A Robust Constrained Reference Governor Approach using Linear Matrix Inequalities

J.L. Guzmán [a]*, T. Álamo [b], M. Berenguel [a], S. Dormido [c], E. F. Camacho [b]

[a] *Dep. Lenguajes y Computación, Universidad de Almería, Ctra. Sacramento s/n 04120 Almería, Spain. E-mail:joguzman@ual.es*
[b] *Dep. Ingeniería de Sistemas y Automática, Universidad de Sevilla, C. de los Descubrimientos s/n 41092 Sevilla, Spain.*
[c] *Dep. Informática y Automática, ETSI Informática, UNED, 28040 Madrid, Spain.*

Abstract

The purpose of this paper is to examine and provide a solution to the output reference tracking problem for uncertain systems subject to input saturation. As well-known, input saturation and modelling errors are very common problems at industry, where control schemes are implemented without accounting for such problems. In many cases, it is sometimes difficult to modify the existing implemented control schemes being necessary to provide them with external supervisory control approaches in order to tackle problems with constraints and modelling errors. In this way, a cascade structure is proposed, combining an inner loop containing any controller with an outer loop where a Generalized Predictive Controller (GPC) provides adequate references for the inner loop considering input saturations and uncertainties. Therefore, the contribution of this paper consists in providing a state space representation for the inner loop and using Linear Matrix Inequalities (LMI) to obtain a predictive state-vector feedback in such a way that the input reference for the inner loop is calculated to satisfy robust tracking specifications considering input saturations. Hence, the final proposed solution consists in solving a regulation problem to a fixed reference value subjected to a set of constraints described by several LMI and Bilinear Matrix Inequalities (BMI). An illustrative numerical example is presented.

Key words: predictive control, robust control, tracking control, constrained control, linear matrix inequalities

1. Introduction

Mathematical models are required at design level in any control system development. Models cannot represent every aspect of reality, so assumptions must be made in order to use them for control purposes. On the other hand, most physical processes are constrained by several reasons, such as physical limits (e.g. valve position), security levels (e.g. pressure levels), or performance criteria (e.g. working near the optimal operating point) [1]. In practice, many control techniques implemented at industry work without taking into account these modelling errors and system constraints. Fixed-structure models and known parameters are used supposing that the model exactly represents the real process, and the imperfections will be removed by means of feedback. Furthermore, detuned control is usually used in order to cope with system saturations.

As well-know, these problems have widely been addressed by the scientific control community. Numerous robust control techniques are available in order to face system uncertainties such as $\mathcal{H}_\infty$ [2], Quantitative Feedback Theory (QFT) [3], or μ-synthesis [4]. In the same way, constrained system problems have been solved from different points of view including anti-windup schemes [5],[6], constrained model predictive control [7], [8], and LMI-based synthesis [9]. Furthermore, combinations of these techniques can be found in order to solve both robustness and input saturation problems [10], [11], [12], [13].

All previous approaches can be used in order to control systems presenting the problems described above. However, many industrial processes are currently controlled by some traditional control schemes such as PID control or a generic two degrees of freedom controller, being difficult or even impossible to modify this primal controller. Therefore, in these cases, an external supervisory control is required in order to face the problems of the implemented inner loop. In this sense, the reference governor approach has been presented in several works as solution to this problem, mainly focused on providing appropriated references to the inner loop in order to ensure nominal stability in presence of constraints [14], [15], [16], [17], [18]. This approach has also been studied for solving robustness problems due to the presence of uncertainties in the inner loop [19], [20],

[21]. However, the existing reference governor approaches present some drawbacks, such as the requirement of knowing the future reference in advance, the use of very conservative uncertainty representations, the heavy computational burden or the difficulty to improve tracking performance.

Therefore, this work presents a robust constrained reference governor approach based on LMI with the aim of solving the previous problems in a systematic way [22]. In this approach a GPC controller [23], [7] provides adequate references for the inner loop considering input saturations and uncertainties. Thus, the proposed solution in order to prove robust stability when constraints are active, is to translate the inner loop problem into a state space representation, and then using LMI to obtain a predictive state-vector feedback in such a way that the input reference to this inner loop is calculated in order to satisfy robust tracking specifications considering input saturation. This work describes the different steps required to obtain such solution and how input saturation in the inner loop can be handled using LMI-based methods. As it will be shown, non-symmetric limits are also allowed.

The final solution consists in solving a set of constraints defined by several LMI and BMI, where a Branch and Bound algorithm has been developed in order to handle the bilinear terms. It is important to notice that the algorithm is implemented for tracking problems where the aim is to regulate to a fixed reference value and not to the origin, in the presence of input constraints in the inner loop of the system what is not usual in the referenced works.

The paper is organized as follows. Section 2 describes the state space representation of the inner loop including the saturation term. The next section presents the LMI-based solution for this approach ensuring constrained robust stability. The problem is formulated by means of LMI in order to ensure constrained robust stability and fulfill specific performance criteria, describing also the proposed Branch and Bound algorithm to handle the bilinear terms. Finally, a numerical example, which can be found in many industrial plants, is presented in Section 4.

2. State space representation of the inner loop.

The first step necessary before applying a LMI-based solution is to obtain the state space representation for the inner loop in the proposed approach (see Figure 1). The inner loop has been considered as a typical control scheme with two degrees of freedom for general purposes. In order to provide optimal references to the inner loop, the GPC algorithm has been selected. GPC is based on CARIMA model, where the following plant representation is considered being the time delay included into the $B(z^{-1})$ polynomial

$$A(z^{-1})y(t) = B(z^{-1})u(t-1) \tag{1}$$

with

$$A(z^{-1}) = \prod_{i=1}^{n}(1+p_i z^{-1}),\ B(z^{-1}) = K_z \prod_{i=1}^{m}(1+c_i z^{-1}) \tag{2}$$

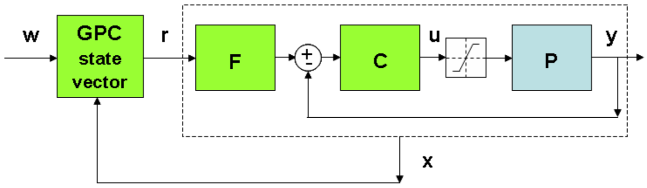

Fig. 1. Control system scheme for LMI-based approach

The plant parameters $K_z, c_1, \ldots, c_m, p_1, \ldots, p_n$ are considered uncertain and supposed to lie within known intervals due to the presence of uncertainties in the plant time domain parameters. That is,

$$K_z \in [K_{z,min}, K_{z,max}]$$
$$c_i \in [c_{i,min}, c_{i,max}], \quad i = 1, \ldots, m$$
$$p_i \in [p_{i,min}, p_{i,max}], \quad i = 1, \ldots, n$$

The polynomials $A(z^{-1})$ and $B(z^{-1})$ can be rewritten as

$$A(z^{-1}) = 1 + a_1 z^{-1} + a_2 z^{-1} + \ldots + a_n z^{-n} \tag{3}$$

$$B(z^{-1}) = b_0 + b_1 z^{-1} + b_2 z^{-1} + \ldots + b_m z^{-m} \tag{4}$$

where the coefficients $a_1, \ldots, a_n$, $b_0, \ldots, b_m$, depend multilineally on the parametric vector

$$\phi = [K_z, c_1, \ldots, c_m, p_1, \ldots, p_n]^T$$

.

That is, each coefficient depends affinely on each element of vector ϕ. In order to express explicitly the dependence on the vector ϕ, the coefficients can be expressed as $b_0(\phi), \ldots, b_m(\phi)$, $a_1(\phi), \ldots, a_n(\phi)$.

Considering the plant dynamics (1) and the polynomials $A(z^{-1})$ and $B(z^{-1})$, it results that

$$y(t) = -\sum_{i=1}^{n} a_i(\phi)y(t-i) + \sum_{i=0}^{m} b_i(\phi)u(t-i-1) \tag{5}$$

or

$$y(t+1) = -\sum_{i=1}^{n} a_i(\phi)y(t-i+1) + \sum_{i=0}^{m} b_i(\phi)u(t-i) \tag{6}$$

The plant dynamics can be represented by a state-space representation, where the proposed state depends on the current output, and the past outputs and inputs in the following way

$$x_p(t) = [y(t)\ \ldots\ y(t-n+1)\ u(t-1)\ \ldots\ u(t-m)]^T \tag{7}$$

This state selection has the advantage that the state $x_p(t)$ is always accessible, that is, the value of $x_p(t)$ is known since it is always possible to access to the output $y(t)$ and input $u(t)$ signals. So, the state space representation is given by

$$x_p^+ = A_p x_p + B_p u \tag{8}$$

$$y = C_p x_p \tag{9}$$

where

$$A_p = \left[\begin{array}{ccccc|ccccc} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n & b_1 & b_2 & \dots & b_{m-1} & b_m \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \hline 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{array}\right], B_p = \left[\begin{array}{c} b_0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \hline 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{array}\right]$$

$$C_p = \left[\begin{array}{ccccc} 1 & 0 & 0 & \dots & 0 \end{array}\right]$$

x_p denotes the state vector, u the system input, and x_p^+ the next state $x_p(t+1)$.

Considering that the coefficients $a_1, \dots, a_n$, $b_0, \dots, b_m$ depend on the parametric vector ϕ, the system dynamics can be rewritten as

$$x_p^+ = A_p(\phi)x_p + B_p(\phi)u \tag{10}$$

$$y = C_p x_p \tag{11}$$

The plant disturbances have not been included along the previous description. Bounded disturbances in the plant represented by the CARIMA model are given as

$$A(z^{-1})y(t) = B(z^{-1})u(t-1) + \mathtt{T}(z^{-1})\frac{\epsilon(t)}{\Delta} \tag{12}$$

where $\Delta = 1 - z^{-1}$. It is assumed that the coefficients of the $\mathtt{T}(z^{-1})$ polynomial depends multilineally on a bounded parametric vector ϕ_T and $\epsilon(t)$ is bounded for all $t > 0$, that is, $\|\epsilon(t)\|_\infty < \epsilon_{max}$, $\forall t$.

