Comparison between two state estimation techniques for linear systems ⋆

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Abstract: This paper presents a comparison in terms of accuracy and complexity between two approaches used for state estimation of linear systems: a classic Kalman filter and a guaranteed set-membership state estimation technique. The main goal of this paper is to analyze the advantages of these techniques and to combine them in the future in a new accurate and simple extension that handles system uncertainties and chance constraints. Two academic examples illustrate the main differences between the compared techniques.

Keywords: Set-membership estimation, Kalman filter, linear systems, ellipsoidal set

1. INTRODUCTION

Generally, process control requires accurate information about the plant. However, the measured variables do not totally describe the behavior of the system. Particularly, the entire system state is not always accessible. This is why it is important to get access to the unknown information using available data/knowledge. Various methods for state estimation are suggested in the literature and they can be divided into two categories. Stochastic approaches such as the Kalman Filter (see Kalman (1960)) assume the prior knowledge of the distribution of the perturbations and the measurement noises (in general Gaussian distribution) taking into account certain characteristics like the mean and the covariance. This assumption can be sometimes unrealistic. Thus deterministic approaches (Bertsekas and Rhodes (1971), Fogel and Huang (1982)) that considers unknown but bounded perturbations and bounded noises have been elaborated. There are several deterministic approaches used for state estimation, like set-membership state estimation (Schweppe (1968)), interval observers (Pourasghar et al. (2016)) or robust filtering methods (El Ghaoui and Calafiore (2001)). In the implementation of set-based deterministic estimation methods, various sets are used: polytopes (Walter and Piet-Lahanier (1989)), zonotopes (Combastel (2003), Alamo et al. (2005), Le et al. (2013)), ellipsoids (Kurzhanski and Vályi (1996), Durieu et al. (2001), Polyak et al. (2004), Daryin et al. (2006), Daryin and Kurzhanski (2012), Chernousko (1994)). The low complexity of ellipsoids makes them widely used compared to polytopes which offer better accuracy of the estimation. Combastel (2015) recently proposed a combination between stochastic and deterministic approaches, more exactly a zonotopic Kalman filter.

In the present paper, a comparison in terms of accuracy and computation complexity is made between two estimation techniques studied in the litterature: an ellipsoidal set-membership state estimation (Ben Chabane et al. (2014a), Ben Chabane et al. (2014b)) and a classical Kalman filter. The results illustrated in this paper are the main motivation to develop a future extension that will combine the advantages of the two compared techniques: better accuracy and less complexity.

The remainder of the paper is organized as follows. Section 2 formulates the state estimation problem for linear systems. Section 3 briefly presents the ellipsoidal set-membership state estimation technique. Section 4 reminds the state estimation using the classical Kalman Filter. Section 5 exposes the comparison between the two techniques. Section 6 proposes two illustrative examples. Finally, conclusions and perspectives are drawn in Section 7.

Notation. An *interval* denoted $[a, b]$ is the set defined by $\{x \in \mathbb{R} : a \leq x \leq b\}$. Thus, $\mathbf{B}$ = [-1,1] can be denoted as unitary interval. A *box* $([a_1, b_1], \dots, [a_n, b_n])^\top$ is an interval vector. A *unitary box* in $\mathbb{R}^n$ is a box composed by n unitary intervals. The identity matrix of size n is defined by I_n. A random variable x normally distributed with mean of $\bar{x}$ and with variance of σ^2 is represented by $x \sim N\left(\bar{x}, \sigma^2\right)$.

2. PRELIMINARIES AND SETUP

Consider the following discrete-time Linear Time Invariant (LTI) system

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$$\begin{cases} x_{k+1} = Ax_k + Bu_k + E_w w_k \\ y_k = Cx_k + Du_k + F_v v_k \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}$ is the state vector of the system, $u_k \in \mathbb{R}^{n_u}$ is the input vector, and $y_k \in \mathbb{R}^{n_y}$ is the measured output vector at sample time k. The matrices A, B, C, D, E_w and F_v have the appropriate dimensions. Here, $w_k \in \mathbb{R}^{n_x}$ is a vector containing the state perturbations, while $v_k \in \mathbb{R}^{n_y}$ contains the measurement noises.

