COMPUTER PROGRAM SOFTWARE FOR DETERMINING FORMAL SYMMETRY OF EVOLUTION EQUATIONS

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ABSTRACT

The existence of formal symmetry of an evolution equation is one of the criteria of the complete integrability or solvability of evolution equations, due to Sokolov and Shabat.

Many evolution equations such as the soliton (solitary equation) of Korteweg-de Vries (KdV) equation have been found recently to have various kinds of explicit integral or solutions. Such evolution equations admit infinitely many symmetries or admit the recursion operator. In this paper we introduce the definition of the formal symmetry. Formal symmetry is the approximation of the recursion operator, which brings us to a convenient way of characterizing equations admitting infinitely many symmetries.

In this research, we developed a program for computing the formal symmetries of evolution equations. To verify the correctness of the program, we apply it to some evolution equations (as testing equations), which have been proved to be formally completely integrable.

The program we obtained can compute the formal symmetry of finite arbitrary order (up to order 18) of the testing equations, which verify the correctness of the program.

INTRODUCTION

Background

From 1960's there has been a great progress in soliton theory, and many evolution equations such as KdV equation have been found to have various kinds of explicit solutions. These evolution equations are often called completely integrable evolution equations because they have many analogous properties to completely integrable Hamiltonian equation of motion in classical mechanics.

Now, we know many kinds of similar definitions or criteria for (formal) complete integrability of evolution equations, and still we do not have enough results on the relation among such criteria. Here, we use the word *formal*, since the definition or the criteria does not directly yield the way to find explicit solutions.

Literature Review

The complete integrability or solvability of equations is one of the criteria of the Hamiltonian system (Dickey, 1991). The definition for the formal complete integrability of evolution equations is due to Sokolov and Shabat (Sokolov and Shabat, 1984). It has close relation with the existense of infinitely many symmetries (Mikhailov, Shabat, and Yamilov, 1987).

It can be proved that if an evolution equation admits infinitely many symmetries then it admits the recursion operator (Olver, 1977). Recursion operator can be approximated by the definition of formal symmetry. Our focus here is to develop a program for computing the formal symmetries of evolution equations.

We will verify the correctness of the program by applying it to some evolution equations obtained by Fujimoto and Watanabe (Fujimoto and Watanabe, 1983), which have been proved to be formally completely integrable in some sense, which is a little bit different from our definition (Fujimoto and Watanabe, 1984).

Method of Research

In this paper, we introduce an operator called recursion operator, which maps symmetry to symmetry. Our method for finding recursion operator is based on the use of fractional powers of the operator. The definition of the formal symmetry, which is the approximation of the recursion operator, brings us to a convenient way characterizing equations admitting infinitely many symmetries.

The theme of this research is to develop program for computing the formal symmetry of arbitrary order in the sense of Sokolov and Shabat on the computer algebra system REDUCE Ver. 3.7. In making the program, we use many of the subroutines for the formal calculus of variations or symbol calculus of the pseudo-differential operators, which are developed by Watanabe and Ito.

We will verify the correctness of the program by applying it to some evolution equations obtained

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by Fujimoto and Watanabe, which have been proved to be formally completely integrable. It is well-known that formally integrable evolution equation admits formal symmetry of arbitrary order. If we can compute the formal symmetry of arbitrary order of those evolution equations using the program, then it is sufficient to conclude that we succeed to verify the correctness of the program.

THEORETICAL BASIS

Symmetry of Evolution Equations

Consider an evolution equation of order m of the form

$$u_t = H(u, u_1, u_2, \ldots, u_m) \quad (1)$$

for unknown function $u(x,t)$ of the independent variables x, t. H is a function of m+1 arguments. Moreover $u_i(i = 1, \ldots, m)$ denote the higher order partial derivatives $\frac{\partial^i u}{\partial x^i}$ and u_t denotes the partial derivative of u with respect to t. We will define the notion of symmetry of evolution equations. Let u be a solution of equation (1) and consider the infinitesimal transformation $\bar{u} = u + \epsilon G$ with an infinitesimal parameter ϵ and a function $G(u, u_1, \ldots, u_n)$. Assume that u becomes a solution of Eq.(1) up to $O(\epsilon^2)$ and derives the equation for G. First, we have

$$\bar{u}_t = u_t + \epsilon G_t, \quad (2)$$

$$H(\bar{u}, \bar{u}_1, \bar{u}_2, \ldots, \bar{u}_m) = H(u + \epsilon G, u_1 + \epsilon DG, u_2 + \epsilon D^2 G, \ldots, u_m + \epsilon D^m G), \quad (3)$$

where D denotes total differentiation with respect to x, that is, we have $u_i = D^i u$.

