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Exact solutions for a class of matrix Riemann-Hilbert problems

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Abstract. Consider the matrix Riemann-Hilbert problem. In contrast to scalar Riemann-Hilbert problems, a general matrix Riemann-Hilbert problem cannot be solved in term of Sokhotskyi-Plemelj integrals. As far as the authors know, the only known exact solutions known are for a class of matrix Riemann-Hilbert problems with commutative and factorable kernel, and a class of homogeneous problems. This article employs the well known Shannon sampling theorem to provide exact solutions for a class of matrix Riemann-Hilbert problems. We consider matrix Riemann-Hilbert problems in which all the partial indices are zero and the logarithm of the components of the kernels and their nonhomogeneous vectors are functions of exponential type (equivalently, band-limited functions). Then, we develop exact solutions for such matrix Riemann-Hilbert problems. Several well known examples along with a remark on the case of functions not of exponential type are given.

Keywords: matrix Riemann-Hilbert problem, Wiener-Hopf factorization, Shannon sampling theorem, exponential type function, fourier transform

Introduction

Roughly speaking, the matrix Riemann-Hilbert problem is the problem of finding a *vector* of functions whose components are analytic and bounded in the upper and lower complex half planes (i.e., $C^+ := \{\lambda \in C : Im(\lambda) \geq 0\}$ and $C^- := \{\lambda \in C : Im(\lambda) \leq 0\}$, respectively) and having a prescribed jump across R. Matrix functions that are analytic in this sense may be termed *sectionally analytical* matrix functions. A homogeneous matrix Riemann-Hilbert problem with all partial indices zero is also known as a matrix *Wiener-Hopf factorization* problem.

The matrix and scalar Riemann-Hilbert problems have proven remarkably useful in solving an enormous variety of model problems in a wide range of branches of physics, mathematics, and engineering. Subjects for which the problem is applicable range from neutron transport (see Noble, 1988), geophysical fluid dynamics (Davis, 1987), di raction theory (Noble 1988), fracture mechanics (Freund 1993), non-destructive evaluation of materials (Achenbach 1973), a wide class of integral equations (Payandeh & Kucerovsky, 2010), acoustics (Abrahams & Wickham, 1990), elasticity (Norris & Achenbach, 1984), electromagnetics (Sautbekov & Nilsson, 2009), water wave phenomena (Chakrabarti & George, 1994), fracture mechanics (Freund, 1998), geophysics (Davis, 1987) and financial mathematics (Fusai et al.. 2006), distribution of extrema in a wide class of Lévy processes (Payandeh & Kucerovsky, 2011a, b), and in statistical decision problem (Kucerovsky, et al.., 2009).

The crucial step in the solution of a *Riemann-Hilbert* problem is to decompose the given kernel into a product of two sectionally analytic matrix functions. Apart from some technical di culties associated with computing the Sokhotskyi-Plemelj integral (see Abrahams, 2000 and Kucerovsky & Payandeh, 2009), such decomposition can be accomplished explicitly for scalar Riemann-Hilbert problems and may be generalized to a class of matrix Riemann-Hilbert problems where:

(i) all partial indices are zero; and
(ii) the upper and lower radial limits of the *Sokhotskyi-Plemelj* integral of the logarithm of the kernels takes values in a commutative family of matrices, in which case we expect a commutative factorization, see Heins (1950) for more detail. For 2-by-2 matrix Riemann-Hilbert problems, Khrapkov (1971a, b) and Daniele (1978), independently, suggested a procedure to determine which matrix kernels have a commutative factorization.

They proved that a sufficient condition for a 2-by-2 matrix kernel **G** to have commutative factors is that: it can be rewritten in the form $\mathbf{G}(a) = k_0(a)\mathbf{I} + k_1\mathbf{J}(a)$, where k_0 and k_1 are two arbitrary functions and $\mathbf{J}(a)$ is an entire matrix which $\mathbf{J}^2(a) = \Delta(a)\mathbf{I}$, where $\Delta^2(a)$ is a polynomial with suitable properties. This finding was generalized to the case of general n-by-n matrix kernels by Jones (1984a, b) and Benjamin et al.. (2007).