Combining the state perturbations and the measurement noises in one vector $\omega_k = [w_k \ v_k]^\top \in \mathbb{R}^{n_x+n_y}$, the system (1) can be rewritten in an equivalent form

$$\begin{cases} x_{k+1} = Ax_k + Bu_k + E\omega_k \\ y_k = Cx_k + Du_k + F\omega_k \end{cases} \tag{2}$$

with the matrices $E = \begin{bmatrix} E_w & 0_{n_x,n_y} \end{bmatrix}$ and $F = \begin{bmatrix} 0_{n_y,n_x} & F_v \end{bmatrix}$.

In this work, we aim to compare an estimate of the state of the system (1) provided by two approaches that are further detailed in Sections 3 and 4.

3. GUARANTEED ELLIPSOIDAL SET-MEMBERSHIP STATE ESTIMATION

This section briefly describes the guaranteed ellipsoidal set-membership state estimation proposed by Ben Chabane et al. (2014a) for the system (2).

In this context, consider that the initial state x_0 belongs to the ellipsoid:

$$\mathcal{E}(P_0, x_0, \rho_0) = \{x \in \mathbb{R}^{n_x} : (x - x_0)^\top P_0 (x - x_0) \leq \rho_0\}$$

with the shape matrix $P_0 = P_0^\top \succ 0$, the center x_0 and the so called radius ρ_0.

Given an ellipsoidal estimation set for x_k, with $\bar{x}_k$ the nominal estimated state, the objective of this technique is to obtain an ellipsoidal set estimation for x_{k+1}. Figure 1 illustrates the 2-step procedure (prediction and correction) to calculate the estimation set. At each sample time k, the green set represents the predicted state set. The yellow strip represents the set of states compatible with the measurements y_{k+1}. The blue ellipsoid (which contains the state estimation set) overapproximates the intersection of the predicted state set and the measurement strip. Repeating the procedure at each time k leads to a guaranteed estimation set that contains the state of the system.

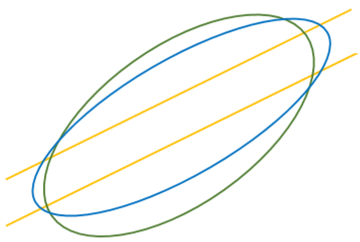

Fig. 1. State estimation using ellipsoids

More precisely, at each time k, the radius of the ellipsoidal set is minimized by solving a Linear Matrix Inequality (LMI) problem (see Ben Chabane et al. (2014a) for more details)

$$\min_{\beta, Y_k, \rho_{k+1}} \ \rho_{k+1}$$

subject to

$$\begin{cases} \begin{bmatrix} \beta P & * & * \\ 0 & \rho_{k+1} - \beta\rho_k & * \\ PA - Y_k C & (PE - Y_k F)\omega_k & P \end{bmatrix} \succ 0 \\ \rho_{k+1} \leq \beta\rho_k + \sigma \\ 0 < \beta < 1 \end{cases} \tag{3}$$

for all $\omega_k \in \mathbf{B}^{n_x+n_y}$, with $Y_k = PL_k$ and the nominal estimated state $\bar{x}_{k+1} = A\bar{x}_k + Bu_k + L_k(y_k - C\bar{x}_k - Du_k)$. The symbol "$*$" denotes symmetrical terms.

In fact, expression (3) guarantees that the system state x_{k+1} belongs to the ellipsoid $\mathcal{E}(P, \bar{x}_{k+1}, \rho_{k+1})$.

