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ONLINE DIMENSION OF PARTIALLY ORDERED SETS

A b s t r a c t. We investigate the online dimension of upgrowing partial orders. The problem is treated as a two-person game. One person builds an order, one point at a time; the other person maintains its online realizer. The value of the game (the number of linear extensions used) is compared with the offline width of the poset. We prove that the online dimension of width 2 upgrowing orders is exactly 2. In general case we prove a lower bound of $\lfloor 7d/4 \rfloor - 2$ for width d upgrowing orders.

A linear order $\mathcal{Q} = (X, \leq_{\mathcal{Q}})$ is called a linear extension of another partial order $\mathcal{P} = (X, \leq_{\mathcal{P}})$ if $x < y$ in $\mathcal{P}$ implies that $x < y$ in $\mathcal{Q}$, for all $x, y \in X$. For a fixed n-element poset $\mathcal{P} = (X, \leq)$ define the family $\mathcal{L}$ of all linear extensions of $\mathcal{P}$. Note that the size of $\mathcal{L}$ may vary from 1 (in case when $\mathcal{P}$ is a chain) to $n!$ (in case when $\mathcal{P}$ as an n-element antichain). It is easy to see that for two incomparable elements x, y of $\mathcal{P}$ there always exists a linear extension L of $\mathcal{P}$ with $x < y$ in L. This fact proves that $\bigcap \mathcal{L} = \mathcal{P}$ and justifies the correctness of the following definition.

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Definition 1 *Let $\mathcal{P} = (X, \leq)$ be a finite partial order. The dimension of $\mathcal{P}$, denoted $\mathsf{dim}(\mathcal{P})$, is the least positive integer d for which there exists a family $\mathcal{R} = \{L_1, \ldots, L_d\}$ of linear extensions of $\mathcal{P}$ such that*

$$\bigcap \mathcal{R} = \bigcap_{i=1}^{d} L_i = \mathcal{P}.$$

Any such family $\mathcal{R}$ of linear extensions is called a realizer of $\mathcal{P}$. A geometric interpretation justifying the name of the term is the following. Let φ be a mapping from X to distinct points in $\mathbb{Z}^d$ such that $x < y$ in $\mathcal{P}$ iff each coordinate of $\varphi(x)$ is less than the corresponding coordinate of $\varphi(y)$. The dimension of $\mathcal{P}$ is the least positive integer d for which there exists such mapping φ from X to $\mathbb{Z}^d$.

The construction of Dushnik and Miller [2] shows there exist posets of arbitrary large dimensions. Yannakakis [3] proved that it is NP-complete to determine if a partial order has dimension at most 3. Determining whether a poset has dimension at most 2 can be done in polynomial time. On the other hand, from Dilworth's theorem it can be deduced that the dimension of an order never exceeds its width, i.e. the maximal size of an antichain.

An online dimension algorithm receives as input an online order, i.e. elements of the order are presented one by one from the externally determined list. With a new element the algorithm learns the comparability status of the existing elements to the new one. After adding a new element to the order the algorithm calculates its online realizer – the new element is put into existing linear extensions, while the order of the previous elements may not be changed. Performance of an online dimension algorithm is measured by comparing the size of presented online realizer with the offline width of the order.

Computing online dimension may be treated as a two-person game. The players are called Alice and Bob. Alice represents an online algorithm, Bob is responsible for presenting the online order. In the online dimension game for orders of width d, Bob presents to Alice an order of width at most d. Alice, in turn, maintains its online realizer. The game is played in rounds. During one round Bob introduces a new point x to the order and reveals comparability status between x and the previous points. Alice responds by adding x to the existing linear extensions (or creates a new linear extension). The value of the game for orders of width d, denoted further by $\mathsf{dim\ on\text{-}line}(d)$, is the largest integer n for which Bob has a strategy that forces Alice to output an online realizer consisting of n linear extensions. Alternatively, it is the least integer n such that Alice has a strategy which for width d orders outputs an online realizer consisting of at most n linear extensions.

The described connection of online problems and games puts online algorithms into the framework of logic. A winning strategy for Alice can be expressed by a sentence of the form $\forall\exists\forall\ldots\forall\exists$, i.e., for each move of Bob there is a response of Alice such that whatever Bob does there is a response of Alice… etc. Similarly, a winning strategy for Bob can be represented by a sentence of the form $\exists\forall\exists\ldots\exists\forall$.

