# Addendum for
# "A Study of Disturbance Observers with Unknown Relative Degree of the Plant"

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**Abstract**

The paper "A Study of Disturbance Observers with Unknown Relative Degree of the Plant" [1] by the authors could not include the proofs for Theorem 5 and Theorem 6 due to the page limit. We provide them in this supplementary document, and an example is included with simulation results.

**Proof of Theorem 5:** We now consider the case where $m_\beta - m_\alpha < 0$. From the Newton diagram of this case (Fig. 1), it is seen that there are two groups of vanishing roots of $\bar{\delta}(s;\tau)$. The first group consists of $k$ roots of the form $s^*(\tau) = \gamma_a \tau^1 + o(\tau^1)$ where $\gamma_a$ is the roots of $\phi_a(\gamma) = b_k + \cdots + b_{\bar{k}+1}\gamma^{k-\bar{k}-1} + a_{\bar{k}}\gamma^{k-\bar{k}} + \cdots + a_0\gamma^k = b_k + \cdots + b_0\gamma^k = \gamma^k N_Q(1/\gamma;1)$. The condition (ii) guarantees that $\phi_a$ is Hurwitz. On the other hand, it is seen from Fig. 1 that the second group has the roots of the form $s^*(\tau) = \gamma_b \tau^{(l-k)/(l+m_\beta-m_\alpha-k)}$, with $\gamma_b$ being the roots of $\phi_b(\gamma) = \beta_{m_\beta} + b_k\alpha_{m_\alpha}\gamma^{l+m_\beta-m_\alpha-k}$. Note that $1 \le l+m_\beta-m_\alpha-k = \text{r.deg}(Q)+\text{r.deg}(P)-\text{r.deg}(P_{\mathsf{n}}) \le 2$ by (i) and by $\text{r.deg}(Q) \ge \text{r.deg}(P_{\mathsf{n}})$. If its value is 1, then (iii) guarantees that $\phi_b$ is Hurwitz (of first order). If its value is 2 (so that $l-k>2$), then two roots of the second group are $s^*(\tau) = (\pm i\bar{\gamma} + \hat{s}(\tau))\tau^{(l-k)/2}$ where $\bar{\gamma} = \sqrt{\beta_{m_\beta}/(b_k\alpha_{m_\alpha})}$ and $\hat{s}$ is a continuous function to be found such that $\hat{s}(0)=0$. Let $\hat{\tau} = \tau^{1/2}$ and $A(\hat{\tau}) = i\bar{\gamma} + \hat{s}(\hat{\tau}^2)$ so that $s^*(\hat{\tau}^2) = A\hat{\tau}^{l-k}$. From (4) of [1], it is seen that the power of $\hat{\tau}$ in each $\hat{\tau}^{2j}\bar{q}_j(A\hat{\tau}^{l-k})$ begins with $(l-k)(l+m_\beta-m_\alpha-j)+2j$ if $0 \le j \le k$, and with $(l-k)(l-j)+2j$ if $k+1 \le j \le l$, and increases by $(l-k)$ in both cases. Since $l-k>2$, the term of the second lowest power in the polynomial $\bar{\delta}(A\hat{\tau}^{l-k},\hat{\tau}^2)$ comes from the lowest power term of $\hat{\tau}^{2(k-1)}\bar{q}_{k-1}(A\hat{\tau}^{l-k})$ and of $\hat{\tau}^{2(l-1)}\bar{q}_{l-1}(A\hat{\tau}^{l-k})$ but not from others. Writing $\bar{\delta}$ in ascending power of $\hat{\tau}$, we have

$$
\begin{aligned}
&\bar{\delta}(s^*;\hat{\tau}^2) = [b_k\alpha_{m_\alpha}A^2 + \beta_{m_\beta}]\hat{\tau}^{2l} \\
&\quad + [b_{k-1}\alpha_{m_\alpha}A^3 + a_{l-1}\beta_{m_\beta}A]\hat{\tau}^{2l-m_\beta+m_\alpha} + \cdots = \\
&[2ib_k\alpha_{m_\alpha}\bar{\gamma}\hat{s} + (\cdots)\hat{s}^2]\hat{\tau}^{2l} + [i\bar{\gamma}(a_{l-1}\beta_{m_\beta} - b_{k-1}\alpha_{m_\alpha}\bar{\gamma}^2) \\
&\quad + (\cdots)\hat{s} + (\cdots)\hat{s}^2 + (\cdots)\hat{s}^3]\hat{\tau}^{2l-m_\beta+m_\alpha} + \cdots \\
&=: \hat{\delta}(\hat{s};\hat{\tau}).
\end{aligned}
$$