An improvement of this method in terms of accuracy is proposed in Ben Chabane et al. (2014b), with the advantage that it can deal with interval uncertainties on the evolution matrix A. The main difference with respect to Ben Chabane et al. (2014a) is the use of the measurement y_{k+1} together with additional quadratic constraints on the perturbations ω_k. This improvement allows us to reduce even more the size of the ellipsoidal estimated state set by solving an additional optimization problem

$$\min_{\rho'_{k+1}, P', \bar{x}'_{k+1}, H, \tau, \mu_i} \ \rho'_{k+1}$$

subject to

$$\begin{cases} \begin{bmatrix} \tau P + C^\top HC & * & * \\ \eta_1 - \tau\bar{x}_{k+1}^\top P & \eta_2 - \sum_{i=1}^{n_x+n_y} \mu_i & * \\ P' & -P'\bar{x}'_{k+1} & P' \end{bmatrix} \succ 0, \\ P' \succ 0, \\ F^\top HF < \sum_{i=1}^{n_x+n_y} \mu_i T_i, \\ \tau \geq 0, \\ \tau < 1, \\ \rho'_{k+1} > \tau\rho_{k+1}, \\ \mu_i \geq 0, \quad i = 1, \ldots, n_x + n_y \end{cases} \tag{4}$$

with $\eta_1 = -(y_{k+1} + Du_{k+1})^\top HC$ and $\eta_2 = \rho'_{k+1} - \tau\rho_{k+1} + \tau\bar{x}_{k+1}^\top P\bar{x}_{k+1} + (y_{k+1} + Du_{k+1})^\top H(y_{k+1} + Du_{k+1})$.

Supposing that $x_{k+1} \in \mathcal{E}(P, \bar{x}_{k+1}, \rho_{k+1})$, the expression (4) offers an improved ellipsoidal state estimation set $\mathcal{E}'(P', \bar{x}'_{k+1}, \rho'_{k+1})$.

4. KALMAN FILTER

Recall the LTI system (1) taking into consideration that w_k and v_k are random, independent white Gaussian noises, with the covariance matrices denoted by G_w and G_v, respectively. Notice that the state is a random Gaussian vector denoted by $x \sim N(\bar{x}, \sigma)$ and particularly the initial state is represented by $x_0 \sim N\left(x_{0|-1}, G_{0|-1}\right)$.

The Kalman filter design is divided into two steps:

- *Prediction.* A previously estimated state $\hat{x}_{k-1|k-1}$ and the linear nominal model (without any perturbation) are used to predict the value of the next estimated state $\hat{x}_{k|k-1}$ as well as the state estimate covariance $G_{k|k-1}$

$$\hat{x}_{k|k-1} = A\hat{x}_{k-1|k-1} + Bu_{k-1} \quad (5)$$
$$G_{k|k-1} = AG_{k-1|k-1}A^\top + E_w G_w E_w^\top \quad (6)$$

- *Correction.* The current output measurements and the statistical properties of the model are used to correct the state estimation, leading to compute the state estimate covariance

$$S_k = CG_{k|k-1}C^\top + F_v G_v F_v^\top \quad (7)$$
$$K_k = G_{k|k-1}C^\top S_k^{-1} \quad (8)$$
$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k(y_k - C\hat{x}_{k|k-1}) \quad (9)$$
$$G_{k|k} = (I - K_k C)G_{k|k-1} \quad (10)$$

with K_k the Kalman gain and S_k the innovation covariance at the sample time k.

5. COMPARISON

The main difference between the approaches presented in Sections 3 and 4 can be mainly spotted in terms of system modeling. The ellipsoidal set-membership state estimation Ben Chabane et al. (2014a) guarantees the state estimation bounds within an ellipsoid for any LTI system (1) or (2), while certain requirements should be met in order to efficiently run the classical Kalman filter. The Kalman filter works properly when the LTI model matrices are fixed and do not present parametric uncertainties. The improved set-membership state estimation Ben Chabane et al. (2014b) offers guaranteed bounds for the state estimation despite the presence of possible interval uncertainties on the evolution matrix A of the system (1) or (2). However, the Kalman filter offers a reduced computation complexity with respect to the considered set-membership estimation method. In fact, the Kalman equations are based on basic matrix operations and the computational complexity can be approximated by the number of multiplications per loop. Using the expressions (5)-(10) and considering the worst case scenario (i.e. full matrices) we can approximate the filter computational complexity to $\mathcal{O}(N^3)$, with $N = max(n_x, n_y)$.

