbol{x},...,\boldsymbol{x})$.

In summary, we have the following:

$$(5.2.4) = k \cdot (5.2.7) + (5.2.13) + (5.2.14),$$

hence

$$(5.2.3) = k \cdot (5.2.7) + (5.2.13) + (5.2.14) + (5.2.5) + (5.2.6) + (5.2.7).$$

Now it remains to show that

$$(5.2.13) + (5.2.14) + (5.2.5) + (5.2.6) = (N - k) \cdot (5.2.7),$$

which proves the theorem.

Let us list the remaining terms first.

$$(5.2.5) \qquad \sum_{p+k+j_1+j_2+1=N} \langle \boldsymbol{x}^{\otimes p}, e_i, \boldsymbol{x}^{\otimes j_1}, \underline{m_k(\vec{\boldsymbol{x}})}, \boldsymbol{x}^{\otimes j_2} \mid \boldsymbol{x} \rangle,$$

$$(5.2.6) \qquad \sum_{p+k+j_1+j_2+1=N} \langle \boldsymbol{x}^{\otimes p}, \underline{m_k(\vec{\boldsymbol{x}})}, \boldsymbol{x}^{\otimes j_1}, e_i, \boldsymbol{x}^{\otimes j_2} \mid \boldsymbol{x} \rangle,$$

$$(5.2.13) \qquad \sum_{p+k+j_1+j_2+1=N} \langle \boldsymbol{x}^{\otimes p}, \underline{\boldsymbol{x}}, \boldsymbol{x}^{\otimes j_1}, m_k(\vec{\boldsymbol{x}}), \boldsymbol{x}^{\otimes j_2} \mid e_i \rangle,$$

$$(5.2.14) \qquad \sum_{p+k+j_1+j_2+1=N} \langle \boldsymbol{x}^{\otimes p}, m_k(\vec{\boldsymbol{x}}), \boldsymbol{x}^{\otimes j_1}, \underline{\boldsymbol{x}}, \boldsymbol{x}^{\otimes j_2} \mid e_i \rangle.$$

Now we use the closed condition with these terms.

1. By applying the closed condition from theorem 2.3.9 to (5.2.6) and (5.2.13), we obtain (here (a_i, a_j, a_k) corresponds to $(e_i, m_k(\vec{x}), \boldsymbol{x})$)

$$\begin{aligned} & \langle \underbrace{\boldsymbol{x}, \cdots, \boldsymbol{x}, \underline{\boldsymbol{x}}, \boldsymbol{x}, \cdots, \boldsymbol{x}}_{s}, m_k(\vec{\boldsymbol{x}}), \boldsymbol{x}^{\otimes r} \mid e_i \rangle \\ + \quad & \langle \boldsymbol{x}, \cdots, \boldsymbol{x}, \underline{m_k(\vec{\boldsymbol{x}})}, \boldsymbol{x}, \cdots, \boldsymbol{x}, e_i, \boldsymbol{x}, \cdots, \boldsymbol{x} \mid \boldsymbol{x} \rangle \\ + \quad & \langle \boldsymbol{x}^{\otimes r}, \underline{e_i}, \boldsymbol{x}^{\otimes s} \mid m_k(\vec{\boldsymbol{x}}) \rangle = 0 \end{aligned}$$

In fact, we obtain s different such equations depending on the position of $\underline{\boldsymbol{x}}$ in the first line. Hence, the sum of expressions (5.2.6) and (5.2.13) produces s times that of (5.2.7) as the last term equals the minus of (5.2.7):

$$\langle \boldsymbol{x}^{\otimes r}, \underline{e_i}, \boldsymbol{x}^{\otimes s} \mid m_k(\vec{\boldsymbol{x}})\rangle = -\langle \boldsymbol{x}^{\otimes s}, \underline{m_k(\vec{\boldsymbol{x}})}, \boldsymbol{x}^{\otimes r} \mid \underline{e_i}\rangle.$$

2. Similarly by applying the closed condition to (5.2.5) and (5.2.14),

$$\begin{aligned} & \langle \boldsymbol{x}^{\otimes s}, m_k(\vec{\boldsymbol{x}}), \underbrace{\boldsymbol{x}, \cdots, \boldsymbol{x}, \underline{\boldsymbol{x}}, \boldsymbol{x}, \cdots, \boldsymbol{x}}_{r} \mid e_i\rangle \\ + \quad & \langle \boldsymbol{x}, \cdots, \boldsymbol{x}, e_i, \boldsymbol{x}, \cdots, \boldsymbol{x}, \underline{m_k(\vec{\boldsymbol{x}})}, \boldsymbol{x}, \cdots, \boldsymbol{x} \mid \boldsymbol{x}\rangle \\ + \quad & \langle \boldsymbol{x}^{\otimes r}, \underline{e_i}, \boldsymbol{x}^{\otimes s} \mid m_k(\vec{\boldsymbol{x}})\rangle \\ = \quad & 0. \end{aligned}$$

we obtain r different such equations depending on the position of $\underline{\boldsymbol{x}}$ in the first line.

Hence we obtain $r + s = N - k$ times the expression of (5.2.7), which proves lemma 5.2.3. □

Chapter 6

Proof of Theorem B

We studied A_∞-algebras via formal noncommutative geometry in chapter 3, because it is very useful in the proof of Theorem B. We will use such noncommutative geometry languages, so we begin from giving explicit correspondences between A_∞-algebras and formal noncommutative manifolds, and then the proof will be based on the correspondence.

6.1 Correspondences between algebra and formal noncommutative geometry

It is useful to develop a "dictionary" between notions in homological algebras and that of formal manifolds. First, it is well-known(originally due to Kontsevich) that cyclic symmetry of an A_∞-algebra can be understood as certain symplecic forms.

Lemma 6.1.1. *For an A_∞-algebra A, if an A_∞-bimodule map $\phi : A \to A^*$ is a cyclic inner product on A, then it is equivalent to a noncommutative constant symplectic two form ω with $\mathcal{L}_Q \omega = 0$.*

Proof. Let $\omega = \sum_{a,b} \omega_{ab}(dx^a dx^b)_c$, where $\phi(e_a, e_b) = \omega_{ab}$. Then

$$
\begin{aligned}
& \mathcal{L}_Q \omega = \mathcal{L}_Q(\sum_{a,b} \omega_{ab}(dx^a dx^b)_c) \\
= \ & \sum_{a,b}((\omega_{ab}(\mathcal{L}_Q dx^a) dx^b)_c + (dx^a (-1)^{|a|'|Q|'} \mathcal{L}_Q dx^b)_c) \\
= \ & \sum_{a,b}(\omega_{ab} \sum_{i_1,\ldots,i_k} \sum_{1 \le l \le k} m^a_{i_1 \cdots i_k}(x^{i_1} \cdots dx^{i_l} \cdots x^{i_k} dx^b)_c) \\
& +(-1)^{|a|'} \sum_{i_1,\ldots,i_k} \sum_{1 \le l \le k} m^b_{i_1 \cdots i_k}(dx^a x^{i_1} \cdots dx^{i_l} \cdots x^{i_k})_c \\
= \ & \sum_{a,b} \omega_{ab} \sum_{i_1,\ldots,i_k} \sum_{1 \le l \le k} m^a_{i_1 \cdots i_k}(x^{i_1} \cdots dx^{i_l} \cdots x^{i_k} dx^b)_c \\
& + \sum_{a,b}(-1)^{|b|'} \omega_{ba} (-1)^{1+|b|'(|i_1|'+\cdots+|i_k|')} \sum_{i_1,\ldots,i_k} \sum_{1 \le l \le k} (x^{i_1} \cdots dx^{i_l} \cdots x^{i_k} dx^b)_c \\
= \ & \sum_{a,b} \sum_{i_1,\ldots,i_k} \sum_{1 \le l \le k} \omega_{ab}(1 + (-1)^{1+|b|'(1+|i_1|'+\cdots+|i_k|')+|a|'|b|'+1}) m^a_{i_1 \cdots i_k}(x^{i_1} \cdots dx^{i_l} \cdots x^{i_k} dx^b)_c \\
= \ & \sum_{a,b} \sum_{i_1,\ldots,i_k} \sum_{1 \le l \le k} 2\omega_{ab} m^a_{i_1 \cdots i_k}(x^{i_1} \cdots dx^{i_l} \cdots x^{i_k} dx^b)_c. \qquad (6.1.1)
\end{aligned}
$$

Note that $(-1)^{1+|b|'(1+|i_1|'+\cdots+|i_k|')+|a|'|b|'+1} = 1$ because

$$|a|' = |i_1|' + \cdots + |i_k|' + 1,$$

by the fact that Q has degree 1. A careful observation on the cyclic monomials in (6.1.1) leads us to the following: $\mathcal{L}_Q \omega = 0$ is equivalent to

$$
\begin{aligned}
& \omega_{ab} m^a_{i_1 \cdots i_k}(x^{i_1} \cdots dx^{i_l} \cdots x^{i_k} dx^b)_c \\
& +\omega_{a i_l} m^a_{i_{l+1} \cdots i_k b i_1 \cdots i_{l-1}}(x^{i_{l+1}} \cdots x^{i_k} dx^b x^{i_1} \cdots x^{i_{l-1}} dx^{i_l})_c \\
= \ & (\omega_{ab} m^a_{i_1 \cdots i_k} + (-1)^p \omega_{a i_l} m^a_{i_{l+1} \cdots i_k b i_1 \cdots i_{l-1}})(x^{i_1} \cdots dx^{i_l} \cdots x^{i_k} dx^b)_c \\
= \ & 0
\end{aligned}
$$

for $p = 1 + (|i_1|' + \cdots + |i_l|')(|i_{l+1}|' + \cdots + |i_k|' + |b|')$, if and only if

$$\omega_{ab} m^a_{i_1 \cdots i_k} = (-1)^{p+1} \omega_{a i_l} m^a_{i_{l+1} \cdots i_k b i_1 \cdots i_{l-1}},$$

i.e.

$$\begin{aligned} & \langle m(e_{i_1}, \cdots, e_{i_k}), e_b \rangle \\ = \; & (-1)^{(|i_1|' + \cdots + |i_l|')(|i_{l+1}|' + \cdots + |i_k|' + |b|')} \langle m(e_{i_{l+1}}, \cdots, e_{i_k}, e_b, e_{i_1}, \cdots, e_{i_{l-1}}), e_l \rangle, \end{aligned}$$

which is the cyclicity. □

Recall that we have $\mathcal{L}_Q \circ \mathcal{L}_Q = 0$ and hence, on de Rham complex $\Omega_{cyc}(X)$, we have two differentials d_{cyc} and $\mathcal{L}_Q$. By the Poincaré lemma, the homology with respect to d_{cyc} is trivial. On the other hand, the differential $\mathcal{L}_Q$ gives an interesting cohomology.

Lemma 6.1.2. *For a unital finite dimensional A_∞-algebra A, $(\Omega^1_{cyc}(X)[1], \mathcal{L}_Q)$ can be identified with Hochschild cochain complex $(C^\bullet(A, A^*), b^*)$, and $(\Omega^0_{cyc}(X)/\boldsymbol{k}, \mathcal{L}_Q)$ can be identified with cyclic cochain complex $((C^\lambda(A))^*, b^*)$.*

Namely, we have the following 1-1 correspondences.

