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Relational Galois connections between transitive digraphs: Characterization and construction

Inma P. Cabrera[a,*], Pablo Cordero[a], Emilio Muñoz-Velasco[a], Manuel Ojeda-Aciego[a], Bernard De Baets[b]

[a] *Dept. Matemática Aplicada, Universidad de Málaga. Blv Louis Pasteur 35, 29071 Málaga, Spain*
[b] *KERMIT, Department of Data Analysis and Mathematical Modelling., Ghent University, Coupure links 653, Gent, 9000, Belgium*

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ABSTRACT

This paper focuses on a twofold relational generalization of the notion of Galois connection. It is twofold because it is defined between sets endowed with arbitrary transitive relations and, moreover, both components of the connection are relations, not necessarily functions. A characterization theorem of the notion of relational Galois connection is provided and, then, it is proved that a suitable notion of closure can be obtained within this framework. Finally, we state a necessary and sufficient condition that allows to build a relational Galois connection starting from a single transitive digraph and a single binary relation.

1. Introduction

Galois connections, together with the equivalent (under duality) residuated mappings, constitute a useful tool to establish semantic foundations in several fields related to data science. From an incidence relation between a set of objects and a set of properties, a natural Galois connection provides dually isomorphic orders on classes of objects and sets of shared properties. This fact has been described from different points of view in the literature. It has been successfully developed in Formal Concept Analysis, Learning Theory, Conceptual Classification, and Relational or Object Databases [11,15,16,21,23–25]. Although Galois connections have been introduced a long time ago, new publications regularly appear dealing with abstract generalizations of this important mathematical construct. At the same time, new applications are being explored, especially in the realm of Formal Concept Analysis and the foundations of Fuzzy Set Theory (see, for instance, [2,3,7,12,17,20]).

In this paper, we deal with a generalization of the notion of Galois connection, more specifically, we focus on its adaptation to a fully relational setting. Our contributions in this direction were initiated with the results in [14], where we aimed to characterize the existence of the residual (or right adjoint, or right part of a Galois connection) of a given mapping between sets with a different structure; it is worth mentioning that precisely this condition of having a different structure is what makes the problem to be outside the scope of Freyd's adjoint functor theorem [22, pg. 120].

Since then, we have obtained additional results in several frameworks. For instance, in [14], given a mapping from a (pre-)ordered set $(A, \le_A)$ into an unstructured set B, we characterized the problem of completing the structure of B, i.e., defining a suitable (pre-)ordering relation $\le_B$ on B, such that there exists a second mapping for which the pair of mappings

* Corresponding author.
E-mail addresses: ipcabrera@uma.es (I.P. Cabrera), pcordero@uma.es (P. Cordero), ejmunoz@uma.es (E. Muñoz-Velasco), aciego@uma.es (M. Ojeda-Aciego), bernard.debaets@ugent.be (B. De Baets).

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forms an isotone Galois connection (or adjunction) between the (pre-)ordered sets $(A, \leq_A)$ and $(B, \leq_B)$. Later, in [4], we extended these results to the fuzzy framework by considering the corresponding problem between a fuzzy preposet (A, ρ_A) and an unstructured B; this work was recently further extended in [5], by assuming that equality is expressed by a fuzzy equivalence relation, thus considering a mapping between a fuzzy preordered structure $(A, \approx_A, \rho_A)$ and a fuzzy structure $(B, \approx_B)$.

The above-mentioned papers [4,5] satisfactorily extend (and solve) the problem to the fuzzy setting for what concerns the domain and the range of the Galois connection, however, in both cases, the components of the Galois connection are (crisp) functions. Hence, the next logical extension is to consider the possibility that those components are actually fuzzy functions. Following this idea, in [6] we introduced the notion of relational fuzzy Galois connection between fuzzy preposets, in which the components of the connection are not fuzzy functions but fuzzy relations satisfying certain conditions.

In this paper, we continue our foundational study of relational versions of the notion of Galois connection in the crisp case by providing a twofold generalization of Galois connection whose components are crisp *relations* between sets equipped with a *transitive relation* (T-digraphs, for short). This framework allows to maintain a great level of abstraction without compromising the ability to reflect the properties of closure [13]. Much to our satisfaction, such properties allow to obtain a necessary and sufficient condition to build a relational Galois connection starting from a single T-digraph and a single binary relation.

The key construction to develop our approach to the relational version is the *powering*, which allows to lift a relation $\mathcal{R} \subseteq A \times B$ to a relation $\mathcal{R}'$ between the powersets 2^A and 2^B. It is worth mentioning that although the Smyth powering works adequately in most cases of lifting a relation to the powersets, we have found that it has to be enriched by an additional condition in order to be able to directly lift the notion of closure operator to the relational case. This has suggested to consider a new ordering on the powersets that has greatly simplified some of the proofs.

In the literature, one can find some earlier approaches to 'relational' versions of the notion of Galois connection, in one sense or another related to the problem we are dealing with in the present paper. For instance, *Essential Galois bonds* between contexts, introduced by Xia [27], are related to our work in the sense that their components are relations; this notion was later renamed as *relational Galois connection* in [10], where a unifying language was developed in order to cope with similar attempts by Domenach and Leclerc [8] and Wille [26].

The structure of this paper is the following. In Section 2, we introduce the necessary preliminaries from the theory of relations and standard Galois connections. Then, in Section 3, we discuss the convenience of using the Smyth powering in the definition of a relational Galois connection. In Section 4, we study some alternative characterizations of our proposed notion of relational Galois connection; as we will base our construction of relational Galois connection on the corresponding generalization of closure operators, in Section 5, we introduce closure relations and study their main properties, showing an adequate balance between generality and preservation of properties from the standard framework. Next, in Section 6 we expound a characterization theorem for the existence of a right adjoint for a given relation between T-digraphs as well as its explicit construction. Furthermore, the proposed characterization is validated by providing a complete construction in order to illustrate that the mechanism works, in the sense that the Galois connection that can be defined is strictly relational (i.e., not functional) and the transitive digraph induced in the codomain is not trivial. Finally, in Section 7, we draw some conclusions and present prospects for future work.

It is worth mentioning that, although the results are stated for Galois connections, they can easily be adapted to any different 'version' of the notion of Galois connection, obtained by considering the dual ordering either in the domain or codomain of the Galois connection.

2. Preliminaries

We will work within the usual framework of (crisp) relations. Namely, a *binary relation* $\mathcal{R}$ between two sets A and B is a subset of the Cartesian product $A \times B$ and it can also be seen as a multivalued function $\mathcal{R}$ from the set A to the powerset 2^B. For an element $(a, b) \in \mathcal{R}$, it is said that *a is related to b*, and is denoted $a\mathcal{R}b$. Given a binary relation $\mathcal{R} \subseteq A \times B$, the *afterset* $a^{\mathcal{R}}$ of an element $a \in A$ is defined as $\{b \in B \mid a\mathcal{R}b\}$. A binary relation $\mathcal{R}$ is said to be *total* if $a^{\mathcal{R}} \neq \varnothing$ for all $a \in A$; the *domain* of $\mathcal{R}$ is defined as $\operatorname{dom}(\mathcal{R}) = \{a \in A \mid a^{\mathcal{R}} \neq \varnothing\}$, the *range* of $\mathcal{R}$ is defined as $\operatorname{rng}(\mathcal{R}) = \bigcup_{a\in A} a^{\mathcal{R}}$.

Given an arbitrary set A and a preorder $\leq$ (reflexive and transitive relation) on A, among several possibilities[1] to lift the preorder on A to the powerset 2^A we recall the following

$$X \ll Y \iff \text{for all } x \in X \text{ there exists } y \in Y \text{ such that } x \leq y$$

$$X \Subset Y \iff \text{for all } y \in Y \text{ there exists } x \in X \text{ such that } x \leq y.$$

Note that the two relations defined above are actually preorders, which can be identified as those used in the construction of the Hoare and Smyth powerdomains [9].

Naturally, each of the powerings above induces a particular notion of isotony, inflation, etc. For instance, given two preordered sets $(A, \leq)$ and $(B, \leq)$,[2] a binary relation $\mathcal{R} \subseteq A \times B$ is said to be:

[1] We will use the generic term *powering* to refer to any lifting of a relation to the corresponding powersets.

[2] Note that, as usual, we use the same symbol to denote both binary relations which need not be equal.

