Heuristic algorithm for portfolio selection with minimum transaction lots

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Abstract. Portfolio selection problem was first formulated in a paper written by Markowitz, where investment diversification can be translated into computing. Mean-variance model he introduced has been used and developed because of it's limitations in the larger constraints found in the real world, as well as it's computational complexity which found when it used in large-scale portfolio. Quadratic programming model complexity given by Markowitz has been overcome with the development of the algorithm research. They introduce a linear risk function which solve the portfolio selection problem with real constraints, i.e. minimum transaction lots. With the Mixed Integer Linear models, proposed a new heuristic algorithm that starts from the solution of the relaxation problems which allow finding close-to-optimal solutions. This algorithm is built on Mixed Integer Linear Programming (MILP) which formulated using nearest integer search method.
Key words: MILP, heuristics, portfolio optimization, minimum transaction lots, nearest integer search.

Introduction

The main objective in financial investments is to combine certain assets into the portfolio which gives the optimal profit. Optimal portfolio provides a balance between return and risk. The mathematical model was first constructed by Markowitz in his article about 50 years ago, is often difficult to use, due to it's limitations in use. Markowitz Mean-Variance model for portfolio selection is one of the best known models in the financial field and is the foundation for modern portfolio theory. But its simplicity in some assumptions, in practice, it can not work well, because this model also ignores the perceived transaction costs, liquidity constraints (and transaction costs that result from non-linear), the minimum transaction lots, and cardinality constraints, namely restrictions on portfolio to be able to be a particular asset. If all of the above assumptions applied in this model, they will produce mixed integer non-linear programming problems which substantially difficult to solve. When it applied to large-scale problems, it's difficult to obtain the exact solution, even its approximation is also becomes not simple and unrealistic.

In the original Markowitz's model assumes that asset class returns is a multivariative normally distribution. Thus the return on assets of portfolio can be described completely by the first two moments, the expectation or the mean of return and the return variance (risk measure). Optimization is attempt to find the set of portfolios that has the lowest level of risk for any particular rate of return or, alternatively, the highest return rate for any particular risk. The set is called the efficient frontier portfolios and can be determined by quadratic programming. Which is usually presented as a plot curve expected portfolio return against the standard deviation for each prediction from the return.

There are two critical assessment on the assumption of mean-variance, both as a consumption preferences of quadratic problem or asset prices are normally distributed. The downside of this model is that it assumes to be normally multivariative. Theoretically, this means that the first two moments, the expected return and variance, is not sufficient to describe the overall portfolio. This model also stated that any investor can determine the value or utility of wealth to invest in a portfolio based on expectations of return and risk of the portfolios. There is an assumption that the first two moments, the expected return and risk, sufficient to determine the investor utility function, which is usually depicted with indifference curve. If returns are not normally distributed the asset's class, investor's utility can be presented with a very different distribution but has a mean and standard deviation, the same. The capital asset pricing model (CAPM) and the arbitrage pricing theory (APT) has shown that the risk of a portfolio is systematic, that is some risk depends only on the market, with the upper limit of the average variance of the portfolio of assets divided by the number of assets in the portfolio, if the number of stock increases, the risk is growing dramatically.

In this paper, will be showed that when multiples of minimum transactions lot taken into account, the problem to determine a feasible solution can be expressed as a linear function of risk. Then a new algorithm proposed as a solution with multiple models of the minimum transaction lots. The proposed heuristic method based on the idea of building and problem solving mixed integer by considering subsets of the various investment options which are available. Subsets are generated by exploiting information obtained from a relaxation heuristics problem.

The research was done by building a portfolio selection model developed from the Markowitz model by adding constraints minimum transaction lots. Then change the quadratic programming problem (Quadratic Programming) obtained from the Markowitz model to be Mixed Integer Linear Programming (MILP) problem. Furthermore this model was developed into a heuristic algorithm.

Literature Review

The portfolio selection process consists of two stages. The first stage starts with observation and experience that end up with beliefs about the performance of securities that available in the future. The second stage starts with the relevant beliefs about future performance and ends with selecting the portfolio. To maximize the return expectations will be easier to use static models of the use of time series. As in the dynamic case if investors want to maximize the return on the portfolio, he will put all his funds into the securities that have a maximum return. Investors should diversify and he also had to maximize expected returns by distributing funds to all securities that provide maximum return expectations. Portfolio with maximum expected return does not need to be a portfolio with maximum variance. There is a level where investors can obtain the expected return by using a variance or reduce variance by reducing the expected return (Markowitz, 1952).

