16th Triennial World Congress, Prague, Czech Republic

ELSEVIER IFAC PUBLICATIONS

INVARIANT SETS FOR A CLASS OF DISCRETE-TIME LUR'E SYSTEMS

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Abstract: We present a method to estimate the domain of attraction of a class of discrete-time Lur'e systems. A new notion of *LNL*-invariance, stronger than the traditional invariance concept, is introduced. An algorithm to determinate the largest *LNL*-invariant set for this class of systems is proposed. Moreover, it is proven that the *LNL*-invariant sets provided by this algorithm are polyhedral convex sets and constitute an estimation of the domain of attraction of the non-linear system.

Keywords: Set invariance, Domain of attraction, Lur'e systems

1. INTRODUCTION

The estimation of stability regions of non-linear systems is important for many fields in engineering. Regions of asymptotic stability are zones of safe operation that can avoid unnecessary operational restrictions if they are non-conservative (Chiang and Thorp, 1989; Gilbert and Tan, 1991; Blanchini, 1999).

The importance of Lur'e systems in the context of control theory stems from the fact that different control schemes appearing in practical applications can be formulated using the Lur'e systems structure (Slotine and Li, 1991). In this paper, we consider Lur'e systems in which the non-linearity appearing in the feedback path has a piecewise affine nature.

The stability analysis of a Lur'e system can be done, for example, by means of Popov and circle criterions (see (Vidyasagar, 1993)). A novel approach to this problem can be found in (Hu *et al.*, 2004) where a procedure to compute invariant ellipsoids for Lur'e systems with piecewise affine nonlinearity is given.

In this paper, a new notion of invariance (*LNL*-invariance) is presented. Based on its geometrical properties, a simple algorithm to obtain the largest *LNL*-invariant set is proposed. *LNL*-invariance is a more conservative concept than traditional invariance, but its geometrical properties allows us to obtain a polyhedric estimation of the domain of attraction of the non-linear system.

The paper is organized as follows. In section 2 the class of piecewise-affine discrete-time Lur'e systems considered is presented. In section 3 some geometrical properties are given. The new notion of *LNL*-invariance is introduced in section 4. This concept is extended to the one of *LNL*-domain of attraction in section 5. An illustrative example is given in section 6. The paper draws to a close with a section of conclusions.

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2. PROBLEM STATEMENT

Consider the following discrete-time Lur'e system:

$$\begin{cases} x_{k+1} = Ax_k - B\phi(y_k) \\ y_k = Fx_k \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^n$ represents the state vector and $y_k = Fx_k \in \mathbb{R}$ the output of the system. The nonlinear function $\phi(\cdot)$ is assumed to satisfy the following conditions:

(i) $\phi(y)$ is piecewise-affine.
(ii) $\phi(y)$ is a continuous odd function.
(iii) $\phi(y)$ is concave in $\mathbb{R}^+$.

The following property characterizes all the functions $\phi(.)$ that satisfy previous assumptions.

Property 1. (Hu *et al.*, 2004) The piecewise-affine function $\phi(y)$ is concave in $\mathbb{R}^+$ if and only if it can be expressed as

$$\phi(y) = \begin{cases} k_0 y & \text{if } y \in [0, b_1) \\ k_1 y + c_1 & \text{if } y \in [b_1, b_2) \\ \vdots & \\ k_N y + c_N & \text{if } y \in [b_N, \infty) \end{cases}, \quad \forall y \geq 0 \tag{2}$$

where the scalars k_i, $i = 0, \ldots, N$, b_i, $i = 1, \ldots, N$ and c_i, $i = 1, \ldots, N$ satisfy:

$$0 < b_1 < b_2 < \ldots < b_N$$

$$k_0 > k_1 > k_2 > \ldots > k_N$$

$$c_i = \begin{cases} (k_0 - k_1)b_1 & \text{if } i = 1 \\ c_{i-1} + (k_{i-1} - k_i)b_i & \text{if } 2 \leq i \leq N \end{cases}$$

See figure 1 for an example of piecewise-affine concave function in $\mathbb{R}^+$ ($N = 3$).