A_∞-*algebra* A	*Formal noncommutative manifold* X
$\eta \in C^\bullet(A, A^*)$	$\alpha_\eta \in \Omega^1_{cyc}(X)$
$b^*\eta$	$\mathcal{L}_Q \alpha_\eta$
$\xi \in (C^\lambda_\bullet(A))^*$	$f_\xi \in \Omega^0_{cyc}(X)$
$b^*\xi$	$\mathcal{L}_Q f_\xi$

Proof. We first check the statement for Hochschild cochains. The degree shifting [1] is the result due to the choice of the chain complex in 4.1.1. If $\eta \in \mathrm{Hom}(A[1]^{\otimes n}, A^*)$ given by $\eta(e_{i_1}, ..., e_{i_n})(e_j) = \eta^j_{i_1,...,i_n}$ for basis elements e_*, it corresponds to the 1-form $\alpha_\eta = \sum \eta^j_{i_1,...,i_n} (x^{i_1} \cdots x^{i_n} dx^j)_c$. We omit the Koszul signs in the following formulas. We verify that $b^*\eta$ corresponds to $\mathcal{L}_Q\alpha_\eta$.

$$\begin{aligned} b^*\eta(e_{i_1},...,e_{i_n})(e_j) &= \sum \eta(e_{i_1},...,m_k(e_{i_l},...,e_{i_{l+k-1}}),...,e_{i_n})(e_j) \\ &\quad + \sum \eta(e_{i_l},...,e_{i_{l+p}})(m_k(e_{i_{l+p+1}},...,e_{i_n},e_j,e_{i_1},...,e_{i_{l-1}})) \\ &= \sum_q \eta^j_{i_1,...,i_{l-1},q,i_{l+k},...,i_n} \cdot m^q_{i_l,...,i_{l+k-1}} \\ &\quad + \sum_q \eta^q_{i_l,...,i_{l+p}} \cdot m^q_{i_{l+p+1},...,i_n,j,i_1,...,i_{l-1}}. \end{aligned}$$

Thus,

$$\begin{aligned} \alpha_{b^*\eta} &= \sum(\sum_q \eta^j_{i_1,...,i_{l-1},q,i_{l+k},...,i_n} \cdot m^q_{i_l,...,i_{l+k-1}} \\ &\quad + \sum_q \eta^q_{i_l,...,i_{l+p}} \cdot m^q_{i_{l+p+1},...,i_n,j,i_1,...,i_{l-1}})x^{i_n}\cdots x^{i_1}dx^j \end{aligned}$$

is the 1-form corresponding to $b^*\eta$.

On the other hand,

$$\begin{aligned} \mathcal{L}_Q\alpha_\eta &= \sum \eta^j_{i_1,...,i_n} x^{i_1}\cdots x^{i_{l-1}}(m^{i_l}_{j_1,...,j_r}x^{j_1}\cdots x^{j_r})x^{i_{l+1}}\cdots x^{i_n}dx^j \\ &\quad + \sum \eta^j_{i_1,...,i_n} x^{i_1}\cdots x^{i_n} d(m^j_{j_1,...,j_r}x^{j_1}\cdots x^{j_r}). \end{aligned}$$

By comparing each coefficients, we obtain $\alpha_{b^*\eta} = \mathcal{L}_Q\alpha_\eta$.

For the cyclic case, the Connes' complex $C^\lambda_\bullet(A) = C_\bullet(A,A)/\mathrm{im}(1-t)$ defines the cyclic homology and similar arguments as above can be used to prove the desired identifications, which we leave for the readers as an exercise.

Later, we will introduce an operation $\widetilde{\ }$, and then show that $\widetilde{b^*\eta}$ corresponds to $d\mathcal{L}_Q\eta$. □

6.2 Explicit relations

In this section, we show that for a negative cyclic cocycle $\phi \in HC^\bullet_-(A,A^*)$ with a suitable non-degeneracy condition, it gives rise to a strong homotopy inner product in a canonical way. Denote the negative cyclic cycle ϕ as $\phi = \sum_{i\geq 0}\phi_i v^i$, where v is a formal parameter of degree -2. Here cocycle condition implies that we have $b^*\phi_i = B^*\phi_{i+1}$ for each i.

First, we make the following observation.

Proposition 6.2.1. *Let $\phi \in C^\bullet(A, A^*)$ be a negative cyclic cocycle. We define*

$$\widetilde{\phi_0}(\vec{a}, \underline{v}, \vec{b})(w) := \phi_0(\vec{a}, v, \vec{b})(w) - \phi_0(\vec{b}, w, \vec{a})(v).$$

Then $\widetilde{\phi_0}$ is an A_∞-bimodule map from A to A^*, satisfying the skew-symmetry and closedness condition in the definition 2.3.7.

For convenience, we write both $\widetilde{\phi_0} = \widetilde{\phi}$ without distinction.

Proof. Recall that $\widetilde{\phi_0}$ is an A_∞-bimodule map from (C, m) to (C^*, m^*) if

$$\widetilde{\phi_0} \circ \widehat{m} = m^* \circ \widehat{\widetilde{\phi_0}}.$$

We will show this in two steps.

Lemma 6.2.2. *We have*

$$\widetilde{\phi_0} \circ \widehat{m} - m^* \circ \widehat{\widetilde{\phi_0}} = \widetilde{B^*\phi_1},$$

where $\widetilde{B^*\phi_1}$ is defined by

$$\widetilde{B^*\phi_1}(\vec{a}, v, \vec{b})(w) = B^*\phi_1(\vec{b}, w, \vec{a})(v) - B^*\phi_1(\vec{a}, v, \vec{b})(w)$$

Lemma 6.2.3. *We have*

$$\widetilde{B^*}\gamma(\vec{a}, v, \vec{b})(w) = B^*\gamma(\vec{b}, w, \vec{a})(v) - B^*\gamma(\vec{a}, v, \vec{b})(w) = 0,$$

for any $\gamma \in C^\bullet(A, A^*)$, and for any $\vec{a}, \vec{b}, v, w$.

Combining the above two lemmas, we obtain the proposition. The skew-symmetry and closedness condition is easy to check and its proof is omitted. □

Proof. We begin the proof of lemma 6.2.2. We first show that

$$(\widetilde{\phi_0} \circ \widehat{m} - m^* \circ \widehat{\widetilde{\phi_0}})(\vec{a}, \underline{v}, \vec{b})(w) = b^*\phi_0(\vec{a}, v, \vec{b})(w) - b^*\phi_0(\vec{b}, w, \vec{a})(v). \quad (6.2.1)$$

And this equals the following as $b^*\phi_0 = B^*\phi_1$ as it is negative cyclic cocycle.

$$B^*\phi_1(\vec{b}, w, \vec{a})(v) - B^*\phi_1(\vec{a}, v, \vec{b})(w).$$

We again omit the Koszul signs in the following formula and express the additional contributions of signs. Let $\vec{a} := (a_1, ..., a_n)$ and $\vec{b} := (b_1, ..., b_m)$.

$$\begin{aligned}
& (\widetilde{\phi_0} \circ \widehat{m})(\vec{a}, \underline{v}, \vec{b})(w) \qquad (6.2.2)\\
= & \sum_{\substack{0 \le i \le n-1 \\ k \ge 1}} \phi_0(a_1, ..., a_i, m_k(a_{i+1}, ..., a_{i+k}), a_{i+k+1}, ..., a_n, v, \vec{b})(w) \\
+ & \sum_{\substack{0 \le i \le n-1, 1 \le j \le m \\ k \ge 1}} \phi_0(a_1, ..., a_i, m_k(a_{i+1}, ..., a_n, v, b_1, ..., b_j), b_{j+1}, ..., b_m)(w) \\
+ & \sum_{\substack{0 \le j \le m-1 \\ k \ge 1}} \phi_0(\vec{a}, v, b_1, ..., b_j, m_k(b_{j+1}, ..., b_{j+k}), b_{j+k+1}, ..., b_m)(w) \\
- & \sum_{\substack{0 \le j \le m \\ k \ge 1}} \phi_0(b_1, ..., b_j, m_k(b_{j+1}, ..., b_{j+k}), b_{j+k+1}, ..., b_m, w, \vec{a})(v) \\
- & \sum_{\substack{0 \le i \le n-1 \\ k \ge 1}} \phi_0(\vec{b}, w, a_1, ..., a_i, m_k(a_{i+1}, ..., a_{i+k}), a_{i+k+1}, ..., a_n)(w) \\
- & \sum_{\substack{0 \le s \le n-1, 1 \le j \le m \\ k \ge 1}} \phi_0(b_{j+1}, ..., b_m, w, a_1, ..., a_s)(m_k(a_{s+1}, ..., a_n, v, b_1, ..., b_j))
\end{aligned}$$

$$\begin{aligned}
& (m^* \circ \widehat{\widehat{\phi_0}})(\vec{a}, \underline{v}, \vec{b})(w) && (6.2.3)\\
= & \sum_{\substack{0 \le j \le m-1, 1 \le i \le n \\ k \ge 1}} \phi_0(b_1, ..., b_j, m_k(b_{j+1}, ..., b_m, w, a_1, ..., b_i), a_{i+1}, ..., a_n)(v) \\
- & \sum_{\substack{1 \le i \le n, 0 \le j \le m-1 \\ k \ge 1}} \phi_0(a_{i+1}, ..., a_n, v, b_1, ..., b_j)(m_k(b_{j+1}, ..., b_m, w, a_1, ..., a_i))
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& b^*\phi_0(\vec{a}, v, \vec{b})(w) - b^*\phi_0(\vec{b}, w, \vec{a})(v) \\
= & (6.2.2) - (6.2.3) \\
+ & \sum \phi_0(a_i, ..., a_l)(m(a_{l+1}, ..., a_k, v, \vec{b}, w, a_1, ..., a_{i-1})) && (6.2.4)\\
+ & \sum \phi_0(b_j, ..., b_p)(m(b_{p+1}, ..., b_n, w, \vec{a}, v, b_1, ..., b_{j-1})) && (6.2.5)\\
- & \sum \phi_0(b_j, ..., b_p)(m(b_{p+1}, ..., b_n, w, \vec{a}, v, b_1, ..., b_{j-1})) && (6.2.6)\\
- & \sum \phi_0(a_i, ..., a_l)(m(a_{l+1}, ..., a_k, v, \vec{b}, w, a_1, ..., a_{i-1})) && (6.2.7)
\end{aligned}$$

Note that the terms (6.2.4)-(6.2.7) cancel out by themselves. By combining the above results, the lemma 6.2.2 is obtained. □

Proof. Now we prove lemma 6.2.3.

$$B^*\gamma(c_1, ..., c_n)(c_{n+1}) \;=\; \sum_{\sigma \in \mathbb{Z}/n\mathbb{Z}} \gamma(c_{\sigma(1)}, ..., c_{\sigma(n+1)})(1)$$

Hence, $B^*\gamma(c_1, ..., c_n)(c_{n+1}) = B^*\gamma(c_{\sigma(1)}, ..., c_{\sigma(n)})(c_{\sigma(n+1)})$ for any $\sigma \in \mathbb{Z}/n\mathbb{Z}$. In particular, $B^*\phi_1(\vec{b}, w, \vec{a})(v) - B^*\phi_1(\vec{a}, v, \vec{b})(w) = 0$. □

Remark 6.2.4. *In the case that we use the (dual of) Tsygan's bicomplex, instead of (b^*, B^*)-complex to define the negative cyclic cohomology, the same proposition holds true: this is because the equation 6.2.1 still holds. If we have $b^*\phi_0 = N^*\phi_1'$ instead for the symmetrization operator N, then the proof above* *shows that $\widetilde{N^*\phi_1}$ also should vanish as in the case of B using the same symmetry argument.*

Hence if $\widetilde{\phi}_{0,0}$ is nondegenerate on $H^\bullet(A)$, then ϕ indeed gives a strong homotopy inner product. We call such a $\phi \in C^\bullet_-(A, A^*)$ be *homologically nondegenerate*(H.N. for short below).