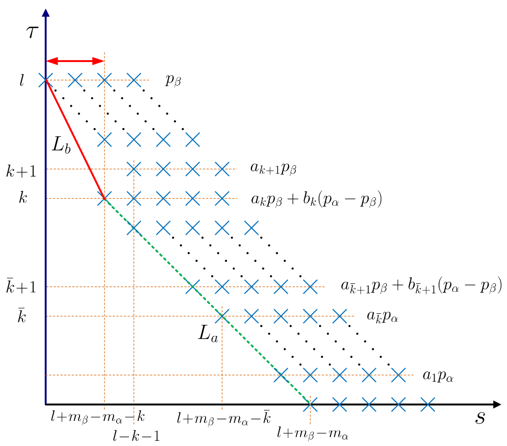


Fig. 1. Newton diagram for the case $\text{r.deg}(P) < \text{r.deg}(P_{\mathsf{n}})$.

The corresponding Newton diagram (Fig. 2) suggests that it has one root $\hat{s}^*(\hat{\tau})$ of the form $\hat{\gamma}\hat{\tau}^c + o(\hat{\tau}^c)$ where $c = m_\alpha - m_\beta$ and $\hat{\gamma}$ is the root of

$$
\begin{aligned}
\hat{\phi}(\hat{\gamma}) &= 2b_k\alpha_{m_\alpha}\hat{\gamma} + (a_{l-1}\beta_{m_\beta} - b_{k-1}\alpha_{m_\alpha}\bar{\gamma}^2) \\
&= 2b_k\alpha_{m_\alpha}\left[\hat{\gamma} + \frac{\beta_{m_\beta}}{2b_k^2\alpha_{m_\alpha}}(a_{l-1}b_k - b_{k-1})\right].
\end{aligned}
$$

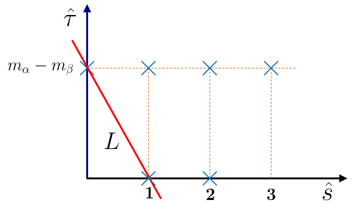


Fig. 2. Newton diagram for $\hat{\delta}(\hat{s};\hat{\tau})/\hat{\tau}^{2l}$ under the condition $\text{r.deg}(P) < \text{r.deg}(P_{\mathsf{n}})$.

The condition (iv) implies that $\hat{\phi}$ is Hurwitz. The result is the same with $A(\hat{\tau}) = -i\bar{\gamma} + \hat{s}(\hat{\tau}^2)$.

**Proof of Theorem 6:** Conclusions of Theorem 6 are easily derived from the proof of Theorem 5. Indeed, the condition (d) implies that $\phi_a(\gamma)$ is not Hurwitz. On the other hand, if the condition (a) holds, then $\phi_b(\gamma)$ is not Hurwitz because $l + m_\beta - m_\alpha - k \geq 3$. Regarding the condition (e), it implies that $\phi_b(\gamma)$ has at least one RHP root. Finally, suppose that $\beta_{m_\beta}/\alpha_{m_\alpha} > 0$ while $\text{r.deg}(Q) = \text{r.deg}(P_{\mathsf{n}}) - \text{r.deg}(P) + 2$. Then, $s^*(\tau)$ is of the form $s^*(\tau) = \hat{\gamma}\tau^{m_\alpha - m_\beta} + o(\tau^{m_\alpha - m_\beta})$. But, $\hat{\gamma}$ is positive because of the condition (f).

**An Illustrative Example:** A numerical example is given to illustrate the method presented in Section 3.1. Let $\text{h.gain}(P)$ denote the high-frequency gain of $P(s)$ and define sets of transfer functions (having finite coefficients and of minimum phase)

$$\begin{aligned}
\mathcal{P}_{12} &= \{P(s) |\ 1 \leq \text{r.deg}(P) \leq 2,\ 0.1 \leq \text{h.gain} \leq 8\} \\
\mathcal{P}_3 &= \{P(s) |\ \text{r.deg}(P) = 3,\ 0.1 \leq \text{h.gain} \leq 8\} \\
\mathcal{P}_4 &= \{P(s) |\ \text{r.deg}(P) = 4,\ 0.1 \leq \text{h.gain} \leq 8,\ \mu(P) \geq 8\}.
\end{aligned}$$