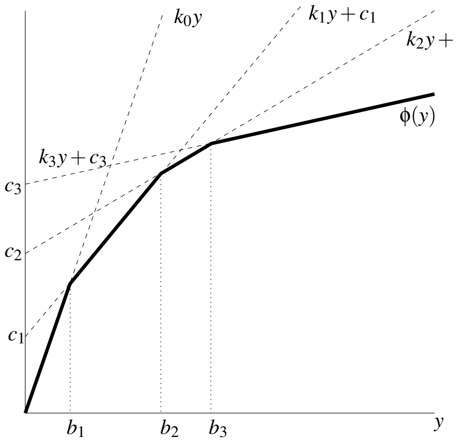

Fig. 1. An example of a piecewise-affine function $\phi(\cdot)$ that is concave in $\mathbb{R}^+$.

Note that the results presented in this paper can also be applied to systems of the form:

$$x_{k+1} = \hat{A}x_k - \hat{B}\hat{\phi}(y_k)$$

where $\hat{\phi}(\cdot)$ is an odd piecewise-affine function convex in $\mathbb{R}^+$ (it suffices to define $\phi(\cdot) = -\hat{\phi}(\cdot)$, $A = \hat{A}$ and $B = -\hat{B}$).

3. ANALYSIS OF THE NON-LINEAR FUNCTION

In this section some properties of function $\phi(\cdot)$ are presented. For that purpose the following definition is introduced.

Definition 1. Given the piecewise-affine odd function:

$$\phi(y) = \begin{cases} k_0 y & \text{if } y \in [0, b_1) \\ k_1 y + c_1 & \text{if } y \in [b_1, b_2) \\ \vdots & \\ k_N y + c_N & \text{if } y \in [b_N, \infty) \end{cases}, \quad \forall y \geq 0,$$

the odd functions $\phi_i(y)$, $i = 1, \ldots, N$ are defined as:

$$\phi_i(y) = \begin{cases} k_0 y & \text{if } y \in [0, d_i) \\ k_i y + c_i & \text{if } y \in [d_i, \infty) \end{cases}, \quad \forall y \geq 0, \tag{3}$$

where $d_i = \dfrac{c_i}{k_0 - k_i}$, $i = 1, \ldots, N$.

Figure 2 depicts functions $\phi_i(\cdot)$, $i = 1, \ldots, 3$ for the function $\phi(\cdot)$ corresponding to figure 1.

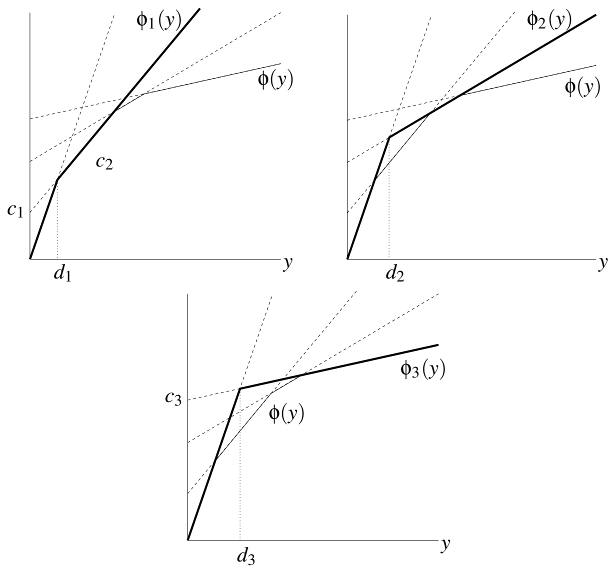

Fig. 2. Three $\phi_i(\cdot)$ functions related to the previous $\phi(\cdot)$ function.

The following lemma shows how function $\phi(\cdot)$ can be expressed in terms of functions $\phi_i(\cdot)$, $i = 1, \ldots, N$.

