Ph.D dissertation

A Passivity-based Nonlinear Observer and a Semi-global Separation Principle

February, 2000

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ABSTRACT

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The main topics of this dissertation are the observer problem and its applications to the output feedback stabilization for nonlinear systems. The observer problem refers to the general problem of reconstructing the state of a system only with the input and output information of the system. While the problem has been solved in depth for linear systems, the nonlinear counterpart has not yet been wholly solved in the general sense. Motivated by this fact, we pursue the general method of observer construction in order to provide much larger classes of systems with the design method. In particular, we propose a new approach to the observer problem via the passivity, which is therefore named the passivity framework for state observer. It begins by considering the observer problem as the static output feedback stabilization for a suitably defined error dynamics. We then make use of the output feedback

passification which is the recent issue in the literature, to the design of observer as a tool for static output feedback stabilization. The proposed framework includes the precise definition of passivity-based state observer (PSO), the design scheme of it, and the redesign technique for a given PSO to have the robust property to the measurement disturbances in the sense of input-to-state stability. Moreover, it is also shown that the framework of PSO provides the unified viewpoint to the earlier works on the nonlinear observer and generalizes them much more.

As well as the new notion of PSO, two other methods of observer design are proposed for the special classes of nonlinear systems. They are, in fact, a part of or an extension of the design scheme of PSO. However, compared to the general design scheme of PSO, these methods specifically utilize the particular structure of the system, which therefore lead to more explicit techniques for the observer design. The first one we present for the special cases is the semi-global observer, which extends with much flexibility the earlier designs of Gauthier's high-gain observer. By introducing the saturation function into the observer design, several difficulties to construct the high-gain observer (e.g. peaking phenomenon, etc.) are effectively eliminated. As the second result, we propose a novel design method for the nonlinear observer, which may be regarded as the *observer backstepping* since the design is recursively carried out similarly to the well-known backstepping control design. It enlarges the class of systems, for which the observer can be designed, to the systems that have the non-uniformly observable modes and detectable modes as well as uniformly observable modes.

The other topic of the dissertation is the output feedback stabilization of nonlinear systems. Our approach to the problem is the state feedback control law *plus* the state observer, therefore, in view of the so-called separation principle. The benefits of the approach via separation principle is that the designs of state feedback law and observer are completely separated so that any state feedback and any observer can be combined to yield the output feedback controller, which is well-known for linear

systems. Unfortunately, it has been pointed out that the separation principle for nonlinear systems does not hold in the global sense, and thus the alternative semi-global separation principle (i.e., the separation principle on a bounded region rather than on the global region) has been studied so far. In this dissertation, we continue that direction of research and establish the semi-global separation principle that shares the more common properties with the linear one than the earlier works do. In particular, it is shown that, for general nonlinear systems, when a state feedback control stabilizes an equilibrium point with a certain bounded region of attraction, it is also stabilized by an output feedback controller with arbitrarily small loss of the region, under uniform observability. The proposed output feedback controller has the dynamic order n which is the same as the order of the plant, which is the essential difference from the earlier works. As a consequence, the nonlinear separation principle enables the state observer of the dissertation to be used in conjunction with any state feedback for the output feedback stabilization, although the observer problem in itself is worthwhile in several practical situations.

Keywords: observer, nonlinear system, passivity, separation principle, output feedback stabilization

Student Number: `95412-806`

To my wife Sehwa and lovely daughter SJ.

Contents

Notation and Symbols

$\mathbb{R}$	fields of real numbers
$\mathbb{R}^n$	real Euclidean space of dimension n
$\mathbb{R}^{m\times n}$	space of $m \times n$ matrices with real entries
A^T	transpose of $A \in \mathbb{R}^{m\times n}$
$\lambda_{\max}(A)(\lambda_{\min}(A))$	the largest (smallest) eigenvalue of $A \in \mathbb{R}^{n\times n}$
$A > 0$ ($A \geq 0$)	implies A is symmetric and positive definite (semidefinite).
I_n	$n \times n$ identity matrix (Subscript n is omitted when there is no confusion.)
$\mathrm{diag}(A_1, \ldots, A_n)$	a block diagonal matrix with A_i as the i-th diagonal matrix
$:=$	defined as
$\blacksquare$	end of proof
$\lVert \cdot \rVert$	Euclidean norm
$\lVert \cdot \rVert_{\mathcal{A}}$	Euclidean norm with respect to a set $\mathcal{A}$, i.e., $\lVert x\rVert_{\mathcal{A}}$ denotes $\inf_{y\in\mathcal{A}} \lVert x - y\rVert$.
∂A	boundary of a set A

- When x (or, $f(x)$) is a vector (or, vector-valued function), x_i (or, $f_i(x)$) represents its i-th element.

- A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *Lipschitz on* $A \subset \mathbb{R}^n$, when there is a constant L such that
$$\|f(x) - f(y)\| \leq L\|x - y\|$$
for all $x, y \in A$.

- A function $f(x, u)$ is said to be *Lipschitz in* x when there is a function $c(u)$ such that $\|f(x, u) - f(z, u)\| \leq c(u)\|x - z\|$. If, furthermore, $c(u)$ is a constant independent of u, then $f(x, u)$ is said to be *Lipschitz in* x *uniformly in* u.

- A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *positive definite* when $f(0) = 0$ and $f(x) > 0$ for $x \neq 0$.

- A function $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be a $\mathcal{K}$ function if it is continuous, strictly increasing, and satisfies $\psi(0) = 0$. It is a $\mathcal{K}_\infty$ function if in addition $\psi(s) \to \infty$ as $s \to \infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be a $\mathcal{KL}$ function if, for each fixed $t \geq 0$, $\beta(\cdot, t)$ is a $\mathcal{K}$ function and for each fixed $s \geq 0$, $\beta(s, \cdot)$ is decreasing to zero.

- For a system $\dot{x} = f(x)$, let $x(0)$ denote the initial condition and $x(t)$ the solution trajectory from $x(0)$.

- A function is said to be C^k if it is continuously differentiable k times, and a smooth function implies C^∞ function.

- Lie derivative of a vector field f with respect to h is defined to be $L_f h = dh \cdot f$ where dh is the gradient of the function h.

- Lie bracket of two vector fields $f(x)$ and $g(x)$, denoted by $[f, g]$, can be calculated as
$$[f, g] = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x).$$

- Higher derivatives are defined as $L_f^i h = L_f(L_f^{i-1} h)$ and $\mathrm{ad}_f^i g = \mathrm{ad}_f(\mathrm{ad}_f^{i-1} g)$ with $\mathrm{ad}_f g = [f, g]$.

- $\mathrm{Span}\{f_1, f_2, \cdots, f_n\}$ denotes the set of vector fields which can be written as linear combinations of the vector fields $f_1, f_2, \cdots, f_n$, with the coefficients being C^∞ functions.

Chapter 1

Introduction

1.1 The Scope

The main topics of this dissertation are the state observer for nonlinear dynamic systems and its application to the output feedback stabilization. Before getting to the specific problems, it has to be pointed out that all the analyses and syntheses in this dissertation are strictly accompanying the underlying region in state-space, on which a certain property is valid. In order to emphasize the importance of the region and simultaneously to guide to the problems of the dissertation, we begin by the comparison between the linear and nonlinear systems.

Linear vs. Nonlinear

A physical system is destined to be linear or nonlinear when it is mathematically modeled. Strictly speaking, most of dynamical systems in nature is intrinsically nonlinear, and a linear system is often the simplified model representing the real system. Once a real system is modeled by a linear one, there are abundant tools and methods to analyze the system and to control it, which yields the benefits for the linear model. On the contrary, a nonlinear model for the real system requires the

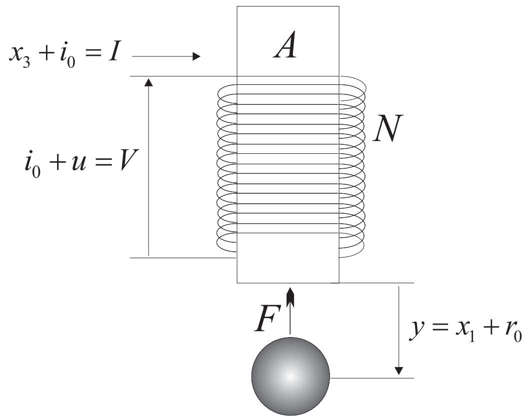


Figure 1.1: A typical magnetic levitation system

complex calculations and difficult techniques for its control, and there are relatively few tools and methods.

Nevertheless, when an engineer makes a nonlinear model, his/her intension is usually to represent an extraordinary phenomenon for which a linear model cannot stand, or to control the real system with higher precision on the larger region of state-space.

For example, consider the mathematical model for the real system (Fig. 1.1):

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{\mu_0 N^2 A}{4m}\left(\frac{x_3 + i_0}{x_1 + r_0}\right)^2 + G \\
\dot{x}_3 &= \frac{x_2(x_3 + i_0)}{x_1 + r_0} - \frac{2R}{\mu_0 N^2 A}(x_1 + r_0)x_3 + \frac{2(x_1 + r_0)}{\mu_0 N^2 A}u
\end{aligned} \tag{1.1.1}$$

where x_1 is the vertical positional deviation of the iron ball from the reference position r_0, x_2 is the vertical velocity of the ball, x_3 is the current deviation of the magnet from the steady-state current i_0, and u is the voltage deviation from the steady-state voltage Ri_0, in which, $i_0 = \sqrt{(4mr_0^2 G)/(\mu_0 N^2 A)}$ is the steady-state current[1].

[1]The model (1.1.1) is obtained from

This nonlinear model represents the real system well enough. Indeed, the model has a singularity when the ball is attached to the upper magnet ($x_1 = -r_0$), and the effect of the input voltage also varies according to the vertical position in a bilinear fashion.

On the other hand, a linear model for the system is given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{bi_0^2}{mr_0^2}x_1 - \frac{bi_0}{mr_0}x_3 \\ \dot{x}_3 &= \frac{i_0}{r_0}x_2 - \frac{R}{b}x_3 + \frac{1}{b}u \end{aligned} \tag{1.1.2}$$

where $b := (\mu_0 N^2 A)/(2r_0)$. This linear model is simple and admits a plentiful methodology for the control. However, even though this model coincides the real system locally around the steady-state, it does not well describe the real system away from the steady-state. For example, even if the stabilizing control for (1.1.2) has the whole region $\mathbb{R}^n$ as the region of attraction, the real situation is not so desirable.

Therefore, a control engineer makes the model after considering merits and demerits of linear and nonlinear model, and this dissertation is devoted to the development of the theory for the observer and output feedback stabilization problem based on the nonlinear model, anticipating the compensation of the demerits—the relative lack of tools and methods for nonlinear models.

- $m\ddot{y} = mG - F$ where m is the mass and G is the acceleration of gravity.
- $F = \frac{\mu_0 N^2 A}{4y^2} I^2$ where A is crosssectional area of the magnet, N is the number of turns, and $\mu_0 = 4 \times 10^7\ \pi$.
- $V = RI + \dot{L}I + L\dot{I} = RI + \frac{\mu_0 N^2 A}{2y}\dot{I} - \frac{\mu_0 N^2 AI}{2y^2}\dot{y}$ since $L = \frac{\mu_0 N^2 A}{2y}$, in which R is the resistance of magnet and V is the input voltage.

and, the translation of the steady-state for the reference position r_0, to the origin.

Local, Semi-global and Global Properties

The tools and methods for nonlinear models (e.g. feedback linearization, observer construction, asymptotic tracking, nonlinear $\mathcal{H}_\infty$, etc.) are said to be local, semi-global or global according to their applicable region in the state-space. For example, the feedback stabilizability can be classified as follows.

Definition 1.1.1. An equilibrium point $x = 0$ of a nonlinear system is *locally state (respectively, output) feedback stabilizable* if there exists a feedback control law using the information of the state x (respectively, the output y) such that the closed-loop system is locally asymptotically stable, more precisely, there is an open region of attraction containing the origin.

Definition 1.1.2. An equilibrium point $x = 0$ of a nonlinear system is *semi-globally state (respectively, output) feedback stabilizable* if, for each compact set $\mathcal{K}$ which is a neighborhood of the origin, there exists a feedback control law using the information of the state x (respectively, the output y) such that the region of attraction contains $\mathcal{K}$.

Definition 1.1.3. An equilibrium point $x = 0$ of a nonlinear system is *globally state (respectively, output) feedback stabilizable* if there exists a feedback control law using the information of the state x (respectively, the output y) such that the closed-loop system is globally asymptotically stable, more precisely, the region of attraction is the whole space of $\mathbb{R}^n$.

By these definitions, it can be seen that when a design method of stabilizing control law has the local property, the effective region of attraction is the *outcome* of the design, thus is not a specification for the design. On the contrary, the global property is desirable because the designer need not consider the valid region of the method since the whole region is valid and his/her region of interest is always contained in the valid region. However, the global methodology often restricts the class of systems to which the method is applicable. Recall the system (1.1.1) which has the singularity at $x_1 = -r_0$. To stabilize this system, the global theory is a bit exaggerated and the local theory may be unsatisfactory in view of the region.

The semi-global approach is now appealing because it provides the tools and methods which are effective on a *bounded* region whose size can be arbitrarily large. It is the remedy to the problems for which the global theory cannot give solutions due to the unboundedness of the underlying region, and the local theory cannot either because the region of interest is so large.

Most of the works in this dissertation are devoted to the semi-global property. By the 'semi-global property', it is meant, in this dissertation, that a property or a method is applicable on any *bounded* region of the state-space.

State Feedback Controller vs. State Observer

Nonlinear control theory has been attracted great deal of attention in the past few decades, but the relatively little attention seems to have been paid for the observer problem for nonlinear systems. In order to develop a control method for nonlinear systems, lots of researchers have been extracting the core properties from the linear theory and generalizing them to the nonlinear systems. In this dissertation, the similar trials are continued for constructing the state observer instead of the controller.

From the linear system theory, the state feedback stabilizing problem and the observer problem are known to be dual. However, since this duality is originated from the structure of linear systems, but not from the physical meanings, this beneficial property diminishes as the research goes towards the nonlinear systems. Therefore, the development of nonlinear control does not accompany the development of the observer in general, and the research on the observer should be independently performed.

In this dissertation, the concept of uniform observability will be introduced and the new methods for the observer construction will be presented using Lipschitz extension, the saturation functions or the derivatives of inputs. These are also the generalizations of the linear observer, but there is no duality for nonlinear controls.

The observer problem has its own interest independently of the state feedback control for a system (e.g. fault detection, diagnostics, GPS, etc.). Motivated by this fact, some of the proposed observers in this dissertation assume the state and the input of the system are *bounded* but the size of the actual bound does not matter. By virtue of this assumption which is reasonable for aforementioned situations, the generalization is obtained in the theoretical level, i.e., the class of systems for the observer exists is extended.

State Feedback Controller plus State Observer

One exception of the above assumption (the boundedness of the state and the input) is when the observer is used in conjunction with the state feedback control law for the stabilization of a system. In this case, the boundedness of the state is one of the objectives and cannot be assumed. This is the topic of the output feedback stabilization problem discussed in Chapter 5, in other words, the problem of state feedback plus observer is considered, which is a natural generalization of the separation principle in the linear system theory.

As a result, an elegant theory is obtained that when a nonlinear system is stabilized in a certain bounded region of attraction with state feedback, then it can also be stabilized by output feedback with arbitrarily small loss of the region if the system is uniformly observable. Again, the boundedness of the region plays the key role in the development.

The result is useful especially for the system like (1.1.1) which has the singularity at some point. This issue will be discussed in Chapter 5.3.

1.2 Contributions of the Dissertation

Major contributions of this dissertation are

- **Multi-output extension of Gauthier's observer [GHO92]** (Chapter 3.4): The observer construction for the uniformly observable nonlinear systems, which has multi-output, is not a trivial extension of the Gauthier's result for single-output case. This is due to the 'peaking phenomenon' [SK91] originated by the high-gain nature of Gauthier's observer. By introducing the saturation function (Lipschitz extension) inspired by the semi-global concept, the Gauthier's observer is successfully reformulated as a semi-global observer for multi-output nonlinear systems. A characterization of the class is also provided using the differential geometric conditions.

- **Recursive design algorithm for nonlinear observer** (Chapter 3.5): As another extension of Gauthier's observer, a design method is presented for a class of multi-output systems which include the uniform, non-uniform and detectable modes in all. The design procedure is recursive and much resembles the backstepping method for designing the state feedback control, thus it can be regarded as the *observer backstepping*[2]. A notable feature of this design is that the boundedness of input (and its derivatives) is assumed *a posteriori* in contrast to the Gauthier's observer. (Gauthier's observer requires *a priori* knowledge of the bound which is utilized at the design stage.)

- **Passivity framework for nonlinear state observer** (Chapter 4): This is a new viewpoint for the observer problem. At first, the observer problem is seen as the static output feedback stabilization problem of the suitably defined error dynamics. Then, the strategy is the output feedback passification to the error dynamics, which is the recent issue in the passivity literature. In

[2]Unfortunately, this term has already been used in the literature with other implications. See [RJ98, KKK95].

order to describe the passivity of the error dynamics effectively, we discuss the extended passivity in Section 2.3, where the standard passivity is reformed for the augmented error dynamics which also includes the plant dynamics. The proposed framework includes the precise definition of Passivity-based State Observer (PSO) and the design scheme of PSO. It is also shown that a PSO has its potential robustness to the measurement disturbance. With these tools, the framework of PSO provides a new viewpoint on the earlier works in the literature and unifies them.

- **Semi-global separation principle** (Chapter 5): It is shown that, for general nonlinear systems, when a state feedback control stabilizes an equilibrium point of a plant with a certain bounded region of attraction, it is also stabilized by an output feedback controller with arbitrarily small loss of the region. Moreover, the proposed output feedback controller has the dynamic order n which is the same as the order of the plant. From any given state feedback, an explicit form of the overall controller is provided. A sufficient condition presented for the result is shown to be necessary and sufficient for regional uniform observability when the system is input affine. Thus, the result can be regarded as a regional separation principle for affine nonlinear systems.

Also, there are several minor contributions.

- The robustness of passivity under unmodeled dynamics is studied. Especially, it is shown that the passivity is not robust under unmodeled dynamics even if it is sufficiently fast, but can be robust if a certain structural condition is assumed. (Chapter 2.2)

- The scattered definitions of uniform observability in the literature are unified and their equivalences are shown. (Chapter 3.1)

- The scheme of Gauthier's observer is enriched with the characterization using differential geometric conditions. (Chapter 3.2)

- The practical realization methods for Lipschitz extension are presented with the construction of an observer in the x-coordinates. (Chapter 3.3)

Chapter 2

Advanced Passivity Formalism

In the context of linear system theory, the passive system is known as positive real system because the real part of its transfer function has nonnegative value when it is evaluated on $j\omega$-axis. The concept is known to have been originated from the linear circuit theory in which the circuit comprises the passive components such as resistors, capacitors and inductors. Since a passive system does not generate the energy in some sense, the concept is related to the stability of the system, and for this reason the passivity has been introduced into the control literature [vdS96, Lin95b, OJH97, FH98, BIW91].

The state-space formulation of the passivity has been introduced by Willems [Wil72] with the general notions of dissipativity, storage function and supply rate. In particular, Byrnes *et al.* [BIW91] solved the problem when and how the given nonlinear system can be made passive by a feedback, i.e. the *passification* problem. Afterwards, the passivity has been actively studied for the stabilization of nonlinear systems.

As well as introducing the basic concepts of passivity and passification, this chapter plays the preliminary role for Chapter 4. Furthermore, there are some stand-alone contributions in this chapter, to the analysis of passivity. More specifically,

- The standard passivity and passification [Kha96, SJK97, BIW91] are extended and some of their properties are discussed which have attracted little attention so far. The extended passivity in this chapter will be utilized in Chapter 4.

- Robustness of the passivity under unmodeled dynamics is discussed and a structural condition is presented for preserving the robustness. Motivated by the fact that the passivity is not preserved with stable unmodeled dynamics—even if it is sufficiently fast—in general, we propose sufficient conditions which guarantee the preservation of passivity of reduced model under sufficiently fast unmodeled dynamics. It is also illustrated that the proposed condition is readily met for a passive electric circuit.

2.1 Passivity and Passification

2.1.1 Passive Systems

Consider a dynamical system modeled by the smooth finite-dimensional ordinary differential equations:

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x) \tag{2.1.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input and $y \in \mathbb{R}^m$ is the output of the system. Without loss of generality, assume that $f(0) = 0$ and $h(0) = 0$. Let $\mathcal{X}$ be a connected open set of $\mathbb{R}^n$ which contains the origin, and $\mathcal{U}$ be an admissible set of input functions $u(\cdot)$. Also assume that $(\partial g/\partial x)(x)$ and $(\partial h/\partial x)(x)$ are of full rank in $\mathcal{X}$.

Definition 2.1.1. A system (2.1.1) is said to be *passive in* $\mathcal{X}$ if there exists a C^0 nonnegative function $V : \mathcal{X} \to \mathbb{R}$, called a *storage function*, such that $V(0) = 0$ and, for all $x_0 \in \mathcal{X}$,

$$V(x(t)) - V(x_0) \leq \int_0^t w(x(\tau), y(\tau), u(\tau))\, d\tau \tag{2.1.2}$$

$$w(x, y, u) = y^T u \tag{2.1.3}$$

for all $u(\cdot) \in \mathcal{U}$ and all $t \geq 0$ such that $x(\tau) \in \mathcal{X}$ for all $\tau \in [0, t]$, where $x(t)$ is a solution trajectory of the system with the input $u(\cdot)$ and the initial state x_0. The function w is called a *supply rate*.

There are various other definitions which can describe the excess, shortage or strictness of the passivity. These can be effectively defined by replacing (2.1.3) in the above definition.

- When $w(x, y, u) = u^T y - S(x)$ with a positive definite $S(x)$, the system (2.1.1) is said to be *Strictly Passive*.
- When $w(x, y, u) = u^T y - \sigma y^T y$ with a constant σ, the system (2.1.1) is said to be *Output Feedback Passive* (OFP(σ)).
- When $w(x, y, u) = u^T y - \sigma u^T u$ with a constant σ, the system (2.1.1) is said to be *Input Feedforward Passive* (IFP(σ)).

Also, when the corresponding storage function $V(x)$ is C^r and positive definite in $\mathcal{X}$, we put the prefix 'C^r PD-' to the above definitions. For example, if the system (2.1.1) is passive with C^r positive definite storage function, we call it C^r *PD-passive*.

For linear systems, it is well-known that there always exists a quadratic storage function when the system is passive (e.g. [FH98]), i.e. $\exists P \geq 0$ such that $V(x) = x^T P x$ in Definition 2.1.1. Moreover, when the system is minimal (controllable and observable), the corresponding P is positive definite [SJK97, p.28].

A sharp contrast between the linear and the nonlinear systems arises when we consider the region of passivity. For linear systems, once a system is passive in some set of $\mathbb{R}^n$ then the system is passive in the whole region $\mathbb{R}^n$. However, this is not the case for nonlinear systems, and therefore we've mentioned the passivity 'in $\mathcal{X}$'. The passivity in a region is effectively characterized by the following Kalman-Yakubovich-Popov lemma.

Nonlinear Kalman-Yakubovich-Popov Lemma

For analyzing and characterizing the passive systems, Definition 2.1.1 is not so useful. Instead, the following Lemma gives more useful way when the storage function is continuously differentiable.

Lemma 2.1.1 (Prop. 2.12 of [BIW91])**.** *A system* (2.1.1) *is C^1 passive in $\mathcal{X}$, if and only if, there exists a C^1 nonnegative function $V : \mathcal{X} \to \mathbb{R}$ such that, in $\mathcal{X}$,*

$$L_f V(x) \leq 0, \tag{2.1.4a}$$

$$L_g V(x) = h^T(x). \tag{2.1.4b}$$

Various passivities are also characterized by replacing (2.1.4a) as follows.

- The system is strictly passive if and only if $\exists$ a positive definite function $S(x)$ such that $L_f V(x) \leq -S(x)$ and $L_g V(x) = h^T(x)$ in $\mathcal{X}$.
- The system is OFP(σ) if and only if $L_f V(x) \leq -\sigma h^T(x) h(x)$ and $L_g V(x) = h^T(x)$ in $\mathcal{X}$.

The exception is for the system which is IFP(σ).

Lemma 2.1.2. *Suppose $\sigma < 0$. Then a system* (2.1.1) *is IFP(σ) in $\mathcal{X}$ if and only if $\exists$ a C^1 nonnegative function $V : \mathcal{X} \to \mathbb{R}$ such that*

$$L_f V(x) \leq \frac{1}{4\sigma}(L_g V(x) - h^T(x))(L_g V(x) - h^T(x))^T \tag{2.1.5}$$

in $\mathcal{X}$.

Proof. By dividing both sides of (2.1.2) by t and taking the limit as $t \to 0$, we obtain

$$L_f V(x) + L_g V(x) u \le y^T u - \sigma u^T u.$$

Let $\phi(x) := (L_g V(x) - h^T(x))^T$. Then,

$$\begin{aligned} 0 &\ge \sigma u^T u + \phi^T u + L_f V \\ &= \sigma(u^T u + \frac{1}{\sigma}\phi^T u + \frac{1}{4\sigma^2}\phi^T\phi) - \frac{1}{4\sigma}\phi^T\phi + L_f V \\ &= \sigma(u + \frac{1}{2\sigma}\phi)^T(u + \frac{1}{2\sigma}\phi) + L_f V - \frac{1}{4\sigma}\phi^T\phi. \end{aligned}$$

Since the above inequality should hold for all x and u, it follows that

$$0 \ge L_f V - \frac{1}{4\sigma}\phi^T\phi.$$

The converse also follows trivially. ∎

Therefore, for $\sigma > 0$ there is no characterization for IFP like Lemma 2.1.1. (Recall that IFP(0) is equivalent to the passivity.) Since a system with IFP(σ) property can be seen as a system $\dot{x} = f(x) + g(x)u$, $y = h(x) + \sigma u$ where the triple $(f(x), g(x), h(x))$ constitutes a passive system, this difficulty is reminiscent of the difficulty raised by the feedthrough term D when the linear Kalman-Yakubovich-Popov Lemma is proved (e.g. [IS96]).

Nevertheless, Lemma 2.1.2 leads to the following corollary which is of help in analyzing the IFP property of linear system.

Corollary 2.1.3. *A linear system (A, B, C) is IFP(σ) with $\sigma < 0$ if and only if the following Riccati-type inequality is solved with $P \ge 0$,*

$$PA + A^T P - \frac{1}{2\sigma}(PB - C^T)(PB - C^T)^T \le 0. \tag{2.1.6}$$

With the help of LMI[1] tools [BGFB96, GNLC95], the corollary is computationally beneficial. The corollary can even be used for synthesizing the suitable input or output matrix (B or C) for IFP property. (See e.g. [PSSS].)

[1] Linear Matrix Inequality

Regional Considerations

Even if a system is passive (or, strictly passive or what so ever) in $\mathcal{X}$, there may exist an input $u(\cdot)$ and an initial x_0 which give rise to the trajectory of the system goes beyond the region $\mathcal{X}$. This is even for the case when $\mathcal{X} = \mathbb{R}^n$. For example, consider

$$\dot{x} = -x + x^3 u, \qquad y = x^4$$

which has an initial condition with the input $u(\cdot) = 1$ which leads to the finite time escape phenomenon. However, this system is passive in $\mathbb{R}^n$ because it satisfies, with $V(x) = \frac{1}{2}x^2$, the conditions (2.1.4). Therefore, it should be kept in mind that the passivity of a system says nothing about the boundedness of its state variables. Note also that, for the characterization of passivity, the advantage of differential form (KYP lemma) over the integral form (Definition 2.1.1) lies in the easiness for the regional analysis, in particular when $\mathcal{X} \neq \mathbb{R}^n$.

Necessary Conditions for Passive Systems

By [BIW91] it is well-known that for a system to be passive, the system should be weakly minimum phase (i.e., the zero dynamics of the system is stable) and should have the relative degree one. We recall these two facts here.

Lemma 2.1.4 (Minimum Phase). *A (strictly) C^2 PD-passive system in $\mathcal{X}$ has Lyapunov (asymptotically) stable zero dynamics.*

Proof. See Proposition 2.46 of [SJK97]. ■

Lemma 2.1.5 (Relative Degree). *A C^2 PD-passive system in $\mathcal{X}$ has the relative degree one at $x = 0$, that is, $L_g h(0)$ is nonsingular.*

Proof. See Proposition 2.44 of [SJK97]. ■

2.1.2 Passification of Non-passive Systems

Suppose, for a system (2.1.1) which is not passive in general, there is a regular feedback

$$u = \alpha(x) + \beta(x)v \tag{2.1.7}$$

where $\beta(x)$ is invertible, with which the system becomes passive from v to y. Then, we call the input (2.1.7) as a *passifying* input, and this procedure is called as *passification* (i.e., the system becomes passive by a feedback).

The followings are recalled from [BIW91] and [JH98], respectively.

State Feedback Passification

For the passification of a system, one usually begin[2] by the *normal form* [Isi95]:

$$\begin{aligned} \dot{z} &= q(z, y) = f(z) + p(z, y)y \\ \dot{y} &= a(z, y) + b(z, y)u \end{aligned} \tag{2.1.8}$$

where $y \in \mathbb{R}^m$ is the output of this system. Suppose the regions Z and Y are the projections[3] of a given region $\mathcal{X}$ ($\subset \mathbb{R}^n$) to $\mathbb{R}^{n-m}$ and $\mathbb{R}^m$, respectively.

Now assume the following minimum phase and relative degree conditions.

(H1) Minimum Phase—There is a continuously differentiable Lyapunov function $W(z)$ such that $L_f W(z) \leq 0$ in Z.

(H2) Relative Degree—$b(z, y)$ is nonsingular in $\mathcal{X}$.

Under these two assumptions, the feedback

$$u = (b(z, y))^{-1} \left(-a(z, y) - (L_{p(z,y)} W(z))^T + v \right) \tag{2.1.9}$$

[2]For the passification method without the normal form, refer to the local result of [BIW91], or the result using the output feedback in [BSS00].

[3]That is, $Z := \{z \in \mathbb{R}^{n-m} \mid {}^\exists y \text{ s.t. } (z, y) \in \mathcal{X}\}$ and $Y := \{y \in \mathbb{R}^m \mid {}^\exists z \text{ s.t. } (z, y) \in \mathcal{X}\}$.

passifies the system (2.1.8) in $\mathcal{X}$. For the full discussions and the verification, consult [BIW91].

Output Feedback Passification

In general, a passification method uses the information of the full state by (2.1.9). However, if some conditions are imposed in addition to (H1) and (H2), the passification of the system (2.1.8) is possible only with the measured information, i.e., the output y.

(H3) $\exists$ two smooth matrix-valued functions $b_1(z) > 0$ and $b_0(y)$ such that $b(z,y) = b_1(z)b_0(y)$. Moreover, $b_0(y)$ is invertible in Y.

(H4) $\exists$ two functions ϕ_1 and ϕ_2 such that

$$\left| L_p^T W(z,y) + b_1^{-1}(z)a(z,y) + \frac{1}{2}\dot{\overline{b_1^{-1}}}(z,y)y \right| \leq \phi_1(y)\|y\| + \phi_2(y)\|L_f W(z)\|^{\frac{1}{2}}$$

where $\dot{\overline{b_1^{-1}}}$ is $m \times m$ matrix-valued function whose (i,j)-th element is

$$\frac{\partial (b_1)_{(i,j)}}{\partial z}(z)q(z,y).$$

Under these assumptions (H1)-(H4), the feedback

$$u = b_0^{-1}(y)(-y\phi_1(y) - y\phi_2^2(y) + v) \tag{2.1.10}$$

passifies the system (2.1.8) in $\mathcal{X}$ using the output only. For the full discussions and the verification, consult to [JH98].

Remark 2.1.1. As already pointed out in [JH98], the conditions (H3) and (H4) are automatically satisfied for linear systems. Therefore, for linear systems, the conditions (H1) and (H2) also solve the output feedback passification problem.

Asymptotic Stabilization via Passification

The asymptotic stabilization of a passive system is achieved extremely easily (i.e., by $u = -y$) when the system is zero-state detectable [BIW91, SJK97]. On the other hand, if (H2) is replaced with $L_f W(z) < 0$ ($z \neq 0$) instead of $L_f W(z) \leq 0$, then applying the passifying input (2.1.9) or (2.1.10) with $v = -y$ also guarantees the asymptotic stability of the closed-loop system. The former is the result of Theorem 3.2 of [BIW91], and the latter easily follows from the procedure of constructing the passifying inputs.

2.2 Passivity under Unmodeled Dynamics

2.2.1 Motivations

In order to model a real system, it is usual to consider a system described by the state space representation

$$(H) \qquad \dot{x} = F(x) + G(x)u, \qquad y = h(x) \tag{2.2.1}$$

where a state x is in a set $X \subset \mathbb{R}^n$, a control $u \in \mathbb{R}^m$ and an output $y \in \mathbb{R}^m$. Assume that F, G and h are continuously differentiable, and $F(0) = 0$, $h(0) = 0$.

However, (H) is not usually a complete representation of the real system since the stable fast (high frequency) dynamics is often neglected in the model in order to reduce the complexity and so on. Suppose that the full dynamics is written[4] as

$$(FH) \qquad \begin{cases} \dot{x} &= f_1(x) + Q_1(x)z + g_1(x)u, \quad y = h(x) \\ \epsilon\dot{z} &= f_2(x) + Q_2(x)z + g_2(x)u \end{cases} \tag{2.2.2}$$

where $z \in \mathbb{R}^l$ is a state of the fast dynamics and ϵ is a small positive constant which represents two-time scale behavior of the full order system. Suppose all vector fields

[4] This is a usual way to represent the unmodeled dynamics. See [KKO86].

and functions are C^1, and $f_1(0) = 0$, $f_2(0) = 0$. Since most neglected high frequency dynamics in the real world is stable, the following assumption is a reasonable one.

Assumption 2.2.1. Assume that the unmodeled dynamics is asymptotically stable for all fixed values of x, that is, $Q_2(x)$ is Hurwitz for a fixed $x \in X$.

By Assumption 2.2.1, Q_2 is invertible, and it can be concluded that the reduced system (the model) (H) has dynamics such that

$$F(x) = f_1(x) - Q_1(x)Q_2^{-1}(x)f_2(x)$$
$$G(x) = g_1(x) - Q_1(x)Q_2^{-1}(x)g_2(x).$$

In the above setting, singular perturbation theory shows that, roughly speaking, "if a reduced system is asymptotically stable, then the full order system is also asymptotically stable for a sufficiently small ϵ" [KKO86, Kha96, TKMK89]. In other words, stability is robust to fast unmodeled dynamics. Now, a question arises naturally,

Is the passivity robust under fast unmodeled dynamics?

i.e., when the reduced system (H) is passive, is the full order system (FH) also passive for a sufficiently small ϵ?

2.2.2 Passivity is Not Robust

The following simple example shows that passivity is not robust under unmodeled dynamics, i.e., the reduced system (H) is passive (even strictly C^∞ PD-passive), but the full order system (FH) is not passive, even for sufficiently small ϵ.

A Counterexample

Suppose the full order system as

$$\begin{aligned} \dot{x} &= x + 2z - u, \qquad y = 9x \\ \epsilon \dot{z} &= -x - z + 5u. \end{aligned} \tag{2.2.3}$$

Letting $\epsilon = 0$, the system is reduced to

$$\dot{x} = -x + 9u \tag{2.2.4}$$

since the slow manifold is $z = -x + 5u$.

Taking $V(x) = \frac{1}{2}x^2$, (2.2.4) is surely passive by Lemma 2.1.1. In fact,

$$\begin{aligned} L_F V(x) &= -x^2 \leq 0 \\ L_G V(x) &= 9x = h(x). \end{aligned}$$

However, the zero dynamics of the full order system is

$$\dot{z} = \frac{9}{\epsilon} z \tag{2.2.5}$$

which is unstable for all positive ϵ. Since the zero dynamics of the full system is unstable, by Lemma 2.1.4, the full order system is not PD-passive for any ϵ. (In order to find zero dynamics, at first, let $y \equiv 0$. Thus, $x \equiv 0$ and $u^* = 2z$. This results in (2.2.5) from (2.2.3).) Moreover, the full order system (2.2.3) has a transfer function from u to y,

$$G(s) = -\frac{9(s - \frac{9}{\epsilon})}{s^2 + (\frac{1}{\epsilon} - 1)s + \frac{1}{\epsilon}}$$

which is not a positive real function, and hence the full order system is not passive[5].

[5] Further discussion about the relation between positive realness and passivity can also be found at [BIW91, Prop. 2.10, Remark 2.9].

