Higher genus Abelian functions associated with algebraic curves

Matthew England

Submitted for the qualification of

Doctor of Philisophy

Heriot Watt University

School of Mathematics and Computer Science

Department of Mathematics

December 2009

Abstract

We investigate the theory of Abelian functions with periodicity properties defined from an associated algebraic curve. A thorough summary of the background material is given, including a synopsis of elliptic function theory, generalisations of the Weierstrass σ and $\wp$-functions and a literature review.

The theory of Abelian functions associated with a tetragonal curve of genus six is considered in detail. Differential equations and addition formula satisfied by the functions are derived and a solution to the Jacobi Inversion Problem is presented. New methods which centre on a series expansion of the σ-function are used and discussions on the large computations involved are included. We construct a solution to the KP equation using these functions and outline how a general class of solutions can be generated from a wider class of curves.

We proceed to present new approaches used to complete results for the lower genus trigonal curves. We also give some details on the the theory of higher genus trigonal curves before finishing with an application of the theory to the Benney moment equations. A reduction is constructed corresponding to Schwartz-Christoffel maps associated with the tetragonal curve. The mapping function is evaluated explicitly using derivatives of the σ-function.

Acknowledgments

Thanks first to my supervisor Professor Chris Eilbeck for his guidance throughout my studies and his proof reading of this document. Professor Eilbeck is my co author in [39], upon which Chapter 3 of this document is based. Many thanks also to Dr. John Gibbons for his patient explanations of some complicated ideas. Dr. Gibbons is my co author in [40], upon which Chapter 6 of this document is based.

Next I would like to acknowledge Dr. Y. Ônishi for his assistance with Lemma 3.5.2, Dr A. Nakayashiki for his suggestion that led to the construction of the basis in equation (3.55) and Dr. V. Enolski for useful conversations. Acknowledgments should also be given to the anonymous referees of my papers [39] and [40], whose comments have in turn improved this document.

Finally, I would like to thank Sue England, Cliff England and Pinar Ozdemir for their continued support.

Contents

Chapter 1

Introduction

Recent times have seen a revival of interest in the theory of Abelian functions associated with algebraic curves. An Abelian function may be defined as one that has multiple independent periods, in this case derived from the periodicity property of an underlying algebraic curve. The topic can be dated back to the Weierstrass theory of elliptic functions which we use as a model.

The elliptic functions have often been described as one of the jewels of nineteenth century mathematics and have been the subject of study from mathematicians including Abel, Gauss, Jacobi, Legendre, Riemann and Weierstrass. They have been of great importance since their original definition and have been applied in a variety of mathematical areas. Over the last three decades their relevance in physics and applied mathematics has also been greatly developed. This has in turn inspired renewed interest in the theory of Abelian functions in which solutions to a number of the challenging problems of mathematical physics occur naturally.

In this chapter we will start in Section 1.1 by describing the motivation for our work. We discuss the theory of the Weierstrass $\wp$-function and introduce the generalisation that we work with. Note that formal definitions and proofs will be presented in Chapter 2 with the material in this chapter just for introductory purposes. We will highlight the important areas of the theory which are studied in detail during the remainder of the document.

In Section 1.2 we will proceed to identify the key contributions that are made to the topic by this document. This includes new classes of problems studied, new techniques and methods used to solve the problems and new interpretations of the generalisation. Finally, in Section 1.3 we present a guide to this document.

1.1 Motivation

Let $\wp(u)$ be the *Weierstrass $\wp$-function*. We define this formally in Chapter 2 and show that it has two complex periods ω_1, ω_2:

$$\wp(u+\omega_1) = \wp(u+\omega_2) = \wp(u), \qquad \text{for all } u \in \mathbb{C}. \tag{1.1}$$

Functions that are doubly periodic are known as *elliptic functions* and have been the subject of much study since their discovery in the 1800s. The $\wp$-function is a particularly important elliptic function which has the simplest possible pole structure for an elliptic function. It has poles of order two which occur only on those points that are a sum of integer multiples of the periods.

The $\wp$-function satisfies a number of interesting properties. For example, it can be used to parametrise an elliptic curve,

$$y^2 = 4x^3 - g_2x - g_3, \tag{1.2}$$

where g_2 and g_3 are constants. It also satisfies the following well-known differential equations,

$$\big(\wp'(u)\big)^2 = 4\wp(u)^3 - g_2\wp(u) - g_3, \tag{1.3}$$

$$\wp''(u) = 6\wp(u)^2 - \tfrac{1}{2}g_2. \tag{1.4}$$

Weierstrass introduced an auxiliary function, $\sigma(u)$, in his theory which satisfied,

$$\wp(u) = -\frac{d^2}{du^2}\log\big[\sigma(u)\big]. \tag{1.5}$$

This σ-function plays a crucial role in the generalisation and in applications of the theory. A particularly interesting result it satisfies is the following two term addition formula.

$$-\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2} = \wp(u) - \wp(v). \tag{1.6}$$

Taking logarithmic derivatives of this will give the standard addition formula for the Weierstrass $\wp$-function. In this document we present generalisations of equations (1.3)$-$(1.6) for new classes of functions.

Klein developed an approach to generalise the Weierstrass $\wp$-function to a function with two variables and four periods. Such functions were defined as *hyperelliptic* and the approach is described in Baker's classic texts [7] and [10] from 1897 and 1907 respectively.

It is centered around a generalisation of the σ-function which is then used to define new $\wp$-functions by equation (1.5). This approach has motivated the general definition of what we now call *Kleinian $\wp$-functions*.

Recall that hyperelliptic curves are algebraic curves with equations,

$$y^2 = f(x),$$

where $f(x)$ is a polynomial of degree greater than four. The original functions of Klein and Baker were associated with the simplest hyperelliptic curve, (with $f(x)$ of degree five). Baker later constructed examples using another hyperelliptic curve, but a theory for the functions associated to an arbitrary hyperelliptic curve did not follow until the 1990s when Buchstaber, Enolskii and Leykin published [19].

This was followed by a general definition for Abelian functions, structured by the underlying algebraic curves to which the functions are associated. In this document we work with *cyclic (n, s)-curves* defined by,

$$y^n = x^s + \lambda_{s-1}x^{s-1} + \dots \lambda_1 x + \lambda_0,$$

where $\lambda_0, \dots, \lambda_{s-1}$ are constants and (n, s) a pair of coprime integers. Note that the original elliptic curve in equation (1.2) was an (n, s)-curve with $(n, s) = (2, 3)$ while the hyperelliptic curves are those (n, s)-curves with $n = 2$ and $s > 4$.

In each case the genus of the curve is a unique integer associated to the curve that plays a key role. In particular, the functions become multivariate with g variables,

$$\sigma = \sigma(\boldsymbol{u}) = \sigma(u_1, u_2, \dots u_g).$$

The elliptic curve has genus one and hence the Weierstrass $\wp$-function has one complex variable. The generalisations however, will have genus greater than one.

We will define the Kleinian $\wp$-functions as logarithmic derivatives of $\sigma(\boldsymbol{u})$ in analogy with equation (1.5). However, we introduce a new subscript notation to make clear which variables are used in the differentiation.

$$\wp_{ij}(\boldsymbol{u}) = -\frac{\partial^2}{\partial u_i \partial u_j} \log \big[\sigma(\boldsymbol{u})\big].$$

Further derivatives are described by adding more subscripts and we may refer to the functions as *n-index $\wp$-functions*.

$$\wp_{i_1,i_2,\dots,i_n}(\boldsymbol{u}) = -\frac{\partial}{\partial u_{i_1}}\frac{\partial}{\partial u_{i_2}}\dots\frac{\partial}{\partial u_{i_n}} \log \big[\sigma(\boldsymbol{u})\big], \quad i_1 \leq \dots \leq i_n \in \{1, \dots, g\}. \tag{1.7}$$

The generalised σ and $\wp$-functions have properties in analogy to the Weierstrass functions.

In particular the $\wp$-functions satisfy a periodicity equation like (1.1). Here the periods are no longer scalar but matrices derived from the curve differentials. The 2-index $\wp$-functions will have poles of order at most two, like the original $\wp$-function, while the poles of the derivatives will have increasing orders.

Since the development of these general definitions there has been renewed interest in the theory of Abelian functions, with mathematicians including Athorne, Baldwin, Eilbeck, Gibbons, Matsutani, Nakayashiki, Onishi and Previato now also working in this area.

In the last few years a good deal of progress has been made on the theory of Abelian functions associated to those (n, s) curve with $n = 3$, (labeled *trigonal curves*). In particular, the two canonical cases of the (3,4) and (3,5)-curves have been examined in [30] and [11] respectively. The class of (n, s)-curves with $n = 4$ are labeled *tetragonal curves* and are considered for the first time in this document.

There are a variety of problems and questions that may be addressed for each class of curves. These often centre around derivations of generalisations for the differential equations (1.3) and (1.4). Other interesting relations include the various addition formula for the functions, such as the formula to generalise (1.6).

The main tool that was used in the hyperelliptic and trigonal cases was a theorem by Klein that linked the $\wp$-functions with points on the underlying algebraic curve. We present this theorem in Section 3.2 and discuss how we may *expand the Kleinian formula* to find differential equations between the $\wp$-functions. It has usually been the case that manipulation of these equations can achieve useful results. However, we find that this tool is not as helpful in the tetragonal cases and that alternative methods must be used.

Another problem for study is explicit descriptions of important substructures of the curves, such as the Jacobian and Kummer varieties. These have been achieved in the cases already studied as expressions using the associated $\wp$-functions.

Finally, the applications in other areas of mathematics are of great interest. Applications of the elliptic $\wp$-function included the construction of solutions to the pendulum equation and the KdV-equation.

The generalised $\wp$-functions have been demonstrated to give solutions to the 2-soliton KdV-equation and the Boussinesq equation. Also, the generalised σ-function has been used in the evaluation of integrals such as those that relate to reductions of the Benney equations. Both these applications are pursued further in this document.

1.2 What is contributed in this thesis

The first major contribution of this document is the investigation of Abelian functions associated with the (4,5)-curve, which is the first tetragonal curve to be considered. It has genus six which makes it also the curve of highest genus to be investigated.

The expansion of the Kleinian formula in the (4,5)-case is considerably more involved than the corresponding calculations in the trigonal and hyperelliptic cases. While we were able to use this method to solve the Jacobi Inversion Problem, it was not applicable for the derivation of differential equations. Instead we implemented new methods that center on the construction of a series expansion for the σ-function.

The computations involved are all considerably more complex than in the lower genus cases. For example, from equation (1.7), the higher genus means there are much larger sets of functions in this case. The extra complexity leads in turn to higher time and memory requirements when the computations are performed. Alongside the development of new mathematical methods, we have needed to create efficient computational procedures.

The computations were performed with the computer algebra package Maple. For several calculations we have rewritten Maple commands so they are more efficient for our problems. Additionally, we have made use of distributed computing by conducting many of the computations in parallel using the Distributed Maple package, [72].

These approaches have allowed us to present a complete set of differential equations that generalise (1.3) for the (4,5)-case. We have also derived a number of other differential equations between the $\wp$-functions and an addition formula to generalise equation (1.6).

The new methods, techniques and Maple procedures can all be applied to other (n, s)-curves with only minor modifications and so represent a significant contribution. For example, we have investigated two higher genus trigonal curves in Chapter 4. Also, the computations themselves are of value as they demonstrate the benefits and possibilities of symbolic computation and are one of only a limited number of serious applications of Distributed Maple.

Other contributions arising from the study of the (4,5)-curve include an understanding of another class of Abelian functions (the n-index Q-functions), which were introduced in Chapter 3 to complete a basis for the simplest functions associated to the (4,5)-curve. We have also constructed a solution to the KP-equation using Abelian functions associated with the (4,5)-curve. Given the similar results for lower genus curves, such an application was to be expected. However, we have presented here a wider class of solutions by outlining the result for all (n, s) curves with $n > 4$.

The work on the (4,5)-curve used a set of *Sato weights* associated to the theory which render all equations homogeneous. A derivation of these weight properties is presented in Section 3.3 for an arbitrary (n, s)-curve. Although no new results are presented here, such a thorough investigation has not been presented in any of the literature and so represented a worthwhile contribution to the understanding of the theory. This is also true for some of the general theory on the functions given in Chapter 2.

The second major contribution of this document is the new approaches detailed in Chapter 5. This includes a discussion of how we should approach the generalisations of the elliptic differential equations, and new methods for deriving relations based on pole cancellations and the σ-expansion. We present a method to derive the complete set of relations bilinear in the 2-index and 3-index $\wp$-functions which may be applied to any (n, s)-curve. We have also managed to generate complete sets of relations that generalise equation (1.3) for lower genus trigonal curves.

Other contributions in this document include the proofs in Appendix B which reconcile the different definitions of the Kleinian σ-function and the application to the Benney equations in Chapter 6. The application involves the construction of reductions of the Benney equations where the mapping function may be expressed using Kleinian functions. We present a specific reduction which uses results on the tetragonal curve considered earlier. The chapter follows the ideas of previous examples but is considerably more complicated and requires the development of a number of different methods. The procedures set out here should now be easily applicable to a wide class of reductions.

1.3 Guide to this document

Chapter 2

Here we present the background material necessary to understand the rest of the document. Section 2.1 gives an overview of elliptic function theory, starting with some general results on elliptic functions and then focusing on the Weierstrass functions. The key results which are generalised in the later chapters are emphasised.

In Section 2.2 we describe the generalised functions which we work with. These are presented using a general notation that may be specified to the cases of individual (n, s)-curves. This section includes the definitions for the Kleinian σ and $\wp$-functions.

In Section 2.3 we give a literature review with includes details on the derivation of the general definition, information on the cases that has already been considered and the other areas of current research.

Chapter 3

Here we present the theory of Abelian functions associated with the (4,5)-curve. We start in Section 3.1 by explicitly deriving the differentials of the curve. We then define the necessary functions including a new set of *Q-functions* which are required in addition to the $\wp$-functions for this case.

In Section 3.2 we introduce the Kleinian equation and describe a procedure which can be used to generate relations between the $\wp$-functions from this theorem. We use these to solve the Jacobi Inversion Problem for the (4,5)-case. In Section 3.3 we introduce a set of weights that render all equations in the theory homogeneous. We describe the idea for an arbitrary (n, s)-curve, giving explicit examples for the (4,5)-curve.

In Section 3.4 we describe how to derive a Taylor series expansion of the σ-function around the origin. This involves large computations performed in Maple and we include a discussion of the steps that may be taken to ensure the computations are efficient. This expansion is used to derive a number of relations between the Abelian functions which we present in Section 3.5.

In Section 3.6 we establish the two-term addition formula for the σ-function, giving details on the construction. Finally, in Section 3.7, we introduce the applications in the KP hierarchy of differential equations.

Chapter 4

Here we apply the techniques and methods described in Chapter 3 to two of the higher genus trigonal curves. We present explicit results including differentials, expansions, addition formula and differential equations. The differences and similarities with the tetragonal and lower genus trigonal results are identified.

Section 4.1 presents the theory of the cyclic (3,7)-curve and Section 4.2 the theory of the cyclic (3,8)-curve.

Chapter 5

Here we describes a number of new approaches and techniques that were developed following the research on the (4,5)-curve. These have been used to derive new results for the lower genus trigonal curves. In Section 5.1 we discuss the different approaches to the generalisation of equations (1.3) and (1.4), including the corresponding problems and results.

In Section 5.2 we describe a new process to derive complete sets of relations that are bilinear in the 2-index and 3-index $\wp$-functions. Then in Section 5.3 we use these along with some other methods to derive generalisations of equation (1.3) for the cyclic (3,4) and (3,5)-curves.

Chapter 6

This chapter deals with an application of the theory of Abelian functions to the Benney moment equations. We consider the reductions of the Benney equations in which the mapping function may be realised using the Kleinian σ-function. A full introduction is given in Section 6.1, followed by the necessary background information in Section 6.2.

The remainder of the chapter performs the explicit calculations for the case that relates to the (4,5)-curve. This example is constructed in Section 6.3 with the integrand of the mapping function evaluated in Section 6.5. This required sets of relations between the derivatives of the σ-function which were derived in Section 6.4. Finally, in Section 6.6 an explicit formula for the mapping is presented.

Appendices

Appendix A contains background mathematics that is used in this thesis, starting with Appendix A.1 which derives some results for general elliptic functions. Appendix A.2 introduces the Jacobi θ-functions and then Appendix A.3 presents the multivariate θ-functions which are used in the realisation of the Kleinian σ-function.

Appendix A.4 gives details on Weierstrass gap sequences which are used in the description of the general theory while Appendix A.5 defines Schur-Weierstrass polynomials which have been shown to act as a limit of the σ-function. Finally, in Appendix A.6 we briefly recap the theory of resultants.

Appendix B explicitly resolves a technical point arising from the slightly different definitions of the Kleinian σ-function. The remaining printed Appendices contain relations that were considered too lengthly to include in the main body of the thesis. Those in Appendix C relate to the (4,5)-curve and those in Appendix D to the (3,5)-curve. Finally, the relations in Appendix E were used in the application to the Benney equations described in Chapter 6.

Extra Appendix of files

Submitted alongside this document is an extra Appendix of files. This takes the form of a CD-rom for the physical version and a folder of files for the electronic version. This extra Appendix is split into two parts. The first contains text files of results that were too large or cumbersome to typeset. These files are organised according to the curves they are associated to and use an obvious notation.

The second part of this Appendix contains the Maple worksheets that were used to derive many of the results in this document. Also included here are all the text files that are referenced by these worksheets. Each worksheet starts with the code,

```
> currentdir("path"):
```

with `path` replaced by the location of the folder containing this worksheet. This code tells Maple the folder to work from.

To run the Maple worksheets copy the files to a machine and replace the start code with the correct path referencing the directory in which the worksheet has been stored. For example,

```
currentdir("C:/Users/Matthew/Documents/Maths/DM-45"):
```

Ensure that all the necessary text files which are referenced are also in this directory.

Some calculations were preformed in parallel using Distributed Maple. This is a free piece of software that may be downloaded for from [72] and will need to also be present in the directory with the worksheet. Distributed Maple opens Maple kernels on a cluster of machines and allows data and commands to be sent from a master kernel to the others. For more information see [72] and [65].

Chapter 2

Background Material

This section is designed to cover the necessary background material for the understanding of this thesis. The new results presented over the coming chapters involve Abelian functions associated with algebraic curves. These are multivariate functions of many periods.

We may think of Abelian functions as a generalisation of the elliptic functions. These were functions of a complex variable which take values that are periodic in two directions. They may be introduced by comparison to the trigonometric functions, which have a single period. While there are no univariate complex functions with more than two periods we can define multivariate functions with many periods, and we refer to these as Abelian. In particular, an Abelian generalisation of the classic Weierstrass elliptic $\wp$-function will be developed.

This chapter is split into three sections. Section 2.1 summarises key parts of elliptic function theory. After giving some general information on elliptic functions it proceeds to consider the Weierstrass $\wp$-function in detail. Emphasis is given to those areas of the theory that are present or relevant in the generalisation.

Section 2.2 introduces the generalised functions. These functions are classified by sets of algebraic curves from which the periods are generated. The section will give the necessary theory of these curves before describing the functions and their core properties. Again, emphasis is given to those areas of the material that are necessary or relevant in the proceeding chapters. These functions are introduced in general terms to avoid repetition throughout the document. It is easy to specialise these definitions to the case of particular curves, as is the case in later chapters.

Finally, Section 2.3 acts as a literature review, putting the theory described in Section 2.2 in a historical context. It clarifies which classes of functions have been already studied at the time of writing, what results have been derives and what other related areas of current research are ongoing.

2.1 Elliptic function theory

Elliptic functions are essentially functions of a complex variable that take values which are periodic in two directions. They were first discovered in the mid 1800s as the inverse functions of elliptic integrals. These integrals were connected with the problem of finding the arc length of an ellipse, and it is from here that the name is derived.

This section starts by recalling some facts from complex analysis, before formally defining elliptic functions and introducing some of their key properties. It then moves on to consider the specific example of the Weierstrass $\wp$-function, which is the subject of the generalisation in the next section. The theory is presented, focusing on those results and equations which appear later in the generalised theory. Unless noted otherwise, the classical theory given here is well known. We loosely follow Chapter 20 of [70] which gives a thorough summary of the properties of elliptic functions (including the $\wp$-function). Other resources used to write this section include [26], [28] and [3]. Additionally, [57] is recommended for a broader examination of some key ideas.

2.1.1 Meromorphic functions

The defining properties of elliptic function are their periodicity properties and a constraint on their singularities. Hence it is worth recalling some basic definitions that will be used throughout this document, regarding the singularities of complex functions.

Let D denote an open subset of the complex plane, p a point in D and f a function defined on an open subset of $\mathbb{C}$ and taking values within $\mathbb{C}$.

Definition 2.1.1. *The function f is **holomorphic over** D if it is complex-differentiable at every point in D. It is a **holomorphic function** if complex-differentiable at every point on which it is defined and an **entire function** if holomorphic over the whole complex plane.*

If f is holomorphic then it is also analytic, and so may be described by its Taylor series about a point. We may categorise singularities of complex functions as follows.

Definition 2.1.2. *Let f be a holomorphic function defined on $D - \{p\}$.*

- *Suppose f is not defined at p but there is a holomorphic function g defined on D with $f(u) = g(u)$ for all $u \in U - \{p\}$. Then p is an **removable singularity**.*
- *The point p is a **pole** of f if there exists a holomorphic function g defined on D and a natural number n such that*

$$f(u) = \frac{g(u)}{(u-p)^n} \qquad \textit{for all} \quad u \in D - \{p\}.$$

*The number n is labeled the **order of the pole**. If $n = 1$ then the pole is **simple**.*

- *The point p is an **essential singularity** if and only if $\lim_{u\to p} f(u)$ does not exist as a complex number, nor equals infinity. The Laurent series of f at p will have infinitely many terms of negative degree.*

*If there exists a disk $\mathcal{D}$ centred on p such that f is holomorphic on $\mathcal{D} - p$ then p is an **isolated singularity**. By definition both removable singularities and poles are isolated. Any isolated singularity that is not removable or a pole is an essential singularity.*

Definition 2.1.3. *The function f is **meromorphic function** on D if it is holomorphic on all of D except a set of isolated points, which are poles of the function.*

Since the poles of a meromorphic function are isolated, they are at most countably infinite. The function f may be expressed as a Laurent series about p.

$$f(u) = \sum_{n=-\infty}^{\infty} \alpha_n (u-p)^n, \qquad u \in \mathbb{C} \tag{2.1}$$

Definition 2.1.4. *The constant α_{-1} is called the **residue** of $f(u)$. If f is holomorphic at p then the residue is zero, (although the converse is not always true). At a simple pole, the residue is given by*

$$\mathrm{Res}(f(u), p) = \lim_{u\to p}(u-p)f(u). \tag{2.2}$$

2.1.2 Elliptic functions

Definition 2.1.5. *An **elliptic function** is a meromorphic function, $f(u)$, defined on $\mathbb{C}$ for which there exist two periods ω_1, ω_2.*

$$f(u+\omega_1) = f(u+\omega_2) = f(u) \qquad \textit{for all } u \in \mathbb{C}. \tag{2.3}$$

The periods are non-zero complex numbers that satisfy $\omega_1/\omega_2 \notin \mathbb{R}$.

The periods ω_1, ω_2 are usually assumed to be the smallest complex numbers in the second quadrant of the Argand diagram which satisfy equation (2.3). In Figure 2.1 we have plotted the points $\{0, \omega_1, \omega_2, \omega_1 + \omega_2\}$ and joined them up to give a parallelogram. This is known as the **fundamental period parallelogram** for the elliptic functions with periods ω_1, ω_2. Note that if the ratio, ω_1/ω_2 was real then the parallelogram would collapse to a line, and the function would either be a constant, or have just one period.

Copies of this parallelogram can be used to span the complex plane $\mathbb{C}$. Each parallelogram is called a **period parallelogram** and together they for a **mesh** over $\mathbb{C}$. (See Figure 2.2.) Two points, say U and U', that occur at the same position in the parallelogram are called **congruent**. We may write this using the notation,

$$U \equiv U' \qquad (\mathrm{mod}\ \omega_1 + \omega_2).$$

Note that an elliptic function will take the same value at congruent points.

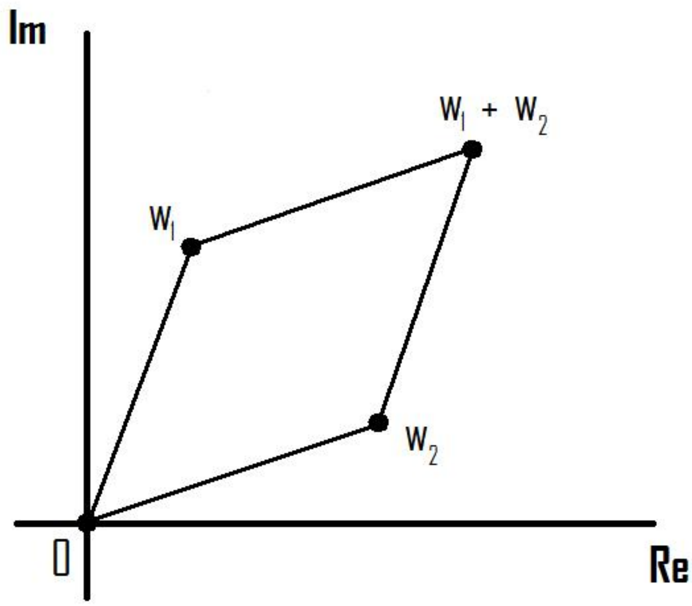


Figure 2.1: The fundamental period parallelogram formed by ω_1 and ω_2.

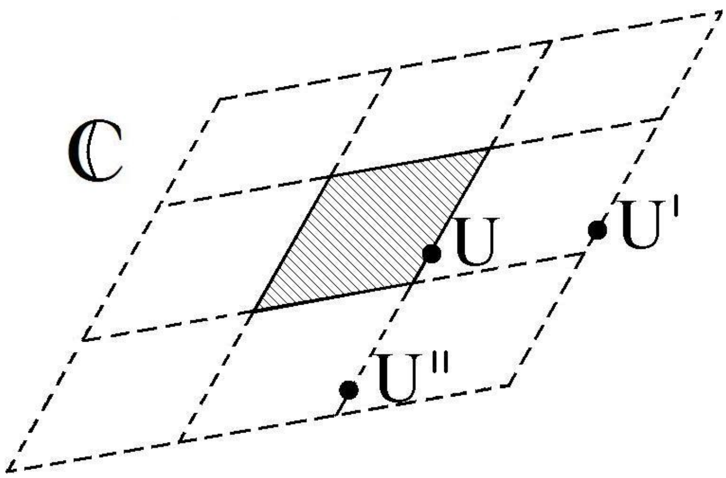


Figure 2.2: A mesh of period parallelograms. The points U, U' and U'' are congruent, so an elliptic function would take the same value at these points.

For integration purposes it is not convenient to deal with meshes if they have singularities of the integrand on the boundaries. However, due to the periodicity properties, no information would be lost if the integral of an elliptic function was taken not over a period parallelogram, but over one of its translations (without rotation) which we label a **cell**. The values assumed by an elliptic function along a cell are a repetition of its values along a period parallelogram.

In Appendix A.1 a number of interesting and useful results for elliptic functions are proved. We summarise these below.

- An elliptic function has a finite number of poles in each cell.
- An elliptic function has a finite number of zeros in each cell.
- Consider the poles of an elliptic function in a cell. The sum of the residues will be zero.
- An elliptic function with no poles is a constant.
- There does not exist an elliptic function with a single simple pole.
- A non-constant elliptic function has exactly as many poles as zeros (when counting multiplicities).
- If $f(u)$ is an elliptic function and c a constant then the number of roots of $f(u) = c$ in any cell is the number of poles of $f(u)$ in a cell. This number is defined as the **order of the elliptic function**.
- Let $a_1, \dots, a_n$ denote the zeros and $b_1, \dots, b_n$ the poles of an elliptic function (when counting multiplicities). Then

$$a_1 + \cdots + a_n \equiv b_1 + \cdots + b_n \pmod{\omega_1 + \omega_2} \tag{2.4}$$

Types of elliptic functions

The two standard forms of elliptic functions are the Jacobi elliptic functions and the Weierstrass elliptic functions. The generalisation given in the next section is based on the $\wp$-function of Weierstrass and so the rest of this section is dedicated to the study of this.