Traditional portfolio optimization problem is to find a plan to invest in the securities exchange that is acceptable between the level of return and risk. The mean-variance model of Markowitz (1952) is a static model of a single period to get a portfolio that can be obtained from the average rate of return given the minimum risk. Markowitz next paper suggest a number of alternative models for the same problem. Its main objective is to overcome the computational complexity of the original quadratic programming problems. For example, the linear approximation in part, Mean Absolute Deviation (MAD), Weighted Goal Programming (WGP) and minimax models (MM) (Kim, et al, 2003).

MAD models proposed by Konno and Yamazaki (1991) is a model of linear programming (LP) where risk is measured by standard deviation instead of the variance. They showed that it is equivalent to the Markowitz model if it's return is multivariative normally distribution. Then Zenios and Kang (1993) analyze a model for other asymmetric distributions and found that the model of MAD does not require a specific type of return distribution. Speranza gives the general form of MAD models using weighted risk function. He showed that the model can be built as a compact equivalent coefficient in the linear combination which is chosen accordingly. Three subsequent papers introduce a more flexible model (Speranza, 1996; Mansini, 1997; Mansini and Speranza, 1997). Furthermore Mansini and Speranza (1999) developed their research beforehand to develop three heuristics. Paper and Wijayanake Konno (2001) and Kellerer et al. (2001) calculate this problem by creating a realistic calculation features such as fixed transaction costs, and minimum transaction lots. Mansini and Ogryczak (2003) introduced a systematic view of the LP model that can be solved with further discussion of their theory.

The largest section in the portfolio selection models which widely recommended in the literature are based on the assumption of perfect division of the investment portfolio so that the distribution of any securities to be presented is a real variable. In the real world, negotiable securities transactions as a multiple of the minimum lot (hereinafter called round). By using the round, the portfolio selection problem resolution requires the determination of mixed integer programming model solution. When it applied to the real problems, it has the tractability that the integer constraints model will be the available for an algorithm that is able to find a good integer solution eventhough it's not optimal in an acceptable time. A simple heuristic has been proposed and tested for the case where there is a minimum transaction lots (Mansini and Speranza, 1997).

Portfolio Optimization's Model and Heuristic Algorithm

Markowitz' Model

Suppose R_j as a random variable rate of return (per period) from assets S_j, $j = 1,...,n$. x_j as the amount of money invested in a fund S_j of total C. Expectations return (per period) of these investments is determined as:

$$r(x_1,...,x_n) = E\left[\sum_{j=1}^{n} R_j x_j\right] = \sum_{j=1}^{n} E[R_j]x_j \tag{1}$$

An investor wants $r(x_1,...,x_n)$ as big as possible. At the same time he wants to make the risk as small as possible.

The standard deviation of return (per period):

$$\sigma(x_1,...,x_n) = \sqrt{E\left[\left\{\sum_{j=1}^{n} R_j x_j - E\left[\sum_{j=1}^{n} R_j x_j\right]\right\}^2\right]} \tag{2}$$

stating the size and risk of the portfolio optimization problema, it is formulated as a quadratic programming problem parameters:

$$\min \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j$$

Subject to

$$\sum_{j=1}^{n} r_j x_j \geq \rho C$$

$$\sum_{j=1}^{n} x_j = C$$

$$0 \leq x_j \leq u_j \qquad j = 1,...,n \tag{3}$$

Where, $r_j = E[R_j]$ and $\sigma_{ij} = E[(R_j - r_i)(R_j - r_j)]$ and ρ is the minimum rate of return required by investors. u_j the maximum amount of money that can be invested in S_j.

This model is valid if:

a. R_j multivariate normal distribution.
b. Investors are risk averse in this case it requires a smaller standard deviation.

Using the Markowitz model in large-scale portfolio optimization (full covariance matrix) is considered impractical not only because of the difficulty in its calculations, but also because of the complexity associated with the implementation of the obtained solution.

Mean – Absolute Deviation Model

Konno and Yamazaki, (1991) suggest a linear programming model of the classic quadratic models. Their approach is based on the observation that the size of different risks, volatility

and risk-related Lj are close enough and that the alternative risk measures are also suitable for portfolio optimization.
Volatility of the portfolio return is

$$\sigma(x) = \sqrt{E\left[(\sum_{j=1}^{n} (R_j - \mu_j) x_j)^2 \right]} \tag{4}$$

Where R_j states random asset returns j, μ_j is the mean.

The risk-L_j of portfolio's return is defined as:

$$w(x) = E\left[\left| \sum_{j=1}^{n} (R_j - \mu_j) x_j \right| \right] \tag{5}$$

Theorem 1 (Konno and Yamazaki) If *(R_1, R_2,..., R_n)* are multivariate normally distributed random variables, then $w(x) = \sqrt{\frac{2}{\pi}}\sigma(x)$.