Computing the online dimension of a poset is closely related to computing its online width. In the game corresponding to determine online width, Bob builds an online order while Alice maintains its online chain partitioning (i.e. a partitioning of the order into pairwise disjoint chains). The value of the game is compared with the number returned by an optimal offline algorithm, i.e. with the width of the order. Denote the value of the game by $\mathsf{width\ upgrowing\ on\text{-}line}(d)$.

In this paper we investigate the online dimension of upgrowing orders. An online order $\mathcal{P}$ is upgrowing if each incoming point is maximal in $\mathcal{P}$ at the time it arrives. In other words, points of the poset are presented in some linear extension of the entire order. The known results about this problem may be summarized in the following

Theorem 2 (Felsner [1]) *For every $d \in \mathbb{N}$ we have*

$$d \le \mathsf{dim\ upgrowing\ on\text{-}line}(d) \le \mathsf{width\ upgrowing\ on\text{-}line}(d) = d(d+1)/2.$$

□

In the rest of the paper we improve upon the above result. First, we show tight upper- and lower-bounds on the class of width 2 orders. In the general case of width d posets we reformulate the problem as a variant of the chain partitioning game and use the equivalent form of the problem to prove a $7d/4$ lower-bound.

Proposition 3 $\mathsf{dim\ upgrowing\ on\text{-}line}(2) = 2$.

Proof. We induct on the number k of elements of a width 2 poset $\mathcal{P} = (P, \le)$. For $k = 1$, i.e., $P = \{p_1\}$, the 2-realizer is defined as follows:

$R_1 = (p_1)$, $R_2 = (p_1)$. Now assume that the thesis holds for all k-element partial orders of width 2. Let $\mathcal{P} = (P \cup \{a\}, \leq)$ denote an arbitrary $(k+1)$-element order of width 2, with a being the last added point. By the induction hypothesis there exists a 2-realizer $\{R'_1, R'_2\}$ of order $(P, \leq)$. If a is comparable with no element in P, we define $R_1 = (a, R'_1)$, $R_2 = (R'_2, a)$. Otherwise, there exists $b \in P$ such that $b \prec a$ (b precedes a in P). Let us partition $P \setminus \{b\}$ into three sets S, I, G of elements: smaller, incomparable and greater than b.

$$S = \{p \in P : \ p < b\}, \ I = \{p \in P : p \parallel b\}, \ G = \{p \in P : p > b\}.$$

Clearly, $a \parallel G$ (if for some $c \in G$ we had $a > c$, then $a > c > b$, which contradicts the definition of b).

Suppose that $I = \emptyset$. Without loss of generality we may assume that the 2-realizer $\{R'_1, R'_2\}$ of $(P, \leq)$ is of the form $R'_1 = (S, b, G)$, $R'_2 = (S, b, G)$. (Here (S, b, G) denotes any linear extension of $\mathcal{P}$ in which elements of S are followed by b and then by elements of G). We define a 2-realizer $\{R_1, R_2\}$ of $(P \cup \{a\})$ as follows:

$$R_1 = (S, b, a, G), \ R_2 = (R'_2, a).$$

Now consider the case when $I \neq \emptyset$. Note that I is a chain in P (if there existed two incomparable elements $c, d \in I$, then set $\{b, c, d\}$ would form a 3-element antichain in P, which would contradict the assumption that P is of width 2). A similar reasoning (with a in the role of b) shows that G, if not empty, is also a chain in P. Taking into account that $b \parallel \max(I)$ and $b \parallel \min(I)$ we may assume without loss of generality that $b < \min(I)$ in R'_1, $b > \max(I)$ in R'_2 and therefore

$$R'_1 = (S, b, I \cup G), \ R'_2 = (S \cup I, b, G).$$

If $a > I$ or set G is empty then we define R_1 and R_2 as follows:

$$R_1 = (R'_1, a), \ R_2 = (S \cup I, b, a, G).$$

If G is not empty and the inequality $a > I$ does not hold, then $G > I$ (as otherwise, the set $\{\max(I), \min(G), a\}$ would be a 3-element antichain in G). Define two sets I_1 and I_2 as follows:

$$I_1 = \{p \in I : a \parallel p\}, \ I_2 = \{p \in I : a > p\}.$$

Clearly, $I_1 > I_2$ and $I_1 \cup I_2 = I$. Again, without loss of generality we may assume that a 2-realizer $\{R'_1, R'_2\}$ for P is of the following form:

$$R'_1 = (S, b, I_2, I_1, G),\ R'_2 = (S \cup I_1 \cup I_2, b, G).$$

R_1 and R_2 are defined in this case as follows.

$$R_1 = (S, b, I_2, a, I_1, G),\ R_2 = (R'_2, a).$$

The proof of the induction step is now complete. □

As we have seen above, it is not very handy to deal with online realizers of posets. In [1] Felsner proposed a variant of the chain partitioning game called the *adaptive chain covering game*. Its rules are the following.