The majority of matrix kernels which appear in practical problems simply do not have commutative factors. Gohberg & Krein (1960) introduced *left standard factorization*, meaning that a sufficiently smooth n-by-n matrix kernel **G** decomposes as

$$\mathbf{G}(\omega) \;=\; \mathbf{G}_+(\omega)\mathrm{diag}\left(\left(\frac{\omega-i}{\omega+i}\right)^{\kappa_1},\cdots,\left(\frac{\omega-i}{\omega+i}\right)^{\kappa_n}\right)\mathbf{G}_-(\omega),\;\; \omega\in\mathbb{R},$$

where the entries of $\mathbf{G}_\pm$ are sectionally analytic in $C_\pm$ and the integers $\kappa_1, \cdots, \kappa_n$ are the partial indices associated with the matrix kernel **G**. They showed that solutions of a homogeneous Riemann-Hilbert problem $\mathbf{\Phi}_+(\omega) = \mathbf{G}(\omega)\mathbf{\Phi}_-(\omega)$, $\omega \in R$ are given by

$$\begin{aligned}\mathbf{\Phi}_+(\omega) &= \mathbf{G}_+(\omega)\\ \mathbf{\Phi}_-(\omega) &= \mathbf{G}_-^{-1}(\omega)\mathrm{diag}\left(\left(\frac{\omega-i}{\omega+i}\right)^{-\kappa_1},\cdots,\left(\frac{\omega-i}{\omega+i}\right)^{-\kappa_n}\right)\end{aligned}$$

Litvinchuk & Speikovskii (1987, Page 35) established that a rational matrix kernel **G** admits a left standard factorization whenever its poles and zeros do not lie on R. Moreover, Litvinchuk & Speikovskii (1987, page 241) established that if a complicated matrix kernel **G** is approximated uniformly by simple and factorable matrix kernels $\mathbf{G}_n{}^*$ such that the partial indices of **G** coincide with the partial indices of $\mathbf{G}_n{}^*$, then, for every factorization of **G**, there can be found a factoriza-tion of $\mathbf{G}_n{}^*$ for which $|\mathbf{G}_\pm - \mathbf{G}^*_\pm| < \varepsilon$. Of course, this result assumes the *a priori* existence of a factorization of **G**, and generally there is no method for deciding if a factorization does in fact exist. Abrahams replaces a complicated matrix kernel **G** with a rational matrix kernel $\mathbf{G}^*$ obtained from a Padé approximant which satisfies conditions due to Litvinchuk & Speikovskii (1987), and thus obtain a approximate solutions of a complicated matrix Riemann-Hilbert problem, see Benjamin et al.. (1987), Abrahams (1996, 1997, 1998, 2000, 2007), and Veitch & Abrahams (2007) for more detail. The process requires a Padé approximation that converges uniformly on a noncompact open strip of the complex plane, including infinity, and this seems problematic in general. The di culty seems to arise because of the interesting fact that meromorphic functions that are not already rational, even if well-behaved at infinity on the real line, generally have essential singularities in the complex plane at infinity.

This article employs the well-known Shannon sampling theorem to provide exact solutions for a class of matrix Riemann-Hilbert problems such that the nonhomogeneous vectors and the logarithm of their kernels have components given by functions of exponential type. A remark on non-exponential type functions has been made. Section 2 provides lemmas for the other sections, and our main results, including an error estimate for our approximate method of solving matrix Riemann-Hilbert problems in Section 3. Section 4 reviews our conclusions and discusses how one may employ our results for matrix Riemann-Hilbert problems with functions not of exponential type.

Preliminaries

The *Sokhotskyi-Plemelj integral* of a function **s** satisfing a Hölder condition is defined by a principal value integral:

$$\phi^s(\lambda) := \frac{1}{2\pi i}\,-\!\!\!\!\!\!\int_{\mathbb{R}}\frac{\mathbf{s}(\omega)}{\omega-\lambda}d\omega, \quad \text{for } \lambda\in\mathbb{C}.$$

The upper and lower radial limits $\phi^{s}_{\pm}(\omega) = \lim_{\lambda \to \omega + i0_{\pm}} \phi^{s}(\lambda)$ have the jump property:

$$\phi^{s}_{\pm}(\omega) = \pm s(\omega)/2 + \phi^{s}(\omega),$$

where $\omega \in$ R (Gakhov, 1990).