Following the same steps as previously, the system dynamics can be rewritten as

$$\begin{aligned} x_p^+ &= A_p(\tilde{\phi})x_p + B_p(\tilde{\phi})u + E_p(\tilde{\phi}) \\ y &= C_p x_p \end{aligned} \tag{13}$$

In this representation, $\tilde{\phi}$ is a parametric vector containing ϕ, ϕ_T and ϵ. Furthermore, it can be assumed that $\tilde{\phi}$ can only take values within a convex set (typically an hyperrectangle). Finally, notice that $A_p(\tilde{\phi})$, $B_p(\tilde{\phi})$ and $E_p(\tilde{\phi})$ depends multilineally on parametric vector $\tilde{\phi}$.

2.0.1. *Controller and prefilter representation.*

Assume available state space descriptions for the prefilter $F(z^{-1})$ and controller $C(z^{-1})$. Denoting x_F as the state vector of the filter $F(z^{-1})$, r the filter input and r_F the filter output, it is supposed that matrices A_F, B_F, C_F, and D_F describe the filter dynamics as follows

$$\begin{aligned} x_F^+ &= A_F x_F + B_F r \\ r_F &= C_F x_F + D_F r \end{aligned} \tag{14}$$

In the same way, x_C denotes the state vector for the controller $C(z^{-1})$ and u the controller output. The matrices A_C, B_C, C_C and D_C describe the controller dynamics as follows

$$\begin{aligned} x_C^+ &= A_C x_C + B_C(r_F - y) \\ u &= C_C x_C + D_C(r_F - y) \end{aligned} \tag{15}$$

Note that the input to the controller is given by the filter output r_F minus the plant output y, and the plant is subjected to uncertainties and disturbances as discussed above.

2.0.2. *Inner loop representation.*

As commented previously, the goal is to design a robust predictive controller considering input saturation in the inner loop. Therefore, the state space representation of the inner loop must be developed including the saturation.

The input saturation in the inner loop is given by

$$\sigma_p(u) = \begin{cases} U_{min} & \text{if } u < U_{min} \\ u & \text{if } U_{min} \le u \le U_{max} \\ U_{max} & \text{if } u > U_{max} \end{cases} \tag{16}$$

where nonsymmetric saturation can be present.

Firstly, the saturation is redefined in order to use a symmetric representation to facilitate the calculations. Therefore, the saturation is obtained as

$$\sigma_p(u) = L_s \sigma(\frac{1}{L_s}(u - u_c)) + u_c \tag{17}$$

where

$$\sigma(u) = \begin{cases} -1 \ if & u < -1 \\ u \ if & -1 \le u \le 1 \\ 1 \ if & u > 1 \end{cases} \tag{18}$$

$$u_c = \frac{U_{max} + U_{min}}{2}, \ L_s = \frac{U_{max} - U_{min}}{2}$$

Then, the plant representation (13) is modified to consider input saturation in the following way

$$\begin{aligned} x_p^+ &= A_p x_p + B_p(L_s \sigma(\frac{1}{L_s}(u - u_c)) + u_c) + E_p \\ y &= C_p x_p \end{aligned} \tag{19}$$

where $A_p = A_p(\tilde{\phi})$, $B_p = B_p(\tilde{\phi})$, and $E_p = E_p(\tilde{\phi})$ will be considered from now on for the sake of simplicity. The proposed extended vector x including the inner loop dynamics is defined as

$$x = \begin{bmatrix} x_p \\ x_C \\ x_F \end{bmatrix} \tag{20}$$

Then, the full system described by the plant, prefilter, and controller has r as input (prefilter input), and y as output (plant output). Then, x_F^+ and x_C^+ can be described as function of x and r as follows

$$x_F^+ = A_F x_F + B_F r = \begin{bmatrix} 0 & 0 & A_F \end{bmatrix} x + B_F r \tag{21}$$

$$\begin{aligned} x_C^+ &= A_C x_C + B_C(r_F - y) = \\ &= A_C x_C + B_C(C_F x_F + D_F r) - B_C C_p x_p = \\ &= -B_C C_P x_p + A_C x_C + B_C C_F x_F + B_C D_F r = \\ &= \begin{bmatrix} -B_C C_p & A_C & B_C C_F \end{bmatrix} x + B_C D_F r \end{aligned} \tag{22}$$

being the control signal u obtained in the following way

$$\begin{aligned} u &= C_C x_C + D_C(r_F - y) = \\ &= C_C x_C + D_C(C_F x_F + D_F r) - D_C C_p x_p = \\ &= -D_C C_p x_p + C_C x_C + D_C C_F x_F + D_C D_F r = \\ &= \begin{bmatrix} -D_C C_p & C_C & D_C C_F \end{bmatrix} x + D_C D_F r = \\ &= C_u x + D_u r \end{aligned} \tag{23}$$

In this way, using the new plant representation (19), and the prefilter and controller state-space dynamics (21-22), the closed-loop state space representation for the inner loop is described as

$$\begin{aligned} x^+ &= Ax + B_u \sigma\left(\frac{C_u}{L_s} x + \frac{D_u r - u_c}{L_s}\right) + E + B_r r \\ y &= C_y x \end{aligned} \tag{24}$$

where

$$A = \begin{bmatrix} A_p & 0 & 0 \\ -B_C C_p & A_C & B_C C_F \\ 0 & 0 & A_F \end{bmatrix}, \; B_u = \begin{bmatrix} L_s B_p \\ 0 \\ 0 \end{bmatrix},$$

$$E = \begin{bmatrix} E_p + B_p u_c \\ 0 \\ 0 \end{bmatrix}, B_r = \begin{bmatrix} 0 \\ B_C D_F \\ B_F \end{bmatrix}, C_y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

3. Robust Constrained LMI-based approach.

3.1. *Tracking problem. Preliminary ideas.*

Most of the results obtained for constrained MPC using LMI have been proposed to regulate the system to the origin. In this way, the results obtained in [24] can be used to calculate a control law $r = K_s x$ for the plant (24) considering the system free of disturbances ($E(\tilde{\phi}) = 0$) and regulating to the origin. Notice that $x = 0$ is an equilibrium for the system and for all value of $\tilde{\phi}$. Therefore, K_s can be calculated ensuring robust stability and in such a way that the control law $r = K_s x$ regulates to the origin for all posible initial conditions and any value of $\tilde{\phi}$ [24], [25]. However, one of the objectives considered in the approach presented in this work is to make the output y reach the reference value w. Therefore, the problem formulation must be oriented to this objective. This work presents some preliminary ideas based on the extensions proposed in [24] for set-point tracking.

Firstly, it is necessary to notice that due to the dependence on the parametric vector $\tilde{\phi}$, it is imposible to find static values for x and r (x_e and r_e) such that the system finds a unique equilibrium for all values of the parametric vector $\tilde{\phi}$. In this work in order to address this problem, the following control law is proposed

$$r = r_e + K_s(x - x_e) \tag{25}$$

where x_e, r_e, and K_s will be obtained in such a way that the performance of the closed-loop system is enhanced and the system evolution is ensured to be inside an invariant ellipsoid containing the problem initial conditions.

Substituting r in equation (24) by the desired control law $r = r_e + K_s(x - x_e)$, the following expression is obtained

$$\begin{aligned} x^+ &= Ax + B_u \sigma\left(\tfrac{C_u x + D_u(r_e + K_s(x - x_e)) - u_c}{L_s}\right) + E + \\ &\quad + B_r(r_e + K_s(x - x_e)) \end{aligned} \tag{26}$$

Then, if the change $\bar{x} = x - x_e$ is considered, and so $\bar{x}^+ = x^+ - x_e$, the system dynamics can be represented as

$$\begin{aligned} \bar{x}^+ &= A(\bar{x} + x_e) + B_u \sigma\left(\tfrac{(C_u + D_u K_s)\bar{x} + C_u x_e + D_u r_e - u_c}{L_s}\right) + E + \\ &\quad + B_r(r_e + K_s \bar{x}) - x_e \\ y &= C_y x = C_y \bar{x} + C_y x_e \end{aligned}$$

Finally, if the following changes are performed

$$K_u = C_u + D_u K_s, \; d_u = C_u x_e + D_u r_e - u_c$$

it is obtained that

$$\begin{aligned} \bar{x}^+ &= A_{Bc}\bar{x} + B_{Bc}\sigma\left(\frac{K_u \bar{x} + d_u}{L_s}\right) + E_{Bc} \\ y &= C_y \bar{x} + C_y x_e \end{aligned} \tag{27}$$

where

$$A_{Bc} = (A + B_r K_s), \; B_{Bc} = B_u, \; E_{Bc} = E + A x_e + B_r r_e - x_e$$

Define $\wp(P_s, \rho) = \bar{x}^T P_s \bar{x} \leq \rho$ as an ellipsoid where $\bar{x}_0 \in \wp(P_s, \rho)$ with $\bar{x}_0 = x_0 - x_e$. In this way, the tracking problem assuring constrained robust stability for the system (27) will be solved using LMI and fulfilling the following objectives:

(i) Firstly, the decision variables x_e, r_e, K_s, P_s, and ρ are calculated in such a way that the ellipsoid $\wp(P_s, \rho)$ is invariant containing the system initial conditions $\bar{x}_0$ and using the control law $r = r_e + K_s(x - x_e)$.

(ii) After that, new constrains will be included in order to fulfill a certain performance criteria.

These objectives will be addressed in next sections, but the way in which the saturation term presented in (27) can be taken into account will be addressed before.