2.2.3 Passivity can be Robust

By the counterexample it has been shown the passivity can be broken when there is unmodeled dynamics even though the dynamics is exponentially stable and sufficiently fast. One of the reasons that the full order system is not passive seems to be $g_2 \neq 0$, in other words, there is a direct path from input to the fast dynamics. For a system to be passive, the passivity inequality (2.1.2) should be satisfied for all admissible input $u(\cdot)$ which may take large value, or may have high frequency. Therefore, a direct path to the integrator of fast dynamics may break the two-time scale assumptions, i.e., the model reductions are no longer valid since the effects of the unmodeled dynamics are not negligible.

Motivated by the above fact, a sufficient structural condition is presented under which the system with sufficiently fast unmodeled dynamics remains passive when the reduced system is passive. The proposed condition guarantees global passivity for linear systems and regional passivity for nonlinear systems.

Linear System Case

Consider a linear system as

$$\begin{aligned} \dot{x} &= F_1 x + Q_1 z + G_1 u, \qquad y = Hx \\ \epsilon \dot{z} &= F_2 x + Q_2 z + G_2 u \end{aligned} \tag{2.2.6}$$

where Q_2 is Hurwitz by Assumption 2.2.1, and all matrices have appropriate dimensions.

The system (2.2.6) is reduced to

$$\dot{x} = Fx + Gu, \qquad y = Hx \tag{2.2.7}$$

where $F = F_1 - Q_1 Q_2^{-1} F_2$ and $G = G_1 - Q_1 Q_2^{-1} G_2$.

In order to guarantee the passivity of the full order system, we present the following structural condition.

Assumption 2.2.2. $\boxed{G_2 \equiv 0 \text{ and } F_2 G_1 \equiv 0}$

With the above condition, the following theorem says that the passivity is preserved under sufficiently fast unmodeled dynamics. The passive property is global due to the linear structure of the system, that is, $X = \mathbb{R}^n$ in the discussions.

Theorem 2.2.1. *Under Assumptions 2.2.1 and 2.2.2, suppose the reduced system* (2.2.7) *is strictly PD-passive. Then there exists a positive constant ϵ^* such that for each $\epsilon \in (0, \epsilon^*]$ the full order system* (2.2.6) *is also strictly PD-passive.*

Proof. Since the linear system (2.2.7) is strictly PD-passive, there exists $V(x) = \frac{1}{2}x^T Px$ which satisfies the KYP lemma, where $P > 0$ and $P^T = P$. By Lemma 2.1.1 and Assumption 2.2.2,

$$PF + F^T P = -\Gamma$$
$$PG = PG_1 = H^T$$

where Γ is a positive definite matrix.

To describe the behavior more easily, the full order system (2.2.6) is transformed, by a change of variable $\eta = z + Q_2^{-1} F_2 x$ and Assumption 2.2.2, into

$$\begin{pmatrix} \dot{x} \\ \dot{\eta} \end{pmatrix} = \bar{f}(x, \eta) + \bar{g}(x, \eta)u, \qquad y = Hx \tag{2.2.8}$$

where

$$\bar{f}(x, \eta) = \begin{pmatrix} Fx + Q_1\eta \\ \frac{1}{\epsilon}Q_2\eta + Q_2^{-1}F_2Fx + Q_2^{-1}F_2Q_1\eta \end{pmatrix}, \qquad \bar{g}(x, \eta) = \begin{pmatrix} G \\ 0 \end{pmatrix}.$$

Consider now the positive definite function

$$W(x, \eta) = V(x) + \frac{1}{2}\eta^T P_\eta \eta = \frac{1}{2}x^T Px + \frac{1}{2}\eta^T P_\eta \eta$$

where P_η is the solution of $P_\eta Q_2 + Q_2^T P_\eta = -I$. Such P_η is well-defined since Q_2 is Hurwitz. A straightforward calculation shows that

$$\begin{aligned}
L_{\bar{f}} W(x,\eta) + L_{\bar{g}} W(x,\eta) u &= \dot{W}(x,\eta) \\
&= -\frac{1}{2} x^T \Gamma x + x^T PGu + x^T PQ_1 \eta \\
&\quad - \frac{1}{2\epsilon} \eta^T \eta + \eta^T P_\eta Q_2^{-1} F_2 F x + \eta^T P_\eta Q_2^{-1} F_2 Q_1 \eta \\
&= -\frac{1}{2} \begin{pmatrix} x \\ \eta \end{pmatrix}^T \begin{pmatrix} \Gamma & -N^T \\ -N & \frac{1}{\epsilon} I - M \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix} + x^T PGu
\end{aligned}$$

where

$$\begin{aligned}
M &= (P_\eta Q_2^{-1} F_2 Q_1) + (P_\eta Q_2^{-1} F_2 Q_1)^T \\
N &= Q_1^T P + P_\eta Q_2^{-1} F_2 F.
\end{aligned}$$

For M, N and Γ are independent of ϵ, there exists a constant ϵ^* such that for each $\epsilon \in (0, \epsilon^*]$

$$\begin{pmatrix} \Gamma & -N^T \\ -N & \frac{1}{\epsilon} I - M \end{pmatrix}$$

is positive definite. For such small ϵ, there exists a positive definite function $S(x,\eta)$ such that,

$$L_{\bar{f}} W(x,\eta) + L_{\bar{g}} W(x,\eta) u = -S(x,\eta) + y^T u \tag{2.2.9}$$

using the fact that $L_{\bar{g}} W(x,\eta) = L_G V(x) = Hx = y$ by Assumption 2.2.2.

Clearly, integrating both sides of (2.2.9) yields Definition 2.1.1 with $w = y^T u - S(x)$. Thus, the full order system is also strictly PD-passive with a storage function $W(x,\eta) = W(x, z + Q_2^{-1} F_2 x)$. ■

Physical Example

Assumption 2.2.2 may seem to be restrictive. Nevertheless, there is a physical passive circuit which does satisfy Assumption 2.2.2. Since it can be conjectured

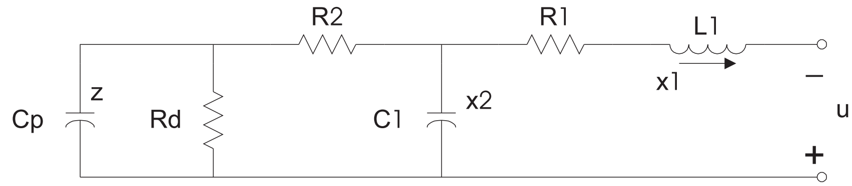


Figure 2.1: Full order model

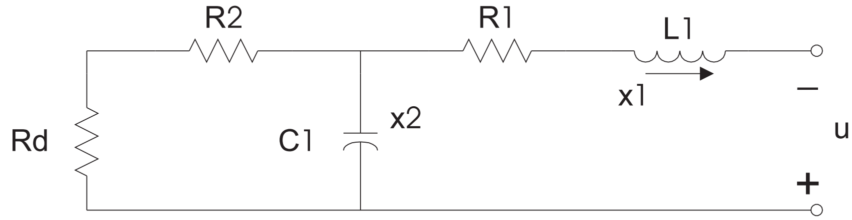


Figure 2.2: Reduced order model

that a circuit, made of passive elements such as resistors, capacitors and inductors, is surely passive, the full order model and reduced order model would be both passive. The following example clarifies this conjecture.

Consider a circuit shown in Figure 2.1. In this circuit, the clear choices of state variables are x_1, x_2 and z, where x_1 is the current of the inductor, x_2 and z are the voltages of the capacitors C_1 and C_p respectively. When the value of C_p is sufficiently small, it can be regarded as a parasitic capacitor, thus, it is neglected and can be viewed as a disconnected line as in Figure 2.2.

The standard circuit theory leads to the following system equation from Figure 2.1,

$$\begin{aligned}
\dot{x} &= \begin{bmatrix} -\frac{R_1}{L_1} & \frac{1}{L_1} \\ -\frac{1}{C_1} & -\frac{1}{C_1 R_2} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{C_1 R_2} \end{bmatrix} z + \begin{bmatrix} \frac{1}{L_1} \\ 0 \end{bmatrix} u \\
\epsilon \dot{z} &= \begin{bmatrix} 0 & \frac{1}{R_2} \end{bmatrix} x - \frac{R_d + R_2}{R_d R_2} z + 0 \quad u \\
y &= x_1
\end{aligned} \tag{2.2.10}$$

where $\epsilon = C_p$, which is assumed to be small. The input and the output are assumed to be the driving voltage u, and the current x_1 respectively.

Similarly, from Figure 2.2, the following reduced system equation is deduced:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L_1} & \frac{1}{L_1} \\ -\frac{1}{C_1} & -\frac{1}{C_1(R_d+R_2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{L_1} \\ 0 \end{bmatrix} u \tag{2.2.11}$$
$$y = x_1$$

which is also obtained by letting $\epsilon = 0$ in (2.2.10).

It is easily checked that the reduced system (2.2.11) is strictly PD-passive. Indeed, a storage function $V(x) = \frac{1}{2}x^T P x$, in which

$$P = \begin{bmatrix} L_1 & 0 \\ 0 & C_1 \end{bmatrix},$$

satisfies Lemma 2.1.1.

On the other hand, Assumption 2.2.2 is indeed satisfied in (2.2.10). Hence the full order system (2.2.10) is also strictly PD-passive for small ϵ. These conclusions coincide with the conjecture.

Nonlinear System Case

The previous results are extended here[6] to nonlinear systems under the following nonlinear version of Assumption 2.2.2.

Assumption 2.2.3. In the representation of the full order system (FH), assume that

$$\boxed{g_2(x) \equiv 0 \text{ and } L_{g_1}(Q_2^{-1} f_2)(x) = \frac{\partial}{\partial x}\left(Q_2^{-1}(x) f_2(x)\right) g_1(x) \equiv 0}$$

for all $x \in X$.

[6] For the clear comparison, we use $|\cdot|$ to indicate the Euclidean norm of a vector or the absolute value of a scalar, and $\|\cdot\|$ for the induced matrix norm of A corresponding to the Euclidean norm.

However, compared to Theorem 2.2.1, the following two theorems are regional rather than global. This is a bit disappointing, but is often the case even when the stability of nonlinear systems is analyzed using singular perturbation method. In particular, the global preservation of stability for nonlinear systems is rare unless the growth of nonlinearity is severely restricted. See e.g. [ZI99].

In what follows, let a compact set imply a bounded and closed set which is connected and contains the origin. The point is that the size of the set can be arbitrarily large only if it is bounded.

Theorem 2.2.2. *Suppose the reduced system (H) is strictly C^1 PD-passive in a compact set X. More specifically, there exist a C^1 storage function $V(x)$ and a positive definite function $\psi(x)$ on X such that*

$$L_F V(x) = -\psi^2(x), \qquad L_G V(x) = h^T(x). \tag{2.2.12}$$

Moreover, assume that, for $x \in X$, with continuous functions c_1 and c_2 on X,

$$\left| \frac{\partial V}{\partial x}(x) \right| \le c_1(x)\psi(x), \qquad |F(x)| \le c_2(x)\psi(x). \tag{2.2.13}$$

Then, under Assumptions 2.2.1 and 2.2.3, there exist a compact set $\Omega (\subset X)$ and a positive constant ϵ^* such that, for each $\epsilon \in (0, \epsilon^*]$, the full order system (FH) is also strictly C^1 PD-passive in $\Omega \times \mathbb{R}^l$.

Remark 2.2.1. In our earlier paper [SBLS98], the condition used instead of (2.2.13) was that there exist c ($c > 0$) and ρ ($0 < \rho \le 1$) such that

$$\psi(x) \ge c|x|^\rho, \qquad \forall x \in X. \tag{2.2.14}$$

The condition of Theorem 2.2.2 is less restrictive than (2.2.14) since it allows locally higher order of x in $\psi(x)$. Indeed, for example, it always holds that $|F(x)| \le c(x)|x|$

because C^1 function $F(x)$ vanishes at zero [NvdS90, p.39]. Therefore (2.2.14) implies that $|F(x)| \leq \frac{c(x)|x|^{1-\rho}}{c} c|x|^\rho \leq c_2(x)\psi(x)$. For $|\frac{\partial V}{\partial x}(x)|$, the argument is similar. In the literature this type of condition is referred as *interconnection condition*. To get a glimpse of the necessity of the interconnection condition in the singular perturbation approach, refer to [CT96, Section VI].

Proof. By a change of coordinates $\eta = z + Q_2^{-1}(x)f_2(x)$ and Assumption 2.2.3, the full order system (FH) is written as

$$\begin{pmatrix} \dot{x} \\ \dot{\eta} \end{pmatrix} = \bar{f}(x,\eta) + \bar{g}(x,\eta)u, \qquad y = h(x) \tag{2.2.15}$$

where

$$\bar{f}(x,\eta) = \begin{bmatrix} F(x) + Q_1(x)\eta \\ \frac{1}{\epsilon}Q_2(x)\eta + \frac{\partial(Q_2^{-1}f_2)}{\partial x}F(x) + \frac{\partial(Q_2^{-1}f_2)}{\partial x}Q_1(x)\eta \end{bmatrix}, \quad \bar{g}(x,\eta) = \begin{bmatrix} G(x) \\ 0 \end{bmatrix}.$$

Since $Q_2(0)$ is Hurwitz from Assumption 2.2.1, there exists the positive definite symmetric matrix P_η such that

$$P_\eta Q_2(0) + Q_2^T(0)P_\eta = -I. \tag{2.2.16}$$

It follows that, by the continuity of $Q_2(x)$, there exists a compact set Ω in X such that

$$\Gamma_\eta(x) := -(P_\eta Q_2(x) + Q_2^T(x)P_\eta) \tag{2.2.17}$$

is positive definite for each $x \in \Omega$.

Now, consider a positive definite storage function for the transformed full order system (2.2.15) as

$$W(x,\eta) = V(x) + \frac{1}{2}\eta^T P_\eta \eta. \tag{2.2.18}$$

Before evaluating the derivative of W, define $M(x)$ and $N(x)$ for notational convenience as

$$(L_{Q_1}V)^T(x) + P_\eta \frac{\partial(Q_2^{-1}f_2)}{\partial x}(x)F(x) =: N(x) \tag{2.2.19}$$
$$P_\eta \frac{\partial(Q_2^{-1}f_2)}{\partial x}(x)Q_1(x) =: M(x).$$

Hence,

$$\begin{aligned}\dot{W}(x,\eta) =& L_F V + L_G V u + L_{Q_1}V\eta - \frac{1}{2\epsilon}\eta^T\Gamma_\eta(x)\eta + \\ & \eta^T P_\eta\Big(\frac{\partial(Q_2^{-1}f_2)}{\partial x}F(x) + \frac{\partial(Q_2^{-1}f_2)}{\partial x}Q_1(x)\eta\Big) \\ =& L_F V + \eta^T N(x) - \frac{1}{2\epsilon}\eta^T\Gamma_\eta(x)\eta + \eta^T M(x)\eta + L_G V u \\ \le& -\psi^2(x) + c(x)|\eta|\ \psi(x) - \frac{\lambda_{\min}(\Gamma_\eta(x))}{2\epsilon}|\eta|^2 + \|M(x)\|\ |\eta|^2 + L_G V u\end{aligned}$$

where $c(x) = \|Q_1(x)\|c_1(x) + \|P_\eta \frac{\partial(Q_2^{-1}f_2)}{\partial x}(x)\|c_2(x)$, with which $\|N(x)\| \le c(x)\psi(x)$ by (2.2.13).

Therefore, it follows that

$$L_{\bar{f}}W(x,\eta) + L_{\bar{g}}W(x,\eta)u = \dot{W}(x,\eta) \le -\begin{pmatrix}\psi(x)\\|\eta|\end{pmatrix}^T \Phi \begin{pmatrix}\psi(x)\\|\eta|\end{pmatrix} + y^T u \tag{2.2.20}$$

where

$$\Phi = \begin{bmatrix} 1 & -\frac{1}{2}c(x) \\ -\frac{1}{2}c(x) & \frac{1}{2\epsilon}\lambda_{\min}(\Gamma_\eta(x)) - \|M(x)\| \end{bmatrix}.$$

Because $c(x)$ and $M(x)$ are bounded on Ω by their continuity and the compactness of Ω, and because $\Gamma_\eta(x)$ is positive definite on Ω by (2.2.17), there exists a positive constant ϵ^* such that for each $\epsilon \in (0, \epsilon^*]$ the matrix Φ is positive definite. Indeed, choose ϵ^* so that

$$\epsilon^* < \frac{2\lambda_{\min}(\Gamma_\eta(x))}{4\|M(x)\| + c^2(x)} \quad \text{in } \Omega.$$

Finally, by letting $u = 0$ in (2.2.20), $L_{\bar{f}}W < 0$ on $\Omega \times \mathbb{R}^l$ except the origin and $L_{\bar{g}}W = L_G V = h^T(x) = y^T$ by Assumption 2.2.3, which imply the full order system is strictly C^1 PD-passive on $\Omega \times \mathbb{R}^l$ by Lemma 2.1.1. ∎

In the above theorem, the guaranteed region Ω of passivity for x-dynamics generally shrinks from X. This is mainly because the constant P_η was used in the storage function W in order that the input u may not appear in the derivative of W. However, if some restriction is imposed on the admissible input set, the given region X can be maintained on which the full order system is passive.

Theorem 2.2.3. *Assume the conditions in Theorem 2.2.2. In addition, the admissible input set consists of norm bounded inputs. Then, for any compact set $Z \in \mathbb{R}^l$ including the origin, there exists a positive constant ϵ^* such that, for each $\epsilon \in (0, \epsilon^*]$, the full order system (FH) is also strictly C^1 PD-passive in $X \times Z$.*

Proof. The proof follows the same track as that of Theorem 2.2.2, except that, instead of (2.2.18), the storage function

$$W(x, \eta) = V(x) + \frac{1}{2}\eta^T P_\eta(x)\eta,$$

is used where $P_\eta(x)$ is the solution on X for

$$P_\eta(x)Q_2(x) + Q_2^T(x)P_\eta(x) = -I.$$

Even though $P_\eta(x)$ depends on the state x, it is well-defined, positive definite and symmetric for a fixed x since $Q_2(x)$ is Hurwitz on X. In addition, the continuous differentiability of $P_\eta(x)$ is inherited from that of $Q_2(x)$ by the above Lyapunov equation.

Now, define $N(x)$ as in (2.2.19) and M' as

$$P_\eta(x)\frac{\partial(Q_2^{-1}f_2)}{\partial x}(x)Q_1(x) + \frac{1}{2}\Psi(x, \eta, u) =: M'(x, \eta, u) \tag{2.2.21}$$

where the (i, j)-th element of Ψ, $\Psi_{i,j} = \frac{\partial(P_\eta)_{i,j}}{\partial x}(F(x) + Q_1(x)\eta + G(x)u)$. Let Z' is the image set of $X \times Z$ by the map $\eta = z + Q_2^{-1}(x)f_2(x)$, which is also compact.

Notice that $M'(x, \eta, u)$ is bounded for $(x, \eta) \in X \times Z'$ since all quantities in (2.2.21) are continuous and u is bounded.

Hence, by calculating the derivative of W similarly to the previous proof, it can be shown that

$$L_{\bar{f}} W(x, \eta) + L_{\bar{g}} W(x, \eta) u = \dot{W}(x, \eta) \leq - \begin{pmatrix} \psi(x) \\ |\eta| \end{pmatrix}^T \Phi \begin{pmatrix} \psi(x) \\ |\eta| \end{pmatrix} + y^T u \quad (2.2.22)$$

where

$$\Phi = \begin{bmatrix} 1 & -\frac{1}{2} c(x) \\ -\frac{1}{2} c(x) & \frac{1}{2\epsilon} - \|M'(x, \eta, u)\| \end{bmatrix}.$$

Here again, by the fact that $c(x)$ and $M'(x, \eta, u)$ are bounded on $X \times Z'$ there exists a positive contant ϵ^* such that for each $\epsilon \in (0, \epsilon^*]$ the matrix Φ is positive definite. Thus, the full order system is strictly C^1 PD-passive on $X \times Z$. ■

2.3 Extended Passivity

In this section, the standard passivity is extended and specialized for the purpose of Chapter 4. Firstly, the uniform passivity is considered which implies the passivity under the external disturbance (or, external input). Secondly, we consider the strict passivity with respect to a set (or, a partial state) rather than a point in the state-space.

2.3.1 Uniform Passivity

Consider a dynamical system modeled by ordinary differential equations, with input vector v, output vector y and time-varying external input $d(t) \in \mathcal{D}$, of the form,

$$\begin{aligned} \dot{x} &= f(x, d) + g(x, d) v \\ y &= h(x, d) \end{aligned} \quad (2.3.1)$$

where $x \in \mathcal{X} = \mathbb{R}^n$.

Definition 2.3.1. The system (2.3.1) is said to be *uniformly C^1 passive*[7] with respect to d if there exists a continuously differentiable nonnegative function $V : \mathcal{X} \to \mathbb{R}$, called a *storage function*, such that $V(0) = 0$ and

$$V(x(t)) - V(x(0)) \leq \int_0^t y(\tau)^T v(\tau)\, d\tau$$

for all $t \geq 0$, or, equivalently,

$$L_f V(x, d) \leq 0, \qquad L_g V(x, d) = h(x, d)^T$$

for $x \in \mathcal{X}$ and $d \in \mathcal{D}$.

The equivalence between the integral form and the differential form in the definition follows from the nonlinear version of KYP lemma (e.g. Lemma 2.1.1).

2.3.2 Passivity with respect to a Set

Now we consider the passivity with respect to a set $\mathcal{A}$ in the state space.

Definition 2.3.2. The system (2.3.1) is said to be *uniformly C^1 passive* with respect to a *pair* $(\mathcal{A}, d)$ if there exist a continuously differentiable function $V : \mathcal{X} \to \mathbb{R}$, $\mathcal{K}_\infty$ functions α_1, α_2 and a continuous positive definite function α_3 satisfying

$$\alpha_1(\|x\|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(\|x\|_{\mathcal{A}}) \tag{2.3.2a}$$

$$L_f V(x, d) \leq -\alpha_3(\|x\|_{\mathcal{A}}) \tag{2.3.2b}$$

$$L_g V(x, d) = h(x, d)^T \tag{2.3.2c}$$

for $x \in \mathcal{X}$ and $d \in \mathcal{D}$.

In a special case that $\mathcal{A} = \{0\}$ and $d \equiv 0$, this definition is equivalent to the strict passivity with a positive definite storage function. On the other hand, when the system (2.3.1) is uniformly C^1 passive with respect to $(\mathcal{A}, d)$ where $\mathcal{A} = \{x =$

[7] Note that the uniformity is with respect to the *external input* not with respect to the *time*.

$(x_1, x_2) \mid x_2 = 0\}$ in which x_1 and x_2 are certain partitions of x, we call the system *partial-state uniformly C^1 passive (PSUP)* with respect to a pair (x_2, d). In this case, $\|x\|_{\mathcal{A}} = \|x_2\|$.

Finally, the following lemma will play the fundamental role in Chapter 4.

Lemma 2.3.1. *Consider* (2.3.1) *with $v \equiv 0$. If the system is PSUP with respect to a pair (x_2, d) and the trajectory $x_1(t)$ exists for all $t \geq 0$, then there is a $\mathcal{KL}$ function $\beta(\cdot, \cdot)$ such that the trajectory $x_2(t)$ of* (2.3.1) *satisfies*

$$\|x_2(t)\| \leq \beta(\|x_2(0)\|, t) \tag{2.3.3}$$

for all $t \geq 0$.

Proof. The proof is omitted since it can be proved similarly to the first part of Section 6 in [LSW96]. ■

Remark 2.3.1. The passification of a system under the extended passivity concept is also possible, following the similar procedures in Section 2.1.2.

2.4 Notes on the Chapter

In Section 2.1, we've introduced the brief expositions of passivity and passification. In particular, the nonlinear KYP lemma is presented for IFP and OFP systems with the special emphasis on the effective region. The presented output feedback passification scheme will also be the foundation of the passivity framework for observer in Chapter 4.

Now, concerning Section 2.2 let us append some backgrounds. Since Willems introduced the state-space approach to dissipativity in terms of an inequality involving the storage function and supply rate [Wil72], passivity has been extensively studied in the control literature. In particular, the relationship between the passivity and the stability of a system has been well established in [HM76, HM80, vdS96], and the design of stabilizing controller, using the passivity or the passification method, has been proposed in [BIW91, Ort89, Lin95b]. While these works have not considered the uncertain system case, several authors [JHF96, SX96] dealt with the robust passification for the system with parametric uncertainties and, more recently, Lin and Shen [LS99] developed the robust passivity framework for structural uncertainties. On the other hand, we've analyzed, in Section 2.2, the passivity when there is unmodeled dynamics rather than the parametric or structural uncertainty. Especially, it was shown that the passivity can be broken under unmodeled dynamics. In the counterexample, it occurs even though the unmodeled dynamics is sufficiently fast. We've also presented a condition which guarantees the robust passivity under unmodeled dynamics. For the linear systems, the proposed condition is sufficient for global passivity. But, in the nonlinear case, global preservation of passivity is not achieved without serious restriction of nonlinearity. Instead, our results are regional, in other words, passivity is preserved on a bounded (arbitrarily large) region of state-space rather than globally.

Section 2.3 is the preliminary for Chapter 4 where the PSUP property will be actively utilized for the analysis of augmented error dynamics between the plant and the observer. More specifically, the plant state x and the estimate error e will be matched with x_1 and x_2 in the definition of PSUP, respectively.

Chapter 3

New Developments for Nonlinear State Observer

The observer problem is to estimate asymptotically the state of a given plant only with the input and output information of the plant. Contrary to linear systems, the problem has not yet been wholly solved in the general sense and there are several particular classes of nonlinear systems [KI83, KR85, WZ87, Tsi89, GHO92, DQC92, GK94, BH96, HP99, SSS99], for which the observer exists.

Roughly speaking, there are two major approaches to the state observer design for nonlinear systems; they are the *observer canonical form* approach and the *observable canonical form* approach, according to the terminology of [Kel87]. In the former case, a given nonlinear system is transformed into the following form:

$$\dot{x} = Ax + \gamma(*), \qquad y = Cx$$

where (A, C) is observable (or detectable) pair, and $\gamma(*)$ is a vector field of which the arguments $(*)$ are the known quantities, like input and output, or sometimes their derivatives. Then, the observer construction becomes straightforward with L such that $(A - LC)$ is Hurwitz, i.e., $\dot{\hat{x}} = A\hat{x} + \gamma(*) - L(C\hat{x} - y)$ is the observer. This is the approach of *Linearized Error Dynamics* prevalent in the literature [BZ83, Kel87, KI83, KR85, BZ88, XG88, XG89, LT86, PM93, HP99]. However, these designs have a bottleneck that one should solve some partial differential equations to find the suitable transformation.

On the contrary, the latter approach has the benefit that the change of coordinates is relatively easy because the transformation is intrinsically based on the mapping $[h(x), L_f h(x), \cdots, L_f^{n-1} h(x)]^T$ which constitutes the *observation space* [NvdS90]. Moreover, the observable canonical form is directly related to the notion of observability, which results in the conceptual coincidence.

In this chapter, we begin by the *uniform observability* and the *uniformly observable canonical form* raised by Gauthier *et al.* [GB81] and present two novel observer design methods for some classes of nonlinear systems. The contributions are

- To tie scattered definitions for uniform observability and to add another characterization via differential geometry
- To give a comprehensive insights to Gauthier's high gain observer
- To provide a practical technique (Lipschitz extension) for relieving the global Lipschitz requirement
- To extend the Gauthier's observer to the multi-output case
- To introduce a novel recursive observer design which enables to incorporate the non-uniform observable modes (observer backstepping)

3.1 Uniform Observability

3.1.1 Definition of Uniform Observability

The observability for nonlinear systems is defined as the existence of a specific input $u(t)$ for any two different initial states, such that the states are distinguishable by the input and the output of the system [NvdS90]. Thus, even for an observable system, it is possible for the two states to be indistinguishable by an input other than the specific $u(t)$. (See, for example, [Vid93, p. 415].) This is not the case of linear systems in which the input u is multiplied by a constant vector field.

Yet, there is another notion of observability for nonlinear systems, which is a bit stronger than the usual observability.

Definition 3.1.1. A system is *uniformly observable* when it is observable for any input. More specifically, for any input $u(t)$ and any finite time inverval $[0, T]$, $T > 0$, such that the trajectory of the system exists, the output $y(t)$, $t \in [0, T]$ determines the initial state $x(0)$.

Further discussions for the uniform observability are well organized in [GB81].

3.1.2 Equivalence of Various Conditions for Uniform Observability

There are several characterizations for the notion in the literature, and each condition has appeared for its own advantage to the specific purpose. However, since the relations of each conditions are rarely discussed, we prove that the conditions are all equivalent assuming that the system is single-input, single-output, smooth and input affine, and that all the conditions are global.

Various Characterizations

Consider a system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{3.1.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $y \in \mathbb{R}$, and f, g, h are all smooth. Define a smooth map $\Phi(x) : \mathbb{R}^n \to \mathbb{R}^n$ as

$$\Phi(x) := \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix}.$$

Also define a notational abbreviation w and v such that

$$w := \begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \\ \vdots \\ y^{(n-1)} \end{bmatrix}, \qquad v := \begin{bmatrix} u \\ \dot{u} \\ \ddot{u} \\ \vdots \\ u^{(n-2)} \end{bmatrix}.$$

Therefore, $w \in \mathbb{R}^n$, $v \in \mathbb{R}^{n-1}$, $w_i = y^{(i-1)}$ and $v_i = u^{(i-1)}$. Define a map $D(x, v)$ from x and v to w (successive derivatives of y) as

$$w := D(x, v)$$

in which (d_i is i-th element of D)

$$\begin{aligned}
d_1(x) &= h(x) \\
d_2(x, u) &= \frac{\partial d_1}{\partial x}(x) \cdot (f(x) + g(x)u) \qquad (= \dot{d}_1) \\
d_3(x, u, \dot{u}) &= \frac{\partial d_2}{\partial x}(x, u) \cdot (f(x) + g(x)u) + \frac{\partial d_2}{\partial u}(x, u) \cdot \dot{u} \qquad (= \dot{d}_2) \\
&\vdots \\
d_i(x, u, \cdots, u^{(i-1)}) &= \frac{\partial d_{i-1}}{\partial x}(x, u, \cdots, u^{(i-2)}) \cdot (f(x) + g(x)u) \\
&\quad + \sum_{k=0}^{i-2} \frac{\partial d_{i-1}}{\partial u^{(k)}}(x, u, \cdots, u^{(i-2)}) \cdot u^{(k+1)} \qquad (= \dot{d}_{i-1}).
\end{aligned}$$

Here, Φ and D are well-defined, but not invertible with respect to x, in general.

Definition 3.1.2 ([Isi95])**.** The system (3.1.1) is uniformly observable if

(i) $\Phi(x)$ is a global diffeomorphism.

(ii) rank $\left(\frac{\partial D}{\partial x}(x, v)\right) = n$.

Definition 3.1.3 ([TP94, Tor92])**.** The system (3.1.1) is uniformly observable if there exists a map T from the successive derivatives of y and u to the state x such that

$$x = T(w, v)$$

and the map from w to x is onto for any v.

The followings are structural conditions.

Definition 3.1.4 ([GB81, GHO92])**.** The system (3.1.1) is uniformly observable if it is diffeomorphic to

$$\dot{z} = \begin{bmatrix} z_2 \\ z_3 \\ \vdots \\ z_n \\ a(z) \end{bmatrix} + \begin{bmatrix} b_1(z_1) \\ b_2(z_1, z_2) \\ \vdots \\ b_{n-1}(z_1, \cdots, z_{n-1}) \\ b_n(z_1, \cdots, z_n) \end{bmatrix} u \tag{3.1.2}$$

$$y = z_1.$$

Definition 3.1.5 ([SS])**.** The system (3.1.1) is uniformly observable if

(i) $\Phi(x)$ is a global diffeomorphism.

(ii) $[g(x), \mathcal{R}_j] \subset \mathcal{R}_j$ for $0 \leq j \leq n-2$ where

$$\mathcal{R}_j := \text{span}\left\{r, ad_f r, \cdots, ad_f^j r\right\}$$

in which r satisfies

$$\frac{\partial \Phi}{\partial x}(x) \cdot r = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Remark 3.1.1. Although we are considering the coincidence of these definitions, each definition also has its own advantage. For example, Definition 3.1.4 and 3.1.5 can characterize the uniform observability of a system even when the input u is not continuous. On the other hand, Definition 3.1.3 can be applied to the multi-output and input non-affine systems. Indeed, suppose a system given by

$$\begin{aligned} \dot{x}_1 &= x_2 + x_2 u^2 \qquad\qquad y = x_1 \\ \dot{x}_2 &= u \end{aligned}$$

which is indeed uniformly observable in the sense that $x_1 = y$ and $x_2 = \dot{y}/(1+u^2)$. However, this system cannot be put in the form similar to (3.1.2) [Tee98].

Equivalences

By assuming the input is differentiable sufficiently many times, the following facts are proved.

(a) Def. 3.1.3 implies Def. 3.1.2.

Proof. Assuming there exists the map $T(w, v)$,

$$w = D(x, v) = D(T(w, v), v).$$

Then, taking partial derivation with respect to w,

$$I = \frac{\partial D}{\partial x}(T(w, v), v) \cdot \frac{\partial T}{\partial w}(w, v).$$

Since the above holds for all (w, v), the map T is onto for any fixed v, and D and T are smooth,

$$\operatorname{rank} \frac{\partial D}{\partial x}(x, v) = n \qquad \text{for any } x.$$

Thus, the condition (ii) of Def. 3.1.2 follows.

Now, it is left to show the condition (i). For that purpose, it is sufficient to show that $\Phi(x)$ is globally one-to-one and $\operatorname{rank} \frac{\partial \Phi}{\partial x} = n$ [Mun91, p. 65]. Since the system

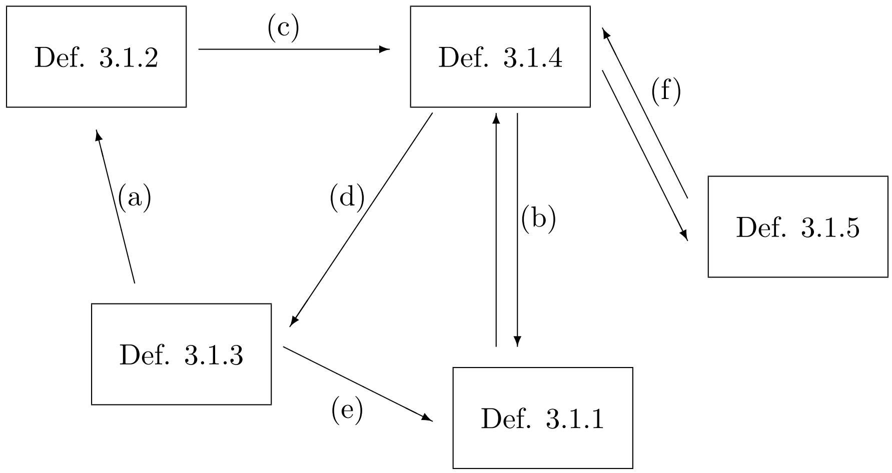


Figure 3.1: Relationships for Various Definitions of Uniform Observability

is affine, $D(x,0) = \Phi(x)$. (When $u \equiv 0$ (i.e. $v = 0$), the system (3.1.1) becomes $\dot{x} = f(x)$.) Thus, $\text{rank}\frac{\partial \Phi}{\partial x} = n$ for all x. Moreover, since the inverse of $T(w,v)$ with respect to w for any v is $D(x,v)$, it follows that $D(x,0)$ is one-to-one by letting $v = 0$. ■

(b) Def. 3.1.1 is equivalent to Def. 3.1.4.

See [GHO92].

(c) Def. 3.1.2 implies Def. 3.1.4.

See the proof of [Isi95, Prop. 9.6.1].

(d) Def. 3.1.4 implies Def. 3.1.3.

Trivial by the form. (Or, [Isi95, Prop. 9.6.1] also claims this.)

(e) Def. 3.1.3 implies Def. 3.1.1.

Since the map $T(w,v)$ determines x for any v, the uniform observability follows.

(f) Def. 3.1.5 is equivalent to Def. 3.1.4.

We defer the proof until Section 3.2.3, in which this claim is proved in the more general settings by Lemma 3.2.3.

The discussions so far are illustrated in Fig. 3.1. From the figure, it can be easily seen that all the definitions for uniform observability coincide.

3.2 Gauthier's High Gain Observer

The uniform observability of the previous section is closely related to the existence of high gain state observer proposed by Gauthier *et al.* [GHO92]. That is, if a system is uniformly observable, then an observer always exists for the system under some technical assumptions. The design of the observer has been presented for input affine systems in [GHO92] and for input non-affine systems in [GK94], respectively. However, since the design of [GK94] does not provide an explicit form of the observer, we discuss the observer in the input non-affine form, but based on the approach of [GHO92]. Note that the observer construction for this section is only applicable to single-output systems.

