Mediterr. J. Math. (2020) 17:40
https://doi.org/10.1007/s00009-020-1472-1

Mediterranean Journal of Mathematics

The Gauge-Natural Bilinear Operators Similar to the Dorfman–Courant Bracket

Włodzimierz M. Mikulski

Abstract. All gauge-natural bilinear operators $A : \Gamma^l_E(TE \oplus T^*E) \times \Gamma^l_E(TE \oplus T^*E) \to \Gamma^l_E(TE \oplus T^*E)$ transforming pairs of linear sections of the "doubled" tangent bundle $TE \oplus T^*E$ of a vector bundle E into linear sections of $TE \oplus T^*E$ are completely described. Then, all such A with the Jacobi identity in Leibniz form are extracted.

Mathematics Subject Classification. 53A55, 53A45, 53A99.

Keywords. Natural operator, linear vector field, linear 1-form, Dorfman–Courant bracket, Jacobi identity in Leibniz form.

1. Introduction

All manifolds considered in the paper are assumed to be Hausdorff, second countable, finite dimensional, without boundary, and smooth (of class $\mathcal{C}^\infty$). Maps between manifolds are assumed to be $\mathcal{C}^\infty$.

In [3,8], the authors completely described bilinear operators on sections of $TN \oplus T^*N \to N$ (for N a smooth manifold), which are $\mathcal{M}f_m$-natural, i.e., invariant under the morphisms in the category $\mathcal{M}f_m$ of m-dimensional manifolds and their submersions. The principal result of [3] is precisely the full classification of such operators which also, like the Courant bracket, satisfy the Jacobi identity in Leibniz form. The Courant bracket, defined in [2], is of particular interest, because it involves in the concept of Dirac structures and in the concept of generalised complex structures on N, see [2,4,5].

This article classifies bilinear operators on the linear sections of the double vector bundles $(TE; E, TM; M)$ and $(T^*E; E, E^*; M)$ (for $E \to M$ a smooth vector bundle), which are gauge-natural, i.e., invariant under the morphisms in the category $\mathcal{VB}_{m,n}$ of rank-n vector bundles over m-dimensional bases and their vector bundle isomorphisms onto images. These double vector bundles are of particular interest, because their direct sum $(TE \oplus T^*E; E, TM \oplus E^*; M)$ is the *standard VB-Courant algebroid*. The Dorfman–Courant bracket is part of this structure and an example of a $\mathcal{VB}_{m,n}$-gauge natural operator $\Gamma^l_E(TE \oplus T^*E) \times \Gamma^l_E(TE \oplus T^*E) \to \Gamma^l_E(TE \oplus T^*E)$, where

Birkhäuser

Published online: 13 February 2020

$\Gamma^l_E(TE \oplus T^*E)$ denotes the space of linear sections of $TE \oplus T^*E \to E$. The Dorfman–Courant bracket is the restriction of the Courant bracket to linear sections of $TE \oplus T^*E \to E$, see [6]. (It can be also interpreted as the bracket of the Omni–Lie algebroid $Der(E^*) \oplus J^1(E^*)$, studied in [1].)

The principal result of the paper is precisely the full classification of such operators which also, like the Dorfman–Courant bracket, satisfy the Jacobi identity in Leibniz form. The article first establishes the general form of $\mathcal{VB}_{m,n}$-gauge natural bilinear operators $A : \Gamma^l_E(TE \oplus T^*E) \times \Gamma^l_E(TE \oplus T^*E) \to \Gamma^l_E(TE \oplus T^*E)$, while its later half is dedicated to establishing which of these operators A satisfy the Jacobi identity in Leibniz form. Thus, the main result of the paper is the following.

Theorem 1.1. *Let $m \geq 2$ and $n \geq 1$. Any $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator $A : \Gamma^l_E(TE \oplus T^*E) \times \Gamma^l_E(TE \oplus T^*E) \to \Gamma^l_E(TE \oplus T^*E)$ is of the form*

$$A(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = a[X^1, X^2] \oplus \{b_1\mathcal{L}_{X^1}\omega^2 + b_2\mathcal{L}_{X^2}\omega^1 + b_3 d i_{X^1}\omega^2 + b_4 d i_{X^2}\omega^1 + b_5\mathcal{L}_{X^1} d i_L\omega^2 + b_6\mathcal{L}_{X^2} d i_L\omega^1\}$$

for arbitrary (uniquely determined by A) real numbers $a, b_1, b_2, b_3, b_4, b_5, b_6$, where $[-,-]$ is the usual bracket on vector fields, $\mathcal{L}$ denotes the Lie derivative, d denotes the exterior derivative, i denotes the insertion derivative and L denotes the Euler vector field.

Moreover, such A satisfies the Jacobi identity in Leibniz form (i.e.

$$A(\nu^1, A(\nu^2, \nu^3)) = A(A(\nu^1, \nu^2), \nu^3) + A(\nu^2, A(\nu^1, \nu^3))$$

for any $\nu^i \in \Gamma^l_E(TE \oplus T^*E)$ for $i = 1, 2, 3$) if and only if $(a, b_1, b_2, b_3, b_4, b_5, b_6)$ is from the following list of 7-tuples:

$$(c, 0, 0, 0, 0, c, 0), (c, 0, 0, 0, 0, c, -c),$$
$$(c, c, 0, 0, 0, -c, 0), (c, c, -c, 0, 0, -c, c),$$
$$(c, 0, 0, 0, 0, 0, 0), (c, c, 0, 0, 0, 0, 0),$$
$$(c, c, 0, 0, 0, 0, -c), (c, c, -c, 0, 0, 0, 0),$$
$$(c, c, -c, 0, c - \lambda, 0, \lambda), (0, 0, 0, \lambda, \mu, -\lambda, -\mu),$$

where c, λ, μ are arbitrary real numbers with $c \neq 0$.

It seems that the case $m = 1$ and $n \geq 1$ is more complicated. It remains open.

Most proofs in the paper hinge the application of the following *multilinear Peetre theorem* in the same manner: This implies that any $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator $\Gamma^l_E(TE \oplus T^*E) \times \Gamma^l_E(TE \oplus T^*E) \to \Gamma^l_E(TE \oplus T^*E)$ is of finite order.

Multilinear Peetre Theorem (Theorem 19.9 in [7]). *Let $L_1, ..., L_k$ be vector bundles over the same base M, $L \to N$ be another vector bundle and let $\pi : N \to M$ be continuous and locally non-constant. If $D : C^\infty(L_1) \times ... \times C^\infty(L_k) \to C^\infty(L)$ is a k-linear π-local operator, then for every compact set $K \subset N$, there is a natural number r such that for every $x \in \pi(K)$ and all sections $s, q \in C^\infty(L_1 \oplus ... \oplus L_k)$, the condition $j^r s(x) = j^r q(x)$ implies $Ds|(\pi^{-1}(x) \cap K) = Dq|(\pi^{-1}(x) \cap K)$.*

From now on, let $\mathbf{R}^{m,n}$ be the trivial vector bundle over $\mathbf{R}^m$ with the standard fibre $\mathbf{R}^n$ and let $x^1, ..., x^m, y^1, ..., y^n$ be the usual coordinates on $\mathbf{R}^{m,n}$.

2. The Gauge-Natural Bilinear Operators Similar to the Dorfman–Courant Bracket

Let $E = (E \to M)$ be a vector bundle.

Applying the tangent and the cotangent functors to $E \to M$, we obtain double vector bundles $(TE; E, TM; M)$ and $(T^*E; E, E^*; M)$.

A vector field X on E is called linear if it is a vector bundle map $X : E \to TE$ between $E \to M$ and $TE \to TM$. Equivalently, a vector field X on E is linear iff it has expression

$$X = \sum_{i=1}^{m} a^i(x^1, ..., x^m) \frac{\partial}{\partial x^i} + \sum_{j,k=1}^{n} b^k_j(x^1, ...x^m) y^j \frac{\partial}{\partial y^k},$$

in any local vector bundle trivialization on E. The Euler vector field L on E is an example of a linear vector field on E. (We recall that the coordinate expression of L is $L = \sum_{j=1}^{n} y^j \frac{\partial}{\partial y^j}$.)

A 1-form ω on E is called linear if it is a vector bundle map $\omega : E \to T^*E$ between $E \to M$ and $T^*E \to E^*$. Equivalently, a 1-form ω on E is linear iff it has expression

$$\omega = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}(x^1, ..., x^m) y^j dx^i + \sum_{j=1}^{n} b_j(x^1, ..., x^m) dy^j,$$

in any local vector bundle trivialization on E.

We need the following definition being respective modification of the general one from the fundamental monograph [7].

Definition 2.1. A $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator $A : \Gamma^l(T \oplus T^*) \times \Gamma^l(T \oplus T^*) \rightsquigarrow \Gamma^l(T \oplus T^*)$ is a $\mathcal{VB}_{m,n}$-invariant family of $\mathbf{R}$-bilinear operators

$$A : \Gamma^l_E(TE \oplus T^*E) \times \Gamma^l_E(TE \oplus T^*E) \to \Gamma^l_E(TE \oplus T^*E)$$

for all $\mathcal{VB}_{m,n}$-objects E, where $\Gamma^l_E(TE \oplus T^*E)$ is the vector space of linear sections of $TE \oplus T^*E$ (i.e. couples $X \oplus \omega$ of linear vector fields X and linear 1-forms ω on E). The $\mathcal{VB}_{m,n}$-invariance of A means that if $(X^1 \oplus \omega^1, X^2 \oplus \omega^2) \in \Gamma^l_E(TE \oplus T^*E) \times \Gamma^l_E(TE \oplus T^*E)$ and $(\overline{X}^1 \oplus \overline{\omega}^1, \overline{X}^2 \oplus \overline{\omega}^2) \in \Gamma^l_{\overline{E}}(T\overline{E} \oplus T^*\overline{E}) \times \Gamma^l_{\overline{E}}(T\overline{E} \oplus T^*\overline{E}))$ are φ-related by an $\mathcal{VB}_{m,n}$-map $\varphi : E \to \overline{E}$ (i.e. $\overline{X}^i \circ \varphi = T\varphi \circ X^i$ and $\overline{\omega}^i \circ \varphi = T^*\varphi \circ \omega^i$ for $i = 1, 2$), then so are $A(X^1 \oplus \omega^1, X^2 \oplus \omega^2)$ and $A(\overline{X}^1 \oplus \overline{\omega}^1, \overline{X}^2 \oplus \overline{\omega}^2)$.

Remark 2.2. Quite similarly, we can define $\mathcal{VB}_{m,n}$-gauge-natural bilinear operators $\Gamma^l(T) \times \Gamma^l(T) \rightsquigarrow \Gamma^l(T)$, $\Gamma^l(T) \times \Gamma^l(T^*) \rightsquigarrow \Gamma^l(T)$, etc. For example, a $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator $A : \Gamma^l(T) \times \Gamma^l(T^*) \rightsquigarrow \Gamma^l(T)$ is a $\mathcal{VB}_{m,n}$-invariant family of $\mathbf{R}$-bilinear operators $A : \Gamma^l_E(TE) \times \Gamma^l_E(T^*E) \to$

$\Gamma^l_E(TE)$ for all $\mathcal{VB}_{m,n}$-objects E, where $\Gamma^l_E(TE)$ is the space of linear vector fields on E and $\Gamma^l_E(T^*E)$ is the space of linear 1-forms on E.

Example 2.3. The usual bracket $[X,Y]$ of (linear) vector fields defines a $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator $[-,-]:\Gamma^l(T)\times\Gamma^l(T)\rightsquigarrow\Gamma^l(T)$.

Example 2.4. The Lie derivative $\mathcal{L}_X\omega$ of linear 1-forms ω with respect to linear vector fields X defines a $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator $\mathcal{L}:\Gamma^l(T)\times\Gamma^l(T^*)\rightsquigarrow\Gamma^l(T^*)$.

Example 2.5. Let ω be a linear 1-form and X be a linear vector field on a vector bundle E. Then, we have linear 1-form $i_Xd\omega$, and we have the corresponding $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator $\Gamma^l(T)\times\Gamma^l(T^*)\rightsquigarrow\Gamma^l(T^*)$, where d denotes the exterior derivative and i denotes the insertion derivative.

Example 2.6. The Dorfman–Courant bracket $[[X^1\oplus\omega^1,X^2\oplus\omega^2]]:=[X^1,X^2]\oplus(\mathcal{L}_{X^1}\omega^2-i_{X^2}d\omega^1)$ gives $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator $[[-,-]]:\Gamma^l(T\oplus T^*)\times\Gamma^l(T\oplus T^*)\rightsquigarrow\Gamma^l(T\oplus T^*)$.

Example 2.7. Let ω be a linear 1-form and X be a linear vector field on a vector bundle E and let L denotes the Euler vector field on E. Then, we have linear 1-form $\mathcal{L}_Xdi_L\omega$, and we have the corresponding $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator $\Gamma^l(T)\times\Gamma^l(T^*)\rightsquigarrow\Gamma^l(T^*)$.

Lemma 2.8. *Any $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator A (in question) is of finite order. It means that there is a finite number r (depending on A) such that*

$$(j^r_x\nu_1=j^r_x\overline{\nu}_1,j^r_x\nu_2=j^r_x\overline{\nu}_2)\Rightarrow A(\nu_1,\nu_2)_{|E_x}=A(\overline{\nu}_1,\overline{\nu}_2)_{|E_x}$$

for any $\mathcal{VB}_{m,n}$-object $E\to M$, any linear sections ν_1 and ν_2 on E and any $x\in M$, where E_x is the fibre of $E\to M$ over $x\in M$, and $j^r_x\nu_1=j^r_x\overline{\nu}_1$ means that $j^r_v\nu_1=j^r_v\overline{\nu}_1$ for any $v\in E_x$ (or equivalently for any v from the basis of E_x).

Proof. The space $\Gamma^l_E(TE\oplus T^*E)$ is a locally free $\mathcal{C}^\infty(M)$-module. Hence, there is a vector bundle $\hat{E}$ over M such that $\Gamma^l_E(TE\oplus T^*E)$ is isomorphic to $\Gamma\hat{E}$ as $\mathcal{C}^\infty(M)$-module. So, we can treat A as bilinear local operator $A:\Gamma\hat{E}\times\Gamma\hat{E}\to\Gamma\hat{E}$. Then, the multi-linear Peetre theorem (cited in Introduction) for $k=2$, $M=N=\mathbf{R}^m$, $\pi=\mathrm{id}$, $K=\{0\}$, $L_1=L_2=L=\hat{\mathbf{R}}^{m,n}$ and $D=A$ implies that there is a natural number r such that for every pairs (ν_1,ν_2) and $(\overline{\nu}_1,\overline{\nu}_2)$ of linear sections, the condition $j^r_0(\nu_1,\nu_2)=j^r_0(\overline{\nu}_1,\overline{\nu}_2)$ implies $A(\nu_1,\nu_2)=A(\overline{\nu}_1,\overline{\nu}_2)$ at 0. Then, using the invariance of A, we complete the proof.

