A lattice Poisson algebra for the Pohlmeyer reduction of the $AdS_5 \times S^5$ superstring

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Abstract. The Poisson algebra of the Lax matrix associated with the Pohlmeyer reduction of the $AdS_5 \times S^5$ superstring is computed from first principles. The resulting non-ultralocality is mild, which enables to write down a corresponding lattice Poisson algebra.

1 Introduction

We recently showed in [1] that the Poisson algebra of the Lax matrix associated with symmetric space sine-Gordon models, defined through a gauged Wess-Zumino-Witten action with an integrable potential [2], admits an integrable lattice discretization. In the present letter we compute the r/s-matrix structure [3] associated with the Pohlmeyer reduction of $AdS_5 \times S^5$ superstring theory [4, 5] directly from its representation in terms of a fermionic extension of a gauged WZW action with an integrable potential. We similarly find that it is precisely of the type which, after regularization as in [6], admits an integrable lattice discretization of the general form identified in [7, 8].

2 Canonical analysis and Hamiltonian

To begin with we briefly recall some usual notations. We refer the reader to [4] for more details concerning this setup. The superalgebra $\mathfrak{f} = \mathfrak{psu}(2,2|4)$ admits a $\mathbb{Z}_4$-grading, $\mathfrak{f} = \mathfrak{f}^{(0)} \oplus \mathfrak{f}^{(1)} \oplus \mathfrak{f}^{(2)} \oplus \mathfrak{f}^{(3)}$ where $\mathfrak{g} = \mathfrak{f}^{(0)} = \mathfrak{so}(4,1) \oplus \mathfrak{so}(5)$. Let G denote the corresponding Lie group. The supertrace is compatible with the $\mathbb{Z}_4$-grading, in the sense that $\mathrm{Str}(A^{(m)}B^{(n)}) = 0$ for $m + n \neq 0$ mod 4. The reduced theory relies on the element $T = \frac{i}{2}\mathrm{diag}(1,1,-1,-1,1,1,-1,-1) \in \mathfrak{f}^{(2)}$. It defines a $\mathbb{Z}_2$-grading of $\mathfrak{f}$ with $\mathfrak{f}^{[0]} = \mathrm{Ker}(\mathrm{Ad}_T)$ and $\mathfrak{f}^{[1]} = \mathrm{Im}(\mathrm{Ad}_T)$. Elements of $\mathfrak{f}^{[0]}$ commute with T while those of $\mathfrak{f}^{[1]}$ anti-commute with T and we have $\mathrm{Str}(A^{[0]}B^{[1]}) = 0$. Finally, projectors on $\mathfrak{f}^{[0]}$ and $\mathfrak{f}^{[1]}$ are given respectively by $P^{[0]} = -[T,[T,\,\cdot\,]_+]_+$ and $P^{[1]} = -[T,[T,\,\cdot\,]]$. Let $\mathfrak{h} = \mathfrak{g}^{[0]}$ be the subalgebra in $\mathfrak{g}$ of elements commuting with T. The corresponding Lie group H is $[SU(2)]^4$.

Our starting point is the field theory introduced in [4]. It corresponds to a fermionic extension of a G/H gauged WZW with a potential term. The action we start with is, taking $\epsilon^{\tau\sigma\xi} = 1$,

$$\begin{aligned}
\mathcal{S} = \tfrac{1}{2}\int d\tau d\sigma\, \mathrm{Str}(g^{-1}\partial_+ g g^{-1}\partial_- g) + \tfrac{1}{3}\int d\tau d\sigma d\xi \epsilon^{\alpha\beta\gamma}\, \mathrm{Str}(g^{-1}\partial_\alpha g g^{-1}\partial_\beta g g^{-1}\partial_\gamma g) \\
- \int d\tau d\sigma\, \mathrm{Str}(A_+\partial_- g g^{-1} - A_- g^{-1}\partial_+ g + g^{-1}A_+ g A_- - A_+A_-) \\
+ \tfrac{1}{2}\int d\tau d\sigma\, \mathrm{Str}(\psi_L[T, D_+\psi_L] + \psi_R[T, D_-\psi_R]) \\
+ \int d\tau d\sigma \big(\mu^2\, \mathrm{Str}(g^{-1}TgT) + \mu\, \mathrm{Str}(g^{-1}\psi_L g\psi_R)\big).
\end{aligned}$$

