Linear Dynamic Systems and Control

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$$
\begin{aligned}
\dot{x}(t) &= A(t)x(t) + B(t)u(t) \\
y(t) &= C(t)x(t) + D(t)u(t),
\end{aligned}
$$

수업 내용

선형 제어 시스템을 학습하기 위한 기초 및 기본 개념을 제공하는 강좌로, system model에 대한 기본 개념을 상태변수 공간에서 학습하고, 선형대수학의 기초를 다시 살펴 본 후, 상태공간 방정식의 해, 시스템의 안정도 (Lyapunov stability), 제어 가능한 시스템 (controllability), 관측 가능한 시스템 (observability), controllable and observable canonidal form, duality, 상태변수 궤환 제어기 설계, 상태변수 관측기 설계, 및 출력 궤환 제어 방법을 학습한다.

Textbook:
Lecture note written by Prof. Jin Heon Seo

References:

W. Brogan, *Modern Control Theory*, 3rd Edition, Prentice-Hall, 1990.

C.T. Chen, *Linear System Theory and Design*, 4th Edition, Oxford publishing, 2012.

Definition (Causality)

A system is called *causal* (or *nonanticipative*) if, for any t, $y(t)$ does not depend on any $u(t_1)$ for $t_1 > t$. A system is called *noncausal* (or *anticipative*) if it is not causal.

Definition (Dynamic Systems)
A system is called *dynamic* (or is said to *have memory*) if, for some t_0, $y(t_0)$ depends on $u(t)$ for $t \neq t_0$. A system is called *instantaneous* (*static*, *memoryless* or is said to *have zero memory*) if, for any t_0, $y(t_0)$ does not depend on $u(t)$ for $t \neq t_0$.

A system is said to be *relaxed (or* at rest) at time t_0 if the output $y_{[t_0,\infty)}$ is solely excited and uniquely determined by the input $u_{[t_0,\infty)}$. We assume that every system is relaxed at time $t = -\infty$.

Definition (State)

The state $x(t)$ of a system at time t is the information at time t that is sufficient to uniquely specify the output $y_{[t,\infty)}$ given the input $u_{[t,\infty)}$.

Definition (Linear State Space Systems)

A system is said to be *linear* if for every t_0 and any admissible two input-state-output pairs

$$\{u^1_{[t_0,\infty)}, x^1(t_0)\} \longrightarrow y^1_{[t_0,\infty)}$$
$$\{u^2_{[t_0,\infty)}, x^2(t_0)\} \longrightarrow y^2_{[t_0,\infty)}$$

and real (complex) scalar α, the property of additivity

$$\{u^1_{[t_0,\infty)} + u^2_{[t_0,\infty)}, x^1(t_0) + x^2(t_0)\} \longrightarrow y^1_{[t_0,\infty)} + y^2_{[t_0,\infty)}$$

and the property of homogeneity

$$\{\alpha u^1_{[t_0,\infty)}, \alpha x^1(t_0)\} \longrightarrow \alpha y^1_{[t_0,\infty)}$$

hold. Otherwise, the system is said to be nonlinear.

Definition (Time Invariance)

A system is said to be *time-invariant* if for every t_0 and every admissible input-state-output pair on the interval $[t_0, \infty)$

$$\{u^1_{[t_0,\infty)}, x^1(t_0) = x^0\} \longrightarrow y^1_{[t_0,\infty)}$$

and any T, the input-state-output pair on the interval $[t_0 + T, \infty)$

$$\{u^2_{[t_0+T,\infty)}, x^2(t_0 + T) = x^0\} \longrightarrow y^2_{[t_0+T,\infty)}$$

where

$$u^2(t) = u^1(t - T), \quad y^2(t) = y^1(t - T), \;\; t \geq t_0 + T$$

is also admissible.

Definition (Fields)

A *field* consists of a set, denoted by $\mathcal{F}$, of elements called scalars and two operations called addition "$+$" and multiplication "$\cdot$" which are defined over $\mathcal{F}$ such that they satisfy the following conditions:

1. To every pair of elements $\alpha, \beta \in \mathcal{F}$, there is an associated unique element $\alpha + \beta$ in $\mathcal{F}$, called the sum of α and β.
2. There exists an element, denoted by 0, such that $\alpha + 0 = \alpha$ for all $\alpha \in \mathcal{F}$.
3. Addition is commutative; $\alpha + \beta = \beta + \alpha$.
4. Addition is associative; $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.

5. For each $\alpha \in \mathcal{F}$, there exists an element β such that $\alpha + \beta = 0$, which is called the additive inverse and denoted by $-\alpha$.
6. To every pair of elements $\alpha, \beta \in \mathcal{F}$, there is associated a unique element $\alpha \cdot \beta$ in $\mathcal{F}$, called the product of α and β.
7. There exists an element, denoted by 1, such that $\alpha \cdot 1 = \alpha$ for all $\alpha \in \mathcal{F}$.
8. Multiplication is commutative; $\alpha \cdot \beta = \beta \cdot \alpha$.
9. Multiplication is associative;
$(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$.
10. For each $\alpha \in \mathcal{F}, \alpha \neq 0$, there exists an element γ such that $\alpha \cdot \gamma = 1$, which is called the multiplicative inverse and denoted by α^{-1}.

11. Multiplication is distributive with respect to addition; $(\alpha + \beta) \cdot \gamma = \alpha \cdot \beta + \alpha \cdot \gamma$.

