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Cost Effectiveness of Postponed Routing Decisions in a Supply Chain

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Abstract

This paper examines the cost-reduction effect of “postponed” routing decisions in a two-echelon supply chain. When a delivery vehicle leaves the warehouse with system replenishment quantities, it sequentially makes routing decisions to choose the next retailer to visit. To measure this effect this paper studies combined vehicle routing and inventory-allocation policies designed to minimize total expected purchasing, inventory-holding and backordering cost/period for a one-warehouse N-retailer “symmetric” distribution system. Using a numerical study, we demonstrate that optimal dynamic routing and allocation policies can significantly reduce the inventory-management costs associated with fixed routing and dynamic allocation in “medium-to-large” customer demand-variance scenarios.

Keywords: SCM, postponement, dynamic routing

INTRODUCTION

One of the key issues in SCM is to answer how to cope with uncertainty in demand. Fisher(1997) suggested some tools to manage demand uncertainty and one of them is *postponement*. A company can avoid uncertainty by cutting lead times and increasing supply chain’s flexibility so that it can respond to not

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forecasted but materialized demand as much as possible. In particular, a company can avoid some demand uncertainty by postponing its routing and allocation decisions later in time so that it can replenish retailers based on their most up-to-dated inventory status. This paper examines the risk-pooling effect of *postponed* routing decisions in a two-echelon supply chain consisted of one warehouse and multiple retailers. To measure this effect this paper studies the routing and allocation policies for managing a one-warehouse N-retailer system facing stochastic demand and managed under a periodic-review policy. *The routing policy* specifies the sequence in which the retailers are visited, and *the inventory-allocation policy* how the system-replenishment quantity is allocated among the retailers. In the specific system examined, the warehouse places a system-replenishment order with an outside supplier every m periods, receiving it after a fixed leadtime of L periods. Upon receipt, a delivery vehicle starts from the warehouse with the system-replenishment quantity(the warehouse holds no inventory), visits each retailer once and only once, allocating its inventory to the retailers along its delivery route. We seek a distribution policy that minimizes the expected system purchasing, inventory-holding, and backorder costs per period over an infinite number of time periods.

We examine two kinds of routing: fixed and dynamic. Under *fixed routing*, the delivery vehicle visits each retailer along a predetermined route that does not change over time. Consequently, the time interval between successive allocations to each retailer is m periods. Under *dynamic routing*, which represents postponed routing decisions, a decision-rule is used to decide which retailer to visit next, based on the inventory status of the retailers not yet visited. We also examine two different types of inventory allocation: static and dynamic. *Static allocations* are determined for the entire route at the moment the delivery vehicle leaves the warehouse, based on the system inventory status at that time. *Dynamic allocations* are determined sequentially upon arrival of the vehicle at each retailer, based on its inventory status and the inventory status of the retailers not yet visited

It is important to note that dynamic routing has the potential to decrease *or* increase the uncertainty of each retailer's inventory process, and, consequently, the expected inventory costs of the system. That is, although dynamic routing allows the vehicle to expedite an allocation to a retailer whose inventory is "low," or to delay an allocation to a retailer whose inventory is "high", one consequence for the *system* is uncertainty in the number of time periods between every retailer's successive allocation. All other things being equal, this uncertainty *increases* the uncertainty of each retailer's net inventory. Hence, the optimal dynamic routing policy can be viewed as balancing the increased uncertainty in retailer inventory(induced by changing routes) against the reduced uncertainty in retailer inventory(resulting from management's ability to expedite or delay allocations). We are not aware of any supply-chain models that examine this tradeoff. Most of the value of a dynamic routing resides in its capability to control the variance of retailer net inventories when they are "out of balance." Therefore, all dynamic routing policies do not guarantee net benefit compared to the fixed routing policy. For example, the dynamic routing policy of choosing randomly the next retailer to visit increases overall uncertainty and costs. But under the dynamic routing policy of visiting the most "under-inventoried" retailer next, the benefit of the reduced uncertainty in retailer inventory by expedition outweighs the cost of increased uncertainty caused by changing routes. The logic behind this is that replenishing the retailer in most need first can quickly restore balance in net inventories among retailers, that is, maximizes the probability that allocated units will be sold in the(near) future and avoids possible backorders(The detailed proof for the N-retailer symmetric case is given in appendix A.).

Our research has two principal objectives: First, to validate the use of dynamic routing to reduce overall system uncertainty(i.e., to assess the potential of dynamic routing and dynamic allocation to reduce cost given a baseline policy involving fixed routing and allocation). Second, to examine how dynamic routing avoid demand uncertainty. In doing so we ignore factors that would normally be important in vehicle routing, such as the "traveling salesman" aspects of minimizing travel distances, as well as any operational benefits from fixed routes. Even with this simplification, dynamic routing is still quite complex to model.

In order to limit this complexity, our analysis focuses on two-retailer and N-retailer “symmetric” systems. “Symmetric” means that the retailers are identical(i.e., have identical costs and face identically-distributed period demand); and, further: (1) the vehicle takes the same number of time periods, a, to travel between each retailer and the warehouse and; (2) that the same number of time periods, b, transpire between successive allocations to any pair of retailers on any route.

Although a system in which travel *distances* are truly symmetric is unrealistic, we believe that to the extent that the retailers are clustered together, a symmetric approximation of travel *times* may be reasonable. Furthermore, to the extent that the unloading and paperwork time at each retailer is fixed and a significant portion of between-retailer travel time, then the number of time periods between successive allocations to any pair of retailers on any given route may be approximately the same even if the travel time is not. Finally, it should be noted that we have ignore transportation costs, or, equivalently, assume that they are not affected by changing routes. Kumar, Schwarz, and Ward(1995) also ignored transportation costs, but since their model employs fixed routes, transportation costs are correspondingly fixed. Our reason for doing so is that asymmetric transportation costs pose the same kind of complexity in the analysis as asymmetries in between-warehouse-and-retailer travel times or asymmetries in the times between successive allocations to any pair of retailers on any given route.

Additional assumptions are as follows: If retailer inventory is not sufficient to meet demand, then shortages are backordered. We further assume that these shortages occur only in the period before each retailer’s next allocation, as in Kumar, Schwarz, and Ward(1995) and Jönsson and Silver(1987a, 1987b). Per-unit acquisition, inventory-holding, and backorder costs are assumed to be constant. We also assume the existence of an information system capable of supporting dynamic allocation and routing.

