공학박사학위논문

Adaptive Output Regulation for Linear Systems with Unknown Sinusoidal Exogenous Inputs

미지의 정현파 외부 입력을 갖는 선형시스템을 위한 적응 출력 제어

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서울대학교 대학원

전기컴퓨터공학부

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ABSTRACT

Adaptive Output Regulation for Linear Systems with Unknown Sinusoidal Exogenous Inputs

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This dissertation investigates the output regulation problem (which is equivalent to the problem of asymptotic tracking and disturbance rejection when the reference inputs and the disturbances are generated by an autonomous differential equation, the so-called *exosystem*) for linear systems driven by unknown sinusoidal exosystems. Unlike previous researches, our ultimate goal is to achieve asymptotic regulation of the plant output to the origin for the sinusoidal exogenous signals (representing the reference inputs and disturbances) generated by the exosystems whose magnitudes, phases, bias, frequencies, and even the number of frequencies are all unknown. Here, the plant is linear time-invariant (LTI) single-input-single-output (SISO) systems (including non-minimum phase systems) without uncertainty.

Before achieving the final control goal, we first start by considering an output regulation problem under the assumption that the number of frequencies contained in the exogenous inputs is known but magnitudes, phases, bias, and frequencies are unknown. To solve this problem, an add-on type output regulator with an adaptive observer is presented. The adaptive observer, based on the persistently exciting (PE) property, is used to estimate the frequencies of sinusoidal exogenous inputs as well as the states of plant and exosystem. Also, by add-on controller we mean an additional controller which runs harmonically with a preinstalled controller that has been in operation for the plant. When the desired performance of the preinstalled controller is not satisfactory, the add-on controller can be used. Some advantages of the proposed add-on controller include that it can be designed without much information about the preinstalled controller and it can be plugged in the feedback loop any time in operation without causing unnecessary transient response. Both simulation and experimental results of the track-following control for commercial optical disc drive (ODD) systems confirm the effectiveness of the proposed method.

As the next step, we deal with the case where, as well as magnitudes, phases, bias, and frequencies, the number of frequencies contained in the exogenous inputs is unknown. To this end, a closed-form solution is given under the assumptions that the plant has hyperbolic zero dynamics (i.e., there is no zero on the imaginary axis of the complex plane), and that the number of unknown frequencies has known upper bound. In particular, the PE property is not necessary for the estimation of the unknown frequencies. For this, an adaptive observer is proposed to estimate the frequencies and the number of frequencies, simultaneously. This is important contribution, because, sufficient persistency of excitation is usually required since the unknown parameters are estimated by the adaptive control. Moreover, we propose a suitable dead-zone function with a computable deadband only using the plant parameters to avoid the singularity problem in the transient-state and, at the same time, to achieve output regulation in the steady-state.

Keywords: output regulation, adaptive observer, persistently exciting, sinusoidal exogenous inputs, add-on, optical disc drive

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Symbols and Acronyms

Symbols

$\mathbb{R}$	field of real numbers
$\mathbb{R}^n$	real Euclidean space of dimension n
$\mathbb{R}^{m \times n}$	space of $m \times n$ matrices with real entries
$\mathrm{Re}(s)$	real part of the complex number s
$\mathbb{C}$	field of complex numbers
$\mathbb{C}_{<0}$	open left-half complex plane; i.e., $\{s \in \mathbb{C} : \mathrm{Re}(s) < 0\}$
$\mathbb{C}_{>0}$	open right-half complex plane; i.e., $\{s \in \mathbb{C} : \mathrm{Re}(s) > 0\}$
$\mathbb{C}_{=0}$	imaginary axis; i.e., $\{s \in \mathbb{C} : \mathrm{Re}(s) = 0\}$
$\mathbb{C}_{\leq 0}$	closed left-half complex plane; i.e., $\mathbb{C}_{<0} \bigcup \mathbb{C}_{=0}$
$\mathbb{C}_{\geq 0}$	closed right-half complex plane; i.e., $\mathbb{C}_{>0} \bigcup \mathbb{C}_{=0}$
$\forall$	for all
I_n	$n \times n$ identity matrix (subscript n is omitted when there is no confusion.)
$0_{m \times n}$	$m \times n$ zero matrix (subscript $m \times n$ is omitted when there is no confusion.)
A^{-1}	inverse of the square matrix A
$A^\top$	transpose of the matrix A
$A^\dagger$	pseudoinverse of the matrix A

vec(A)	stacking the columns of the $m \times n$ matrix A; i.e., $[a_{1,1}, \cdots, a_{m,1}, a_{1,2}, \cdots, a_{m,2}, \cdots, a_{1,n}, \cdots, a_{m,n}]^\top$ where $a_{i,j}$ represents the (i,j)-th element of A
diag$(a_1, \ldots, a_k)$	diagonal matrix with diagonal elements a_1 to a_k
blockdiag$(A_1, \ldots, A_k)$	block diagonal matrix with diagonal blocks A_1 to A_k
rank(A)	rank of the matrix A
det(A)	determinant of the square matrix A
adj(A)	adjoint of the square matrix A
sgn(x)	signum function of $x \in \mathbb{R}$; i.e., -1 if $x < 0$, 0 if $x = 0$, and 1 if $x > 0$
card(X)	cardinal number of the set X
$A \otimes B$	Kronecker product of matrices A and B
$\sum_{i=m}^{n} x_i$	summation of the sequence x_i; i.e., $x_m + x_{m+1} + \cdots + x_{n-1} + x_n$ if $m < n$, x_m if $m = n$, and 0 if $m > n$
$\prod_{i=m}^{n} x_i$	product of the sequence x_i; i.e., $x_m \cdot x_{m+1} \cdot \cdots \cdot x_{n-1} \cdot x_n$ if $m < n$, x_m if $m = n$, and 1 if $m > n$
$\|x\|$	Euclidean norm of the vector x
$\lvert s \rvert$	absolute value of the number s
$\exp(\cdot)$	exponential function
$\min\{a_1, \cdots, a_k\}$	minimum value among $a_1, a_2, \cdots, a_k$
$:=$	defined as
$\equiv$	identically equal
$\Rightarrow$	implies
$\Diamond$	designation of the end of theorem, lemma, proposition, assumption, remark, and so on
$\square$	designation of the end of proof

- A square matrix A is said to be Hurwitz (matrix) if every eigenvalue λ of A has strictly negative real parts, i.e., $\mathrm{Re}(\lambda) < 0$.
- For any state variable $x(t)$, its initial condition will be denoted by $x(0)$.
- In order to messy notation, the time symbol t is omitted when there in no confusion.

Acronyms

LTI	Linear time-invariant
SISO	Single-input-single-output
MIMO	Multi-input-multi-output
IMP	Internal model principle
PE	Persistently exciting
ISS	Input-to-state stable
ODD	Optical disc drive

Chapter 1

Introduction

1.1 Research Background

Over the past several decades, the problem of controlling the output of a system to achieve tracking of prescribed references and/or rejection of undesired disturbances has been one of the central topics in control theory. It falls into the domain of the problem depicted in Figure 1.1. Here, a plant is given which is subject to a disturbance $d(t)$, and a controller is to be designed so that the closed-loop system is stable and the plant output $y(t)$ asymptotically tracks a given reference input $r(t)$. In particular, it is critical and significant that the reference input and disturbance are sinusoids because they commonly appear in a variety of engineering applications in the industry. Examples of such applications include disc drives control with repeatable runout error [JDRC98, BDCS02, CYC$^+$03, Kim05, LY08, LRLS12, KSJ14], tracking control in mechanical systems subject to repetitive tasks [KHCH98, CH00, KH00], noise and vibration control for the

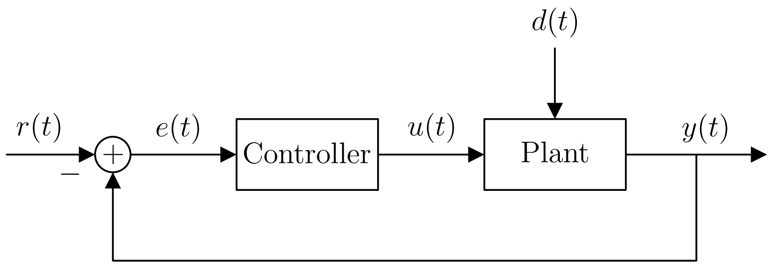

Figure 1.1: Unity feedback control.

rotary engine of a helicopter [BM94], manufacturing process (milling and steel casting) [SS94, MTBR96], active vibration control (AVC) [LAS+13], and active noise control (ANC) [BJD01, WB04].

For this reason, many researches have been carried out and there are several approaches for the disturbance rejection problem. For example, (i) adaptive feedforward cancellation (AFC) based on the phase-locked loop technique is able to reject periodic disturbances [MB94, BSK94]. In [BD97, Bod05], Bodson and Douglas proposed an advanced method to reject sinusoidal disturbances with uncertain frequency. However, to reject sinusoidal disturbances by AFC, it is necessary to compute the gain of the plant at all estimated frequencies. In practice, it is difficult to determine the gain, therefore, if the plant is complicated then this method may be difficult to implement. (ii) Repetitive control has also been shown to be highly effective for rejecting repetitive or periodic disturbances [HYON88, KMTH93, DSvdHS95, Ste02, DWZ03, LROLDS10, CZQ13, DR13, EMHGMR14]. Four different repetitive algorithms have been compared in [KMTH93]. Although repetitive control based on the internal model principle (IMP) enables perfect or near perfect cancellation of periodic disturbances, it requires an exact knowledge of the period of the repetitive disturbance. Several solutions have been proposed to relax this requirement, most of them using a supervisory adaptive scheme by the estimating period from the closed-loop response [DSvdHS95]. In contrast to such approaches, a new structure for repetitive control has been introduced in [Ste02], which is robust to changes in the period. Although this method requires additional memory space, it is very effective for rejecting disturbances when the period varies within a narrow range. (iii) Disturbance observer (DOB), since the introduction in [Ohn87], has been widely applied to industrial applications because of its simple structure and powerful ability for disturbance attenuation [YCC05, KSC13, YSKK13, KR13]. Furthermore, theoretical analysis on the DOB has been presented in [SJ07, BS08, SJ09]. Although DOB guarantees the robust stability under plant uncertainties, it is not very effective to reject a disturbance of specific frequency. In order to solve this problem, the authors of [PJS14, JPBS] have recently proposed a disturbance observer with internal model which can asymptotically reject the disturbances of sinusoidal or

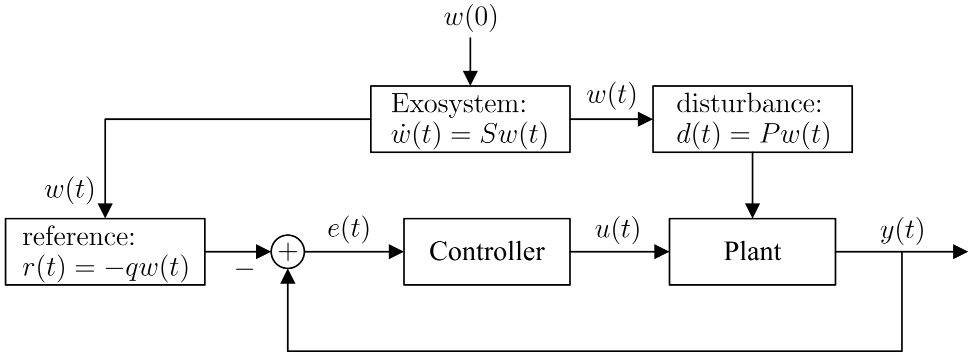

Figure 1.2: Output regulation problem with linear exosystem.

polynomial-in-time type. However, it requires the exact knowledge of the frequency of the disturbances. (iv) In addition, there are various studies for the disturbance rejection [Tom87, TT87, ÅW97, KH00, BDCS02, BZ04, OMI$^+$06].

The problem of asymptotic tracking and disturbance rejection is simply called the *output regulation* problem or the *servomechanism* problem in the literature (for example, [Fra77, KIF93, BPI97, Hua04]) when the disturbances and the reference inputs are generated by an autonomous differential equation, the so-called *exosystem*, as shown in Figure 1.2. The key idea of the output regulation is to embed the exosystem into the controller, in the perspective of internal model principle. In this dissertation, the term 'exogenous inputs' will be used to refer to both reference inputs and disturbances when there is no need to distinguish them. Since the output regulation approach is simpler and more natural to handle the sinusoidal exogenous inputs compared to those other approaches, some effort has been devoted to introduce frequency estimation within the output regulation framework. Serrani et al. [SIM01] presented a generalized method for uncertain nonlinear minimum phase systems, although its practical applicability is limited due to its specialized design procedure with less flexibility. For a class of nonlinear systems transformable into the strict feedback form or the output feedback form, the problem of complete adaptive compensation has been solved for unknown external disturbances generated by the linear exosystem of the known order but with unknown parameters [Nik98, Din06]. Furthermore, the unmodelled exogenous system is allowed in [Nik01]. Beyond the common nonlinear models such as the strict feedback form and the output feedback form, Ding [Din07] proposed an asymptotic disturbance rejection algorithm for nonlinear systems whose control design for disturbance-free case is available. The disturbance is assumed to be sinusoidal with completely unknown phases, amplitude, and frequencies, but the number of distinct frequencies or the order of the corresponding unknown linear exosystem is known. Also, dynamic systems with time delays is studied in [DL10]. Unlike the traditional controllers designed by using measurement of states or outputs, the authors of [BK13] presented an adaptive controller to cancel matched unknown sinusoidal disturbances for a linear time-invariant (LTI) system by using only measurement of state-derivatives. Also uncertain LTI plants are dealt with in [BK14]. On the other hand, Marino et al. [MT03, MS07, MT07, MT13a] nicely combined an adaptive observer in the output regulator, which can be considered as more natural solution when the system is linear and has no uncertainty, and uncertain minimum phase linear systems is addressed in [MT11]. Moreover, nonlinear systems with known output dependent nonlinearities is allowed in [MS05] while minimum phase uncertain nonlinear systems is studied in [MT05, MT13b]. Of these, the output regulation problem for the sinusoidal exosystems of the unknown parameters and order is addressed in [MS07, MT07, MT13a, MT13b], but the researches require the following assumption: minimum phase plant [MT07, MT13b] or known lower bound of the unknown frequencies [MT13a]. On the other hand, the controller proposed in [MS07] has the singularity problems related to the computation of the controller.

This dissertation deals with two topics in regard to the output regulation problem with unknown sinusoid exogenous inputs. One is to design an add-on type adaptive regulator under the assumption that the number of frequencies contained in the sinusoids is known but magnitudes, phases, bias, and frequencies are unknown. The controller not only achieves asymptotic tracking and disturbance rejection, but also runs harmonically with a preinstalled controller. It is a useful feature for industrial application such as optical disc drive (ODD) systems to be introduced in Chapter 4. The other is to propose an adaptive regulator under the assumption that even the number of frequencies is unknown but the upper is known. It does not require the persistently exciting (PE) property to achieve the estimation of the unknown frequencies.

1.2 Contributions and Outline of the Dissertation

The following outlines the dissertation and summarizes the contributions of each chapter.

Chapter 2. Reviews of Related Prior Studies
In this chapter, we review various important control methods for known and unknown frequencies of disturbance: adaptive feedforward cancellation (AFC) [MB94, BSK94, BD97], repetitive control [HYON88, Ste02], and disturbance observer (DOB) [Ohn87, OSM96, SJ07, JPBS]. Moreover, the frequency estimation methods are introduced for the indirect approach (where the frequency of the disturbance is estimated and then the estimate is used in another adaptive algorithm that adjusts the magnitude and phase of the input needed to cancel the effect of the disturbance): adaptive notch filtering [Reg91, BD97], phase-locked loops [Ste97, WB01], extended Kalman filtering [SB96], and Marino's frequency estimator [MT02].

Chapter 3. Highlights of Output Regulation for Linear Systems
As a preliminary of the subsequent chapters, we recall some background materials on output regulation theories for linear systems. The precise definition of the output regulation problem is given and the solvability conditions of the problem through both state feedback and error feedback are also presented. Parts of this chapter are based on [KIF93, Hua04].

Chapter 4. Adaptive Add-on Output Regulator for Unknown Sinusoidal Exogenous Inputs
This chapter deals with the output regulation problem of linear systems subject to unknown sinusoidal exogenous inputs under the assumption that the frequencies, amplitudes, and phases of the sinusoids are unknown but the number of frequencies is known. To solve this problem, we present an add-on type adaptive output regulator which guarantees the asymptotic tracking and disturbance rejection. Most of this chapter is based on [KSJ14] and the contributions of the chapter are summarized as follows:

i) The proposed controller can be designed without requiring the transfer function (or state space realization) of the preinstalled controller so that it can be plugged into any existing control systems.

ii) It can be freely turned on and off without disturbing the overall stability of the closed-loop system. Hence, it may be a good idea to turn on the add-on controller only when the disturbance attenuation performance is not acceptable, since the add-on controller may degrade the nominal performance provided by the primary controller.

iii) By utilizing the adaptive observer technique in [Zha02], an adaptive version, based on the persistently exciting property, is developed in order to achieve regulation of the plant output for the sinusoidal exogenous inputs whose frequencies are unknown.

iv) It is applied to the track-following control for commercial optical disc drive (ODD) systems, and its effectiveness is confirmed via simulations and experiments. Also, through the comparison with adaptive feedforward cancellation and repetitive control of [BD97, Ste02], it shows that the performance of the proposed control method is better than them.

Chapter 5. Adaptive Output Regulator for Unknown Number of Unknown Sinusoidal Exogenous Inputs

In this chapter, we focus on the output regulation problem of sinusoidal exogenous inputs whose not only magnitudes, phases, bias, frequencies are unknown but also the number of frequencies is unknown. To this end, an adaptive error feedback controller is proposed for estimating the frequencies and the number of frequencies at the same time. In particular, without the persistently exciting property, we claim any linear system (including non-minimum phase systems) with hyperbolic zero dynamics (i.e., there is no zero on the imaginary axis of the complex plane) admits the proposed controller. Most of this chapter is based on [KS15] and a few more contributions of this chapter are listed as follows:

i) It provides a dead-zone function that can avoid any singularity problem (which is already done in [MT13a]). In particular, we propose a formula using only the plant parameters in order to compute a suitable value of the width of the dead-zone.

ii) Observability assumption of (S, γ) (to be seen in Chapter 5) is not necessary which has been in other previous researches.

iii) The employed adaptive observer in this chapter has more simplified structure than similar researches [MT07, MS07, MT13a] because it does not rely on the filtered transformation, which enabled the relatively simpler analysis.

Chapter 6. Conclusions and Further Issues

This chapter concludes the dissertation with some concluding remarks and with future directions of research.

Chapter 2

Reviews of Related Prior Studies

As were mentioned in Chapter 1, the important results of the control method for the rejection of sinusoidal disturbances are reviewed in this chapter. This chapter is composed of two parts: control methods for rejecting disturbances and frequency estimation algorithms for indirect approach.

2.1 Control Methods for Rejecting of Sinusoidal Disturbance

The well-known controllers for rejecting sinusoidal disturbances are introduced in this section such as adaptive feedforward cancellation, repetitive control, and disturbance observer.

2.1.1 Adaptive Feedforward Cancellation (AFC)

As shown in Figure 2.1, AFC guarantees that the disturbance is simply cancelled at the input of the plant by adding the negative of its value at all times [MB94, BSK94, BD97, Bod05, PB10, MBK12]. First of all, we consider the problem of disturbance rejection for sinusoidal signals of known frequency. The plant $P(s)$ is a linear time-invariant (LTI) system, and the disturbance, for convenience, is expressed in terms of its sine and cosine components as

$$d(t) = A\sin(\omega_1 t + \phi) = \theta_1 \cos(\omega_1 t) + \theta_2 \sin(\omega_1 t).$$

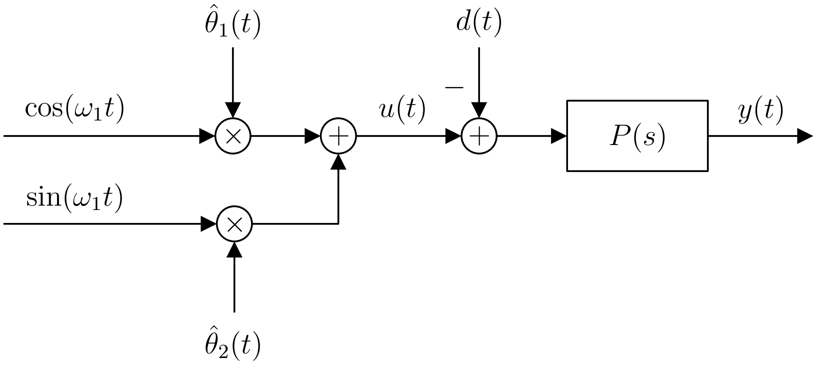

Figure 2.1: Feedforward cancellation of a sinusoidal disturbance of known frequency.

The control input is selected to be

$$u(t) = \hat\theta_1(t)\cos(\omega_1 t) + \hat\theta_2(t)\sin(\omega_1 t)$$

so that the disturbance is exactly rejected when the parameters have the actual values, i.e., $\hat\theta_1(t) = \theta_1$ and $\hat\theta_2(t) = \theta_2$. Then, the control goal is to find a suitable algorithm in order that the adaptive parameters $\hat\theta_1(t)$ and $\hat\theta_2(t)$ converge to the actual values.

Let

$$\theta := \begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}, \quad \hat\theta(t) := \begin{bmatrix}\hat\theta_1(t)\\ \hat\theta_2(t)\end{bmatrix}, \quad w(t) := \begin{bmatrix}\cos(\omega_1 t)\\ \sin(\omega_1 t)\end{bmatrix},$$

then the plant output can be written as

$$y(t) = P(s)\left[\left(\hat\theta(t) - \theta\right)^\top w(t)\right],$$

where the vector $w(t)$ is called the *regressor vector*. This equation provides the framework of standard adaptive control theory [SB89, IS96]. The *pseudo-gradient algorithm* is the simplest update law given by

$$\dot{\hat\theta}(t) = -gy(t)w(t),$$

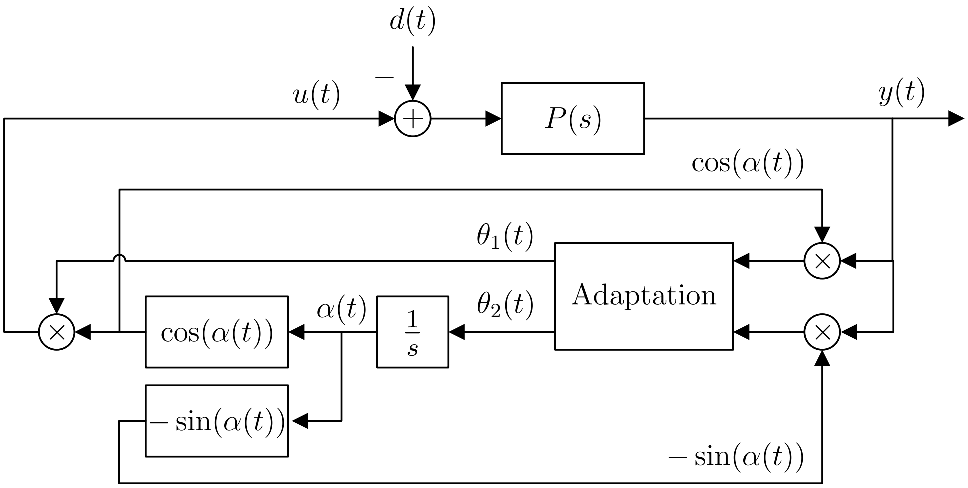

Figure 2.2: Direct approach to sinusoidal disturbance cancellation.

where g is a positive real number called the adaptive gain. Since A is not zero in $d(t)$, the regressor vector $w(t)$ is persistently exciting (PE). Then, the algorithm guarantees the exponential convergence of the adaptive parameter $\hat{\theta}(t)$ to its actual value θ as time tends to infinity if the plant $P(s)$ is strictly positive real (SPR) [Kha01]. It is easy to extend to the case when higher-order harmonics need to be rejected. One just adds components θ_3, θ_4, $\cdots$ to θ and $\cos(\omega_2 t)$, $\sin(\omega_2 t)$, $\cdots$ to w. In this case, the stability is not affected by the number of frequency components, and the PE property still remains [CP90]. However, the range of the allowable adaptation gain may be more restricted, if the plant is not SPR.

Now we consider the problem for rejecting the sinusoidal disturbance with unknown frequency proposed in [BD97]. In particular, the direct approach is shown in Figure 2.2, and consists of the following equations:

$$\text{Adaptation} : \begin{cases} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 2 \begin{bmatrix} \hat{P}_{\mathrm{R}}(\omega_d) & -\hat{P}_{\mathrm{I}}(\omega_d) \\ \hat{P}_{\mathrm{I}}(\omega_d) & \hat{P}_{\mathrm{R}}(\omega_d) \end{bmatrix}^{-1} \begin{bmatrix} y(t)\cos(\alpha(t)) \\ -y(t)\sin(\alpha(t)) \end{bmatrix}, \\ \begin{bmatrix} \theta_1(s) \\ \theta_2(s) \end{bmatrix} = -\dfrac{1}{s} \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \dfrac{s+a}{s+b} \end{bmatrix} \begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix}, \end{cases}$$

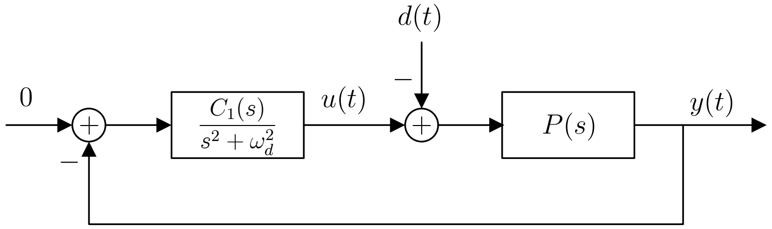

Figure 2.3: Controller based on the internal model principle.

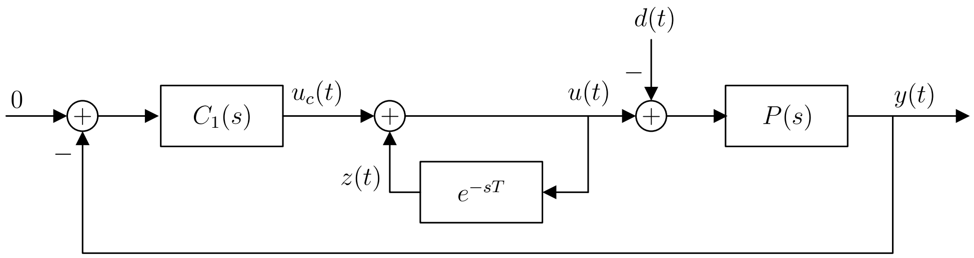

Figure 2.4: Repetitive controller.

$$\alpha(s) = \frac{1}{s}\theta_2(s),$$
$$u(t) = \theta_1 \cos(\alpha(t)),$$

where $\hat{P}_{\mathrm{R}}(\omega_d)$ and $\hat{P}_{\mathrm{I}}(\omega_d)$ are the estimated real and imaginary parts of the frequency response of the plant at the disturbance frequency ω_d, and α, θ_1, and θ_2 are the estimate of the phase, magnitude, and frequency of the disturbance signal, respectively. (See [BD97, Bod05] for more details.)

2.1.2 Repetitive Control

Repetitive control is a compensator based on the concept of internal model principle (IMP) [HYON88, Ste02, DWZ03, LROLDS10, CZQ13, DR13, EMHGMR14]. As shown in Figure 2.3, the IMP presents that the controller should include a model of the sinusoidal disturbance, and hence its poles are located on the $j\omega$-axis at positions corresponding to the disturbance frequency ω_d. Here, the compensator $C_1(s)$ is designed such that the closed-loop system is stable. It is easy to extend to the case when the disturbance is the sum of two or more sinusoids, that is to say, poles are simply added at all the frequencies.

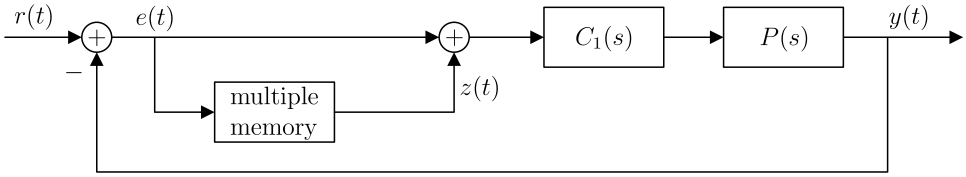

Figure 2.5: Repetitive controller with multiple memory loops.

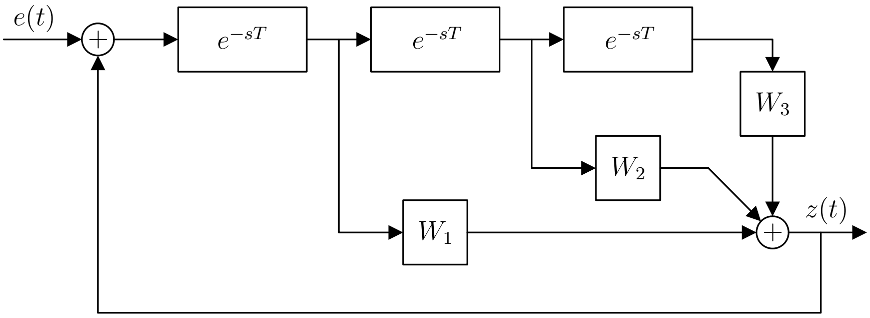

Figure 2.6: Multiple memory loops.

Figure 2.4 shows the system diagram of the repetitive controller. Note that the control input

$$u(t) = u(t - T) + u_c(t)$$

so that the control input signal repeats itself every period T except for a term $u_c(t)$. Here, the design parameter T is chosen to be equal to the period of the disturbance, i.e., $T = 2\pi/\omega_d$. Then, the overall controller is

$$C(s) = \frac{C_1(s)}{1 - e^{-sT}},$$

and the transfer function has poles at $\pm j\omega_d,\ \pm 2j\omega_d, \cdots$. Hence, repetitive control is closely related to IMP method, and the fundamental component of the sinusoidal disturbance and all its harmonics are asymptotically eliminated. However, it requires exact knowledge of the period T of the sinusoidal disturbance.

In order to handle the disturbances with uncertain periods, the repetitive control is equipped with *multiple memory loops* in [Ste02]. As shown in Figures

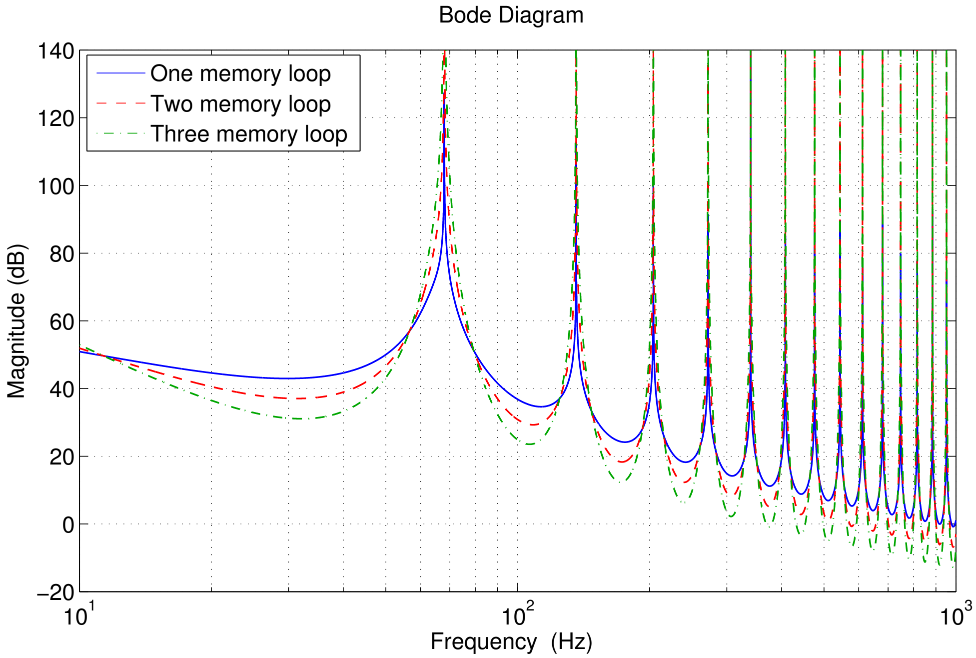

Figure 2.7: Magnitudes of the loop gains with the repetitive controls with multiple memory loops.