- $\Subset$ -isotone if $a_1 \le a_2$ implies $a_1^{\mathcal{R}} \Subset a_2^{\mathcal{R}}$, for all $a_1, a_2 \in \mathrm{dom}(\mathcal{R})$;
- $\Subset$ -antitone if $a_1 \le a_2$ implies $a_2^{\mathcal{R}} \Subset a_1^{\mathcal{R}}$, for all $a_1, a_2 \in \mathrm{dom}(\mathcal{R})$.

A binary relation $\mathcal{R} \subseteq A \times A$ is said to be:

- $\Subset$ -inflationary if $\{a\} \Subset a^{\mathcal{R}}$, for all $a \in \mathrm{dom}(\mathcal{R})$;
- $\Subset$ -deflationary if $a^{\mathcal{R}} \Subset \{a\}$, for all $a \in \mathrm{dom}(\mathcal{R})$;
- $\Subset$ -idempotent if $a^{\mathcal{R}\circ\mathcal{R}} \Subset a^{\mathcal{R}}$ and $a^{\mathcal{R}} \Subset a^{\mathcal{R}\circ\mathcal{R}}$, for all $a \in \mathrm{dom}(\mathcal{R})$.

We use the prefix to distinguish the powering used in the different definitions.

Note that the definitions above consider just elements in the domain of the relation, in the same way as for partial functions. For the sake of convenience, and without loss of generality, we will consider hereafter that all the relations are total.

Let $\mathcal{R}$ be a binary relation between A and B and $\mathcal{S}$ be a binary relation between B and C. The composition of $\mathcal{R}$ and $\mathcal{S}$ is defined as follows

$$\mathcal{R} \circ \mathcal{S} = \{(x, z) \in A \times C \mid \text{ there exists } b \in B \text{ such that } x\mathcal{R}b \text{ and } b\mathcal{S}z\}.$$

Observe that for $a \in A$, the afterset $a^{\mathcal{R}\circ\mathcal{S}}$ can be written as $\bigcup\limits_{b \in a^{\mathcal{R}}} b^{\mathcal{S}}$.

In the classical setting, a Galois connection between two posets is a pair of antitone mappings whose compositions are inflationary. A well-known characterization of a Galois connection (f, g) between two posets is the so-called Galois condition

$$a \le g(b) \quad \iff \quad b \le f(a).$$

Once again, in our general framework there are several possible choices, which we will distinguish through the use of a prefix. For instance, the $\ll$ -Galois condition is

$$\{a\} \ll b^{\mathcal{S}} \quad \iff \quad \{b\} \ll a^{\mathcal{R}}.$$

In previous works, we have studied the extensions obtained in terms of the powerings $\ll$ and $\Subset$ and the relation of the corresponding Galois condition with the properties of antitony and inflation. We continue this line of work by focusing our attention on another convenient property of classical Galois connections, namely that their compositions are isotone, inflationary and idempotent, i.e., they are closure operators. The following example shows that the definition based on the Hoare powering $\ll$ does not behave satisfactorily. As a result, hereafter, we will work essentially with the Smyth powering $\Subset$.

Example 1. Consider the set of natural numbers together with the discrete ordering given by the identity relation $(\mathbb{N}, =)$, and consider the relation $\mathcal{R}$ given by $n^{\mathcal{R}} = \{0, \dots, n+1\}$. The relation $\mathcal{R}$ is trivially $\ll$ -antitone, and $\mathcal{R} \circ \mathcal{R}$ is obviously $\ll$ -inflationary; however, it does not make sense to consider $(\mathcal{R}, \mathcal{R})$ as an extended Galois connection, since $\mathcal{R} \circ \mathcal{R}$ is not a $\ll$ -closure operator (it fails to be $\ll$ -idempotent) and, hence, the $\ll$ -Galois condition does not hold either.

3. Relational Galois connections between T-digraphs

As stated in the introduction, our goal is to define a relational Galois connection as a pair of relations between sets with the least possible structure. Our general framework will be that of sets endowed with a transitive relation: we will refer to a couple $\mathbb{A} = (A, \tau)$, with a transitive relation $\tau \subseteq A \times A$, as a *T-digraph*.

Definition 1. A relational Galois connection between two T-digraphs $\mathbb{A}$ and $\mathbb{B}$ is a pair of relations $(\mathcal{R}, \mathcal{S})$ where $\mathcal{R} \subseteq A \times B$ and $\mathcal{S} \subseteq B \times A$ such that the following properties hold:

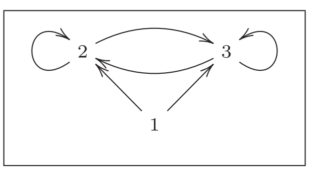
(i) $\mathcal{R}$ and $\mathcal{S}$ are $\Subset$ -antitone.
(ii) $\mathcal{R} \circ \mathcal{S}$ and $\mathcal{S} \circ \mathcal{R}$ are $\Subset$ -inflationary.

Note that we can consider the $\Subset$-extension even if the underlying relation τ is not a preorder. In this case, the lifted relation need not be a preorder but, anyway, it inherits the transitivity.

Observe that $(\mathcal{R}, \mathcal{S})$ is a relational Galois connection between $\mathbb{A}$ and $\mathbb{B}$ if and only if $(\mathcal{S}, \mathcal{R})$ is a relational Galois connection between $\mathbb{B}$ and $\mathbb{A}$. Next, we show an example in which both $\mathcal{R}$ and $\mathcal{S}$ are proper (non-functional) relations.

Example 2. Consider $\mathbb{A} = (A, \tau)$ where $A = \{1, 2, 3\}$ and τ is the transitive relation $\{(1, 2), (1, 3), (2, 2), (2, 3), (3, 2), (3, 3)\}$. The pair of relations $(\mathcal{R}, \mathcal{S})$ given by the tables below constitutes a relational Galois connection between $\mathbb{A}$ and $\mathbb{A}$.

$\mathbb{A}$



x	$x^{\mathcal{R}}$
1	$\{2,3\}$
2	$\{2\}$
3	$\{3\}$

x	$x^{\mathcal{S}}$
1	$\{2,3\}$
2	$\{2\}$
3	$\{2,3\}$

Given a relation $\mathcal{R} \subseteq A \times B$, the direct and subdirect images of A under the relation $\mathcal{R}$ define two mappings between the powersets 2^A and 2^B.

- *Direct, Upper* extension of $\mathcal{R}$, denoted by $\mathcal{R}(\cdot) : 2^A \to 2^B$, and defined as follows:

$$\mathcal{R}(X) = \bigcup_{x \in X} x^{\mathcal{R}};$$

- *Subdirect, Lower* extension of $\mathcal{R}$, denoted by $(\cdot)^{\mathcal{R}} : 2^A \to 2^B$, and defined as follows:

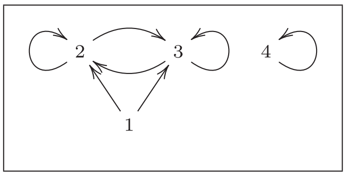
$$X^{\mathcal{R}} = \bigcap_{x \in X} x^{\mathcal{R}}.$$

Given (A, τ) and (B, τ), two relations $\mathcal{R} \subseteq A \times B$ and $\mathcal{S} \subseteq B \times A$ can be extended to mappings between the corresponding powersets 2^A and 2^B. In this framework, it is worth studying the possible relationship between the standard notion of Galois connection and the notion of relational Galois connection introduced above. We show that the standard notion neither implies nor is implied by our notion of relational Galois connection.

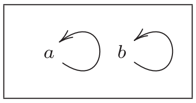
The following example shows a relational Galois connection whose direct extension to the powerset (with the Smyth powering) is not a classical Galois connection.

Example 3. Let $\mathbb{A}$ and $\mathbb{B}$ be the T-digraphs shown below, and $\mathcal{R} \subseteq A \times B$ and $\mathcal{S} \subseteq B \times A$ the relations defined as follows:

$\mathbb{A}$



$\mathbb{B}$



x	$x^{\mathcal{R}}$
1	$\{a\}$
2	$\{a\}$
3	$\{a\}$
4	$\{b\}$

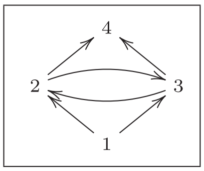
x	$x^{\mathcal{S}}$
a	$\{2,3\}$
b	$\{4\}$

It is straightforward to see that $(\mathcal{R}, \mathcal{S})$ is a relational Galois connection, but its direct extension to the powersets is not a Galois connection. Observe that $\{1, 4\} \Subset \mathcal{S}(\{a\}) = \{2, 3\}$, however, $\{a\} \not\Subset \mathcal{R}(\{1, 4\}) = \{a, b\}$.