Definition (Vector Spaces)

A *vector space* over a field $\mathcal{F}$, denoted by $(\mathcal{X}, \mathcal{F})$, consists of a set, denoted by $\mathcal{X}$, of elements called vectors, a field $\mathcal{F}$, and two operations called vector addition "$+$" and scalar multiplication "$\cdot$", which are defined over $\mathcal{X}$ and $\mathcal{F}$ such that they satisfy the following conditions:

1. To every pair of elements $x_1, x_2 \in \mathcal{X}$, there is an associated unique element $x_1 + x_2$ in $\mathcal{X}$, called the sum of x_1 and x_2.
2. Addition is commutative; $x_1 + x_2 = x_2 + x_1$.
3. Addition is associative; $(x_1 + x_2) + x_3 = x_1 + (x_2 + x_3)$.
4. There exists a vector, denoted by 0, such that $x + 0 = x$ for all $x \in \mathcal{X}$.

5. For each $x \in \mathcal{X}$, there exists an element $y \in \mathcal{X}$ such that $x + y = 0$, which is denoted by $-x$.
6. To every scalar $\alpha \in \mathcal{F}$ and vector $x \in \mathcal{X}$, there is associated a unique element $\alpha \cdot x \in \mathcal{X}$, called the scalar product of α and x.
7. Scalar multiplication is associative; $(\alpha\beta) \cdot x = \alpha \cdot (\beta \cdot x)$.
8. Scalar multiplication is distributive with respect to vector addition; $\alpha \cdot (x_1 + x_2) = \alpha \cdot x_1 + \alpha \cdot x_2$.
9. Scalar multiplication is distributive with respect to scalar addition; $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$.
10. For $1 \in \mathcal{F}$ and $x \in \mathcal{X}$, $1 \cdot x = x$.

Definition (Subspaces)
Let $(\mathcal{X}, \mathcal{F})$ be a vector space and $\mathcal{Y} \subset \mathcal{X}$. Then, $(\mathcal{Y}, \mathcal{F})$ is said to be a *subspace* of $(\mathcal{X}, \mathcal{F})$ if under the operations of $(\mathcal{X}, \mathcal{F})$, $\mathcal{Y}$ forms a vector space over $\mathcal{F}$, that is, for each pair of elements $x_1, x_2 \in \mathcal{Y}$, $x_1 + x_2 \in \mathcal{Y}$, and for each scalar $\alpha \in \mathcal{F}$ and vector $x \in \mathcal{Y}$, $\alpha x \in \mathcal{Y}$.

Definition (Sum of Subspaces)
Let $(\mathcal{X}, \mathcal{F})$ be a vector space and $\mathcal{Y}, \mathcal{Z} \subset \mathcal{X}$ be subspaces of $\mathcal{X}$. Then, the *sum* of two subspaces $\mathcal{Y}, \mathcal{Z} \subset \mathcal{X}$ is the set

$$\mathcal{Y} + \mathcal{Z} = \{x : x = y + z, y \in \mathcal{Y}, z \in \mathcal{Z}\}.$$

Definition (Intersection of Subspaces)
Let $(\mathcal{X}, \mathcal{F})$ be a vector space and $\mathcal{Y}, \mathcal{Z} \subset \mathcal{X}$ be subspaces of $\mathcal{X}$. Then, the *intersection* of two subspaces $\mathcal{Y}, \mathcal{Z} \subset \mathcal{X}$ is the set

$$\mathcal{Y} \cap \mathcal{Z} = \{x : x \in \mathcal{Y}, x \in \mathcal{Z}\}.$$

Definition (Direct Sum of Subspaces)
Let $(\mathcal{X}, \mathcal{F})$ be a vector space and $\mathcal{Y}, \mathcal{Z} \subset \mathcal{X}$ be subspaces of $\mathcal{X}$ such that

$$\mathcal{Y} + \mathcal{Z} = \mathcal{X}$$
$$\mathcal{Y} \cap \mathcal{Z} = \{0\}$$

In this case, $\mathcal{X}$ is said to be a *direct sum* of $\mathcal{Y}$ and $\mathcal{Z}$, and we write $\mathcal{X} = \mathcal{Y} \oplus \mathcal{Z}$.

Definition (Linear Combinations)
Let A be a set in a vector space $(\mathcal{X}, \mathcal{F})$. A vector $x \in \mathcal{X}$ is said to be a *linear combination* of elements in A if there exist a finite set of vectors $\{x_1, x_2, \ldots, x_n\}$ in A and a finite set of scalars $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ such that

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n.$$

Definition (Span)
Let A be a nonempty subset in a vector space $(\mathcal{X}, \mathcal{F})$. The *span* of A, denoted by $\mathrm{span}(A)$, is the set consisting of all linear combinations of elements in A. For convenience, $\mathrm{span}(\emptyset) = \{0\}$.