3.2.1 Some Insights

Contrary to the linearized error dynamics approach which is based on the exact canceling of nonlinear terms, Gauthier's observer suppress some nonlinearity by the high injection gain. However, it is different from the approach of [RC95, Raj98] in which the growth of nonlinearity is restricted. In that approach, the observer gain is selected independently of the Lipschitz coefficient of the nonlinearity. Therefore, one should verify the convergence of the error after assigning the gain and it may happen that the appropriate gain cannot be found. On the other hand, in the Gauthier's approach, the gain is chosen by the knowledge of the Lipschitz coefficient of the nonlinearity. Thus, once the nonlinearity is known to be Lipschitz, there always exists a suitable gain.

Nested-High-Gain

For the intuition, consider a system

$$\dot{x} = (A - lC)x + \gamma(x)$$

where $\gamma(x)$ is Lipschitz and

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{3.2.1}$$

with appropriate dimensions. In the approach of [RC95, Raj98], one has to check, after choosing l such that $(A - lC)$ is Hurwitz, whether the Lipschitz coefficient of $\gamma(x)$ is less than a certain bound resulted from the chosen gain l. (There exists a similar method for robust control. See [Kha96, Exam. 5.1].) If it is not satisfied, then one should choose another gain l. On the other hand, if $\gamma(x)$ has the form

$$\gamma(x) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \gamma_n(x) \end{bmatrix},$$

then, the method of [CMG93] always provides a gain l which guarantees the asymptotic stability of the system using Vandermonde transformation (see also [SBS99]). Now, if $\gamma(x)$ has the lower triangular form, then there always exists a gain l which guarantees the asymptotic stability as follows.

Lemma 3.2.1. *Consider a system*

$$\dot{x} = (A - lC)x + \gamma(x, d) \tag{3.2.2}$$

where (A, C) is given in (3.2.1), and γ has the lower triangular structure and is globally Lipschitz[1] *in x uniformly in d. Then, there exists $l \in \mathbb{R}^{n\times 1}$ such that (3.2.2) is globally asymptotically stable with a quadratic Lyapunov function $V(x) = \frac{1}{2}x^T Px$.*

More specifically, the appropriate l and P are obtained by

$$l_i = \theta^i a_i \tag{3.2.3}$$

and $(P)_{i,j} = (P_0)_{i,j}/\theta^{i+j-1}$, where $\theta > 1$ is sufficiently large and P_0 is the solution of $P_0 A_c + A_c^T P_0 = -Q$ with $Q > 0$ and a Hurwitz matrix

$$A_c = A - \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} C. \tag{3.2.4}$$

Proof. Let $z_i = \theta^{1-i}x_i$ for $1 \le i \le n$. Then, in z-coordinates, (3.2.2) becomes

$$\begin{aligned}
\dot{z}_1 &= \theta z_2 - l_1 z_1 + \gamma_1(z_1, d) \\
\dot{z}_2 &= \theta z_3 - \frac{1}{\theta} l_2 z_1 + \frac{1}{\theta}\gamma_2(z_1, \theta z_2, d) \\
&\vdots \\
\dot{z}_n &= -\frac{1}{\theta^{n-1}} l_n z_1 + \frac{1}{\theta^{n-1}}\gamma_n(z_1, \cdots, \theta^{n-1} z_n, d).
\end{aligned}$$

By the Lipschitz property and the triangular structure, there exists a constant ρ_i which is independent of θ, such that

$$\left|\frac{1}{\theta^{i-1}}\gamma_i(z_1, \cdots, \theta^{i-1} z_i, d)\right| \le \rho_i \|z\|, \qquad 1 \le i \le n.$$

[1] Recall that a function $f(x, u)$ is said to be Lipschitz in x when there is a function $c(u)$ such that $\|f(x, u) - f(z, u)\| \le c(u)\|x - z\|$. If, furthermore, $c(u)$ is a constant independent of u, then $f(x, u)$ is Lipschitz in x uniformly in u.

Therefore, with $l_i = \theta^i a_i$, the system can be written as

$$\dot{z} = \theta A_c z + \tilde{\gamma}(z, d, \theta)$$

where $\tilde{\gamma}$ satisfies $\|\tilde{\gamma}(z, d, \theta)\| \leq \rho \|z\|$ with $\rho = n \cdot \max \rho_i$.

Finally, the derivative of $V(z) = \frac{1}{2\theta} z^T P_0 z$ along the trajectory is

$$\dot{V} \leq -\frac{1}{2}\lambda_{\min}(Q)\|z\|^2 + \frac{\rho}{\theta}\|P_0\|\|z\|^2$$

which is negative definite for $\theta > 2\rho\|P_0\|/\lambda_{\min}(Q)$. ■

Therefore, the gain selection of (3.2.3) solves the stabilization problem of (3.2.2) with the triangular structure of γ. Note that the gain selection is nested-high-gain type, i.e., a_i's are coefficients of Hurwitz polynomial and the coefficients are multiplied by the powers of θ. At this point, let us investigate what this type of gains imply.

Zoomed Pole Location

The characteristic polynomial of A_c in Lemma 3.2.1 is given by

$$s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n = 0. \tag{3.2.5}$$

On the other hand, the characteristic polynomial of $(A - lC)$ where l is chosen as (3.2.3), is given by

$$s^n + \theta a_1 s^{n-1} + \theta^2 a_2 s^{n-2} + \cdots + \theta^n a_n = 0. \tag{3.2.6}$$

Here, if p is a root of (3.2.5), then θp becomes a root of (3.2.6). Therefore, the pole placement scheme of (3.2.3) can be interpreted as the placement of the poles at the θ-times zoomed location from the original pole location of A_c. See Fig. 3.2.

Remark 3.2.1. Another way to choose the stabilizing gain l is to define $l = P^{-1}C^T$ where P is the unique solution of

$$0 = -\theta P - PA - A^T P + C^T C \tag{3.2.7}$$

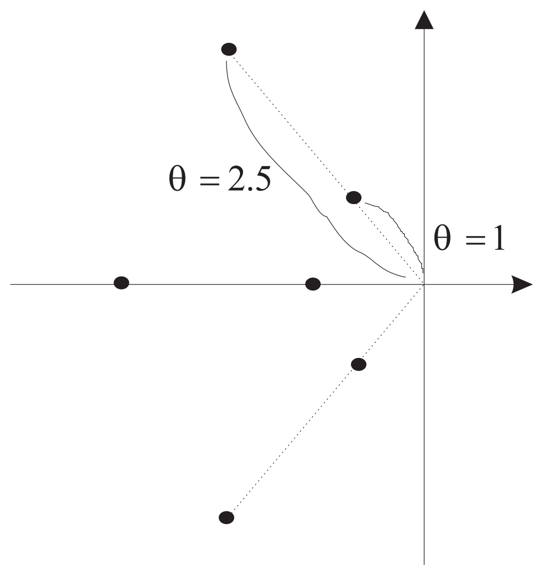


Figure 3.2: Pole locations are zoomed θ-times.

with sufficiently large $\theta > 1$. Indeed, let P_0 be the solution of (3.2.7) when $\theta = 1$. Then, it is clear that $A_c = A - P_0^{-1}C^TC$ is Hurwitz (i.e. $[a_1, \cdots, a_n]^T = P_0^{-1}C^T$), since

$$P_0 A_c + A_c^{-1} P_0 = -P_0 - C^T C \leq -P_0.$$

Furthermore, it can be shown that the solution of (3.2.7) is given by $P = \frac{1}{\theta}\Lambda P_0 \Lambda$ where $\Lambda := \mathrm{diag}(1, 1/\theta, \cdots, 1/\theta^{n-1})$, because $\Lambda A \Lambda^{-1} = \theta A$ and $C\Lambda^{-1} = C$. Thus, it follows that $l_i = \theta^i a_i$ and $(P)_{i,j} = (P_0)_{i,j}/\theta^{i+j-1}$.

3.2.2 Gauthier's Observer

Consider the single-output and input non-affine system:

$$\boxed{\begin{aligned} \dot{z} &= a(z) + b(z,u) = \begin{bmatrix} z_2 \\ z_3 \\ \vdots \\ z_n \\ a_n(z) \end{bmatrix} + \begin{bmatrix} b_1(z_1,u) \\ b_2(z_1,z_2,u) \\ \vdots \\ b_{n-1}(z_1,\cdots,z_{n-1},u) \\ b_n(z_1,\cdots,z_n,u) \end{bmatrix} \\ y &= z_1 \end{aligned}} \tag{3.2.8}$$

where $b(z,0)=0$. Note that this system is uniformly observable in the sense of Definition 3.1.3, but the class of uniformly observable input non-affine systems is not always written in this form. See Remark 3.1.1.

Assumption 3.2.1. $a_n(z)$ is globally Lipschitz in z, and $b_i(z,u)$ is globally Lipschitz in z uniformly in u.

A sufficient conditions for the assumption that $b_i(z,u)$ is Lipschitz in z uniformly in u, is that $b_i(z,u)$ is Lipschitz in z and at the same time the input u is bounded. Therefore, we can assume the alternative:

Assumption 3.2.2. $a_n(z)$ and $b_i(z,u)$ are globally Lipschitz in z, and the input u is bounded.

Here, the value of the Lipschitz coefficient is not restricted and can be arbitrarily large. Then, the Gauthier's observer is given by

$$\begin{aligned} \dot{\hat{z}} &= a(\hat{z}) + b(\hat{z},u) - G^{-1}C^T(C\hat{z}-y) \\ &= \begin{bmatrix} \hat{z}_2 \\ \hat{z}_3 \\ \vdots \\ \hat{z}_n \\ a_n(\hat{z}) \end{bmatrix} + \begin{bmatrix} b_1(\hat{z}_1,u) \\ b_2(\hat{z}_1,\hat{z}_2,u) \\ \vdots \\ b_{n-1}(\hat{z}_1,\cdots,\hat{z}_{n-1},u) \\ b_n(\hat{z}_1,\cdots,\hat{z}_n,u) \end{bmatrix} - G^{-1}C^T(C\hat{z}-y) \end{aligned} \tag{3.2.9}$$

where $\hat{z}$ is the estimated state of z and G is the unique positive definite solution of

$$0 = -\theta G - A^T G - GA + C^T C \tag{3.2.10}$$

in which A and C is the same as (3.2.1), and a positive constant θ is to be chosen in the following lemma.

Lemma 3.2.2 (Exponential Observer [GHO92])**.** *Consider the plant and the observer are given by* (3.2.8) *and* (3.2.9)*, respectively. Under Assumption 3.2.1 or 3.2.2, there exists a constant* $\theta_1^* \geq 1$ *such that for any* $\theta > \theta_1^*$*, the observer* (3.2.9) *guarantees*

$$\|\hat{z}(t) - z(t)\| \leq K(\theta)\exp(-\frac{\theta}{4}t)\|\hat{z}(0) - z(0)\|. \tag{3.2.11}$$

for $t \geq 0$ *with some function* $K(\theta)$*. Moreover, for a fixed time* $\tau > 0$*,*

$$K(\theta)\exp(-\frac{\theta}{4}\tau) \to 0 \qquad as \qquad \theta \to \infty. \tag{3.2.12}$$

Remark 3.2.2. In the above lemma, the property (3.2.12) will be utilized in Chapter 5.

Proof. With $e := \hat{z} - z$, the error dynamics is obtained as

$$\dot{e} = Ae + \Gamma(z, \hat{z}, u) - G^{-1}C^T Ce$$

where the i-th element of Γ is defined as

$$\begin{aligned} \Gamma_i(z, \hat{z}, u) &= b_i(\hat{z}_1, \cdots, \hat{z}_i, u) - b_i(z_1, \cdots, z_i, u), \qquad 1 \leq i \leq n-1, \\ \Gamma_n(z, \hat{z}, u) &= b_n(\hat{z}, u) - b_n(z, u) + a_n(\hat{z}) - a_n(z). \end{aligned}$$

By defining $\xi := \Lambda(\theta)e$ where $\Lambda := \mathrm{diag}(1, 1/\theta, \cdots, 1/\theta^{n-1})$, it can be shown, for $\theta \geq 1$,

$$\|\Lambda(\theta)\Gamma\| \;\leq\; L_\Lambda \,\|\xi\| \tag{3.2.13}$$

where the constant L_Λ is independent of θ. Indeed, by the Lipschitz property of b_i and the boundedness of u,

$$\begin{aligned}\frac{1}{\theta^{i-1}}\big|b_i(z_1+\xi_1,\cdots,&z_i+\theta^{i-1}\xi_i,u)-b_i(z_1,\cdots,z_i,u)\big|\\&\le \frac{1}{\theta^{i-1}}L_{\Lambda,i}\big\|[\xi_1,\cdots,\theta^{i-1}\xi_i]^T\big\|\\&\le\ L_{\Lambda,i}\ \big\|\xi\big\|\end{aligned}$$

for $1\le i\le n-1$, and L_Λ is the n times of the maximum of $L_{\Lambda,i}$ $(1\le i\le n)$.

Now define $\tilde{G}:=\theta\Lambda^{-1}G\Lambda^{-1}$. Then θ-independent $\tilde{G}$ satisfies the equality

$$0=-\tilde{G}-\tilde{G}A-A^T\tilde{G}+C^TC \tag{3.2.14}$$

from (3.2.10), since $\Lambda A\Lambda^{-1}=\theta A$ and $C\Lambda^{-1}=C$. Also,

$$\begin{aligned}\dot{\xi}&=\Lambda A\Lambda^{-1}\xi+\Lambda\Gamma-\Lambda G^{-1}C^TC\Lambda^{-1}\xi\\&=\theta A\xi+\Lambda\Gamma-\theta\tilde{G}^{-1}C^TC\xi.\end{aligned} \tag{3.2.15}$$

Hence, by (3.2.10) and (3.2.13) and by denoting $\|\xi\|_{\tilde{G}}=\sqrt{\xi^T\tilde{G}\xi}$,

$$\begin{aligned}\frac{d}{dt}\big(\xi^T\tilde{G}\xi\big)&=2\theta\xi^T\tilde{G}A\xi-2\theta\xi^TC^TC\xi+2\xi^T\tilde{G}\Lambda\Gamma\\&=-\theta\xi^T\tilde{G}\xi-\theta\xi^TC^TC\xi+2\xi^T\tilde{G}\Lambda\Gamma\\&\le-\theta\|\xi\|^2_{\tilde{G}}+2\|\xi\|_{\tilde{G}}\|\Lambda\Gamma\|_{\tilde{G}}\\&\le-\theta\|\xi\|^2_{\tilde{G}}+2L_\Lambda\sqrt{\frac{\lambda_M}{\lambda_m}}\|\xi\|^2_{\tilde{G}}\end{aligned}$$

where λ_M (λ_m) denotes the maximum (minimum) eigenvalue of $\tilde{G}$, respectively. By choosing $\theta_1^*=4L_\Lambda\sqrt{\lambda_M/\lambda_m}$, for any $\theta>\theta_1^*$,

$$\frac{d}{dt}\|\xi\|^2_{\tilde{G}}\le-\frac{\theta}{2}\|\xi\|^2_{\tilde{G}} \tag{3.2.16}$$

which implies

$$\|\xi(t)\|_{\tilde{G}}\le\exp(-\frac{\theta}{4}t)\|\xi(0)\|_{\tilde{G}}.$$

Thus, by the fact $\sqrt{\lambda_m}\|\cdot\| \le \|\cdot\|_{\tilde{G}} \le \sqrt{\lambda_M}\|\cdot\|$,

$$\|e(t)\| \le \theta^{n-1}\sqrt{\frac{\lambda_M}{\lambda_m}}\exp(-\frac{\theta}{4}t)\|e(0)\|$$

which shows the claim (3.2.11), and for fixed $t = \tau$, the claim (3.2.12) also follows.

■

3.2.3 Characterization via Differential Geometric Condition

Suppose a system is given in x-coordinates as

$$\begin{aligned}\dot{x} &= f(x) + g(x,u)\\ y &= h(x)\end{aligned} \tag{3.2.17}$$

where the state $x \in \mathbb{R}^n$, the input $u \in \mathbb{R}^m$ and the output $y \in \mathbb{R}$; the vector fields f and g, and the function h are smooth; $g(x,0) = 0$.

Then, one may wonder if the given system (3.2.17) can be transformed to the lower triangular form (3.2.8). Here we present a necessary and sufficient condition for the system (3.2.17) to be transformed into (3.2.8). Therefore, at the early stage of the design, one can check the given system with this condition to know whether the Gauthier's approach is applicable to the system.

Lemma 3.2.3 (Uniform Observability)**.** *There exists a coordinate transformation $z = T(x)$ on $\mathbb{R}^n$ which transforms* (3.2.17) *to* (3.2.8)*, if and only if the following conditions hold;*

(i) $\Phi(x) := [h(x), L_f h(x), \cdots, L_f^{n-1}h(x)]^T$ *is a global diffeomorphism.*

(ii) $[g(x,u), \mathcal{R}_j] \subset \mathcal{R}_j$, $0 \le j \le n-2$, $\forall u$, *where*

$$\mathcal{R}_j := \text{span}\{r, ad_f r, ad_f^2 r, \cdots, ad_f^j r\}$$

in which r satisfies

$$\frac{\partial \Phi}{\partial x}(x) \cdot r = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}. \tag{3.2.18}$$

Moreover, when there is such a transformation $T(x)$, the mapping $T(x)$ is the same as $\Phi(x)$.

Proof. (*Sufficiency.*) Suppose a system $\dot{x} = f(x) + r(x)u$, $y = h(x)$. Then the relative degree of this system is n by (3.2.18), which implies the pair (f, r) is feedback linearizable with $z = \Phi(x)$ on U. Thus, the distribution $\mathcal{R}_i$ becomes

$$\mathcal{R}_i = \operatorname{span}\{\frac{\partial}{\partial z_n}, \cdots, \frac{\partial}{\partial z_{n-i}}\}, \qquad 0 \le i \le n-1 \tag{3.2.19}$$

in z-coordinates [MT95, Thm. 2.2.1]. In z-coordinates, the system (3.2.17) is

$$\begin{aligned} \dot{z}_j &= z_{j+1} + L_g L_f^{j-1} h, \qquad 1 \le j \le n-1 \\ \dot{z}_n &= L_f^n h + L_g L_f^{n-1} h \\ y &= z_1 \end{aligned}$$

and b is written as

$$b = \sum_{j=1}^{n} L_g L_f^{j-1} h \frac{\partial}{\partial z_j} = \sum_{j=1}^{n} b_j \frac{\partial}{\partial z_j}$$

by defining $b_j := L_g L_f^{j-1} h$.

The condition (ii) becomes in z-coordinates

$$[\sum_{j=1}^{n} b_j \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}] \in \operatorname{span}\{\frac{\partial}{\partial z_n}, \cdots, \frac{\partial}{\partial z_{n-i}}\} \tag{3.2.20}$$

for each $0 \le i \le n-2$ and $n-i \le k \le n$. Then, since $[\sum_{j=1}^{n} b_j(\partial/\partial z_j), (\partial/\partial z_k)] = \sum_{j=1}^{n} (\partial b_j/\partial z_k)(\partial/\partial z_j)$, (3.2.20) implies $(\partial b_j/\partial z_k) = 0$ for each $0 \le i \le n-2$, $1 \le j \le n-i-1$ and $n-i \le k \le n$. By simplifying the indices, this implies that

$$\frac{\partial b_j}{\partial z_k} = 0$$

for each $1 \leq j \leq n-1$ and $1+j \leq k \leq n$, which concludes that

$$b_j = b_j(z_1, \cdots, z_j, u), \qquad 1 \leq j \leq n.$$

(*Necessity.*) The assumption that $z = T(x)$ transforms (3.2.17) to (3.2.8) implies that $dT(x)$ is nonsingular for each $x \in \mathbb{R}^n$ [Mun91, p. 60] and $T : \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one. On the other hand, it is easy to show that if the system (3.2.17) is diffeomorphic to (3.2.8) via a certain transformation $T(x)$ it should be $\Phi(x)$. In fact, $z_1 = h(x)$ follows directly from the comparison of the outputs between (3.2.17) and (3.2.8), and $z_i = L_f^{i-1} h(x)$ also follows by letting $u = 0$ in (3.2.17) and (3.2.8). Thus, (i) holds since $T(x) = \Phi(x)$ on U.

For (ii), the condition is coordinate-free and thus it is verified in z-coordinates. In the coordinates,

$$f(z) = \begin{pmatrix} z_2 \\ \vdots \\ z_n \\ \psi(z) \end{pmatrix} \quad \text{and} \quad r(z) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \tag{3.2.21}$$

by (3.2.18) and the fact $(\partial\Phi/\partial x) = I$ in the coordinates. With these f and r, (3.2.19) again follows. Now, by a direct calculation, it can be easily shown that (ii) holds with

$$g(z, u) = [g_1(z_1, u), g_2(z_1, z_2, u), \cdots, g_n(z, u)]^T.$$

■

3.3 Lipschitz Extension

3.3.1 Lipschitz Extension

Although the Gauthier's observer is a good solution for the observer problem of uniformly observable systems, Assumption 3.2.1 (or 3.2.2) is the main restriction of the approach. In order to overcome the restriction, we present the *Lipschitz extension* technique in this section. For the technique to be applicable, we assume the boundedness of the trajectory $z(t)$ of the plant:

$$\dot{z} = a(z) + b(z, u). \tag{3.3.1}$$

Assumption 3.3.1. There is a compact set $Z \in \mathbb{R}^n$ such that the trajectory $z(t)$ of (3.3.1) is contained in Z for all $t \geq 0$.

This is a sacrifice for relieving the Lipschitz property of the vector fields a and b. Now since $z(t)$ is confined to the bounded set Z, the dynamics (3.3.1) can be modified outside the region Z. That is, when the system (3.3.1) satisfies Assumption 3.3.1, it doesn't make any difference that we regard the system model, instead of (3.3.1), as

$$\dot{z} = \hat{a}(z) + \hat{b}(z, u) \tag{3.3.2}$$

where $\hat{a}$ and $\hat{b}$ are modified outside Z to be globally Lipschitz in z. Then, since $\hat{a}$ and $\hat{b}$ are Lipschitz in z, the system (3.3.2) (equivalently (3.3.1)) always admits the Gauthier's observer when the input u is bounded.

There is another advantage of the extension. Suppose that the system model (3.3.1) is only valid on a bounded region Z, in particular, that the model (3.3.1) has a singularity outside Z. This is often the case because the modeling is usually done in the operating region. For example, the usual modeling process for magnetic levitation system gives singularity when the magnet is attached to the iron deck. (See [JBSS94] or Chapter 1.) Now suppose there is an observer which traces the true state asymptotically. Even though the estimate converges to the true state, it may overshoot at the initial stage. (See Fig. 3.3.) That is, it may go through the region outside Z where the vector fields a or b may not be well defined. Therefore, the modification of the system model outside the region Z is also a good solution to this accident. As a result, the observer uses the modified model to estimate the true state.

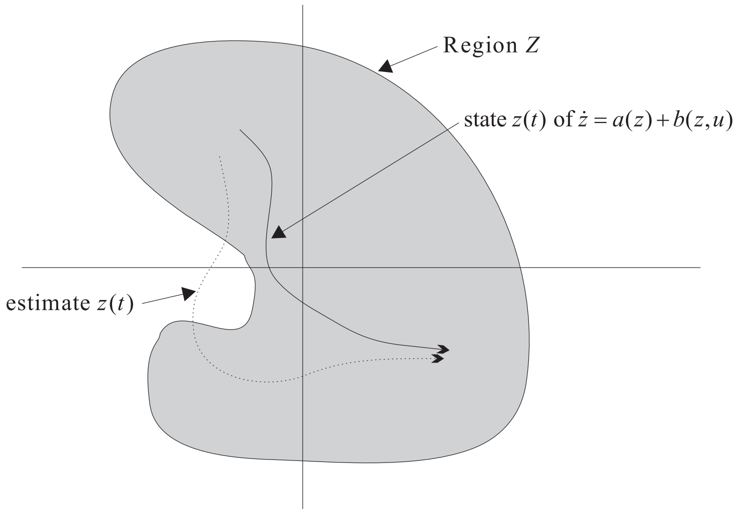


Figure 3.3: $\dot{z} = a(z) + b(z, u)$ may have singularity outside the region Z.

From now on, we precisely define this modification as the *Lipschitz extension*, and show the practical methods for the extension.

Preliminary and Definition

Lemma 3.3.1. *If a function is Lipschitz on a set A and continuous on $A \cup \partial A$, it is also Lipschitz on the closure $A \cup \partial A$ with the same Lipschitz constant.*

Proof. Trivial by the continuity. ∎

Lemma 3.3.2. *Product of two Lipschitz functions on a compact set is also Lipschitz on the set.*

Proof. Let f and g are Lipshitz on a compact set A. Then,

$$\begin{aligned}
&\|f(x)g(x) - f(y)g(y)\| \\
&\qquad \leq \|f(x)g(x) - f(y)g(x)\| + \|f(y)g(x) - f(y)g(y)\| \\
&\qquad \leq L_f\|g(x)\|\|x-y\| + L_g\|f(y)\|\|x-y\| \\
&\qquad \leq L\|x-y\|
\end{aligned}$$

where L is the maximum value of $L_f\|g(x)\|$ and $L_g\|f(x)\|$ on a compact set A. ■

Lemma 3.3.3. *Suppose two sets $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^n$ are disjoint and $A \cup B$ is convex. When a function f is continuous on $\mathbb{R}^n$ and Lipschitz on each set, then f is also Lipschitz on $A \cup B$.*

Proof. Choose $x \in A$, $y \in B$ and $z \in \partial A$ where z is on the line connecting x and y. Such z exists since $A \cup B$ is convex. Then,

$$\begin{aligned}
\|f(x) - f(y)\| &\leq \|f(x) - f(z)\| + \|f(z) - f(y)\| \\
&\leq L_A\|x-z\| + L_B\|z-y\| \\
&\leq L(\|x-z\| + \|z-y\|) = L\|x-y\|
\end{aligned}$$

where L_A and L_B is a Lipschitz constant on A and B, respectively, and $L = \max\{L_A, L_B\}$. ■

Definition 3.3.1 (Lipschitz Extension)**.** For a C^1 function $f(x,u) : \bar{X}(\subset \mathbb{R}^n) \times \mathbb{R}^p \to \mathbb{R}^m$ and a compact set $X \subset \bar{X}$, a function $\hat{f}(x,u) : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^m$ is called *Lipschitz extension* of $f(x,u)$ from X if

(i) $\hat{f}(x,u)$ is continuous and globally well-defined.

(ii) $\hat{f}(x,u) = f(x,u)$ for $^\forall(x,u) \in X \times \mathbb{R}^p$.

(iii) $\hat{f}(x,u)$ is globally Lipschitz in x.

Remark 3.3.1. If $\hat{f}(x,u)$ is bounded for all $(x,u) \in \mathbb{R}^n \times \mathbb{R}^p$ instead of (iii), then we call $\hat{f}$ a *bounded extension.* This bounded extension will be extensively used in Chapter 5.

Lipschitz Extension for the Observer Construction

For a given function, a Lipschitz extension can be found in general without severe difficulty. However, in our case, the extension also should preserve the lower triangular structure of $b(z,u)$. The next lemma solves the question whether there always exists an extension preserving the structure, even for the arbitrarily shaped set Z.

Lemma 3.3.4. *Suppose $b(\cdot,u)$ is defined on an open set $\bar{Z} \subset \mathbb{R}^n$ and a compact set $Z \subset \bar{Z}$. Then, there exists a Lipschitz extension of $b(\cdot,u)$ from Z preserving the structure of $b(\cdot,u)$. In particular, for each element $b_i(z_1,\cdots,z_i,u)$ of $b(z,u)$, there exists an extension $\hat{b}_i(z_1,\cdots,z_i,u)$ such that*

(i) $\hat{b}_i(\cdot,u)$ *is defined on* $\mathbb{R}^i$.

(ii) $\hat{b}_i(\cdot,u) = b_i(\cdot,u)$ *on the projection of Z into $\mathbb{R}^i$, i.e., on the set* $\{z \in \mathbb{R}^i \mid {}^{\exists}w \text{ s.t. } (z,w) \in Z\}$.

(iii) $\hat{b}_i(\cdot,u)$ *is globally Lipschitz and bounded on* $\mathbb{R}^i$.

Remark 3.3.2. In this lemma, the input u is regarded as a parameter. Therefore, the properties (i)-(iii) should hold for each u with the actual bound and the Lipschitz coefficient depending on the value of u.

Proof. Let z_a and z_b be state partitions as $z_a = (z_1,\cdots,z_i)$ and $z_b = (z_{i+1},\cdots,z_n)$. Define the *projection of Z into* $\mathbb{R}^i$ as

$$\mathcal{P} := \{z_a \in \mathbb{R}^i \mid {}^{\exists}z_b \text{ s.t. } (z_a,z_b) \in Z\}. \tag{3.3.3}$$

Similarly, let $\mathcal{Q}$ be the projection of $\bar{Z}$ into $\mathbb{R}^i$. Then, it can be easily shown that $\mathcal{P} \subset \mathcal{Q}$ where $\mathcal{P}$ is compact and $\mathcal{Q}$ is open.

Define a distance function $\rho(\cdot) : \mathbb{R}^i \to \mathbb{R}_{\geq 0}$ as

$$\rho(z_a) := \min_{z_p \in \mathcal{P}} \|z_a - z_p\|. \tag{3.3.4}$$

Then, it can be shown that $\rho(\cdot)$ is globally Lipschitz in $\mathbb{R}^i$ and there is $\delta > 0$ such that $\mathcal{P} \subset \{z_a \in \mathbb{R}^i \mid \rho(z_a) \leq \delta\} \subset \mathcal{Q}$ (see, e.g. [Mun91, Thm. 4.6]).

Finally, define

$$\hat{b}_i(z_a, u) := \begin{cases} (1 - \frac{1}{\delta}\rho(z_a))b_i(z_a, u), & \text{if } 0 \leq \rho(z_a) \leq \delta \\ 0, & \text{if } \delta < \rho(z_a). \end{cases} \tag{3.3.5}$$

Then, $\hat{b}_i(\cdot, u)$ is globally well-defined and continuous. Since $\hat{b}_i(\cdot, u)$ is continuous on $\mathbb{R}^i$ and $\rho(\cdot)$ and $b_i(\cdot, u)$ are Lipschitz on the compact region $\{z_a \in \mathbb{R}^i \mid 0 \leq \rho(z_a) \leq \delta\}$, the extension $\hat{b}_i$ is globally bounded and Lipschitz by Lemma 3.3.2 and 3.3.3. ■

Remark 3.3.3. While the extension $\hat{b}$ in Lemma 3.3.4 is not smooth, there is even a smooth extension which satisfies global Lipschitz and bounded properties, and its existence can be proved by slightly modifying the proof of Lemma 3.3.4 and by [Con92, Thm. 2.6.2]. However, the proof is based on partitions of unity and is therefore nonconstructive.

Lipschitz Extensions from a Rectangular Region

Since it is not so practical to find the extension by Lemma 3.3.4, more practical extensions are implemented here when the region Z is rectangular, i.e.,

$$Z = \{z \in \mathbb{R}^n \mid -\rho_i \leq x_i \leq \rho_i, 1 \leq i \leq n\} \tag{3.3.6}$$

where ρ_i's are certain positive constants. For simplicity, let $\bar{Z}$ be sufficiently large in the following discussions. The methods are classified according to their smoothness, assuming that the given function (e.g. $b(z, u)$) is smooth.

(a) Smooth Lipschitz Extension

Define a smooth function $\psi(\cdot,\rho):\mathbb{R}\to\mathbb{R}$ such that

$$\psi(s,\rho) := \begin{cases} 1, & \text{if } \ |s|\le\rho \\ 1-\frac{1}{c}\int_\rho^{|s|}\exp\left(\frac{-1}{\tau-\rho}\right)\exp\left(\frac{-1}{\rho+1-\tau}\right)\,d\tau, & \\ & \text{if } \ \rho<|s|<\rho+1 \\ 0, & \text{if } \ \rho+1\le|s| \end{cases}$$

where $c := \int_\rho^{\rho+1}\exp\left(\frac{-1}{\tau-\rho}\right)\exp\left(\frac{-1}{\rho+1-\tau}\right)\,d\tau$, and define the smooth function $\chi_i(\cdot):\mathbb{R}\to\mathbb{R}$ such that

$$\chi_i(s) := \int_0^s \psi(\tau,\rho_i)\,d\tau.$$

Let

$$\hat{b}_i := b_i(\chi_1(z_1),\cdots,\chi_i(z_i),u). \tag{3.3.7}$$

Then $\hat{b}_i$, $(1\le i\le n)$, becomes the smooth extension of b_i (the i-th element of $b(z,u)$). The verification is left to the reader.

(b) C^2 Lipschitz Extension

Define $\chi_i(s):\mathbb{R}\to\mathbb{R}$ such that

$$\chi_i(s) := \begin{cases} s, & \text{if } -\rho_i<s<\rho_i \\ \tanh(s-\rho_i)+\rho_i, & \text{if } \rho_i\le s \\ \tanh(s+\rho_i)-\rho_i, & \text{if } s\le-\rho_i \end{cases}$$

then, χ_i is globally bounded (i.e. $-\rho_i-1\le\chi_i(s)\le\rho_i+1$), and C^2. Indeed, $\chi_i(s)$ is C^∞ on $(-\rho_i,\rho_i)$, (ρ_i,∞) and $(-\infty,-\rho_i)$. At $s=\rho_i$,

$$\frac{\partial\chi_i}{\partial s}(\rho_i+) = \frac{\partial\chi_i}{\partial s}(\rho_i-) = 1$$
$$\frac{\partial^2\chi_i}{\partial s^2}(\rho_i+) = \frac{\partial^2\chi_i}{\partial s^2}(\rho_i-) = 0,$$

but,

$$\frac{\partial^3\chi_i}{\partial s^3}(\rho_i+) = -2, \quad \frac{\partial^3\chi_i}{\partial s^3}(\rho_i-) = 0.$$

Moreover,

$$\frac{\partial \chi_i}{\partial s}(s) = \begin{cases} 1, & \text{if } -\rho_i < s < \rho_i \\ 1 - \tanh^2(s - \rho_i), & \text{if } \rho_i \leq s \\ 1 - \tanh^2(s + \rho_i), & \text{if } s \leq -\rho_i \end{cases}$$

hence, $\frac{\partial \chi_i}{\partial s}$ is globally bounded.

Now, define $\hat{b}_i$ as (3.3.7). Then $\hat{b}_i$ is globally Lipschitz in z since for all $1 \leq j \leq i$,

$$\frac{\partial \hat{b}_i(x)}{\partial x_j} = \frac{\partial b_i}{\partial \chi_j}(\chi_1(x_1), \chi_2(x_2), \cdots, \chi_i(x_i), u)\,\frac{\partial \chi_j}{\partial x_j}(x_j)$$

which is bounded by a function of u from the boundedness of $\frac{\partial \chi_j}{\partial s}$ and χ_j's. Moreover, it can be checked that $\hat{b}(x)$ is also C^2.

(c) Continuous Lipschitz Extension I

Define the saturation function $\chi_i(s) : \mathbb{R} \to \mathbb{R}$ is such that

$$\chi_i(s) := \begin{cases} s, & \text{if } -\rho_i < s < \rho_i \\ \rho_i, & \text{if } \rho_i \leq s \\ -\rho_i, & \text{if } s \leq -\rho_i. \end{cases}$$

Now, define $\hat{b}_i$ as (3.3.7). Then the verification follows from the next lemma.

Lemma 3.3.5. *Consider a C^1 function $f(x, y) : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}$ which is continuous and well-defined on $X \times \mathbb{R}^q$ where $X = \{x \in \mathbb{R}^p \mid |x_i| \leq \rho_i, 1 \leq i \leq p\}$ with $\rho_i > 0$. Then, $f(\sigma(x), y)$ is globally well-defined, equals to $f(x, y)$ if $x \in X$, and there exists $L(y)$ such that*

$$|f(\sigma(x_2), y) - f(\sigma(x_1), y)| \leq L(y)\|x_2 - x_1\|, \qquad {}^\forall x_1, x_2 \in \mathbb{R}^p, \quad {}^\forall y \in \mathbb{R}^q \quad (3.3.8)$$

where $\sigma(x)$ is a component-wise saturation function which is saturated outside X.