Computing the index or partial indices of a scalar or matrix Riemann-Hilbert problem is usually a *key step* in determining the existence and number of solutions of a scalar (or matrix) Riemann-Hilbert problem. The index of a complex-valued and smooth scalar kernel G on a smooth oriented curve Γ is defined to be the winding number of $G(\Gamma)$ about the origin. In contrast to the case of a scalar kernel, the indices of a matrix kernel apparently cannot be determined by any *a priori* mthod: one must solve the corresponding homogeneous Riemann-Hilbert problem. Thus, to evaluate the partial indices of an n-by-n matrix kernel **G** on a smooth oriented curve Γ, one has to find a fundamental solution matrix **X** that is componentwise sectionally analytical in the upper and lower complex half-planes, and such that the lower and upper radial limits, say $\mathbf{X}_{\pm}$, respectively, satisfy $\mathbf{G}(t) = \mathbf{X}_{+}(t)\mathbf{X}^{-}{}_{-}{}^{1}(t)$, for all $t \in \Gamma$. Then, the partial indices $\kappa_1 \cdots, \kappa_n$ are defined by investigating behavior of $X_{11}(t), \cdots, X_{nn}(t)$ at infinity. Some limited *a priori* information on partial indices can be found using the fact that

$$\kappa = \sum_{i=1}^{n} \kappa_i = \mathrm{ind}_{\Gamma} \det(\mathbf{G}(t))$$

more detail can be found in Ablowitz & Fokas (1997). One can thus at least find the sum of the partial indices in a simple way. We are primarily interested in the matrix Riemann-Hilbert problems with all partial indices zero. A scalar kernel **G** which is positive (or negative), continuous function, and goes to zero faster than some power has zero index on R, see Gakhov (1990, page 86) for more detail. A matrix kernel **G** has zero partial indices on R, whenever the real or imaginary part of **G** is (positive or negative) definite, det $\mathbf{G}(k)$ is nowhere zero, and and all entries of the matrix function **G** are in the algebraic ring of functions

$$\Lambda \ := \ \{c + \int_{\mathbb{R}} f(x) e^{-ix\omega} dx, \text{ where } f \in L^1(\mathbb{R}) \cap C(\mathbb{R})\},$$

see Gohberg & Krein (1960) for more detail.

The scalar Riemann-Hilbert problem is the function-theoretical problem of finding a single function Φ which is sectionally analytic in $C_{\pm}$, bounded, and its corresponding upper and lower radial limits, say $\Phi_{\pm}$, having a prescribed jump discontinuity on the real line R, i.e.,

$$\Phi_{+}(\omega) = G(\omega)\Phi_{-}(\omega) + F(\omega), \quad \text{for } \omega \in \mathrm{R},$$

where kernel G and nonhomogeneous part F are two given complex-valued and continuous functions which satisfy a Hölder condition on R, and G does not vanish on R.
If a scalar Riemann-Hilbert problem with index υ has solutions, they can be found in the form

$$\Phi_{\pm}(\omega) = X_{\pm}(\omega) \ \varphi^{h}{}_{\pm}(\omega) + P_{\upsilon}(\omega) \ ,$$

where $P_{\upsilon}(\cdot)$ is a polynomial of degree υ, with arbitrary coe cients, and $P_0(\omega) = 0$, $X_{-}(\omega) = \omega^{-\upsilon} \exp\{\varphi^{k}{}_{\pm}(\omega)\}$, $X_{+}(\omega) = \exp\{\varphi^{k}{}_{\pm}(\omega)\}$, and $\varphi^{k}{}_{\pm}(\cdot)$ and $\varphi^{h}{}_{\pm}(\cdot)$ stand for the lower and upper ra-dial limits of the Sokhotskyi-Plemelj integral of two functions $k(\omega) := \ln(\omega^{-\upsilon}G(\omega))$ and $h(\omega) := F(\omega)/X^{+}(\omega)$, respectively, see Gakhov (1990, page 97) for more detail.