3.1.1. *Linear Difference Inclusion of the saturation term*

Notice that due to the input saturation, a nonlinear term appears in the system dynamics, $\sigma(\frac{K_u\bar{x}+d_u}{L_s})$. This nonlinear term can be approximated using the *Linear Difference Inclusion* (LDI) results obtained in [26] and [27] where it is shown that, if $b \in \mathbb{R}$ satisfies $|b| \leq 1$ then

$$\sigma(a) \ \in \ Co\{a, b\}, \ \forall a \in \mathbb{R}$$

being Co the convex hull. In particular, if $|H_s\bar{x} + h| \leq 1$ $\forall \bar{x} \in \wp(P_s, \rho)$ then

$$\sigma(\tfrac{K_u\bar{x}+d_u}{L_s}) \ \in \ Co\{\tfrac{K_u\bar{x}+d_u}{L_s}, H_s\bar{x}+h\}, \ \forall(\tfrac{K_u\bar{x}+d_u}{L_s}) \in \mathbb{R}, \forall \bar{x} \in \wp(P_s,\rho)$$

Therefore, each objective commented above will be translated to analyze if it satisfies the extremes of the convex hull

$$\bar{x}^+ = A_{Bc}\bar{x} + B_{Bc}\left(\frac{K_u\bar{x} + d_u}{L_s}\right) + E_{Bc} \tag{28}$$

$$\bar{x}^+ = A_{Bc}\bar{x} + B_{Bc}(H_s\bar{x} + h) + E_{Bc} \tag{29}$$

as will be shown in next sections.

On the other hand, the inequality $|H_s\bar{x} + h| \leq 1$ must be considered. This inequality can be translated to a LMI in order to be included in the final optimization problem. The inequality can be expressed as two inequalities in the following way

$$H_s\bar{x} + h \leq 1 \ \Rightarrow \ H_s\bar{x} \leq 1 - h, \ \ \forall \bar{x} \in \wp(P_s, \rho) \tag{30}$$

$$H_s\bar{x} + h \geq -1 \ \Rightarrow \ H_s\bar{x} \geq -1 - h, \ \ \forall \bar{x} \in \wp(P_s, \rho) \tag{31}$$

where these inequalities must be satisfied in the ellipsoid $\wp(P_s, \rho)$. In the next section, this ellipsoid will be forced to be invariant containing the system initial conditions.

Considering the first inequality (30) and using the $S-procedure$, it is equivalent to study the existence of $\lambda_2 \geq 0$ such that (Farkas lemma [25])

$$H_s\bar{x} + h + \lambda_2(\rho - \bar{x}^T P_s\bar{x}) \leq 1, \ \ \forall \bar{x} \tag{32}$$

This can be expressed as

$$\begin{bmatrix} 1 \\ -\bar{x} \end{bmatrix}^T \begin{bmatrix} \lambda_2\rho + h - 1 & -\frac{1}{2}H_s \\ -\frac{1}{2}H_s^T & -\lambda_2 P_s \end{bmatrix} \begin{bmatrix} 1 \\ -\bar{x} \end{bmatrix} \leq 0, \ \forall \bar{x} \tag{33}$$

or equivalently

$$\begin{bmatrix} 1 - h - \lambda_2\rho & \frac{1}{2}H_s \\ \frac{1}{2}H_s^T & \lambda_2 P_s \end{bmatrix} > 0 \tag{34}$$

Then, pre- and post-multiplying by $diag[I \ \ 2P_s^{-1}]$ and making $W = P_s^{-1}$, $V = H_sW$, it results

$$\begin{bmatrix} 1 - h - \lambda_2\rho & V \\ V^T & 4\lambda_2 W \end{bmatrix} > 0 \tag{35}$$

The same procedure can be applied for the second inequality (31) obtaining

$$\begin{bmatrix} 1 + h - \lambda_2\rho & V \\ V^T & 4\lambda_2 W \end{bmatrix} > 0 \tag{36}$$

For a fixed value of λ_2, notice that the obtained matrix inequalities are LMI in the decision variables ρ, P_s, H_s, and h. The procedure to find correct values for λ_2 will be described later. As consequence of the previous results, the following property is proposed:

Property 1 *Suppose that there exits $\lambda_2 \geq 0$ such that LMI (35) and (36) are fulfilled, then:*

$$\sigma(\frac{K_u\bar{x} + d_u}{L_s}) \ \in \ Co\{\frac{K_u\bar{x} + d_u}{L_s}, H_s\bar{x} + h\}, \ \forall \bar{x} \in \wp(P_s, \rho),$$

where $P = W^{-1}$ and $H = W^{-1}V$.

3.2. *Robust invariant ellipsoid*

As commented previously, one of the objectives is to calculate the decision variables x_e, r_e, K_s, P_s, and ρ in such a way that the ellipsoid $\wp(P_s, \rho)$ is invariant including the system initial conditions and using the control law $r = r_e + K_s(x - x_e)$. Therefore, in order to ensure the ellipsoid being invariant the following inequality must be fulfilled

$$(\bar{x}^+)^T P_s(\bar{x}^+) \leq \rho, \ \ \forall \bar{x} \in \wp(P_s, \rho) \tag{37}$$

This problem can be reformulated using $S-procedure$ as follows:

$$(\bar{x}^+)^T P_s(\bar{x}^+) + \lambda_1(\rho - \bar{x}^T P_s\bar{x}) \leq \rho, \ \ \forall \bar{x}, \ \lambda_1 \geq 0 \tag{38}$$

or equivalently

$$(\bar{x}^+)^T P_s(\bar{x}^+) - \lambda_1\bar{x}^T P_s\bar{x} + \rho(\lambda_1 - 1) \leq 0, \ \ \forall \bar{x}, \ \lambda_1 \geq 0 \tag{39}$$

Lets consider the following property:

Property 2 *Suppose that $P_s > 0$, then*

$$\vartheta^T P_s\vartheta \geq v^T P_s v + 2v^T P_s(\vartheta - v) = -v^T P_s v + 2v^T P_s\vartheta$$

and

$$\vartheta^T P_s\vartheta = \max_v\{-v^T P_s v + 2v^T P_s\vartheta\} \tag{40}$$

Therefore, using the previous property and the closed-loop system dynamics (27), the inequality (39) results

$$\begin{aligned} -v^T P_s v + 2v^T P_s(A_{Bc}\bar{x} + B_{Bc}\sigma(\tfrac{K_u\bar{x}+d_u}{L_s}) + E_{Bc}) - \\ -\lambda_1\bar{x}^T P_s\bar{x} + \rho(\lambda_1 - 1) \leq 0 \end{aligned} \tag{41}$$

where this inequality must be satisfied $\forall \bar{x}$ and $\forall v$.

Notice that in order to address this problem and demonstrate that the system evolution belongs to an invariant ellipsoid, it is necessary to obtain a LDI of the saturation term as shown in the previous section (see Property 1). Therefore, the inequality (41) must be satisfied for the extremes of the convex hull, $\frac{K_u\bar{x}+d_u}{L_s}$ and $H_s\bar{x}+h$.

3.2.1. Case $\frac{K_u\bar{x}+d_u}{L_s}$

Using Property 2 (40) and (28) in (39) is obtained that

$$-v^TP_sv+2v^TP_s(A_{Bc}\bar{x}+B_{Bc}\frac{K_u\bar{x}+d_u}{L_s}+E_{Bc})-$$
$$-\lambda_1\bar{x}^TP_s\bar{x}+\rho(\lambda_1-1)\le 0, \forall\bar{x},\ \forall v$$

The matrix representation of the previous inequality is given by

$$\nu^T\begin{bmatrix}\rho(\lambda_1-1) & * & * \\ 0 & -\lambda_1P_s & * \\ -P_s(E_{Bc}+B_{LBc}d_u) & -P_s(A_{Bc}+B_{LBc}K_u) & -P_s\end{bmatrix}\nu\le 0 \quad (42)$$

$\forall\bar{x},\ \forall v$, where $B_{LBc}=\frac{1}{L_s}B_{Bc}$, $\nu=\begin{bmatrix}1 & \bar{x} & -v\end{bmatrix}^T$, and $*$ represents the transpose of the symmetric term. So,

$$\begin{bmatrix}\rho(1-\lambda_1) & * & * \\ 0 & \lambda_1P_s & * \\ P_s(E_{Bc}+B_{LBc}d_u) & P_s(A_{Bc}+B_{LBc}K_u) & P_s\end{bmatrix}>0 \quad (43)$$

Pre- and post-multiplying by $diag[I\ P_s^{-1}\ P_s^{-1}]$ and making $W=P_s^{-1}$

$$\begin{bmatrix}\rho(1-\lambda_1) & * & * \\ 0 & \lambda_1W & * \\ (E_{Bc}+B_{LBc}d_u) & (A_{Bc}+B_{LBc}K_u)W & W\end{bmatrix}>0 \quad (44)$$

Considering that $E_{Bc}=E+Ax_e+B_rr_e-x_e$, $B_{LBc}=\frac{1}{L_s}B_{Bc}=\frac{1}{L_s}B_u$ and $d_u=C_ux_e+D_ur_e-u_c$, it results that

$$E_{Bc}+B_{LBc}d_u=A_{x_e}x_e+B_{r_e}r_e+E_e$$

where $A_{x_e}=A-I+(\frac{1}{L_s})B_uC_u$, $B_{r_e}=B_r+(\frac{1}{L_s})B_uD_u$ and $E_e=E-(\frac{1}{L_s})B_uu_c$. On the other hand, reminding that $A_{Bc}=A+B_rK_s$ and $K_u=C_u+D_uK_s$, it is obtained that

$$A_{Bc}W+B_{LBc}K_uW=A_wW+B_yY$$

where $A_w=A+\frac{1}{L_s}B_uC_u$, $B_y=B_r+\frac{1}{L_s}B_uD_u$, and $Y=K_sW$.