Lemma 6.2.5. *We have the following 1-1 correspondences.*

A_∞*-algebra* A	*Formal noncommutative manifold* X
skew-sym. A_∞*-bimod. map* $\psi : A \to A^*$	$\omega_\psi \in \Omega^2_{cyc}(X)$ *with* $L_Q\omega_\psi = 0$
$\eta \in C^\bullet(A, A^*)$	$\alpha_\eta \in \Omega^1_{cyc}(X)$
$\widetilde{\eta}$	$d\alpha_\eta$
S.H.I.P. $\phi : A \to A^*$	*H.N.* $\omega_\phi \in \Omega^2_{cyc}(X), d\omega_\phi = 0 = \mathcal{L}_Q\omega_\phi$

Proof. Given a collection of maps $\psi_{k,l} : A^{\otimes k} \otimes \underline{A} \otimes A^{\otimes l} \to A^*$, we assign a cyclic 2-form

$$\omega_\psi = \sum(\psi_{k,l}(e_{i_1}, ..., e_{i_k}, \underline{e_j}, e_{j_1}, ..., e_{j_l})(e_n))x^{i_1} \cdots x^{i_k} dx^j x^{j_1} \cdots x^{j_l} dx^n$$

for basis elements e_* (as in [C1]). Skew-symmetry is needed as we cannot tell the order of dx^j, dx^n in the expression for cyclic forms.

We omit the proof of the correspondence of L_Q-closedness and A_∞-bimodule property. This can be carried out similarly as in the proof of Prop 6.2.1 and Lemma 6.1.2 and it is tedious but elementary computations.

We show that $\omega_{\widetilde{\eta}} = d\alpha_\eta$. Observe that

$$\begin{aligned}\widetilde{\eta}(e_{i_1}, ..., e_{i_k}, \underline{e_j}, e_{j_1}, ..., e_{j_l})(e_n) &= \eta(e_{i_1}, ..., e_k, e_j, e_{j_1}, ..., e_{j_l})(e_n) \\ &\quad -\eta(e_{j_1}, ..., e_{j_l}, e_n, e_{i_1}, ..., e_{i_k})(e_j) \\ &= \eta^n_{i_1,...,i_k,j,j_1,...,j_l} - \eta^j_{j_1,...,j_l,n,i_1,...,i_k},\end{aligned}$$

so

$$\omega_{\widetilde{\eta}} = \sum(\eta^n_{i_1,...,i_k,j,j_1,...,j_l} - \eta^j_{j_1,...,j_l,n,i_1,...,i_k})x^{i_1} \cdots x^{i_k} dx^j x^{j_1} \cdots x^{j_l} dx^n$$

is the 2-form corresponding to $\widetilde{\eta}$.

By definition,

$$d\alpha_\eta = \sum_l \sum \eta^j_{i_1,\dots,i_n} x^{i_1} \cdots dx^{i_l} \cdots x^{i_n} dx^j. \tag{6.2.8}$$

Note that in Ω_{cyc}, we have (up to Koszul sign)

$$x^{i_{l+1}} \cdots x^{i_n} dx^j x^{i_1} \cdots x^{i_{l-1}} dx^{i_l} = -x^{i_1} \cdots dx^{i_l} \cdots x^{i_n} dx^j,$$

and hence (6.2.8) reduces to

$$d\alpha_\eta = \sum_l \sum (\eta^j_{i_1,\dots,i_n} - \eta^{i_l}_{i_{l+1},\dots,i_n,j,i_1,\dots,i_{l-1}}) x^{i_1} \cdots dx^{i_l} \cdots x^{i_n} dx^j.$$

Then we have $\omega_\eta = d\alpha_\eta$ by rearranging indices above.

Suppose that we are given a strong homotopy inner product $\phi : A \to A^*$. Consider the corresponding two form ω_ϕ from the above. It is not hard to check that the closedness condition is equivalent to $d_{cyc}\omega_\phi = 0$. Hence, as we proved that $\mathcal{L}_Q$-closedness of ω_ϕ is equivalent to ϕ being A_∞-bimodule map, so we obtain the last claim. □

6.3 Construction of an automorphism

In this section, we prove that two strong homotopy inner products obtained two negative cyclic cocycles in the same homology class are indeed equivalent to each other in the sense of 2.3.12 (see also the comments at the end of the section 3.3).

First, we construct A_∞-automorphisms from certain kinds of vector fields.

Lemma 6.3.1. *A formal vector field v which satisfies $[Q, v] = 0$ provides an A_∞-automorphism. Here v is assumed to have length ≥ 2. (i.e. any non-trivial component of v which is given by $f(x)\frac{\partial}{\partial x^i}$ satisfies $order(f(x)) \geq 2$).*

Proof. A formal vector field v (as a derivation) corresponds to a coderivation, which we also call v, of tensor coalgebra $TV[1]$. Such v is represented by a family of maps $v_k : A^{\otimes k} \to A$, and denote by $\widehat{v}$ the coderivation

$$\widehat{v} : TV[1] \to TV[1], \widehat{v} = \sum_k \widehat{v_k}.$$

where $\widehat{v_k}$ is defined as in the definition of A_∞-operation $\widehat{m_k}$. Corresponding to the condition $[Q, v] = 0$ is the identity

$$\widehat{d} \circ \widehat{v} = \widehat{v} \circ \widehat{d}. \tag{6.3.1}$$

We define its exponential $e^{\widehat{v}}$ as

$$e^{\widehat{v}} = 1 + \widehat{v} + \frac{1}{2!}\widehat{v} \circ \widehat{v} + \frac{1}{3!}\widehat{v} \circ \widehat{v} \circ \widehat{v} + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}(\widehat{v})^k \tag{6.3.2}$$

One can check that the infinite sum makes sense due to the assumption on v. Let $\pi : TV[1] \to V[1]$ be the natural projection to its component of tensor length one. Then, we define

$$f := \pi \circ e^{\widehat{v}} : TV[1] \to V[1]. \tag{6.3.3}$$

It is easy to check that one may write

$$f = \mathrm{id} \circ \pi + v(\sum \frac{1}{k!}(\widehat{v})^{k-1}).$$

In fact, by the assumption on v, $f_1 : V[1] \to V[1]$ is given by identity.

For example, we have

$$\begin{aligned}
& e^{\widehat{v}}(x_1 \otimes x_2 \otimes x_3) \\
= \; & x_1 \otimes x_2 \otimes x_3 + v(x_1 \otimes x_2) \otimes x_3 + x_1 \otimes v(x_2 \otimes x_3) \\
& + v(x_1 \otimes x_2 \otimes x_3) + \frac{v(v(x_1 \otimes x_2) \otimes x_3) + v(x_1 \otimes v(x_2 \otimes x_3))}{2} \\
= \; & f_1(x_1) \otimes f_1(x_2) \otimes f_1(x_3) + f_2(x_1 \otimes x_2) \otimes f_1(x_3) + f_1(x_1) \otimes f_2(x_2 \otimes x_3) \\
& + f_3(x_1 \otimes x_2 \otimes x_3) \\
= \; & \widehat{f}(x_1 \otimes x_2 \otimes x_3).
\end{aligned}$$

In general, we have $\widehat{f} = e^{\widehat{v}}$, which we prove in the following lemma. Now, the proof of the Lemma 6.3.1 follows from the following lemma. □

Lemma 6.3.2. *f defines an A_∞-automorphism. More precisely, we have*

$$\widehat{f} = e^{\widehat{v}}, \quad \widehat{d}\widehat{f} = \widehat{f}\widehat{d}.$$

Proof. We first show that $e^{\widehat{v}} : TV \to TV$ satisfies the following identity

$$(e^{\widehat{v}} \otimes e^{\widehat{v}}) \circ \Delta = \Delta \circ e^{\widehat{v}} \tag{6.3.4}$$

This would imply that $e^{\widehat{v}}$ is a cohomomorphism, and it is well-known that such a cohomomorphism is completely determined by its projection (6.3.3) (see for example [T]) and satisfies the identity $\widehat{f} = e^{\widehat{v}}$.

To prove the identity, we apply (6.3.4) to an expression $x_1 \otimes \cdots \otimes x_k$. The left hand side of (6.3.4) becomes

$$(e^{\widehat{v}} \otimes e^{\widehat{v}}) \circ \Delta(x_1 \otimes \cdots \otimes x_k) = \sum_{i=1}^{k} \big(e^{\widehat{v}}(x_1 \otimes \cdots \otimes x_i) \otimes e^{\widehat{v}}(x_{i+1} \otimes \cdots \otimes x_k)\big).$$

The right hand side becomes

$$\Delta \circ e^{\widehat{v}}(x_1 \otimes \cdots \otimes x_k) = \Delta\big(\sum_{j=0}^{\infty} \frac{1}{j!}(\underbrace{\widehat{v} \circ \cdots \circ \widehat{v}}_{j}(x_1 \otimes \cdots \otimes x_k))\big)$$

$$= \sum_{j=0}^{\infty} \frac{1}{j!} \sum_{\substack{(j_1,j_2) \text{ shuffle} \\ j_1+j_2=j}} \big(\underbrace{\widehat{v} \circ \cdots \widehat{v}}_{j_1}\big) \otimes \big(\underbrace{\widehat{v} \circ \cdots \widehat{v}}_{j_2}\big) \circ \Delta(x_1 \otimes \cdots \otimes x_k).$$

The equality here is obtained by noting that Δ divides the tensor product into two parts. Recall that the number of such shuffles are $\frac{j!}{j_1!j_2!}$ and hence the above expression becomes

$$= \sum_{j=0}^{\infty} \sum_{j_1+j_2=j} \big(\frac{1}{j_1!}(\widehat{v})^{j_1} \otimes \frac{1}{j_2!}(\widehat{v})^{j_2}\big) \circ \Delta(x_1 \otimes \cdots \otimes x_k).$$

This proves the claim.

From this, we have

$$\widehat{d}\widehat{f} = \widehat{d} \circ e^{\widehat{v}} = e^{\widehat{v}} \circ \widehat{d} = \widehat{f}\widehat{d}.$$

by the identity (6.3.1) above. □

Remark 6.3.3. *The automorphism just defined is* not *the automorphism to transform*

$$\omega + \mathcal{L}_Q(d\alpha) \mapsto \omega$$

that is suggested in the Lemma 3.3.1. In fact it is a first order approximation of the correct automorphism, and in the next proposition, we show how to find the actual automorphism which transforms $\omega + \mathcal{L}_Q(d\alpha)$ to ω.

