Supplementary material of

Back-and-forth Operation of State Observers and Norm Estimation of Estimation Error

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***Abstract*— This is a supplementary material for the paper "Back-and-forth Operation of State Observers and Norm Estimation of Estimation Error" that will be presented at the 51st IEEE Conference on Decision and Control, December, 2012.**

Consider the linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

with $x(0) = x_0$.

Proposition 1: If (A, C) is an observable pair, then, for any $d > 0$ and α such that $0 < \alpha < 1$, there exist gain matrices L_f and L_b such that

$$\|\exp((A - L_f C)t)\| \le \alpha, \qquad {}^{\forall} t \in [d/2, d], \tag{2}$$

and

$$\|\exp(-(A - L_b C)t)\| \le \alpha, \qquad {}^{\forall} t \in [d/2, d]. \tag{3}$$

Then, the forward observer is given by

$$\frac{d}{dt}\hat{x}_f = A\hat{x}_f + Bu(t) + L_f(y(t) - C\hat{x}_f) \tag{4}$$

while the backward observer is

$$\frac{d}{ds}\hat{x}_b = -A\hat{x}_b - Bu(d-s) - L_b(y(d-s) - C\hat{x}_b). \tag{5}$$

In fact, the backward observer is based on the backward-time description of the system (1) written as

$$\frac{d}{ds}\bar{x} = -A\bar{x} - Bu(d-s), \quad y(d-s) = C\bar{x}, \quad \bar{x}(0) = x(d),$$

with $\bar{x}(s) = x(d-s)$ for $s = d - t \in [0, d]$.

I. Real-time Applications

A. State Estimation of a Switched System

Consider a switched system given in Fig. 1, which has two modes of operation described by

$$\Sigma_1 : \begin{cases} \dot{x}_1 = A_1 x_1 + B_1 u, & y = C_1 x_1, \\ \dot{x}_2 = A_2 x_2 + A_{21} x_1, \end{cases} \tag{6}$$

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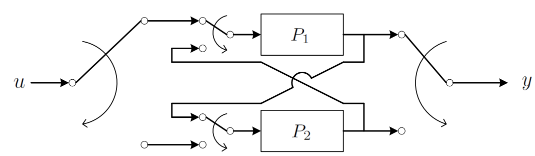

Fig. 1. All switches are synchronized.

for mode 1, and

$$\Sigma_2 : \begin{cases} \dot{x}_1 = A_1 x_1 + A_{12} x_2 \\ \dot{x}_2 = A_2 x_2 + B_2 u, & y = C_2 x_2, \end{cases} \tag{7}$$

for mode 2, where the pair (A_i, C_i), $i = 1, 2$, is observable. Now suppose that the system configuration switches between modes 1 and 2 (i.e., between (6) and (7)) after every T seconds, and we want to estimate the states x_1 and x_2 completely. Note that, at each mode, the system is not completely observable. For example, at mode 1, the state x_2 is unobservable.

One can design a separate observer for each mode of operation as follows:

$$\dot{\hat{x}}_1 = A_1\hat{x}_1 + B_1 u - L_1 C_1 \hat{x}_1 + L_1 y \tag{8a}$$
$$\dot{\hat{x}}_2 = A_2\hat{x}_2 + A_{21}\hat{x}_1 \tag{8b}$$

for Σ_1, and

$$\dot{\hat{x}}_1 = A_1\hat{x}_1 + A_{12}\hat{x}_2 \tag{9a}$$
$$\dot{\hat{x}}_2 = A_2\hat{x}_2 + B_2 u - L_2 C_2 \hat{x}_2 + L_2 y \tag{9b}$$

for Σ_2, where L_1 and L_2 are large enough so that some meaningful estimates $\hat{x}_1$ and $\hat{x}_2$ are obtained over the interval of length T. In fact, at the end of the first period $[0, T)$, one can obtain the estimate $\hat{x}_1(T)$ by (8a) for mode 1. For the second interval $[T, 2T)$, this estimate serves as the initial condition of (9a), and the observer (9b) starts to estimate $x_2(t)$. However, the observer (9b) will exhibit some transients during the initial period of the interval $[T, 2T)$, which may corrupt the estimate $\hat{x}_1(t)$ being obtained through the observer (9a) because of initially large error between $\hat{x}_2(t)$ and $x_2(t)$.