For the other operators mentioned in Remark 2.2, the proofs are similar. □

A linear vector field X on $\mathbf{R}^{m,n}$ is monomial if it is of the form $x^\alpha\frac{\partial}{\partial x^i}$ or $x^\alpha y^j\frac{\partial}{\partial y^k}$, where $\alpha=(\alpha^1,...,\alpha^m)$ is a m-tuple of non-negative integers and $i=1,...,m$, $j,k=1,...,n$, Of course, $x^\alpha:=(x^1)^{\alpha^1}\cdot...\cdot(x^m)^{\alpha^m}$.

Similarly, a linear 1-form ω on $\mathbf{R}^{m,n}$ is monomial if it is of the form $x^\alpha dy^j$ or $x^\alpha y^j dx^i$.

A linear section $X \oplus \omega$ is called monomial if (X is monomial and $\omega = 0$) or ($X = 0$ and ω is monomial).

Lemma 2.9. *Let A be a $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator (in question) such that $A(\nu_1, \nu_2) = 0$ over $0 \in \mathbf{R}^m$ for all monomial linear sections ν_1 and ν_2 on $\mathbf{R}^{m,n}$. Then $A = 0$.*

Proof. Because of the invariance of A with respect to local trivialization, it suffices to show that $A(\nu_1, \nu_2) = 0$ over $0 \in \mathbf{R}^m$ for any linear sections ν_1 and ν_2 on $\mathbf{R}^{m,n}$. Since A is of finite order r (because of Lemma 2.8), we may assume that ν_1 and ν_2 are polynomial of degree not more than r. Then, since A is bilinear, we may assume that ν_1 and ν_2 are monomial. □

Lemma 2.10. *Let $A : \Gamma^l(T) \times \Gamma^l(T) \rightsquigarrow \Gamma^l(T)$ (or $A : \Gamma^l(T) \times \Gamma^l(T) \rightsquigarrow \Gamma^l(T^*)$) be a $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator. Assume that $m \geq 2$ and $A(\frac{\partial}{\partial x^1}, (x^1)^q \frac{\partial}{\partial x^2}) = 0$ over $0 \in \mathbf{R}^m$ for all $q = 0, 1, \ldots$. Then $A = 0$.*

Proof. Because of the invariance of A with respect to local trivialization, it suffices to show that $A(X, Y) = 0$ over $0 \in \mathbf{R}^m$ for any linear vector fields X and Y on $\mathbf{R}^{m,n}$. We can assume X is not vertical over 0. Then by the Frobenius theorem and the invariance of A, we can assume $X = \frac{\partial}{\partial x^1}$. Next, by the similar arguments as in the proof of Lemma 2.9, we may assume that Y is monomial.

So, let $\beta = (\beta_1, \beta_2, \ldots, \beta_m) \in (\mathbf{N} \cup \{0\})^m$ and $j, k = 1, \ldots, n$ and $i = 1, \ldots, m$.

There exists a $\mathcal{VB}_{m,n}$-map $\psi : \mathbf{R}^{m,n} \to \mathbf{R}^{m,n}$ preserving x^1 and $\frac{\partial}{\partial x^1}$ and sending the germ at $0 \in \mathbf{R}^m$ of $\frac{\partial}{\partial x^2}$ into the germ at $0 \in \mathbf{R}^m$ of $\frac{\partial}{\partial x^2} + (x^2)^{\beta_2} \cdot \ldots \cdot (x^m)^{\beta_m} y^j \frac{\partial}{\partial y^k}$. Then, by the invariance of A with respect to ψ, from assumption $A(\frac{\partial}{\partial x^1}, (x^1)^{\beta_1} \frac{\partial}{\partial x^2}) = 0$ over $0 \in \mathbf{R}^m$, we get $A(\frac{\partial}{\partial x^1}, (x^1)^{\beta_1} \frac{\partial}{\partial x^2} + x^\beta y^j \frac{\partial}{\partial y^k}) = 0$ over $0 \in \mathbf{R}^m$. Then $A(\frac{\partial}{\partial x^1}, x^\beta y^j \frac{\partial}{\partial y^k}) = 0$ over $0 \in \mathbf{R}^m$.

If $i = 2, \ldots, m$, there exists a $\mathcal{VB}_{m,n}$-map $\varphi : \mathbf{R}^{m,n} \to \mathbf{R}^{m,n}$ preserving x^1 and $\frac{\partial}{\partial x^1}$ and sending the germ at $0 \in \mathbf{R}^m$ of $\frac{\partial}{\partial x^2}$ into the germ at $0 \in \mathbf{R}^m$ of $\frac{\partial}{\partial x^2} + (x^2)^{\beta_2} \cdot \ldots \cdot (x^m)^{\beta_m} \frac{\partial}{\partial x^i}$. Then, similarly as above (using the invariance with respect to φ), we get $A(\frac{\partial}{\partial x^1}, x^\beta \frac{\partial}{\partial x^i}) = 0$ over $0 \in \mathbf{R}^m$.

Then, using the invariance of A with respect to $(x^1 + \tau x^2, x^2, \ldots, x^m, y^1, \ldots, y^n)$, we get $A(\frac{\partial}{\partial x^1}, (x^1 - \tau x^2)^{\beta_1} (x^2)^{\beta_2} \cdot \ldots \cdot (x^m)^{\beta_m} (\frac{\partial}{\partial x^2} + \tau \frac{\partial}{\partial x^1}))(e) = 0$ for any $\tau \in \mathbf{R}$ and any e in the fibre of $\mathbf{R}^{m,n}$ over $0 \in \mathbf{R}^m$. Considering the coefficient on τ, we get $A(\frac{\partial}{\partial x^1}, x^\beta \frac{\partial}{\partial x^1})(e) - \beta_1 A(\frac{\partial}{\partial x^1}, (x^1)^{\beta_1 - 1} (x^2)^{\beta_2 + 1} (x^3)^{\beta_3} \cdot \ldots \cdot (x^m)^{\beta_m} \frac{\partial}{\partial x^2})(e) = 0$. Then $A(\frac{\partial}{\partial x^1}, x^\beta \frac{\partial}{\partial x^1}) = 0$ over $0 \in \mathbf{R}^m$.

The lemma is complete □

Lemma 2.11. *Let $m \geq 1$ and $n \geq 1$. Let $A : \Gamma^l(T) \times \Gamma^l(T^*) \rightsquigarrow \Gamma^l(T)$ be a $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator. Assume that $A(\frac{\partial}{\partial x^1}, \omega) = 0$ over $0 \in \mathbf{R}^m$ for all monomial linear 1-forms ω on $\mathbf{R}^{m,n}$. Then $A = 0$.*

Proof. It suffices to show that $A(X, \omega) = 0$ over $0 \in \mathbf{R}^m$ for any linear vector field X and any linear 1-form ω on $\mathbf{R}^{m,n}$. We can assume $X = \frac{\partial}{\partial x^1}$ and ω is monomial. □

Lemma 2.12. *Let $m \geq 1$ and $n \geq 1$. Let $A : \Gamma^l(T^*) \times \Gamma^l(T^*) \rightsquigarrow \Gamma^l(T)$ (or $A : \Gamma^l(T^*) \times \Gamma^l(T^*) \rightsquigarrow \Gamma^l(T^*)$) be a $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator. Assume that $A(\omega^1, \omega^2) = 0$ over $0 \in \mathbf{R}^m$ for all monomial linear 1-forms ω^1, ω^2 on $\mathbf{R}^{m,n}$. Then $A = 0$.*

Proof. It is the particular case of Lemma 2.9. □

Lemma 2.13. *Let $A : \Gamma^l(T) \times \Gamma^l(T^*) \rightsquigarrow \Gamma^l(T^*)$ be a $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator. Assume that $m \geq 2$ and*

$$A\left(\frac{\partial}{\partial x^1}, (x^1)^k dy^1\right)(e_1) = 0 \text{ and } A\left(\frac{\partial}{\partial x^1}, (x^1)^k y^1 dx^1\right)(e_1) = 0$$

for all $k = 0, 1, \ldots$, where $e_1 = (1, 0, \ldots, 0) \in \mathbf{R}^n$ is the element in the fibre over $0 \in \mathbf{R}^m$ of $\mathbf{R}^{m,n}$. Then $A = 0$.

Proof. It suffices to show that $A(\frac{\partial}{\partial x^1}, \omega) = 0$ over $0 \in \mathbf{R}^m$ for any monomial linear 1-form ω on $\mathbf{R}^{m,n}$.

(I) At first, we prove that $A(\frac{\partial}{\partial x^1}, x^\beta dy^j) = 0$ over $0 \in \mathbf{R}^m$ for all m-tuples $\beta = (\beta_1, \ldots, \beta_m)$ of non-negative integers and all $j = 1, \ldots, n$.

Consider a m-tuple $\beta = (\beta_1, \ldots, \beta_m)$ of non-negative integers. Let $j = 1, \ldots, n$ and $e = (\xi_1, \ldots, \xi_n)$ be a point from the fibre over $0 \in \mathbf{R}^m$ of $\mathbf{R}^{m,n}$. We may assume $\xi_j \neq 0$. Let $k = |\beta|$. Using the invariance of A with respect to $\mathcal{VB}_{m,n}$-maps

$$(\tau_1 x^1 + \tau_2 x^2 + \ldots + \tau_m x^m, x^2, \ldots, x^m, y^1, \ldots, y^n)^{-1}$$

for $\tau_1 \neq 0, \tau_2, \ldots, \tau_m$ (sending $\frac{\partial}{\partial x^1}$ into $\frac{1}{\tau_1}\frac{\partial}{\partial x^1}$ and preserving e_1), from the assumption of the lemma, we get

$$A\left(\frac{\partial}{\partial x^1}, (\tau_1 x^1 + \tau_2 x^2 + \ldots + \tau_m x^m)^k dy^1\right)(e_1) = 0.$$

Then, considering the coefficients on $(\tau_1)^{\beta_1} \cdot \ldots \cdot (\tau_m)^{\beta_m}$ of these polynomials in $\tau_1, \ldots, \tau_m$, we get $A(\frac{\partial}{\partial x^1}, x^\beta dy^1)(e_1) = 0$. Since $\xi_j \neq 0$, there is a linear isomorphism $\varphi : \mathbf{R}^n \to \mathbf{R}^n$ sending y^1 into $\frac{1}{\xi_j} y^j$ and e_1 into e. Then, using the invariance of A with respect to $(x^1, \ldots, x^m, \varphi(y^1, .., y^n))$, we get

$$\frac{1}{\xi_j} A\left(\frac{\partial}{\partial x^1}, x^\beta dy^j\right)(e) = 0 \ ,$$

i.e. $A(\frac{\partial}{\partial x^1}, x^\beta dy^j)(e) = 0$.

(II) Now, we prove $A(\frac{\partial}{\partial x^1}, x^\beta y^j dx^i) = 0$ over $0 \in \mathbf{R}^m$ for all m-tuples $\beta = (\beta_1, \ldots, \beta_m)$ of non-negative integers and $j = 1, \ldots, n$ and $i = 1, \ldots, m$.

Let $\beta = (\beta_1, \ldots, \beta_m)$ be an m-tuple of non-negative integers, $j = 1, \ldots, n$, $i = 1, \ldots, m$ and e be from the fibre over $0 \in \mathbf{R}^m$ of $\mathbf{R}^{m,n}$. If $\beta_2 + \ldots + \beta_m \geq 1$, using the invariance of A with respect to $\mathcal{VB}_{m,n}$-map

$$(x^1, x^2, \ldots, x^m, y^1 + (x^2)^{\beta_2} \cdot \ldots \cdot (x^m)^{\beta_m} y^1, y^2 \ldots, y^n)^{-1}$$

(preserving $\frac{\partial}{\partial x^1}$ and e_1), from the assumption $A(\frac{\partial}{\partial x^1}, (x^1)^{\beta_1} y^1 dx^1)(e_1) = 0$ we get

$$A\left(\frac{\partial}{\partial x^1}, (x^1)^{\beta_1} (y^1 + (x^2)^{\beta_2} \cdot \ldots \cdot (x^m)^{\beta_m} y^1) dx^1\right)(e_1) = 0.$$

Consequently, $A(\frac{\partial}{\partial x^1}, x^\beta y^1 dx^1)(e_1) = 0$ (for $\beta_2 + ... + \beta_m = 0$, too). So, if $i = 2, ..., m$, then, by the invariance of A with respect to $(x^1 + \tau x^i, x^2, ..., x^m, y^1, y^2 ..., y^n)^{-1}$ (preserving $\frac{\partial}{\partial x^1}$ and e_1), we get

$$A\left(\frac{\partial}{\partial x^1}, (x^1 + \tau x^i)^{\beta_1}(x^2)^{\beta_2} \cdot ... \cdot (x^m)^{\beta_m} y^1 d(x^1 + \tau x^i)\right)(e_1) = 0.$$

Then, considering the coefficient on τ, we get

$$\beta_1 B + A\left(\frac{\partial}{\partial x^1}, x^\beta y^1 dx^i\right)(e_1) = 0 \ ,$$

where $B := A(\frac{\partial}{\partial x^1}, (x^1)^{\beta_1 - 1}(x^2)^{\beta_2} \cdot ... \cdot (x^i)^{\beta_i + 1} \cdot ... \cdot (x^m)^{\beta_m} y^1 dx^1)(e_1)$. If $\beta_1 \neq 0$, $B = 0$ (it is proved above). If $\beta_1 = 0$, the term $\beta_1 B$ does not occur. Consequently, $A(\frac{\partial}{\partial x^1}, x^\beta y^1 dx^i)(e_1) = 0$ (for $i = 1$, too). Then (using similar arguments to the one of the end of the part (I) of the proof) $A(\frac{\partial}{\partial x^1}, x^\beta y^j dx^i)(e) = 0$. □

Lemma 2.14. *The collection of $\mathcal{VB}_{m,n}$-gauge-natural operators $A^i : \Gamma^l(T) \times \Gamma^l(T^*) \rightsquigarrow \Gamma^l(T^*)$ for $i = 1, 2, 3$ given by $A^1(X, \omega) = \mathcal{L}_X \omega$, $A^2(X, \omega) = i_X d\omega$ and $A^3(X, \omega) = \mathcal{L}_X d i_L \omega$ is $\mathbf{R}$-linearly independent.*

Proof. We know that $L = \sum_{j=1}^n y^j \frac{\partial}{\partial y^j}$ end $e_1 = (1, 0..., 0) \in \mathbf{R}^n$ is the element over the fibre over $0 \in \mathbf{R}^m$ of $\mathbf{R}^{m,n}$. Then, it is easy to compute that $A^1(\frac{\partial}{\partial x^1}, y^1 dx^1)(e_1) = 0$, $A^2(\frac{\partial}{\partial x^1}, y^1 dx^1)(e_1) = -d_{e_1} y^1$, $A^3(\frac{\partial}{\partial x^1}, y^1 dx^1)(e_1) = 0$, $A^1(\frac{\partial}{\partial x^1}, x^1 dy^1)(e_1) = d_{e_1} y^1$, $A^2(\frac{\partial}{\partial x^1}, x^1 dy^1)(e_1) = d_{e_1} y^1$, $A^3(\frac{\partial}{\partial x^1}, x^1 dy^1)(e_1) = d_{e_1} y^1$, $A^1(\frac{\partial}{\partial x^1}, (x^1)^2 dy^1)(e_1) = 0$, $A^2(\frac{\partial}{\partial x^1}, (x^1)^2 dy^1)(e_1) = 0$, $A^3(\frac{\partial}{\partial x^1}, (x^1)^2 dy^1)(e_1) = 2 d_{e_1} x^1$. Now, the lemma is clear. □

Proposition 2.15. *Let $m \geq 2$ and $n \geq 1$. Any $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator $A : \Gamma^l(T) \times \Gamma^l(T) \rightsquigarrow \Gamma^l(T)$ is the constant multiple of the usual bracket $[-, -]$ on (linear) vector fields.*

Proof. Let k be a non-negative integer. We can write

$$A\left(\frac{\partial}{\partial x^1}, (x^1)^k \frac{\partial}{\partial x^2}\right)(e) = \sum_{i=1}^m f^{[k,i]} \frac{\partial}{\partial x^i}_{|e} + \sum_{j,l=1}^n g_j^{[k,l]} e^j \frac{\partial}{\partial y^l}_{|e}$$

for any $e = (e^1, ..., e^n)$ from the fibre $\mathbf{R}^n$ at $0 \in \mathbf{R}^m$ of $\mathbf{R}^{m,n}$, where $f^{[k,i]}$ and $g_j^{[k,l]}$ are the real numbers (independent of e).