In the section 6.2, we assigned a strong homotopy inner product to a negative cyclic cocycle. Now we prove that the assignment is also well-defined on the cohomology level up to equivalence of strong homotopy inner products.

Proposition 6.3.4. *Let A be a weakly unital compact A_∞-algebra. If two negative cyclic cocycles ϕ and ϕ' give the same cohomology class, then $\widetilde{\phi}$ and $\widetilde{\phi'}$ are equivalent as strong homotopy inner products.*

Proof. First, we pullback all the related notions to the minimal model $H^\bullet(A, m_1)$, which is unital and finite dimensional. By using the decomposition theorem of an A_∞-algebra, suppose we have $A = H \oplus A_{lc}$, where H is the minimal part and A_{lc} is the linear contractible part of A. Let $i : H \to A$ be the inclusion, which is also an A_∞-quasi-isomorphism.

First,in the unital case, as the two cycles ϕ, ϕ' are cohomologous, we may write $\phi^{'} = \phi + (b^* + vB^*)\psi$. Hence may write for some η, $\gamma \in C^{\bullet}_{-}(A, A^*)$

$$\phi_0^{'} = \phi_0 + b^*\eta + B^*\gamma.$$

Hence, the induced A_∞-bimodule maps from the Prop. 6.2.1 satisfy $\widetilde{\phi}' = \widetilde{\phi} + \widetilde{b^*\eta} + \widetilde{B^*\gamma}$, but by lemma 6.2.3, we have $\widetilde{B^*\gamma} \equiv 0$, so $\widetilde{\phi}' = \widetilde{\phi} + \widetilde{b^*\eta}$. In the weakly unital case, one can proceed similarly using Tsygan's bicomplex using the remark 6.2.4.

Now, using the A_∞-quasimorphism $i : H \to A$, we pull back $\widetilde{\phi}$ and $\widetilde{\phi}'$ to H by

$$\begin{array}{ccc} A & \xleftarrow{\tilde{i}} & H \\ {\scriptstyle \widetilde{\phi}}\downarrow & & \downarrow{\scriptstyle i^*\widetilde{\phi}} \\ A^* & \xrightarrow{\tilde{i}^*} & H^* \end{array} \tag{6.3.5}$$

to obtain $i^*\widetilde{\phi}$ and $i^*\widetilde{\phi}'$. From the definition of the equivalence of strong homotopy inner products, it is enough to prove the equivalence between $i^*\widetilde{\phi}$ and $i^*\widetilde{\phi}'$.

Using i, we can also pullback the Hochschild cohomology classes by $i^* : C^{\bullet}(A, A^*) \to C^*(H, H^*)$. We claim that

$$i^*\widetilde{b_A^*\eta} = \widetilde{i^*b_A^*\eta} = \widetilde{b_H^* i^*\eta}$$

Here, the first i^* was used to pullback an infinity inner product, while the other i^*'s are for Hochschild cochains. The first equality is almost trivial, and the second one is given by following:

$$
\begin{aligned}
& i^*b_A^*\eta(a_1,...,a_k)(a_{k+1}) \\
= & \sum b_A^*\eta(i(a_1),...,i(a_k))(i(a_{k+1})) \\
= & \sum \eta(i(a_1),...,i(a_l),m_A(i(a_{l+1}),...,i(a_p)),i(a_{p+1}),...,i(a_k))(i(a_{k+1})) \\
& + \sum \eta(i(a_j),...,i(a_p))(m_A(i(a_{p+1}),...,i(a_{k+1}),i(a_1),...,i(a_{j-1}))) \\
= & \sum \eta(i(a_1),...,i(a_l),i(m_H(a_{l+1},...,a_p)),i(a_{p+1}),...,i(a_k))(i(a_{k+1})) \\
& + \sum \eta(i(a_j),...,i(a_p))(i(m_H(a_{p+1}),...,i(a_{j-1}))) \\
= & b_H^* i^*\eta(a_1,...,a_k)(a_{k+1}).
\end{aligned}
$$

Observe that in the third equality we used the fact $i \circ \widehat{m_H} = m_A \circ \widehat{i}$, i.e. i is an A_∞-homomorphism.

By using the results of [C1], in fact, we can pull them back further similarly via the diagram

$$
\begin{array}{ccc}
H & \xrightarrow{\tilde{g}} & H \\
{\scriptstyle cyc}\downarrow & & \downarrow{\scriptstyle \widetilde{\phi}} \\
H^* & \xleftarrow{\tilde{g}^*} & H^*
\end{array}
\tag{6.3.6}
$$

to assume that the strong homotopy inner product $\widetilde{\phi}$ is in fact cyclic inner product.

Therefore, it is enough to prove the proposition for the minimal model H with the cyclic inner product $\widetilde{\phi}$ and it suffices to find an A_∞-automorphism f with the following commutative diagram:

$$
\begin{array}{ccc}
H & \overset{f}{\dashrightarrow} & H \\
{\scriptstyle \widetilde{\phi}}\downarrow & & \downarrow{\scriptstyle \widetilde{\phi}+\widetilde{b^*\eta}} \\
H^* & \underset{f^*}{\dashleftarrow} & H^*
\end{array}
\tag{6.3.7}
$$

It is very hard to get such an automorphism f at once, so we need to construct it recursively. The construction becomes more natural if we use the dual notion of all above, namely formal noncommutative calculus. In the dual context, H corresponds to a formal noncommutative affine manifold X, and A_∞-automorphism f corresponds to the coordinate change of X preserving Q which is a vector field corresponding to the A_∞-structure of H as before.

Let $\omega = \sum \omega_{ij} dx^i dx^j$ be a closed cyclic 2-form on X which corresponds to the cyclic inner product $\widetilde{\phi}$. We denote

$$d\mathcal{L}_Q \eta = \sum a_{ij} dx^j dx^i + \sum_{|I \cup J| \geq 1} a_{ij,IJ} x^I dx^i x^J dx^j.$$

We claim that the coefficients $a_{ij} = 0$ for all i, j: By minimality of H, $Q = 0 + O(x^2)$, i.e. the constant and the linear part of Q is zero, and this implies the claim.

By a simple-minded idea, we might be tempted to solve an equation

$$\mathcal{L}_{v'} \omega = -d\mathcal{L}_Q \eta$$

or

$$i_{v'} \omega = -\mathcal{L}_Q \eta$$

by the nondegeneracy of ω, to get a vector field $v' = \sum v_i'(x) \frac{\partial}{\partial x^i}$. But such v' may not be the desired solution, in the sense that it may not satisfy suitable length condition. Instead of solving the above equation, we solve

$$i_v \omega = -L_Q \eta_{\geq 1},$$

where we write

$$\eta = \sum_i a_i dx_i + \sum_{|I| \geq 1} a_{Ij} x_I dx_j$$

and $\eta_{\geq 1} := \sum_{|I| \geq 1} a_{Ij} x_I dx_j$. It is straightforward to check that $L_v \omega = -dL_Q \eta$. But the important feature of v is that $v = 0 + O(x^2)$. The vector field v' from the simple-minded equation does not satisfy this in general.

Note that ω and $\omega + d\mathcal{L}_Q \eta$ have the same constant part, or

$$\omega + d\mathcal{L}_Q \equiv \omega + O(x^2).$$

Using v, we constructed the automorphism f in the previous lemma. To check how much f has transformed $\omega + d\mathcal{L}_Q$, we proceed as follows using noncommutative calculus.

First, we denote

$$e^{\mathcal{L}_v} := Id + \mathcal{L}_v + \frac{(\mathcal{L}_v)^2}{2!} + \frac{(\mathcal{L}_v)^3}{3!} + \cdots .$$

Lemma 6.3.5. *Under change of coordinates $x^i \mapsto e^{\mathcal{L}_v} x^i$, any differential form β transforms as*

$$\beta \mapsto e^{\mathcal{L}_v}\beta.$$

In fact the coordinate change here corresponds to an A_∞-isomorphism of the lemma 6.3.1 in the sense of the (5.1.3).

Proof. This is easily seen as follows. Since this is trivial for coordinate functions and $e^{\mathcal{L}_v}$ commutes with d, it suffices to show that $e^{\mathcal{L}_v}(\alpha \cdot \beta) = e^{\mathcal{L}_v}\alpha \cdot e^{\mathcal{L}_v}\beta$ for any two differential forms α and β.

$$\begin{aligned} e^{\mathcal{L}_v}\alpha \cdot e^{\mathcal{L}_v}\beta &= \sum_{k\geq 0} \frac{(\mathcal{L}_v)^k}{k!}\alpha \cdot \sum_{l\geq 0} \frac{(\mathcal{L}_v)^l}{l!}\beta \\ &= \sum_{k,l\geq 0} \frac{(\mathcal{L}_v)^k}{k!}\alpha \cdot \frac{(\mathcal{L}_v)^l}{l!}\beta \end{aligned}$$

and

$$\begin{aligned} e^{\mathcal{L}_v}(\alpha \cdot \beta) &= \sum_{k\geq 0} \frac{(\mathcal{L}_v)^k}{k!}\alpha \cdot \beta \\ &= \sum_{k\geq l\geq 0} \frac{1}{k!}(\mathcal{L}_v)^{k-l}\alpha \cdot (\mathcal{L}_v)^l\beta \cdot \frac{k!}{(k-l)!l!}. \end{aligned} \tag{6.3.8}$$

In (6.3.8), we used that $\mathcal{L}_v$ is a derivation, and $\dfrac{k!}{(k-l)!l!}$ means the number of $(k-l, l)$-shuffles. Hence we get the desired result.

To prove the second claim, let $f : H \to H$ be an A_∞-automorphism such that

$$f(e_{i_1}, ..., e_{i_k}) = \sum_j f^j_{i_1,...,i_k} e_j.$$

Then the coordinate change associated to f is given by

$$x^j \mapsto \sum f^j_{i_1,...,i_k} x^{i_1} \cdots x^{i_k},$$

where $\{x^j\}$ is the dual coordinate of $\{e_j\}$.

