Multi-dimensional Limiting Process for Two- and Three-dimensional Flow Physics Analyses

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<Abstract> In this paper, we derive a limiting condition for three-dimensional compressible flows and present the multi-dimensional limiting process for three-dimensions. The basic idea of the multi-dimensional limiting condition is that the vertex values interpolated at a grid point should be within the maximum and minimum cell-average values of neighboring cells for the monotonic distribution. By applying the MLP (Multi-dimensional Limiting Process), we can achieve monotonic characteristic, which results in the substantial enhancement of solution accuracy and convergence behavior.

1. Introduction

Since the late 1970s, numerous ways to control oscillations have been studied and several limiting concepts have been proposed. Most representatives would be TVD [1, 2], TVB [4] and ENO [3]. The concept of TVD (Total Variation Diminishing) was proven to be extremely successful in solving hyperbolic conservation laws. Most oscillation-free schemes have been based on the mathematical analysis of one-dimensional convection equation and applied to systems of equations with the help of some linearization step. They are also applied to multi-dimensional applications with dimensional splitting. Though they may work successfully in many cases, it is insufficient or almost impossible to control oscillations near shock discontinuity in multiple space dimensions. For that reason, the need of oscillation control method for multi-dimensional applications is obvious.

In order to find out the criterion of oscillation control for multiple dimensions, Kim and Kim [5] extended the one-dimensional monotonic condition to two dimensions and presented the two-dimensional limiting condition successfully. With the limiting condition, a multi-dimensional limiting process (MLP) is proposed which gives more accurate results for two-dimensional Euler and Navier-Stokes equations. It is this approach which prompts the work of the present paper. Basically, it extends the idea of MLP to three dimensions. Thus, in this paper, we derive a three-dimensional limiting condition and present the multi-dimensional limiting process for two- and three-dimensional situations.

2. Multi-dimensional Limiting Process (MLP)

(1) Multi-dimensional Limiting Condition

In view of Godunov-type approach, the steps to construct a numerical flux at a cell-interface usually consist of interpolation stage and evolution stage. It is known that interpolation stage is generally independent of evolution stage where the local Riemann problem is solved at a cell-interface. Thus, for higher order spatial accuracy, interpolation stage can be modified without changing a Riemann solver. This method for the generation of second-order upwind schemes is often referred as the MUSCL approach [6].

One-dimensional limiting condition using TVD constraint can be written as follows. [2]

$$0 \le \phi(r) \le \min(2r, 2). \tag{1}$$

Since the extension of Eq.(1) in a dimensional splitting manner is insufficient to prevent oscillations in multi-dimensional flow, it needs to be modified and/or extended with appropriate consideration of multi-dimensional situation.

However, the dimensional splitting extension does not possess any information on property distribution at cell vertex points, which would be essential when property gradient is not aligned with grid lines. Thus, as an extended condition including the missing information, we require multi-dimensional limiting condition as following.

$$\overline{\mathbf{\Phi}}^{\min}_{\text{neighbor}} \le \mathbf{\Phi} \le \overline{\mathbf{\Phi}}^{\max}_{\text{neighbor}} \tag{2}$$

In order to realize Eq.(2) in three-dimensional situation, the values at vertex points are required to satisfy the following condition.

$$\overline{\mathbf{\Phi}}^{\min}_{p,q,r} \le \mathbf{\Phi}_{i+p/2,\, j+q/2,\, k+r/2} \le \overline{\mathbf{\Phi}}^{\max}_{p,q,r} \tag{3}$$

where $\overline{\mathbf{\Phi}}_{i+p/2,\, j+q/2, k+r/2}$ is a vertex point value and $\overline{\mathbf{\Phi}}_{i+p,\, j+q, k+r}$ is a cell-averaged value. The values of index variables, p, q, and r, can be positive or negative one, which indicate each vertex point value and neighboring cell-averaged value in three-dimensional case. $\overline{\mathbf{\Phi}}^{\min}_{p,q,r}$ and $\overline{\mathbf{\Phi}}^{\max}_{p,q,r}$ are the minimum and maximum cell-averaged values among neighboring candidates, respectively.