Our major results for N symmetric retailers are: (1) there exists an optimal routing and allocation policy that incorporates routing the vehicle to the retailer with the least inventory first (LIF); (2) under the “allocation assumption”, myopic replenishment and allocation policies are optimal; (3) our numerical experiments indicate that LIF routing and dynamic allocation can significantly reduce the inventory-management costs associated with fixed routing and dynamic allocation (Kumar, Schwarz, and Ward's policy) in "medium-to-large" customer demand-variance scenarios, which validates the cost effectiveness of postponed routing decisions. Although our model is clearly stylized, we believe that it contributes to the supply-chain literature by examining the tradeoff between increased/decreased variance in retailer net inventory brought about by dynamic routing, and developing some intuition about it.

RELATED RESEARCH

There have been many articles on replenishment and allocation policies for multi-echelon distribution systems. Graves (1996) provides a brief review of the works on the multi-echelon distribution systems with both deterministic and stochastic demand. Research directly related to ours is as follows:

Federgruen and Zipkin(1984b), Anily and Federgruen(1990, 1993), and Gallego and Simchi-Levi(1990) integrate inventory decisions with routing considerations. In particular, Federgruen and Zipkin(1984b) analyze a combined vehicle-routing and inventory-allocation problem with stochastic demand. In their model, allocation is static and routing is fixed. Their objective is to determine a joint route-allocation strategy that minimizes the sum of expected inventory cost and transportation cost for the entire system. In their model the interdependence between routing and allocation arises from the fact that while the optimal allocation may prescribe a positive allocation to some particular retailer, the cost of routing the vehicle through that retailer may exceed the savings achieved by that allocation. Another source of interdependence is the vehicle capacities. Overall savings accruing from the joint consideration of the inventory-allocation and routing decisions, of 5-6% is reported. Anily and Federgruen (1990) study the dynamic vehicle-routing and inventory problem in one-warehouse multiple-retailer systems when demand is deterministic.

Most dynamic-routing research focuses on dynamic vehicle-routing problem(VRP), wherein, as in our model, delivery routes are determined dynamically based on real-time information. Its application areas include fleet management(see Powell(1986)), traffic assignment(Friesz, Luque, and Wie 1989), air traffic control(Vranas, Bertsimas, and Odoni 1994). See Bertsimas and Simchi-Levi(1996) for a complete review of VRP. What differentiates our work from dynamic VRP is that dynamic VRP dynamically decides a set of customers served by a specific route, equivalently, a specific vehicle, while dynamic routing in our problem dynamically decides a sequence in which a given set of retailers are visited.

Campbell et al.(1997) describe some of the challenges in modeling combined routing and inventory-allocation scenarios. The scenario they examine is both more complex — involving "asymmetric" transportation costs and vehicle capacity — and simpler — retailer demand is known and constant and routes, once determined, are fixed. Nonetheless, they observe what we observe: "This long-term control problem is already hard to formulate, it is almost impossible to solve". Fortunately, as we demonstrate, the symmetric version of the problem is amenable to formulation and solution.

Kumar, Schwarz, and Ward(1995) examine static and dynamic policies for replenishing and allocating inventories amongst N retailers located along a fixed route. Their major analytical results, under the appropriate dynamic(static) allocation assumption, are: (1) optimal allocations under each policy involve bringing each retailer's "normalized-inventory" to a corresponding "normalized" system inventory; (2) optimal system replenishments employ base-stock policies; (3) the minimum expected cost per cycle of the dynamic(static) policy can be derived from an equivalent dynamic(static) "composite retailer". Given this, they prove that the "risk-pooling incentive", a simple measure of the benefit from adopting dynamic allocation policies, is always positive. Simulation tests confirm that dynamic-allocation policies yield lower costs than static policies, regardless of whether or not their respective allocation assumptions are valid. The magnitude of the cost savings, however, is sensitive to some system parameters.

This paper is organized as follows: Section 3 describes the N-retailer symmetric system and establishes the optimality of LIF routing. In Section 4 we describe some of the complexity in modeling the infinite-horizon problem. In Section 5, we formulate the corresponding myopic replenishment-allocation problem, and show that under the allocation assumption, the optimal myopic policy is optimal in the infinite horizon. Section 6 derives some important properties of the optimal myopic inventory-allocation policy in the two-retailer case. Section 7 compares the computer-simulated performance of the optimal myopic-allocation policy with the baseline policy. Finally, Section 8 summarizes our results, and provides insights and guidelines.

A ONE-WAREHOUSE N-RETAILER SYMMETRIC SYSTEM

A one-warehouse N-retailer "symmetric" system is defined to be one in which: (1) all retailers face identical demand distributions and experience identical marginal inventory-holding and backordering costs; and (2) the delivery vehicle requires the same number of time periods, a, to travel from/to the warehouse, and the same number of time periods, b, transpire between successive allocations to pair of retailers on any route. Let R^i represent retailer i, $i = 1, 2$. Figure 1 shows the system when $N = 2$.

The warehouse places a system-replenishment order every m periods, which arrives after a fixed leadtime L. Without loss of

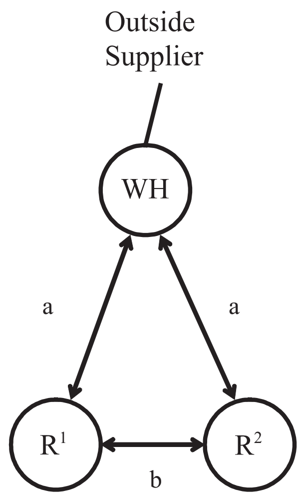

Figure 1. The Two-Retailer Symmetric System

generality, we assume that the first system-replenishment order is placed at time 0. Correspondingly, the tth system-replenishment order will be placed at time $(t-1)m$. Upon receipt of each system-replenishment order at the warehouse, the first routing decision is made(i.e., which retailer to go to first), and the vehicle begins its route. Given any realized route, let R_i denote the ith retailer on the route. The vehicle arrives at R_1 a periods after the first routing decision, allocates part or all of its inventory, and makes the second routing decision(i.e., which retailer to go to next). After an additional b periods, the vehicle arrives at R_2 allocates part or all of its remaining inventory, and makes the next routing decision. This continues until the allocation at $R_{N\text{-}1}$. Once $R_{N\text{-}1}$ receives its allocation, all remaining inventory is, in effect, allocated to R_N, although it is delivered b periods later. The same sequence of decisions is repeated every m periods. Figure 2 shows a hypothetical time-line using $N = 2$, $L = 0$, $m = 4$, $a = 2$, and $b = 1$. Note that the identities of R_1 and R_2 are route specific, and, in general, will change over time. We assume $m \geq (N-1)b$, which guarantees that retailer-replenishments do not cross; that is, the tth allocation quantity is delivered to a given retailer before(or at the same time as) the $(t+1)$st allocation quantity. Order-crossing, even in a single-location inventory setting, considerably complicates the analysis. (Kaplan 1970; Nahmias 1979; Ehrhardt 1984)