- to each new point p a non-empty set $C(p)$ of numbers (chains) is assigned,
- the set $C(p)$ may shrink in the subsequent moves yet it may not become empty,
- for every chain k the set $\{p \in P : k \in C(p)\}$ is a chain in $\mathcal{P}$.

Note, that the chain partitioning problem is exactly the adaptive chain covering game with additional restriction that $|C(p)| = 1$ for every point p of the poset. On the other hand, Felsner in [1] claimed that on the class of upgrowing orders, the two games — adaptive chain covering and the online dimension problem are equivalent. The proof of this fact turned out to be incomplete. Below, we restate Felsner's theorem and include its complete proof.

Theorem 4 *On the class of upgrowing online orders the two games — online dimension and the adaptive chain covering game are equivalent, i.e.*

$$\mathsf{adaptive\ chain\ covering}(d) = \mathsf{dim\ upgrowing\ on\text{-}line}(d).$$

Proof. To prove $\geq$-inequality we show how to solve the online dimension problem basing on the solution of the adaptive chain covering problem. Let $\mathcal{P} = (P, \leq)$ be an arbitrary upgrowing order. Suppose that for $\mathcal{P}$ we have solved the adaptive chain covering problem with k chains $L_1, L_2, \ldots, L_k$. With each chain L_i we associate a linear extension R_i according to the following rules. For every incoming new point p do the following:

1. For every $j \in C(p)$ append p at the end of R_j.

2. In the remaining linear extensions R_i let p go as deep in R_i as possible, i.e., place p after the last (biggest) element smaller than p.

It remains to show that $R_1 \cap R_2 \cap \ldots \cap R_k = \mathcal{P}$. The latter is a consequence of the following fact: for every x and every $j \in C(x)$ all elements dominating x in R_i also dominate x in P. For a pair of incomparable elements $a, b \in X$ we have $a < b$ in R_j for all $j \in C(b)$. This proves that $R_1 \cap R_2 \cap \ldots \cap R_k = \mathcal{P}$.

Before proving the $\leq$-inequality we need some preparations.

Definition 5 *Let $\mathcal{R} = \{R_1, R_2, \ldots, R_n\}$ be a family of linear extensions of an order $\mathcal{P} = (P, \leq)$. Choose an element $p \in P$. We will say that the property $(\star)$ holds for p in R_i, if*

$$\forall q \in P : (q > p \text{ in } R_i) \Rightarrow (q > p \text{ in } \mathcal{P}). \qquad (\star)$$

We will say that the property $(\star)$ holds for x in $\mathcal{R}$, if there exists $i \in \{1, 2, \ldots, n\}$ such that $(\star)$ holds for x in R_i.

We will say that the property $(\star)$ holds for $\mathcal{R}$, if for every $x \in X$ property $(\star)$ holds for x in $\mathcal{R}$.

Let $\mathcal{P} = (P, \leq)$ be an arbitrary upgrowing online order, let $\mathcal{R}$ be its online realizer (constructed by Alice). We will say that Alice's strategy fulfills the $(\star)$-property if property $(\star)$ holds for $\mathcal{R}$ at any time in $\mathcal{P}$.

Lemma 6 *Let $d \in \mathbb{N}$. The value* $\mathsf{dim\ upgrowing\ on\text{-}line}(d)$ *remains unchanged if we additionally impose that Alice's strategy fulfills the $(\star)$ property.*

Proof. Choose $d \in \mathbb{N}$ and let $\mathsf{dim\ upgrowing\ on\text{-}line}(d) = k$ (we know from Theorem 2 that k is finite and $k \leq d(d+1)/2$). Let $\mathcal{S}$ denote Alice's strategy which maintains an online realizer of size at most k on the class of width d upgrowing orders. We present a strategy $\mathcal{S}_\star$ which additionally fulfills the $(\star)$ property.

From the game duality property we know that Bob has the strategy $\mathcal{K}$ which forces Alice to use exactly k linear extensions on a certain order $(P_k, \leq)$. We now describe the strategy $\mathcal{S}_\star$ for an arbitrary order $\mathcal{P}$.

1. Before Bob starts revealing the order $\mathcal{P}$ Alice builds k linear extensions $R_1, R_2, \ldots, R_k$ of $(P_k, \leq)$; $R_1, R_2, \ldots, R_k$ are exactly the extensions to which Alice would have been "forced" when Bob presented $(P_k, \leq)$ online.