Proof:

Let $(\mu_1, \mu_2, ..., \mu_n)$ are mean of $(R_1, R_2, ..., R_n)$. Let $\Sigma = (\sigma_{ij}) \in R^{nxn}$ as covarians matrix of $(R_1, R_2, ..., R_n)$. Then $\sum R_i x_i$ normally distributed where its mean is $\sum \mu_i x_i$ and its standard deviation is $\sigma(x) = \sqrt{\sum_i \sum_j \sigma_{ij} x_i x_j}$.

Thus, $w(x) = E[|U|]$ where $U \sim N(0, \sigma)$.

$$\begin{aligned} w(x) &= \frac{1}{\sqrt{2\pi}\sigma(x)} \int_{-\infty}^{\infty} |u| e^{-\frac{u^2}{2\sigma^2(x)}} du \\ &= \frac{2}{\sqrt{2\pi}\sigma(x)} \int_{-\infty}^{\infty} u e^{-\frac{u^2}{2\sigma^2(x)}} du \\ &= \sqrt{\frac{2}{\pi}}\sigma(x) \end{aligned}$$

This theorem imply that minimizing $\sigma(x)$ equivalent to minimizing *w(x)* when $(R_1, R_2, ..., R_n)$ are multivariate normally distributed.
By this assumption, Markowitz' model can be formulated as:

$$\min \quad w(x) = E\left[\left| \sum_{j=1}^{n} R_j x_j - E\left[\sum_{j=1}^{n} R_j x_j \right] \right| \right]$$

Subject to:

$$\begin{aligned} &\sum_{j=1}^{n} E[x] x_j \geq \rho C \\ &\sum_{j=1}^{n} x_j = C \\ &0 \leq x_j \leq u_j, \ j = 1, ..., n \end{aligned} \tag{6}$$

Either $(R_1, R_2, ..., R_n)$ are multivariative normally distributed or not, *mean-absolute deviation model* above will remain to form efficient portfolio to measure risk-L_j. let r_{jt} as the realization of random variable R_j during t period, for t = 1, ..., T which assumed available from the historical datum or of some future projections. It is also assumed that expectation value of random variables can be approximated by the average obtained from these datum.

Especially, let

$$r_j = E[R_j] = \frac{1}{T}\sum_{j=1}^{T} r_{jt} \tag{7}$$

Next,

$$w(x) = E\left[\left|\sum_{j=1}^{n}\left(R_j - \mu_j\right)x_j\right|\right]$$
$$= \frac{1}{T}\sum_{t=1}^{T}\left|\sum_{j=1}^{n}\left(r_{jt} - \mu_j\right)x_j\right|$$

(8)

The absolute value in this equation makes it non-linear. But, it can be linearizes by using additional variables.

Let $y_t = \left|\sum_{j=1}^{n}\left(r_{jt} - r_j\right)x_j\right|$; $y_t \geq 0$, then obtained:

$$y_t = \begin{cases} -\sum_{j=1}^{n}\left(r_{jt} - r_j\right)x_j \\ +\sum_{j=1}^{n}\left(r_{jt} - r_j\right)x_j \end{cases}$$

Further, it can be written as:

$$y_t + \sum_{j=1}^{n}\left(r_{jt} - r_j\right)x_j \geq 0 \text{ dan } y_t - \sum_{j=1}^{n}\left(r_{jt} - r_j\right)x_j \geq 0 \tag{9}$$

Then let

$a_{jt} = r_{jt} - r_j$, $j = 1, \ldots, n$, $t = 1, \ldots, T$

then (6) can be made as the following minimization problems:

$$\min \frac{\sum_{t=1}^{T}\left|\sum_{j=1}^{n} a_{jt}x_j\right|}{T}$$

subject to:

$$\sum_{j=1}^{n} r_j x_j \geq \rho\, C$$
$$\sum_{j=1}^{n} x_j = C$$
$$0 \leq x_j \leq u_j,\ j = 1,\ldots,n \tag{10}$$

Thus, from (9) and (10) obtained the following approach:

$$\min \frac{\sum_{t=1}^{T} y_t}{T}$$

subject to:

$$y_t + \sum_{j=1}^{n} a_{jt} x_j \geq 0, \ t = 1,\ldots, T$$

$$y_t - \sum_{j=1}^{n} a_{jt} x_j \geq 0, \ t = 1,\ldots, T$$

$$\sum_{j=1}^{n} r_j x_j \geq \rho \ C$$

$$\sum_{j=1}^{n} x_j = C$$

$$0 \leq x_j \leq u_j, j = 1,\ldots,n \tag{11}$$

This approach is a linear programming. Therefore it can be used to solve the large scale of portfolio optimization problems.