Define the tth replenishment cycle as the m period cycle between successive replenishment orders. Figure 2 illustrates. Denote by R^i, $i = 1, \ldots, N$, the permanent identity of the N retailers(Note that the identity of Ri is not route dependent, as is R_i.). Define C_{it}, $i = 1, \ldots, N$ to be the set of contiguous time periods between the vehicle's tth and $(t+1)$st visits to R^i, and define $\{C_{1t}, \ldots, C_{Nt}\}$ as the tth *allocation cycle*. Under a fixed routing policy, each C_{it} contains exactly m periods for all t values). However, under dynamic routing, the number of time periods in each C_{it}, and which particular periods are first and last, depends on the tth and $(t+1)$st routing decisions. Note, for example, the first and second allocation cycles for each retailer in Figure 2 differ in the number of periods they contain.

Note further that the allocation cycle is, in general, not contained in the replenishment cycle. For example, in figure 2, period 5 is in the first allocation cycle(for one of the retailers),

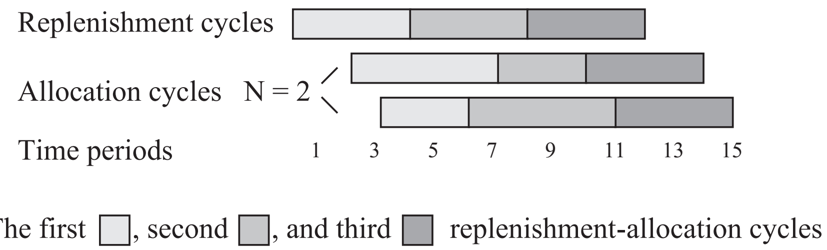

Figure 2. Replenishment-allocation cycles

but in the second replenishment cycle. Also note that the sequence of replenishment cycles is a partition of all periods and, similarly, that the sequence of allocation cycles partitions all periods for each retailer. Finally, define the tth *replenishment-allocation cycle* as the union of the tth allocation cycle and the tth replenishment cycle. Hence, the sequence of replenishment-allocation cycles is a partition of all periods for the system. The myopic policies to be examined below are based on replenishment-allocation cycles.

Least-Inventory-First(LIF) Routing

Define the least-inventory-first routing policy(LIF) as the policy under which the delivery vehicle goes next to the not-yet-visted retailer with the smallest inventory position.

Theorem 1: *There exists an optimal routing and allocation policy for N symmetric retailers that incorporates LIF routing.*

Proof: See appendix A.

Since an optimal routing and allocation policy can be found among those with LIF routing, we will henceforth limit all routing to be LIF.

THE COMPLEXITY OF SOLVING THE INFINITE-HORIZON PROBLEM

Given LIF routing, we would like to find a routing and allocation policy that minimizes total expected cost/period over an infinite number of replenishment-allocation cycles. Unfortunately, solving the infinite horizon problem is not practical(Park(1997) provides a dynamic-programming formulation of the total expected discounted cost, infinite-horizon problem.). In order to understand the major difficulties, note that in the fixed-routing case, each retailer's allocation cycle has a fixed number of time periods. Hence, it is known at the time of each allocation decision when each retailer will receive its *next* allocation. In contrast, under LIF routing, each retailer's allocation cycle has a yet-to-be-determined number of periods, depending on the route realized during the *next* replenishment-allocation cycle, which will be determined by the current allocations and the demand realizations at all retailers between the current allocation decision and the last routing decision of the *next* allocation cycle. Hence, to compute the expected costs of the tth allocation cycle, expectations must be taken not only with respect to future demand realizations but also with respect to the *next* realized route.

A second complication is that the probability distribution of each retailer's demand during its allocation cycle is generally not a "standard" distribution, even if period-demand is. Again, for comparison's sake: in the fixed-routing case(Kumar, Schwarz, and Ward 1995), where retailer period-demand is normally distributed, the distribution of each retailer's demand during its allocation cycle is also normal, since the sum of a fixed number of normal observations is normal. However, under LIF routing, even though retailer demand each period is normally distributed, the distribution of each retailer's demand during its allocation cycle is not normal, since the probabilistically-weighted sum of a random number of normal observations is not normal.

Given the complexity of the infinite-horizon problem, we turn to the corresponding "myopic"(i.e., single replenishment-allocation cycle) problem, and show that, under the well-known allocation assumption, its solution is optimal in the infinite-horizon problem.

THE MYOPIC REPLENISHMENT-ALLOCATION PROBLEM

The *myopic replenishment-allocation problem* is the problem in which the system-replenishment and allocation decisions during *a specific* replenishment-allocation cycle are chosen to minimize the expectation of the sum of the purchasing, inventory-holding and backorder costs assigned to that cycle, without regard to their impact on costs in subsequent cycles. For simplicity of presentation, we will formulate the case when $N = 2$, the system-replenishment leadtime is zero(i.e., $L = 0$) and $m \geq a$. For $m < a$, at the time of system-replenishment decision, the previous route and allocations have not yet been determined. Hence, one has to define additional state variables for the dynamic program that represent the subsystem of the retailers to be visited on the previous route and the amount of inventory left on the vehicle. This only makes the presentation more difficult to understand, without changing the nature of the optimal policy. Nonetheless, all of the results derived in this section, particularly Theorem 2, hold in the general case.