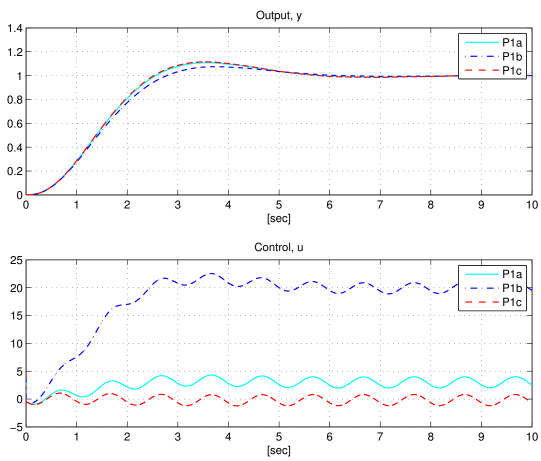


Fig. 3. Simulation results for $P_{1,a}, P_{1,b}, P_{1,c}$ (plants having relative degree 1) in the presence of disturbance $d(t) = \sin(2\pi t)$.

It is assumed that the primary control goal is to achieve zero steady-state error (to step response) with overshoot less than 15% and settling time less than 6 seconds. We will show that, for any plant $P(s) \in \mathcal{P} := \mathcal{P}_{12} \cup \mathcal{P}_3 \cup \mathcal{P}_4$, a robust controller can be designed in order to achieve the control goal. As discussed in Section 3.1, we first choose

$$\begin{aligned}
P_{\mathsf{n}}(s) &= \frac{1}{s(s+2)(s+3)}, \\
Q(s) &= \frac{1}{(\tau s)^3 + 3(\tau s)^2 + 3(\tau s) + 1},
\end{aligned} \tag{1}$$

which guarantees that, for $P \in \mathcal{P}_4$,

$$\mu(P) - \mu(P_{\mathsf{n}}) \geq 8 - 5 > \frac{8}{3} = \frac{\bar{K}_{\mathsf{p}}}{K_{\mathsf{n}}} \frac{a_0 a_2}{a_1^2}. \tag{2}$$

Next, select

$$C(s) = 5 \tag{3}$$

so that the unity feedback control system composed of $P_{\mathsf{n}}(s)$ and $C(s)$ achieves the primary control goal. Then, according to Theorems 3 and 5, the DOB control system (with small $\tau$) will be stable for any $P \in \mathcal{P}$.

To verify the stability as well as the performance, the computer simulations are carried out using the DOB controller with (1), (3), and $\tau = 0.01$. In addition, the disturbance and the reference inputs are chosen as $d(t) = \sin(2\pi t)$ and $r(t) = 1$. For simulation purpose, we consider the following plants of variation:

$$\begin{aligned}
&P_{1,a} = \frac{2}{s+6},\ P_{1,b} = \frac{0.2}{s+4},\ P_{1,c} = \frac{5}{s-1}, \\
&P_{2,a} = \frac{2}{(s+2)(s+4)},\ P_{2,b} = \frac{0.2}{(s+1)(s+3)}, \\
&P_{2,c} = \frac{5}{(s+2)(s-1)}, \\
&P_{3,a} = \frac{1}{s}P_{2,a},\ P_{3,b} = \frac{1}{s}P_{2,b},\ P_{3,c} = \frac{1}{s}P_{2,c}, \\
&P_{4,a} = \frac{1}{s}P_{3,a},\ P_{4,b} = \frac{1}{s}P_{3,b},\ P_{4,c} = \frac{s+1}{s(s+8)}P_{3,c}.
\end{aligned}$$

It is observed that (a) all the plants except $P_{4,b}$ belong to $\mathcal{P}$, (b) all the plants have different high-frequency gains from $P_{\mathsf{n}}(s)$, (c) (2) is satisfied by $P_{1,a}, P_{2,a}, P_{3,a}, P_{4,a}$ and $P_{4,c}$ but not by the others, and (d) $P_{1,c}$, $P_{2,c}$ and $P_{3,c}$ are unstable.