Definition (Linear Independence)

A nonempty set A in a vector space $(\mathcal{X}, \mathcal{F})$ is said to be *linearly dependent* if there exists a finite set of distinct elements $\{x_1, x_2, \dots, x_n\}$ in A and a finite set of scalars $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$, not all zero, such that

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0.$$

A set A in a vector space $(\mathcal{X}, \mathcal{F})$ is said to be *linearly independent* if it is not linearly dependent. In other words, a set A in a vector space $(\mathcal{X}, \mathcal{F})$ is linearly independent if for each nonempty finite subset of distinct elements in A, say, $\{x_1, x_2, \dots, x_n\}$ in A, the only n-tuple of scalars satisfying the equation

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0$$

is the trivial solution $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

Definition (Basis)

A set B of linearly independent vectors in a vector space $(\mathcal{X}, \mathcal{F})$ is said to be a *basis* of $\mathcal{X}$ if every vector in $\mathcal{X}$ can be expressed as a linear combination of vectors in B, that is, $\mathrm{span}(B) = \mathcal{X}$.

Theorem

Let $(\mathcal{X}, \mathcal{F})$ be a vector space and $V = \{v_1, v_2, \ldots, v_n\}$ be a subset of $\mathcal{X}$. Then, V is a basis for $\mathcal{X}$ if and only if each vector x in $\mathcal{X}$ can be uniquely expressed as a linear combination of vectors in V, that is, can be expressed in the form

$$x = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

for unique scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Theorem (Replacement Theorem)

Let $(\mathcal{X}, \mathcal{F})$ *be a vector space and* $V = \{v_1, v_2, \ldots, v_n\} \subset \mathcal{X}$ *be a basis for* $\mathcal{X}$*. Let* $U = \{u_1, u_2, \ldots, u_m\} \subset \mathcal{X}$ *be linearly independent. Then,* $m \leq n$*.*

Theorem

Let $(\mathcal{X}, \mathcal{F})$ *be a vector space and* $V = \{v_1, v_2, \ldots, v_n\} \subset \mathcal{X}$ *be a basis for* $\mathcal{X}$*. Let* $U = \{u_1, u_2, \ldots, u_m\} \subset \mathcal{X}$ *be another basis for* $\mathcal{X}$*. Then,* $m = n$*.*

Definition (Dimension)

A vector space $(\mathcal{X}, \mathcal{F})$ is called *finite-dimensional* if it has a basis B consisting of a finite number of elements. The unique number of elements in each basis for $\mathcal{X}$ is called the *dimension* of $\mathcal{X}$ and is denoted by $\dim(\mathcal{X})$. A vector space that is not finite-dimensional is called *infinite-dimensional*.

Definition (Linear Transformation)
Let $(\mathcal{X}, \mathcal{F})$ and $(\mathcal{Y}, \mathcal{F})$ be vector spaces. A function L which maps $(\mathcal{X}, \mathcal{F})$ into $(\mathcal{Y}, \mathcal{F})$ is called a *linear transformation* if

$$L(\alpha x) = \alpha L x \qquad \text{for all } x \in \mathcal{X} \text{ and all } \alpha \in \mathcal{F}$$
$$L(x_1 + x_2) = L x_1 + L x_2 \qquad \text{for all } x_1, x_2 \in \mathcal{X}$$

Definition (Range space)
The *range space* $\mathcal{R}(L)$ of a linear transformation L from $(\mathcal{X}, \mathcal{F})$ into $(\mathcal{Y}, \mathcal{F})$ is a subset of $(\mathcal{Y}, \mathcal{F})$ defined by

$$\mathcal{R}(L) = \{y \in \mathcal{Y} : y = Lx \text{ for some } x \in \mathcal{X}\}.$$

Definition (Null space)
The *null space* $\mathcal{N}(L)$ of a linear transformation L from $(\mathcal{X}, \mathcal{F})$ into $(\mathcal{Y}, \mathcal{F})$ is a subset of $(\mathcal{X}, \mathcal{F})$ defined by

$$\mathcal{N}(L) = \{x \in \mathcal{X} : Lx = 0\}.$$

Theorem

Consider a linear transformation L *from* $(\mathcal{X}, \mathcal{F})$ *into* $(\mathcal{Y}, \mathcal{F})$. *The following statements are equivalent.*

1. *The mapping* L *is one-to-one.*
2. *Null space of* L *is trivial, that is,* $\mathcal{N}(L) = \{0\}$.
3. L *maps linearly independent vectors in* $\mathcal{X}$ *into a linearly independent vectors in* $\mathcal{Y}$.

Definition (Nullity and rank)
Let L be a linear transformation from $(\mathcal{X}, \mathcal{F})$ into $(\mathcal{Y}, \mathcal{F})$. If $\mathcal{N}(L)$ and $\mathcal{R}(L)$ are finite dimensional, we define the *nullity* of L, denoted by $\mathrm{nullity}(L)$, and *rank* of L, denoted by $\mathrm{rank}(L)$, to be the dimensions of $\mathcal{N}(L)$ and $\mathcal{R}(L)$, respectively.

Theorem (Dimension Theorem)

Let L be a linear transformation from $(\mathcal{X}, \mathcal{F})$ into $(\mathcal{Y}, \mathcal{F})$. If $\mathcal{X}$ is finite dimensional, then $\text{nullity}(L) + \text{rank}(L) = \dim(\mathcal{X})$.

Definition (Similarity transformation)
Two square matrices A and B are said to be *similar* if a nonsingular matrix T exists such that $A = T^{-1}BT$. The matrix T is called a *similarity transformation*.