The following two examples involve preordered structures as particular cases of T-digraphs; in both cases, the depicted graphs induce the preorders via the reflexive and transitive closure. The first example shows a relational Galois connection whose subdirect extension to the powerset (with the Smyth powering) is not a classical Galois connection.

Example 4. Let $\mathbb{A}$ be the preordered set induced by the graph below, and $\mathcal{R} \subseteq A \times A$ be the relation defined as follows:

$\mathbb{A}$



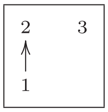
x	$x^{\mathcal{R}}$
1	$\{4\}$
2	$\{2\}$
3	$\{3\}$
4	$\{1\}$

It is straightforward to check that $(\mathcal{R}, \mathcal{R})$ is a relational Galois connection between $\mathbb{A}$ and $\mathbb{A}$, but its subdirect extension to the powersets is not a Galois connection. Observe that $\{4\} \Subset \{2, 3\}^{\mathcal{R}} = \{2\}^{\mathcal{R}} \cap \{3\}^{\mathcal{R}} = \varnothing$, however, $\{2, 3\} \not\Subset \{4\}^{\mathcal{S}} = \{1\}$.

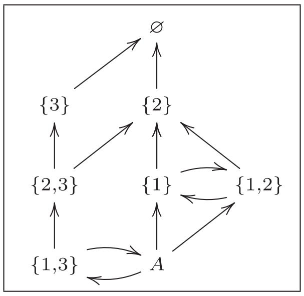
The second example shows a classical Galois connection between powersets, whose restriction to singletons is not a relational Galois connection.

Example 5. Given the preordered set $\mathbb{A} = (A, \leq)$, its Smyth extension and the mapping f: $2^A \to 2^A$ depicted below, the pair (f, f) is a Galois connection between $2^{\mathbb{A}}_{\Subset} = (2^A, \Subset)$ and itself.

$\mathbb{A}$



$2^{\mathbb{A}}_{\Subset}$



X	$f(X)$
$\varnothing$	A
$\{3\}$	$\{1,2\}$
$\{2\}$	$\{2,3\}$
$\{2,3\}$	$\{2\}$
$\{1\}$	$\{3\}$
$\{1,2\}$	$\{3\}$
$\{1,3\}$	$\varnothing$
A	$\varnothing$

The corresponding restriction of the above mapping between powersets to singletons is the relation $\mathcal{R}$ on A given by

x	$x^{\mathcal{R}}$
1	$\{3\}$
2	$\{2,3\}$
3	$\{1,2\}$

The pair $(\mathcal{R}, \mathcal{R})$ is not a relational Galois connection because $2 \notin 2^{\mathcal{R}\circ\mathcal{R}}$.

4. Characterization of relational Galois connections

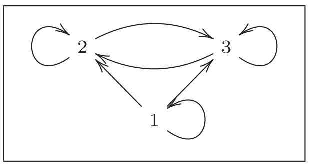
Having in mind the characterization of classical Galois connections between posets, the definition of a relational Galois connection given above *might* be equivalent to the corresponding Galois condition, namely:

$$\{a\} \Subset b^{\mathcal{S}} \iff \{b\} \Subset a^{\mathcal{R}}, \text{ for all } a \in A, \text{ and } b \in B\,. \tag{1}$$

However, the following example shows that Galois condition (1) does not imply $(\mathcal{R}, \mathcal{S})$ being a relational Galois connection.

Example 6. Consider the T-digraph $\mathbb{A}$ and the relations $\mathcal{R}$ and $\mathcal{S}$ depicted below.

$\mathbb{A}$



x	$x^{\mathcal{R}} = x^{\mathcal{S}}$
1	$\{3\}$
2	$\{1,2\}$
3	$\{1,3\}$

It is routine to prove that $(\mathcal{R}, \mathcal{S})$ verifies the Galois condition (1), but it does not verify Definition 1, because, for instance, $\{2\} \not\Subset_A \{2\}^{\mathcal{R}\circ\mathcal{S}}$.

Let us introduce the following extension of a transitive relation τ to the powersets:

$$X \propto Y \iff \text{ for all } x \in X \text{ and for all } y \in Y \text{ we have that } x\,\tau\,y\,.$$

We will later see in Section 5 that $\propto$ is indeed the powering to be used in order to define the closure relation, i.e., the relational extension of the notion of closure operator.

Remark 1. Note that $\propto$ neither needs to be reflexive nor transitive. Nevertheless, it satisfies the following weakened version of transitivity:

$$\text{For any } Y \neq \varnothing, \text{if } X \propto Y \text{ and } Y \propto Z, \text{ then } X \propto Z\,. \tag{2}$$

We will prove that the equivalence between our definition of a relational Galois connection and the Galois condition does not hold unless an extra condition condition is assumed, as expressed in Proposition 1 below.

Definition 2. Let (A, τ) be a T-digraph and $X \subseteq A$. It is said that X is a clique if $X \propto X$.

Now, we can prove the following technical result.

Lemma 1. *Let (A, τ) be a T-digraph and $x \in X \subseteq A$. If X is a clique then, for all $Y \subseteq A$, the following statements hold:*

(i) $Y \propto \{x\}$ *implies* $Y \propto X$.
(ii) $\{x\} \propto Y$ *implies* $X \propto Y$.
(iii) $X \Subset Y$ *if and only if* $X \propto Y$.

Proof. We only prove (i), since the other items follow easily from this one. Since $X \propto X$, it holds that $\{x\} \propto X$, which, together with $Y \propto \{x\}$ and (2), implies $Y \propto X$. □

Remark 2. Note that there exists a tight relation between $\Subset$ and $\propto$, since for all $x \in A$ and all $Y \subseteq A$ we have that

$$\{x\} \Subset Y \quad \Longleftrightarrow \quad \{x\} \propto Y\,.$$

In particular, the notions of $\Subset$-inflation and $\propto$-inflation are equivalent and, moreover, the corresponding versions of the Galois condition are also equivalent.

Relational Galois connections can be characterized as follows.

Proposition 1. *$(\mathcal{R}, \mathcal{S})$ is a relational Galois connection between $\mathbb{A}$ and $\mathbb{B}$ iff the following properties hold:*

(i) $\{a\} \Subset b^{\mathcal{S}}$ *iff* $\{b\} \Subset a^{\mathcal{R}}$ *for all* $a \in A$ *and* $b \in B$.
(ii) $a^{\mathcal{R}}$ *and* $b^{\mathcal{S}}$ *are cliques for all* $a \in A$ *and* $b \in B$.

Proof. Assume that $(\mathcal{R}, \mathcal{S})$ is a relational Galois connection. To prove property (i), suppose $\{a\} \Subset_A b^{\mathcal{S}}$, i.e., $a\ \tau\ x$, for all $x \in b^{\mathcal{S}}$. Since $\mathcal{R}$ is $\Subset$-antitone, we obtain $x^{\mathcal{R}} \Subset_B a^{\mathcal{R}}$. Thus, for all $y \in a^{\mathcal{R}}$ there exists $y' \in x^{\mathcal{R}}$ such that $y'\ \tau\ y$. Furthermore, as $\mathcal{S} \circ \mathcal{R}$ is $\Subset$-inflationary, we have $b\ \tau\ y' \in b^{\mathcal{S} \circ \mathcal{R}}$. Hence, by transitivity, $b\ \tau\ y$, for all $y \in a^{\mathcal{R}}$, that is, $\{b\} \Subset_B a^{\mathcal{R}}$, proving one implication of the Galois condition. The converse implication can be proved similarly.

Let us prove now that $a^{\mathcal{R}}$ is a clique, that is, $a^{\mathcal{R}} \propto a^{\mathcal{R}}$, for all $a \in A$. As $\mathcal{R} \circ \mathcal{S}$ is $\Subset$-inflationary, it holds that $\{a\} \Subset_A a^{\mathcal{R} \circ \mathcal{S}}$, which means that $\{a\} \Subset_A b^{\mathcal{S}}$, for all $b \in a^{\mathcal{R}}$. This is also equivalent, by property (i) already proven, to $\{b\} \Subset_B a^{\mathcal{R}}$, for all $b \in a^{\mathcal{R}}$; hence $a^{\mathcal{R}} \propto a^{\mathcal{R}}$. The proof of $b^{\mathcal{S}} \propto b^{\mathcal{S}}$, for all $b \in B$, is similar.