Therefore, the final LMI results as follows

$$\begin{bmatrix}\rho(1-\lambda_1) & * & * \\ 0 & \lambda_1W & * \\ A_{x_e}x_e+B_{r_e}r_e+E_e & A_wW+B_yY & W\end{bmatrix}>0 \quad (45)$$

3.2.2. *Case $H_s\bar{x}+h$*

In order to solve the case for $H_s\bar{x}+h$, it is easy to see that the same previous LMI is obtained only substituting K_u by H_s, d_u by h, and B_{LBc} by B_{Bc}

$$\begin{bmatrix}\rho(1-\lambda_1) & * & * \\ 0 & \lambda_1W & * \\ (E_{Bc}+B_{Bc}h) & (A_{Bc}+B_{Bc}H_s)W & W\end{bmatrix}>0 \quad (46)$$

Then, knowing that $E_{Bc}=E+Ax_e+B_rr_e-x_e$, $B_{Bc}=B_u$ and $A_{Bc}=A+B_rK_s$:

$$E_{Bc}+B_{Bc}h=A_{nlxe}x_e+B_{nlre}r_e+B_hh+E_{nle}$$

$$(A_{Bc}+B_{Bc}H_s)W=A_{nlW}W+B_{nlY}Y+B_vV$$

where $A_{nlxe}=(A-I)$, $B_{nlre}=B_r$, $B_h=B_u$, $E_{nle}=E$, $A_{nlW}=A$, $B_{nlY}=B_r$, $B_v=B_u$, $Y=K_sW$, and $V=H_sW$.

So, the resulting LMI is

$$\begin{bmatrix}\rho(1-\lambda_1) & * & * \\ 0 & \lambda_1W & * \\ A_{nlxe}x_e+B_{nlre}r_e+B_hh+E_{nle} & A_{nlW}W+B_{nlY}Y+B_vV & W\end{bmatrix}>0 \quad (47)$$

For a fixed value of λ_1, notice that the obtained matrix inequalities are LMI in the decision variables x_e, r_e, ρ, P_s, K_s, H_s, and h. The procedure to find correct values for λ_1 will be described later.

Property 3 *Suppose that there exists $\lambda_1\ge 0$ and $\lambda_2\ge 0$ such that the LMI (35), (36), (45) and (47) are fulfilled. Then, $\wp(P_s,\rho)$ is an invariant ellipsoid using the control law $r=r_e+K_s(x-x_e)$ and containing the system initial conditions, where $P=W^{-1}$ and $K=W^{-1}Y$.*

Remark 1 *Notice that the previous LMI depends multilineally on the parametric vector $\tilde{\phi}\in\Phi$ due to the dependence of $A_p=A_p(\tilde{\phi})$, $B_p=B_p(\tilde{\phi})$, and $E_p=E_p(\tilde{\phi})$. Then, properties 1 and 3 must be satisfied for all extreme plants of the hyperrectangle Φ.*

3.3. *Including performance inequality.*

Consider the representation of system (27) for the instant time k

$$\bar{x}_k^+=A_{Bc}\bar{x}_k+B_{Bc}\sigma\left(\frac{K_u\bar{x}_k+d_u}{L_s}\right)+E_{Bc} \quad (48)$$
$$y_k=C_y\bar{x}_k+C_yx_e$$

and suppose the following equality

$$w=C_yx_e \quad (49)$$

For an initial condition x_0 and the reference w, it is desired to calculate the system input r_k by the law $r_k=$

$r_e + K_s(x - x_e)$ such that the the following functional is minimized

$$J = \sum_{k=0}^{N} (y_k - w)^\top Q (y_k - w) + \bar{x}^T K_u^\top R_u K_u \bar{x} \tag{50}$$

where $Q > 0$ and $R_u > 0$ are symmetric matrices positive semi-defined.

From the equality (49), it results that the functional J can be rewritten as

$$J = \sum_{k=0}^{N} \bar{x}_k^\top C_y^\top Q C_y \bar{x}_k + \bar{x}^T K_u^\top R_u K_u \bar{x} \tag{51}$$

Defining $L_J(\bar{x}_k) = \bar{x}_k^\top C_y^\top Q C_y \bar{x}_k + \bar{x}^T K_u^\top R_u K_u \bar{x}$, it results that

$$J = \sum_{k=0}^{N} L_J(\bar{x}_k) \tag{52}$$

In the following property a strategy is defined for a correct selection of K_s, x_e and r_e in order to fulfill the performance criteria (50).

Property 4 *Suppose that*

$$\bar{x}_{k+1}^\top P_s \bar{x}_{k+1} - \bar{x}_k^\top P_s \bar{x}_k \leq -L_J(\bar{x}_k) + \gamma, \quad \forall \tilde{\phi} \in \Phi, \ \forall \bar{x}$$

and that an initial condition is equal to x_0. Suppose also that the control law $r_k = r_e + K_s(x_k - x_e)$ is applied to the system, then

$$J \leq \bar{x}_0^\top P_s \bar{x}_0 + N\gamma$$

where $\bar{x}_0 = x_0 - x_e$.

Proof 1 *The assumption of the property leads to*

$$\bar{x}_{k+1}^\top P_s \bar{x}_{k+1} - \bar{x}_k^\top P_s \bar{x}_k \leq -L_J(\bar{x}_k) + \gamma, \quad \forall \tilde{\phi} \in \Phi, \ \forall k \geq 0$$

Therefore,

$$\begin{aligned}
\bar{x}_1^\top P_s \bar{x}_1 - \bar{x}_0^\top P_s \bar{x}_0 &\leq -L_J(\bar{x}_0) + \gamma \\
\bar{x}_2^\top P_s \bar{x}_2 - \bar{x}_1^\top P_s \bar{x}_1 &\leq -L_J(\bar{x}_1) + \gamma \\
&\vdots \\
\bar{x}_N^\top P_s \bar{x}_N - \bar{x}_{N-1}^\top P_s \bar{x}_{N-1} &\leq -L_J(\bar{x}_{N-1}) + \gamma \\
\bar{x}_{N+1}^\top P_s \bar{x}_{N+1} - \bar{x}_N^\top P_s \bar{x}_N &\leq -L_J(\bar{x}_N) + \gamma
\end{aligned}$$

If the previous inequalities are added, it is obtained that

$$\begin{aligned}
\bar{x}_{N+1}^\top P_s \bar{x}_{N+1} - \bar{x}_0^\top P_s \bar{x}_0 &\leq -J + N\gamma \\
J &\leq \bar{x}_0^\top P_s \bar{x}_0 + N\gamma - \bar{x}_{N+1}^\top P_s \bar{x}_{N+1} \\
J &\leq \bar{x}_0^\top P_s \bar{x}_0 + N\gamma
\end{aligned}$$

So, from the previous property the following optimization problem can be proposed

$$\begin{aligned}
\min_{P_s, K_s, x_e, r_e, \gamma} \quad & \bar{x}_0^\top P_s \bar{x}_0 + N\gamma \qquad \forall \tilde{\phi} \in \Phi, \ \forall \bar{x} \\
s.t. \quad & (\bar{x}^+)^T P_s (\bar{x}^+) - \bar{x} P_s \bar{x} < -\bar{x}^\top C_y^\top Q C_y \bar{x} - \bar{x}^T K_u^\top R_u K_u \bar{x} + \gamma
\end{aligned} \tag{53}$$

in order to calculate the control law that minimizes an upper limit of the functional. Notice that N is a design parameter absent from problem constraints. If N is very large, the problem solution will tend to minimize γ. The following interpretation can be considered for γ

$$\lim_{N\to\infty} \frac{J}{N} \leq \lim_{N\to\infty} \frac{\bar{x}_0^\top P_s \bar{x}_0 + N\gamma}{N} = \gamma \tag{54}$$

Therefore, if N is very large the initial transitory is almost not considered (the initial condition $\bar{x}_0$ doesn't play a relevant role) and the emphasis is placed on improving the future behavior. On the other hand, if N takes very small values, the initial condition and the initial transitory gains relevance.

Then, the problem (53) can be reformulated as

$$\begin{aligned}
\min_{P_s, K_s, x_e, r_e, \gamma, \alpha_s} \quad & \alpha_s \\
s.a. \quad & \bar{x}_0^T P_s \bar{x}_0 + N\gamma < \alpha_s \\
& (\bar{x}^+)^T P_s (\bar{x}^+) - \bar{x}^T P_s \bar{x} < -\bar{x}^T C_y^T Q C_y \bar{x} - \\
& -\bar{x}^T K_u^T R_u K_u \bar{x} + \gamma
\end{aligned} \tag{55}$$

The problem inequalities will be translated to LMI form in order to address the optimization problem. Firstly, the upper inequality is considered

$$\bar{x}_0^T P_s \bar{x}_0 + N\gamma < \alpha_s \tag{56}$$

This can be easily expressed as a LMI using the *Schur* complement [25] in the form

$$\begin{bmatrix} \alpha_s - N\gamma & \bar{x}(0)^T \\ \bar{x}(0) & W \end{bmatrix} \geq 0 \tag{57}$$

On the other hand, and remembering the presence of the saturation term in (48), the another inequality

$$(\bar{x}^+)^T P_s (\bar{x}^+) - \bar{x}^T P_s \bar{x} < -\bar{x}^T C_y^T Q C_y \bar{x} - \bar{x}^T K_u^T R_u K_u \bar{x} + \gamma \tag{58}$$

must be satisfied for two extreme vertices of the LDI, $\frac{K_s \bar{x} + d_u}{L_s}$ and $H_s \bar{x} + h$, in the same way that for the invariant ellipsoid. The first step consists in using Property 2 on the previous inequality, where it is obtained that

$$-v^T P_s v + 2v^T P_s (A_{Bc} \bar{x} + B_{Bc} \sigma(\tfrac{K_u \bar{x} + d_u}{L}) + E_{Bc}) - \bar{x}^T P_s \bar{x} < -\bar{x}^T C_y^T Q C_y \bar{x} - \bar{x}^T K_u^T R_u K_u \bar{x} + \gamma$$