A matrix Riemann-Hilbert problem is, in general, far more complicated than the scalar Riemann-Hilbert problems. It is the function-theoretical problem of finding a vector of functions **Φ** which are sectionally analytic, bounded, and corresponding upper and lower radial limits, say $\mathbf{\Phi}_{\pm}$, having a prescribed jump discontinuity on R, i.e.,

$\mathbf{\Phi}_{+}(\omega) = \mathbf{G}(\omega)\mathbf{\Phi}_{-}(\omega) + \mathbf{F}(\omega)$, for $\omega \in$ R, where the kernel **G** and the nonhomogeneous vector **F** are two given complex-valued and continuous matrix functions whose elements satisfy a Hölder condition on R and det $\mathbf{G}(t)$ does not vanish on R.

For both scalar and matrix Riemann-Hilbert problems, the continuity and non-vanishing properties are quite restrictive conditions. In some cases, the Riemann-Hilbert problem can be extended to handle cases with vanishing **G** or jump discontinuities of **F**, see Gakhov (1990, page 107) for the case of the scalar Riemann-Hilbert problem and Muskhelishvili (1977, page 391) for the case of the matrix Riemann-Hilbert problem.

In general, the solutions of a matrix Riemann-Hilbert problems cannot be found in closed form, *i.e.*, they cannot be expressed in terms of explicit Sokhotskyi-Plemelj integrals. To solve a matrix Riemann-Hilbert problem with zero indices one has to find a fundamental solution **X** such the components are sectionally analytical and the corresponding upper and lower radial limits, say $\mathbf{X}_{\pm}$, satisfies $\mathbf{G}(\omega) = \mathbf{X}_{+}(\omega)\mathbf{X}^{-}{}_{-}{}^{1}(\omega)$. Then, solutions of equation 4 are given by

$$\mathbf{\Phi}_{\pm}(\omega) = \mathbf{X}_{\pm}(\omega)\varphi^{\mathbf{h}}{}_{\pm}(\omega),$$

where $\varphi^{\mathbf{h}}{}_{\pm}(\cdot)$ stands for the lower and upper radial limits of the Sokhotskyi-Plemelj integral of $\mathbf{h}(\omega) = \mathbf{X}^{-}{}_{+}{}^{1}(\omega)\mathbf{F}(\omega)$, see Ablowitz & Fokas (1997) for more detail.
Due to (i) numerical problems caused by singularities near the real line in the Sokhotskyi-Plemelj integral $\varphi^{\mathbf{h}}$; and (ii) nonexistence of an explicit formula to compute $\mathbf{X}_{\pm}$, the above formula is di cult to employ in practice. An alternative method, well known as *Carlemann's method*, is based on simply observing that the unique solution of a zero partial indices matrix Riemann-Hilbert problem as in equation (4), when the partial indices are zero, is

$$\mathbf{\Phi}_{\pm}(\omega) = \mathbf{G}_{\pm}(\omega)\mathbf{H}_{\pm}(\omega),$$

where sectionally analytic matrix functions $\mathbf{G}_{\pm}$ and $\mathbf{H}_{\pm}$ respectively satisfy $\mathbf{G}(\omega) = \mathbf{G}_{+}(\omega)\mathbf{G}^{-}{}_{-}{}^{1}(\omega)$ and $\mathbf{H}(\omega) = \mathbf{H}_{+}(\omega) + \mathbf{H}_{-}(\omega)$ and $\mathbf{H}(\omega) = \mathbf{G}^{-}{}_{+}{}^{1}(\omega)\mathbf{F}(\omega)$. The above functions must be found by inspection or by some other method, hence Carelemann's method is really just a restatement of the Riemann-Hilbert problem.