Conversely, assume that properties (i) and (ii) hold. Let us prove first that $\mathcal{R} \circ \mathcal{S}$ is $\Subset$-inflationary. For all $a \in A$, given $x \in a^{\mathcal{R} \circ \mathcal{S}} = \bigcup_{b \in a^{\mathcal{R}}} b^{\mathcal{S}}$, there exists $b \in a^{\mathcal{R}}$ such that $x \in b^{\mathcal{S}}$. Since $a^{\mathcal{R}}$ is a clique by hypothesis, we have that $\{b\} \Subset_B a^{\mathcal{R}}$, which is equivalent to $\{a\} \Subset_A b^{\mathcal{S}}$. In particular, $a\ \tau\ x$, which proves that $a \Subset_A a^{\mathcal{R} \circ \mathcal{S}}$, for all $a \in A$. The proof that $\mathcal{S} \circ \mathcal{R}$ is $\Subset$-inflationary is similar.

Finally, instead of proving that $\mathcal{R}$ is $\Subset$-antitone, we will prove that it is $\propto$ -antitone. Assume that $a_1\ \tau\ a_2$, by the previous paragraph and Remark 2, we have that $\{a_2\} \propto a_2^{\mathcal{R} \circ \mathcal{S}}$. Therefore, we obtain that $\{a_1\} \propto a_2^{\mathcal{R} \circ \mathcal{S}}$. Now, given an arbitrary $b \in a_2^{\mathcal{R}}$ and $a_3 \in b^{\mathcal{S}}$ (so $a_3 \in a_2^{\mathcal{R} \circ \mathcal{S}}$), we have $a_1\ \tau\ a_3$, which implies $\{a_1\} \propto b^{\mathcal{S}}$ by Lemma 1. By our hypothesis, this is equivalent to $\{b\} \propto a_1^{\mathcal{R}}$. Therefore, we obtain $a_2^{\mathcal{R}} \propto a_1^{\mathcal{R}}$. □

The following result shows that our definition of $\Subset$-based relational Galois connection is equivalent to the corresponding $\propto$-version.

Proposition 2. *$(\mathcal{R}, \mathcal{S})$ is a relational Galois connection between $\mathbb{A}$ and $\mathbb{B}$ iff the following properties hold:*

(i) $\mathcal{R}$ *and* $\mathcal{S}$ *are* $\propto$*-antitone.*
(ii) $\mathcal{R} \circ \mathcal{S}$ *and* $\mathcal{S} \circ \mathcal{R}$ *are* $\propto$*-inflationary.*

Proof. Assume that $(\mathcal{R}, \mathcal{S})$ is a relational Galois connection, then clearly $\mathcal{R} \circ \mathcal{S}$ and $\mathcal{S} \circ \mathcal{R}$ are $\propto$-inflationary by Remark 2. As $\mathcal{R}$ is $\Subset$-antitone and, by Proposition 1, $a^{\mathcal{R}}$ is a clique for all $a \in A$, Lemma 1(iii) states that $\mathcal{R}$ is $\propto$-antitone as well (similarly for $\mathcal{S}$).

The other implication is straightforward. □

The crisp version of the definition of fuzzy adjunction presented in [6] suggests the following property for a relational Galois connection:

$$a\ \tau\ x \iff b\ \tau\ y, \qquad \text{for all } a \in A, b \in B, x \in b^{\mathcal{S}} \text{ and } y \in a^{\mathcal{R}}\,. \tag{3}$$

It is easy to check that (3) implies condition (1). However, the converse is not true, as Example 6 shows because, for instance, $3\ \tau\ 3$ and $3 \in 3^{\mathcal{S}}$, while $3 \not\tau\ 1$ and $1 \in 3^{\mathcal{R}}$.

Nevertheless, we have the following result.

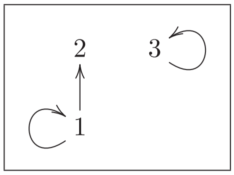
Proposition 3. *If $(\mathcal{R}, \mathcal{S})$ is a relational Galois connection between $\mathbb{A}$ and $\mathbb{B}$, then property (3) holds.*

Proof. Assume that $(\mathcal{R}, \mathcal{S})$ is a relational Galois connection, and consider $a \in A$, $b \in B$, $x \in b^{\mathcal{S}}$ and $y \in a^{\mathcal{R}}$, such that $a\ \tau\ x$. By antitonicity, we have $x^{\mathcal{R}} \Subset_A a^{\mathcal{R}}$. Then, for all $y \in a^{\mathcal{R}}$, there exists $y' \in x^{\mathcal{R}}$ such that $y'\ \tau\ y$; note that $y' \in b^{\mathcal{S} \circ \mathcal{R}}$ and, by inflation, we have that $b\ \tau\ y'$. By transitivity, we obtain that $b\ \tau\ y$. The other implication of (3) can be proved similarly. □

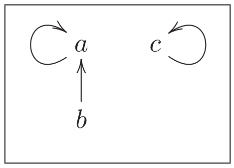
The converse of Proposition 3 does not hold, as the following example shows.

Example 7. Consider the T-digraphs $\mathbb{A}$, $\mathbb{B}$, and the relations $\mathcal{R}$ and $\mathcal{S}$ given as follows:

$\mathbb{A}$



$\mathbb{B}$



x	$x^{\mathcal{R}}$	x	$x^{\mathcal{S}}$
1	$\{a\}$	a	$\{1\}$
2	$\{b\}$	b	$\{1,2\}$
3	$\{c\}$	c	$\{3\}$

It is easy to check that $(\mathcal{R},\mathcal{S})$ verifies property (3), but it is not a relational Galois connection, because $\{1\} \in 2^{\mathcal{R}\circ\mathcal{S}}$, while $2 \not\tau 1$, which contradicts $\{2\} \Subset_A 2^{\mathcal{R}\circ\mathcal{S}}$.

Once again, another characterization of a relational Galois connection can be given in terms of property (3) if the additional clique condition is imposed.

Corollary 1. *$(\mathcal{R},\mathcal{S})$ is a relational Galois connection between $\mathbb{A}$ and $\mathbb{B}$ iff the following properties hold:*

(i) *$a\,\tau\,x$ iff $b\,\tau\,y$, for all $a \in A$, $b \in B$, $x \in b^{\mathcal{S}}$ and $y \in a^{\mathcal{R}}$.*
(ii) *$a^{\mathcal{R}}$ and $b^{\mathcal{S}}$ are cliques for all $a \in A$ and $b \in B$.*

Proof. Consequence of Propositions 1 and 3. □

It is interesting to note that the clique condition can be substituted by the inflationary property of both compositions $\mathcal{R}\circ\mathcal{S}$ and $\mathcal{S}\circ\mathcal{R}$. More specifically, we have the following result.

Proposition 4. *$(\mathcal{R},\mathcal{S})$ is a relational Galois connection between $\mathbb{A}$ and $\mathbb{B}$ iff the following properties hold:*

(i) *$a\,\tau\,x$ iff $b\,\tau\,y$, for all $a \in A$, $b \in B$, $x \in b^{\mathcal{S}}$ and $y \in a^{\mathcal{R}}$.*
(ii) *$\mathcal{R}\circ\mathcal{S}$ and $\mathcal{S}\circ\mathcal{R}$ are $\Subset$ -inflationary.*

Proof. The definition of a relational Galois connection includes property (ii); and Proposition 3 allows to obtain property (i).

Conversely, assume that properties (i) and (ii) hold. We will use Proposition 1 to prove that $(\mathcal{R},\mathcal{S})$ is a relational Galois connection. To begin with, it is straightforward that property (i) implies the Galois condition; consequently, we just have to prove that both $a^{\mathcal{R}}$ and $b^{\mathcal{S}}$ are cliques for all $a \in A$ and $b \in B$.

Due to property (ii) it holds that $a \Subset_A a^{\mathcal{R}\circ\mathcal{S}}$, hence for all $b \in a^{\mathcal{R}}$ and for all $x \in b^{\mathcal{S}}$, we have that $a\,\tau\,x$. Now, by property (i), this is equivalent to $b\,\tau\,y$ for all $y \in a^{\mathcal{R}}$. Hence, $a^{\mathcal{R}}$ is a clique. The proof that $b^{\mathcal{S}}$ is a clique for all $b \in B$ is similar. □

5. On closure relations

It is important to recall that the construction of the classical right adjoint can be stated in terms of closure operators [13,14]. As this has some advantages, in this section, we elaborate on a relational approach to the notion of closure operator and its link with the relational Galois connections, showing an adequate equilibrium between generality and preservation of properties from the standard framework.