We use the following notation. Recall that R_1 and R_2 denote the retailers visited first and second, respectively, in the cycle.

c = purchasing cost per unit;
h = inventory-holding cost per unit per period for units held either at any retailer or on the delivery vehicle;
p = backorder cost per unit per period;
q = system-replenishment quantity;
x_i = net inventory at R_i at the instant of the system-replenishment, and (since $L = 0$) routing decision; Note that since LIF routing is employed, $x_1 \leq x_2$.
$\overline{X} = (x_1, x_2)$
$x = x_1 + x_2$;
y = system inventory position at the instant after the system-replenishment decision. $y = x + q$.
z_i = allocation to R_i;
$\overline{Z} = (z_1, z_2)$;
v_i = inventory position at R_i at the instant of the allocation decision;

$\overline{V} = (v_1, v_2)$;
μ = mean demand/period at each retailer;
σ = standard deviation of demand/period at each retailer;
δ_i^k = k-period demand at R_i with probability density function $\phi^k(.)$ and cumulative distribution function $\Phi^k(,)$;
$\Delta^k = \sum_{i=1}^{2}\delta_i^k$, system demand over k periods;
$E[.]$ = expectation function as viewed from time of replenishment

The key to the formulation of the myopic problem is the assignment of costs to specific replenishment-allocation cycles. Purchasing occurs once every m periods, but we assign the purchasing cost of each unit to the replenishment cycle in which the demand for that unit occurs. All backorders are assigned to the allocation cycle in which they occur. Initially, we assign all holding costs to the replenishment cycle in which they are incurred. System-wide(positive) inventory at the end of the kth period of the replenishment cycle is given by $x + q - \Delta^k + s_k$, where s_k is the backorders at the end of the period. We now modify the assignment of holding costs as follows: First, partition the backorders during the allocation cycle into two sets. Let U denote the sum of the backorders that occur during periods that are in both the replenishment cycle and its corresponding allocation cycle. Let V^P denote the sum of the backorders that occur during the replenishment cycle, but not during its corresponding allocation cycle. Note that V^P must have occurred during the *previous* allocation cycle. Similarly, let V^N denote backorders that occur in the *current* allocation cycle, but the *next* replenishment cycle. The total holding cost assigned to the replenishment-allocation cycle is $h[m(x + q) - \sum_{i=k}^{m}\Delta^k + U + V^P]$. However, we now reassign the holding cost hV^P from the current replenishment cycle to the previous replenishment cycle. Similarly, we shift hV^N from the next replenishment cycle to the current replenishment cycle. This reallocation of holding costs preserves the property that all holding costs are allocated to exactly one replenishment cycle, and makes the holding cost assigned to a replenishment-allocation cycle equal to $h[m(x+q) - \sum_{i=1}^{m}\Delta^k + U + V^N]$. Finally, note that $U + V^N$ is the total backorder-periods during the allocation cycle, that the expected value of $m(x + q) - \sum_{i=1}^{m} \Delta^k$ equals $m(x + q) - m(m + 1)\mu$, and that the expected demand over a replenishment cycle is $2m\mu$.

Given the cost-assignments described above, the myopic replenishment-allocation problem is:

$$S(\overline{X}) = 2cm\mu + \min_{q \geq 0}\{hm(x + q - \mu(m+1)) + (h+p)E[\min_{z_1}\{g(v_1,\ v_2)]\} \tag{1}$$

$$\text{Subject to:} \quad z_1 + z_2 = q \tag{2}$$

$$x_i + z_i - \delta_i^a = v_i \qquad i = 1, 2 \tag{3}$$

$$z_i \geq 0 \qquad i = 1, 2 \tag{4}$$

where $g(v_1,\ v_2)$ is the expected backorders over the allocation cycle given inventory positions of v_1 and v_2 at the time of allocation. An exact specification of $g(.)$ is given in the next section. Constraint (2) requires that the sum of the allocations to the retailers equals to the system-replenishment quantity, and constraint (4), that both allocations must be non-negative. Constraint (3) provides the inventory-balance equations.

The allocation assumption removes constraint (4), thereby, in effect, permitting negative allocations. Under the allocation assumption, and using $y = x + q$, the myopic replenishment-allocation problem can be written as MP:

$$MP: \quad \min_{y \geq x}\{M(y) = hm(y - \mu(m+1)) + (h+p)E[\min_{v_1}\{g(v_1,\ v_2)]\} \tag{5}$$

$$\text{Subject to: } v_1 + v_2 = y - \Delta^a \tag{6}$$

Note that the purchasing cost has been dropped since it plays no role in the optimization.

Under the allocation assumption, note that the allocation decision in any particular cycle will not affect the value of x or any costs in subsequent replenishment-allocation cycles. Further, MP depends on x only through the constraint $y \geq x$ and, given non-negative demand and stable problem parameters, $y \geq x$ is unnecessary. As a consequence, the infinite-horizon problem is completely separable into a series of essentially identical myopic problems.

Theorem 2: *Under the allocation assumption and the cost-allocations described above, MP solves the infinite-horizon problem.*

Proof: As in Kumar, Schwarz, and Ward(1995), Federgruen and Zipkin(1984a), etc..

Although the allocation assumption appears to be a strong one, our computational tests indicate it is seldom invoked. For example, in the simulation study to be described in Section 8, negative allocations were prescribed on average in less than 1.25% of the allocation cycles(5% maximum).

THE MYOPIC ALLOCATION PROBLEM (MAP)

In this section we examine the allocation subproblem imbedded in MP. We call this the myopic allocation problem (MAP). Let υ be the system net inventory at the time of the allocation decision and $\bar{\upsilon} = (\upsilon_1, \upsilon_2)$. From MP we have

$$MAP: \quad \hat{M}(\upsilon) = \min_{\upsilon_1}\{g(\upsilon_1, \ \upsilon_2)\}, \quad s.t. \ \ \upsilon_1 + \upsilon_2 = \upsilon \tag{7}$$

Where $g(\upsilon_1, \upsilon_2)$ is the expected shortages during the allocation cycle.

In the two-retailer case, only one allocation decision is made in each allocation cycle; i.e., the amount allocated to the first retailer, in effect, determines the amount to be allocated to the second retailer. Recall, also, that the subsequent route determines the end-periods of the current allocation cycle. Under LIF routing, the selection of the subsequent route depends on each retailer's inventory position at the instant of the allocation, υ_i, less its demand, $\delta_i^{m\text{-}a}$, during the $(m - a)$ periods between this cycle's allocation decision and the next replenishment cycle's routing decision.

Define $p_i(\bar{\upsilon}, \ \delta_i^{m\text{-}a})$ as the probability that R_i will be the first retailer on the next route given $\bar{\upsilon}$ and $\delta_i^{m\text{-}a}$. Under LIF routing, R_1 will be visited first on the next route if and only if R_1 has the smaller inventory position at that time; that is, if and only if $\upsilon_1 - \delta_1^{m\text{-}a} \le \upsilon_2 - \delta_2^{m\text{-}a}$, which is equivalent to $\delta_2^{m\text{-}a} \le \delta_1^{m\text{-}a} - \upsilon_1 + \upsilon_2$.