Lemma 1. (Hu *et al.*, 2004) Suppose that $\phi(\cdot)$ is an odd piecewise-affine function concave in $\mathbb{R}^+$. Then:

$$\phi(y) = \min_{1 \leq i \leq N} \phi_i(y), \quad \forall y > 0$$

The previous lemma can be justified from a graphical point of view (see figure 2). It can be observed in that figure that $\phi(y)$ can be obtained from the minimum of $\phi_1(y)$, $\phi_2(y)$ and $\phi_3(y)$.

4. THE *LNL*-INVARIANCE NOTION

In this section, the concept of *LNL*-invariance is presented. This new notion of invariance is stronger than the classical one. However, the *LNL*-invariance enjoys from a number of geometrical properties that makes it possible the computation of the greatest *LNL*-invariance set by means of a simple algorithm.

Definition 2. Consider system $x_{k+1} = Ax - B\phi(Fx)$ and let function $\phi(\cdot)$ be defined as in equation (2), $f(x)$ and $f_L(x)$ are defined as:

$$\begin{aligned} f(x) &= Ax - B\phi(Fx) \\ f_L(x) &= Ax - Bk_0Fx \end{aligned} \tag{4}$$

The notion of *LNL*-invariance is introduced in the following definition:

Definition 3. A set Ω is said to be *LNL*-invariant for system $x_{k+1} = Ax - B\phi(Fx)$ if $x \in \Omega$ implies:

$$\begin{aligned} f(x) &= Ax - B\phi(Fx) \in \Omega \\ f_L(x) &= Ax - Bk_0Fx \in \Omega \end{aligned}$$

This concept is stronger than simple invariance, that is, if Ω is *LNL*-invariant it is also invariant, but the opposite is not true.

Remark 1. *LNL* stands for *Linear* and *Non − Linear*. Note that the new constraint $f_L(x) \in \Omega$ added to the concept of *LNL*-invariance is not a very strong constraint as there is a neighborhood of the origin where $f(x)$ equals $f_L(x)$.

Definition 4. We say that $S_0, S_1, \ldots, S_k$ is an admissible sequence if $S_i \in \{1, -1\}$, $i = 0, \ldots, k$.

Definition 5. Given x and $S \in \{1, -1\}$, function $G(x, S)$ is defined as follows:

$$G(x,S) = \begin{cases} f(x) & \text{if } S = 1 \\ f_L(x) & \text{if } S = -1 \end{cases}$$

Definition 6. We say that x belong to the *LNL*-domain of attraction of system $x_{k+1} = Ax - B\phi(Kx)$ if the recursion:

$$x_{k+1} = G(x_k, S_k), \;\; x_0 = x$$

converges to the origin for every admissible infinite sequence $\{S_0, S_1, S_2, \ldots\}$.

4.1 The one step function

Definition 7. Given a set Ω and the system $x_{k+1} = Ax_k - B\phi(Fx_k)$, where $\phi(\cdot)$ is defined as in definition 2, the operators $Q_{NL}(\cdot)$, $Q_L(\cdot)$ and $Q_{LNL}(\cdot)$ are defined as follows:

$$\begin{aligned} Q_{NL}(\Omega) &= \{\, x \,:\, Ax - B\phi(Fx) \in \Omega \,\}. \\ Q_L(\Omega) &= \{\, x \,:\, Ax - Bk_0Fx \in \Omega \,\}. \\ Q_{LNL}(\Omega) &= Q_L(\Omega) \bigcap Q_{NL}(\Omega). \end{aligned}$$

Given a convex set Ω, the one step set $Q_{NL}(\Omega)$ is not necessarily convex due to the non-linear nature of function $\phi(\cdot)$. The non-convex nature of $Q_{NL}(\Omega)$ makes it difficult the use of operator $Q_{NL}(\cdot)$ in the computation of invariant sets the class of Lur'e systems under consideration. The most remarkable property of $Q_{LNL}(\cdot)$ is that given a convex polyhedral set Ω, $Q_{LNL}(\Omega)$ is a convex polyhedron. This fact is one of the main contributions of this paper and will be proved in this section. For that purpose, the following auxiliary operators $Q_{LNL,i}(\cdot)$ are defined.