Definition 3. Let (A,τ) be a T-digraph and $\mathcal{R} \subseteq A \times A$. The relation $\mathcal{R}$ is said to be a *closure relation* if the following properties hold:

(i) $\mathcal{R}$ is $\propto$-inflationary.
(ii) $\mathcal{R}$ is $\propto$-isotone.
(iii) $a^{\mathcal{R}\circ\mathcal{R}} \propto a^{\mathcal{R}}$ for all $a \in A$.

Note that from the above definition we get that $a^{\mathcal{R}} \propto a^{\mathcal{R}\circ\mathcal{R}}$ for all $a \in A$, as a consequence of $\mathcal{R}$ being $\propto$-inflationary and $\propto$-isotone. As a result, we would have both $a^{\mathcal{R}} \propto a^{\mathcal{R}\circ\mathcal{R}}$ and $a^{\mathcal{R}\circ\mathcal{R}} \propto a^{\mathcal{R}}$, i.e., $\mathcal{R}$ is $\propto$-idempotent. Hence, condition (3) in Definition 3 can be replaced by $\mathcal{R}$ being $\propto$-idempotent.

The notion of closure relation could have been introduced as well by using the relation $\Subset$ but, in this case, we should require explicitly $a^{\mathcal{R}}$ to be a clique, for all $a \in A$. In Definition 3, this condition trivially follows from $\mathcal{R}$ being $\propto$-idempotent.

Proposition 5. *Let $\mathbb{A}$ be a T-digraph and $\mathcal{R} \subseteq A \times A$. The relation $\mathcal{R}$ is a closure relation if and only if it satisfies the following properties:*

(i) *$\mathcal{R}$ is $\Subset$-inflationary.*
(ii) *$\mathcal{R}$ is $\Subset$-isotone.*
(iii) *$a^{\mathcal{R}\circ\mathcal{R}} \Subset a^{\mathcal{R}}$, for all $a \in A$.*
(iv) *$a^{\mathcal{R}}$ is a clique, for all $a \in A$.*

Proof. Suppose that $\mathcal{R}$ is a closure relation. Since $\propto$ is stronger than $\Subset$, $\mathcal{R}$ is trivially $\Subset$-inflationary, $\Subset$-isotone and $a^{\mathcal{R}\circ\mathcal{R}} \Subset a^{\mathcal{R}}$. Moreover, $a^{\mathcal{R}}$ is a clique because $\mathcal{R}$ is $\propto$-idempotent.

Conversely, assume that $\mathcal{R}$ satisfies properties (i)–(iv). Firstly, $\mathcal{R}$ is trivially $\propto$-inflationary; moreover, since $a^{\mathcal{R}}$ is a clique for all $a \in A$, by Lemma 1(iii), it holds that $\mathcal{R}$ is $\Subset$-isotone if and only if $\mathcal{R}$ is $\propto$-isotone. Let us now prove that $a^{\mathcal{R}\circ\mathcal{R}}$ is a clique: given $x_1, x_2 \in a^{\mathcal{R}\circ\mathcal{R}}$, there exist $a_1, a_2 \in a^{\mathcal{R}}$ such that $x_i \in a_i^{\mathcal{R}}$, for $i \in \{1, 2\}$. As $a^{\mathcal{R}}$ is a clique, it holds that $a_1 \ \tau\ a_2$ which implies $a_1^{\mathcal{R}} \propto a_2^{\mathcal{R}}$; hence, $x_1 \ \tau\ x_2$. Similarly, $a^{\mathcal{R}\circ\mathcal{R}}$ being a clique, it is straightforward that property (iii) is equivalent to $a^{\mathcal{R}\circ\mathcal{R}} \propto a^{\mathcal{R}}$ and thus, we conclude that $\mathcal{R}$ is a closure relation. $\square$

The previous proposition shows that $\propto$ is precisely the powering that allows to lift the notion of closure operator to the relational framework. The following example shows that $\mathcal{R}$ being $\Subset$-inflationary, $\Subset$-isotone and $\Subset$-idempotent does not guarantee that $\mathcal{R}$ is a closure relation.

Example 8. Consider $A = [0, 1]$ with the usual linear order, and $\mathcal{R} \subseteq A \times A$ defined by $0^{\mathcal{R}} = \,]0, 1]$ and $x^{\mathcal{R}} = [x, 1]$, for all $x \in \,]0, 1]$. It is easy to check that $\mathcal{R}$ is $\Subset$-inflationary, $\Subset$-isotone and $\Subset$-idempotent. However, $\mathcal{R}$ is not a closure relation because, for instance, $0^{\mathcal{R}}$ is not a clique.

We can now prove the following technical result, which reflects the usual behaviour of Galois connections in the standard framework.

Lemma 2. *Let $(\mathcal{R}, \mathcal{S})$ be a relational Galois connection, then $a^{\mathcal{R}} \approx a^{\mathcal{R}\circ\mathcal{S}\circ\mathcal{R}}$ and $b^{\mathcal{S}} \approx b^{\mathcal{S}\circ\mathcal{R}\circ\mathcal{S}}$ for all $a \in A$ and $b \in B$.*

Proof. Let us prove that $a^{\mathcal{R}} \propto a^{\mathcal{R}\circ\mathcal{S}\circ\mathcal{R}}$ and $a^{\mathcal{R}\circ\mathcal{S}\circ\mathcal{R}} \propto a^{\mathcal{R}}$ for all $a \in A$.

Given $y \in a^{\mathcal{R}}$, since $\mathcal{S}\circ\mathcal{R}$ is $\propto$-inflationary, we have $\{y\} \propto y^{\mathcal{S}\circ\mathcal{R}}$ and this, together with $a^{\mathcal{R}} \propto \{y\}$, implies $a^{\mathcal{R}} \propto y^{\mathcal{S}\circ\mathcal{R}}$ for all $y \in a^{\mathcal{R}}$. Now, by definition of $a^{\mathcal{R}\circ\mathcal{S}\circ\mathcal{R}}$, this means that $a^{\mathcal{R}} \propto a^{\mathcal{R}\circ\mathcal{S}\circ\mathcal{R}}$. Conversely, given $y_1 \in a^{\mathcal{R}}$ and $y_2 \in a^{\mathcal{R}\circ\mathcal{S}\circ\mathcal{R}}$, it holds that $y_2 \in x_2^{\mathcal{R}}$, for some $x_2 \in a^{\mathcal{R}\circ\mathcal{S}}$. Now, since $\mathcal{R}\circ\mathcal{S}$ is $\propto$-inflationary, we have that $a \ \tau\ x_2$, which implies, since $\mathcal{R}$ is antitone, that $x_2^{\mathcal{R}} \propto a^{\mathcal{R}}$ and, hence, $y_2 \ \tau\ y_1$, which yields $a^{\mathcal{R}\circ\mathcal{S}\circ\mathcal{R}} \propto a^{\mathcal{R}}$. $\square$

Theorem 1. *Let $(\mathcal{R}, \mathcal{S})$ be a relational Galois connection between two T-digraphs (A, τ) and (B, τ). Then, $\mathcal{R}\circ\mathcal{S}$ and $\mathcal{S}\circ\mathcal{R}$ are closure relations.*

Proof. We prove that $\mathcal{R}\circ\mathcal{S}$ is a closure relation by using the definition, namely, by showing that it is $\propto$-inflationary, $\propto$-isotone and $\propto$-idempotent. The proofs are similar for the other composition.

