ACCURATE AND EFFICIENT RIEMANN SOLVERS FOR EULER AND NAVIER-STOKES EQUATIONS

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ABSTRACT

The present paper deals with an improvement of efficient and accurate numerical flux schemes for the hyperbolic conservation laws of aerodynamics. Due to numerical approximation and linearization, many flux schemes suffer shock instability and unwanted oscillations. The proposed schemes, called AUSMPW+ and RoeM, cure these problems and numerical tests show the robustness, accuracy and efficiency of both flux schemes.

INTRODUCTION

In order to demonstrate the advective features of aerodynamics, hyperbolic conservation laws, especially the Euler and Navier-Stokes equations, are used as the governing equations. Unfortunately, due to the high non-linearity of these equations, it is alomost impossible to adopt analytical approach for practical problems, thus the discretization of these equations for computational approach is important. Among many discretization methods, the finite volume method (FVM) is very popular, because it properly reflects the flow physics involving physical discontinuities. With the pioneering work of Godunov [1], the FVM of non-linear hyperbolic conservation laws can be tractable by solving a local Riemann problem at a cell interface. To illustrate this, let us consider one-dimensional hyperbolic conservation law,

$$w_t + f_x(w) = 0. \tag{1}$$

Godunov method of this equation can be written as follows.

$$W_i^{n+1} = W_i^n + \Delta t/\Delta x\left[F_{i+1/2} - F_{i-1/2}\right], F_{i+1/2} = f\left(W_{i+1/2}(0)\right), \tag{2}$$

where W_i^n is the averaged value of *i-th* cell at *n-th* time step and $F_{i+1/2}$ is the numerical flux function at interface between *i-th* cell and *i+1-th* cell. $W_{i+1/2}\left(\bar{x}/\bar{t}\right)$ is a local Riemann solution. In order to avoid tremendous numerical efforts for the exact Riemann solver, the efficient numerical flux is required, and two types of approach have been developed: one is flux difference splitting (FDS) approach; the other is flux vector splitting (FVS) approach. Though some numerical fluxes of both approaches are applied successfully in many problems, these flux functions often suffer from shock instability, typically known as a carbuncle problem, or unwanted oscillation near the wall or post strong shock region.

In present paper, we propose two numerical fluxes, called RoeM [2] and AUSMPW+ [3]. Originating from Roe-type FDS scheme [4], RoeM scheme cures carbuncle phenomenon by balancing feeding and damping rate with Mach number-based functions. AUSMPW+ scheme is an improvement of AUSM-type schemes [5] by introducing pressure-based weighting functions to restrict oscillations and to prevent shock instability. Both flux schemes enhance robustness of original schemes and maintain high accuracy and good efficiency.

IMPROVEMENT OF ROE-TYPE SCHEMES: ROEM SCHEME

The Baseline Scheme: Roe's FDS

FDS approach solves a local Riemann problem approximately with proper assumption. Roe scheme, one of the most widely used FDS flux schemes, relies on the linearization of flux by parameterized state and flux vector, thus it is efficient to solve the local Riemann problem and has good shock capturing characteristics. Formulation of this scheme can be written as follows.

$$F_{i+1/2} = 1/2\left(f_i + f_{i+1}\right) - 1/2\sum\left|\hat{A}\right|\Delta W \,, \tag{3}$$

where $\hat{A}$ is the flux jacobian matrix estimated at the cell interface. Although Roe scheme shows remarkable accuracy, it does not distinguish a shock from an expansion discontinuity, which violates the entropy condition, and more seriously, suffers from carbuncle phenomena, which hampers robustness and accuracy. In order to overcome these problems, RoeM scheme is proposed.

RoeM Scheme

The reason of carbuncle phenomena is not clearly explained until now, but several hypotheses have been proposed. Among them, the improper numerical dissipation is regarded as a cause of these phenomena. By examining the mass flux of the HLLE scheme, which is known as a carbuncle-free scheme, it suggests that the pressure difference term in mass dissipation is not damped properly, especially in the region where normal Mach number goes to zero. In order to control this term, the Mach number based function *f* is designed, and it switch off pressure difference term near problematic region.

$$f = \left|\hat{M}\right|^{h} \ \text{except} \ \hat{u}^2 + \hat{v}^2 = 0 \,, \tag{4}$$

where $\hat{M}$ is the estimated Mach number at the cell interface, and the function *h* detects the shock discontinuity near the cell.

Though most numerical tests show that the function *f* prevents shock instability successfully, it is not sufficient in some test cases, where damping rate is small compared with the feeding rate of pressure perturbation. In order to balance these rates, another Mach number based function *g* is introduced, which is written as follows.

$$g = \left|\hat{M}\right|^{1-\min\left(p_j/p_{j+1},\, p_{j+1}/p_j\right)} \ \text{except} \ \hat{M} = 0 \,. \tag{5}$$

Moreover, the diffusion term of energy should be modified to preserve the total enthalpy in inviscid steady flow. With these modifications, the RoeM scheme can be formulated by following form.