The computational complexity of the ellipsoidal state estimation method relies on solving a LMI optimization problem. The `mincx` solver of the Matlab Robust Control Toolbox is based on the interior point method Nesterov and Nemirovski (1994) which is an iterative technique solving a least square problem at each iteration. The complexity of the method in the worst case scenario can be approximated to $\mathcal{O}(m^{2.75}l^{1.5})$ with m the number of decision variables and l the number of constraints Vandenberghe and Boyd (1994). Notice that $m = (n_x + n_y)^2 + n_x n_y + 2$ and $l = 2^{n_x+ny} + 3$ for the optimization problem (3) and $m = 0.5(n_x^2 + n_y^2) + 2.5n_x + 1.5n_y + 2$ and $l = n_x + n_y + 6$ for the optimization problem (4).

The comparison allows us to conclude that the Kalman filter offers us a better result in terms of complexity, thus faster computations. In terms of accuracy, and for each iteration, the ellipsoidal method computes an ellipsoidal set to which the real state is guaranteed to belong.

The set-membership estimation setup (Section 3) offers the possibility to use correlated/uncorrelated perturbations and measurement noises, however the choice of the perturbation bounds needs good knowledge of the plant. The Kalman filter uses the assumption of Gaussian noises, which can be difficult to verify for some real plants.

Starting from this results, the aim of our future research work is to combine the advantages of the two presented techniques in order to propose an extented method that handles systems uncertainties (i.e. interval uncertainties in the system matrices) and chance constraints.

6. ILLUSTRATIVE EXAMPLES

Two numerical examples are considered in this section to illustrate the comparison of the presented state estimation techniques.

Example 1. Consider the following stable LTI system

$$\begin{cases} x_{k+1} = \begin{bmatrix} -0.8 & 0.2 \\ -0.3 & 0.1 \end{bmatrix} x_k + \begin{bmatrix} -0.12 \\ 0.02 \end{bmatrix} w_k \\ y_k = [-2 \ \ 1]\, x_k + 0.2v_k \end{cases} \quad (11)$$

In this example, we present the results obtained by the improved guaranteeed elipsoidal set-membership state estimation (4) compared to the results obtained by Kalman filter.

In order to make a valid comparison between these two techniques, appropriate assumptions should be taken regarding the initial state, and noises. In fact, we consider that $x_0 \sim N\left(x_{0|-1}, G_{0|-1}\right)$ and $w_k \sim N(0,1)$, $v_k \sim N(0,1)$ for the Kalman filter. For the ellipsoidal set-membership approach, the initial state x_0 belongs to $\mathcal{E}(P_0, x_{0|-1}, \rho_0)$, and the perturbations and measurement noises are bounded, i.e. $|w_k| \leq 1$ and $|v_k| \leq 1$. Notice that $x_{0|-1} = [5 \ 5]^\top$, $G_{0|-1} = I_2$, $P_0 = 10^{-9}I_2$ and $\rho_0 = 2 \cdot 10^{-8}$.

Figure 2 and 3 show the bounds of x_1 and x_2 respectively after 10 iterations obtained by the ellipsoidal set-membership state estimation method (4) and the Kalman filter. The real state x (red asterix) is always inside the guaranteed bounds (in dashed blue) calculated by the ellipsoidal set-membership method (4). It can be noticed that, in this example, the state estimated with the Kalman filter (black asterix) has a slower convergence and it is not always inside the guaranteed bounds obtained with the improved set-membership method Ben Chabane et al. (2014b).

Concerning the computational complexity, the classic Kalman filter takes around 0.21ms per iteration, while the set-membership estimation technique (LMIs (3) and (4)) spends around 9ms to determine the estimation bounds.

Example 2. The aim of using an unstable system, in which the states do not converge to 0 is to prove the efficiency of the ellipsoidal method. In this context, we consider the previous example with a different evolution matrix A, while keeping the same perturbations, noises and initial conditions

$$\begin{cases} x_{k+1} = \begin{bmatrix} -1.5 & 0.2 \\ -0.3 & 0.1 \end{bmatrix} x_k + \begin{bmatrix} -0.12 \\ 0.02 \end{bmatrix} \omega_k \\ y_k = [-2 \ \ 1]\, x_k + 0.2v_k \end{cases} \quad (12)$$