Since u is the solution of Eq.(1), we have

$$G_t = \frac{\partial G}{\partial u} u_t + \frac{\partial G}{\partial u} (u_1)_t + \ldots + \frac{\partial G}{\partial u_n} (u_n)_t$$
$$= \frac{\partial G}{\partial u} H + \frac{\partial G}{\partial u} DH + \ldots + \frac{\partial G}{\partial u_n} D^n(H).$$

If we introduce the operator ∂_H by

$$\partial_H = \sum_{i=0}^{\infty} D^i(H) \frac{\partial}{\partial u_i}, \quad (4)$$

where we write u_0 for u. Then we have $G_t = \partial_H G$, that is, ∂_H denotes the total differentiation with respect to t on Eq.(1). Then we have

$$\bar{u}_t = u_t + \epsilon \partial_H G. \quad (5)$$

On the other hand, introducing the Frechet Jacobian H_* by

$$H_* = \sum_{i=0}^{m} \frac{\partial H}{\partial u_i} D^i, \quad (6)$$

we compute the right-hand side of (3) as follows:

$$H(u, u_1, u_2, \ldots, u_m) + \epsilon\left(\frac{\partial H}{\partial u} G + \frac{\partial H}{\partial u_1} DG + \frac{\partial H}{\partial u_2} D^2 G + \ldots + \frac{\partial H}{\partial u_m} D^m G\right) + O(\epsilon^2)$$
$$= H(u, u_1, u_2, \ldots, u_m) + \epsilon \sum_{i=0}^{m} \frac{\partial H}{\partial u_i} D^i G + O(\epsilon^2)$$
$$= H(u, u_1, u_2, \ldots, u_m) + \epsilon H_* G + O(\epsilon^2). \quad (7)$$

Substituting (5) and (7) into the equation

$$\bar{u}_t = H(\bar{u}, \bar{u}_1, \bar{u}_2, \ldots, \bar{u}_m) + O(\epsilon^2)$$

and using $u_t = H(u, u_1, u_2, \ldots, u_m)$, we have the equation for G:

$$(\partial_H - H_*)G = 0. \quad (8)$$

We call the function G satisfying Eq.(8) as *Lie-Bäclund* symmetry or simply as symmetry. It is easily proved that an arbitrary evolution equation (1) admit u_l and H as symmetries. They are called trivial symmetries.

Example 1 *The well-known KdV equation*

$$u_t = H(u, u_1, u_2, \ldots, u_m) = u_3 + 6u_1 u.$$

The KdV equation is known to admit infinitely many symmetries, some of them are given as follows:

$$G_1 = u_1,\ G_3 = H = u_3 + 6u_1 u = (D^2 + 4u + 2u_1 D^{-1})\, G_1,$$
$$G_5 = (D^2 + 4u + 2u_1 D^{-1})\, G_3 = \ldots, \ldots$$

Let $A((D^{-1}))$ be the set of all formal Laurent series whose coefficients are differentiable functions of u_i (i = 0, 1, …) of the form

$$P = p_n D^n + p_{n-1} D^{n-1} + \ldots + p_0 + p_{-1} D^{-1} + \ldots \quad (9)$$

If $p_n \neq 0$ then, we call the degree of P is n and write $\deg(P) = n$. We introduce the multiplication law in $A((D^{-1}))$. For monomial fD^k, $gD^l \in A((D^{-1}))$, we define

$$fD^k . gD^l = \sum_{i=0}^{\infty} f \binom{k}{i} g^{(i)} D^{k+l-i} \quad (10)$$

where $\binom{k}{i}$ with an integer k and a non-negative integer i denotes the general binomial coefficient:

$$\binom{k}{i} = \frac{k(k-1)...(k-i+1)}{i!}$$

and we use the notation as $g^{(i)} = D^{(i)}(g)$. We note that if k is negative then the sum in (10) becomes infinite. $A((D^{-1}))$ becomes an algebra with the multiplication law defined by (10).

Let us define an operator $A((D^{-1}))$ which maps symmetries to symmetries in a formal sense. Such an operator will be called a recursion operator (Olver, 1977). We consider an operator $L \in A((D^{-1}))$ satisfying the operator equation

$$L_t - [H_*, L] = 0 \qquad (11)$$

Where L_t is obtained from L by differentiating each coefficient of L with respect to t, that is applying ∂_H. Also, $[P,Q]$ denotes the commutator $P.Q - Q.P$. Let L be an operator satisfying Eq.(11). Then we can prove that if G is a symmetry of Eq.(11), then LG also becomes a symmetry, if LG is well-defined. So, an element $L \in A((D^{-1}))$ satisfying Eq.(11) is called a recursion operator or strong symmetry of Eq.(1) (Sokolov, 1984).