Proof. By the Mean-Value Theorem, there exists $z \in \mathbb{R}^p$ such that

$$f(\sigma(x_2), y) - f(\sigma(x_1), y) = \frac{\partial f}{\partial x}(z, y)(\sigma(x_2) - \sigma(x_1))$$

which implies

$$|f(\sigma(x_2), y) - f(\sigma(x_1), y)| \leq L(y)\|\sigma(x_2) - \sigma(x_1)\|$$

where $L(y)$ is the maximum of $\|\frac{\partial f}{\partial x}(z, y)\|$ with respect to z over the compact range of saturation function. Then the claim (3.3.8) follows from the fact that $\|\sigma(x_2) - \sigma(x_1)\| \leq \|x_2 - x_1\|$. ■

(d) Continuous Lipschitz Extension II

Define

$$\chi_i(s) = \begin{cases} 1 & \text{when} \quad |s| \leq \rho_i \\ \rho_i + 1 - |s| & \text{when} \quad \rho_i \leq |s| < \rho_i + 1 \\ 0 & \text{when} \quad \rho_i + 1 \leq |s| \end{cases} \tag{3.3.9}$$

and let $w_i : \mathbb{R}^i \to \mathbb{R}$ as

$$w_i(s_1, \cdots, s_i) = \chi_1(s_1)\chi_2(s_2)\cdots\chi_i(s_i). \tag{3.3.10}$$

Then, the extension $\hat{b}_i$ is

$$\hat{b}_i := w_i(z_1, \cdots, z_i)b_i(z_1, \cdots, z_i, u). \tag{3.3.11}$$

Let us verify this claim utilizing the following lemma.

Lemma 3.3.6. *For $1 \leq i \leq n$, the function $w_i(s_1, \cdots, s_i)$ is Lipschitz on $\mathbb{R}^i$.*

Proof. It can be easily shown that the function $\chi_i(s_i)$ is globally Lipschitz, i.e., there is a constant L_i such that $|\chi_i(x_i) - \chi_i(y_i)| \leq L_i|x_i - y_i|$.

The rest of the proof is simple.[2] Suppose, for example, the case when $i = 3$.

$$\begin{aligned}|w_3(x) - w_3(y)| &= |\chi_1(x_1)\chi_2(x_2)\chi_3(x_3) - \chi_1(y_1)\chi_2(y_2)\chi_3(y_3)| \\ &\le |\chi_1(x_1)\chi_2(x_2)\chi_3(x_3) - \chi_1(x_1)\chi_2(x_2)\chi_3(y_3)| \\ &\qquad + |\chi_1(x_1)\chi_2(x_2)\chi_3(y_3) - \chi_1(x_1)\chi_2(y_2)\chi_3(y_3)| \\ &\qquad + |\chi_1(x_1)\chi_2(y_2)\chi_3(x_3) - \chi_1(y_1)\chi_2(y_2)\chi_3(y_3)| \\ &\le \chi_1(x_1)\chi_2(x_2)|\chi_3(x_3) - \chi_3(y_3)| + \chi_1(x_1)\chi_3(y_3)|\chi_2(x_2) - \chi_2(y_2)| \\ &\qquad + \chi_2(y_2)\chi_3(y_3)|\chi_1(x_1) - \chi_1(y_1)| \\ &\le L_3|x_3 - y_3| + L_2|x_2 - y_2| + L_1|x_1 - y_1| \\ &\le L\|x - y\|\end{aligned}$$

The other cases are similar. ■

Now, consider $\Omega_i = \{(z_1, \cdots, z_i) \mid |z_j| \le \rho_j + 1, 1 \le j \le i\}$ which is compact in $\mathbb{R}^i$. On Ω_i, Lemmas 3.3.6 and 3.3.2 yield that $\hat{b}_i$ is Lipschitz in z. On its complement Ω_i^c, $\hat{b}_i$ is also Lipschitz in z since $\hat{b}_i \equiv 0$. Then, Lemma 3.3.3 leads to that $\hat{b}_i$ is globally Lipschitz in z. Therefore, $\hat{b} = [w_1 b_1, \cdots, w_n b_n]^T$ gives the desired extension. Note that Ω_i is the projection of $\{z \in \mathbb{R}^n \mid |z_j| \le \rho_j + 1, 1 \le j \le n\}$ into $\mathbb{R}^i$.

Remark 3.3.4. All the constructions of the Lipschitz extension in this section also give the bounded extensions.

3.3.2 x-coordinate Observer

Even though a system is given in x-coordinates as

$$\dot{x} = f(x) + g(x, u), \qquad y = h(x), \tag{3.3.12}$$

the nonlinear observer for this system is usually designed in a special coordinates like the observer canonical form or the observable canonical form. Thus, the nonlinear

[2] But, not trivial. In general, product of two Lipschitz functions are not Lipschitz.

coordinate transformation $z = \Phi(x)$ is inevitable. Nevertheless, the calculation of the inverse $\Phi^{-1}(z)$ can be avoided for the observer designs. For example, suppose (3.3.12) is transformed to $\dot{z} = Az + \gamma(y, u)$ and $y = Cz$ and the observer is designed as

$$\dot{\hat{z}} = A\hat{z} + \gamma(y, u) - L(C\hat{z} - y).$$

Then, the observer in x-coordinates is obtained by

$$\dot{\hat{x}} = f(\hat{x}) + g(\hat{x}, u) - \big[\frac{\partial \Phi}{\partial \hat{x}}(\hat{x})\big]^{-1} L(h(\hat{x}) - y)$$

where $[\frac{\partial \Phi}{\partial \hat{x}}(\hat{x})]^{-1}$ is easily calculated by a computer, and thus, there is no need to have $\Phi^{-1}(z)$.

However, in our case of Lipschitz extension, the extension is performed in z-coordinates and therefore the dynamics is modified with respect to the state z rather than x. Hence, the analytic calculation of $\Phi^{-1}(z)$ is necessary in general. Fortunately, if the extension method (d) of the previous discussions is used, we can have the x-coordinate observer form without the knowledge of $\Phi^{-1}(z)$.

The key to the success is the modification by the multiplication. We provide the x-coordinate observer for (3.3.12) without further proof:

$$\dot{\hat{x}} = W_f(\hat{x}) f(\hat{x}) + W_g(\hat{x}) g(\hat{x}, u) - \big[\frac{\partial \Phi}{\partial x}(\hat{x})\big]^{-1} G^{-1} C^T (Ch(\hat{x}) - y) \tag{3.3.13}$$

where G is the solution of (3.2.10) for sufficiently large θ and

$$W_f(x) = \big[\frac{\partial \Phi}{\partial x}(x)\big]^{-1} \cdot \mathrm{diag}[1, \cdots, 1, w_0(\Phi(x))] \cdot \big[\frac{\partial \Phi}{\partial x}(x)\big]$$

$$W_g(x) = \big[\frac{\partial \Phi}{\partial x}(x)\big]^{-1} \cdot \mathrm{diag}[w_1(\Phi_1(x)), w_2(\Phi_1(x), \Phi_2(x)), \cdots, \\ \cdots, w_n(\Phi_1(x), \cdots, \Phi_n(x))] \cdot \big[\frac{\partial \Phi}{\partial x}(x)\big]$$

where Φ_i implies the i-th element of Φ.

3.4 Multi-output Extension I: Semi-global Observer

Contrary to the Gauthier's observer in Section 3.2 which is applicable only to single-output systems, we present an explicit form of nonlinear observer for a class of multi-output systems in this section.

Observer construction for multi-output nonlinear systems is not a trivial extension of single-output case, especially when the Gauthier's approach is utilized. A recent paper [DBB$^+$93] has already extended the Gauthier's observer to the multi-output systems which have the lower triangular structure like (3.2.8) for each output y_i. However, the proposed structure[3] does not allow the interconnection between each *channels* arisen from each outputs, and therefore is supposed to be just the multiple parallel connection of (3.2.8).

We consider a class of systems in which the subsystem for each output has a triangular dependence on the states of that subsystem itself, and the overall system has a block triangular form for each subsystem. Hence, the contribution is to extend the result of [DBB$^+$93] because interconnections between the subsystems are allowed.

For the class, we design a semi-global observer with global error convergence. Here by *semi-global observer* we mean an observer which guarantees the error system $(\hat{x} - x)$ is globally asymptotically stable at the origin as long as the state of plant x remains in a compact region $X \in \mathbb{R}^n$ whose size can be arbitrarily large. Since our interest is semi-global observer, we assume the boundedness of the state.

---

[3]In [DBB$^+$93], further extension was tried which allows some state dependence on A as $A^i_{j,k} = \delta_{j+1,k} \cdot a^i_j(x^i_1, \cdots, x^i_j; u)$. However, the proof of error convergence is not completed because, for even the simple system $\dot{x}_1 = (\sin(x_1) + 2)x_2, \dot{x}_2 = -(\sin(x_1) + 2)x_1 - x_2$ which satisfies Assumption H1, H2 and H3, the equation (2) of [DBB$^+$93] does not hold. Indeed, in the proof of [DBB$^+$93], $z^T \tilde{S} \Delta \Gamma$ is third order of z, but $B(\tilde{S}) P_u(1/\Theta) z^T \tilde{S} z$ second order of z, which contradicts eq. (2) of [DBB$^+$93].

Assumption 3.4.1. The state $x(t)$ and control $u(t)$ are bounded, i.e. $x(t) \in X$, $u(t) \in U$ for $t \geq 0$ where X and U are compact sets in $\mathbb{R}^n$ and $\mathbb{R}^p$, respectively.

Remark 3.4.1. In the above assumption, the boundedness is assumed for all non-negative time t. If it is assumed for some finite time interval, then it will be seen that the error convergence is guaranteed only for that time interval. Nevertheless, since the error convergence of the proposed observer is exponential and the rate of convergence is assignable, the arbitrarily small error at the end of time interval can be guaranteed if we put the initial of observer $\hat{x}(0)$ in X. This property can be used, e.g., for the purpose of semi-global output feedback stabilization in Chapter 5.

3.4.1 Class of Block Triangular Structure

Consider a class of smooth MIMO systems which are diffeomorphic to

$$\begin{aligned} \dot{x} &= Ax + B(x,u), \qquad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^p, \\ y &= Cx = \begin{pmatrix} C_1 x^1 \\ C_2 x^2 \\ \vdots \\ C_m x^m \end{pmatrix}, \quad y \in \mathbb{R}^m, \end{aligned} \tag{3.4.1}$$

where $x^i \in \mathbb{R}^{\lambda_i}$ is the i-th partition of the state x so that $x = [(x^1)^T, \cdots, (x^m)^T]^T$ and $\sum_{i=1}^m \lambda_i = n$; $A = \text{diag}(A_1, \cdots, A_m)$ where A_i is $\lambda_i \times \lambda_i$ matrix of Brunovsky form (see (3.2.1)); $C = \text{diag}(C_1, \cdots, C_m)$ where $C_i = [1, 0, \cdots, 0] \in \mathbb{R}^{\lambda_i}$, and the vector field $B(x,u) = [b^1(x,u)^T, \cdots, b^m(x,u)^T]^T$ in which the j-th element of b^i, b^i_j, has the following structural dependence on the state

$$b^i_j = b^i_j(x^1, \cdots, x^{i-1};\ x^i_1, \cdots, x^i_j;\ u;\ y_{i+1}, \cdots, y_m) \tag{3.4.2}$$

for all $1 \leq i \leq m$ and $1 \leq j \leq \lambda_i$. Therefore, b^i_j is independent of the lower states $(x^i_{j+1}, \cdots, x^i_{\lambda_i})$ of the i-th block and the unmeasured states of the lower blocks $(x^{i+1}, \cdots, x^m)$. In other words, the i-th block of (3.4.1) can be written as

$$\boxed{\begin{aligned} \begin{pmatrix} \dot{x}^i_1 \\ \dot{x}^i_2 \\ \vdots \\ \vdots \\ \dot{x}^i_{\lambda_i} \end{pmatrix} &= \begin{pmatrix} x^i_2 \\ x^i_3 \\ \vdots \\ x^i_{\lambda_i} \\ 0 \end{pmatrix} + \begin{pmatrix} b^i_1(x^{[1,i-1]};\ x^i_1;\ u;\ y_{[i+1,m]}) \\ \vdots \\ b^i_j(x^{[1,i-1]};\ x^i_{[1,j]};\ u;\ y_{[i+1,m]}) \\ \vdots \\ b^i_{\lambda_i}(x^{[1,i-1]}; x^i_{[1,\lambda_i]}; u; y_{[i+1,m]}) \end{pmatrix} \\ y_i &= x^i_1 \end{aligned}} \tag{3.4.3}$$

where x^i_j is the j-th element of the i-th block x^i. For the notational simplicity, we use the abbreviation $x^{[1,k]} := [(x^1)^T, \cdots, (x^k)^T]^T$, $x^i_{[1,j]} := [x^i_1, \cdots, x^i_j]^T$ and $y_{[i+1,m]} = [y_{i+1}, \cdots, y_m]^T$; and also write y instead of $y_{[i+1,m]}$ when there is no confusion. Finally, assume that

$$b^i_j(x, u) = 0 \text{ when } u = 0, \qquad 1 \leq i \leq m, \quad 1 \leq j \leq \lambda_i - 1. \tag{3.4.4}$$

By Assumption 3.4.1, we can restrict to $X \times U$ the region where the system (3.4.1) is valid. Outside the region, we will modify the dynamics so that the model is well-defined and globally Lipschitz (Lipschitz extension). Therefore, if the system (3.4.1) is transformed from a given system in x-coordinates, the change of coordinate need not be defined globally. We will consider the regional coordinate transformation in Section 3.4.3

Remark 3.4.2. As already mentioned, there is a canonical form of uniform observability for single-output input affine systems (i.e. (3.1.2)), or for single-output input non-affine systems ([GK94]). Unfortunately, there is no such canonical form for multi-output systems. However, according to Definition 3.1.3, the system (3.4.1) is contained in the class of uniformly observable systems.

3.4.2 Observer Construction

Lipschitz Extension

Suppose the vector field $B(x,u)$ of (3.4.1) is modified to $\hat{B}(x,u)$ using the Lipschitz extension technique of Section 3.3. Therefore, $\hat{B}(x,u)$ is continuous and well-defined in $\mathbb{R}^n \times U$, $\hat{B}(x,u) = B(x,u)$ for $^\forall(x,u) \in X \times U$, and every element $\hat{b}^i_j$ is globally Lipschitz in the states of its own block[4], i.e., there exist Lipschitz coefficients L^i_j, possibly depending on $(x^{[1,i-1]}, u, y)$ in a continuous manner, such that

$$|\hat{b}^i_j(x^{[1,i-1]};\ x^i_{[1,j]};\ u;\ y) - \hat{b}^i_j(x^{[1,i-1]};\ \bar{x}^i_{[1,j]};\ u;\ y)| \le L^i_j(x^{[1,i-1]}, u, y)\ \|x^i_{[1,j]} - \bar{x}^i_{[1,j]}\| \tag{3.4.5}$$

for $1 \le i \le m$ and $1 \le j \le \lambda_i$.

For example, if Assumption 3.4.1 holds with a rectangular region X, a Lipschitz extension of $B(x,u)$ is $\hat{B}(x,u) = B(\sigma(x),u)$ where σ is a component-wise saturation function which saturates outside X. Although we mentioned saturation technique for the Lipschitz extension, hereafter, the extended $\hat{B}$ by another method is also allowed.

Now, due to Assumption 3.4.1, it doesn't make any difference that we regard the given system as

$$\dot{x} = Ax + \hat{B}(x,u) \tag{3.4.6}$$

instead of (3.4.1). Therefore, in what follows, we present an observer for (3.4.6).

Observer Construction

By Assumption 3.4.1, there are constants ρ^i_j such that

$$|x^i_j(t)| \le \rho^i_j, \qquad ^\forall t \ge 0$$

[4]This point differs slightly from the definition of Lipschitz extension in Section 3.3.

for $1 \le i \le m$, $1 \le j \le \lambda_i$. Define saturation function σ_j^i as

$$\sigma_j^i(s) = \begin{cases} \rho_j^i, & \text{if } \rho_j^i < s \\ s, & \text{if } -\rho_j^i \le s \le \rho_j^i \\ -\rho_j^i, & \text{if } s < -\rho_j^i. \end{cases} \tag{3.4.7}$$

Then, the i-th block of the proposed observer is

$$\boxed{\begin{pmatrix} \dot{\hat{x}}_1^i \\ \dot{\hat{x}}_2^i \\ \vdots \\ \vdots \\ \dot{\hat{x}}_{\lambda_i}^i \end{pmatrix} = \begin{pmatrix} \hat{x}_2^i \\ \hat{x}_3^i \\ \vdots \\ \hat{x}_{\lambda_i}^i \\ 0 \end{pmatrix} + \begin{pmatrix} \hat{b}_1^i(\sigma^1(\hat{x}^1), \cdots, \sigma^{i-1}(\hat{x}^{i-1});\ \hat{x}_1^i;\ u;\ y_{[i+1,m]}) \\ \vdots \\ \hat{b}_j^i(\sigma^1(\hat{x}^1), \cdots, \sigma^{i-1}(\hat{x}^{i-1});\ \hat{x}_{[1,j]}^i;\ u;\ y_{[i+1,m]}) \\ \vdots \\ \hat{b}_{\lambda_i}^i(\sigma^1(\hat{x}^1), \cdots, \sigma^{i-1}(\hat{x}^{i-1}); \hat{x}_{[1,\lambda_i]}^i; u; y_{[i+1,m]}) \end{pmatrix} - S_i^{-1} C_i^T \big(C_i \hat{x}^i - y_i\big)} \tag{3.4.8}$$

where S_i is the unique solution of

$$0 = -\theta_i S_i - S_i A_i - A_i^T S_i + C_i^T C_i \tag{3.4.9}$$

in which θ_i is a tuning parameter to be specified shortly, and

$$\sigma^i(x^i) = [\sigma_1^i(x_1^i),\ \sigma_2^i(x_2^i),\ \cdots,\ \sigma_{\lambda_i}^i(x_{\lambda_i}^i)]^T.$$

By the knowledge of the bound for x, we know the estimates are wrong when $\hat{x}$ is beyond the bound of x. Therefore, by using the saturation σ, we limit the mal-effect of wrong estimates to the i-th block, since the large error of the previous $(i-1)$-th block makes the observing x^i difficult in the i-th block. With the help of saturation technique, the following theorem shows the global error convergence of the observer (3.4.8).

Remark 3.4.3. The idea of saturation is originated by [EK92, KE93], in which, they studied the semi-global output feedback stabilization problem, and used a saturation function in the control law in order to prevent the mal-effect due to the wrong estimates of the observer.

It is left to show the existence of the value of θ_i with which the global error convergence is guaranteed.

Theorem 3.4.1. *Suppose the system* (3.4.6) *(equivalent to* (3.4.1) *under Assumption 3.4.1) satisfies Lipschitz property* (3.4.5)*. Then, under Assumption 3.4.1, there exist sufficiently large θ_i's ($1 \leq i \leq m$) satisfying*

$$1 \leq \theta_1 \qquad \text{and} \qquad \theta_{i-1}^{\lambda_{i-1}} \leq \theta_i \qquad \text{for} \quad 2 \leq i \leq m \tag{3.4.10}$$

such that, for any initial $\hat{x}(0)$, the estimate $\hat{x}(t)$ of the observer (3.4.8) *converges to the true state $x(t)$ of the plant* (3.4.1) *exponentially.*

Remark 3.4.4. In fact, the error system is globally exponentially stable which is justified by (3.4.14). Moreover, by assigning θ_i's sufficiently large such that (3.4.10) holds, the convergence rate can be made arbitrarily fast. Indeed, by (3.4.14) it can be shown that

$$\|\epsilon(t)\| \leq c_1 \, \theta_m^{\lambda_m} \exp(-c_2 \Theta t) \|\epsilon(0)\|$$

with some constant c_1, c_2 and error $\epsilon = \hat{x} - x$, where Θ is defined from (3.4.13). Then, keeping in mind that $\theta^\lambda \exp(-\theta) \to 0$ as $\theta \to \infty$ the claim can be verified from (3.4.13) and (3.4.14).

Proof of Theorem 3.4.1

Define the error $\epsilon := \hat{x} - x$. Then the error dynamics of the i-th block become

$$\dot{\epsilon}^i = A_i \epsilon^i + \Gamma^i(x, \hat{x}, u) - S_i^{-1} C_i^T C_i \epsilon^i, \quad 1 \leq i \leq m$$

where $\Gamma^i(x, \hat{x}, u) = [\Gamma_1^i, \ \Gamma_2^i, \ \cdots, \ \Gamma_{\lambda_i}^i]^T$ in which, by abuse of notation $\sigma(\cdot)$,

$$\Gamma_j^i\big(x, \hat{x}, u\big) \ := \ \hat{b}_j^i\big(\sigma(\hat{x}^{[1,i-1]}); \ \hat{x}_{[1,j]}^i; \ u; \ y_{[i+1,m]}\big) - \ \hat{b}_j^i\big(x^{[1,i-1]}; \ x_{[1,j]}^i; \ u; \ y_{[i+1,m]}\big).$$

Define additional variable $\xi^i := \Delta^i(\theta_i)\epsilon^i$ where $\Delta^i(\theta_i) := \mathrm{diag}(\frac{1}{\theta_i}, \frac{1}{\theta_i^2}, \cdots, \frac{1}{\theta_i^{\lambda_i}})$.

Lemma 3.4.2. *There exist constants k^i and γ^i for each i-th block, which are independent of the value of θ_i $(1 \leq i \leq m)$, such that*

$$\|\Delta^i \Gamma^i\| \ \leq \ k^i \ \|\xi^i\| + \gamma^i \ \sum_{q=1}^{i-1} \|\xi^q\|$$

if

$$1 \leq \theta_1 \quad and \quad \theta_{i-1}^{\lambda_{i-1}} \leq \theta_i \quad for \quad 2 \leq i \leq m. \tag{3.4.11}$$

Proof. For the j-th element of $\Delta^i \Gamma^i$,

$$\begin{aligned}\frac{1}{\theta_i^j}\big|\Gamma_j^i(x,\hat{x},u)\big| \ \leq \ & \frac{1}{\theta_i^j}\big|\hat{b}_j^i(\sigma(\hat{x}^{[1,i-1]});\ \hat{x}_{[1,j]}^i;\ u;\ y) - \hat{b}_j^i(\sigma(\hat{x}^{[1,i-1]});\ x_{[1,j]}^i;\ u;\ y)\big| \\ & + \frac{1}{\theta_i^j}\big|\hat{b}_j^i(\sigma(\hat{x}^{[1,i-1]});\ x_{[1,j]}^i;\ u;\ y) - \hat{b}_j^i(x^{[1,i-1]};\ x_{[1,j]}^i;\ u;\ y)\big|.\end{aligned}$$

By the Lipschitz property (3.4.5) and by the boundedness of u, y and the saturation function $\sigma(\cdot)$, the first term of the right hand of above inequality satisfies

$$\begin{aligned}\frac{1}{\theta_i^j}\big|\hat{b}_j^i\big(\sigma(\hat{x}^{[1,i-1]}); x_1^i + \theta_i\xi_1^i, \cdots, x_j^i + \theta_i^j\xi_j^i; u; y\big) - \hat{b}_j^i\big(\sigma(\hat{x}^{[1,i-1]}); x_1^i, \cdots, x_j^i; u; y\big)\big| \\ \leq \ \frac{1}{\theta_i^j}\ k_j^i\ \big\|[\theta_i\xi_1^i,\ \cdots,\ \theta_i^j\xi_j^i]^T\big\| \\ \leq \ k_j^i\ \big\|\xi_{[1,j]}^i\big\|\end{aligned}$$

where k_j^i is the maximum value of $L_j^i(\sigma(\cdot); u; y_{[i+1,m]})$ over the compact range of $(\sigma(\cdot), u, y_{[i+1,m]})$.

Also for the second term, it can be shown that there exists a constant γ_j^i which satisfies the following inequality, by Lemma 3.3.5 and the fact $\sigma(x) = x$ for $^\forall x \in X$.

$$\begin{aligned}\frac{1}{\theta_i^j}\ \big|\ \hat{b}_j^i\big(\sigma^1(x^1 + (\Delta^1)^{-1}\xi^1), \cdots, \sigma^{i-1}(x^{i-1} + (\Delta^{i-1})^{-1}\xi^{i-1});\ x_{[1,j]}^i;\ u;\ y\big) \\ - \hat{b}_j^i\big(x^1, \cdots, x^{i-1};\ x_{[1,j]}^i;\ u;\ y\big)\ \big| \\ \leq \ \frac{1}{\theta_i^j}\ \gamma_j^i\ \big\|\big[\big((\Delta^1)^{-1}\xi^1\big)^T,\ \cdots,\ \big((\Delta^{i-1})^{-1}\xi^{i-1}\big)^T\big]^T\big\| \\ \leq \ \gamma_j^i\ \sum_{q=1}^{i-1}\|\xi^q\|\end{aligned}$$

where the last inequality follows from the fact that $\|(\Delta^{i-1})^{-1}\| \leq \theta_i$.

By choosing $k^i = \lambda_i \cdot \max\{k_1^i, \cdots, k_{\lambda_i}^i\}$ and $\gamma^i = \lambda_i \cdot \max\{\gamma_1^i, \cdots, \gamma_{\lambda_i}^i\}$, the proof is completed. ■

Now define $\tilde{S}_i := \frac{1}{\theta_i}(\Delta^i)^{-1} S_i (\Delta^i)^{-1}$. Then $\tilde{S}_i$ satisfies the equality

$$0 = -\tilde{S}_i - \tilde{S}_i A_i - A_i^T \tilde{S}_i + C_i^T C_i \tag{3.4.12}$$

from (3.4.9), because $\Delta^i A_i (\Delta^i)^{-1} = \theta_i A_i$ and $C_i(\Delta^i)^{-1} = \theta_i C_i$. Notice that the unique positive definite solution $\tilde{S}_i$ is independent of the value θ_i.

These equalities also yield

$$\begin{aligned}
\dot{\xi}^i &= \Delta^i A_i (\Delta^i)^{-1} \xi^i + \Delta^i \Gamma^i - \Delta^i S_i^{-1} C_i^T C_i (\Delta^i)^{-1} \xi^i \\
&= \theta_i A_i \xi^i + \Delta^i \Gamma^i - \theta_i \tilde{S}_i^{-1} C_i^T C_i \xi^i.
\end{aligned}$$

Hence, by (3.4.12),

$$\begin{aligned}
\frac{d}{dt}\big((\xi^i)^T \tilde{S}_i \xi^i\big) &= 2\theta_i (\xi^i)^T \tilde{S}_i A_i \xi^i - 2\theta_i (\xi^i)^T C_i^T C_i \xi^i + 2(\xi^i)^T \tilde{S}_i \Delta^i \Gamma^i \\
&= -\theta_i (\xi^i)^T \tilde{S}_i \xi^i - \theta_i (\xi^i)^T C_i^T C_i \xi^i + 2(\xi^i)^T \tilde{S}_i \Delta^i \Gamma^i \\
&\leq -\theta_i \underline{\mu}_i \|\xi^i\|^2 + 2\overline{\mu}_i \|\xi^i\| \, \|\Delta^i \Gamma^i\|
\end{aligned}$$

where $\overline{\mu}_i$ denotes the maximum eigenvalue of $\tilde{S}_i$, and $\underline{\mu}_i$ the minimum eigenvalue of $\tilde{S}_i$. Applying Lemma 3.4.2, and by the inequality $2xy \leq \|x\|^2 + \|y\|^2$,

$$\begin{aligned}
\frac{d}{dt}\big((\xi^i)^T \tilde{S}_i \xi^i\big) &\leq -\theta_i \underline{\mu}_i \|\xi^i\|^2 + 2\overline{\mu}_i k^i \|\xi^i\|^2 + 2\overline{\mu}_i \gamma^i \sum_{q=1}^{i-1} \|\xi^q\| \, \|\xi^i\| \\
&\leq -\theta_i \underline{\mu}_i \|\xi^i\|^2 + 2\overline{\mu}_i k^i \|\xi^i\|^2 + \overline{\mu}_i \gamma^i \sum_{q=1}^{i-1} (\|\xi^q\|^2 + \|\xi^i\|^2) \\
&= -\theta_i \underline{\mu}_i \|\xi^i\|^2 + \big(2\overline{\mu}_i k^i + \overline{\mu}_i \gamma^i (i-1)\big) \|\xi^i\|^2 + \overline{\mu}_i \gamma^i \sum_{q=1}^{i-1} \|\xi^q\|^2.
\end{aligned}$$

These inequalities can be summed up to

$$
\begin{aligned}
\frac{d}{dt}\Big(\sum_{i=1}^{m}(\xi^i)^T\tilde{S}_i\xi^i\Big) \le \big(&-\theta_1\underline{\mu}_1 + 2\overline{\mu}_1 k^1 + \overline{\mu}_2\gamma^2 + \overline{\mu}_3\gamma^3 + \overline{\mu}_4\gamma^4 + \cdots + \overline{\mu}_m\gamma^m\big)\|\xi^1\|^2 \\
&+ \big(-\theta_2\underline{\mu}_2 + 2\overline{\mu}_2 k^2 + \overline{\mu}_2\gamma^2 + \overline{\mu}_3\gamma^3 + \overline{\mu}_4\gamma^4 + \cdots + \overline{\mu}_m\gamma^m\big)\|\xi^2\|^2 \\
&+ \big(-\theta_3\underline{\mu}_3 + 2\overline{\mu}_3 k^3 + 2\overline{\mu}_3\gamma^3 + \overline{\mu}_4\gamma^4 + \cdots + \overline{\mu}_m\gamma^m\big)\|\xi^3\|^2 \\
&\qquad\vdots \\
&+ \big(-\theta_m\underline{\mu}_m + 2\overline{\mu}_m k^m + (m-1)\overline{\mu}_m\gamma^m\big)\|\xi^m\|^2.
\end{aligned}
\tag{3.4.13}
$$

Finally, θ_i's are chosen to satisfy the relation (3.4.11) and to make the right hand of (3.4.13) negative definite. By the chosen θ_i's, there is a positive constant Θ such that

$$
\frac{d}{dt}\Big(\sum_{i=1}^{m}(\xi^i)^T\tilde{S}_i\xi^i\Big) \le -\Theta\|\xi\|^2 \tag{3.4.14}
$$

in which $\frac{d}{dt}\big(\sum_{i=1}^{m}(\xi^i)^T\tilde{S}_i\xi^i\big)$ can be seen as a quadratic Lyapunov function. Thus exponential convergence of the error ϵ follows. (See [Kha96, Corollary 3.4].)

3.4.3 Characterization of the Class

This section plays the same role as Section 3.2.3, that is, we consider the condition under which a given nonlinear system

$$
\dot{z} = F(z,u), \qquad y = h(z) \tag{3.4.15a}
$$

is transformed to the block triangular form (3.4.1). In particular, a necessary and sufficient condition is obtained for the system (3.4.15a) to be transformed into (3.4.1).

Now, instead of the description (3.4.15a), we regard the given system equivalently as

$$
\dot{z} = f(z) + g(z,u), \qquad y = h(z) \tag{3.4.15b}
$$

where $f(z) = F(z,0)$ and $g(z,u) = F(z,u) - F(z,0)$. Suppose (3.4.15) is defined on $Z \times U \subset \mathbb{R}^n \times \mathbb{R}^p$.

The condition comprises two statements. The first one is related to the observability indices for the system, and the second one is for the block triangular structure. The conditions are regional rather than global, which gives much flexibility. In fact, the vector field $F(z,u)$ is required to be well-defined only on some region $Z \times U$ where the system operates, rather than on the global region.

Condition 3.4.1 (Observability)**.** There are integer numbers $\lambda_1, \cdots, \lambda_m$ such that $\sum_{i=1}^m \lambda_i = n$ and that $\Phi(z) : Z \to \Phi(Z)$ is a diffeomorphism where

$$\Phi(z) := \begin{pmatrix} h_1(z) \\ L_f h_1(z) \\ \vdots \\ L_f^{\lambda_1 - 1} h_1(z) \\ h_2(z) \\ \vdots \\ L_f^{\lambda_2 - 1} h_2(z) \\ \vdots \\ L_f^{\lambda_m - 1} h_m(z) \end{pmatrix}. \tag{3.4.16}$$

Remark 3.4.5. In order to check whether $\Phi(z)$ is diffeomorphism on a region Z, it is necessary and sufficient that the Jacobian $(\partial\Phi/\partial z)(z)$ is nonsingular on Z and $\Phi(z)$ is one-to-one from Z to $\Phi(Z)$. See [Mun91, p.65].

Under this condition, the Jacobian $d\Phi(z)$ is invertible on Z. Let $r_i(z)$ be the i-th column (vector field) of $[(\partial\Phi/\partial z)(z)]^{-1}$ and $\mathcal{R}(i)_j := \text{span}\{r_{\nu_i - j}, \cdots, r_{\nu_i}\}$ where $\nu_i = \sum_{j=1}^i \lambda_j$.

Condition 3.4.2 (Triangular Structure). On Z,

(i) for $2 \leq i \leq m$, $[f(z), \mathcal{R}(i)_{\lambda_i - 2}] \subset \mathcal{R}(i)_{\lambda_i - 1} + \mathcal{R}(i+1)_{\lambda_{i+1}-1} + \cdots + \mathcal{R}(m)_{\lambda_m - 1}$.

(ii) for $1 \leq i \leq m$ and $0 \leq j \leq \lambda_i - 2$, $[g(z,u), \mathcal{R}(i)_j] \subset \mathcal{R}(i)_j + \mathcal{R}(i+1)_{\lambda_{i+1}-1} + \cdots + \mathcal{R}(m)_{\lambda_m - 1}$, ${}^\forall u \in U$.

Remark 3.4.6. Exchanging and re-ordering the outputs of the given system (3.4.15) increases the possibility that Conditions 3.4.1 and 3.4.2 hold.

Theorem 3.4.3. *Consider a connected open set $Z \subset \mathbb{R}^n$ containing the origin. Then, there exists a coordinate transformation $x = T(z)$ on Z which transforms* (3.4.15) *to* (3.4.1)*, if and only if Conditions 3.4.1 and 3.4.2 hold on Z. Moreover, when there is such a transformation $T(z)$, $\Phi(z)$ defined in Condition 3.4.1 is one of such mappings.*

Proof. In this proof, the states are indexed sequentially for convenience. Thus, x^i_j in (3.4.3) is equivalent to $x_{\nu_{i-1}+j}$ (with $\nu_0 = 0$) in this proof.

(*Sufficiency:*) By the transformation $x = \Phi(z)$, the system (3.4.15) becomes

$$\begin{aligned}
\dot{x}_{\nu_{i-1}+j} &= x_{\nu_{i-1}+j+1} + L_g L_f^{j-1} h_i, \qquad 1 \leq j \leq \lambda_i - 1 \\
\dot{x}_{\nu_i} &= L_f^{\lambda_i} h_i + L_g L_f^{\lambda_i - 1} h_i \\
y_i &= x_{\nu_{i-1}+1}
\end{aligned}$$

for $1 \leq i \leq m$ in x-coordinate. Let

$$b_g := \sum_{i=1}^{m} \sum_{j=1}^{\lambda_i} L_g L_f^{j-1} h_i \frac{\partial}{\partial x_{\nu_{i-1}+j}} = \sum_{l=1}^{n} b_{g,l} \frac{\partial}{\partial x_l} \qquad \text{and} \qquad b_f := \sum_{i=1}^{m} L_f^{\lambda_i} h_i \frac{\partial}{\partial x_{\nu_i}}. \tag{3.4.17}$$

Now, we show, under Condition 3.4.2, the vector fields b_g and b_f have the block triangular structure so that $b = b_g + b_f$ exhibits the structure of (3.4.2).

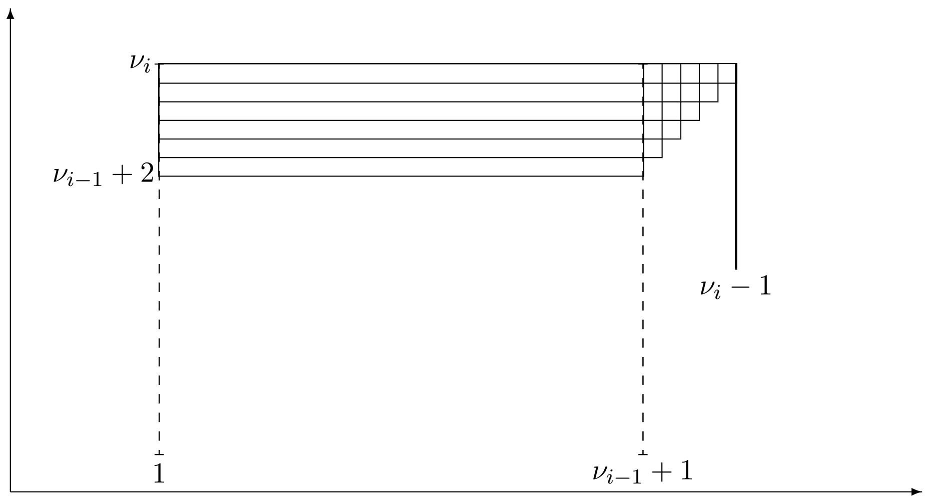


Figure 3.4: Helpful Diagram for the Proof of Sufficiency

By the construction,

$$\mathcal{R}(i)_j = \text{span}\{\frac{\partial}{\partial x_{\nu_i}}, \cdots, \frac{\partial}{\partial x_{\nu_i - j}}\}, \qquad 1 \le i \le m, \quad 0 \le j \le \lambda_i - 1$$

in x-coordinate on $\Phi(Z)$. In this coordinate, Condition 3.4.2.(ii) also becomes

$$[\sum_{l=1}^{n} b_{g,l}\frac{\partial}{\partial x_l}, \frac{\partial}{\partial x_k}] \in \text{span}\{\frac{\partial}{\partial x_{\nu_m}}, \frac{\partial}{\partial x_{\nu_m - 1}}, \cdots, \frac{\partial}{\partial x_{\nu_i}}, \cdots, \frac{\partial}{\partial x_{\nu_i - j}}\} \tag{3.4.18}$$

for each $1 \le i \le m$, $0 \le j \le \lambda_i - 2$ and $\nu_i - j \le k \le \nu_i$. Then, since $\big[\sum_{l=1}^{n} b_{g,l}\frac{\partial}{\partial x_l}, \frac{\partial}{\partial x_k}\big] = \sum_{l=1}^{n} \frac{\partial b_{g,l}}{\partial x_k}\frac{\partial}{\partial x_l}$, (3.4.18) implies $\frac{\partial b_{g,l}}{\partial x_k} = 0$ for each $1 \le i \le m$, $0 \le j \le \lambda_i - 2$, $1 \le l \le \nu_i - j - 1$ and $\nu_i - j \le k \le \nu_i$. By simplifying the indices, this implies that

$$\frac{\partial b_{g,l}}{\partial x_k} = 0 \tag{3.4.19}$$

for each $1 \le i \le m$, $1 \le l \le \nu_i - 1$ and $\max(l+1, \nu_{i-1}+2) \le k \le \nu_i$.