Following a similar procedure that in section 3.2, this inequality results in the following LMIs for the two extreme vertices of the LDI (the detailed procedure can be found in Appendix A [22])

$$\begin{bmatrix}
\gamma & * & * & * & * \\
0 & W & * & * & * \\
A_{xe} x_e + B_{re} r_e + E_e & A_w W + B_y Y & W & * & * \\
0 & Q^{1/2} C_y W & 0 & I & * \\
0 & R_W W + R_Y Y & 0 & 0 & I
\end{bmatrix} > 0 \tag{59}$$

$$\begin{bmatrix} \gamma & * & * & * & * \\ 0 & W & * & * & * \\ A_{nlxe}x_e+B_{nlre}r_e+B_h h+E_{nle} & A_{nlW}W+B_{nlY}Y+B_vV & W & * & * \\ 0 & Q^{1/2}C_yW & 0 & I & * \\ 0 & R_vV & 0 & 0 & I \end{bmatrix}>0 \quad (60)$$

where $A_{x_e} = A - I + (\frac{1}{L_s})B_uC_u$, $B_{r_e} = B_r + (\frac{1}{L_s})B_uD_u$, $E_e = E - (\frac{1}{L_s})B_uu_c$, $A_w = A + \frac{1}{L_s}B_uC_u$, $B_y = B_r + \frac{1}{L_s}B_uD_u$, $Y = K_sW$ and

$$R_u^{1/2}\,K_u\,W = R_u^{1/2}(C_u + D_uK_s)W = \\ = R_u^{1/2}C_uW + R_u^{1/2}D_uK_sW = R_wW + R_yY$$

Finally, the equality (49), which was supposed before, must be included in the optimization problem. Therefore, the optimization problem has been reformulated to minimize the value of α_s subject to a set of LMI. The following section describes the final optimization problem and the different obtained LMI.

3.4. *Final optimization problem*

Notice that in section 2.0.2, it was considered that $A_p = A_p(\tilde{\phi})$, $B_p = B_p(\tilde{\phi})$, and $E_p = E_p(\tilde{\phi})$ for simplifying reasons. That is, it is necessary to remind that the matrices of the plant depend multilineally on the parametric vector $\tilde{\phi}$. In this way, the previous LMI that were formulated for the nominal case, must be satisfied for all extreme values of the hyperrectangle Φ. Hence, the final problem can be formulated to calculate the decision variables x_e, r_e, K_s, P_s, ρ, H_s, and h, in such a way that using the control law $r = r_e + K_s(x - x_e)$, $\wp(P_s, \rho)$ is an invariant ellipsoid and the system fulfills the performance criteria given by J (50). The final optimization problem is given by

$$\min_{P_s,K_s,x_e,r_e,\gamma,\alpha_s} \quad \alpha_s \quad (61)$$
$$s.a.\ \bar{x}_0^T P_s\bar{x}_0 + N\gamma < \alpha_s$$
$$(\bar{x}^+)^T P_s(\bar{x}^+) - \bar{x}^T P_s\bar{x} < -\bar{x}^T C_y^T QC_y\bar{x} - \\ -\bar{x}^T K_u^T R_uK_u\bar{x} + \gamma$$

Then, considering the results obtained in previous section, a conservative way to solve the optimization problem consists in solving the following constraints

$$w = C_yx_e \quad (62)$$

$$\begin{bmatrix} \alpha_s - N\gamma & \bar{x}(0)^\top \\ \bar{x}(0) & W \end{bmatrix} > 0 \quad (63)$$

$$\begin{bmatrix} \rho & \bar{x}(0)^\top \\ \bar{x}(0) & W \end{bmatrix} > 0 \quad (64)$$

$$\begin{bmatrix} \gamma & * & * & * & * \\ 0 & W & * & * & * \\ A_{xe}(\tilde{\phi})x_e+B_{re}(\tilde{\phi})r_e+E_e(\tilde{\phi}) & A_w(\tilde{\phi})W+B_y(\tilde{\phi})Y & W & * & * \\ 0 & Q^{1/2}C_y(\tilde{\phi})W & 0 & I & * \\ 0 & R_W(\tilde{\phi})W+R_YY & 0 & 0 & I \end{bmatrix}>0 \quad (65)$$

$$\begin{bmatrix} \rho(1-\lambda_1) & * & * \\ 0 & \lambda_1W & * \\ A_{x_e}(\tilde{\phi})x_e+B_{r_e}(\tilde{\phi})r_e+E_e(\tilde{\phi}) & A_w(\tilde{\phi})W+B_YY & W \end{bmatrix}>0 \quad (66)$$

$$\begin{bmatrix} \gamma & * & * & * & * \\ 0 & W & * & * & * \\ \begin{matrix} A_{nlxe}(\tilde{\phi})x_e+B_{nlre}(\tilde{\phi})r_e+ \\ +B_h(\tilde{\phi})h+E_{nle}(\tilde{\phi}) \end{matrix} & \begin{matrix} A_{nlW}(\tilde{\phi})W+B_{nlY}Y+ \\ +B_v(\tilde{\phi})V \end{matrix} & W & * & * \\ 0 & Q^{1/2}C_y(\tilde{\phi})W & 0 & I & * \\ 0 & R_vV & 0 & 0 & I \end{bmatrix}>0 \quad (67)$$

$$\begin{bmatrix} \rho(1-\lambda_1) & * & * \\ 0 & \lambda_1W & * \\ \begin{matrix} A_{nlxe}(\tilde{\phi})x_e+B_{nlre}(\tilde{\phi})r_e+ \\ +B_h(\tilde{\phi})h+E_{nle}(\tilde{\phi}) \end{matrix} & \begin{matrix} A_{nlW}(\tilde{\phi})W+B_{nlY}Y+ \\ +B_v(\tilde{\phi})V \end{matrix} & W \end{bmatrix}>0 \quad (68)$$

$$\begin{bmatrix} 1 - h - \lambda_2\rho & V \\ V^\top & 4\lambda_2W \end{bmatrix} \geq 0 \quad (69)$$

$$\begin{bmatrix} 1 + h - \lambda_2\rho & V \\ V^\top & 4\lambda_2W \end{bmatrix} \geq 0 \quad (70)$$

where it is necessary to incorporate constrains for each extreme value of the hypercube Φ. Also, as observed from the resulting constraints, some of them are BMI (Bilinear Matrix Inequalities) containing different bilinear terms $\rho(1 - \lambda_1)$, λ_1W, $\lambda_2\rho$), and $4\lambda_2W$. So, in order to obtain a stable MPC controller with good performance, it is necessary to choose λ_1 and λ_2 in a convenient way.

Property 5 *Suppose that there exist $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ such that the constraints (62), (63), (64), (65), (66), (67), (68), (69), and (70) are feasible for every extreme of the* *hypercube Φ. Then, there exits a control law $r = r_e + K_s(x - x_e)$ providing that $\wp(P_s, \rho)$ is an invariant ellipsoid and the system fulfills the performance criteria given by J (50), where $P = W^{-1}$ and $K = W^{-1}Y$.*

3.5. *Branch & Bound algorithm for bilinear terms*

Due to the presence of bilinear terms in the final optimization problem, a *Branch & Bound* algorithm is proposed to find the optimal solution. Before describing the algorithm, some aspects must be considered [22]:

- From the constraints it is easy to see that $\lambda_1 \in [\underline{\lambda_1}, \overline{\lambda_1}) = [0,1)$ and $\lambda_2 \in [\underline{\lambda_2}, \overline{\lambda_2}) = [0,\infty)$, respectively. Notice that $\underline{\lambda_i}$ and $\overline{\lambda_i}$ represent the minimum and maximum values for λ_i with $i = 1,2$.
- A *lower bound* solution is that obtained considering that there exist $\lambda_1 \in [\underline{\lambda_1}, \overline{\lambda_1})$ and $\lambda_2 \in [\underline{\lambda_2}, \overline{\lambda_2})$ such that the constraints (62), (63), (64), (65), and (67) are feasible, and also the following constrains are satisfied

$$\begin{bmatrix} \rho(1-\underline{\lambda_1}) & * & * \\ 0 & \overline{\lambda_1}W & * \\ A_{x_e}(\tilde{\phi})x_e + B_{r_e}(\tilde{\phi})r_e + E_e(\tilde{\phi}) & A_w(\tilde{\phi})W + B_Y Y & W \end{bmatrix} > 0 \tag{71}$$

$$\begin{bmatrix} \rho(1-\underline{\lambda_1}) & * & * \\ 0 & \overline{\lambda_1}W & * \\ \begin{matrix} A_{nlxe}(\tilde{\phi})x_e + B_{nlre}(\tilde{\phi})r_e + \\ + B_h(\tilde{\phi})h + E_{nle}(\tilde{\phi}) \end{matrix} & \begin{matrix} A_{nlW}(\tilde{\phi})W + B_{nlY}Y + \\ + B_v(\tilde{\phi})V \end{matrix} & W \end{bmatrix} > 0 \tag{72}$$

$$\begin{bmatrix} 1 - h - \underline{\lambda_2}\rho & V \\ V^T & 4\overline{\lambda_2}W \end{bmatrix} > 0 \tag{73}$$

$$\begin{bmatrix} 1 + h - \underline{\lambda_2}\rho & V \\ V^T & 4\overline{\lambda_2}W \end{bmatrix} > 0 \tag{74}$$

The list of lower bound solutions is called O_l.

- An *upper bound* solution is that obtained for specific values of λ_1 and λ_2. Given a lower bound solution defined by $[\underline{\lambda_1}, \overline{\lambda_1})$ and $[\underline{\lambda_2}, \overline{\lambda_2})$, specific values for λ_1 and λ_2 are obtained as $\lambda_1 = (\underline{\lambda_1} + \overline{\lambda_1})/2$ and $\lambda_2 = (\underline{\lambda_2} + \overline{\lambda_2})/2$. The list of upper bound solutions is called P_l.
- The ranges $[\underline{\lambda_1}, \overline{\lambda_1})$ and $[\underline{\lambda_2}, \overline{\lambda_2})$ define an initial square which determines the search space. Such square will successively be divided until the optimal solution is obtained.