Example 2 *For the KdV equation, one of the recursion operators is given by*

$$L = D^2 + 4u + 2u_1D^{-1}.$$

It is known that infinitely many symmetries of the so-called completely integrable evolution equation is generated from trivial symmetries by a suitable recursion operator.

We must note that if L is a recursion operator of Eq.(1) then a power L^k with an integer k becomes a recursion operator. We are interested in evolution equations admitting infinitely many symmetries. But, it is not easy to characterize such an evolution equation. Thus, we search for evolution equations admitting recursion operator instead of searching for equations admitting infinitely many symmetries.

Formal Symmetry of Evolution Equations

In this section, we try to find a recursion operator of Eq.(1) starting with the Frechet Jacobian H_*. First, we define the notion of formal symmetry, which is considered to be an approximation of a recursion operator in some sense (Sokolov, 1984). An operator $L \in A((D^{-1}))$ is said to be a formal symmetry of Eq.(1) of order N if it satisfies

$$L_t - [H_*, L] = O(m+n-N). \qquad (12)$$

Here the symbol $O(N)$ means that it is an element of $L \in A((D^{-1}))$ whose degree is equal or less than the integer N. Since it can be easily verified that $[A,B] = O(m+n-N)$ holds for $A,B \in A((D^{-1}))$ with $\deg(A) = m$, $\deg(B) = n$, we see that an arbitrary $L \in A((D^{-1}))$ becomes a formal symmetry of order 1, that is, we have

$$L_t - [H_*, L] = O(m+n-1).$$

Put $L = H_*$, than we have

$$\begin{aligned} L_t - [H_*, L] &= (H_*)_t - [H_*, H_*] = (H_*)_t - 0 \\ &= O(m) = O(m+m-m). \end{aligned}$$

This means that H_* becomes a formal symmetry of order m. Moreover, by the similar argument, it can be proved that the first m-1 terms of H_*:

$$\frac{\partial G}{\partial u_m} D^m + \frac{\partial G}{\partial u_{m-1}} D^{m-1} + \ldots + \frac{\partial G}{\partial u_2} D^2$$

becomes a formal symmetry of order m of Eq.(1).

It is well-known that we can compute the $1/m$-th fractional power $(H_*)^{1/m}$ of H_* in the algebra $A((D^{-1}))$ if the quantity is computable. Where $M = (H_*)^{1/m}$ means the element of $A((D^{-1}))$ satisfying $M^m = H_*$ in the algebra $A((D^{-1}))$. The important thing we have to note is that the first m-1 terms of the fractional power:

$$(H_*)^{1/m}_{m-1} = L_m = l_1D + l_0 + l_{-1}D^{-1} + \ldots + l_{3-m}D^{3-m} \qquad (13)$$

with $l_1 = (\frac{\partial H}{\partial u_m})^{1/m}$ etc. becomes a formal symmetry of order m. Then, starting with L_m given by (13), we try to find higher order formal symmetry by adding lower order terms to L_m.

Condition for the formal symmetry of order m+1

First, we put

$$L_{m+1} = L_m + l_{2-m}D^{2-m} \qquad (14)$$

and try to compute the coefficient l_{2-m} so that L_{m+1} becomes the formal symmetry of order m+1, that is, L_{m+1} must satisfy

$$(L_{m+1})_t - [H_*, L_{m+1}] = O(m+1-(m+1)) = O(0) \qquad (15)$$

Multiplying both sides of the above equation by L_{m+1}^{-1} from the left and right, we see that (15) is equivalent to

$$(L_{m+1}^{-1})_t - [H_*, L_{m+1}^{-1}] = O(-2).$$

It is easily proved that $L = L_{m+1}$ with arbitrary l_{2-m} satisfies the equation

$$(L^{-1})_t - [H_*, L^{-1}] = O(-1).$$

So, the additional coefficient l_{2-m} must be determined by the equation:

$$\text{Res}\,((L_{m+1}^{-1})_t - [H_*, L_{m+1}^{-1}])) = 0 \qquad (16)$$

where symbol Res (residue) applies to an element of $A((D^{-1}))$ as (9) and gives the coefficient p_{-1} of D^{-1}. We will examine the condition (16) in more detail. As is well-known, the residue of the commutator of $A, B \in A((D^{-1}))$ can be written as Res $[A,B] = D(f)$ with some function f. We can write it as Res $[H_*, L_{m+1}^{-1}] = D(\sigma_{-1})$, with some function σ_{-1}. So, define $\rho_{-1} = \text{Res}\,(L_{m+1}^{-1})$, then the condition (16) is reduced to