Therefore, $p_1(\overline{v},\ \delta_1^{m\text{-}a}) = \Phi^{m\text{-}a}(\delta_1^{m\text{-}a} - v_1 + v_2)$. Similarly, $p_2(\overline{v},\ \delta_2^{m\text{-}a}) = \Phi^{m\text{-}a}(\delta_2^{m\text{-}a} + v_1 - v_2)$. Hence,

$$g(\overline{v}) = \sum_{i=1}^{2} \int_0^{\infty} [p_i(\overline{v},\delta)(L^a(v_i - \delta)) + (1 - p_i(\overline{v},\delta))(L^{a+b}(v_i - \delta))]\phi^{m-a}(\delta)d\delta \tag{8}$$

where $L^k(u)$ are the expected shortages after k periods of demand, given an initial inventory of u. Note that if $m \leq a$(or if fixed routing is used), then at the time of the allocation decision, the next route is already known(or decided simultaneously). Correspondingly, the p_i's in (8) will be known at the time of allocation. In particular, if R_1 is the first retailer on the next route, then $p_1 = 1$ and $p_2 = 0$, for $\forall\delta$; and, if R_1 is the second retailer on the next route, $p_1 = 0$ and $p_2 = 1$, for $\forall\delta$. Therefore, MAP becomes the static-route allocation problem as in Kumar, Schwarz, and Ward(1995). For $m \geq a$ the p_i's are not known at the time of current allocation.

Nonetheless, the optimal allocation under both fixed and LIF routing satisfies the same first-order condition: that both retailers have the same stock-out probability.

Theorem 3: *Under the last-period-backorder assumption, if $g(\overline{v})$ is continuous and differentiable, then under the optimal myopic allocation, $\overline{v}^* = (v_1{}^*, v_2{}^*)$ satisfies*

$$P_1(\overline{v}^*) = P_2(\overline{v}^*) \tag{9}$$

where $P_i(\overline{v})$ is the probability that there will be no leftovers at R_i at the end of the allocation cycle.

Proof: See appendix B.

Under fixed routing, $g(\overline{v})$, in (8), is strictly convex. Hence, the allocation that equalizes the retailers' stock-out probabilities (9) is a global minimum. However, under LIF routing, $g(\overline{v})$ is not necessarily convex. In particular, despite the fact that for any given δ, both $L^a(.)$ and $L^{a+b}(.)$ are convex, the products $p_i(.) \cdot L^a(.)$ and $p_i(.) \cdot L^{a+b}(.)$ are not necessarily convex. Indeed, it is possible to construct parameterizations where $g(.)$ isn't convex. $g(\overline{v})$ is not necessarily even unimodal in v_1 on the interval $[-\infty, \infty]$. If it were, then, of course, (9) defines the global minimum. If not, it defines a local minimum or maximum. Campbell, et al.(1997) illustrated a similar possible non-convexity/non-unimodality in the inventory-management costs associated with dynamic routing in a two-retailer inventory-allocation problem with deterministic customer demand.

It is interesting to note that under LIF routing, the equal-allocation heuristic satisfies condition (9). Hence, in those parameterizations when $g(\overline{v})$ is unimodal under LIF routing, equal allocation is the globally-minimizing allocation. However, based on our numerical study(described in Section 7), $g(\overline{v})$ is typically bimodal.

All this necessitates the use of numerical search for the optimal allocation under LIF routing. This search is simplified somewhat because $g(.)$ is symmetric with respect to $v_1 = v/2$; i.e., $g(v_1, v - v_1) = g(v - v_1, v_1)$ for all v_1. Hence, one only needs to search the first half-interval $[-\infty, v/2]$ for an optimal v_1.

Theorem 4 further narrows the search interval for the optimal inventory value, v_1*.

Theorem 4: *Let v_s be the optimal inventory position at R_1 the instant after the allocation decision under fixed routing. Then, under LIF routing v_1* satisfies:*

$$v_s \le v_1^* \le \frac{v}{2} \tag{10}$$

Proof: See appendix C.

In managerial terms, Theorem 4 states that, under LIF routing, the optimal allocation to the first retailer is at least as large as that under fixed routing, but never more than equal allocation.

NUMERICAL STUDY

In this section, we compare the computer-simulated performance of the optimal myopic routing and allocation policies to assess the cost/cycle savings of LIF routing over fixed routing. Kumar, Schwarz, and Ward's base-stock replenishment policy — which assumes fixed routing and dynamic allocation — was used to determine system-basestock in each parameter set for every policy.

A total of 128 different scenarios(i.e., parameter sets) were simulated, all with $m = 4$, $\mu = 100$, and $h = 1$. The remaining four system parameters were varied as follows: $a = 0, 1, 2, 3$, $b = 1, 2, 3, 4$, $\sigma = 20, 50, 70, 100$, and $p = 10, 15$. Using data provided by the simulations, we estimated the cost/cycle and the probability that a negative allocation is prescribed. Note that although negative allocations are allowed under the allocation assumption, negative allocations were not allowed in the simulation; that is, whenever a negative allocation was prescribed for one of the retailers, this retailer was allocated zero units instead. Each simulation was run for 300,200 allocation cycles. The first 200 cycles were used to eliminate the effect of any initial conditions, and next 300,000 allocation cycles were used to collect data. Every policy was simulated using the same demand realizations. Finally, despite the end-of-cycle-backorders assumption, backorder costs were charged in any period when backorders occurred.

The results are summarized in table 1. All entries are, of course, estimates, but, for brevity's sake, we will omit this word in the following discussion. Our results are summarized according to the standard deviation of period demand, σ (Col. 1). The body of the table provides maximum, minimum, and average cost/time percentage differences.

Static vs. Dynamic Allocation Under Fixed Routing

Column 3 reports the cost/cycle savings of dynamic allocation (A=DYN) over static allocation (A=STAT) under a fixed routing policy (R=FIXED). These results are similar to those reported by Kumar, Schwarz, and Ward(1995), and are provided here as a benchmark. The average and maximum savings are 2.9% and 10.8%, respectively, and, as might be expected, increase as σ increases; i.e., as the variance of retailer net inventory increases.