Therefore, the proposed *Branch & Bound* algorithm presents the following steps:

(i) Initialize $O_l = \{\}$, $P_l = \{\}$, $[\underline{\lambda_1}, \overline{\lambda_1}) = [0,1)$, and $[\underline{\lambda_2}, \overline{\lambda_2}) = [0,\infty)$.

(ii) Obtain the initial node n_o as an optimistic solution using the initial values for $[\underline{\lambda_1}, \overline{\lambda_1})$ and $[\underline{\lambda_2}, \overline{\lambda_2})$; $O_l = \{n_o\}$.

(iii) The best node from the optimistic list O_l is selected obtaining $[\underline{\lambda_1}', \overline{\lambda_1}')$ and $[\underline{\lambda_2}', \overline{\lambda_2}')$:
  (a) For odd iterations, the node will be that containing the square with bigger area.
  (b) For even iterations, the node will be that with better performance.

(iv) New upper bound solution is calculated using the best node calculated in the previous step, $[\underline{\lambda_1}', \overline{\lambda_1}')$ and $[\underline{\lambda_2}', \overline{\lambda_2}')$. The solution is calculated trying to solve the optimization problem for $\lambda_1 = \lambda'_{1m} = (\underline{\lambda_1}' + \overline{\lambda_1}')/2$ and $\lambda_2 = \lambda'_{2m} = (\underline{\lambda_2}' + \overline{\lambda_2}')/2$. If the optimization problem has solution for these values of λ_1 and λ_2, the new value is included in the list of pessimistic solutions P_l. Then a local search is performed trying to improve the found solution. The local search is based on the following heuristic steps:
  (a) The values obtained for λ_1 and λ_2, and the obtained performance solution are considered as initial variables.
  (b) Solve the optimization problem (61) using λ_1 and λ_2 and obtaining values for W and ρ. If a better performance solution is obtained, then the new upper bound solution is included in the upper bound list P_l. The best performance solution is updated with the new one.
  (c) Reformulate the LMI considering λ_1 and λ_2 as decision variables and, W and ρ as constant values. Solve the optimization problem obtaining new values for λ_1 and λ_2.
  (d) Return to step (b) while better performance solutions are obtained or until a finite number of iterations is reached.

(v) Remove worst lower bound solutions. The lower bound solutions with worst performance than the best upper bound one are removed from the lower bound list O_l.

(vi) New lower bound solutions are obtained. That is, four new nodes are calculated from the node obtained in the previous step as follows

$$n_1 = \{[\underline{\lambda_1}', \lambda'_{1m}], [\underline{\lambda_2}', \lambda'_{2m}]\},$$
$$n_2 = \{[\underline{\lambda_1}', \lambda'_{1m}], [\lambda'_{2m}, \overline{\lambda_2}']\},$$
$$n_3 = \{[\lambda'_{1m}, \overline{\lambda_1}'], [\underline{\lambda_2}', \lambda'_{2m}]\},$$
$$n_4 = \{[\lambda'_{1m}, \overline{\lambda_1}'], [\lambda'_{2m}, \overline{\lambda_2}']\}.$$

The lower bound solution calculated for n_i, $i = 1,..,4$ will be removed if the solution is empty (the optimization problem is not feasible), the obtained performance is worst than the best upper bound solution, or the square defined by the associated ranges on λ_1 and λ_2 is too small. Otherwise, it will be included in the lower bound list O_l.

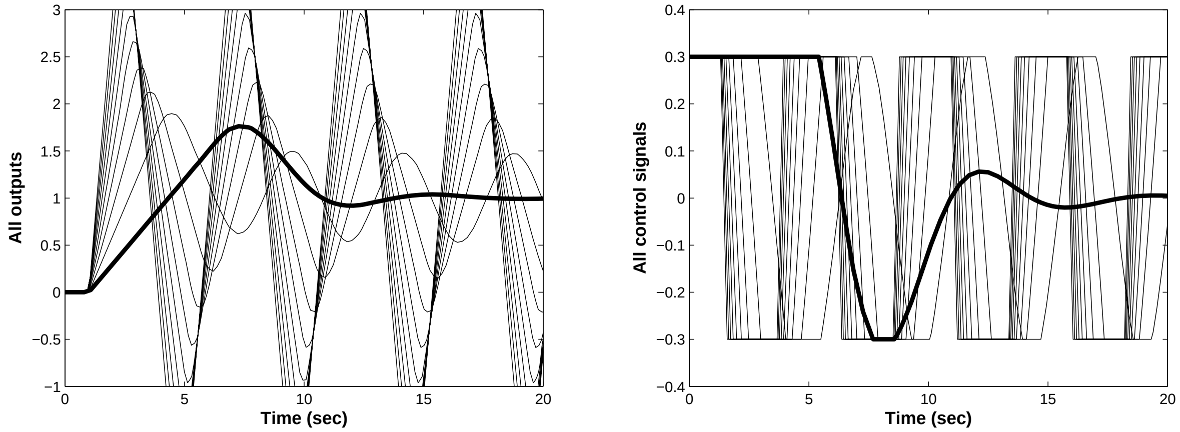

Fig. 2. PI control using Ziegler-Nichols tuning rules. $K_p = 0.71$, $T_i = 3.33$.

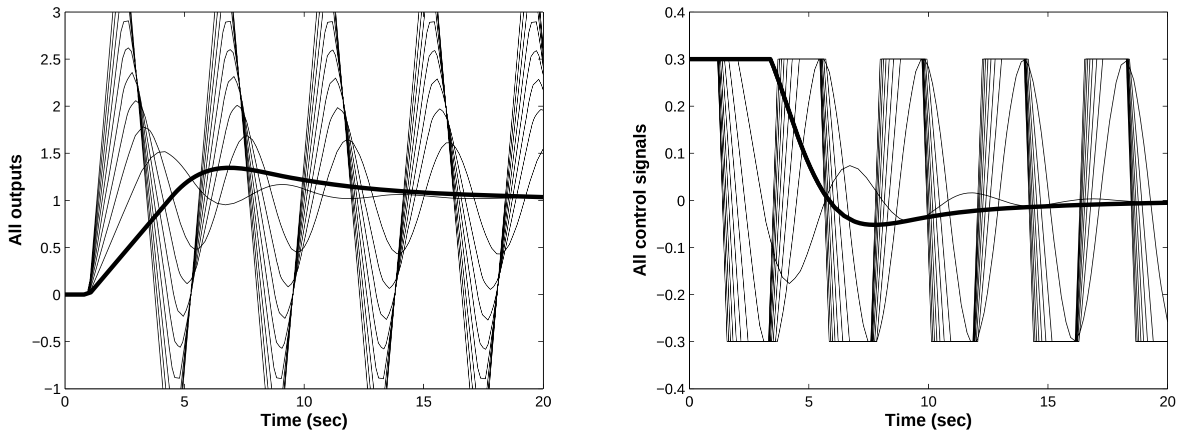

Fig. 3. PI control using Skogestad tuning rules. $K_p = 0.5$, $T_i = 8$.

(vii) Return to 3 until the optimal solution is reached or while there exist lower bound solutions.

Notice that the convergence of the optimal solution is ensured using the previous algorithm. This is due to the fact that the search is not always performed looking for the best solution, since in odd iterations the solution is chosen based on the greatest search space of λ_1 and λ_2.

4. Numerical Example

As a process representative of a common industrial problem, an integrator plant plus time delay (as well-known a wide range of industrial processes can be represented by this transfer function [1]) with uncertainty in the gain has been selected in order to show the main features of the proposed control structure, which can be easily applied to more complex plants. The plant is given by $P(s) = \frac{K_p}{s}e^{-s}$ where $K_p \in [1, 10]$ and being the control signal limited to $[-0.3, 0.3]$. This plant is very similar to that presented in [28],[11] where robust and input saturation problems are also studied.

Suppose that this plant has been attempted to be controlled by a typical PI controller. For the PI tuning, two different methods have been considered in this example. First, it is supposed that the PI parameters have been obtained using the well-known Zielger-Nichols rules [29], and on the other hand, a more robust and recent method developed by Skogestad [30] is used. In both cases, $P(s) = \frac{1}{s}e^{-s}$ has been chosen as nominal plant in order to show better the features of the control architecture presented in this work.

Figures 2 and 3 show the control results. The thick lines represent the results for the free uncertainty case where acceptable results are obtained in both cases. As expected, a more aggressive response is provided by the Ziegler-Nichols method where the control signal reaches the two saturation limits during the transitory period.