For a short summary of Jacobi's approach see Chapter 21 of [70], while a clear detailed description may be found in [54]. The three basic functions, denoted $cn(u), dn(u)$ and $sn(u)$ arise from the inversion of the elliptic integral of the first kind. They are doubly periodic generalisations of the three main trigonometric functions. (See Figure 2.3). While they are still used in a wide variety of applications, they are not convenient for generalising the theory.

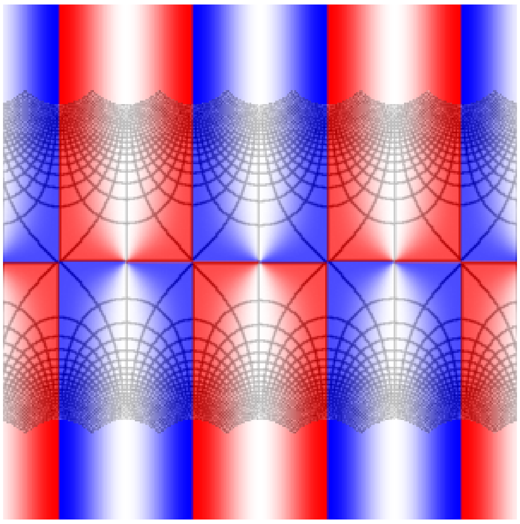
It is easy to switch between the notation of the Jacobi and Weierstrass approaches using θ-functions, another component of Jacobi's theory. While the Jacobi elliptic functions are not easy to generalise, the θ-functions are and the generalised θ-functions are used to prove parts of the general theory in the next section. The definitions and core properties of the Jacobi θ-functions are summarised in Appendix A.2. It may be possible to use θ-functions as an alternative to the Weierstrass function approach of this document, however they are not considered as advantageous.

One of the major benefits of the Weierstrass $\wp$-functions is that the pole structure is as simple as possible, with one double pole in each cell. (Recall from Theorem A.1.1 that a holomorphic elliptic function is a constant and there does not exist an elliptic function with a single simple pole). Further, these poles occur exactly at the corners of the period parallelograms. See Figure 2.4 for a comparison of a Jacobi elliptic function with a Weierstrass elliptic function. Note that the Weierstrass $\wp$-function and its first derivative span the field of elliptic functions and so the theory of elliptic functions may be constructed using only these.

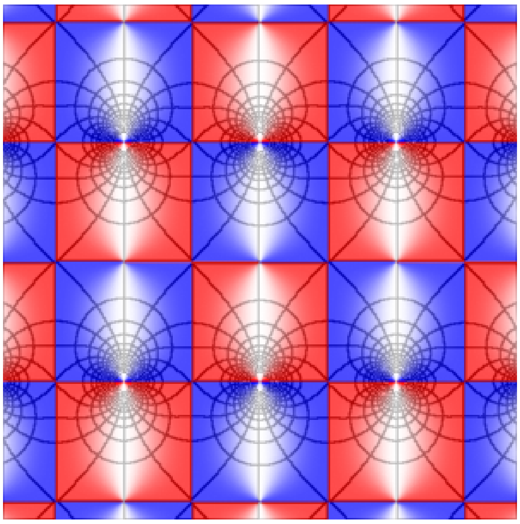
Visualising elliptic functions

The plots displayed over the coming pages were obtained from [55] and give visualisations of periodic complex functions. They are included to clarify some of the points made earlier in this section. Each plot is a square in the complex plane centred at the origin with corners at $\pm 6 \pm 6$i. The blue regions indicate where the function in question has positive imaginary part, while the red regions are where it has negative imaginary part. Along the boundaries of these regions the function takes real values. The white lines indicate that the real part of the function is zero, while the grey grid lines are lines of constant real or imaginary parts. Since the functions are periodic the patterns will continue to repeat outside the region shown.

Figure 2.3: **Comparison of a trigonometric function with an elliptic function**



(a) The sine function



(b) A Jacobi sn-function

In Figure 2.3 we compare a trigonometric function with an elliptic function. The values given by both the sine and sn-function repeat as the real part of the input variable is increased or decreased. Hence the pattern in the plots repeats in the horizontal direction. The sn-function will also repeat in the vertical direction as the imaginary part is varied.

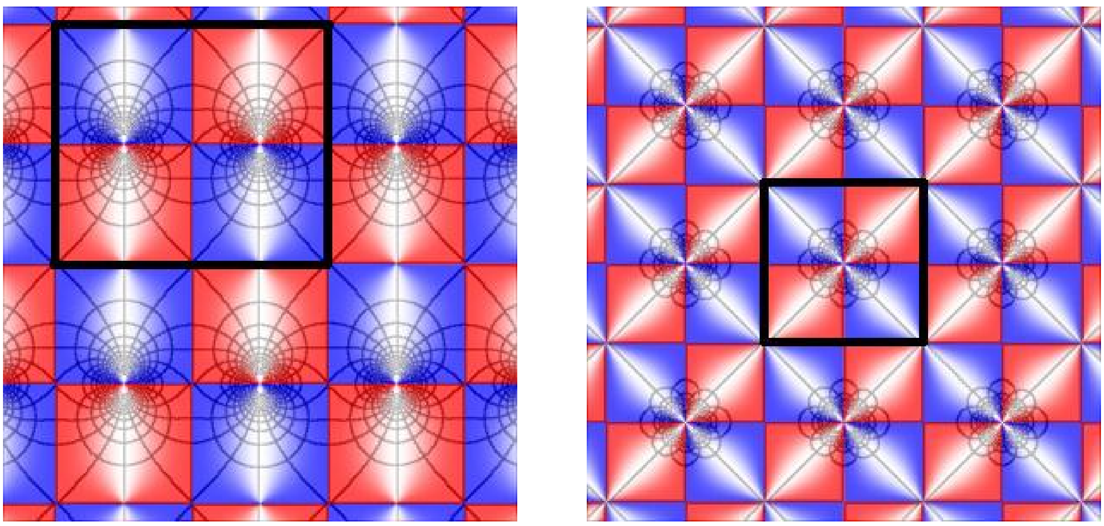
The plots are very similar close to the real axis, but differ in the rest of the complex plane. Changing from $sin(u)$ to $sn(u)$ causes the vertical edges of the strips to bend inward and enclose rectangles. Where the tips meet, the function has simple poles, as indicated by rapid change in grid lines.

In Figure 2.4 we compare a Jacobi elliptic function with a Weierstrass elliptic function. Both of these functions are elliptic and hence the patterns repeat in two different directions due to the double periodicity.

Note that these plots of elliptic function are given for specific values of the periods. The patterns and values the function take may change with different periods, but the core properties do not.

In each of the plots a black line has been drawn around a cell (region which repeats over $\mathbb{C}$). Note that in each cell the Jacobi function has two simple poles, while the Weierstrass function has a single double pole.

Figure 2.4: **Comparison of a Jacobi elliptic function with a Weierstrass elliptic function**

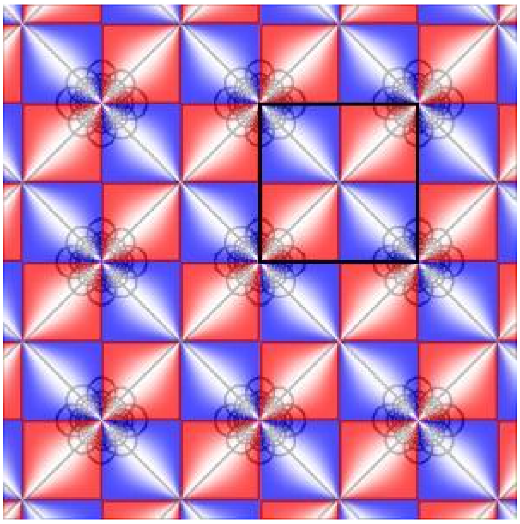


(a) A Jacobi sn-function

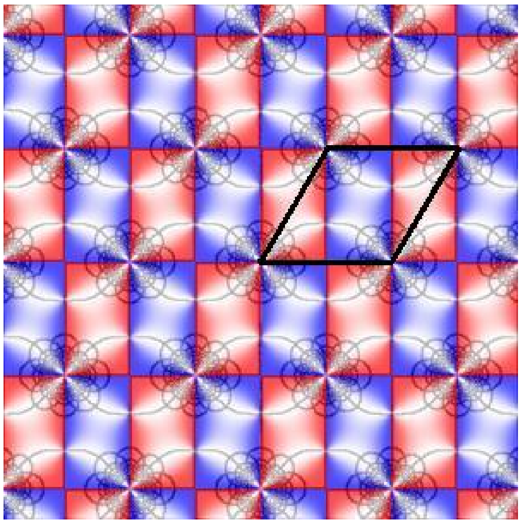
(b) A Weierstrass $\wp$-function

Finally, in Figure 2.5, we compare two different special cases of the Weierstrass $\wp$-function. The difference arises from the values of the two periods which were used. In the first plot it can be seen that the period parallelograms are in fact squares. In the second plot the period parallelograms are made up from two equilateral triangles. This is not as clear from the plot and so in each case the fundamental period parallelogram has been marked on in black.

Figure 2.5: **Comparison of two different Weierstrass $\wp$-functions**

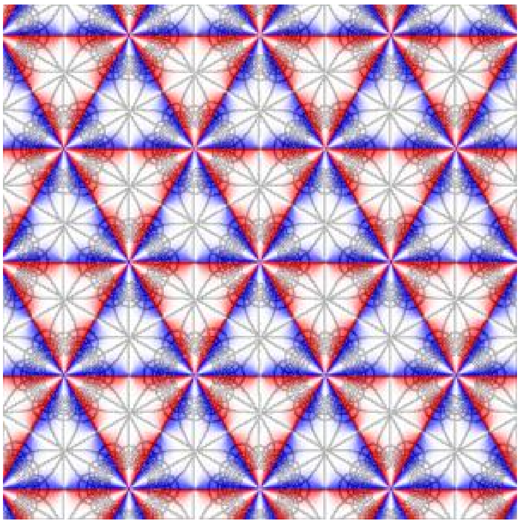


(a) The lemniscatic $\wp$-function



(b) The equianharmonic $\wp$-function

Note that although the patterns are different, the pole properties remain the same. That is, the only singularities are double poles at the corners of the period parallelograms.
The third plot is the derivative of the function used for the second plot. It shows the triangles that make up the period parallelograms more clearly. The derivative has triple poles on the parallelogram corners.



(c) The derivative of the equianharmonic $\wp$-function

2.1.3 The Weierstrass $\wp$-function

As with an arbitrary elliptic function the $\wp$-function is defined using a complex variable u and two complex periods ω_1, ω_2. There are several alternative definitions which can be given. The definition given below is probably the most common, although it will be equation (2.21) that is extended in the next section to give a definition for the higher genus $\wp$-functions. When dealing with problems that require numerical results then it is most efficient to use the definition for the $\wp$-function in terms of θ-functions. These are discussed in Appendix A.2 with the definition given by equation (A.13).

Definition 2.1.6. *The **Weierstrass $\wp$-function** with respect to the periods ω_1, ω_2 is given by*

$$\wp(u) = \wp(u;\omega_1,\omega_2) = \frac{1}{u^2} + \sum_{\substack{m,n\in\mathbb{Z}\\ m^2+n^2\neq 0}} \left\{ \frac{1}{(u-m\omega_1-n\omega_2)^2} - \frac{1}{(m\omega_1+n\omega_2)^2} \right\}.$$

Remark 2.1.7.

(i) The infinite double sum in the definition has m, n ranging over the integers, with the entry where $n = m = 0$ excluded. For simplicity the notation $\sum'_{m,n}$ is often used to denote this.

(ii) Note that if $n, m \neq 0$ then $m\omega_1 + n\omega_2 \neq 0$. (If this were not the case then $n = -m\frac{\omega_1}{\omega_2}$. But recall from Definition 2.1.5 that $\frac{\omega_1}{\omega_2} \notin \mathbb{R}$. Hence there is a contradiction since $n, m \in \mathbb{Z}$.)

(iii) It is now clear that $\wp(u)$ is holomorphic for all values of $u \in \mathbb{C}$ except those equal to $m\omega_1 + n\omega_2$ for some n, m. At these points the function has a double pole. Hence $\wp(u)$ is meromorphic.

Definition 2.1.8. *Denote the **period lattice** formed from ω_1, ω_2 by Λ. This is the set of points*

$$\Lambda_{m,n} = m\omega_1 + n\omega_2, \qquad m, n \in \mathbb{Z}.$$

This can be used to simplify the definition of $\wp(u)$ to

$$\wp(u) = \wp(u;\Lambda) = u^{-2} + \sum_{m,n}{}' \Big[(u-\Lambda_{m,n})^{-2} - \Lambda_{m,n}^{-2} \Big]. \tag{2.5}$$

This series defining $\wp(u)$ is absolutely and uniformly convergent (see Chapter 3.4 in [70]), and is constructed from entire functions. Hence it is appropriate to use term by term differentiation to define the derivatives of the $\wp$-function. We use the prime notation to indicate the first derivative of the $\wp$-function with respect to its variable.

$$\begin{aligned}\wp'(u) = \frac{d}{du}\wp(u) &= -2u^3 - 2\sum_{m,n}{}'(u-\Lambda_{m,n})^{-3} \\ &= -2\sum_{m,n}(u-\Lambda_{m,n})^{-3}. \end{aligned} \tag{2.6}$$

Lemma 2.1.9. *The function $\wp'(u)$ is odd:* $\quad \wp'(-u) = -\wp'(u)$.

Proof. Consider

$$\wp'(-u) = -2\sum_{m,n}(-u-\Lambda_{m,n})^{-3} = 2\sum_{m,n}(u+\Lambda_{m,n})^{-3}.$$
$$-\wp'(-u) = -2\sum_{m,n}(u+\Lambda_{m,n})^{-3}.$$

The set of points $-\Lambda_{m,n}$ is the same as the set $\Lambda_{m,n}$ and so the terms in $-\wp(-u)$ will be the same as those in $\wp(u)$ except in a different order. Now since the series for $\wp'(u)$ is absolutely convergent the order will not matter, and hence $\wp'(u)$ is odd.

■

Lemma 2.1.10. *The function $\wp(u)$ is even:* $\quad \wp(-u) = \wp(u)$.

Proof. The proof is in an identical manner to the previous lemma.

■

Lemma 2.1.11. *The function $\wp'(u)$ is doubly periodic with respect to ω_1, ω_2.*

Proof. Consider

$$\wp'(u+\omega_1) = -2\sum_{m,n}(u-\Lambda_{m,n}+\omega_1)^{-3}.$$

The set of points $\Lambda_{m,n}+\omega_1$ is the same as the set $\Lambda_{m,n}$ so the series for $\wp'(u+\omega_1)$ is a rearrangement of the series for $\wp'(u)$. Hence, since the series is convergent,

$$\wp'(u+\omega_1) = \wp'(u). \tag{2.7}$$

So $\wp'(u)$ has period ω_1 and using an identical method also period ω_2.

■

Lemma 2.1.12. *The function $\wp(u)$ is doubly periodic with respect to ω_1, ω_2.*

Proof. Integrate equation (2.7) to give $\wp(u+\omega_1) = \wp(u) + A$. Substitute in

$$u = -\frac{\omega_1}{2} \qquad \text{to give} \quad \wp\left(\frac{\omega_1}{2}\right) = \wp\left(-\frac{\omega_1}{2}\right) + A.$$

Recall that $\wp(u)$ is even and hence $A = 0$. Therefore $\wp(u)$ is periodic with period ω_1. An identical method shows it is also periodic with period ω_2.

■

Remark 2.1.13. From Remark 2.1.7 we know $\wp(u)$ is a meromorphic function and it has just been shown in Lemma 2.1.12 that it is periodic with respect to ω_1, ω_2. Hence by Definition 2.1.5, $\boldsymbol{\wp(u)}$ **is an elliptic function**.

Similarly, from equation (2.6) it can be seen that $\wp'(u)$ is homomorphic everywhere except its poles and is hence meromorphic. Coupled with Lemma 2.1.11 this allows us to conclude that $\boldsymbol{\wp'(u)}$ **is an elliptic function**.

The differential equation satisfied by $\wp(u)$

The Weierstrass $\wp$-function satisfies a differential equation linking the function with its first derivative. This equation can be used to specify the function and is the core of several applications. The equation can be derived by considering the series expansion of the $\wp$-function.

Theorem 2.1.14. *The function $\wp(u)$ satisfies*

$$[\wp'(u)]^2 = 4\wp(u)^3 - g_2\wp(u) - g_3, \tag{2.8}$$

where g_2 and g_3 are defined as the ***elliptic invariants*** *and are given by*

$$g_2 = 60\sum_{m,n}{}' \Lambda_{m,n}^{-4}, \qquad g_3 = 140\sum_{m,n}{}' \Lambda_{m,n}^{-6}. \tag{2.9}$$

Proof. Consider the function, $f(u) = \wp(u) - u^{-2}$. From equation (2.5)

$$f(u) = \sum_{m,n}{}' \left[(u - \Lambda_{m,n})^{-2} - \Lambda_{m,n}^{-2}\right]. \tag{2.10}$$

Using Lemma 2.1.10 the function $f(u)$ can be concluded even. It will be holomorphic in a region about the origin and so Taylor's theorem may be applied here. (Recall that an even function will have an Taylor series expansion with only even powers.)

$$f(u) = f(0) + \frac{f''(0)}{2!}u^2 + \frac{f^{(4)}(0)}{4!}u^4 + O(u^6).$$

From equation (2.10) it is obvious that $f(0) = 0$. Differentiating gives

$$f''(u) = -6\sum_{m,n}{}'[u - \Lambda_{m,n}]^{-4}, \qquad \Longrightarrow \qquad f''(0) = 6\sum_{m,n}{}' \Lambda_{m,n}^{-4},$$
$$f^{(4)}(u) = -120\sum_{m,n}{}'[u - \Lambda_{m,n}]^{-6}, \qquad \Longrightarrow \qquad f^{(4)}(0) = 120\sum_{m,n}{}' \Lambda_{m,n}^{-6}.$$

Then substituting back gives

$$\begin{aligned} f(u) &= 3\left[\sum_{m,n}{}' \Lambda_{m,n}^{-4}\right]u^2 + 5\left[\sum_{m,n}{}' \Lambda_{m,n}^{-6}\right]u^4 + O(u^6) \\ &= \frac{1}{20}g_2u^2 + \frac{1}{28}g_3u^4 + O(u^6), \end{aligned}$$

where

$$g_2 = 60\sum_{m,n}{}' \Lambda_{m,n}^{-4}, \qquad g_3 = 140\sum_{m,n}{}' \Lambda_{m,n}^{-6}.$$

So the series expansion for $\wp(u)$ is

$$\wp(u) = u^{-2} + \tfrac{1}{20}g_2u^2 + \tfrac{1}{28}g_3u^4 + O(u^6). \tag{2.11}$$

Differentiating term by term gives a similar series for $\wp'(u)$.

$$\wp'(u) = -2u^{-3} + \tfrac{1}{10}g_2u + \tfrac{1}{7}g_3u^3 + O(u^5). \tag{2.12}$$

Respectively cube and square these results to get

$$\wp(u)^3 = u^{-6} + \tfrac{3}{20}g_2u^{-2} + \tfrac{3}{28}g_3 + O(u^2),$$
$$[\wp'(u)]^2 = 4u^{-6} - \tfrac{2}{5}g_2u^{-2} - \tfrac{4}{7}g_3 + O(u^2).$$

Hence

$$[\wp'(u)]^2 - 4\wp(u)^3 = -g_2u^{-2} - g_3 + O(u^2).$$

Then, using equation (2.11),

$$[\wp'(u)]^2 - 4\wp(u)^3 + g_2\wp(u) + g_3 = O(u^2).$$

This means that the function on the left hand side is holomorphic at the origin. Further, this function is constructed from elliptic functions, and so is elliptic itself. The function is holomorphic at all points congruent to the origin, however these are the only possible singularities. Therefore this is an elliptic function with no singularities and by Theorem A.1.1(v) is a constant. Finally, since the expansion of the function is $O(u^2)$, this constant must be set to zero.

$$[\wp'(u)]^2 - 4\wp(u)^3 + g_2\wp(u) + g_3 = 0.$$

Therefore $\wp(u)$ satisfies the differential equation given in the theorem for the given values of g_2, g_3.

■

Corollary 2.1.15. *The Weierstrass $\wp$-function satisfies*

$$\wp''(u) = 6\wp(u)^2 - \tfrac{1}{2}g_2. \tag{2.13}$$

Proof. Differentiate equation (2.8) derived above with respect to u.

$$2\wp'(u)\wp''(u) = 12\wp(u)^2\wp'(u) - g_2\wp'(u).$$

Then dividing both sides by $2\wp'(u)$ will give the desired result.

■

Corollary 2.1.16. *Consider the differential equation*

$$\left(\frac{d}{dz}y(z)\right)^2 = 4y(z)^3 - G_2y(y) - G_3(y).$$

A solution is given by

$$y = \wp(\pm z + \alpha), \qquad \alpha \text{ constant},$$

provided that periods ω_1, ω_2 can be determined such that G_2 and G_3 equal the elliptic invariants as given in equation (2.9).

Proof. Let $y = \wp(\pm z + \alpha)$ and consider

$$\left(\frac{d}{dz}y(z)\right)^2 = \left((\pm 1)\wp'(\pm z + \alpha)\right)^2 = \wp'(\pm z + \alpha)^2.$$

Now apply Theorem 2.1.14 to conclude

$$\left(\frac{d}{dz}y(z)\right)^2 = 4\wp(\pm z + \alpha)^3 - g_2\wp(\pm z + \alpha) - g_3 = 4y(z)^3 - G_2y(z) - G_3$$

as required.

■

Note that so long as $G_2^3 \neq 27G_3^2$ it will be possible to find periods ω_1, ω_2 such that G_2 and G_3 equal the elliptic invariants. (See Chapter 21.73 in [70].) This result can be used to derive the following *integral formula* for the $\wp$-function.

Lemma 2.1.17. *The equation*

$$u = \int_\xi^\infty \left(4t^3 - g_2t - g_3\right)^{-\frac{1}{2}} \tag{2.14}$$

is equivalent to the statement $\xi = \wp(u)$.

Proof. Differentiate equation (2.14) with respect to ξ.

$$\frac{du}{d\xi} = \left(4\xi^3 - g_2\xi - g_3\right)^{-\frac{1}{2}}, \qquad \Longrightarrow \qquad \left(\frac{d\xi}{du}\right)^2 = 4\xi^3 - g_2\xi - g_3.$$

From Corollary 2.1.16 above, this gives $\xi = \wp(\pm u + \alpha)$ for some constant α. To determine α we let $\xi \to \infty$. From equation (2.14) this implies $u \to 0$ and hence α is pole of the $\wp$-function. Therefore $\alpha = \Lambda_{m,n}$ for some m, n and so

$$\xi = \wp(\pm u + \alpha) = \wp(\pm u + \Lambda_{m,n}) = \wp(\pm u) = \wp(u)$$

as required.

■

This result is sometimes abbreviated to

$$u = \int_{\wp(u)}^{\infty} (4t^3 - g_2 t - g_3)^{-\frac{1}{2}} dt. \tag{2.15}$$

Elliptic curves

Much of the theory of elliptic functions is linked to the properties of a certain class of algebraic curves, which is introduced below.

Definition 2.1.18. *An elliptic curve is a non-singular algebraic curve which may be written in the form*

$$y^2 = x^3 + ax + b. \tag{2.16}$$

We may instead consider a general cubic polynomial on the right. However, as long as it has no repeated roots, we can make a change of variables to obtain equation (2.16).

We now demonstrate the link between elliptic curves and elliptic functions. Substitute x for $4^{\frac{1}{3}}\bar{x}$ in equation (2.16) to obtain

$$y^2 = 4(\bar{x})^3 + 4^{\frac{1}{3}} a\bar{x} + b.$$

By Corollary 2.1.16 one solution will be $y = \wp(u+\alpha)$ providing periods can be determined so that the invariants g_2, g_3 are equal to $4^{\frac{1}{3}}a, b$ respectively. (Note that this condition will always be satisfied due to a condition on a and b arising from the curve being non-singular.) Hence the elliptic curve may be parameterised by the Weierstrass $\wp$-function.

It is well known that the surface mapped by an elliptic curve is topologically a torus. In fact, the Weierstrass $\wp$-function describes how to get from a torus giving the solutions of an elliptic curve to the algebraic form of the elliptic curve. A torus, T, may be expressed as the quotient of the complex plane and a lattice, Λ.

$$T = \mathbb{C}/\Lambda.$$

(The complex plane with those points at the same position of the lattice 'glued together'). Then this torus may be embedded in the complex projective plane by means of the map

$$u \mapsto \big(1, \wp(u;\Lambda), \wp'(u;\Lambda)\big).$$

Given an elliptic curve with equation $y^2 = 4x^3 - g_2x - g_3$, equation (2.15) could be rewritten as

$$u = \int_{\wp(u)}^{\infty} \frac{dx}{y}. \tag{2.17}$$

2.1.4 Weierstrass' quasi-periodic functions

Weierstrass defined other functions within his theory which were associated to the period lattice Λ. While these functions are not elliptic they do satisfy *quasi-periodic* properties which are demonstrated below. The σ-function in particular plays an important role both in the elliptic case and in the generalisation of the next section.

Definition 2.1.19. *The* ***Weierstrass σ-function*** *and the* ***Weierstrass ζ-function*** *are defined below using the complex variable u and the period lattice Λ.*

$$\sigma(u) = \sigma(u;\Lambda_{m,n}) = u\prod_{m,n}{}' \left[\left(1-\frac{u}{\Lambda_{m,n}}\right)\exp\left(\frac{u}{\Lambda_{m,n}}+\frac{1}{2}\left[\frac{u}{\Lambda_{m,n}}\right]^2\right)\right]. \tag{2.18}$$

$$\zeta(u) = \zeta(u;\Lambda_{m,n}) = \frac{1}{u}+\sum_{m,n}{}' \left[\frac{1}{u-\Lambda_{m,n}}+\frac{1}{\Lambda_{m,n}}+\frac{u}{\Lambda_{m,n}^2}\right]. \tag{2.19}$$

As discussed in Remark 2.1.7 the $'$ means the term in the series with $m=n=0$ is excluded.

From these definitions it is clear that $\sigma(u)$ is an entire function, with simple zeros at each of the points $\Lambda_{m,n}$. These are key properties of the function which are present in the generalisation. The ζ-function is holomorphic everywhere except at the points $\Lambda_{m,n}$, which are simple poles of the function. Both these series are absolutely and uniformly convergent.

Lemma 2.1.20. *Both $\zeta(u)$ and $\sigma(u)$ are odd functions of u.*

Proof. First consider the ζ-function.

$$\begin{aligned}-\zeta(-u) &= \frac{1}{u}+\sum_{m,n}{}' \left[\frac{1}{u+\Lambda_{m,n}}+\frac{1}{\Lambda_{m,n}}+\frac{u}{\Lambda_{m,n}^2}\right]\\ &= \zeta(u)-\sum_{m,n}{}' \left[\frac{1}{u-\Lambda_{m,n}}\right]+\sum_{m,n}{}' \left[\frac{1}{u+\Lambda_{m,n}}\right]\end{aligned}$$

Both the sums on the final line run over all the integers, and so consist of the same terms in a different order. Since the series is absolutely convergent the two sums can be concluded equal and hence the ζ-function is odd.