2.5 and 2.6, the transfer function of a generalized repetitive controller, consisting of N multiple memory loops, can be written as

$$\frac{z(s)}{e(s)} = \frac{H(s)}{1 - H(s)},$$

where the loop transfer function $H(s) = \sum_{i=1}^{N} W_i e^{-isT}$ and the weighting factors $W_1,\ W_2, \cdots$ are designed such that $\sum_{i=1}^{N} W_i = 1$ and $\sum_{i=1}^{N} W_i i = 0$ for $N > 1$. Then, we can state that the proposed robust repetitive controller with N multiple memory loops is given by

$$\frac{z(s)}{e(s)} = \frac{1 - (1 - e^{-sT})^N}{(1 - e^{-sT})^N}.$$

Figure 2.7 shows the frequency response of the loop gain with the repetitive controller having one, two, and three memory loops, respectively, for optical disc drive (ODD) systems (to be introduced in Section 4.3). Although the magnitude of the loop gains in Figure 2.7 increases (for the limited range of interest) as

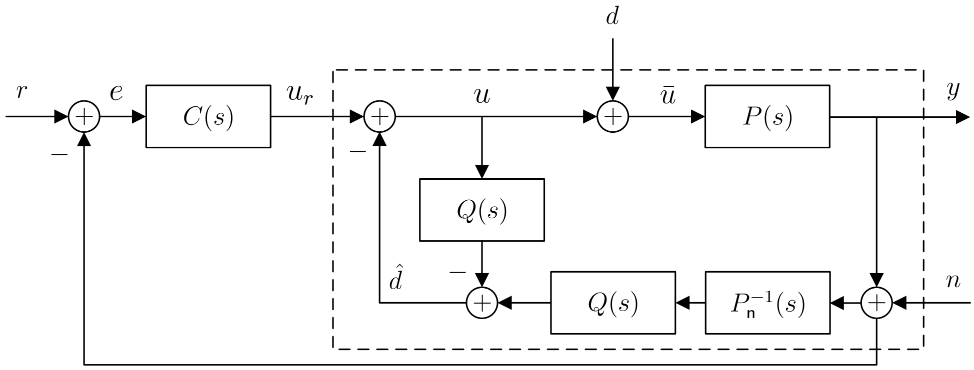

Figure 2.8: Closed-loop system with DOB structure (dotted-block). $P(s)$ is the plant, $P_{\mathsf{n}}(s)$ is the nominal plant, $C(s)$ is a controller designed for $P_{\mathsf{n}}(s)$, and $Q(s)$ is low-pass filters called *Q-filters*.

the number of memory loop gets larger, the performance enhancement always has limitation if the frequency deviation is large from its nominal value and its harmonics.

2.1.3 Disturbance Observer (DOB) with Internal Model

Disturbance observer approach [Ohn87, OSM96] as a tool for robust control has been widely employed in industry due to its powerful ability for disturbance rejection and robustness under plant uncertainties. In [SJ07], the authors analyzed the classical linear disturbance observer (DOB) approach in the state space using the singular perturbation theory [Kha01]. Also, in [SJ09], an almost necessary and sufficient stability condition is presented when the time constant of the Q-filter is sufficiently small in accordance with the performance enhancement. However, DOB is not very effective to cancel out a disturbance of specific frequency. Recently, a DOB with internal model proposed in [PJS14, JPBS] asymptotically eliminates sinusoidal disturbances of known frequencies for minimum-phase plant by embedding the generating model of the disturbances into the DOB structure. We first introduce the conventional DOB, and then review the DOB with internal model.

Figure 2.8 shows the DOB structure where the plant $P(s)$ is a single-input single-output, minimum phase, linear time-invariant system whose relative degree is $\nu \geq 1$ and the nominal model for $P(s)$ is denoted by $P_{\mathsf{n}}(s)$. The outer-loop controller $C(s)$ is designed for $P_{\mathsf{n}}(s)$ to achieve the given specification. The system $Q(s)$, known as the Q-filter, is a stable low-pass filter which is generally designed as

$$Q(s) = \frac{c_k(\tau s)^k + c_{k-1}(\tau s)^{k-1} + \cdots + c_0}{(\tau s)^l + a_{l-1}(\tau s)^{l-1} + \cdots + a_1(\tau s) + a_0},$$

where $l \geq k+r$, $c_0 = a_0$, all a_i's are chosen such that $s^l + a_{l-1}s^{l-1} + \cdots + a_1 s + a_0$ is a Hurwitz polynomial (i.e., all the roots have negative real parts), and the parameter τ is a positive constant, which determines the cutoff frequency of the low-pass filter $Q(s)$. The reference r, the input disturbance d, and the measurement noise n are the input signals of the closed-loop system. Then the output y of the plant, with this configuration, becomes

$$y(s) = T_{yr}(s)r(s) + T_{yd}(s)d(s) + T_{yn}(s)n(s), \tag{2.1.1}$$

where

$$\begin{aligned} T_{yr}(s) &:= \frac{P(s)P_{\mathsf{n}}(s)C(s)}{Q(s)(P(s) - P_{\mathsf{n}}(s)) + P_{\mathsf{n}}(s)(1 + P(s)C(s))}, \\ T_{yd}(s) &:= \frac{P(s)P_{\mathsf{n}}(s)(1 - Q(s))}{Q(s)(P(s) - P_{\mathsf{n}}(s)) + P_{\mathsf{n}}(s)(1 + P(s)C(s))}, \\ T_{yn}(s) &:= -\frac{P(s)(P_{\mathsf{n}}(s)C(s) + Q(s))}{Q(s)(P(s) - P_{\mathsf{n}}(s)) + P_{\mathsf{n}}(s)(1 + P(s)C(s))}. \end{aligned}$$

By the property of the low-pass filter $Q(s)$, $Q(j\omega) \approx 1$ in the low frequency range, and thus it follows that $T_{yr}(j\omega) = \frac{P_{\mathsf{n}}(j\omega)C(j\omega)}{1+P_{\mathsf{n}}(j\omega)C(j\omega)}$ and $T_{yd}(j\omega) \approx 0$. Since the noise $n(j\omega)$ is small in the frequency range, the equation (2.1.1) is approximated as

$$y(j\omega) \approx \frac{P_{\mathsf{n}}(j\omega)C(j\omega)}{1 + P_{\mathsf{n}}(j\omega)C(j\omega)} r(j\omega).$$

This implies that, in the low frequency range, the closed-loop system with the disturbance observer structure behaves as the nominal closed-loop system in the absence of uncertainties and disturbance. In other words, in spite of the existence of disturbance and uncertainties, the disturbance observer recovers the nominal performance. Here, the nominal performance means the performance of the nominal closed-loop system $\frac{P_{\mathsf{n}}(s)C(s)}{1+P_{\mathsf{n}}(s)C(s)}$ without the input disturbance. It is important to notice that the above property is only valid when the closed-loop system is internally stable. (See [SJ07, BS08, SJ09] for more details.)

Now we briefly introduce a DOB with internal model proposed in [JPBS] for minimum phase systems under uncertainties. The DOB can reject not only approximately the unmodelled disturbance, but also asymptotically the disturbances of sinusoidal or polynomial in-time type (such as $d_0 + d_1 t + \cdots$). To this end, the authors pose some assumptions on which the proposed DOB is based.

Assumption 2.1.1. The plant $P(s)$ and its nominal model $P_{\mathsf{n}}(s)$ belong to a set $\mathcal{P}$ defined by

$$\mathcal{P} = \left\{ \frac{\beta_{n-\nu}s^{n-\nu} + \beta_{n-\nu-1}s^{n-\nu-1} + \cdots + \beta_0}{\alpha_n s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_0} : \alpha_i \in [\underline{\alpha}_i, \bar{\alpha}_i], \beta_j \in \left[\underline{\beta}_j, \bar{\beta}_j\right], \right.$$
$$\left. i = 0, \cdots, n, \qquad j = 0, \cdots, n-\nu \right\},$$

where $\underline{\alpha}_i$, $\bar{\alpha}_i$, $\underline{\beta}_j$, and $\bar{\beta}_j$ are known constants, the intervals $[\underline{\alpha}_n, \bar{\alpha}_n]$ and $\left[\underline{\beta}_{n-\nu}, \bar{\beta}_{n-\nu}\right]$ do not contain zero, and β_i's are such that $\beta_{n-\nu}s^{n-\nu} + \beta_{n-\nu-1}s^{n-\nu-1} + \cdots + \beta_0$ is Hurwitz (i.e., $\mathcal{P}$ consists of minimum phase plants). $\Diamond$

Assumption 2.1.2.

$$d(t) = \bar{d}(t) + \sum_{i=0}^{k_t} d_i t^i + \sum_{j=1}^{k_s} \sigma_j \sin(\omega_j t + \phi_j) =: \bar{d}(t) + \tilde{d}(t),$$

where $k_t \geq 0$ and $k_s \geq 1$ are known integers, d_i, σ_j, and ϕ_j are unknown constants while the frequencies $\omega_j > 0$ are known such that $\omega_j \neq \omega_{\bar{j}}$ for $j \neq \bar{j}$, and $\bar{d}(t)$ is an unknown but bounded signal whose time derivative is also bounded. $\Diamond$

To solve the problem, a new Q-filter under the same DOB of Figure 2.8 is proposed as follows:

$$Q(s) = \frac{c_k(\tau)(\tau s)^k + \cdots + c_{k_t+1}(\tau)(\tau s)^{k_t+1} + c_{k_t}(\tau s)^{k_t} + \cdots + c_0}{(\tau s)^l + a_{l-1}(\tau s)^{l-1} + \cdots + a_1(\tau s) + a_0}, \qquad (2.1.2)$$

where the parameters a_i's, c_i's, and $c_i(\tau)$'s are designed as follows:

- a_i's are chosen such that the following equation $p_f^\star(s)$ is Hurwitz for all $P(s) \in \mathcal{P}$.

$$p_f^\star(s) = s^l + a_{l-1}s^{l-1} + \cdots + a_{k+1}s^{k+1} + \frac{g}{g_\mathsf{n}}a_k s^k + \cdots + \frac{g}{g_\mathsf{n}}a_1 s + + \frac{g}{g_\mathsf{n}}a_0,$$

where

$$g := \frac{\beta_{n-\nu}}{\alpha_n} \quad \text{and} \quad g_\mathsf{n} := \frac{\beta^\mathsf{n}_{n-\nu}}{\alpha^\mathsf{n}_n},$$

with

$$P(s) = \frac{\sum_{j=0}^{n-\nu}\beta_j s^j}{\sum_{i=0}^{n}\alpha_i s^i} \quad \text{and} \quad P_\mathsf{n}(s) = \frac{\sum_{j=0}^{n-\nu}\beta^\mathsf{n}_j s^j}{\sum_{i=0}^{n}\alpha^\mathsf{n}_i s^i},$$

which are in fact the high-frequency gains of $P(s)$ and $P_\mathsf{n}(s)$, respectively.

- The parameters $c_0, c_1, \cdots, c_{k_t}$ are given by

$$c_i = a_i, \qquad i = 0, \cdots, k_t.$$

The rest $c_{k_t+1}(\tau), \cdots, c_k(\tau)$ are defined as

$$\begin{aligned}
[c_{k_t+1}(\tau), c_{k_t+3}(\tau), \cdots, c_{k-1}(\tau)]^\top &= V_{k_s-1}^{-1} V_{\frac{1}{2}(l-k_t-2+k^*)} A_{\mathrm{Re}}, \\
[c_{k_t+2}(\tau), c_{k_t+4}(\tau), \cdots, c_{k}(\tau)]^\top &= V_{k_s-1}^{-1} V_{\frac{1}{2}(l-k_t-2-k^*)} A_{\mathrm{Im}},
\end{aligned}$$

where V_i's are Vandermonde matrices [Kai80] given by

$$V_i := \begin{bmatrix} 1 & (-\tau^2\omega_1^2)^1 & \cdots & (-\tau^2\omega_1^2)^i \\ 1 & (-\tau^2\omega_2^2)^1 & \cdots & (-\tau^2\omega_2^2)^i \\ \vdots & \vdots & \ddots & \vdots \\ 1 & (-\tau^2\omega_{k_s}^2)^1 & \cdots & (-\tau^2\omega_{k_s}^2)^i \end{bmatrix} \in \mathbb{R}^{k_s \times (i+1)},$$

in which, τ is a small positive number, and
if $l - k_t$ is even,

$$\begin{aligned}
A_{\mathrm{Re}} &= [a_{k_t+1}, a_{k_t+3}, \cdots, a_{l-1}]^\top \in \mathbb{R}^{(l-k_t)/2}, \\
A_{\mathrm{Im}} &= [a_{k_t+2}, a_{k_t+4}, \cdots, a_{l-2}, 1]^\top \in \mathbb{R}^{(l-k_t)/2}, \\
k^* &= 0,
\end{aligned}$$

if $l - k_t$ is odd,

$$\begin{aligned}
A_{\mathrm{Re}} &= [a_{k_t+1}, a_{k_t+3}, \cdots, a_{l-2}, 1]^\top \in \mathbb{R}^{(l-k_t+1)/2}, \\
A_{\mathrm{Im}} &= [a_{k_t+2}, a_{k_t+4}, \cdots, a_{l-1}]^\top \in \mathbb{R}^{(l-k_t-1)/2}, \\
k^* &= 1.
\end{aligned}$$

Now the following result presents a condition which ensures robust stability of the closed-loop system in Figure 2.8 (see [JPBS] for the proof and more details).

Theorem 2.1.1. Under Assumptions 2.1.1 and 2.1.2, suppose that the Q-filter are designed as in (2.1.2) and the outer loop controller $C(s)$ is designed such that $P_{\mathsf{n}}(s)C(s)/(1 + P_{\mathsf{n}}(s)C(s))$ is stable. Then, there exists a constant $\bar{\tau} > 0$ such that, for all $0 < \tau \leq \bar{\tau}$, the closed-loop system is robustly internally stable. $\diamondsuit$

2.2 Frequency Estimation Algorithms for Indirect Approach

To reject the sinusoidal disturbances of unknown frequency, it may be considered an alternative method, the so-called *indirect approach*. The basic idea is that the frequency of sinusoidal disturbance is estimated in real time and then the estimated value is used in a disturbance rejection scheme designed for a known frequency. In this regard, we introduce several frequency estimation methods for the indirect approach such as adaptive notch filtering, phase-locked loops, and extended Kalman filtering. In addition, Marino's frequency estimator focused on globally convergent for all initial conditions is also presented. As a matter of fact, there are several other algorithms [HOD99, OPCTL02, SK08, Hou12].

2.2.1 Adaptive Notch Filtering

An adaptive notch filter is proposed in [Reg91], and then it is used to estimate the frequency of a sinusoidal signal in [BD97]. In the continuous-time, the notch filter is given by

$$N(s) = k\frac{s^2 + \omega^2}{s^2 + 2\zeta\omega s + \omega^2},$$

where k is the filter gain and ζ is the damping factor that determines the bandwidth of the filter's notch. While the component ω (which is called the notch frequency) is adapted to minimize the output of the notch filter, the sinusoidal signal is filtered through the notch filter. In order to estimate the unknown frequency, the adaptive notch filter is expressed by the following equation:

$$\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -2\zeta\hat{\omega}x_2 - \hat{\omega}^2 x_1 + k y_1, \\
\dot{\hat{\omega}} &= -g_1(k y_1 - 2\zeta\hat{\omega}x_2)x_1,
\end{aligned}$$

where g_1 is the adaptive gain, which is positive-valued. Under the assumption that g_1 is a small value, the algorithm's behavior is explained through an averaging analysis in [SB89, BD97], justified for small values of g_1 and for a periodic signal $y_1(t)$, and it is seen that the frequency estimate $\hat{\omega}$ converges to the actual value. When the filter input $y_1(t)$ contained multiple sinusoidal signals, the estimated value $\hat{\omega}$ converges to the most dominant frequency that was in the neighborhood of the initial value $\hat{\omega}(0)$. Here, the estimation error depends on the damping factor ζ, that is, the error converges to zero as ζ tends to zero.

2.2.2 Phase-Locked Loops

In [Ste97, WB01], the phased-locked loops algorithm is effective and simple in order to estimate and track the time-varying frequency. As shown in Figure 2.9, a phase-locked loop is presented by

$$\begin{aligned}
\alpha(t) &= K_f\hat{\omega}(t) + \int_0^t \hat{\omega}(\tau)d\tau, \\
\dot{\hat{\omega}}(t) &= 2g_{\hat{\omega}} y_1(t)\big(-\sin(\alpha(t))\big),
\end{aligned}$$

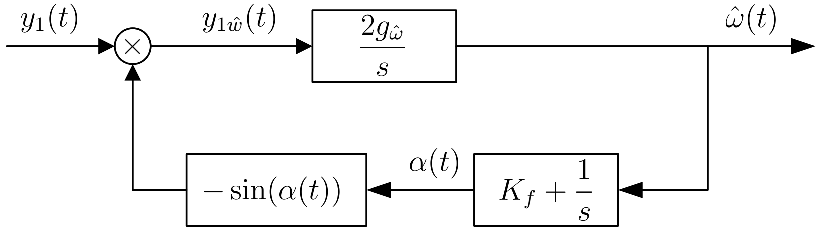

Figure 2.9: Frequency estimation based on phase-locked loops.

where the design parameters K_f and $g_{\hat{\omega}}$ are positive numbers. Note that the phase $\alpha(t)$ is not just integrated from the estimated frequency $\hat{\omega}$, as is generally done in phase-locked loops [Ste97]. Also, the additional proportional term K_f provides the phase lead which is usually incorporated through a lead filter.

Here, if the high frequency terms of $y_{1\hat{\omega}}(t)(:= y_1(t)(-\sin(\alpha(t))))$ is discarded and the estimated frequency $\hat{\omega}$ is closed enough to the fundamental frequency ω_d, then the loop dynamics of the linear approximate are those of a second-order system with poles determined by the roots of the following equation:

$$s^2 + g_{\hat{\omega}} m_{1d} K_f s + g_{\hat{\omega}} m_{1d} = 0,$$

where m_{1d} is the magnitude of the fundamental component of the signal. When the parameters K_f and $g_{\hat{\omega}} m_{1d}$ are both positive numbers, stability is guaranteed.

In addition, a magnitude/phase-locked loop approach is proposed in [WB03]. As the name indicates, the scheme is similar to a phase-locked loop, but a major difference is that it enables the tracking of the magnitude, phase, and frequency of an incoming sinusoidal signal simultaneously. The detail description of the algorithm scheme is omitted, instead, please refer to [WB03].

2.2.3 Extended Kalman Filtering

The extended Kalman filter (EKF) is proposed by Bitmead and co-authors in [SBJ95, SB96, SBQ96] for the design of a frequency tracker. Especially, the EKF frequency tracker is proposed in [SB96] and some heuristic guidelines for the tuning of its design parameters are presented. Also, its stability analysis is developed in [SBJ95] and the case of high-noise environments is considered in [SBQ96]. In this section, we introduce the basic idea of the EKF frequency tracker.

Suppose that the measured signal $y(k) = A\cos(\omega k + \varphi) + n(k)$ where $n(k)$ is a broad-band stationary signal (e.g., a white noise). Then, the signal $y(k)$ is expressed by the following third-order state space model:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} \cos(x_3(k)) & -\sin(x_3(k)) & 0 \\ \sin(x_3(k)) & \cos(x_3(k)) & 0 \\ 0 & 0 & 1-\epsilon \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ w(k) \end{bmatrix},$$
$$y(k) = x_1(k) + v(k),$$

where $w(k)$ and $v(k)$ are zero-mean uncorrected white noises, having variances q and r, respectively. The design parameter $\epsilon \geq 0$ (where, typically, $\epsilon \ll 1$) has the goal of enforcing stability into the third equation. The state $x_3(k)$ represents the unknown frequency ω. Under the state-space model, the EKF frequency tracker is recalled (see [SBJ95, SB96, SBQ96] for more details)

$$\begin{aligned} \hat{x}(k|k) &= f(\hat{x}(k-1|k-1)) + K(k)\big(y(k) - Hf(\hat{x}(k-1|k-1))\big), \\ \hat{\omega}(k) &= C\hat{x}(k|k), \\ K(k) &= P(k)H^\top \left(HP(k)H^\top + r\right)^{-1}, \\ P(k+1) &= F(k)\left[P(k) - P(k)H^\top\left(HP(k)H^\top + r\right)^{-1} HP(k)\right] F^\top(k) + q\bar{I}_3, \end{aligned}$$

where

$$\hat{x}(k|k) = \begin{bmatrix} \hat{x}_1(k|k) \\ \hat{x}_2(k|k) \\ \hat{x}_3(k|k) \end{bmatrix}, \; f(\hat{x}(k|k)) = \begin{bmatrix} \cos(\hat{x}_3(k|k))\hat{x}_1(k|k) - \sin(\hat{x}_3(k|k))\hat{x}_2(k|k) \\ \sin(\hat{x}_3(k|k))\hat{x}_1(k|k) + \cos(\hat{x}_3(k|k))\hat{x}_2(k|k) \\ (1-\epsilon)\hat{x}_3(k|k) \end{bmatrix},$$
$$H = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \quad F(k) = \left.\frac{\partial f(x)}{\partial x}\right|_{x=\hat{x}(k-1|k-1)},$$
$$P(k) = \begin{bmatrix} p_{11}(k) & p_{12}(k) & p_{13}(k) \\ p_{12}(k) & p_{22}(k) & p_{23}(k) \\ p_{13}(k) & p_{23}(k) & p_{33}(k) \end{bmatrix}, \quad \bar{I}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

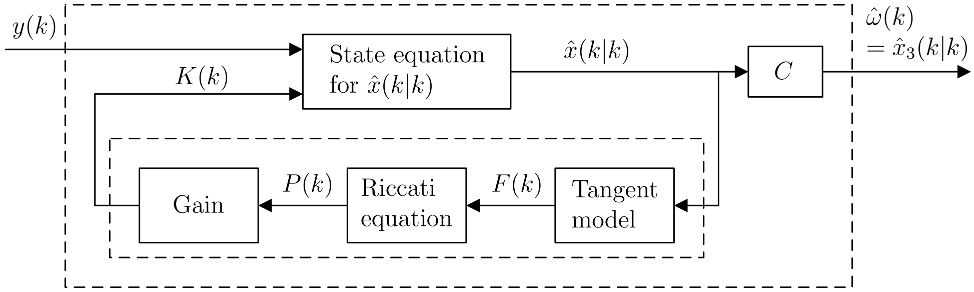

Figure 2.10: A representation of the EKF frequency tracker.

Here, the fourth equation, giving a recursion for the auxiliary matrix $P(k) \in \mathbb{R}^{3\times 3}$, is the celebrated Riccati equation [AM89, LS95]. Figure 2.10 shows the structure of the EKF frequency tracker with the measured signal $y(k)$ as input and the estimated frequency $\hat{w}(k) = \hat{x}_3(k|k)$ as output.

Note that q, r, and ϵ are the design parameters by means of which the designer can obtain suitable tracking performance. A preliminary analysis of the roles played by the parameters is given in [SB96, SBJ95], and some guidelines for their tuning are proposed. In order to explain the effects of each parameter on the EKF performance, the proposed analysis results are mainly based upon Monte Carlo simulation trials. Although some useful and interesting indications have been obtained in this way, the tuning of the parameters still remains a difficult task because of the unclear cross-relationships between such parameters.

2.2.4 Marino's Frequency Estimator

Authors in [MT02] focused on deriving a globally convergent algorithm for the estimation of n unknown frequency. Consider a signal as the following:

$$y(t) = \sum_{i=1}^{n} A_i \sin(\omega_i t + \phi_i),$$

where the magnitudes $A_i \neq 0$, the phases ϕ_i, and the frequencies $\omega_i > 0$, $1 \leq i \leq n$, $\omega_i \neq \omega_j$ for $i \neq j$, are unknown constant. Under the assumptions that the signal $y(t)$ is measurable and the positive integer n is known, we deal with the problem of designing a dynamic algorithm which asymptotically recovers the unknown frequencies $\omega_1, \omega_2, \cdots, \omega_n$. Now, we can consider that the signal $y(t)$ is generated by the linear observer model of order $2n$

$$\begin{aligned} \dot{w}_{i1} &= \omega_{i2}, \\ \dot{w}_{i2} &= -\omega_i^2 w_{i1}, \qquad 1 \le i \le n, \\ y &= \sum_{i=1}^{n} w_{i1} \end{aligned} \tag{2.2.1}$$

with unknown initial conditions $w_{i1}(0) = A_i \sin \phi_i$ and $w_{i2}(0) = A_i \cos \phi_i$. Since the system is observable, it is transformable by a linear change of coordinates into an observer canonical form ($x \in \mathbb{R}^{2n}$)

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ -\theta_1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\theta_n & 0 & 0 & \cdots & 0 \end{bmatrix} x = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} x - \left(\theta_1 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \cdots + \theta_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right) y \\ &=: A_c x - \left(\sum_{i=1}^{n} \theta_i e_i \right) y, \\ y &= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} x =: C_c x, \end{aligned} \tag{2.2.2}$$

where $e_i = [0, \ \cdots, \ 0, \ 1, \ 0, \ \cdots, \ 0]^\top \in \mathbb{R}^{2n \times 1}$ is a vector whose only nonzero entry is the $2i$th one and $\theta_1, \cdots, \theta_n$ are the coefficients of the characteristic polynomial of the system (2.2.1) as the following form:

$$\begin{aligned} \prod_{i=1}^{n} (s^2 + \omega_i^2) &= s^{2n} + \sum_{i=1}^{n} \omega_i^2 s^{2(n-1)} + \cdots + \prod_{i=1}^{n} \omega_i^2 \\ &=: s^{2n} + \theta_1 s^{2(n-1)} + \cdots + \theta_n. \end{aligned}$$

Given an arbitrary Hurwitz vector $d = [1, \ d_2, \ \cdots, \ d_{2n}]^\top$, i.e., a vector such that all the $2n-1$ zeros of the polynomial $s^{2n-1} + d_2 s^{2n-2} + \cdots + d_{2n-1} s + d_{2n}$ have negative real parts. Consider the filtered transformation

$$
\begin{aligned}
\dot{\xi}_i &= \Gamma \xi_i + e_i y, \qquad 1 \le i \le n, \quad \xi \in \mathbb{R}^{2n-1}, \\
\mu_i &= \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \xi_i, \\
z_1 &:= x_1 = y, \\
z_j &:= x_j - \sum_{i=1}^{n} \xi_{i,j-1} \theta_i, \qquad 2 \le j \le 2n,
\end{aligned}
$$

where

$$
\Gamma := \begin{bmatrix} -d_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -d_{2n-1} & 0 & \cdots & 1 \\ -d_{2n} & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{(2n-1)\times(2n-1)}
$$

is Hurwitz matrix, i.e., all eigenvalues in $\mathbb{C}_{<0}$. Then, by the filtered transformation, the system (2.2.2) is transformed into an adaptive observer form ($z \in \mathbb{R}^{2n}$, $\mu := [\mu_1, \ \cdots, \ \mu_n]^\top$, $\theta := [\theta_1, \ \cdots, \ \theta_n]^\top$)

$$
\dot{z} = A_c z + d\mu^\top \theta, \qquad y = C_c z. \tag{2.2.3}
$$

Defining $\eta_j := z_{j+1} - d_{j+1} z_1$, $1 \le j \le 2n-1$, then the system (2.2.3) is equivalently expressed as ($\eta := [\eta_1, \ \cdots, \ \eta_{2n-1}]^\top$)

$$
\begin{aligned}
\dot{y} &= \eta_1 + d_2 y + \mu^\top \theta, \\
\dot{\eta} &= \Gamma \eta + \beta y
\end{aligned} \tag{2.2.4}
$$

with $\beta = [d_3 - d_2^2, \ d_4 - d_3 d_2, \ \cdots, d_{2n} - d_{2n-1} d_2, -d_{2n} d_2]^\top$. Now, the adaptive observer for (2.2.4) is given by

$$
\begin{aligned}
\dot{\hat{y}} &= \hat{\eta}_1 + d_2 y + \mu^\top \hat{\theta}, \\
\dot{\hat{\eta}} &= \Gamma \hat{\eta} + \beta y, \\
\dot{\hat{\theta}} &= G\mu (y - \hat{y}),
\end{aligned} \tag{2.2.5}
$$

where $G := \mathrm{diag}(g_1, \cdots, g_n)$, in which, $g_1, \cdots, g_n$ are the positive adaptation gains.

The proposed adaptive observer (2.2.5) guarantees that if the persistently exciting (PE) condition for $\mu(t)$ is satisfied then the estimated state $\hat{\theta}(t)$ tends exponentially to the true value θ as time goes to infinity (see [MT02] for more details). Also, it is possible to extend to the case when the signal $y(t)$ contains an unknown bias A_0, i.e., $y(t) = A_0 + \sum_{i=1}^{n} A_i \sin(\omega_i t + \phi_i)$.

Chapter 3

Highlights of Output Regulation for Linear Systems

In this chapter, we briefly review, without proof, some established results about output regulation for linear systems, which are closely related to the topics that will be studied in this dissertation. The chapter introduces two results: output regulation via full information and error feedback. While the full information implies that all the states are measurable, the output error is only available in the error feedback. Most of this chapter is based on [KIF93, Hua04].

3.1 Problem Formulation

Consider a linear time-invariant (LTI) system described by equations of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Pw(t), \\ e(t) &= Cx(t) + Qw(t), \end{aligned} \tag{3.1.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $e \in \mathbb{R}^p$ is the error output to be regulated to zero, and $w \in \mathbb{R}^q$ is the exogenous input vector that yields the disturbance vector Pw to be rejected and the reference signal $-Qw$ to be tracked by the plant output Cx. The exogenous inputs are generated by a linear autonomous differential equation (which is called *exosystem*) of the form

$$\dot{w}(t) = Sw(t). \tag{3.1.2}$$

The control goal is to obtain that the overall closed-loop system is asymptotically stable and that the output error $e(t)$ converges to zero as time tends to infinity. The problems of designing controllers achieving the goal can be formally stated as following terms.

- **Problem of output regulation via full information**: for the given plant (3.1.1) and exosystem (3.1.2), find a control input of the form $u = Kx + Lw$ such that
  $(\mathrm{S})_{fi}$: the matrix $A + BK$ is Hurwitz, i.e., all eigenvalues in $\mathbb{C}_{<0}$,
  $(\mathrm{R})_{fi}$: for any initial condition $(x(0), w(0))$, the solution $(x(t), w(t))$ of

$$\begin{aligned} \dot{x} &= (A + BK)x + (P + BL)w, \\ \dot{w} &= Sw \end{aligned}$$

  satisfies $\lim_{t\to\infty} e(t) = \lim_{t\to\infty}(Cx(t) + Qw(t)) = 0$.