Using the invariance of A with respect to $(x^1, tx^2, x^3, ..., x^m, y^1, ..., y^n)$ for $t > 0$, we get the conditions $t f^{[k,i]} = f^{[k,i]}$ for $i = 1, 3, ..., m$ (i.e. for $i \neq 2$) and $t g_j^{[k,l]} = g_j^{[k,l]}$ for $j, l = 1, ..., n$. Then $f^{[k,i]} = 0$ for $i = 1, 3, ..., m$ and $g_j^{[k,l]} = 0$ for $j, l = 1, ..., n$.

Similarly, by the invariance of A with respect to $(\frac{1}{t} x^1, x^2, ..., x^m, y^1, ..., y^n)$, we get $t^{k-1} f^{[k,2]} = f^{[k,2]}$, i.e. $f^{[k,2]} = 0$ if $k \neq 1$.

Then by Lemma 2.10, A is determined by the value $f^{[1,2]} \in \mathbf{R}$. Consequently, the vector space of all such A is of dimension not more than 1. Then, the dimension argument completes the proof of the proposition. □

Proposition 2.16. *Let $m \geq 1$ and $n \geq 1$. Any $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator $A : \Gamma^l(T) \times \Gamma^l(T^*) \rightsquigarrow \Gamma^l(T)$ is zero.*

Proof. Let $\alpha = (\alpha^1, ..., \alpha^m) \in (\mathbf{N} \cup \{0\})^m$ be an m-tuple of non-negative integers and let $i_o \in \{1, ..., m\}$ and $j_o \in \{1, ..., n\}$.

We can write

$$A\left(\frac{\partial}{\partial x^1}, x^\alpha y^{j_o} dx^{i_o}\right)(e) = \sum_{i=1}^m f^i \frac{\partial}{\partial x^i}_{|e} + \sum_{j,l=1}^n g^l_j e^j \frac{\partial}{\partial y^l}_{|e}$$

for any $e = (e^1, ..., e^n)$ from the fibre $\mathbf{R}^n$ at $0 \in \mathbf{R}^m$ of $\mathbf{R}^{m,n}$, where f^i and g^l_j are the real numbers (depending on α, i_o and j_o and independent of e).

Using the invariance of A with respect to $(x^1, ..., x^m, \frac{1}{t}y^1, ..., \frac{1}{t}y^n)$ for $t > 0$, we get

$$\sum_{i=1}^m t f^i \frac{\partial}{\partial x^i}_{|\frac{1}{t}e} + \sum_{j,l=1}^n t g^l_j \frac{1}{t} e^j \frac{\partial}{\partial y^l}_{|\frac{1}{t}e} = \sum_{i=1}^m f^i \frac{\partial}{\partial x^i}_{|\frac{1}{t}e} + \sum_{j,l=1}^n g^l_j e^j \frac{1}{t}\frac{\partial}{\partial y^l}_{|\frac{1}{t}e}.$$

Then $A(\frac{\partial}{\partial x^1}, x^\alpha y^{j_o} dx^{i_o}) = 0$ over $0 \in \mathbf{R}^m$ for $i_o = 1, ..., m$, $j_o = 1, ..., n$, $\alpha \in (\mathbf{N} \cup \{0\})^m$. By the same argument (replacing $x^\alpha y^{j_o} dx^{i_o}$ by $x^\alpha dy^{j_o}$), we derive $A(\frac{\partial}{\partial x^1}, x^\alpha dy^{j_o}) = 0$ over $0 \in \mathbf{R}^m$ for $j_o = 1, ..., n$ and $\alpha \in (\mathbf{N} \cup \{0\})^m$. Consequently, $A(\frac{\partial}{\partial x^1}, \omega) = 0$ over $0 \in \mathbf{R}^m$ for any linear 1-form ω on $\mathbf{R}^{m,n}$.

Now, applying Lemma 2.11, we complete the proof. □

Proposition 2.17. *Let $m \geq 1$ and $n \geq 1$. Any $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator $A : \Gamma^l(T^*) \times \Gamma^l(T^*) \rightsquigarrow \Gamma^l(T)$ is zero.*

Proof. We proceed quite similar as for Proposition 2.16. Let $\omega^1 = x^\beta y^{j_1} dx^{i_1}$ or $\omega^1 = x^\beta dy^{j_1}$ and let $\omega^2 = x^\alpha y^{j_o} dx^{i_o}$ or $\omega^2 = x^\alpha dy^{j_o}$. We can write

$$A(\omega^1, \omega^2)(e) = \sum_{i=1}^m f^i \frac{\partial}{\partial x^i}_{|e} + \sum_{j,l=1}^n g^l_j e^j \frac{\partial}{\partial y^l}_{|e}$$

for any $e = (e^1, ..., e^n)$ from the fibre $\mathbf{R}^n$ at $0 \in \mathbf{R}^m$ of $\mathbf{R}^{m,n}$, where f^i and g^l_j are the real numbers (depending on ω^1 and ω^2 and independent of e). Using the invariance of A with respect to $(x^1, ..., x^m, \frac{1}{t}y^1, ..., \frac{1}{t}y^n)$ for $t > 0$, we get

$$\sum_{i=1}^m t^2 f^i \frac{\partial}{\partial x^i}_{|\frac{1}{t}e} + \sum_{j,l=1}^n t^2 g^l_j \frac{1}{t} e^j \frac{\partial}{\partial y^l}_{|\frac{1}{t}e} = \sum_{i=1}^m f^i \frac{\partial}{\partial x^i}_{|\frac{1}{t}e} + \sum_{j,l=1}^n g^l_j e^j \frac{1}{t}\frac{\partial}{\partial y^l}_{|\frac{1}{t}e}.$$

Then $A(\omega^1, \omega^2) = 0$ over 0. $\mathbf{R}^{m,n}$. Then $A = 0$ because of Lemma 2.12. □.

Proposition 2.18. *Let $m \geq 2$ and $n \geq 1$. Any $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator $A : \Gamma^l(T) \times \Gamma^l(T) \rightsquigarrow \Gamma^l(T^*)$ is zero.*

Proof. Let k be a non-negative integer. We can write

$$A\left(\frac{\partial}{\partial x^1}, (x^1)^k \frac{\partial}{\partial x^2}\right)(e) = \sum_{j=1}^n f^{[k]}_j d_e y^j + \sum_{i=1}^m \sum_{l=1}^n g^{[k]}_{il} e^l d_e x^i$$

for any $e = (e^1, ..., e^n)$ from the fibre $\mathbf{R}^n$ at $0 \in \mathbf{R}^m$ of $\mathbf{R}^{m,n}$, where $f_j^{[k]}$ and $g_{il}^{[k]}$ are the real numbers (independent of e). By the invariance of A with respect to $(x^1, tx^2, ..., x^m, y^1, ..., y^n)$ for $t > 0$, we get

$$\sum_{j=1}^{n} t f_j^{[k]} d_e y^j + \sum_{i=1}^{m}\sum_{l=1}^{n} t g_{il}^{[k]} e^l d_e x^i = \sum_{j=1}^{n} f_j^{[k]} d_e y^j + \sum_{i=1}^{m}\sum_{l=1}^{n} \frac{1}{t^{\delta_{i2}}} g_{il}^{[k]} e^l d_e x^i$$

(the Kronecker delta). Then $A(\frac{\partial}{\partial x^1}, (x^1)^k \frac{\partial}{\partial x^2}) = 0$ over 0 for $k = 0, 1, ...$. Then $A = 0$ because of Lemma 2.10. □

Proposition 2.19. *Let $m \geq 1$ and $n \geq 1$. Any $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator $A : \Gamma^l(T^*) \times \Gamma^l(T^*) \rightsquigarrow \Gamma^l(T^*)$ is the zero one.*

Proof. Let $\omega^1 = x^\beta y^{j_1} dx^{i_1}$ or $\omega^1 = x^\beta dy^{j_1}$ and let $\omega = x^\alpha y^{j_o} dx^{i_o}$ or $\omega = x^\alpha dy^{j_o}$. We can write

$$A(\omega^1, \omega^2)(e) = \sum_{j=1}^{n} f_j d_e y^j + \sum_{i=1}^{m}\sum_{l=1}^{n} g_{il} e^l d_e x^i$$

for any $e = (e^1, ..., e^n)$ from the fibre $\mathbf{R}^n$ at $0 \in \mathbf{R}^m$ of $\mathbf{R}^{m,n}$, where f_j and g_{il} are the real numbers (independent of e). Using the invariance of A with respect to $(x^1, ..., x^m, \frac{1}{t}y^1, ..., \frac{1}{t}y^n)$, we get

$$\sum_{j=1}^{n} t^2 f_j d_{\frac{1}{t}e} y^j + \sum_{i=1}^{m}\sum_{l=1}^{n} t^2 g_{il} \frac{1}{t} e^l d_{\frac{1}{t}e} x^i = \sum_{j=1}^{n} t f_j d_{\frac{1}{t}e} y^j + \sum_{i=1}^{m}\sum_{l=1}^{n} g_{il} e^l d_{\frac{1}{t}e} x^i.$$

Then $A(\omega^1, \omega^2) = 0$ over 0. Then $A = 0$ because of Lemma 2.12. □

Proposition 2.20. *Let $m \geq 2$ and $n \geq 1$. Any $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator $A : \Gamma^l(T) \times \Gamma^l(T^*) \rightsquigarrow \Gamma^l(T^*)$ is of the form*

$$A(X, \omega) = c_1 \mathcal{L}_X \omega + c_2 i_X d\omega + c_3 \mathcal{L}_X d i_L \omega$$

for the (uniquely determined by A) real numbers c_1, c_2, c_3.

Proof. Let k be a non-negative integer and $e_1 = (1, 0, ..., 0) \in \mathbf{R}^n$. We can write

$$A\left(\frac{\partial}{\partial x^1}, (x^1)^k dy^1\right)(e_1) = \sum_{j=1}^{n} f_j^{(k)} d_{e_1} y^j + \sum_{i=1}^{m} g_i^{(k)} d_{e_1} x^i,$$

where $f_j^{(k)}$ and $g_i^{(k)}$ are the real numbers. Using the invariance of A with respect to $(\frac{1}{t}x^1, ..., \frac{1}{t}x^m, y^1, ..., y^n)$, we get $t^{k-1} f_j^{(k)} = f_j^{(k)}$ and $t^{k-1} g_i^{(k)} = g_i^{(k)} t$. Then $f_j^{(k)} = 0$ for $j = 1, ..., n$ if $k \neq 1$, and if $k \neq 2$ then $g_i^{(k)} = 0$ for $i = 1, ..., m$. Hence,

$$A\left(\frac{\partial}{\partial x^1}, (x^1)^k dy^1\right)(e_1) = 0 \text{ if } k = 0, 3, 4, 5, ...,$$

$A(\frac{\partial}{\partial x^1}, x^1 dy^1)(e_1) = \sum_{j=1}^{n} f_j^{(1)} d_{e_1} y^j$ and $A(\frac{\partial}{\partial x^1}, (x^1)^2 dy^1)(e_1) = \sum_{i=1}^{m} g_i^{(2)} d_{e_1} x^i$. Now, using the invariance of A with respect to $(x^1, tx^2, ..., tx^m, y^1, ty^2, ..., ty^n)$ for $t > 0$, we deduce that

$$A\left(\frac{\partial}{\partial x^1}, x^1dy^1\right)(e_1) = f_1^{(1)}d_{e_1}y^1 \text{ and } A\left(\frac{\partial}{\partial x^1}, (x^1)^2dy^1\right)(e_1) = g_1^{(2)}d_{e_1}x^1.$$

Similarly, we can write

$$A\left(\frac{\partial}{\partial x^1}, (x^1)^k y^1 dx^1\right)(e_1) = \sum_{j=1}^{n} \tilde{f}_j^{(k)} d_{e_1}y^j + \sum_{i=1}^{m} \tilde{g}_i^{(k)} d_{e_1}x^i,$$

where $\tilde{f}_j^{(k)}$ and $\tilde{g}_i^{(k)}$ are the real numbers. Then quite similarly as above, we get

$$A\left(\frac{\partial}{\partial x^1}, y^1dx^1\right)(e_1) = \tilde{f}_1^{(0)}d_{e_1}y^1 \text{ and } A\left(\frac{\partial}{\partial x^1}, x^1y^1dx^1\right)(e_1) = \tilde{g}_1^{(1)}d_{e_1}x^1$$

and $A(\frac{\partial}{\partial x^1}, (x^1)^k y^1 dx^1)(e_1) = 0$ if $k = 2, 3, 4, 5, ...$

We prove that

$$2\tilde{g}_1^{(1)} + g_1^{(2)} = 2(f_1^{(1)} + \tilde{f}_1^{(0)}).$$

We know (see, above) that

$$A\left(\frac{\partial}{\partial x^1}, d(x^1y^1)\right)(e_1) = A\left(\frac{\partial}{\partial x^1}, x^1dy^1\right)(e_1) + A\left(\frac{\partial}{\partial x^1}, y^1dx^1\right)(e_1)$$
$$= (f_1^{(1)} + \tilde{f}_1^{(0)})d_{e_1}y^1.$$

Consequently, by the invariance of A with respect to $(x^1, ..., x^m, y^1+\tau x^1y^1, y^2, ..., y^n)$ preserving e_1 and sending $\frac{\partial}{\partial x^1}$ into $\frac{\partial}{\partial x^1} + \frac{\tau}{1+\tau x^1}y^1\frac{\partial}{\partial y^1}$ and y^1 into $y^1 - \tau x^1y^1 + \tau^2(x^1)^2y^1 - ....$, we get

$$A\left(\frac{\partial}{\partial x^1} + \tau y^1\frac{\partial}{\partial y^1} - \tau^2x^1y^1\frac{\partial}{\partial y^1} + ... + (-1)^{r+1}\tau^{r+2}(x^1)^{r+1}y^1\frac{\partial}{\partial y^1},\right.$$
$$\left. d(x^1(y^1 - \tau x^1y^1 + ... + (-1)^{r+2}\tau^{r+2}(x^1)^{r+2}y^1))\right)(e_1)$$
$$= (f_1^{(1)} + \tilde{f}_1^{(0)})(d_{e_1}y^1 - \tau d_{e_1}x^1)$$

where $r = \max(2, \tilde{r})$ and $\tilde{r}$ is the finite order of A. Considering the coefficients on τ, we get

$$A\left(\frac{\partial}{\partial x^1}, d((x^1)^2y^1)\right)(e_1) - A\left(y^1\frac{\partial}{\partial y^1}, d(x^1y^1)\right)(e_1) = (f_1^{(1)} + \tilde{f}_1^{(0)})(d_{e_1}x^1)\ .$$