Next consider the σ-function.

$$\begin{aligned}\sigma(-u) &= -u\prod_{m,n}{}' \left[\left(1+\frac{u}{\Lambda_{m,n}}\right)\exp\left(-\frac{u}{\Lambda_{m,n}}+\frac{1}{2}\left[-\frac{u}{\Lambda_{m,n}}\right]^2\right)\right].\\ -\sigma(-u) &= u\prod_{m,n}{}' \left[\left(1+\frac{u}{\Lambda_{m,n}}\right)\exp\left(-\frac{u}{\Lambda_{m,n}}+\frac{1}{2}\left[\frac{u}{\Lambda_{m,n}}\right]^2\right)\right].\end{aligned}$$

Once again, this infinite product will have the same terms as $\sigma(u)$ but in a different order and so the σ-function can be concluded to be odd.

■

Lemma 2.1.21. *The Weierstrass functions are connected as follows.*

$$\frac{d}{du}\log\big[\sigma(u)\big] = \zeta(u), \qquad \frac{d}{du}\zeta(u) = -\wp(u). \tag{2.20}$$

Proof. Taking logs of equation (2.18) gives

$$\log\big[\sigma(u)\big] = \log(u) + {\sum_{m,n}}' \left[\log\left(1-\frac{u}{\Lambda_{m,n}}\right) + \frac{u}{\Lambda_{m,n}} + \frac{1}{2}\left(\frac{u}{\Lambda_{m,n}}\right)^2\right].$$

Then differentiate once to show

$$\begin{aligned}\frac{d}{du}\log\big[\sigma(u)\big] &= \frac{1}{u} + {\sum_{m,n}}'\left[\left(\frac{\Lambda_{m,n}}{\Lambda_{m,n}-u}\right)\left(\frac{-1}{\Lambda_{m,n}}\right) + \frac{1}{\Lambda_{m,n}} + \frac{u}{\Lambda_{m,n}^2}\right] \\ &= \frac{1}{u} + {\sum_{m,n}}'\left[\left(\frac{1}{u-\Lambda_{m,n}}\right) + \frac{1}{\Lambda_{m,n}} + \frac{u}{\Lambda_{m,n}^2}\right] = \zeta(u),\end{aligned}$$

as required. Now differentiate again to obtain the second result.

$$\frac{d}{du}\zeta(u) = -\frac{1}{u^2} + {\sum_{m,n}}'\left[\frac{-1}{(u-\Lambda_{m,n})^2} + \frac{1}{\Lambda_{m,n}^2}\right] = -\wp(u).$$

■

Corollary 2.1.22. *From the previous lemma it is obvious that*

$$\wp(u) = -\frac{d^2}{du^2}\log\big[\sigma(u)\big]. \tag{2.21}$$

In the next section it is the σ-function that is generalised first. Then the $\wp$-functions are defined by satisfying a generalisation of equation (2.21).

Quasi-periodicity properties

Next the quasi-periodicity properties of these functions are derived from the periodicity property of $\wp(u)$ given in Lemma 2.1.12. First, integrate $\wp(u+\omega_1) = \wp(u)$ to give

$$\zeta(u+\omega_1) = \zeta(u) + \eta_1,$$

where η_1 is a constant of integration. Substitute $u = -\frac{\omega_1}{2}$ into the previous equation to find

$$\zeta\left(\frac{\omega_1}{2}\right) = \zeta\left(-\frac{\omega_1}{2}\right) + \eta_1.$$

Then use the fact that $\zeta(u)$ is odd to give

$$\eta_1 = 2\zeta\left(\frac{\omega_1}{2}\right).$$

Similarly,

$$\zeta(u+\omega_2) = \zeta(u) + \eta_2 \qquad \text{where} \quad \eta_2 = 2\zeta\left(\frac{\omega_2}{2}\right).$$

Theorem 2.1.23 (Legendre's Formula). *The formula* $\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i$ *is satisfied by these functions.*

Proof. Consider $\int_C \zeta(u)du$ where C is the boundary of the cell. There is one pole inside each cell, with residue +1. Hence $\int_C \zeta(u)du = 2\pi i$ by Cauchy's residue theorem. Now split up the integral to the four sides as discussed in Theorem A.1.1(iii) to find

$$\begin{aligned} 2\pi i &= \int_t^{t+\omega_1} \big[\zeta(u) - \zeta(u+\omega_2)\big]du - \int_{t+\omega_2}^{t} \big[\zeta(u) - \zeta(u+\omega_1)\big]du \\ &= -\eta_2 \int_t^{t+w_1} dt + \eta_1 \int_t^{t+\omega_2} dt = \eta_1\omega_2 - \eta_2\omega_1. \end{aligned}$$

■

Next the quasi-periodicity of $\sigma(u)$ is derived by integrating the property for the ζ-function.

$$\begin{aligned} &\int \zeta(u+\omega_1)du = \int \big[\zeta(u) + \eta_1\big]du \\ &\log[\sigma(u+\omega_1)] = \log[\sigma(u)] + \eta_1 u + k \quad \Longrightarrow \quad \sigma(u+\omega_1) = c\cdot e^{\eta_1 u}\sigma(u). \end{aligned}$$

Here k was the constant of integration and $c = e^k$. To find c first set $u = -\frac{\omega_1}{2}$,

$$\sigma\left(\frac{\omega_1}{2}\right) = c\cdot e^{-\eta_1\frac{\omega_1}{2}}\sigma\left(-\frac{\omega_1}{2}\right).$$

Then recall that $\sigma(u)$ is an odd function to give

$$c = -e^{\eta_1\omega_1/2}.$$

This can be repeated for ω_2 giving the following quasi-periodicity properties for $\sigma(u)$.

$$\sigma(u+\omega_i) = -\exp\left[\eta_1\left(u + \frac{\omega_i}{2}\right)\right]\sigma(u), \qquad i = 1, 2. \tag{2.22}$$

The generalised σ-function is defined later to satisfy an analogue of this property.

Series expansions

As an entire function, $\sigma(u)$ can be expressed using its Taylor expansion. The derivation of such an expansion plays a key role in the following chapters and is a powerful tool in the investigation of the generalised theory. The series expansion in the elliptic case can be derived easily using the Laurent expansion for $\wp(u)$ that was constructed earlier and given in equation (2.11).

$$\wp(u) = u^{-2} + \tfrac{1}{20}g_2u^2 + \tfrac{1}{28}g_3u^4 + O(u^6).$$

Integrating and changing signs gives an expansion for the ζ-function.

$$\zeta(u) = u^{-1} - \tfrac{1}{60}g_2u^3 - \tfrac{1}{140}g_3u^5 - O(u^7).$$

Integrate again and take exponents to give the expansion for the σ-function.

$$\begin{aligned}\sigma(u) &= \exp\big[\log(u) - \tfrac{1}{240}g_2u^4 - \tfrac{1}{840}g_3u^6 - O(u^8)\big]\\ &= u\cdot\exp\big[-\tfrac{1}{240}g_2u^4 - \tfrac{1}{840}g_3u^6 - O(u^8)\big].\end{aligned}$$

We then use the expansion for the exponential function, $e^x = 1 + x + \frac{x^2}{2} + O(x^3)$, to show

$$\sigma(u) = u - \tfrac{1}{240}g_2u^5 - \tfrac{1}{840}g_3u^7 + O(u^9). \tag{2.23}$$

In fact, both the elliptic $\wp$ and σ-functions can be given as power series with coefficients that satisfy a recursive argument. The σ-function can be given by the double sum,

$$\sigma(u) = \sum_{m,n=0}^{\infty} a_{m,n}\left(\frac{1}{2}g_2\right)^m (2g_3)^n \frac{u^{4m+6n+1}}{(4m+6n+1)!}, \tag{2.24}$$

where $a_{0,0} = 1$ and $a_{m,n} = 0$ for either m or n negative. The other values are given by the recurrence relation

$$\begin{aligned}a_{m,n} = 3(m+1)a_{m+1,n+1} &+ \frac{16}{3}(n+1)a_{m-2,n+1}\\ &- \frac{1}{3}(2m+3n-1)(4m+6n-1)a_{m-1,n}.\end{aligned}$$

Similarly, the Laurent expansion of $\wp(u)$ at $u = 0$ can be given by

$$\wp(u) = \frac{1}{u^2} + \sum_{n=1}^{\infty} b_n u^{2n}, \qquad \text{where} \quad b_1 = \frac{g_2}{20}, \quad b_2 = \frac{g_3}{28}$$

$$\text{and} \quad b_n = \frac{3}{(2n+3)(n-2)}\sum_{k=1}^{n-2} b_k b_{n-k-1}, \quad \text{for } n > 2.$$

(See for example [26] page 30.)

Building blocks of elliptic functions

The generalisation of the σ-function will play a key role in the following chapters. One of the main reasons for this is that any Abelian function can be expressed using σ-functions with the same periods. We prove this now for the elliptic case.

Theorem 2.1.24. *Any elliptic function $f(u)$ can be expressed as a quotient of σ-functions with the same periods as follows.*

$$f(u) = K \cdot \prod_{r=1}^{n} \frac{\sigma(u-a_r)}{\sigma(u-b_r)}, \tag{2.25}$$

where $a_1, \ldots, a_n$ are the set of irreducible zeros of the function, $b_1, \ldots, b_n$ are a set of poles of $f(u)$ such that all poles of $f(u)$ are congruent to one of them, and K is a constant.

Proof. This proof follows Section 20.53 of [70]. Suppose $f(u)$ is an elliptic function with periods ω_1, ω_2 as normal. By Theorem A.1.1(ii) there will be a finite set of irreducible zeros of $f(u)$, labeled here as $a_1, \ldots, a_n$. By Theorem A.1.3 there will be a set of poles $b_1, \ldots, b_n$ such that all poles of $f(u)$ are congruent to one of them. (Recall that poles and zeros are counted according to multiplicity). Further by Theorem A.1.5,

$$a_1 + \cdots + a_n = b_1 + \cdots + b_n. \tag{2.26}$$

Now, consider the function

$$g(u) = \prod_{r=1}^{n} \frac{\sigma(u-a_r)}{\sigma(u-b_r)}.$$

The function $g(u)$ will clearly have the same poles and zeros as $f(u)$. Consider the effect of increasing u by ω_1.

$$\begin{aligned} g(u+\omega_1) = \prod_{r=1}^{n} \frac{\sigma(u-a_r+\omega_1)}{\sigma(u-b_r+\omega_1)} &= \prod_{r=1}^{n} \frac{\exp[(u-a_r+\frac{\omega_1}{2})\eta_1]}{\exp[(u-b_r+\frac{\omega_1}{2})\eta_1]} \frac{\sigma(u-a_r)}{\sigma(u-b_r)} \\ &= \left[\prod_{r=1}^{n} \frac{\exp[u-a_r+\frac{\omega_1}{2}]}{\exp[u-b_r+\frac{\omega_1}{2}]}\right] \left[\prod_{r=1}^{n} \frac{\sigma(u-a_r)}{\sigma(u-b_r)}\right]. \end{aligned}$$

However

$$\begin{aligned} \prod_{r=1}^{n} \frac{\exp[u-a_r+\frac{\omega_1}{2}]}{\exp[u-b_r+\frac{\omega_1}{2}]} = \prod_{r=1}^{n} \frac{\exp(u)\exp(-a_r)\exp(\frac{\omega_1}{2})}{\exp(u)\exp(-b_r)\exp(\frac{\omega_1}{2})} &= \prod_{r=1}^{n} \frac{\exp[-a_r]}{\exp[-b_r]} \\ = \frac{\exp[-(a_1+\cdots+a_r)]}{\exp[-(b_1+\cdots+b_r)]} &= 1, \end{aligned}$$

where the final equality follows from equation (2.26). Hence $g(u+\omega_1) = g(u)$ and similarly for ω_2. Finally consider the quotient,

$$\frac{f(u)}{g(u)}.$$

This is is an elliptic function with no zeros or poles. By Theorem A.1.1(v) it is a constant, say K. Therefore the arbitrary elliptic function $f(u)$ can be expressed in the form

$$f(u) = K \cdot \prod_{r=1}^{n} \frac{\sigma(u-a_r)}{\sigma(u-b_r)},$$

as required.

■

2.1.5 The addition formulae

One of the most celebrated features of the Weierstrass $\wp$-function is that it satisfies an addition formula. That is, $\wp(u+v)$ can be expressed as a rational function of $\wp(u), \wp(v), \wp'(u)$ and $\wp'(v)$ for general u, v.

Theorem 2.1.25. *The following addition formula is true for two arbitrary complex variables u, v such that $u \neq \pm v$ mod($\omega_1 + \omega_2$).*

$$\wp(u+v) = \frac{1}{4}\left[\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}\right]^2 - \wp(u) - \wp(v). \tag{2.27}$$

If $u = \pm v$ mod($\omega_1 + \omega_2$), but $2u$ is not a period then the following duplication formula is satisfied.

$$\wp(2u) = \frac{1}{4}\left[\frac{\wp''(u)}{\wp'(u)}\right]^2 - 2\wp(u). \tag{2.28}$$

Proof. Consider the equations

$$\wp'(u) = A\wp(u) + B, \qquad \wp'(v) = A\wp(v) + B \tag{2.29}$$

and solve simultaneously to give

$$A = \frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}. \tag{2.30}$$

This is valid for $u \neq \pm v$ mod$(\omega_1 + \omega_2)$ which was specified in the theorem. Next consider the function below, defined with this same A, B.

$$f(\kappa) = \wp'(\kappa) - A\wp(\kappa) - B.$$

The function $f(\kappa)$ is clearly elliptic, with a triple pole at $\kappa = 0$. Therefore by Theorem A.1.3 the function has three zeros and by Theorem A.1.5 the sum of these will be zero, (the sum of the poles). From equations (2.29) it can easily be seen that $\kappa = u$ and $\kappa = v$ are zeros of $f(\kappa)$. Then since the sum is zero the third zero must be $\kappa = -u - v$.

Next consider the function

$$g(\kappa) = \wp'(\kappa)^2 - [A\wp(\kappa) + B]^2.$$

When κ is congruent to u, v or $-u-v$,

$$f(\kappa) = 0 \quad \Longrightarrow \quad \wp'(\kappa) = A\wp(\kappa) + B \quad \Longrightarrow \quad \wp'(\kappa)^2 = \big[A\wp(\kappa) + B\big]^2,$$

and so at these points the function $g(\kappa)$ will vanish. Expanding the bracket and using the differential equation for the $\wp$-function, (2.8), gives

$$g(\kappa) = 4\wp(\kappa)^3 - A^2\wp(\kappa)^2 - (2AB + g_2)\wp(\kappa) - (B^2 + g_3),$$

which vanishes when $\wp(\kappa)$ is equal to any one of $\wp(u), \wp(v)$ or $\wp(u+v)$. For general u, v these are unequal and so they are all roots of the general equation

$$4X^3 - A^2X^2 - (2AB + g_3)X - (B^2 + g_3) = 0.$$

Now, recall that in general the sum of the roots of a cubic equation is given by $-c_2/c_3$, where c_3 and c_2 are the coefficients of the cubic and quadratic terms respectively. Therefore

$$\wp(u) + \wp(v) + \wp(u+v) = \tfrac{1}{4}A^2.$$

Finally, substituting from equation (2.30) and rearranging give the desired addition formula.

$$\wp(u+v) = \frac{1}{4}\left[\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}\right]^2 - \wp(u) - \wp(v)$$

To derive the duplication formula take the limit when v approaches u.

$$\lim_{v\to u}\wp(u+v) = \frac{1}{4}\lim_{v\to u}\left[\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}\right]^2 - \wp(v) - \lim_{v\to u}\wp(v).$$

So long as $2u$ is not a period, this will reduce to

$$\wp(2u) = \frac{1}{4}\lim_{h\to 0}\left[\frac{\wp'(u) - \wp'(u+h)}{\wp(u) - \wp(u+h)}\right]^2 - 2\wp(u).$$

Now apply Taylor's theorem to $\wp(u+h)$ and $\wp'(u+h)$.

$$\wp(2u) = \frac{1}{4}\lim_{h\to 0}\left[\frac{-h\wp''(u) + O(h^2)}{-h\wp'(u) + O(h^2)}\right]^2 - 2\wp(u).$$

Therefore, so long as $2u$ is not a period,

$$\wp(2u) = \frac{1}{4}\left[\frac{\wp''(u)}{\wp'(u)}\right]^2 - 2\wp(u)$$

as required.

■

These addition and duplication formulae for the $\wp$-function are in fact the algebraic form of the addition law that can be defined for points on an elliptic curve. It is this property of elliptic curves which makes them so important in areas such as cryptography. For a detailed description of elliptic curves from the perspective of their use in cryptography see [67]. Additionally, [27] gives full details of how such schemes are implemented and details of similar properties for the hyperelliptic generalisation discussed in Section 2.3.1.

Now we consider the corresponding addition formula for the σ-function.

Theorem 2.1.26. *The following addition formula holds for any two complex variables u, v.*

$$-\frac{\sigma(u+v)\sigma(u-v)}{\sigma(v)^2\sigma(u^2)} = \wp(u) - \wp(v). \tag{2.31}$$

Proof. Consider the elliptic function

$$f(u,v) = \wp(u) - \wp(v).$$

Let us describe this as a function of one variable, $z = u + v$,

$$f(z) = \wp(z-v) - \wp(z-u).$$

The function $f(u,v)$ will have double poles when $u = 0$ or $v = 0$ (and at congruent points). Equivalently $f(z)$ has double poles at $z = u$ and $z = v$. Next, $f(u,v)$ will have zeros when $\wp(u) = \wp(v)$. Clearly this is when $v = u$ and also when $v = -u$, recalling that the $\wp$-function is even. Equivalently the function $f(z)$ has zeros when $z = 0$ and $z = 2v$. Therefore apply the result of Theorem 2.1.24 to give

$$f(z) = K\frac{\sigma(z-0)\sigma(z-2v)}{\sigma(z-u)^2\sigma(z-v)^2} \quad \Longrightarrow \quad f(u,v) = K\frac{\sigma(u+v)\sigma(u-v)}{\sigma(v)^2\sigma(u^2)}$$

for some constant K. Returning from z to u, v and recalling our original definition of $f(u,v)$ gives

$$\wp(u) - \wp(v) = K\frac{\sigma(u+v)\sigma(u-v)}{\sigma(v)^2\sigma(u^2)}.$$

To determine the constant K use the series expansions given in equations (2.11) and (2.23). Substituting in the expansions up to order six and expanding gives

$$0 = \big(464486400000(1+K)\big)u^4v^2 - \big(464486400000(K+1)\big)u^2v^4 \\ + \text{ higher degree terms.}$$

Hence we must have $K = -1$, and so equation (2.31) has been derived as required. ■

While these addition formulae for the $\wp$ and σ-functions are related, in the higher genus cases it will be the σ-function addition formula which can be generalised. In fact it can be generalised to whole families of addition formula depending on the symmetries present.

2.2 Abelian function theory

At the end of this section a class of multivariate functions with multiple periods will be defined. These will be an analogue of the Weierstrass $\wp$-function, but instead of using two scalar periods the function will use two matrices of periods. These will be derived from particular classes of algebraic curves, which give us a framework to classify the functions. This section first introduces the curves considered and describes how period matrices are derived from them. We then proceed to define a generalisation of the Weierstrass σ-function and uses that to define the generalised $\wp$-functions.

2.2.1 Curves, surfaces and differentials

We will consider the following set of algebraic curves, classified using the notation of [23].

Definition 2.2.1. *For two coprime integers (n, s) with $s > n$ we define a **general (n, s)-curve** as an algebraic curve,*

$$f(x, y) = 0, \qquad f(x, y) = y^n - x^s - \sum_{\alpha,\beta} \mu_{[ns-\alpha n-\beta s]} x^\alpha y^\beta. \tag{2.32}$$

In this equation x, y are two complex variables while the μ_j are a set of curve constants. The subscripts of these constants are just labels chosen to match the weight properties that are discussed in Section 3.3. The α and β are integers restricted by $\alpha \in (0, s-1), \beta \in (0, n-1)$ and $\alpha n + \beta s < ns$.

These curves have several properties which make them simpler to work with than an arbitrary algebraic curve. They are non-singular, non-degenerate (have no multiple points) and have one of their branch points at infinity in the projective plane, (denoted ∞). In the literature these curves are sometimes referred to as *canonical curves*. (See Chapter 3 of [35] for some details regarding the theory associated to a less restrictive class of algebraic curves).

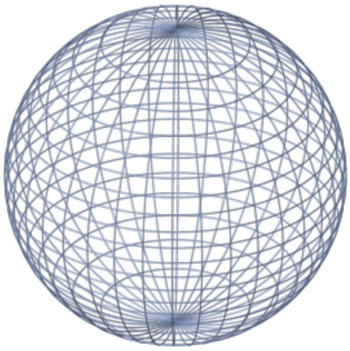
We can think of these curves as compact Riemann surfaces by introducing a local parametrisation of the curve. We describe the point (x, y) in the vicinity of the point (a, b) using the local parameter ξ at this point.

$$(x, y) = \begin{cases} \big(a + \xi, b + \xi\big) & \text{if} \quad (a, b) \text{ is a regular point.} \\ \left(\dfrac{1}{\xi^n}, \dfrac{1}{\xi^s}\right) & \text{if} \quad (a, b) = (\infty, \infty) \text{ is a branching point at } \infty. \\ \big(a + \xi^m, b + \xi\big) & \text{if} \quad (a, b) \text{ is another branching point of order } m. \end{cases}$$

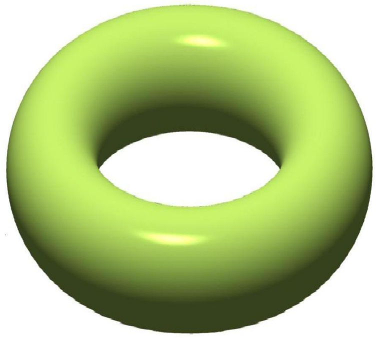
Recall that Riemann surfaces look like the complex plane locally near every point, but the global topology can be quite different. They are one dimensional complex manifolds.

The genus of such a surface is a unique integer associated to the surface which represents the maximum number of cuts along closed simple curves that can be made without rendering the resulting manifold disconnected. It is topologically invariant and may be calculated using the degree and singularity properties of the curve. It may be equivalently thought of as the number of handles of the surface. (See Figure 2.6.)

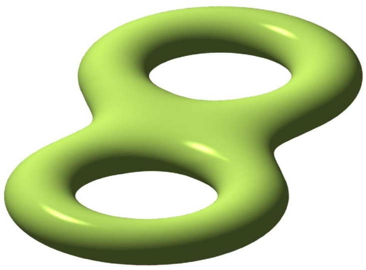
Figure 2.6: **The genus of mathematical surfaces**



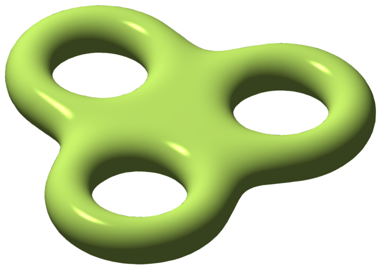
(a) A surface with genus zero



(b) A surface with genus one



(c) A surface with genus two



(d) A surface with genus three

From [35] we see that the general (n,s)-curves have genus given by the particularly simple formula,

$$g = \frac{1}{2}(n-1)(s-1). \tag{2.33}$$

Throughout this document we will use g to represent the genus of the curve we are working with. It can be thought informally as a measure of the complexity of the theory. Note that the classic elliptic curve could be labeled a $(2,3)$-curve and has genus one as predicted by equation (2.33). We now define a subset of the general (n,s) curves obtained by setting $\beta = 0$ in Definition 2.2.1.

Definition 2.2.2. *For two coprime integers (n,s) with $s > n$ we define a* ***cyclic*** ***(n,s)-curve*** *as an algebraic curve,*

$$f(x,y) = 0, \qquad f(x,y) = y^n - (x^s + \lambda_{s-1}x^{s-1} + \cdots + \lambda_1 x + \lambda_0). \tag{2.34}$$

In keeping with tradition we use λ_j for the curve constants in the cyclic case.

We use the name cyclic because these curves are invariant under

$$[\zeta]: \quad (x, y) \to (x, \zeta y), \tag{2.35}$$

where ζ is a nth root of unity. The cyclic (n, s)-curve will also have one of its branch points at ∞ and genus given by equation (2.33). The material presented in this section is applicable to both the general and cyclic curves. However, it is computationally much easier to work with the cyclic curves and in later chapters most results are limited to these cases. For brevity the label (n, s)-curves is used when it is not necessary to specify.

We denote the Riemann surface defined by an (n, s)-curve with C. The key to identifying the properties of these Riemann surfaces and defining the associated period matrices is the differentials associated to the surfaces. We will describe how these can be derived for an arbitrary (n, s)-curve using the Weierstrass gap-sequence as a tool.

The Weierstrass gap sequence

Definition 2.2.3. *Let (n, s) be a pair of coprime integers such that $s > n \geq 2$. Then the natural numbers not representable in the form*

$$an + bs \qquad \textit{where} \qquad a, b \in \mathbb{N} = \{0, 1, 2, \dots\} \tag{2.36}$$

*form a **Weierstrass gap sequence**, $W_{n,s}$. The numbers in the sequence are called **gaps**, while the numbers that are representable in the form of (2.36) are labeled the **non-gaps**.*

These sequences, first introduced by Weierstrass, are of great importance in several areas of mathematics. Appendix A.4 proves and demonstrates the core properties of the sequence. These include the property that the length of $W_{n,s}$ is $g = \frac{1}{2}(n-1)(s-1)$, the genus of the corresponding (n, s)-curve. We usually denote the sequence of non-gaps by $\overline{W}_{n,s}$, include zero as a non-gap and write both sequences in ascending order as below.

$$W_{n,s} = \{\omega_1, \omega_2, \omega_3, \dots, \omega_g\} \qquad \overline{W}_{n,s} = \{0, \overline{w}_2, \overline{w}_3, \dots\}.$$

We summarise the rest of the results proved in Appendix A.4 below.

- An element of the gap sequence $w \in W_{n,s}$ can be expressed as $w = -\alpha n + \beta s$ where $\alpha, \beta \in \mathbb{Z}, \alpha > 0, 0 < \beta < n$. The integers α, β are determined uniquely.
- The maximal element of the sequence, w_g, is equal to $2g - 1$.
- If $w \in W_{n,s}$, then $(w_g - w) \notin W_{n,s}$.
- If $w > \overline{w}$ where $w \in W_{n,s}$ and $\overline{w} \notin W_{n,s}$, then $(w - \overline{w}) \in W_{n,s}$.

- Each element in the gap sequence satisfies $i \leq w_i \leq 2i - 1$.

Some examples of Weierstrass gap-sequences are given below.

$$W_{2,3} = \{1\}, \quad W_{2,5} = \{1,3\}, \quad W_{3,4} = \{1,2,5\}, \quad W_{3,5} = \{1,2,4,7\}$$

The differentials associated with C

When working with Riemann surfaces there are traditionally three types of differential 1-forms that are considered. (See for example [58] or [16] for a more detailed treatment).

- *The differentials of the first kind*, are the differential 1-forms that are regular everywhere. We refer to them as the *holomorphic differentials*.
- *The differentials of the second kind* are meromorphic, with zero residue at each pole.
- *The differentials of the third kind* are meromorphic with the sum of the residues equal to zero.

We will first discuss the holomorphic differentials, which are the building blocks of those functions that are holomorphic on the surface. They can be represented locally at every point (x, y) on C in the form

$$du = h(\xi)d\xi,$$

where h is a holomorphic function and ξ the local parameter in the vicinity of (x, y). For an algebraic curve of genus g there exist exactly g independent holomorphic differentials, which form a basis. The following proposition, from [35], describes how we can construct the basis of holomorphic differentials for an arbitrary (n, s) curve using the gap sequence.