$$(\rho_{-1})_t = D(\sigma_{-1}). \qquad (17)$$

By an easy computation, we find

$$\rho_{-1} = \text{Res}\,(L_m^{-1}) = \text{Res}\,(L_{m+1}^{-1}) = \frac{1}{l_1} = \left(\frac{\partial H}{\partial u_m}\right)^{-1/m}.$$

Moreover, by a straightforward computation, we see that σ_{-1} can be expressed as

$$\begin{aligned}\sigma_{-1} &= D^{-1}(\text{Res}\,[H_*, L_{m+1}^{-1}]) \\ &= -m\, l_1^{m-2}\, l_{2-m} + \Delta_{-1}(l_1, \ldots, l_{3-m}) \qquad (18)\end{aligned}$$

with a suitable function Δ_{-1} of known quantities $l_1, \ldots, l_{3-m}$.

A function $\rho(u, u_1, \ldots, u_k)$ is said to be a conserved density of Eq.(1) if there exists σ such that

$$\rho_t = \partial_H(\rho) = D(\sigma) \qquad (19)$$

holds. From the argument above, we see that if ρ_{-1} is a conserved density of Eq.(1), then we can get the expression of l_{2-m} by solving the linear equation $\sigma_{-1} = D^{-1}(\rho_{-1})_t$. Thus, if ρ_{-1} is a conserved density of Eq.(1), then Eq.(1) admits the formal symmetry of order $m+1$.

Condition for the formal symmetry of order $m+k$

We assume that L_{m+k-1} is a formal symmetry of order $m+k-1$. Putting

$$L_{m+k} = L_{m+k-1} + l_{-m-k-3}D^{-m-k-3}, \qquad (20)$$

we try to find l_{-m-k-3} so that $L = L_{m+k}$ becomes the formal symmetry of order $m+k$. The defining equation for $L = L_{m+k}$ is

$$L_t - [H_*, L] = O(m+1-(m+k)) = O(-k+1). \qquad (21)$$

Multiplying (21) by L^{k-i-3} from the left and by L^i from the right and summing for $i = 0, \ldots, k-3$, we have

$$(L^{k-2})_t = [H_*, L^{k-2}] = O(-2) \qquad (22)$$

which is equivalent to

$$\text{Res}\,(L^{k-2} - [H_*, L^{k-2}]) = 0. \qquad (23)$$

We introduce ρ_{k-2} and σ_{k-2} by

$$\rho_{k-2} = \text{Res}\, L^{k-2},\ D(\sigma_{k-2}) = \text{Res}\,[H_*, L^{k-2}]. \qquad (24)$$

Then (22) is reduced to the equation

$$(\rho_{k-2})_t - D(\sigma_{k-2}) = 0. \qquad (25)$$

If ρ_{k-2} is a conserved density of Eq.(1), then we have the equation $\sigma_{k-2} = D^{-1}(\rho_{k-2})_t$, from which we get the coefficient l_{-m-k-3} so that L_{m+k} becomes the formal symmetry of order $m+k$. Here we must note that σ_{k-2} depends linearly on l_{-m-k-3}.

So far, we have defined a series of quantities ρ_i, σ_i (i = -1, 1, 2, ...) by

$$\rho_i = \text{Res}\, L^i, \qquad (26)$$

$$\sigma_i = D^{-1}\,\text{Res}\,[H_*, L^i], \qquad (27)$$

It is proved that if $\rho_{-1}, \ldots, \rho_k$ become conserved densities of Eq.(1), then Eq.(1) admits the formal symmetry of order $m+k+2$. However, the quantities ρ_i which can be computed from H_* by Res $(H_*^{i/m})$ in advance are only $\rho_{-1}, \ldots, \rho_{m-3}$ and other ρ_k for $k \geq m-2$ depends on the quantities σ_j ($j = -1, \ldots, k-m+1$), which can be obtained by way of the integration $\sigma_j = D^{-1}(\rho_j)_t$.

In closing this section, we make some comments, which is useful in computing the explicit expressions of ρ_k.