2. Bob presents order $(P, \leq)$. In response Alice follows the strategy $\mathcal{S}$ by adding the incoming points in $R_1, \ldots, R_k$ and additionally assuming that $(P, \leq)$ dominates entire $(P_k, \leq)$.

3. As a result Alice presents extensions $R'_1, \ldots, R'_k$. They are exactly $R_1, \ldots, R_k$ constrained to the elements from $(P, \leq)$.

The idea of the algorithm is to force Alice to use k linear extensions *before* presenting the order $(P, \leq)$. Thanks to this constraint, the property $(\star)$ holds for $\{R'_1, R'_2, \ldots, R'_k\}$.

Assume to the contrary that the property $(\star)$ does not hold for $\{R'_1, R'_2, \ldots, R'_k\}$. Let a be the first point from $(P, \leq)$ whose addition spoiled the $(\star)$-property, i.e., property $(\star)$ holds in the set $P_0 \subseteq P$, and fails in the set $P' = P_0 \cup \{a\}$.

We deduce there exists $b \in P'$ such that

$$\forall\ i = 1, 2, \ldots, k \ \exists\ p_i \in P' : p_i \parallel b \text{ in } \mathcal{P}, \text{ but } p_i > b \text{ in } R'_i. \qquad (\ddagger)$$

Define the set $b\!\uparrow$ of elements greater than or equal b.

$$b\!\uparrow = \{p \in P_0 : b \leq p \text{ in } \mathcal{P}\}.$$

Note that $a \parallel b\!\uparrow$ — if we had $a > c \geq b$ for some $c \in b\!\uparrow$ then every p_i from $(\ddagger)$ would be different from a; this in turn would contradict the assumption that a is the first point violating $(\star)$. We deduce that width$(b\!\uparrow) < d$. We will show that Bob may add point z to P' in a way which will force Alice to use $k+1$ linear extensions on $P' \cup \{z\}$. This fact, together with the inequality width$(P' \cup \{z\}) \leq d$ will contradict the fact that $\mathsf{dim\ upgrowing\ on\text{-}line}(d) = k$. It suffices now to declare z incomparable to $b\!\uparrow$ and greater than the rest of points, i.e.

$$z \parallel p \text{ for every } p \in b\!\uparrow, \ z > p \text{ for every } p \in P' \setminus b\!\uparrow.$$

Note, that width$(X' \cup \{z\}) \leq d$, because width$(b\!\uparrow) < d$. Moreover, for witnesses $p_1, p_2, \ldots, p_k$ from $(\ddagger)$ we have $z > p_i$, $i = 1, 2, \ldots, k$. This together with $p_i >_{R'_i} b$ implies that $z >_{R'_i} b$ for any i. If extensions $R'_1, R'_2, \dots, R'_k$ realized $\mathcal{P}$ then we would have $z >_{\mathcal{P}} b$, despite the assumption that $z \parallel b$. We have shown that the order $P' \cup \{z\}$ of width not exceeding d cannot be realized by k extensions. The contradiction proves Lemma 6. $\square$

We return to the proof of Theorem 4. It remains to show that

$$\textsf{adaptive chain covering}(d) \leq \textsf{dim upgrowing on-line}(d).$$

Let $\mathcal{R} = \{R_1, \dots, R_k\}$ be an online realizer of an upgrowing order $\mathcal{P}$. We may assume, due to Lemma 6, that the property $(\star)$ holds for $\mathcal{R}$. For $p \in P$ define $C(p)$ as follows.

$$C(p) = \{i : \text{ property } (\star) \text{ holds for } p \text{ in } R_i,\ i = 1, 2, \dots, k\}.$$

Property $(\star)$ for $\mathcal{R}$ guarantees that $C(p)$ is never empty. Moreover, for $i = 1, 2, \dots, k$ the set $\{p : i \in C(p),\ p \in P\}$ is a chain in $\mathcal{P}$ (if $i \in C(p) \cap C(q)$ and $p < q$ in R_i then $(\star)$ holds for p in R_i, hence $p < q$ in $\mathcal{P}$). The family $\{C(p) : p \in P\}$ solves the adaptive chain covering problem for $\mathcal{P}$. $\square$

Fact 7 $\textsf{dim upgrowing on-line}(3) \geq 4$.

Proof. Consider the adaptive chain covering game for the following order. At the beginning Bob presents a partial order $\mathcal{P}$ as seen on Figure 1.