Definition 8. Given functions $\phi_i(\cdot)$, $i = 1, \ldots, N$, defined as in equation (3), the operators $Q_i(\cdot)$ $i = 1, \ldots, N$ are defined as:

$$Q_{LNL,i}(\Omega) = Q_L(\Omega) \bigcap \{\, x \,:\, Ax - B\phi_i(Fx) \in \Omega \,\}.$$

The following theorem states that $Q_{LNL,i}(\cdot)$ is a convex operator.

Theorem 1. Let Ω be a polyhedric set given by $\Omega = \{x : \; Hx \preceq g\}$, $Q_{LNL,i}(\Omega)$ is a convex polyhedron that can be obtained from the equality:

$$Q_{LNL,i}(\Omega) = P_i(\Omega), \quad i = 1, \ldots, N$$

where:

$$P_i(\Omega) = Q_L(\Omega) \bigcap \{\, x \,:\, H(A - Bk_iF)x \preceq g + |c_iHB| \,\}.$$

Proof: Let us suppose that there is $x \in P_i(\Omega)$ such that $x \notin Q_{LNL,i}(\Omega)$. In this case it results that $x \notin \bigcap \{\, x \,:\, Ax - B\phi_i(Fx) \in \Omega \,\}$. That is, there exists j such that denoting H_j and g_j the j-th row of H and j-th component of g_j respectively:

$$H_j(Ax - B\phi_i(Fx)) > g_j$$

Using the inequality: $a\phi_i(y) \leq max(ak_0y, ak_iy - |ac_i|)$ (see lemma (2) in appendix A):

$$\begin{aligned} & H_j(Ax - B\phi_i(Fx)) \leq \\ & \leq H_jAx + max(-H_jBk_0Fx, -H_jBk_iFx - |H_jBc_i|). \end{aligned}$$

Two different cases must be taken into account:

(1) $-H_jBk_0Fx > -H_jBk_iFx - |H_jBc_i|$:
In this case:

$$\begin{aligned} & g_j < H_j(Ax - B\phi_i(Fx)) \leq \\ & \leq H_jAx - H_jBk_0Fx = H_j(A - Bk_0F)x. \end{aligned}$$

This contradicts the fact that $H(A - Bk_0F)x \preceq g$ (see the definition of $P_i(\Omega)$).

(2) $-H_jBk_0Fx < -H_jBk_iFx - |H_jBc_i|$

In this case:

$$g_j < H_j(Ax - B\phi_i(Fx)) \leq$$
$$\leq H_jAx - H_jBk_1Fx - |H_jBc_i|.$$

This contradicts the fact that $H(A - Bk_1F)x \preceq g + |c_iHB|$ (see the definition of $P_i(\Omega)$).

There is no $x \notin Q_i(\Omega)$ such that $x \in P_i(\Omega)$. This proves the first part of the claim.

To conclude the proof it will be shown that $Q_{LNL,i}(\Omega) \subseteq P_i(\Omega)$. In effect, due to the fact that $-|c_iHB| \preceq -HB\phi(Fx)$:

$$HAx - |c_iHB| \preceq H(Ax - B\phi(Fx)) \preceq g.$$

■

In the following theorem it is shown that the operator $Q_{LNL}(\cdot)$ can be obtained from the intersection of operators $Q_{LNL,i}(\cdot)$, $i = 1, \ldots, N$.