However, if this control loop is studied in presence of uncertainties, poor performance results and stability problems appear for both design methods. This fact is presented in Figures 2 and 3, where thin lines represent the evolution of the system due to the uncertainty influence. At this point, a robust control strategy, such as those commented in the introduction section, could be used in order to solve this problem. However, in the following, these problems are

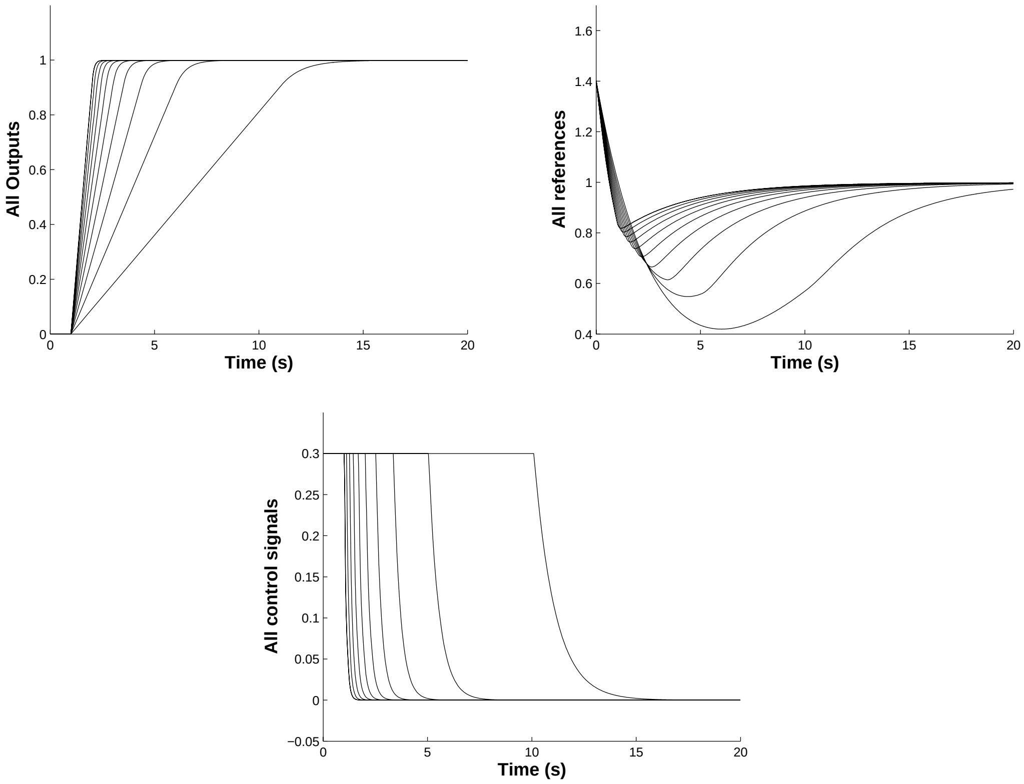

Fig. 4. LMI-based approach for the inner loop designed with Ziegler-Nichols method.

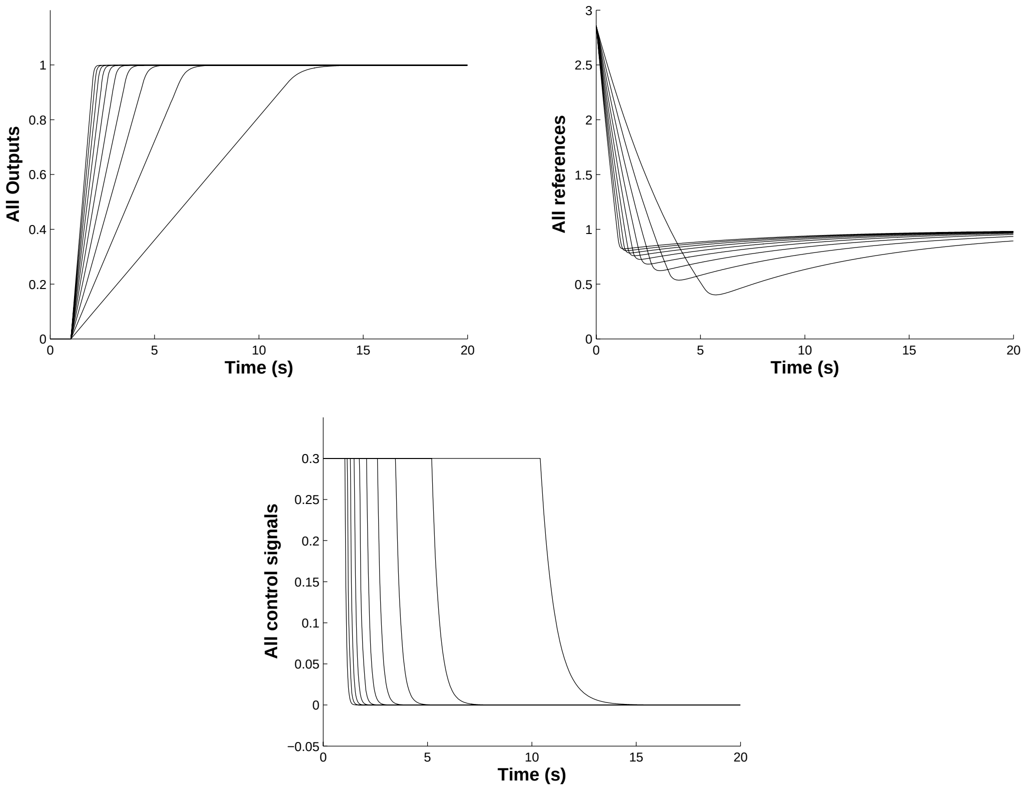

Fig. 5. LMI-based approach for the inner loop designed with Skogestad method.

solved using the LMI-based solution presented in this paper in order to control the proposed inner loop considering input saturation and ensuring constrained robust stability, while keeping the existing inner control loop. Consider the sample time $T_m = 0.01$, $N = 20$, $R_u = 1$, $Q = 1$, $w = 1$, and $x_0 = [0\ 0\ 0\ 0]^T$. So, for the inner loop designed with the Ziegler-Nichols method, the Branch and Bound algorithm found an optimal solution for $\lambda_1 = 0.99707$ and $\lambda_2 = 0.0051$ obtaining

$$K_s = \begin{bmatrix} -0.4030 & -0.003 \end{bmatrix}$$

Figure 4 shows the results of applying the obtained solution considering all plants of the family. It can be seen how the system reaches the proposed reference $w = 1$ obtaining good performance. Comparing the results with the previous ones, the proposed LMI-based approach presents acceptable performance results, but also ensuring constrained robust stability.

Consider now the control loop for the Skogestad method. The same design parameters that in the previous case are used where for this inner loop the obtained solution is given by $\lambda_1 = 0.9981$ and $\lambda_2 = 0.001123$ being

$$K_s = \begin{bmatrix} -1.860623 - 0.00125 \end{bmatrix}$$

Figure 5 shows the results where it can be observed how similar results that in the Ziegler-Nichols case are obtained. Notice that, for both cases, the proposed architecture present very similar performance results, but the references provided for the inner loops are different in each case.

On the other hand, the Branch and Bound algorithm has presented a good behavior in finding optimal values for λ_1 and λ_2. This fact can be observed from Figure 6 where it is shown how, for the previous example, the algorithm divides correctly the search space in order to find optimal values.

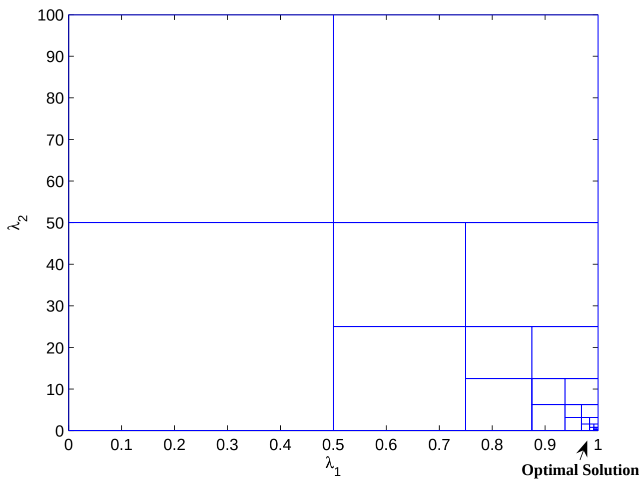

Fig. 6. Search space division by the Branch and Bound algorithm.

Notice that, the LMI-based approach is implemented with low computational load where the feedback gain K_s is calculated off-line. However, the optimization problem could be solved on-line at each instant time in order to obtain more optimal results, where the feedback gain K_s is always calculated based on the current state of the system. Then, in this case, the LMI-based approach would require bigger computational load than some of the approaches presented in [31], being this fact the main drawback with respect to the other reference governor techniques.

5. Conclusions

A robust constrained LMI-based approach has been developed as solution to the problem of controlling an uncertain system subject to input saturation. In this way an existing control loop has been translated into state space representation, and LMI have been used to obtain a state-vector feedback in such a way that the input reference to the inner loop is calculated in order to satisfy robust tracking specifications considering input saturation. The proposed solution consists in solving a set of constraints described by several LMI and BMI, where a Branch and Bound algorithm has been developed in order to account for the bilinear terms. Notice that the algorithm is implemented for tracking problems where the aim is to regulate to a fixed reference value and not to the origin, and also input constraints are present in the inner loop of the system. The algorithm is very useful for a wide range of industrial processes controlled by classical control algorithms where the presence of input constraints and/or uncertainties cause problems in the stability and performance of the system.

Acknowledgements

This work was supported by the Spanish CICYT and FEDER funds under grant DPI2004-07444-C04-01/04.

Appendix A

The second inequality of (55)

$$(\bar{x}^+)^T P_s (\bar{x}^+) - \bar{x}^T P_s \bar{x} < -\bar{x}^T C_y^T Q C_y \bar{x} - \bar{x}^T K_u^T R_u K_u \bar{x} + \gamma \tag{A.1}$$

must be satisfied for two extreme vertices of the LDI, $\frac{K_s\bar{x}+d_u}{L_s}$ and $H_s\bar{x} + h$.