$$F_{i+1/2} = \frac{b_1 \times F_i + b_2 \times F_{i+1}}{b_1 - b_2} + \frac{b_1 \times b_2}{b_1 - b_2}\Delta Q^* - g\frac{b_1 \times b_2}{b_1 - b_2} \times \frac{1}{1+\left|\hat{M}\right|} B\Delta Q \,, \tag{6}$$

$$\Delta Q^* = \Delta\begin{pmatrix}\rho \\ \rho u \\ \rho v \\ \rho H\end{pmatrix},\ B\Delta Q = \left(\Delta\rho - f\frac{\Delta p}{\hat{c}^2}\right)\begin{pmatrix}1 \\ \hat{u} \\ \hat{v} \\ \hat{H}\end{pmatrix} + \hat{\rho}\begin{pmatrix}0 \\ \Delta u - n_x\Delta U \\ \Delta v - n_y\Delta U \\ \Delta H\end{pmatrix}. \tag{7}$$

In order to avoid expansion shock and capture contact and shock discontinuity accurately, the following estimation of wave speed is considered for the RoeM scheme

$$b_1 = \max\left(0, \hat{U} + \hat{c}, U_{i+1} + \hat{c}\right),\ b_1 = \max\left(0, \hat{U} - \hat{c}, U_{i-1} - \hat{c}\right). \tag{8}$$

IMPROVEMENT OF AUSM-TYPE SCHEMES: AUSMPW+ SCHEME

AUSM-type Approach

Starting from upwind schemes of linear hyperbolic conservation system, FVS-type flux at the interface can be decomposed as follows.

$$F(W) = F^{+}(W) + F^{-}(W), \tag{9}$$

$$\lambda^{+} = \lambda\left(\partial F^{+}/\partial x\right) \geq 0\,, \lambda^{-} = \lambda\left(\partial F^{-}/\partial x\right) \geq 0\,, \tag{10}$$

where $\lambda(A)$ is the eigenvalue of the matrix A. Several schemes have been developed by properly choosing a splitting of eigenvalue. AUSM-type schemes consider the flow characteristics of convection and pressure separately, thus the numerical flux can be expressed by convection term and pressure term as follows.

$$F_{i+1/2} = F^{(c)}_{i+1/2} + P_{i+1/2} = m_{1/2} c_{1/2} \Phi_{L/R} + P_{i+1/2}\,, \tag{11}$$

where $m_{1/2}$ and $c_{1/2}$ is the estimated Mach number and speed of sound at the cell interface, and $\Phi_{L/R}$ is the one-sided convection quantity. Although AUSM-type scheme is originated from FVS approach, the final form of AUSM shares some similarities with FDS. Thus, it often called hybrid-type of FDS and FVS approach.

AUSMPW+ Scheme

AUSM+ scheme improves the accuracy and shock instability of the original AUSM scheme, but it suffers from unwanted oscillation near the wall and overshoots behind strong shock. On the other hand, AUSMD/V scheme does not show oscillations and overshoots but it has carbuncle phenomena. Comparing the mass fluxes of both schemes, the pressure-based weight function *f* is introduced to remove oscillations.

$$\rho u_{1/2,AUSMPW+} = \left(M^{+}_{L} c_{1/2} \rho_L + M^{-}_{R} c_{1/2} \rho_L\right)_{AUSM+} + f_L M^{+}_{L} c_{1/2} \rho_L + + f_R M^{-}_{R} c_{1/2} \rho_R\,. \tag{12}$$

Unlike AUSM+ scheme, AUSMD/V and AUSMPW+ have pressure difference term on the mass flux, which can potentially induce shock instability, thus the operation of the function *f* should be limited near strong shock and stagnation point.

$$f_{L,R} = \left(p_{L,R} / p_s - 1\right) \min\left(1, \min\left(p_{1,L}, p_{1,R}, p_{2,L}, p_{2,R}\right) / \min\left(p_L, p_R\right)\right)^2. \tag{13}$$

Unfortunately, since AUSM+ only uses one-sided property, the function *f* is not sufficient to get rid of overshoots behind shock. Thus, we introduce another pressure-based weight function *w*, in order to consider both-side properties.

$$w(p_L, p_R) = 1 - \min\left(p_L / p_R\,, p_R / p_L\right)^3. \tag{14}$$

With two pressure-based weight functions, the numerical flux of AUSMPW+ is written as

$$F_{1/2} = \overline{M}_L^+ c_{1/2} \Phi_L + \overline{M}_R^- c_{1/2} \Phi_R + \left(P_L^+ p_L + P_R^- p_R\right), \tag{15}$$

where $m_{1/2} \geq 0$,

$$\overline{M}_L^+ = M_L^+ + M_R^- \left[(1-w)\cdot(1+f_R) - f_L\right],\ \overline{M}_R^- = M_R^- \cdot w \cdot (1+f_R). \tag{16}$$

In AUSM-type schemes, the choice of the numerical speed of sound is crucial to capture discontinuity. In order to improve resolution of oblique shock and to prevent entropy violating solution, the new definition of the numerical speed of sound is introduced satisfying the Prandtl relation across an oblique shock.

$$c_{1/2} = c_s^2 / \max\left(\left|U_{L/R}\right|, c_s\right),\ c_s = \sqrt{2(\gamma-1)/(\gamma+1) H_{normal}}\,. \tag{17}$$

CONCLUSION

In this paper, we introduce two numerical flux schemes for the hyperbolic conservation laws of aerodynamics: AUSMPW+ and RoeM schemes. Both schemes are very effective to remove unwanted oscillations and they maintain shock stability by the action of additional weighting functions. In the presentation, several numerical results which confirm the robustness, accuracy and efficiency of these schemes will be presented.

ACKNOWLEDGEMENTS

The author appreciates the financial supports provided by the second stage of the Brain Korea 21 Project for the Mechanical and Aerospace Engineering Research at Seoul National University, and by the Agency of Defense Development.

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