For b_f, Condition 3.4.2.(i) becomes

$$\begin{aligned}[b_f + \sum_{l=1}^{m} \sum_{j=1}^{\lambda_l - 1} x_{\nu_{l-1}+j+1} \frac{\partial}{\partial x_{\nu_{l-1}+j}}, \frac{\partial}{\partial x_k}] \in \\ \mathrm{span}\{\frac{\partial}{\partial x_{\nu_m}}, \frac{\partial}{\partial x_{\nu_m - 1}}, \cdots, \frac{\partial}{\partial x_{\nu_i}}, \cdots, \frac{\partial}{\partial x_{\nu_{i-1}+1}}\}\end{aligned} \tag{3.4.20}$$

for $2 \leq i \leq m$ and $\nu_{i-1} + 2 \leq k \leq \nu_i$, which implies that by (3.4.17) and (3.4.20)

$$\frac{\partial b_{f,\lambda_l}}{\partial x_k} + \sum_{j=1}^{\lambda_l - 1} \left(\frac{\partial x_{\nu_{l-1}+j+1}}{\partial x_k} \right) = 0$$

for each $2 \leq i \leq m$, $\nu_{i-1} + 2 \leq k \leq \nu_i$ and $1 \leq l \leq i - 1$. Therefore, since $\sum_{j=1}^{\lambda_l - 1} \left(\frac{\partial x_{\nu_{l-1}+j+1}}{\partial x_k} \right) = 0$ for each index,

$$\frac{\partial b_{f,\lambda_l}}{\partial x_k} = 0, \tag{3.4.21}$$

for each $2 \leq i \leq m$, $\nu_{i-1} + 2 \leq k \leq \nu_i$ and $1 \leq l \leq i - 1$. (See Fig. 3.4 which may be helpful.)

From (3.4.19) and (3.4.21), the block triangular structure of (3.4.2) follows.

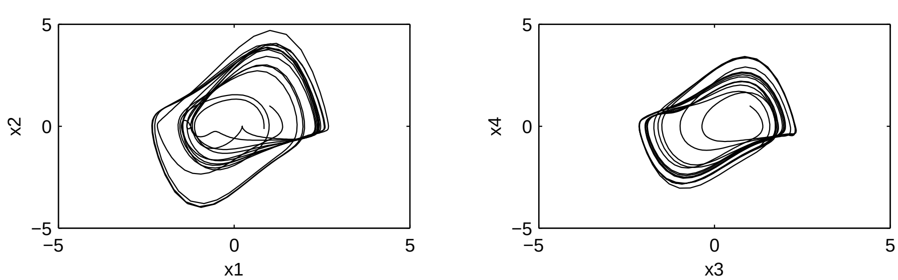
(*Necessity:*) Let $u = 0$. Then, transformability from (3.4.15b) ($\dot{z} = f(z)$) to (3.4.1) ($\dot{x} = Ax + B(x, 0)$) on Z implies Condition 3.4.1 by (3.4.4). Moreover, Condition 3.4.2 can be verified from the structure of (3.4.1) in x-coordinate because it is coordinate-free. ■

3.4.4 Illustrative Example

An example of constructing observer is provided in order to show the effectiveness of the proposed method. Consider a multi-output nonlinear system having the form of (3.4.1):

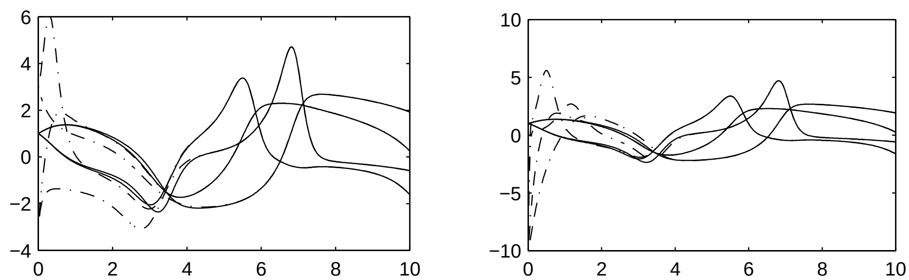
$$
\begin{aligned}
\dot{x}_1 &= x_2 + 0.01 x_1 u \\
\dot{x}_2 &= -x_1 + (1 - x_1^2) x_2 + x_3 u \\
\dot{x}_3 &= x_4 + 0.01 x_2 x_3 \exp(u) \\
\dot{x}_4 &= -x_3 + (1 - x_3^2) x_4 + \frac{1}{1 + (x_2 x_4)^2} u \\
y_1 &= x_1 \\
y_2 &= x_3
\end{aligned}
\tag{3.4.22}
$$

where the first block (y_1) consists of x_1 and x_2, and the second block (y_2) consists of the rest. The system has some interconnections between the blocks, i.e. x_3 in the first block and x_2 in the second block, which is not the case of [DBB$^+$93]. With $x(0) = [1, 1, 1, 1]$ and $u(t) = \sin(0.3t)$, the solution trajectory of the system is bounded and behaves like the Van der Pol oscillator. The phase planes of x_1 and x_2, and of x_3 and x_4 are depicted in Fig. 3.5. In these settings, Assumption 3.4.1 holds for the system.



(a) x_1 and x_2 (b) x_3 and x_4

Figure 3.5: Trajectories of (3.4.22) for 100 seconds from $[1, 1, 1, 1]$ by $u(t) = \sin(0.3t)$



(a) $\hat{x}(0) = (3, -3, -3, 3)$ (b) $\hat{x}(0) = (-10, -10, -10, -10)$

Figure 3.6: Time simulation for 10 seconds: Solid lines represent the true states and dotted lines represent the estimates.

By (3.4.8), the proposed observer is

$$\begin{pmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{pmatrix} = \begin{pmatrix} \hat{x}_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0.01\sigma(\hat{x}_1)u \\ -\hat{x}_1 + (1-\sigma(\hat{x}_1)^2)\sigma(\hat{x}_2) + y_2 u \end{pmatrix} - S_1^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}(\hat{x}_1 - y_1)$$

$$\begin{pmatrix} \dot{\hat{x}}_3 \\ \dot{\hat{x}}_4 \end{pmatrix} = \begin{pmatrix} \hat{x}_4 \\ 0 \end{pmatrix} + \begin{pmatrix} 0.01\sigma(\hat{x}_2)\sigma(\hat{x}_3)\exp(u) \\ -\hat{x}_3 + (1-\sigma(\hat{x}_3)^2)\sigma(\hat{x}_4) + \frac{1}{1+(\sigma(\hat{x}_2)\sigma(\hat{x}_4))^2}u \end{pmatrix} - S_2^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}(\hat{x}_3 - y_2) \tag{3.4.23}$$

where S_1 and S_2 is the solutions of (3.4.9) with $\theta_1 = 1$ and $\theta_2 = 2.5$, respectively, and σ is the saturation function of value 5 (i.e. $\rho = 5$ in (3.4.7)). This value is chosen with Fig. 3.5. Since the given system does not have the Lipschitz property of (3.4.5), some of the saturation functions in (3.4.23) are introduced for the Lipschitz extension according to Lemma 3.3.5 and the others are used by the prototype of the observer (3.4.8).

In Fig. 3.6, the results of simulations are shown with the initial condition of the observer $\hat{x}(0) = [3, -3, -3, 3]$ and $\hat{x}(0) = [-10, -10, -10, -10]$ for 10 seconds, respectively. By the saturation function of (3.4.7), the global error convergence is guaranteed.

3.5 Multi-output Extension II: beyond the Uniform Observability

In this section, we present another novel design method of state observer for multi-output nonlinear systems. Compared to the previous section, the class of system in this section has no structural priority in each blocks and is a plain generalization of the lower triangular structure (3.2.8). However, the class even includes non-uniformly observable mode, or detectable mode in the system, and thus the design can be thought as the observer construction beyond the uniform observability. In addition, the design is recursively performed, which resembles the well-known *back-stepping* procedure [KKM91] in the control literature.

The class of systems is described by

$$\boxed{\begin{aligned} \dot{x}_1 &= x_2 + g_1(x_1, u) \\ \dot{x}_2 &= x_3 + g_2(x_1, x_2, u) \\ &\vdots \\ \dot{x}_{r-1} &= x_r + g_{r-1}(x_1, \cdots, x_{r-1}, u) \\ \dot{x}_r &= g_r(x_1, \cdots, x_r, \eta, u) \\ \dot{\eta} &= f(x_1, \cdots, x_r, \eta, u), \qquad y = x_1 \end{aligned}} \tag{3.5.1}$$

where $x_i \in \mathbb{R}^p$, $\eta \in \mathbb{R}^l$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ such that $pr + l = n$.

Assumption 3.5.1. g_i's are Lipschitz in x, and f and g_r are Lipschitz in (x, η).

This assumption also can be relieved by the Lipschitz extension technique when the semi-global observer is of interest. The class of systems which are diffeomorphic to (3.5.1) is a generalization of the class (3.2.8) in several aspects as follows.

(i) *Multi-output System*

The class of (3.5.1) also includes some interconnection between the channels since each x_i is not a scalar but the p-valued vector. If there is no η-dynamics, it is another multi-output structure which guarantees the uniform observability, in the sense of Definition 3.1.3.

(ii) *Non-uniformly Observable System*

Suppose the system (3.2.8) is input affine and some g_i does not satisfy the triangular structure (i.e. $\exists j > i$ such that g_i depends on the variable x_j). Then, the uniform observability does not hold any more by the counterexample in the proof of [GHO92, Thm. 2]. Nevertheless, this case is contained in the class of (3.5.1) by taking $r = i$ and $\eta = (x_{i+1}, \cdots, x_n)^T$.

A possibility to design an observer for non-uniform observable systems has been already presented in [BH96, Bes99]. In those articles, a conjecture has been made that, in order to design an observer for non-uniformly observable system, the injection gain would depend on the value of input u. The proposed observer in this paper also use the information of input and its derivatives as well in the injection gain. Moreover, our viewpoint is more specific than that of [BH96, Bes99]. We regard x and η are uniformly observable and non-uniformly observable mode, respectively, and the assumption will be made only on the non-uniformly observable mode.

(iii) *Detectable System*

It is well-known that under the observability rank condition a nonlinear system has its local decomposition in which the observable state and the unobservable state are separated [NvdS90]. Analogously, suppose the given system has the global decomposition, i.e., the system (3.5.1) admits the form:

$$\begin{aligned}
&\vdots \\
\dot{x}_{r-1} &= x_r + g_{r-1}(x_1, \cdots, x_{r-1}, u) \\
\dot{x}_r &= g_r(x_1, \cdots, x_r, \eta_1, u) \\
\dot{\eta}_1 &= f_1(x_1, \cdots, x_r, \eta_1, u) \\
\dot{\eta}_2 &= f_2(x_1, \cdots, x_r, \eta_1, \eta_2, u), \qquad y = x_1.
\end{aligned} \tag{3.5.2}$$

Then, the state x would be uniformly observable, η_1 observable but non-uniformly observable, and η_2 unobservable state. In this case if η_2 is detectable in some sense, then the design of state observer may be possible. There are several notions of nonlinear detectability [SW97, Alv97, Bes99], but it will be seen that our proposed assumption is another version of detectability.

(iv) *Lipschitz Property*

Most of works [GHO92, DG91, DBGR92, DBB+93, SSS99, BFH97, GK94] assumed the boundedness of u, and *a priori* knowledge of the actual bound is incorporated in the design of the gain. In this section, the requirement is just the Lipschitz property in x, and the boundedness of input (and its derivatives) is assumed *a posteriori*, in order to guarantee the error convergence, after completing the design. This is of interest in its own right even for the uniform observable system (3.2.8).

Notations in This Section

Throughout this section, the following notations are used.

- For the partial derivative of f, $D_x f(x)$ are used.
- For the notational simplicity, let $u_0 = 0$, $u_1 = u$, $u_2 = (u, \dot{u})$, $u_3 = (u, \dot{u}, \ddot{u})$ and so on.

- For a given function $f(x,u)$, the capital F is defined as

$$F(e;x,u) := f(e+x,u) - f(x,u). \tag{3.5.3}$$

- Positive function $\psi(u)$ means that $\psi(u) > 0$ for any u.

- A function $V(x,e,u_i)$ is said to be *quadratic in e with u_i* when there are positive functions $\psi_1(u_i)$, $\psi_2(u_i)$ and $\psi_3(u_i)$ such that

$$\psi_1(u_i)\|e\|^2 \le V(x,e,u_i) \le \psi_2(u_i)\|e\|^2 \quad \text{and} \quad \|D_e V(x,e,u_i)\| \le \psi_3(u_i)\|e\|. \tag{3.5.4}$$

- For a system $(S) : \dot{x} = f(x,u)$ and a function $V(x,u)$, $\dot{V}|_{(S)}$ implies the time derivative of V along the trajectory of (S), i.e.,

$$\dot{V}|_{(S)} = D_x V \cdot f(x,u) + D_u V \cdot \dot{u}.$$

3.5.1 One-step Propagation

As a preliminary we derive a state observer in the generalized framework. Consider a system generally described by:

$$\begin{aligned} \dot{x} &= \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} f_1(x_1,x_2,u) \\ f_2(x_1,x_2,u) \end{pmatrix} = f(x,u) \\ y &= x_1 \end{aligned} \tag{3.5.5}$$

where $u \in \mathbb{R}^m$, the state $x \in \mathbb{R}^n$ is partitioned as $x_1 \in \mathbb{R}^p$ and $x_2 \in \mathbb{R}^{n-p}$ according to the order of output $y \in \mathbb{R}^p$, and the vector field $f(x,u)$ is Lipschitz in x. We seek an observer of the form:

$$\begin{aligned} \dot{z}_1 &= f_1(z_1,z_2,u) + \gamma(*) = f_1(z_1,z_2,u) + v \\ \dot{z}_2 &= f_2(z_1,z_2,u) + L(u)\gamma(*) = f_2(z_1,z_2,u) + L(u)v \end{aligned} \tag{3.5.6}$$

where $z = (z_1^T, z_2^T)^T$ is the estimate of x, $L(u) \in \mathbb{R}^{(n-p)\times p}$ is a matrix-valued injection gain which is continuously differentiable with respect to u and $\gamma(*)$ is some function of known quantities such as the output of the plant, the estimate z, the input u and its derivatives. Note that γ is replaced with the virtual *control* v in the above equation.

Then, the augmented[5] error dynamics ($e := z - x$) is obtained as

$$\begin{aligned}
\dot{x} &= f(x, u) \\
\dot{e}_1 &= F_1(e_1, e_2; x_1, x_2, u) + v \\
\dot{e}_2 &= F_2(e_1, e_2; x_1, x_2, u) + L(u)v \\
y_a &= z_1 - y = e_1.
\end{aligned} \tag{3.5.7}$$

In this description, we regard the error dynamics has the input v and the output y_a. Then, the observer construction problem becomes equivalent to find $L(u)$ and γ with which the error e is *controlled* to be asymptotically stable by the feedback $v = \gamma(*)$.

The requirement which the gain $L(u)$ should satisfy is given as follows. (Since the argument of this section is for the recursive design of next section, we use the input and its derivatives (u_{j+1}) instead of u in what follows. On the first reading, just suppose $j = 0$.)

Assumption 3.5.2. There exist a C^1 function $V(x, e_2, u_j)$ which is quadratic in e_2 with u_j where $j \geq 0$, a C^1 matrix-valued function $L(u_{j+1}) \in \mathbb{R}^{(n-p)\times p}$ and a positive function $\alpha_0(u_j)$ such that

$$\begin{aligned}
D_x V \cdot f(x, u) + D_{e_2} V \cdot [F_2(0, e_2; x, u) - L(u_{j+1}) F_1(0, e_2; x, u)] \\
+ D_{u_j} V \cdot \dot{u}_j \quad \leq \quad -\alpha_0(u_j) \|e_2\|^2
\end{aligned}$$

for all $x \in \mathbb{R}^n$, $e_2 \in \mathbb{R}^{n-p}$ and $u, \dot{u}, \cdots, u^{(j)} \in \mathbb{R}^m$.

[5] We added the term 'augmented' since there is x-dynamics in the error dynamics description.

Remark 3.5.1. When u_j is bounded, Assumption 3.5.2 implies that the augmented error dynamics is minimum phase with respect to e_2 in some sense, because the zero dynamics of (3.5.7) is obtained as

$$\begin{aligned} \dot{x} &= f(x,u) \\ \dot{e}_2 &= F_2(0,e_2;x_1,x_2,u) - L(u_{j+1})F_1(0,e_2;x_1,x_2,u). \end{aligned} \tag{3.5.8}$$

Note that the stability of the plant dynamics is not required, which is not of concern in observer problem.

Finally, the suitable $\gamma(*)$ is constructed in the following theorem.

Theorem 3.5.1. *Under Assumption 3.5.2, there are a C^1 function $W(x,e_1,e_2,u_{j+1})$ which is quadratic in (e_1,e_2) with u_{j+1}, a C^1 function $\phi(u_{j+2})$, and a positive function $\alpha_1(u_{j+1})$ such that*

$$\dot{W}|_{(S)} \le -\alpha_1(u_{j+1})\|e\|^2$$

where (S) is the system (3.5.7) *with $\gamma(*) = -\phi(u_{j+2})(z_1 - y)$.*

Proof. By change of coordinates $\xi = T(u_{j+1})e$ where

$$T(u_{j+1}) = \begin{bmatrix} I & 0 \\ -L(u_{j+1}) & I \end{bmatrix},$$

the augmented error dynamics (3.5.7) becomes

$$\begin{aligned} \dot{x} &= f(x,u) \\ \dot{\xi}_1 &= F_1(\xi_1, \xi_2 + L(u_{j+1})\xi_1; x_1, x_2, u) + v \\ \dot{\xi}_2 &= F_2(\xi_1, \xi_2 + L(u_{j+1})\xi_1; x_1, x_2, u) \\ &\qquad - L(u_{j+1})F_1(\xi_1, \xi_2 + L(u_{j+1})\xi_1; x_1, x_2, u) - \tilde{L}(u_{j+2})\xi_1 \\ y_a &= \xi_1 \end{aligned} \tag{3.5.9}$$

where $\tilde{L}$ is $(n-p) \times p$ matrix-valued function whose (i,k)-th element is

$$D_{u_{j+1}} L_{i,k}(u_{j+1}) \cdot \dot{u}_{j+1}.$$

In this coordinates, it is clear that the zero dynamics is obtained as

$$\begin{aligned}
\dot{x} &= f(x,u) \\
\dot{\xi}_2 &= F_2(0,\xi_2;x_1,x_2,u) - L(u_{j+1})F_1(0,\xi_2;x_1,x_2,u) \\
&=: f_2^*(\xi_2,x,u_{j+1})
\end{aligned}$$

which is the same representation as (3.5.8). Using the abbreviation f_2^*, ξ_2-dynamics of (3.5.9) is rewritten as

$$\begin{aligned}
\dot{\xi}_2 = f_2^*(\xi_2,x,u_{j+1}) &+ F_2(\xi_1, L(u_{j+1})\xi_1; x_1, \xi_2 + x_2, u) \\
&- L(u_{j+1})F_1(\xi_1, L(u_{j+1})\xi_1; x_1, \xi_2 + x_2, u) - \tilde{L}(u_{j+2})\xi_1
\end{aligned}$$

where the right-hand terms except f_2^* vanish when $\xi_1 = 0$.

Here, recall that f_1 and f_2 is Lipschitz in x, which yields the existence of a function $\rho(u_{j+1})$ such that

$$\|(F_2 - L(u_{j+1})F_1)(\xi_1, L(u_{j+1})\xi_1; x_1, \xi_2 + x_2, u)\| \leq \rho(u_{j+1})\|\xi_1\|.$$

A conservative choice of ρ would be $(c_2(u) + \|L(u_{j+1})\|c_1(u)) \cdot (1 + \|L(u_{j+1})\|)$ where c_1 and c_2 are Lipschitz coefficients of f_1 and f_2, respectively. Also define $\delta(u_{j+2}) := \|\tilde{L}(u_{j+2})\|$ and $\sigma(u_{j+1}) := \|T^{-1}(u_{j+1})\|$.

Now, let $\overline{W}(x,\xi,u_j) := V(x,\xi_2,u_j) + \frac{1}{2}\xi_1^T\xi_1$. It can be shown that $\overline{W}(x,\xi,u_j)$ is quadratic in ξ with u_j from the quadraticity of $V(x,\xi_2,u_j)$. Then, by Assumption

3.5.2,

$$\begin{aligned}\dot{\overline{W}}|_{(3.5.9)} &= D_x V f(x,u) + D_{\xi_2} V f_2^*(\xi_2, x, u) \\ &\qquad + D_{\xi_2} V (F_2 - L(u_{j+1})F_1)(\xi_1, L(u_{j+1})\xi_1; x_1, \xi_2 + x_2, u) - D_{\xi_2} V \tilde{L}(u_{j+2})\xi_1 \\ &\qquad + D_{u_j} V \cdot \dot{u}_j + \xi_1^T F_1(\xi_1, \xi_2 + L(u_{j+1})\xi_1; x_1, x_2, u) + \xi_1^T v \\ &\le -\alpha_0(u_j)\|\xi_2\|^2 + (\beta(u_j)[\rho(u_{j+1}) + \delta(u_{j+2})] + c_1(u)\sigma(u_{j+1}))\,\|\xi_1\|\,\|\xi_2\| \\ &\qquad\quad + c_1(u)\sigma(u_{j+1})\|\xi_1\|^2 + \xi_1^T v \\ &\le -\frac{3}{4}\alpha_0(u_j)\|\xi_2\|^2 + \frac{1}{\alpha_0(u_j)}\left(\beta(u_j)[\rho + \delta] + c_1(u)\sigma(u_{j+1})\right)^2 \|\xi_1\|^2 \\ &\qquad\quad + c_1(u)\sigma(u_{j+1})\|\xi_1\|^2 + \xi_1^T v\end{aligned}$$

where $\beta(u_j)$ is such that $\|D_{\xi_2} V(x, \xi_2, u_j)\| \le \beta(u_j)\|\xi_2\|$. Finally, by choosing a C^1 function $\phi(u_{j+2})$ such that

$$\phi(u_{j+2}) \ge c_1(u)\sigma(u_{j+1}) + \frac{1}{\alpha_0(u_j)}\left(\beta(u_j)[\rho(u_{j+1}) + \delta(u_{j+2})] + c_1(u)\sigma(u_{j+1})\right)^2 + \kappa$$

with $\kappa > 0$ and by applying $v = -\phi(u_{j+2})\xi_1$, there exists a function $\bar{\alpha}_1(u_j)$ such that

$$\dot{\overline{W}}|_{(3.5.9)} \le -\frac{3}{4}\alpha_0(u_j)\|\xi_2\|^2 - \kappa\|\xi_1\|^2 \le -\bar{\alpha}_1(u_j)\|\xi\|^2.$$

Let $W(x, e, u_{j+1}) = \overline{W}(x, \xi, u_j)|_{\xi = T(u_{j+1})e}$. Then, from the quadraticity of $\overline{W}$ it follows that $W(x, e, u_{j+1})$ is also quadratic in e with u_{j+1}. Indeed, since $\overline{W}$ is quadratic, $\exists \bar{\psi}_i(u_j)$, $1 \le i \le 3$, such that $\bar{\psi}_1(u_j)\|\xi\|^2 \le \overline{W}(x, \xi, u_j) \le \bar{\psi}_2(u_j)\|\xi\|^2$ and $\|D_\xi \overline{W}\| \le \bar{\psi}_3(u_j)\|\xi\|$, which leads to

$$\frac{\bar{\psi}_1(u_j)}{\sigma^2(u_{j+1})}\|e\|^2 \le W(x, e, u_{j+1}) \le \bar{\psi}_2(u_j)\|T(u_{j+1})\|^2\|e\|^2$$

and

$$\|D_e W(x, e, u_{j+1})\| \le \|D_\xi \overline{W}\|\,\|T(u_{j+1})\| \le \bar{\psi}_3(u_j)\|T(u_{j+1})\|^2\|e\|.$$

Moreover, with $v = -\phi(u_{j+2})\xi_1 = -\phi(u_{j+2})e_1$,

$$\dot{W}|_{(3.5.7)} = \dot{\overline{W}}|_{(3.5.9)} \le -\frac{\bar{\alpha}_1(u_j)}{\sigma^2(u_{j+1})}\|e\|^2,$$

which implies the existence of $\alpha_1(u_{j+1})$. ■

It should be noted that the error convergence is not yet guaranteed at this stage, because the quadratic function $W(x, e, u_{j+1})$ is not *decrescent* [Kha96] in the sense that the function ψ_i in (3.5.4) is not upper or lower bounded uniformly in u_j.

Corollary 3.5.2. *If $\|u_{j+1}\|$ is bounded, the dynamic system*

$$\begin{aligned} \dot{z}_1 &= f_1(z_1, z_2, u) - \phi(u_{j+2})(z_1 - y) \\ \dot{z}_2 &= f_2(z_1, z_2, u) - \phi(u_{j+2})L(u_{j+1})(z_1 - y) \end{aligned} \tag{3.5.10}$$

where ϕ is obtained by Theorem 3.5.1 under Assumption 3.5.2, is an exponential state observer for (3.5.5).

Proof. From the boundedness, there are positive constants ψ_i such that $\psi_1\|e\|^2 \leq W \leq \psi_2\|e\|^2$ and

$$\dot{W}|_{(3.5.10)-(3.5.5)} \leq -\psi_3\|e\|^2,$$

which shows the exponential stability of the error dynamics. ■

3.5.2 Recursive Design Algorithm

Consider the following observer prototype for (3.5.1),

$$\begin{aligned} \dot{z}_1 &= z_2 + g_1(z_1, u) + l_r(*)v \\ \dot{z}_2 &= z_3 + g_2(z_1, z_2, u) + l_{r-1}(*)v \\ &\vdots \\ \dot{z}_{r-1} &= z_r + g_{r-1}(z_1, \cdots, z_{r-1}, u) + l_2(*)v \\ \dot{z}_r &= g_r(z, \mu, u) + l_1(*)v \\ \dot{\mu} &= f(z, \mu, u) + l_0(*)v \end{aligned} \tag{3.5.11}$$

where z and μ are the estimates of x and η, respectively, and the terms $l_i(*)$ and v will be designed only with the available information.

Let $e := z - x$ and $\epsilon := \mu - \eta$. By omitting the x-dynamics for simplicity, the augmented error system is written as

$$\begin{aligned}
\dot{e}_1 &= e_2 + G_1(e_1; x_1, u) + l_r(*)v \\
\dot{e}_2 &= e_3 + G_2(e_1, e_2; x_1, x_2, u) + l_{r-1}(*)v \\
&\vdots \\
\dot{e}_{r-1} &= e_r + G_{r-1}(e_1, \cdots, e_{r-1}; x_1, \cdots, x_{r-1}, u) + l_2(*)v \\
\dot{e}_r &= G_r(e, \epsilon; x, \eta, u) + l_1(*)v \\
\dot{\epsilon} &= F(e, \epsilon; x, \eta, u) + l_0(*)v, \qquad y_a = e_1.
\end{aligned} \tag{3.5.12}$$

Assumption 3.5.3. There are C^1 functions $V_0(x, \epsilon)$ and $\phi_0(u) \in \mathbb{R}^{l \times p}$ such that

$$\psi_1 \|\epsilon\|^2 \leq V_0(x, \epsilon) \leq \psi_2 \|\epsilon\|^2, \quad \|D_\epsilon V_0(x, \epsilon)\| \leq \psi_3 \|\epsilon\|$$
$$D_x V_0 \cdot f(x, u) + D_\epsilon V_0 \cdot [F(0, \epsilon; x, \eta, u) - \phi_0(u) G_r(0, \epsilon; x, \eta, u)] \leq -\psi_4 \|\epsilon\|^2$$

where ψ_i's are positive constants.

Step 1:

Consider a system which is obtained from the last two equations of (3.5.12) by letting $e_1 = e_2 = \cdots = e_{r-1} = 0$, $l_0 = \phi_0$, $l_1 = I$ and $y_a = e_r$:

$$\begin{aligned}
\dot{e}_r &= G_r(0, \cdots, 0, e_r, \epsilon; x, \eta, u) + v \\
\dot{\epsilon} &= F(0, \cdots, 0, e_r, \epsilon; x, \eta, u) + \phi_0(u_1) v.
\end{aligned}$$

By Assumption 3.5.3, Theorem 3.5.1 gives $\phi_1(u_2)$ and $V_1(x, e_r, \epsilon, u_1)$ which is quadratic in (e_r, ϵ) with u_1 such that

$$\dot{V}_1|_{(S1)} \leq -\alpha_1(u_1) \|(e_r, \epsilon)\|^2$$

where $\alpha_1(u_1)$ is a positive function, and

$$(S1): \begin{cases} \dot{e}_r & = & G_r(0,\cdots,0,e_r,\epsilon;x,\eta,u) - \phi_1(u_2)e_r \\ \dot{\epsilon} & = & F(0,\cdots,0,e_r,\epsilon;x,\eta,u) - \phi_1(u_2)\phi_0(u_1)e_r. \end{cases}$$

Step 2:

Now consider the system from the last three equations of (3.5.12) and let $e_1 = e_2 = \cdots = e_{r-2} = 0$, $l_0 = \phi_1\phi_0$, $l_1 = \phi_1$, $l_2 = I$ and $y_a = e_{r-1}$:

$$\begin{aligned} \dot{e}_{r-1} &= e_r + G_{r-1}(0,\cdots,0,e_{r-1};x,u) + v \\ \dot{e}_r &= G_r(0,\cdots,0,e_{r-1},e_r,\epsilon;x,\eta,u) + \phi_1(u_2)v \\ \dot{\epsilon} &= F(0,\cdots,0,e_{r-1},e_r,\epsilon;x,\eta,u) + \phi_1(u_2)\phi_0(u_1)v. \end{aligned} \tag{3.5.13}$$

The result of Step 1, i.e. the existence of ϕ_1 and V_1, guarantees Assumption 3.5.2, because the zero dynamics of (3.5.13) is $(S1)$. Then, Theorem 3.5.1 again gives $\phi_2(u_3)$ and $V_2(x,e_{r-1},e_r,\epsilon,u_2)$ which is quadratic in (e_{r-1},e_r,ϵ) with u_2 such that

$$\dot{V}_2|_{(S2)} \leq -\alpha_2(u_2)\|(e_{r-1},e_r,\epsilon)\|^2$$

where $\alpha_2(u_2)$ is a positive function, and

$$(S2): \begin{cases} \dot{e}_{r-1} & = & e_r + G_{r-1}(0,\cdots,0,e_{r-1};x,u) - \phi_2(u_3)e_{r-1} \\ \dot{e}_r & = & G_r(0,\cdots,0,e_{r-1},e_r,\epsilon;x,\eta,u) - \phi_2(u_3)\phi_1(u_2)e_{r-1} \\ \dot{\epsilon} & = & F(0,\cdots,0,e_{r-1},e_r,\epsilon;x,\eta,u) - \phi_2(u_3)\phi_1(u_2)\phi_0(u_1)e_{r-1}. \end{cases}$$

Step 3:

Similarly, consider the following system with $e_1 = e_2 = \cdots = e_{r-3} = 0$, $l_0 = \phi_2\phi_1\phi_0$, $l_1 = \phi_2\phi_1$, $l_2 = \phi_2$, $l_3 = I$ and $y_a = e_{r-2}$.

$$\begin{aligned} \dot{e}_{r-2} &= e_{r-1} + G_{r-2}(0,\cdots,0,e_{r-2};x,u) + v \\ \dot{e}_{r-1} &= e_r + G_{r-1}(0,\cdots,0,e_{r-2},e_{r-1};x,u) + \phi_2 v \\ \dot{e}_r &= G_r(0,\cdots,0,e_{r-2},e_{r-1},e_r,\epsilon;x,\eta,u) + \phi_2\phi_1 v \\ \dot{\epsilon} &= F(0,\cdots,0,e_{r-2},e_{r-1},e_r,\epsilon;x,\eta,u) + \phi_2\phi_1\phi_0 v \end{aligned} \tag{3.5.14}$$

The previous step guarantees Assumption 3.5.2 for this system, and Theorem 3.5.1 gives $\phi_3(u_4)$ and the quadratic $V_3(x, e_{r-2}, e_{r-1}, e_r, \epsilon, u_3)$ such that, with a positive function α_3,

$$\dot{V}_3|_{(S3)} \leq -\alpha_3(u_3)\|(e_{r-2}, e_{r-1}, e_r, \epsilon)\|^2$$

where

$$(S3): \begin{cases} \dot{e}_{r-2} &=& e_{r-1} + G_{r-2}(0, \cdots, 0, e_{r-2}; x, u) - \phi_3 e_{r-2} \\ \dot{e}_{r-1} &=& e_r + G_{r-1}(0, \cdots, 0, e_{r-2}, e_{r-1}; x, u) - \phi_3\phi_2 e_{r-2} \\ \dot{e}_r &=& G_r(0, \cdots, 0, e_{r-2}, e_{r-1}, e_r, \epsilon; x, \eta, u) - \phi_3\phi_2\phi_1 e_{r-2} \\ \dot{\epsilon} &=& F(0, \cdots, 0, e_{r-2}, e_{r-1}, e_r, \epsilon; x, \eta, u) - \phi_3\phi_2\phi_1\phi_0 e_{r-2}. \end{cases}$$

In this way, the suitable l_i's and v can be found step by step. At the last step r, we finally get $V_r(x, e, \epsilon, u_r)$ which is quadratic in (e, ϵ) with u_r such that

$$\dot{V}_r|_{(Sr)} \leq -\alpha_r(u_r)\|(e, \epsilon)\|^2$$

where (Sr) is the system (3.5.12) with $l_k = \phi_r \cdots \phi_k$ $(0 \leq k \leq r)$ and $v = -e_1$. Now, under *a posteriori* assumption that the norm of u_r is bounded, the obtained system (3.5.11) becomes the exponential observer for (3.5.1) by Corollary 3.5.2.

3.6 Notes on the Chapter

For the multi-output extension I in Section 3.4, the saturation function has been used for the observer construction. The reasons are to eliminate the peaking phenomenon of some state and also to relax the global Lipschitz condition. The idea is originated by [EK92, KE93] where the semi-global control problem is solved with the saturation functions. The class considered is fairly general since it includes the classes of [GHO92, DG91, DBGR92, DBB$^+$93, RZ94] and the class of observer canonical form. For the class, we have proposed an explicit form of nonlinear observer and showed the global exponential convergence of the error by choosing appropriate values θ_i.

Section 3.5 presents a new procedure of designing an observer. The procedure resembles the well-known control method 'backstepping' in that the Lyapunov function is constructed with the virtual control (the output injection in our case) at each step. There have been some trials in the literature to apply backstepping-like method for the observer construction [RJ98, KKK95]. However, their concerns are the observer construction *with* the state feedback for the stabilization problem. The proposed method considers the observer *only*, therefore, it is purely the dual concept of the backstepping control. Regarding to this procedure, we have to mention the followings.

- The design assumes that the system is globally Lipschitz. However, without this assumption, it can also be reformulated as the semi-global observer utilizing the Lipschitz extension. Readers may now understand how this can be possible.