Now let f be given as in lemma 7.1, i.e. $f = id \circ \pi + v(\sum \frac{1}{k!}(\widehat{v})^{k-1})$. As usual, let

$$v(e_{i_1}, ..., e_{i_k}) = \sum_j v^j_{i_1,...,i_k} e_j,$$

and let $v^j(e_{i_1}, ..., e_{i_k}) := v^j_{i_1,...,i_k} e_j$. Then

$$f_k(e_{i_1}, ..., e_{i_k}) = \sum_{1 \le l \le k-1} \frac{1}{l!} v \circ \widehat{v}^{l-1}(e_{i_1}, ..., e_{i_k}).$$

As above, let

$$f^j_k(e_{i_1}, ..., e_{i_k}) := \sum_{1 \le l \le k-1} \frac{1}{l!} v^j \circ \widehat{v}^{l-1}(e_{i_1}, ..., e_{i_k}). \tag{6.3.9}$$

Finally, compare the coefficient of the l-th summand of (6.3.9) and that of $\frac{(\mathcal{L}_v)^l}{l!} x^j$, then we will easily get the result. □

Hence, this coordinate change gives us

$$\omega^{(1)} := \omega + d\mathcal{L}_Q \eta \mapsto \omega^{(2)} := e^{\mathcal{L}_v} \omega + e^{\mathcal{L}_v}(d\mathcal{L}_Q \eta),$$

$$
\begin{aligned}
e^{\mathcal{L}_v}\omega + e^{\mathcal{L}_v} d\mathcal{L}_Q\eta &= (\omega + \mathcal{L}_v\omega + \sum_{k\geq 2}\frac{1}{k!}(\mathcal{L}_v)^k\omega) + (d\mathcal{L}_Q\eta + \sum_{k\geq 1}\frac{1}{k!}(\mathcal{L}_v)^k d\mathcal{L}_Q\eta) \\
&= \omega + \sum_{k\geq 2}\frac{1}{k!}(\mathcal{L}_v)^{k-1}\mathcal{L}_v\omega + d\mathcal{L}_Q\sum_{k\geq 1}\frac{1}{k!}(\mathcal{L}_v)^k\eta \\
&= \omega + \sum_{k\geq 2}\frac{1}{k!}(\mathcal{L}_v)^{k-1}(-d\mathcal{L}_Q\eta) + d\mathcal{L}_Q\sum_{k\geq 1}\frac{1}{k!}(\mathcal{L}_v)^k\eta \\
&= \omega + d\mathcal{L}_Q\sum_{k\geq 2}(-\frac{1}{k!}(\mathcal{L}_v)^{k-1}\eta) + d\mathcal{L}_Q\sum_{k\geq 1}\frac{1}{k!}(\mathcal{L}_v)^k\eta \\
&= \omega + d\mathcal{L}_Q\sum_{k\geq 1}a_k(\mathcal{L}_v)^k\eta \\
&= \omega + \sum_{k\geq 1}a_k(\mathcal{L}_v)^k(d\mathcal{L}_Q\eta)
\end{aligned}
$$

for some numbers $a_k \in k$. We emphasize that for the second and the fourth identities, we used lemma 3.3.1 so that $[\mathcal{L}_Q, \mathcal{L}_v] = \mathcal{L}_{[Q,v]} = 0$.

Note that the term $d\mathcal{L}_Q\eta$ changed into $\sum_{k\geq 1} a_k(\mathcal{L}_v)^k(d\mathcal{L}_Q\eta)$. The operation $\mathcal{L}_v = d\circ i_v + i_v \circ d$ increase the number of formal variable x^i's in the expression at least by one.

Hence, we have

$$\omega^{(2)} \equiv \omega + O(x^3).$$

By repeating the same procedure, we can transforms $\omega + d\mathcal{L}_Q\eta$ into ω via countably many procedures. We remark that the infinite composition of such automorphism is well-defined as the automorphism at the step (k) will fix the tensor product of length up to (k). This proves the proposition. □

Summarizing this provides proof of the Theorem B.

6.4 A connection to the Kontsevich-Soibelman's result

In [KS1], Kontsevich-Soibelman has provided the formula for the cyclic inner product on the minimal model using the trace, and we show that it agrees with our formula.

Namely, we have two ways to get cyclic inner products on $H^\bullet(A)$ from given a homologically nondegenerate negative cyclic cocycle ϕ. Namely, for $a, b \in H^\bullet(A)$, we may consider $\widetilde{\phi}(a)(b) = \phi(a)(b) - \phi(b)(a)$ as in proposition 6.2.1, or consider $\omega(a)(b) = Tr_{c[\phi]}(m_2(a,b)) = Tr_{[B^*\phi_0]}(m_2(a,b))$ as in [KS1]. $Tr_{[\eta]} : A/[A,A] \to k$ for $[\eta] \in HC^\bullet$ is given by

$$Tr_{[\eta]}(a) = \eta_0|_{A^*}(a)$$

(Recall that $\eta_0 \in C^\bullet(A, A^*) = \bigoplus_{n \geq 0} \mathrm{Hom}(A^{\otimes n}, A^*) = \bigoplus_{n \geq 1} \mathrm{Hom}(A^{\otimes n}, k)$).

Proposition 6.4.1. *Let ϕ be a negative cyclic cocycle of A with whose zeroth column part is ϕ_0.*

Then $Tr_{[B^*\phi_0]}(m_2(\cdot,\cdot)) = \widetilde{\phi}(\cdot)(\cdot)$.

Proof. We identify cocycles in $\bigoplus_{n\geq 1} Hom(A^{\otimes n}, k)$. For $a, b \in H^\bullet(A)$,

$$\begin{aligned} & Tr_{[B^*\phi_0]}(m_2(a,b)) \\ = \; & B^*\phi_0(m_2(a,b)) \\ = \; & \phi_0(1, m_2(a,b)). \end{aligned}$$

A priori, we have $b^*\phi(1,a,b) = B^*\psi(1,a,b)$ for some hochschild cochain ψ

because ϕ is a cocycle. Clearly the right hand side is zero. On the other hand,

$$\begin{aligned}
& b^*\phi_0(1,a,b) \\
= \; & \phi_0(\widehat{b}(1,a,b)) \\
= \; & \phi_0(m_2(1,a),b) + (-1)^{1\cdot 1}\phi_0(1,m_2(a,b)) + (-1)^{|b|'(|a|'+1)}\phi_0(m_2(b,1),a) \\
= \; & \phi_0(a,b) - \phi_0(1,m_2(a,b)) + (-1)^{|a|'|b|'+|b|'+|b|}\phi_0(b,a).
\end{aligned}$$

Hence $Tr_{[B^*\phi]}(m_2(a,b)) = \phi_0(1,m_2(a,b)) = \phi_0(a,b) - (-1)^{|a|'|b|'}\phi_0(b,a)$ as we desired. □

We remark that a minimal model of an A_∞-algebra also has many automorphisms which do not preserve the A_∞-structure and the cyclic structure, hence given an arbitrary minimal model, one can*not* assume that the trace as above provides the cyclic inner product of the given minimal model. Rather, [KS1] proves the existence of one minimal model which is cyclic with respect to the trace. Our formula provides a diagram to connect cyclic structure, (negative) cyclic cohomology class and the related A_∞-structures.

We also remark that the homological nondegeneracy of cyclic cohomology class ϕ does *not* imply that ϕ is a nontrivial cohomology class. For example, there exists an A_∞-algebra with trivial m_1-homology, but equipped with cyclic inner product. In such a case, cyclic cohomology can be shown to be trivial using the spectral sequence arguments with the length filtration.

6.5 Gapped filtered cases

Gapped filtered A_∞-algebras are introduced by Fukaya, Oh, Ohta and Ono in their construction of gapped filtered A_∞-algebra of Lagrangian submanifold. For the gapped filtered A_∞-algebras, many of the results in this paper remain true as it will be explained. But there exists some subtlety in filtered notions, as sometimes non-negativity of the energy from the filtration is needed. For example, the Darboux theorem in the general form does not hold true, but only for non-negative symplectic forms.

6.5.1 Filtered A_∞-algebras

We recall the notion of gapped filtered A_∞-algebra, and we refer readers to [FOOO1] for full details. To consider A_∞-algebras arising from the study of Lagrangian submanifolds or in general pseudo-holomorphic curves, one considers filtered A_∞-algebras over Novikov rings, where the filtration is given by the energy of pseudo-holomorphic curves. Here Novikov rings are, for a ring R (here T and e are formal parameters)

$$\Lambda_{nov} = \{\sum_{i=0}^{\infty} a_i T^{\lambda_i} e^{q_i} \,|\, a_i \in R,\ \lambda_i \in \mathbb{R},\ q_i \in \mathbb{Z},\ \lim_{i\to\infty} \lambda_i = \infty\}$$

$$\Lambda_{nov,0} = \{\sum_i a_i T^{\lambda_i} e^{q_i} \in \Lambda_{nov} | \lambda_i \geq 0\}.$$

When we take dualizations, it is convenient to work with Novikov fields. The above rings $\Lambda_{nov}, \Lambda_{nov,0}$ are not fields but one can forget the formal parameter e (and work with $\mathbb{Z}/2$ grading only) and work with the following Novikov fields

$$\Lambda = \{\sum_{i=0}^{\infty} a_i T^{\lambda_i} \,|\, a_i \in \boldsymbol{k},\ \lambda_i \in \mathbb{R},\ \lim_{i\to\infty} \lambda_i = \infty\},\ \ \Lambda_0 = \{\sum_{i=0}^{\infty} a_i T^{\lambda_i} \in \Lambda | \lambda_i \geq 0\}. \tag{6.5.1}$$

Here, we consider a field $\boldsymbol{k}$ containing $\mathbb{Q}$, and there also exist another choice $\Lambda_{nov}^{(e)}$ in [C1]. We remark that in most of the construction of [FOOO1], they work with $\Lambda_{nov,0}$ and only when one needs to work with Λ_{nov}, they take tensor product $\otimes\Lambda_{nov}$ to work with Λ_{nov} coefficients. We take a similar approach for Λ and Λ_0.

The gapped condition is defined as follows. The monoid $G \subset \mathbb{R}_{\geq 0} \times 2\mathbb{Z}$ is assumed to satisfy the following conditions

1. The projection $\pi_1(G) \subset \mathbb{R}_{\geq 0}$ is discrete.
2. $G \cap (\{0\} \times 2\mathbb{Z}) = \{(0,0)\}$
3. $G \cap (\{\lambda\} \times 2\mathbb{Z})$ is a finite set for any λ.

Consider a free graded $\Lambda_{nov,0}$ module C, and let $\overline{C}$ be an $\boldsymbol{k}$-vector space such that $C = \overline{C} \otimes_{\boldsymbol{k}} \Lambda_{nov,0}$. Then $(C, m_{\geq 0})$ is said to be G-gapped if there exists homomorphisms $m_{k,\beta} : (\overline{C}[1])^{\otimes k} \to \overline{C}[1]$ for $k = 0, 1, \cdots, \beta = (\lambda(\beta), \mu(\beta)) \in G$ such that

$$m_k = \sum_{\beta \in G} T^{\lambda(\beta)} e^{\mu(\beta)/2} m_{k,\beta}.$$

One defines filtered gapped A_∞-algebras as in the definition 2.1.2, by considering the same equation (2.1.4) for $k = 0, 1, \cdots$.

Recall that these m_k operations may be considered as coderivations by defining

$$\widehat{m}_k(x_1 \otimes \cdots \otimes x_n) = \sum_{i=1}^{n-k+1} (-1)^{|x_1|' + \cdots + |x_{i-1}|'} x_1 \otimes \cdots \otimes m_k(x_i, \cdots, x_{i+k-1}) \otimes \cdots \otimes x_n \tag{6.5.2}$$

for $k \leq n$ and $\widehat{m}_k(x_1 \otimes \cdots \otimes x_n) = 0$ for $k > n$. If we set $\widehat{d} = \sum_{k=0}^{\infty} \widehat{m}_k$, the A_∞-equations are equivalent to the equality $\widehat{d} \circ \widehat{d} = 0$.

We recall cyclic A_∞-algebras in the gapped filtered case.