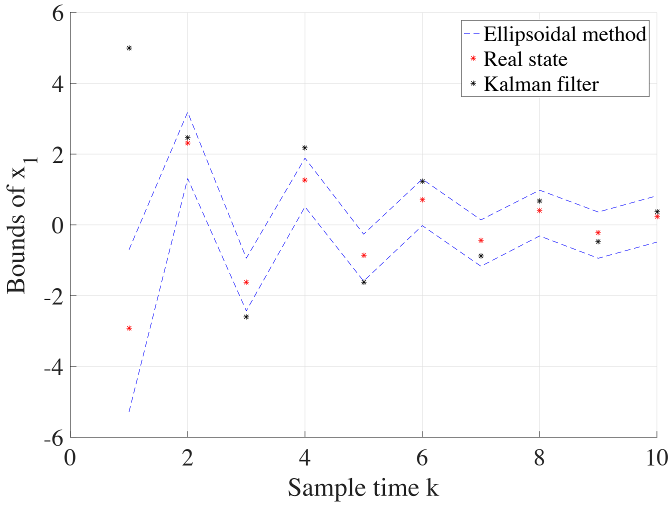

Fig. 2. Bounds of x_1

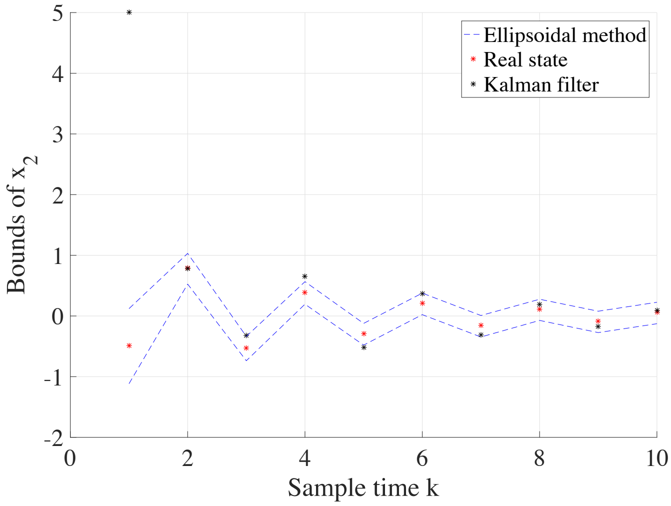

Fig. 3. Bounds of x_2

Figure 4 and its zoom (Fig. 6) show the bounds of x_1 (in dashed blue) obtained with the ellipsoidal estimation method. The bounds of x_2 are illustrated in Fig. 5 and its zoom (Fig. 7). It can be noticed that the real state (red asterix) is guaranteed to be inside these bounds, while it is not the case for the classical Kalman filter (black asterix).

7. CONCLUSION

In this paper, a brief comparison has been made between two methodogies used for the state estimation of discrete-time linear time invariant systems, subject to perturbations and measurement noises. The guaranteed ellipsoidal set-membership estimation method Ben Chabane et al. (2014b) is compared to the classic Kalman Filter, in terms of accuracy and complexity. The best estimation results (i.e. guaranteed bounds) are obtained with the improved estimation method Ben Chabane et al. (2014b). The main advantage of the Kalman filter is its lower computational complexity. In order to take advantage of the benefits of the two proposed methodologies, future work will consist on finding a new estimation technique that guarantees high accuracy, with a small computational charge. Additionally,