Mean-semi-absolute deviation model

If the risk is measure by using mean-semi-absolute deviation instead of mean-absolute deviation, then the objective function is:

$$\frac{1}{T} \sum \left| \min \left\{ 0, \sum_j \left(r_{jt} - r_j \right) x_j \right\} \right| \tag{12}$$

And can be written as:

$$\min \frac{\sum_{t=1}^{T} y_t}{T}$$

$$y_t + \sum_j \left(r_{jt} - r_j \right) x_j \geq 0, \ t = 1,\ldots,T \tag{13}$$

i.e with the smaller constraints. So, it can be seen from (11) that (10) is equivalent with (12). Thus, variance is under assumption that its return is multivariate normally distributed.

Because the model is based on semi-absolute deviation, then the risk function is linear, so that it can be introduced a spesification that obtained from the market structure such applied constraints.

Portfolio Selection Model with Minimum Transaction Lot

Briefly, the following is the notation for mixed integer model with mínimum transaction lot. The purchase Price for securities with mínimum lot j is denoted as c_j. thus, for every securities, mínimum lot can be described as Money and its equivalent with $\boldsymbol{c_j = N_j p_j}$. Where p_j is the market Price for j securities needed as mínimum quantities. So, $\boldsymbol{c_j = p_j}$ when the asset j is traded without mínimum lot. Next, C_0 and C_1 as the mínimum amount of money and the máximum that is available to be invested by an investor.

Integer variables $x_j, \ \forall j \in S$ showed the mínimum lot quantities for every j securities which bécame parts of the optimum portfolio. The quantities $c_j x_j$ showed parts of the total money to be invested in securities j. The constants d_j are varies according to market conditions and types of the agreements, showed the transaction cost proportion of purchasing, because of the transaction cost proportion can be directly included into the price. Next, assume that the price c_j is included all the possible transaction cost proportion.

Thus, the mixed integer linear programming for portfolio selection problems with minimum transaction lots is formulated as follows:

$$\min \frac{\sum_{t=1}^{T} y_t}{T}$$

Subject to $$y_t + \sum_j (r_{jt} - r_j) c_j x_j \geq 0, \quad t = 1,\dots,T$$
$$C = \sum_j c_j x_j$$
$$\sum_j r_j c_j x_j \geq \rho C$$
$$C_0 \leq C \leq C_1$$
$$0 \leq x_j \leq u_j, \ j \in S$$
$$y_t \geq 0, \ t = 1,\dots,T \tag{14}$$

Portfolio Selection with The Nearest Integer Search Method
According to the solution obtained from the model above, next a heuristic is developed to select the portfolios with the basic ideas is follows:

Assume a mixed integer linear programming (MILP):]

Min $P = C^T x$

Subject to $Ax \leq b$

$x \geq 0$

x_j integer, for all $j \in J$

Note that the feasible basis vector of MILP wich is solved as the continuous problems, and it can be written as: $(x_B)_k = \beta_k - \alpha_{ki}(x_N)_i - \ldots - \alpha_{kj*}(x_N)_{j*} - \ldots - \alpha_{k,n-m}(x_N)_{n-m}$.

Let $(x_B)_k$ be a natural valued variables, β_k is partitioned into integer and fraction components $\beta_k = [\beta_k] + f_k$. If $(x_B)_k$ is increased to the closest integer, $([\beta]+1)$ can be increased to be non basis variables, let $(x_N)_{j*}$ above di upper limit as long as α_{kj*}, i.e. one of the element vector α_{j*}, is negative.

Take Δ_{j*} one of non-basis variables $(x_N)_{j*}$ then numerical and scalar value of $(x_B)_k$ is integer, then Δ_{j*} can be stated as: $\Delta_{j*} = \dfrac{1-f_k}{-\alpha_{kj*}}$, The other non-basis variables is remain zero. So, after it is substited into Δ_{j*} for $(x_N)_{j*}$ obtained $(x_B)_k = [\beta]+1$. Now $(x_B)_k$ is an integer. It is now clear that the non-basis variables is important in rounding the basis variables values. This basic idea is also used to solve the mixed integer stochastic problems.