- **Problem of output regulation via error feedback**: for the given plant (3.1.1) and exosystem (3.1.2), find a dynamic controller of the form

$$\begin{aligned} \dot{\xi} &= F\xi + Ge, \\ u &= H\xi \end{aligned}$$

  such that
  $(\mathrm{S})_{ef}$: the closed-loop matrix A_{cl} is Hurwitz, where

$$A_{cl} := \begin{bmatrix} A & BH \\ GC & F \end{bmatrix},$$

  $(\mathrm{R})_{ef}$: for any initial condition $(x(0), \xi(0), w(0))$, the solution $(x(t), \xi(t), w(t))$ of

$$\begin{aligned} \dot{x} &= Ax + BH\xi + Pw, \\ \dot{\xi} &= GCx + F\xi + GQw, \\ \dot{w} &= Sw \end{aligned}$$

satisfies $\lim_{t\to\infty} e(t) = \lim_{t\to\infty}(Cx(t) + Qw(t)) = 0$.

Now, it will be assumed that the exosystem (3.1.2) satisfies the following assumption.

Assumption 3.1.1. The exosystem (3.1.2) is antistable, i.e., all the eigenvalues of S have nonnegative real parts. $\Diamond$

In fact, if Assumption 3.1.1 does not hold, the components of the exosystem (3.1.2) corresponding to the modes associated with the eigenvalues of S with negative real parts will exponentially decay to zero as time tends to infinity. Moreover, if the closed-loop system is asymptotically stable, as required, the output error corresponding to this kind of exogenous input will also exponentially decay to zero as time tends to infinity. Therefore, Assumption 3.1.1 is reasonable and constitutes no loss of generality.

3.2 Output Regulation via Full Information

In this section, it shows that the problem of output regulation via state feedback can be solved. First of all, we introduce the following simple but significant result which later on will provide the key idea to the solution of the problem.

Lemma 3.2.1. [KIF93, Lemma 1.3.1] Under Assumption 3.1.1, suppose that there exists $u = Kx + Lw$ for which condition $(\mathrm{S})_{fi}$ holds. Then, condition $(\mathrm{R})_{fi}$ also holds if and only if there exists a matrix Π which solves the following linear equations

$$\begin{aligned} \Pi S &= (A + BK)\Pi + (P + BL), \\ 0 &= C\Pi + Q. \end{aligned}$$

$\Diamond$

Now we present a controller synthesis method for the problem. From the condition $(\mathrm{S})_{fi}$, it is necessary that the pair (A, B) be stabilizable and thus we impose this property in the form of an explicit assumption.

Assumption 3.2.1. The matrix pair (A, B) is stabilizable. $\Diamond$

Then, we can provide the following result.

Theorem 3.2.2. [KIF93, Theorem 1.3.1] Consider the plant (3.1.1) with exosystem (3.1.2). Then, under Assumptions 3.1.1 and 3.2.1 with

$$u = Kx + (\Gamma - K\Pi)w = \Gamma w + K(x - \Pi w),$$

the problem of output regulation via full information can be solved if and only if there exist matrices Π and Γ which solve the following linear matrix equations

$$\begin{aligned} \Pi S &= A\Pi + B\Gamma + P, \\ 0 &= C\Pi + Q. \end{aligned} \tag{3.2.1}$$

$\Diamond$

Equations (3.2.1) are known as the *regulator equations*. From Theorem 3.2.2, an easily testable condition can be given with regard to the solvability of the regulator equations (3.2.1) as shown below.

Theorem 3.2.3. [Hua04, Theorem 1.9] For any matrices P and Q, the regulator equations (3.2.1) are solvable if and only if the following assumption holds. $\Diamond$

Assumption 3.2.2. For each λ which is an eigenvalue of S,

$$\text{rank}\left(\begin{bmatrix} A - \lambda I & B \\ C & 0 \end{bmatrix}\right) = n + p.$$

$\Diamond$

From Theorem 3.2.2, Theorem 3.2.3 directly leads to the following sufficient condition for the solvability of the output regulation via full information.

Corollary 3.2.4. [Hua04, Corollary 1.10] Under Assumptions 3.1.1, 3.2.1, and 3.2.2, the problem of output regulation via full information can be solved. $\Diamond$

Remark 3.2.1. [Hua04, Remark 1.11] If the triple (A, B, C) is controllable and observable, then those values of λ at which the matrix

$$\begin{bmatrix} A - \lambda I & B \\ C & 0 \end{bmatrix}$$

is not full rank are called the *transmission zeros* of the system. (It is a generalization of the notion of zeros of the single-input-single-output (SISO) systems to multi-input-multi-output (MIMO) systems.) Therefore, Assumption 3.2.2 can be paraphrased by saying that the transmission zeros of the plant (3.1.1) do not coincide with the eigenvalues of the exosystem (3.1.2), and it is often simply called the *transmission zeros condition.* As a special case, if all transmission zeros of the plant (3.1.1) are on the open left-half complex plane then the plant is called a minimum phase system, and thus it follows from Assumption 3.1.1 that a minimum phase system always satisfies the transmission zeros condition. ◊

3.3 Output Regulation via Error Feedback

In this section, we deal with the case where the state x and the exogenous input vector w are not available but the output error e is possible. Analogous to Lemma 3.2.1, we first introduce a preliminary result which will also be useful later on.

Lemma 3.3.1. [KIF93, Lemma 1.4.1] Under Assumption 3.1.1, suppose that there exists an error feedback controller

$$\begin{aligned} \dot{\xi} &= F\xi + Ge, \\ u &= H\xi, \end{aligned}$$

for which condition $(\mathrm{S})_{ef}$ holds. Then, condition $(\mathrm{R})_{ef}$ also holds if and only if there exists matrices Π and Σ which solve the following linear equations

$$\begin{aligned} \Pi S &= A\Pi + BH\Sigma + P, \\ \Sigma S &= F\Sigma, \\ 0 &= C\Pi + Q. \end{aligned}$$

◊

Now we present a controller synthesis method for the problem. Since we already know how to synthesize a state feedback controller in Section 3.2 which takes the plant state x and the exosystem state w, we naturally seek to synthesize an error feedback controller by estimating the state x and the exogenous input w. To this end, combining (3.1.1) and (3.1.2), we obtain

$$\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A & P \\ 0 & S \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u,$$
$$e = \begin{bmatrix} C & Q \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}.$$

Then, employing the well-known Luenberger observer theory [Lue64], we construct an observer driven by the measured error e and the control u. This observer-based controller has the form

$$\begin{aligned} \begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{w}} \end{bmatrix} &= \begin{bmatrix} A & P \\ 0 & S \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} L_x \\ L_w \end{bmatrix} \left(e - \begin{bmatrix} C & Q \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix} \right), \\ u &= \begin{bmatrix} K & \Gamma - K\Pi \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix}, \end{aligned} \tag{3.3.1}$$

where Γ and Π are the solution of the regulator equations (3.2.1), the matrices K and $\begin{bmatrix} L_x \\ L_w \end{bmatrix}$ are chosen such that the matrices

$$A + BK, \qquad \begin{bmatrix} A & P \\ 0 & S \end{bmatrix} - \begin{bmatrix} L_x \\ L_w \end{bmatrix} \begin{bmatrix} C & Q \end{bmatrix} \tag{3.3.2}$$

are Hurwitz, respectively. For (3.3.2), it is necessary not only that the matrix pair (A, B) be stabilizable but also that the matrix pair $\left(\begin{bmatrix} A & P \\ 0 & S \end{bmatrix}, \begin{bmatrix} C & Q \end{bmatrix} \right)$ be detectable, and thus we impose this property in the form of an additional explicit assumption.

Assumption 3.3.1. The matrix pair (A^e, C^e) is detectable where

$$A^e := \begin{bmatrix} A & P \\ 0 & S \end{bmatrix}, \qquad C^e := \begin{bmatrix} C & Q \end{bmatrix}.$$

◊

We now show that the observer-based controller (3.3.1) solves the problem of output regulation via error feedback.

Theorem 3.3.2. [KIF93, Theorem 1.4.1] Consider the plant (3.1.1) with exosystem (3.1.2). Suppose that, under Assumptions 3.1.1, 3.2.1, and 3.3.1, there exists matrices Π and Γ which solve the regulator equations (3.2.1). Then, for any initial condition $(x(0), w(0), \hat{x}(0), \hat{w}(0))$, the observer-based controller (3.3.1) guarantees that all the states of the closed-loop system are bounded and $\lim_{t\to\infty} e(t) = \lim_{t\to\infty}(Cx(t) + Qw(t)) = 0$. ◊

The detectability of (A^e, C^e) in Assumption 3.3.1 is stronger than the detectability of (A, C), but does not involve loss of generality. The reason is a little more technical and depends on the following fact.

Proposition 3.3.3. [KIF93, Proposition 1.4.1] Suppose (A, C) is detectable, while Assumption 3.3.1 does not hold. Consider the augmented system

$$\begin{aligned} \dot{x}^e &= A^e x^e + B^e u, \\ e &= C^e x^e, \end{aligned}$$

where $x^e := [x^\top \quad w^\top]^\top$ and $B^e := [B^\top \quad 0]^\top$. Then, there exists a coordinate transformation $\tilde{x}^e := T^e x^e$ such that

$$\tilde{A}^e = T^e A^e (T^e)^{-1} = \left[\begin{array}{cc|c} A & P_1 & 0 \\ 0 & S_{11} & 0 \\ \hline 0 & S_{21} & S_{22} \end{array}\right], \quad \tilde{B}^e = T^e B^e = \left[\begin{array}{c} B \\ 0 \\ \hline 0 \end{array}\right],$$

$$\tilde{C}^e = C^e (T^e)^{-1} = \left[\begin{array}{cc|c} C & Q_1 & 0 \end{array}\right].$$

Moreover, the pair

$$\left(\begin{bmatrix} A & P_1 \\ 0 & S_{11} \end{bmatrix}, \begin{bmatrix} C & Q_1 \end{bmatrix}\right)$$

is detectable (i.e., satisfies Assumption 3.3.1). ◊

Now, let new variables $\tilde{x}$ and $\tilde{w}$ defined as

$$\begin{bmatrix} \tilde{x} \\ \tilde{w} \end{bmatrix} := T^e \begin{bmatrix} x \\ w \end{bmatrix},$$

and partition $\tilde{w}$ in two blocks as

$$\tilde{w} = \begin{bmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{bmatrix}.$$

Then, from Proposition 3.3.3, the original plant equations (3.1.1) are replaced by equations of the form

$$\begin{aligned} \dot{\tilde{x}} &= A\tilde{x} + Bu + P_1\tilde{w}_1, \\ e &= C\tilde{x} + Q_1\tilde{w}_1, \end{aligned}$$

and the exosystem (3.1.2) is also replaced by equations of the form

$$\begin{aligned} \dot{\tilde{w}}_1 &= S_{11}\tilde{w}_1, \\ \dot{\tilde{w}}_2 &= S_{21}\tilde{w}_1 + S_{22}\tilde{w}_2. \end{aligned}$$

It is seen from the replaced equations that the output error e is not affected at all by the new state $\tilde{w}_2$ of the exosystem. Therefore, the output regulation problem for the original plant (3.1.1) and exosystem (3.1.2) is equivalent to the output regulation problem for the following plant

$$\begin{aligned} \dot{\tilde{x}} &= A\tilde{x} + Bu + P_1\tilde{w}_1, \\ e &= C\tilde{x} + Q_1\tilde{w}_1 \end{aligned} \tag{3.3.3}$$

driven by a reduced exosystem

$$\dot{\tilde{w}}_1 = S_{11}\tilde{w}_1, \tag{3.3.4}$$

and, Assumption 3.3.1 now holds for this plant and exosystem.

Hence, if Assumption 3.3.1 does not hold and (A, C) is detectable, then the observer-based controller (3.3.1) can be replaced by the following controller:

$$\begin{aligned}\begin{bmatrix}\dot{\hat{x}} \\ \dot{\hat{\tilde{w}}}_1\end{bmatrix} &= \begin{bmatrix}A & P_1 \\ 0 & S_{11}\end{bmatrix}\begin{bmatrix}\hat{x} \\ \hat{\tilde{w}}_1\end{bmatrix} + \begin{bmatrix}B \\ 0\end{bmatrix}u + \begin{bmatrix}\tilde{L}_x \\ \tilde{L}_w\end{bmatrix}\left(e - \begin{bmatrix}C & Q_1\end{bmatrix}\begin{bmatrix}\hat{x} \\ \hat{\tilde{w}}_1\end{bmatrix}\right), \\ u &= \begin{bmatrix}K & \tilde{\Gamma} - K\tilde{\Pi}\end{bmatrix}\begin{bmatrix}\hat{x} \\ \hat{\tilde{w}}_1\end{bmatrix},\end{aligned} \tag{3.3.5}$$

where $\tilde{\Gamma}$ and $\tilde{\Pi}$ are the solution of the following regulator equations

$$\begin{aligned}\tilde{\Pi}S_{11} &= A\tilde{\Pi} + B\tilde{\Gamma} + P_1, \\ 0 &= C\tilde{\Pi} + Q_1,\end{aligned}$$

and the matrices K and $\begin{bmatrix}\tilde{L}_x \\ \tilde{L}_w\end{bmatrix}$ are chosen such that the matrices

$$A + BK, \qquad \begin{bmatrix}A & P_1 \\ 0 & S_{11}\end{bmatrix} - \begin{bmatrix}\tilde{L}_x \\ \tilde{L}_w\end{bmatrix}\begin{bmatrix}C & Q_1\end{bmatrix}$$

are Hurwitz, respectively. Then, we obtain the following corollary.

Corollary 3.3.4. [KIF93, Corollary 1.4.1] Under Assumptions 3.1.1, 3.2.1, and 3.2.2, the problem of output regulation via error feedback is solvable by the controller (3.3.5) if the matrix pair (A, C) is detectable. ♢

Chapter 4

Adaptive Add-on Output Regulator for Unknown Sinusoidal Exogenous Inputs

In this chapter, we extend the result of [SKC04] to the case where the frequencies of reference inputs and disturbances are unknown by resorting to the adaptive observer proposed in [Zha02], while the number of frequencies contained in the exogenous inputs is known. In this chapter, the reference signals are also viewed as external disturbances.

Controller design for industrial applications often confronts many constraints and tight design specifications, which are not easily satisfied, and therefore, the controller used in commercial products is the outcome of many design iterations. As a result, when the performance specification is upgraded for better precision, it is a burden to go through all the iterations again. Motivated by this fact, an add-on controller has been proposed in [SKC04] which improves the performance of sinusoidal disturbance rejection while the stability of the closed-loop system with the preinstalled controller (called a *primary* controller in this chapter) is preserved. Its benefit also includes the following.

- Design of the add-on controller does not require much information about the preinstalled controller (such as internal states, structure, and gains). It just uses the output signal of the preinstalled controller (which is in fact the control input to the plant without the add-on controller). This feature could be beneficial in some cases where the details about the preinstalled controller are not available.

- The add-on controller can be inserted *smoothly* into the feedback loop in order not to damage the transient response (which is supposed to be satisfied with the preinstalled controller). This feature is achieved by the *slow-start* of the add-on controller; that is, although the add-on controller start to run from the beginning, it is introduced into the feedback loop *after* the nominal transient period. Moreover, the control signal from the add-on controller is gradually appended to the control signal of the preinstalled controller in order to avoid unnecessary transient (which might have been caused by the abrupt change of controller structure). For this, it has been shown that the stability of the closed-loop system is guaranteed with any fractional addition of the control signals from the add-on controller and the preinstalled controller.

However, the disturbance rejection of [SKC04] is limited to the case where the frequencies of disturbances are known. In some applications, the frequency of the disturbance is not known and needs to be estimated. The same problem has been considered in [KKCS05, KKCT11], in which, instead of adaptive observer, the frequency identifier is used. Because of this, it is required to know the upper and lower bounds of unknown frequencies, and even with the knowledge of such bounds, it is not clear whether the controller design is always possible. Compared to [KKCS05, KKCT11], the proposed method in this chapter is always applicable and the bounds of unknown frequencies are not necessary. In addition, the design procedure is simpler (although the stability analysis of this chapter is a little more complicated than [KKCS05, KKCT11]).

In terms of disturbance rejection or attenuation, there are also several other approaches. For example, disturbance observer (DOB) is known to be effective to compensate disturbances [OSM96, CYC$^+$03, YCC05, BSPS10, KSC13, YSKK13, KR13]. Conventionally, the disturbance observer is designed based on the inverse dynamics of a plant with a low-pass filter, whose effect on robust stability and disturbance rejection has been analyzed in [SJ07, BS08, SJ09]. However, disturbance observer is not very effective to cancel out disturbances of specific frequencies. Repetitive control, e.g., in [DWZ03, LROLDS10, DR13, EMHGMR14], has also been shown to be effective for rejecting repetitive disturbances. Although repetitive control approach achieves almost perfect cancellation of periodic disturbances, it requires exact knowledge of the period of the external signals. To relax this requirement, a robust repetitive control has been introduced in [Ste02]. It is however not very effective to cancel out disturbances when the frequency varies in a wide range. To deal with disturbances with unknown frequencies, adaptive feedforward cancellation has been presented in [BD97, MBK12]. However, it requires a computation of plant gains at all estimated frequencies if the plant is stable (or the gains of the closed-loop system with another controller that stabilizes the plant if the plant is unstable). On the other hand, motivated by the theoretical developments in [SIM01, MT03], our tool for dealing with the problem is the output regulation theory [FW76, Hua04], which enables to overcome such drawbacks.

The rest of the chapter is organized as follows. In Section 4.1, we review the add-on output regulator which eliminates disturbances of known frequencies. Section 4.2 is devoted to develop an adaptive output regulation of add-on type in order to reject disturbances of unknown frequencies. In Section 4.3, the proposed controller is experimentally tested for the track-following problem of optical disc drive (ODD) systems.

4.1 Add-on Output Regulator

In this section, an add-on type output feedback controller is discussed that can eliminate disturbances with known frequencies.

4.1.1 Problem Formulation

We consider a linear time-invariant (LTI) single-input-single-output (SISO) system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + bu(t) + Pw(t), \\ e(t) &= cx(t) + qw(t), \end{aligned} \tag{4.1.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the control input, and $e \in \mathbb{R}$ is the output to be regulated to zero. The variable $w \in \mathbb{R}^r$ represents biased sinusoidal disturbances which are generated by the exosystem

$$\dot{w}(t) = Sw(t), \tag{4.1.2}$$

where the matrix S has one eigenvalue in zero or distinct pairs of pure imaginary eigenvalues. It is assumed that e and u are measurable while x and w are not available for feedback. We also assume that a primary controller $C_{pr}(s)$ has already been installed and in operation for (4.1.1) and its realization is given by

$$\begin{aligned} \dot{z}(t) &= A_p z(t) + B_p e(t), \\ u_c(t) &= C_p z(t) + D_p e(t). \end{aligned} \tag{4.1.3}$$

Our control goal is to design an add-on output regulator such that, for any initial conditions $x(0)$ and $w(0)$, all the closed-loop signals are bounded and $\lim_{t\to\infty} e(t) = 0$.

In order to propose the add-on output regulator, we need several assumptions. When $u = u_c$ and $w \equiv 0$, the primary controller leads to the following closed-loop system

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A + bD_p c & bC_p \\ B_p c & A_p \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} =: A_{cl} \begin{bmatrix} x \\ z \end{bmatrix}.$$

For this system, we assume the following.

Assumption 4.1.1. The matrix A_{cl} is Hurwitz (that is, the system under the primary controller is stable). ◊

Assumption 4.1.2. There exist matrices $\Pi \in \mathbb{R}^{n\times r}$ and $\gamma \in \mathbb{R}^{1\times r}$ such that

$$\begin{aligned} \Pi S &= A\Pi + b\gamma + P, \\ 0 &= c\Pi + q. \end{aligned} \tag{4.1.4}$$

◊

If the system does not have zeros on the imaginary axis, then Assumption 4.1.2 always holds (see Theorem 3.2.3 and Remark 3.2.1), and it is seen from

(4.1.4) that $\Pi \in \mathbb{R}^{n\times r}$ and $\gamma \in \mathbb{R}^{1\times r}$ can be computed as follows:

$$\begin{aligned}
&\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \Pi \\ \gamma \end{bmatrix} S - \begin{bmatrix} A & b \\ c & 0 \end{bmatrix}\begin{bmatrix} \Pi \\ \gamma \end{bmatrix} I = \begin{bmatrix} P \\ q \end{bmatrix} \\
\Rightarrow\ & \operatorname{vec}\left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \Pi \\ \gamma \end{bmatrix} S\right) - \operatorname{vec}\left(\begin{bmatrix} A & b \\ c & 0 \end{bmatrix}\begin{bmatrix} \Pi \\ \gamma \end{bmatrix} I\right) = \operatorname{vec}\left(\begin{bmatrix} P \\ q \end{bmatrix}\right) \\
\Rightarrow\ & \left(S^\top \otimes \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\right)\operatorname{vec}\left(\begin{bmatrix} \Pi \\ \gamma \end{bmatrix}\right) - \left(I \otimes \begin{bmatrix} A & b \\ c & 0 \end{bmatrix}\right)\operatorname{vec}\left(\begin{bmatrix} \Pi \\ \gamma \end{bmatrix}\right) = \operatorname{vec}\left(\begin{bmatrix} P \\ q \end{bmatrix}\right) \\
\Rightarrow\ & \operatorname{vec}\left(\begin{bmatrix} \Pi \\ \gamma \end{bmatrix}\right) = \left(S^\top \otimes \begin{bmatrix} I_n & 0_{n\times 1} \\ 0_{1\times n} & 0 \end{bmatrix} - I_r \otimes \begin{bmatrix} A & b \\ c & 0 \end{bmatrix}\right)^{-1} \operatorname{vec}\left(\begin{bmatrix} P \\ q \end{bmatrix}\right),
\end{aligned}$$

where $\otimes$ and vec$(\cdot)$ denote the *Kronecker product* and the *stacking operator*, respectively [Hua04, Appendix A][1].

From Assumption 4.1.1, it follows that (A, c) is detectable (see Appendix A.1), and thus, if the matrix pair $\left(\begin{bmatrix} A & P \\ 0 & S \end{bmatrix}, \begin{bmatrix} c & q \end{bmatrix}\right)$ is not detectable then it is clear from Proposition 3.3.3 that the output regulation problem for the original plant (4.1.1) and exosystem (4.1.2) is equivalent to the output regulation problem for a new reduced system, such as (3.3.3) and (3.3.4) in Chapter 3. Hence, the following assumption can be made without loss of generality.

Assumption 4.1.3. The matrix pair $\left(\begin{bmatrix} A & P \\ 0 & S \end{bmatrix}, \begin{bmatrix} c & q \end{bmatrix}\right)$ is detectable. $\Diamond$

4.1.2 Controller Design and Stability Analysis

Now, we present an add-on type output regulator when the frequency of disturbance is known.

Theorem 4.1.1. Consider the system (4.1.1), (4.1.2), the primary controller

[1] For this, we use the following property: for any matrices $A \in \mathbb{R}^{m\times q}$, $B \in \mathbb{R}^{p\times n}$, and $X \in \mathbb{R}^{n\times m}$, $\operatorname{vec}(BXA) = (A^\top \otimes B)\operatorname{vec}(X)$ (see [Hua04, Proposition A.1.(ii)] for the proof).

(4.1.3), and the following add-on type output regulator:

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{w}} \end{bmatrix} = \begin{bmatrix} A - K_1 c & P - K_1 q \\ -K_2 c & S - K_2 q \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix} + \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} e + \begin{bmatrix} b \\ 0 \end{bmatrix} u, \tag{4.1.5}$$

$$u_r = \begin{bmatrix} 0 & \gamma \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix}, \tag{4.1.6}$$

where K_1 and K_2 are chosen such that

$$\left(\begin{bmatrix} A & P \\ 0 & S \end{bmatrix} - \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \begin{bmatrix} c & q \end{bmatrix} \right) \quad \text{is Hurwitz.}$$

Under Assumptions 4.1.1–4.1.3, the control law

$$u = \rho(t) u_r + u_c$$

guarantees that all the states of the closed-loop system (4.1.1)–(4.1.3), (4.1.5), and (4.1.6) are bounded for any time-varying bounded function $\rho(t)$, and that the output error $e(t)$ converges to zero when $\rho(t) = 1$. ♢

Remark 4.1.1. The $\rho(t)$ is a switching function that can be utilized to achieve a bumpless transition. It determines whether the add-on output regulator is included in the feedback loop or not. In particular, when $\rho = 0$, only the primary controller is running, whereas, if $\rho = 1$, the add-on controller takes part in the feedback as well as $C_{pr}(s)$. In a typical situation, the primary controller works with $\rho(t) = 0$. On the contrary, when it is detected that disturbances are not sufficiently removed, we set $\rho(t) = 1$ for the add-on controller to do its job. However, if $\rho(t)$ is abruptly switched from zero to one, the plant input changes suddenly from $u_c(t)$ to $u_c(t) + u_r(t)$, which may cause undesirable transient response such as large overshoot or undershoot. In order to avoid this, we suggest to select $\rho(t)$ such that it increases slowly from zero to one, and Theorem 4.1.1 guarantees such a slow-start does not damage stability. In fact, it will be seen in Section 4.3 that the slow-start indeed suppresses the unexpected transient. ♢

Proof. From Assumption 4.1.3, the selection of K_1 and K_2 is always possible.

Now, by taking $e_x := \hat{x} - x$ and $e_w := \hat{w} - w$, we have

$$\begin{bmatrix} \dot{e}_x \\ \dot{e}_w \end{bmatrix} = \begin{bmatrix} A - K_1 c & P - K_1 q \\ -K_2 c & S - K_2 q \end{bmatrix} \begin{bmatrix} e_x \\ e_w \end{bmatrix},$$

which is exponentially stable. Therefore, it follows that

$$e_w(t) \to 0 \qquad \text{as} \qquad t \to \infty.$$

Now the plant (4.1.1), the exosystem (4.1.2), and the primary controller (4.1.3) can be written as

$$\begin{aligned} \dot{x} &= Ax + b(\rho u_r + u_c) + Pw \\ &= Ax + b(\rho\gamma\hat{w} + C_p z + D_p e) + Pw \\ &= (A + bD_p c)x + bC_p z + (P + bD_p q)w + \rho b\gamma(w + e_w), \\ \dot{z} &= A_p z + B_p(cx + qw) = B_p cx + A_p z + B_p qw, \\ \dot{w} &= Sw. \end{aligned}$$

With the matrix Π of Assumption 4.1.2, we define $\tilde{x} := x - \Pi w$. Then, in a new coordinates $(\tilde{x}, z, w)$ the above system becomes

$$\begin{aligned} \dot{\tilde{x}} &= (A + bD_p c)\tilde{x} + (A + bD_p c)\Pi w + bC_p z + (P + bD_p q)w \\ &\qquad + b\gamma w + \rho b\gamma e_w - (1-\rho)b\gamma w - \Pi S w \\ &= (A + bD_p c)\tilde{x} + bC_p z + (A\Pi + b\gamma + P - \Pi S + bD_p(c\Pi + q))\, w \\ &\qquad + b\gamma\left(\rho e_w - (1-\rho)w\right) \\ &= (A + bD_p c)\tilde{x} + bC_p z + b\gamma\left(\rho e_w - (1-\rho)w\right), \\ \dot{z} &= B_p c\tilde{x} + A_p z + B_p(c\Pi + q)w = B_p c\tilde{x} + A_p z, \\ \dot{w} &= Sw, \end{aligned}$$

in which (4.1.4) has been used. Thus, we have

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{z} \end{bmatrix} = A_{cl} \begin{bmatrix} \tilde{x} \\ z \end{bmatrix} + \begin{bmatrix} b\gamma \\ 0 \end{bmatrix} \left(\rho e_w - (1-\rho)w\right) =: A_{cl} \begin{bmatrix} \tilde{x} \\ z \end{bmatrix} + M(t),$$

which is input-to-state stable (ISS) [Kha01] with $M(t)$ viewed as the input since the matrix A_{cl} is Hurwitz by Assumption 4.1.1. Therefore, the states $\tilde{x}$ and z are bounded for any bounded ρ since e_w and w are bounded.

Finally, when $\rho(t) = 1$, the state $\tilde{x}(t)$ and $z(t)$ go to zero since the input to the system becomes $e_w(t)$ which decays to zero. Thus, the error $e(t)$ converges to zero because

$$e(t) = cx(t) + qw(t) = c\tilde{x}(t) + (c\Pi + q)w(t) = c\tilde{x}(t).$$

□

4.2 Adaptive Add-on Output Regulator

In the previous section, an add-on type output regulator has been discussed for the case where the frequencies of disturbances are known. Now, by utilizing the adaptive observer in [Zha02], we propose an adaptive version of the add-on controller when the frequencies of disturbances are unknown.

4.2.1 Problem Formulation

Without loss of generality, we assume that the exosystem is given by, with some positive integer m,

$$\dot{w}(t) = Sw(t), \quad w \in \mathbb{R}^{2m+1}, \tag{4.2.1}$$

$$S := \text{blockdiag}(\sigma_1 S_o, \sigma_2 S_o, \cdots, \sigma_m S_o, 0_{1\times 1}), \quad S_o := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

in which $\sigma_1, \sigma_2, \cdots, \sigma_m$ are unknown distinct positive constants. The following assumption is required in order to identify the unknown frequencies $\sigma_1, \cdots, \sigma_m$.

Assumption 4.2.1. All oscillatory modes of the exosystem (4.2.1) are excited by the initial condition $w(0)$, that is, the 2×1 vector $[w_{2i-1}(0) \;\; w_{2i}(0)]^\top$ is a nonzero vector for all $i = 1, 2, \cdots, m$. ◊

A simple sufficient condition for Assumption 4.2.1 is that $w(t)$ contains exactly m different sinusoids. As a matter of fact, Assumption 4.2.1 is not a restrictive one. Suppose that Assumption 4.2.1 is not satisfied, for instance, assume that only $m-1$ sinusoidal signals are generated by the exosystem (4.2.1). Then, if we construct another matrix $\tilde{S} \in \mathbb{R}^{(2m-1)\times(2m-1)}$ by removing the unexcited mode from $S \in \mathbb{R}^{(2m+1)\times(2m+1)}$, Assumption 4.2.1 holds for the new exosystem with $\tilde{S}$ (and suitably adjusted P and q).

We now note that, with unknown frequencies σ_i, checking Assumption 4.1.2 is not easy because the matrix S of (4.2.1) is not known. However, since the eigenvalues of S still lie on the imaginary axis, we assume the following instead of Assumption 4.1.2.

Assumption 4.2.2. The plant (4.1.1) (with $w \equiv 0$) has no zero on the imaginary axis of the complex plane. ◊

A consequence of Assumption 4.2.2 is that, for any eigenvalue λ of the matrix S, we obtain

$$\operatorname{rank}\left(\begin{bmatrix} A-\lambda I & b \\ c & 0 \end{bmatrix}\right) = n+1, \tag{4.2.2}$$

which guarantees that Assumption 4.1.2 holds (see Theorem 3.2.3 and Remark 3.2.1 or [KIF93, Lemma 1.5.1]). Thus, there exist matrices $\Pi \in \mathbb{R}^{n\times(2m+1)}$ and $\gamma \in \mathbb{R}^{1\times(2m+1)}$ satisfying (4.1.4).

Here, if the matrix pair (S,γ) is not observable, then the 'output regulation' theory [Hua04] implies that the unobservable mode of $w(t)$ will not appear in the output $e(t)$ even if there is no control action for that mode. Thus, we impose the following assumption in order to deal with a nontrivial regulation problem.

Assumption 4.2.3. The matrix pair (S,γ) is observable for any positive constants $\sigma_1, \sigma_2, \cdots, \sigma_m$. ◊

In general, it is not easy to check Assumption 4.2.3 since S and γ depend on unknown frequencies $\sigma_1, \cdots, \sigma_m$. The following lemma may be useful for many cases.