Definition 6.5.1. *A filtered gapped A_∞-algebra $(C, \{m_*\})$ is said to have a* cyclic inner product *if there exists a skew-symmetric non-degenerate, bilinear map*

$$\langle , \rangle : \overline{C}[1] \otimes \overline{C}[1] \to \boldsymbol{k},$$

which is extended linearly over C, such that for all integer $k \geq 0$, $\beta \in G$,

$$\langle m_{k,\beta}(x_1, \cdots, x_k), x_{k+1} \rangle = (-1)^K \langle m_{k,\beta}(x_2, \cdots, x_{k+1}), x_1 \rangle. \tag{6.5.3}$$

where $K = |x_1|'(|x_2|' + \cdots + |x_{k+1}|')$. For short, we will call such an algebra, cyclic (filtered) A_∞-algebra.

6.5.2 Weakly filtered A_∞-bimodule homomorphisms

The usual notions of filtered A_∞-homomorphisms, and filtered bimodule maps require the maps to preserve filtrations. But the map obtained via differential forms in the Lemma 6.2.5 do not always preserve the filtration, but provides so called, weakly filtered A_∞-bimodule homomorphisms in [FOOO1].

First we recall the notion of filtered A_∞-homomorphism between two filtered A_∞-algebras. The family of maps of degree 0

$$f_k : B_k(C_1) \to C_2[1] \ \text{ for } k = 0, 1, \cdots$$

induce the coalgebra map $\hat{f} : \hat{B}C_1 \to \hat{B}C_2$, which for $x_1 \otimes \cdots \otimes x_k \in B_k C_1$ is defined by the formula

$$\hat{f}(x_1 \otimes \cdots \otimes x_k) = \sum_{0 \le k_1 \le \cdots \le k_n \le k} f_{k_1}(x_1, \cdots, x_{k_1}) \otimes \cdots \otimes f_{k-k_n}(x_{k_n+1}, \cdots, x_k).$$

We remark that the above can be an infinite sum due to the possible existence of $f_0(1)$. In particular, $\hat{f}(1) = e^{f_0(1)}$. It is assumed that

$$\begin{cases} f_k(F^\lambda B_k(C_1)) \subset F^\lambda C_2[1], \ \text{ and} \\ f_0(1) \in F^{\lambda'} C_2[1] \ \text{ for some } \lambda' > 0. \end{cases} \tag{6.5.4}$$

The map $\hat{f}$ is called a filtered A_∞-homomorphism if

$$\hat{d} \circ \hat{f} = \hat{f} \circ \hat{d}.$$

We recall the definition of weakly filtered A_∞-bimodule homomorphisms from [FOOO1] in a simple case of A-bimodules for an A_∞-algebra $A = (C, \{m\})$. Let $\widetilde{M}$ and $\widetilde{M}'$ be filtered (A, A) A_∞-bimodules over Λ_{nov}. A *weakly filtered A_∞-bimodule homomorphism* $\widetilde{M} \to \widetilde{M}'$ is a family of Λ_{nov}-module homomorphisms

$$\phi_{k_1,k_0} : B_{k_1}(C) \hat{\otimes} \widetilde{M} \hat{\otimes} B_{k_0}(C) \to \widetilde{M}'$$

with the following properties:

1. There exists $c \ge 0$ independent of k_0, k_1 such that

$$\phi_{k_1,k_0}\big(F^{\lambda_1} B_{k_1}(C) \hat{\otimes} F^\lambda \widetilde{M} \hat{\otimes} F^{\lambda_0} B_{k_0}(C)\big) \subset F^{\lambda_1+\lambda+\lambda_0-c} \widetilde{M}'$$

2. $\hat{\phi} \circ \hat{d} = \hat{d}' \circ \hat{\phi}$

Weakly filtered homomorphisms arise when we study the invariance property of the Floer cohomology $HF(L_0, L_1) \cong HF(L_0, \phi(L_1))$ where the constant c is related to the Hofer norm of the Hamiltonian isotopy ϕ.

6.5.3 Formal manifolds

The bar complex $\widehat{B}C$ in the filtered case is obtained by taking a completion with respect to energy. Hence, the Hochschild complex $C_{\bullet}(A, A)$ is similarly defined but also has to be completed. To consider dualization of the bar complex $\widehat{B}C$, we consider only Novikov fields Λ, and also assume that C is a finite dimensional vector space. And then, we can take topological dual spaces as in [C1]: Let V be a vector space over the field Λ with finitely many generators $\{e_i\}_{i=1}^n$. Consider V as a topological vector space by defining a fundamental system of neighborhoods of V at 0: first define the filtrations $F^{>\lambda}V$ as

$$F^{>\lambda}V = \{\sum_{j=1}^{k} a_j v_{i_j} | a_i \in \Lambda, \tau(a_i) > \lambda, \ \forall i\}.$$

Here τ is the valuation of Λ which gives the minimal exponent of T. We regard $F^{>\lambda}V$ for $\lambda = 0, 1, 2, \cdots$ as fundamental system of neighborhoods at 0. The completion with respect to energy can be also considered as a completion using the Cauchy sequences in V with the above topology.

Now, consider the following topological dual space

$$\mathcal{O}(X) = \mathrm{Hom}_{cont}(\widehat{B}C, \Lambda).$$

Consider the dual basis $\{x_i\}_{i=1}^n$ considered as elements in $\hat{V}^* = \mathrm{Hom}_{cont}(\hat{V}, \Lambda)$. Then, the Lemma 9.1 of [C1] may be translated as

Lemma 6.5.2. *We have*

$$\mathcal{O}(X) = \Lambda\langle\langle x_1, \cdots, x_n \rangle\rangle, \tag{6.5.5}$$

where the right hand side is the set of all formal power series of variables $x_1, \cdots, x_n$ *whose coefficients in* Λ *are bounded below.*

In particular, $\mathcal{O}(X)$ does not contain formal power series whose energy of the coefficients converging to $-\infty$. Intuitively, the dual elements are allowed to have infinite sums with bounded energy since the inputs for the evaluation already has energy converging to infinity in its infinite sum.

One can also possibly use (6.5.5) as a definition with several different choices of coefficient rings $\Lambda_{nov}, \Lambda_{nov,0}, \Lambda, \Lambda_0$. From now on, we will work with Λ but other coefficients can be used for the rest of the paper also with little modification.

6.5.4 Darboux theorem

First, we define de Rham complex $\Omega_{cyc}(X)$, vector field Q as before. Note that the coefficients of the vector field Q always have non-negative energy from the definition of A_∞-structure. Also note that Q may have a component of constant vector field which corresponds to the term m_0.

We show that the Darboux theorem in general does not hold in the filtered case, and one should restrict to symplectic forms with non-negative energy. Let $\omega \in \Omega^2_{cyc}(X)$ be a closed non-degenerate two form in the filtered setting as above. Suppose the symplectic form can be written as $\omega = \omega_{ij} dx^i dx^j + \omega'$ for $\omega' \in \Omega^2_{cyc}(X)$ such that each term of ω' has either positive energy (T^λ for $\lambda > 0$) or positive length (with possibly negative energy).

Theorem 6.5.3 (Darboux theorem)**.** *Consider the symplectic form* $\omega = \omega_{ij} dx^i dx^j + \omega'$ *as above. If* ω' *does not contain a term with negative energy, then there exist filtered* A_∞*-isomorphism* f *which solves Darboux theorem.*

$$i.e. \ \ f^*\omega = \omega_{ij} dx^i dx^j.$$

But if ω' *contains a term with negative energy with positive length, then there does* not *exist any filtered* A_∞*-isomorphism* f *solving the Darboux theorem.*

Proof. For the first claim, we follow the proof of unfiltered case in the theorem 4.15 of [Kaj]. In the gapped filtered case, the induction should be run over the sum of two indices. As $\pi_1(G)$ is discrete, we can find an increasing sequence λ_j with $\lim \lambda_j = \infty$ which covers the image of $\pi_1(G) \subset \mathbb{R}_{\geq 0}$. We run the induction over the sum $k + j = N$, where k is the power of x^i's and j is for the energy level λ_j.

Now, assume that ω satisfies the assumption, and ω is transformed to the constant up to level N. Then, we consider the transformation of the form

$$x^i \mapsto x^i + f^i, \quad f^i = \sum_{j+k=N} T^{\lambda_j} x^{i_1} \cdots x^{i_k}.$$

By this transformation, ω is transformed as

$$\big(\omega_{ij} dx^i dx^j + \omega_N + \cdots\big) \longmapsto \big(\omega_{ij} dx^i dx^j + \omega_N + \omega_{ij} 2 d_{cycl}((f^i) dx^j)_c + \cdots\big).$$

But as $\omega_{ij} dx^i dx^j + \omega_N + \cdots$ is d_{cycl}-closed, hence ω_N is d_{cycl}-closed and hence d_{cycl}-exact. So, by appropriate choice of f^i, ω_N can be cancelled out as ω_{ij} is non-degenerate. Thus ω is transformed to be constant up to $(N+1)$-level. Repeating this process completes the proof.

For the second statement, it is enough to show that a filtered isomorphism preserve the minimal negative exponent of the given symplectic form. Note that as f is an isomorphism, f_1 is an isomorphism. Then it is not hard to see as in the above that from the contribution of f_1, the change of coordinate by filtered A_∞-map f preserve the minimal negative exponent of the given symplectic form. □

We remark that there does not exist a notion of weakly filtered A_∞-homomorphism. Namely, a component f_k of the filtered A_∞-map f cannot decrease the energy. If f_k does decrease the energy, $\widehat{f_k}$ for the bar complex would provide sequence of terms with the energy converging to $-\infty$, but such elements do not exist in the bar complex $\widehat{BC}$.

6.5.5 Correspondences

First, the definition of Hochschild (co)homology of filtered A_∞-algebra can be given in a similar way. But to consider its homological algebra, one has to be careful to deal with m_0 terms, which we refer readers to [C1]. (For example, the standard contracting homotopy for the bar complex has to be modified.) One define also the Hochschild cochain complex $(C^\bullet(A, A^*), b^*)$ by taking the topological dual of $(C_\bullet(A, A), b)$.

As in the lemma 6.1.2, we have

Lemma 6.5.4. *For a unital finite dimensional filtered gapped A_∞-algebra A, the complex $(\Omega^1_{cycl}(X)[1], \mathcal{L}_Q)$ can be identified with Hochschild cochain complex $(C^\bullet(A, A^*), b^*)$, and $(\Omega^0_{cycl}(X)/\Lambda, \mathcal{L}_Q)$ can be identified with cyclic cochain complex $((C^\lambda(A))^*, b^*)$.*

Also the Prop. 6.2.1 holds true in the gapped filtered case. The lemma 6.2.5 has to be modified as follows. First, we denote by

$$\Omega^2_{cyc,+}(X) \subset \Omega^2_{cyc}(X),$$

the subset consisting of formal sums each term of which has non-negative energy coefficient.

Lemma 6.5.5. *We have the following 1-1 correspondences.*

1. *A filtered (resp. weakly filtered) skew-symmetric A_∞-bimodule map $\psi : A \to A^*$ corresponds to a two form $\omega_\psi \in \Omega^2_{cyc,+}(X)$ (resp. $\in \Omega^2_{cyc}(X)$) with $L_Q\omega_\psi = 0$.*

2. *The relation between $\widetilde{\eta}$ and $d\alpha_\eta$ is as before.*

3. *The strong homotopy inner product $\phi : A \to A^*$ corresponds to the homologically nondegenerate $\omega_\phi \in \Omega^2_{cyc,+}(X)$ with $d\omega_\phi = 0 = \mathcal{L}_Q\omega_\phi$.*

We remark that weakly filtered A_∞-bimodule maps can be used to prove the following lemma, which is proved in [C1].