Now, we collect some useful elements for the rest of the paper.
Definition 1. A function f in $L_1(R) \cap L_2(R)$ is said to be of exponential type T on the domain D if there are positive constants M and T such that $|f(\omega)| \le M \exp\{T|\omega|\}$, for $\omega \in D$. An n × m matrix function **f** is said to be of exponential type T in a domain D if the components are all of exponential type T or better.

The well known Paley-Wiener theorem states that the Fourier transform of an L^2(R) function vanishes outside of an interval $[-T, T]$, if and only if it is of exponential type T, see Dym & McKean (1972, page 158) for more detail. The exponential type functions are continuous functions which are infinitely di erentiable everywhere and have a Taylor series expansion over every interval, see Champeney (1987, page 77) and Walnut (2002, page 81). These functions are also called band-limited functions, see Bracewell (2000, page 119) for more detail on bandlimited functions (which are equivalent to exponential type functions by the above stated Paley-Wiener theorem).

The following Baker-Campbell-Hausdor theorem plays a crucial role in the rest of this article.
Theorem 1. (Baker-Campbell-Hausdor) Suppose A and B are two n-by-n matrices. Then, $\exp\{A\}\exp\{B\} = \exp\{A + B\}$, *whenever A and B commute, i.e., AB = BA.*
A proof, along with more detail, can be found in Hall (2004, page 68).

The Poisson summation formula is an equation that allows us to relate the Fourier series coe cients of the periodic summation of a function to the values of the function's continuous Fourier transform, more information can be found in Grafakos (2004) and Pinsky (2002), among others.

Lemma 1. (Poisson summation formula) Suppose g is a function in $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$*. Then*

$$\sum_{k=-\infty}^{\infty} g(x+2k\pi) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\hat{g}(n)e^{inx},$$

where $\hat{g}$ *is the Fourier transform of g.*

Main results

Using the *Shannon sampling theorem*, the following result from Kucerovsky & Payandeh (2009) is an elegant scheme that allows decomposing an exponential-type function as a sum of two sectionally analytic functions.

Lemma 2. (Kucerovsky & Payandeh, 2009) Suppose f is a function of exponential type T and is in $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$*. Then, f can be decomposed as* $f(\omega) = f_+(\omega) + f_-(\omega)$*, where* $f^{\pm}$ *and* f_+ *are sectionally analytic in* $\mathbb{C}_\pm$ *and given by*

$$f_\pm(\omega) = \pm \sum_{n=-\infty}^{+\infty} f(\frac{2n}{T})\frac{\exp\{\pm i\pi(T\omega-2n)\}-1}{2i\pi(T\omega-2n)}.$$

In many situations it is desired to approximate a complicated exponential type function f by a presumably simpler exponential type function $f^{(m)}$. The following lemma gives error bounds for the Shannon factorization under this process

Lemma 3. Suppose $f^{(m)}$ *and f are functions in* $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ *that are of exponential type T. Then,* $f^{(m)}$ *and* $f_\pm$ *given by Lemma 2 satisfy* $|f_\pm^{(m)} - f_\pm| \le A\|\hat{f} - \hat{f}^{(m)}\|_1$ *and* $|f_\pm^{(m)} - f_\pm| \le B\|f - f^{(m)}\|_2$ i.e., $f_\pm^{(m)}$ *converges uniformly to* $f_\pm$.

Proof. Choose m. We note that by the uniform convergence, the limit function f is also of exponential type T, and is in $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$. Let $g(\omega) := f - f^{(m)}$, $s(\omega) := (-\exp\{2i\pi\omega\}-1)/(4i\pi\omega)$, and $y := Tw/2$. Then, using Lemma 2, we have

$$g_+(y) = \sum_{n=-\infty}^{\infty} g(n)s(n-y).$$

Since the product gs is in $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$, we may, using the Poisson summation formula (Lemma 1) and denoting convolution by $*$, rewrite g_+ as follows:

$$\begin{aligned} g_+(y) &= \sum_{-\infty}^{\infty}(\hat{g} * (e^{i\lambda y}\hat{s}))(k) \\ &= \sum_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{g}(k-\lambda)e^{i\lambda y}\hat{s}(\lambda)d\lambda \end{aligned}$$