But $A(\frac{\partial}{\partial x^1}, d((x^1)^2y^1))(e_1) = 2A(\frac{\partial}{\partial x^1}, x^1y^1dx^1)(e_1) + A(\frac{\partial}{\partial x^1}, (x^1)^2dy^1)(e_1)$. Then

$$2\tilde{g}_1^{(1)}d_{e_1}x^1 + g_1^{(2)}d_{e_1}x^1 = A(y^1\frac{\partial}{\partial y^1}, d(x^1y^1))(e_1) + (f_1^{(1)} + \tilde{f}_1^{(0)})(d_{e_1}x^1)\ .$$

Further, we have observed above that $A(\frac{\partial}{\partial x^1}, dy^1)(e_1) = 0$. Then, using the invariance of A with respect to $(x^1, ..., x^m, y^1 + \tau x^1y^1, y^2, ..., y^n)$, we deduce that

$$A\left(\frac{\partial}{\partial x^1} + \tau y^1\frac{\partial}{\partial y^1} - \tau^2x^1y^1\frac{\partial}{\partial y^1} + ... + (-1)^{r+1}\tau^{r+2}(x^1)^{r+1}y^1\frac{\partial}{\partial y^1},\right.$$
$$\left. d(y^1 - \tau x^1y^1 + ... + (-1)^{r+2}\tau^{r+2}(x^1)^{r+2}y^1)\right)(e_1) = 0\ .$$

Then, considering the coefficients on τ, we get

$$A\left(y^1\frac{\partial}{\partial y^1}, dy^1\right)(e_1) = A\left(\frac{\partial}{\partial x^1}, d(x^1y^1)\right)(e_1) = (f_1^{(1)} + \tilde{f}_1^{(0)})d_{e_1}y^1.$$

Then, using the invariance of A with respect to $(x^1, ..., x^m, y^1+\tau x^1y^1, y^2, ..., y^n)$ preserving $y^1\frac{\partial}{\partial y^1}$ (as it is the Euler vector field in $\mathcal{VB}_{1,1}$ and then it is $\mathcal{VB}_{1,1}$-invariant), we deduce that

$$A\left(y^1\frac{\partial}{\partial y^1}, d(y^1 - \tau x^1y^1 + ... + (-1)^{r+2}\tau^{r+2}(x^1)^{r+2}y^1)\right)(e_1)$$
$$= (f_1^{(1)} + \tilde{f}_1^{(0)})(d_{e_1}y^1 - \tau d_{e_1}x^1).$$

So, considering the coefficients on τ, we get

$$A\left(y^1\frac{\partial}{\partial y^1}, d(x^1y^1)\right)(e_1) = (f_1^{(1)} + \tilde{f}_1^{(0)})d_{e_1}x^1.$$

That is why, $2\tilde{g}_1^{(1)} + g_1^{(2)} = 2(f_1^{(1)} + \tilde{f}_1^{(0)})$.

So, by Lemma 2.13, A is determined by the real numbers $f_1^{(1)}, g_1^{(2)}$ and $\tilde{f}_1^{(0)}$. Then the dimension of vector space of all A in question is not more than 3. So, the dimension argument (Lemma 2.14) completes the proposition. □

Now, we are in position to obtain the following theorem corresponding to the first part of Theorem 1.1.

Theorem 2.21. *Let $m \geq 2$ and $n \geq 1$. Any $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator $A : \Gamma^l(T \oplus T^*) \times \Gamma^l(T \oplus T^*) \rightsquigarrow \Gamma^l(T \oplus T^*)$ is of the form*

$$A(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = a[X^1, X^2] \oplus \{b_1\mathcal{L}_{X^1}\omega^2 + b_2\mathcal{L}_{X^2}\omega^1 + b_3di_{X^1}\omega^2 + b_4di_{X^2}\omega^1 + b_5\mathcal{L}_{X^1}di_L\omega^2 + b_6\mathcal{L}_{X^2}di_L\omega^1\} \quad (1)$$

for arbitrary (uniquely determined by A) real numbers $a, b_1, b_2, b_3, b_4, b_5, b_6$.

Proof. Let $A : \Gamma^l(T \oplus T^*) \times \Gamma^l(T \oplus T^*) \rightsquigarrow \Gamma^l(T \oplus T^*)$ be a $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator. Let $X^1 \oplus \omega^1, X^2 \oplus \omega^2 \in \Gamma^l_E(TE \oplus T^*E)$. We can write

$$A(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = \hat{A}(X^1 \oplus \omega^1, X^2 \oplus \omega^2) \oplus \check{A}(X^1 \oplus \omega^1, X^2 \oplus \omega^2),$$

where $\hat{A}(X^1 \oplus \omega^1, X^2 \oplus \omega^2) \in \Gamma^l_E(TE)$ and $\check{A}(X^1 \oplus \omega^1, X^2 \oplus \omega^2) \in \Gamma^l_E(T^*E)$. Next,

$$\hat{A}(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = \hat{A}^{(1)}(X^1, X^2) + \hat{A}^{(2)}(X^1, \omega^2) + \hat{A}^{(3)}(\omega^1, X^2) + \hat{A}^{(4)}(\omega^1, \omega^2),$$

where $\hat{A}^{(1)}(X^1, X^2) = \hat{A}(X^1 \oplus 0, X^2 \oplus 0)$, $\hat{A}^{(2)}(X^1, \omega^2) = \hat{A}(X^1 \oplus 0, 0 \oplus \omega^2)$, *etc.* , and similarly for $\check{A}$ instead of $\hat{A}$. Hence, A defines (is determined by) eight $\mathcal{VB}_{m,n}$-gauge-natural bilinear operators $\hat{A}^{(1)} : \Gamma^l(T) \times \Gamma^l(T) \rightsquigarrow \Gamma^l(T)$, $\hat{A}^{(2)} : \Gamma^l(T) \times \Gamma^l(T^*) \rightsquigarrow \Gamma^l(T)$, ... , $\check{A}^{(4)} : \Gamma^l(T^*) \times \Gamma^l(T^*) \rightsquigarrow \Gamma^l(T^*)$. Further, the $\mathcal{VB}_{m,n}$-gauge-natural bilinear operators $B : \Gamma^l(T^*) \times \Gamma^l(T) \rightsquigarrow \Gamma^l(T)$ are in bijection with the $\mathcal{VB}_{m,n}$-gauge-natural bilinear operators B^{op} :

$\Gamma^l(T) \times \Gamma^l(T^*) \rightsquigarrow \Gamma^l(T)$ by $B^{op}(X,\omega) = B(\omega, X)$, etc. So, our theorem is a immediate consequence of Propositions 2.15–2.20 and the expression $\mathcal{L}_X = i_X d + d i_X$. □

We end this section by the following two lemmas.

Lemma 2.22. *For any linear vector field X and any linear 1-form ω on a vector bundle E (with the basis of dimension ≥ 2), we have*

$$d i_L \mathcal{L}_X \omega = \mathcal{L}_X d i_L \omega = d i_L \mathcal{L}_X d i_L \omega, \tag{2}$$

where L is the Euler vector field on E.

Proof. We have three $\mathcal{VB}_{m,n}$-gauge-natural bilinear operators $A^1, A^2, A^3 : \Gamma^l(T) \times \Gamma^l(T^*) \rightsquigarrow \Gamma^l(T^*)$ given by

$$A^1(X,\omega) = d i_L \mathcal{L}_X \omega \ , \ A^2(X,\omega) = \mathcal{L}_X d i_L \omega \ , \ A^3(X,\omega) = d i_L \mathcal{L}_X d i_L \omega.$$

It remains to show that $A^1 = A^2 = A^3$.

By Lemma 2.13, it is sufficient to verify that

$$A^1\left(\frac{\partial}{\partial x^1}, (x^1)^k dy^1\right) = A^2\left(\frac{\partial}{\partial x^1}, (x^1)^k dy^1\right) = A^3\left(\frac{\partial}{\partial x^1}, (x^1)^k dy^1\right)$$
$$= k(x^1)^{k-1} dy^1 + k(k-1)(x^1)^{k-2} y^1 dx^1;$$

and $A^1(\frac{\partial}{\partial x^1}, (x^1)^k y^1 dx^1) = A^2(\frac{\partial}{\partial x^1}, (x^1)^k y^1 dx^1) = A^3(\frac{\partial}{\partial x^1}, (x^1)^k y^1 dx^1) = 0.$ □

Lemma 2.23. *For any linear vector field X and any linear 1-form ω on a vector bundle E (with the basis of dimension ≥ 2), we have*

$$d i_L d i_X \omega = d i_X \omega. \tag{3}$$

Proof. We have two $\mathcal{VB}_{m,n}$-gauge-natural bilinear operators $A^1, A^2 : \Gamma^l(T) \times \Gamma^l(T^*) \rightsquigarrow \Gamma^l(T^*)$ given by $A^1(X,\omega) = d i_L d i_X \omega$, $A^2(X,\omega) = d i_X \omega$. It remains to show that $A^1 = A^2$.

By Lemma 2.13, it is sufficient to see that $A^1(\frac{\partial}{\partial x^1}, (x^1)^k dy^1) = A^2(\frac{\partial}{\partial x^1}, (x^1)^k dy^1) = 0$; and $A^1(\frac{\partial}{\partial x^1}, (x^1)^k y^1 dx^1) = A^2(\frac{\partial}{\partial x^1}, (x^1)^k y^1 dx^1) = k(x^1)^{k-1} y^1 dx^1 + (x^1)^k dy^1$. □

3. The Complete Description of All $\mathcal{VB}_{m,n}$-Gauge-Natural Operators $A : \Gamma^l(T \oplus T^*) \times \Gamma^l(T \oplus T^*) \rightsquigarrow \Gamma^l(T \oplus T^*)$ Satisfying the Jacobi Identity in Leibniz Form

Let $A : \Gamma^l(T \oplus T^*) \times \Gamma^l(T \oplus T^*) \rightsquigarrow \Gamma^l(T \oplus T^*)$ be a $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator in the sense of Definition 2.1.

Definition 3.1. We say that A satisfies the Jacobi identity in Leibniz form if

$$A(\nu^1, A(\nu^2, \nu^3)) = A(A(\nu^1, \nu^2), \nu^3) + A(\nu^2, A(\nu^1, \nu^3)) \tag{4}$$

for any linear sections $\nu^i = X^i \oplus \omega^i \in \Gamma^l_E(TE \oplus T^*E)$ for $i = 1, 2, 3$ and any $\mathcal{VB}_{m,n}$-object E.

By Theorem 2.21, A is of the form (1) for (uniquely determined by A) real numbers $a, b_1, b_2, ..., b_6$. We are going to obtain some conditions on the numbers $a, b_1, b_2, b_3, b_4, b_5, b_6$ equivalent to the Jacobi identity in Leibniz form of A.

Lemma 3.2. *For any linear vector fields X^1, X^2, X^3 on $\mathbf{R}^{m,n}$ and any linear 1-forms $\omega^1, \omega^2, \omega^3$ on $\mathbf{R}^{m,n}$, we can write*

$$\begin{aligned} A(X^1 \oplus \omega^1, A(X^2 \oplus \omega^2, X^3 \oplus \omega^3)) &= a^2[X^1, [X^2, X^3]] \oplus \Omega, \\ A(A(X^1 \oplus \omega^1, X^2 \oplus \omega^2), X^3 \oplus \omega^3) &= a^2[[X^1, X^2], X^3] \oplus \Theta, \\ A(X^2 \oplus \omega^2, A(X^1 \oplus \omega^1, X^3 \oplus \omega^3)) &= a^2[X^2, [X^1, X^3]] \oplus \mathcal{T}, \end{aligned}$$

where

$$\begin{aligned} \Omega = {} & b_1 \mathcal{L}_{X^1}\{b_1 \mathcal{L}_{X^2}\omega^3 + b_2 \mathcal{L}_{X^3}\omega^2 + b_3 d i_{X^2}\omega^3 + b_4 d i_{X^3}\omega^2 \\ & + b_5 \mathcal{L}_{X^2} d i_L \omega^3 + b_6 \mathcal{L}_{X^3} d i_L \omega^2\} + b_2 \mathcal{L}_{a[X^2,X^3]}\omega^1 + b_3 d i_{X^1}\{b_1 \mathcal{L}_{X^2}\omega^3 \\ & + b_2 \mathcal{L}_{X^3}\omega^2 + b_3 d i_{X^2}\omega^3 + b_4 d i_{X^3}\omega^2 + b_5 \mathcal{L}_{X^2} d i_L \omega^3 + b_6 \mathcal{L}_{X^3} d i_L \omega^2\} \\ & + b_4 d i_{a[X^2,X^3]}\omega^1 + b_5 \mathcal{L}_{X^1} d i_L\{b_1 \mathcal{L}_{X^2}\omega^3 \\ & + b_2 \mathcal{L}_{X^3}\omega^2 + b_3 d i_{X^2}\omega^3 + b_4 d i_{X^3}\omega^2 \\ & + b_5 \mathcal{L}_{X^2} d i_L \omega^3 + b_6 \mathcal{L}_{X^3} d i_L \omega^2\} + b_6 \mathcal{L}_{a[X^2,X^3]} d i_L \omega^1, \\ \Theta = {} & b_1 \mathcal{L}_{a[X^1,X^2]}\omega^3 + b_2 \mathcal{L}_{X^3}\{b_1 \mathcal{L}_{X^1}\omega^2 + b_2 \mathcal{L}_{X^2}\omega^1 + b_3 d i_{X^1}\omega^2 \\ & + b_4 d i_{X^2}\omega^1 + b_5 \mathcal{L}_{X^1} d i_L \omega^2 + b_6 \mathcal{L}_{X^2} d i_L \omega^1\} + \ b_3 d i_{a[X^1,X^2]}\omega^3 \\ & + b_4 d i_{X^3}\{b_1 \mathcal{L}_{X^1}\omega^2 + b_2 \mathcal{L}_{X^2}\omega^1 + b_3 d i_{X^1}\omega^2 + b_4 d i_{X^2}\omega^1 + b_5 \mathcal{L}_{X^1} d i_L \omega^2 \\ & + b_6 \mathcal{L}_{X^2} d i_L \omega^1\} + b_5 \mathcal{L}_{a[X^1,X^2]} d i_L \omega^3 + b_6 \mathcal{L}_{X^3} d i_L\{b_1 \mathcal{L}_{X^1}\omega^2 + b_2 \mathcal{L}_{X^2}\omega^1 \\ & + b_3 d i_{X^1}\omega^2 + b_4 d i_{X^2}\omega^1 + b_5 \mathcal{L}_{X^1} d i_L \omega^2 + b_6 \mathcal{L}_{X^2} d i_L \omega^1\}\ , \\ \mathcal{T} = {} & b_1 \mathcal{L}_{X^2}\{b_1 \mathcal{L}_{X^1}\omega^3 + b_2 \mathcal{L}_{X^3}\omega^1 + b_3 d i_{X^1}\omega^3 + b_4 d i_{X^3}\omega^1 \\ & + b_5 \mathcal{L}_{X^1} d i_L \omega^3 + b_6 \mathcal{L}_{X^3} d i_L \omega^1\} + b_2 \mathcal{L}_{a[X^1,X^3]}\omega^2 + b_3 d i_{X^2}\{b_1 \mathcal{L}_{X^1}\omega^3 \\ & + b_2 \mathcal{L}_{X^3}\omega^1 + b_3 d i_{X^1}\omega^3 + b_4 d i_{X^3}\omega^1 + b_5 \mathcal{L}_{X^1} d i_L \omega^3 + b_6 \mathcal{L}_{X^3} d i_L \omega^1\} \\ & + b_4 d i_{a[X^1,X^3]}\omega^2 + b_5 \mathcal{L}_{X^2} d i_L\{b_1 \mathcal{L}_{X^1}\omega^3 \\ & + b_2 \mathcal{L}_{X^3}\omega^1 + b_3 d i_{X^1}\omega^3 + b_4 d i_{X^3} d\omega^1 \\ & + b_5 \mathcal{L}_{X^1} d i_L \omega^3 + b_6 \mathcal{L}_{X^3} d i_L \omega^1\} + b_6 \mathcal{L}_{a[X^1,X^3]} d i_L \omega^2. \end{aligned}$$