Proposition 2.2.4. *Let C be an (n, s)-curve of genus g and $W_{n,s}$ the Weierstrass gap sequence generated by (n, s). Then a basis for the space of holomorphic differentials upon the curve is*

$$du_i(x,y) = \frac{x^{P_i} y^{Q_i}}{f_y(x,y)} dx, \qquad i = 1, \dots, g \tag{2.37}$$

where P_i, Q_i are the integers in the decomposition of the first g non-gaps, (Lemma A.4.1).

$$nP_i + sQ_i = \overline{\omega}_i \qquad i = 1, \dots, g.$$

This basis is sometimes called the standard or canonical basis of holomorphic differentials. We denote the row vector containing the basis by $\boldsymbol{du} = (du_i, du_2, \dots, du_g)$.

Example 2.2.5. We will construct a basis of holomorphic differentials for the cyclic $(3, 4)$-curve which has genus $g = 3$. In Example A.4.3 the gap sequence was calculated as $W_{3,4} = \{1, 2, 5\}$. Then the first three non gaps can be decomposed as follows.

$$\begin{array}{lclll} \overline{w}_1 = 0 = 0\cdot 3 + 0\cdot 4 & \Longrightarrow & P_1 = 0, & Q_1 = 0 \\ \overline{w}_2 = 3 = 1\cdot 3 + 0\cdot 4 & \Longrightarrow & P_2 = 1, & Q_2 = 0 \\ \overline{w}_3 = 4 = 0\cdot 3 + 1\cdot 4 & \Longrightarrow & P_3 = 0, & Q_3 = 1 \end{array}$$

The cyclic (3,4)-curve is defined by

$$f(x,y) = 0, \quad \text{where} \quad f(x,y) = y^3 - (x^4 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0). \tag{2.38}$$

Therefore $f_y(x,y) = 3y^2$ and a basis for the space of holomorphic differentials is

$$du_1(x,y) = \frac{dx}{3y^2}, \qquad du_2(x,y) = \frac{xdx}{3y^2}, \qquad du_3(x,y) = \frac{ydx}{3y^2}.$$

□

We now define a particular second kind differential.

Definition 2.2.6. *The 2-form $\Omega\big((x,y),(z,w)\big)$ on $C \times C$ is a **fundamental differential of the second kind** if*

1. *It is symmetric:* $\Omega\big((x,y),(z,w)\big) = \Omega\big((z,w),(x,y)\big)$
2. *The only pole of second order is along the diagonal of $C \times C$ (where $x = z$)*
3. *It can be expanded in a power series as*

$$\Omega\big((x,y),(z,w)\big) = \left(\frac{1}{(\xi - \xi')^2} + O(1)\right) d\xi d\xi' \quad \textit{as } (x,y) \to (z,w)$$

   where ξ and ξ' are the local coordinates of (x,y) and (z,w).

We will construct a generalisation of Klein's explicit realisation of the fundamental differential in Proposition 2.2.8. This generalisation was developed for the (n,s)-curves in [35]. This realisation is used in the Kleinian formula discussed in Section 3.2 and also allows us to construct a basis for the differentials of the second kind. These are denoted by

$$dr_j(x,y) = \frac{h_j(x,y)}{f_y(x,y)}dx, \qquad j = 1,\dots,g \tag{2.39}$$

and they are determined modulo the space spanned by the $\boldsymbol{du}$. Denote the row vector containing this basis by $\boldsymbol{dr} = (dr_1, dr_2, \dots, dr_g)$.

Definition 2.2.7. *Define the following meromorphic function on $C \times C$, where $[\quad]_w$ means that we remove any terms which have negative powers of w.*

$$\Sigma\big((x,y),(z,w)\big) = \frac{1}{(x-z)f_y(x,y)} \cdot \sum_{k=1}^{n} y^{n-k} \left[\frac{f(z,w)}{w^{n-k+1}}\right]_w. \tag{2.40}$$

Proposition 2.2.8. *The fundamental differential of the second kind can be expressed as*

$$\Omega\big((x,y),(z,w)\big) = R\big((x,y),(z,w)\big)dxdz \tag{2.41}$$

where

$$R\big((x,y),(z,w)\big) = \frac{\partial}{\partial z}\Sigma\big((x,y),(z,w)\big) + \sum_{j=1}^{g} \frac{du_j(x,y)}{dx}\cdot\frac{dr_j(z,w)}{dz}. \tag{2.42}$$

The polynomials $h_j(x,y)$ contained within $dr_j(x,y)$ should be constructed so that Ω satisfies the symmetry condition in Definition 2.2.6.

For the proof see for example [30] page 9. This proposition will allow us to write the fundamental differential of the second kind as

$$\Omega\big((x,y),(z,w)\big) = \frac{F\big((x,y),(z,w)\big)dxdz}{(x-z)^2 f_y(x,y) f_w(x,y)}, \tag{2.43}$$

where F is a polynomial of its variables.

Example 2.2.9. Consider again the cyclic (3,4)-curve defined by equation (2.38). Since the curve is cyclic,

$$\left[\frac{f(z,w)}{w^{n-k+1}}\right]_w = \frac{w^3}{w^{3-k+1}}, \quad k=1,\ldots,3.$$

Therefore the function given in Definition 2.2.7 is

$$\Sigma\big((x,y),(z,w)\big) = \frac{1}{(x-z)\cdot 3y^2}\cdot\sum_{k=1}^{3} y^{3-k}\frac{w^3}{w^{3-k+1}} = \frac{y^2+yw+w^2}{3y^2(x-z)}.$$

Hence equation (2.42) becomes

$$R\big((x,y),(z,w)\big) = \frac{d}{dz}\left(\frac{y^2+yw+w^2}{x-z}\right)\frac{1}{3y^2} + \left(\frac{h_1(z,w)+xh_2(z,w)+yh_3(z,w)}{9y^2w^2}\right),$$

where the polynomials h_j have yet to be determined. Note that from the curve equation

$$w_z = \frac{1}{3w^2}\big(4z^3 + 4\lambda_3 z^2 + 2\lambda_2 z + \lambda_1\big),$$

and so we can evaluate the derivative in $R((x,y),(z,w))$ and substitute for w_z, (along with any higher powers of w, using the curve equation). We can find the h_j by imposing the symmetry condition on Ω and hence $R((x,y),(z,w))$.

$$R\big((x,y),(z,w)\big) = R\big((z,w),(x,y)\big).$$

We find that the condition holds for

$$h_1(x,y) = (5x^2 + 3x\lambda_3 + \lambda_2)y, \qquad h_2(x,y) = 2xy, \qquad h_3(x,y) = x^2.$$

Note that this set of polynomials is not unique. For example the λ_2 term could have been included in h_2 instead of h_1, however, this would have made no difference to the end results.

So given these values of h_j we see that a basis for the differentials of the second kind associated to the cyclic (3,4)-curve is given by

$$dr_1(x,y) = \frac{y(5x^2 + 3x\lambda_3 + \lambda_2)dx}{3y^2}dx,$$
$$dr_2(x,y) = \frac{2xydx}{3y^2}dx, \qquad dr_3(x,y) = \frac{x^2dx}{3y^2}dx.$$

Further, the fundamental differential of the second kind is given by

$$\Omega\big((x,y),(z,w)\big) = \frac{F\big((x,y),(z,w)\big)dxdz}{9(x-z)^2y^2w^2},$$

where $F((x,y),(z,w))$ is the following symmetric polynomial

$$F((x,y),(z,w)) = (3\lambda_0 + 2z^3x + 3\lambda_3z^2x + 2\lambda_1z + z^2x^2 + 2\lambda_2zx + \lambda_1x + \lambda_2z^2)y$$
$$+ 3y^2w^2 + (3\lambda_0 + 2x^3z + 3\lambda_3x^2z + 2\lambda_1x + x^2z^2 + 2\lambda_2xz + \lambda_1z + \lambda_2x^2)w$$

□

In Section 3.1.1 this calculation is performed for the cyclic (4,5)-curve with details of all the steps shown in full.

2.2.2 Abelian functions

The next step is to describe how period matrices can be generated from these curves. After that we introduce the notion of a function that is periodic with respect to these matrices — an Abelian function. This subsection ends with a discussion on the Jacobian of the curve. This is an important substructure of the curve that has algebraic group properties and can be described by the associated Abelian functions.

Let C be the surface defied by an (n,s)-curve, and $\boldsymbol{du} = (du_1,\dots,du_g)$ the vector of holomorphic differentials defined in the previous section. We will usually denote a point in $\mathbb{C}^g$ by $\boldsymbol{u}$, a row vector with coordinates $(u_1,\dots,u_g)$. Any point $\boldsymbol{u} \in \mathbb{C}^g$ may be written as

$$\boldsymbol{u} = \sum_{i=1}^{g} \int_{\infty}^{P_i} \boldsymbol{du}, \tag{2.44}$$

where the P_i are a set of g variable points upon C.

Now we must choose a basis of cycles (closed paths) upon the surface C. Denote them

$$\alpha_i, \beta_j, \qquad 1 \le i,j \le g,$$

and ensure they have intersection numbers

$$\alpha_i \cdot \alpha_j = 0, \qquad \beta_i \cdot \beta_j = 0, \qquad \alpha_i \cdot \beta_j = \delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}. \tag{2.45}$$

(Note that these cycles are the first homology basis of the Riemann surface.) We now define the periods associated to the surface as the integrals of the differentials along these paths. We combine them into the following two $g \times g$ period matrices.

$$\omega' = \left(\oint_{\alpha_k} du_\ell \right)_{k,\ell=1,\dots,g} \qquad \omega'' = \left(\oint_{\beta_k} du_\ell \right)_{k,\ell=1,\dots,g} \tag{2.46}$$

Note for future use that the matrix $(\omega')^{-1}\omega''$ is symmetric with

$$\mathrm{Im}\big((\omega')^{-1}\omega''\big) \quad \text{positive definite.} \tag{2.47}$$

(See [19] for details). These matrices play the role of the scalar periods ω_1, ω_2 in the elliptic theory. They are used below to construct the period lattice Λ, defined in analogy to Definition 2.1.8 from the elliptic case.

Definition 2.2.10. *Denote the* ***period lattice*** *formed from ω', ω'' by Λ. These are the points*

$$\Lambda = \big\{ \omega' \boldsymbol{m} + \omega'' \boldsymbol{n}, \quad \boldsymbol{m}, \boldsymbol{n} \in \mathbb{Z}^g \big\}.$$

We can now give a definition for those function that are periodic with respect to these matrices, or equivalently, invariant under translations by this period lattice.

Definition 2.2.11. *Let $\mathfrak{M}(\boldsymbol{u})$ be a meromorphic function of $\boldsymbol{u} \in \mathbb{C}^g$. Then $\mathfrak{M}$ is an* ***Abelian function associated with*** *C if*

$$\mathfrak{M}(\boldsymbol{u} + \omega' \boldsymbol{m} + \omega'' \boldsymbol{n}) = \mathfrak{M}(\boldsymbol{u}), \tag{2.48}$$

for all integer column vectors $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{Z}^g$.

The Jacobian

Recall that an *algebraic variety* is the set of solutions to a system of polynomial equations. An *Abelian variety* is an algebraic variety that is also an algebraic commutative group. (I.e. a group law for points in the variety can be defined by regular functions.) Topologically an Abelian variety is a complex g-handled torus.

Every algebraic curve C of genus $g > 1$ is associated with an Abelian variety of dimension g, called the *Jacobian* of the curve and denoted J. They are associated by means of an holomorphic map from C to J. Any point in the Jacobian can be generated by a g-tuple of points in C. As with any complex torus of dimension g, the Jacobian may be obtained as the quotient of a g-dimensional complex vector space by a lattice of rank $2g$.

Definition 2.2.12. *Let C be defined by an (n,s)-curve and Λ the lattice of its periods. Then the manifold $\mathbb{C}^g/\Lambda$ is the* ***Jacobian*** *of C, denoted by J.*

Intuitively speaking, the Jacobian is $\mathbb{C}^g$ with those points on equivalent positions in the lattice 'glued together'. Let κ denote the quotient map of modulo Λ over $\mathbb{C}$.

$$\kappa : \mathbb{C}^g \to \mathbb{C}^g/\Lambda = J.$$

Then $\Lambda = \kappa^{-1}((0,\dots,0))$. Note that the Jacobian of an elliptic curve is isomorphic to the curve itself. Hence in the elliptic case equation (2.44) simplifies to equation (2.17), with dx/y as the sole differential in the basis.

Next, the map that is used to move from the curve to the Jacobian is defined.

Definition 2.2.13. *For $k = 1,2,\dots$ define $\mathfrak{A}$, the* ***Abel map****, as below. This is a map from the kth symmetric product of the curve to the Jacobian of the curve.*

$$\begin{aligned} &\mathfrak{A} : \mathit{Sym}^k(C) \to J \\ &(P_1,\dots,P_k) \mapsto \left(\int_\infty^{P_1} \boldsymbol{du} + \dots + \int_\infty^{P_k} \boldsymbol{du} \right) \pmod{\Lambda}. \end{aligned} \tag{2.49}$$

Here the P_i are points upon the curve C.

When $k=1$ the Abel map gives an embedding, or a one dimensional image, of C.

The *Jacobi Inversion Problem* is, given a point $u \in J$ to find the preimage of this point under the Abel map (2.49). Denote the image of the kth Abel map by $W^{[k]}$. (For $k \geq g$ the image $W[k] = J$ by the Abel-Jacobi theorem.)

The next definition is for the strata of the Jacobian. These spaces decrease in dimension and are the natural location to work with lower dimensional problems (as is the case in Chapter 6). First note that we write, $[-1](u_1,\dots,u_g) = (-u_1,\dots,-u_g)$.

Definition 2.2.14. *Denote the the* ***kth strata*** *of the Jacobian by $\Theta^{[k]}$ and define it as below.*

$$\Theta^{[k]} = W^{[k]} \cup [-1]W^{[k]}. \tag{2.50}$$

*This is also referred in the literature as the k*th standard theta subset *or k*th stratum*.*

These strata are all subsets of the Jacobian and are arranged as follows.

$$J = \Theta^{[g]} \supset \Theta^{[g-1]} \supset \dots \supset \Theta^{[1]} \supset \mathbf{0} \tag{2.51}$$

where $\mathbf{0}$ is the origin, and in the literature is sometimes assigned to be $\Theta^{[0]}$. When the (n,s)-curve has n even the process is simplified and $W^{[k]} = [-1]W^{[k]} = \Theta^{[k]}$.

2.2.3 The Kleinian σ-function

In Section 2.2.4 the class of Abelian functions that generalise the Weierstrass $\wp$-function is defined. This is done using an auxiliary function which we discuss in this section.

First define two further period matrices, (in addition to those given in equation (2.46)), by integrating the basis of meromorphic differentials, $\boldsymbol{dr}$. (This basis was defined in equation (2.39) and should be explicitly determined through the construction of the fundamental differential of the second kind detailed in Proposition 2.2.8.)

$$\eta' = \left(\oint_{\alpha_k} dr_\ell \right)_{k,\ell=1,\dots,g} \qquad \eta'' = \left(\oint_{\beta_k} dr_\ell \right)_{k,\ell=1,\dots,g} \tag{2.52}$$

We combine the period matrices into

$$M = \begin{pmatrix} \omega' & \omega'' \\ \eta' & \eta'' \end{pmatrix}.$$

The matrix M satisfies

$$M \begin{pmatrix} 0_g & -I_g \\ I_g & 0_g \end{pmatrix} M^T = 2\pi \mathrm{i} \begin{pmatrix} 0_g & -I_g \\ I_g & 0_g \end{pmatrix}, \tag{2.53}$$

where I_g is the identity matrix of size g and 0_g is the zero matrix of size g. Equation (2.53) is known as the *generalised Legendre equation* and we refer the reader to page 11 of [19] for more details.

The Kleinian σ-function will be a function associated to the curve C that generalises the Weierstrass σ-function discussed in Section 2.1.4. (When C is the classical elliptic curve then the Kleinian σ-function will equal the classical function.) It will be a function of g variables where g is the genus of the curve.

$$\sigma = \sigma(\boldsymbol{u}) = \sigma(u_1, u_2, \dots, u_g).$$

The function was first introduced by Klein in his research on hyperelliptic functions, (see Section 2.3.1). This approach is developed for the general (n, s)-curves in [19], [35] and [60]. The definition was developed so that the function would generalise the classical σ-function, maintaining those properties that are necessary for the successful development of the theory of Abelian functions. These conditions were that the function be entire, satisfy a quasi-periodicity property, have a specific set of zeros and a particular series expansion.

Given these properties it was shown that the Kleinian σ-function could be expressed using multivariate θ-functions. These are a generalisation of the Jacobi θ-functions, (discussed in Appendix A.2). The essential definitions and properties that are important for this document are summarised in Appendix A.3. We refer the reader to [59], [42] and [53] for a detailed treatment of the multivariate θ-functions.

In a number of applications authors have developed theory purely in the θ-functions, see for example [61]. However, the σ-function approach pursued in this document is considered the most advantageous method for developing the theory of Abelian functions.

We now give the definition for the Kleinian σ-function associated to C using θ-functions and equivalently as an infinite sum. We then present the key properties that the function was designed to incorporate. For a more detailed study of the construction and properties of the multivariate σ-function we refer the reader to [19] and [60].

Definition 2.2.15. *Let C be an (n,s)-curve and M the matrix of periods discussed above. Then the* ***Kleinian σ-function*** *associated to C is defined using a multivariate θ-function with characteristic $[\delta]$ as*

$$\sigma(\boldsymbol{u}) = \sigma(\boldsymbol{u}; M) = c \exp\left(\tfrac{1}{2}\boldsymbol{u}\eta'(\omega')^{-1}\boldsymbol{u}^T\right) \cdot \theta[\delta]\big((\omega')^{-1}\boldsymbol{u}^T \bigm| (\omega')^{-1}\omega''\big).$$

Using Definition A.3.2 for the θ-function, we can write it as an infinite sum.

$$\sigma(\boldsymbol{u}) = c \exp\left(\tfrac{1}{2}\boldsymbol{u}\eta'(\omega')^{-1}\boldsymbol{u}^T\right) \times \sum_{\boldsymbol{m}\in\mathbb{Z}^g} \exp\Bigg[2\pi i\Big\{ \tfrac{1}{2}(\boldsymbol{m}+\boldsymbol{\delta'})^T(\omega')^{-1}\omega''(\boldsymbol{m}+\boldsymbol{\delta'}) + (\boldsymbol{m}+\boldsymbol{\delta'})^T((\omega')^{-1}\boldsymbol{u}^T+\boldsymbol{\delta''})\Big\}\Bigg]. \tag{2.54}$$

The θ-function characteristic, $[\delta]$, is the $2g$ vector of half integers

$$[\delta] = \begin{bmatrix} \boldsymbol{\delta'} \\ \boldsymbol{\delta''} \end{bmatrix}$$

that is related to the Riemann constant with base point ∞. (See Appendix A.3 for the definitions or [19] for full details). The constant c is dependent upon the curve parameters and the choice of cycles $\{\alpha_i, \beta_i\}$. It is fixed later in Remark 2.2.23.

Remark 2.2.16.

(i) When the (n,s)-curve is chosen to be the classic elliptic curve then the Kleinian σ-function coincides with the Weierstrass σ-function.

(ii) For brevity we will now just refer to the Kleinian σ-function as *the σ-function* and specify if we are referring to the classical function.

(iii) The derivatives of the σ-function with respect to the variables in $\boldsymbol{u}$ are often used and are referred to as *σ-derivatives*.

They are denoted in this document by adding subscripts as follows.

$$\sigma_{i_1,i_2,\dots,i_n}(\boldsymbol{u}) = \frac{\partial}{\partial u_{i_1}}\frac{\partial}{\partial u_{i_2}}\cdots\frac{\partial}{\partial u_{i_n}}\sigma(\boldsymbol{u}), \qquad i_1 \le \cdots \le i_n \in \{1,\dots,g\}.$$

If appropriate then the commas in the indices may be removed for simplicity.

(iv) If the vector of variables to which we are referring is obvious or not relevant then we may omit it. For example we may write σ_{ij} instead of $\sigma_{ij}(\boldsymbol{u})$.

We now summarise the most important properties of the σ-function.

Lemma 2.2.17. *The Kleinian σ-function and the σ-derivatives are all entire functions of $\boldsymbol{u}$ over $\mathbb{C}^g$.*

Proof. This is clear from equation (2.54). ∎

Next we make precise the periodicity properties of $\sigma(\boldsymbol{u})$.

Lemma 2.2.18. *Given $\boldsymbol{u} \in \mathbb{C}^g$, denote by $\boldsymbol{u'}$ and $\boldsymbol{u''}$ the unique elements in $\mathbb{R}^g$ such that*

$$\boldsymbol{u} = \omega'\boldsymbol{u'} + \omega''\boldsymbol{u''}.$$

Let $\boldsymbol{\ell}$ represent a point on the period lattice. Then

$$\boldsymbol{\ell} = \omega'\boldsymbol{\ell'} + \omega''\boldsymbol{\ell''} \in \Lambda, \qquad \boldsymbol{\ell'}, \boldsymbol{\ell''} \in \mathbb{Z}^g. \tag{2.55}$$

For $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{C}^g$ and $\boldsymbol{\ell} \in \Lambda$, define $L(\boldsymbol{u}, \boldsymbol{v})$ and $\chi(\boldsymbol{\ell})$ as follows:

$$\begin{aligned} L(\boldsymbol{u}, \boldsymbol{v}) &= \boldsymbol{u}^T\big(\eta'\boldsymbol{v'} + \eta''\boldsymbol{v''}\big), \\ \chi(\boldsymbol{\ell}) &= \exp\big[2\pi i\big\{(\boldsymbol{\ell'})^T\delta'' - (\boldsymbol{\ell''})^T\delta' + \tfrac{1}{2}(\boldsymbol{\ell'})^T\boldsymbol{\ell''}\big\}\big]. \end{aligned}$$

Then for all $\boldsymbol{u} \in \mathbb{C}^g, \boldsymbol{\ell} \in \Lambda$ the function $\sigma(\boldsymbol{u})$ has the following quasi-periodicity property.

$$\sigma(\boldsymbol{u} + \boldsymbol{\ell}) = \chi(\boldsymbol{\ell})\exp\left[L\left(\boldsymbol{u} + \frac{\boldsymbol{\ell}}{2}, \boldsymbol{\ell}\right)\right]\sigma(\boldsymbol{u}). \tag{2.56}$$

(Note that if $\sigma(\boldsymbol{u}) = 0$ then clearly $\sigma(\boldsymbol{u} + \boldsymbol{\ell}) = 0$ also. Therefore, while the σ-function itself is not Λ-periodic, the zero set of the function is.)

Additionally, the σ-function is modular invariant:

$$\sigma(\boldsymbol{u}; \Upsilon M) = \sigma(\boldsymbol{u}; M), \qquad \textit{for } \Upsilon \in Sp(2g, \mathbb{Z}). \tag{2.57}$$

($Sp(2g, \mathbb{Z})$ is the group of size $2g$ symplectic matrices with elements in $\mathbb{Z}$).

Proof. The quasi-periodicity property (2.56) is a classical result, first discussed in [7], that was fundamental to the original definition of the multivariate σ-function. It is a generalisation of equation (2.22) in the classical case and follows from the properties of the multivariate θ-function.

Although both c and the θ-function are not independent of the basis of cycles, the σ-function is, so long as the restriction (2.45) is satisfied. Such a change is equivalent to multiplying M by another symplectic matrix.

See [19] or [60] for the construction of the σ-function to satisfy these properties.

■

Recall that the elliptic σ-function has zeros on each of the lattice points. This is not the case in general, however, we are still able to specify where the σ-function has its zeros. (The extra simplicity in the elliptic case is due to the fact that an elliptic curve is isomorphic to its Jacobian.)

Lemma 2.2.19. *The function $\sigma(\boldsymbol{u})$ has zeros of order one when $\boldsymbol{u} \in \Theta^{[g-1]}$. Further, for all other $\boldsymbol{u}$ we have $\sigma(\boldsymbol{u}) \neq 0$.*

Proof. This is a classical result first discussed in [7], generalised in [19] and formalised with this notation in [14]. The σ-function was defined using a Riemann θ-function with base point ∞. By Corollary A.3.7 the θ-function, and hence the σ-function, would vanish on the theta divisor, which when generated by this base point coincides with our definition of $\Theta^{[g-1]}$.

■

Recall that the Weierstrass σ-function was an odd function of the variable u. In general the Kleinian σ-function has definite parity with respect to the change of variables, $\boldsymbol{u} \mapsto [-1]\boldsymbol{u}$ and is odd or even depending on the choice of (n, s).

Lemma 2.2.20. *The σ-function associated with an (n, s)-curve has definite parity which may be predicted by*

$$\sigma(-\boldsymbol{u}) = (-1)^{\frac{1}{24}(n^2-1)(s^2-1)}\sigma(\boldsymbol{u}). \tag{2.58}$$

Proof. This is Proposition 4(iv) in [60].

■

Consider the Taylor series expansion of $\sigma(\boldsymbol{u})$ about the origin, $\mathbf{0} = (0, \ldots, 0)$. This will be multivariate in the variables $u_1, \ldots, u_g$ and will also depend on the parameters of the underlying (n, s)-curve. These were denoted by μs or λs depending on the type of curve, (Definitions 2.2.1 and 2.2.2). The final property of the σ-function that is presented in this section allows the part of the expansion that does not depend on the curve parameters to be specified up to a multiplicative constant.

Theorem 2.2.21. *The Taylor series expansion of $\sigma(\boldsymbol{u})$ about the origin may be written as*

$$\sigma(\boldsymbol{u}) = K \cdot SW_{n,s}(\boldsymbol{u}) + \textit{ terms dependent on curve parameters}, \tag{2.59}$$

where K is a constant and $SW_{n,s}$ the Schur-Weierstrass polynomial generated by (n, s).

Proof. The result was first stated in [20], with an alternative proof now available in [60]. ■

The Schur-Weierstrass polynomial is a finite polynomial in the variables $\boldsymbol{u}$ that is not dependent on the curve parameters. Appendix A.5 defines the Schur-Weierstrass polynomial and give details on how it is constructed from the integers (n, s). Some examples are given below for specific values of (n, s).

$$\begin{aligned}
W_{2,3} &= u_1\\
W_{2,5} &= \tfrac{1}{2}(-u_2^3 + 3u_1)\\
W_{3,4} &= \tfrac{1}{20}(u_3^5 + 20u_1 - 20u_2^2u_3)\\
W_{3,5} &= \tfrac{1}{448}(u_4^8 + 448u_3u_4^2u_2 - 56u_3^2u_4^4 - 112u_3^4 + 448u_2^2 - 448u_4u_1)
\end{aligned}$$

Remark 2.2.22.

(i) In the elliptic case we have $(n, s) = (2, 3)$ and so the σ-function expansion should contain only one term, u_1 multiplied by a constant that does not depend on the curve parameters. Check equations (2.23) and (2.24) to see that this was indeed the case. (In the elliptic case the curve parameters were the elliptic invariants g_2, g_3.)

(ii) The polynomial $SW_{n,s}$ will have, with respect to the variable u_g, a leading term given by a constant multiplied by
$$u_g^{\frac{1}{24}(n^2-1)(s^2-1)}.$$
(Note that this is the case in the four examples above). This follows from Proposition 4 of [60] and matches the parity property of the σ-function given in Lemma 2.2.20.

This section concludes by fixing the constant c in the definition of the Kleinian σ-function.

Remark 2.2.23. We will fix the constant c in Definition 2.2.15 to be the value that makes $K = 1$ in Theorem 2.2.21. Some other authors working in this area may instead set

$$c = \left(\frac{\pi^g}{\det(w')}\right)^{\frac{1}{2}} \cdot \frac{1}{D^{\frac{1}{2n}}},$$

where D is the discriminant of the curve C. In general these two choices of c are not equivalent. However the constant c will cancel in the definitions of all the Abelian functions used in this paper, (see Appendix B). Hence almost all results derived in later chapters of this document are independent of c. (The only exceptions are the two-term addition formula in Sections 3.6 and 4.1 which we discuss when they occur.)

Note that this choice of c ensures that the Kleinian σ-function matches the Weierstrass σ-function when the (n, s)-curve is chosen to be the classic elliptic curve.