By a calculation one may observe that the Fourier transform of s is zero everywhere except on an interval of length 2π, where it is constant. This observation together with the Hölder inequality with exponent 2 on $[0, 2\pi]$ lets us obtain an estimate as follows:

$$
\begin{aligned}
|g_+| &\leq \sum_{-\infty}^{\infty} \int_0^{2\pi} |\hat{g}(k-\lambda)\hat{s}(\lambda)|d\lambda \\
&\leq ||\hat{s}||_2 \sum_{-\infty}^{\infty} \left\{ \int_0^{2\pi} |\hat{g}(k-\lambda)|^2 d\lambda \right\}^{1/2} \\
&\leq A \left\{ \int_{-\infty}^{\infty} |\hat{g}(\lambda)|^2 d\lambda \right\}^{1/2} \\
&\leq A||g||_2.
\end{aligned}
$$

The last step uses the Hausdor -Young inequality**.** Thus we have the second of the claimed estimates. For the first estimate, we proceed similarly, using Hölder's inequality with exponents 1 and ∞, to obtain

$$|g^+(y)| \leq \sum_{-\infty}^{\infty} \int_0^{2\pi} |\hat{g}(k-\lambda)|d\lambda = B \int_{-\infty}^{\infty} |\hat{g}(\lambda)|d\lambda$$

Next we use the above to find a commutative factorization, using the functional calculus, for an exponential type *T* matrix function **F***.*

Lemma 4. Suppose **F**(*ω*) *is an exponential type T matrix function. Then,* **F** *can be decomposed into a sum of two mutually commuting, sectionally analytic, and bounded matrix functions*

$$\mathbf{F}_{\pm}(\omega) \quad = \quad \pm \sum_{n=-\infty}^{+\infty} \mathbf{F}(\frac{2n}{T}) \frac{\exp\{\pm i\pi(T\omega - 2n)\} - 1}{2i\pi(T\omega - 2n)}.$$

Proof. The sectionally analytic and bounded properties of $\mathbf{F}_{\pm}$ come from Lemma 2 and the fact that finite products and sums of sectionally analytic functions are sectionally analytic. To establish that the two factors commute compute as follows:

$$
\begin{aligned}
\mathbf{F}_+(\omega)\mathbf{F}_-(\omega) &= -\sum_{n=-\infty}^{\infty} \mathbf{F}(\frac{2n}{T}) \left[\frac{\exp\{i\pi(T\omega - 2n)\} - 1}{2i\pi(T\omega - 2n)}\right] \\
&\quad \times \sum_{m=-\infty}^{\infty} \mathbf{F}(\frac{2m}{T}) \left[\frac{\exp\{-i\pi(T\omega - 2m)\} - 1}{2i\pi(T\omega - 2m)}\right] \\
&= -\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \mathbf{F}(\frac{2n}{T})\mathbf{F}(\frac{2m}{T}) \\
&\quad \times \left[\frac{1 - \exp\{-i\pi(T\omega - 2n)\}}{2i\pi(T\omega - 2n)}\right] \left[\frac{1 - \exp\{i\pi(T\omega - 2m)\}}{2i\pi(T\omega - 2m)}\right] e^{i\pi(T\omega-2n)-i\pi(T\omega-2m)} \\
&= -\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \mathbf{F}(\frac{2n}{T})\mathbf{F}(\frac{2m}{T}) \cos(2\pi(m-n)) \\
&\quad \times \left[\frac{\exp\{i\pi(T\omega - 2m)\} - 1}{2i\pi(T\omega - 2m)}\right] \left[\frac{\exp\{-i\pi(T\omega - 2n)\} - 1}{2i\pi(T\omega - 2n)}\right] \\
&= -\sum_{n=-\infty}^{\infty} \mathbf{F}(\frac{2n}{T})) \left[\frac{\exp\{-i\pi(T\omega - 2n)\} - 1}{2i\pi(T\omega - 2n)}\right] \\
&\quad \times \sum_{m=-\infty}^{\infty} \mathbf{F}(\frac{2m}{T}) \left[\frac{\exp\{i\pi(T\omega - 2m)\} - 1}{2i\pi(T\omega - 2m)}\right] \\
&= \mathbf{F}_-(\omega)\mathbf{F}_+(\omega). \ \Box
\end{aligned}
$$

The following theorem provides an explicit solution for a wide class of matrix Riemann-Hilbert problems.