Table 1. Summary of Simulation Study for N=2 Retailer System

Standard Deviation of Period Demand (σ)	Observed Difference	Percent Reduction in Expected Cost/Cycle from Changing Routing and /or Allocation Policy	
		R=FIXED/A=DYN Reduces Cost/Cycle of R=FIXED/A=STAT	R=LIF/A=DYN Further Reduces Cost/Cycle of R=FIXED/A=DYN
all	maximum	10.85%	11.97%
	minimum	0.00%	0.00%
	average	2.92%	1.92%
When σ=20	maximum	2.79%	0.53%
	minimum	0.00%	0.00%
	average	1.05%	0.04%
When σ=50	maximum	6.30%	6.98%
	minimum	0.00%	0.00%
	average	2.59%	1.17%
When σ=70	maximum	8.47%	9.16%
	minimum	0.00%	0.00%
	average	3.50%	2.24%
When σ=100	maximum	10.85%	11.97%
	minimum	0.00%	0.04%
	average	4.54%	4.21%

Dynamic vs. Fixed Routing Under Dynamic Allocation

Column 4 reports the additional cost/cycle savings from combining LIF routing with dynamic allocation; i.e., the addition savings in changing from R = FIXED/A = DYN to R = LIF/A = DYN. These savings — a maximum of 11.97% and an average of 1.92% — are comparable in magnitude to those provided by changing from static to dynamic allocation under a fixed routing policy. As expected, both the maximum and average savings increase as α increases. Note that the cost reductions reported in Cols. 3 and 4 are cumulative; that is, for example, in adopting R = LIF/A = DYN policy instead of a R = FIXED/A = STAT policy, the average cost saving is approximately 4.84% (= 2.92+1.92), and the maximum saving is approximately 22.82%

Although not detailed here, we observed that these savings were greater for scenarios with small a, the from/to warehouse travel time. This is because, under LIF routing, the next route will be determined ($m - a$) periods after the current route's allocation decision. To the extent that LIF routing can be viewed as pooling system uncertainties by postponing the next routing decision these ($m - a$) time periods, the smaller the value of a, the greater the postponement, the greater the risk-pooling.

Unlike the effect of a, the benefits from LIF routing are not monotonic in b, the between-retailer allocation time. Instead, it is observed to be largest for intermediate values of b; e.g., at $b = 1$ in some cases and at $b = 2$ in the other cases. We interpret this as follows: Under optimal or near-optimal allocation, routes are fairly stable. As b increases, (1) the probability that the next route will differ from the current route tends to decrease, and (2) when a route change does occur, the resulting benefit tends to be greater.

N-Retailer Symmetric System

In $N > 2$ cases, since it is impractical to numerically find optimal dynamic allocations given LIF, LIF is combined with a heuristic allocation policy, which will determine the lower bound on the cost-savings of R = LIF/A = DYN. Assume that a heuristic A = FIXED makes allocations decisions dynamically as if the route during the next allocation will be the same as during the current allocation cycle. In order to estimate the lower bound on the benefit of LIF routing for $N > 2$ retailers, we compared the simulated performance of R = LIF/A = FIXED with R = FIXED/A = DYN(i.e., Kumar, Schwarz, and Ward's policy) for $N = 2, 4, 9, 25$, and 49 with $m = N$; $a = b = 1$; $\sigma\sqrt{N}/N\mu = 0.1, 0.3, 0.5$; and $p = 1.5m$, $2m$. All other conditions are the same as when $N = 0$. On average, LIF routing reduced total costs an average of 4.0%, and a maximum of 11.8%(in the case when $m = N = 49$, $\sigma\sqrt{N}/N\mu = 0.5$, and $p = 2m$).

We also compared R = FIXED/A = DYN with R = FIXED/A = STAT policy to estimate the risk-pooling benefit from dynamic allocation alone. Dynamic allocation reduced total cost/cycle an average of 2.8 %, and a maximum at 5.7%(in the case when $m = N = 49$, $\sigma\sqrt{N}/N\mu = 0.5$, and $p = 2m$). Combining these observations with those of the two-retailer case, we believe that in the *N*-retailer symmetric scenario, and given a baseline policy of R = FIXED/A = STAT, the improvement from adopting R = LIF/A = FIXED is larger than the benefit of adopting R = FIXED/A = DYN

DISCUSSION

The research reported here had two principal goals. The first, to assess the potential of dynamic routing and dynamic allocation to reduce costs given a baseline scenario involving fixed routing and allocation. Our numerical tests indicate that for a symmetric system using a R = FIXED/A = DYN policy, LIF routing(and its corresponding allocation) provides cost reductions comparable to, *and in addition to*, those from adopting dynamic allocation. In other words, given a symmetric system operating with fixed routing and static allocation, by adopting dynamic allocation, cost/cycle can be reduced an average of 2.5 to 3%. Then, by adopting LIF routing and its corresponding allocation policy, average cost/cycle can be reduced an additional 2 to 4%.

Of course, these savings must be traded off against any increased travel cost. It is difficult to make general statements about the relative magnitude of these cost reductions/increases. As the value of the SKU being managed increases, the savings in inventory-related costs described here could become substantial. However, as the material-handling difficulty(e.g., mass) of the SKU increases, transportation costs increase. On the other hand, at first glance, commodity items seem to be inappropriate candidates for either dynamic routing or allocation. However, to the extent that the shortages of these items yield lost market share, the corresponding cost of running out of these SKUs is very high. Correspondingly, a small reduction in shortages for these items may represent a very substantial savings. Although not detailed here, our simulation tests indicated that most of the inventory-cost reduction from LIF routing was due to a reduction in shortages. Therefore, if the extra travel cost associated with dynamic routing for such SKUs is small, then dynamic routing and allocation might be cost effective even for commodity items.

The second goal of our research was to examine how dynamic routing avoids demand uncertainty. Our analysis suggests that most of the value of a LIF routing and allocation resides in the capability of LIF routing to control the variance of retailer net inventories when they are "out of balance." In summary, based on the tests reported here, we believe that a routing and allocation policy that employs dynamic routing to expedite (postpone) deliveries to retailers whose inventories are low (high), can provide significant cost savings in scenarios with symmetric retailers.

The implication of this research to the real-world application is somewhat limited. Due to the high complexity of the problem we analyzed the simple symmetric problem and the results of our analysis can not be directly applied to the non-symmetric cases. But the numerical result implies that even in non-symmetric cases the dynamic routing policy of visiting the retailer in most need next can provide significant cost reduction when retailer net inventories are significantly unbalanced. For the non-symmetric case, LIF routing should be revised to decide which retailer is in most need. For example, route can be decided based on normalized net inventory. We are planning the future research on the non-symmetric cases, which will significantly improve the applicability of our research to real-world problems.