2.2.4 The Kleinian $\wp$-functions

We are now in a position to define the generalisation of the $\wp$-function. We define it using the Kleinian σ-function by taking logarithmic derivatives, in analogy to equation (2.21) in the elliptic case. Since in general there is more that one variable, there will be a number of possible $\wp$-functions we can define depending on which variable is chosen for the differentiation. We introduce some new notation to overcome this ambiguity.

Definition 2.2.24. *Define the **2-index Kleinian $\wp$-functions** as*

$$\wp_{ij}(\boldsymbol{u}) = -\frac{\partial^2}{\partial u_i \partial u_j} \log\big[\sigma(\boldsymbol{u})\big], \qquad i \leq j \in \{1, \dots, g\}. \tag{2.60}$$

Lemma 2.2.25. *The 2-index $\wp$-functions are meromorphic functions with poles of order two when $\sigma(\boldsymbol{u}) = 0$.*

Proof. Expand equation (2.60) to give

$$\wp_{ij}(\boldsymbol{u}) = \frac{\sigma_i(\boldsymbol{u})\sigma_j(\boldsymbol{u}) - \sigma(\boldsymbol{u})\sigma_{ij}(\boldsymbol{u})}{\sigma(\boldsymbol{u})^2}. \tag{2.61}$$

Recall that the σ-function is entire and so $\wp_{ij}(\boldsymbol{u})$ will be meromorphic, with poles of order two only when $\sigma(\boldsymbol{u}) = 0$.

■

Lemma 2.2.26. *The 2-index Kleinian $\wp$-functions are Abelian functions.*

Proof. We need to prove that $\wp_{ij}(\boldsymbol{u})$ satisfies the periodicity property (2.48). This is equivalent to proving

$$\wp_{ij}(\boldsymbol{u} + \boldsymbol{\ell}) = \wp_{ij}(\boldsymbol{u}), \tag{2.62}$$

where $\boldsymbol{\ell}$ is an arbitrary point on the period lattice as defined in equation (2.55). We can prove this using the quasi-periodicity property of the σ-function, given in equation (2.56).

First recall the function $L(\boldsymbol{u}, \boldsymbol{v})$ defined in Lemma 2.2.18. We will examine the function $L(\boldsymbol{u} + \frac{\boldsymbol{\ell}}{2}, \boldsymbol{\ell})$ which we label $\hat{L}(\boldsymbol{u})$ for simplicity.

$$\hat{L}(\boldsymbol{u}) = L\left(\boldsymbol{u} + \frac{\boldsymbol{\ell}}{2}, \boldsymbol{\ell}\right) = \left(\boldsymbol{u} + \frac{\boldsymbol{\ell}}{2}\right)^T \big(\eta' \boldsymbol{\ell'} + \eta'' \boldsymbol{\ell''}\big).$$

Using the subscripts to indicate the components of the vectors and matrices we see

$$\begin{aligned}
\hat{L}(\boldsymbol{u}) &= \begin{pmatrix} u_1 + \frac{\ell_1}{2} \\ \vdots \\ u_g + \frac{\ell_g}{2} \end{pmatrix}^T \begin{pmatrix} \eta'_{11}\ell'_1 + \eta'_{12}\ell'_2 + \dots + \eta'_{1g}\ell'_g + \eta''_{11}\ell''_1 + \eta''_{12}\ell''_2 + \dots + \eta''_{1g}\ell''_g \\ \vdots \\ \eta'_{g1}\ell'_1 + \eta'_{g2}\ell'_2 + \dots + \eta'_{gg}\ell'_g + \eta''_{g1}\ell''_1 + \eta''_{g2}\ell''_2 + \dots + \eta''_{gg}\ell''_g \end{pmatrix} \\
&= \big(u_1 + \tfrac{\ell_1}{2}\big)\big(\eta'_{11}\ell'_1 + \eta'_{12}\ell'_2 + \dots + \eta'_{1g}\ell'_g + \eta''_{11}\ell''_1 + \eta''_{12}\ell''_2 + \dots + \eta''_{1g}\ell''_g\big) + \dots \\
&\quad \dots + \big(u_g + \tfrac{\ell_g}{2}\big)\big(\eta'_{g1}\ell'_1 + \eta'_{g2}\ell'_2 + \dots + \eta'_{gg}\ell'_g + \eta''_{g1}\ell''_1 + \eta''_{g2}\ell''_2 + \dots + \eta''_{gg}\ell''_g\big).
\end{aligned}$$

Therefore, the derivatives of $\hat{L}(\boldsymbol{u})$ are

$$\begin{aligned}
\hat{L}_i(\boldsymbol{u}) &= \frac{\partial}{\partial u_i}\hat{L}(\boldsymbol{u}) = \eta'_{i1}\ell'_1 + \eta'_{i2}\ell'_2 + \cdots + \eta'_{ig}\ell'_g + \eta''_{i1}\ell''_1 + \eta''_{i2}\ell''_2 + \cdots + \eta''_{ig}\ell''_g, \\
\hat{L}_{ij}(\boldsymbol{u}) &= \frac{\partial}{\partial u_i \partial u_j}\hat{L}(\boldsymbol{u}) = 0.
\end{aligned} \tag{2.63}$$

We can now examine the quasi-periodicity of the first and second σ-derivatives, noting that the latter simplifies due to equation (2.63).

$$\begin{aligned}
\sigma(\boldsymbol{u}+\boldsymbol{\ell}) &= \chi(\boldsymbol{\ell}) \exp\left[\hat{L}(\boldsymbol{u})\right] \cdot \sigma(\boldsymbol{u}) \\
\sigma_i(\boldsymbol{u}+\boldsymbol{\ell}) &= \chi(\boldsymbol{\ell}) \exp\left[\hat{L}(\boldsymbol{u})\right] \cdot \left[\hat{L}_i(\boldsymbol{u})\sigma(\boldsymbol{u}) + \sigma_i(\boldsymbol{u})\right] \\
\sigma_{ij}(\boldsymbol{u}+\boldsymbol{\ell}) &= \chi(\boldsymbol{\ell}) \exp\left[\hat{L}(\boldsymbol{u})\right] \cdot \left[\hat{L}_i(\boldsymbol{u})\hat{L}_j(\boldsymbol{u})\sigma(\boldsymbol{u}) + \hat{L}_j(\boldsymbol{u})\sigma_i(\boldsymbol{u}) \right. \\
&\qquad\qquad \left. + \hat{L}_i(\boldsymbol{u})\sigma_j(\boldsymbol{u}) + \sigma_{ij}(\boldsymbol{u})\right]
\end{aligned} \tag{2.64}$$

Now substitute $\boldsymbol{u}$ for $\boldsymbol{u}+\boldsymbol{\ell}$ in equation (2.61) and use equations (2.64).

$$\begin{aligned}
\wp_{ij}(\boldsymbol{u}+\boldsymbol{\ell}) &= \frac{\sigma_i(\boldsymbol{u}+\boldsymbol{\ell})\sigma_j(\boldsymbol{u}+\boldsymbol{\ell}) - \sigma(\boldsymbol{u}+\boldsymbol{\ell})\sigma_{ij}(\boldsymbol{u}+\boldsymbol{\ell})}{\sigma(\boldsymbol{u}+\boldsymbol{\ell})^2} \\
&= \frac{\chi(\boldsymbol{\ell})^2 \exp\left[\hat{L}(\boldsymbol{u})\right]^2}{\chi(\boldsymbol{\ell})^2 \exp\left[\hat{L}(\boldsymbol{u})\right]^2} \cdot \left(\frac{\left[\hat{L}_i(\boldsymbol{u})\sigma(\boldsymbol{u}) + \sigma_i(\boldsymbol{u})\right] \cdot \left[\hat{L}_j(\boldsymbol{u})\sigma(\boldsymbol{u}) + \sigma_j(\boldsymbol{u})\right]}{\sigma(\boldsymbol{u})^2} \right. \\
&\quad \left. - \frac{\sigma(\boldsymbol{u}) \cdot \left[\hat{L}_i(\boldsymbol{u})\hat{L}_j(\boldsymbol{u})\sigma(\boldsymbol{u}) + \hat{L}_j(\boldsymbol{u})\sigma_i(\boldsymbol{u}) + \hat{L}_i(\boldsymbol{u})\sigma_j(\boldsymbol{u}) + \sigma_{ij}(\boldsymbol{u})\right]}{\sigma(\boldsymbol{u})^2} \right) \\
&= \frac{\sigma_i(\boldsymbol{u})\sigma_j(\boldsymbol{u}) - \sigma(\boldsymbol{u})\sigma_{ij}(\boldsymbol{u})}{\sigma(\boldsymbol{u})^2} = \wp_{ij}(\boldsymbol{u}).
\end{aligned}$$

So $\wp_{ij}(\boldsymbol{u})$ satisfies the periodicity condition and by Lemma 2.2.25 the function is meromorphic. Hence it is an Abelian function.

■

We extend this new notation to define the derivatives of these functions.

Definition 2.2.27. *For $n \geq 2$ define **n-index Kleinian $\wp$-functions** as*

$$\wp_{i_1,i_2,\ldots,i_n}(\boldsymbol{u}) = -\frac{\partial}{\partial u_{i_1}}\frac{\partial}{\partial u_{i_2}} \cdots \frac{\partial}{\partial u_{i_n}} \log\left[\sigma(\boldsymbol{u})\right], \quad i_1 \leq \cdots \leq i_n \in \{1, \ldots, g\}.$$

Note that the n-index $\wp$ functions are all independent of the constant c used in the definition of the σ-function. (See Appendix B for the proof.)

We now use similar techniques to show that these functions are also Abelian.

Lemma 2.2.28. *The n-index $\wp$-functions are meromorphic functions with poles of order n when $\sigma(\boldsymbol{u}) = 0$.*

Proof. Use proof by induction. The statement has already been proved for the case $n = 2$ in Lemma 2.2.25. Suppose it is true for $n = p$. Then we could write

$$\wp_{i_1,i_2,\dots,i_p}(\boldsymbol{u}) = \frac{f(\boldsymbol{u})}{\sigma(\boldsymbol{u})^p}$$

where $f(p)$ is an entire function. Differentiating gives

$$\wp_{i_1,i_2,\dots,i_p,i_{p+1}}(\boldsymbol{u}) = \frac{\sigma(\boldsymbol{u})^p\left[\frac{\partial}{\partial u_{i_{p+1}}}f(\boldsymbol{u})\right] + f(\boldsymbol{u})\left[p\sigma(\boldsymbol{u})^{p-1}\sigma_{i_{p+1}}(\boldsymbol{u})\right]}{\sigma(\boldsymbol{u})^{2p}}$$
$$= \frac{\sigma(\boldsymbol{u})\left[\frac{\partial}{\partial u_{i_{p+1}}}f(\boldsymbol{u})\right] + pf(\boldsymbol{u})\sigma_{i_{p+1}}(\boldsymbol{u})}{\sigma(\boldsymbol{u})^{2p-(p-1)}} = \frac{\sigma(\boldsymbol{u})\left[\frac{\partial}{\partial u_{i_{p+1}}}f(\boldsymbol{u})\right] + pf(\boldsymbol{u})\sigma_{i_{p+1}}(\boldsymbol{u})}{\sigma(\boldsymbol{u})^{p+1}}.$$

Since both $f(\boldsymbol{u})$ and $\sigma(\boldsymbol{u})$ are entire functions we know the numerator is entire. Hence the function has poles of order $p + 1$ when $\sigma(\boldsymbol{u}) = 0$. Therefore, if the statement is true for $n = p$ then it is true for $n = p + 1$ as required.

■

Lemma 2.2.29. *The n-index Kleinian $\wp$-functions are all Abelian functions.*

Proof. Given that equation (2.62) is true we can just differentiate to conclude that all the n-index $\wp$-functions satisfy the periodicity condition. Coupled with Lemma 2.2.28 this shows that the n-index Kleinian $\wp$-functions are all Abelian.

■

We now give some remarks on how these functions are related to the Weierstrass $\wp$-function.

Remark 2.2.30. Definition 2.2.24 was given as an analogy to equation (2.21) in the elliptic case. It was remarked earlier that when the (n, s)-curve is chosen to be the classic elliptic curve then the Kleinian σ-function coincides with the classic σ-function. In this case, since $g = 1$, there is only one 2-index Kleinian $\wp$-function and it coincides with the Weierstrass $\wp$-function. The only difference would be the notation with

$$\wp_{11}(\boldsymbol{u}) \equiv \wp(u), \quad \wp_{111}(\boldsymbol{u}) \equiv \wp'(u), \quad \wp_{1111}(\boldsymbol{u}) \equiv \wp''(u).$$

So to emphasise, 2-index $\wp$-functions play the role of the classic $\wp$-function, a 3-index the role of the first derivative $\wp'$ and a 4-index the role of the second derivative $\wp''$.

Remark 2.2.31.

(i) Both the Weierstrass $\wp$-function and the Kleinian $\wp$-functions have poles of order two when the σ-function is zero. In the elliptic case this occurred on the lattice points, but in general it occurs for $\boldsymbol{u} \in \Theta^{[g-1]}$.

(ii) In the elliptic case the functions $\wp'$ and $\wp''$ had poles of order three and four respectively, occurring again at the lattice points. Similarly the n-index Kleinian $\wp$-functions have poles of order n when $\boldsymbol{u} \in \Theta^{[g-1]}$.

Recall that the Weierstrass $\wp$-function is an even function of its variable while the derivative $\wp'$ is odd. A similar property is present for the $\wp$-functions associated to a general (n,s)-curve.

Lemma 2.2.32. *The n-index $\wp$-functions have definite parity with respect to the change of variables $\boldsymbol{u} \to [-1]\boldsymbol{u}$. The functions are odd if n is odd and even if n is even.*

Proof. Recall Lemma 2.2.20 which stated that the σ-function had definite parity. Suppose first that the σ-function is even. Then the odd index σ-derivatives will be odd functions while the even index σ-derivatives are even functions.

$$\sigma([-1]\boldsymbol{u}) = \sigma(\boldsymbol{u}) \quad \Longrightarrow \quad \begin{aligned} -\sigma_i([-1]\boldsymbol{u}) &= \sigma_i(\boldsymbol{u}) \\ \sigma_{ij}([-1]\boldsymbol{u}) &= \sigma_{ij}(\boldsymbol{u}) \end{aligned}$$

Then consider the effect of the change of variables on $\wp_{ij}(\boldsymbol{u})$ as given by equation (2.61).

$$\begin{aligned} \wp_{ij}([-1]\boldsymbol{u}) &= \frac{\sigma_i([-1]\boldsymbol{u})\sigma_j([-1]\boldsymbol{u}) - \sigma([-1]\boldsymbol{u})\sigma_{ij}([-1]\boldsymbol{u})}{\sigma([-1]\boldsymbol{u})^2} \\ &= \frac{(-1)^2\sigma_i(\boldsymbol{u})\sigma_j(\boldsymbol{u}) - \sigma(\boldsymbol{u})\sigma_{ij}(\boldsymbol{u})}{\sigma(\boldsymbol{u})^2} = \wp_{ij}(\boldsymbol{u}). \end{aligned}$$

Hence the 2-index $\wp$-functions would be even. Now suppose that the σ-function is odd. Then the odd index σ-derivatives will be even functions while the even index σ-derivatives are odd functions.

$$\sigma([-1]\boldsymbol{u}) = -\sigma(\boldsymbol{u}) \quad \Longrightarrow \quad \begin{aligned} \sigma_i([-1]\boldsymbol{u}) &= \sigma_i(\boldsymbol{u}) \\ -\sigma_{ij}([-1]\boldsymbol{u}) &= \sigma_{ij}(\boldsymbol{u}) \end{aligned}$$

Consider again the effect of the change of variables on $\wp_{ij}(\boldsymbol{u})$.

$$\begin{aligned} \wp_{ij}([-1]\boldsymbol{u}) &= \frac{\sigma_i([-1]\boldsymbol{u})\sigma_j([-1]\boldsymbol{u}) - \sigma([-1]\boldsymbol{u})\sigma_{ij}([-1]\boldsymbol{u})}{\sigma([-1]\boldsymbol{u})^2} \\ &= \frac{\sigma_i(\boldsymbol{u})\sigma_j(\boldsymbol{u}) - (-1)^2\sigma(\boldsymbol{u})\sigma_{ij}(\boldsymbol{u})}{[(-1)\sigma(\boldsymbol{u})]^2} = \wp_{ij}(\boldsymbol{u}). \end{aligned}$$

Therefore, regardless of the parity of the σ-function, the 2-index $\wp$-functions are even. It clearly follows from their definition that the n-index $\wp$-functions must then alternate between having odd and even parity.

■

We now give some remarks on the notation used in this document.

Remark 2.2.33.

(i) From now on we may just refer to *the $\wp$-functions*, specifying if we are referring to the Weierstrass function or to $\wp$-functions of a particular index.

(ii) The $\wp$-function indices are usually single digits such as $\wp_{ij}$. Occasionally commas are used for clarity, as in $\wp_{i_1,\dots,i_n}$.

(iii) The order of the indices is irrelevant. For simplicity we always use ascending numerical order, as indicated in the definitions.

(iv) If the vector of variables to which we are referring is obvious or not relevant then we may omit it. For example we may write $\wp_{ij}$ instead of $\wp_{ij}(\boldsymbol{u})$.

Note that as the genus of the curve increases so do the number of associated $\wp$-functions. It was noted above that when $g = 1$ there is only one $\wp$-function of each index, coinciding with the Weierstrass $\wp$-function and its derivatives. However, when $g = 3$ for example, there will be six 2-index $\wp$-functions, ten 3-index $\wp$-functions and fifteen 4-index $\wp$-functions.

$$\begin{aligned}
\{\wp_{ij}\} &= \{\wp_{11}, \wp_{12}, \wp_{13}, \wp_{22}, \wp_{23}, \wp_{33}\} \\
\{\wp_{ijk}\} &= \{\wp_{111}, \wp_{112}, \wp_{113}, \wp_{122}, \wp_{123}, \wp_{133}, \wp_{222}, \wp_{223}, \wp_{233}, \wp_{333}\} \\
\{\wp_{ijkl}\} &= \{\wp_{1111}, \wp_{1112}, \wp_{1113}, \wp_{1122}, \wp_{1123}, \wp_{1133}, \wp_{1222}, \wp_{1223}, \wp_{1233}, \wp_{1333}, \\
&\qquad \wp_{2222}, \wp_{2223}, \wp_{2233}, \wp_{2333}, \wp_{3333}\}
\end{aligned}$$

In general, the number of r-index $\wp$-functions associated with a genus g curve is

$$\frac{(g+r-1)!}{r!(g-1)!}.$$

(Since the problem is choosing r indices from g possible indices with repetition allowed and order not important.)

2.3 Literature review

The definitions of the previous section were given for an arbitrary (n, s)-curve of genus g, generating infinite classes of Abelian functions with increasing genus. Such a definition was not developed as an immediate generalisation of the Weierstrass elliptic function theory. In this section we discuss in chronological order the cases that have been considered. We present some of the more important results which motivate the work in the coming chapters.

The first generalisation of elliptic functions were labeled *hyperelliptic functions* and had two variables and four periods. They could be defined as functions on the Jacobian of a hyperelliptic curve.

Definition 2.3.1. *A* ***hyperelliptic curve*** *is an algebraic curve given by an equation of the form*

$$y^2 = f(x),$$

where f(x) is a polynomial of degree $n > 4$ *with* n *distinct roots.*

From Definition 2.2.2 all the the cyclic $(2, s)$-curve are hyperelliptic curves.

There were several alternative approaches taken to developing the theory of hyperelliptic functions, before Klein developed the definitions that were generalised in Section 2.2. These originated from the German language paper, [51]. The following quote is taken from the introduction of [19], with the citations replaced with the numbers used in the Bibliography of this document.

"The paper of Klein [51], was an alternative to the developments of Weierstrass [68], [69] (the hyperelliptic generalisation of the Jacobi elliptic functions sn, cn, dn) and the purely θ-functional theory of Göppel [45] and Rosenhain [64] for genus 2, generalised further by Riemann. The σ-approach was contributed by Burkhardt [24], Wiltheiss [71], Bolza [18], Baker [8] and others."

It is the work of Baker in particular that has motivated the recent developments in Abelian function theory.

2.3.1 The genus two generalisation

In 1897 Baker published a classic monograph, [7], detailing Klein's theory. Later in 1907 he published a second monograph, [10] which derives in detail theory related to the genus two hyperelliptic curve. Both books have now been reprinted with the publication details available in the Bibliography.

The simplest hyperelliptic curve is, in our notation, the cyclic (2,5)-curve which has genus $g = 2$.

$$C: \qquad y^2 = x^5 + \lambda_4 x^4 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0$$

A basis for the space of holomorphic differentials upon C may be calculated as $\boldsymbol{du} = (du_1, du_2)$ where

$$du_1 = \frac{dx}{y}, \qquad du_2 = \frac{xdx}{y}.$$

The associated meromorphic differentials of the second kind are calculated as $\boldsymbol{dr} = (dr_1, dr_2)$ where

$$dr_1 = \frac{\lambda_3 x + 2\lambda_4 x^2 + 12x^3}{4y} dx, \qquad dr_2 = \frac{x^2}{y} dx.$$

We can then define the period matrices and the hyperelliptic σ-function as in the general case.

We define the hyperelliptic $\wp$-functions as logarithmic derivatives of the hyperelliptic σ-function. Since the genus is two there will be three generalisation of the Weierstrass $\wp$-function, $\{\wp_{11}, \wp_{12}, \wp_{22}\}$. Similarly

$$\begin{aligned}
\{\wp_{ijk}\} &= \{\wp_{111}, \wp_{112}, \wp_{122}, \wp_{222}\}, \\
\{\wp_{ijk}\} &= \{\wp_{1111}, \wp_{1112}, \wp_{1122}, \wp_{1222}, \wp_{2222}\}.
\end{aligned}$$

In [10] Baker derived many results that generalised the Weierstrass theory. We detail three examples below which motivate the new results in this document. These results are even more impressive given that matrix theory was still in its infancy and all calculations were performed by hand.

- In the elliptic case the $\wp$-function satisfied the following differential equation below, expressing the square of the first derivative of the $\wp$-function in terms of the $\wp$-function itself. (See Theorem 2.1.14.)

  $$[\wp'(u)]^2 = 4\wp(u)^3 - g_2\wp(u) - g_3.$$

  We may interpret $[\wp'(\boldsymbol{u})]^2$ as a pair of 3-index $\wp$-functions multiplied together. In the genus two case there are 10 such combinations.

  $$\begin{aligned}
  \{\wp_{ijk}\wp_{lmn}\} = \{\wp_{111}^2, \quad & \wp_{111}\wp_{112}, \quad \wp_{111}\wp_{122}, \quad \wp_{111}\wp_{222}, \\
  & \wp_{112}^2, \quad \wp_{112}\wp_{122}, \quad \wp_{112}\wp_{222}, \quad \wp_{122}^2, \quad \wp_{122}\wp_{222}, \quad \wp_{222}^2\}.
  \end{aligned}$$

  Baker derived expressions for these 10 products as polynomials in the

2-index $\wp$-functions of total degree no greater than three.

$$\begin{aligned}
\wp_{222}^2 &= 4\wp_{22}^3 + 4\wp_{12}\wp_{22} + 4\wp_{11} + \lambda_4\wp_{22}^2 + \lambda_2 \\
\wp_{122}\wp_{222} &= 4\wp_{22}^2\wp_{12} + \lambda_4\wp_{22}\wp_{12} + 2\wp_{12}^2 - 2\wp_{11}\wp_{22} + \tfrac{1}{2}\lambda_3\wp_{22} + \tfrac{1}{2}\lambda_1 \\
&\vdots
\end{aligned} \tag{2.65}$$

- Consider next the second differential equation of the elliptic case, expressing the second derivative $\wp''(u)$ using the $\wp$-function itself. (See Corollary 2.1.15.)

$$\wp''(u) = 6\wp(u)^2 - \tfrac{1}{2}g_2.$$

Baker derived expressions for the five $\wp_{ijkl}$ as polynomials of 2-index $\wp$-functions of total degree no greater than two.

$$\begin{aligned}
\wp_{2222} &= 6\wp_{22}^2 + \tfrac{1}{2}\lambda_3 + \lambda_4\wp_{22} + 4\wp_{12} \\
\wp_{1222} &= 6\wp_{22}\wp_{12} + \lambda_4\wp_{12} - 2\wp_{11} \\
&\vdots
\end{aligned} \tag{2.66}$$

- Finally, consider the addition formula for the Weierstrass σ-function derived in Theorem 2.1.26.

$$-\frac{\sigma(u+v)\sigma(u-v)}{\sigma(v)^2\sigma(u^2)} = \wp(u) - \wp(v).$$

The following hyperelliptic σ-function addition formula has the same ratio of σ-functions on the left hand side, but an extra cross-product on the right.

$$-\frac{\sigma(\mathbf{u}+\mathbf{v})\sigma(\mathbf{u}-\mathbf{v})}{\sigma(\mathbf{u})^2\sigma(\mathbf{v})^2} = \wp_{11}(\mathbf{u}) + \wp_{21}(\mathbf{u})\wp_{22}(\mathbf{v}) - \wp_{22}(\mathbf{u})\wp_{21}(\mathbf{v}) - \wp_{11}(\mathbf{v}). \tag{2.67}$$

2.3.2 Developing the general definitions

In [9] Baker developed results for a genus three hyperelliptic curve, but after the research on hyperelliptic functions in the late 1800s there was little progress on a further generalisation for almost a century. The classical elliptic functions have been of great importance in mathematics since their original definition. However, over that last three decades their relevance in physics and applied mathematics has been greatly developed. This inspired renewed interest in the theory of Abelian functions and led to the general definitions introduced in [19] and [35].

In 1997 Buchstaber, Enolskii and Leykin published a theory, [19], for the functions associated to hyperelliptic curves of arbitrary genus, (the $(2,s)$-curves). They develop Klein's construction of the fundamental differential and specify the σ-function, using results on the

Riemann θ-function.

A key result is the solution of the Jacobi Inversion Problem for an arbitrary hyperelliptic curve of genus g, (Chapter 3 of [19]). The authors prove that the Abel preimage of $\boldsymbol{u} \in J$ is given by $\{(x_1, y_1), \dots (x_g, y_g)\}$ with $x_1, \dots, x_g$ the zeros of $\mathcal{P}(x, \boldsymbol{u}) = 0$ where

$$\mathcal{P}(x, \boldsymbol{u}) = x^g - x^{g-1}\wp_{g,g}(\boldsymbol{u}) - x^{g-2}\wp_{g,g-1}(\boldsymbol{u}) - \dots - x\wp_{g,2}(\boldsymbol{u}) - \wp_{g,1}(\boldsymbol{u}).$$

The corresponding $y_1, \dots, y_g$ are then given by

$$y_k = -\frac{\partial}{\partial u_g}\mathcal{P}(x, \boldsymbol{u})\Big|_{x=x_k}.$$

Also derived are the relations that connect the functions $\wp_{gi}$ with their derivatives, (Chapter 4 of [19]). In particular a set of equations expressing the $\wp_{gggi}$ as a polynomial of 2-index $\wp$-functions with total degree at most two.

$$\wp_{gggi} = (6\wp_{g,g} + \lambda_{2g})\wp_{g,i} + 6\wp_{g,i-1} - 2\wp_{g-1,i} + \tfrac{1}{2}\delta_{g,i}\lambda_{2g-1}.$$

These generalise Corollary 2.1.15 in the elliptic case and equations (2.66) derived by Baker. Similarly there is a generalisation of the key elliptic equation, (Theorem 2.1.14) generalised by Baker in equations (2.65). These express products of 3-index $\wp$-functions as a polynomial of 2-index $\wp$-functions with total degree at most two.

$$\begin{aligned}\wp_{ggi}\wp_{ggk} &= 4\wp_{g,g}\wp_{g,i}\wp_{g,k} - 2(\wp_{g,i}\wp_{g-1,k} + \wp_{g,k}\wp_{g-1,i}) + 4\wp_{k-1,i-1}\\ &+ 4(\wp_{g,k}\wp_{g,i-1} + \wp_{g,i}\wp_{g,k-1}) - 2(\wp_{k,i-2} + \wp_{i,k-2}) + \lambda_{2g}\wp_{g,k}\wp_{g,i}\\ &+ \tfrac{1}{2}\lambda_{2g-1}(\delta_{ig}\wp_{k,g} + \delta_{k,g}\wp_{i,g}) + \lambda_{2i-2}\delta_{i,k} + \tfrac{1}{2}(\lambda_{2i-1}\delta_{k,i+1} + \lambda_{2k-1}\delta_{i,k+1}).\end{aligned}$$

In Chapter 5 of [19] matrix formulations of these equations are developed. The paper also introduces applications of the results to the Sine-Gordan equation, the Schrödinger equation and the KdV system.