1. $\mathcal{R}\circ\mathcal{S}$ is $\propto$-inflationary by Proposition 2.
2. $\mathcal{R}\circ\mathcal{S}$ is $\propto$-isotone, i.e., if $a_1 \ \tau\ a_2$, then $a_1^{\mathcal{R}\circ\mathcal{S}} \propto a_2^{\mathcal{R}\circ\mathcal{S}}$, for all $a_1, a_2 \in A$. Assume that $a_1 \ \tau\ a_2$ and consider $x_1 \in a_1^{\mathcal{R}\circ\mathcal{S}}$ and $x_2 \in a_2^{\mathcal{R}\circ\mathcal{S}}$. This means that $x_1 \in b_1^{\mathcal{S}}$ and $x_2 \in b_2^{\mathcal{S}}$, for some $b_1 \in a_1^{\mathcal{R}}$ and $b_2 \in a_2^{\mathcal{R}}$. As $\mathcal{R}$ is antitone, we obtain $a_2^{\mathcal{R}} \propto a_1^{\mathcal{R}}$, hence $b_2 \ \tau\ b_1$. Similarly, as $\mathcal{S}$ is antitone, we obtain $b_1^{\mathcal{S}} \propto b_2^{\mathcal{S}}$ and $x_1 \ \tau\ x_2$.
3. Now, given two elements $x_1 \in a^{\mathcal{R}\circ\mathcal{S}}$ and $x_2 \in a^{\mathcal{R}\circ\mathcal{S}\circ\mathcal{R}\circ\mathcal{S}}$, there exist $b_1 \in a^{\mathcal{R}}$ and $b_2 \in a^{\mathcal{R}\circ\mathcal{S}\circ\mathcal{R}}$ such that $x_i \in b_i^{\mathcal{S}}$ for $i \in \{1, 2\}$. By Lemma 2, we have $b_1 \tau b_2$ and $b_2 \tau b_1$ which, since $\mathcal{S}$ is antitone, leads to $b_2^{\mathcal{S}} \propto b_1^{\mathcal{S}}$ and $b_1^{\mathcal{S}} \propto b_2^{\mathcal{S}}$. This shows that $\mathcal{R}\circ\mathcal{S}$ is idempotent.

$\square$

To conclude this section, we introduce the corresponding notion of closure system in our framework. First of all, given a T-digraph (A, τ) and $X \subseteq A$, we denote

$$m(X) = \{a \in X \mid a \propto X\}.$$

It is not difficult to check that $m(X)$ is a clique for all $X \subseteq A$.

Definition 4. Let (A, τ) be a T-digraph. A subset $C \subseteq A$ is said to be a *relational closure system* in A if $m(a^{\tau} \cap C)$ is non-empty, for all $a \in A$.

In order to study the properties of relational closure systems and how they relate to closure relations in the framework of T-digraphs, we introduce the following auxiliary notions.

Definition 5. Let $\mathbb{A} = (A, \tau)$ be a T-digraph

- Given $C \subseteq A$, the *reflexive kernel of* C is defined as $C^{\circ} = \{x \in C \mid x \ \tau\ x\}$.
- The *symmetric kernel relation* on $\mathbb{A}$ is the relation $\approx$ on 2^A defined as follows[3] for $X, Y \subseteq A$:

$$X \approx Y \quad \text{if} \quad X \propto Y \ \text{ and } \ Y \propto X.$$

Note that $X \approx Y$ is equivalent to $x \ \tau\ y$ and $y \ \tau\ x$, for all $x \in X$ and $y \in Y$.

The following theorem states that the notions of relational closure system and closure relation keep being interdefinable in the framework of T-digraphs.

Theorem 2. *Let (A, τ) be a T-digraph, $C \subseteq A$ and $\mathcal{R} \subseteq A \times A$.*

(i) *If C is a relational closure system, then $\mathcal{R}_C \subseteq A \times A$ defined by $a^{\mathcal{R}_C} = m(a^{\tau} \cap C)$ is a closure relation.*

[3] Note that $\approx$ depends ultimately on the relation τ.

(ii) *If $\mathcal{R}$ is a closure relation, then $C_{\mathcal{R}} = \left\{x \in A \mid x^{\mathcal{R}} \approx \{x\}\right\}$ is a relational closure system. Moreover, it holds that $\mathcal{R} \subseteq \mathcal{R}_{C_{\mathcal{R}}}$.*

(iii) *If C is a relational closure system, then $C_{\mathcal{R}_C} = \bigcup_{z \in C} \left\{x \in A \mid \{x\} \approx \{z\}\right\}$. Furthermore, it holds that $C^\circ \subseteq C_{\mathcal{R}_C} \subseteq A^\circ$.*

(iv) *If $\mathcal{R}$ is a closure relation, then $a^{\mathcal{R}_{C_{\mathcal{R}}}} = \{x \in A \mid \{x\} \approx a^{\mathcal{R}}\}$, for all $a \in A$. In particular, $a^{\mathcal{R}_{C_{\mathcal{R}}}} \approx a^{\mathcal{R}}$, for all $a \in A$.*

Proof.

(i) It is obvious that $\mathcal{R}_C$ is inflationary, since $a^{\mathcal{R}_C} \subseteq a^\tau$. Now, in order to prove that $\mathcal{R}_C$ is isotone, take $a_1 \;\tau\; a_2$ and $x_i \in m(a_i^\tau \cap C)$ for $i \in \{1, 2\}$. We have that $a_1 \;\tau\; a_2 \;\tau\; x_2$, which implies $a_1 \;\tau\; x_2$ by transitivity. Then, $x_2 \in a_1^\tau \cap C$ and therefore $x_1 \;\tau\; x_2$, i.e., $a_1^{\mathcal{R}} \propto a_2^{\mathcal{R}}$.
Let us show that $\mathcal{R}_C$ is idempotent: for all $y \in a^{\mathcal{R}_C \circ \mathcal{R}_C}$ there exists $x \in a^{\mathcal{R}_C}$ such that $y \in x^{\mathcal{R}_C}$. Since, by definition, $a^{\mathcal{R}_C}$ and $x^{\mathcal{R}_C}$ are cliques, we have that $x \in x^{\mathcal{R}_C}$, whence $\{x\} \approx \{y\}$; Lemma 1 implies that $y \propto a^{\mathcal{R}_C}$, proving that $a^{\mathcal{R}_C \circ \mathcal{R}_C} \propto a^{\mathcal{R}_C}$ for all $a \in A$.

(ii) We will prove that $\varnothing \neq m(a^\tau \cap C_{\mathcal{R}})$ by showing that $\mathcal{R} \subseteq \mathcal{R}_{C_{\mathcal{R}}}$. More specifically, we will show that $\varnothing \neq a^{\mathcal{R}} \subseteq m(a^\tau \cap C_{\mathcal{R}})$.
Since $\mathcal{R}$ is inflationary, we have that $a^{\mathcal{R}} \subseteq a^\tau$. For any element $y \in a^{\mathcal{R}}$, since $\mathcal{R}$ is idempotent, we have $y^{\mathcal{R}} \approx a^{\mathcal{R}}$ and, in particular, $y^{\mathcal{R}} \approx \{y\}$, whence $y \in C_{\mathcal{R}}$. Consider now $z \in a^\tau \cap C_{\mathcal{R}}$ and let us prove that $a^{\mathcal{R}} \propto \{z\}$; from $a \;\tau\; z$, we obtain $a^{\mathcal{R}} \propto z^{\mathcal{R}}$, and from $z \in C_{\mathcal{R}}$ we deduce that $a^{\mathcal{R}} \propto \{z\}$ by Remark 1.

(iii) Given $x \in C_{\mathcal{R}_C}$, by definition, $\{x\} \approx m(x^\tau \cap C) = x^{\mathcal{R}_C} \subseteq C$. Then there exists an element $z \in C$ such that $\{x\} \approx \{z\}$.
On the other hand, consider $x \in A$ such that there exists $z \in C$ with $\{x\} \approx \{z\}$. Since $x \;\tau\; z$, we have that $z \in x^\tau \cap C$; now, for all $y \in x^\tau \cap C$, we have $z \;\tau\; x$ and $x \;\tau\; y$, therefore $z \;\tau\; y$ and $z \in m(x^\tau \cap C)$. Now, as $m(x^\tau \cap C)$ is a clique, again by using Lemma 1, we get $x \in C_{\mathcal{R}_C}$.
The proof of the chain of inclusions is straightforward.