Definition (Invariant subspace)
Let L be a linear transformation of $(\mathcal{X}, \mathcal{F})$ into itself. A subspace $\mathcal{Y}$ of $\mathcal{X}$ is said to be an *invariant subspace* of $\mathcal{X}$ under L, or an L*-invariant subspace* of $\mathcal{X}$, if $L(\mathcal{Y}) \subset \mathcal{Y}$, which implies that $Ly \in \mathcal{Y}$ for all $y \in \mathcal{Y}$.

Definition (Eigenvalue and eigenvector)

Let A be a linear transformation from $(\mathsf{C}^n, \mathsf{C})$ into $(\mathsf{C}^n, \mathsf{C})$. A scalar λ in C is called an *eigenvalue* of A if there exists a nonzero vector x in C^n such that $Ax = \lambda x$. Any nonzero vector x satisfying $Ax = \lambda x$ is called an *eigenvector* of A associated with the eigenvalue λ.

Theorem

Let $\lambda_1, \lambda_2, \ldots, \lambda_d$ be distinct eigenvalues of A and let v_i be an eigenvector of A associated with λ_i. Then, the set $\{v_1, v_2, \ldots, v_d\}$ is linearly independent.

Theorem

Let $\lambda_1, \lambda_2, \ldots, \lambda_d$ be distinct eigenvalues of A and let $\{v_{i1}, v_{i2}, \ldots, v_{ig_i}\}$ be a set of linearly independent eigenvectors of A associated with λ_i. Then, the set

$$\{v_{11}, \ldots, v_{1g_1}, v_{21}, \ldots, v_{2g_2}, \ldots, v_{d1}, \ldots, v_{dg_d}\}$$

is linearly independent.

Theorem

A is diagonalizable if and only if it has n linearly independent eigenvectors, that is, $g_i = m_i$, $i = 1, 2, \ldots, d$.

Definition (Generalized eigenspace)
The *generalized eigenspace* of A corresponding to the eigenvalue λ is the subset defined by

$$\{x : (\lambda I - A)^k x = 0 \text{ for some } k \geq 1\}.$$

Theorem

The dimension of the generalized eigenspace of A *corresponding to the eigenvalue* λ_i *is equal to* m_i*.*

Definition (Generalized eigenvector)

A vector v is called a *generalized eigenvector* of *grade* k $(k \geq 1)$ of A associated with eigenvalue λ if

$$(A - \lambda I)^k v = 0$$
$$(A - \lambda I)^{k-1} v \neq 0.$$

Theorem

The union over all the eigenvalues of the set of generalized eigenvectors associated with each different eigenvalue is linearly independent.

$$A = \begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ -4 & 1 & -3 & 2 & 1 \\ -2 & -1 & 0 & 1 & 1 \\ -3 & -1 & -3 & 4 & 1 \\ -8 & -2 & -7 & 5 & 4 \end{bmatrix}, \quad \det(\lambda I - A) = (\lambda - 2)^5$$

$\dim(\mathcal{N}(A - 2I)) = 2$, $\dim(\mathcal{N}(A - 2I)^2) = 4$,
$\dim(\mathcal{N}(A - 2I)^3) = 5$

$$u_{11} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, u_{12} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, u_{21} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, u_{22} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$u_{31} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = v_{31}, \;\; v_{21} = (A - 2I)v_{31} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 5 \end{bmatrix}$$

Choose v_{22} s.t. $\{u_{11}, u_{12}, v_{21}, v_{22}\}$ is a basis of $\mathcal{N}(A - 2I)^2$:

$$v_{22} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$v_{11} = (A - 2I)v_{21}, \quad v_{12} = (A - 2I)v_{22}$$

$$A = \begin{bmatrix} 5 & 19 & 9 \\ 0 & 0 & 1 \\ -1 & -4 & -2 \end{bmatrix}, \ \det(\lambda I - A) = (\lambda - 1)^3$$

$$\dim(\mathcal{N}(A - I)) = 1$$

$v_1 = [7, -1, -1]^T$

$$(A - I)v_2 = v_1$$

$v_2 = [-3, 1, 0]^T$

$$(A - I)v_3 = v_2$$

$v_3 = [-3, 0, 1]^T$

$$A = \begin{bmatrix} -2 & -9 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix}, \ \det(\lambda I - A) = (\lambda - 1)^3$$

$$\dim(\mathcal{N}(A - I)) = 2$$

$v_{11} = [3, -1, 0]^T$, $v_{12} = [4, 0, -1]^T$

$$(A - I)v_2 = ?$$

Theorem (Cayley-Hamilton THM)

Let
$\gamma(\lambda) = \det(\lambda I - A) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$
be the characteristic polynomial of A. Then,

$$\gamma(A) = A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I = 0.$$

Definition (Minimal Polynomial)

The *minimal polynomial* of a matrix A is the monic polynomial $\mu(\lambda)$ of the least degree such that $\mu(A) = 0$.

Theorem

The minimal polynomial $\mu(\lambda)$ of J is given as

$$\mu(\lambda) = \prod_{i=1}^{d} (\lambda - \lambda_i)^{\eta_i}$$

where η_i is the index of λ_i in J.

Definition (Function of a matrix)
Let $f(\lambda)$ be a function that is defined on the spectrum of A. If $p(\lambda)$ is a polynomial that has the same values as $f(\lambda)$ on the spectrum of A, then $f(A)$ is defined as $f(A) = p(A)$.