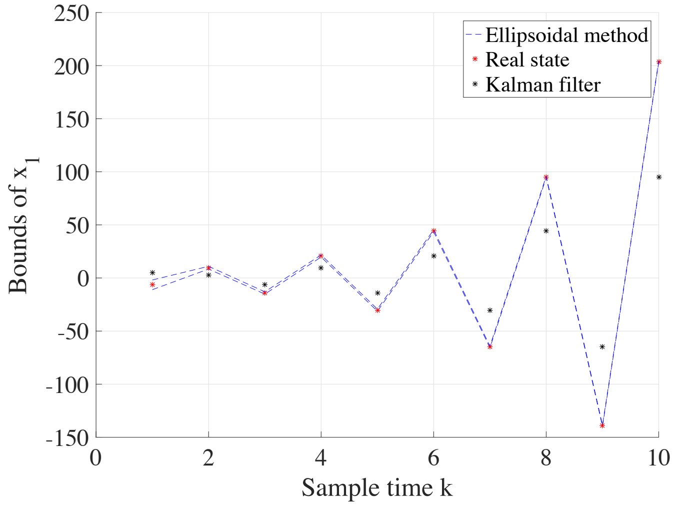

Fig. 4. Bounds of x_1

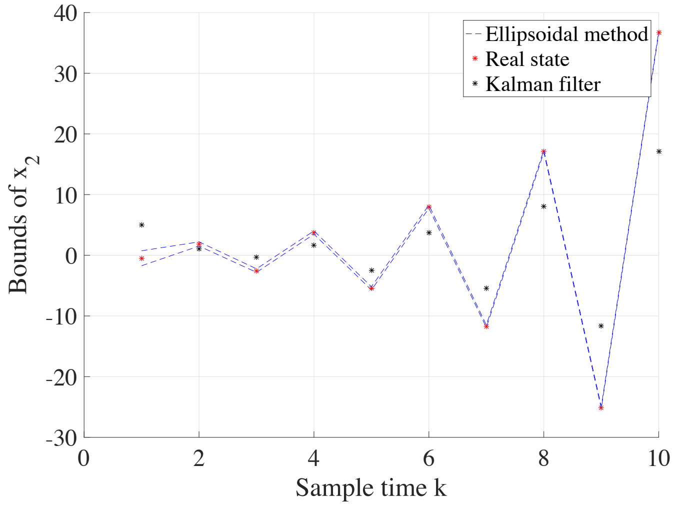

Fig. 5. Bounds of x_2

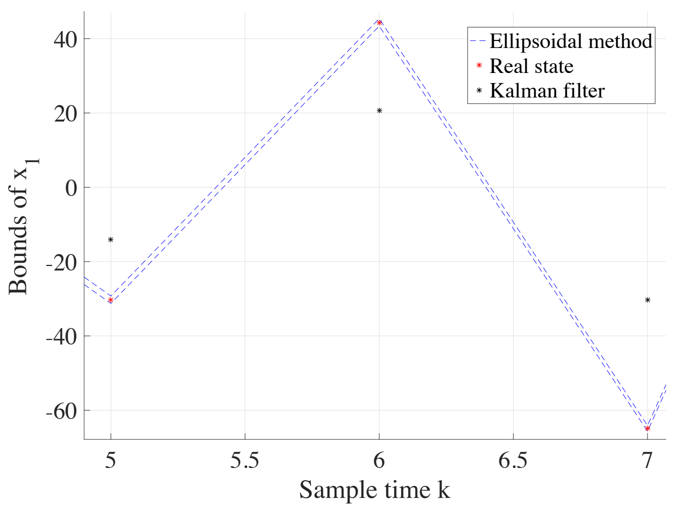

Fig. 6. Zoom on the bounds of x_1

parametric uncertainties and chance constraints will be considered in future developments.

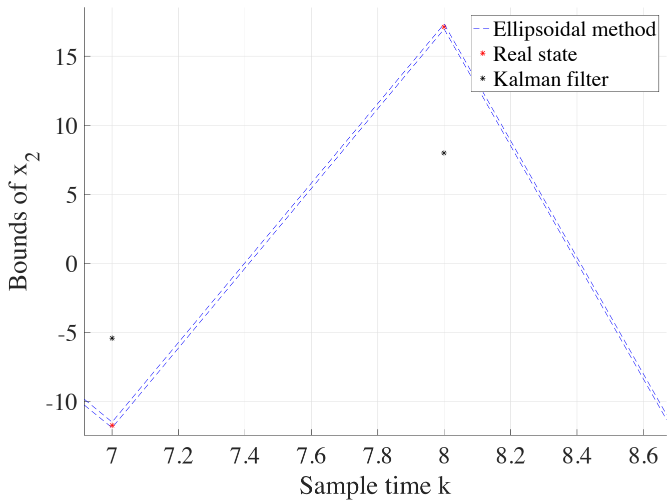

Fig. 7. Zoom on the bounds of x_2

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