Lemma 4.2.1. Let γ_i denote the i-th component of γ. Then, Assumption 4.2.3 holds if $\gamma_{2m+1} \neq 0$ and all the vectors $[\gamma_1 \;\; \gamma_2], \cdots, [\gamma_{2m-1} \;\; \gamma_{2m}]$ are nonzero for any positive σ_i's. ♢

Proof. We first note that, for any positive real number σ^*, any real number ω^*, and any non-zero real vector $\gamma^* = [\gamma_1^* \;\; \gamma_2^*]$, the matrix

$$\begin{bmatrix} \sigma^* S_o - j\omega^* I \\ \gamma^* \end{bmatrix} = \begin{bmatrix} -j\omega^* & \sigma^* \\ -\sigma^* & -j\omega^* \\ \gamma_1^* & \gamma_2^* \end{bmatrix}$$

has full column rank. Thus, by the structure of S, the matrix $\begin{bmatrix} S - \lambda I \\ \gamma \end{bmatrix}$ has full column rank for any eigenvalue λ of S. Therefore, by the PBH (Popov-Belevitch-Hautus) rank test [Kai80, Hes09], the pair (S, γ) is observable. □

Remark 4.2.1. While Assumption 4.2.1 is trivial, Assumption 4.2.3 is not realistic even though Lemma 4.2.1 is very useful for many applications, including optical disc drive (ODD) systems to be introduced in this chapter. To solve this problem, we will provide a solution in Chapter 5 to relax Assumption 4.2.1 and 4.2.3. ♢

4.2.2 Controller Design and Analysis

Now, we are ready to present an adaptive add-on regulator. From Appendix A.1 (or [TSH01, Theorem 3.40]), under Assumption 4.1.1, it can be assumed that the matrix pair (A, b) is stabilizable and (A, c) is observable. Thus, we can assume that A, b, and c have the form of

$$A = \begin{bmatrix} -a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & \cdots & 1 \\ -a_n & 0 & \cdots & 0 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^\top$$

for some constants $a_1, \cdots, a_{n-1}, a_n, b_1, \cdots, b_{n-1}, b_n$. In fact, Assumption 4.1.1 implies the detectability instead of the observability. However, even though the system is detectable but not observable, the same conclusion can be obtained using the Kalman decomposition [Che99]. The design methodology with the decomposition can be found in Chapter 5. Here, the observability assumption is made only for the sake of simple exposition.

In addition, define $\bar{A} \in \mathbb{R}^{(n+2m+1)\times(n+2m+1)}$, $\bar{b} \in \mathbb{R}^{(n+2m+1)\times 1}$, and $\bar{c} \in \mathbb{R}^{1\times(n+2m+1)}$ as

$$\bar{A} := \begin{bmatrix} -a_1 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -a_n & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}, \ \bar{b} := \begin{bmatrix} b \\ 0_{(2m+1)\times 1} \end{bmatrix}, \ \bar{c} := \begin{bmatrix} c & 0_{1\times(2m+1)} \end{bmatrix},$$

and let

$$\Psi(e,u) := -\begin{bmatrix} a[1], a[2], \cdots, a[m] \end{bmatrix} e + \begin{bmatrix} b[1], b[2], \cdots, b[m] \end{bmatrix} u,$$

where $a[i] \in \mathbb{R}^{(n+2m+1)\times 1}$ and $b[i] \in \mathbb{R}^{(n+2m+1)\times 1}$, $i = 1, 2, \cdots, m$, are given by

$$\begin{aligned} a[i] &:= \begin{bmatrix} 0_{1\times(2i-1)} & 1 & a_1 & \cdots & a_n & 0_{1\times(2m-2i+1)} \end{bmatrix}^\top, \\ b[i] &:= \begin{bmatrix} 0_{1\times 2i} & b_1 & \cdots & b_n & 0_{1\times(2m-2i+1)} \end{bmatrix}^\top. \end{aligned}$$

For given $\theta := [\theta_1 \ \ \theta_2 \ \ \cdots \ \ \theta_m]^\top$, define $T(\theta) \in \mathbb{R}^{(n+2m+1)\times(n+2m+1)}$ by

$$T(\theta) := \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ \bar{\alpha}_1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \bar{\alpha}_{n+2m-1} & \bar{\alpha}_{n+2m-2} & \cdots & 1 & 0 \\ \bar{\alpha}_{n+2m} & \bar{\alpha}_{n+2m-1} & \cdots & \bar{\alpha}_1 & 1 \end{bmatrix} \begin{bmatrix} \bar{c} \\ \bar{c}A_e(\theta) \\ \vdots \\ \bar{c}A_e^{n+2m-1}(\theta) \\ \bar{c}A_e^{n+2m}(\theta) \end{bmatrix},$$

where $\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_{n+2m}$ are chosen such that

$$\begin{aligned}&s^{n+2m+1} + \bar{\alpha}_1 s^{n+2m} + \cdots + \bar{\alpha}_{n+2m} s \\ &= (s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n) \cdot (s^{2m+1} + \theta_1 s^{2m-1} + \theta_2 s^{2m-3} + \cdots + \theta_m s),\end{aligned}$$

and $A_e(\theta) \in \mathbb{R}^{(n+2m+1)\times(n+2m+1)}$ is given by

$$A_e(\theta) := \begin{bmatrix} A & -b\bar{\gamma} \\ 0 & \bar{S}(\theta) \end{bmatrix},$$

in which, $\bar{S}(\theta) \in \mathbb{R}^{(2m+1)\times(2m+1)}$ and $\bar{\gamma} \in \mathbb{R}^{1\times(2m+1)}$ are defined as

$$\bar{S}(\theta) := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ -\theta_1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ -\theta_m & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \qquad \bar{\gamma} := \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}^\top. \tag{4.2.3}$$

Theorem 4.2.2. Consider the system (4.1.1), (4.2.1), the primary controller (4.1.3), and the following adaptive add-on regulator:

$$\dot{\hat{\xi}} = \bar{A}\hat{\xi} + \bar{b}u + \Psi(e,u)\hat{\theta} + (L + \Xi K \Xi^\top \bar{c}^\top)(e - \bar{c}\hat{\xi}), \qquad \hat{\xi} \in \mathbb{R}^{n+2m+1}, \tag{4.2.4}$$

$$\dot{\hat{\theta}} = K\Xi^\top \bar{c}^\top (e - \bar{c}\hat{\xi}), \qquad \hat{\theta} \in \mathbb{R}^m, \tag{4.2.5}$$

$$\dot{\Xi} = (\bar{A} - L\bar{c})\Xi + \Psi(e,u), \qquad \Xi \in \mathbb{R}^{(n+2m+1)\times m}, \tag{4.2.6}$$

$$\dot{p} = -a_p p + a_p \big(\bar{c}\hat{\xi} - e\big)^2, \qquad p \in \mathbb{R}, \tag{4.2.7}$$

$$u_r = \begin{bmatrix} 0_{1\times n} & 1 & 0_{1\times 2m} \end{bmatrix} \left(\frac{\det\big(T(\hat{\theta})\big)}{p + \det^2\big(T(\hat{\theta})\big)} \operatorname{adj}\big(T(\hat{\theta})\big) \right) \hat{\xi}, \tag{4.2.8}$$

where $L \in \mathbb{R}^{n+2m+1}$ is chosen such that $\bar{A} - L\bar{c}$ is Hurwitz, $K \in \mathbb{R}^{m\times m}$ is any symmetric positive definite matrix, the initial condition $p(0)$ and a_p are any positive constants, and $\operatorname{adj}(T(\hat{\theta}))$ denotes the adjoint matrix of $T(\hat{\theta})$. Under Assumptions 4.1.1 and 4.2.1–4.2.3, control law $u = u_c + u_r$ guarantees that $\lim_{t\to\infty} \hat{\theta}(t) = \theta$

and $\lim_{t\to\infty} e(t) = 0$ and all the states of the closed-loop system composed of (4.1.1), (4.1.3), (4.2.1), and (4.2.4)–(4.2.8) are bounded. ◊

Remark 4.2.2. Note that the matrix γ (or its estimate) does not appear in the controller (4.2.4)–(4.2.8), while the controller in [KKCS05, KKCT11] depends on γ. Since γ is unknown when the exosystem is unknown, it is replaced by its estimate in [KKCS05, KKCT11]. ◊

Proof. Let Π and γ be the solution to the regulator equation (4.1.4) and

$$\Phi(p,\hat{\theta}) := \frac{\det\big(T(\hat{\theta})\big)}{p+\det^2\big(T(\hat{\theta})\big)}\operatorname{adj}\big(T(\hat{\theta})\big).$$

Then, defining $x_r := \Pi w$, $u_w := \gamma w$, and $\tilde{x} := x - x_r$ yields

$$\begin{aligned}\dot{\tilde{x}} &= A\tilde{x} - bu_w + bu,\\ e &= c\tilde{x} + cx_r + qw = c\tilde{x}.\end{aligned}\tag{4.2.9}$$

Moreover, (4.2.8) can be written as $u_r = \bar{\gamma}[0_{(2m+1)\times n} \;\; I_{2m+1}]\Phi(p,\hat{\theta})\hat{\xi}$ and, as a result,

$$\begin{aligned}u = u_c + u_r &= C_p z + D_p e + \bar{\gamma}[0 \;\; I]\Phi(p,\hat{\theta})\hat{\xi}\\ &= C_p z + D_p c\tilde{x} + \bar{\gamma}[0 \;\; I]\Phi(p,\hat{\theta})\hat{\xi}.\end{aligned}$$

Thus, we have

$$\begin{aligned}\dot{\tilde{x}} &= A\tilde{x} - b\gamma w + bC_p z + bD_p c\tilde{x} + b\bar{\gamma}[0 \;\; I]\Phi(p,\hat{\theta})\hat{\xi}\\ &= (A + bD_p c)\tilde{x} + bC_p z + b\bar{\gamma}[0 \;\; I]\Phi(p,\hat{\theta})\hat{\xi} - b\gamma w,\\ e &= c\tilde{x}.\end{aligned}\tag{4.2.10}$$

Let $\alpha_1, \alpha_2, \cdots, \alpha_{2m}$ be the coefficients of the characteristic polynomial of S, i.e., $\det(sI - S) = s^{2m+1} + \alpha_1 s^{2m} + \alpha_2 s^{2m-1} + \cdots + \alpha_{2m} s$. Then, it is easily seen that $\alpha_1, \alpha_2, \cdots, \alpha_{2m}$ are given by

$$\alpha_{2i-1} = 0, \quad \alpha_{2i} = \sum_{j_1<j_2<\cdots<j_i} \sigma_{j_1}^2 \sigma_{j_2}^2 \cdots \sigma_{j_i}^2,$$

where $i = 1, 2, \cdots, m$ and $j_1, j_2, \cdots, j_i \in \{1, 2, \cdots, m\}$. Let

$$T_e := \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ \alpha_1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{2m-1} & \alpha_{2m-2} & \cdots & 1 & 0 \\ \alpha_{2m} & \alpha_{2m-1} & \cdots & \alpha_1 & 1 \end{bmatrix} \begin{bmatrix} \gamma \\ \gamma S \\ \vdots \\ \gamma S^{2m-1} \\ \gamma S^{2m} \end{bmatrix}.$$

Then, T_e is nonsingular for any $\sigma_1, \sigma_2, \cdots, \sigma_m$ because of Assumption 4.2.3. Using the state transformation $\bar{w} := T_e w$, the exosystem (4.2.1) and $u_w = \gamma w$ are transformed into an observable canonical form

$$\begin{aligned} \dot{\bar{w}} &= \bar{S}(\theta)\bar{w}, \\ u_w &= \bar{\gamma}\bar{w}, \end{aligned} \tag{4.2.11}$$

where $\bar{S}(\theta)$ and $\bar{\gamma}$ are given by (4.2.3) with $\theta_1 = \alpha_2, \theta_2 = \alpha_4, \cdots, \theta_m = \alpha_{2m}$. Thus, combining (4.2.9) and (4.2.11), we obtain

$$\begin{aligned} \begin{bmatrix} \dot{\tilde{x}} \\ \dot{\bar{w}} \end{bmatrix} &= A_e(\theta) \begin{bmatrix} \tilde{x} \\ \bar{w} \end{bmatrix} + \bar{b}u, \\ e &= \bar{c} \begin{bmatrix} \tilde{x} \\ \bar{w} \end{bmatrix}. \end{aligned} \tag{4.2.12}$$

Now, we consider another state transformation $\xi := T(\theta) \begin{bmatrix} \tilde{x}^\top & \bar{w}^\top \end{bmatrix}^\top$. According to Lemma A.2.1 in the Appendix A.2, the matrix $T(\theta)$ is nonsingular because Assumption 4.2.2 holds and (A, c) is observable. Using the fact that $\det(sI - A_e) = \det(sI - A) \cdot \det(sI - \bar{S}) = s^{n+2m+1} + \bar{\alpha}_1 s^{n+2m} + \cdots + \bar{\alpha}_{n+2m}s$, the system (4.2.12) can be transformed into

$$\begin{aligned} \dot{\xi} &= \bar{A}\xi + \bar{b}u + \Psi(e, u)\theta, \\ e &= \bar{c}\xi. \end{aligned}$$

Let

$$\tilde{\xi} := \hat{\xi} - \xi, \qquad \tilde{\theta} := \hat{\theta} - \theta, \qquad \eta := \tilde{\xi} - \Xi\tilde{\theta}.$$

Then, we have

$$\begin{aligned}\dot{\tilde{\xi}} &= (\bar{A} - L\bar{c})\tilde{\xi} + \Xi\dot{\hat{\theta}} + \Psi(e,u)\tilde{\theta},\\ \dot{\tilde{\theta}} &= \dot{\hat{\theta}} = -K\Xi^\top \bar{c}^\top \bar{c}\tilde{\xi}\\ &= -K\Xi^\top \bar{c}^\top \bar{c}\Xi\tilde{\theta} - K\Xi^\top \bar{c}^\top \bar{c}\eta,\\ \dot{\eta} &= (\bar{A} - L\bar{c})\tilde{\xi} + \Xi\dot{\hat{\theta}} + \Psi(e,u)\tilde{\theta} - \dot{\Xi}\tilde{\theta} - \Xi\dot{\tilde{\theta}}\\ &= (\bar{A} - L\bar{c})\eta.\end{aligned} \tag{4.2.13}$$

Define $\Xi_i \in \mathbb{R}^{n+2m+1}$ by $\Xi = [\Xi_1 \;\; \Xi_2 \;\; \cdots \;\; \Xi_m]$, and let

$$N_i := [0_{2i\times n}^\top \;\; I_n \;\; 0_{(2(m-i)+1)\times n}^\top]^\top \qquad \text{and} \qquad \chi_i := \Xi_i - N_i\tilde{x}.$$

Then, since $\dot{\Xi}_i = (\bar{A} - L\bar{c})\Xi_i + (-a[i]e + b[i]u)$, we have

$$\begin{aligned}\dot{\chi}_i &= (\bar{A} - L\bar{c})(\chi_i + N_i\tilde{x}) + (-a[i]c\tilde{x} + b[i]u) - N_i\dot{\tilde{x}}\\ &= (\bar{A} - L\bar{c})\chi_i + \left((\bar{A} - L\bar{c})N_i - N_iA - a[i]c\right)\tilde{x} + (b[i] - N_ib)u + N_ib\bar{\gamma}\bar{w}\\ &= (\bar{A} - L\bar{c})\chi_i + N_ib\bar{w}_1,\end{aligned}$$

where the last equality follows from $(\bar{A}-L\bar{c})N_i - N_iA - a[i]c = 0$ and $b[i] - N_ib = 0$. Thus, $\chi_i(t)$ is bounded because $\bar{w}(t)$ is bounded and $\bar{A}-L\bar{c}$ is Hurwitz. Moreover,

$$\mu(t) := \Xi^\top(t)\bar{c}^\top$$

is also bounded since $\bar{c}\Xi_i = \bar{c}\chi_i + \bar{c}N_i\tilde{x} = \bar{c}\chi_i$. Furthermore, it follows from Assumption 4.2.1 and (4.2.11) that the scalar variable $\bar{w}_1(t)$ is sufficiently rich of order m [IS96, Definition 5.2.3], i.e., $\bar{w}_1(t)$ contains m distinct frequencies. Thus, by virtue of [IS96, Theorem 5.2.1], the vector $\Xi^\top(t)\bar{c}^\top$ is persistently exciting (PE), i.e., there exist $\kappa_1 > 0$, $\kappa_2 > 0$, and $T > 0$ such that

$$\kappa_1 I \le \int_t^{t+T} \Xi^\top(\tau)\bar{c}^\top \bar{c}\Xi(\tau)d\tau \le \kappa_2 I$$

for all $t \ge 0$. Hence, according to [Zha02, Lemma 1], the homogeneous part with $\tilde{\theta}$ in (4.2.13), $\dot{\tilde{\theta}} = -K\Xi^\top\bar{c}^\top\bar{c}\Xi\tilde{\theta}$, is exponentially stable. This implies that $\tilde{\theta}(t)$

converges exponentially to zero since so does $\eta(t)$ and $\Xi^\top(t)\bar{c}^\top = \mu(t)$ is bounded. Therefore, since $\bar{c}\tilde{\xi} = \bar{c}\eta + \bar{c}\Xi\tilde{\theta} = \bar{c}\eta + \mu^\top\tilde{\theta}$, it follows that $\bar{c}\hat{\xi}(t) - e(t)$ converges to zero, which implies $\lim_{t\to\infty} p(t) = 0$ from (4.2.7) and

$$\lim_{t\to\infty} \Phi\big(p(t), \hat{\theta}(t)\big)T(\theta) = I. \tag{4.2.14}$$

From (4.2.10), we have

$$\begin{aligned}
\dot{\tilde{x}} &= (A + bD_pc)\tilde{x} + bC_pz + b\bar{\gamma}\left([0\ I]\Phi(p,\hat{\theta})\hat{\xi} - \bar{w}\right) \\
&= (A + bD_pc)\tilde{x} + bC_pz + b\bar{\gamma}[0\ I]\left(\Phi(p,\hat{\theta})(\xi + \tilde{\xi}) - T^{-1}(\theta)\xi\right) \\
&= (A + bD_pc)\tilde{x} + bC_pz + b\bar{\gamma}[0\ I]\left(\left(\Phi(p,\hat{\theta}) - T^{-1}(\theta)\right)T(\theta)\begin{bmatrix}\tilde{x}\\ \bar{w}\end{bmatrix} + \Phi(p,\hat{\theta})\tilde{\xi}\right) \\
&= (A + bD_pc)\tilde{x} + bC_pz + b\bar{\gamma}[0\ I]\Bigg(\left(\Phi(p,\hat{\theta})T(\theta) - I\right)\begin{bmatrix}\tilde{x}\\ \bar{w}\end{bmatrix} \\
&\qquad + \Phi(p,\hat{\theta})\left(\eta + \sum_{i=1}^{m}(\chi_i + N_i\tilde{x})\tilde{\theta}_i\right)\Bigg), \\
\dot{z} &= B_pc\tilde{x} + A_pz,
\end{aligned}$$

which implies

$$\begin{bmatrix}\dot{\tilde{x}}\\ \dot{z}\end{bmatrix} = A_{cl}\begin{bmatrix}\tilde{x}\\ z\end{bmatrix} + M_1(t)\begin{bmatrix}\tilde{x}\\ z\end{bmatrix} + M_2(t), \tag{4.2.15}$$

where

$$M_1(t) := \begin{bmatrix} b\bar{\gamma}\begin{bmatrix}0 & I\end{bmatrix}\left(\left(\Phi(p,\hat{\theta})T(\theta) - I\right)\begin{bmatrix}I\\ 0\end{bmatrix} + \Phi(p,\hat{\theta})\sum_{i=1}^{m}(N_i\tilde{\theta}_i)\right) & 0 \\ 0 & 0\end{bmatrix},$$

$$M_2(t) := \begin{bmatrix} b\bar{\gamma}\begin{bmatrix}0 & I\end{bmatrix}\left(\left(\Phi(p,\hat{\theta})T(\theta) - I\right)\begin{bmatrix}0\\ I\end{bmatrix}\bar{w} + \Phi(p,\hat{\theta})\left(\eta + \sum_{i=1}^{m}(\chi_i\tilde{\theta}_i)\right)\right) \\ 0\end{bmatrix}.$$

It follows from (4.2.14) that

$$\lim_{t\to\infty} M_1(t) = 0 \qquad \text{and} \qquad \lim_{t\to\infty} M_2(t) = 0.$$

because $\eta, \tilde{\theta}_i$ converge exponentially to zero and $\hat{\theta}, \bar{w}, \chi_i$ are bounded. Moreover, by virtue of Assumption 4.1.1 (see [Kha01, Lemma 4.6 and Corollary 9.1]), the system (4.2.15) is input-to-state stable (ISS) with $M_2(t)$ viewed as the input. Therefore, $\lim_{t\to\infty} \tilde{x}(t) = 0$ and $\lim_{t\to\infty} z(t) = 0$, and as a result

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty} c\tilde{x}(t) = 0$$

and Ξ is bounded since $\chi_i = \Xi_i - N_i\tilde{x}$ and χ_i is bounded. Since $\eta = \tilde{\xi} - \Xi\tilde{\theta}$, it follows that $\lim_{t\to\infty} \tilde{\xi}(t) = 0$ and $\hat{\xi}$ $(= \tilde{\xi} + \xi)$ is bounded, and hence the proof is complete. □

If $T(\hat{\theta})$ is guaranteed to be nonsingular for any $\hat{\theta}$, then, instead of (4.2.7) and (4.2.8), a simpler form

$$u_r = \begin{bmatrix} 0_{1\times n} & 1 & 0_{1\times 2m} \end{bmatrix} T^{-1}(\hat{\theta})\hat{\xi} \tag{4.2.16}$$

can be used. The reason why (4.2.7) and (4.2.8) are used is to avoid the situation where $T(\hat{\theta})$ becomes singular[2] during the initial transient period in which $\hat{\theta}$ is not close to θ. Therefore, if the plant (4.1.1) has no zero dynamics, (4.2.2) always holds for any λ, which implies that $T(\hat{\theta})$ is nonsingular for any $\hat{\theta}$ by virtue of Lemma A.2.1 in the Appendix A.2 (see the proof of Lemma A.2.1), and thus, (4.2.16) is used instead of (4.2.7) and (4.2.8).

For the bumpless transfer, we propose to use, instead of $u = u_c + u_r$,

$$u = u_c + \rho(t)u_r, \tag{4.2.17}$$

where $\rho(t)$ is a switching function that has a value between 0 and 1. The following corollary establishes that the introduction of $\rho(t)$ does not affect adversely the boundedness property.

[2]This idea has been taken from [MT03].

Corollary 4.2.3. Under Assumptions 4.1.1 and 4.2.1–4.2.3, all the states of the closed-loop system composed of (4.1.1)–(4.1.3), (4.2.4)–(4.2.8), and (4.2.17) are bounded for any bounded function $\rho(t)$. $\Diamond$

Proof. Since the insertion of ρ does not affect $\tilde{\theta}$ and η dynamics, $\tilde{\theta}$ and η tend exponentially to zero, regardless of ρ. It is clear that Ξ, $\hat{\xi}$ and e are bounded if $\tilde{x}$ is bounded. Thus, it is enough to show that $\tilde{x}$ and z of (4.2.15) are bounded. To proceed further, we consider (4.2.15) with $\Phi(p, \hat{\theta})$ of $M_1(t)$ and $M_2(t)$ replaced with $\rho\Phi(p, \hat{\theta})$. Denote them by $\bar{M}_1(t)$ and $\bar{M}_2(t)$, respectively. Then, since $\lim_{t\to\infty} \bar{M}_1(t) = 0$ and $\bar{M}_2(t)$ is bounded for any bounded ρ, it is easy to see that $\tilde{x}$ and z are bounded. $\square$

4.3 Industrial Application: Optical Disc Drive (ODD) Systems

In this section, the application to an ODD is performed in order to verify the ability of the proposed controller to eliminate the periodic signal with an unknown bias, magnitude, phase, and frequency.

4.3.1 Introduction of ODD Systems

In general, the internal mechanism of optical disc drives (ODD) such as CD-ROM, DVD-ROM, or Blu-ray consists of an optical pick-up reading data recorded on a disc, a spindle motor for rotating the disc, and other mechanical elements to sustain them. Especially, the pick-up is made up of an objective lens, a fine actuator of a voice coil motor (VCM), and a coarse actuator of a step motor, which are briefly depicted in Figure 4.1.

In order to read the data on a track of disc, the optical spot produced by the objective lens needs to be on the disc track. For controlling the optical spot, the pick-up positioning system operates in two modes one after the other, which are the track seeking mode and the track following mode. In most cases, the track following mode for data retrieval is launched after completion of the seeking mode. These modes are controlled by cooperation of the fine actuator and the

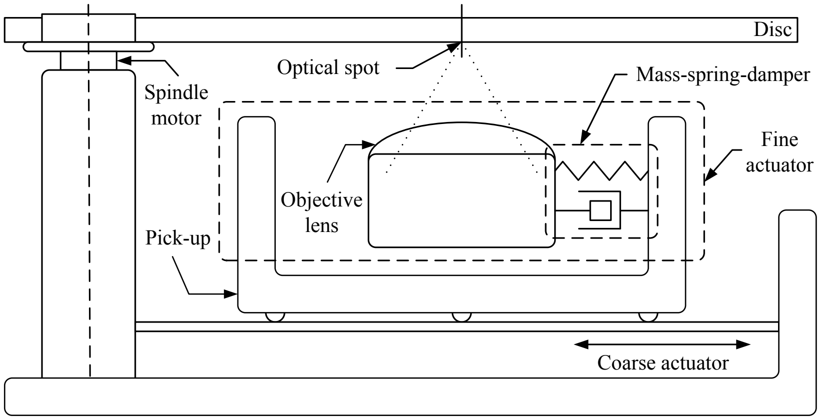

Figure 4.1: Diagram of optical disc drive (ODD) systems.

coarse actuator. While the coarse actuator moves slowly across the entire disc radius, the fine actuator mounted on top of the large coarse actuator has faster response for a small displacement.

As shown in Figure 4.2, the track following problem for ODD is to control the position of optical pick-up (more precisely, optical spot) so that it follows the desired track of optical disc media which is usually deviated from the concentric circles due to the disc eccentricity [OMI$^+$06, Kan95]. (The eccentricity means a distance between the center of the disc and its rotational axis as shown in Figure 4.2.) For CD-ROM drive, the optical spot must follow the track within 0.1μm while the displacement error amounts to more than 280μm in the worst case. (For DVD-ROM drive and Blu-ray drive, the allowable tolerance are 0.074μm and 0.032μm, respectively.) Although the disturbance is relatively large, the fine actuator should take care of it because the coarse actuator, which has a lower bandwidth (e.g. 35Hz in [LY08]) than the disturbance signal, is suitable only for a large range of movement. Therefore, only the fine actuator plays the central role in the track following mode.

The fine actuator can be modeled by a mass-spring-damper system and the transfer function can be described as [Kim05, LRLS12]

$$\frac{Y(s)}{V(s)} = \frac{1}{R_m + L_m s} \cdot \frac{\mu_0 \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \tag{4.3.1}$$

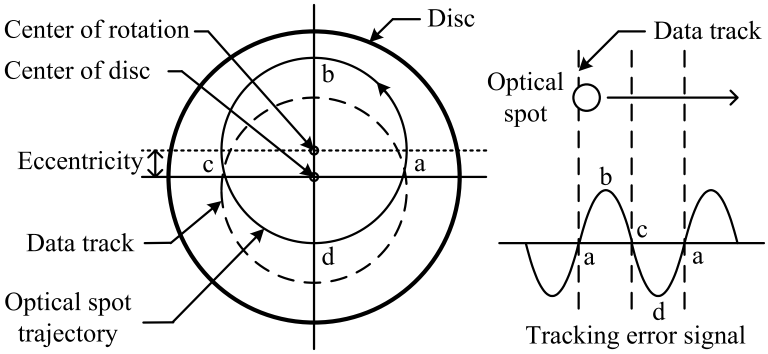

Figure 4.2: Disc eccentricity and tracking error signal.

where R_m and L_m are resistance and inductance of voice coil, respectively, ω_n, ζ, and μ_0 are some positive constants, $Y(s)$ represents the position of the pick-up, and $V(s)$ is the input voltage. Since L_m is sufficiently small compared to R_m, and $1/(R_m + L_m s)$ is stable, (4.3.1) can be approximated as (with $\mu := \mu_0/R_m$)

$$\frac{Y(s)}{V(s)} = \frac{\mu\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

Based on this form, a model of MAX ×52 CD-ROM drive (manufactured by LG Electronics Co.) can be obtained experimentally as follows:

$$G(s) := \frac{Y(s)}{V(s)} = \frac{818.22}{s^2 + 64.73s + 166800}(m/V), \tag{4.3.2}$$

which can be realized as

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -a_1 & 1 \\ -a_2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ b_2 \end{bmatrix} v, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x, \end{aligned} \tag{4.3.3}$$

where $a_1 = 64.73$, $a_2 = 166800$, and $b_2 = 818.22$.

The track following control system usually undergoes some periodic signal fluctuation that mainly comes from an eccentricity of a disc. The eccentricity disturbance can be regarded as a biased sinusoidal signal with unknown bias, magnitude, and phase, which can be represented by

$$d(t) = A_0 + A_1 \sin(\sigma_1 t + \phi_1), \tag{4.3.4}$$

where A_0, A_1, and ϕ_1 are some constants, and σ_1 is the frequency that depends on the rotation speed of the disc. The ODD system does not measure the absolute position of the pick-up (i.e., y) but measures the tracking error

$$e = K_{opt}(y - d),$$

where K_{opt} is a sensor gain that converts the position displacement into voltage. (For the model of MAX $\times 52$ CD-ROM drive, $K_{opt} = 1.25 \times 10^6 V/m$.) With $w := [w_1 \;\; w_2 \;\; w_3]^\top$, the tracking error can be expressed as

$$\begin{aligned} e &= \begin{bmatrix} K_{opt} & 0 \end{bmatrix} x + \begin{bmatrix} -K_{opt} & 0 & -K_{opt} \end{bmatrix} w, \\ \dot{w} &= \begin{bmatrix} 0 & \sigma_1 & 0 \\ -\sigma_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} w, \end{aligned} \tag{4.3.5}$$

where the initial condition $w(0)$ is related to A_0, A_1, and ϕ_1.

Now we assume that a controller $C_{pr}(s)$ has already been designed and kept in operation, in such a way that the closed-loop system is stable and achieves a good nominal performance. The nominal performance is usually given in terms of settling time, bandwidth, phase margin, and so on. Taking into consideration such requirements, we suppose that the following lead-lag compensator, for example, is designed for the ODD system (4.3.2)

$$C_{pr}(s) = \frac{V(s)}{E(s)} = -\frac{0.4178s^2 + 1316s + 188000}{s^2 + 41860s + 3134000}.$$

Although the controller $C_{pr}(s)$ may attain a certain level of disturbance attenuation, its design becomes more difficult as the range of disturbance frequency becomes wider. Such a situation may happen in a constant linear velocity (CLV) or a zoned constant linear velocity (ZCLV) operating mode, where the rotational

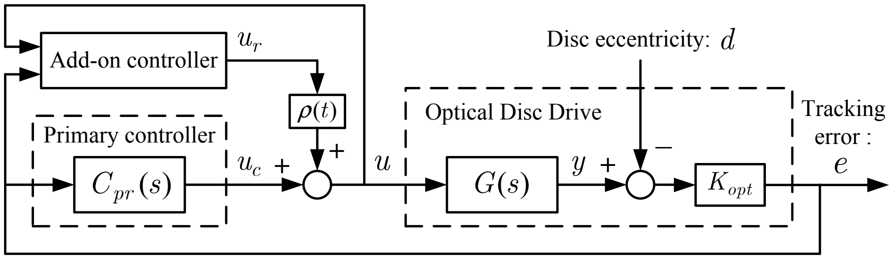

Figure 4.3: Add-on controller for ODD system. The role of $\rho(t)$ is discussed in Remark 4.1.1.

frequency increases as the optical pickup moves from the outer track of the optical disc to its inner track. In fact, the rotational frequency under the $\times 2$ CLV mode has values ranging from 6 Hz (outermost) to 16 Hz (innermost), which directly changes the frequency of the eccentricity disturbance. In such a case, it would be very helpful if we are able to design an additional controller that can be combined with the primary controller (as shown in Figure 4.3) in order to effectively remove the disturbance. In addition, by making the add-on controller contain the internal model of the disturbance, perfect cancellation of the disturbance becomes possible.