Heuristic algorithm of the feasible solution search method
After solving the relaxation problems with the methods before for the linear stochastics program, the searching procedure for integer region solutions can be described as follows:
Let $x = [x] + f, \quad 0 \leq f < 1$, the continuous solution of the relaxation problems are:

Step 1 choose basis i* the smallest infeasible integer, so that $\delta_{i*} = \min\{f_i, 1-f_i\}$

Step 2 Do the pricing operations, i.e. compute $v_{i*}^T = \ell_{i*}^T B^{-1}$

Step 3 Compute $\sigma_{ij} = v_{i*}^T a_j$ with j obtained from $\min_j \left\{ \left| \dfrac{\ell_i}{\sigma_{ij}} \right| \right\}$

I. for non-basis j lower limitted

- If σ_{ij} < 0 and δ_{i*} = f_i then compute $\Delta = \frac{(1-\delta_{i*})}{-\sigma_{ij}}$
- If σ_{ij} > 0 and δ_{i*} = 1 - f_i then compute $\Delta = \frac{(1-\delta_{i*})}{\sigma_{ij}}$
- If σ_{ij} < 0 and δ_{i*} = 1 - f_i then compute $\Delta = \frac{\delta_{i*}}{-\sigma_{ij}}$
- If σ_{ij} > 0 and δ_{i*} = f_i then compute $\Delta = \frac{\delta_{i*}}{\sigma_{ij}}$

II. for non-basis j upper limitted

- If σ_{ij} < 0 and δ_{i*} = 1 - f_i then compute $\Delta = \frac{(1-\delta_{i*})}{-\sigma_{ij}}$
- If σ_{ij} > 0 and δ_{i*} = f_i then compute $\Delta = \frac{(1-\delta_{i*})}{\sigma_{ij}}$
- If σ_{ij} > 0 and δ_{i*} = 1 - f_i then compute $\Delta = \frac{\delta_{i*}}{\sigma_{ij}}$
- If σ_{ij} < 0 and δ_{i*} = f_i then compute $\Delta = \frac{\delta_{i*}}{-\sigma_{ij}}$

Else go to the upper non-basis of the next superbasis j (if there is exist). So that column *j** is increased from its lower limit or decreased from its Upper limit. Else, go to the next *i**.

Step 4 Compute $\alpha_{j*} = B^{-1}a_{j*}$ i.e. find $B\alpha_{j*} = a_{j*}$ for α_{j*}

Step 5 On the feasibility test, there are 3 possible fixed feasible variables since the releasing of the non-basis from its limit.

- if *j** lower limitted then take

$$A = \underset{i' \neq i^* | \alpha_{ij^*} > 0}{\mathrm{Min}} \left\{ \frac{x_{B_{i'}} - \ell_{i'}}{\alpha_{ij^*}} \right\}$$

$$B = \underset{i' \neq i^* | \alpha_{ij^*} < 0}{\mathrm{Min}} \left\{ \frac{a_{i'} - x_{B_{i'}}}{-\alpha_{ij^*}} \right\}$$

$$C = \Delta$$

Máximum movement of *j** depend on $\theta^* = \min(A, B, C)$

- if *j** upper limitted, then take

$$A' = \underset{i' \neq i^* | \alpha_{ij^*} > 0}{\mathrm{Min}} \left\{ \frac{x_{B_{i'}} - \ell_{i'}}{-\alpha_{ij^*}} \right\}$$

$$B' = \underset{i' \neq i^* | \alpha_{ij^*} < 0}{\mathrm{Min}} \left\{ \frac{a_{i'} - x_{B_{i'}}}{\alpha_{ij^*}} \right\}$$

$$C' = \Delta$$

Máximum movement of *j** depend on $\theta^* = \min(A', B', C')$

Step 6 keep the basis for these 3 possibility

1. if *A* or *A′*, then
    - $x_{B_{i'}}$ become non-basis lower limitted $\ell_{i'}$

- x_{j*} become basis (replacing $x_{B_{i'}}$)
- x_{i*} still basis (not integer).

2. if *B* or *B′*
   - $x_{B_{i'}}$ become nonbasis upper limitted $a_{i'}$
   - x_{j*} become basis (replacing $x_{B_{i'}}$)
   - x_{i*} still basis (not integer).
3. if C or C′
   - x_{j*} become basis (replacing x_{i*})
   - x_{i*} become integer superbasis.

Repeat from step 1

Conclusions

This paper concludes that with the developed model, the covarians matrix are no longer needed to build portfolio selection model as it in the classical model of Markowitz. It is easier to solve linear programming than non-linear programming, so that it would reduce the computation time to solve it. Changes in the input data would not make a significant change on the whole model.
The T variable can be used as a control variable to limit the amount of assets in the portfolio. This method can be applied on portfolio containing any types of assets, as long as the return and the risk forecasting are available.

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