Fig. 3 and Fig. 4 show the simulation results for $P_{1,a}$, $P_{1,b}$, $P_{1,c}$ and $P_{2,a}$, $P_{2,b}$, $P_{2,c}$, respectively. Although

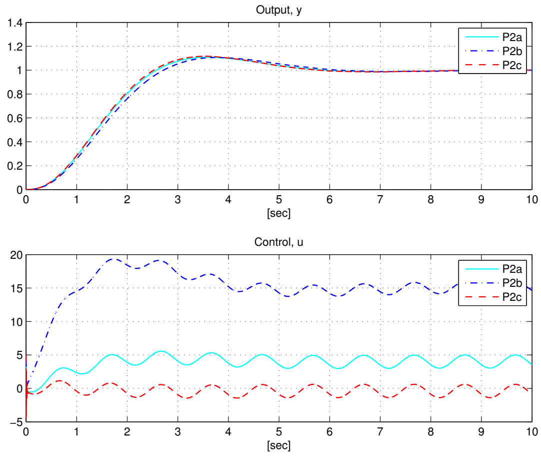


Fig. 4. Simulation results for $P_{2,a}, P_{2,b}, P_{2,c}$ (plants having relative degree 2) in the presence of disturbance $d(t) = \sin(2\pi t)$.

there is the disturbance signal $d(t)$, it seems that plant outputs are not affected by $d(t)$. In addition, it is seen that the performance of each plant can be recovered to that of nominal one so that the primary control goal is achieved for any plant belonging to $\mathcal{P}_{12}$. The simulation results for $P_{3,a}$, $P_{3,b}$, $P_{3,c}$ are depicted in Fig. 5. It is also seen that the recovery of the nominal closed-loop system performance is achieved. From Figs. 3–5, it is verified that, when $\text{r.deg}(P) = \text{r.deg}(P_{\mathsf{n}})$ or $1 \leq \text{r.deg}(P) \leq 2$, the control system can be stabilized regardless of whether or not the condition (2) (i.e., (iii) of Theorem 3) is satisfied.

Fig. 6 shows the simulation results for $P_{4,a}, P_{4,b}, P_{4,c}$. It is seen that $P_{4,a}$ and $P_{4,c}$ can by stabilized by the DOB controller and the nominal performance is recovered. On the other hand, the instability occurs for $P_{4,b} \notin \mathcal{P}$, which indicates that the condition (2) is very critical when

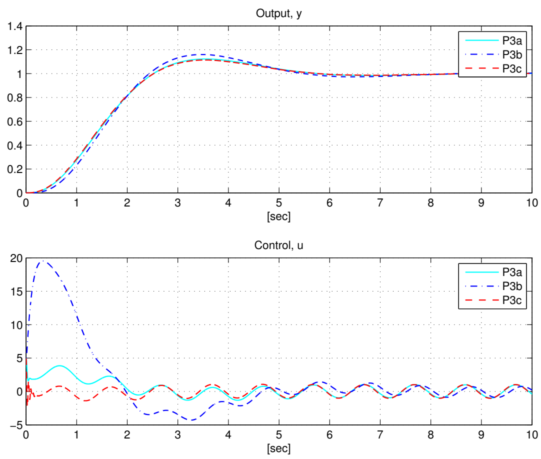


Fig. 5. Simulation results for $P_{3,a}, P_{3,b}, P_{3,c}$ (plants having relative degree 3) in the presence of disturbance $d(t) = \sin(2\pi t)$.

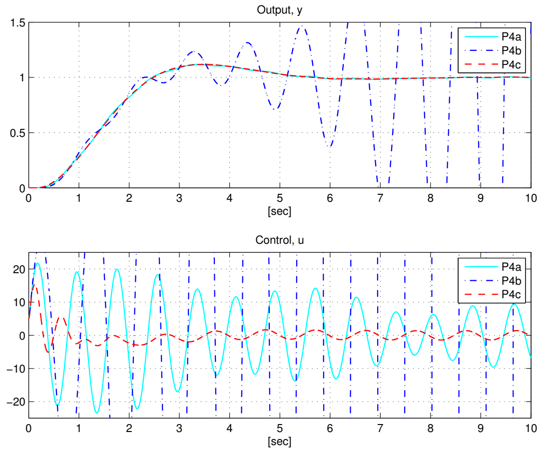


Fig. 6. Simulation results for $P_{4,a}, P_{4,b}, P_{4,c}$ (plants having relative degree 4) in the presence of disturbance $d(t) = \sin(2\pi t)$.

$\text{r.deg}(P) = \text{r.deg}(P_{\mathsf{n}}) + 1$.

Finally, it should be remarked that, with the help of the DOB controller, plant output of any $P(s) \in \mathcal{P}$ is almost indistinguishable from that of nominal plant in the absence of disturbance input.

## References

[1] N.H. Jo, Y. Joo, and H. Shim, "A Study of Disturbance Observers with Unknown Relative Degree of the Plant," to appear at *Automatica*.