To overcome this problem, the following hybrid-type observer may be utilized instead of (8) and (9)

$$\hat{\Sigma}_1 : \begin{cases} \dot{\hat{x}}_1 = A_1\hat{x}_1 + B_1 u \\ \dot{\hat{x}}_2 = A_2\hat{x}_2 + A_{21}\hat{x}_1 \end{cases} \tag{10}$$

$$\hat{\Sigma}_2 : \begin{cases} \dot{\hat{x}}_1 = A_1\hat{x}_1 + A_{12}\hat{x}_2 \\ \dot{\hat{x}}_2 = A_2\hat{x}_2 + B_2 u \end{cases} \tag{11}$$

$$\begin{pmatrix} \hat{x}_1(kT) \\ \hat{x}_2(kT) \end{pmatrix} = \begin{pmatrix} \hat{\xi}_1(kT^-) \\ \hat{\xi}_2(kT^-) \end{pmatrix}, \qquad k \geq 1, \tag{12}$$

where the variables $\hat{\xi}_1$ and $\hat{\xi}_2$ will be obtained shortly using the back-and-forth operation such that the inequality,

$$|\hat{x}((k+2)T) - x((k+2)T)| \leq \gamma |\hat{x}(kT) - x(kT)|, \tag{13}$$

holds for all $k \geq 1$, $x := (x_1^T, x_2^T)^T$, and a desired parameter $\gamma < 1$. The inequality (13) guarantees the convergence of estimation error to zero due to the fact that $\sup_{t\in[kT,(k+1)T)} |\hat{x}(t) - x(t)| \leq M|\hat{x}(kT) - x(kT)|$ where M is a constant. The latter inequality holds because the dynamics for $\hat{x} - x$ are linear and their growth is bounded over a finite interval.

In order to obtain $\hat{\xi}_1$ and $\hat{\xi}_2$, we prepare the back-and-forth observer (4) and (5) for the x_1-subsystem of (6) and for the x_2-subsystem of (7), respectively. For each subsystem, the injection gains L_f and L_b are designed such that (2) and (3) hold with $d = T/2$ and

$$\alpha = \left(\frac{\gamma}{\sqrt{2}\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}} \right)^{\frac{1}{R}}$$

where $\alpha_1 = M_1 + L_1L_2 + L_2$, $\alpha_2 = L_1M_2 + M_2$, $\alpha_3 = M_2 + L_1L_2 + L_1$, $\alpha_4 = L_2M_1 + M_1$ and

$$M_i := \|e^{A_iT}\|, \;\; L_i := \left\| \int_0^T e^{A_is} ds A_{ij} \right\|, \;\; i,j = 1,2, i \neq j,$$

and let $R+1$ be the number of round-trips of numerical back-and-forth integrations that are possible within the interval of length $T/2$. Clearly, the number R relies on the computation power.

Let the initial condition $\hat{\xi}_1(0^-)$ and $\hat{\xi}_2(0^-)$ be arbitrary. Fig. 2 illustrates the strategy to obtain $\hat{\xi}_1(kT^-)$ and $\hat{\xi}_2(kT^-)$, for $k = 2$, over the interval $[T, 2T)$ when mode 2 is active. At time $t = T$, the estimate $\hat{x}_1(T)$ and $\hat{x}_2(T)$ are set to $\hat{\xi}_1(T^-)$ and $\hat{\xi}_2(T^-)$ respectively, and they are integrated in the real time by (11). At the same time, the initial condition of the forward observer (for estimating x_2) is set by $\hat{\xi}_2(T^-)$, and this forward observer runs in the real time first until $T + T/2$. At $T + T/2$, the backward observer is employed with the terminal state of the forward observer as its initial condition. The round-trip of back-and-forth operation continues R times with the input-output data of the interval $[T, T + T/2]$, after which the forward observer is finally integrated from T to $2T$. Since the time elapsed by the back-and-forth operation and the last forward operation does not exceed $T/2$, the last forward integration will 'catch up' with the real time, as indicated in Fig. 2. While these operations are performed, the information about

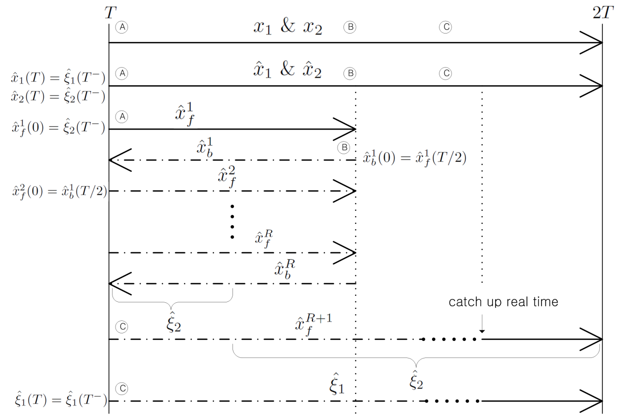

Fig. 2. Operation time chart for mode 2 in the interval $[T, 2T)$. Solid arrow implies real time frame, and solid-dot means numerical integration time frame that is faster than real time. Circles with letters help identifying the same times.