4.3.2 Simulation Results

Here, the simulations are performed in Matlab/Simulink with ODE15s and the maximum step size was 1×10^{-3}. For this, we note that the ODD system (4.3.3) and (4.3.5) satisfies all the Assumptions 4.1.1 – 4.1.3 and 4.2.1 – 4.2.3 because both the controllability and the observability matrices for (4.3.3) are nonsingular (Assumption 4.1.1), the equation (4.1.4) is satisfied with

$$\Pi = \begin{bmatrix} 1 & 0 & 1 \\ a_1 & \sigma_1 & a_1 \end{bmatrix}, \quad \gamma = \frac{1}{b_2}\left[a_2 - \sigma_1^2, a_1\sigma_1, a_2\right]$$

(Assumption 4.1.2), the matrix pair

$$\left(\text{blockdiag}\left(\begin{bmatrix} -a_1 & 1 \\ -a_2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{bmatrix}, 0_{1\times 1}\right), \left[K_{opt} \quad 0 \quad -K_{opt} \quad 0 \quad -K_{opt}\right]\right)$$

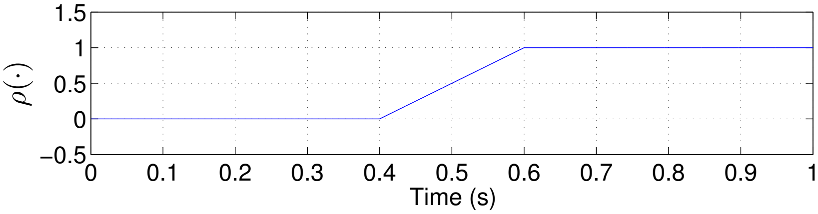

Figure 4.4: Switching function.

Table 4.1: Design parameters of the add-on controller applied to the ODD systems.

Parameter	Value
Observer gain L	$\left[199, 1.8\times10^5, 4.51\times10^7, 2.74\times10^{10}, 2.69\times10^{11}\right]^\top$
Adaptive gain K	2.0×10^9

is detectable (Assumption 4.1.3), the constant A_1 is non-zero in (4.3.4) (Assumption 4.2.1), there is no zero in (4.3.2) (Assumption 4.2.2), and the assumption of Lemma 4.2.1 holds (Assumption 4.2.3).

Since the ODD system (4.3.3) has no zero dynamics, Theorem 4.2.2 with (4.2.7) and (4.2.8) replaced by (4.2.16) can be applied. For simulation purpose, the unknown disc rotation frequency and the initial value of the frequency estimate are assumed 68Hz and 60Hz, respectively, which implies that $\sigma_1 = 2\pi\cdot 68$ rad/s and the initial condition of $\hat{\theta}_1$ $(=: \hat{\sigma}_1^2)$ in (4.2.5) is $(2\pi\cdot 60)^2$ rad/s. The design parameters for the proposed controller are given in Table 4.1, and the switching function $\rho(t)$ is chosen as in Figure 4.4. The simulation results are shown in Figure 4.5. It is clear that the tracking error converges to zero after 0.6 seconds and the frequency estimate ($\hat{\sigma}_1$) approaches the true value after 0.15 seconds. In order to emphasize the importance of selecting $\rho(t)$ as in Figure 4.4, we also carry out simulation without switching function (i.e., $\rho(t)\equiv 1$ for all $t\geq 0$). Comparison between with and without switching function is depicted in Figure 4.6. It is seen that, in the absence of the switching function, an overshoot is observed while it could be avoided by utilizing the switching function. In addition, as shown in Figure 4.7, we carry out a simulation with measurement noises in order to reflect the actual situation. The measured signals e and u are disturbed by Gaussian

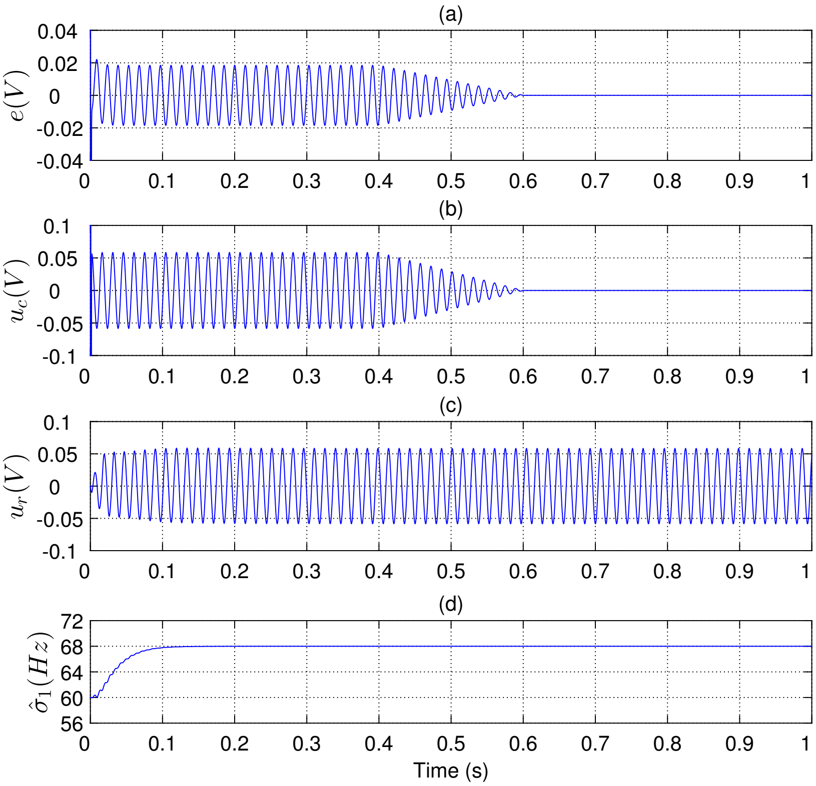

Figure 4.5: Simulation results. (a) Tracking error e. (b) Output of the primary controller $C_{pr}(s)$. (c) Output of the add-on adaptive regulator. (d) Frequency estimation.

noise whose the variance is 5×10^{-8}. Due to the measurement noises, the estimated frequency randomly oscillates around 68Hz instead of tending to 68Hz.

The comparison of stability margins may clarify the role of the add-on controller more clearly. However, by the introduction of the adaptive algorithm as in (4.2.5), the controller became a nonlinear system that does not admit transfer functions (and so, considering phase margins is not an easy task). Instead, we plot the frequency response of the sensitivity function without the adaptive algorithm (that is, with known frequencies) in Figure 4.8. It describes the behavior of the proposed controller after the frequency estimation is completed. In Figure 4.8, the sensitivity function without the add-on controller is also depicted. It is seen that the add-on controller achieves perfect rejection against the disturbance with disc rotation frequency while two sensitivity functions are not very different

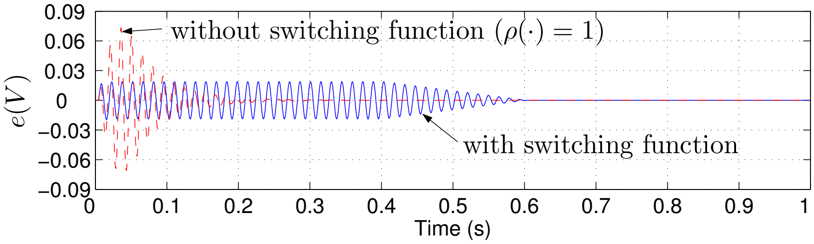

Figure 4.6: Comparison between with and without switching function.

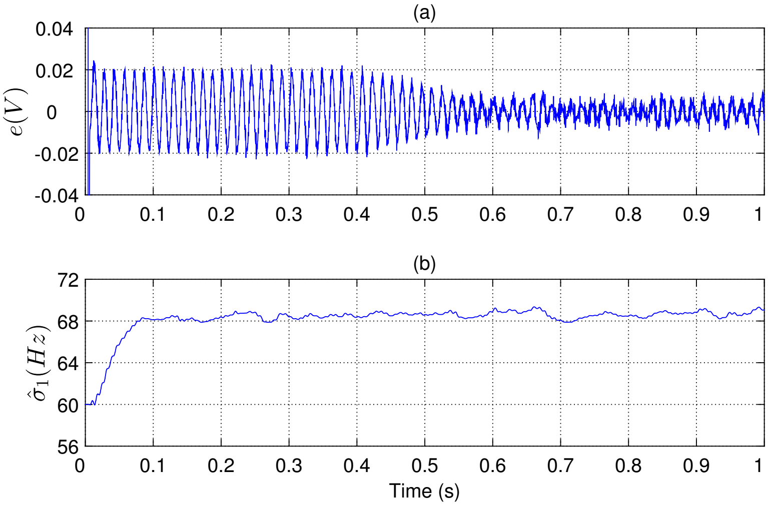

Figure 4.7: Simulation results with measurement noises. (a) Tracking error e. (b) Frequency estimation.

except at those disturbance frequencies.

Now, we perform some simulations to compare the proposed adaptive regulator with repetitive control and adaptive feedforward cancellation (AFC) introduced in Chapter 2. In order to handle the disturbances with uncertain periods, the repetitive control is equipped with 'multiple memory loops' in [Ste02]. The simulation results, depicted in Figures 4.9.(a) and 4.9.(b), show that, when the fundamental frequency of the disturbance is different from its nominal value, the regulation performance is not good. Although it can be improved by adding more memory loops, its performance is still far from perfect regulation. On the other hand, our method does not have any limit so that the actual frequency can be much different from its nominal value (see Figure 4.9.(c)). Although the proposed

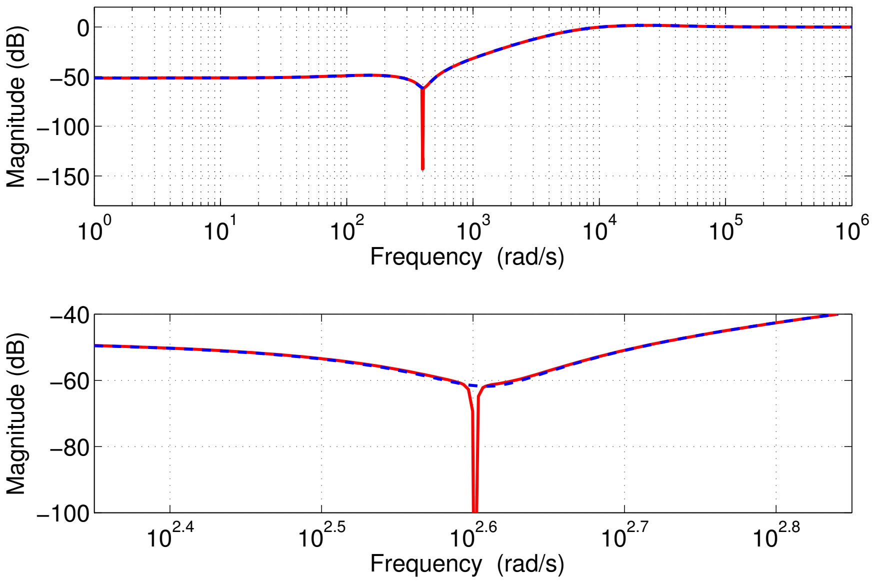

Figure 4.8: Bode magnitude plot of sensitivity function without add-on controller (blue dashed) and with add-on controller (red solid). The bottom one is the enlarged version of the top.

controller requires a longer settling time than the repetitive control, it is trivial because our approach requires the frequency estimation time of about 0.5s, as shown in Figure 4.5. The adaptive feedforward cancellation proposed in [BD97] requires a computation of plant gains within a limited frequency range of interest. (It is impractical to compute plant gains at all disturbance frequencies.) As a result, the regulation performance can be very poor when the frequency of disturbance does not lie in the range. To verify this, the simulation is carried out for the case where the uncertain frequency is out of the range. It is seen from Figure 4.10 that the performance of the proposed adaptive regulator is much superior to AFC of [BD97].

Finally, although the stability analysis was performed for disturbances with constant frequencies, the disturbance rejection ability is tested for slowly varying frequencies. Figure 4.11 shows the simulation results, where it is seen that the proposed scheme works quite well even for disturbances with slowly varying frequencies.

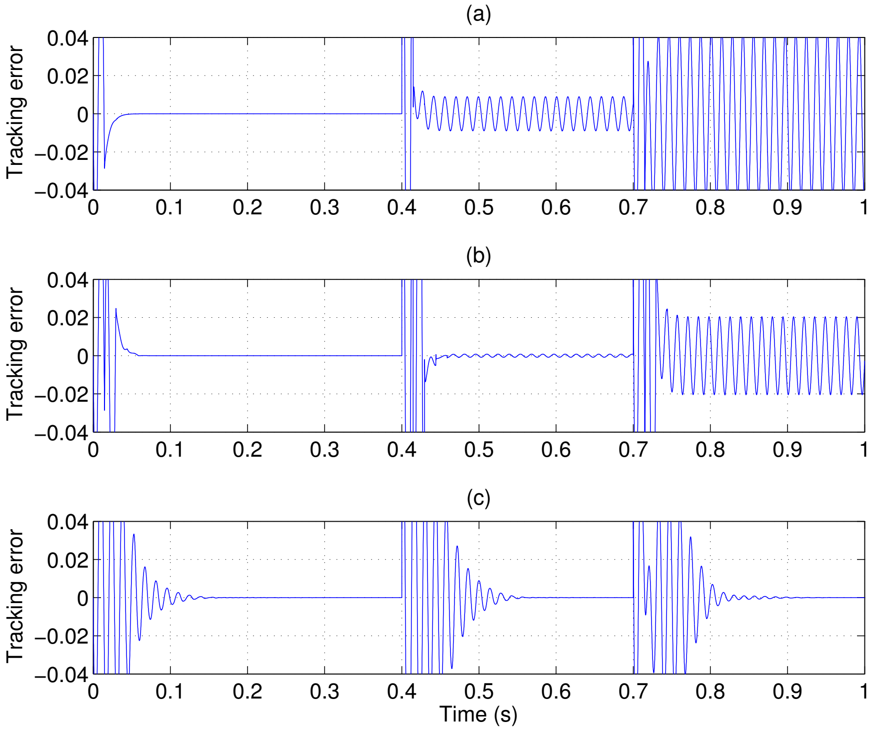

Figure 4.9: Comparison of disturbance rejection performance (disturbance frequency: 68Hz (0.0–0.4s), 69Hz (0.4–0.7s), 73Hz (0.7–1.0s)). (a) Standard repetitive control whose nominal frequency is set for 68Hz (it has only one memory loop). (b) Repetitive control (nominal frequency is set to 68Hz) having two memory loops [Ste02]. (c) The adaptive add-on controller proposed in this chapter (which does not require nominal frequency setting).

4.3.3 Experimental Results

The proposed adaptive add-on regulator has been implemented for a commercial high-speed (MAX ×52) CD-ROM disc drive (manufactured by LG Electronics Co.) with a TMS320C6701 32bit floating-point DSP (manufactured by TI Co.), as shown in Figure 4.12. The features of the analog to digital (A/D) and digital to analog (D/A) converter used in the experiment are given in Table 4.2. The configuration of the experimental setup is shown in Figure 4.13, which is the same as in Figure 4.3 except that low-pass filters (LPF) are placed in front of the A/D converter because the measured tracking error e and control input u contains high frequency noise. Here, the features of the operational amplifier (Op Amp, LM6172

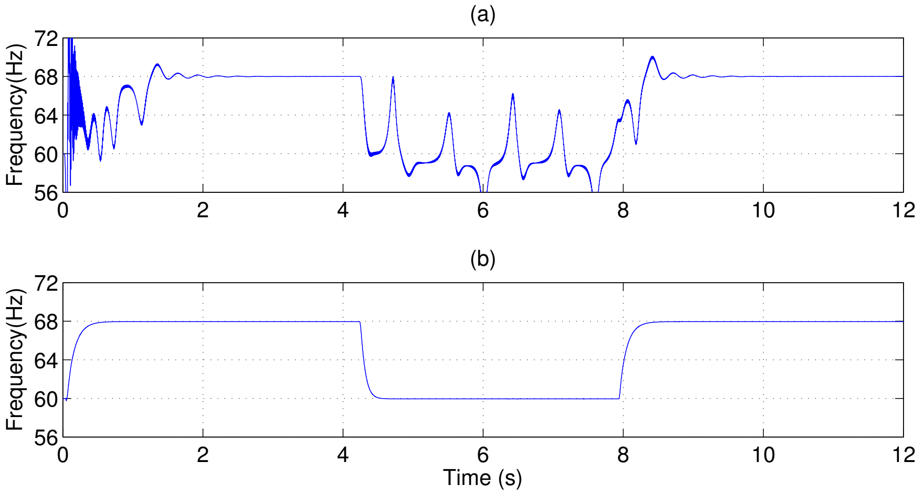

Figure 4.10: Comparison of the frequency estimation (disturbance frequency: 68Hz (0–4s), 60Hz (4–8s), 68Hz (8–12s)). (a) The adaptive feed-forward cancellation (AFC) of [BD97]. (b) The proposed add-on controller.

manufactured by TI Co.) used in the low-pass filters and the signal sum board is given in Table 4.3. The pseudo code implemented on the DSP board is listed in Appendix A.3, which is executed every sampling time. The disturbance rejection performance is evaluated using an audio disc with about $\pm 150\mu m$ eccentricity. This amount of eccentricity has prohibited the CD-ROM drive under experiment from playing audio continuously. Then, by introducing the proposed controller, the drive could play the audio without any interruption. Figure 4.14 shows the signals measured from the experimental set, in which it is observed that the error

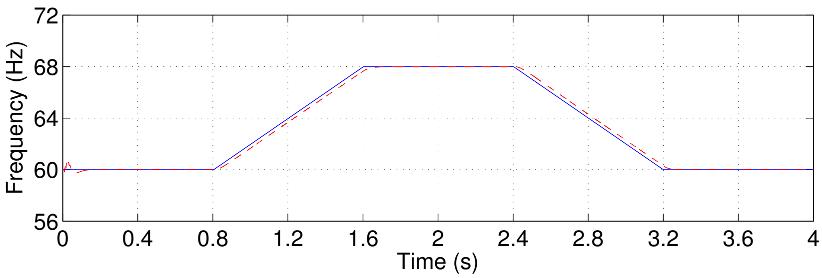

Figure 4.11: Simulation result for slowly varying frequency. Actual frequency (blue solid) and frequency estimation (red dashed).

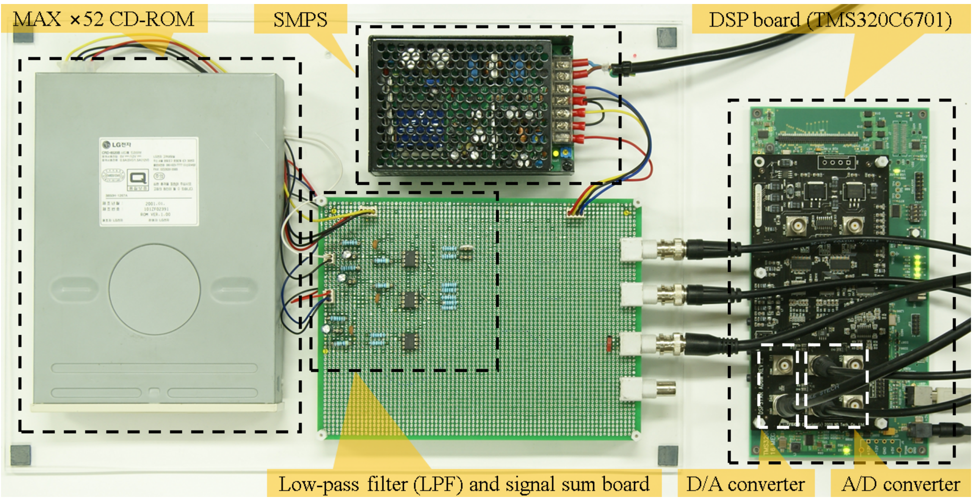

Figure 4.12: Photograph of the experimental setup of ODD systems.

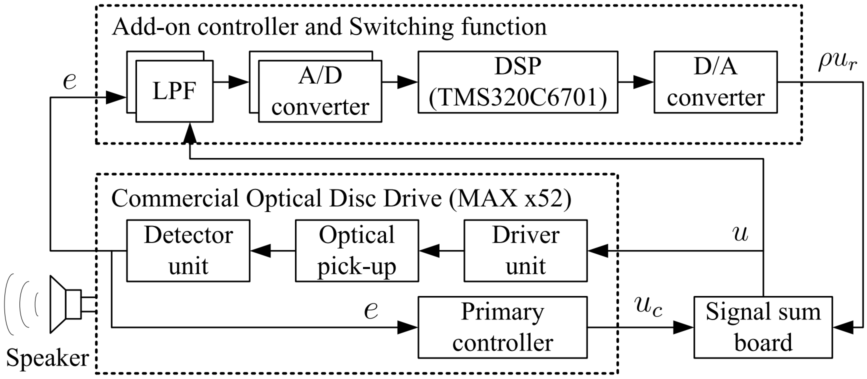

Figure 4.13: Schematic diagram of the experimental setup.

has been reduced by half. In contrast to the simulation results, the eccentricity disturbance is not perfectly removed, which might result from the measurement noise, the low-pass filter, and the quantization error. Figures 4.15 and 4.16 show the fast Fourier transform (FFT) of the tracking error and the histogram of the sampled error data, respectively, which indicate that disturbance rejection performance is enhanced.

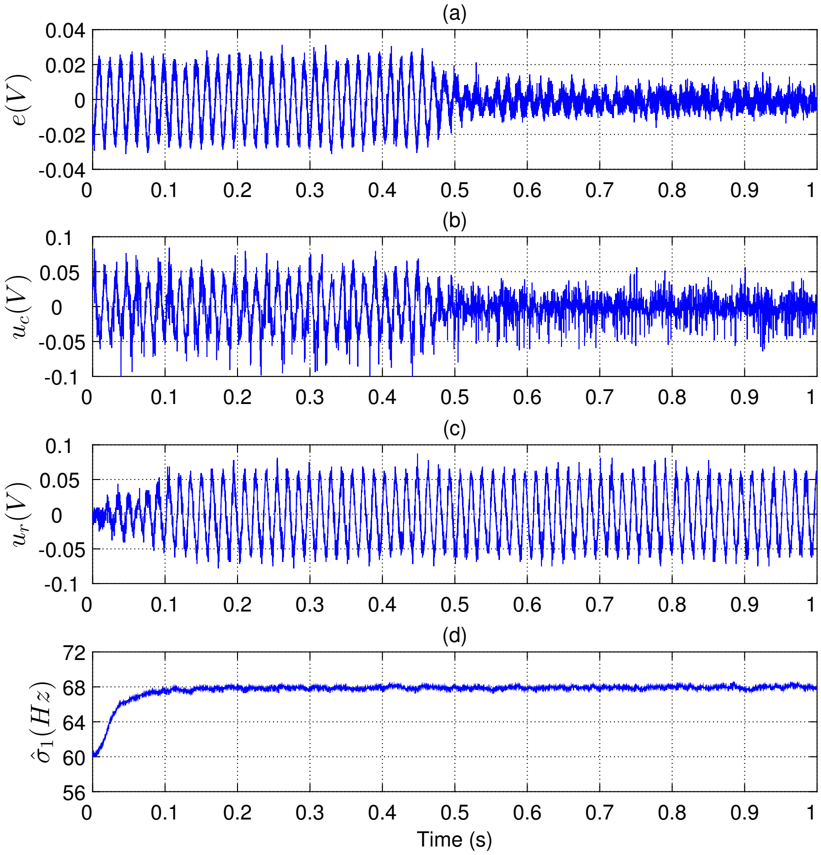

Figure 4.14: Experimental results. (a) Tracking error e. (b) Output of the primary controller $C_{pr}(s)$. (c) Output of the add-on adaptive regulator. (d) Frequency estimation.

Table 4.2: Specifications of A/D and D/A converter.

Parameter	Value	Unit
Sample rate	88.2	kHz
Resolution	16	Bits
Input range	± 2.5	V

Table 4.3: Specifications of operational amplifier (Op Amp).

Parameter	Value	Unit
Operation voltage	±15	V
Slew rate	3000	V/μs
Unit gain band width	100	MHz
Common mode rejection ratio (CMRR)	110	dB

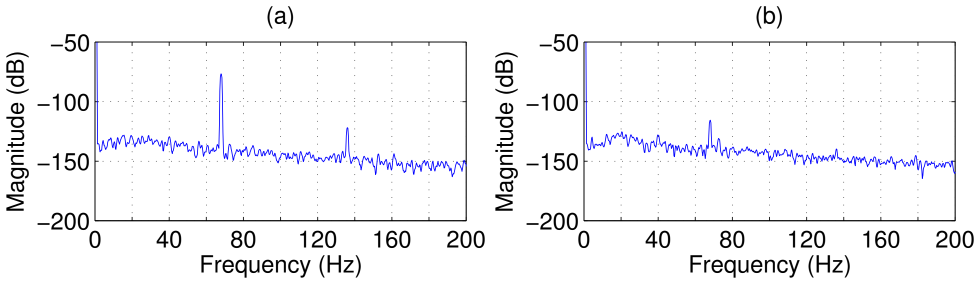

Figure 4.15: FFT of the tracking error. (a) Without adaptive add-on regulator ($\rho = 0$). (b) With adaptive add-on regulator ($\rho = 1$).

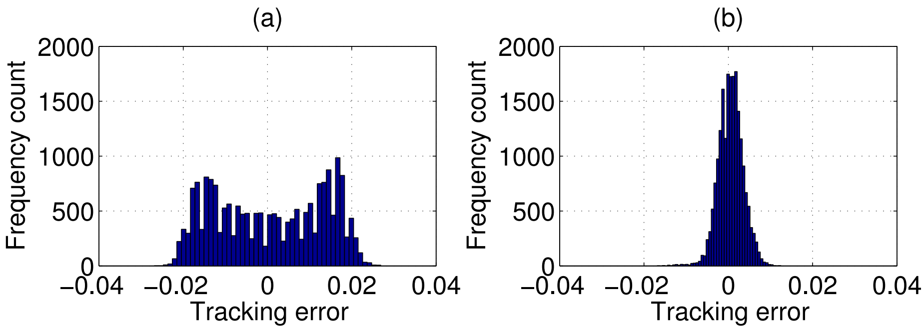

Figure 4.16: Histogram of the tracking error. (a) Without adaptive add-on regulator ($\rho = 0$). (b) With adaptive add-on regulator ($\rho = 1$).

Chapter 5

Adaptive Output Regulator for Unknown Number of Unknown Sinusoidal Exogenous Inputs

In Chapter 4, we have proposed an adaptive regulator that achieves output regulation when the frequencies of the exogenous inputs (or the disturbances) are unknown but the number of them is known. Under this condition, solutions are given in terms of adaptive internal model in [MT03, KKCS05, KKCT11, KSJ14, SIM01, Din06, Din07, DL10, BD97, BZ04, Nik98, Nik01, FF13, BK13, BK14].

On the other hand, this chapter mainly focuses on the case where the number of unknown frequencies is also unknown (in addition to unknown magnitude, phase, bias, and frequency). This problem has been studied by Marino, Tomei, and Santosuosso in [MT07, MS07, MT11, MT13a], and some important findings are made. In particular, for the minimum phase linear systems, [MT07] proposed an elegant solution, in which the singularity problem (to be discussed) does not occur thanks to the minimum phase property of the plant, and even uncertainty of the plant model is treated in [MT11]. For the non-minimum phase systems, some progress has been made in [MS07] and [MT13a] for the plant that has no uncertainty. In [MS07], another adaptive algorithm that can estimate the number of unknown frequencies is proposed with a remedy to avoid the singularity problem (which is however effective only for the initial transient). An improved treatment for the singularity problem has appeared in [MT13a] where the case when the actual number of unknown frequencies exceeds its presumed upper bound is also analyzed. However, the unknown frequencies are assumed to have their lower bounds and the information of the lower bounds are used in the design in [MT13a].

Motivated by the progress made in [MT07, MS07, MT13a], a *closed-form* output regulator is presented in this chapter for the plant that has hyperbolic zero dynamics (i.e., there is no zero of the plant on the imaginary axis of the complex plane), under the assumptions that the plant have no parametric uncertainty and that the upper bound on the number of unknown frequencies is known. A few more contributions are made in this chapter.

- First, while a dead-zone function is used to prevent the singularity problem (which is already done in [MT13a]), we present a formula to compute a suitable value of the width of dead-zone. This is an important contribution (over [MT13a]), because, if the width of the dead-zone is unnecessarily large, then complete reference tracking and disturbance rejection are not possible in the steady-state.

- Moreover, we remove the observability assumption of (S, γ) (to be seen shortly) that has been used in [KSJ14, MT03, MT07, MS07, MT13a, Din06, Din07, DL10, Nik98, Nik01, BK13]. We will discuss (in Remark 5.3.1) that assuming observability of (S, γ) *a priori* may not make sense due to uncertainty in S and γ, but this assumption is easily removed in our framework by *tightening* the exosystem in the design procedure.

- Finally, we employ different adaptive observer that has been proposed in [Zha02]. While it is intrinsically the same as those in [MT07, MS07, MT13a], it does not rely on the filtered transformation, which enabled the relatively simpler analysis performed in this chapter.

The chapter is organized as follows. The problem and the proposed solution (i.e., the controller) are presented in Section 5.1 and 5.2, respectively, where the reader who does not need the underlying proof can find everything. Section 5.3 provides constructive proof with more detailed explanation about the proposed adaptive controller. Finally, simulation results are found in Section 5.4.

5.1 Problem Formulation

We consider a general linear time-invariant (LTI) single-input-single-output (SISO) system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + bu(t) + Pw(t), \\ e(t) &= cx(t) + qw(t), \end{aligned} \tag{5.1.1}$$

where $x = [x_1, \cdots, x_n]^\top \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the control input, $e \in \mathbb{R}$ is the error output to be regulated to zero, and w is the exogenous input vector that yields the disturbance vector Pw to be rejected and the reference signal $-qw$ to be tracked by the plant output cx. We assume that the error signal e is measurable while x and w are not. It is also supposed that P and q, as well as w, are unknown, but A, b, and c are known. The pair (A, b) is stabilizable and (A, c) is detectable. The disturbance Pw and the reference $-qw$ consist of sinusoidal signals, but their bias, magnitudes, phases, and frequencies are all unknown. Moreover, the number of unknown frequencies is also unknown. In order to concisely describe the situation, we suppose that there is a generator (which is called *exosystem*) of the vector $w \in \mathbb{R}^{2r+1}$, written as

$$\dot{w}(t) = Sw(t), \tag{5.1.2}$$

where

$$S := \text{blockdiag}(\sigma_1 S_o, \sigma_2 S_o, \cdots, \sigma_r S_o, 0_{1\times1}), \quad S_o := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

so that, each pair of $[w_{2i-1}(t), w_{2i}(t)]^\top$ consists of sinusoidal function of frequency σ_i for $i = 1, \cdots, r$, and $w_{2r+1}(t)$ is a constant (which will generate unknown biases in Pw and $-qw$ with uncertain matrices P and q). We assume that the non-negative integer r and the distinct positive constants $\sigma_1, \sigma_2, \cdots, \sigma_r$ are unknown, and that the initial condition $w(0)$ is also unknown. Since r can be taken as any value, it is assumed, without loss of generality, that the initial condition $w(0)$ excites all oscillatory modes of the exosystem (5.1.2), i.e., $[w_{2i-1}(0), w_{2i}(0)]^\top \neq$

$[0, 0]^\top$ for all $i = 1, 2, \cdots, r$. From the structure of S in (5.1.2), r indicates the number of unknown frequencies $\sigma_1, \sigma_2, \cdots, \sigma_r$.