Theorem 2. Let Ω be a convex polyhedron given by $\Omega = \{x : Hx \preceq g\}$, then

$$Q_{LNL}(\Omega) = \bigcap_{i=1}^{N} Q_{LNL,i}(\Omega)$$

Proof: First it will be shown that $Q_{LNL}(\Omega) \subseteq \bigcap_{i=1}^{N} Q_{LNL,i}(\Omega)$. Let us suppose that $x \in Q_{LNL}(\Omega)$ and $Fx \geq 0$. Then

$$Ax - B\phi(Fx) \in \Omega$$
$$Ax - Bk_0Fx \in \Omega.$$

Note that lemma 1 guarantees that $\phi(Fx) \leq \phi_i(Fx) \leq k_0Fx$, $i = 1, \ldots, N$. From this, and the fact that Ω is a convex set, it can be shown that

$$Ax - B\phi_i(Fx) \in \Omega, \;\; i = 1, \ldots, N$$

That is, $x \in Q_{LNL,i}(\Omega)$. To prove that $\bigcap_{i=1}^{N} Q_{LNL,i}(\Omega) \subseteq Q_{LNL}(\Omega)$ let us suppose that $x \in \cap Q_{LNL,i}(\Omega)$ and $Fx \geq 0$. Let j be such that

$$\phi_j(Fx) = min(\phi_i(Fx) : \;\; i \in 1, \ldots, N).$$

then as $x \in Q_{LNL,j}(\Omega)$, it is obtained that

$$Ax - B\phi_j(Fx) = Ax - B\phi(Fx) \in \Omega.$$

Therefore, this and the fact that $Ax - Bk_0Fx \in \Omega$ leads to $x \in Q_{LNL}(\Omega)$, prove the claim.

Similar analysis can be made if $Fx < 0$. ■

Theorem 3. Let Ω be a convex polyhedron given by $\Omega = \{x : Hx \preceq g\}$. Then $Q_{LNL}(\Omega)$ is a convex polyhedron that can be obtained from the following equality:

$$Q_{LNL}(\Omega) = \bigcap_{i=1}^{N} P_i(\Omega).$$

Proof: The proof stems from a direct application of theorems (1) and (2). ■

5. *LNL*-DOMAIN OF ATTRACTION

In this section, it is proposed a recursion and shown it properties that allows to create an algorithm to obtain an *LNL*-invariant set that it is also *LNL*-domain of attraction. This invariant set is characterized by a convex polyhedron.

Theorem 4. Denote $L(F)$ the region of linear behaviour of system (1), that is, $L(F) = \{x \in \mathbb{R}^n : |Fx| \leq b_1\}$. Suppose that $\Phi \in L(F)$ is a convex polyhedric invariant set, with non zero volume, corresponding to the asymptotically stable system $x^+ = (A - Bk_0F)x$. Denote now $C_0 = \Phi$ and consider the following recursion:

$$C_{k+1} = Q_{LNL}(C_k).$$

Then:

(1) C_k is a convex polyhedron, $\forall k \geq 0$.
(2) C_k is an *LNL*-invariant set, $\forall k \geq 0$.
(3) C_k belongs to the LNL-domain of attraction of the system, $\forall k \geq 0$.
(4) The sequence $\{C_0, C_1, \ldots\}$ converges to the *LNL*-domain of attraction of system (1).
(5) The *LNL*-domain of attraction of system (1) is a convex set.

Proof:

(1) Theorem (3) states that if Ω is a convex polyhedron then $Q_{LNL}(\Omega)$ is also a convex polyhedron. This, and the fact that C_0 is a convex polyhedron, prove that the recursion $C_{k+1} = Q_{LNL}(\Omega)$ always yields convex polyhedrons.
(2) As C_0 belongs to $L(F)$ it results that $x^+ = Ax - Bk_0Fx = Ax - B\phi(Fx)$, $\forall x \in C_0$. Therefore, C_0 is not only an invariant set for the linear system $x^+ = Ax - Bk_0Fx$, but also for the nonlinear one: $x^+ = Ax - B\phi(Fx)$. This is equivalent to say that C_0 is an LNL-invariant set.
Let us now suppose that C_{k-1} is LNL-invariant, then $C_{k-1} \subseteq Q_{LNL}(C_{k-1}) = C_k$. Therefore, if $x \in C_k$ then $Ax - Bk_0Fx \in C_{k-1} \subseteq C_k$ and $Ax - B\phi(Fx) \in C_{k-1} \subseteq C_k$. This proves the claim.
(3) From the LNL-invariance of $C_0 \subseteq L(F)$ and the asymptotically stability of the non saturated system it is inferred that C_0 belongs to the LNL-domain of attraction of the system. Note that if C_{k-1} belongs to the LNL-domain of attraction then $C_k = Q_{LNL}(C_{k-1})$ also belongs to the