Lemma 6.5.6. *The homologically non-degenerate weakly filtered A_∞-bimodule map $\phi : A \to A^*$ provides an isomorphism of Hochschild homology $H_\bullet(A, A)$ with $H_\bullet(A, A^*)$.*

6.5.6 Theorem B in the filtered case

Now, we prove the main theorem for gapped filtered A_∞-algebras. First, we call a filtered A_∞-algebra (C, m) *compact* if the homology $H^\bullet(C, m_{1,0})$ is finite dimensional, and is called *canonical* if $m_{1,0} \equiv 0$. In [FOOO1], the canonical model theorem (similar to minimal model theorem) is proved.

Theorem 6.5.7. *For a weakly unital compact gapped filtered A_∞-algebra A, a homologically nondegenerate negative cyclic cohomology class $[\phi]$, each term of which has non-negative energy, gives rise to an isomorphism class of strong homotopy inner products on A. In particular, from $[\phi]$, we construct a strong homotopy inner product $\widetilde{\phi_0} : A \to A^*$ explicitly using the Proposition 6.2.1.*

In particular, we have a quasi-isomorphic cyclic gapped filtered A_∞-algebra B with $\psi : B \to B^*$ satisfying the commuting diagram

$$\begin{array}{ccc} A & \xleftarrow{\ \tilde{\iota}\ } & B \\ {\scriptstyle \widetilde{\phi_0}}\downarrow & & \downarrow{\scriptstyle \psi\ cyc} \\ A^* & \xrightarrow{\ \tilde{\iota}^*\ } & B^* \end{array} \tag{6.5.6}$$

Proof. The correspondence can be proved using the lemma in the previous section. Hence it is enough to prove that cohomologous negative cyclic homology classes provide equivalent strong homotopy inner products. As before, we pullback all the related notions to the canonical model $H^\bullet(C, m_{1,0})$, which is unital and finite dimensional.

In the unital case, as the two cycles ϕ, ϕ' are cohomologous, we may write $\phi' = \phi + (b^* + vB^*)\psi$. Hence may write for some η, $\gamma \in C^\bullet_-(A, A^*)$

$$\phi_0' = \phi_0 + b^*\eta + B^*\gamma.$$

Here we also assume that each term of η and γ also has non-negative energy. So we have $\widetilde{\phi'} = \widetilde{\phi} + \widetilde{b^*}\eta$ as before. To find a filtered A_∞-automorphism f satisfying the diagram 6.3.6, we proceed as before but only modify the inductive argument using sum of order and energy.

In fact, we run the induction over the sum $k + 2j = N$, where k is the power of x^i's and j is for the energy level λ_j as in the proof of the theorem 6.5.3. The reason that we use $2j$ instead of j is as follows.

First, given a differential form

$$cT^{\lambda_j} x_{i_{11}} \cdots x_{i_{1a_1}} dx_{j_1} x_{i_{21}} \cdots x_{i_{2a_2}} dx_{j_2} \cdots dx_{j_{m-1}} x_{i_{m1}} \cdots x_{i_{ma_m}}$$

we define its order to be $2j + a_1 + \cdots a_m$. One can note that d decrease the order by one, and i_Q for the canonical model, increase the order by at least two. Hence, the Lie derivative $\mathcal{L}_Q = d \circ i_Q + i_Q \circ d$ increases the order by one.

Now, we will work with formal vector field v such that $\mathcal{L}_v \omega = -d\mathcal{L}_Q \eta$ as before. Note that such a v can be chosen without a constant vector field term. Then, the following can be proved analogously:

Lemma 6.5.8. *In the filtered case, a formal vector field v which satisfies $[Q, v] = 0$ provides an A_∞-automorphism. Here v is assumed to have order ≥ 2 and no constant vector field term. (i.e. any non-trivial component of v which is given by $T^{\lambda_j} f_i(x) \frac{\partial}{\partial x^i}$ satisfies $\big(order(f_i(x)) + 2j \geq 2\big)$ and $f_i(x)$ is not constant.)*

The rest of proof works as in the unfiltered case. In this case also, the automorphism f constructed above will change the symplectic form as

$$\omega + d\mathcal{L}_Q \eta \longmapsto \omega + \sum_{k \geq 1} a_k (\mathcal{L}_v)^k (d\mathcal{L}_Q \eta)$$

But, note that v has at least have order two. Hence, $\mathcal{L}_v = d \circ i_v + i_v \circ d$ increase the order at least by one. Hence even in the gapped filtered case, the induction works as in the unfiltered case. $\square$

Chapter 7

Proof of Theorem C

Now we can study another kind of potentials for a unital homotopy cyclic A_∞-algebra A. We discuss its gauge invariance and its relationship with the algebraic analogue of generalized holonomy map in [ATZ].

We assume that the strong homotopy inner product $\phi : A \to A^*$ is a unital A_∞-bimodule map, or $\phi_{k,l}(\vec{a}, v, \vec{b})(w)$ vanishes if one of a_i's or b_i's is a constant multiple of I.

We also recall the Maurer-Cartan elements and its gauge equivalences.

Definition 7.0.9. *Let $A = (C, m)$ be an A_∞-algebra. An element $b \in C^1$ satisfying $m(e^b) = \sum_k m_k(b, ..., b) = 0$ is called a* Maurer-Cartan element *and we denote by $MC(A)$ the set of all Maurer-Cartan elements. Let $\mathcal{MC} := MC/\sim$ be the moduli space of Maurer-Cartan elements, whose gauge equivalence is defined as follows(definition 2.3 of [Fu2]): b is* gauge equivalent *to $\widetilde{b}$ if there are one-parameter families $b(t) \in A^1[t], c(t) \in A^0[t]$ such that*

- *$b(0) = b, b(1) = \widetilde{b}$, and*
- $\dfrac{d}{dt}b(t) = \displaystyle\sum_{k\geq 1} m_k(b(t), ..., b(t), c(t), b(t), ..., b(t))$.

We remark that $b(t)$ is also a Maurer-Cartan element for any t (Lemma 4.3.7 of [FOOO1]). Now, we prove the gauge invariance of the potential Ψ for Maurer-Cartan elements.

Proposition 7.0.10. *The potential* $\Psi(x) = \sum_{p,q\geq 0} \frac{1}{p+q+1}\langle x^{\otimes p} \otimes \underline{x} \otimes x^{\otimes q} \mid I\rangle$ *which is restricted to the Maurer-Cartan elements* MC *is invariant under gauge equivalences. i.e. if* $x(t)$ *is a one-parameter family in the Maurer-Cartan solution space, then*

$$\frac{d}{dt}\Psi(x(t)) = 0.$$

Proof. We prove this proposition with the help of following lemmas.

Lemma 7.0.11. $\Psi(x) = \sum_{k\geq 0}\langle \underline{x} \otimes x^{\otimes k} \mid I\rangle$.

Proof. By the closedness condition of ϕ, for any p and q we have

$$\begin{aligned}\langle x^{\otimes p} \otimes \underline{x} \otimes x^{\otimes q} \mid I\rangle \quad &+ \quad \langle x^{\otimes p+q} \otimes \underline{I} \mid x\rangle \\ &+ \quad \langle x^{\otimes q} \otimes I \otimes \underline{x} \otimes x^{\otimes p-1} \mid x\rangle = 0.\end{aligned}$$

By definition of unital A_∞-bimodule homomorphisms, we have

$$\langle x^{\otimes q} \otimes I \otimes \underline{x} \otimes x^{\otimes p-1} \mid x\rangle = 0,$$

and the above equation gives

$$\langle x^{\otimes p} \otimes \underline{x} \otimes x^{\otimes q} \mid I\rangle = -\langle x^{\otimes p+q} \otimes \underline{I} \mid x\rangle = \langle \underline{x} \otimes x^{\otimes p+q} \mid I\rangle,$$

where the last equality follows from the skew-symmetry of ϕ. This proves the lemma. □

Lemma 7.0.12. $\sum_{\sigma\in\mathbb{Z}/n\mathbb{Z}}\langle \underline{a_{\sigma(1)}}, a_{\sigma(2)}, ..., a_{\sigma(n-1)} \mid a_{\sigma(n)}\rangle = 0.$

Proof. Fix $a_1, \cdots, a_n$ and denote $[i,j] := \langle ..., \underline{a_i}, ... \mid a_j\rangle$. Then what we need to prove is

$$[1,n] + [2,1] + \cdots + [n,n-1] = 0.$$

The closedness condition of strong homotopy inner products gives

$$[i,j] + [j,k] = [i,k].$$

Hence, it follows that

$$[1, n] + [n, n-1] + \cdots + [2, 1] = [1, n] + [n, 1] = 0.$$

□

Now we prove the above proposition. First, assume

$$\frac{d}{dt}x(t) = \sum_{i+j=k\geq 0} m_{k+1}(x(t)^{\otimes i} \otimes c(t) \otimes x(t)^{\otimes j}).$$

We denote x by $x(t)$ and c by $c(t)$, for it causes no problem in this proof.

Applying lemma 7.0.11, the fraction disappears and we get

$$\frac{d}{dt}\Psi(x) = \sum_{l\geq 0} \langle \underline{\sum_{i+j=k\geq 0} m_{k+1}(x^{\otimes i} \otimes c \otimes x^{\otimes j}) \otimes x^{\otimes l}} \mid I \rangle \tag{7.0.1}$$

$$+ \sum_{l,m\geq 0} \langle \underline{x} \otimes x^{\otimes l} \otimes \sum_{i+j=k\geq 0} m_{k+1}(x^{\otimes i} \otimes c \otimes x^{\otimes j}) \otimes x^{\otimes m} \mid I \rangle \tag{7.0.2}$$

To prove that it is zero, we use the A_∞-bimodule equation. Namely, we compute

$$(\phi \circ \widehat{m} - m^* \circ \widehat{\phi})(\sum_{l\geq 0} \underline{c} \otimes x^{\otimes l} + \sum_{l,m\geq 0} \underline{x} \otimes x^{\otimes l} \otimes c \otimes x^{\otimes m})(I),$$

which is a priori zero.

$$(\phi \circ \widehat{m})(\sum_{i\geq 0} \underline{c} \otimes x^{\otimes i})(I) = \sum_{l\geq 0} \langle \underline{\sum_{k\geq 0} m_{k+1}(c \otimes x^{\otimes k}) \otimes x^{\otimes l}} \mid I \rangle \tag{7.0.3}$$

$$+ \sum_{l,m\geq 0} \langle \underline{c} \otimes x^{\otimes l} \otimes (\sum_{k\geq 1} m_k(x^{\otimes k})) \otimes x^{\otimes m} \mid I \rangle \tag{7.0.4}$$

and (7.0.4) is zero by Maurer-Cartan equation.

$$
\begin{aligned}
& (\phi \circ \widehat{m})(\sum_{i,j\geq 0} \underline{x} \otimes x^{\otimes i} \otimes c \otimes x^{\otimes j})(I) \\
= & \sum_{l,m\geq 0} \langle \underline{\sum_{k\geq 1} m_k(x^{\otimes k})} \otimes x^{\otimes l} \otimes c \otimes x^{\otimes m} \mid I \rangle \qquad (7.0.5)\\
+ & \sum_{l\geq 0} \langle \underline{\sum_{i\geq 1, j\geq 0} m_k(x^{\otimes i} \otimes c \otimes x^{\otimes j})} \otimes x^{\otimes l} \mid I \rangle \qquad (7.0.6)\\
+ & \sum_{l,m\geq 0} \langle \underline{x} \otimes x^{\otimes l} \otimes \sum_{i+j=k\geq 0} m_{k+1}(x^{\otimes i} \otimes c \otimes x^{\otimes j}) \otimes x^{\otimes m} \mid I \rangle \qquad (7.0.7)\\
+ & \sum_{l,m,n\geq 0} \langle \underline{x} \otimes x^{\otimes l} \otimes c \otimes x^{\otimes m} \otimes \sum_{k\geq 1} m_k(x^{\otimes k}) \otimes x^{\otimes n} \mid I \rangle \qquad (7.0.8)\\
+ & \sum_{l,m,n\geq 0} \langle \underline{x} \otimes x^{\otimes l} \otimes \sum_{k\geq 1} m_k(x^{\otimes k}) \otimes x^{\otimes m} \otimes c \otimes x^{\otimes n} \mid I \rangle. \qquad (7.0.9)
\end{aligned}
$$

Remark again, that (7.0.5), (7.0.8) and (7.0.9) vanish by Maurer-Cartan equation. Observe also that

- (7.0.3)+(7.0.6)=(7.0.1),
- (7.0.7)=(7.0.2).