The problem considered in this chapter can be stated as follows. Given system (5.1.1) and exosystem (5.1.2), find a dynamic error feedback controller of the form

$$\begin{aligned} \dot{z}(t) &= f(t, z, u, e), \\ u(t) &= h(t, z), \end{aligned} \tag{5.1.3}$$

such that $\lim_{t\to\infty} e(t) = 0$ and all the states of the closed-loop system are bounded. To solve the problem, we pose some conditions on which the proposed controller is based.

Assumption 5.1.1. The zero dynamics of plant (5.1.1) is hyperbolic. ◊

Assumption 5.1.2. The upper bound of r (which is the number of unknown frequencies) is known, say, m. ◊

Remark 5.1.1. The idea behind Assumption 5.1.1 is to avoid the case where the eigenvalues of (5.1.2) coincide with the zeros of the plant (5.1.1) because, if it happens, then the so-called *regulator equations* (i.e., (3.2.1) in Chapter 3 or (5.3.2) to appear) may not have any solution. (See Theorem 3.2.3 and Remark 3.2.1 based on [Hua04] for more details.) Since the eigenvalues of (5.1.2) are not known but lie in the imaginary axis, Assumption 5.1.1 suffices for avoiding the pathological case. ◊

5.2 Adaptive Output Regulator

A solution to the problem stated in the previous section is given in terms of an adaptive error feedback regulator, which will be described below. For this, we perform the Kalman decomposition [Che99, Theorem 6.O6]) to the matrix pair (A, b, c) of the plant (5.1.1) so that

$$A \longleftrightarrow \begin{bmatrix} A_{\bar{o}} & A_{o\bar{o}} \\ 0 & A_o \end{bmatrix}, \qquad b \longleftrightarrow \begin{bmatrix} b_{\bar{o}} \\ b_o \end{bmatrix}, \qquad c \longleftrightarrow \begin{bmatrix} 0 & c_o \end{bmatrix}$$

where $A_o \in \mathbb{R}^{\nu\times\nu}$ with $\nu \leq n$, $A_{\bar{o}}$ is Hurwitz, and the pair (A_o, c_o) is observable. Without loss of generality, it is supposed that

$$A_o := \begin{bmatrix} -a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{\nu-1} & 0 & \dots & 1 \\ -a_\nu & 0 & \cdots & 0 \end{bmatrix}, \quad b_o := \begin{bmatrix} b_1 \\ \vdots \\ b_{\nu-1} \\ b_\nu \end{bmatrix}, \quad c_o := \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^\top$$

with some constants a_i's and b_i's.

The proposed error feedback regulator (5.1.3) consists of two parts. The first part is an adaptive observer given by

$$\begin{aligned} \dot{\hat{\xi}} &= A_c\hat{\xi} + b_c u + \Psi(e,u)\hat{\theta} + L(e - c_c\hat{\xi}) + \Xi\dot{\hat{\theta}} \\ &= A_c\hat{\xi} + b_c u + \Psi(e,u)\hat{\theta} + (\Xi\lambda_a\Xi^\top c_c^\top + L)(e - c_c\hat{\xi}) \\ &\quad - \Xi\lambda_a \cdot \mathrm{diag}\Big(\exp\left(-\det{}^2(\Omega_1)t\right), \cdots, \exp\left(-\det{}^2(\Omega_m)t\right) \Big) \cdot \hat{\theta}, \quad \hat{\xi} \in \mathbb{R}^{\nu+2m+1}, \end{aligned} \tag{5.2.1a}$$

$$\dot{\Xi} = (A_c - Lc_c)\Xi + \Psi(e,u), \quad \Xi \in \mathbb{R}^{(\nu+2m+1)\times m}, \tag{5.2.1b}$$

$$\begin{aligned} \dot{\hat{\theta}} &= \lambda_a\Xi^\top c_c^\top(e - c_c\hat{\xi}) \\ &\quad - \lambda_a \cdot \mathrm{diag}\Big(\exp\left(-\det{}^2(\Omega_1)t\right), \cdots, \exp\left(-\det{}^2(\Omega_m)t\right) \Big) \cdot \hat{\theta}, \quad \hat{\theta} \in \mathbb{R}^m, \end{aligned} \tag{5.2.1c}$$

$$\dot{\Omega} = -\lambda_b\Omega + \lambda_c\Xi^\top c_c^\top c_c\Xi, \quad \Omega \in \mathbb{R}^{m\times m}, \tag{5.2.1d}$$

where

$$A_c := \begin{bmatrix} -a_1 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -a_\nu & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad b_c := \begin{bmatrix} b_1 \\ \vdots \\ b_\nu \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad c_c := \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix},$$

$$\Psi(e,u) := -\begin{bmatrix} a[1] & a[2] & \cdots & a[m] \end{bmatrix} e + \begin{bmatrix} b[1] & b[2] & \cdots & b[m] \end{bmatrix} u,$$

in which, $a[i] \in \mathbb{R}^{\nu+2m+1}$ and $b[i] \in \mathbb{R}^{\nu+2m+1}$, $i = 1, 2, \cdots, m$, are given by

$$a[i] := \begin{bmatrix} 0_{1\times(2i-1)} & 1 & a_1 & a_2 & \cdots & a_\nu & 0_{1\times(2m-2i+1)} \end{bmatrix}^\top,$$
$$b[i] := \begin{bmatrix} 0_{1\times(2i-1)} & 0 & b_1 & b_2 & \cdots & b_\nu & 0_{1\times(2m-2i+1)} \end{bmatrix}^\top,$$

the design parameters λ_a, λ_b, and λ_c are any positive numbers, $\det(\Omega_i)$ is the determinant of Ω_i where $\Omega_i := [I_i, 0_{i\times(m-i)}]\Omega[I_i, 0_{i\times(m-i)}]^\top \in \mathbb{R}^{i\times i}$ for $i = 1, \cdots, m$, the observer gain $L \in \mathbb{R}^{(\nu+2m+1)\times 1}$ is chosen such that $A_c - Lc_c$ is Hurwitz, and the initial condition $\Omega(0)$ is any positive definite symmetric matrix.

The second part is a feedback control law written as

$$u = \begin{bmatrix} K & \bar{\gamma} \end{bmatrix} \frac{\mathrm{adj}\big(T_c(\hat{\theta})\big) \cdot \det\big(T_c(\hat{\theta})\big)}{\delta + \bar{d}\left(\det^2\big(T_c(\hat{\theta})\big)\right)} \hat{\xi} =: \begin{bmatrix} K & \bar{\gamma} \end{bmatrix} \mathcal{T}_c(\hat{\theta})\hat{\xi}, \tag{5.2.2}$$

where $\bar{\gamma} = [1, 0, \cdots, 0] \in \mathbb{R}^{1\times(2m+1)}$, $K \in \mathbb{R}^{1\times\nu}$ is a matrix such that $A_o + b_o K$ is Hurwitz[1], $T_c(\hat{\theta}) \in \mathbb{R}^{(\nu+2m+1)\times(\nu+2m+1)}$ has the form of Sylvester matrix [Che84] given by

$$T_c(\hat{\theta}) := \begin{bmatrix} \hat{\theta}[1] & \hat{\theta}[2] & \cdots & \hat{\theta}[\nu] & \bar{b}[1] & \bar{b}[2] & \cdots & \bar{b}[2m+1] \end{bmatrix}, \tag{5.2.3}$$

in which, with $\hat{\theta} = [\hat{\theta}_1, \cdots, \hat{\theta}_m]^\top$, $\hat{\theta}[i] \in \mathbb{R}^{\nu+2m+1}$ and $\bar{b}[j] \in \mathbb{R}^{\nu+2m+1}$ are defined as

$$\hat{\theta}[i] := \begin{bmatrix} 0_{1\times(i-1)} & 1 & 0 & \hat{\theta}_1 & 0 & \hat{\theta}_2 & \cdots & 0 & \hat{\theta}_m & 0_{1\times(\nu-i+1)} \end{bmatrix}^\top,$$
$$\bar{b}[j] := \begin{bmatrix} 0_{1\times j} & -b_1 & -b_2 & \cdots & -b_\nu & 0_{1\times(2m-j+1)} \end{bmatrix}^\top,$$

the symbol $\mathrm{adj}(T_c(\hat{\theta}))$ implies adjoint of $T_c(\hat{\theta})$, and $\bar{d}(\cdot)$ is a dead-zone function

[1] Because the pair (A, b) is stabilizable, it is clear from the Kalman decomposition that the selection of K is always possible.

defined as

$$\bar{d}(v) := \begin{cases} v - \delta \cdot \mathrm{sgn}(v), & \text{if } |v| \geq \delta, \\ 0, & \text{if } |v| < \delta, \end{cases} \tag{5.2.4}$$

where δ is a dead-band parameter given by

$$\delta := b_\rho^{4m+2} \prod_{k=1}^{\nu-\rho} \Big(\mathrm{Re}(\zeta_k)\Big)^{4m+2}, \tag{5.2.5}$$

in which, ρ and $\zeta_1, \cdots, \zeta_{\nu-\rho}$ are the relative degree and the zeros of the transfer function $c(sI-A)^{-1}b = c_o(sI-A_o)^{-1}b_o$, respectively, and $\mathrm{Re}(\zeta_k)$ is the real part of ζ_k.

Now, we state the main result of this chapter, whose constructive proof is given in the next section with detailed explanation about the proposed controller.

Theorem 5.2.1. Under Assumptions 5.1.1 and 5.1.2, the error feedback controller (5.2.1) and (5.2.2) guarantees that $\lim_{t\to\infty} e(t) = 0$ and all the states of the closed-loop system composed of (5.1.1), (5.1.2), (5.2.1), and (5.2.2) are bounded. ◊

5.3 Constructive Proof of Theorem 5.2.1

The proof is composed of a few parts. The first part is to show that the plant (5.1.1) and the exosystem (5.1.2) are transformed into the standard form for the adaptive observer in [Zha02], which also explains the equations (5.2.1a), (5.2.1b), and the first term of (5.2.1c). Then, we will briefly discuss the role of the second term in (5.2.1c) and the equation (5.2.1d), which are taken from [MS07]. With them, it will become clear why the feedback control law has the form of (5.2.2), which is followed by a justification of the choice of δ in (5.2.5). Putting all together, we finally prove the Theorem 5.2.1.

Let us begin with a direct consequence of Assumption 5.1.1, that is,

$$\mathrm{rank}\left(\begin{bmatrix} A-\lambda I & b \\ c & 0 \end{bmatrix}\right) = n+1, \quad {}^\forall\lambda \in \{0, \pm j\sigma_1, \pm j\sigma_2, \cdots, \pm j\sigma_r\}, \tag{5.3.1}$$

(see, e.g., Remark 3.2.1 or [KIF93, Lemma 1.5.1]) where σ_i's are the frequencies of (5.1.2), which are unknown. By virtue of (5.3.1), it follows from Theorem 3.2.3 (or [Hua04, Theorem 1.9]) that there exist matrices $\Pi \in \mathbb{R}^{n\times(2r+1)}$ and $\gamma \in \mathbb{R}^{1\times(2r+1)}$ which are the unique solution to the regulator equations

$$\begin{aligned} \Pi S &= A\Pi + b\gamma + P, \\ 0 &= c\Pi + q, \end{aligned} \tag{5.3.2}$$

where the solution Π and γ are unknown since S, P, and q are unknown.

From (5.1.1), (5.1.2), and (5.3.2), defining $\tilde{x} := x - \Pi w$ and $u_w := \gamma w$ yields

$$\begin{aligned} \dot{\tilde{x}} &= A\tilde{x} + b(u - u_w), \\ e &= c\tilde{x}. \end{aligned} \tag{5.3.3}$$

This system is equivalently written in the coordinates of observable canonical form as

$$\dot{\tilde{x}}_{\bar{o}} = A_{\bar{o}}\tilde{x}_{\bar{o}} + A_{o\bar{o}}\tilde{x}_o + b_{\bar{o}}(u - u_w), \tag{5.3.4a}$$

$$\dot{\tilde{x}}_o = A_o\tilde{x}_o + b_o(u - u_w), \tag{5.3.4b}$$

$$e = c_o\tilde{x}_o, \tag{5.3.4c}$$

where $\tilde{x}_{\bar{o}} \in \mathbb{R}^{n-\nu}$ and $\tilde{x}_o \in \mathbb{R}^{\nu}$ represent the unobservable and observable states of the system (5.3.3), respectively. Since $A_{\bar{o}}$ is Hurwitz and $u_w(t) = \gamma w(t)$ are bounded (since S is neutrally stable), $\tilde{x}_{\bar{o}}(t)$ is also bounded if $u(t)$ and $\tilde{x}_o(t)$ are bounded. So the proof is done if we show that the closed-loop system (5.1.2), (5.2.1), (5.2.2), and (5.3.4b) with $u_w = \gamma w$ has the property that the error in (5.3.4c) tends to zero as time goes to infinity and the states of (5.2.1) and (5.3.4b) are bounded.

Here we note that the pair (S, γ) may not be observable. Indeed, with γ_i being the i-th component of γ, it is seen from the structure of (5.1.2) that if (and only if) $[\gamma_{2k-1}, \gamma_{2k}]$ is a zero vector for some $k \in \{1, \cdots, r\}$, then u_w does not contain the sinusoid of the frequency σ_k and so the partial states w_{2k-1} and w_{2k} of w corresponding to the frequency σ_k become unobservable. (Proof of this claim is given in the Appendix A.4.) Let $l(\leq r)$ be the number of (unknown) frequencies observed in $u_w = \gamma w$ (i.e., $l := \mathrm{card}\left(\{k : [\gamma_{2k-1}, \gamma_{2k}] \neq 0, 1 \leq k \leq r\}\right)$. Then, we can replace the exosystem $u_w = \gamma w$ and $\dot{w} = Sw$ with its *tightened* version

$$\begin{aligned} u_w &= \tilde{\gamma}\tilde{w}, \\ \dot{\tilde{w}} &= \tilde{S}\tilde{w}, \end{aligned} \tag{5.3.5}$$

where $\tilde{w} \in \mathbb{R}^{2l+1}$ is the vector w with its unobservable components eliminated, and $\tilde{S} \in \mathbb{R}^{(2l+1)\times(2l+1)}$ and $\tilde{\gamma} \in \mathbb{R}^{1\times(2l+1)}$ are the matrix S and the vector γ whose components corresponding to the unobservable frequency are absent. Now the pair $(\tilde{S}, \tilde{\gamma})$ becomes observable.

Remark 5.3.1. This replacement, which we call *tightening*, is possible since the number of unknown frequency r is unknown, and so, replacing it with $l \leq r$ does not alter the design procedure in our framework. (It seems that the similar replacement could have been possible in [MT07, MS07, MT11, MT13a] because they also deal with unknown number of frequencies.) On the other hand, observability of (S, γ) is assumed in [MT03, Din06, Din07, DL10, Nik98, Nik01, BK13, BK14] where the number of unknown frequencies is fixed. However, it is noted that assuming observability of (S, γ) does not make much sense because σ_i's in S are unknown (and P and q are unknown as well in some references) so that γ, which is the solution of (5.3.2), becomes unknown. $\diamondsuit$

Now let $\theta := [\theta_1, \theta_2, \cdots, \theta_m]^\top \in \mathbb{R}^m$ where $\theta_1, \theta_2, \cdots, \theta_m$ are chosen such that

$$\begin{aligned} & s^{2(m-l)+1}\Pi_{1\leq j\leq r, [\gamma_{2j-1},\gamma_{2j}]\neq 0}(s^2 + \sigma_j^2) \\ &\quad = s^{2m+1} + \theta_1 s^{2m-1} + \theta_2 s^{2m-3} + \cdots + \theta_{m-1}s^3 + \theta_m s. \end{aligned} \tag{5.3.6}$$

Then, it follows that $\Pi_{1\leq j\leq r, [\gamma_{2j-1},\gamma_{2j}]\neq 0}(s^2 + \sigma_j^2) = s^{2l} + \theta_1 s^{2l-2} + \cdots \theta_{l-1}s^2 + \theta_l$ and that $\theta_j = 0$ for $l < j \leq m$. Since information about θ is equivalent to know the frequencies σ_j and the number l, the adaptive observer to be presented will estimate this vector θ. In order to construct the adaptive observer, we suppose that the exosystem in (5.3.5) is embedded into the m dimensional $\bar{w}$-dynamics as follows. Let

$$T_w(\theta) := \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \theta_{l-1} & 0 & \cdots & 0 & 0 \\ 0 & \theta_{l-1} & \cdots & 1 & 0 \\ \theta_l & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{\gamma} \\ \tilde{\gamma}\tilde{S} \\ \vdots \\ \gamma\tilde{S}^{2l-2} \\ \tilde{\gamma}\tilde{S}^{2l-1} \\ \tilde{\gamma}\tilde{S}^{2l} \end{bmatrix} \in \mathbb{R}^{(2l+1)\times(2l+1)},$$

$$\bar{T}_w := \begin{bmatrix} I_{2l+1} \\ 0_{(2m-2l)\times(2l+1)} \end{bmatrix} \in \mathbb{R}^{(2m+1)\times(2l+1)},$$

and let $\bar{w} := \bar{T}_w T_w(\theta)\tilde{w} \in \mathbb{R}^{2m+1}$. Then, it can be shown that

$$\dot{\bar{w}} = \bar{T}_w T_w(\theta)\dot{\tilde{w}} = \bar{T}_w T_w(\theta)\tilde{S}\tilde{w} = \bar{S}_m(\theta)\bar{T}_w T_w(\theta)\tilde{w} = \bar{S}_m(\theta)\bar{w}, \tag{5.3.7a}$$

$$u_w = \tilde{\gamma}\tilde{w} = \bar{\gamma}\bar{T}_w T_w(\theta)\tilde{w} = \bar{\gamma}\bar{w}, \tag{5.3.7b}$$

where

$$\bar{S}_m(\theta) := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ -\theta_1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ -\theta_m & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathbb{R}^{(2m+1)\times(2m+1)}, \quad \bar{\gamma} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}^\top \in \mathbb{R}^{1\times(2m+1)}$$

with the initial condition $\bar{w}(0) = \bar{T}_w T_w(\theta)\tilde{w}(0)$. Indeed, by the standard equivalence transformation to the observable canonical form [Che84, p. 329], or by the brief discussion in the Appendix A.5, it is seen that $T_w(\theta)\tilde{S}T_w^{-1}(\theta) = \bar{S}_l(\theta) \in \mathbb{R}^{(2l+1)\times(2l+1)}$ and $\tilde{\gamma}T_w^{-1}(\theta) = [1, 0, \cdots, 0] \in \mathbb{R}^{1\times(2l+1)}$. And, from the fact $\theta_{l+1} = \cdots = \theta_m = 0$ and from the structure of $\bar{T}_w$, it follows that

$$\bar{T}_w T_w(\theta)\tilde{S} = \bar{S}_m(\theta)\bar{T}_w T_w(\theta) \qquad \text{and} \qquad \bar{\gamma}\bar{T}_w T_w(\theta) = \tilde{\gamma}.$$

Therefore, we can now compactly rewrite equations (5.3.4b), (5.3.4c), and

(5.3.7a) as

$$\begin{aligned}
\begin{bmatrix} \dot{\tilde{x}}_o \\ \dot{\bar{w}} \end{bmatrix} &= \begin{bmatrix} A_o & -b_o\bar{\gamma} \\ 0 & \bar{S}_m(\theta) \end{bmatrix} \begin{bmatrix} \tilde{x}_o \\ \bar{w} \end{bmatrix} + \begin{bmatrix} b_o \\ 0 \end{bmatrix} u =: \bar{A}(\theta) \begin{bmatrix} \tilde{x}_o \\ \bar{w} \end{bmatrix} + \bar{b}u, \\
e &= \begin{bmatrix} c_o & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_o \\ \bar{w} \end{bmatrix} =: \bar{c} \begin{bmatrix} \tilde{x}_o \\ \bar{w} \end{bmatrix}.
\end{aligned} \tag{5.3.8}$$

Let $T_c(\theta)$ be $T_c(\hat{\theta})$ of (5.2.3) with θ instead of $\hat{\theta}$. Then it should be noted that $T_c(\theta)$ can also be written as

$$T_c(\theta) = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ \alpha_1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{\nu+2m-1} & \alpha_{\nu+2m-2} & \cdots & 1 & 0 \\ \alpha_{\nu+2m} & \alpha_{\nu+2m-1} & \cdots & \alpha_1 & 1 \end{bmatrix} \begin{bmatrix} \bar{c} \\ \bar{c}\bar{A}(\theta) \\ \vdots \\ \bar{c}\bar{A}^{\nu+2m-1}(\theta) \\ \bar{c}\bar{A}^{\nu+2m}(\theta) \end{bmatrix}, \tag{5.3.9}$$

where $\alpha_1, \alpha_2, \cdots, \alpha_{\nu+2m}$ are the coefficients of the characteristic polynomial of $\bar{A}(\theta)$ given by

$$\begin{aligned}
s^{\nu+2m+1} + \alpha_1 s^{\nu+2m} + \cdots + \alpha_{\nu+2m} s &= \det(sI - \bar{A}(\theta)) = \det(sI - A_o) \cdot \det(sI - \bar{S}_m(\theta)) \\
&= (s^\nu + a_1 s^{\nu-1} + a_2 s^{\nu-2} + \cdots + a_\nu) \cdot (s^{2m+1} + \theta_1 s^{2m-1} + \theta_2 s^{2m-3} + \cdots + \theta_m s).
\end{aligned} \tag{5.3.10}$$

(Proof of this claim is given in the Appendix A.5.) Note that the matrix $T_c(\theta)$ in (5.3.9) is the matrix of the equivalence transformation for observable canonical form of the system (5.3.8) (again, see the Appendix A.5) if the pair $(\bar{A}(\theta), \bar{c})$ is observable (which is true as can also be seen from the following lemma). In particular, under the hyperbolic zero dynamics condition, the following lemma shows an important fact that, although $T_c(\theta)$ is not known (due to the unknown vector θ), its determinant is lower bounded by a known quantity.

Lemma 5.3.1. $\det^2(T_c(\theta)) \geq \delta > 0$ (where δ is given in (5.2.5)). ◊

Proof. Recalling that the matrix $T_c(\theta) \in \mathbb{R}^{(\nu+2m+1)\times(\nu+2m+1)}$ (in (5.2.3) with $\hat{\theta}$

replaced by θ) has the Sylvester matrix form

$$
T_c(\theta) = \begin{array}{c}
\begin{array}{cc} \overbrace{\hphantom{\theta_1 \quad 0 \quad \ddots \quad 1}}^{\nu} & \overbrace{\hphantom{-b_{\nu-1} \quad -b_{\nu-2} \quad \ddots \quad -b_1}}^{2m+1} \end{array} \\
\left[\begin{array}{cccccccc}
1 & & & & 0 & & & \\
0 & 1 & & & -b_1 & 0 & & \\
\theta_1 & 0 & & & -b_2 & -b_1 & & \\
\vdots & \vdots & \ddots & & \vdots & \vdots & \ddots & \\
\theta_m & 0 & & 1 & -b_{\nu-1} & -b_{\nu-2} & & 0 \\
0 & \theta_m & & 0 & -b_\nu & -b_{\nu-1} & & -b_1 \\
 & 0 & & \vdots & & -b_\nu & & \vdots \\
 & & & \theta_m & & & & -b_{\nu-1} \\
 & & & 0 & & & & -b_\nu
\end{array}\right]
\end{array}
\tag{5.3.11}
$$

(where the remaining entries are zero), we employ the formula for the determinant of Sylvester matrix.[2] For convenience of computation, we note that $b_1 = \cdots = b_{\rho-1} = 0$ and $b_\rho \neq 0$ with ρ being the relative degree of the plant $c(sI-A)^{-1}b = c_o(sI-A_o)^{-1}b_o$, so that the $\rho \times (\nu - \rho + 2m + 1)$ upper right block of $T_c(\theta)$ is zero. Then, since the $\rho \times \rho$ upper left block of $T_c(\theta)$ is lower triangular whose diagonal elements are all 1, the determinant is the same as the determinant of the $(\nu - \rho + 2m + 1) \times (\nu - \rho + 2m + 1)$ lower right submatrix which we denote by $\bar{T}_c(\theta)$. To compute the determinant of $\bar{T}_c(\theta)$ which is still a Sylvester matrix, we employ the formula and obtain that

$$
\det\left(T_c(\theta)\right) = \det\left(\bar{T}_c(\theta)\right) = 1^{\nu-\rho}(-b_\rho)^{2m+1} \prod_{i=1}^{2m+1} \prod_{k=1}^{\nu-\rho} (\psi_i - \zeta_k), \tag{5.3.12}
$$

where ψ_i's are the roots of the polynomial (5.3.6) and ζ_i's are the roots of the polynomial $-b_\rho s^{\nu-\rho} - b_{\rho+1}s^{\nu-\rho-1} - \cdots - b_{\nu-1}s - b_\nu$ (i.e., the zeros of $c_o(sI - A_o)^{-1}b_o$). Noting that any complex ψ_i's or ζ_i's occur in conjugate pairs, we

[2]For a Sylvester matrix of two polynomial $f(s) = \sum_{i=0}^{n} a_i s^i$ and $g(s) = \sum_{j=0}^{m} b_j s^j$, its determinant is given by $a_n^m b_m^n \Pi_{i=0}^{n} \Pi_{j=0}^{m} (s_{f,i} - s_{g,j})$ where $s_{f,i}$ and $s_{g,j}$ are roots of $f(s)$ and $g(s)$, respectively. See, e.g., [Apo70, Section 2] or [vdW03] for the proof and more details.

obtain that

$$\det^2\left(T_c(\theta)\right) = b_\rho^{4m+2}\prod_{i=1}^{2m+1}\prod_{k=1}^{\nu-\rho}|\psi_i - \zeta_k|^2 \geq b_\rho^{4m+2}\prod_{i=1}^{2m+1}\prod_{k=1}^{\nu-\rho}|\mathrm{Re}(\psi_i - \zeta_k)|^2$$

and since $\mathrm{Re}(\psi_i) = 0$, it turns out that

$$\det^2\left(T_c(\theta)\right) \geq b_\rho^{4m+2}\prod_{k=1}^{\nu-\rho}\left(\mathrm{Re}(\zeta_k)\right)^{4m+2} = \delta,$$

in which, $\delta > 0$ because of the hyperbolic condition of the zero dynamics. □

Finally, the system (5.3.8) is converted, by the equivalence transformation $\xi := T_c(\theta)[\tilde{x}_o^\top, \bar{w}^\top]^\top$, $\xi \in \mathbb{R}^{\nu+2m+1}$, into its observable canonical form written as

$$\dot{\xi} = A_c\xi + b_c u + \Psi(e,u)\theta, \quad (5.3.13a)$$

$$e = c_c\xi. \quad (5.3.13b)$$

Hence, the closed-loop system is given also by (5.2.1), (5.2.2), and (5.3.13) where (5.3.13) is equivalent to (5.3.8) (or, (5.3.4b), (5.3.4c), and (5.3.7a)). Note that the system (5.3.13) is in the standard adaptive observer form of [Zha02], and thus, we obtain the following.

Lemma 5.3.2. For the systems (5.2.1) and (5.3.13) (or, (5.3.8)) under any continuous signal $u(t)$, $\lim_{t\to\infty}\hat{\theta}(t) = \theta$. ◊

Proof. Let $\Xi_i \in \mathbb{R}^{\nu+2m+1}$ be the i-th column of Ξ and let μ_i be the first element of Ξ_i. In addition, let

$$\mu := \begin{bmatrix}\mu_1\\ \vdots\\ \mu_m\end{bmatrix} \quad \text{and} \quad \chi_i := \Xi_i - N_i\tilde{x}_o \quad \text{where} \quad N_i := \begin{bmatrix}0_{2i\times\nu}\\ I_\nu\\ 0_{(2m-2i+1)\times\nu}\end{bmatrix}.$$

Since $\dot{\Xi}_i = (A_c - Lc_c)\Xi_i + (-a[i]e + b[i]u)$, we have from (5.3.4b) and (5.3.4c) that

$$\begin{aligned}
\dot{\chi}_i &= (A_c - Lc_c)\Xi_i + (-a[i]e + b[i]u) - N_i\dot{\tilde{x}}_o \\
&= (A_c - Lc_c)\chi_i + ((A_c - Lc_c)N_i - N_iA_o - a[i]c_o)\,\tilde{x}_o + (b[i] - N_ib_o)u + N_ib_ou_w \\
&= (A_c - Lc_c)\chi_i + N_ib_ou_w, \qquad (5.3.14a) \\
\mu_i &= \begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}\Xi_i = \begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}(\chi_i + N_i\tilde{x}_o) = \begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}\chi_i \\
&= c_c\chi_i \qquad (5.3.14b)
\end{aligned}$$

from the fact that $(A_c - Lc_c)N_i - N_iA_o - a[i]c_o = 0$ and $b[i] - N_ib_o = 0$ for $i = 1, \cdots, m$. Thus,

$$\mu(t) = \Xi^\top(t)c_c^\top$$

is bounded because u_w is bounded and $A_c - Lc_c$ is Hurwitz. By [IS96, Theorem 5.2.1], we see that the vector $[\mu_1, \cdots, \mu_l]^\top$ is persistently exciting (PE) (but, $[\mu_1, \cdots, \mu_k]^\top$, $k \geq l+1$, is not) because u_w of (5.3.5) contains the sinusoids of l distinct frequencies, i.e., there exist $\kappa_1 > 0$, $\kappa_2 > 0$, and $T > 0$ such that, for all $t \geq 0$,

$$\kappa_1 I_i \leq \int_t^{t+T} \begin{bmatrix}\mu_1(\tau) & \cdots & \mu_i(\tau)\end{bmatrix}^\top \begin{bmatrix}\mu_1(\tau) & \cdots & \mu_i(\tau)\end{bmatrix} d\tau \leq \kappa_2 I_i, \quad i = 1, \cdots, l. \qquad (5.3.15)$$

From (5.2.1d) and $\Omega_i = [I_i, 0_{i\times(m-i)}]\Omega[I_i, 0_{i\times(m-i)}]^\top$, it is seen that

$$\begin{aligned}
\Omega_i(t) &= \exp(-\lambda_b t)\Omega_i(0) \\
&\quad + \lambda_c \int_0^t \Big(\exp\big(-\lambda_b(t-\tau)\big) \cdot [I_i, 0_{i\times(m-i)}]\Xi(\tau)^\top c_c^\top c_c \Xi(\tau)[I_i, 0_{i\times(m-i)}]^\top\Big)\, d\tau, \qquad (5.3.16)
\end{aligned}$$

where $[I_i, 0_{i\times(m-i)}]\Xi^\top c_c^\top c_c\Xi[I_i, 0_{i\times(m-i)}]^\top = [\mu_1 \ \cdots \ \mu_i]^\top[\mu_1 \ \cdots \ \mu_i]$ by $\mu(t) = \Xi^\top(t)c_c^\top$. Therefore, since $\Omega(0)$ is positive definite, it follows from Appendix A.6 that $\det^2(\Omega_i(t)) \geq \vartheta > 0$ for $i = 1, \cdots, l$ where ϑ is a positive number and that $\det^2(\Omega_{l+1}(t)), \cdots, \det^2(\Omega_m(t))$ tend exponentially to zero as time goes to infinity. Then, we have

$$\lim_{t\to\infty} \exp\left(-\det^2(\Omega_i(t))\, t\right) = \begin{cases} 0, & \text{for } i = 1, \cdots, l, \\ 1, & \text{for } i = l+1, \cdots, m. \end{cases} \tag{5.3.17}$$

Furthermore, it follows from $\theta = [\theta_1, \cdots, \theta_l, 0, \cdots, 0]^\top \in \mathbb{R}^m$ (see (5.3.6)) that

$$\begin{aligned} &\lim_{t\to\infty} \operatorname{diag}\Big(\exp\left(-\det^2(\Omega_1(t))\, t\right), \cdots, \exp\left(-\det^2(\Omega_m(t))\, t\right)\Big) \cdot \theta \\ &\qquad =: \lim_{t\to\infty} D(t)\cdot\theta = 0, \end{aligned} \tag{5.3.18}$$

in which, the convergence is exponential.