Theorem 2. Suppose all the partial indices of matrix Riemann-Hilbert problem (equation 4) are zero. Moreover, suppose ln(**G**) *and* **F***, in the matrix Riemann-Hilbert problem (4) are two exponential type T and $T^{\square}$ matrix functions. Then, unique solutions of the matrix Riemann-Hilbert problem (equation 4) can be explicitly determined by*

$$\Phi_{\pm}(\omega) \;=\; \pm\exp\left\{\pm\sum_{n=-\infty}^{\infty}\ln(\mathbf{G}(\frac{2n}{T}))\frac{e^{\pm i\pi(T\omega-2n)}-1}{2i\pi(T\omega-2n)}\right\}\left[\sum_{m=-\infty}^{\infty}\mathbf{H}(\frac{2m}{T^{*}})\frac{e^{\pm i\pi(T^{*}\omega-2m)}-1}{2i\pi(T^{*}\omega-2m)}\}\right],$$

Where

$$\mathbf{H}(\omega) := \exp\{-\textstyle\sum_{n=-\infty}^{\infty}\ln(\mathbf{G}(\frac{2n}{T}))\frac{\exp\{i\pi(T\omega-2n)\}-1}{2i\pi(T\omega-2n)}\}\mathbf{F}(m).$$

Proof. Using the fact that ln(**G**) is an exponential type T function and Lemma 2, one can decompose $\ln(\mathbf{G}(\omega))$ as $\ln(\mathbf{G}(\omega)) = \mathbf{K}_{+}(\omega) + \mathbf{K}_{-}(\omega)$, where

$$\mathbf{K}_{\pm}(\omega) \;=\; \pm\sum_{n=-\infty}^{\infty}\ln(\mathbf{G}(\frac{2n}{T}))\frac{\exp\{\pm i\pi(T\omega-2n)\}-1}{2i\pi(T\omega-2n)}.$$

The fact that $\mathbf{K}_{+}\mathbf{K}_{-} \equiv \mathbf{K}_{-}\mathbf{K}_{+}$, as shown in Lemma 4, along with the Baker-Campbell-Hausdor Theorem leads us to conclude that $\mathbf{G} \equiv \exp\{\mathbf{K}_{+}\}\exp\{\mathbf{K}_{-}\}$, where $e^{\mathbf{K}_{\pm}}$ are two sectionally analytic matrix functions in $C_{\pm}$, respectively.

The matrix Riemann-Hilbert problem 4 may be restated as

$$e^{-\mathbf{K}_{+}(\omega)}\mathbf{\Phi}_{+}(\omega) - e^{\mathbf{K}_{-}(\omega)}\mathbf{\Phi}_{-}(\omega) \;=\; \mathbf{H}(\omega),\;\; \omega \in \mathrm{R},$$

where $\mathbf{H} \equiv e^{-\mathbf{K}_{+}}\mathbf{F}$. Now, using the fact that $e^{in(T\omega-n)}$ is a bounded complex-valued function for all $\omega \in$ R and $n \in$ Z, one can conclude that **H** is an exponential type T^{*} matrix function. It is clear that Lemma 2 gives the required decomposition, and thus we in e ect may apply Carleman's method.

Remark 1. The solutions of the matrix Riemann-Hilbert problem given by Theorem 2 for the special case of the homogeneous matrix Riemann-Hilbert problem, i.e., **F** ≡ 0*, reduce to*

$$\Phi_{\pm}(\omega) \;=\; \exp\left\{\pm\sum_{n=-\infty}^{\infty}\ln(\mathbf{G}(\frac{2n}{T}))\frac{e^{\pm i\pi(T\omega-2n)}-1}{2i\pi(T\omega-2n)}\right\}$$

The following theorem gives the error bound for approximate solution of the matrix Riemann-Hilbert problem.