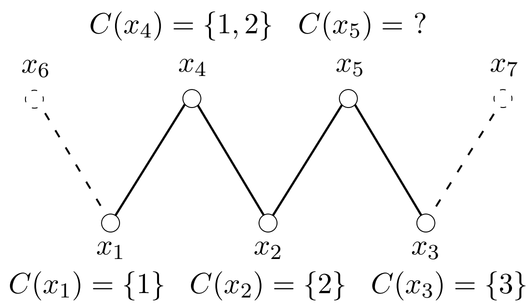


Figure 1.

Alice has two possibilities in defining $C(x_5)$. She may define either $C(x_5) = \{3\}$ or $C(x_5) = \{2, 3\}$, $C(x_4) = \{1\}$. In the former case Bob responds with x_7, in the latter with x_6. Either way, Alice is forced to use the fourth chain (declaring $C(x_7) = \{4\}$ or $C(x_6) = \{4\}$). The proof of the Fact follows now from Theorem 4. $\square$

We now generalize above schema on orders of arbitrary width. Still, instead of constructing an online realizer we will solve the adaptive chain covering problem.

Note that in the proof of Fact 7 every point y added above x_6 (alternatively above x_7) may be designated exclusively to chain number 4. In other words, chain number 4 is the only chain used so far which can be put into set $C(y)$ for $y > x_6$ (alt. x_7). Three such points could be used to repeat the construction from the Fact 7 (see Figure 2).

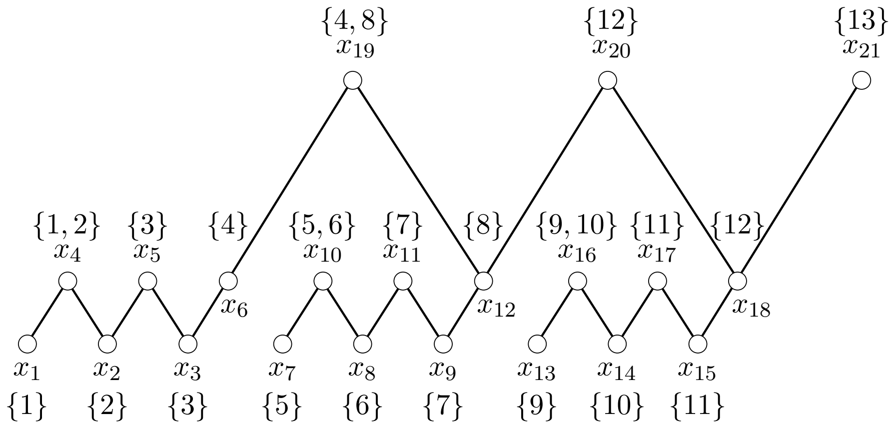


Figure 2.

For triples of elements (x_1, x_2, x_3), (x_7, x_8, x_9), (x_{13}, x_{14}, x_{15}) we simply repeat the algorithm from the Fact 7, forcing Alice to use three new chains 4, 8, 12. In the end we repeat the procedure for the triple (x_6, x_{12}, x_{18}), forcing Alice to use the 13th chain.

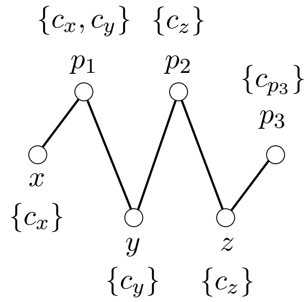
In general we build an order of width 3^k consisting of k layers. Layer number j has width 3^j and consists of 3^{j-1} zigzags from Example 7. Every zigzag forces one new chain. For orders of width $d = 3^k$ we obtain the following number of colors used.

$$3^k + 3^{k-1} + \ldots + 3^1 + 1 = \lceil 3d/2 \rceil - 1.$$

We now generalize the above result to orders of arbitrary width. Next we outline the construction which raises the result to the value $7d/4 - 2$.

The $3d/2$ algorithm looks as follows.

Construct $d-$element antichain $A = (x_1, x_2, \ldots, x_d)$;
while $|A| \geq 3$ **do**
  build a zigzag on three arbitrary points $x, y, z \in A$
  i.e. build an order of the form:



  $A := A \setminus \{x, y, z\} \cup \{p_3\}$;
    (p_3 − point recycled from the built zigzag)

For $d = 7$ the resulting poset looks as on Figure 3.