Lemma

Given distinct numbers $\lambda_1, \lambda_2, \ldots, \lambda_d$*, positive integers* $\eta_1, \eta_2, \ldots, \eta_d$ *with* $\eta = \sum_{i=1}^{d} \eta_i$*, and a set of numbers*

$$f_{i,0}, f_{i,1}, \ldots, f_{i,\eta_i-1}, \qquad i = 1, 2, \ldots, d$$

there exists a polynomial $p(\lambda)$ *of degree less than* η *such that*

$$p(\lambda_i) = f_{i,0},\ p^{(1)}(\lambda_i) = f_{i,1},\ \ldots,\ p^{(\eta_i-1)}(\lambda_i) = f_{i,\eta_i-1},$$

Furthermore, such a polynomial $p(\lambda)$ *is unique.*

$a_1 = \frac{1}{4}(e^{-t} - e^{-5t})$ and $a_0 = \frac{5}{4}e^{-t} - \frac{1}{4}e^{-5t}$

$$
\begin{aligned}
e^{At} &= (\frac{5}{4}e^{-t} - \frac{1}{4}e^{-5t})I + \frac{1}{4}(e^{-t} - e^{-5t})A \\
&= \begin{bmatrix} 2e^{-t} - e^{-5t} & e^{-t} - e^{-5t} \\ \frac{1}{2}(e^{-t} - e^{-5t}) & \frac{3}{2}e^{-t} - \frac{1}{2}e^{-5t} \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
f(1) = p(1) &\qquad e^t = a_0 + a_1 + a_2 \\
f^{(1)}(1) = p^{(1)}(1) &\qquad te^t = a_1 + 2a_2 \\
f(2) = p(2) &\qquad e^{2t} = a_0 + 2a_1 + 4a_2
\end{aligned}
$$

$$
e^{At} = (-2te^t + e^{2t})I + (3te^t + 2e^t - 2e^{2t})A + (e^{2t} - e^t - te^t)A^2
$$

$$e^t = a_0 + a_1$$
$$e^{2t} = a_0 + 2a_1$$

$$e^{At} = (2e^t - e^{2t})I + (e^{2t} - e^t)A$$

cf.

$$e^{At} = (-2te^t + e^{2t})I + (3te^t + 2e^t - 2e^{2t})A + (e^{2t} - e^t - te^t)A^2$$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix} = PJP^{-1} = P \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} P^{-1}$$

$$P = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$f(\lambda) = e^{\lambda t}, f^{(1)}(\lambda) = te^{\lambda t}$$

$$f(J) = e^{Jt} = \begin{bmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}$$

$$f(A) = e^{At} = Pe^{Jt}P^{-1} = P \begin{bmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} P^{-1}$$

Definition (Inner Product)
Let $(\mathcal{X}, \mathcal{F})$ be a vector space. The *inner product* of two vectors x and y in $\mathcal{X}$ denoted by $< x, y >$ takes the value in $\mathcal{F}$ and satisfies the following properties for all $a, b \in \mathcal{F}$:

1. $< x, y >= \overline{< y, x >}$
2. $< x, ay + bz >= a < x, y > +b < x, z >$
3. $< x, x > \geq \ 0$ for all x, and $< x, x >= 0$ if and only if $x = 0$.

Definition (Orthogonality)
Two vectors x and y in an inner product space $\mathcal{X}$ are said to be *orthogonal* if their inner product is zero, that is, $< x, y >= 0$. If x and y are orthogonal, this is denoted by $x \perp y$.
Two subsets A and B in an inner product space $\mathcal{X}$ are said to be *orthogonal* if $x \perp y$ for all x in A and y in B. If A and B are orthogonal, this is denoted by $A \perp B$.

Definition (Orthogonal Complement)

Let $\mathcal{X}$ be an inner product space and $\mathcal{Y}$ be any subspace of $\mathcal{X}$. The set

$$\mathcal{Y}^{\perp} = \{x :< x, y >= 0 \text{ for all } y \in \mathcal{Y}\}$$

is called the *orthogonal complement* of $\mathcal{Y}$.

Definition (Norm)

A real-valued function $\|x\|$ defined on a vector space $(\mathcal{X}, \mathcal{F})$, where $x \in \mathcal{X}$, is called a *norm* if for all $x, y \in \mathcal{X}$ and $\alpha \in \mathcal{F}$

$$
\begin{aligned}
(i) \quad & \|x\| \geq 0 \text{ for all } x \in \mathcal{X}, \text{ and } \|x\| = 0 \\
& \text{if and only if } x = 0 \\
(ii) \quad & \|\alpha x\| = |\alpha| \|x\| \text{ for all } x \in \mathcal{X} \text{ and } \alpha \in \mathcal{F} \\
(iii) \quad & \|x + y\| \leq \|x\| + \|y\| \text{ for all } x, y \in \mathcal{X}.
\end{aligned}
$$

Theorem (Schwarz Inequality)

If, on an inner product space, $\|x\|$ is defined by $\|x\| =< x, x >^{1/2}$, then

$$| < x, y > | \leq \|x\| \|y\|.$$

Definition (Orthonormal Basis)

Let $\mathcal{X}$ be an inner product space. A subset B is an *orthonormal basis* if it is a basis that is orthonormal.