In order to derive the multi-dimensional limiting function from Eq.(3), we need to express the vertex point value in terms of variations at the cell-interface. Here we assume that the variations in a cell are linear without loss of generality.

(2) General Form of Multi-dimensional Limiting Process (MLP)

With the multi-dimensional limiting function, a new family of limiting process to control oscillations in a multi-dimensional flow can be developed. For three-dimensional flows,

$$\begin{aligned}\mathbf{\Phi}^{L}_{i+1/2,\,j,\,k} &= \overline{\mathbf{\Phi}}_{i,\,j,\,k} + 0.5\,\phi(r^{\xi}_{L,i,j,k}, \alpha_L, \beta_L)\,\Delta\mathbf{\Phi}_{i-1/2,\,j,\,k} \\ &= \overline{\mathbf{\Phi}}_{i,\,j,\,k} + 0.5\max(0, \min(\alpha_L, \alpha_L r^{\xi}_{L,i,j,k}, \beta_L))\,\Delta\mathbf{\Phi}_{i-1/2,\,j,\,k}\end{aligned} \tag{4a}$$

$$\begin{aligned}\mathbf{\Phi}^{R}_{i+1/2,\,j,\,k} &= \overline{\mathbf{\Phi}}_{i+1,\,j,\,k} - 0.5\,\phi(r^{\xi}_{R,i+1,j,k}, \alpha_R, \beta_R)\,\Delta\mathbf{\Phi}_{i+3/2,\,j,\,k} \\ &= \overline{\mathbf{\Phi}}_{i+1,\,j,\,k} - 0.5\max(0, \min(\alpha_R, \alpha_R r^{\xi}_{R,i+1,j,k}, \beta_R))\,\Delta\mathbf{\Phi}_{i+3/2,\,j,\,k}\end{aligned} \tag{4b}$$

where α is the multi-dimensional restriction coefficient which determines the baseline region of MLP and β is the local slope evaluated by higher order polynomial interpolation.

The interpolated values of $\mathbf{\Phi}^{L}_{i+1/2,\,j,\,k}$ and $\mathbf{\Phi}^{R}_{i+1/2,\,j,\,k}$ are based on the final form of MLP. Since the calculations of interpolated values are independent of numerical flux scheme, MLP can be combined with any numerical flux. Values of $\alpha_{L,R}$ and $\beta_{L,R}$ in Eq.(4) are summarized as follows.

Along the ξ-direction, if $\Delta\Phi_{\xi}^{p} \geq 0$,

$$\alpha_L = g\left[\frac{2\max\left(1,\ r_{L,i,j,k}^{\xi}\right)\left(\overline{\Phi}_{p,q,r}^{max} - \overline{\Phi}_{i,j,k}\right)}{\left(1 + \frac{\Delta\Phi_{\eta}^{q}}{\Delta\Phi_{\xi}^{p}} + \frac{\Delta\Phi_{\zeta}^{r}}{\Delta\Phi_{\xi}^{p}}\right)_{i,j,k} \Delta\Phi_{i+1/2,j,k}}\right],$$

$$\alpha_R = g\left[\frac{2\max\left(1,\ \frac{1}{r_{R,i+1,j,k}^{\xi}}\right)\left(\overline{\Phi}_{p,q,r}^{min} - \overline{\Phi}_{i+1,j,k}\right)}{\left(1 + \frac{\Delta\Phi_{\eta}^{q}}{\Delta\Phi_{\xi}^{p}} + \frac{\Delta\Phi_{\zeta}^{r}}{\Delta\Phi_{\xi}^{p}}\right)_{i+1,j,k} \Delta\Phi_{i+3/2,j,k}}\right]. \qquad (5)$$

where $r_{L,i,j,k}^{\xi} = \frac{\Delta\Phi_{i+1/2,j,k}}{\Delta\Phi_{i-1/2,j,k}}$, $r_{R,i+1,j,k}^{\xi} = \frac{\Delta\Phi_{i+1/2,j,k}}{\Delta\Phi_{i+3/2,j,k}}$ and $g(x) = \max(1,\ \min(2,\ x))$. Along the η- and ζ-direction, the left and right values at the cell-interface can be calculated in the same way.