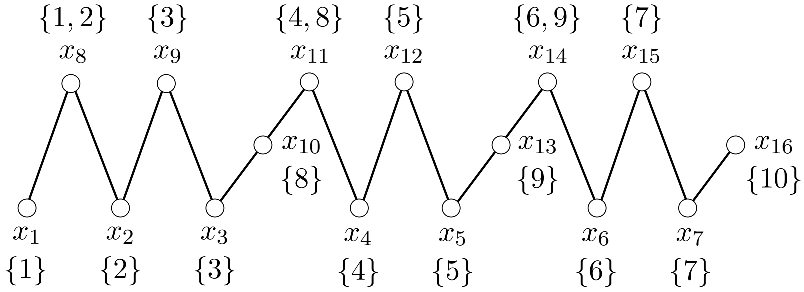


Figure 3.

Denote by $f(d)$ the result returned by Alice for orders of width d. From the above algorithm we get

$$\begin{aligned} f(d+2) &= 3 + f(d) \text{ for } d \geq 3, \\ f(1) &= 1, \\ f(2) &= 2. \end{aligned}$$

Simple transformations lead us to the desired formula

$$f(d) = \lceil 3d/2 \rceil - 1.$$

Now we give a better lower bound of the value dim upgrowing on-line(d). The new construction is much more technical than the previous one and therefore we will only illustrate its main idea on the poset from Figure 3.

Note that points x_8 and x_9 may not be used as the base of the new zigzag. We show how to reuse some of such points.

First Bob adds points x_{17} i x_{18} (see Figure 4).

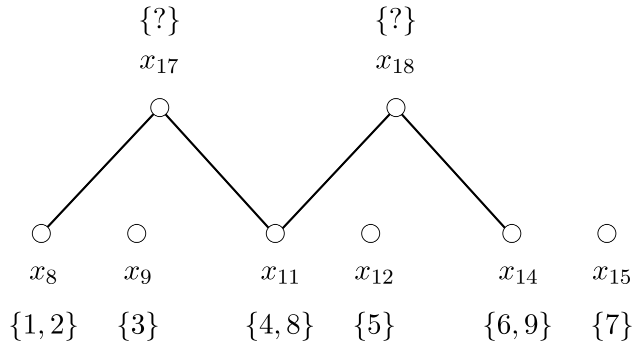


Figure 4.

Alice has (up to equivalent renumbering of points and chains) two possible answers:

i. $C(x_{17}) = \{1, 2, 4, 8\}$, $C(x_{18}) = \{6, 9\}$, or

ii. $C(x_{17}) = \{1, 2, 4\}$, $C(x_{18}) = \{8, 6, 9\}$.

In the first case Bob adds point x_{19} (see Figure 5).

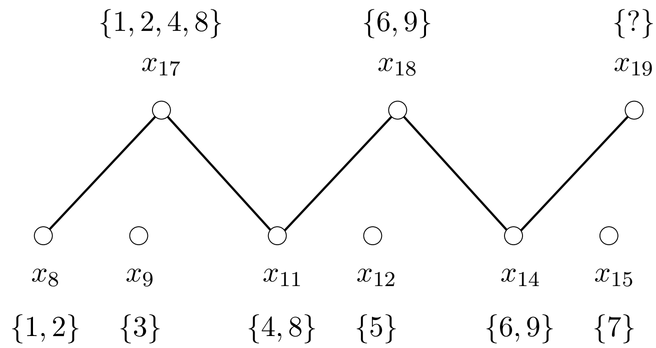


Figure 5.

We may assume that Alice's response would be of the following form: $C(x_{18}) = \{6\}$, $C(x_{19}) = \{9\}$ (response $C(x_{18}) = \{9\}$, $C(x_{19}) = \{6\}$ is symmetric, while adding a new chain is not an optimal move). Bob has produced two points x_{19} i x_{15} which may be used in the future as the base of the zigzag (above the point x_{19} Alice can use only chain number 9, above the point x_{15} Alice can only use chain number 7).

Consider the second case, when $C(x_{17}) = \{1, 2, 4\}$, $C(x_{18}) = \{8, 6, 9\}$. Bob adds points x_{19} and x_{20} (see Figure 6).

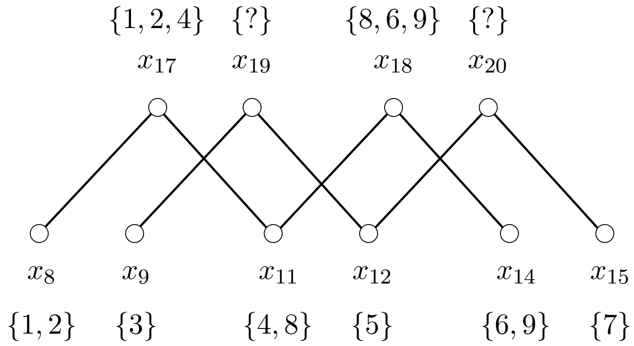


Figure 6.