These developments of the hyperelliptic theory inspired new applications such as [41] and [33] and an interest in the development of a general theory. In 2000 Eilbeck, Enolskii and Leykin published [35] which developed the definitions in [19] from hyperelliptic curve to (n, s)-curves. In particular they relaxed the cyclic restriction, allowing additional powers of y in the curve equations as specified by Definition 2.2.1. The fundamental differential, σ-function and $\wp$-functions were introduced for the general cases.

In 1999 Buchstaber, Enolski and Leykin published [20] and [22] which introduced the connection of the σ-function to the Schur-Weierstrass polynomials, as set out in Theorem 2.2.21. This idea was developed further in the recent paper by Nakayashiki, [60].

The next step was to investigate the theory for (n, s)-curves with $n = 3$.

2.3.3 Abelian functions associated to trigonal curves

Definition 2.3.2. *Define those (n, s)-curves with $n = 3$ as* ***trigonal curves****.*

In [21] Buchstaber, Enolskii and Leykin furthered their methods in [19] for the hyperelliptic case to obtain realisations of the Jacobian and another important structure, the Kummer variety, for the case of an arbitrary trigonal curve of genus g. Over recent years several groups of authors have begun to investigate other aspects of the theory of Abelian functions associated to trigonal curves, both general and cyclic. (Note that in the literature the cyclic trigonal cases are sometimes referred to as *strictly trigonal* or *purely trigonal*.)

The two canonical classes of trigonal curves are the (3,4) and (3,5)-curves of genus three and four respectively. Detailed studies are given in [30] by Eilbeck, Enolski, Matsutani, Ônishi and Previato for functions associated to the general (3,4)-curve, and in [11] by Baldwin, Eilbeck, Gibbons and Ônishi for functions associated to the cyclic (3,5)-curve.

Both papers explicitly construct the fundamental differential, solve the Jacobi Inversion Problem and obtains sets of differential equations between the $\wp$-functions. These include complete sets of relations to generalise Corollary 2.1.15 and the beginnings of sets to generalise Theorem 2.1.14. Also obtained are previously unconsidered sets of relations that are bilinear in the 2 and 3-index $\wp$-functions and two term addition formulae for the σ-function that generalise Theorem 2.1.26. The sets of differential equations in these papers were recently completed using some new approaches as discussed in Chapter 5.

In [30] a second addition formula is derived, for the case of the cyclic curve only. This expresses a similar ratio of σ-functions,

$$\frac{\sigma(\boldsymbol{u}+\boldsymbol{v})\sigma(\boldsymbol{u}+[\zeta]\boldsymbol{v})\sigma(\boldsymbol{u}+[\zeta^2]\boldsymbol{v})}{\sigma(\boldsymbol{u})^3\sigma(\boldsymbol{v})^3}, \qquad \zeta^3 = 1, \tag{2.68}$$

as a polynomial of Abelian functions and is related to the invariance in equation (2.34). In [11] a duplication formula for the σ-function is presented.

The results of [30] and [11] have influenced the research into tetragonal curves that is presented in Chapter 3 of this document. In particular, both papers required the introduction of an additional set of Abelian functions associated with the curve in order to derive the full theory. This idea is introduced and extended in Section 3.1.3 for use in the rest of the document.

Another important contribution of these papers were the results on the σ-function expansion. In [30] there is a proof that this expansion will have coefficients in $\boldsymbol{u}$ that are polynomial in the curve coefficients. Both papers derive such an expansion and use it to generate results. It should be noted however that in these papers most of the results were obtained independent of this σ-expansion, through the *Kleinian expansion* technique outlined in Section 3.2. As discussed in Section 3.2 this technique is not always as fruitful and so alternative approaches using the σ-expansion have been developed in this document.

In fact, the first use of the σ-expansion was in [14] where Baldwin and Gibbons use results on the Abelian functions associated to the $(3,5)$-curve to evaluate an integral that characterises reductions of the Benney moment equations. This was part of a series of papers and in Chapter 6 a similar application of the results for the tetragonal curve is presented. A summary of the key literature in this topic is given in the introduction to Chapter 6.

2.3.4 Recent and additional contributions

Research on this area is continuing in several different directions. This document starts by investigating the next logical class of (n,s)-curves; those which have $n = 4$. It then uses ideas generated from this work to further the canonical trigonal cases. Meanwhile in the opposite direction, the new addition formula (2.68) found for trigonal curves has inspired new relations for the Weierstrass $\wp$-function in [36].

Other areas that have been explored include determinant expressions for the functions. In [63] Ônishi develops formula to express ratios of σ-functions associated to a trigonal curve as determinants involving the curve variables. (This was preceded by similar results for hyperelliptic functions in [62]). In [34] the topic of Frobenius-Stickelberger formula is explored with applications of such formula presented in [31].

A particularly interesting new contribution to the theory of Abelian functions has involved a new methodology which simplifies the derivation of the relations between Abelian functions using representation theory. The recent paper [4] presents this approach for hyperelliptic curves of genus one, two and three.

This new approach is based on the observation that the underlying algebraic curves belong to families permuted under the $SL(2)$ action. This has been interpreted in [5], [6] as a covariance property of the $\wp$-functions. In [4] each of the identities between the $\wp$-functions belong to a finite dimensional representation of $SL(2)$, with it only necessary to generate one identity in each representation.

These covariant ideas were considered originally by Baker in [7] and later in [25], but without such success. A requirement of the new approach is that the theory be developed for a generic family of curves, as opposed to the (n,s)-curves used in this document with one branch point to infinity. An alternative covariant definition of the $\wp$-functions is used. However, it is possible to translate all the results generated this way into the language of the $\wp$-functions used in this document.

The benefit of this approach is the minimal amount of computer algebra required. If the methods can be developed for a more general class of curve then they may be useful in the further development of the theory of Abelian functions.

Chapter 3

Abelian functions associated with a cyclic tetragonal curve of genus six

In Section 2.3 we discussed those classes of (n, s)-curve that had been already studied in relation to their Abelian functions. The theory for the cases where $n = 2$ was well established and much progress had been made for the cases where $n = 3$, although research on some aspects is still continuing. The next logical class to consider would be those curves where $n = 4$.

Definition 3.0.3. *We define those (n, s)-curves with $n = 4$ to be* ***tetragonal curves****.*

In this Chapter we investigate Abelian functions associated with the (4,5)-curve. This is the first curve from the tetragonal class to be studied in detail and is the simplest of these curves.

We construct the Abelian functions using the multivariate σ-function associated to the curve, as discussed in general in Section 2.2. We then derive a number of results for these functions including a basis for the space of fundamental Abelian functions, associated sets of differential equations, the solution to the Jacobi Inversion Problem and an addition formula for the σ-function. At the end of this chapter we demonstrate that such functions can be used to construct a solution to the KP-equation and we outline how a general class of solutions could be generated using a wider class of curves.

This chapter was recently summarised and published in [39], which was co authored by Professor Chris Eilbeck. The Maple worksheets that were used in the derivation of the results can be found in the extra Appendix of files and some of the key results have been made available online at [38].

The (4,5)-curve has genus $g = 6$ which is considerably higher that any curve previously considered. This leads to larger classes of associated Abelian functions and much larger computations in all areas of the theory. In [21] it was the opinion of the authors that certain results could not be derived for the tetragonal curves. This prediction was made not because of the computation complexity but because of a number of other complications which hinder methods used in lower genus cases. This is characterised by the fact the inequality

$$n \leq g = \tfrac{1}{2}(n-1)(s-1) < s$$

is not satisfied for $n > 3$. One of the major differences in this work was the extent to which the series expansion of the σ-function was used to obtain results for the $\wp$-functions. The derivation of this expansion is one of the major results in this chapter, involving considerable computing power and requiring efficient programming.

This chapter is organised as follows. In Section 3.1 we start by introducing the curves and functions we use, drawing on the definitions and properties that were given in general in the previous chapter. Then in Section 3.2 we discuss a key theorem satisfied by the $\wp$-functions that connects the $\wp$-functions to a point of the curve they are associated to. We are able to manipulate this result to derive relations between the $\wp$-functions, although we find this process is much more involved than in previous examples. While it is not as useful in deriving the key differential equations between the $\wp$-functions, it does allow us to construct a solution to the Jacobi Inversion Problem.

In Section 3.3 we discuss the set of weights that can be defined for the theory of Abelian functions and which are necessary for the computations that follow. We present these for a general (n, s)-curve giving explicit examples for the (4,5)-case.

In Section 3.4 we derive properties of the σ-function including the series expansion. This involves some heavy calculations and we discuss the computational difficulties and how they can be overcome. Next, in Section 3.5, we use the σ-expansion to derive differential equations between the Abelian functions.

Section 3.6 establishes and constructs the addition formula for the σ-function and finally in Section 3.7 we give details on the application to the KP-equation. Appendix C contains some of the results obtained in this chapter that were considered too large to keep in the main text. Other results too large to typeset, along with the Maple worksheets used to derive the results can be found in the extra Appendix of files.

3.1 The (4,5)-curve and associated functions

The tetragonal curves have not been the subject of detailed study before, so it is worthwhile to establish what properties exist and how the associated Abelian functions behave. The simplest tetragonal curve is, in the notation of the (n,s)-curves, a $(4,5)$-curve. Using Definition 2.2.1 the general (4,5)-curve is given by $g(x,y)=0$ where

$$g(x,y) = y^4 + \big(\mu_1 x + \mu_5\big)y^3 + \big(\mu_2 x^2 + \mu_6 x + \mu_{10}\big)y^2 + \big(\mu_3 x^3 + \mu_7 x^2 + \mu_{11} x + \mu_{15}\big)y \\ - \big(x^5 + \mu_4 x^4 + \mu_8 x^3 + \mu_{12} x^2 + \mu_{16} x + \mu_{20}\big) \qquad (\mu_j \text{ constants}).$$

We will simplify further by considering the cyclic subclass of this family. By Definition 2.2.2 these are the curves C, given by $f(x,y)=0$ where

$$f(x,y) = y^4 - \big(x^5 + \lambda_4 x^4 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0\big), \qquad (\lambda_j \text{ constants}). \tag{3.1}$$

Using equation (2.33) we can check that this curve has genus

$$g = \tfrac{1}{2}\big(n-1\big)\big(s-1\big) = 6.$$

It contains an extra level of symmetry demonstrated by the fact that it is invariant under

$$[\zeta]: \quad (x,y) \to (x, \zeta y), \tag{3.2}$$

where ζ is a 4th root of unity.

We start by constructing the differentials on the Riemann surface defined by C. Recall the Weierstrass gap sequence given in Definition 2.2.3. We must formulate this sequence for $(n,s)=(4,5)$. Take each natural number in turn and see if it can be represented as

$$4a + 5b \quad \text{for some } a,b \in \mathbb{N} = \{0,1,2,\dots\}.$$

If so it is a nongap and we overline it. We stop once we have found $g=6$ gaps.

$$\overline{0}, 1, 2, 3, \overline{4}, \overline{5}, 6, 7, \overline{8, 9, 10}, 11, \overline{12, 13, \dots}$$

So the sequences of gaps and non-gaps are given by

$$W_{4,5} = \{1,2,3,6,7,11\}, \qquad \overline{W}_{4,5} = \{0,4,5,8,9,10,12,13,\dots\}. \tag{3.3}$$

Now we follow Proposition 2.2.4 to construct the standard basis of holomorphic differentials upon C.

$$du_i(x,y) = \frac{x^{P_i} y^{Q_i}}{f_y(x,y)} dx, \qquad i = 1,\dots,g.$$

The integers $\{P_i, Q_i\}$ are the integers used in the decomposition of the first g non-gaps.

$$\begin{array}{llll}
\overline{w}_1 = 0 \;\; = 0\cdot 4 + 0\cdot 5 & \Longrightarrow & P_1 = 0, & Q_1 = 0 \\
\overline{w}_2 = 4 \;\; = 1\cdot 4 + 0\cdot 5 & \Longrightarrow & P_2 = 1, & Q_2 = 0 \\
\overline{w}_3 = 5 \;\; = 0\cdot 4 + 1\cdot 5 & \Longrightarrow & P_3 = 0, & Q_3 = 1 \\
\overline{w}_4 = 8 \;\; = 2\cdot 4 + 0\cdot 5 & \Longrightarrow & P_4 = 2, & Q_4 = 0 \\
\overline{w}_5 = 9 \;\; = 1\cdot 4 + 1\cdot 5 & \Longrightarrow & P_5 = 1, & Q_5 = 1 \\
\overline{w}_6 = 10 = 0\cdot 4 + 2\cdot 5 & \Longrightarrow & P_6 = 0, & Q_6 = 2
\end{array}$$

From equation (3.1) we have $f_y(x,y) = 4y^3$ and therefore the standard basis of holomorphic differentials upon C is

$$\begin{array}{c}
\boldsymbol{du} = (du_1, \dots, du_6), \qquad \text{where} \quad du_i(x,y) = \dfrac{g_i(x,y)}{4y^3}dx, \\
\text{with} \quad \begin{array}{lll} g_1(x,y) = 1, & g_2(x,y) = x, & g_3(x,y) = y, \\ g_4(x,y) = x^2, & g_5(x,y) = xy, & g_6(x,y) = y^2. \end{array}
\end{array} \tag{3.4}$$

We know from the general theory that any point $\boldsymbol{u} \in \mathbb{C}^6$ can be expressed as

$$\boldsymbol{u} = (u_1, u_2, u_3, u_4, u_5, u_6) = \sum_{i=1}^{6} \int_{\infty}^{P_i} \boldsymbol{du}, \tag{3.5}$$

where the P_i are six variable points upon C.

3.1.1 Constructing the fundamental differential

Recall Definition 2.2.6 for the fundamental differential of the second kind associated to an (n,s)-curve. Recall also Klein's explicit realisation of this differential which was set out in Proposition 2.2.8. We will now construct the fundamental differential associated to the (4,5)-curve. This will involve a basis of meromorphic differentials which have their only pole at ∞. Following equation (2.39) in the general case, we will denote this basis by

$$\boldsymbol{dr} = (dr_1, \dots, dr_6), \qquad \text{where} \qquad dr_j(x,y) = \frac{h_j(x,y)}{4y^3}dx. \tag{3.6}$$

Start by constructing the meromorphic function $\Sigma((x,y),(z,w))$ that was given in Definition 2.2.7. Using the curve equation (3.1) we have

$$\begin{aligned}
\left[\frac{f(z,w)}{w^{n-k+1}}\right]_w &= \frac{w^4}{w^{4-k+1}}, \quad k = 1, \dots, 4 \\
\sum_{k=1}^{4} y^{4-k}\left[\frac{f(z,w)}{w^{n-k+1}}\right]_w &= \frac{y^3w^4}{w^4} + \frac{y^2w^4}{w^3} + \frac{yw^4}{w^2} + \frac{w^4}{w} = y^3 + y^2w + yw^2 + w^3.
\end{aligned}$$

Hence, from Definition 2.2.7, the meromorphic function is

$$\Sigma\big((x,y),(z,w)\big) = \frac{1}{(x-z)\cdot 4y^3}\cdot\sum_{k=1}^{4} y^{4-k}\frac{w^4}{w^{4-k+1}} = \frac{y^3+y^2w+yw^2+w^3}{4y^3(x-z)}.$$

Next we form the function $R((x,y),(z,w))$ given in Proposition 2.2.8 by equation (2.42).

$$\begin{aligned} R\big((x,y),(z,w)\big) &= \frac{\partial}{\partial z}\Sigma\big((x,y),(z,w)\big) + \sum_{j=1}^{6}\frac{du_j(x,y)}{dx}\cdot\frac{dr_j(z,w)}{dz} \\ &= \frac{d}{dz}\left(\frac{y^3+y^2w+yw^2+w^3}{x-z}\right)\frac{1}{4y^3} + \frac{1}{16y^3w^3}\bigg(h_1(z,w)+xh_2(z,w) \\ &\qquad + yh_3(z,w)+x^2h_4(z,w)+xyh_5(z,w)+y^2h_6(z,w)\bigg), \end{aligned}$$

where the polynomials h_j are from equation (3.6) and have yet to be determined. Multiplying up by $16y^3w^3$ allows us to write

$$16y^3w^3R\big((x,y),(z,w)\big) = Q_1\big((x,y),(z,w)\big) + Q_2\big((x,y),(z,w)\big), \tag{3.7}$$

where

$$Q_1\big((x,y),(z,w)\big) = \frac{d}{dz}\left(\frac{y^3+y^2w+yw^2+w^3}{x-z}\right)4w^3, \tag{3.8}$$

$$Q_2\big((x,y),(z,w)\big) = h_1(z,w)+xh_2(z,w)+\cdots+y^2h_6(z,w). \tag{3.9}$$

By Definition 2.2.6 the fundamental differential must be symmetric and hence by Proposition 2.2.8 so must the function $R((x,y),(z,w))$. We will determine the h_j by imposing this condition which may be written as follows.

$$\begin{aligned} R\big((x,y),(z,w)\big) &= R\big((z,w),(x,y)\big) \\ 16y^3w^3R\big((x,y),(z,w)\big) &= 16w^3y^3R\big((z,w),(x,y)\big) \\ Q_1\big((x,y),(z,w)\big)+Q_2\big((x,y),(z,w)\big) &= Q_1\big((z,w),(x,y)\big)+Q_2((z,w),(x,y)) \end{aligned}$$

Rearrange this to give the following equation that we will ensure is satisfied.

$$Q_1\big((x,y),(z,w)\big) - Q_1\big((z,w),(x,y)\big) = Q_2\big((z,w),(x,y)\big) - Q_2\big((x,y),(z,w)\big). \tag{3.10}$$

First we must perform the differentiation in equation (3.8) to obtain

$$Q_1\big((x,y),(z,w)\big) = \left(\frac{y^2w_z+2yww_z+3w^2w_z}{(x-z)} - \frac{(y^3+y^2w+yw^2+w^3)}{(x-z)^2}\right)4w^3. \tag{3.11}$$

We simplify this by using the curve equation,

$$w^4 = z^5 + +\lambda_4 z^4 + \lambda_3 z^3 + \lambda_2 z^2 + \lambda_1 z + \lambda_0. \tag{3.12}$$

Differentiating the curve equation with respect to z gives

$$\begin{aligned} 4w^3 w_z &= 5z^4 + 4\lambda_4 z^3 + 3\lambda_3 z^2 + 2\lambda_2 z + \lambda_1 \\ w_z &= \frac{5z^4 + 4\lambda_4 z^3 + 3\lambda_3 z^2 + 2\lambda_2 z + \lambda_1}{4w^3}. \end{aligned} \tag{3.13}$$

Then substituting (3.12) and (3.13) into (3.11) gives

$$\begin{aligned} Q_1\big((x,y),(z,w)\big) = \left[\frac{1}{(x-z)^2}\right] &\big[4y^3w^3 + w^2\big(\lambda_1 z + 9\lambda_3 z^2 x - 8\lambda_4 z^4 - 2\lambda_2 z^2 \\ &+ 3\lambda_1 x + 12\lambda_4 z^3 x - 5\lambda_3 z^3 + 4\lambda_0 - 11z^5 + 15z^4 x + 6\lambda_2 zx\big) + w\big(2y\lambda_1 z \\ &+ 4y\lambda_2 zx + 8y\lambda_4 z^3 x + 2y\lambda_1 x - 4y\lambda_4 z^4 - 2y\lambda_3 z^3 + 4y\lambda_0 - 6yz^5 + 6y\lambda_3 z^2 x \\ &+ 10yz^4 x\big) + 2y^2\lambda_2 z^2 + 2y^2\lambda_2 zx + 5y^2 z^4 x + 3y^2\lambda_1 z + y^2\lambda_3 z^3 + 4y^2\lambda_0 \\ &- y^2 z^5 + y^2\lambda_1 x + 4y^2\lambda_4 z^3 x + 3y^2\lambda_3 z^2 x\big]. \end{aligned} \tag{3.14}$$

We can now form the left hand side of equation (3.10) by reversing the roles in equation (3.14) and subtracting the new equation from the old. After considerable canceling of terms this leaves us with

$$\begin{aligned} \text{LHS}(3.10) = wy&\big(2x\lambda_3 - 2z\lambda_3 - 2z^2 x + 2x^2 z - 4z^2\lambda_4 + 4x^2\lambda_4 - 6z^3 + 6x^3\big) \\ &+ y^2\big(2\lambda_2 + 8x^2\lambda_4 + 7x^2 z + 3z^2 x + 11x^3 + 5x\lambda_3 + z\lambda_3 + 4xz\lambda_4 - z^3\big) \\ &- w^2\big(2\lambda_2 + 8z^2\lambda_4 + 7z^2 x + 3x^2 z + 11z^3 + 5z\lambda_3 + x\lambda_3 + 4xz\lambda_4 - x^3\big). \end{aligned}$$

We have been able to collect this into three parts; the terms which contain w^2, the terms which contain y^2 and the terms which contain wy. The right hand side of equation (3.10) is simply

$$\begin{aligned} \text{RHS}(3.10) = h_1(x,y) + zh_2(x,y) + wh_3(x,y) + z^2h_4(x,y) + zwh_5(x,y) + w^2h_6(x,y) \\ - h_1(z,w) - xh_2(z,w) - yh_3(z,w) - x^2h_4(z,w) - xyh_5(z,w) - y^2h_6(z,w). \end{aligned}$$

We can now assign values to the h_j so that equation (3.10) holds. First set $h_1(x,y)$ to be the terms in LHS(3.10) that do not contain z or w.

$$h_1(x,y) = y^2\big(11x^3 + 5x\lambda_3 + 8x^2\lambda_4 + 2\lambda_2\big).$$

These must all come from the section of terms that contain y^2. Similarly, the terms on the left hand side of equation (3.10) that do not contain w but do contain a single power of z

must also come from the y^2 section. We set these to be $zh_2(x,y)$.

$$zh_2(x,y) = y^2\big(z\lambda_3 + 7x^2z + 4xz\lambda_4\big) \qquad \Longrightarrow \qquad h_2(x,y) = y^2\big(\lambda_3 + 7x^2 + 4x\lambda_4\big).$$

Next set $z^2h_4(x,y)$ to be those terms on the left hand side of equation (3.10) that do not contain w but do contain a single power of z^2.

$$z^2h_4(x,y) = y^2\big(3z^2x\big), \qquad \Longrightarrow \qquad h_4(x,y) = 3y^2x.$$

We have now determined values for three of the six h_j. Substitute these values into equation (3.10) and cancel terms to obtain

$$\begin{aligned} &wy\big(2x\lambda_3 - 2z\lambda_3 - 2z^2x + 2x^2z - 4z^2\lambda_4 + 4x^2\lambda_4 - 6z^3 + 6x^3\big) + w^2x^3 - y^2z^3 \\ &\quad = wh_3(x,y) + zwh_5(x,y) + w^2h_6(x,y) - yh_3(z,w) - xyh_5(z,w) - y^2h_6(z,w). \end{aligned} \tag{3.15}$$

Now, the only term on the left of equation (3.15) containing w^2 is w^2x^3 so set

$$h_6(x,y) = x^3.$$

Then set those terms on the left hand side of equation (3.15) with a single power of w and no z to be $wh_3(x,y)$.

$$wh_3(x,y) = wy\big(2x\lambda_3 + 4x^2\lambda_4 + 6x^3\big) \qquad \Longrightarrow \qquad h_3(x,y) = 2xy\big(\lambda_3 + 2x\lambda_4 + 3x^2\big).$$

Substituting the expressions for h_3 and h_6 into equation (3.15) leaves us with

$$wy\big(-2z^2x + 2x^2z\big) = zwh_5(x,y) - xyh_5(z,w).$$

Hence we must have

$$zwh_5(x,y) = wy(2x^2z), \qquad \Longrightarrow \qquad h_5(x,y) = 2x^2y.$$

Note that this set of polynomials is not unique. If we had proceeded in a different order then some of the terms may have been put into different polynomials h_j. However, this would make no difference to the end results.

Substituting the expressions for the h_j into equation (3.8) gives

$$\begin{aligned} Q_2\big((x,y),(z,w)\big) &= \big(11w^2z^3 + 5w^2z\lambda_3 + 8w^2\lambda_4z^2 + 2w^2\lambda_2\big) + 3w^2x^2z + y^2z^3 \\ &\quad + 2wyz^2x + x\big(w^2\lambda_3 + 7w^2z^2 + 4zw^2\lambda_4\big) + \big(6wz^3 + 4z^2w\lambda_4 + 2zw\lambda_3\big)y. \end{aligned} \tag{3.16}$$

Then substituting equations (3.14) and (3.16) into equation (3.7) gives an explicit equation for $R((x,y),(z,w))$.

$$R\big((x,y),(z,w)\big) = \left[\frac{1}{16w^3y^3(x-z)^2}\right]\Big[3w^2x^4z + 3\lambda_1w^2x + \lambda_3w^2x^3 + 4y^2\lambda_0 + 3y^2z^4x + w^2\lambda_1z + 4wy\lambda_2zx + 2wy\lambda_3z^2x + 4y^3w^3 + w^2x^3z^2 + y^2z^3x^2 + 2y^2\lambda_2zx + 3y^2\lambda_3z^2x + 4y^2\lambda_4z^3x + y^2\lambda_1x + 3y^2\lambda_1z + 2y^2\lambda_2z^2 + y^2\lambda_3z^3 + 4wy\lambda_0 + 2w^2\lambda_2x^2 + 4z^2wy\lambda_4x^2 + 2zwy\lambda_3x^2 + 2wy\lambda_1z + 2wy\lambda_1x + 2z^3wyx^2 + 2wyz^2x^3 + 4w^2\lambda_0 + 2w^2\lambda_2xz + 4w^2\lambda_4x^3z + 3w^2\lambda_3x^2z\Big].$$

We can substitute this into equation (2.41) to give an explicit realisation of the fundamental differential presented in the summary below. In the extra Appendix of files there is a Maple worksheet in which these calculations were performed.