1. We define $\sigma_k = 0$ when ρ_k is constant. For many evolution equations in soliton theory, the highest order derivative with respect to x is linear with constant coefficient. In such a case, ρ_{-1} becomes constant, so that we may put $\sigma_{-1} = 0$. Hence ρ_i can be computed by Res $(H_*^{i/m})$ from $i = 0$ to $i = m-2$.
2. For many evolution equations in soliton theory such as the KdV equation, H does not involve the second highest order derivative with respect to x. In such cases, we get $l_0 = 0$ so that ρ_0 must be

zero. Then we conclude that $\sigma_0 = 0$. So, we can compute ρ_i by Res $(H_*^{i/m})$ from $i = 1$ to $i = m-1$.

Result and Discussion

In this research, we developed a program for determining formal symmetry of required order. We have made the program by the computer algebra system Reduce 3.7. The main program is named **`symmetry`**. The program **`symmetry`** uses many functions that are made by Y. Watanabe. Functions that we use in the main program are contained in various program files named **`ddiff.red`**, **`diffop.rl`**, **`seki.red`**, **`fpow.red`**. So, in the **`symmetry`** we load such files before using the functions in **`symmetry`**. The file **`ddiff.red`** contains fundamental functions which treat the derivation with respect to x. Especially, the variable u_i is expressed as `u(i)`, for example, the right-hand side of the KdV equation $u_3 + 6uu_1$ is expressed as

```
u(3)+6*u(0)*u(1)
```

The differentiation rule such as `df(u(i),x)=u(i+1)` is defined and we can compute the x derivatives of polynomials of the variables `u(i)`.

Program file **`diffop.rl`** contains the functions on the various kinds of differential operators. For example, if we want to apply the evolution derivation of r with respect to the time evolution along the KdV equation, we use the command **`dfev`** with two arguments and type

```
dfev(r,u(3)+6*u(0)*u(1));
```

If we want to make the Frechet Jacobian of `u(3)+6*u(0)*u(1)` we use the command **`fj`** with one argument and type

```
fj(u(3)+6*u(0)*u(1));
```

Also, if we want to apply the Euler operator, we use the command **`eul`** with one argument.

Program files **`seki.red`** or **`fpow.red`** consists of the functions for computing the multiplication in $A((D^{-1}))$ or the fractional power (or the power) of the elements in $A((D^{-1}))$, respectively.

The main program file **`symmetry`** consists of commands **`formsym(f,n)`** with two arguments. In this function, the first argument is the right-hand side of the target evolution equation and the second argument is the order of the formal symmetry, which we are going to compute. In order to use the program, we first type

```
1: in "symmetry";
```

If we are going to compute for the KdV equation, then we type

```
2: f:=(u(3)+6*u(0)*u(1));
```

For the computation of the formal symmetry of order 1, we type

```
3: formsym(u(3)+6*u(0)*u(1),1);
```

then the result will be

```
***** the answer is arbitrary
```

This corresponds to the result that an arbitrary operator becomes the formal symmetry of order 1. If we want to compute the formal symmetry of order 7, then type

```
4: formsym(f,7);
```

The returned value is

```
4*u(1)*u(0)*id**4-
2*u(0)**2*id**3+2*u(0)*id+d
```

which is the operator with 6-terms. Here, the number of terms in the above expression must be counted as 6, because the fourth term and the constant term is 0. In the result expression, **`d**i`** and **`id**i`** means D^i and D^{-i}, respectively.

In this research, we succeeded to compute the formal symmetry of the KdV equation of order 18, by using our computer program. Moreover, we succeeded to compute the higher order formal symmetry of some types of the fifth order evolution equations similar to the fifth order KdV equation or the fifth order modified KdV equation, as testing equations. Those evolution equations are obtained by Fujimoto and Watanabe, which have been proved to be formally completely integrable in some sense, which is a little bit different from our definition. We have known those formally integrable evolution equations admit formal symmetry of arbitrary order, as we previously noted at Sec.1. So, because the program succeeded to compute the formal symmetry of finite arbitrary order (up to order 18) of those testing evolution equations, then it is sufficient to conclude that we succeed to verify the correctness of the program.

CONCLUSIONS

The program we developed is used for computing the formal symmetry of required order for single evolution equation. Of course, the program can only compute the formal symmetry of finite arbitrary order of evolution equations. It failed to compute the formal symmetry of order more than 18, because of the lack of computer's stack memory and the program consumed a lot of time. However, it is impossible to make a program that can compute the formal symmetry of arbitrary order of evolution equations. We can only make a program computing the formal symmetry of arbitrary as higher order as it is possible.

In the future research, we would like to develop the program so that it can compute the formal symmetry of some order more efficiently and in higher speed.

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