Alice may answer in one of the following ways:

i. $C(x_{19}) = \{3, 5\}$, $C(x_{20}) = \{7\}$.

ii. $C(x_{19}) = \{3, 5\}$, $C(x_{20}) = \{7, 6\}$.

iii. $C(x_{19}) = \{3, 5\}$, $C(x_{20}) = \{7, 4\}$.

iv. $C(x_{19}) = \{3, 5\}$, $C(x_{20}) = \{7, 6, 4\}$.

v. $C(x_{19}) = \{3\}$, $C(x_{20})$ is anything legal.

Note that we do not consider cases when Alice brings to existence new chains; it can be easily shown that such an action leads eventually to values at least as high as the the values in cases (i)–(v).

In cases (i) and (ii) Bob adds points $x_{21} > x_{14}$ and $x_{22} > x_{15}$. Alice's response has to be of the following form.

In case (i) (see Figure 7):

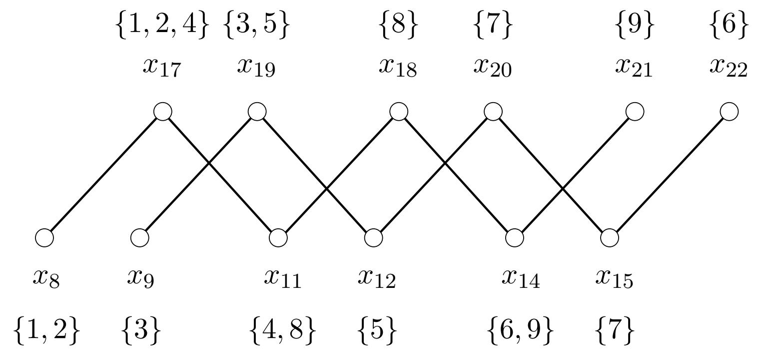


Figure 7.

In case (ii) (see Figure 8):

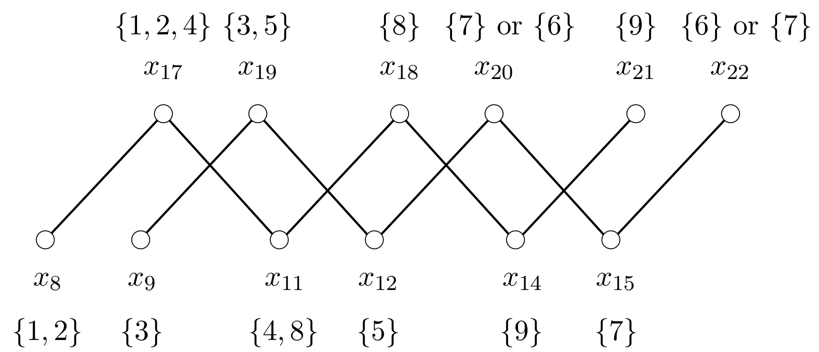


Figure 8.

In cases (iii), (iv), (v) Bob adds points $x_{21} > x_8$ and $x_{22} > x_9$. Alice's response has to be of the following form.

In case (iii) (see Figure 9):

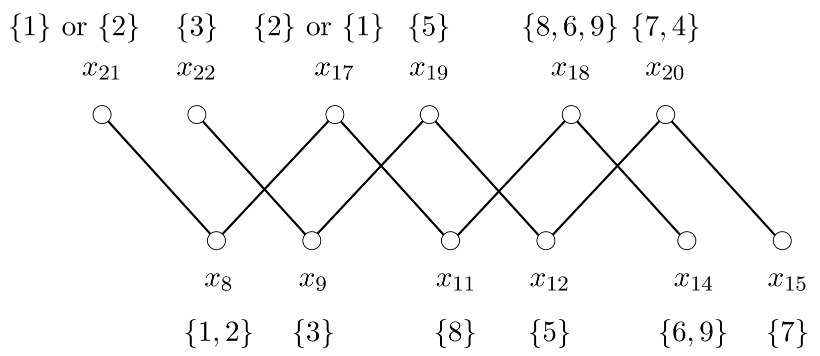


Figure 9.