Theorem (Gram-Schmidt Orthonormalization Process)

Let $\mathcal{X}$ *be an inner product space, and* $W = \{w_1, w_2, \ldots, w_n\}$ *be a linearly independent subset of* $\mathcal{X}$*. Then, there exists an orthonormal set* $V = \{v_1, v_2, \ldots, v_n\}$ *such that* $v_k = \sum_{i=1}^{k} a_{ik} w_i$*,* $k = 1, 2, \ldots, n$*.*

Let $w_1 = [1, 1, 0]^T$, $w_2 = [2, 0, 1]^T$, $w_3 = [2, 2, 1]^T$.
Then,

$$v_1 = \frac{1}{\|w_1\|} w_1 = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right]^T$$

$$u_2 = w_2 - < v_1, w_2 > v_1 = [2, 0, 1]^T - [1, 1, 0]^T$$

$$= [1, -1, 1]^T$$

$$v_2 = \left[\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]^T$$

$$u_3 = w_3 - < v_1, w_3 > v_1 - < v_2, w_3 > v_2$$

$$v_3 = \frac{1}{\|u_3\|} u_3 = \left[-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right]^T$$

Corollary

Let $\mathcal{X}$ be an inner product space and $\mathcal{Y}$ be a finite dimensional subspace of $\mathcal{X}$. For each $x \in \mathcal{X}$, there exist unique vectors $y \in \mathcal{Y}$ and $z \in \mathcal{Y}^\perp$ such that $x = y + z$. Furthermore, if $\{v_1, v_2, \ldots, v_k\}$ is an orthonormal basis of $\mathcal{Y}$, then

$$y = \sum_{i=1}^{k} < v_i, x > v_i.$$

The vector y is the unique vector such that $\|x - y\| < \|x - u\|$ for any $u \in \mathcal{Y}$ such that $u \neq y$.

Theorem (Properties of induced norm)

For $m \times n$ *matrices* A *and* B *mapping normed space* $\mathcal{R}^n$ *into normed space* $\mathcal{R}^m$*, and* $n \times l$ *matrix* C *mapping normed space* $\mathcal{R}^l$ *into normed space* $\mathcal{R}^n$*, the following hold:*

1. $\|A\| \geq 0$*, and* $\|A\| = 0$ *if and only if* $A = 0$*,*
2. $\|\alpha A\| = |\alpha| \|A\|$*,*
3. $\|A + B\| \leq \|A\| + \|B\|$*,*
4. $\|AC\| \leq \|A\| \|C\|$*.*

Theorem

Let A *be an* $m \times n$ *matrix mapping* $\mathcal{R}^n$ *into* $\mathcal{R}^m$ *defined by* $y = Ax$ *with the adjoint transformation given by* $x = A^T y$.

1. $\mathcal{N}(A^T)$ *is an orthogonal complement of* $\mathcal{R}(A)$, *that is,* $\mathcal{N}(A^T) = \mathcal{R}(A)^\perp$.
2. $\mathcal{R}(A)$ *is an orthogonal complement of* $\mathcal{N}(A^T)$, *that is,* $\mathcal{R}(A) = \mathcal{N}(A^T)^\perp$.

Theorem

For a Hermitian matrix H,

1. *all the eigenvalues are real,*
2. *there are n linearly independent eigenvectors,*
3. *the eigenvectors corresponding to different eigenvalues are orthogonal,*
4. *$\exists$ a unitary matrix Q such that*

$$H = Q\Lambda Q^{-1} = Q\Lambda\bar{Q}^T$$

where Λ is a diagonal matrix.

Definition (Positive definiteness)
A quadratic form $x^T Mx$ is said to be

1. *positive definite* if $x^T Mx > 0$ for all $x \neq 0$,
2. *positive semidefinite* if $x^T Mx \geq 0$ for all x,
3. *negative definite* if $x^T Mx < 0$ for all $x \neq 0$,
4. *negative semidefinite* if $x^T Mx \leq 0$ for all x.

A symmetric matrix M is said to be *positive definite (positive semidefinite, negative definite, negative semidefinite, respectively)* if the quadratic form $x^T Mx$ is so.

Theorem

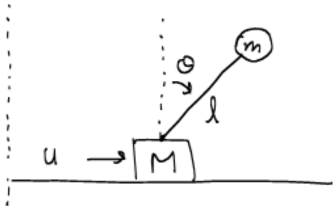
A symmetric matrix M is positive definite (positive semidefinite) if and only if all the eigenvalues of M are positive (nonnegative).

Theorem

If a symmetric matrix M is positive definite (positive semidefinite), then $\det(M) > 0$ ($\det(M) \geq 0$).

Theorem

A symmetric matrix M is positive definite (positive semidefinite) if and only if all the leading principal minors (all the principal minors) of M are positive (nonnegative).



$$(M+m)\ddot{x} + ml(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) = u$$

$$m\left(\ddot{x}\cos\theta + l\ddot{\theta} - g\sin\theta\right) = 0$$

$$\Downarrow$$

$$(M+m)\ddot{x} + ml\ddot{\theta} = u$$

$$\ddot{x} + l\ddot{\theta} - g\theta = 0$$

$$(M+m)\ddot{x} + ml(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) = u$$

$$m\left(\ddot{x}\cos\theta + l\ddot{\theta} - g\sin\theta\right) = 0$$

$$\Downarrow$$

$$\begin{bmatrix} M+m & ml \\ 1 & l \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} u \\ g\theta \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \frac{1}{Ml} \begin{bmatrix} l & -ml \\ -1 & M+m \end{bmatrix} \begin{bmatrix} u \\ g\theta \end{bmatrix}$$