(iv) By items (i) and (ii) above, we have that $\mathcal{R}_{C_{\mathcal{R}}}$ is a closure relation and $a^{\mathcal{R}} \subseteq a^{\mathcal{R}_{C_{\mathcal{R}}}}$. Then, by Proposition 5, we have that $a^{\mathcal{R}_{C_{\mathcal{R}}}}$ is a clique and, therefore, for all $x \in a^{\mathcal{R}_{C_{\mathcal{R}}}}$, we have that $\{x\} \approx a^{\mathcal{R}}$.
Conversely, assuming that $\{y\} \approx a^{\mathcal{R}}$, since $\mathcal{R}$ is isotone, we get $y^{\mathcal{R}} \approx a^{\mathcal{R} \circ \mathcal{R}}$. Now, as $\mathcal{R}$ is idempotent, we have that $a^{\mathcal{R} \circ \mathcal{R}} \approx a^{\mathcal{R}}$, hence $y^{\mathcal{R}} \approx \{y\}$, which means that $y \in C_{\mathcal{R}}$. On the other hand, as $\mathcal{R}$ is inflationary, we have that $a^{\mathcal{R}} \subseteq a^\tau$, which implies, by using the hypothesis, that $y \in a^\tau$. Finally, let us prove that $y \in m(a^\tau \cap C_{\mathcal{R}})$. Consider $z \in a^\tau \cap C_{\mathcal{R}}$, which implies that $a^{\mathcal{R}} \propto z^{\mathcal{R}}$ and $z^{\mathcal{R}} \propto \{z\}$. This, together with the fact that $\{y\} \approx a^{\mathcal{R}}$, implies, by transitivity, that $y \;\tau\; z$, as desired.

□

The following corollaries show some interesting properties concerning the iteration of closure systems and closure relations.

Corollary 2. *Let (A, τ) be a T-digraph, $C \subseteq A$ be a relational closure system and $\mathcal{R} \subseteq A \times A$ be a closure relation.*

(i) *$\mathcal{R}_{C_{\mathcal{R}}} = \mathcal{R}$ if and only if $\{x\} \approx a^{\mathcal{R}}$ implies $x \in a^{\mathcal{R}}$ for all $x, a \in A$.*
(ii) *$C_{\mathcal{R}_C} = C^\circ$ if and only if for all $x \in A$ and $z \in C$, $\{x\} \approx \{z\}$ implies $x \in C$.*

Proof. Straightforward. □

Corollary 3. *Let (A, τ) be a T-digraph.*

(i) *Let $\mathcal{R}_1, \mathcal{R}_2 \subseteq A \times A$ be two closure relations, then*

$$C_{\mathcal{R}_1} = C_{\mathcal{R}_2} \quad \text{iff} \quad \mathcal{R}_1 \subseteq \mathcal{R}_{C_{\mathcal{R}_2}} \quad \text{iff} \quad \mathcal{R}_2 \subseteq \mathcal{R}_{C_{\mathcal{R}_1}}.$$

(ii) *Let $C_1, C_2 \subseteq A$ be two relational closure systems, then $\mathcal{R}_{C_1} = \mathcal{R}_{C_2}$ iff $C_1^\circ = C_2^\circ$.*
(iv) *Let $\mathcal{R} \subseteq A \times A$ be a closure relation, then $C_{\mathcal{R}_{C_{\mathcal{R}}}} = C_{\mathcal{R}}$.*
(iii) *Let $C \subseteq A$ be a relational closure system, then $\mathcal{R}_C = \mathcal{R}_{C_{\mathcal{R}_C}}$ iff $C_{\mathcal{R}_C} \subseteq C$.*

Proof.

(i) Note that $a^{\mathcal{R}_1} \subseteq a^{\mathcal{R}_{C_{\mathcal{R}_2}}} = \{x \in A \mid \{x\} \propto a^{\mathcal{R}_2}\}$ for all $a \in A$ is equivalent to $a^{\mathcal{R}_1} \approx a^{\mathcal{R}_2}$ for all $a \in A$. From the definition of $C_{\mathcal{R}_i}$ and transitivity, it is straightforward that $a^{\mathcal{R}_1} \approx a^{\mathcal{R}_2}$ for all $a \in A$ if and only if $C_{\mathcal{R}_1} = C_{\mathcal{R}_2}$.

(ii) $\mathcal{R}_{C_1} \subseteq \mathcal{R}_{C_2}$ implies $C_1^\circ \subseteq C_2^\circ$ because, for all $x \in C_1^\circ$, we have $x \;\tau\; x$, hence it holds that $x \in m(x^\tau \cap C_1) \subseteq m(x^\tau \cap C_2)$ and, therefore, $x \in C_2^\circ$.
Assume that $C_1^\circ = C_2^\circ$ and let us prove that $a^{\mathcal{R}_{C_1}} \subseteq a^{\mathcal{R}_{C_2}}$ for all $a \in A$. If $x \in m(a^\tau \cap C_1)$, then we have that $a \;\tau\; x \;\tau\; x$ and $x \in C_1^\circ$. Therefore, by hypothesis, $x \in C_2^\circ \subseteq C_2$. In addition, for all $z \in a^\tau \cap C_2$ and all $y \in m(a^\tau \cap C_2)$, we have that $y \;\tau\; z$. Now, since $y \in C_2^\circ = C_1^\circ$, we have that $x \;\tau\; y \;\tau\; z$. Therefore, $x \in m(a^\tau \cap C_2) = a^{\mathcal{R}_{C_2}}$.

(iii) Following item (i), we have $C_{\mathcal{R}_{C_{\mathcal{R}}}} = C_{\mathcal{R}}$ if and only if $\mathcal{R}_{C_{\mathcal{R}}} \subseteq \mathcal{R}_{C_{\mathcal{R}}}$, which trivially holds.

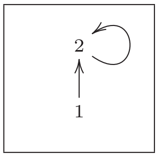
(iv) It is straightforward from Theorem 2(iii) and item (ii) above.

□

The following examples show that neither $C_{\mathcal{R}_C} \subseteq C$ nor $C \subseteq C_{\mathcal{R}_C}$ hold in general for a given relational closure system $C \subseteq A$.

Example 9. Consider $A = \{1, 2\}$ and the relation

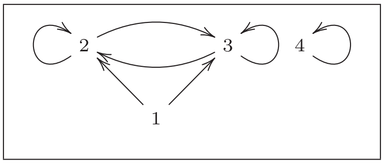
$\mathcal{T}$



For the relational closure system $C = \{1, 2\}$, we have that $\mathcal{R}_C = \{(1, 2), (2, 2)\}$ and $C \not\subseteq C_{\mathcal{R}_C} = \{2\}$.

Example 10. Consider $A = \{1, 2, 3, 4\}$ and the relation

$\mathcal{T}$



The subset $C = \{3, 4\}$ is a relational closure system, and it is routine to prove that $C_{\mathcal{R}_C} = \{2, 3, 4\}$ and $C_{\mathcal{R}_C} \not\subseteq C$.

6. Closure-based construction of relational Galois connections between T-digraphs

In this section, we introduce a necessary and sufficient condition to build a relational Galois connection given a T-digraph and a binary relation. The construction is given in terms of relational closure systems, following the line of [13].

Theorem 3. *Let $\mathbb{A} = (A, \tau)$ be a T-digraph and $\mathcal{R} \subseteq A \times B$ be a relation. Then there exists a transitive relation τ' on B, and a relation $\mathcal{S} \subseteq B \times A$ such that $(\mathcal{R}, \mathcal{S})$ is a relational Galois connection between (A, τ) and (B, τ') if and only if there exists a relational closure system $C \subseteq A$ such that the following condition holds:*

$$\text{if} \quad a_1^{\mathcal{R}} \cap a_2^{\mathcal{R}} \neq \varnothing \quad \text{then} \quad a_1^{\tau} \cap C = a_2^{\tau} \cap C. \tag{4}$$

Proof. Assume that there exists a transitive relation τ' on B, and a relation $\mathcal{S} \subseteq B \times A$ such that $(\mathcal{R}, \mathcal{S})$ is a relational Galois connection between (A, τ) and (B, τ'). From Theorems 1 and 2(ii), we have that $C_{\mathcal{R}\circ\mathcal{S}} = \left\{x \in A \mid x^{\mathcal{R}\circ\mathcal{S}} \approx \{x\}\right\} \subseteq A$ is a relational closure system.