- The procedure yields some complexity because there appear the derivatives of inputs. This complexity disappears when the input u is bounded *a priori*. That is, if the input is bounded, the gains at each step can be obtained as a constant resulting the final constant gain observer like the Gauthier's.

- Assumption 3.5.3 is a version of detectability condition *with* u. For further insights, see Section 4.4 comparing Assumption 3.5.3 with Condition (C1).

Before closing this chapter, we append a useful tip here. In Lemma 3.2.3 and Condition 3.4.1, it has to be checked whether the map $\Phi(x)$ is diffeomorphism on $\mathbb{R}^n$ and on a bounded set Z, respectively. This can be also checked by the fact that $\Phi(x)$ is diffeomorphism on X ($X = \mathbb{R}^n$ or $X = Z$) if and only if $\frac{\partial \Phi}{\partial x}(x)$ is nonsingular on X and $\Phi(x)$ is one-to-one from X to $\Phi(X)$. Nevertheless, if one feels it difficult to check the one-to-oneness of a map, then refer to [HSM83]. For completeness, we adopt two theorems which guarantee the one-to-oneness on $\mathbb{R}^n$ or on a bounded region, respectively.

Theorem 3.6.1 ([HSM83]). *Let the leading principal minors of* $\frac{\partial \Phi}{\partial x}(x)$ *be* $\Delta_1(x)$, $\Delta_2(x)$, $\cdots$, $\Delta_n(x)$. *If there exists a constant* $\epsilon > 0$ *such that*

$$|\Delta_1(x)| \geq \epsilon, \frac{|\Delta_2(x)|}{|\Delta_1(x)|} \geq \epsilon, \cdots, \frac{|\Delta_n|}{|\Delta_{n-1}|} \geq \epsilon$$

for all $x \in \mathbb{R}^n$, *then* Φ *is one-to-one from* $\mathbb{R}^n$ *onto* $\mathbb{R}^n$.

Theorem 3.6.2 ([KET73]). *Consider a map* $\Phi : X \subset \mathbb{R}^n \to \mathbb{R}^n$ *where* X *is an open convex bounded region and* Φ *is continuously differentiable. If* $\det\left(\frac{\partial \Phi}{\partial x}(x)\right) > 0$ *for all* $x \in X$, *and* $\frac{\partial \Phi}{\partial x}(x) + (\frac{\partial \Phi}{\partial x}(x))^T$ *has nonnegative principal minors for all* $x \in X$, *then* Φ *is one-to-one from* X *onto* $\Phi(X)$.

Chapter 4

Passivity Framework for Nonlinear Observer

In this chapter, we solve the general problem of reconstructing the state of a plant only with the input and output information of the plant, referred to as the observer problem. Our strategy of observer design is the output feedback passification to the error dynamics, which is the recent issue in the passivity literature. In order to describe the passivity of the error dynamics effectively, we've already discussed the extended passivity in Section 2.3, where the standard passivity is reformed for the augmented error dynamics which also includes the plant dynamics. The proposed framework includes the precise definition of Passivity-based State Observer (PSO) and the design scheme of PSO. It is also shown that a PSO has its potential robustness to the measurement disturbance. With these tools, the framework of PSO provides a novel viewpoint on the earlier works in the literature and unifies them.

4.1 Motivations

Contrary to Chapter 3, the observer problem is viewed, in this chapter, as a static output feedback stabilization of the error dynamics between the plant and the prototype of observer. As a solution of the static output feedback stabilization problem, we utilize the recent developments of output feedback passification [FH98, JH98, BSS00] recalling that the (state feedback) passification [BIW91] has been a good tool for nonlinear stabilization in the control literature. However, while the Lyapunov method has been much employed to design nonlinear state observers, the alternative passivity approach has rarely been explored [SF99, AK99] and it has not been studied to make relation between the passification and the observer problem. Motivated by this fact, we propose the passivity framework for general observer problem. The framework includes the precise definition of Passivity-based State Observer (PSO) and a design scheme of PSO.

Once the framework is established, the concept of PSO enjoys some advantages. It enables to tie various results in the literature and generalize them. Therefore, we can have a unified point of view for various earlier works for nonlinear observers. As will be seen, a PSO also has its potential robustness to the measurement disturbance.

Linear Systems

As an introductory study, we sketch the concept of PSO for linear systems. By this sketch, it follows that the Kalman filter is in fact a PSO.

Consider a linear plant

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Hx. \end{aligned} \tag{4.1.1}$$

When (A, H) is detectable pair, the well-known Luenberger state observer of the form

$$\dot{z} = Az + Bu + L(y - Hz) \tag{4.1.2}$$

leads to the error equation ($e = z - x$):

$$\dot{e} = Ae - LHe \tag{4.1.3}$$

where L is chosen such that $A - LH$ is Hurwitz.

Although the error system (4.1.3) has no input and output, a new viewpoint arises from the following error system with the virtual input v and output y_a:

$$\begin{aligned} \dot{e} &= Ae + Lv \\ y_a &= He \end{aligned} \tag{4.1.4a}$$

and the feedback:

$$v = -Ky_a + \bar{v}. \tag{4.1.4b}$$

Note that when $K = I$ and $\bar{v} = 0$ the closed-loop of the error system (4.1.4) is equivalent to (4.1.3).

Now, the state observation problem is solved with $\bar{v} = 0$, if, with suitably chosen L and K, the error system (4.1.4) is strictly passive with positive definite storage function (see e.g. [BIW91]) from $\bar{v}$ to y_a, in other words, if there is a positive definite matrix P such that

$$P(A - LKH) + (A - LKH)^T P < 0 \tag{4.1.5}$$

$$PL = H^T. \tag{4.1.6}$$

The process of making (4.1.4a) strictly passive by a feedback (4.1.4b) can be viewed as an 'output feedback passification' devised to stabilize the given plant by static output feedback [JH98, FH98, BSS00]. That is, a state observer is constructed when we make (4.1.4a) strictly passive, via output feedback passification method, from the output injection term to the error (y_a) between the outputs of the plant and observer[1].

[1] An observer obtained in this way will be called a PSO, whose precise definition will be given shortly.

Clearly, this passivity-based viewpoint is stronger concept than Luenberger observer construction. In fact, (4.1.5), without (4.1.6), is sufficient for linear state observer. Nevertheless, the passivity analysis for the observer problem is useful especially for nonlinear systems. In addition, it is remarkable that the detectability of linear systems is equivalent to the existence of the passivity-based state observer. As the first example, we show that an observer by optimal pole placement method is actually of this type.

When (A, H) is detectable pair, one way to find L which makes $(A-LH)$ Hurwitz is to solve for the unique positive definite matrix[2] P which satisfies

$$PA + A^T P + PQP - H^T R^{-1} H = 0 \tag{4.1.7}$$

where $R > 0$ and $Q \geq 0$ such that $(A, \sqrt{Q})$ is controllable. Then letting $L^* = P^{-1} H^T R^{-1}$ makes $(A - L^* H)$ Hurwitz. (Indeed, let $A_c = A - L^* H = A - P^{-1} H^T R^{-1} H$. Then

$$PA_c + A_c^T P = PA + A^T P - 2H^T R^{-1} H = -PQP - H^T R^{-1} H \leq 0$$

which shows the marginal stability of A_c. Moreover, by the detectability of (A, H), it follows that[3] $PA_c + A_c^T P < 0$.) Now, with $L = P^{-1} H^T$ and $K = R^{-1}$ the closed-loop error system (4.1.4) becomes strictly passive from $\bar{v}$ to y_a by (4.1.5) and (4.1.6). Therefore, an observer with optimally placed poles is a passivity-based state observer and when (A, H) is detectable there always exists a passivity-based state observer.

Proposition 4.1.1. *A linear system is detectable if and only if there exists a passivity-based state observer (PSO) for the system.*

[2]When $(A, \sqrt{Q})$ is controllable and (A, H) is detectable, there exists the unique positive definite solution $\bar{P}$ of $A\bar{P} + \bar{P}A^T + Q - \bar{P}H^T R^{-1} H \bar{P} = 0$ [AM89]. Therefore, existence and uniqueness of $P = \bar{P}^{-1}$ follow.

[3]Let $A_c w = \lambda w$ with $w \neq 0$. Since $Re(\lambda) \leq 0$, we need to show $Re(\lambda) < 0$. Now, since $(\lambda + \lambda^*) w^* P w = -w^* H^T R^{-1} H w - w^* PQP w$, if $(\lambda + \lambda^*) = 0$ then $Hw = 0$ and $Aw = 0$ which contradicts the detectability.

Furthermore, the observation gain $L^* = LK$ is optimal for a stochastic system,

$$\begin{aligned} \dot{x} &= Ax + Bu + w \\ y &= Hx + v \end{aligned} \tag{4.1.8}$$

where w and v are white noise processes with zero means and covariances Q and $R > 0$, respectively. Therefore,

$$\boxed{\text{The Kalman filter is a passivity-based state observer.}}$$

Nomenclature

The followings are actively used in this chapter, in addition to the basic notations in the dissertation.

- $\|x\|_{[0,t]} := \sup_{0 \le \tau \le t} \|x(\tau)\|$.
- For the partial derivative of f, $D_x f(x)$ are used. For the partitioned state x_1 such that $x = [x_1^T, x_2^T]^T$, $D_1 f(x_1, x_2) = \frac{\partial}{\partial x_1} f(x_1, x_2)$.
- When matrices A and h has the following structure of $n \times n$ and $1 \times n$, respectively,

  $$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad h = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

  it is said that (A, h) *represents n-th order integrator chain*, due to the fact that $\dot{x} = Ax$, $y = hx$ represents recursive integrators.
- For a given function $f(x, u)$, the capital F is defined as

  $$F(e; x, u) := f(e + x, u) - f(x, u). \tag{4.1.9}$$

4.2 Definition of Passivity-based State Observer

Fundamental Assumption

Since we are interested in the asymptotic observer, it is assumed that the solution $x(t)$ of the plant exists for all positive time. Although assuming the forward completeness [AS99] of the plant vector field $f(x,u)$ is sufficient for guaranteeing the case, we just require the existence of solution $x(t)$ for all positive time. In other words, we consider the pair of initial and input $(x(0), u(\cdot))$ which does not result in the finite time escape of the trajectory $x(t)$. Since the forward completeness requires the existence of solution for all initials and all inputs, our requirement is weaker than the completeness of the plant.

Passivity-based State Observer

Consider a continuously differentiable system given by

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned} \tag{4.2.1}$$

where $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}^p$ is the output, and u is the input which is contained in the admissible input set U consisting of the signals $u(\cdot) : \mathbb{R} \to \mathcal{U} \subset \mathbb{R}^m$.

If a continuously differentiable state observer exists for the plant (4.2.1), then without loss of generality the observer takes the following form,

$$\dot{z} = f(z,u) + l(z,u,h(z,u)-y)k(u,y,h(z,u)-y)(y-h(z,u)) \tag{4.2.2}$$

where z is the estimate of x and the term $l \cdot k$ is the output injection gain with a nonsingular square function k. Indeed, a continuously differentiable state observer generally has the form

$$\dot{z} = \varpi(z,u,h(x,u)) \tag{4.2.3}$$

with the property that if $z(0) = x(0)$ then $z(t)$ should be the same as $x(t)$ for $t \geq 0$. This leads to $\varpi(z,u,h(x,u)) = f(z,u) + l^*(z,u,h(x,u))$ where $l^*(z,u,h(z,u)) = 0$.

Define $\bar{l}(z,u,s) := l^*(z,u,s+h(z,u))$. Then, since $\bar{l}(z,u,0) = 0$ there is $\tilde{l}(z,u,s)$ such that $\bar{l}(z,u,s) = \tilde{l}(z,u,s)s$ by the continuous differentiability of $\bar{l}$. Let $k : \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^{p\times p}$ be any nonsingular function[4]. By letting $l(z,u,-s) := \tilde{l}(z,u,s)k^{-1}(u,s+h(z,u),-s)$ and $s = h(x,u) - h(z,u)$, (4.2.3) becomes (4.2.2).

Consider the system given by

$$\boxed{\begin{aligned} \dot{x} &= f(x,u) \\ \dot{e} &= F(e;x,u) + l(e+x,u,H(e;x,u))v \\ y_a &= H(e;x,u) \end{aligned}} \tag{4.2.4}$$

where $e := z - x$. Note that F and H are used according to the nomenclature of this chapter. Now if we let $v = -k(u,y,y_a)y_a$, then this system becomes exactly the same as the error system between the plant (4.2.1) and the observer (4.2.2), plus the plant dynamics. Therefore, we call the system (4.2.4) the *augmented error dynamics*.

Definition 4.2.1. The system (4.2.2) is *passivity-based state observer (PSO)* for the plant (4.2.1) if, with the feedback

$$\boxed{v = -k(u,y,y_a)y_a + \bar{v},} \tag{4.2.5}$$

the corresponding augmented error dynamics (4.2.4) is PSUP[5] with respect to (e,u) from $\bar{v}$ to y_a.

A PSO does play the role of observer that guarantees $z(t) \to x(t)$ as $t \to \infty$ because, for the corresponding augmented error dynamics with feedback, the point $e = 0$ is globally asymptotically stable by Lemma 2.3.1[6].

The advantages of the concept of PSO is that it provides a unified viewpoint to the various results for nonlinear observers. This point will be emphasized in later sections. Here let us discuss another advantage of PSO.

[4]On the first reading, one can regard $k \equiv I$ at this stage.
[5]See Section 2.3.
[6]To show the convergence of the estimate, regard x and e as to x_1 and x_2 in Lemma 2.3.1, respectively.

4.3 Robust Redesign for Measurement Disturbance

Although (4.2.2) works well without any disturbance, it frequently happens that the measure of y is corrupted by the inaccurate sensing or measurement noise. Thus, the observer may have the corrupted value $y_m := y + d$ instead of the true output y, where d is the measurement disturbance. Especially for nonlinear observers, this type of disturbance can cause the instability or even the finite time escape of the estimate. (See e.g. [Fre95].)

Fortunately, PSO can be modified to have the *input-to-state stability (ISS)* [Son89] when d and e are viewed as the input and the state, respectively. The key to the modification is the passive property of the observer and the following Lemma.

Lemma 4.3.1. *For locally Lipschitz $k(u, y, y_a)$, there are a continuous function $\kappa : \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}_{\geq 0}$ and a $\mathcal{K}$ function ρ such that, for all $u \in \mathbb{R}^m$ and y, y_a, $d \in \mathbb{R}^p$,*

$$\|k(u, y + d, y_a - d)(y_a - d) - k(u, y, y_a)y_a\| \leq \kappa(u, y + d, y_a - d)\rho(\|d\|).$$

Proof. In fact, the claim follows from [FK93, Lemma 2] or [FK96, Appendix] with a simple trick. Let $\mu(u, y, y_a) := k(u, y, y_a)y_a$. Then, by [FK93, Lemma 2], it follows that there are a continuous function κ and a $\mathcal{K}$ function $\tilde{\rho}$ such that

$$\|\mu(u + d_1, y + d_2, y_a + d_3) - \mu(u, y, y_a)\| \leq \kappa(u + d_1, y + d_2, y_a + d_3)\tilde{\rho}(\|(d_1, d_2, d_3)\|)$$

for all u, y, y_a, d_1, d_2 and d_3. Taking $d_1 = 0$, $d_2 = d$, $d_3 = -d$ and $\rho(s) = \tilde{\rho}(\sqrt{2}s)$ proves the claim. ∎

Now we redesign the gain $k(u, y, y_a)$ with which the observer is still PSO and exhibits the ISS property from d to e. Suppose the observer (4.2.2) is PSO for the plant (4.2.1) with the gain l independent of the system output y. Then by Definition 4.2.1 there is a storage function $V(x, e)$ which satisfies $\alpha_1(\|e\|) \leq V(x, e) \leq \alpha_2(\|e\|)$,

$$D_e V \cdot l(e + x, u) = y_a^T \tag{4.3.1}$$

and

$$D_xV \cdot f(x,u) + D_eV \cdot (F(e;x,u) - l(e+x,u)k(u,y,y_a)y_a) \leq -\alpha_3(\|e\|) \qquad (4.3.2)$$

with $\mathcal{K}_\infty$ functions α_1, α_2 and a continuous positive definite function α_3.

Lemma 4.3.2. *For a PSO (4.2.2), assume that the gain l is independent of y, the gain $k(u,y,y_a)$ is locally Lipschitz continuous and the corresponding α_3 is $\mathcal{K}_\infty$ function. Then, for the modified PSO with the disturbed measurement y_m:*

$$\dot{z} = f(z,u) + l(z,u)k^*(u, y_m, h(z,u) - y_m)(y_m - h(z,u)) \qquad (4.3.3)$$

where $k^*(u,y,y_a) = k(u,y,y_a) + \kappa(u,y,y_a)I$, the following property holds

$$\|e(t)\| \leq \beta(\|e(0)\|, t) + \gamma(\|u\|_{[0,t]}, \|y\|_{[0,t]}, \|d\|_{[0,t]}) \qquad (4.3.4)$$

where β is $\mathcal{KL}$ function and the function $\gamma : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is nondecreasing for each argument and $\gamma(\cdot,\cdot,0) = 0$.

Proof. The proof is similar to [Fre97]. With the modified observer (4.3.3),

$$\begin{aligned}
\dot{V}(x,e) &= D_xV \cdot f(x,u) + D_eV \cdot (F(e;x,u) - l(e+x,u)k^*(u, y+d, y_a - d)(y_a - d)) \\
&= D_xV \cdot f(x,u) + D_eV \cdot (F(e;x,u) - l(e+x,u)k(u,y,y_a)y_a) \\
&\quad - D_eV \cdot l(e+x,u)[k(u,y+d,y_a-d)(y_a-d) - k(u,y,y_a)y_a] \\
&\quad - D_eV \cdot l(e+x,u)\kappa(u,y+d,y_a-d)(y_a-d).
\end{aligned}$$

By (4.3.1) and (4.3.2),

$$\begin{aligned}
\dot{V}(x,e) &\leq -\alpha_3(\|e\|) + \|y_a\| \, \|k(u,y+d,y_a-d)(y_a-d) - k(u,y,y_a)y_a\| \\
&\qquad - \kappa(u,y+d,y_a-d)\|y_a\|^2 + \kappa(u,y+d,y_a-d)\|y_a\| \, \|d\| \\
&= -\alpha_3(\|e\|) + \kappa(u,y+d,y_a-d)\|y_a\|(\rho(\|d\|) + \|d\| - \|y_a\|)
\end{aligned}$$

Let $\tilde{\gamma} : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a function which is nondecreasing for each argument and $\tilde{\gamma}(\cdot, \cdot, 0) = 0$ such that

$$\tilde{\gamma}(s_1, s_2, s_3) \geq \max_{\substack{\|y_a\| \leq \rho(s_3) + s_3 \\ \|u\| \leq s_1, \|y\| \leq s_2, \|d\| \leq s_3}} \kappa(u, y + d, y_a - d)\|y_a\|(\rho(\|d\|) + \|d\| - \|y_a\|).$$

Then,

$$\|e\| \geq \alpha_3^{-1}(2\tilde{\gamma}(\|u\|, \|y\|, \|d\|)) \qquad \Rightarrow \qquad \dot{V} \leq -\frac{1}{2}\alpha_3(\|e\|).$$

Therefore, (4.3.4) follows with $\gamma(s_1, s_2, s_3) = \alpha_1^{-1}(\alpha_2(\alpha_3^{-1}(2\tilde{\gamma}(s_1, s_2, s_3))))$ [Son89]. ■

From Lemma 4.3.2, it is finally concluded that if the input u and the output y of the plant are bounded, then the ISS property follows. Note that we do not need to know the actual bounds of u and y.

Remark 4.3.1. The requirement of the boundedness of u and y arises from the fact that $k(u, y, y_a)$ is generally unbounded with respect to u and y. Therefore, if $k(u, y, y_a)$ is independent of u or y then the requirement of the boundedness of u or y can be removed, respectively. (See the proof.) By this remark, we would endow the robust property with the well-known observers of [GHO92] and [Tsi89] in Section 4.5.

4.4 Design of Passivity-based State Observer

This section considers the design aspect of PSO. We restrict ourselves to consider the system without feedthrough term:

$$\begin{aligned} \dot{x} &= f(x, u) = \begin{pmatrix} f_1(x_1, x_2, u) \\ f_2(x_1, x_2, u) \end{pmatrix} \\ y &= Hx = [0 \ I]x = x_2 \end{aligned} \tag{4.4.1}$$

where the partial state $x_2 \in \mathbb{R}^p$ is the output. Possibly, this can be achieved by the coordinate transformation using $h(x)$ as the partial coordinates.

For the construction, we seek an observer of the form (4.2.2) where the gain $l(z, u, Hz - y)$ is an $n \times p$ matrix L and $k(u, y, Hz - y)$ is a scalar function. Thus, the augmented error dynamics is obtained as

$$\boxed{\begin{aligned} \dot{x} &= f(x, u) \\ \dot{e}_1 &= F_1(e_1, e_2; x_1, x_2, u) + L_1 v \\ \dot{e}_2 &= F_2(e_1, e_2; x_1, x_2, u) + L_2 v \\ y_a &= He = e_2 \end{aligned} \qquad (4.4.2)}$$

where $e_1 = z_1 - x_1$ and $e_2 = z_2 - x_2$. Note that $F_i(0, 0; x_1, x_2, u) = 0$ for $i = 1, 2$.

Now our task is to choose L and k so that the augmented error dynamics (4.4.2) is PSUP with respect to (e, u) by the feedback $v = -k(u, y, y_a)y_a + \bar{v}$. At this point, an important observation is that this problem is reminiscent of the 'output feedback passification' in the recent paper [JH98], which has been devised to stabilize the nonlinear plant by static output feedback. Motivated by the work, we provide sufficient conditions for the construction of PSO.

For the passification, there are two well-known necessary conditions (see Section 2.1.2); the system has relative degree one and is weakly minimum phase. For the system (4.4.2), the relative degree condition implies the invertibility of L_2 and the minimum phase condition says the zero dynamics of (4.4.2), which is obtained as

$$\begin{aligned} \dot{x} &= f(x, u) \\ \dot{e}_1 &= F_1(e_1, 0; x_1, x_2, u) - L_1 L_2^{-1} F_2(e_1, 0; x_1, x_2, u), \end{aligned} \qquad (4.4.3)$$

is stable. However, since requiring the stability of (4.4.3) also implies the stability of the plant which is not of concern in observer problem, our proposed condition does not mention about the stability of x-state as follows.

Condition C1 (Generalized Min. Phase and Rel. Deg.)**.** There exist a C^1 function $V(x, e_1) : \mathbb{R}^n \times \mathbb{R}^{n-p} \to \mathbb{R}_{\geq 0}$, a continuous positive definite function ψ_3 and two $\mathcal{K}_\infty$ functions ψ_1 and ψ_2 such that

(i) $\psi_1(\|e_1\|) \leq V(x, e_1) \leq \psi_2(\|e_1\|)$

(ii) $^\exists L_1 \in \mathbb{R}^{(n-p)\times p}$ and $L_2 \in \mathbb{R}^{p\times p}$ which is invertible s.t.

$$D_x V \cdot f(x, u) + D_{e_1} V \cdot \left[F_1(e_1, 0; x_1, x_2, u) - L_1 L_2^{-1} F_2(e_1, 0; x_1, x_2, u)\right] \leq -\psi_3(\|e_1\|) \quad (4.4.4)$$

for all $x \in \mathbb{R}^n$, $e_1 \in \mathbb{R}^{n-p}$ and $u \in \mathcal{U}$.

Remark 4.4.1. If the plant (4.4.1) is forward complete and the input set $\mathcal{U}$ is compact, then the existence of V satisfying C1 is necessary and sufficient for the uniformly global asymptotic stability of (4.4.3) with respect to the set $\{(x, e_1) \mid e_1 = 0\}$. For details, refer to [LSW96, Thm. 2.8].

When the function V does not contain the plant state x, C1 can be rewritten as follows, which is stronger but would be frequently used.

Condition C1*. There exist a C^1 function $V(e_1) : \mathbb{R}^{n-p} \to \mathbb{R}_{\geq 0}$, a continuous positive definite function ψ_3 and two $\mathcal{K}_\infty$ functions ψ_1 and ψ_2 such that

(i) $\psi_1(\|e_1\|) \leq V(e_1) \leq \psi_2(\|e_1\|)$

(ii) $^\exists L_1 \in \mathbb{R}^{(n-p)\times p}$ and $L_2 \in \mathbb{R}^{p\times p}$ which is invertible s.t.

$$D_{e_1} V \cdot \left[F_1(e_1, 0; x_1, x_2, u) - L_1 L_2^{-1} F_2(e_1, 0; x_1, x_2, u)\right] \leq -\psi_3(\|e_1\|) \quad (4.4.5)$$

for all $x \in \mathbb{R}^n$, $e_1 \in \mathbb{R}^{n-p}$ and $u \in \mathcal{U}$.

Remark 4.4.2. By an example, it can be seen that C1* is stronger condition than C1. Suppose a second-order system $\dot{x}_1 = (x_2^2 - 1)x_1$, $\dot{x}_2 = -2\exp(x_2^2)x_2 + x_2$ and $y = x_2$. Since $F_1(e_1, 0; x, u) = (x_2^2 - 1)e_1$ and $F_2(e_1, 0; x, u) = 0$, C1* is never met. Nonetheless, C1 holds for this case with $L_1 = 0$, $L_2 = 1$ and $V(x, e_1) = (1 - \frac{1}{2}\exp(-x_2^2))e_1^2$. Indeed,

$$\begin{aligned}\dot{V} &= e_1^2[x_2^2\exp(-x_2^2)(1 - 2\exp(x_2^2)) + 2(1 - \frac{1}{2}\exp(-x_2^2))(x_2^2 - 1)] \\ &= e_1^2[-2 + \exp(-x_2^2)] \le -e_1^2.\end{aligned}$$

In case of state feedback passification, the minimum phase and relative degree condition (i.e., C1) is also sufficient [BIW91]. However, since our concern is *output feedback* passification raised by the observer problem, an additional condition is required recalling Section 2.1.2.

Condition C2 (Nonlinear Growth)**.** There are nonnegative functions ϕ_1 and ϕ_2 such that

$$\begin{aligned}\big|D_{e_1}V(x, e_1)\big(F_1(L_1L_2^{-1}e_2, e_2; e_1 + x_1, x_2, u) - L_1L_2^{-1}F_2(L_1L_2^{-1}e_2, e_2; e_1 + x_1, x_2, u)\big) \\ + e_2^T L_2^{-1}F_2(e_1 + L_1L_2^{-1}e_2, e_2; x_1, x_2, u)\big| \\ \le \phi_1(u, x_2, e_2)\|e_2\|^2 + \phi_2(u, x_2, e_2)\psi_3^{\frac{1}{2}}(\|e_1\|)\|e_2\|\end{aligned} \tag{4.4.6}$$

for all x, e and u.

The next condition is another version of C2 which is useful for later discussions. Actually, the assumption of [JH98] is the type of C2* rather than C2.

Condition C2*. There are nonnegative functions ϕ_1 and ϕ_2 such that

$$\begin{aligned}\big\|D_{e_1}V(x,e_1)\big(\tilde{F}_1(e,x,u)-L_1L_2^{-1}\tilde{F}_2(e,x,u)\big)+F_2^T(e_1+L_1L_2^{-1}e_2,e_2;x_1,x_2,u)L_2^{-T}\big\| \\ \leq \phi_1(u,x_2,e_2)\|e_2\|+\phi_2(u,x_2,e_2)\psi_3^{\frac{1}{2}}(\|e_1\|)\end{aligned} \tag{4.4.7}$$

for all x, e and u, where

$$\tilde{F}_i(e,x,u):=\int_0^1\big(D_{\beta_1}f_i(\beta_1,\beta_2,u)L_1L_2^{-1}+D_{\beta_2}f_i(\beta_1,\beta_2,u)\big)\big|_{\substack{\beta_1=\theta L_1L_2^{-1}e_2+e_1+x_1\\ \beta_2=\theta e_2+x_2}}\,d\theta \tag{4.4.8}$$

for $i=1,2$.

Remark 4.4.3. The definition of (4.4.8) leads to, by the chain rule,

$$\begin{aligned}\tilde{F}_i(e,x,u)e_2&=\int_0^1 D_\theta f_i(\theta L_1L_2^{-1}e_2+e_1+x_1,\theta e_2+x_2,u)\,d\theta\\ &=f_i(L_1L_2^{-1}e_2+e_1+x_1,e_2+x_2,u)-f_i(e_1+x_1,x_2,u)\\ &=F_i(L_1L_2^{-1}e_2,e_2;e_1+x_1,x_2,u)\end{aligned}$$

for $i=1,2$. Therefore, by multiplying with $\|e_2\|$ both sides of (4.4.7), it follows that C2* implies C2.

Remark 4.4.4. It can be seen that the condition C2(C2*) is trivially satisfied when $V(x,e_1)$ is quadratic[7] with respect to e_1 and $f_1(x,u)$ and $f_2(x,u)$ is globally Lipschitz[8] in x. Therefore, a bilinear system with bounded input always satisfies C2(C2*). For linear systems, C2(C2*) does not impose any restriction.

Now we are ready to show that the conditions C1 and C2 are sufficient to the construction of PSO.

[7] Quadraticity of $V(x,e_1)$ with respect to e_1 is understood in the usual sense, i.e., in C1, $\psi_i(s)=\psi_i\|s\|^2$, $i=1,2,3$, and $\|D_{e_1}V\|\leq\psi_4\|e_1\|$ with positive constants ψ_i.

[8] That is, for $i=1,2$, $\exists\Pi_i$ s.t. $\|F_i(e;x,u)\|\leq\Pi_i(u)\|e\|$.

Theorem 4.4.1. *Under Conditions C1 and C2, there is a PSO for the system* (4.4.1)*. More specifically, with* $k(u, y, y_a) = \epsilon + \phi_1(u, y, y_a) + \phi_2^2(u, y, y_a)$ *for any* $\epsilon > 0$*, the augmented error dynamics* (4.4.2) *is PSUP with respect to a pair* (e, u) *by the feedback* (4.2.5)*.*

Proof. By change of coordinates $\xi_1 = e_1 - L_1 L_2^{-1} e_2$ and $\xi_2 = e_2$, the augmented error dynamics (4.4.2) becomes

$$\begin{aligned} \dot{x} &= f(x, u) \\ \dot{\xi}_1 &= F_1(\xi_1 + L_1 L_2^{-1}\xi_2, \xi_2; x_1, x_2, u) - L_1 L_2^{-1} F_2(\xi_1 + L_1 L_2^{-1}\xi_2, \xi_2; x_1, x_2, u) \\ \dot{\xi}_2 &= F_2(\xi_1 + L_1 L_2^{-1}\xi_2, \xi_2; x_1, x_2, u) + L_2 v \\ y_a &= \xi_2 \end{aligned} \tag{4.4.9}$$

In this coordinates, it is clear that the zero dynamics is

$$\begin{aligned} \dot{x} &= f(x, u) \\ \dot{\xi}_1 &= F_1(\xi_1, 0; x_1, x_2, u) - L_1 L_2^{-1} F_2(\xi_1, 0; x_1, x_2, u) \\ &=: f_1^*(\xi_1, x, u) \end{aligned} \tag{4.4.10}$$

which are the same representation as (4.4.3). Using the abbreviation f_1^*, ξ_1-dynamics of (4.4.9) is rewritten as

$$\dot{\xi}_1 = f_1^*(\xi_1, x, u) + F_1(L_1 L_2^{-1}\xi_2, \xi_2; \xi_1 + x_1, x_2, u) - L_1 L_2^{-1} F_2(L_1 L_2^{-1}\xi_2, \xi_2; \xi_1 + x_1, x_2, u)$$

where the term $F_1 - L_1 L_2^{-1} F_2$ vanishes when $\xi_2 = 0$.

Now, let a storage function be

$$W(x, \xi) := V(x, \xi_1) + \frac{1}{2}\xi_2^T L_2^{-1} \xi_2 \tag{4.4.11}$$

which clearly satisfies

$$\alpha_1(\|\xi\|) \le W(x, \xi) \le \alpha_2(\|\xi\|) \tag{4.4.12}$$

where α_1 and α_2 are $\mathcal{K}_\infty$ functions. By Conditions C1 and C2, the time derivative of W along the trajectory of (4.4.9) satisfies

$$\begin{aligned}\dot{W} &= D_x V f(x,u) + D_{\xi_1} V f_1^*(\xi_1, x, u) \\ &\quad + D_{\xi_1} V (F_1 - L_1 L_2^{-1} F_2)(L_1 L_2^{-1}\xi_2, \xi_2; \xi_1 + x_1, x_2, u) \\ &\quad + \xi_2^T L_2^{-1} F_2(\xi_1 + L_1 L_2^{-1}\xi_2, \xi_2; x_1, x_2, u) + \xi_2^T v \\ &\le -\psi_3(\|\xi_1\|) + \phi_1(u, x_2, \xi_2)\|\xi_2\|^2 + \phi_2(u, x_2, \xi_2)\psi_3^{\frac{1}{2}}(\|\xi_1\|)\|\xi_2\| + \xi_2^T v\end{aligned}$$

Therefore, by applying $v = -k(u, y, y_a)y_a + \bar{v}$,

$$\begin{aligned}\dot{W} \le -\frac{3}{4}\psi_3(\|\xi_1\|) - \epsilon\|\xi_2\|^2 \\ + (\phi_1 - \phi_1)\|\xi_2\|^2 - (\frac{1}{2}\psi_3^{\frac{1}{2}} - \phi_2\|\xi_2\|)^2 + \xi_2^T \bar{v}.\end{aligned}$$

This leads to the PSUP with respect to (ξ, u), that is with respect to (e, u). ■

Remark 4.4.5. We can always choose ϕ_1 and ϕ_2 to be locally Lipschitz in C2. Therefore, if ψ_3 of C1 is a $\mathcal{K}_\infty$ function, it can be seen that all the assumptions of Lemma 4.3.2 are satisfied. The robust redesign is then possible for the PSO of Theorem 4.4.1.

4.5 Application to the Existing Results

In this section, we investigate several well-known observers (e.g. [GHO92] and [Tsi89]) to show the observers are in fact PSO. Moreover, it is shown that the assumptions required for the observers in [GHO92] and [Tsi89] imply our conditions C1 and C2, which says the design scheme of previous section is a generalization of their results.

It should be also noted that by just showing an observer is PSO, the observer is given the robust redesign scheme of Section 4.3. The observers of [GHO92, Tsi89] readily satisfy all the assumptions of Lemma 4.3.2. Consequently, they have the robust property to the measurement disturbance. (Also refer to Remarks 4.3.1 and 4.4.5.)

4.5.1 Gauthier's Observer is a Passive Observer

In Section 3.2, we introduced the Gauthier's observer. In particular, under Assumption 3.2.1 or 3.2.2, the system (3.2.8) has an observer (3.2.9) by Lemma 3.2.2.

If we look into the proof of Lemma 3.2.2, it can be seen that the error dynamics (3.2.15), which is equivalent to the error dynamics between the plant (3.2.8) and the observer (3.2.9), is globally asymptotically stabilized by the injection gain $\theta\tilde{G}^{-1}C^T$. The stability is shown by (3.2.16) and this can be equivalently interpreted as that the stability of the system (3.2.15) is shown in (3.2.16) by the Lyapunov function $V(\xi) = \frac{1}{2}\theta^{-1}\xi^T\tilde{G}\xi$.

However, in our context of PSO, this implies that the augmented error dynamics is PSUP with the observer gain $L = \theta\tilde{G}^{-1}C^T$, $k = I$ and the storage function $V(\xi) = \frac{1}{2}\theta^{-1}\xi^T\tilde{G}\xi$. Therefore[9],

$$\boxed{\text{The Gauthier's observer (3.2.9) is a PSO.}}$$

Gauthier's Assumptions vs. C1* and C2*

A more interesting point is that the assumptions for Gauthier's observer (i.e. the triangular structure and Assumption 3.2.1) are sufficient for the proposed conditions C1* and C2*. Let us clarify this point.

Consider the system (3.2.8) which is re-written here as:

$$\begin{aligned} \dot{x} &= Ax + \begin{bmatrix} g_1(x_1, u) \\ g_2(x_1, x_2, u) \\ \vdots \\ g_n(x_1, x_2, \cdots, x_n, u) \end{bmatrix} \\ y &= hx = x_1 \end{aligned} \tag{4.5.1}$$

where g_i is globally Lipschitz in $(x_1, \cdots, x_i)$ uniformly in u by Assumption 3.2.1.

[9]This also follows from the Lemma 3.2.1 and Remark 3.2.1.

