**More proofs for "Determination of Stability with respect to Positive Orthant for a Class of Positive Nonlinear Systems"**

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*Abstract:* In the published paper "Determination of Stability with respect to Positive Orthant for a Class of Positive Nonlinear Systems," *IEEE Trans. on Automatic Control,* vol. 53, no. 5, pp. 1329–1334, 2008, by the authors, some proofs are omitted due to the space limitation of the journal. In this note, we present those omitted proofs.

**Proof of Claim 1:** Define

$$\bar{T} := \begin{bmatrix} 1 & 0 \\ \bar{A}_{22}^{-1}\bar{A}_{21} & I \end{bmatrix}, \text{ then } \bar{T}^{-1} = \begin{bmatrix} 1 & 0 \\ -\bar{A}_{22}^{-1}\bar{A}_{21} & I \end{bmatrix}.$$

With $\bar{T}^{-1}$ and (10), it is verified that

$$(E_k A(0) E_k^{-1})\bar{T}^{-1} = \bar{T}^{-1}\begin{bmatrix} 0 & 0 \\ 0 & \bar{A}_{22} \end{bmatrix}.$$

With $T_0 := \bar{T}E_k$, it follows that

$$T_0 A(0) T_0^{-1} = (\bar{T}E_k)A(0)(\bar{T}E_k)^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & \bar{A}_{22} \end{bmatrix}.$$

Also, it holds that $(T_0)_{(1)} = (\bar{T}E_k)_{(1)} = e_k^T$ because the first and the $k$-th columns of $\bar{T}$ are exchanged in $\bar{T}E_k$. In addition, since $E_k^{-1} = E_k$, we obtain that

$$(T_0^{-1})^{(1)} = (E_k^{-1}\bar{T}^{-1})^{(1)} = E_k(\bar{T}^{-1})^{(1)} = E_k\begin{bmatrix} 1 \\ -\bar{A}_{22}^{-1}\bar{A}_{21} \end{bmatrix}.$$

Let $[\bar{T}_1, \bar{T}_2] := T_0^{-1}$ where $\bar{T}_1 \in \mathbb{R}^{n\times 1}$ and $\bar{T}_2 \in \mathbb{R}^{n\times(n-1)}$, and let $A_1(u) \in \mathbb{R}$ and $t_1(u) \in \mathbb{R}^{n\times 1}$ be $C^2$ functions of $u$ such that $A_1(0) = 0$, $t_1(0) = \bar{T}_1$, and

$$A(u)t_1(u) = A_1(u)t_1(u) \tag{31}$$

for each $u \in [0, \bar{u}]$. (Then, $A_1(u)$ and $t_1(u)$ are an eigenvalue and the corresponding eigenvector of $A(u)$, respectively.)

Let $\mathcal{T}(u) := [t_1(u), \bar{T}_2]$. Then, because $\mathcal{T}(0) = T_0^{-1}$, there exists a positive $\hat{u}_1 \le \bar{u}$ such that $\mathcal{T}(u)$ is nonsingular for $u \in [0, \hat{u}_1]$. Now let $D(u) := \mathcal{T}^{-1}(u)A(u)\mathcal{T}(u)$. Since $\mathcal{T}^{-1}(u)\mathcal{T}(u) = \mathcal{T}^{-1}(u)[t_1(u), \bar{T}_2] = I$, we have $\mathcal{T}^{-1}(u)t_1(u) = e_1$, which leads to

$$\begin{aligned} D(u) &= \mathcal{T}^{-1}(u)[A(u)t_1(u), A(u)\bar{T}_2] \\ &= [A_1(u)\mathcal{T}^{-1}(u)t_1(u), \mathcal{T}^{-1}(u)A(u)\bar{T}_2] \\ &= \begin{bmatrix} A_1(u) & A_{12}(u) \\ 0 & A_2(u) \end{bmatrix}, \end{aligned} \tag{32}$$

with some $A_{12}(u) : [0, \hat{u}_1] \to \mathbb{R}^{1\times(n-1)}$ and $A_2(u) : [0, \hat{u}_1] \to \mathbb{R}^{(n-1)\times(n-1)}$. It follows that

$$D(0) = \begin{bmatrix} 0 & 0 \\ 0 & \bar{A}_{22} \end{bmatrix},$$

which implies $A_{12}(0) = [0, 0, \cdots, 0]$. Since $A_1(0) = 0$ and $A_2(u)$ is Hurwitz for each $u \in [0, \hat{u}_1]$, there exists a $\tilde{u}_1 < \hat{u}_1$ such that $A_1(u)$ and $-A_2(u)$ have distinct eigenvalues for $u \in [0, \tilde{u}_1)$. As a result, there exists a unique (continuous) solution $S(u) : [0, \tilde{u}_1] \to \mathbb{R}^{1\times(n-1)}$ to the Sylvester equation

$$A_1(u)S(u) - S(u)A_2(u) = -A_{12}(u). \tag{33}$$

Here, it is also seen that $S(0) = [0, 0, \cdots, 0]$. Let

$$t_2(u) := \mathcal{T}(u)\begin{bmatrix} S(u) \\ I \end{bmatrix}. \tag{34}$$

Then, it follows that $t_2(0) = \bar{T}_2$. Therefore, we have $[t_1(0), t_2(0)] = [\bar{T}_1, \bar{T}_2] = T_0^{-1}$. This implies that there exists an $\bar{u}_1 \le \tilde{u}_1$ such that $[t_1(u), t_2(u)]$ is nonsingular for $u \in [0, \bar{u}_1]$.