Let us now prove that condition (4) holds. Consider $a_1, a_2 \in A$ such that $a_1^{\mathcal{R}} \cap a_2^{\mathcal{R}} \neq \varnothing$ and $x \in a_1^{\tau} \cap C_{\mathcal{R}\circ\mathcal{S}}$. Then, $a_1 \, \tau \, x$, which implies $x^{\mathcal{R}} \propto a_1^{\mathcal{R}}$. For any element $b \in a_1^{\mathcal{R}} \cap a_2^{\mathcal{R}}$, it holds that $x^{\mathcal{R}} \propto \{b\}$. By Proposition 1, we have that $a_2^{\mathcal{R}}$ is a clique and, by applying Lemma 1, we obtain $x^{\mathcal{R}} \propto a_2^{\mathcal{R}}$. Therefore, for all $y \in x^{\mathcal{R}}$, we have $\{y\} \propto a_2^{\mathcal{R}}$, which is equivalent to $\{a_2\} \propto y^{\mathcal{S}}$. Since $y \in x^{\mathcal{R}}$, on the one hand, by definition, $y^{\mathcal{S}} \subseteq x^{\mathcal{R}\circ\mathcal{S}}$; on the other hand, $x \in C_{\mathcal{R}\circ\mathcal{S}}$ implies $x^{\mathcal{R}\circ\mathcal{S}} \propto \{x\}$, hence, $y^{\mathcal{S}} \propto \{x\}$ which implies $a_2 \, \tau \, x$. This proves that $x \in a_2^{\tau} \cap C_{\mathcal{R}\circ\mathcal{S}}$.

Conversely, assume now that there exists a relational closure system $C \subseteq A$ that satisfies condition (4). In order to construct the relations $\mathcal{S} \subseteq B \times A$ and τ' on B, we fix an arbitrary element $a_0 \in A$; by the axiom of choice, there exists a mapping $\xi\colon B \to A$ such that the following condition holds:

$$\text{If } b \notin \operatorname{rng}(\mathcal{R}) \text{ then } \xi(b) = a_0, \text{ otherwise } b \in \xi(b)^{\mathcal{R}}. \tag{5}$$

The relation $\mathcal{S} \subseteq B \times A$ is defined as follows:

$$b^{\mathcal{S}} = m(\xi(b)^{\tau} \cap C) = \xi(b)^{\mathcal{R}_C}. \tag{6}$$

The definition of $\mathcal{S}$ does not depend on the choice of the mapping ξ, because if $b \in a_1^{\mathcal{R}} \cap a_2^{\mathcal{R}}$, then condition (4) guarantees that $a_1^{\tau} \cap C = a_2^{\tau} \cap C$. Note that $\mathcal{S}$ is total because C is a relational closure system.

We now define the relation τ' as follows:

$$b_1 \tau' b_2 \qquad \text{iff} \qquad b_2^{\mathcal{S}} \propto b_1^{\mathcal{S}}. \tag{7}$$

Let us prove that τ' is transitive: if $b_1 \tau' b_2$ and $b_2 \tau' b_3$ it holds that $b_2^{\mathcal{S}} \propto b_1^{\mathcal{S}}$ and $b_3^{\mathcal{S}} \propto b_2^{\mathcal{S}}$. By Remark 1, we obtain $b_3^{\mathcal{S}} \propto b_1^{\mathcal{S}}$.

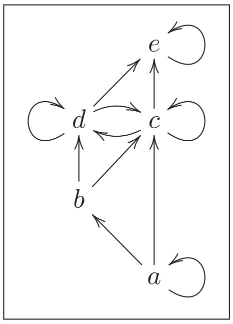
To show that $(\mathcal{R}, \mathcal{S})$ is a relational Galois connection, we will check the conditions of Proposition 1. By definition of $\mathcal{S}$, see (6) above, trivially $b^{\mathcal{S}}$ is a clique for all $b \in B$. In addition, taking into account the definition of τ' and the fact that $b^{\mathcal{S}}$ is a clique, we have that $a^{\mathcal{R}}$ is a clique for all $a \in A$.

Now, we just have to prove the Galois condition. Assume that $\{a\} \propto b^{\mathcal{S}}$, and let us prove $\{b\} \propto a^{\mathcal{R}}$ which, by (7), is equivalent to $a^{\mathcal{R}_C} \propto b^{\mathcal{S}}$. From $a \propto b^{\mathcal{S}}$, we have that $a \, \tau \, x$ for all $x \in b^{\mathcal{S}}$, and by isotonicity of $\mathcal{R}_C$ (Theorem 2) we have $a^{\mathcal{R}_C} \propto x^{\mathcal{R}_C}$. Now, since $x \in x^{\mathcal{R}_C}$, we have $a^{\mathcal{R}_C} \propto b^{\mathcal{S}}$. Conversely, suppose $b \propto a^{\mathcal{R}}$, i.e., for any $y \in a^{\mathcal{R}}$, it holds that $y^{\mathcal{S}} = m(a^{\tau} \cap C) \propto b^{\mathcal{S}}$. Observe that for any $z \in m(a^{\tau} \cap C)$, we have $a \, \tau \, z$ and $z \propto b^{\mathcal{S}}$. By transitivity, $\{a\} \propto b^{\mathcal{S}}$. □

In the remainder of this section, we conclude the validation of the characterization by expounding a situation in which the relation obtained is not functional, and the transitive digraph induced in the codomain is not trivial.

Consider the sets $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3\}$, and the T-digraph (A, τ) and the relation $\mathcal{R} \subseteq A \times B$ where τ and $\mathcal{R}$ are defined below:

(A, τ)



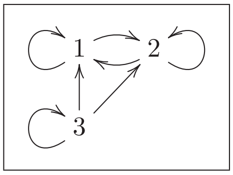
x	$x^{\mathcal{R}}$
a	$\{1, 2\}$
b	$\{2\}$
c	$\{1\}$
d	$\{1, 2\}$
e	$\{3\}$

It is straightforward to check that $C = \{c, d, e\}$ is a relational closure system (namely $m(z^\tau \cap C) \neq \varnothing$ for all $z \in A$). It is also clear that if $z_1^{\mathcal{R}} \cap z_2^{\mathcal{R}} \neq \varnothing$, then $z_1^\tau \cap C = z_2^\tau \cap C$.

For the construction of $\mathcal{S}$ we choose $\xi(1) = \xi(2) = a$ and $\xi(3) = e$, and we obtain $1^{\mathcal{S}} = 2^{\mathcal{S}} = \{c, d\}$ and $3^{\mathcal{S}} = \{e\}$.

Finally, the construction of τ' is done by the equivalence $x\tau'y \iff y^{\mathcal{S}} \propto x^{\mathcal{S}}$, and we obtain $1^{\tau'} = 2^{\tau'} = \{1, 2\}$ and $3^{\tau'} = \{1, 2, 3\}$.

(B, τ')



x	$x^{\mathcal{S}}$
1	$\{c, d\}$
2	$\{c, d\}$
3	$\{e\}$

Certainly, as a result we obtain that $(\mathcal{R}, \mathcal{S})$ is a relational Galois connection between the original T-digraph (A, τ) and the construted T-digraph (B, τ').

7. Conclusions and future extensions

There are a number of possible options to extend the notion of Galois connection to a relational setting, and choosing the most adequate one requires a trade-off between generality and preservation of properties from the standard framework. In this work, our first contribution has been the definition of the notion of relational Galois connection between transitive digraphs. We have shown the convenience of using the Smyth powering in the definition of a relational Galois connection since in this case both compositions generate closure relations. Secondly, some alternative characterizations have been obtained in terms of the relation $\propto$ which turns out to be the adequate one to straightforwardly lift the notion of closure operator to the relational case. The main contribution of this paper is the characterization theorem of the existence of a right adjoint for a given relation between T-digraphs, together with an explicit construction of this right adjoint. A validating example has been included in order to illustrate the construction.

As stated in the introduction, this paper is a two-fold extension of [4–6], because we characterize the existence of the residual (or right adjoint, or right part of a Galois connection) of a given (crisp) *relation* between *a T-digraph* and an unstructured set. Moreover, it is also an extension of the approaches developed in [8,10,26,27] where some attempts were made in order to deal with relations instead of functions.

The results in this paper can be further extended, firstly, by considering *fuzzy* T-digraphs, and maintaining the (crisp) relation; and also by considering *fuzzy* T-digraphs, and a *fuzzy* relation. These generalizations will pave the way to new approaches to Formal Concept Analysis based on the new alternative definitions of relational Galois connection. Last but not least, this approach could also contribute to further advances in the study of generalized Chu correspondences, in order to use the approach given in [18] to analyze more structures related to quantum logics, such as those in [1,19].

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

Inma P. Cabrera, Pablo Cordero, Emilio Muñoz-Velasco, Manuel Ojeda-Aciego, Bernard De Baets: Conceptualization, Methodology, Investigation, Writing - review & editing, Visualization, Supervision, Project administration, Funding acquisition.

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