From the prototype of PSO, $\dot{z} = Az + g(z,u) + Lv$ with $L = [1, l^T]^T$, the error equation is obtained as

$$\begin{aligned} \dot{e}_a &= h_1 e_b + G_a(e_a; x_1, u) + v \\ \dot{e}_b &= A_1 e_b + G_b(e_a, e_b; x_1, x_2, u) + lv \end{aligned} \tag{4.5.2}$$

where $e_a = z_1 - x_1$, $e_b = [z_2 - x_2, \cdots, z_n - x_n]^T$, (A_1, h_1) represents $(n-1)$-th order integrator chain, and $G_a(e_a; x_1, u) = g_1(z_1, u) - g_1(x_1, u)$, $G_b(e_a, e_b; x_1, x_2, u) = [g_2, \cdots, g_n]^T(z, u) - [g_2, \cdots, g_n]^T(x, u)$.

To see C1* holds, the generalized zero dynamics is obtained as

$$\begin{aligned} \dot{x} &= Ax + g(x,u) \\ \dot{e}_b &= (A_1 - lh_1)e_b + G_b(0, e_b; x_1, x_2, u). \end{aligned} \tag{4.5.3}$$

Then, with the help of Lemma 3.2.1, it can be seen that the system (4.5.3) satisfies C1* with $L = [1, l^T]^T$ and $V(e_b) = \frac{1}{2} e_b^T P e_b$ where l and P are obtained in the Lemma.

Finally, since the system (4.5.1) is globally Lipschitz and $V(e_b)$ is quadratic, C2* naturally follows by Remark 4.4.4.

4.5.2 Tsinias' Observer is a Passive Observer

A system which Tsinias [Tsi89] dealt with is

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= Hx \end{aligned} \tag{4.5.4}$$

where f is C^1 and H is a constant matrix. Without loss of generality, we assume that $H = [0,\ I]$ by linear change of coordinates[10] when H has full rank.

In order to construct an observer, Tsinias [Tsi89] assumed the following three conditions.

[10]This change of coordinates does not alter the class of systems to which the conditions proposed by Tsinias [Tsi89] are applicable. See Remark 2 of [Tsi89].

T1: There exist a positive definite symmetric matrix P and a positive constant k_1 such that

$$x^T P D_x f(q,u) x \leq -k_1 \|x\|^2 \tag{4.5.5}$$

for all $q \in \mathbb{R}^n$, $x \in \text{Ker } H$ and $u \in \mathbb{R}^m$.

T2: Moreover, for each non-zero $x \in \text{Ker } H$, there is a neighborhood S_x of x such that, for all $v \in S_x$, (4.5.5) holds with $x = v$.

T3: There exists a continuous function $p : \mathbb{R}^m \to \mathbb{R}$ and a constant $k_2 > 0$ such that $p(u) \geq k_2$ for all u and

$$|x^T P D_x f(q,u) x| \leq p(u) \|x\|^2 \tag{4.5.6}$$

for all $q, x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$.

Using P and $p(u)$ in these conditions, they showed that for an observer of the form

$$\dot{z} = f(z,u) + cp(u) P^{-1} H^T (y - Hz), \tag{4.5.7}$$

there is a constant value c with which the global error convergence ($z - x \ \to \ 0$) is guaranteed [Tsi89, Thm. 1]. Specifically, by the fact that the error dynamics becomes

$$\dot{e} = F(e; x, u) - cp(u) P^{-1} H^T (He),$$

they utilized the Lyapunov function $V(e) = \frac{1}{2} e^T P e$ whose derivative becomes negative by choosing a suitable value of c. This leads to the fact that the corresponding augmented error dynamics is PSUP with respect to the pair (e, u) if we take $L = P^{-1} H^T$, $k(u, y, y_a) = cp(u)$ and the storage function $V(e) = \frac{1}{2} e^T P e$. That is,

$$\boxed{\text{Tsinias' observer (4.5.7) is a PSO.}}$$

T1 and T3 imply C1* and C2

Now we relate T1-T3 to our proposed conditions C1* and C2. Especially, we show that T1 and T3 imply C1* and C2, respectively. This means that T2 is redundant for the observer construction, although it plays an active role in the proof of [Tsi89, Thm. 1].

Let $x = [x_1^T, x_2^T]^T$ according to the dimension of y such that $y = x_2$. For simplicity, the proof is performed in the transformed coordinates. In other words, without loss of generality, we suppose that the system

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, u) \\ \dot{x}_2 &= f_2(x_1, x_2, u) \\ y &= x_2 \end{aligned} \tag{4.5.8}$$

satisfies the conditions T1 and T3 with a positive definite matrix

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}. \tag{4.5.9}$$

Indeed, if the given system (4.5.4) satisfies T1 and T3 with

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

then, by the linear change of coordinates

$$T := \begin{bmatrix} I & P_{11}^{-1} P_{12} \\ 0 & I \end{bmatrix},$$

and by redefining the transformed system as (4.5.8), it can be shown that (4.5.8) satisfies T1 and T3 with $P_1 = P_{11}$ and $P_2 = P_{22} - P_{21} P_{11}^{-1} P_{12}$.

Now the condition T1 becomes that

$$e_1^T P_1 D_{x_1} f_1(q, u) e_1 \le -k_1 \|e_1\|^2, \qquad e_1 \in \mathbb{R}^p,\ q \in \mathbb{R}^n,\ u \in \mathbb{R}^m. \tag{4.5.10}$$

Let $V(e_1) = \frac{1}{2}e_1^T P_1 e_1$, $L_1 = 0$ and $L_2 = P_2^{-1}$, and compare C1* with (4.5.10). It can be proved that, for each e_1, x and u, there is q such that[11]

$$e_1^T P_1 F_1(e_1, 0; x_1, x_2, u) = e_1^T P_1 D_{x_1} f_1(q, u) e_1.$$

Then, since (4.5.10) holds for all q by T1, C1* follows naturally.

Now it is left to show that T3 implies C2. Suppose T3 holds with (4.5.8) and (4.5.9), i.e., there is a continuous function $p(u)$ such that

$$\begin{aligned} |x_1^T P_1 D_1 f_1(q,u)x_1 + x_2^T P_2 D_1 f_2(q,u)x_1 + x_1^T P_1 D_2 f_1(q,u)x_2 + x_2^T P_2 D_2 f_2(q,u)x_2| \\ \le p(u)\|x_1\|^2 + p(u)\|x_2\|^2 \end{aligned}$$

for all $q, x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Then, it follows by the next Lemma 4.5.1 that

$$\begin{aligned} |x_2^T P_2 D_1 f_2(q,u)x_1 + x_1^T P_1 D_2 f_1(q,u)x_2 + x_2^T P_2 D_2 f_2(q,u)x_2| \\ \le p(u)\|x_2\|^2 + r(u)\|x_1\|\,\|x_2\| \end{aligned} \tag{4.5.11}$$

for all $q, x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$.

Lemma 4.5.1. *Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $z \in \mathbb{R}^k$ and $u \in \mathbb{R}^p$, and let L, M and N be matrix-valued functions with appropriate dimensions. If there exists a continuous function $c(u)$ such that*

$$|x^T L(z,u)x + x^T M(z,u)y + y^T N(z,u)y| \le c(u)\|y\|^2 + c(u)\|x\|^2 \tag{4.5.12}$$

for all x, y, z and u, then there exists a continuous function $d(u)$ such that

$$|x^T M(z,u)y + y^T N(z,u)y| \le c(u)\|y\|^2 + d(u)\|x\|\,\|y\| \tag{4.5.13}$$

for all x, y, z and u.

[11] This can be proved by the technique used in the forthcoming Lemma 4.5.2.

Proof. First, by letting $x = 0$ in (4.5.12), it holds that $|y^T N(z,u)y| \leq c(u)\|y\|^2$. Moreover, it is easy to see that the diagonal terms of $N(z,u)$ is bounded with respect to z by fixing $y = \delta_i$, where δ_i is such that its i-th element is 1 and others are 0. Now, we show element-wisely that $M(z,u)$ is bounded with respect to z. That is, (i,j)-th element of M, say $M_{ij}(z,u)$, is bounded since, when $x = \delta_i$, $y = \delta_j$, the equation (4.5.12) becomes

$$|M_{ij}(z,u) + N_{jj}(z,u)| \leq 2c(u) \tag{4.5.14}$$

where N_{jj} is bounded. Using their boundedness, it is clear that (4.5.13) holds. ∎

Now, consider the condition C2. Since $V(e_1) = \frac{1}{2}e_1^T P_1 e_1$, $L_1 = 0$ and $L_2 = P_2^{-1}$, C2 is simplified as

$$\begin{aligned} &|e_1^T P_1 F_1(0, e_2; e_1 + x_1, x_2, u) + e_2^T P_2 F_2(e_1, e_2; x_1, x_2, u)| \\ &\qquad \leq \phi_1(u, x_2, e_2)\|e_2\|^2 + \phi_2(u, x_2, e_2)\sqrt{k_1}\|e_1\|\|e_2\|. \end{aligned} \tag{4.5.15}$$

Then, again, for each (e, x, u), there are $q \in \mathbb{R}^n$ such that

$$\begin{aligned} &e_1^T P_1 F_1(0, e_2; e_1{+}x_1, x_2, u) + e_2^T P_2 F_2(e_1, e_2; x_1, x_2, u) \\ &\qquad = e_1^T P_1 D_2 f_1(q,u) e_2 + e_2^T P_2 D_1 f_2(q,u) e_1 + e_2^T P_2 D_2 f_2(q,u) e_2. \end{aligned}$$

(For the completeness, we present the strict proof of this fact in Lemma 4.5.2.)

Finally, it is easy to see that (4.5.11) implies (4.5.15) which claims that T3 implies C2, with $\phi_1 = p(u)$ and $\phi_2 = r(u)/\sqrt{k_1}$.

Lemma 4.5.2. [12] *For each e, x and u, there exists $q \in \mathbb{R}^n$ such that*

$$\begin{aligned} &e_1^T P_1 \left(f_1(e_1 + x_1, e_2 + x_2, u) - f_1(e_1 + x_1, x_2, u)\right) \\ &\qquad\qquad + e_2^T P_2 \left(f_2(e_1 + x_1, e_2 + x_2, u) - f_2(x_1, x_2, u)\right) \\ &\quad = e_1^T P_1 D_2 f_1(q,u) e_2 + e_2^T P_2 D_1 f_2(q,u) e_1 + e_2^T P_2 D_2 f_2(q,u) e_2. \end{aligned}$$

[12] Lemma 4.5.2 is not a trivial consequence of the Mean-Value Theorem. For $f : \mathbb{R}^n \to \mathbb{R}^n$ $(n > 1)$ such that $f(0) = 0$, it is not true in general that, for each x, there is a q such that $f(x) = Df(q)x$. This can be seen by a counterexample $f(x_1, x_2) = [x_1^2, \exp(x_1) - 1]^T$ when $x_1 = x_2 = 1$.

Proof. Let

$$f^*(x, z, u, p) := \begin{bmatrix} p_1^T P_1 & p_2^T P_2 \end{bmatrix} \begin{bmatrix} f_1(z_1, z_2, u) - f_1(z_1, z_2 - x_2, u) \\ f_2(z_1, z_2, u) - f_2(z_1 - x_1, z_2 - x_2, u) \end{bmatrix} \tag{4.5.16}$$

where (z, u, p) is supposed to be parameters. Then, since $f^*(0, z, u, p) = 0$ and $f^* : \mathbb{R}^n \to \mathbb{R}$ for each (z, u, p), by the mean-value theorem [Mun91, p.59], for each x, z, u and p, there exists q such that

$$f^*(x, z, u, p) = D_x f^*(q, z, u, p)x = \begin{bmatrix} p_1^T P_1 & p_2^T P_2 \end{bmatrix} \begin{bmatrix} 0 & D_2 f_1(q, u) \\ D_1 f_2(q, u) & D_2 f_2(q, u) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Finally, substituting $x = e$, $p = e$ and $z = e + x$ proves the claim. ∎

Thanks to the arguments so far, we claims the following theorem.

Theorem 4.5.3. *For the system* (4.5.4)*, if T1 and T3 hold, then there is a state observer of the form* (4.5.7)*. Furthermore, the observer is a PSO.*

Remark 4.5.1. Due to [Tsi89, Thm. 2] a detectable linear system satisfies T1. Furthermore, a linear system always satisfies C2 by Remark 4.4.5. Thus, we recover Proposition 4.1.1.

4.5.3 More Applications

From 1980's, nonlinear observer has been studied in the framework of linearizable error dynamics [KI83, KR85, HP99]. This approach uses the pole placement technique for linearized error dynamics, in other words, it chooses a gain L such that $(A - LC)$ is Hurwitz for a detectable pair (A, C). However, as sketched in Section 4.1, if the poles are optimally located (i.e. using the Riccati equation (4.1.7) so that the gain L has the form of $P^{-1}C^T$), then the corresponding observer becomes a PSO. Therefore, a PSO can always be obtained by the approach of linearizable error dynamics.

On the other hand, Walcott and Zak [WZ87] and Dawson *et al.* [DQC92] considered a nonlinear observer for the system given by

$$\begin{aligned} \dot{x} &= Ax + f(x,u) \\ y &= Cx \end{aligned} \tag{4.5.17}$$

where C is of full rank, under the following assumptions:

A1: $^\exists K$ and $P > 0$ s.t. $A_c := A - KC$ is Hurwitz, $A_c^T P + PA_c < 0$ and $Pf(x,u) = C^T h(x,u)$.

A2: $^\exists \rho(y,u)$ s.t. $\|h(x,u)\| \leq \rho(y,u)$.

Under these assumptions, Walcott and Zak [WZ87] constructed the state observer for (4.5.17), but it is variable-structure type and is discontinous around $e = z - x = 0$. Since the discontinuity usually leads to some practical problems such as chattering, Dawson *et al.* [DQC92] improved the observer as a continuous one under the same assumptions. (For [DQC92], let $u = t$.) However, their observer has the following time-varying term ([DQC92, Eq.(3.7)], we recall it here for convenience)

$$\frac{P^{-1}C^T Ce\rho^2}{\|Ce\|\rho + \epsilon \exp(-\beta t)}.$$

Although this term is continuous for all e and t, it is still discontinuous in the practical sense because as time goes ($t \to \infty$) the term $\epsilon \exp(-\beta t)$ becomes negligible. We suppose this discontinuity is induced by the weak assumption A2. Thus, we slightly modify these assumptions, and show that a PSO can be constructed which does not have any discontinuity.

B1: $^\exists P > 0$ s.t. $Pf(x,u) = C^T h(x,u)$ and $PA + A^T P + PQP - C^T R^{-1} C = 0$ with $R > 0$ and $Q > 0$.

B2: $^{\exists}\rho$ s.t. $\|h(z,u) - h(x,u)\| \leq \rho(Cz, Cx, u)\|Cz - Cx\|$.

The condition B1 is modified from A1 so that the gain K in A1 has the form of $P^{-1}C^T$, thus, it can be proved that $A - KC = A - P^{-1}C^TC$ is Hurwitz with P of B1.

Theorem 4.5.4. *If B1 and B2 hold for the system* (4.5.17)*, then there is a state observer (in fact a PSO) in which all terms are continuous.*

Proof. The conditions B1–B2 can be easily shown to be invariant under the linear change of coordinates. Therefore, we assume, without any loss of generality, the given system (4.5.17) has the output $y = Cx = x_2$ where $C = [0, I]$ and the system satisfies B1 and B2 with the block diagonal matrix $P > 0$, i.e.

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}.$$

This can be verified by two step transformations of coordinates ($T = T_2T_1$). In fact, after transforming into the intermediate coordinates with $T_1 = [D^T \; C^T]^T$, apply the second transformation

$$T_2 = \begin{bmatrix} I & P_{11}^{-1}P_{12} \\ 0 & I \end{bmatrix},$$

where P_{ij} is the (i,j)-th block element of P which satisfies B1 for the intermediate coordinates.

Then the system (4.5.17) can be written as

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 \\ \dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + P_2^{-1}h(x_1, x_2, u). \end{aligned}$$

On the other hand, note that $Q > 0$ implies $Q_{11} > 0$, by which and B1, it follows that $P_1A_{11} + A_{11}^TP_1 < 0$. Now it can be easily checked that this system satisfies C1* with $V(e_1) = \frac{1}{2}e_1^TP_1e_1$, $L_1 = 0$ and $L_2 = P_2^{-1}$. The condition C2 also follows from B2. Therefore, Theorem 4.4.1 provides the state observer (PSO) for (4.5.17). ∎

4.6 Reduced Order Observer

Suppose a situation that the given system (4.4.1) satisfies the condition C1, but does not satisfy C2. Even in this case, we can have a (reduced order) state observer if we disregard the PSO. In other words, C1 is sufficient for the construction of the reduced order observer which is, however, no longer a PSO in our definition.

Theorem 4.6.1. *When the system* (4.4.1) *satisfies C1 with appropriately chosen* L_1 *and* L_2*, the system, with* $L^* = L_1 L_2^{-1}$,

$$\dot{z} = f_1(z + L^* y, y, u) - L^* f_2(z + L^* y, y, u) \tag{4.6.1}$$

$$\hat{x}_1 = z + L^* y, \qquad \hat{x}_2 = y$$

is a reduced order observer of (4.4.1).

Proof. By $\xi = x_1 - L^* x_2$ and $x_2 = x_2$, the given system (4.4.1) is transformed into

$$\dot{\xi} = f_1(\xi + L^* x_2, x_2, u) - L^* f_2(\xi + L^* x_2, x_2, u) \tag{4.6.2a}$$

$$\dot{x}_2 = f_2(\xi + L^* x_2, x_2, u) \tag{4.6.2b}$$

$$y = x_2. \tag{4.6.2c}$$

Since x_2 is measurable, we discard x_2-dynamics and construct an observer for (4.6.2a) as in (4.6.1). Then, the augmented error dynamics becomes ($e = z - \xi = z - (x_1 - L^* x_2)$)

$$\dot{x} = f(x, u)$$

$$\dot{e} = F_1(e, 0; x_1, x_2, u) - L^* F_2(e, 0; x_1, x_2, u).$$

Finally, it can be shown that $z(t) \to \xi(t) = x_1(t) - L^* x_2(t)$ as $t \to \infty$ by utilizing Lemma 2.3.1 with C1. ∎

Generalization of [BH96]

Besançon and Hammouri [BH96] also considered the reduced order observer for a class of nonlinear systems. In [BH96], they developed a reasonable concept of nonlinear detectability and related the concept to the construction of the reduced order observer. Here, as a final application of our condition C1, we show the main assumption of [BH96] implies C1*. Hence,

$$\boxed{\text{Theorem 4.6.1 is an extension of the main work of [BH96].}}$$

The class of systems considered in [BH96] has the form of (4.5.17), which is transformed into the partitioned form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} f_1(x_1, x_2, u) \\ f_2(x_1, x_2, u) \end{bmatrix}, \qquad y = x_2.$$

Then, the assumptions of [BH96, Thm. 8] can be written as

C: $^\exists K$, $P > 0$ and $Q > 0$ such that

$$P(A_{11} + KA_{21}) + (A_{11} + KA_{21})^T P = -Q \tag{4.6.3}$$

and, for all x, e_1 and $u \in \mathcal{U} \subset \mathbb{R}^m$,

$$|e_1^T P(F_1(e_1, 0; x_1, x_2, u) + KF_2(e_1, 0; x_1, x_2, u))| \leq \gamma \|e_1\|^2 \tag{4.6.4}$$

where $\gamma < \frac{1}{2}\lambda_{\min}(Q)$.

Since C1* for this system becomes

$$D_{e_1}V(e_1)[(A_{11}e_1 + F_1(e_1, 0; x_1, x_2, u)) - L_1 L_2^{-1}(A_{21}e_1 + F_2(e_1, 0; x_1, x_2, u))] \\ \leq -\psi_3(\|e_1\|),$$

with $V(e_1) = \frac{1}{2}e_1^T P e_1$, $L_1 = -K$ and $L_2 = I$, it follows by (4.6.3) and (4.6.4) that

$$\begin{aligned} e_1^T P[(A_{11} + KA_{21})e_1 + F_1 + KF_2] &\leq -\frac{1}{2}\lambda_{\min}(Q)\|e_1\|^2 + \gamma\|e_1\|^2 \\ &= -\psi_3\|e_1\|^2. \end{aligned}$$

Therefore, we recover the claim of [BH96, Thm. 8].

4.7 Illustrative Examples

Example 4.7.1. Consider a simple system of Remark 3.1.1:

$$\begin{aligned} \dot{x}_1 &= u \\ \dot{x}_2 &= x_1 + x_1 u^2 \qquad\qquad y = x_2 \end{aligned} \tag{4.7.1}$$

which is uniformly observable by Definition 3.1.3. Since this system is uniformly observable, the high-gain observer of the type in [Tor92, TP94] exists but requires the knowledge of $\dot{u}$. Even though [GK94] gives an answer without $\dot{u}$ in this case, design of PSO is another easier answer.

Suppose an observer for (4.7.1) as

$$\dot{z}_1 = u + l_1 v \qquad \dot{z}_2 = z_1 + z_1 u^2 + l_2 v \tag{4.7.2}$$

with which the error dynamics is obtained as

$$\begin{aligned} \dot{e}_1 &= l_1 v \\ \dot{e}_2 &= e_1 + e_1 u^2 + l_2 v \qquad y_a = e_2. \end{aligned}$$

Therefore, the zero dynamics becomes

$$\dot{e}_1 = -\frac{l_1}{l_2}(1 + u^2) e_1$$

for which the condition C1* holds with $V(e_1) = \frac{1}{2} e_1^2$, $l_1 = 1$ and $l_2 = 1$. Moreover, C2 holds with $\phi_1 = \phi_2 = 1 + u^2$ in (4.4.6). Indeed, (4.4.6) becomes

$$|e_1 \left(-(1 + u^2) e_2\right) + e_2 (1 + u^2)(e_1 + e_2)| \leq (1 + u^2) \|e_1\|^2.$$

Thus, Theorem 4.4.1 gives a PSO for (4.7.1).

Example 4.7.2. Consider a system

$$\begin{aligned} \dot{x}_0 &= -(1 + x_0^2) x_0 + (x_1 - 1) u \\ \dot{x}_1 &= -x_1 u^2 \\ \dot{x}_2 &= -x_2^3 - 2 x_1 (1 + u^2) \qquad y = x_2, \end{aligned} \tag{4.7.3}$$

to which the several approaches in Section 4.5 are not applicable.

For (4.7.3), the vector fields of error dynamics are obtained as

$$
\begin{aligned}
F_1(e;x,u) &= \begin{bmatrix} -(e_0+x_0)\left(1+(e_0+x_0)^2\right)+x_0(1+x_0^2)+e_1u \\ -e_1u^2 \end{bmatrix} \\
F_2(e;x,u) &= -(e_2+x_2)^3+x_2^3-2e_1(1+u^2).
\end{aligned}
$$

Let $L_1=[l_0,l_1]^T$ and $L_2=1$, then the left-hand term of (4.4.5) becomes

$$
\begin{aligned}
e_0[-(e_0+x_0)\left(1+(e_0+x_0)^2\right)+x_0(1+x_0^2)+l_0(2e_1(1+u^2))]& \\
+e_0e_1u+e_1[-e_1u^2+l_1(2e_1(1+u^2))]&
\end{aligned}
$$

with $V(e_0,e_1)=\frac{1}{2}(e_0^2+e_1^2)$. Here, since

$$
\begin{aligned}
e_0[-(e_0+x_0)\left(1+(e_0+x_0)^2\right)+x_0(1+x_0^2)]& \\
=-e_0^2\left(e_0^2+3x_0e_0+(3x_0^2+1)\right)\le -e_0^2,&
\end{aligned}
$$

choose $l_0=0$. Similarly, by taking $l_1=-1$ the last term becomes $-(2+3u^2)e_1^2$. Therefore, (4.4.5) becomes that

$$
\begin{aligned}
D_{(e_0,e_1)}V\left[F_1-L_1L_2^{-1}F_2\right] &\le -e_0^2+e_0e_1u-2e_1^2-3u^2e_1^2 \\
&= -\frac{1}{2}e_0^2-2e_1^2-\frac{5}{2}u^2e_1^2-\frac{1}{2}(e_0-ue_1)^2 \\
&\le -\frac{1}{2}(e_0^2+e_1^2)
\end{aligned}
$$

which shows that the condition C1* holds.

For the condition C2, (4.4.6) is written as

$$\begin{aligned}
&\Bigg| [e_0\ e_1] \left\{ \begin{bmatrix} -e_2 u \\ e_2 u^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left[-(e_2 + x_2)^3 + x_2^3 + 2e_2(1+u^2) \right] \right\} \\
&\qquad\qquad + e_2\{-(e_2+x_2)^3 + x_2^3 - 2(e_1 - e_2)(1+u^2)\} \Bigg| \\
&= | - e_0 e_2 u + e_1\{e_2 u^2 - (e_2 + x_2)^3 + x_2^3\} \\
&\qquad\qquad + e_2\{-(e_2+x_2)^3 + x_2^3 + 2e_2(1+u^2)\}| \\
&\leq |u|\,|e_0|\,|e_2| + |e_2^2 + 3x_2 e_2 + 3x_2^2 - u^2|\,|e_1|\,|e_2| \\
&\qquad\qquad + |e_2^2 + 3x_2 e_2 + 3x_2^2 - 2 - 2u^2|\,|e_2|^2.
\end{aligned}$$

Thus, C2 is satisfied with $\phi_1(u, y, y_a) = |e_2^2 + 3x_2e_2 + 3x_2^2 - 2 - 2u^2|$ and $\phi_2(u, y, y_a) = 2(|u| + |e_2^2 + 3x_2e_2 + 3x_2^2 - u^2|)$. Then, Theorem 4.4.1 gives a PSO for (4.7.3).

4.8 Notes on the Chapter

In this chapter, we have proposed the concept of passivity-based state observer (PSO) and two conditions (C1 and C2) with which a PSO is designed. Even though the conditions are for constructing the PSO, they also provide a new viewpoint for general observer problem. In particular, seemingly un-related two works [GHO92, Tsi89] have been related and interpreted via the framework of PSO. The approach has eliminated some redundant assumptions in the existing works and extended them. The proposed robust redesign method is also useful because, once a given observer is shown to be PSO, then the method is applicable to the observer.

The condition C1 can be viewed as the minimum phase condition of the augmented error dynamics. We conjecture that this condition has some relationship to the nonlinear detectability based on the viewpoint of [BH96]. Especially, we've shown that C1 is a generalized version of the detectability assumption in [BH96].

One restriction of the design scheme of PSO in Section 4.4 is the *linear* measurement $y = [0\ I]x$ of (4.4.1). However, there are several cases when the given output $y = h(x)$ can be made $y = x_2$ by a change of coordinates. For example, if the function $h(x)$ has full rank in $\mathbb{R}^n$ and if we find additional coordinate component $\phi(x)$ such that $[h^T(x)\ \phi^T(x)]^T$ is one-to-one on $\mathbb{R}^n$ and its Jacobian is nonsingular on $\mathbb{R}^n$, then the map $[h^T(x)\ \phi^T(x)]^T$ becomes the coordinate transform that we need. The linear case, that the output is given as $y = Hx$ where H is of full rank, is the very example.

On the other hand, there is another possible modification which is motivated by [SS85]. Suppose that the given output $y = h(x)$ is independent of x_1 and is *modifiable*, i.e., there is a function ϑ such that

$$h(z_2) - h(x_2) = \vartheta(z_2, h(x_2))(z_2 - x_2),$$

which leads to the augmented error output $y_a = \vartheta(z_2, h(x_2))e_2$. Then, it may be possible to yield the similar claims to the proposed design scheme. This is left currently as a future research topic.

Chapter 5

Semi-global Separation Principle

Until now, the observer problem is discussed in detail. The observer construction is of interest in its own right, and frequently used in practice. However, it is also of great interest to use the observer in conjunction with a state feedback control law in order to result in the overall *output* feedback controller. Unfortunately, unlike the linear case, it is not so straightforward to combine the observer with the pre-designed state feedback law for the stability of nonlinear systems.

This chapter is devoted to that combination, i.e., the state feedback law *plus* the observer. Especially, it is shown that, for multi-input single-output non-affine nonlinear systems, when a state feedback control stabilizes an equilibrium point of a plant with a certain bounded region of attraction, it is also stabilized by an output feedback controller with arbitrarily small loss of the region. Moreover, the proposed output feedback controller has the dynamic order n which is the same as the order of the plant. From any given state feedback, an explicit form of the overall controller is provided. A sufficient condition presented for the result is shown to be necessary and sufficient for regional uniform observability when the system is input affine. Thus, the result can be regarded as a regional separation principle for affine nonlinear systems.

5.1 Motivations

For linear systems, stabilizability and detectability of the system guarantee the existence of output feedback stabilizing controller, i.e., any pole-placement state feedback and any Luenberger observer can be combined to construct an output feedback controller (*separation principle*). However, for nonlinear systems, it has been understood that such a desirable property does not hold *in general*, even though there is a version of separation principle for a class of Lyapunov-stable nonlinear systems [GK92, Lin95a]. Especially, Mazenc *et al.* [MPD94] presented a counterexample which shows that global stabilizability and global observability are not sufficient for global output feedback stabilization. As a consequence, research activities in the literature can be classified into two major categories. One is imposing additional conditions on the system for global output feedback stabilization, for example, differential geometric conditions on the system structure [MT95], or the existence of a certain Lyapunov function [Tsi91]. The other approach is focused on the semi-global output feedback stabilization instead of the global stabilization [EK92, KE93, AK97]. The most general and satisfiable result in this direction is the work of Teel and Praly [TP94, TP95], where only global (semi-global) stabilizability and global uniform observability are assumed for the system. However, their result used the 'dynamic extension' technique and the high-gain observer [Tor92] which estimates the derivatives of the output y. As a result, the order of controller is greater than that of the plant in general, which is unnecessary in the case of linear output feedback stabilization.

In this chapter, we consider the problem of output feedback stabilization for a multi-input single-output nonlinear system given by

$$\begin{aligned} \dot{x} &= f(x) + g(x,u) \\ y &= h(x) \end{aligned} \tag{5.1.1}$$

where the state $x \in \mathbb{R}^n$, the input $u \in \mathbb{R}^m$ and the output $y \in \mathbb{R}$; the vector fields f and g, and the function h are smooth at every $(x,u) \in \mathcal{U} \times \mathbb{R}^m$ where $\mathcal{U}$ is a connected open set in $\mathbb{R}^n$ containing the origin, and $g(x,0) = 0$.

To establish nonlinear output feedback stabilization, which is a natural extension of linear one, some crucial properties of linear version should be pointed out.

P1: Only stabilizability and observability are sufficient for output feedback stabilization. No more conditions are needed.

P2: If the system is observable when $u \equiv 0$, it is also observable for every known u.

P3: The order of observer is the same as that of the plant. Thus, the order of output feedback controller is n.

P4: The procedures to design output feedback controller is completely separated, that is, any state feedback controller and any observer can be combined.

From now on, three aspects of output feedback are presented. These give some motivation and justification of the treatment in this chapter.

(1) *Uniform Observability.* The fact that, in nonlinear systems, the observability can be destroyed by an input u [Vid93, p.415], is an obstacle because our purpose is the separation (P4) but the specific control u may break the observability. Therefore, we require the uniform observability (P2). Then, we will use the Gauthier's observer for output feedback because it fits the purpose stated in (P3) and has useful properties for output feedback, which will be further studied. By saturating the input and using the semi-global concept, the

assumption for Gauthier's observer (Assumption 3.2.2) will also be eliminated for (P1).

(2) *Feedback Control using Estimated States.* Another key obstruction for global output feedback is 'finite escape time' phenomenon which is well discussed in [MPD94]. Suppose a system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_2^3 + u$$
$$y = x_1$$

which is globally state feedback stabilizable with $u(x) = -x_1 - x_2 - x_2^3$, and uniformly observable. Suppose also that a global observer is constructed which estimates the true state asymptotically, that is,

$$u(\hat{x}(t)) \to u(x(t)) \qquad \text{as} \qquad t \to \infty. \tag{5.1.2}$$

Hence, the system with output feedback is

$$\dot{x}_1 = x_2 \tag{5.1.3a}$$
$$\dot{x}_2 = x_2^3 + u(\hat{x}) = \frac{1}{2}x_2^3 + [\frac{1}{2}x_2^3 + u(\hat{x})]. \tag{5.1.3b}$$

Though the global observer guarantees the convergence of (5.1.2), it takes some time for the control value $u(\hat{x}(t))$ to converge to the true control $u(x(t))$. During that time interval, some state may escape to infinity. For the example in (5.1.3) with $x_1(0) = 0$, $x_2(0) = 10$ and $\hat{x}(0) = 0$, the state x_2 goes to infinity within 0.01 seconds unless the second term of (5.1.3b) $([\frac{1}{2}x_2^3(t) + u(\hat{x}(t))])$ becomes negative during that time. This facts shows that, for the output feedback stabilization, the convergence rate of the observer should be sufficiently fast.

However, at this point, there arise another two obstacles. The first one is the fact that no matter how fast the convergence rate of observer is, there always exists an initial condition of x_2 whose trajectory blows up in finite time. Indeed, for a system $\dot{z} = \frac{1}{2}z^3$, the solution $z(t)$ from $z(0) = z_0 > 0$ blows up at $t = \frac{1}{z_0^2}$, which can be made arbitrarily small by increasing the initial z_0 [MPD94]. The semi-global approach is now appealing since it restricts possible initial conditions, which is practically reasonable. The second obstacle is the so-called 'peaking phenomenon' [SK91] which is generally inevitable when the convergence of observer is forced to be sufficiently fast. For fast convergence rate, most observers use high-gain, or place their poles far left. This ensures fast convergence but may generate initial peaking, i.e. large mismatched value between $u(x(t))$ and $u(\hat{x}(t))$ for the short initial period. This mismatching again may reduce the escape time of the system, thus, the observer needs to converge faster. A remedy for this vicious cycle is saturating the value of control $u(\hat{x})$, which is based on the idea of [EK92].

(3) *Region.* Another advantage of the semi-global approach is for the fact that, in many cases, the model (5.1.1) does not coincide with the real plant on the whole state space $\mathbb{R}^n$ since it is often simplified by a designer outside the region of interest. It even happens that some functions in (5.1.1) have singularity on a point of $\mathbb{R}^n$ or aren't defined outside the region. In many practical situations, finding a valid model of a plant, like (5.1.1), in the whole area of $\mathbb{R}^n$ is rather difficult or is not necessary since the state of the plant is usually bounded when the plant is well operating. Hence, global control problems can be viewed to be a bit exaggerated although it has been actively studied (see, e.g. [KKK95] and the references therein). On the other hands, local control problems may not be so satisfactory either, because in the local problems, the size of operating region is not at the designer's choice and it is often too small to contain the region of interest.

Based on these considerations, a dynamic output feedback controller is presented in this chapter when the system (5.1.1) admits a stabilizing state feedback with a certain bounded region of attraction and satisfies a strong notion of observability (uniform observability) on the region. The construction of such a controller is based on a state observer and does not restrict the form of state feedback controller. Thus any state feedback can be used with the proposed observer-based controller. In addition, with the proposed output feedback controller, the region of attraction is nearly preserved on which the system is asymptotically stabilized by a state feedback.

Since the sufficient conditions for our result are stabilizability and observability, and the resulting controller has the same dynamic order n with the plant, the contribution of this chapter is regarded as the *separation principle* for nonlinear systems.

5.2 Separation Principle

5.2.1 Separation Principle on a Bounded Region of Attraction

As already discussed in Section 3.3, in many cases, the system description (5.1.1) is not a globally valid model for the real plant and, thus, a globally stabilizing state feedback control law is neither necessary. Inspired by these facts, the starting point of this paper is,

> *Suppose that the origin of the system* (5.1.1) *is asymptotically stabilized by a state feedback control* $\alpha(x)$ *with a guaranteed region of attraction* $\mathcal{U}_c$.

More specifically, we assume the following condition.

Condition (C1). For the system (5.1.1), there are a state feedback control $\alpha(x)$ and a C^1 positive definite function $V(x) : \mathbb{R}^n \to \mathbb{R}$ such that, with some positive constant c,
(a) the set

$$\mathcal{U}_c := \{x \in \mathbb{R}^n \mid V(x) < c\} \tag{5.2.1}$$

is connected, bounded, open and contained in $\mathcal{U}$.
(b) on $\mathcal{U}_c$ except the origin,

$$\frac{\partial V}{\partial x}(x)\left(f(x) + g(x, \alpha(x))\right) < 0. \tag{5.2.2}$$

(c) α is Lipschitz on $\mathcal{U}_c$ and $\alpha(0) = 0$.

A sufficient condition for (C1) is *global asymptotic stabilizability* of the system (5.1.1), which is defined by the existence of a smooth feedback $\alpha(x)$ that globally asymptotically stabilizes the system (see, e.g. [Isi95, Kha96]). In that case, the region of attraction is the whole region $\mathbb{R}^n$ and the converse Lyapunov theorem guarantees the existence of $V(x)$ which satisfies (C1). *Semi-global state feedback stabilizability* [TP94] is also a sufficient condition for (C1). Although these two conditions are often assumed in the literature, they are not necessary in our discussion. The requirements are just the existence of a feedback control $\alpha(x)$, a function $V(x)$ and a region $\mathcal{U}_c$ which satisfy (C1).