The fields g, ψ_R, ψ_L and the gauge fields $A_\pm$ respectively take values in G, $\mathfrak{f}^{(1)[1]}$, $\mathfrak{f}^{(3)[1]}$ and in $\mathfrak{h}$. The covariant derivatives are $D_\pm = \partial_\pm - [A_\pm,]$ with $\partial_\pm = \partial_\tau \pm \partial_\sigma$.

Generalizing the analysis of [9] to the case considered here, one finds that the phase space is spanned by the fields $(g, \mathcal{J}_L, A_\pm, P_\pm, \psi_L, \psi_R)$. The field $\mathcal{J}_L$ corresponds to the left-invariant WZW current. Alternatively, one can use instead the right-invariant current $\mathcal{J}_R$, related to $\mathcal{J}_L$ by

$$\mathcal{J}_R = -2\partial_\sigma g g^{-1} + g\mathcal{J}_L g^{-1}.$$

The fields $P_\pm$ are the canonical momenta of $A_\pm$. The non-vanishing Poisson brackets are

$$\begin{aligned}
\{\mathcal{J}_{L\underline{1}}(\sigma), \mathcal{J}_{L\underline{2}}(\sigma')\} &= [C^{(00)}_{\underline{\mathbf{12}}}, \mathcal{J}_{L\underline{2}}]\delta_{\sigma\sigma'} + 2C^{(00)}_{\underline{\mathbf{12}}}\partial_\sigma\delta_{\sigma\sigma'}, \\
\{\mathcal{J}_{R\underline{1}}(\sigma), \mathcal{J}_{R\underline{2}}(\sigma')\} &= -[C^{(00)}_{\underline{\mathbf{12}}}, \mathcal{J}_{R\underline{2}}]\delta_{\sigma\sigma'} - 2C^{(00)}_{\underline{\mathbf{12}}}\partial_\sigma\delta_{\sigma\sigma'}, \\
\{\mathcal{J}_{L\underline{1}}(\sigma), g_{\underline{2}}(\sigma')\} &= -g_{\underline{2}}C^{(00)}_{\underline{\mathbf{12}}}\delta_{\sigma\sigma'} \\
\{\mathcal{J}_{R\underline{1}}(\sigma), g_{\underline{2}}(\sigma')\} &= -C^{(00)}_{\underline{\mathbf{12}}}g_{\underline{2}}\delta_{\sigma\sigma'} \\
\{A_{\pm\underline{1}}(\sigma), P_{\pm\underline{2}}(\sigma')\} &= C^{(00)[00]}_{\underline{\mathbf{12}}}\delta_{\sigma\sigma'}, \\
\{\psi_{R\underline{1}}(\sigma), \psi_{R\underline{2}}(\sigma')\} &= \big[T_{\underline{2}}, C^{(13)}_{\underline{\mathbf{12}}}\big]\delta_{\sigma\sigma'}, \\
\{\psi_{L\underline{1}}(\sigma), \psi_{L\underline{2}}(\sigma')\} &= \big[T_{\underline{2}}, C^{(31)}_{\underline{\mathbf{12}}}\big]\delta_{\sigma\sigma'}.
\end{aligned}$$

In these expressions $C^{(ij)}_{\underline{\mathbf{12}}} \in \mathfrak{f}^{(i)} \otimes \mathfrak{f}^{(j)}$ are the components of the tensor Casimir (see [10] for its properties) in the decomposition $C_{\underline{\mathbf{12}}} = C^{(00)}_{\underline{\mathbf{12}}} + C^{(13)}_{\underline{\mathbf{12}}} + C^{(22)}_{\underline{\mathbf{12}}} + C^{(31)}_{\underline{\mathbf{12}}}$ with respect to the $\mathbb{Z}_4$-grading. The component $C^{(00)[00]}_{\underline{\mathbf{12}}}$ is defined in a similar way relative to the $\mathbb{Z}_2$-grading.