In case (iv) (see Figure 10):

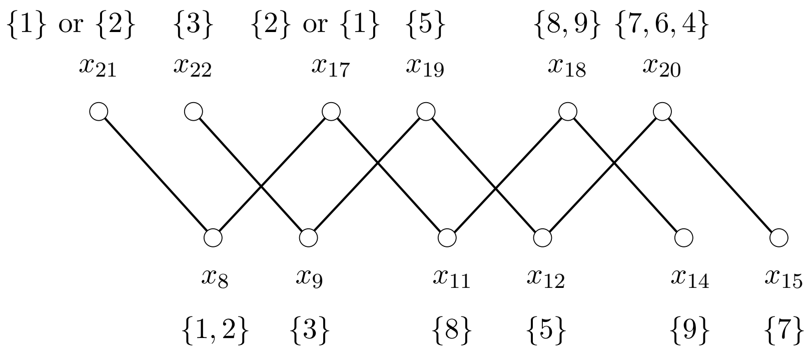


Figure 10.

In case (v) (see Figure 11):

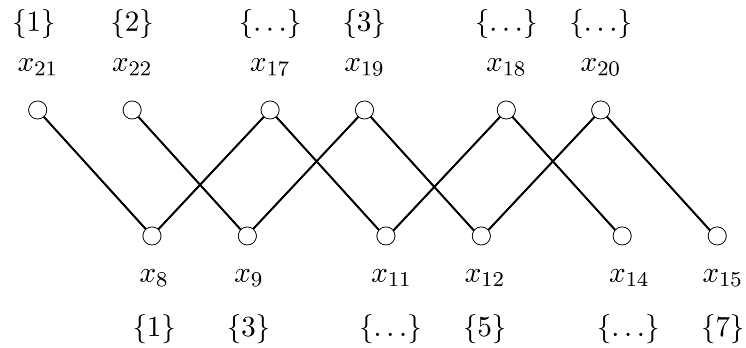


Figure 11.

In all possible cases points x_{21} i x_{22} may be used by Bob as the base of further zigzags. A formalized version of the algorithm looks as follows.

Construct $d-$element antichain $A = (x_1, x_2, \ldots, x_d)$;
$B := \emptyset$; $\{B$ *contains points which are used to build zigzags*$\}$
while $(|A| \geq 3$ or $|B| \geq 3)$ **do**
  if $(|A| \geq 3)$
    build a zigzag on 3 arbitrary points $x, y, z \in A$;
    $A := A \setminus \{x, y, z\} \cup \{p_3\}$; $B := B \cup \{(p_1, p_2)\}$;
  if $(|B| \geq 3)$
    extract from B three pairs of points $(p_1, p_2), (q_1, q_2), (r_1, r_2)$;
    add points x_1, y_1 such that $x_1 > p_1, x_1 > q_1, y_1 > q_1, y_1 > r_1$;
  if Alice responds according to the case (i)

add point $z_1 > r_1$; $\{z_1 \approx x_{19}\}$
$A := A \cup \{z_1, r_2\}$;
else
add points x_2, y_2 such that $x_2 > p_2, x_2 > q_2, y_2 > q_2, y_2 > r_2$; $\{x_2 \approx x_{19},\ y_2 \approx x_{20}\}$
if Alice responds according to the case (i) or (ii)
add points $z_1 > r_1,\ z_2 > r_2$; $\{z_1 \approx x_{21},\ z_2 \approx x_{22}\}$
else (scheme (iii), (iv), (v))
add points $z_1 > p_1,\ z_2 > p_2$; $\{z_1 \approx x_{21},\ z_2 \approx x_{22}\}$
$A := A \cup \{z_1, z_2\}$;

Denote by $f(m,n)$ the number of zigzags which can be constructed when $|A| = m,\ |B| = n$. From the above algorithm we deduce the following recursive formula:

$$f(m,n) = \begin{cases} 1 + f(m-2, n+1), & \text{for } m \geq 3, \\ f(m+2, n-3), & \text{for } m < 3, n \geq 3, \\ 0, & \text{for } m < 3, n < 3. \end{cases}$$

After simple transformations we get

$$f(m,0) = \begin{cases} \lfloor 3m/4 \rfloor - 1, & \text{for odd } m > 2, \\ \lfloor 3m/4 \rfloor - 2, & \text{for even } m > 2. \end{cases}$$

Alice returns for the constructed order of width d value not less than $d + f(d,0)$. We have proved the following

Theorem 8 *For every $d \in \mathbb{N}$ we have*

$$\mathsf{dim\ upgrowing\ on\text{-}line}(d) \geq \lfloor 7d/4 \rfloor - 2.$$

□

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