Appendix A: Proof of Theorem 1

The proof below is constructed for $N = 2$. It is easily extended to N retailers by applying the same logic to any two of the N retailers at a time.

Proof:

For the first part of the proof, assume that the sequence of system-replenishment quantities and retailer allocations over time are fixed. Observe that the system-wide inventory position in any period is invariant to all routing decision. Through an analysis of costs similar to that presented in Section 6, it can be shown that the only holding or backorder costs that are affect by the tth routing decision equal $(h + p)$, the holding plus penalty costs per unit, times the backorders that occur at the end of the $t - 1$st allocation cycle(The tth route determines the end periods of the $t - 1$st allocation cycle.). Define the tth *single-cycle routing problem*(SCRP) as the problem in which the tth route is chosen to minimize backorder in the $t - 1$st allocation cycle.

Let R^1 and R^2 denote the two retailers and let s^1 and s^2 be the inventory positions at R^1 and R^2 at the moment of the tth routing decision, respectively. W. L. G. assume $s^1 \leq s^2$. Under LIF R^1 is visited first, and R^1's $t - 1$st allocation cycle ends a periods later and R^2's allocation cycle ends $(a + b)$ periods later. The end periods will be reversed if R^2 is first, and R^1 second. Let δ^{a1} (δ^{a2}) denote the demand at R^1 (R^2) during the first a periods after the routing decision, and let δ^b denote the demand that occurs during the subsequent b periods at the retailer that is visited second on the route. Let $BLIF(\delta^{a1}, \delta^{a2}, \delta^b)$ equal the backorders during the $t - 1$st allocation cycle given R^1 is visited first on the tth route. Let $\mathrm{BNOT}(\delta^{a1}, \delta^{a2}, \delta^b)$ be the backorders if R^2 is visited first. For any real number, r, let $[r]^+ = r$ if $r > 0$, otherwise $[r]^+ = 0$.

$$\mathrm{BLIF}(\delta^{a1}, \delta^{a2}, \delta^b) = [\delta^{a1} - s^1]^+ + [\delta^{a2} + \delta^b - s^2]^+$$
$$\mathrm{BNOT}(\delta^{a1}, \delta^{a2}, \delta^b) = [\delta^{a1} - s^2]^+ + [\delta^{a2} + \delta^b - s^1]^+$$

We need to show that $E\{\mathrm{BLIF}(\delta^{a1}, \delta^{a2}, \delta^b)\} \leq E\{\mathrm{BNOT}(\delta^{a1}, \delta^{a2}, \delta^b)\}$ where the expectation is over $(\delta^{a1}, \delta^{a2}, \delta^b)$. We use the two

Table 2. All Possible Cases

Condition	Result
$s^1 \le s^2 \le \delta^{a1} \le \delta^{a1} + \delta^b$	BLIF = BNOT = $2\delta^{a1} + \delta^b - s^1 - s^2$
$\delta^{a1} \le \delta^{a1} + \delta^b \le s^1 \le s^2$	BLIF = BNOT = 0
$s^1 \le \delta^{a1} \le s^2 \le \delta^{a1} + \delta^b$	BNOT - BLIF = $s^2 - \delta^{a1} \ge 0$
$\delta^{a1} \le s^1 \le \delta^{a1} + \delta^b \le s^2$	BLIF = 0, BNOT = $\delta^{a1} + \delta^b - s^1 \ge 0$
$s^1 \le \delta^{a1} \le \delta^{a1} + \delta^b \le s^2$	BNOT - BLIF = $\delta^b \ge 0$
$\delta^{a1} \le s^1 \le s^2 \le \delta^{a1} + \delta^b$	BNOT - BLIF = $s^2 - s^1 \ge 0$

properties below.

Property A1: If $\delta^{a1} = \delta^{a2}$ then BLIF $\le$ BNOT for all $\delta^b \ge 0$.

Proof: Table 2 considers all possible cases.

For any demand realization (δ^{a1}, δ^{a2}, δ^b), construct its "symmetric demand" (δ^{a2}, δ^{a1}, δ^b) by interchanging the values of δ^{a1} and δ^{a2}.

Property A2: For any demand realiztion (δ^{a1}, δ^{a2}, δ^b)
BLIF(δ^{a1}, δ^{a2}, δ^b) + BLIF(δ^{a2}, δ^{a1}, δ^b)
$\le$ BNOT(δ^{a1}, δ^{a2}, δ^b) + BNOT(δ^{a2}, δ^{a1}, δ^b). (A1-1)

Proof: From Property A1
$[\delta^{a1} - s^1]^+ + [\delta^{a1} + \delta^b - s2]^+ \le [\delta^{a1} - s2]^+ + [\delta^{a1} + \delta^b - s1]^+$ and
$[\delta^{a2} - s^1]^+ + [\delta^{a2} + \delta^b - s2]^+ \le [\delta^{a2} - s2]^+ + [\delta^{a2} + \delta^b - s1]^+$.
Adding these two inequalities yields (A1-1).

Since retailer demand is identically distributed, the probability (or probability density) of (δ^{a1}, δ^{a2}, δ^b) and (δ^{a2}, δ^{a1}, δ^b) are equal. It follows that the expectation of the left (right) side of (A1-1) equals twice the expectation of BLIF (BNOT), so that the E{BLIF(δ^{a1}, δ^{a2}, δ^b)} $\le$ E{BNOT(δ^{a1}, δ^{a2}, δ^b)}.

Now, we prove that there must exist a distribution policy that minimizes average per-period cost over the infinite horizon that uses LIF routing. Suppose that (O, A, R) is an optimal distribution policy(= a joint system-replenishment (O), allocation (A), and routing (R) policy) over the infinite horizon. Suppose that for some demand realization, (O, A, R) does not follow LIF.

Specifically, suppose that the tth route does not follow LIF, given the set of demand realizations D to that point in time. Let (O, A, R') be the distribution policy, which makes the same system-replenishment, allocation, and routing decisions as (O, A, R) except that (O, A, R') follows LIF in the tth routing decision. As noted above, given D, total expected costs associated with the backorders in the t – 1st allocation cycle are equal to or less than that of (O, A, R), while all other future expected costs remain the same. Therefore, either (O, A, R) is not optimal(a contradiction) or the use of LIF routing leads to the same optimal expected cost per period as (O, A, R).