LNL-domain of attraction. This is due to the fact that $G(x,S) \in C_{k-1}$, for all $x \in C_k$ and for all $S \in \{1,-1\}$. Therefore, the recursion $C_{k+1} = Q_{LNL}(C_k)$ with $C_0 = \Phi$ yields LNL-invariant sets that belong to the LNL-domain of attraction.

(4) Suppose now that x belongs to the LNL-domain of attraction of the system. As Φ is an invariant set with nonzero volume, there exists p such that the recursion $x_{k+1} = G(x_k, S_k)$ with $x_0 = x$ satisfies $x_p \in \Phi = C_0$ for all admissible sequence $S_0, S_1, \ldots, S_p$. This is equivalent to say that x is included in C_p. That is, if x belongs to the LNL-domain of attraction then there exists a finite integer p such that x is included into the p-th LNL-invariant set provided by the algorithm.

(5) This is directly inferred from the convexity of the obtained sets $\{C_1, C_2 \ldots,\}$ and the fact that the sequence $\{C_0,\ C_1,\ \ldots,\}$ converges to the *LNL*-domain of attraction.

■

The recursion presented in the previous theorem requires an invariant set of the linear system $x^+ = (A - Bk_0F)x$, included in $L(F)$. This admissible invariant set can be obtained by standard algorithms (see (Gilbert and Tan, 1991; Blanchini, 1999)).

6. NUMERICAL EXAMPLE

In this section an *LNL*-invariant set for a numerical example is obtained. We will compare this set with the one obtained using the results of (Hu and Lin, 2004).

Let us consider the system $x^+ = Ax - B\phi(Fx)$ with

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \quad F = [0.6167 \quad 1.2703]. \tag{5}$$

and the odd function $\phi(\cdot)$:

$$\phi(y) = \begin{cases} y, & \text{if } y \in [0, 0.5) \\ 0.5y + 0.25, & \text{if } y \in [0.5, 1.5) \\ 1, & \text{if } y \in [1.5, \infty) \end{cases} \quad y > 0. \tag{6}$$

This function is represented in figure (3).

Theorem 5 shows how to obtain a sequence of LNL-invariant sets that constitute an estimation of the domain of attraction of the nonlinear system. This sequence has been calculated for system (5) and it is shown in figure (4).

In that figure, the most inner set is an invariant set of the linear system corresponding to the zone of linear behavior of the system. The *LNL*-domain of attraction of the system is also depicted.

This is not the only method to determinate invariant sets for piecewise-affine feedback systems. In (Hu and Lin, 2004), the authors propose an algorithm to obtain