Theorem

If A is an $n \times n$ matrix function whose entries are continuous functions of time on the interval $I = [t_l, t_u]$, then there exists the unique solution to the initial value problem

$$\dot{x}(t) = A(t)x(t), \quad x(t_0) = x^0, \quad t_0 \in I = [t_l, t_u].$$

$$\frac{d}{dt}x(t) = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} x(t) = A(t)x(t)$$

Show

$$\Phi(t, t_0) = \begin{bmatrix} e^{(t-t_0)} & 0 \\ \frac{1}{2}(t^2 - t_0^2)e^{(t-t_0)} & e^{(t-t_0)} \end{bmatrix}$$

is the state transition matrix. Answer: $\Phi(t_0, t_0) = I$ and

$$\begin{aligned} \frac{d}{dt}\Phi(t, t_0) &= \begin{bmatrix} e^{(t-t_0)} & 0 \\ \frac{1}{2}(t^2 - t_0^2)e^{(t-t_0)} + te^{(t-t_0)} & e^{(t-t_0)} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \begin{bmatrix} e^{(t-t_0)} & 0 \\ \frac{1}{2}(t^2 - t_0^2)e^{(t-t_0)} & e^{(t-t_0)} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \Phi(t, t_0) = A(t)\Phi(t, t_0) \end{aligned}$$

Definition (Stability)

For $\dot{x} = f(t, x)$, the equilibrium point x_e (i.e., $f(t, x_e) = 0$, $^{\forall}t$) is

1. *stable* i.s.L. (in the sense of Lyapunov) if for each t_0 and each $\epsilon > 0$, there exists $\delta(\epsilon, t_0) > 0$ s.t. if $\|x(t_0) - x_e\| < \delta$ then

$$\|x(t) - x_e\| < \epsilon \qquad \text{for all } t \geq t_0$$

2. *uniformly stable* i.s.L. if, $^{\forall}\epsilon > 0$, $\exists\delta = \delta(\epsilon) > 0$ s.t. if $\|x(t_0) - x_e\| < \delta$ then

$$\|x(t) - x_e\| < \epsilon \qquad \text{for all } t \geq t_0$$

3. *unstable* if it is not stable

4. *asymptotically stable* if it is stable i.s.L. and for each t_0, there is a positive constant c such that if $\|x(t_0) - x_e\| < c$, then $x(t) \to x_e$ as $t \to \infty$.
5. *globally asymptotically stable* if it is stable i.s.L. and for each t_0 and each $x(t_0)$, $x(t) \to x_e$ as $t \to \infty$.

$$\dot{x} = \left[\begin{array}{cc} 0 & \omega \\ -\omega & 0 \end{array} \right] x(t)$$

$$\Phi(t, t_0) = \left[\begin{array}{cc} \cos \omega(t - t_0) & \sin \omega(t - t_0) \\ -\sin \omega(t - t_0) & \cos \omega(t - t_0) \end{array} \right]$$

$$\begin{aligned} x_1(t) &= \cos \omega(t - t_0) x_1(t_0) + \sin \omega(t - t_0) x_2(t_0) \\ x_2(t) &= -\sin \omega(t - t_0) x_1(t_0) + \cos \omega(t - t_0) x_2(t_0) \end{aligned}$$

Theorem

For linear continuous time system, the equilibrium point at the origin is

1. *stable if and only if for each* t_0*, there exists a constant* $\kappa(t_0)$ *such that*

$$\|\Phi(t, t_0)\| \leq \kappa(t_0) \qquad \text{for all } t \geq t_0,$$

2. *asymptotically stable if and only if for each* t_0*,* $\|\Phi(t, t_0)\| \to 0$ *as* $t \to \infty$*.*

$$\dot{x} = \begin{bmatrix} -1 & e^{2t} \\ 0 & -1 \end{bmatrix} x$$

$$\Phi(t, t_0) = \begin{bmatrix} e^{-(t-t_0)} & \frac{1}{2} e^{-(t-t_0)} (e^{2t} - e^{2t_0}) \\ 0 & e^{-(t-t_0)} \end{bmatrix}$$

Theorem

For the linear time invariant system with system matrix A, the equilibrium point at the origin is

1. *asymptotically stable if and only if all the eigenvalues of A have negative real parts,*
2. *stable if and only if all the eigenvalues of A have nonpositive real parts, and those eigenvalues with zero real parts are distinct roots of the minimal polynomial of A (or, equivalently, have indices equal to 1),*
3. *unstable if there exist eigenvalues with positive real parts or eigenvalues with zero real parts which are not distinct roots of the minimal polynomial of A (or equivalently, have indices greater than 1).*

Definition (BIBO stability)
The input-output system is said to be *bounded-input-bounded-output (BIBO) stable* if for any bounded input $u(t), \|u(t)\| \leq M$ for all t, there exists a finite constant $N(M)$ such that $\|y(t)\| \leq N$ for all t.

Theorem

For a linear time-invariant system, the zero-state response is BIBO stable if and only if all the poles of the transfer function are located in the open left-half complex plane.

Theorem

For a linear time-invariant system, the zero-state response is BIBO stable if for all the eigenvalues λ_i *of the system matrix* A, $\mathrm{Re}\,\lambda_i < 0$.