Appendix B: Proof of Theorem 3

Proof:
(i) If $m \le a$ - see Kumar, Schwarz, and Ward (1995).
(ii) If $m > a$

$$g(\bar{v}) = \sum_{i=1}^{2} \int_0^\infty [\Phi^{m-a}(\delta + v - 2v_i)(L^a(v_i - \delta)) + (1 - \Phi^{m-a}(\delta + v - 2v_i))(L^{a+b}(v_i - \delta))]\phi^{m-a}(\delta)d\delta \qquad \text{(A2-1)}$$

where $L^k(u) = \int_u^\infty (\eta - u)\phi_k(\eta)d\eta$ is the expected loss function after k periods of demand, given an initial inventory of u. Using $v_1 + v_2 = v$ the first derivative of g(.) with respect to v_1 is

$$\begin{aligned}\frac{\partial g(\bar{v})}{\partial v_1} = \int_0^\infty \{ & -\Phi^{m-a}(\delta - v_1 + v_2)(1 - \Phi^a(v_1 - \delta)) \\ & - 2\phi^{m-a}(\delta - v_1 + v_2)\int_{v_1-\delta}^\infty (\eta - (v_1 - \delta))\phi^a(\eta)d\eta \\ & - (1 - \Phi^{m-a}(\delta - v_1 + v_2))(1 - \Phi^{a+b}(v_1 - \delta)) \\ & + 2\phi^{m-a}(\delta - v_1 + v_2)\int_{v_1-\delta}^\infty (\eta - (v_1 - \delta))\phi^{a+b}(\eta)d\eta \\ & + \Phi^{m-a}(\delta + v_1 - v_2)(1 - \Phi^a(v_2 - \delta)) \\ & + 2\phi^{m-a}(\delta + v_1 - v_2)\int_{v_2-\delta}^\infty (\eta - (v_2 - \delta))\phi^a(\eta)d\eta \\ & + (1 - \Phi^{m-a}(\delta + v_1 - v_2))(1 - \Phi^{a+b}(v_2 - \delta))\end{aligned}$$

$$-2\phi^{m-a}(\delta+\upsilon_1-\upsilon_2)\int_{\upsilon_2-\delta}^{\infty}(\eta-(\upsilon_2-\delta))\phi^{a+b}(\eta)d\eta\}\phi^{m-a}(\delta)d\delta \tag{A2-2}$$

Applying the outermost integration to each line separately, the sixth line of (A2-2) is

$$\int_0^{\infty}\{2\phi^{m-a}(\delta+\upsilon_1-\upsilon_2)\int_{\upsilon_2-\delta}^{\infty}(\eta-(\upsilon_2-\delta))\phi^{a}(\eta)d\eta\}\phi^{m-a}(\delta)d\delta.$$

Applying a change of variable ($\delta = \delta' - \upsilon_1 + \upsilon_2$) the sixth line becomes

$$\int_0^{\infty}2\phi^{m-a}(\delta')\int_{\upsilon_1-\delta'}^{\infty}(\eta-(\upsilon_1-\delta'))\phi^{a}(\eta)d\eta\phi^{m-a}(\delta'-\upsilon_1+\upsilon_2)d\delta'.$$

Hence the sixth line equals the negation of the second line of (A2-2). Similarly, the fourth and eighth lines cancel. Using $p_1(\overline{\upsilon},\delta)=\Phi^{m-a}(\delta-\upsilon_1+\upsilon_2)$ and $p_2(\overline{\upsilon},\delta)=\Phi^{m-a}(\delta+\upsilon_1-\upsilon_2)$, the respective probabilities that R_1 and R_2 are first on the next route, given that the other retailer's demand over the first m-a periods is δ, the derivative of expected shortages is

$$\frac{\partial g(\overline{\upsilon})}{\partial \upsilon_1} = -\int_0^{\infty}\{p_1(\overline{\upsilon},\delta)(1-\Phi^{a}(\upsilon_1-\delta))-(1-p_1(\overline{\upsilon},\delta))(1-\Phi^{a+b}(\upsilon_1-\delta))d\delta$$
$$+\int_0^{\infty}\{p_2(\overline{\upsilon},\delta)(1-\Phi^{a}(\upsilon_2-\delta))-(1-p_2(\overline{\upsilon},\delta))(1-\Phi^{a+b}(\upsilon_2-\delta))d\delta \tag{A2-3}$$

Or equivalently, $\dfrac{\partial g(\overline{\upsilon})}{\partial \upsilon_1} = -P_1(\overline{\upsilon}^*)+P_2(\overline{\upsilon}^*)$, where $P_1(\overline{\upsilon}^*)$ and $P_2(\overline{\upsilon}^*)$ are the probabilities that R_1 and R_2 will have a shortage in this cycle, respectively. This proves the Lemma.

Appendix C: Proof of Theorem 4

Proof:
(i) $\upsilon_1^* \le \upsilon/2$

Because of the symmetry of optimal allocations, there always exists υ_1^* which is less than or equal to $\upsilon/2$.

(ii) $\upsilon_1^* \geq \upsilon_s$

We prove this by contradiction. Suppose that $\hat{\upsilon}_1 < \upsilon_s$ and $\hat{\upsilon}_1$ is an optimal allocation to R_1. Let $\overline{\upsilon} = (\hat{\upsilon}_1, \upsilon - \hat{\upsilon}_1)$. In the static-route case, the total expected costs function is convex and its value goes to infinity as υ_1 goes to infinity or minus infinity. Since its first derivative is $P_1(\overline{\upsilon}) + P_2(\overline{\upsilon})$ like in the dynamic-route case, $-P_1(\overline{\upsilon}) + P_2(\overline{\upsilon}) < 0$ when the static route is used; that is, $P_1(\overline{\upsilon}) > P_2(\overline{\upsilon})$ in the static-route case. Compared to the static-route case, under dynamic routing, $P_1(\overline{\upsilon})$ will increase and $P_2(\overline{\upsilon})$ will decrease: When LIF prescribes no route change, the probability of stockout at the end of the allocation cycle will remain the same at both retailers, but when LIF prescribes a route change, that probability at R_1 (R_2) will increase(decrease). Therefore, under dynamic routing, $P_1(\overline{\upsilon}) > P_2(\overline{\upsilon})$. $\hat{\upsilon}_1$ can not be optimal since it does not satisfy the first-order condition.

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