We finally define

$$T(u) := [t_1(u), t_2(u)]^{-1} \tag{35}$$

for $u \in [0, \bar{u}_1]$, and obtain from (32) and (33) that

$$D(u)\begin{bmatrix} S(u) \\ I \end{bmatrix} = \begin{bmatrix} S(u) \\ I \end{bmatrix} A_2(u).$$

By virtue of (34), this leads to $D(u)\mathcal{T}^{-1}(u)t_2(u) = \mathcal{T}^{-1}(u)t_2(u)A_2(u)$, which implies that $A(u)t_2(u) = t_2(u)A_2(u)$. With (31), we have

$$A(u)[t_1(u), t_2(u)] = [t_1(u), t_2(u)]\begin{bmatrix} A_1(u) & 0 \\ 0 & A_2(u) \end{bmatrix},$$

which, together with (35), completes the proof. $\Diamond$

**Proof of Claim 2:** We begin with the idea taken from [3], where the existence of center manifold for a *parameter-dependent* system is proved. System (18) is regarded as

$$\begin{aligned} \dot{z}_1 &= g_1(z_1, z_2, u) \\ \dot{z}_2 &= \bar{A}_{22}z_2 + g_2(z_1, z_2, u) \\ \dot{u} &= 0. \end{aligned} \tag{36}$$

Then the $(z_1, u)$-dynamics has its Jacobian matrix at the origin whose eigenvalues have zero real parts. Therefore, the standard existence proof of center manifold gives a function $(z_1, u) \mapsto \pi(z_1, u)$ such that $\pi(0, 0) = 0$, $\frac{\partial \pi}{\partial z_1}(0, 0) = 0$, and (21) holds for all $0 \le u \le \bar{u}_2$ and $|z_1| \le \bar{r}_1$ with some $\bar{r}_1 > 0$ and $0 < \bar{u}_2 \le \bar{u}_1$. However, to complete the proof, we have to show that $\pi(0, u) = 0$ for $0 \le u \le \bar{u}_2$. This is obtained with a slight modification of the standard proof. Instead of repeating the whole proof here, we refer to the proof of [6, Lemma C.6] with the following key ingredient.

Let $S$ be the set of functions $\pi(\cdot, \cdot)$ such that (1) it is continuously differentiable, (2) $\pi(0, u) = 0$, $\frac{\partial \pi}{\partial z_1}(0, 0) = 0$, and $\frac{\partial \pi}{\partial u}(0, 0) = 0$, (3) $|\pi(z_1, u)| \le c_1$, (4) $\left|\frac{\partial \pi}{\partial (z_1, u)}(z_1, u)\right| \le c_2$, and (5) $\left|\frac{\partial \pi}{\partial (z_1, u)}(z_1, u) - \frac{\partial \pi}{\partial (z_1, u)}(\hat{z}_1, \hat{u})\right| \le c_3|(z_1, u) - (\hat{z}_1, \hat{u})|$ with some positive constants $c_1$, $c_2$ and $c_3$. Now define a map on the set $S$ as

$$\begin{aligned} (P\pi)(z_1, u) := \int_{-\infty}^{0} &\exp(-\bar{A}_{22}s) \\ &\times g_2\left(\phi(s; (z_1, u), \pi), \pi(\phi(s; (z_1, u), \pi), u), u\right) ds, \end{aligned} \tag{37}$$

where $\phi(s; (\xi, u), \pi)$ is the solution[2] of

$$\dot{z}_1 = g_1(z_1, \pi(z_1, u), u), \qquad z_1(0) = \xi. \tag{38}$$

Then, the existence of the function $\pi$ of our purpose is proved by the contraction mapping theorem if we prove that those properties (1)–(5) also hold for the function $(z_1, u) \mapsto (P\pi)(z_1, u)$ (i.e., the set $S$ is mapped into $S$ by (37)), and that the map (37) is contracting so that it has a fixed point $\pi$ (i.e., $\pi = (P\pi)$). In fact, the contracting property and the properties (1), (3), (4) and (5) are easily proved just as in [6]. In order to prove (2), that is, $(P\pi)(0, u) = 0$, we note that $\phi(s; (0, u), \pi) = 0$ for $s \ge 0$, that is, $z_1 = 0$ is an equilibrium of

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[2]In the original proof, the function $g_1$ and $g_2$ are modified to have zero values outside a local neighborhood so that the solution exists for all positive time and the integral (37) is well-defined. We do not get into details here, but assume just that $g_1$ and $g_2$ have such nice properties.

(38) since $g_1(0, 0, u) = 0$ and $\pi \in S$ so that $\pi(0, u) = 0$. Then, it is obvious that $(P\pi)(0, u) = 0$. ♢

**Proof of Theorem 1:** Since system (3) can be viewed as system (7) with $u = 0$, it directly follows from Theorem 2 that $x^*$ is locally asymptotically stable w.r.t. $\mathbb{R}_+^n$ when $\frac{\partial^2\psi}{\partial s^2}(0) < 0$.

In order to prove the unstability, note that system (22) with $u = 0$ (that is equivalent to (3)) is rewritten as

$$\begin{aligned} \dot{z}_1 &= c_1(0)z_1^2 + o(z_1^2) + N_1(z_1, w, 0) \\ \dot{w} &= \bar{A}_{22}w + N_2(z_1, w, 0) \end{aligned} \quad (39)$$

in which the origin corresponds to the equilibrium $x^*$ of (3), and $c_1(0) > 0$ from the proof of Claim 3. Then, there exists a $\bar{z}_1 > 0$ such that the solution of (39) initiated from $(\bar{z}_1, 0)$ diverges from the origin, which implies that $x^*$ is unstable w.r.t. $\mathbb{R}_+^n$. ♢

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