A.1. Case $\frac{K_s\bar{x}+d_u}{L_s}$

Considering that $\bar{x}^+$ is described by (28), the previous inequality results as

$$-v^T P_s v + 2v^T P_s (A_{Bc}\bar{x} + E_{Bc}) - \bar{x}^T P_s \bar{x} + \bar{x}^T C_y^T Q C_y \bar{x} +$$
$$+\bar{x}^T K_u^T R_u K_u \bar{x} - \gamma + 2v^T P_s B_{Bc} \left(\frac{K_u\bar{x} + d_u}{L_s}\right) \leq 0$$

The matrix representation is given by

$$\nu^T \begin{bmatrix} -\gamma & * & * \\ 0 & -P_s+C_y^TQC_y+K_u^TR_uK_u & * \\ -(P_sE_{Bc}+P_sB_{LBc}d_u) & -(P_sA_{Bc}+P_sB_{LBc}K_u) & -P_s \end{bmatrix} \nu \leq 0 \tag{A.2}$$

where $B_{LBc} = \frac{1}{L_s}B_{Bc}$, $\nu = \begin{bmatrix} 1 & \bar{x} & -v \end{bmatrix}^T$, and the previous LMI must be satisfied $\forall \bar{x}$ and $\forall v$. Therefore, it is obtained that

$$\begin{bmatrix} \gamma & * & * \\ 0 & P_s - C_y^TQC_y - K_u^TR_uK_u & * \\ P_s(E_{Bc}+B_{LBc}d_u) & P_s(A_{Bc}+B_{LBc}K_u) & P_s \end{bmatrix} > 0 \tag{A.3}$$

where pre- and post-multiplying by $diag[I \ P_s^{-1} \ P_s^{-1}]$ and making $W = P_s^{-1}$, it is obtained that

$$\begin{bmatrix} \gamma & * & * \\ 0 & W-WC_y^TQC_yW-WK_u^TR_uK_uW & * \\ (E_{Bc}+B_{LBc}d_u) & (A_{Bc}+B_{LBc}K_u)W & W \end{bmatrix} > 0 \tag{A.4}$$

From the Schur complement the LMI results as follows

$$\begin{bmatrix} \gamma & * & * & * \\ 0 & W - WK_u^TR_uK_uW & * & * \\ (E_{Bc}+B_{LBc}du) & (A_{Bc}+B_{LBc}K_u)W & W & * \\ 0 & Q^{1/2}C_yW & 0 & I \end{bmatrix} > 0 \tag{A.5}$$

Using the Schur complement again, it is obtained that

$$\begin{bmatrix} \gamma & * & * & * & * \\ 0 & W & * & * & * \\ (E_{Bc}+B_{LBc}d_u) & (A_{Bc}+B_{LBc}K_u)W & W & * & * \\ 0 & Q^{1/2}C_yW & 0 & I & * \\ 0 & R_u^{1/2}K_uW & 0 & 0 & I \end{bmatrix} > 0 \tag{A.6}$$

or equivalently

$$\begin{bmatrix} \gamma & * & * & * & * \\ 0 & W & * & * & * \\ A_{xe}x_e + B_{re}r_e + E_e & A_wW + B_yY & W & * & * \\ 0 & Q^{1/2}C_yW & 0 & I & * \\ 0 & R_WW + R_YY & 0 & 0 & I \end{bmatrix} > 0 \tag{A.7}$$

where $A_{x_e} = A - I + (\frac{1}{L_s})B_uC_u$, $B_{r_e} = B_r + (\frac{1}{L_s})B_uD_u$, $E_e = E - (\frac{1}{L_s})B_uu_c$, $A_w = A + \frac{1}{L_s}B_uC_u$, $B_y = B_r + \frac{1}{L_s}B_uD_u$, $Y = K_sW$ and

$$R_u^{1/2}K_uW = R_u^{1/2}(C_u + D_uK_s)W = \\ = R_u^{1/2}C_uW + R_u^{1/2}D_uK_sW = R_wW + R_yY$$

A.2. Case $H_s\bar{x} + h$

In this case, it is easy to see that the same LMI (A.7) is obtained only replacing K_u by H_s, d_u by h, and B_{LBc} by B_{Bc}. So, it is obtained that

$$\begin{bmatrix} \gamma & * & * & * & * \\ 0 & W & * & * & * \\ (E_{Bc}+B_{Bc}h) & (A_{Bc}+B_{Bc}H_s)W & W & * & * \\ 0 & Q^{1/2}C_yW & 0 & I & * \\ 0 & R_u^{1/2}K_uW & 0 & 0 & I \end{bmatrix} > 0 \tag{A.8}$$

Then, using the same notation that in previous sections where, $E_{Bc} + B_{Bc}h = A_{nlxe}x_e + B_{nlre}r_e + B_hh + E_{nle}$, $A_{Bc} + B_{Bc}H_s = A_{nlW}W + B_{nlY}Y + B_vV$, $R_u^{1/2}H_sW = R^{1/2}V = R_vV$, $Y = K_sW$, and $V = H_sW$, the LMI can be represented as follows

$$\begin{bmatrix} \gamma & * & * & * & * \\ 0 & W & * & * & * \\ A_{nlxe}x_e+B_{nlre}r_e+B_hh+E_{nle} & A_{nlW}W+B_{nlY}Y+B_vV & W & * & * \\ 0 & Q^{1/2}C_yW & 0 & I & * \\ 0 & R_vV & 0 & 0 & I \end{bmatrix} > 0 \tag{A.9}$$

References

[1] K. Åström, T. Hägglund, Advanced PID Control, ISA - The Instrumentation, Systems, and Automation Society. Research Triangle Park, NC 27709, 2005.

[2] G. Zames, B. Francis, Feedback, minimax sensitivity, and optimal robustness, IEEE Transactions on Automatic Control 28 (5) (1983) 585–601.

[3] I. Horowitz, Quantitative Feedback Design Theory (QFT), QFT Publications, Boulder, Colorado, 1993.

[4] J. Doyle, Analysis of feedback systems with structured uncertainties, IEE Proceedings. Part D-Control Theory and Applications 129 (6) (1982) 242–250.

[5] K. Åstrom̈, L. Runqwist, Integrator windup and how to avoid it, Proceedings of American Control ConferencePittsburgh, PA.

[6] M. V. Kothare, P. J. Campo, M. Morari, C. N. Nett, Multiplier Theory for Stability Analisys of Anit-Windup Contorl Systems, Automatica 35 (5) (1999) 917–928.

[7] E. F. Camacho, C. Bordóns, Model Predictive Control, Springer, 2004.

[8] J. Maciejowski, Constrained Predictive Control, Academic Press, 2002.

[9] J. M.G. da Silva, S. Tarbouriech, Antiwindup design with guaranteed regions of stability: an LMI-based approach, IEEE Transactions on Automatic Control 50 (1) (1982) 106–111.

[10] P. O. M. Scokaert, D. Q. Mayne, Min-max feedback model predictive control for constrained linear systems, IEEE Transactions on Automatic Control 43 (8) (1998) 1136–1142.

[11] J. C. Moreno, A. Baños, M. Berenguel, A synthesis theory for a class of uncertain linear systems with amplitude saturation, in: Proceedings of the 4th IFAC Symposium on Robust Control Design (ROCOND03), Milan (Italy), 2003.

[12] P. Gahinet, P. Apkarian, A linear matrix inequality approach to H_∞ control, International Journal of Robust and Nonlinear Control 4 (1994) 421–448.

[13] C. M. Turner, G. Herrmann, I. Postlethwaite, Accounting for uncertainty in anti-windup synthesis, Proceedings of American Control ConferenceBoston, US.

[14] A. Bemporad, Reference governor for constrained nonlinear systems, IEEE Transactions on Automatic Control 43 (3) (1998) 415–419.

[15] A. Bemporad, F. Borreli, M. Morari, Model predictive control based on linear programming. The explicit solution, IEEE Transactions on Automatic Control 47 (12) (2002) 1974–1985.

[16] K. Hirata, K. Kogiso, A performance improving off-line reference management for systems with state and control constraints, in: Proceedings of the 40th IEEE Conference on Decision and Control, Orlando, USA, 2001.

[17] E. G. Gilbert, I. Kolmanovsky, A generalized reference governor for nonlin-ear systems, in: Proceedings of the 40th IEEE Conference on Decision and Control, Orlando, USA, 2001.

[18] J. A. Rossiter, B. Kouvaritakis, Reference governors and predictive control, in: Proceedings of the American Control Conference ACC98, AACC, Philadelphia, Pennsylvania (USA), 1998, pp. 3692–3693.

[19] A. Bemporad, E. Mosca, Fulfilling hard constraints in uncertain linear systems by reference managing, Automatica 34 (4) (1998) 451–461.

[20] A. Casavola, E. Mosca, D. Angeli, Robust command governors for constrained linear systems, IEEE Transactions on Automatic Control 45 (11) (2000) 2071–2077.

[21] T. Sugie, H. Suzuki, Robust reference shaping of periodic trajectories for systems with state/input constraints using impulse and step responses, in: Proceedings of the 43rd IEEE Conference on Decision and Control, Atlantis, Paradise Island, Bahamas, 2004.

[22] J. L. Guzmán, Interactive Control System Design, Phd thesis, University of Almería, Spain (2006).

[23] D. W. Clarke, C. Mohtadi, P. S. Tuffs, Generalized predictive control - parts I and II, Automatica 23 (2) (1987) 137–160.

[24] M. V. Kothare, V. Balakrishnan, M. Morari, Robust constrained model predictive control using linear matrix inequalities, Automatica 32 (10) (1996) 1361–1379.

[25] S. Boyd, L. El Ghaoui, E. Feron, V. Balakrishnan, Linear Matrix Inequalities in System and Control Theory, Vol. 15, Society for Industrial and Applied Mathematics, SIAM, 1994.

[26] Q. Hu, G. Rangaiah, Anti-windup schemes for uncertain nonlinear systems, IEE Proceedings Part D, Control Theory and Applications 147 (3) (2000) 321–329.

[27] H. Fang, Z. Lin, T. Hu, Analysis of linear systems in the presence of actuator saturation and L_2-disturbances, Automatica 40 (2004) 1229–1238.

[28] I. M. Horowitz, A synthesis theory for a class of saturating systems, International Journal of Control 38 (1983) 169–187.

[29] J. Ziegler, N. Nichols, Optimum settings for automatic controllers, Transactions of the ASME 64 (1942) 759–768.

[30] S. Skogestad, Simple analytic rules for model reduction and PID controller tuning, Journal of Process Control 13 (4) (2003) 291–309.

[31] J. L. Guzmán, M. Berenguel, S. Dormido, T. Hägglund, A robust constrained predictive GPC-QFT approach, UAL/UNED/LUND Internal Report.