Combining Eq.(5) with β in the form of third order polynomial and fifth order polynomial, we finally obtain MLP3 and MLP5, respectively. Detailed explanation is given in Ref. [6].

MLP with using 3rd order polynomial (MLP3) :

$$\beta_L = \frac{1 + 2r_{L,i,j,k}^{\xi}}{3} \qquad (6a)$$

$$\beta_R = \frac{1 + 2r_{R,i+1,j,k}^{\xi}}{3} \qquad (6b)$$

MLP with using 5th order polynomial (MLP5) :

$$\beta_L = \frac{-2/r_{L,i-1,j,k}^{\xi} + 11 + 24r_{L,i,j,k}^{\xi} - 3r_{L,i,j,k}^{\xi} r_{L,i+1,j,k}^{\xi}}{30} \qquad (7a)$$

$$\beta_R = \frac{-2/r_{R,i+2,j,k}^{\xi} + 11 + 24r_{R,i+1,j,k}^{\xi} - 3r_{R,i+1,j,k}^{\xi} r_{R,i,j,k}^{\xi}}{30} \qquad (7b)$$

3. Numerical Results

Here, we consider a three-dimensional normal shock discontinuity in order to investigate the shock-capturing characteristics of TVD MUSCL limiters and MLP. This test shows the advantages of MLP clearly in terms of monotonicity and convergence.

We can observe wiggles which indicate that there are considerable oscillations across the shock discontinuity. Figure 1 shows density contours and the error history of convectional limiter. CFL number is 5.0 and LU-SGS is used for time integration. Even if the test case is relatively simple, van leer limiter is never converged due to oscillatory behavior

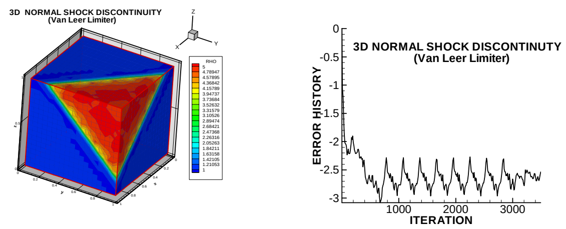

Fig .1 Density contours and error history : van leer limiter

across the normal shock discontinuity. On the other hand, MLP-van leer limiter shows smooth contours in the post-shock region and good convergence characteristics as in Fig.2.

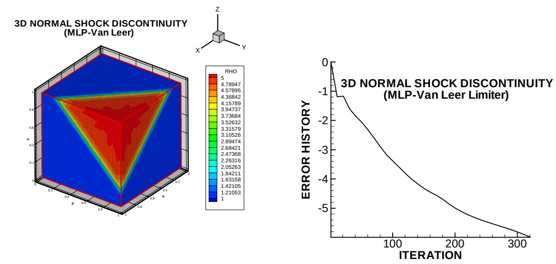

Fig .2 Density contours and error history : MLP-van leer limiter

4. Conclusion

The multi-dimensional limiting process (MLP) is developed by combining the multi-dimensional limiting function with a higher order polynomial interpolation. The newly developed method turns out to have several desirable characteristics such as multi-dimensional monotonicity across a discontinuity, robust convergence. In addition, higher order interpolation can be easily incorporated.

The most distinguishable property of MLP is to provide non-oscillatory profiles in multi-dimensional flows and, as a result, exhibits a good convergence characteristic. Through several test cases, it is verified that MLP can control numerical oscillations in multiple space dimensions very effectively. From the numerical results, MLP provides substantial accuracy improvement, compared with TVD MUSCL approach using popular flux functions.

As the outcomes in two-dimensional flows [5], MLP is also proved to bring enhanced convergence and accuracy improvement simultaneously in three-dimensional compressible flows.

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