The standard analysis shows that there is a total of four constraints,

$$\chi_1 = P_+, \qquad \chi_2 = P_-, \tag{2.2a}$$

$$\chi_3 = \mathcal{J}_R^{[0]} + A_+ - A_- - \tfrac{1}{2}[\psi_L,[T,\psi_L]], \qquad \chi_4 = \mathcal{J}_L^{[0]} + A_+ - A_- + \tfrac{1}{2}[\psi_R,[T,\psi_R]]. \tag{2.2b}$$

The extended Hamiltonian, which has weakly vanishing Poisson brackets with the constraints (2.2), is

$$\begin{aligned} H = \int d\sigma \Big(& \tfrac{1}{4}\operatorname{Str}\big(\mathcal{J}_L{}^2 + \mathcal{J}_R{}^2\big) + \operatorname{Str}\big(\mathcal{J}_R^{[0]}A_+ - \mathcal{J}_L^{[0]}A_-\big) + \tfrac{1}{2}\operatorname{Str}\big[(A_+ - A_-)^2\big] \\ & - \tfrac{1}{2}\operatorname{Str}\big(\psi_L[T,\partial_\sigma\psi_L - [A_+,\psi_L]]\big) - \tfrac{1}{2}\operatorname{Str}\big(\psi_R[T,-\partial_\sigma\psi_R - [A_-,\psi_R]]\big) \\ & - \mu^2\operatorname{Str}(g^{-1}TgT) - \mu\operatorname{Str}(g^{-1}\psi_L g\psi_R) + v_+P_+ + v_-P_- + \lambda(\chi_3 - \chi_4)\Big) \end{aligned} \tag{2.3}$$

with $v_+ - v_- = \partial_\sigma(A_+ + A_-) - [A_+, A_-]$. The combination $\chi_3 - \chi_4$ of the constraints generates a gauge invariance.

3 Continuum and lattice Poisson algebras

Up to a gauge transformation, the equations of motion for the fields $(\mathcal{J}_L, g, \psi_L, \psi_R)$ under the Hamiltonian (2.3) are equivalent to the zero curvature equation $\{\mathcal{L}, H\} = \partial_\sigma\mathcal{M} + [\mathcal{M},\mathcal{L}]$ for the following Lax connection [4]

$$\mathcal{L}(z) = -\tfrac{1}{2}\mathcal{J}_L - \tfrac{1}{2}z\sqrt{\mu}\psi_R - \tfrac{1}{2}z^2\mu T + \tfrac{1}{2}z^{-1}\sqrt{\mu}g^{-1}\psi_L g + \tfrac{1}{2}z^{-2}\mu g^{-1}Tg, \tag{3.1a}$$

$$\mathcal{M}(z) = -\tfrac{1}{2}\mathcal{J}_L + A_- - \tfrac{1}{2}z\sqrt{\mu}\psi_R - \tfrac{1}{2}z^2\mu T - \tfrac{1}{2}z^{-1}\sqrt{\mu}g^{-1}\psi_L g - \tfrac{1}{2}z^{-2}\mu g^{-1}Tg. \tag{3.1b}$$

The field A_+ entering the equations appears as an arbitrary element of $\mathfrak{h}$. We now have all the ingredients needed to compute the Poisson bracket of the Lax matrix (3.1a). The result reads

$$\begin{aligned} 4\{\mathcal{L}_{\underline{\mathbf{1}}}(z_1), \mathcal{L}_{\underline{\mathbf{2}}}(z_2)\} = {} & [r_{\underline{\mathbf{12}}}(z_1,z_2), \mathcal{L}_{\underline{\mathbf{1}}}(z_1) + \mathcal{L}_{\underline{\mathbf{2}}}(z_2)]\delta_{\sigma\sigma'} \\ & + [s_{\underline{\mathbf{12}}}(z_1,z_2), \mathcal{L}_{\underline{\mathbf{1}}}(z_1) - \mathcal{L}_{\underline{\mathbf{2}}}(z_2)]\delta_{\sigma\sigma'} + 2s_{\underline{\mathbf{12}}}(z_1,z_2)\partial_\sigma\delta_{\sigma\sigma'}, \end{aligned} \tag{3.2}$$