Theorem

The matrix A is Hurwitz, or equivalently, the zero state of $\dot{x} = Ax$ is asymptotically stable if and only if for any given symmetric positive definite matrix Q, the matrix equation

$$A^T P + PA = -Q$$

has a unique symmetric positive definite solution P.

$$A = \begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix}, \qquad Q = I$$

$$P = \begin{bmatrix} 23/60 & -7/60 \\ -7/60 & 11/60 \end{bmatrix}$$

Definition (Controllability)

A linear system (or the pair $(A(t), B(t))$) is said to be *controllable* on the interval $[t_0, t_1]$ if for any x^0 in the state space $\mathcal{S}$ and any x^1 in $\mathcal{S}$, there exists an input $u_{[t_0,t_1]}$ which transfers the state $x(t_0) = x^0$ to the state $x(t_1) = x^1$ at time t_1.

Theorem

A linear system (or the pair $(A(t), B(t))$) is controllable on the interval $[t_0, t_1]$ if and only if the controllability Gramian

$$G_c(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, t)B(t)B^T(t)\Phi^T(t_0, t)dt$$

is nonsingular.

Theorem

A linear system is controllable on the interval $[t_0, t_1]$ *if there exists* $\tau \in [t_0, t_1]$ *such that*

$$\operatorname{rank}\left(\left[\begin{array}{cccc} M_0(\tau) & M_1(\tau) & \cdots & M_{n-1}(\tau) \end{array}\right]\right) = n$$

where $M_0(\tau) = B(\tau)$ *and*

$$M_j(\tau) = -A(\tau)M_{j-1}(\tau) + \frac{d}{d\tau}M_{j-1}(\tau)$$

where $j = 1, 2, \ldots, n-1$.

Theorem

A linear time-invariant system (or the pair (A, B)) is controllable on the interval $[t_0, t_1]$ if and only if the controllability matrix

$$\mathcal{P} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

has rank n.

Lemma

The range space and null space of

$$G_c(t_0, t_1) = \int_{t_0}^{t_1} e^{A(t_0 - t)} B B^T e^{A^T (t_0 - t)} dt$$

coincide with the range space and null space of $\mathcal{P}\mathcal{P}^T$.

Definition (Observability)

A linear system (or the pair $(C(t), A(t))$) is said to be *observable* on the interval $[t_0, t_1]$ if for any initial state $x(t_0)$ in the state space $\mathcal{S}$, the knowledge of the input $u_{[t_0,t_1]}$ and the output $y_{[t_0,t_1]}$ is sufficient to uniquely solve for $x(t_0)$.

Theorem

A linear system is observable on the interval $[t_0, t_1]$ *if there exists* $\tau \in [t_0, t_1]$ *such that*

$$\operatorname{rank}\left(\begin{bmatrix} N_0(\tau) \\ N_1(\tau) \\ \vdots \\ N_{n-1}(\tau) \end{bmatrix}\right) = n$$

where $N_0(\tau) = C(\tau)$ *and*

$$N_j(\tau) = N_{j-1}(\tau)A(\tau) + \frac{d}{d\tau}N_{j-1}(\tau).$$

Theorem

For a single-input LTI system

$$\dot{x} = Ax + bu$$

with $\det(sI - A) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$*,*
there exists a change of coordinates $x_c = Tx$ *s.t.*

$$\dot{x}_c = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} x_c + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

if and only if the pair (A, b) *is controllable.*

Theorem

For a single-output LTI system

$$\dot{x} = Ax, \qquad y = cx$$

with $\det(sI - A) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$*,*
there exists a change of coordinates $x_o = Tx$ *s.t.*

$$\dot{x}_o = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{n-2} \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix} x_o$$
$$y = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} x_o$$

if and only if the pair (c, A) *is observable.*

Definition (Controllable subspace)
The set of the initial states that can be transferred to the zero state in finite time is called the *controllable subspace* and is denoted by $\mathcal{C}$.

Definition (Stabilizability)

The linear time invariant system is said to be *stabilizable* if its unstable subspace is contained in its controllable subspace, that is, any vector x in the unstable subspace is also in the controllable subspace.

Definition (Unobservable subspace)
The set of the initial states that produce zero-input responses $\bar{y}(t)$ which are identically zero on any finite time interval is called the *unobservable subspace* and is denoted by $\mathcal{O}$.

Definition (Detectability)
The linear time invariant system is said to be *detectable* if its unobservable subspace is contained in its stable subspace, that is, any vector x in the unobservable subspace is also in the stable subspace.

Theorem

The pair (A, B) *is controllable if and only if by the state feedback* $u(t) = Kx(t) + r(t)$*, the eigenvalues of* $(A + BK)$ *can be arbitrarily assigned provided that complex conjugate eigenvalues appear in pair.*

Theorem

The pair (C, A) *is observable if and only if the eigenvalues of* $A - LC$ *can be arbitrarily assigned by a proper choice of the matrix* L *provided that complex conjugate eigenvalues appear in pair.*

Definition (Realization)

A *realization* of a transfer matrix $G(s)$ is any state space model (A, B, C, D) such that $G(s) = C(sI - A)^{-1}B + D$. If such a state space model exists, then $G(s)$ is said to be *realizable*.

Definition (Minimal realization)
A realization (A, B, C, D) of a transfer matrix $G(s)$ is called a *minimal realization* if there exists no other realization with state space of smaller dimension.