$\hat{\xi}_2(t)$ is collected as illustrated in the figure. At the same time with the start of the last forward observer, we begin the integration of

$$\dot{\hat{\xi}}_1 = A_1\hat{\xi}_1 + A_{12}\hat{\xi}_2 \tag{14}$$

with the initial condition $\hat{\xi}_1(T) = \hat{\xi}_1(T^-)$ and with the signal $\hat{\xi}_2(t)$ obtained by back-and-forth operation over the interval $[T, 2T)$. By this procedure, we obtain $\hat{\xi}_1(2T^-)$ and $\hat{\xi}_2(2T^-)$. This procedure repeats in the next interval, with the role for x_1 and x_2 being switched, and instead of (14), the following equation is used to compute $\hat{\xi}_2$:

$$\dot{\hat{\xi}}_2 = A_2\hat{\xi}_2 + A_{21}\hat{\xi}_1. \tag{15}$$

We now proceed with the error analysis. Let $\epsilon := \hat{x} - x$. Then, $\epsilon(kT) = \hat{x}(kT) - x(kT) = \hat{\xi}(kT^-) - x(kT)$. We note that, for the $(2k+1)$-th interval with k being nonnegative integer, $\hat{\xi}_1$ is the estimate from the back-and-forth observer while $\hat{\xi}_2$ is the state of (15), and for the $2k$-th interval, their roles are reversed. Therefore, in the first interval $[0, T)$, we have that

$$|\epsilon_1(T)| \leq \alpha^R |\epsilon_1(0)|$$
$$|\epsilon_2(T)| \leq \|e^{A_2T}\| |\epsilon_2(0)| + \left\| \int_0^T e^{A_2s} ds A_{21} \right\| \left[\alpha^R |\epsilon_1(0)| \right].$$

Similarly, we can derive the following expressions for the second interval $[T, 2T)$,

$$\begin{aligned} |\epsilon_1(2T)| &\leq \|e^{A_1T}\| |\epsilon_1(T)| + \left\| \int_0^T e^{A_1s} ds A_{12} \right\| \left[\alpha^R |\epsilon_2(T)| \right] \\ &\leq \alpha^R \|e^{A_1T}\| |\epsilon_1(0)| + \alpha^R \left\| \int_0^T e^{A_1s} ds A_{12} \right\| \|e^{A_2T}\| |\epsilon_2(0)| \\ &\quad + \alpha^{2R} \left\| \int_0^T e^{A_1s} ds A_{12} \right\| \left\| \int_0^T e^{A_2s} ds A_{21} \right\| |\epsilon_1(0)| \end{aligned}$$

and

$$
\begin{aligned}
|\epsilon_2(2T)| &\le \alpha^R |\epsilon_2(T)| \\
&\le \alpha^R \|e^{A_2 T}\| |\epsilon_2(0)| + \alpha^{2R} \left\| \int_0^T e^{A_2 s} ds A_{21} \right\| |\epsilon_1(0)|.
\end{aligned}
$$

The terms within the brackets, $[\cdot]$, are due to the back-and-forth observer, which yields a rich estimation on the entire interval of length T, including the initial period. Finally, it is seen that

$$
\begin{aligned}
|\epsilon(2T)| &\le |\epsilon_1(2T)| + |\epsilon_2(2T)| \\
&\le \alpha^R (M_1 + L_1 L_2 + L_2)|\epsilon_1(0)| + \alpha^R (L_1 M_2 + M_2) \\
&\quad \times |\epsilon_2(0)| \\
&\le \frac{\gamma}{\sqrt{2}} (|\epsilon_1(0)| + |\epsilon_2(0)|) \le \gamma |\epsilon(0)|.
\end{aligned}
$$

This proves the claim (13) for even k. For odd k, the proof is similar and thus omitted.

An underlying reasoning is that the use of the back-and-forth observer has improved the transient response of the state estimation error, thus leading to quality estimates of the state variable over the entire interval.

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