Theorem 3. Suppose all the partial indices of matrix Riemann-Hilbert problem (4) are zero and ln(**G**) *and* **F***, given by the matrix Riemann-Hilbert problem of equation 4, are two matrix functions of exponential type T and T^{*} respectively. Moreover, suppose that $\mathbf{G}^{(m)}$ and $\mathbf{F}^{(m)}$, respectively, are exponential type T and T^{*} matrix functions. Then, approximate solutions of matrix Riemann-Hilbert problem (4) can be explicitly determined by*

$$\Phi_{\pm}^{(m)}(\omega) \;=\; \pm\exp\left\{\pm\sum_{n=-\infty}^{\infty}\ln(\mathbf{G}^{(m)}(\frac{2n}{T}))\frac{e^{\pm i\pi(T\omega-2n)}-1}{2i\pi(T\omega-2n)}\right\}\left[\sum_{k=-\infty}^{\infty}\mathbf{H}^{(m)}(\frac{2m}{T^{*}})\frac{e^{\pm i\pi(T^{*}\omega-2k)}-1}{2i\pi(T^{*}\omega-2k)}\}\right],$$

where $\mathbf{H}^{(m)}(\omega) := \exp\{-\sum_{n=-\infty}^{\infty} \ln(\mathbf{G}^{(m)}(\frac{2n}{T}))\frac{\exp\{i\pi(T\omega-2n)\}-1}{2i\pi(T\omega-2n)}\}\mathbf{F}(k);$ *and satisfy the error bound* $|\mathbf{\Phi}_{\pm}^{(m)} - \mathbf{\Phi}_{\pm}| \leq \|\ln(\mathbf{G}^{(m)}) - \ln(\mathbf{G})\|\|\mathbf{H}^{(m)} - \mathbf{H}\|,$
where the norm is defined by

$$\|M\| := \sup_{ij}\left\{\int_{-\infty}^{\infty} |M_{ij}(x)|^2\,dx\right\}^{1/2}.$$

Proof. The proof is straightforward by a double application of Lemma 3 and Theorem 2.

The above theorem states that if matrix functions ln **G** and **F** in the matrix Riemann-Hilbert problem 4 are replace by a sequence of matrix functions which converge in L^2 to the original functions, then, the solutions of their corresponding matrix Riemann-Hilbert problems uniformly converge to solutions of the original matrix Riemann-Hilbert problem.

Conclusion and Suggestions

This article considers a class of zero partial indices matrix Riemann-Hilbert problems which loga-rithm of their kernels and their nonhomogeneous vectors are the exponential type matrix functions. Then, it provides exact solutions for such class of matrix Riemann-Hilbert problems. In literature there are several methods to solve (explicitly or approximately) a matrix Riemann-Hilbert problems which were developed for special cases. In contrast, the technique proposed in this article should find broad application. It is worth mentioning that in the presence of a zero element in the kernel of a matrix Riemann-Hilbert problem, one can replace this element by a sequence of functions which converge to zero. In the case where the kernel of a matrix Riemann-Hilbert problem is either an upper or lower triangular matrix the matrix Riemann-Hilbert problem can be reduced by back-substitution to a system of scalar Riemann-Hilbert problems which can be solved separately, using techniques for scalar Riemann-Hilbert problems, see Ablowitz & Fokas (1997) for more detail. In the situation where some element of a matrix function is a function not of exponential type, it may still be approximated by band-limited functions. We therefore suggest to approximate such function with an exponential type function which pointwise converges to such function, see Payandeh & Kucerovsky (2009) for more detail

Remark 2. In the case that t is a non-exponential type function, it can be approximated by

$$t_n(\omega) = \sum_{i=-n}^{n} t(\frac{n}{T})\frac{\sin(\pi(T\omega-n))}{\pi(T\omega-n)},$$

Where $\lim_{n\to\infty} t_n = t.$ If f is in L^2 then the t_n converge to t in the L^2 sense.

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