Now, let

$$\tilde{\xi} := \hat{\xi} - \xi, \qquad \tilde{\theta} := \hat{\theta} - \theta, \qquad \eta := \tilde{\xi} - \Xi\tilde{\theta}. \tag{5.3.19}$$

Then, we have

$$\begin{aligned} \dot{\tilde{\xi}} &= (A_c - Lc_c)\tilde{\xi} + \Xi\dot{\hat{\theta}} + \Psi(e,u)\tilde{\theta}, \\ \dot{\eta} &= (A_c - Lc_c)\tilde{\xi} + \Xi\dot{\hat{\theta}} + \Psi(e,u)\tilde{\theta} - \dot{\Xi}\tilde{\theta} - \Xi\dot{\tilde{\theta}} \\ &= (A_c - Lc_c)\eta \end{aligned} \tag{5.3.20}$$

by (5.2.1b) and $\dot{\hat{\theta}} = \dot{\tilde{\theta}}$. Also, by (5.3.13b) and $\mu = \Xi^\top c_c^\top$, we obtain from (5.2.1c),

$$\begin{aligned} \dot{\tilde{\theta}} = \dot{\hat{\theta}} &= -\lambda_a \Xi^\top c_c^\top c_c \tilde{\xi} - \lambda_a D(\tilde{\theta} + \theta) \\ &= -\lambda_a \Xi^\top c_c^\top c_c \Xi\tilde{\theta} - \lambda_a \Xi^\top c_c^\top c_c \eta - \lambda_a D(\tilde{\theta} + \theta) \\ &= -\lambda_a \left(\mu\mu^\top + D\right)\tilde{\theta} - \lambda_a \mu\eta_1 - \lambda_a D\theta, \end{aligned} \tag{5.3.21}$$

where $\eta_1 = c_c\eta$ with $\eta = [\eta_1, \cdots, \eta_{\nu+2m+1}]^\top$. Let

$$\Theta(t) := \frac{1}{2}\tilde{\theta}^\top(t)\tilde{\theta}(t) + \frac{\lambda_a}{4}\int_t^\infty \eta_1^2(\tau)d\tau + \frac{\lambda_a}{4}\int_t^\infty \theta^\top D(\tau)\theta d\tau$$

which is a well-defined continuously differentiable function with respect to t, with the solution of (5.2.1) and (5.3.13) under (5.3.19). Then, the time derivative of

Θ satisfies that

$$\begin{aligned}\dot{\Theta}(t) &= \left(-\lambda_a(\tilde{\theta}^\top \mu)^2 - \lambda_a(\tilde{\theta}^\top \mu)\eta_1 - \frac{\lambda_a}{4}\eta_1^2\right) + \left(-\lambda_a\tilde{\theta}^\top D\tilde{\theta} - \lambda_a\tilde{\theta}^\top D\theta - \frac{\lambda_a}{4}\theta^\top D\theta\right) \\ &= -\lambda_a\left(\tilde{\theta}^\top \mu + \frac{\eta_1}{2}\right)^2 - \lambda_a\left(\tilde{\theta} + \frac{\theta}{2}\right)^\top D\left(\tilde{\theta} + \frac{\theta}{2}\right) \le 0. \end{aligned} \tag{5.3.22}$$

This implies that $\frac{1}{2}\tilde{\theta}^\top(t)\tilde{\theta}(t) \le \Theta(t) \le \Theta(0)$ for all $t \ge 0$, which implies that $\tilde{\theta}(t)$ is bounded. In addition, $\dot{\Theta}(t)$ is uniformly continuous with respect to t because

$$\begin{aligned}\ddot{\Theta}(t) = &-2\lambda_a\left(\tilde{\theta}^\top \mu + \frac{\eta_1}{2}\right)\left(\dot{\tilde{\theta}}^\top \mu + \tilde{\theta}^\top \dot{\mu} + \frac{\dot{\eta}_1}{2}\right) \\ &- \lambda_a\dot{\tilde{\theta}}^\top D\left(\tilde{\theta} + \frac{\theta}{2}\right) - \lambda_a\left(\tilde{\theta} + \frac{\theta}{2}\right)^\top \dot{D}\left(\tilde{\theta} + \frac{\theta}{2}\right) - \lambda_a\left(\tilde{\theta} + \frac{\theta}{2}\right)^\top D\dot{\tilde{\theta}}\end{aligned}$$

and because $\tilde{\theta}(t)$, $\mu(t)$, $\eta_1(t)$, $\dot{\tilde{\theta}}(t)$, $\dot{\mu}(t)$, $\dot{\eta}_1(t)$, $D(t)$, and $\dot{D}(t)$ are all bounded as can be seen from (5.3.14), (5.3.17), (5.3.18), (5.3.20), and (5.3.21), which ensures that $\ddot{\Theta}(t)$ is bounded as well. Since $-\dot{\Theta}(t) \ge 0$ and $\int_0^\infty(-\dot{\Theta}(t))dt \le \Theta(0)$, Barbalat's lemma [Kha01] yields that $\lim_{t\to\infty}\dot{\Theta}(t) = 0$. Then, by $\lim_{t\to\infty}\eta(t) = 0$ and $\lim_{t\to\infty} D(t)\theta = 0$, it follows from (5.3.22) that

$$\lim_{t\to\infty}\tilde{\theta}^\top(t)\mu(t) = 0 \qquad \text{and} \qquad \lim_{t\to\infty} D(t)\tilde{\theta}(t) = 0. \tag{5.3.23}$$

Therefore, from (5.3.17), we have

$$\lim_{t\to\infty}\begin{bmatrix}\tilde{\theta}_{l+1}(t) & \cdots & \tilde{\theta}_m(t)\end{bmatrix}^\top = 0. \tag{5.3.24}$$

For the rest of $\tilde{\theta}$, we have from (5.3.21) that

$$\begin{aligned}\begin{bmatrix}\dot{\tilde{\theta}}_1 \\ \vdots \\ \dot{\tilde{\theta}}_l\end{bmatrix} = &-\lambda_a\begin{bmatrix}\mu_1 \\ \vdots \\ \mu_l\end{bmatrix}\begin{bmatrix}\mu_1 & \cdots & \mu_l\end{bmatrix}\begin{bmatrix}\tilde{\theta}_1 \\ \vdots \\ \tilde{\theta}_l\end{bmatrix} - \lambda_a\begin{bmatrix}\mu_1 \\ \vdots \\ \mu_l\end{bmatrix}\begin{bmatrix}\mu_{l+1} & \cdots & \mu_m\end{bmatrix}\begin{bmatrix}\tilde{\theta}_{l+1} \\ \vdots \\ \tilde{\theta}_m\end{bmatrix} \\ &- \lambda_a\begin{bmatrix}I_l & 0_{l\times(m-l)}\end{bmatrix}D\tilde{\theta} - \lambda_a\begin{bmatrix}\mu_1 \\ \vdots \\ \mu_l\end{bmatrix}\eta_1 - \lambda_a\begin{bmatrix}I_l & 0_{l\times(m-l)}\end{bmatrix}D\theta. \end{aligned} \tag{5.3.25}$$

Here, we think of this system as a perturbation of the nominal system

$$\begin{bmatrix} \dot{\tilde{\theta}}_1 \\ \vdots \\ \dot{\tilde{\theta}}_l \end{bmatrix} = -\lambda_a \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_l \end{bmatrix} \begin{bmatrix} \mu_1 & \cdots & \mu_l \end{bmatrix} \begin{bmatrix} \tilde{\theta}_1 \\ \vdots \\ \tilde{\theta}_l \end{bmatrix}.$$

Then, by [And77, Theorem 1] and (5.3.15), it follows that the nominal system is exponentially stable. Therefore, since $\mu = [\mu_1 \quad \cdots \quad \mu_m]^\top$ is bounded, by virtue of [Kha01, Lemma 9.6.(3)] with (5.3.18), (5.3.20), (5.3.23), and (5.3.24) (which imply the perturbation terms decay to zero), we have

$$\lim_{t\to\infty} \begin{bmatrix} \tilde{\theta}_1(t) & \cdots & \tilde{\theta}_l(t) \end{bmatrix}^\top = 0 \tag{5.3.26}$$

for the perturbed system (5.3.25), and hence the proof is complete by (5.3.24) and (5.3.26). □

Remark 5.3.2. In order to ensure that $u(t)$ is well-defined and continuous for all $t \geq 0$, which is a premise of Lemma 5.3.2, the control law given in (5.2.2) has the protection which prevents $u(t)$ from becoming infinity at some time t. In fact, since $\xi = T_c(\theta)[\tilde{x}_o^\top, \bar{w}^\top]^\top$, in order to have $u = K\hat{\tilde{x}}_o + \bar{\gamma}\hat{\bar{w}}$ (where $\hat{\tilde{x}}_o$ and $\hat{\bar{w}}$ are the estimates of $\tilde{x}_o$ and $\bar{w}$, respectively), one may come up with $u = [K, \bar{\gamma}]T_c^{-1}(\hat{\theta})\hat{\xi}$. While this could be true since $\hat{\theta}(t) \to \theta$ and $\hat{\xi}(t) \to \xi(t)$ (to be shown shortly), the matrix $T_c(\hat{\theta}(t))$ may become singular during the transient of $\hat{\theta}(t)$ (note that $T_c(\theta)$ is nonsingular by Lemma 5.3.1). To see this possibility, we note from (5.3.12) that $\det(T_c(\hat{\theta})) = 0$ if and only if $s(s^{2m} + \hat{\theta}_1 s^{2(m-1)} + \hat{\theta}_2 s^{2(m-2)} + \cdots + \hat{\theta}_{m-1}s^2 + \hat{\theta}_m)$ and $b_\rho s^{\nu-\rho} + \cdots + b_{\nu-1}s + b_\nu$ have a common root. (i) Suppose that they have a common real root β (which is not 0 by Assumption 5.1.1). Then, $s^{2m} + \hat{\theta}_1 s^{2(m-1)} + \cdots + \hat{\theta}_m$ should be written as $(s^{2(m-1)} + f_1 s^{2(m-2)} + \cdots + f_{m-2}s^2 + f_{m-1})(s^2 - \beta^2)$ with $f_i \in \mathbb{R}$ (the term $s^2 - \beta^2$ instead of $s - \beta$ appears because the polynomial consists of even powers of s). This means that $\hat{\theta} \in \mathbb{R}^m$ belongs to the $m-1$ dimensional affine set

$$\Theta_\beta := \left\{ \left[-\beta^2 + f_1, -\beta^2 f_1 + f_2, \cdots, -\beta^2 f_{m-2} + f_{m-1}, -\beta^2 f_{m-1}\right]^\top : f_i \in \mathbb{R} \right\}.$$

(ii) Similarly, if β is a complex number, then, by the structure, $s^{2m}+\hat{\theta}_1 s^{2(m-1)}+\cdots+\hat{\theta}_m$ should be written as $(s^{2(m-2)}+f_1 s^{2(m-3)}+\cdots+f_{m-2})(s-\beta)(s-\bar{\beta})(s+\beta)(s+\bar{\beta}) = (s^{2(m-2)}+f_1 s^{2(m-3)}+\cdots+f_{m-2})(s^4-(\beta^2+\bar{\beta}^2)s^2+\beta^2\bar{\beta}^2)$ with $f_i\in\mathbb{R}$ where $\bar{\beta}$ is the complex conjugate of β. Hence, $\hat{\theta}$ belongs to the $m-2$ dimensional affine set

$$\Theta_\beta := \left\{\left[-(\beta^2+\bar{\beta}^2)+f_1, \beta^2\bar{\beta}^2-(\beta^2+\bar{\beta}^2)f_1+f_2, \cdots, \beta^2\bar{\beta}^2 f_{m-2}\right]^\top : f_i\in\mathbb{R}\right\}.$$

Therefore, with $\Theta_{\mathcal{Z}} := \cup_{\beta\in\mathcal{Z}}\Theta_\beta$ where $\mathcal{Z}$ is the set of zeros of the plant, the matrix $T_c(\hat{\theta}(t))$ becomes singular whenever $\hat{\theta}(t)$ passes through the set $\Theta_{\mathcal{Z}}$. To avoid this, instead of $T_c^{-1}(\hat{\theta})$, $\mathcal{T}_c(\hat{\theta})$ is introduced in (5.2.2) using the dead-zone function (5.2.4), which is different from $T_c^{-1}(\hat{\theta})$ only when $\det^2(T_c(\hat{\theta}))$ is smaller than δ. By virtue of Lemma 5.3.1, the positive level δ of the dead-zone function in (5.2.5) guarantees that $\mathcal{T}_c(\theta) = T_c^{-1}(\theta)$. Then, it follows from Lemma 5.3.2 that

$$\lim_{t\to\infty}\mathcal{T}_c\big(\hat{\theta}(t)\big) = \mathcal{T}_c(\theta) = T_c^{-1}(\theta),$$

and $u(t)$ is continuous for all $t\geq 0$. ◊

Lemma 5.3.3. The state $\tilde{x}_o(t)$ of (5.3.4b) is bounded and $\lim_{t\to\infty}\tilde{x}_o(t) = 0$ under the control of (5.2.1) and (5.2.2). ◊

Proof. It is observed from $\chi_i = \Xi_i - N_i\tilde{x}_o$ that $\Xi\tilde{\theta} = \sum_{i=1}^m(\chi_i+N_i\tilde{x}_o)\tilde{\theta}_i$. Then, from (5.2.2), (5.3.4b), (5.3.7b), and (5.3.19) with $\xi = T_c(\theta)[\tilde{x}_o^\top, \bar{w}^\top]^\top$ and with $\hat{\xi} = \xi+\tilde{\xi} = \xi+\eta+\Xi\tilde{\theta}$, we have

$$\begin{aligned}
\dot{\tilde{x}}_o &= A_o\tilde{x}_o - b_o\bar{\gamma}\bar{w} + b_o\left[K\ \ \bar{\gamma}\right]\mathcal{T}_c(\hat{\theta})\hat{\xi} + (b_oK\tilde{x}_o - b_oK\tilde{x}_o)\\
&= (A_o+b_oK)\tilde{x}_o + b_o\left[K\ \ \bar{\gamma}\right]\left(-\begin{bmatrix}\tilde{x}_o\\ \bar{w}\end{bmatrix} + \mathcal{T}_c(\hat{\theta})\hat{\xi}\right)\\
&= (A_o+b_oK)\tilde{x}_o\\
&\qquad + b_o\left[K\ \ \bar{\gamma}\right]\left(-\begin{bmatrix}\tilde{x}_o\\ \bar{w}\end{bmatrix} + \mathcal{T}_c(\hat{\theta})T_c(\theta)\begin{bmatrix}\tilde{x}_o\\ \bar{w}\end{bmatrix} + \mathcal{T}_c(\hat{\theta})\Big(\eta+\sum_{i=1}^m(\chi_i+N_i\tilde{x}_o)\tilde{\theta}_i\Big)\right)
\end{aligned}$$

$$= (A_o + b_o K)\tilde{x}_o + b_o \left[K \;\; \bar{\gamma}\right] \left(\left(\mathcal{T}_c(\hat{\theta}) T_c(\hat{\theta}) - I \right) \begin{bmatrix} \tilde{x}_o \\ \bar{w} \end{bmatrix} \right.$$

$$\left. + \mathcal{T}_c(\hat{\theta}) \left(T_c(\theta) - T_c(\hat{\theta}) \right) \begin{bmatrix} \tilde{x}_o \\ \bar{w} \end{bmatrix} + \mathcal{T}_c(\hat{\theta}) \left(\eta + \sum_{i=1}^{m} \tilde{\theta}_i \chi_i \right) + \mathcal{T}_c(\hat{\theta}) \sum_{i=1}^{m} (\tilde{\theta}_i N_i) \tilde{x}_o \right).$$

Here, it is observed from the structure of $T_c(\cdot)$ in (5.2.3) (or, (5.3.11)) that

$$\left(T_c(\theta) - T_c(\hat{\theta}) \right) \begin{bmatrix} \tilde{x}_o \\ \bar{w} \end{bmatrix} + \sum_{i=1}^{m} (\tilde{\theta}_i N_i) \tilde{x}_o = 0.$$

Therefore, with $\mathcal{T}_c(\hat{\theta}) T_c(\hat{\theta}) = \dfrac{\det^2\left(T_c(\hat{\theta})\right)}{\delta + \bar{d}\left(\det^2\left(T_c(\hat{\theta})\right)\right)} I =: \varphi(\hat{\theta}) I$, we obtain

$$\begin{aligned} \dot{\tilde{x}}_o &= \left((A_o + b_o K) + b_o K (\varphi(\hat{\theta}) - 1) \right) \tilde{x}_o \\ &\quad + \left(b_o \bar{\gamma} (\varphi(\hat{\theta}) - 1) \bar{w} + b_o \left[K \;\; \bar{\gamma}\right] \mathcal{T}_c(\hat{\theta}) \left(\eta + \sum_{i=1}^{m} \tilde{\theta}_i \chi_i \right) \right) \\ &=: \left((A_o + b_o K) + M_1(t) \right) \tilde{x}_o + M_2(t). \end{aligned} \tag{5.3.27}$$

By virtue of Lemmas 5.3.1 and 5.3.2, $\lim_{t\to\infty} \varphi(\hat{\theta}(t)) = \varphi(\theta) = 1$, and as a result,

$$\begin{aligned} \lim_{t\to\infty} M_1(t) &= \lim_{t\to\infty} \left(b_o K \left(\varphi(\hat{\theta}(t)) - 1 \right) \right) = 0 \quad \text{and} \\ \lim_{t\to\infty} M_2(t) &= \lim_{t\to\infty} \left(b_o \bar{\gamma} \left(\varphi(\hat{\theta}(t)) - 1 \right) \bar{w}(t) \right. \\ &\qquad \left. + b_o \left[K \;\; \bar{\gamma}\right] \mathcal{T}_c(\hat{\theta}(t)) \left(\eta(t) + \sum_{i=1}^{m} \tilde{\theta}_i(t) \chi_i(t) \right) \right) = 0 \end{aligned}$$

since $\lim_{t\to\infty} \eta(t) = 0$ and $\lim_{t\to\infty} \tilde{\theta}_i(t) = 0$ for $i = 1, \cdots, m$ while $\bar{w}(t)$ and $\chi_i(t)$'s are bounded by (5.3.14). Here, we think of the system (5.3.27) as a perturbation of the nominal system

$$\dot{\tilde{x}}_o = \left((A_o + b_o K) + M_1(t) \right) \tilde{x}_o.$$

By virtue of [Kha01, Theorem 4.12, Corollary 9.1, and Lemma 9.5.(2)], the nominal system is exponentially stable since $A_o + b_o K$ is Hurwitz and $\lim_{t\to\infty} M_1(t) = 0$. Then, it follows from [Kha01, Lemma 9.6.(3)] that $\lim_{t\to\infty} \tilde{x}_o(t) = 0$ for the perturbed system (5.3.27) since the perturbation term $\lim_{t\to\infty} M_2(t) = 0$. □

Putting all together, we obtain the following.

- $\tilde{x}_o(t)$ of (5.3.4b) is bounded and $\lim_{t\to\infty} \tilde{x}_o(t) = 0$ by Lemma 5.3.3, and so, $\xi(t) = T_c(\theta)[\tilde{x}_o^\top(t), \varpi^\top(t)]^\top$ is bounded as well.

- $\lim_{t\to\infty} e(t) = \lim_{t\to\infty} c_o\tilde{x}_o(t) = 0$.

- $\Xi(t)$ is bounded since $\chi_i = \Xi_i - N_i\tilde{x}_o$ where $\chi_i(t)$ and $\tilde{x}_o(t)$ are bounded.

- $\Omega(t)$ is bounded by (5.2.1d) since $\Xi(t)$ is bounded.

- $\hat{\theta}(t)$ is bounded since $\tilde{\theta}(t)$ is bounded (and $\lim_{t\to\infty} \tilde{\theta}(t) = 0$) by Lemma 5.3.2.

- $\hat{\xi}(t)$ is bounded because $\eta = \tilde{\xi} - \Xi\tilde{\theta}$ and $\lim_{t\to\infty} \eta(t) = 0$ so that $\lim_{t\to\infty} \tilde{\xi}(t) = 0$, and because $\hat{\xi} = \tilde{\xi} + \xi$.

Finally, the unobservable part $\tilde{x}_{\bar{o}}$ of (5.3.4a) is also bounded since $\tilde{x}_o$, u, and w are bounded and $A_{\bar{o}}$ is Hurwitz, and hence the state x is bounded by $\tilde{x} = x - \Pi w$ with Kalman decomposition.

5.4 Numerical Examples

We consider an unstable non-minimum phase system (which has a zero at 2 and poles at 12 and 13)

$$\begin{aligned}\dot{x}_1 &= 25x_1 + x_2 + u + \omega_1,\\ \dot{x}_2 &= -156x_1 - 2u + \omega_2,\\ e &= x_1 + \omega_3,\end{aligned} \tag{5.4.1}$$

Table 5.1: Design parameters of the proposed adaptive output regulator.

Parameter	Value
Observer gain L	$[30, -90, 66, 90.25, 77, 37.5, 9]^\top$
Feedback gain K	$[-71.6, 31.5]$
Initial condition $\Omega(0)$	I_2
Other gains	$\lambda_a = 12,\ \lambda_b = 3,\ \lambda_c = 120$

where $\omega_1(t)$, $\omega_2(t)$, and $\omega_3(t)$ are defined as

$$\text{first time interval: } 0 \le t \le 40 = \begin{cases} \omega_1(t) = -1 + \sin(\sigma_1 t) + \sin(\sigma_2 t), \\ \omega_2(t) = 1 + 2\sin(\sigma_1 t) - 3\sin(\sigma_2 t), \\ \omega_3(t) = 1 - \sin(\sigma_1 t) + \sin(\sigma_2 t), \end{cases}$$

$$\text{second time interval: } 40 < t \le 80 = \begin{cases} \omega_1(t) = -1 + \sin(\sigma_2 t), \\ \omega_2(t) = 1 + 2\sin(\sigma_2 t), \\ \omega_3(t) = 1 - \sin(\sigma_2 t), \end{cases}$$

in which, $\sigma_1 = 1$ and $\sigma_2 = 2$. Then, the number of unknown frequencies is $r = 2$ for $0 \le t \le 40$ and $r = 1$ for $40 < t \le 80$. Also, θ_1 and θ_2 of (5.3.6) are written as

$$\theta = [\theta_1, \theta_2]^\top = \begin{cases} [5, 4]^\top, & \text{for} \quad 0 \le t \le 40, \\ [4, 0]^\top, & \text{for} \quad 40 < t \le 80. \end{cases}$$

With the upper bound on the number of unknown frequencies $m = 2$, the adaptive observer (5.2.1) and the control law (5.2.2) are designed, and the dead-band parameter δ is selected 1024 by (5.2.5). Also, the design parameters of (5.2.1) and (5.2.2) are selected as shown in Table 5.1.

Figure 5.1 shows simulation results using Matlab/Simulink with ODE15s under the proposed controller, in which the error tends to zero at the end of the first and second time intervals, i.e., 40s and 80s, and the estimated parameters $\hat{\theta}_1$ and $\hat{\theta}_2$ also converge to the true values θ_1 and θ_2, respectively. Moreover, as shown in

Figure 5.1–(d), it is seen that $\exp\left(-\det^2\left(\Omega_1(t)\right)t\right)$ and $\exp\left(-\det^2\left(\Omega_2(t)\right)t\right)$ of (5.2.1c) converge to 0 and 0 in the first time interval and 0 and 1 in the second time interval, respectively. In 0–1 and 45–47 seconds of Figure 5.1–(e), it is shown that the dead-zone function $\bar{d}(\cdot)$ of (5.2.2) is activated to avoid the problem of division by zero when $\det^2(T_c(\hat{\theta}))$ approaches zero. In addition, it is clear from Remark 5.3.2 that the matrix $T_c(\hat{\theta}(t))$ becomes singular whenever $\hat{\theta}(t)$ passes through the set $\Theta_{\mathcal{Z}}$. Figure 5.2 shows that the trajectory of $\hat{\theta}(t)$ passes through the set

$$\Theta_{\mathcal{Z}} = \left\{[-4+f_1, -4f_1]^\top : f_1 \in \mathbb{R}\right\} \tag{5.4.2}$$

for (5.4.1) (with $\beta = 2$) in 0–1 and 45–47 seconds.

Remark 5.4.1. In the control law (5.2.2), we adopt the matrix $\mathcal{T}_c(\hat{\theta}(t))$ instead of $T_c^{-1}(\hat{\theta}(t))$ in order to prevent $u(t)$ from becoming infinity at some time t. Here, we may also consider the pseudoinverse $T_c^\dagger(\hat{\theta})$ of $T_c(\hat{\theta})$ because it always has finite elements and also has the following property from Lemma 5.3.1 and 5.3.2:

$$\lim_{t\to\infty} T_c^\dagger(\hat{\theta}(t)) = T_c^\dagger(\theta) = T_c^{-1}(\theta).$$

The drawback of this approach lie in the problem related to the computation of the controller when the singular values of $T_c(\hat{\theta}(t))$ cross zero [FIS03, Theorem 6.29]. Nevertheless, we can take into account the pseudoinverse with any suitable tolerance in Matlab function `pinv`. The computation of the function is based on singular value decomposition and any singular values less than the tolerance are treated as zero. Therefore, in order to set the appropriate tolerance, it is necessary to obtain information about the smallest singular value of the unknown matrix $T_c(\theta)$ shown in the equation (5.3.11). $\Diamond$

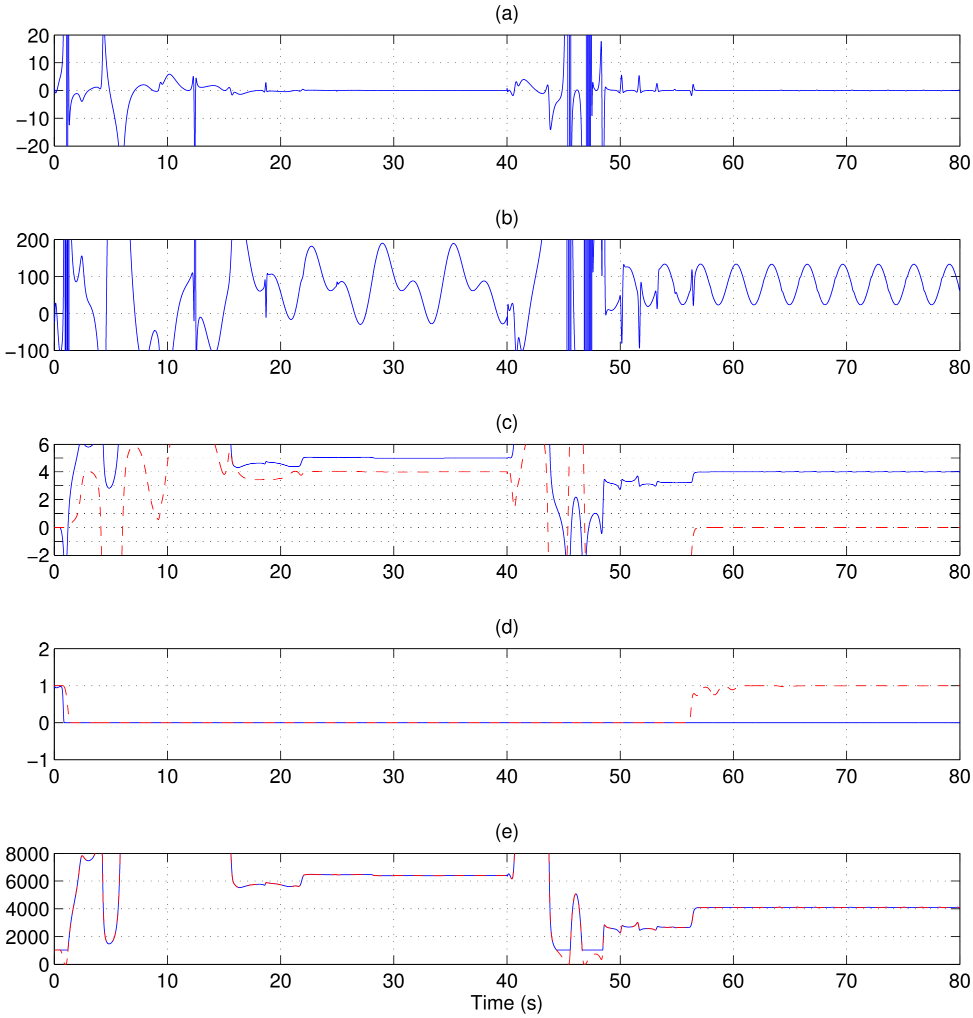

Figure 5.1: Simulation results. (a) Error output e. (b) Control input u. (c) Estimated parameters $\hat{\theta}_1$ (blue solid) and $\hat{\theta}_2$ (red dashed) of (5.2.1c). (d) $\exp\left(-\det^2\left(\Omega_1(t)\right)t\right)$ (blue solid) and $\exp\left(-\det^2\left(\Omega_2(t)\right)t\right)$ (red dashed) of (5.3.17). (e) $\delta + \bar{d}\big(\det^2(T_c(\hat{\theta}))\big)$ (blue solid) and $\det^2(T_c(\hat{\theta}))$ (red dashed) of (5.2.2).

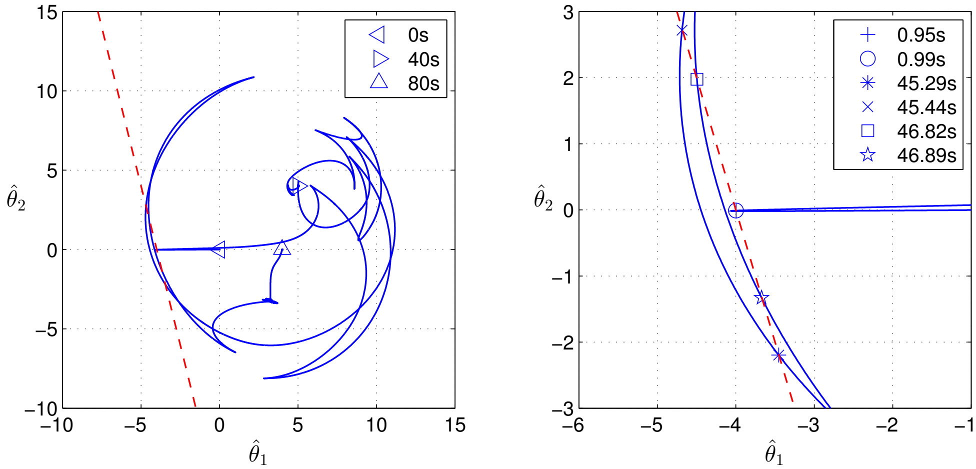

Figure 5.2: Trajectory (blue solid) of the estimated parameters $\hat{\theta}_1(t)$ and $\hat{\theta}_2(t)$. The red dashed is the set $\Theta_{\mathcal{Z}}$ for (5.4.1) (see Remark 5.3.2 and (5.4.2)). The right one is the enlarged version of the left.

Chapter 6

Conclusions and Further Issues

This chapter summarizes the main results of this dissertation that have been addressed so far, and suggests the future issues.