where the kernels of the r/s-matrices are given by

$$r_{\underline{\mathbf{12}}}(z_1,z_2) = \frac{z_2^4 + z_1^4}{z_2^4 - z_1^4}C_{\underline{\mathbf{12}}}^{(00)} + \frac{2z_1z_2^3}{z_2^4 - z_1^4}C_{\underline{\mathbf{12}}}^{(13)} + \frac{2z_1^2z_2^2}{z_2^4 - z_1^4}C_{\underline{\mathbf{12}}}^{(22)} + \frac{2z_1^3z_2}{z_2^4 - z_1^4}C_{\underline{\mathbf{12}}}^{(31)}, \tag{3.3a}$$

$$s_{\underline{\mathbf{12}}}(z_1,z_2) = C_{\underline{\mathbf{12}}}^{(00)}. \tag{3.3b}$$

One can check explicitly that the kernels (3.3) coincide exactly with the ones that would be obtained from the generalization of the alleviation procedure proposed in [1] to semi-symmetric space σ-models. This is simply a matter of replacing the twisted inner product on the twisted loop algebra considered in [11] by the trigonometric one and to compute the corresponding kernels as explained in [1].

An important property of the above r/s-matrix structure is that s is simply the projection onto the subalgebra $\mathfrak{g}$. In this case, the corresponding Poisson algebra (3.2) can be discretized following [6] by introducing a skew-symmetric solution $\alpha \in \operatorname{End}\mathfrak{g}$ of the modified classical Yang-Baxter equation on $\mathfrak{g}$. Then the matrices

$$a_{\underline{\mathbf{12}}} = (r+\alpha)_{\underline{\mathbf{12}}}, \qquad b_{\underline{\mathbf{12}}} = (-s-\alpha)_{\underline{\mathbf{12}}}, \qquad c_{\underline{\mathbf{12}}} = (-s+\alpha)_{\underline{\mathbf{12}}}, \qquad d_{\underline{\mathbf{12}}} = (r-\alpha)_{\underline{\mathbf{12}}},$$

satisfy all the requirements of [7, 8] in order to define the following consistent lattice algebra,

$$4\{\mathcal{L}^n_{\underline{\mathbf{1}}}, \mathcal{L}^m_{\underline{\mathbf{2}}}\} = a_{\underline{\mathbf{12}}}\mathcal{L}^n_{\underline{\mathbf{1}}}\mathcal{L}^m_{\underline{\mathbf{2}}}\delta_{mn} - \mathcal{L}^n_{\underline{\mathbf{1}}}\mathcal{L}^m_{\underline{\mathbf{2}}}d_{\underline{\mathbf{12}}}\delta_{mn} + \mathcal{L}^n_{\underline{\mathbf{1}}}b_{\underline{\mathbf{12}}}\mathcal{L}^m_{\underline{\mathbf{2}}}\delta_{m+1,n} - \mathcal{L}^m_{\underline{\mathbf{2}}}c_{\underline{\mathbf{12}}}\mathcal{L}^n_{\underline{\mathbf{1}}}\delta_{m,n+1}.$$

This algebra reduces to (3.2) in the continuum limit (see [1]). The corresponding algebra for the monodromy may be found in [1].

4 Conclusion

We have constructed a quadratic lattice Poisson algebra associated with the fermionic extension of the $(SO(4,1)\times SO(5))/[SU(2)]^4$ gauged WZW model with an integrable potential. The fact that one is able to write down such a lattice algebra is quite appealing and in sharp contrast with what happens for the canonical Poisson structure of the $AdS_5 \times S^5$ superstring [10]. Indeed, it brings hope of being able to construct a lattice quantum algebra related to the Pohlmeyer reduction of the $AdS_5 \times S^5$ superstring. The precise link of this Pohlmeyer reduction with the alleviation procedure presented in [1] is under study.

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