Summary

The fundamental differential of the second kind associated to the (4,5)-curve is given by

$$\Omega\big((x,y),(z,w)\big) = \frac{F\big((x,y),(z,w)\big)dxdz}{16(x-z)^2y^3w^3}, \tag{3.17}$$

where F is the following symmetric polynomial

$$F\big((x,y),(z,w)\big) = 4y^3w^3 + \big(3xz^4 + z^3\lambda_3 + z^3x^2 + 2\lambda_2z^2 + 3x\lambda_3z^2 + 4z^3x\lambda_4 + 4\lambda_0 + \lambda_1x + 2\lambda_2xz + 3\lambda_1z\big)y^2 + \big(2\lambda_1z + 4\lambda_2xz + 4\lambda_0 + 2\lambda_1x + 4x^2\lambda_4z^2 + 2\lambda_3x^2z + 2x^3z^2 + 2z^3x^2 + 2x\lambda_3z^2\big)wy + \big(\lambda_3x^3 + 4\lambda_0 + 3\lambda_1x + 2\lambda_2x^2 + \lambda_1z + x^3z^2 + 3x^4z + 2\lambda_2xz + 3\lambda_3x^2z + 4\lambda_4x^3z\big)w^2. \tag{3.18}$$

In obtaining this realisation, the following explicit basis for the differentials of the second kind associated to the cyclic (4,5)-curve was derived.

$$\boldsymbol{dr} = (dr_1,\dots,dr_6), \qquad \text{where} \quad dr_j(x,y) = \frac{h_j(x,y)}{4y^3}dx, \tag{3.19}$$

with

$$h_1(x,y) = y^2\big(11x^3 + 5x\lambda_3 + 8x^2\lambda_4 + 2\lambda_2\big) \qquad h_4(x,y) = 3y^2x$$
$$h_2(x,y) = y^2\big(\lambda_3 + 7x^2 + 4x\lambda_4\big) \qquad h_5(x,y) = 2x^2y$$
$$h_3(x,y) = 2xy\big(\lambda_3 + 2x\lambda_4 + 3x^2\big) \qquad h_6(x,y) = x^3.$$

3.1.2 Abelian functions associated with the (4,5)-curve

We can now proceed to define Abelian functions as in the general case. Start by choosing an appropriate basis of cycles upon the surface defined by C, (ensuring condition (2.45) on the intersection numbers is satisfied.) We label these α_i, β_j, for $1 \le i,j \le 6$ and define the period matrices by integrating (3.4) and (3.19) around these cycles.

$$\begin{array}{ll} \omega' = \left(\oint_{\alpha_k} du_\ell \right)_{k,\ell=1,\dots,6} & \omega'' = \left(\oint_{\beta_k} du_\ell \right)_{k,\ell=1,\dots,6} \\ \eta' = \left(\oint_{\alpha_k} dr_\ell \right)_{k,\ell=1,\dots,6} & \eta'' = \left(\oint_{\beta_k} dr_\ell \right)_{k,\ell=1,\dots,6} \end{array}.$$

These can be combined into the matrix M which satisfies the generalised Legendre equation given earlier in (2.53).

$$M = \begin{pmatrix} \omega' & \omega'' \\ \eta' & \eta'' \end{pmatrix}.$$

Let Λ denote the lattice generated by the first pair of period matrices, (Definition 2.2.10), and define an Abelian function associated to C as a meromorphic function that is periodic over this lattice, (Definition 2.2.11).

By Definition 2.2.12 the Jacobian of C is $J = \mathbb{C}^6/\Lambda$ and by Definition 2.2.13 the Abel map associated to the $(4,5)$-curve is defined by

$$\begin{aligned} \mathfrak{A} : \mathrm{Sym}^k(C) &\to J \\ (P_1, \dots, P_k) &\mapsto \left(\int_\infty^{P_1} \boldsymbol{du} + \dots + \int_\infty^{P_k} \boldsymbol{du} \right) \pmod{\Lambda}, \end{aligned} \tag{3.20}$$

where the P_i are points upon C. Finally by Definition 2.2.14 the strata of the Jacobian are the images of the Abel map. When $k = 1$ the Abel map gives an embedding of the curve C upon which we define ξ as the local parameter at the origin, $\mathfrak{A}_1(\infty)$.

$$\xi = x^{-\frac{1}{4}}. \tag{3.21}$$

The local parameter at infinity

Many calculations are easily performed at ∞ with the aid of the local parameter ξ. We derive here series expansion in ξ for the main variables. First, note that in terms of ξ,

$$x = \frac{1}{\xi^4}, \qquad \frac{dx}{d\xi} = -\frac{4}{\xi^5}, \tag{3.22}$$

and

$$\begin{aligned} y &= \left(x^5 + \lambda_4 x^4 + \dots + \lambda_1 x + \lambda_0 \right)^{\frac{1}{4}} = \left(\xi^{-20} + \lambda_4 \xi^{-16} + \dots + \lambda_1 \xi^{-4} + \lambda_0 \right)^{\frac{1}{4}} \\ &= \frac{1}{\xi^5} + \left(\frac{\lambda_4}{4} \right) \frac{1}{\xi} + \left(\frac{\lambda_3}{4} - \frac{3\lambda_4^2}{32} \right) \xi^3 + \left(\frac{\lambda_2}{4} - \frac{3\lambda_4\lambda_3}{16} + \frac{7\lambda_4^3}{128} \right) \xi^7 \\ &\quad + \left(\frac{\lambda_1}{4} - \frac{3\lambda_4\lambda_3}{16} - \frac{3\lambda_3^2}{32} + \frac{21\lambda_4^2\lambda_3}{128} - \frac{77\lambda_4^4}{2048} \right) \xi^{11} + O(\xi^{15}). \end{aligned} \tag{3.23}$$

(See Appendix C.1 for a longer expansion.) Using these substitutions we can calculate series expansions for the six holomorphic differentials in the basis $\boldsymbol{du}$ given by equation (3.4).

$$\begin{aligned} du_1 &= [-\xi^{10} + O(\xi^{14})]d\xi & du_4 &= [-\xi^2 + O(\xi^6)]d\xi \\ du_2 &= [-\xi^6 + O(\xi^{10})]d\xi & du_5 &= [-\xi^1 + O(\xi^5)]d\xi \\ du_3 &= [-\xi^5 + O(\xi^9)]d\xi & du_6 &= [-1 + O(\xi^4)]d\xi. \end{aligned} \tag{3.24}$$

Integrating these gives us a set of series expansions for the variables $\boldsymbol{u}$.

$$\begin{aligned} u_1 &= -\tfrac{1}{11}\xi^{11} + O(\xi^{15}) & u_3 &= -\tfrac{1}{6}\xi^6 + O(\xi^{10}) & u_5 &= -\tfrac{1}{2}\xi^2 + O(\xi^6) \\ u_2 &= -\tfrac{1}{7}\xi^7 + O(\xi^{11}) & u_4 &= -\tfrac{1}{3}\xi^3 + O(\xi^7) & u_6 &= -\xi + O(\xi^5). \end{aligned} \tag{3.25}$$

Note that the higher order terms in (3.24) and (3.25) will all depend on the curve parameters $\lambda_0, \dots, \lambda_4$. (See Appendix C.1 for larger expansions).

Abelian functions

Define the Kleinian σ-function associated to the $(4,5)$-curve as in Definition 2.2.15 and note that it is a function of $g = 6$ variables.

$$\sigma = \sigma(\boldsymbol{u}) = \sigma(u_1, u_2, u_3, u_4, u_5, u_6).$$

Define the 2-index $\wp$-functions associated to the $(4,5)$-curve as in Definition 2.2.24.

$$\wp_{i,j}(\boldsymbol{u}) = -\frac{\partial}{\partial u_i}\frac{\partial}{\partial u_j}\log\big[\sigma(\boldsymbol{u})\big], \quad i \le j \in \{1, \dots, 6\}.$$

There will be 21 of the 2-index $\wp$-functions,

$$\begin{aligned} \{\wp_{ij}\} = \{&\wp_{11}, \wp_{12}, \wp_{13}, \wp_{14}, \wp_{15}, \wp_{16}, \wp_{22}, \wp_{23}, \wp_{24}, \wp_{25}, \wp_{26}, \\ &\wp_{33}, \wp_{34}, \wp_{35}, \wp_{36}, \wp_{44}, \wp_{45}, \wp_{46}, \wp_{55}, \wp_{56}, \wp_{66}\}. \end{aligned}$$

Similarly we can define the n-index $\wp$-functions as in Definition 2.2.27.

$$\wp_{i_1,i_2,\dots,i_n}(\boldsymbol{u}) = -\frac{\partial}{\partial u_{i_1}}\frac{\partial}{\partial u_{i_2}}\cdots\frac{\partial}{\partial u_{i_n}}\log\big[\sigma(\boldsymbol{u})\big], \quad i_1 \le \dots \le i_n \in \{1, \dots, 6\}.$$

There will be 56 of the 3-index $\wp$-functions and 126 of the 4-index $\wp$-functions.

In Lemma 2.2.28 the n-index $\wp$-functions were shown to have poles of order n when $\sigma(\boldsymbol{u}) = 0$. By Lemma 2.2.19 this occurs when $\boldsymbol{u} \in \Theta^{[5]}$. Any Abelian function that is holomorphic must be constant (by a generalisation of Liouville's theorem) and it is not possible for an Abelian function to have simple poles. Hence those Abelian functions with poles of order two have the simplest pole structure possible and are sometimes referred to as *fundamental Abelian functions*. (Note that the Weierstrass $\wp$-function is in this category.)

3.1.3 The Q-functions

We find in Section 3.5 that the 2-index $\wp$-functions are not sufficient to construct a basis for the fundamental Abelian functions associated to the (4,5)-curve. We overcome this problem by considering another class of Abelian functions, alongside the $\wp$-functions. These can also be defined using the Kleinian σ-functions and the following operator.

Definition 3.1.1. *Define the operator Δ_i as below. This is now known as **Hirota's bilinear operator**, although it was used much earlier by Baker in [10].*

$$\Delta_i = \frac{\partial}{\partial u_i} - \frac{\partial}{\partial v_i}.$$

We now check that an alternative, equivalent definition of the 2-index Kleinian $\wp$-functions is given by

$$\wp_{ij}(\boldsymbol{u}) = -\frac{1}{2\sigma(\boldsymbol{u})^2}\Delta_i\Delta_j\sigma(\boldsymbol{u})\sigma(\boldsymbol{v}) \Bigg|_{\boldsymbol{v}=\boldsymbol{u}} \qquad i \leq j \in \{1,\ldots,6\}. \tag{3.26}$$

Applying the operator as defined above we see that

$$\Delta_j\big[\sigma(\boldsymbol{u})\sigma(\boldsymbol{v})\big] = \frac{\partial}{\partial u_j}\left[\sigma(\boldsymbol{u})\sigma(\boldsymbol{v})\right] - \frac{\partial}{\partial v_j}\left[\sigma(\boldsymbol{u})\sigma(\boldsymbol{v})\right] = \sigma(\boldsymbol{v})\sigma_j(\boldsymbol{u}) - \sigma(\boldsymbol{u})\sigma_j(\boldsymbol{v}),$$
$$\Delta_i\Delta_j\big[\sigma(\boldsymbol{u})\sigma(\boldsymbol{v})\big] = \sigma(\boldsymbol{v})\sigma_{ij}(\boldsymbol{u}) - \sigma_i(\boldsymbol{u})\sigma_j(\boldsymbol{v}) - \sigma_i(\boldsymbol{v})\sigma_j(\boldsymbol{u}) + \sigma(\boldsymbol{u})\sigma_{ij}(\boldsymbol{v}).$$

Hence equation (3.26) may be evaluated as follows:

$$\begin{aligned}
\wp_{ij}(\boldsymbol{u}) &= -\frac{1}{2\sigma(\boldsymbol{u})^2}\Delta_i\Delta_j\sigma(\boldsymbol{u})\sigma(\boldsymbol{v}) \Bigg|_{\boldsymbol{v}=\boldsymbol{u}} \\
&= -\frac{1}{2\sigma(\boldsymbol{u})^2}\big[\sigma(\boldsymbol{v})\sigma_{ij}(\boldsymbol{u}) - \sigma_i(\boldsymbol{u})\sigma_j(\boldsymbol{v}) - \sigma_i(\boldsymbol{v})\sigma_j(\boldsymbol{u}) + \sigma(\boldsymbol{u})\sigma_{ij}(\boldsymbol{v})\big]\Big|_{\boldsymbol{v}=\boldsymbol{u}} \\
&= -\frac{1}{2\sigma(\boldsymbol{u})^2}\big[2\sigma(\boldsymbol{u})\sigma_{ij}(\boldsymbol{u}) - 2\sigma_i(\boldsymbol{u})\sigma_j(\boldsymbol{u})\big] = \frac{\sigma_i(\boldsymbol{u})\sigma_j(\boldsymbol{u}) - \sigma(\boldsymbol{u})\sigma_{ij}(\boldsymbol{u})}{\sigma(\boldsymbol{u})^2}
\end{aligned}$$

This is the same ratio of σ-derivatives as we obtained in equation (2.61) from the original definition of $\wp_{ij}(\boldsymbol{u})$. Hence the two definitions are indeed equivalent. We now extend this approach to define a new class of functions.

Definition 3.1.2. *Define the **n-index Q-functions**, for n even, associated to an (n,s)-curve with genus g by*

$$Q_{i_1,i_2,\ldots,i_n}(\boldsymbol{u}) = \frac{(-1)}{2\sigma(\boldsymbol{u})^2}\Delta_{i_1}\Delta_{i_2}\ldots\Delta_{i_n}\sigma(\boldsymbol{u})\sigma(\boldsymbol{v}) \Bigg|_{\boldsymbol{v}=\boldsymbol{u}} \tag{3.27}$$

where $i_1 \leq \cdots \leq i_n \in \{1,\ldots,g\}$.

Note that these functions are all independent of the constant c used in the definition of the σ-function. (See Appendix B for the proof.)

Lemma 3.1.3. *The n-index Q-functions are meromorphic functions with poles of order two when $\sigma(\boldsymbol{u}) = 0$.*

Proof. Applying the Hirota operator on $\sigma(\boldsymbol{u})\sigma(\boldsymbol{v})$ repeatedly yields a polynomial of σ-derivatives, all of which are entire functions. Hence a Q-function is an entire function divided by $\sigma(\boldsymbol{u})^2$ and so will have poles of order two only when $\sigma(\boldsymbol{u}) = 0$. ■

These functions are a generalisation of the Q-functions used by Baker in his work on the (2,5)-case. The 4-index Q-functions have been considered before when similar problems occurred in research on the trigonal curves. In these papers the 4-index Q-functions are just defined as *Q-functions* and are sufficient to complete the basis of fundamental Abelian functions. However, in the $(4,5)$-case we will find that it is necessary to use a 6-index Q-function and in higher genus cases it is inevitable that even higher index Q-functions would be necessary. Hence we have given this general definition.

Remark 3.1.4.

(i) From now on we may write just refer to *the Q-functions*, specifying if we are working with functions of a particular index.

(ii) The subscripts of the $\wp$-functions denote differentiation

$$\frac{\partial}{\partial u_{i_n+1}} \wp_{i_1,i_2,\dots,i_n} = \wp_{i_1,i_2,\dots,i_n,i_{n+1}},$$

but this is not the case for the Q-functions. Here the indices refer to which Hirota operators were used.

Lemma 3.1.5. *If Definition 3.1.2 were applied with n odd then the resulting function is identically zero.*

Proof. Let $f(\boldsymbol{u}, \boldsymbol{v})$ be a symmetric function.

$$f(\boldsymbol{u}, \boldsymbol{v}) = f(\boldsymbol{v}, \boldsymbol{u}).$$

Consider the Hirota operator applied to such a function.

$$\Delta_i f(\boldsymbol{u}, \boldsymbol{v}) = \frac{\partial}{\partial u_i} f(\boldsymbol{u}, \boldsymbol{v}) - \frac{\partial}{\partial v_i} f(\boldsymbol{u}, \boldsymbol{v}) = \frac{\partial}{\partial u_i} f(\boldsymbol{u}, \boldsymbol{v}) - \frac{\partial}{\partial v_i} f(\boldsymbol{v}, \boldsymbol{u}).$$

Let $\hat{f}$ be the derivative of f with respect to the ith component of its first variable. Then

$$\Delta_i f(\boldsymbol{u}, \boldsymbol{v}) = \hat{f}(\boldsymbol{u}, \boldsymbol{v}) - \hat{f}(\boldsymbol{v}, \boldsymbol{u}). \tag{3.28}$$

Now consider the Hirota operator applied a second time.

$$\Delta_j \Delta_i f(\boldsymbol{u}, \boldsymbol{v}) = \frac{\partial}{\partial u_j} \hat{f}(\boldsymbol{u}, \boldsymbol{v}) - \frac{\partial}{\partial u_j} \hat{f}(\boldsymbol{v}, \boldsymbol{u}) - \frac{\partial}{\partial v_j} \hat{f}(\boldsymbol{u}, \boldsymbol{v}) + \frac{\partial}{\partial v_j} f(\boldsymbol{v}, \boldsymbol{u}).$$

Let f_1 and f_2 be the derivative of $\hat{f}$ with respect to the jth component of its first variable and second variables respectively.

$$\Delta_j \Delta_i f(\boldsymbol{u}, \boldsymbol{v}) = f_1(\boldsymbol{u}, \boldsymbol{v}) - f_2(\boldsymbol{v}, \boldsymbol{u}) - f_2(\boldsymbol{u}, \boldsymbol{v}) + f_1(\boldsymbol{v}, \boldsymbol{u}). \tag{3.29}$$

Note that the right hand side is symmetric.

$$\Delta_j \Delta_i f(\boldsymbol{v}, \boldsymbol{u}) = f_1(\boldsymbol{v}, \boldsymbol{u}) - f_2(\boldsymbol{u}, \boldsymbol{v}) - f_2(\boldsymbol{v}, \boldsymbol{u}) + f_1(\boldsymbol{u}, \boldsymbol{v}) = \Delta_j \Delta_i f(\boldsymbol{u}, \boldsymbol{v}). \tag{3.30}$$

Now consider the effect on equation (3.28) of setting $\boldsymbol{v} = \boldsymbol{u}$.

$$\Delta_i f(\boldsymbol{u}, \boldsymbol{v}) \Big|_{\boldsymbol{v}=\boldsymbol{u}} = \hat{f}(\boldsymbol{u}, \boldsymbol{u}) - \hat{f}(\boldsymbol{u}, \boldsymbol{u}) = 0. \tag{3.31}$$

An n-index Q-function is constructed by applying n Hirota derivatives to $\sigma(\boldsymbol{u})\sigma(\boldsymbol{v})$ and then setting $\boldsymbol{v} = \boldsymbol{u}$. Since $\sigma(\boldsymbol{u})\sigma(\boldsymbol{v})$ is symmetric, we can conclude that a 1-index Q-function must be zero by equation (3.29).

By equation (3.30), the application of two Hirota operators on $\sigma(\boldsymbol{u})\sigma(\boldsymbol{v})$ will generate a symmetric function. Similarly any even application of Hirota operators on $\sigma(\boldsymbol{u})\sigma(\boldsymbol{v})$ would also generate a symmetric function. Finally, we can apply equation (3.31) to conclude that any n-index Q-function with n odd must be zero.

■

In [30] it was shown that the 4-index Q-functions could be expressed using the Kleinian $\wp$-functions as follows.

$$Q_{ijk\ell} = \wp_{ijk\ell} - 2\wp_{ij}\wp_{k\ell} - 2\wp_{ik}\wp_{j\ell} - 2\wp_{i\ell}\wp_{jk}. \tag{3.32}$$

This specialises to

$$\begin{aligned} Q_{ijkk} &= \wp_{ijkk} - 2\wp_{ij}\wp_{kk} - 4\wp_{ik}\wp_{jk}, & Q_{iijj} &= \wp_{iijj} - 2\wp_{ii}\wp_{jj} - 4\wp_{ij}^2, \\ Q_{ijjj} &= \wp_{ijjj} - 6\wp_{ij}\wp_{jj}, & Q_{iiii} &= \wp_{iiii} - 6\wp_{ii}^2. \end{aligned}$$

A similar result is presented here for the 6-index Q-functions.

Proposition 3.1.6. *The 6-index Q-functions may be written as*

$$\begin{aligned} Q_{ijklmn} = \wp_{ijklmn} &- 2\Big[\big(\wp_{ij}\wp_{klmn} + \wp_{ik}\wp_{jlmn} + \wp_{il}\wp_{jkmn} + \wp_{im}\wp_{jkln} + \wp_{in}\wp_{jklm}\big) \\ &+ \big(\wp_{jk}\wp_{ilmn} + \wp_{jl}\wp_{ikmn} + \wp_{jm}\wp_{ikln} + \wp_{jn}\wp_{iklm}\big) + \big(\wp_{kl}\wp_{ijmn} + \wp_{km}\wp_{ijln} + \wp_{kn}\wp_{ijlm}\big) \\ &+ \big(\wp_{lm}\wp_{ijkn} + \wp_{ln}\wp_{ijkm}\big) + \wp_{mn}\wp_{ijkl}\Big] + 4\Big[\big(\wp_{ij}\wp_{kl}\wp_{mn} + \wp_{ij}\wp_{km}\wp_{ln} + \wp_{ij}\wp_{kn}\wp_{lm}\big) \\ &+ \big(\wp_{ik}\wp_{jl}\wp_{mn} + \wp_{ik}\wp_{jm}\wp_{ln} + \wp_{ik}\wp_{jn}\wp_{lm}\big) + \big(\wp_{il}\wp_{jk}\wp_{mn} + \wp_{il}\wp_{jm}\wp_{kn} + \wp_{il}\wp_{jn}\wp_{km}\big) \\ &+ \big(\wp_{im}\wp_{jk}\wp_{ln} + \wp_{im}\wp_{jl}\wp_{kn} + \wp_{im}\wp_{jn}\wp_{kl}\big) + \big(\wp_{in}\wp_{jk}\wp_{lm} + \wp_{in}\wp_{jl}\wp_{km} + \wp_{in}\wp_{jm}\wp_{kl}\big)\Big]. \end{aligned}$$

Proof. Apply Definitions 2.2.27 and 3.1.2 to reduce the equation to a sum of rational functions involving σ-derivatives. We find that they all cancel (Maple is useful here). The actual structure of the sum was prompted by considering the existing result for the 4-index Q-functions.

■

Corollary 3.1.7. *The 4-index and 6-index Q-functions are even Abelian functions.*

Proof. Equations (3.32) and Proposition 3.1.6 demonstrate that these functions may be expressed as a sum of even Abelian functions, and hence they are themselves even Abelian functions.

■

The proof that all n-index Q-functions are even and Abelian will involve a lengthly induction. This document uses only 4 and 6-index Q-function so Corollary 3.1.7 is sufficient.

Note that Proposition 3.1.6 specialises to the following set of formulae.

$$
\begin{aligned}
Q_{ijklmm} &= \wp_{ijklmm} - 2\wp_{ij}\wp_{klmm} - 2\wp_{ik}\wp_{jlmm} - 2\wp_{il}\wp_{jkmm} - 4\wp_{im}\wp_{jklm} - 2\wp_{jk}\wp_{ilmm} \\
&\quad - 2\wp_{jl}\wp_{ikmm} - 4\wp_{jm}\wp_{iklm} - 2\wp_{kl}\wp_{ijmm} - 4\wp_{km}\wp_{ijlm} - 4\wp_{lm}\wp_{ijkm} \\
&\quad - 2\wp_{mm}\wp_{ijkl} + 8\wp_{ik}\wp_{jm}\wp_{lm} + 8\wp_{im}\wp_{jl}\wp_{km} + 8\wp_{ij}\wp_{km}\wp_{lm} + 8\wp_{il}\wp_{jm}\wp_{km} \\
&\quad + 4\wp_{ij}\wp_{kl}\wp_{mm} + 4\wp_{ik}\wp_{jl}\wp_{mm} + 4\wp_{il}\wp_{jk}\wp_{mm} + 8\wp_{im}\wp_{jm}\wp_{kl} + 8\wp_{im}\wp_{jk}\wp_{lm} \\
Q_{ijklll} &= \wp_{ijklll} - 2\wp_{ij}\wp_{klll} - 2\wp_{ik}\wp_{jlll} - 6\wp_{il}\wp_{jkll} - 2\wp_{jk}\wp_{illl} - 6\wp_{jl}\wp_{ikll} - 6\wp_{kl}\wp_{ijll} \\
&\quad - 6\wp_{ll}\wp_{ijkl} + 12\wp_{ij}\wp_{kn}\wp_{ll} + 12\wp_{ik}\wp_{jl}\wp_{ll} + 12\wp_{il}\wp_{jk}\wp_{ll} + 24\wp_{il}\wp_{jl}\wp_{kl} \\
Q_{ijkkll} &= \wp_{ijkkll} + 16\wp_{ik}\wp_{jl}\wp_{kl} + 8\wp_{il}\wp_{jl}\wp_{kk} - 4\wp_{jl}\wp_{ikkl} + 16\wp_{il}\wp_{jk}\wp_{kl} + 8\wp_{ij}\wp_{kl}^2 \\
&\quad - 2\wp_{ij}\wp_{kkll} - 4\wp_{jk}\wp_{ikll} + 4\wp_{ij}\wp_{kk}\wp_{ll} + 8\wp_{ik}\wp_{jk}\wp_{ll} - 2\wp_{kk}\wp_{ijll} - 4\wp_{ik}\wp_{jkll} \\
&\quad - 8\wp_{kl}\wp_{ijkl} - 2\wp_{ll}\wp_{ijkk} - 4\wp_{il}\wp_{jkkl} \\
Q_{ijkkkk} &= \wp_{ijkkkk} - 2\wp_{ij}\wp_{kkkk} - 8\wp_{ik}\wp_{jkkk} - 8\wp_{jk}\wp_{ikkk} + 48\wp_{ik}\wp_{jk}\wp_{kk} \\
&\quad + 12\wp_{ij}\wp_{kk}^2 - 12\wp_{kk}\wp_{ijkk} \\
Q_{ijjkkk} &= \wp_{ijjkkk} + 12\wp_{ik}\wp_{jj}\wp_{kk} + 24\wp_{ik}\wp_{jk}^2 + 24\wp_{ij}\wp_{jk}\wp_{kk} - 4\wp_{ij}\wp_{jkkk} - 6\wp_{kk}\wp_{ijjk} \\
&\quad - 6\wp_{ik}\wp_{jjkk} - 2\wp_{jj}\wp_{ikkk} - 12\wp_{jk}\wp_{ijkk} \\
Q_{iijjkk} &= \wp_{iijjkk} + 4\wp_{ii}\wp_{jj}\wp_{kk} + 32\wp_{ik}\wp_{ij}\wp_{jk} + 8\wp_{ij}^2\wp_{kk} - 2\wp_{ii}\wp_{jjkk} - 2\wp_{jj}\wp_{iikk} \\
&\quad + 8\wp_{ik}^2\wp_{jj} + 8\wp_{ii}\wp_{jk}^2 - 8\wp_{jk}\wp_{iijk} - 2\wp_{kk}\wp_{iijj} - 8\wp_{ij}\wp_{ijkk} - 8\wp_{ik}\wp_{ijjk} \\
Q_{ijjjjj} &= \wp_{ijjjjj} - 10\wp_{ij}\wp_{jjjj} + 60\wp_{ij}\wp_{jj}^2 - 20\wp_{jj}\wp_{ijjj} \\
Q_{iijjjj} &= \wp_{iijjjj} - 2\wp_{ii}\wp_{jjjj} - 12\wp_{jj}\wp_{iijj} - 16\wp_{ij}\wp_{ijjj} + 48\wp_{ij}^2\wp_{jj} + 12\wp_{ii}\wp_{jj}^2 \\
Q_{iiijjj} &= \wp_{iiijjj} - 6\wp_{jj}\wp_{iiij} + 36\wp_{ij}\wp_{ii}\wp_{jj} - 6\wp_{ii}\wp_{ijjj} - 18\wp_{ij}\wp_{iijj} + 24\wp_{ij}^3 \\
Q_{iiiiii} &= \wp_{iiiiii} - 30\wp_{ii}\wp_{iiii} + 60\wp_{ii}^3
\end{aligned}
$$

3.2 Expanding the Kleinian formula

This section is based upon the following Theorem, originally by Klein and given for a general (n,s)-curve as Theorem 3.4 in [35]. The result is derived from the Riemann Vanishing Theorem for the θ-function, in which the σ-function and hence the $\wp$-function can be expressed. In this section we will use the theorem applied to the (4,5)-case to solve the Jacobi Inversion Problem, as well as generate relations between the $\wp$-functions.

Theorem 3.2.1. *Let $\{P_1,\dots,P_6\}$ be an arbitrary set of distinct points on C, and (z,w) any point of this set. Then for an arbitrary point (x,y) and the base point ∞ on C we have*

$$\sum_{i,j=1}^{6} \wp_{ij}\left(\int_{\infty}^{(x,y)} \boldsymbol{du} - \sum_{k=1}^{6}\int_{\infty}^{P_k} \boldsymbol{du}\right) g_i(x,y)g_j(z,w) = \frac{F\big((x,y),(z,w)\big)}{(x-z)^2}. \tag{3.33}$$

Here g_i is the numerator of du_i from the standard basis of holomorphic differentials and $F((x,y),(z,w))$ the symmetric function that appeared in Klein's realisation of the fundamental differential. These have already been derived for the cyclic (4,5)-case with the g_i presented in equation (3.4) the symmetric function given by equation (3.18).