The Jacobi identity in Leibniz form of A is equivalent to

$$\Omega = \Theta + \mathcal{T}. \tag{5}$$

Proof. The lemma is obvious. □

Lemma 3.3. *The Jacobi identity in Leibniz form of A is equivalent to the system of equalities* (6), (7) *and* (8) *for all linear vector fields X^1, X^2, X^3 and all linear 1-forms $\omega^1, \omega^2, \omega^3$ on $\mathbf{R}^{m,n}$, where*

$$\begin{aligned} & b_1\mathcal{L}_{X^1}\{b_1\mathcal{L}_{X^2}\omega^3 + b_3 di_{X^2}\omega^3 + b_5\mathcal{L}_{X^2}di_L\omega^3\} \\ &\quad + b_3 di_{X^1}\{b_1\mathcal{L}_{X^2}\omega^3 + b_3 di_{X^2}\omega^3 + b_5\mathcal{L}_{X^2}di_L\omega^3\} \\ &\quad + b_5\mathcal{L}_{X^1}di_L\{b_1\mathcal{L}_{X^2}\omega^3 + b_3 di_{X^2}\omega^3 + b_5\mathcal{L}_{X^2}di_L\omega^3\} \\ &= b_1\mathcal{L}_{a[X^1,X^2]}\omega^3 + b_3 di_{a[X^1,X^2]}\omega^3 + b_5\mathcal{L}_{a[X^1,X^2]}di_L\omega^3 \\ &\quad + b_1\mathcal{L}_{X^2}\{b_1\mathcal{L}_{X^1}\omega^3 + b_3 di_{X^1}\omega^3 + b_5\mathcal{L}_{X^1}di_L\omega^3\} \\ &\quad + b_3 di_{X^2}\{b_1\mathcal{L}_{X^1}\omega^3 + b_3 di_{X^1}\omega^3 + b_5\mathcal{L}_{X^1}di_L\omega^3\} \\ &\quad + b_5\mathcal{L}_{X^2}di_L\{b_1\mathcal{L}_{X^1}\omega^3 + b_3 di_{X^1}\omega^3 + b_5\mathcal{L}_{X^1}di_L\omega^3\}, \end{aligned} \tag{6}$$

$$\begin{aligned} & b_1\mathcal{L}_{X^1}\{b_2\mathcal{L}_{X^3}\omega^2 + b_4 di_{X^3}\omega^2 + b_6\mathcal{L}_{X^3}di_L\omega^2\} \\ &\quad + b_3 di_{X^1}\{b_2\mathcal{L}_{X^3}\omega^2 + b_4 di_{X^3}\omega^2 + b_6\mathcal{L}_{X^3}di_L\omega^2\} \\ &\quad + b_5\mathcal{L}_{X^1}di_L\{b_2\mathcal{L}_{X^3}\omega^2 + b_4 di_{X^3}\omega^2 + b_6\mathcal{L}_{X^3}di_L\omega^2\} \\ &= b_2\mathcal{L}_{X^3}\{b_1\mathcal{L}_{X^1}\omega^2 + b_3 di_{X^1}\omega^2 + b_5\mathcal{L}_{X^1}di_L\omega^2\} \\ &\quad + b_4 di_{X^3}\{b_1\mathcal{L}_{X^1}\omega^2 + b_3 di_{X^1}\omega^2 + b_5\mathcal{L}_{X^1}di_L\omega^2\} \\ &\quad + b_6\mathcal{L}_{X^3}di_L\{b_1\mathcal{L}_{X^1}\omega^2 + b_3 di_{X^1}\omega^2 + b_5\mathcal{L}_{X^1}di_L\omega^2\} \\ &\quad + b_2\mathcal{L}_{a[X^1,X^3]}\omega^2 + b_4 di_{a[X^1,X^3]}\omega^2 + b_6\mathcal{L}_{a[X^1,X^3]}di_L\omega^2, \end{aligned} \tag{7}$$

$$\begin{aligned} & b_2\mathcal{L}_{a[X^2,X^3]}\omega^1 + b_4 di_{a[X^2,X^3]}\omega^1 + b_6\mathcal{L}_{a[X^2,X^3]}di_L\omega^1 \\ &= b_2\mathcal{L}_{X^3}\{b_2\mathcal{L}_{X^2}\omega^1 + b_4 di_{X^2}\omega^1 + b_6\mathcal{L}_{X^2}di_L\omega^1\} \\ &\quad + b_4 di_{X^3}\{b_2\mathcal{L}_{X^2}\omega^1 + b_4 di_{X^2}\omega^1 + b_6\mathcal{L}_{X^2}di_L\omega^1\} \\ &\quad + b_6\mathcal{L}_{X^3}di_L\{b_2\mathcal{L}_{X^2}\omega^1 + b_4 di_{X^2}\omega^1 + b_6\mathcal{L}_{X^2}di_L\omega^1\} \\ &\quad + b_1\mathcal{L}_{X^2}\{b_2\mathcal{L}_{X^3}\omega^1 + b_4 di_{X^3}\omega^1 + b_6\mathcal{L}_{X^3}di_L\omega^1\} \\ &\quad + b_3 di_{X^2}\{b_2\mathcal{L}_{X^3}\omega^1 + b_4 di_{X^3}\omega^1 + b_6\mathcal{L}_{X^3}di_L\omega^1\} \\ &\quad + b_5\mathcal{L}_{X^2}di_L\{b_2\mathcal{L}_{X^3}\omega^1 + b_4 di_{X^3}\omega^1 + b_6\mathcal{L}_{X^3}di_L\omega^1\}. \end{aligned} \tag{8}$$

Proof. If we put $\omega^1 = \omega^2 = 0$ (in 5), we get (6). Similarly, if we put $\omega^1 = \omega^3 = 0$, we get (7). Similarly, if we put $\omega^2 = \omega^3 = 0$, we get (8). Conversely, adding the above equalities (6)–(8), we get (5). The lemma is complete. □

Proposition 3.4. *The Jacobi identity in Leibniz form of A is equivalent to the system consisting of conditions* (9) *and* (10)–(12) *for all linear vector fields X^1, X^2, X^3 and all linear 1-forms $\omega^1, \omega^2, \omega^3$ on $\mathbf{R}^{m,n}$, where*

$$(b_2, b_1) = (0, 0) \text{ or } (b_2, b_1) = (0, a) \text{ or } (b_2, b_1) = (-a, a), \tag{9}$$

$$\begin{aligned} & b_1\mathcal{L}_{X^1}\{b_3 di_{X^2}\omega^3 + b_5\mathcal{L}_{X^2}di_L\omega^3\} \\ &\quad + b_3 di_{X^1}\{b_1\mathcal{L}_{X^2}\omega^3 + b_3 di_{X^2}\omega^3 + b_5\mathcal{L}_{X^2}di_L\omega^3\} \\ &\quad + b_5\mathcal{L}_{X^1}di_L\{b_1\mathcal{L}_{X^2}\omega^3 + b_3 di_{X^2}\omega^3 b_5\mathcal{L}_{X^2}di_L\omega^3\} \\ &= b_3 di_{a[X^1,X^2]}\omega^3 + b_5\mathcal{L}_{a[X^1,X^2]}di_L\omega^3 \\ &\quad + b_1\mathcal{L}_{X^2}\{b_3 di_{X^1}\omega^3 + b_5\mathcal{L}_{X^1}di_L\omega^3\} \\ &\quad + b_3 di_{X^2}\{b_1\mathcal{L}_{X^1}\omega^3 + b_3 di_{X^1}\omega^3 + b_5\mathcal{L}_{X^1}di_L\omega^3\} \\ &\quad + b_5\mathcal{L}_{X^2}di_L\{b_1\mathcal{L}_{X^1}\omega^3 + b_3 di_{X^1}\omega^3 + b_5\mathcal{L}_{X^1}di_L\omega^3\}, \end{aligned} \tag{10}$$

$$\begin{aligned}
&b_1\mathcal{L}_{X^1}\{b_4 di_{X^3}\omega^2 + b_6\mathcal{L}_{X^3}di_L\omega^2\}\\
&\quad +b_3 di_{X^1}\{b_2\mathcal{L}_{X^3}\omega^2 + b_4 di_{X^3}\omega^2 + b_6\mathcal{L}_{X^3}di_L\omega^2\}\\
&\quad +b_5\mathcal{L}_{X^1}di_L\{b_2\mathcal{L}_{X^3}\omega^2 + b_4 di_{X^3}\omega^2 + b_6\mathcal{L}_{X^3}di_L\omega^2\}\\
&= b_2\mathcal{L}_{X^3}\{b_3 di_{X^1}\omega^2 + b_5\mathcal{L}_{X^1}di_L\omega^2\}\\
&\quad +b_4 di_{X^3}\{b_1\mathcal{L}_{X^1}\omega^2 + b_3 di_{X^1}\omega^2 + b_5\mathcal{L}_{X^1}di_L\omega^2\}\\
&\quad +b_6\mathcal{L}_{X^3}di_L\{b_1\mathcal{L}_{X^1}\omega^2 + b_3 di_{X^1}\omega^2 + b_5\mathcal{L}_{X^1}di_L\omega^2\}\\
&\quad +b_4 di_{a[X^1,X^3]}\omega^2 + b_6\mathcal{L}_{a[X^1,X^3]}di_L\omega^2, \qquad (11)\\
&b_4 di_{a[X^2,X^3]}\omega^1 + b_6\mathcal{L}_{a[X^2,X^3]}di_L\omega^1\\
&= b_2\mathcal{L}_{X^3}\{b_4 di_{X^2}\omega^1 + b_6\mathcal{L}_{X^2}di_L\omega^1\}\\
&\quad +b_4 di_{X^3}\{b_2\mathcal{L}_{X^2}\omega^1 + b_4 di_{X^2}\omega^1 + b_6\mathcal{L}_{X^2}di_L\omega^1\}\\
&\quad +b_6\mathcal{L}_{X^3}di_L\{b_2\mathcal{L}_{X^2}\omega^1 + b_4 di_{X^2}\omega^1 + b_6\mathcal{L}_{X^2}di_L\omega^1\}\\
&\quad +b_1\mathcal{L}_{X^2}\{b_4 di_{X^3}\omega^1 + b_6\mathcal{L}_{X^3}di_L\omega^1\}\\
&\quad +b_3 di_{X^2}\{b_2\mathcal{L}_{X^3}\omega^1 + b_4 di_{X^3}\omega^1 + b_6\mathcal{L}_{X^3}di_L\omega^1\}\\
&\quad +b_5\mathcal{L}_{X^2}di_L\{b_2\mathcal{L}_{X^3}\omega^1 + b_4 di_{X^3}\omega^1 + b_6\mathcal{L}_{X^3}di_L\omega^1\}. \qquad (12)
\end{aligned}$$

Proof. Applying the differential d to both sides of the equalities (6)–(8) and applying the formulas $d^2 = 0$ and $d\mathcal{L}_X = \mathcal{L}_X d$, we immediately obtain

$$b_1^2\mathcal{L}_{X^1}\mathcal{L}_{X^2}d\omega^3 = b_1 a\mathcal{L}_{[X^1,X^2]}d\omega^3 + b_1^2\mathcal{L}_{X^2}\mathcal{L}_{X^1}d\omega^3 \qquad (13)$$

for all linear X^1, X^2, ω^3, and

$$b_1 b_2\mathcal{L}_{X^1}\mathcal{L}_{X^3}d\omega^2 = b_2 b_1\mathcal{L}_{X^3}\mathcal{L}_{X^1}d\omega^2 + b_2 a\mathcal{L}_{[X^1,X^3]}d\omega^2 \qquad (14)$$

for all linear X^1, X^3, ω^2, and

$$b_2 a\mathcal{L}_{[X^2,X^3]}d\omega^1 = b_2^2\mathcal{L}_{X^3}\mathcal{L}_{X^2}d\omega^1 + b_1 b_2\mathcal{L}_{X^2}\mathcal{L}_{X^3}d\omega^1 \qquad (15)$$

for all linear X^2, X^3, ω^1.

Then by the formula $\mathcal{L}_{[X,Y]} = \mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X$, we get

$$(b_1^2 - b_1 a)\mathcal{L}_{[X^1,X^2]}d\omega^3 = 0 \qquad (16)$$

for all linear X^1, X^2, ω^3, and

$$(b_1 b_2 - b_2 a)\mathcal{L}_{[X^1,X^3]}d\omega^2 = 0 \qquad (17)$$

for all linear X^1, X^3, ω^2, and

$$(b_1 b_2 - b_2 a)\mathcal{L}_{[X^2,X^3]}d\omega^1 + (b_2^2 + b_2 b_1)\mathcal{L}_{X^3}\mathcal{L}_{X^2}d\omega^1 = 0 \qquad (18)$$

for all linear X^2, X^3, ω^1.

Considering linear X^1, X^2, ω^3 such that $\mathcal{L}_{[X^1,X^2]}d\omega^3 \neq 0$ (for example $X^1 = \frac{\partial}{\partial x^1}$ and $X^2 = x^1\frac{\partial}{\partial x^1}$ and $\omega^3 = (x^1)^2 dy^1$), from (16) we get

$$b_1^2 - b_1 a = 0. \qquad (19)$$

Similarly, considering linear X^1, X^3, ω^2 with $\mathcal{L}_{[X^1,X^3]}d\omega^2 \neq 0$ (for example, $X^1 = \frac{\partial}{\partial x^1}$ and $X^3 = x^1\frac{\partial}{\partial x^1}$ and $\omega^2 = (x^1)^2dy^1$), from (17) we get

$$b_1b_2 - b_2a = 0. \tag{20}$$

Similarly, considering linear X^2, X^3, ω^1 with $\mathcal{L}_{X^3}\mathcal{L}_{X^2}d\omega^1 \neq 0$ (for example, $X^3 = \frac{\partial}{\partial x^1}$ and $X^2 = x^1\frac{\partial}{\partial x^1}$ and $\omega^1 = (x^1)^2dy^1$), from (18) and (20) we get

$$b_2^2 + b_2b_1 = 0. \tag{21}$$

Consequently, we obtain (9).