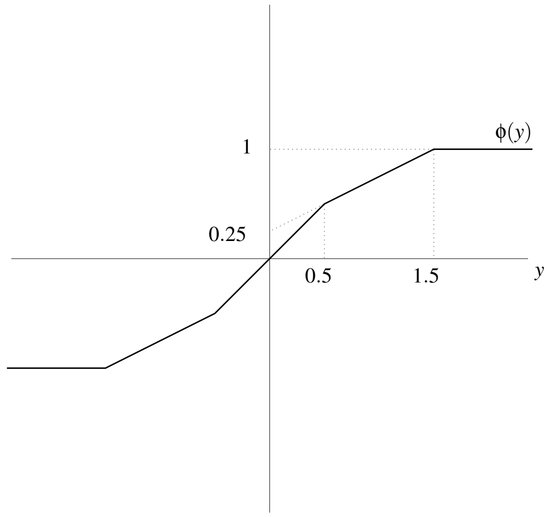

Fig. 3. $\phi(\cdot)$ function of the example

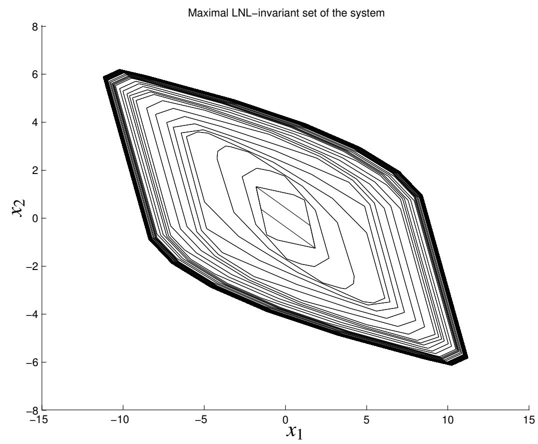

Fig. 4. *LNL*-invariant set of the system

ellipsoidal invariant sets for saturated feedback systems. This approach has been generalized in (Cao and Lin, 2003) to the class of Lur'e systems considered in this paper.

Figure 5 shows the comparison between the ellipsoidal invariant set obtained by means of the results presented in (Hu and Lin, 2004), and the polyhedric *LNL*-invariant obtained by means of the algorithm proposed in this paper.

7. CONCLUSIONS

In this paper we consider the problem of estimating the domain of attraction of a given class of Lur'e systems. A new notion of invariance is presented. This new notion of invariance has an interesting property: it leads to convex estimations of the domain of attraction of the nonlinear system. A simple algorithm for determining an estimation of the domain of attraction and an invariant set of the system is provided. An illustrative example is given.

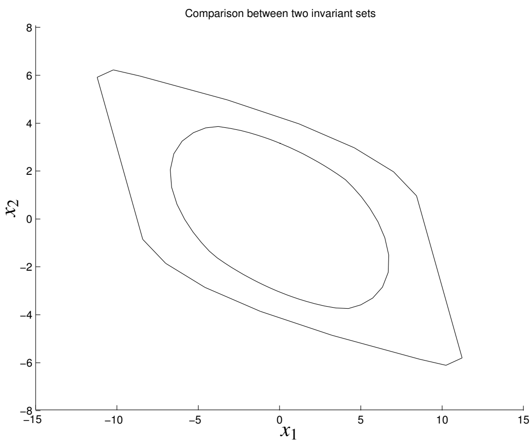

Fig. 5. *LNL*-invariant set and ellipsodial invariant set.

Appendix A

Lemma 2. Given an odd piecewise-affine function $\phi(\cdot)$, convex in $\mathbb{R}^+$, consider functions $\phi_i(\cdot)$, $i = 1,\ldots,N$ defined as in definition (3). Then:

$$a\phi_i(y) \le max(ak_0y, ak_iy - |ac_i|)$$

Proof: There are two different possibilities.

- $|y| \le d_i$:
  If $|y| \le d_i$ then $\phi_i(y) = k_0y$ and the inequality holds.
- $|y| > d_i$:
  In this case, $\phi_i(y) = k_iy + sign(y)c_i$. Note that in virtue of property (1): $k_i < k_0$, $c_i > 0$ and $d_i = \frac{c_i}{k_o - k_i} > 0$.
  There are now four different possibilities:
  · $a > 0$ and $y > d_i$. In this case: $a\phi_i(y) = ak_iy + ac_i < ak_0y$.
  · $a > 0$ and $y < -d_i$. In this case: $a\phi_i(y) = ak_iy - ac_i = ak_iy - |ac_i|$.
  · $a < 0$ and $y > d_i$. In this case: $a\phi_i(y) = ak_iy + ac_i = ak_iy - |ac_i|$.
  · $a < 0$ and $y < -d_i$. In this case: $a\phi_i(y) = ak_iy - ac_i < ak_0y$.

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