6.1 Conclusions

In this dissertation, we have dealt with the output regulation problems for linear systems with unknown sinusoidal exogenous inputs which consist of references and disturbances. The author summarizes the two results as follows:

i) In Chapter 4, we proposed an add-on adaptive regulator using the adaptive observer in [Zha02], based on the persistently exciting (PE) condition, for linear time-invariant (LTI) single-input-single-output (SISO) systems to achieve asymptotic tracking and disturbance rejection when the reference inputs and disturbances are sinusoids with unknown bias, magnitude, phase, and frequency. The proposed controller can be designed independent of the preinstalled controller, and thus we only need a knowledge of the plant under control. Without disturbing the overall stability of the closed-loop system, it can be freely used when the performance of disturbance rejection needs to be enhanced. Furthermore, the results of simulation and experiment for a commercial optical disc drive (ODD) systems have confirmed the effectiveness of the proposed controller.

ii) In Chapter 5, we presented an adaptive error feedback controller in the closed-form of (5.2.1) and (5.2.2), which asymptotically rejects the sinusoidal exogenous inputs whose not only magnitudes, phase, bias, frequencies are unknown but also the number of frequencies is unknown. In particular, without assuming the persistently exciting condition, we have claimed any linear system with hyperbolic zero dynamics admits the proposed controller. By introducing the Ω-dynamics given in (5.2.1d) and carefully designing the adaptive law in (5.2.1c), which has been inspired by [MS07, MT13a], the parameter estimate $\hat{\theta}(t)$ converges to the true value θ. While $\det(T_c(\theta))$ cannot be known since θ is unknown, a lower bound of it can be computed by employing the determinant formula of Sylvester matrix in Lemma 5.3.1. The design parameters of the controller in (5.2.1) and (5.2.2) are simply λ_a, λ_b, λ_c, and the gain matrices L and K, with the allowable upper bound m on the number of unknown frequencies.

6.2 Further Issues

The following issues related to this dissertation seem to be further investigated.

i) Up to now, we have studied stability and performance of reference tracking and disturbance rejection with a nominal plant. However, some uncertainty in the plant model is unavoidable because the plant models in industrial field are usually obtained experimentally, and thus the proposed theory needs to be extended. Indeed, some theoretical studies have been carried out in this respect (see for example [MT11, MT05, MT13b]), but they are available under the assumption that the uncertain plant is minimum phase.

ii) Throughout the dissertation, asymptotic reference tracking and disturbance rejection have been intensively studied for linear systems, even when the frequencies of the sinusoidal exogenous inputs are unknown. In fact, researches on the nonlinear systems with known output dependent nonlinearities have been studied in [MS05, MT05, Din06], and a design method proposed in [Din07] makes it possible to globally completely reject unknown sinusoidal disturbances for general nonlinear systems which may not be in the strict feedback form nor in the output feedback form. However, they all require the number of the sinusoids contained in the exogenous inputs.

iii) In this dissertation, one of the main assumptions is that the exogenous inputs are constrained to be sinusoidal. Although the frequencies of the sinusoids are known, a finite dimensional compensator can not achieve the control goal when the periodic exogenous inputs are not purely sinusoidal and contain infinite harmonics in its Fourier series expansion according to the internal model principle formulated in [FW76]. In fact, an indirect adaptive output feedback controller has been proposed in [MT14] for the known stable LTI-SISO systems with no zeros on the imaginary axis when the disturbances are matched by the control input.

iv) In Chapter 5, we assumed that the upper bound on the number of unknown frequencies is known. If the actual number of frequencies exceeds its presumed upper bound, the proposed adaptive regulator cannot guarantee the frequency estimation and the overall stability of the closed-loop system. To solve this problem, the authors of [MT13a] proposed a regulator which incorporates two observers and an adaptive internal model. It is shown that the regulation error is bounded and tends exponentially into a ball. However, it needs to be investigated in our framework because the controller of [MT13a] has several constraints as mentioned in Chapter 5.

APPENDIX

A.1 Stabilizability and Detectability of the Plant in Chapter 4.

Claim: When $u = u_c$ and $w = 0$, consider the plant

$$\begin{aligned} \dot{x} &= Ax + bu, \qquad && x \in \mathbb{R}^n,\ u \in \mathbb{R}, \\ e &= cx, && e \in \mathbb{R}, \end{aligned}$$

and the primary controller

$$\begin{aligned} \dot{z} &= A_p z + B_p e, \qquad && z \in \mathbb{R}^p, \\ u &= C_p z + D_p e. \end{aligned}$$

Then, the triple (A, b, c) is stabilizable and detectable if and only if the closed-loop system

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A + bD_p c & bC_p \\ B_p c & A_p \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} =: A_{cl} \begin{bmatrix} x \\ z \end{bmatrix}$$

is asymptotically stable, i.e., the matrix A_{cl} is Hurwitz.

Proof. Suppose that the matrix $\begin{bmatrix} A + bD_p c & bC_p \\ B_p c & A_p \end{bmatrix}$ is Hurwitz. Let $\tilde{A} := \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{(n+p)\times(n+p)}$, $\tilde{B} := \begin{bmatrix} 0 & b \\ I & 0 \end{bmatrix} \in \mathbb{R}^{(n+p)\times(p+1)}$, $\tilde{C} := \begin{bmatrix} 0 & I \\ c & 0 \end{bmatrix} \in \mathbb{R}^{(p+1)\times(n+p)}$, and

$\tilde{P} := \begin{bmatrix} A_p & B_p \\ C_p & D_p \end{bmatrix} \in \mathbb{R}^{(p+1)\times(p+1)}$. Then, with $\tilde{x} := [x^\top \ z^\top]^\top$, we can be written as

$$\dot{\tilde{x}} = \begin{bmatrix} A + bD_p c & bC_p \\ B_p c & A_p \end{bmatrix} \tilde{x} = (\tilde{A} + \tilde{B}\tilde{P}\tilde{C})\tilde{x},$$

and thus the triple $(\tilde{A}, \tilde{B}, \tilde{C})$ is stabilizable and detectable since the matrix $\tilde{A} + \tilde{B}\tilde{P}\tilde{C}$ is Hurwitz. For each $\lambda \in \mathbb{C}$ which is an eigenvalue of $\tilde{A}$ and has a nonnegative real part (or for all $\lambda \in \mathbb{C}_{\geq 0}$), the matrices $\begin{bmatrix} \lambda I - \tilde{A} & \tilde{B} \end{bmatrix}$ and $\begin{bmatrix} \lambda I - \tilde{A} \\ \tilde{C} \end{bmatrix}$ have $n + p$ rank by the PBH (Popov-Belevitch-Hautus) rank test [Hes09]. Therefore, since every eigenvalue of A is an eigenvalue of $\tilde{A}$, it follows from the equations

$$\operatorname{rank}\left(\begin{bmatrix} \lambda I - \tilde{A} & \tilde{B} \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} \lambda I - A & 0 & 0 & b \\ 0 & \lambda I & I & 0 \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} \lambda I - A & b \end{bmatrix}\right) + p,$$

$$\operatorname{rank}\left(\begin{bmatrix} \lambda I - \tilde{A} \\ \tilde{C} \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} \lambda I - A & 0 \\ 0 & \lambda I \\ 0 & I \\ c & 0 \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} \lambda I - A \\ c \end{bmatrix}\right) + p$$

that the triple (A, b, c) is both stabilizable and detectable.

Conversely, suppose that the triple (A, b, c) is stabilizable and detectable, so that there exist K and L such that $A + bK$ and $A + Lc$ are Hurwitz, respectively. If the primary controller have $p = n$ and

$$A_p = A + Lc + bK, \quad B_p = -L, \quad C_p = K, \quad \text{and} \quad D_p = 0,$$

then we have

$$\begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}^{-1} \begin{bmatrix} A + bD_p c & bC_p \\ B_p c & A_p \end{bmatrix} \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} = \begin{bmatrix} A + bK & -bK \\ 0 & A + Lc \end{bmatrix},$$

and hence the matrix $\begin{bmatrix} A + bD_p c & bC_p \\ B_p c & A_p \end{bmatrix}$ is Hurwitz. □

A.2 Nonsingularity of the Matrix $T(\theta)$ in Chapter 4.

Lemma A.2.1. Suppose that (A, c) is observable and $\theta = [\theta_1 \ \ \theta_2 \ \ \cdots \ \ \theta_m]^\top$ satisfies

$$\theta_i = \sum_{j_1 < j_2 < \cdots < j_i} \sigma_{j_1}^2 \sigma_{j_2}^2 \cdots \sigma_{j_i}^2, \tag{A.2.1}$$

where $i = 1, 2, \cdots, m$ and $j_1, j_2, \cdots, j_i \in \{1, 2, \cdots, m\}$. Then, under Assumption 4.2.2, the matrix $T(\theta)$ is nonsingular for θ of (A.2.1) with any σ_j's. $\Diamond$

Proof. From the definition of $T(\theta)$, it suffices to show that $(A_e(\theta), \bar{c})$ is observable, which is equivalent that the matrix

$$\begin{bmatrix} \begin{bmatrix} A - \lambda I \\ c \end{bmatrix} & \begin{bmatrix} -b & 0 \\ 0 & 0 \end{bmatrix} \\ 0_{(2m+1)\times n} & \bar{S}(\theta) - \lambda I \end{bmatrix} \tag{A.2.2}$$

has full column rank for each eigenvalue λ of either A or $\bar{S}(\theta)$. We first consider the case where λ is an eigenvalue of $\bar{S}(\theta)$. Because of Assumption 4.2.2, $\begin{bmatrix} A - sI & b \\ c & 0 \end{bmatrix}$ has full column rank for each s that is either zero or purely imaginary. Moreover, it can be shown from (A.2.1) that λ is purely imaginary, and hence $\begin{bmatrix} A - \lambda I & b \\ c & 0 \end{bmatrix}$ has full column rank for each λ. Thus, it follows from the structure of $\bar{S}(\theta)$ that (A.2.2) has full column rank for each λ. On the other hand, suppose that λ is an eigenvalue of A but not of $\bar{S}(\theta)$. Then, $\begin{bmatrix} A - \lambda I \\ c \end{bmatrix}$ has full column rank for each λ and so does (A.2.2). $\square$

A.3 Pseudo Code Implemented on the DSP Board in Chapter 4.

```
////////////////////////////////////////////////////////////////////////////////
// px1,px2,...,px11 and cx1,cx2,...,cx11 are state variables.
// Ts is a sampling time of AD and DA converter.
```

```
// a1, a2, and b2 are given ODD parameters.
// invb2 = 1/b2 (precalculated).
// L1,L2,...,L5 are designed observer gains.
// K is a designed adaptive gain.


// Calculate the differential equations
cx1 = px1 + Ts * ( -a1*px1 + px2 + (L1 + px7*K*px7)*(e - px1) );
cx2 = px2 + Ts * ( -a2*px1 + px3 + b2*u - e*px6 + (L2 + px8*K*px7)*(e - px1) );
cx3 = px3 + Ts * ( px4 + (-a1*e)*px6 + (L3 + px8*K*px7)*(e - px1) );
cx4 = px4 + Ts * ( px5 + (-a2*e + b2*u)*px6 + (L4 + px9*K*px7)*(e - px1) );
cx5 = px5 + Ts * ( (L5 + px9*K*px7)*(e - px1) );
cx6 = px6 + Ts * ( K*px7*(e - px1) );
cx7 = px7 + Ts * ( -a1*px7 + px8 - L1*px7 );
cx8 = px8 + Ts * ( -a2*px7 + px9 - L2*px7 - e);
cx9 = px9 + Ts * ( px10 - L3*px7 -a1*e );
cx10 = px10 + Ts * ( px11 - L4*px7 -a2*e + b2*u );
cx11 = px11 + Ts * ( -L5*px7 );


// Calculate the control u_r
ur = px6*invb2*px1 - invb2*px3;


// Update states
px1 = cx1; px2 = cx2;
px3 = cx3; px4 = cx4;
px5 = cx5; px6 = cx6;
px7 = cx7; px8 = cx8;
px9 = cx9; px10 = cx10;
px11 = cx11;

///////////////////////////////////////////////////////////////////////
```

A.4 Observability Property of the Pair (S, γ) in Chapter 5.

Claim: For some $k \in \{1, \cdots, r\}$, u_w does not contain the sinusoid of the frequency σ_k if and only if $[\gamma_{2k-1}, \gamma_{2k}]$ is a zero vector.

Proof. Consider the following system corresponding to σ_k of S

$$\begin{bmatrix} \dot{w}_{2k-1} \\ \dot{w}_{2k} \end{bmatrix} = \begin{bmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{bmatrix} \begin{bmatrix} w_{2k-1} \\ w_{2k} \end{bmatrix},$$

where $k \in \{1, \cdots, r\}$. Then we obtain

$$\begin{aligned} w_{2k-1}(t) &= A_k \sin(\sigma_k t + \phi_k), \\ w_{2k}(t) &= A_k \cos(\sigma_k t + \phi_k), \end{aligned} \tag{A.4.1}$$

where A_k and ϕ_k are some constants that depend on the initial conditions $w_{2k-1}(0)$ and $w_{2k}(0)$. From (A.4.1), we have

$$\begin{aligned} \begin{bmatrix} \gamma_{2k-1} & \gamma_{2k} \end{bmatrix} \begin{bmatrix} w_{2k-1}(t) \\ w_{2k}(t) \end{bmatrix} &= \gamma_{2k-1} A_k \sin(\sigma_k t + \phi_k) + \gamma_{2k} A_k \cos(\sigma_k t + \phi_k) \\ &= A_k \sqrt{\gamma_{2k-1}^2 + \gamma_{2k}^2} \sin\left(\sigma_k t + \phi_k + \arctan(\gamma_{2k}/\gamma_{2k-1})\right), \end{aligned}$$

where A_k is nonzero since the initial condition $[w_{2k-1}(0), w_{2k}(0)]^\top \neq 0$, and thus it is clear that

$$\begin{aligned} u_w &= \gamma w \\ &= \gamma_{2r+1} w_{2r+1} + \sum_{k=1}^{r} \left(A_k \sqrt{\gamma_{2k-1}^2 + \gamma_{2k}^2} \sin\left(\sigma_k t + \phi_k + \arctan(\gamma_{2k}/\gamma_{2k-1})\right) \right), \end{aligned}$$

where $\gamma_{2r+1} w_{2r+1}$ is a constant real number that depends on the initial condition $w_{2r+1}(0)$. Therefore, from $A_k \neq 0$ and the orthogonality of trigonometric functions [Ful96], it follows that the proof is complete. □

A.5 Structure of the Matrix $T_c(\theta)$ in Chapter 5.

If a system $\dot{x} = Ax \in \mathbb{R}^n$ has the output $y = cx \in \mathbb{R}$ with $\det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$, then,

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_1 & 1 & 0 & \cdots & 0 \\ a_2 & a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & 1 \end{bmatrix} \begin{bmatrix} c \\ cA \\ cA^2 \\ \vdots \\ cA^{n-1} \end{bmatrix} x$$
$$= \begin{bmatrix} cx \\ a_1 cx + cAx \\ a_2 cx + a_1 cAx + cA^2 x \\ \vdots \\ a_{n-1}cx + a_{n-2}cAx + \cdots + cA^{n-1}x \end{bmatrix}$$

the system is converted into the following form

$$\begin{aligned} \dot{z}_1 &= cAx = z_2 - a_1 z_1, \\ \dot{z}_2 &= a_1 cAx + cA^2 x = z_3 - a_2 z_1, \\ &\vdots \\ \dot{z}_n &= a_{n-1}cAx + a_{n-2}cA^2 x + \cdots + cA^n x = c(a_{n-1}A + a_{n-2}A^2 + \cdots + A^n)x \\ &= -a_n cx = -a_n z_1 \end{aligned} \tag{A.5.1}$$

with $y = z_1$, in which Cayley-Hamilton theorem [Che84, Corollary 2-12] is used. In this form, the state z is observable from y (regardless whether (A, c) is observable or not). If (A, c) is observable, then the inverse map from z to x exists so that x can be recovered from y.

Now, it is shown that the matrix $T_c(\theta)$ (in (5.2.3) with $\hat{\theta}$ replaced by θ, or (5.3.11)) can also be written as (5.3.9). For convenience, we rewrite (5.3.10) as $s^{\nu+2m+1} + \alpha_1 s^{\nu+2m} + \cdots + \alpha_{\nu+2m}s + \alpha_{\nu+2m+1} = (a_0 s^\nu + a_1 s^{\nu-1} + \cdots + a_\nu) \cdot$

$(\phi_0 s^{2m+1} + \phi_1 s^{2m} + \cdots + \phi_{2m} s + \phi_{2m+1})$, in which $\alpha_{\nu+2m+1} = 0$, $a_0 = \phi_0 = 1$, $\phi_j = 0$ for odd j, and $\phi_{2k} = \theta_k$ for $k = 1, \cdots, m$ (compare with (5.3.10)). Then,

$$\alpha_k = \sum_{i+j=k, i\geq 0, j\geq 0} a_j \phi_i.$$

For more convenience, let us define the convention $\phi_j = 0$ for $j < 0$ and $j > 2m+1$, $a_j = 0$ for $j < 0$ and $j > \nu$, and $b_j = 0$ for $j < 1$ and $j > \nu$. Let s_k be a row vector such that

$$T_c(\theta) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \alpha_1 & 1 & 0 & \cdots & 0 \\ \alpha_2 & \alpha_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{\nu+2m} & \alpha_{\nu+2m-1} & \alpha_{\nu+2m-2} & \cdots & 1 \end{bmatrix} \begin{bmatrix} \bar{c} \\ \bar{c}\bar{A}(\theta) \\ \bar{c}\bar{A}^2(\theta) \\ \vdots \\ \bar{c}\bar{A}^{\nu+2m}(\theta) \end{bmatrix} =: \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ \vdots \\ s_{\nu+2m+1} \end{bmatrix}$$

with

$$\bar{A}(\theta) = \begin{bmatrix} \overbrace{\begin{matrix} -a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{\nu-1} & 0 & \cdots & 1 \\ -a_\nu & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{matrix}}^{\nu} & \overbrace{\begin{matrix} -b_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -b_{\nu-1} & 0 & \cdots & 0 \\ -b_\nu & 0 & \cdots & 0 \\ -\phi_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\phi_{2m} & 0 & \cdots & 1 \\ -\phi_{2m+1} & 0 & \cdots & 0 \end{matrix}}^{2m+1} \end{bmatrix}, \quad \bar{c} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^\top \in \mathbb{R}^{1\times(\nu+2m+1)}$$

as in (5.3.8). Then, using the convention, it is enough to show the following.

Claim: $s_k \in \mathbb{R}^{1\times(\nu+2m+1)}$ has the form

$$s_k = [\phi_{k-1}, \phi_{k-2}, \cdots, \phi_{k-\nu}, -b_{k-1}, -b_{k-2}, \cdots, -b_{k-2m-1}] \tag{A.5.2}$$

for $k = 1, \cdots, \nu + 2m + 1$.

Proof. Clearly, it holds with $k = 1$ since $s_1 = \bar{c} = [1, 0, \cdots, 0]$. Suppose that (A.5.2) holds for k, and we show it holds also for $k+1$. The discussion around (A.5.1) yields that

$$s_{k+1} = s_k \bar{A}(\theta) + \alpha_k s_1.$$

In particular, by the structure of $\bar{A}(\theta)$,

$$\begin{aligned} & s_k \bar{A}(\theta) + \alpha_k s_1 \\ & \quad = \Bigg[-\sum_{j=1}^{\nu} a_j \phi_{k-j},\ \phi_{k-1},\ \phi_{k-2},\ \cdots,\ \phi_{k-\nu+1},\ -\sum_{j=1}^{\nu} b_j \phi_{k-j} + \sum_{i=1}^{2m+1} \phi_i b_{k-i}, \\ & \qquad\qquad -b_{k-1},\ \cdots,\ -b_{k-2m} \Bigg] + \Bigg[\sum_{i+j=k, i\geq 0, j\geq 0} a_j \phi_i,\ 0,\ \cdots,\ 0 \Bigg] \end{aligned}$$

and we note that the first element is written as

$$\begin{aligned} -\sum_{j=1}^{\nu} a_j \phi_{k-j} + \sum_{i+j=k, i\geq 0, j\geq 0} a_j \phi_i &= -\sum_{j=1}^{\nu} a_j \phi_{k-j} + \sum_{j=0}^{k} a_j \phi_{k-j} \\ &= -\sum_{j=1}^{\nu} a_j \phi_{k-j} + \sum_{j=1}^{k} a_j \phi_{k-j} + a_0 \phi_k \\ &= a_0 \phi_k = \phi_k. \end{aligned}$$

Moreover, by the convention, the $\nu + 1$ element satisfies that

$$-\sum_{j=1}^{\nu} b_j \phi_{k-j} + \sum_{i=1}^{2m+1} \phi_i b_{k-i} = -\sum_{j=-\infty}^{\infty} b_j \phi_{k-j} + \left(\sum_{i=-\infty}^{\infty} \phi_i b_{k-i} - \phi_0 b_k \right) = -b_k.$$

□

A.6 Convergence Property of $\det^2(\Omega_i(t))$ in Lemma 5.3.2.

Claim 1: In the equation (5.3.16), $\det^2(\Omega_i(t)) \geq \vartheta > 0$ for $i = 1, \cdots, l$ where ϑ is a positive number.

Proof. It follows from the equations (5.3.15) and (5.3.16) that

$$\begin{aligned}
\Omega_i(t+T) &= e^{-\lambda_b(t+T)}\Omega_i(0) \\
&\quad + \lambda_c \int_0^{t+T} \left(e^{-\lambda_b(t+T-\tau)} \cdot [\mu_1(\tau), \cdots, \mu_i(\tau)]^\top [\mu_1(\tau), \cdots, \mu_i(\tau)] \right) d\tau \\
&\geq \lambda_c e^{-\lambda_b T} \int_t^{t+T} \left(e^{-\lambda_b(t-\tau)} \cdot [\mu_1(\tau), \cdots, \mu_i(\tau)]^\top [\mu_1(\tau), \cdots, \mu_i(\tau)] \right) d\tau \\
&\geq \lambda_c e^{-\lambda_b T} \int_t^{t+T} \left([\mu_1(\tau), \cdots, \mu_i(\tau)]^\top [\mu_1(\tau), \cdots, \mu_i(\tau)] \right) d\tau \\
&\geq \lambda_c e^{-\lambda_b T} \kappa_1 I_i.
\end{aligned}$$

Then, we obtain

$$\Omega_i(t) \geq \begin{cases} e^{-\lambda_b T}\Omega_i(0) > 0, & \text{if } 0 \leq t \leq T, \\ \lambda_c e^{-\lambda_b T}\kappa_1 I_i > 0, & \text{if } t > T. \end{cases}$$

Since $\Omega(0)$ is positive definite, $e^{-\lambda_b T}\Omega_i(0)$ is positive definite as well as $\lambda_c e^{-\lambda_b T}\kappa_1 I_i$. Therefore, for $i = 1, \cdots, l$, we have

$$\begin{aligned}
\det^2(\Omega_i(t)) &\geq \min\left\{ \det^2\left(e^{-\lambda_b T}\Omega_i(0)\right),\ \det^2\left(\lambda_c e^{-\lambda_b T}\kappa_1 I_i\right)\right\} \\
&=: \vartheta > 0 \quad \text{for all } t \geq 0
\end{aligned}$$

because if $M \geq N$ then $\det^2(M) \geq \det^2(N)$ for any positive definite matrices M and N (due to the fact that the k-th largest eigenvalue of M is greater than the k-th largest eigenvalue of N and the determinant of any matrix is the product of its all eigenvalues). □

Claim 2: In the equation (5.3.16), $\det^2(\Omega_{l+1}(t)), \cdots, \det^2(\Omega_m(t))$ tend exponentially to zero as time goes to infinity.

Proof. From (5.3.7), suppose that

$$u_w = \tilde{\gamma}\tilde{w} = c_0 + \sum_{1 \leq j \leq r, [\gamma_{2j-1}, \gamma_{2j}] \neq 0} c_j \cos(\sigma_j t + \varphi_j),$$

where constants c_0 and φ_j depend on the initial condition $\tilde{w}(0)$. Referring to (5.3.14), it is seen that the transfer function from u_w to μ_i is given by

$$\begin{aligned} H_m(s) &= \frac{b_1 s^\nu + \cdots + b_{\nu-1}s^2 + b_\nu s}{\det(sI - (A_c - Lc_c))}, \quad && i = m, \\ H_i(s) &= s^{2(m-i)} H_m(s), && i < m. \end{aligned}$$

Since $H_i(s)$ is a stable LTI system, the signal $\mu_i(t)$ converges exponentially to its steady-state

$$\mu_i^{ss}(t) = \sum_{1 \le j \le r, [\gamma_{2j-1}, \gamma_{2j}] \neq 0} (-\sigma_j^2)^{m-i} \cdot \bar{c}_j \cdot \cos(\sigma_j t + \bar{\varphi}_j), \quad 1 \le i \le m,$$

where $\bar{c}_j$ and $\bar{\varphi}_j$ are determined by $H_m(s)$ from c_j, φ_j, and σ_j. The bias term c_0 disappears since $H_i(s)$ has a zero at the origin.

Now, we prove that $\det^2(\Omega_m(t))$ tends exponentially to zero as time goes to infinity. Let $\mu^{ss} := [\mu_1^{ss}, \cdots, \mu_m^{ss}]^\top$, then we obtain

$$\mu^{ss}(t)\mu^{ss}(t)^\top = V(\sigma)D(\bar{c})P_\sigma(t)D(\bar{c})V(\sigma)^\top, \tag{A.6.1}$$

where

$$\begin{aligned} V(\sigma) &= \begin{bmatrix} (-\sigma_1^2)^{m-1} & (-\sigma_2^2)^{m-1} & \cdots & (-\sigma_m^2)^{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ (-\sigma_1^2) & (-\sigma_2^2) & \cdots & (-\sigma_m^2) \\ 1 & 1 & \cdots & 1 \end{bmatrix}, \\ D(\bar{c}) &= \mathrm{diag}(\bar{c}_1, \cdots, \bar{c}_m), \\ P_\sigma(t) &= \begin{bmatrix} \cos(\sigma_1 t + \bar{\varphi}_1) \\ \vdots \\ \cos(\sigma_{m-1} t + \bar{\varphi}_{m-1}) \\ \cos(\sigma_m t + \bar{\varphi}_m) \end{bmatrix} \begin{bmatrix} \cos(\sigma_1 t + \bar{\varphi}_1) \\ \vdots \\ \cos(\sigma_{m-1} t + \bar{\varphi}_{m-1}) \\ \cos(\sigma_m t + \bar{\varphi}_m) \end{bmatrix}^\top. \end{aligned}$$

Note that $\bar{c}_k$, $k = 1, \cdots, m$, is zero if u_w does not contain the sinusoid of frequency σ_k, and the Vandermonde matrix $V(\sigma)$ has a nonzero determinant [Kai80]. It follows from the equation (5.3.16) and (A.6.1) that the steady-state of $\Omega_m(t)$ is given by

$$\Omega_m^{ss}(t) = \Omega^{ss}(t) = \lambda_c \cdot V(\sigma)D(\bar{c}) \cdot \left(\int_0^t e^{-\lambda_b(t-\tau)} P_\sigma(\tau) d\tau \right) \cdot D(\bar{c})V(\sigma)^\top$$

and its determinant is obtained as follows:

$$\begin{aligned} &\det\left(\Omega_m^{ss}(t)\right) \\ &= \lambda_c \cdot \det\left(V(\sigma)\right) \det\left(D(\bar{c})\right) \det\left(\int_0^t e^{-\lambda_b(t-\tau)} P_\sigma(\tau) d\tau \right) \det\left(D(\bar{c})\right) \det\left(V(\sigma)^\top \right). \end{aligned}$$

Because l is the number of distinct frequencies observed in u_w, if $m > l$ then $\det\left(D(\bar{c})\right) = 0$, which implies $\det\left(\Omega_m^{ss}(t)\right) = 0$. Similarly, it is easy to see that $\det\left(\Omega_i^{ss}(t)\right) = 0$ for $l + 1 \leq i < m$. □

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국문초록

Adaptive Output Regulation for Linear Systems with Unknown Sinusoidal Exogenous Inputs

미지의 정현파 외부 입력을 갖는 선형시스템을 위한 적응 출력 제어

본 논문은 미지 정현파의 외부시스템(exosystem)에 의해 영향을 받는 선형시스템에 대한 출력제어문제(output regulation problem) (기준입력과 외란이 자율미분방정식 (autonomous differential equation)에 의해 생성될 때, 그 기준입력의 점진적인(asymptotic) 추적과 외란 제거에 대한 문제)를 연구한다. 이전의 연구들과 달리, 우리의 궁극적인 목표는 크기, 위상, 바이어스(bias), 주파수, 그리고 그 주파수 개수도 알 수 없는 외부시스템에 의해 생성되는 정현파의 외부입력에 대해 플랜트(plant) 출력을 원점으로 점진적인 제어를 달성하는 것이다. 여기서, 플랜트는 모델 불확실성이 없는 (비 최소위상(non-minimum phase)을 포함하는) 선형 시 불변 단일입력 단일출력 시스템이다.

최종 제어목표를 달성하기 전에, 우리는 미지의 외부입력에 대한 주파수 개수를 알고 있다는 가정 하에 출력제어문제를 우선 생각하기로 한다. 대신 그 외부입력의 크기, 위상, 바이어스, 그리고 주파수는 모른다고 가정한다. 그 문제를 해결하기 위하여, 적응관측기(adaptive observer)와 함께하는 애드온(add-on)의 출력제어기가 제안된다. 지속적으로 가진되는(persistently exciting) 특징에 기반을 둔 그 적응관측기는 정현파의 외부입력에 대한 주파수를 추정하기 위하여 사용된다. 그와 동시에 플랜트와 외부시스템의 상태도 추정한다. 또한, 애드온형태의 제어기는 이미 플랜트와 동작 중에 있는 기존에 설치된 제어기와 조화롭게 동작하는 특성을 갖는다. 기존에 설치된 제어기로 원하는 제어성능을 만족시킬 수 없을 때, 제시된 애드온제어기가 사용될 수 있다. 그 제안된 제어기는 기존의 제어기에 대한 어떤 정보도 없이 설계가능하며, 불필요한 과도응답을 야기하지 않으면서 언제든 되먹임 루프에 연결될 수 있다. 상용 광디스크드라이브 시스템의 트랙추종 제어에 대한 모의실험과 실제실험은 제안한 제어방법이 효과적임을 보여준다.

다음단계로, 우리는 그 외부입력의 크기, 위상, 바이어스, 주파수뿐만 아니라 그 주파수의 개수도 모르는 경우를 다룬다. 이를 위하여, 플랜트가 복소평면의 허수축에 영점(zeros)을 가지지 않고 미지의 주파수 개수에 대한 상계(upper bound)

를 알고 있다는 가정아래에 폐형식 (closed-form) 의 해법이 제시된다. 특히, 제시된 제어기는 그 미지의 주파수를 추정하기 위하여 지속적으로 가진되는 특징을 필요로 하지 않는다. 이를 위하여, 그 주파수와 주파수 개수를 동시에 추정하는 적응관측기가 제시되며, 이것은 중요한 특징이다. 왜냐하면, 미지의 매개변수는 적응제어로 추정되기 때문에 일반적으로 충분히 지속적으로 가진되는 특징이 요구되기 때문이다. 게다가, 우리는 과도상태에서 특이점문제 (singularity problem) 를 피하고 동시에 정상상태에서 출력제어를 달성하기 위하여 오직 플랜트의 매개변수를 사용해서 계산할 수 있는 데드밴드 (dead-band) 를 가진 적절한 데드존 (dead-zone) 함수를 제시한다.

주요어: 출력 제어, 적응관측기, 지속적인 가진, 정현파의 외부입력, 애드온, 광디스크드라이브

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