Conversely, if b_1, b_2 and a satisfy (9), then using the formula $\mathcal{L}_X\mathcal{L}_Y\omega - \mathcal{L}_Y\mathcal{L}_X\omega = \mathcal{L}_{[X,Y]}\omega$, we get

$$b_1^2\mathcal{L}_{X^1}\mathcal{L}_{X^2}\omega^3 = b_1a\mathcal{L}_{[X^1,X^2]}\omega^3 + b_1^2\mathcal{L}_{X^2}\mathcal{L}_{X^1}\omega^3 \tag{22}$$

for all linear X^1, X^2, ω^3, and

$$b_1b_2\mathcal{L}_{X^1}\mathcal{L}_{X^3}\omega^2 = b_2b_1\mathcal{L}_{X^3}\mathcal{L}_{X^1}\omega^2 + b_2a\mathcal{L}_{[X^1,X^3]}\omega^2 \tag{23}$$

for all linear X^1, X^3, ω^2, and

$$b_2a\mathcal{L}_{[X^2,X^3]}\omega^1 = b_2^2\mathcal{L}_{X^3}\mathcal{L}_{X^2}\omega^1 + b_1b_2\mathcal{L}_{X^2}\mathcal{L}_{X^3}\omega^1 \tag{24}$$

for all linear X^2, X^3, ω^1.

Now, we can easily see that the proposition is a simple consequence of Lemma 3.3. □

We prove the following theorem corresponding to the second part of Theorem 1.1.

Theorem 3.5. *Let $m \geq 2$ and $n \geq 1$ be natural numbers. Any $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator $A : \Gamma^l(T \oplus T^*) \times \Gamma^l(T \oplus T^*) \rightsquigarrow \Gamma^l(T \oplus T^*)$ of the form (1) satisfies the Jacobi identity in Leibniz form if and only if $(a, b_1, b_2, b_3, b_4, b_5, b_6)$ is from the following list of 7-tuples:*

$$\begin{array}{l} (c,0,0,0,0,c,0)\ ,\ (c,0,0,0,0,c,-c), \\ (c,c,0,0,0,-c,0)\ ,\ (c,c,-c,0,0,-c,c), \\ (c,0,0,0,0,0,0)\ ,\ (c,c,0,0,0,0,0), \\ (c,c,0,0,0,0,-c)\ ,\ (c,c,-c,0,0,0,0), \\ (c,c,-c,0,c-\lambda,0,\lambda)\ ,\ (0,0,0,\lambda,\mu,-\lambda,-\mu), \end{array} \tag{25}$$

where c, λ, μ are arbitrary real numbers with $c \neq 0$.

Proof. At first we prove the part " ⇒ " of the theorem.

Let $A : \Gamma^l(T \oplus T^*) \times \Gamma^l(T \oplus T^*) \rightsquigarrow \Gamma^l(T \oplus T^*)$ be a $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator of the form (1). Assume that A satisfies the Jacobi identity in Leibniz form.

If we put linear vector fields $X^1 = \frac{\partial}{\partial x^1}$ and $X^2 = \frac{\partial}{\partial x^2}$ and linear 1-form $\omega^3 = x^2 y^1 dx^1$ into (10), we get

$$\begin{aligned} &b_1b_30 + b_1b_50 + b_3b_1dy^1 + b_3^20 + b_3b_50 + b_5b_10 + b_5b_30 + b_5^20 \\ &\quad = b_3a0 + b_5a0 + b_1b_3dy^1 + b_1b_50 + b_3b_10 \\ &\qquad + b_3^2dy^1 + b_3b_50 + b_5b_10 + b_5b_3dy^1 + b_5^20. \end{aligned}$$

Then $b_5b_3dy^1 + b_3^2dy^1 = 0$. Then

$$b_3 = 0 \text{ or } b_3 + b_5 = 0. \tag{26}$$

If we put linear vector fields $X^1 = \frac{\partial}{\partial x^1}$ and $X^2 = x^1\frac{\partial}{\partial x^1}$ and linear 1-form $\omega^3 = y^1 dx^1$ into (10), we get

$$\begin{aligned} &b_1b_3dy^1 + b_1b_50 + b_3b_1dy^1 + b_3^2dy^1 + b_3b_50 + b_5b_10 + b_5b_3dy^1 + b_5^20 \\ &\quad = b_3ady^1 + b_5a0 + b_1b_30 + b_1b_50 + b_3b_10 \\ &\qquad + b_3^20 + b_3b_50 + b_5b_10 + b_5b_30 + b_5^20. \end{aligned}$$

Then $b_1b_3 + b_3b_1 + b_3^2 + b_5b_3 = b_3a$, i.e. $b_3(2b_1 + b_3 + b_5 - a) = 0$. Then

$$2b_1 + b_3 + b_5 - a = 0 \text{ or } b_3 = 0. \tag{27}$$

If we put linear vector fields $X^1 = \frac{\partial}{\partial x^1}$ and $X^2 = x^1\frac{\partial}{\partial x^1}$ and linear 1-form $\omega^3 = x^1 dy^1$ into (10), we get

$$\begin{aligned} &b_1b_30 + b_1b_5dy^1 + b_3b_10 + b_3^20 + b_3b_5dy^1 + b_5b_1dy^1 + b_5b_30 + b_5^2dy^1 \\ &\quad = b_3a0 + b_5ady^1 + b_1b_30 + b_1b_50 + b_3b_10 \\ &\qquad + b_3^20 + b_3b_50 + b_5b_10 + b_5b_30 + b_5^20. \end{aligned}$$

Then, $b_1b_5 + b_3b_5 + b_5b_1 + b_5^2 = b_5a$, i.e. $b_5(2b_1 + b_3 + b_5 - a) = 0$. Then,

$$b_5 = 0 \text{ or } 2b_1 + b_3 + b_5 - a = 0. \tag{28}$$

If we put linear vector fields $X^2 = \frac{\partial}{\partial x^1}$ and $X^3 = x^1\frac{\partial}{\partial x^1}$ and linear 1-form $\omega^1 = y^1 dx^1$ into (12), we get

$$\begin{aligned} b_4ady^1 + b_6a0 = {} & b_2b_40 + b_2b_60 + b_4b_20 + b_4^20 + b_4b_60 + b_6b_20 + b_6b_40 \\ & + b_6^20 + b_1b_4dy^1 + b_1b_60 + b_3b_2dy^1 + b_3b_4dy^1 + b_3b_60 \\ & + b_5b_20 + b_5b_4dy^1 + b_5b_60. \end{aligned}$$

Then $b_4a = b_1b_4 + b_3b_2 + b_3b_4 + b_5b_4$. Then

$$b_4(a - b_1 - b_5) = b_3(b_2 + b_4). \tag{29}$$

If we put linear vector fields $X^2 = \frac{\partial}{\partial x^1}$ and $X^3 = x^1\frac{\partial}{\partial x^1}$ and linear 1-form $\omega^1 = x^1 dy^1$ into (12), we get

$$\begin{aligned} b_4a0 + b_6ady^1 = {} & b_2b_40 + b_2b_60 + b_4b_20 + b_4^20 + b_4b_60 + b_6b_20 + b_6b_40 \\ & + b_6^20 + b_1b_40 + b_1b_6dy^1 + b_3b_20 + b_3b_40 + b_3b_6dy^1 \\ & + b_5b_2dy^1 + b_5b_40 + b_5b_6dy^1. \end{aligned}$$

Then $b_6a = b_1b_6 + b_3b_6 + b_5b_2 + b_5b_6$. Then

$$b_6(a - b_1 - b_3) = b_5(b_2 + b_6). \tag{30}$$

If we put linear vector fields $X^2 = \frac{\partial}{\partial x^1}$ and $X^3 = \frac{\partial}{\partial x^2}$ and linear 1-form $\omega^1 = x^2y^1dx^1$ into (12), we get

$$\begin{aligned} b_4a0 + b_6a0 = {} & b_2b_4dy^1 + b_2b_60 + b_4b_20 + b_4^2dy^1 + b_4b_60 + b_6b_20 + b_6b_4dy^1 \\ & + b_6^20 + b_1b_40 + b_1b_60 + b_3b_2dy^1 + b_3b_40 + b_3b_60 + b_5b_20 \\ & + b_5b_40 + b_5b_60. \end{aligned}$$

Then $0 = b_2b_4 + b_4^2 + b_6b_4 + b_3b_2$. Then

$$b_4(b_4 + b_6) = -b_2(b_4 + b_3). \tag{31}$$

If we put linear vector fields $X^2 = x^1\frac{\partial}{\partial x^1}$ and $X^3 = \frac{\partial}{\partial x^1}$ and linear 1-form $\omega^1 = x^1dy^1$ into (12), we get

$$\begin{aligned} b_4a0 + b_6a(-dy^1) = {} & b_2b_40 + b_2b_6dy^1 + b_4b_20 + b_4^20 + b_4b_6dy^1 \\ & + b_6b_2dy^1 + b_6b_40 \\ & + b_6^2dy^1 + b_1b_40 + b_1b_60 + b_3b_20 + b_3b_40 + b_3b_60 \\ & + b_5b_20 + b_5b_40 + b_5b_60. \end{aligned}$$

Then $-b_6a = b_2b_6 + b_4b_6 + b_6b_2 + b_6^2$. Then $b_6(2b_2 + b_4 + b_6 + a) = 0$, i.e.

$$b_6 = 0 \text{ or } 2b_2 + b_4 + b_6 + a = 0. \tag{32}$$

It remains to consider two cases consisting of several subcases and sub-subcases.

Case I. $a \neq 0$.

If $b_3 \neq 0$, then by (26) $b_3 + b_5 = 0$, and then by (27), $2b_1 = a$. So, using (9), we get $a = 0$. Contradiction. So, in our case

$$b_3 = 0. \tag{33}$$

We consider two subcases.

Subcase I.1. $b_5 \neq 0$.

By (9), we have three sub-subcases.

Sub-subcase I.1.1. $(b_1, b_2) = (0, 0)$.

Since $b_1 = 0$ and $b_3 = 0$ and $b_5 \neq 0$, then by (28) we have $b_5 = a$.

Since $b_2 = 0$, then by (31), $b_4(b_4 + b_6) = 0$, i.e. $b_4 = 0$ or $b_4 + b_6 = 0$.

If $b_4 = 0$, then (since $b_2 = 0$) by (32), $b_6 = 0$ or $b_6 = -a$.

If $b_4 + b_6 = 0$, then (since $b_2 = 0$) by (32), $b_6 = 0$ (as $a \neq 0$), and then $b_4 = -b_6 = 0$.

Summing up, in our sub-subcase, we have

$$\begin{aligned} & (b_1, b_2, b_3, b_4, b_5, b_6) = (0, 0, 0, 0, a, 0) \\ & \quad \text{or} \;\; (b_1, b_2, b_3, b_4, b_5, b_6) = (0, 0, 0, 0, a, -a). \end{aligned} \tag{34}$$

Sub-subcase I.1.2. $(b_1, b_2) = (a, 0)$.

By (28), since $b_5 \neq 0$ and $b_1 = a$ and $b_3 = 0$ (see (33)), $2a + 0 + b_5 - a = 0$, i.e. $b_5 = -a$.

Since $b_2 = 0$, then by (31), $b_4(b_4 + b_6) = 0$, i.e. $b_4 = 0$ or $b_4 + b_6 = 0$.

If $b_4 = 0$, then by (32) since $b_2 = 0$, we get $b_6 = 0$ or $b_6 = -a$. So, since $(b_1, b_2, b_3, b_4, b_5, b_6) = (a, 0, 0, 0, -a, -a)$ do not satisfy (30), then $b_6 = 0$.

If $b_4 + b_6 = 0$, then by (32) and $b_2 = 0$, we get $b_6 = 0$ (as $a \neq 0$), and then and $b_4 = -b_6 = 0$.

Summing up, in our sub-subcase, we have

$$(b_1, b_2, b_3, b_4, b_5, b_6) = (a, 0, 0, 0, -a, 0). \tag{35}$$

Sub-case I.1.3. $(b_1, b_2) = (a, -a)$.

By (28), since $b_5 \neq 0$ and $b_1 = a$ and $b_3 = 0$ (see (33)), $2a+0+b_5-a = 0$, i.e. $b_5 = -a$.

Then, by (30), we have $b_6(a-a-0) = (-a)(-a+b_6)$, i.e. $0 = -a(-a+b_6)$. Then $b_6 = a$.

Moreover, by (29), we have $b_4(a - a - (-a)) = 0$, i.e. $b_4 a = 0$. Then $b_4 = 0$.

Summing up, in our sub-subcase, we have

$$(b_1, b_2, b_3, b_4, b_5, b_6) = (a, -a, 0, 0, -a, a). \tag{36}$$

Subcase I. 2. $b_5 = 0$.

By (9) we have three sub-subcases.

Sub-subcase I.2.1. $(b_1, b_2) = (0, 0)$.

By (30), we have $b_6(a - 0 - 0) = 0$, i.e. $b_6 = 0$. Then by (31) we have $b_4(b_4 + 0) = 0$, i.e. $b_4 = 0$.

Summing up, in our sub-subcase, we have

$$(b_1, b_2, b_3, b_4, b_5, b_6) = (0, 0, 0, 0, 0, 0). \tag{37}$$

Sub-subcase I.2.2. $(b_1, b_2) = (a, 0)$.

Suppose $b_4 \neq 0$. By (31), $b_4(b_4 + b_6) = 0$. Then $b_4 + b_6 = 0$. Then by (32), $b_6 = 0$, and then and $b_4 = -b_6 = 0$. Contradiction. So, $b_4 = 0$.

Then by (32), $b_6 = 0$ or $b_6 = -a$.

Summing up, in our sub-subcase, we have

$$\begin{aligned} &(b_1, b_2, b_3, b_4, b_5, b_6) = (a, 0, 0, 0, 0, 0) \\ &\quad \text{or} \;\; (b_1, b_2, b_3, b_4, b_5, b_6) = (a, 0, 0, 0, 0, -a). \end{aligned} \tag{38}$$

Sub-subcase I.2.3. $(b_1, b_2) = (a, -a)$.

By (31), we have $b_4^2 + b_4 b_6 = a b_4$, i.e. $b_4 = 0$ or $b_4 + b_6 = a$.

If $b_4 = 0$ then by (32), $b_6 = 0$ or $b_6 = a$.

Summing up, in our sub-subcase, we have

$$\begin{aligned} &(b_1, b_2, b_3, b_4, b_5, b_6) = (a, -a, 0, a - \lambda, 0, \lambda) \text{ for } \lambda \in \mathbf{R} \\ &\quad \text{or} \;\; (b_1, b_2, b_3, b_4, b_5, b_6) = (a, -a, 0, 0, 0, 0). \end{aligned} \tag{39}$$

For $\lambda = a$ we realise the case with $b_4 = 0$ and $b_6 = a$. That is why we do not expose separately this in above.

Case II. $a = 0$.