While condition (C1) is related to the regional stabilizability of the system, the following assumption is a kind of regional observability for the system.

Condition (C2). Suppose that a system (5.1.1) and a region $\mathcal{U}_c$ are given. Define

$$\Phi(x) := \begin{pmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{pmatrix}. \tag{5.2.3}$$

(a) $\Phi(x) : \mathcal{U}_c \to \Phi(\mathcal{U}_c)$ is one-to-one.
(b) rank $\frac{\partial \Phi}{\partial x}(x) = n$ for every $x \in \mathcal{U}_c$.
Also define r as the vector field solution of

$$\frac{\partial \Phi}{\partial x}(x) \cdot r = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}. \tag{5.2.4}$$

(c) $[g(x,u), \mathcal{R}_j] \subset \mathcal{R}_j$, $0 \le j \le n-2$ on $\mathcal{U}_c$, $\forall u$ where

$$\mathcal{R}_j := \text{span}\{r, ad_f r, ad_f^2 r, \cdots, ad_f^j r\}.$$

Remark 5.2.1. Comparing (C1) with the assumption of Lemma 3.2.3, it can be seen that this condition asserts that the system (5.1.1) is uniformly observable *in the region* $\mathcal{U}_c$. In fact, (C1) is equivalent to the transformability of (5.1.1) to (3.2.8) *in the region* $\mathcal{U}_c$. (The proof is similar to Lemma 3.2.3.) When the system (5.1.1) is input affine, (C1) is *equivalent* to the regional uniform observability of (5.1.1) by the discussions in Section 3.1.2.

Under these conditions the main results are summarized by the following theorem, whose proof is deferred to Section 5.2.3.

Theorem 5.2.1. *Suppose the plant* (5.1.1) *satisfies conditions* (C1) *and* (C2). *Then, for any constant c_0 such that $0 < c_0 < c$, there is an output feedback stabilizing controller of order n with the guaranteed region of attraction Ω_{c_0} for the closed-loop plant, where*

$$\Omega_{c_0} := \{x \in \mathbb{R}^n \mid V(x) \leq c_0\}. \tag{5.2.5}$$

Since the plant has $\mathcal{U}_c = \{x \in \mathbb{R}^n \mid V(x) < c\}$ as a guaranteed region of attraction when a state feedback is used, it can be interpreted that the system loses some region $\mathcal{U}_c - \Omega_{c_0}$ by the output feedback. But, the loss can be made arbitrarily small by choosing c_0 close to the value of c. Hence, it can be claimed that when a system (5.1.1) satisfies (C1) and (C2), it is output feedback stabilizable with arbitrarily small loss of its guaranteed region of attraction.

For input affine systems, the same result (but without dynamic extension) as [TP94] is obtained by combining from Theorem 5.2.1 by Remark 5.2.1.

Corollary 5.2.2. *If the system* (5.1.1), *with $\mathcal{U} = \mathbb{R}^n$, is input affine, globally asymptotically stabilizable and globally uniformly observable, then the system is semi-globally output feedback stabilizable (with a dynamic controller of order n).*

Proof. To show the semi-global stabilizability, choose a compact set $\mathcal{K}$ having arbitrary size, such that the initial state $x(0)$ of the plant is located in $\mathcal{K}$. By the global asymptotic stabilizability of the system, there exists a globally stabilizing smooth control $\alpha(x)$ with the region of attraction $\mathbb{R}^n$ (whole region). By the converse Lyapunov theorem, there is a radially unbounded, positive definite smooth Lyapunov function $V(x)$. Then, by the radial unboundedness of $V(x)$, there is c such that

(C1) holds with $\mathcal{K} \subset \mathcal{U}_c$. Since $\mathcal{K}$ is compact and $\mathcal{U}_c$ is open, by [Mun91, Thm. 4.6], there is c_0 such that $\mathcal{K} \subset \Omega_{c_0} \subset \mathcal{U}_c$. The condition (C2) also holds by the uniform observability. Finally, by Theorem 5.2.1, there is an output feedback controller (of order n) which asymptotically stabilizes the origin of the plant, and the region of attraction contains $\mathcal{K}$. Therefore, the semi-global stabilization is achieved. ■

The main difference between the Corollary 5.2.2 and the result of [TP94] is the order of controller. For example, even for a simple linear system

$$\begin{aligned} \dot{x}_1 &= x_2 + u \\ \dot{x}_2 &= x_1 \\ y &= x_1 \end{aligned}$$

the strategy of their paper gives a controller of order 4, but Theorem 5.2.1 gives order 2. This is mainly due to the 'dynamic extension' used in their paper. For this example, the dynamic extension technique causes order 2 in addition to the order 2 of the observer. Also, note that their result requires a globally defined model and global observability which are stronger than the regional conditions (C1) and (C2).

5.2.2 Realization of the Output Feedback Controller

Now an output feedback controller is explicitly constructed under (C1) and (C2). Before that, choose c_1 and c_2 such that $c_0 < c_1 < c_2 < c$. Then, $\Omega_{c_0} \subset \Omega_{c_1} \subset \Omega_{c_2} \subset \mathcal{U}_c$, in which compact sets Ω_{c_1} and Ω_{c_2} are defined similarly as (5.2.5). In what follows, some functions are made to be globally Lipschitz or globally bounded by modifying them outside the region Ω_{c_2} (see the Lipschitz or bounded extension of Section 3.3). Then, it will be shown that the trajectory of the plant (5.1.1), with an initial state in Ω_{c_0}, stays in Ω_{c_1} and converges to the origin. See Fig. 5.1.

The proposed output feedback controller is,

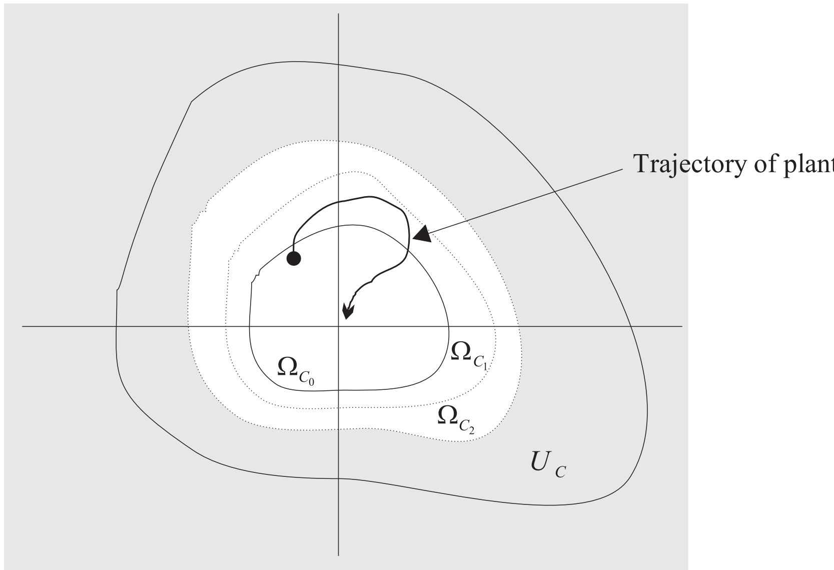


Figure 5.1: Diagram of the sets Ω_{c_0}, Ω_{c_1}, Ω_{c_2} and $\mathcal{U}_c$, and the concept of the controlled state trajectory of the plant

$$\dot{\hat{z}} = \hat{a}(\hat{z}) + \hat{b}(\hat{z}, u) - G^{-1}C^T(C\hat{z} - y)$$

$$= \begin{bmatrix} \hat{z}_2 \\ \hat{z}_3 \\ \vdots \\ \hat{z}_n \\ \hat{a}_n(\hat{z}) \end{bmatrix} + \begin{bmatrix} \hat{b}_1(\hat{z}_1, u) \\ \hat{b}_2(\hat{z}_1, \hat{z}_2, u) \\ \vdots \\ \hat{b}_{n-1}(\hat{z}_1, \cdots, \hat{z}_{n-1}, u) \\ \hat{b}_n(\hat{z}_1, \cdots, \hat{z}_n, u) \end{bmatrix} - G^{-1}C^T(C\hat{z} - y) \tag{5.2.6a}$$

$$u = \hat{\alpha}_z(\hat{z}) \tag{5.2.6b}$$

where

- $\hat{z}$ is the estimated state of $z(= \Phi(x))$.
- $\hat{\alpha}_z(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ is a global bounded extension of $\alpha_z(\cdot)$ from $\Phi(\Omega_{c_2})$ where $\alpha_z(z) := \alpha(\Phi^{-1}(z))$.

- $\hat{a}_n(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is a global Lipschitz extension of $a_n(\cdot)$ from $\Phi(\Omega_{c_2})$.
- $\hat{b}(\cdot, u) : \mathbb{R}^n \to \mathbb{R}^n$ is a global Lipschitz extension of $b(\cdot, u)$ from $\Phi(\Omega_{c_2})$ so that $\hat{b}$ preserves the structural state dependence of b, that is, $\hat{b}_i = \hat{b}_i(\hat{z}_1, \cdots, \hat{z}_i, u)$ for $1 \leq i \leq n$.
- G is the unique positive definite solution of

$$0 = -\theta G - A^T G - GA + C^T C \tag{5.2.7}$$

where, with appropriate dimensions,

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

and a positive constant θ is to be chosen in the next section.

Note that $\alpha_z(\cdot)$, $a_n(\cdot)$ and $b(\cdot, u)$ is well-defined on $\Phi(\mathcal{U}_c)$.

5.2.3 Proof of the Separation Principle (Thm. 5.2.1)

The proof proceeds in z-coordinates. By Lemma 3.2.3, the system dynamics (5.1.1) is equivalent to (3.2.8) on $\Phi(\mathcal{U}_c)$, which is re-written here for convenience;

$$\begin{aligned} \dot{z} &= a(z) + b(z, u) = \begin{bmatrix} z_2 \\ z_3 \\ \vdots \\ z_n \\ a_n(z) \end{bmatrix} + \begin{bmatrix} b_1(z_1, u) \\ b_2(z_1, z_2, u) \\ \vdots \\ b_{n-1}(z_1, \cdots, z_{n-1}, u) \\ b_n(z_1, \cdots, z_n, u) \end{bmatrix} \\ y &= z_1. \end{aligned} \tag{5.2.8}$$

However, outside the region $\Phi(\mathcal{U}_c)$, the model (5.2.8) is no longer valid since the transformation Φ is only guaranteed on $\mathcal{U}_c$ by (C2). This is one of the reasons that the extensions are necessary for constructing the observer (5.2.6a) in Section 5.2.2. Since the proposed observer is of high-gain type, the so-called peaking phenomenon [SK91] often occurs and thus the globally valid model is inevitable.

The proof is composed of two stages. The first stage has already been proved by Lemma 3.2.2. In the lemma, it is shown that the constructed observer (5.2.6a) in the previous section guarantees the convergence to zero of the error $\hat{z} - z$, with a sufficiently large value of θ. Moreover, it is also shown that the convergence is exponential and the rate of convergence can be made arbitrarily fast with θ. By utilizing them, the claim in Theorem 5.2.1 is proved from now on (the second stage).

Lemma 5.2.3. *Suppose that $z(t)$ of (5.2.8) is contained in $\Phi(\Omega_{c_1})$, for all $t \geq 0$, with $z(0) \in \Phi(\Omega_{c_0})$ and that $\hat{z}(0) \in \Phi(\Omega_{c_0})$. For any given $\tau > 0$ and $\epsilon > 0$, there exists $\theta_2^* > 0$ such that, for any $\theta > \theta_2^*$,*

$$\|b(z(t), \hat{\alpha}_z(\hat{z}(t))) - b(z(t), \alpha_z(z(t)))\| \leq \epsilon \exp(-\frac{\theta}{4}(t - \tau))$$

for all $t \geq \tau$.

Proof. Define $d_0 := \sup \|\hat{z} - z\|$ subject to $z \in \Phi(\Omega_{c_0})$ and $\hat{z} \in \Phi(\Omega_{c_0})$. Also, define $D := \inf \|\hat{z} - z\|$ subject to $z \in \Phi(\Omega_{c_1})$ and $\hat{z} \in \mathbb{R}^n - \Phi(\Omega_{c_2})$. Then, it can be shown that $D > 0$ (by the continuity of V).

Now, it follows from Lemma 3.2.2 that there exists a $\tilde{\theta}(\geq \theta_1^*)$ such that for any $\theta > \tilde{\theta}$,

$$\begin{aligned} \|\hat{z}(\tau) - z(\tau)\| &\leq K(\theta) \exp(-\frac{\theta}{4}\tau)\|\hat{z}(0) - z(0)\| \\ &\leq K(\theta) \exp(-\frac{\theta}{4}\tau) d_0 \\ &< D \end{aligned}$$

and thus, $\|\hat{z}(t) - z(t)\| < D$ for all $t \geq \tau$. This means that $\hat{z}(t) \in \Phi(\Omega_{c_2})$ for $t \geq \tau$. Therefore, the control $u = \hat{\alpha}_z(\hat{z}) = \alpha_z(\hat{z})$ after the time τ.

Since the function b is continuously differentiable in its second argument, and $\alpha_z(w)$ is Lipschitz from (C1).(c), there is a Lipschitz constant L of $b(z, \alpha_z(w))$ for w when $z \in \Phi(\Omega_{c_1})$ and $w \in \Phi(\Omega_{c_2})$. Choose a $\theta_2^*(\geq \tilde{\theta})$ such that $LK(\theta)\exp(-\frac{1}{4}\theta\tau)d_0 \leq \epsilon$ for any $\theta > \theta_2^*$. Then, for any $\theta > \theta_2^*$ and $t \geq \tau$,

$$
\begin{aligned}
\|b(z(t),&\hat{\alpha}_z(\hat{z}(t))) - b(z(t), \alpha_z(z(t)))\| \\
&= \|b(z(t), \alpha_z(\hat{z}(t))) - b(z(t), \alpha_z(z(t)))\| \\
&\leq L\|\hat{z}(t) - z(t)\| \\
&\leq LK(\theta)\exp(-\frac{\theta}{4}t)\|\hat{z}(0) - z(0)\| \\
&= LK(\theta)\exp(-\frac{\theta}{4}\tau)\|\hat{z}(0) - z(0)\|\exp(-\frac{\theta}{4}(t-\tau)) \\
&\leq \epsilon\exp(-\frac{\theta}{4}(t-\tau)).
\end{aligned}
$$

■

Now the following theorem proves Theorem 5.2.1.

Theorem 5.2.4. *Consider the closed-loop system* (5.2.8)*,* (5.2.6a) *and* (5.2.6b) *under Conditions* (C1) *and* (C2)*. There exists a positive constant θ^* such that, for any $\theta > \theta^*$ and for any initial states $z(0)$ and $\hat{z}(0)$ in $\Phi(\Omega_{c_0})$, the trajectories $z(t)$ and $\hat{z}(t)$ are bounded and converge to the origin. Moreover, the origin of the overall system is stable with such chosen θ.*

Proof. The closed-loop system with u as in (5.2.6b) can be written as

$$\dot{z} = a(z) + b(z, \hat{\alpha}_z(\hat{z})) \tag{5.2.9}$$

$$\dot{\hat{z}} = \hat{a}(\hat{z}) + \hat{b}(\hat{z}, \hat{\alpha}_z(\hat{z})) - G^{-1}C^T(C\hat{z} - Cz) \tag{5.2.10}$$

with $z(0) \in \Phi(\Omega_{c_0})$ and $\hat{z}(0) \in \Phi(\Omega_{c_0})$. Note that (5.2.9) is well-defined on $\Phi(\mathcal{U}_c)$ while (5.2.10) and $\hat{\alpha}_z$ are globally defined by the extensions. Letting $W(z) := V(\Phi^{-1}(z))$,

$$\dot{W}(z) = \frac{\partial W}{\partial z} F_z(z, \alpha_z(z)) + \frac{\partial W}{\partial z} \Delta_b(z, \hat{z}) \tag{5.2.11}$$

where $F_z(z, u) = a(z) + b(z, u)$ and $\Delta_b(z, \hat{z}) := b(z, \hat{\alpha}_z(\hat{z})) - b(z, \alpha_z(z))$.

For Δ_b, it can be seen that while z is contained in the compact set $\Phi(\Omega_{c_1})$, $\|\Delta_b(z, \hat{z})\|$ is bounded by a constant μ since the values of $\hat{\alpha}_z$ and z are bounded. Similarly, there is a constant $\delta > 0$ such that

$$\left| \frac{\partial W}{\partial z} F_z(z, \alpha_z(z)) + \frac{\partial W}{\partial z} v \right| \leq \delta, \qquad {}^{\forall} z \in \Phi(\Omega_{c_1}) \text{ and } {}^{\forall} \|v\| \leq \mu. \tag{5.2.12}$$

Now let $\tau = (c_1 - c_0)/(2\delta)$. From (5.2.11) and (5.2.12) it follows that, for every initial condition $z(0) \in \Phi(\Omega_{c_0})$,

$$W(z(t)) \leq c_1, \qquad \text{for } 0 \leq t \leq \tau,$$

since $\|\Delta_b\| \leq \mu$ during that time interval.

Next, from the fact that $(\partial W/\partial z) F_z(z, \alpha_z(z))$ is strictly negative for $c_0 \leq W(z) \leq c_1$, there is an $\epsilon > 0$ such that $(\partial W/\partial z)(F_z(z, \alpha_z(z)) + v) < 0$ whenever $c_0 \leq W(z) \leq c_1$ and $\|v\| \leq \epsilon$. By Lemma 5.2.3 with the τ and ϵ, there exists θ^* such that $\|\Delta_b(t)\| \leq \epsilon$ for $t \geq \tau$ for any $\theta > \theta^*$. Therefore, for any $z(0) \in \Phi(\Omega_{c_0})$, $\hat{z}(0) \in \Phi(\Omega_{c_0})$,

$$W(z(t)) \leq c_1, \qquad \text{for } t \geq 0 \tag{5.2.13}$$

since, for all $t \geq \tau$, $\dot{W} < 0$ on $\{z | c_0 \leq W(z) \leq c_1\}$. This fact, with the equation (3.2.11), shows the boundedness of the state $z(t)$ and $\hat{z}(t)$.

Now we show that $z(t) \to 0$ as $t \to \infty$. This argument is similar to the one used by [SK91, p.434]. Pick an $\epsilon_1 > 0$. Then there exists an $\epsilon_1' > 0$ such that $W(z) \leq \epsilon_1'$ implies $\|z\| \leq \epsilon_1$ by the positive definiteness of $W(z)$. By the continuity, there are positive constants δ_1 and μ_1 such that

$$\frac{\partial W}{\partial z} F_z(z, \alpha_z(z)) + \frac{\partial W}{\partial z}\, v \leq -\delta_1, \qquad {}^\forall z : \epsilon_1' \leq W(z) \leq c_1$$
$$ {}^\forall v : \|v\| \leq \mu_1.$$

Let $T \geq \tau$ be such that $\epsilon \exp(-\frac{1}{4}\theta^*(T-\tau)) \leq \mu_1$, and let T' be such that $\delta_1(T'-T) > c_1$. By (5.2.13), $W(z(t)) \leq c_1$ for $0 \leq t \leq T$. By Lemma 5.2.3, $\|\Delta_b(t)\| \leq \mu_1$ for $T \leq t < \infty$. Thus, $\dot{W}(z(t)) \leq -\delta_1$ as long as $W(z(t)) \geq \epsilon_1'$. It then follows that there is a $\hat{t}$ such that $T \leq \hat{t} \leq T'$ and $W(z(\hat{t})) \leq \epsilon_1'$. Now, it is clear that, if $W(z(\hat{t})) \leq \epsilon_1'$ for some $\hat{t}$ such that $\hat{t} \geq T$ then $W(z(t)) \leq \epsilon_1'$ for all larger t. This shows the convergence of $z(t)$ to the origin. Again, by (3.2.11), $\hat{z}(t)$ also converges to the origin.

Finally, it is shown that the origin of the closed-loop system, with the θ selected such that $\theta > \theta^*$, is stable. In fact, it is shown that for any given $\epsilon_2 > 0$, there exists $\delta_2 > 0$ such that

$$\|z(0)\| \leq \delta_2 \text{ and } \|\hat{z}(0)\| \leq \delta_2 \qquad \Rightarrow \qquad \|z(t)\| \leq \epsilon_2 \text{ and } \|\hat{z}(t)\| \leq \epsilon_2. \tag{5.2.14}$$

Without loss of generality, ϵ_2 is assumed to be small so that $\|z\| \leq \epsilon_2 \Rightarrow z \in \Phi(\Omega_{c_0})$.

For given ϵ_2, choose $\delta_2' > 0$ such that $W(z) \leq \delta_2' \Rightarrow \|z\| \leq \frac{1}{2}\epsilon_2$, and $\epsilon_2' > 0$ such that $\|z\| \leq \epsilon_2' \Rightarrow W(z) \leq \frac{1}{2}\delta_2'$. By the continuity again, there exists $\mu_2 > 0$ such that

$$\begin{aligned} \frac{\partial W}{\partial z}\, F_z(z, \alpha_z(z)) + \frac{\partial W}{\partial z}\, v < 0, \qquad & {}^\forall z : \frac{\delta_2'}{2} \leq W(z) \leq \delta_2' \\ & {}^\forall v : \|v\| \leq \mu_2. \end{aligned} \tag{5.2.15}$$

Now, choose δ_2 as

$$\delta_2 = \min\left\{\epsilon_2',\ \frac{\epsilon_2}{4K(\theta)},\ \frac{\mu_2}{2LK(\theta)}\right\} \tag{5.2.16}$$

where L is in the proof of Lemma 5.2.3 and $K(\theta)$ is of Lemma 3.2.2. Then, $\|z\| \leq \delta_2$ implies $z \in \Phi(\Omega_{c_0})$.

From now on, it is shown that (5.2.14) holds for all $t \geq 0$ with the δ_2 chosen as (5.2.16). Suppose that there exists a time T_1 such that $\hat{z}(T_1) \in \partial\Phi(\Omega_{c_1})$ and $\hat{z}(t) \in \Phi(\Omega_{c_1})$ for $0 \leq t < T_1$. Clearly, $T_1 > 0$ since $\hat{z}(t)$ is continuous with respect to t, and T_1 may be ∞. For the state $z(t)$, it is contained in $\Phi(\Omega_{c_1})$ for all $t \geq 0$ by the previous argument. Then, during the time interval $(0 \leq t < T_1)$,

$$\begin{aligned}
\|\Delta_b(t)\| &= \|b(z(t), \alpha_z(\hat{z}(t))) - b(z(t), \alpha_z(z(t)))\| \\
&\leq L\|\hat{z}(t) - z(t)\| \\
&\leq LK(\theta)\|\hat{z}(0) - z(0)\| \\
&\leq LK(\theta)(2\delta_2) \\
&\leq \mu_2,
\end{aligned}$$

which, with (5.2.15), implies that $z(t)$ is captured in the region $\{z : W(z) \leq \delta_2'\}$. Thus, for the interval,

$$\|z(t)\| \leq \frac{\epsilon_2}{2} \tag{5.2.17}$$

and

$$\begin{aligned}
\|\hat{z}(t)\| &\leq \|z(t)\| + K(\theta)\exp(-\frac{\theta}{4}t)\|\hat{z}(0) - z(0)\| \\
&\leq \frac{\epsilon_2}{2} + K(\theta)(2\delta_2) \\
&\leq \epsilon_2
\end{aligned} \tag{5.2.18}$$

by (5.2.16) and (5.2.17). However, since $\hat{z}(t) \in \Phi(\Omega_{c_0})$ for $0 \leq t < T_1$ by (5.2.18), the temporary assumption $\hat{z}(T_1) \in \partial\Phi(\Omega_{c_1})$ is impossible. Thus, T_1 should be ∞, that is, (5.2.17) and (5.2.18) hold for all $t \geq 0$. ■

Remark 5.2.2. The key in the proof is that we used the Lyapunov function $W(z)$ only for the plant dynamics. Thus, the controller dynamics is not monitored in the function $W(z)$, but considered as an external disturbance to the plant. Since the full dynamics is not captured in the Lyapunov function, we have separately proved the boundedness, attractivity and stability of the state z and $\hat{z}$. This approach prevents us to use the complex Lyapunov function like the one in [TP94] for semi-global analysis.

5.3 Illustrative Example: Magnetic Levitation System

In order to illustrate the proposed scheme, an output feedback stabilization problem is solved for a system

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\left(\frac{x_3+1}{x_1+1}\right)^2 + 1 \\
\dot{x}_3 &= \frac{x_2(x_3+1)}{x_1+1} - (x_1+1)x_3 + (x_1+1)u \\
y &= x_1
\end{aligned}$$

which describes the magnetic levitation system (Fig. 1.1) with all constants taken to be unity[1]. In this equation x_1 is the position of levitated body which is measurable quantity; x_2 represents the velocity of the body and x_3 is the current in the electromagnet. Note that, when $x_1 = -1$, the system description is not valid. (The physical meaning is that the body is attached to the magnet of upper deck.)

To get the asymptotically stabilizing state feedback control law, feedback linearizability of the system is utilized. By the following coordinate transformation ($z = \Phi(x)$),

$$\begin{aligned}
z_1 &= x_1 & \qquad x_1 &= z_1 \\
z_2 &= x_2 & \qquad x_2 &= z_2 \\
z_3 &= -\frac{(x_3+1)^2}{(x_1+1)^2} + 1 & \qquad x_3 &= \sqrt{1-z_3}(z_1+1) - 1
\end{aligned}$$

[1]The modeling of the system can be found in [JBSS94] or in Chapter 1.

the system is transformed to

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
\dot{z}_3 &= a(z) + b(z)u \\
y &= z_1
\end{aligned}$$

where

$$\begin{aligned}
a &= 2\frac{(x_3+1)x_3}{x_1+1} = 2\left((z_1+1)(1-z_3) - \sqrt{1-z_3}\right) \\
b &= -2\frac{x_3+1}{x_1+1} = -2\sqrt{1-z_3}.
\end{aligned}$$

The effective state feedback stabilizing control is

$$\alpha_z(z) = \frac{1}{b(z)}(-a(z) + Kz)$$

where K is chosen as $[-\frac{5}{4},\ -\frac{9}{4},\ -2]$ in order to place the closed-loop poles at $(-1, -\frac{1}{2}+i, -\frac{1}{2}-i)$.

Notice that the coordinate transformation is not globally effective. It loses the rank of $(\partial\Phi/\partial x)(x)$ at $x_3 = -1$ and it is not even defined at $x_1 = -1$. Thus, our obtained state feedback law is not globally stabilizing. In particular, the region of attraction for the state feedback should not include the points of the manifold $z_1 = -1$ or $z_3 = 1$ (i.e. $x_1 = -1$ or $x_3 = -1$).

Since the closed-loop system is linear in z-coordinates, a Lyapunov function $V(x)$ which satisfies (C1) is easily found from $W(z) = z^T Pz$ (i.e. $V(x) = W(\Phi(x))$) where P is the solution of $PA_c + A_c^T P = -I$ in which

$$A_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{5}{4} & -\frac{9}{4} & -2 \end{bmatrix}.$$

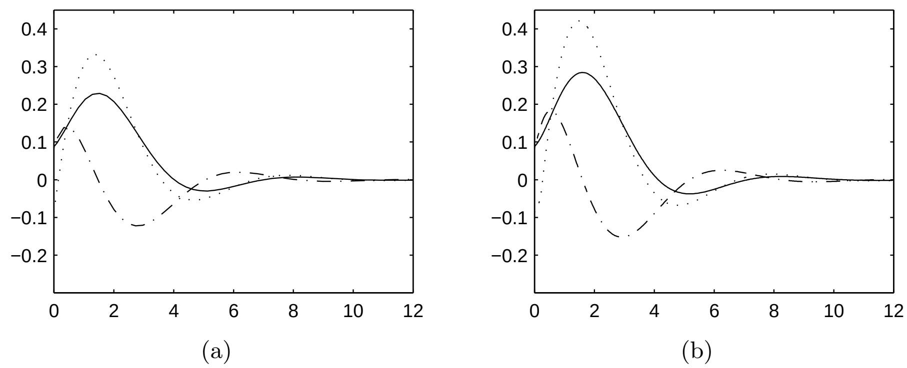


Figure 5.2: Comparison of trajectories between (a) state feedback and (b) output feedback, (x_1:solid, x_2:dotted, x_3:dashdot).

By some calculations using MATLAB[2], the region $\mathcal{U}_c$ with $c = 0.43$ in (5.2.1) is a region of attraction which is disjoint from $x_1 = -1$ and $x_3 = -1$. Take $\mathcal{U}^*$ as

$$\mathcal{U}^* = \{x \in \mathbb{R}^3 : -0.61 < x_1 < 0.61,\ -0.61 < x_2 < 0.61,\ -0.99 < x_3 < 1.28\}.$$

Then, it can be verified that the convex set $\mathcal{U}^*$ contains $\mathcal{U}_c$.

Note that, in this example, the linearizing transformation Φ and the transformation in (5.2.3) is actually identical. With the map Φ and the set $\mathcal{U}^*$, Condition (C2*) is satisfied.

Now, suppose that c_0 and c_2 are chosen as 0.2 and 0.3 respectively. Then, by setting $\rho_1 = 0.51$, $\rho_2 = 0.51$ and $\rho_3 = 0.84$, it can be checked that $\Phi(\Omega_{c_2}) \subset Z \subset \Phi(\mathcal{U}^*)$ where Z is defined as (3.3.6). Therefore, by the continuous Lipschitz extension I presented in Section 3.3, the proposed output feedback controller is

[2]MATLAB is a registered trademark of The Math Works, Inc.

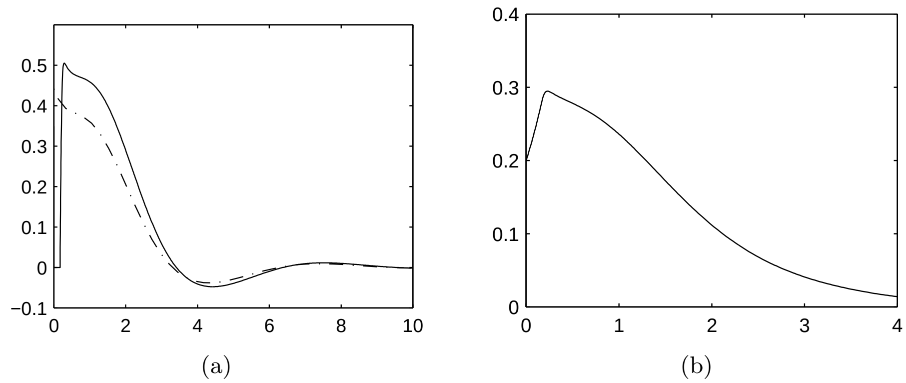


Figure 5.3: (a) Comparison of controls between state feedback (dashdot) and output feedback (solid). (b) Plot of the value $V(x(t))$.

obtained as

$$\begin{bmatrix} \dot{\hat{z}}_1 \\ \dot{\hat{z}}_2 \\ \dot{\hat{z}}_3 \end{bmatrix} = \begin{bmatrix} \hat{z}_2 \\ \hat{z}_3 \\ \hat{a}(\hat{z}) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \hat{b}(\hat{z})u \end{bmatrix} - G^{-1} \begin{bmatrix} \hat{z}_1 - y \\ 0 \\ 0 \end{bmatrix} \tag{5.3.1}$$

$$u = \hat{\alpha}_z = \alpha_z(\chi_1(\hat{z}_1), \chi_2(\hat{z}_2), \chi_3(\hat{z}_3)) \tag{5.3.2}$$

where G satisfies (5.2.7) and

$$\hat{a}(\hat{z}) = a(\chi_1(\hat{z}_1), \chi_2(\hat{z}_2), \chi_3(\hat{z}_3))$$
$$\hat{b}(\hat{z}) = b(\chi_1(\hat{z}_1), \chi_2(\hat{z}_2), \chi_3(\hat{z}_3)).$$

Finally, we should choose an appropriate value of θ. However, calculation of θ through the arguments of Theorem 5.2.4 is impractical. Instead, we've found it by repeated simulations, which is not a heavy burden since Theorem 5.2.4 guarantees the existence of the desired θ and the search for θ is just one parameter tuning. In our trial, $\theta = 300$ is sufficient and any trajectories with initial conditions $x(0) \in \Omega_{c_0}$ and $\hat{z}(0) \in \Phi(\Omega_{c_0})$ converge to the origin.

For this example, another bounded extension of α_z is possible instead of (5.3.2).

Let $c_2' = 0.35$ and the extension be

$$\hat{\alpha}_z(\hat{z}) = \begin{cases} \alpha_z(\hat{z}) & \text{if } W(\hat{z}) \le c_2 \\ \frac{c_2' - W(\hat{z})}{c_2' - c_2}\alpha_z(\hat{z}) & \text{if } c_2 < W(\hat{z}) \le c_2' \\ 0 & \text{if } c_2' < W(\hat{z}) \end{cases} \tag{5.3.3}$$

For this extension, repeated simulations show that the value $\theta = 50$ guarantees the convergence. For example, let $x(0) = [0.088,\ 0.084,\ -0.086]$ and $\hat{z}(0) = 0$, which gives $V(x(0)) = 0.2$. With these initials, Fig. 5.2 compares the trajectories of the plant between the state feedback and the proposed output feedback. Comparison of the control is also depicted in Fig. 5.3.(a). Due to the extension of (5.3.3), the output feedback control becomes zero in the initial short period, by which the value of $V(x(t))$ does not exceed the upper bound 0.3. Indeed, this point is shown in Fig. 5.3.(b), where $V(x(t))$ starts at 0.2 and peaks during the initial period because of the initial mismatch between the true state and the estimated state.

5.4 Notes on the Chapter

In this chapter, for a multi-input single-output, input non-affine nonlinear system, an output feedback stabilization problem is solved when a stabilizing state feedback controller is given with a bounded (arbitrarily large) region of attraction. The key to the regional output feedback stabilization is the existence of state observer, whose convergence is exponential and the rate of convergence is assignable. Therefore, similar result can be obtained for multi-output systems, only when there is a general method constructing an observer satisfying (3.2.11) and (3.2.12).

We have utilized the Lipschitz extension. The technique gives not only a globally defined model, but also some additional properties such as global boundedness or global Lipschitz property. These properties enable the construction of exponentially converging observer (Gauthier's observer). In addition, the bounded extension of the control forces the control not to peak to high value which may be induced in the initial period by the wrong estimates of the observer, and thus prevents the possible finite escape time phenomenon for the closed-loop system. The phenomenon is known as the main cause for the frustrating fact that the global separation principle for nonlinear systems does not hold in general, which was pointed out by [MPD94]. The idea of bounded extension is motivated by [EK92] which is also used in [TP94, TP95].

The proposed scheme satisfies the properties (P1), (P2) and (P3) of Section 5.1, which are inherited from the linear output feedback stabilization. On the other hands, (P4) is not exactly satisfied, since there should be a procedure to select appropriate θ generally depending on the chosen state feedback law $\alpha(x)$. This is because that the observer should be sufficiently faster than the plant dynamics, which is unnecessary for linear output feedback. However, remembering that, for good performance, convergence of observer should be faster than that of the plant even for linear systems, it can be thought as a reasonable drawback.

Chapter 6

Conclusions

This chapter briefly summarizes the whole contents of this dissertation. The concluding remarks, technical remarks and notes in detail can be found in the section of 'Notes on the Chapter' at the end of each chapter.

In this dissertation we have presented several methodologies for the construction of nonlinear observer that produce asymptotic convergence of the estimate to the true state, in pursuit of the generalization to the earlier works in the literature. In particular, we have enlarged the class of systems for which the state observer can be designed, which is illustrated in Section 3.4.4 and 4.7 with several examples.

The observer constructions in Chapter 3 are basically originated from the lower triangular structure of Gauthier's approach, in the conviction that the structure includes fairly general classes of nonlinear systems (e.g., a system which is linear up to output injection (linearized error dynamics approach), a system which has the relative degree n, or even a system which does not have the well-defined relative degree, etc.). Moreover, the structure is directly related to the notion of observability (uniform observability), which gives the theoretical elegance.

The passivity framework in Chapter 4 is a new viewpoint for the state observer. In the framework, we've proposed the precise definition of Passivity-based State

Observer and its design method. This viewpoint is especially useful for nonlinear systems and it explains the prior nonlinear observers in a unified way and extends them. In addition, it has been shown that the passivity-based state observer has a good nature—the robustness to the measurement disturbances.

The separation principle in Chapter 5 also pursues the theoretical elegance. Eventually, the claim is proved that if a nonlinear system is stabilizable and uniformly observable in a certain bounded region, then there always exists an output feedback stabilizing controller for the system—the generalization of linear separation principle except the *boundedness* of underlying region.

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