It remains to show that

$$(m^* \circ \widehat{\phi})(\sum_{l\geq 0} \underline{c} \otimes x^{\otimes l} + \sum_{l,m\geq 0} \underline{x} \otimes x^{\otimes l} \otimes c \otimes x^{\otimes m})(I) = 0.$$

Since I is the unit, we may easily verify that

$$(m^* \circ \widehat{\phi})(\sum_{l\geq 0} \underline{c} \otimes x^{\otimes l})(I) = \sum_{l\geq 0} \langle \underline{c} \otimes x^{\otimes l} \mid x \rangle, \qquad (7.0.10)$$

$$(m^* \circ \widehat{\phi})(\sum_{l\geq 0} \underline{x} \otimes x^{\otimes l} \otimes c)(I) = \sum_{l\geq 0} \langle \underline{x} \otimes x^{\otimes l} \mid c \rangle, \qquad (7.0.11)$$

$$(m^* \circ \widehat{\phi})(\sum_{l\geq 0, m\geq 1} \underline{x} \otimes x^{\otimes l} \otimes c \otimes x^{\otimes m})(I) = \sum_{l,m\geq 0} \langle \underline{x} \otimes x^{\otimes l} \otimes c \otimes x^{\otimes m} \mid x \rangle.$$

In (7.0.10) and (7.0.11), for $l = 0$, we have

$$\langle c \mid x\rangle + \langle x \mid c\rangle = 0$$

by skew-symmetry. For remaining parts, we collect terms appropriately and use closedness condition to show that they all vanish. More precisely, for $k \geq 1$, we claim that

$$\langle \underline{c} \otimes x^{\otimes k} \mid x\rangle + \langle \underline{x} \otimes x^{\otimes k} \mid c\rangle + \sum_{l+m=k-1} \langle \underline{x} \otimes x^{\otimes l} \otimes c \otimes x^{\otimes m} \mid x\rangle = 0$$

But this follows from the previous lemma 7.0.12, by setting

$$a_1 = c, a_2 = \cdots = a_{k+2} = x.$$

□

Lemma 7.0.13. *Let $\phi : B \to B^*$ be a strong homotopy inner product, and let $f : A \to B$ an A_∞-quasi-isomorphism, with pullback strong homotopy inner product $f^*\phi : A \to A^*$. Given a Maurer-Cartan element $x \in A$, denote by $f_*(x) = \sum_k f_k(x, \cdots, x)$ the corresponding Maurer-Cartan element of B. Then, we have*

$$\Psi^A(x) = \Psi^B(f_*(x))$$

Proof. This can be checked from the Lemma 7.0.11 as in the case of the potential Φ. We leave the details to the readers. □

Now, we discuss the potential Ψ and the algebraic generalized holonomy map. We refer readers to [ATZ] and section 4.2 for the relevant definitions of this construction.

First, recall from proposition 6.2.1 that given a negative cyclic cohomology class α of an A_∞-algebra A, one obtains a bimodule map $\widetilde{\alpha_0} : A \to A^*$. This provides a strong homotopy inner product, if α is in addition homologically non-degenerate. The equation (1.0.2) thus provides a definition of the potential Ψ^α using α. Combined with the above proposition, we prove

Theorem 7.0.14. *The potential Ψ provides a map $\Psi : HC^{\bullet}_{-}(A) \to \mathcal{O}(\mathcal{MC})$ defined by $\alpha \mapsto \Psi^{\alpha}|_{MC}$. Furthermore, this agrees with the algebraic analogue of generalized holonomy map of Abbaspour, Tradler and Zeinalian [ATZ].*

Proof. We only need to prove the relation with that of [ATZ] and we recall the construction of a map $\rho : HC^{\bullet}_{-}(A) \to \mathcal{O}(\mathcal{MC})$. Here we always work with reduced versions of negative cyclic or Hochschild (co)homologies.

Given a Maurer-Cartan element a of a unital A_∞-algebra A, consider the expression (Definition 8 of [ATZ])

$$P(a) := \sum_{i \geq 0} I \otimes a^{\otimes i} = (I \otimes I) + (I \otimes a) + (I \otimes a \otimes a) + \cdots .$$

One can check that $P(a)$ is a Hochschild homology cycle from the unital property of I and the Maurer-Cartan equation. Note that Connes-Tsygan operator B of $P(a)$ vanishes on the reduced complex, due to the unit I. Hence, $P(a)$ can be considered as a negative cyclic homology cycle.

Hence, given a negative cyclic cohomology cycle $\alpha \in HC^{\bullet}_{-}(A)$, one can use the pairing $\langle , \rangle : HC^{\bullet}_{-}(A) \otimes HC^{-}_{\bullet}(A) \to \boldsymbol{k}$ to define the map ρ as

$$\rho([\alpha])([a]) := \langle \alpha, \sum_{i \geq 0} I \otimes a^{\otimes i} \rangle \tag{7.0.12}$$

Now, we compare the above expression with that of Lemma 7.0.11. We recall again, proposition 6.2.1. Negative cyclic cocycle lies in 2nd and 3rd quadrant of (b^*, B^*)-bicomplex (4.2.3) including 0-th column(y-axis), and by α_0, we mean a 0-th column of α in that (b^*, B^*)-bicomplex . It is easy to see that Hochschild cocycles $\mathrm{Ker}(b^*)$, given at 0-th column, becomes a negative cyclic cocycle. For a general negative cyclic cocycle α, $b^*\alpha_0$ may not vanish, but equals $B^*\alpha_1$, and it is shown above that $\widetilde{B^*\alpha_1} = 0$ in lemma 6.2.3.

Also, from the unital property, we have

$$\widetilde{\alpha_0}(\underline{a}, a, \cdots, a)(I) = \alpha_0(a, \cdots, a)(I) - \alpha_0(a, \cdots, a, I)(a) = \alpha_0(a, \cdots, a)(I)$$

Hence,

$$\langle \alpha, I \otimes a^{\otimes i} \rangle = \langle \alpha_0, I \otimes a^{\otimes i} \rangle = \alpha_0(a, \cdots, a)(I) = \widetilde{\alpha_0}(\underline{a}, a, \cdots, a)(I) = \langle \underline{a}, a, \cdots, a \mid I \rangle$$

where the second equality follows from the identification

$$\mathrm{Hom}(A \otimes (A[1]/k \cdot 1)^{\otimes n}, k) \cong \mathrm{Hom}((A[1]/k \cdot 1)^{\otimes n}, A^*).$$

Hence, each term of the function ρ of [ATZ] equals the potential Ψ in the paper given in the Lemma 7.0.11. This proves the theorem. □

Remark 7.0.15. *The homological non-degeneracy condition is well-defined for negative cyclic cohomology classes (independent of coboundary), and we know that homologically nondegenerate negative cyclic cohomology elements(not cocycles in the cochain level) determines an equivalence class of strong homotopy inner products. The value of potential at Maurer-Cartan elements are well-defined up to equivalence classes of strong homotopy inner product from the Lemma 7.0.13. Thus the map $\Psi : HC^{\bullet}_{-}(A) \to \mathcal{O}(\mathcal{MC})$ when restricted to the subset with homological non-degeneracy conditions, factors through the equivalence classes of strong homotopy inner products.*

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국문초록

A-무한대수가 주어져 있을때, 콘체비치의 순환내적을 호모토피적으로 일반화한 강호모토피 내적을 정의할 수 있다. 이 논문에서는 강호모토피 내적으로부터 얻어지는 "잠재함수"라 불리는 몇가지 함수들의 호모토피적 불변성과 게이지 불변성에 대해 다룬다. 또한 A-무한대수의 음순환 코호몰로지와 강호모토피 내적의 동치류가 분명한 대응관계로 주어짐을 증명한다.

주요어휘: A-무한대수, 강호모토피 내적, 잠재함수, 음순환 코호몰로지, 비가환 다양체
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감사의 글

지난 5년여의 시간 동안 많은 격려와 가르침으로 저를 지도해 주신 조철현 선생님께 깊은 감사의 마음을 표합니다. 선생님과 함께 공부할수 있었음이 행운이었고 소중한 기회였습니다. 너무나 당연한 이야기이지만, 선생님의 아낌없는 조언과 도우심이 없었다면 이 논문이 탄생하지 못하였을 것입니다.

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형석, 한솔, 태수, 하균형, 한울이와 함께 공부하고 토론한 내용이 제 박사과정 시간 동안 많은 자양분이 되었음을, 졸업이 가까워 오면서 더욱 크게 느끼게 됩니다. 졸업 이후에도 서로가 서로에게 많은 도움과 자극이 되어줄 수 있기를 진심으로 바랍니다. 또한 같이 사교기하를 공부한 영진이와 정수, Urs Frauenfelder 교수님께도 감사의 마음을 전합니다.

오래도록 공부하는 남편을 뒷바라지하고 믿음으로 지켜보아 준 아내 마리아에게 늘 미안하고 고마웠습니다. 아내의 신뢰와 사랑 덕분에 지금까지 걸어올수 있었고, 앞으로도 걸어갈수 있음을 믿습니다. 자식들 위해 늘 기도해 주시고 여러모로 도와 주신 저희들의 부모님들께도 깊은 존경과 감사를 드립니다. 누님 부부와 동생들 또한 큰 힘이 되어주어서 고맙습니다.

마땅히 감사드려야 하나 여기에 미처 기록하지 못한, 제 인생에 떼어놓을 수 없는 모든 분들께 제 감사와 축복의 마음이 전하여 닿기를 바랍니다.

"... the LORD will be your everlasting light, and your God will be your glory." (Isaiah 60:19)