Then by (9), $b_1 = b_2 = 0$. So, if $b_3 \neq 0$ or $b_5 \neq 0$, then by (26) and (28) we have $b_3 + b_5 = 0$. If $b_3 = b_5 = 0$, then we also have $b_3 + b_5 = 0$. Similarly, if $b_4 \neq 0$ or $b_6 \neq 0$, then by (31) and (32) we have $b_4 + b_6 = 0$. If $b_4 = b_6 = 0$ then of course $b_4 + b_6 = 0$.

Summing up, in our case, we have

$$(b_1, b_2, b_3, b_4, b_5, b_6) = (0, 0, \lambda, \mu, -\lambda, -\mu) \text{ for } \lambda, \mu \in \mathbf{R}. \tag{40}$$

The part " $\Rightarrow$ " of the theorem is complete.

Now, we are going to prove the part " $\Leftarrow$ " of the theorem.

Let $(a, b_1, b_2, b_3, b_4, b_5, b_6)$ be a arbitrary 7-tuple from the list (25). By Proposition 3.4, it is sufficient to show that $(a, b_1, b_2, b_3, b_4, b_5, b_6)$ satisfies conditions (9)–(12) for all linear vector fields X^1, X^2, X^3 and all linear 1-forms $\omega^1, \omega^2, \omega^3$ on $\mathbf{R}^{m,n}$.

We consider two cases and several subcases.

Case 1. $a \neq 0$.

Subcase 1.1. $(b_1, b_2, b_3, b_4, b_5, b_6) = (0, 0, 0, 0, 0, 0)$.

The condition (9) holds as $(b_2, b_1) = (0, 0)$. The equalities (10)–(12) are $0 = 0$.

Subcase 1.2. $(b_1, b_2, b_3, b_4, b_5, b_6) = (a, 0, 0, 0, 0, 0)$.

The condition (9) holds as $(b_2, b_1) = (0, a)$. The equalities (10)–(12) are $0 = 0$.

Subcase 1.3. $(b_1, b_2, b_3, b_4, b_5, b_6) = (0, 0, 0, 0, a, 0)$.

The condition (9) holds as $(b_2, b_1) = (0, 0)$. The equalities (11) and (12) are $0 = 0$. Using (2), the equality (10) can be written as

$$a^2 \mathcal{L}_{X^1} \mathcal{L}_{X^2} d i_L \omega^3 = a^2 \mathcal{L}_{[X^1, X^2]} d i_L \omega^3 + a^2 \mathcal{L}_{X^2} \mathcal{L}_{X^1} d i_L \omega^3.$$

It is satisfied (by $(b_1, b_2, b_3, b_4, b_5, b_6)$) because $\mathcal{L}_{[X,Y]}\omega = \mathcal{L}_X \mathcal{L}_Y \omega - \mathcal{L}_Y \mathcal{L}_X \omega$.

Subcase 1.4. $(b_1, b_2, b_3, b_4, b_5, b_6) = (a, -a, 0, 0, 0, 0)$.

The condition (9) holds as $(b_2, b_1) = (-a, a)$. The equalities (10)–(12) are $0 = 0$.

Subcase 1.5. $(b_1, b_2, b_3, b_4, b_5, b_6) = (0, 0, 0, 0, a, -a)$.

The condition (9) holds as $(b_2, b_1) = (0, 0)$. Using (2), the equalities (10)–(12) can be written as

$$\begin{aligned} a^2 \mathcal{L}_{X^1} \mathcal{L}_{X^2} d i_L \omega^3 &= a^2 \mathcal{L}_{[X^1, X^2]} d i_L \omega^3 + a^2 \mathcal{L}_{X^2} \mathcal{L}_{X^1} d i_L \omega^3, \\ -a^2 \mathcal{L}_{X^1} \mathcal{L}_{X^3} d i_L \omega^2 &= -a^2 \mathcal{L}_{X^3} \mathcal{L}_{X^1} d i_L \omega^2 - a^2 \mathcal{L}_{[X^1, X^3]} d i_L \omega^2, \\ -a^2 \mathcal{L}_{[X^2, X^3]} d i_L \omega^1 &= a^2 \mathcal{L}_{X^3} \mathcal{L}_{X^2} d i_L \omega^1 - a^2 \mathcal{L}_{X^2} \mathcal{L}_{X^3} d i_L \omega^1. \end{aligned}$$

They are satisfied because $\mathcal{L}_{[X,Y]}\omega = \mathcal{L}_X \mathcal{L}_Y \omega - \mathcal{L}_Y \mathcal{L}_X \omega$.

Subcase 1.6. $(b_1, b_2, b_3, b_4, b_5, b_6) = (a, 0, 0, 0, 0, -a)$.

The condition (9) holds as $(b_2, b_1) = (0, a)$. The equality (10) is $0 = 0$. Using (2), equalities (11) and (12) can be written as

$$\begin{aligned} -a^2 \mathcal{L}_{X^1} \mathcal{L}_{X^3} d i_L \omega^2 &= -a^2 \mathcal{L}_{X^3} \mathcal{L}_{X^1} d i_L \omega^2 - a^2 \mathcal{L}_{[X^1, X^3]} d i_L \omega^2, \\ -a^2 \mathcal{L}_{[X^2, X^3]} d i_L \omega^1 &= a^2 \mathcal{L}_{X^3} \mathcal{L}_{X^2} d i_L \omega^1 - a^2 \mathcal{L}_{X^2} \mathcal{L}_{X^3} d i_L \omega^1. \end{aligned}$$

They are satisfied because $\mathcal{L}_{[X,Y]}\omega = \mathcal{L}_X \mathcal{L}_Y \omega - \mathcal{L}_Y \mathcal{L}_X \omega$.

Subcase 1.7. $(b_1, b_2, b_3, b_4, b_5, b_6) = (a, 0, 0, 0, -a, 0)$.

The condition (9) holds as $(b_2, b_1) = (0, a)$. Equalities (11) and (12) are $0 = 0$. Using (2), equality (10) can be written as

$$\begin{aligned} &-a^2 \mathcal{L}_{X^1} \mathcal{L}_{X^2} d i_L \omega^3 - a^2 \mathcal{L}_{X^1} \mathcal{L}_{X^2} d i_L \omega^3 + a^2 \mathcal{L}_{X^1} \mathcal{L}_{X^2} d i_L \omega^3 \\ &\quad = -a^2 \mathcal{L}_{[X^1, X^2]} d i_L \omega^3 - a^2 \mathcal{L}_{X^2} \mathcal{L}_{X^1} d i_L \omega^3 - a^2 \mathcal{L}_{X^2} \mathcal{L}_{X^1} d i_L \omega^3 \\ &\qquad + a^2 \mathcal{L}_{X^2} \mathcal{L}_{X^1} d i_L \omega^3, \end{aligned}$$

or (after reduction of similar terms) as

$$-a^2 \mathcal{L}_{X^1} \mathcal{L}_{X^2} d i_L \omega^3 = -a^2 \mathcal{L}_{[X^1, X^2]} d i_L \omega^3 - a^2 \mathcal{L}_{X^2} \mathcal{L}_{X^1} d i_L \omega^3.$$

It is satisfied because $\mathcal{L}_{[X,Y]}\omega = \mathcal{L}_X\mathcal{L}_Y\omega - \mathcal{L}_Y\mathcal{L}_X\omega$.

Subcase 1.8. $(b_1, b_2, b_3, b_4, b_5, b_6) = (a, -a, 0, 0, -a, a)$.

The condition (9) holds as $(b_2, b_1) = (-a, a)$. Using (2), equality (10) can be written as

$$\begin{aligned}&-a^2\mathcal{L}_{X^1}\mathcal{L}_{X^2}di_L\omega^3 - a^2\mathcal{L}_{X^1}\mathcal{L}_{X^2}di_L\omega^3 + a^2\mathcal{L}_{X^1}\mathcal{L}_{X^2}di_L\omega^3\\ &\quad = -a^2\mathcal{L}_{[X^1,X^2]}di_L\omega^3 - a^2\mathcal{L}_{X^2}\mathcal{L}_{X^1}di_L\omega^3 - a^2\mathcal{L}_{X^2}\mathcal{L}_{X^1}di_L\omega^3\\ &\qquad + a^2\mathcal{L}_{X^2}\mathcal{L}_{X^1}di_L\omega^3,\end{aligned}$$

or (after reduction of similar terms) as

$$-a^2\mathcal{L}_{X^1}\mathcal{L}_{X^2}di_L\omega^3 = -a^2\mathcal{L}_{[X^1,X^2]}di_L\omega^3 - a^2\mathcal{L}_{X^2}\mathcal{L}_{X^1}di_L\omega^3.$$

Similarly, (11) can be written as

$$\begin{aligned}&a^2\mathcal{L}_{X^1}\mathcal{L}_{X^3}di_L\omega^2 + a^2\mathcal{L}_{X^1}\mathcal{L}_{X^3}di_L\omega^2 - a^2\mathcal{L}_{X^1}\mathcal{L}_{X^3}di_L\omega^2\\ &\quad = a^2\mathcal{L}_{X^3}\mathcal{L}_{X^1}di_L\omega^2 + a^2\mathcal{L}_{X^3}\mathcal{L}_{X^1}di_L\omega^2 - a^2\mathcal{L}_{X^3}\mathcal{L}_{X^1}di_L\omega^2\\ &\qquad + a^2\mathcal{L}_{[X^1,X^3]}di_L\omega^2,\end{aligned}$$

or (after reduction of similar terms) as

$$a^2\mathcal{L}_{X^1}\mathcal{L}_{X^3}di_L\omega^2 = a^2\mathcal{L}_{X^3}\mathcal{L}_{X^1}di_L\omega^2 + a^2\mathcal{L}_{[X^1,X^3]}di_L\omega^2.$$

Similarly, (12) can be written as

$$\begin{aligned}&a^2\mathcal{L}_{[X^2,X^3]}di_L\omega^1 = -a^2\mathcal{L}_{X^3}\mathcal{L}_{X^2}di_L\omega^1 - a^2\mathcal{L}_{X^3}\mathcal{L}_{X^2}di_L\omega^1\\ &\quad + a^2\mathcal{L}_{X^3}\mathcal{L}_{X^2}di_L\omega^1 + a^2\mathcal{L}_{X^2}\mathcal{L}_{X^3}di_L\omega^1 + a^2\mathcal{L}_{X^2}\mathcal{L}_{X^3}di_L\omega^1\\ &\qquad - a^2\mathcal{L}_{X^2}\mathcal{L}_{X^3}di_L\omega^1,\end{aligned}$$

or (after reduction of similar terms) as

$$a^2\mathcal{L}_{[X^2,X^3]}di_L\omega^1 = -a^2\mathcal{L}_{X^3}\mathcal{L}_{X^2}di_L\omega^1 + a^2\mathcal{L}_{X^2}\mathcal{L}_{X^3}di_L\omega^1.$$

So, (10)–(12) are satisfied because of $\mathcal{L}_{[X,Y]}\omega = \mathcal{L}_X\mathcal{L}_Y\omega - \mathcal{L}_Y\mathcal{L}_X\omega$.

Subcase 1.9. $(b_1, b_2, b_3, b_4, b_5, b_6) = (a, -a, 0, a-\lambda, 0, \lambda)$.

The condition (9) holds as $(b_2, b_1) = (-a, a)$. Condition (10) is $0 = 0$. Using (2), (11) can be written as

$$\begin{aligned}&a(a-\lambda)\mathcal{L}_{X^1}di_{X^3}\omega^2 + a\lambda\mathcal{L}_{X^1}\mathcal{L}_{X^3}di_L\omega^2 = (a-\lambda)adi_{X^3}\mathcal{L}_{X^1}\omega^2\\ &\quad + \lambda a\mathcal{L}_{X^3}\mathcal{L}_{X^1}di_L\omega^2 + (a-\lambda)adi_{[X^1,X^3]}\omega^2 + \lambda a\mathcal{L}_{[X^1,X^3]}di_L\omega^2.\end{aligned}$$

Then, using formulas $\mathcal{L}_{[X,Y]}\omega = \mathcal{L}_X\mathcal{L}_Y\omega - \mathcal{L}_Y\mathcal{L}_X\omega$ and $d\mathcal{L}_X = \mathcal{L}_Xd$, condition (11) can be written as

$$a(a-\lambda)d\mathcal{L}_{X^1}i_{X^3}\omega^2 = (a-\lambda)adi_{X^3}\mathcal{L}_{X^1}\omega^2 + (a-\lambda)adi_{[X^1,X^3]}\omega^2.$$

So, (11) is satisfied because $\mathcal{L}_Xi_Y - i_Y\mathcal{L}_X = i_{[X,Y]}$.

Using (2) and $d\mathcal{L}_X = \mathcal{L}_X d$, (12) is

$$\begin{aligned}(a-\lambda)adi_{[X^2,X^3]}\omega^1 &+ \lambda a\mathcal{L}_{[X^2,X^3]}di_L\omega^1 \\ &= -(a-\lambda)ad\mathcal{L}_{X^3}i_{X^2}\omega^1 - a\lambda\mathcal{L}_{X^3}\mathcal{L}_{X^2}di_L\omega^1 \\ &\quad -(a-\lambda)adi_{X^3}\mathcal{L}_{X^2}\omega^1 + (a-\lambda)^2di_{X^3}di_{X^2}\omega^1 \\ &\quad +(a-\lambda)\lambda di_{X^3}\mathcal{L}_{X^2}di_L\omega^1 - \lambda a\mathcal{L}_{X^3}\mathcal{L}_{X^2}di_L\omega^1 \\ &\quad +\lambda(a-\lambda)\mathcal{L}_{X^3}di_Ldi_{X^2}\omega^1 + \lambda^2\mathcal{L}_{X^3}\mathcal{L}_{X^2}di_L\omega^1 \\ &\quad +a(a-\lambda)d\mathcal{L}_{X^2}i_{X^3}\omega^1 + a\lambda\mathcal{L}_{X^2}\mathcal{L}_{X^3}di_L\omega^1. \end{aligned} \tag{41}$$

So, to prove that (12) is satisfied, it is sufficient to show that the coefficients on λ^0 of both sides of (41) are equal, and the coefficients on λ^1 of both sides of (41) are equal, and the coefficients on λ^2 of both sides of (41) are equal.

Comparing the coefficients on λ^0 in (41), we obtain

$$\begin{aligned}a^2di_{[X^2,X^3]}\omega^1 = -a^2d\mathcal{L}_{X^3}i_{X^2}\omega^1 - a^2di_{X^3}\mathcal{L}_{X^2}\omega^1 + a^2di_{X^3}di_{X^2}\omega^1 \\ +a^2d\mathcal{L}_{X^2}i_{X^3}\omega^1.\end{aligned}$$

This condition is satisfied because

$$\begin{aligned}di_{[X^2,X^3]} &= d(\mathcal{L}_{X^2}i_{X^3} - i_{X^3}\mathcal{L}_{X^2}) = d\mathcal{L}_{X^2}i_{X^3} - di_{X^3}\mathcal{L}_{X^2} \\ &= (d\mathcal{L}_{X^2}i_{X^3} - di_{X^3}\mathcal{L}_{X^2}) + (di_{X^3}di_{X^2} - d\mathcal{L}_{X^3}i_{X^2}) \\ &= -d\mathcal{L}_{X^3}i_{X^2} - di_{X^3}\mathcal{L}_{X^2} + di_{X^3}di_{X^2} + d\mathcal{L}_{X^2}i_{X^3}\end{aligned}$$

as $di_{X^3}di_{X^2} = d(di_{X^3} + i_{X^3}d)i_{X^2} = d\mathcal{L}_{X^3}i_{X^2}$.

Comparing the coefficients on λ in (41) and using $d\mathcal{L}_X = \mathcal{L}_X d$, we obtain

$$\begin{aligned}-adi_{[X^2,X^3]}\omega^1 &+ a\mathcal{L}_{[X^2,X^3]}di_L\omega^1 \\ &= a\mathcal{L}_{X^3}di_{X^2}\omega^1 - a\mathcal{L}_{X^3}\mathcal{L}_{X^2}di_L\omega^1 + adi_{X^3}\mathcal{L}_{X^2}\omega^1 - 2adi_{X^3}di_{X^2}\omega^1 \\ &\quad +adi_{X^3}\mathcal{L}_{X^2}di_L\omega^1 - a\mathcal{L}_{X^3}\mathcal{L}_{X^2}di_L\omega^1 \\ &\quad +a\mathcal{L}_{X^3}di_Ldi_{X^2}\omega^1 - ad\mathcal{L}_{X^2}i_{X^3}\omega^1 + a\mathcal{L}_{X^2}\mathcal{L}_{X^3}di_L\omega^1.\end{aligned} \tag{42}$$

Using the formulas $\mathcal{L}_{[X^2,X^3]} = \mathcal{L}_{X^2}\mathcal{L}_{X^3} - \mathcal{L}_{X^3}\mathcal{L}_{X^2}$ and $i_{[X^2,X^3]} = \mathcal{L}_{X^2}i_{X^3} - i_{X^3}\mathcal{L}_{X^2}$ we can short equivalently (42) to

$$\begin{aligned}0 = a\mathcal{L}_{X^3}di_{X^2}\omega^1 - 2adi_{X^3}di_{X^2}\omega^1 + adi_{X^3}\mathcal{L}_{X^2}di_L\omega^1 \\ -a\mathcal{L}_{X^3}\mathcal{L}_{X^2}di_L\omega^1 + a\mathcal{L}_{X^3}di_Ldi_{X^2}\omega^1.\end{aligned} \tag{43}$$

By (3), we have $di_Ldi_{X^2}\omega^1 = di_{X^2}\omega^1$. Then $\mathcal{L}_{X^3}di_Ldi_{X^2}\omega^1 = \mathcal{L}_{X^3}di_{X^2}\omega^1$. Moreover, by the formulas $\mathcal{L}_X = i_Xd + di_X$ and $d^2 = 0$ and $\mathcal{L}_Xd = d\mathcal{L}_X$, we have

$$di_{X^3}di_{X^2}\omega^1 = (di_{X^3} + i_{X^3}d)di_{X^2}\omega^1 = \mathcal{L}_{X^3}di_{X^2}\omega^1. \tag{44}$$

Also $di_{X^3}\mathcal{L}_{X^2}di_L\omega^1 = (di_{X^3} + i_{X^3}d)d\mathcal{L}_{X^2}i_L\omega^1 = \mathcal{L}_{X^3}\mathcal{L}_{X^2}di_L\omega^1$, i.e.

$$di_{X^3}\mathcal{L}_{X^2}di_L\omega^1 = \mathcal{L}_{X^3}\mathcal{L}_{X^2}di_L\omega^1. \tag{45}$$

So, our equality (43) can be equivalently rewritten as

$$\begin{aligned}0 = a\mathcal{L}_{X^3}di_{X^2}\omega^1 - 2a\mathcal{L}_{X^3}di_{X^2}\omega^1 + a\mathcal{L}_{X^3}\mathcal{L}_{X^2}di_L\omega^1 \\ -a\mathcal{L}_{X^3}\mathcal{L}_{X^2}di_L\omega^1 + a\mathcal{L}_{X^3}di_{X^2}\omega^1\ ,\end{aligned}$$

i.e. as $0 = 0$. So, (42) holds.

Comparing the coefficients on λ^2 in (41), we get

$$0 = di_{X^3}di_{X^2}\omega^1 - di_{X^3}\mathcal{L}_{X^2}di_L\omega^1 - \mathcal{L}_{X^3}di_Ldi_{X^2}\omega^1 + \mathcal{L}_{X^3}\mathcal{L}_{X^2}di_L\omega^1.$$

This condition holds because of (44) and (2) and (45) it can be rewritten as

$$0 = \mathcal{L}_{X^3}di_{X^2}\omega^1 - \mathcal{L}_{X^3}\mathcal{L}_{X^2}di_L\omega^1 - \mathcal{L}_{X^3}di_{X^2}\omega^1 + \mathcal{L}_{X^3}\mathcal{L}_{X^2}di_L\omega^1.$$

Case 2. $a = 0$ and $(b_1, b_2, b_3, b_4, b_5, b_6) = (0, 0, \lambda, \mu, -\lambda, -\mu)$.

The condition (9) holds as $(b_2, b_1) = (0, 0)$.

Condition (10) is

$$\begin{aligned}&\lambda^2 di_{X^1}di_{X^2}\omega^3 - \lambda^2 di_{X^1}\mathcal{L}_{X^2}di_L\omega^3 - \lambda^2\mathcal{L}_{X^1}di_Ldi_{X^2}\omega^3\\ &\quad+\lambda^2\mathcal{L}_{X^1}di_L\mathcal{L}_{X^2}di_L\omega^3 = \lambda^2 di_{X^2}di_{X^1}\omega^3\\ &\quad-\lambda^2 di_{X^2}\mathcal{L}_{X^1}di_L\omega^3 - \lambda^2\mathcal{L}_{X^2}di_Ldi_{X^1}\omega^3 + \lambda^2\mathcal{L}_{X^2}di_L\mathcal{L}_{X^1}di_L\omega^3.\end{aligned}$$

It is satisfied because by (44) and (45) and (2) and (3) it can be rewritten as

$$\begin{aligned}&\lambda^2\mathcal{L}_{X^1}di_{X^2}\omega^3 - \lambda^2\mathcal{L}_{X^1}\mathcal{L}_{X^2}di_L\omega^3 - \lambda^2\mathcal{L}_{X^1}di_{X^2}\omega^3\\ &\quad+\lambda^2\mathcal{L}_{X^1}\mathcal{L}_{X^2}di_L\omega^3 = \lambda^2\mathcal{L}_{X^2}di_{X^1}\omega^3\\ &\quad-\lambda^2\mathcal{L}_{X^2}\mathcal{L}_{X^1}di_L\omega^3 - \lambda^2\mathcal{L}_{X^2}di_{X^1}\omega^3 + \lambda^2\mathcal{L}_{X^2}\mathcal{L}_{X^1}di_L\omega^3.\end{aligned}$$

So, it can be reduced to $0 = 0$.

Condition (11) is

$$\begin{aligned}&\lambda\mu di_{X^1}di_{X^3}\omega^2 - \lambda\mu di_{X^1}\mathcal{L}_{X^3}di_L\omega^2 - \lambda\mu\mathcal{L}_{X^1}di_Ldi_{X^3}\omega^2\\ &\quad+\lambda\mu\mathcal{L}_{X^1}d_L\mathcal{L}_{X^3}di_L\omega^2 = \mu\lambda di_{X^3}di_{X^1}\omega^2\\ &\quad-\mu\lambda di_{X^3}\mathcal{L}_{X^1}di_L\omega^2 - \mu\lambda\mathcal{L}_{X^3}di_Ldi_{X^1}\omega^2 + \mu\lambda\mathcal{L}_{X^3}d_L\mathcal{L}_{X^1}di_L\omega^2.\end{aligned}$$

It is satisfied because by (44) and (45) and (2) and (3) it can be rewritten as

$$\begin{aligned}&\lambda\mu\mathcal{L}_{X^1}di_{X^3}\omega^2 - \lambda\mu\mathcal{L}_{X^1}\mathcal{L}_{X^3}di_L\omega^2 - \lambda\mu\mathcal{L}_{X^1}di_{X^3}\omega^2\\ &\quad+\lambda\mu\mathcal{L}_{X^1}\mathcal{L}_{X^3}di_L\omega^2 = \mu\lambda\mathcal{L}_{X^3}di_{X^1}\omega^2\\ &\quad-\mu\lambda\mathcal{L}_{X^3}\mathcal{L}_{X^1}di_L\omega^2 - \mu\lambda\mathcal{L}_{X^3}di_{X^1}\omega^2 + \mu\lambda\mathcal{L}_{X^3}\mathcal{L}_{X^1}di_L\omega^2,\end{aligned}$$

i.e as $0 = 0$.

Condition (12) is

$$\begin{aligned}0 &= \mu^2 di_{X^3}di_{X^2}\omega^1 - \mu^2 di_{X^3}\mathcal{L}_{X^2}di_L\omega^1\\ &\quad-\mu^2\mathcal{L}_{X^3}di_Ldi_{X^2}\omega^1 + \mu^2\mathcal{L}_{X^3}di_L\mathcal{L}_{X^2}\omega^1\\ &\quad+\lambda\mu di_{X^2}di_{X^3}\omega^1 - \lambda\mu di_{X^2}\mathcal{L}_{X^3}di_L\omega^1 - \lambda\mu\mathcal{L}_{X^2}di_Ldi_{X^3}\omega^1\\ &\quad+\lambda\mu\mathcal{L}_{X^2}di_L\mathcal{L}_{X^3}\omega^1.\end{aligned}$$

It is satisfied because by (44) and (45) and (2) and (3) it can be rewritten as

$$\begin{aligned}0 &= \mu^2\mathcal{L}_{X^3}di_{X^2}\omega^1 - \mu^2\mathcal{L}_{X^3}\mathcal{L}_{X^2}di_L\omega^1 - \mu^2\mathcal{L}_{X^3}di_{X^2}\omega^1\\ &\quad+\mu^2\mathcal{L}_{X^3}\mathcal{L}_{X^2}di_L\omega^1 + \lambda\mu\mathcal{L}_{X^2}di_{X^3}\omega^1\\ &\quad-\lambda\mu\mathcal{L}_{X^2}\mathcal{L}_{X^3}di_L\omega^1 - \lambda\mu\mathcal{L}_{X^2}di_{X^3}\omega^1 + \lambda\mu\mathcal{L}_{X^2}\mathcal{L}_{X^3}di_L\omega^1,\end{aligned}$$

i.e. $0 = 0$.

The theorem is complete. □

Remark 3.6. The space $\Gamma^l_E(TE \oplus T^*E)$ is a locally free $\mathcal{C}^\infty(M)$-module. Hence, there is a vector bundle $\hat{E}$ over M such that $\Gamma^l_E(TE \oplus T^*E)$ is isomorphic to $\Gamma\hat{E}$ as $\mathcal{C}^\infty(M)$-modules. The vector bundle $\hat{E}$ is called the fat vector bundle. It is isomorphic to the Omni–Lie algebroid $\mathcal{A}(E) := Der(E^*) \oplus J^1(E^*)$, studied in [1], where $Der(E^*)$ is the bundle of derivations on E^*, and $J^1(E^*)$ the first jet prolongation bundle, see [6]. Denote $\mathcal{A}(E) := \hat{E}$. Any $\mathcal{VB}_{m,n}$-map $f : E \to E_1$ with the base map $\underline{f} : M \to M_1$ induces in obvious (functor) way the vector bundle map $\mathcal{A}(f) : \mathcal{A}(E) \to \mathcal{A}(E_1)$ covering $\underline{f}$. In other words, we have a so-called vector gauge bundle functor $\mathcal{A} : \mathcal{VB}_{m,n} \to \mathcal{VB}$. Thus a $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator $A : \Gamma^l(T \oplus T^*) \times \Gamma^l(T \oplus T^*) \rightsquigarrow \Gamma^l(T \oplus T^*)$ is a (usual) $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator $A : \mathcal{A} \times \mathcal{A} \rightsquigarrow \mathcal{A}$ (in the sense of [7]). Thus, Theorem 3.5 gives the full description of all $\mathcal{VB}_{m,n}$-gauge-natural bilinear brackets satisfying the Jacobi identity in Leibniz form on sections of the Omni–Lie algebroid of E.

Definition 3.7. A natural Lie bracket on $\Gamma^l_E(TE \oplus T^*E)$ is a $\mathcal{VB}_{m,n}$-gauge-natural bilinear skew-symmetric operator $A : \Gamma^l(T \oplus T^*) \times \Gamma^l(T \oplus T^*) \rightsquigarrow \Gamma^l(T \oplus T^*)$ satisfying the Jacobi identity in Leibniz form.

We have the following immediate consequence of Theorems 2.21 and 3.5.

Corollary 3.8. *Let $m \geq 2$ and $n \geq 1$ be natural numbers. Let $A : \Gamma^l(T \oplus T^*) \times \Gamma^l(T \oplus T^*) \rightsquigarrow \Gamma^l(T \oplus T^*)$ be a $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator. Then, A is skew-symmetric if and only if it is of the form (1) for arbitrary (uniquely determined by A) real numbers $a, b_1, b_2, b_3, b_4, b_5, b_6$ satisfying*

$$b_1 = -b_2, \ b_3 = -b_4, \ b_5 = -b_6.$$

Moreover, such A is a Lie bracket if and only if $(a, b_1, b_2, b_3, b_4, b_5, b_6)$ is from the following list of 7-tuples:

$$\begin{aligned}&(c,0,0,0,0,c,-c), \ (c,c,-c,0,0,-c,c), \ (c,0,0,0,0,0,0),\\ &(c,c,-c,0,0,0,0), \ (0,0,0,\lambda,-\lambda,-\lambda,\lambda),\end{aligned} \tag{46}$$

where c, λ are arbitrary real numbers with $c \neq 0$.

Acknowledgements

I would like to thank the reviewer for valuable suggestions to help improve the article structure.

References

[1] Chen, Z., Liu, Z.: Omni-Lie algebroids. J. Geom. Phys. **60**(5), 799–808 (2010)

[2] Courant, T.: Dirac manifolds. T. Am. Math. Soc. **319**, 631–661 (1990)

[3] Doupovec, M., Kurek, J., Mikulski, W.M.: The natural brackets on couples of vector fields and 1-forms. Turk. Math. J. **42**, 1853–1862 (2018)

[4] Gualtieri, M.: Generalized complex geometry. Ann. Math. **174**(1), 75–123 (2011)

[5] Hitchin, N.: Generalized Calabi–Yau manifolds. Q. J. Math. **54**, 281–308 (2003)

[6] Jotz Lean, M., Kirchhoff-Lukat, C.: Natural lifts of Dorfman brackets. arXiv:1610.05986v2 [math.DG] (14 Jul 2017)

[7] Kolář, I., Michor, P.W., Slovák, J.: Natural Operations in Differential Geometry. Springer-Verlag, Berlin (1993)

[8] Mikulski, W.M.: The natural operators similar to the twisted Courant bracket one. Mediter J Math **16**, 101 (2019)

Włodzimierz M. Mikulski
Institute of Mathematics
Jagiellonian University
Cracow
Poland
e-mail: wlodzimierz.mikulski@im.uj.edu.pl

Received: March 21, 2019.
Revised: January 8, 2020.
Accepted: January 10, 2020.