Equation (3.33) is often referred to as *the Kleinian formula*. We now describe how equations between the $\wp$-functions can be derived from this theorem. We will describe the steps taken, although it is recommended the calculations be performed in a computer algebra package such as Maple. Define

$$\hat{\boldsymbol{u}} = \int_{\infty}^{(x,y)} \boldsymbol{du} - \sum_{k=1}^{6}\int_{\infty}^{P_k} \boldsymbol{du}$$

and consider the effect on $\hat{\boldsymbol{u}}$ as the point (x,y) tends to infinity. We may describe this by writing $\hat{\boldsymbol{u}} = \boldsymbol{u} + \boldsymbol{u}_\xi$ where $\boldsymbol{u} \in \mathbb{C}^6$ and $\hat{\boldsymbol{u}}$ is a vector containing the series expansions in ξ of the variables. (These expansions were derived in equation (3.25) with longer expansions given in Appendix C.1.)

Taking the Taylor series series expansion in ξ we see that

$$\begin{aligned}
\wp_{ij}(\hat{\boldsymbol{u}}) = \Big[&\wp_{ij} - \wp_{6ij}\xi + \left(\tfrac{1}{2}\wp_{66ij} - \tfrac{1}{2}\wp_{5ij}\right)\xi^2 - \left(\tfrac{1}{6}\wp_{666ij} + \tfrac{1}{3}\wp_{4ij} - \tfrac{1}{2}\wp_{56ij}\right)\xi^3 \\
&+ \left(\tfrac{1}{24}\wp_{6666ij} + \tfrac{1}{8}\wp_{55ij} - \tfrac{1}{4}\wp_{566ij} + \tfrac{1}{3}\wp_{46ij}\right)\xi^4 - \Big(\tfrac{1}{120}\wp_{66666ij} - \tfrac{1}{20}\wp_{6ij}\lambda_4 \\
&+ \tfrac{1}{8}\wp_{556ij} - \tfrac{1}{12}\wp_{5666ij} - \tfrac{1}{6}\wp_{45ij} + \tfrac{1}{6}\wp_{466ij}\Big)\xi^5 - \Big(\tfrac{1}{720}\wp_{666666ij} + \tfrac{1}{48}\wp_{555ij} \\
&+ \tfrac{1}{18}\wp_{44ij} + \tfrac{1}{12}\wp_{5ij}\lambda_4 - \tfrac{1}{20}\wp_{66ij}\lambda_4 - \tfrac{1}{6}\wp_{456ij} - \tfrac{1}{48}\wp_{56666ij} + \tfrac{1}{18}\wp_{4666ij} - \tfrac{1}{6}\wp_{3ij} \\
&+ \tfrac{1}{16}\wp_{5566ij}\Big)\xi^6 - \Big(\tfrac{1}{5040}\wp_{6666666ij} + \tfrac{1}{7}\wp_{2ij} - \tfrac{3}{28}\wp_{4ij}\lambda_4 - \tfrac{1}{40}\wp_{666ij}\lambda_4 + \tfrac{1}{18}\wp_{446ij} \\
&+ \tfrac{13}{120}\lambda_4\wp_{56ij} + \tfrac{1}{48}\wp_{55666ij} + \tfrac{1}{72}\wp_{46666ij} - \tfrac{1}{6}\wp_{36ij} - \tfrac{1}{240}\wp_{566666ij} - \tfrac{1}{12}\wp_{4566ij} + \dots
\end{aligned}$$

$$
\begin{aligned}
&\cdots + \tfrac{1}{24}\wp_{455ij} - \tfrac{1}{48}\wp_{5556ij}\big)\xi^7 + \big(\tfrac{1}{40320}\wp_{66666666ij} + \tfrac{1}{192}\wp_{556666ij} + \tfrac{1}{384}\wp_{5555ij} \\
&+ \tfrac{1}{12}\wp_{35ij} + \tfrac{1}{7}\wp_{26ij} - \tfrac{1}{96}\wp_{55566ij} + \tfrac{1}{360}\wp_{466666ij} - \tfrac{1}{120}\wp_{6666ij}\lambda_4 + \tfrac{1}{15}\wp_{566ij}\lambda_4 \\
&- \tfrac{1}{36}\wp_{445ij} - \tfrac{13}{105}\wp_{46ij}\lambda_4 - \tfrac{1}{24}\wp_{55ij}\lambda_4 - \tfrac{1}{1440}\wp_{5666666ij} - \tfrac{1}{12}\wp_{366ij} - \tfrac{1}{36}\wp_{45666ij} \\
&+ \tfrac{1}{36}\wp_{4466ij} + \tfrac{1}{24}\wp_{4556ij}\big)\xi^8 + \big(- \tfrac{1}{362880}\wp_{666666666ij} - \tfrac{1}{384}\wp_{55556ij} - \tfrac{1}{12}\wp_{356ij} \\
&+ \tfrac{1}{10080}\wp_{56666666ij} + \tfrac{1}{144}\wp_{456666ij} - \tfrac{1}{48}\wp_{45566ij} - \tfrac{1}{108}\wp_{44666ij} + \tfrac{1}{144}\wp_{4555ij} \\
&+ \tfrac{1}{36}\wp_{6ij}\lambda_3 - \tfrac{5}{288}\wp_{6ij}{\lambda_4}^2 + \tfrac{1}{18}\wp_{34ij} + \tfrac{1}{288}\wp_{555666ij} - \tfrac{1}{2160}\wp_{4666666ij} + \tfrac{1}{36}\wp_{3666ij} \\
&- \tfrac{1}{960}\wp_{5566666ij} + \tfrac{1}{36}\wp_{4456ij} + \tfrac{1}{14}\wp_{25ij} - \tfrac{1}{14}\wp_{266ij} - \tfrac{1}{162}\wp_{444ij} - \tfrac{41}{504}\wp_{45ij}\lambda_4 \\
&+ \tfrac{59}{840}\lambda_4\wp_{466ij} + \tfrac{23}{480}\wp_{556ij}\lambda_4 - \tfrac{19}{720}\lambda_4\wp_{5666ij} + \tfrac{1}{480}\wp_{66666ij}\lambda_4\big)\xi^9 + O(\xi^{10})\Big](\boldsymbol{u}).
\end{aligned}
$$

(Note that the final $(\boldsymbol{u})$ is notation to indicate that the right hand side is a function of $\boldsymbol{u} \in \mathbb{C}^6$). Now let us construct a series expansion in ξ for equation (3.33) as (x, y) tends to infinity.

Start by forming the sum on the left hand side using the expansion above and the polynomials g_i in equation (3.4). The sum runs for all $i, j \in (1, \dots, 6)$. Multiply this by $(x-z)^2$ and subtract the polynomial F given in equation (3.18). Substitute for the variables (x, y) using their expansions in ξ. Then multiply out to form a final series expansion in ξ that must equal zero.

The coefficients of the ξ will be polynomials in $\wp$-functions and the variables (z, w). It follows that each coefficient with respect to ξ must be zero for any $\boldsymbol{u}$ and some (z, w) on C. This gives us a potentially infinite sequence of equations, starting with the five equations given below.

$$
0 = \rho_1 = -z^3 + \wp_{46}z^2 + (\wp_{56}w + \wp_{26})\,z + \wp_{66}w^2 + \wp_{36}w + \wp_{16} \tag{3.34}
$$

$$
\begin{aligned}
0 = \rho_2 &= (\wp_{45} - \wp_{466} - 2w)\,z^2 + ((\wp_{55} - \wp_{566})\,w + \wp_{25} - \wp_{266})\,z \\
&+ (\wp_{56} - \wp_{666})\,w^2 + (\wp_{35} - \wp_{366})\,w + \wp_{15} - \wp_{166}
\end{aligned} \tag{3.35}
$$

$$
\begin{aligned}
0 = \rho_3 &= \big(\wp_{44} - \tfrac{3}{2}\wp_{456} + \tfrac{1}{2}\wp_{4666}\big)\,z^2 + \big(- 3w^2\,\big(\wp_{45} - \tfrac{3}{2}\wp_{556} + \tfrac{1}{2}\wp_{5666}\big)\,w - \tfrac{3}{2}\wp_{256} \\
&+ \wp_{24} + \tfrac{1}{2}\wp_{2666}\big)z + \big(\tfrac{1}{2}\wp_{6666} - \tfrac{3}{2}\wp_{566} + \wp_{46}\big)\,w^2 + \big(\tfrac{1}{2}\wp_{3666} + \wp_{34} - \tfrac{3}{2}\wp_{356}\big)w \\
&+ \wp_{14} + \tfrac{1}{2}\wp_{1666} - \tfrac{3}{2}\wp_{156} \\
0 = \rho_4 &= \big(\wp_{4566} - \tfrac{4}{3}\wp_{446} - \tfrac{1}{6}\wp_{46666} - \tfrac{1}{2}\wp_{455}\big)\,z^2 + \big(\big(\wp_{5566} - \tfrac{4}{3}\wp_{456} - \tfrac{1}{6}\wp_{56666} - \tfrac{1}{2}\wp_{555}\big)w \\
&- \tfrac{4}{3}\wp_{246} - \tfrac{1}{2}\wp_{255} + \wp_{2566} - \tfrac{1}{6}\wp_{26666}\big)z - 4w^3 + \big(\wp_{5666} - \tfrac{1}{2}\wp_{556} - \tfrac{1}{6}\wp_{66666} - \tfrac{4}{3}\wp_{466}\big)w^2 \\
&+ \big(- \tfrac{1}{6}\wp_{36666} + \wp_{3566} - \tfrac{4}{3}\wp_{346} - \tfrac{1}{2}\wp_{355}\big)w - \tfrac{1}{6}\wp_{16666} + \wp_{1566} - \tfrac{4}{3}\wp_{146} - \tfrac{1}{2}\wp_{155} \\
0 = \rho_5 &= -3z^4 - \big(2\wp_{46} + \tfrac{9}{2}\lambda_4\big)\,z^3 + \big(\tfrac{5}{8}\wp_{4556} - \tfrac{5}{6}\wp_{445} - \tfrac{5}{12}\wp_{45666} - 2\wp_{56}w - 2\wp_{26} - 3\lambda_3 \\
&+ \tfrac{5}{6}\wp_{4466} + \tfrac{1}{2}\wp_{46}\lambda_4 + \tfrac{1}{24}\wp_{466666}\big)z^2 + \big(- 2\wp_{66}w^2 + \big(\tfrac{1}{24}\wp_{566666} - \tfrac{5}{6}\wp_{455} + \tfrac{5}{6}\wp_{4566} \\
&- \tfrac{5}{12}\wp_{55666} + \tfrac{1}{2}\wp_{56}\lambda_4 - 2\wp_{36} + \tfrac{5}{8}\wp_{5556}\big)w - \tfrac{5}{12}\wp_{25666} - \tfrac{5}{6}\wp_{245} - 2\lambda_2 + \tfrac{5}{6}\wp_{2466} + \dots
\end{aligned}
$$

$$\cdots + \tfrac{5}{8}\wp_{2556} + \tfrac{1}{24}\wp_{266666} - 2\wp_{16} + \tfrac{1}{2}\wp_{26}\lambda_4\big)z + \big(\tfrac{1}{2}\wp_{66}\lambda_4 - \tfrac{5}{12}\wp_{56666} + \tfrac{1}{24}\wp_{666666}$$
$$- \tfrac{5}{6}\wp_{456} + \tfrac{5}{6}\wp_{4666} + \tfrac{5}{8}\wp_{5566}\big)w^2 + \big(\tfrac{1}{24}\wp_{366666} + \tfrac{5}{6}\wp_{3466} + \tfrac{5}{8}\wp_{3556} + \tfrac{1}{2}\wp_{36}\lambda_4 - \tfrac{5}{6}\wp_{345}$$
$$- \tfrac{5}{12}\wp_{35666}\big)w + \tfrac{5}{8}\wp_{1556} - \tfrac{5}{12}\wp_{15666} + \tfrac{1}{24}\wp_{166666} - \lambda_1 + \tfrac{5}{6}\wp_{1466} - \tfrac{5}{6}\wp_{145} + \tfrac{1}{2}\wp_{16}\lambda_4$$

These equations all contain $\wp$-functions and the variables z, w. They are valid for any $\boldsymbol{u} \in \mathbb{C}^6$, with (z, w) one of the points on C that are used in equation (3.5) to represent $\boldsymbol{u}$. We have labeled the polynomials ρ_i for later use. They are presented in ascending order (as the coefficients of ξ). The polynomials have been calculated explicitly up to ρ_{14} (using Maple). They get increasingly larger in size and can be found in the extra Appendix of files or online at [38]. The Maple worksheet in which they are derived may also be found in the extra Appendix of files.

3.2.1 Generating relations between the $\wp$-functions

We now describe how these equations may be manipulated in order to generate relations between the $\wp$-functions. We achieve this by eliminating variables between pairs of equations using the method of resultants. Appendix A.6 gives a definition of the resultant, details on its construction and a simple example to demonstrate its use. For more information we refer the reader to [43] Chapter 12. The calculations were performed in the computer algebra package Maple using the inbuilt `resultant` function.

The first step is to take pairs of the equations derived above and take resultants to eliminate the variable w. (Note that equivalently we could have started by eliminating the variable z.) Denote the resultant of polynomials ρ_i and ρ_j by $\rho_{i,j}$. Since ρ_i and ρ_j were equal to zero, so must the resultant that is obtained.

$$\rho_{i,j} = \text{Res}(\rho_i, \rho_j) = 0.$$

This gives us a set of new equations in z and the $\wp$-functions. These equations are quite large and so are not printed here. Instead we present Table 3.1 detailing the number of terms they contain once expanded and the degree in z.

Note that when examining combinations that involved the higher ρ_i a polynomial with degree in z that was less than seven was never found. In general, the higher the ρ_i involved in the resultant, the more terms the resultant would have.

The next step is to combine these to find polynomials of degree five in z. By the following theorem such polynomials would be zero. This would allow us to set each of the coefficients of the polynomial to zero, giving relations between $\wp$-functions.

Table 3.1: Details on the polynomials $\rho_{i,j}$

resultant	# terms	degree in z
$\rho_{1,2}$	317	7
$\rho_{1,3}$	744	8
$\rho_{1,4}$	1526	9
$\rho_{1,5}$	2080	8
$\rho_{2,3}$	2670	7
$\rho_{2,4}$	9479	8
$\rho_{2,5}$	13943	9
$\rho_{3,4}$	30756	7
$\rho_{3,5}$	31592	10
$\rho_{4,5}$	311250	12

Theorem 3.2.2. *Consider an equation in z with coefficients in the $\wp$-functions evaluated at an arbitrary point $\boldsymbol{u} \in \mathbb{C}^g$. If the equation has degree in z less than g then it must be identically zero.*

Proof. Such a polynomial would have at most $g-1$ independent roots, but must be satisfied for g arbitrary variables $\boldsymbol{u} \in \mathbb{C}^g$. Hence the polynomial must be identically zero. ∎

So we start by selecting one of the polynomials with degree seven in z and rearrange it to give an equation for z^7. The simplest approach is to use $\rho_{1,2}$ as it is the smallest such polynomial. The coefficient of z^7 in $\rho_{1,2}$ is $-4\wp_{66}$ so we may use this equation to write

$$z^7 = -\frac{1}{4\wp_{66}}\big(\text{degree six polynomial in } z \text{ with coefficients in } \wp\text{-functions}\big). \tag{3.36}$$

Then since $\rho_{1,3}$ has degree eight in z we can substitute equation (3.36) twice into $\rho_{1,3}$ to leave an equation of degree six in z. We label this equation (T1) for future use. (Note that equation (T1) can be found in the extra Appendix of files or online at [38].)

The next step is to rearrange (T1) to give an equation for z^6. Although (T1) is the simplest such equation, it still has the following non-trivial coefficient of z^6.

$$\begin{aligned}
\frac{1}{16\wp_{66}^2}\Big(&72\wp_{66}\wp_{55}\wp_{666}^2 - 36\wp_{66}\wp_{566}\wp_{666}^2 - 240\wp_{46}\wp_{55}\wp_{66}^2 + 48\wp_{46}\wp_{566}\wp_{66}^2 - 72\wp_{55}\wp_{66}^2\wp_{566} \\
&+ 72\wp_{66}^2\wp_{466}\wp_{666} + 24\wp_{66}\wp_{46}\wp_{56}^2 + 12\wp_{66}\wp_{6666}\wp_{56}^2 - 12\wp_{66}\wp_{6666}\wp_{666}^2 + 72\wp_{66}\wp_{36}\wp_{666} \\
&- 48\wp_{6666}\wp_{66}^2\wp_{55} + 96\wp_{66}^2\wp_{45}\wp_{56} - 36\wp_{66}\wp_{55}\wp_{56}^2 + 72\wp_{556}\wp_{56}\wp_{66}^2 + 64\wp_{46}\wp_{6666}\wp_{66}^2 \\
&+ 24\wp_{6666}\wp_{566}\wp_{66}^2 - 24\wp_{5666}\wp_{56}\wp_{66}^2 - 96\wp_{66}\wp_{666}^2\wp_{46} - 144\wp_{66}^2\wp_{466}\wp_{56} - 72\wp_{66}^2\wp_{45}\wp_{666} \\
&+ 9\wp_{56}^4 + 9\wp_{666}^4 + 72\wp_{66}\wp_{56}\wp_{46}\wp_{666} + 36\wp_{66}\wp_{566}\wp_{56}\wp_{666} - 36\wp_{66}\wp_{55}\wp_{56}\wp_{666} \\
&- 72\wp_{66}\wp_{36}\wp_{56} + 4\wp_{6666}^2\wp_{66}^2 + 144\wp_{66}^2\wp_{35} + 144\wp_{66}^3\wp_{456} - 18\wp_{666}^2\wp_{56}^2 - 96\wp_{66}^3\wp_{44} \\
&- 144\wp_{66}^2\wp_{366} + 108\wp_{55}^2\wp_{66}^2 - 48\wp_{66}^3\wp_{4666} + 112\wp_{46}^2\wp_{66}^2 - 144\wp_{26}\wp_{66}^2\Big).
\end{aligned} \tag{3.37}$$

We rearrange (T1) to obtain an equation of the following form for z^6.

$$z^6 = \frac{1}{(3.37)}\big(\text{degree five polynomial in } z \text{ with coefficients in } \wp\text{-functions}\big) \qquad (3.38)$$

We may now take any of the other $\rho_{i,j}$ and repeatedly substitute for z^7 and z^6 using equations (3.36) and (3.38) until we have a polynomial of degree five in z. By Theorem 3.2.2 the coefficients with respect to z of such an equation must be zero. Further, since they are zero we can take the numerator of the coefficients leaving us with six polynomial equations between the $\wp$-functions from each $\rho_{i,j}$.

There is one further simplification to be made by recalling Lemma 2.2.32 which stated that all the $\wp$-functions have definite parity. This means that we separate each of these six relations into their odd and even parts, each of which must independently be zero. We will denote the polynomial equations between $\wp$-functions that we generate using this method as follows.

Definition 3.2.3. *Reduce equation $\rho_{i,j}$ to degree five in z using equations (3.36) and (3.38). Then define*

$$\mathcal{K}(\rho_{i,j}, n, \pm)$$

to be the relation achieved by selecting the coefficient with respect to z^n, taking the numerator and selecting either the even or odd parts as indicated by $+$ or $-$ respectively.

The simplest of these relations came from the reductions of $\rho_{2,3}$ and $\rho_{1,5}$, however these relations are still very long and complex. We do not print them here but we do present Table 3.2 which indicates the number of terms in each.

Table 3.2: The polynomials $\mathcal{K}(\rho_{i,j}, n, \pm)$

equation	# terms	equation	# terms
$\mathcal{K}(\rho_{2,3}, 5, +)$	1907	$\mathcal{K}(\rho_{1,5}, 5, +)$	6631
$\mathcal{K}(\rho_{2,3}, 5, -)$	1788	$\mathcal{K}(\rho_{1,5}, 5, -)$	6145
$\mathcal{K}(\rho_{2,3}, 4, +)$	6595	$\mathcal{K}(\rho_{1,5}, 4, +)$	18062
$\mathcal{K}(\rho_{2,3}, 4, -)$	6405	$\mathcal{K}(\rho_{1,5}, 4, -)$	17105
$\mathcal{K}(\rho_{2,3}, 3, +)$	13113	$\mathcal{K}(\rho_{1,5}, 3, +)$	28442
$\mathcal{K}(\rho_{2,3}, 3, -)$	12835	$\mathcal{K}(\rho_{1,5}, 3, -)$	27226
$\mathcal{K}(\rho_{2,3}, 2, +)$	17152	$\mathcal{K}(\rho_{1,5}, 2, +)$	33892
$\mathcal{K}(\rho_{2,3}, 2, -)$	16820	$\mathcal{K}(\rho_{1,5}, 2, -)$	32424
$\mathcal{K}(\rho_{2,3}, 1, +)$	12387	$\mathcal{K}(\rho_{1,5}, 1, +)$	23614
$\mathcal{K}(\rho_{2,3}, 1, -)$	12192	$\mathcal{K}(\rho_{1,5}, 1, -)$	22723
$\mathcal{K}(\rho_{2,3}, 0, +)$	5214	$\mathcal{K}(\rho_{1,5}, 0, +)$	10334
$\mathcal{K}(\rho_{2,3}, 0, -)$	5070	$\mathcal{K}(\rho_{1,5}, 0, -)$	9765

So the simplest relation between the $\wp$-functions that may be derived this way has 1907 terms, with the others rising in size considerably. (Note that in general, as the indices of $\rho_{i,j}$ increase so do the size of the relations obtained.)

In the lower genus cases the equations ρ_i could be reduced to give expressions involving linear $\wp$-functions. These could be manipulated to derive relations of interest, such as those that generalise the elliptic differential equations. This was not possible here since the terms in the $\mathcal{K}(\rho_{i,j}, n, \pm)$ are multiples of several $\wp$-functions. However, many of the interesting relations have been derived using alternative methods in Section 3.5.

The reason for the extra complexity is not just the increased genus giving a greater number of $\wp$-functions. It is also due to the fact that two rounds of substitution were needed to achieve a polynomial in z of degree $g-1=5$, while in all the lower genus cases only one round of substitution was necessary. Further, in the lower genus cases the coefficient of z^g in the equation that was used for the substitution was a constant, not a $\wp$-function. Hence it was possible to end up with linear terms in the $\wp$-functions within the end equations.

While these equations do not seem to be of interest themselves, they are an essential component in the construction of the σ-function expansion in Section 3.4. Further, they allow us to give an explicit solution to the Jacobi Inversion Problem.

3.2.2 Solving the Jacobi Inversion Problem

Recall that the Jacobi Inversion Problem is, given a point $\boldsymbol{u} \in J$, to find the preimage of this point under the Abel map, given in equation (3.20) for the (4,5)-curve.

Theorem 3.2.4. *Suppose we are given $\{u_1, \ldots, u_6\} = \boldsymbol{u} \in J$. Then we could solve the Jacobi Inversion Problem explicitly using the equations derived from the expansion of the Kleinian formula, (3.33).*

Proof. Consider equation (T1) defined in the discussion above. This is a polynomial equation constructed from $\wp$-functions and the variable z. The equation has degree six in z so denote by $(z_1, \ldots, z_6)$ the six zeros of the polynomial. Next, rearrange equation (3.34) to give an equation for w^2. Substitute this into equation (3.35) and multiply all terms by $\wp_{66}$ to give the following equation of degree one with respect to w.

$$\begin{aligned} 0 = w\big(&- 2z^2\wp_{66} + z(\wp_{66}\wp_{55} - \wp_{66}\wp_{566} + \wp_{666}\wp_{56} - \wp_{56}^2) + \wp_{36}\wp_{666} + \wp_{66}\wp_{35} - \wp_{66}\wp_{366} \\ &- \wp_{36}\wp_{56}\big) + z^3(\wp_{56} - \wp_{666}) + z^2(\wp_{66}\wp_{45} - \wp_{66}\wp_{466} + \wp_{666}\wp_{46} - \wp_{56}\wp_{46}) + z(\wp_{66}\wp_{25} \\ &- \wp_{66}\wp_{266} - \wp_{56}\wp_{26} + \wp_{666}\wp_{26}) + \wp_{15}\wp_{66} - \wp_{166}\wp_{66} + \wp_{666}\wp_{16} - \wp_{56}\wp_{16}. \end{aligned} \tag{3.39}$$

Substitute each z_i into (equation 3.39) in turn and solve to find the corresponding w_i. Therefore the set of points $\{(z_1, w_1), \ldots, (z_6, w_6)\}$ on the curve C which are the Abel preimage of $\boldsymbol{u}$ have been identified as expressions in $\wp$-functions.

■

3.3 The Sato weights

For any (n, s)-curve we can define a set of weights for all the objects in the theory such that all equations are homogeneous with respect to these weights. These are known as Sato weights, although we will often just write *weight* in this document. The weights have played an important role in the research on lower genus cases and are used regularly in the following sections.

In this section we will define and derive the weights associated to the theory of an arbitrary (n, s)-curve, giving explicit examples for the cyclic (4,5)-curve. Such a justification for the weights in the general case has not been presented in the literature, but follows logically from the general definitions.

We start by defining what we mean by weights.

Definition 3.3.1. *Define the* ***weight*** *of an object χ as an integer α_χ such that under the mapping*

$$\chi \mapsto t^{\alpha_\chi}\tilde{\chi} \tag{3.40}$$

all equations are homogeneous in the variable t. (An equation is ***homogeneous*** *in t if each term has the same degree in t.)*

From this definition it is clear that the following statements must hold.

$$\begin{aligned} \text{weight}\big[a \cdot b\big] &= \text{weight}(a) + \text{weight}(b), \\ \text{weight}\left[\frac{a}{b}\right] &= \text{weight}(a) - \text{weight}(b), \\ \text{weight}\big[a^n\big] &= n \cdot \text{weight}(a) \end{aligned}$$

Given this definition let us derive weights for the variables and parameters of a cyclic (n, s)-curve such that the curve equation is homogeneous. Start by applying the mapping (3.40) to the curve equation (2.34).

$$\begin{aligned} y^n &= x^s + \lambda_{s-1}x^{s-1} + \cdots + \lambda_1 x + \lambda_0 \\ \big(t^{\alpha_y}\tilde{y}\big)^n &= \big(t^{\alpha_x}\tilde{x}\big)^s + \big(t^{\alpha_{\lambda_{s-1}}}\tilde{\lambda}_{s-1}\big)\big(t^{\alpha_x}\tilde{x}\big)^{s-1} + \cdots + \big(t^{\alpha_{\lambda_1}}\tilde{\lambda}_1\big)\big(t^{\alpha_x}\tilde{x}\big) + \big(t^{\alpha_{\lambda_0}}\tilde{\lambda}_0\big) \\ t^{n\alpha_y}\tilde{y}^n &= t^{s\alpha_x}\tilde{x}^s + t^{\alpha_{\lambda_{s-1}}+(s-1)\alpha_x}\tilde{\lambda}_{s-1}\tilde{x}^{s-1} + \cdots + t^{\alpha_{\lambda_1}+\alpha_x}\tilde{\lambda}_1\tilde{x} + t^{\alpha_{\lambda_0}}\tilde{\lambda}_0. \end{aligned}$$

We need this to be homogeneous with respect to t and hence the weights of x and y are determined up to a constant by

$$n\alpha_y = s\alpha_x.$$

To keep with the convention in the literature we choose these weights so that they are the largest negative integers satisfying the condition. That is

$$\alpha_x = -n, \qquad \alpha_y = -s.$$

The weights of the curve constants are then determined uniquely by ensuring that the other terms in the equation are also homogeneous. We define this choice as the Sato weights.

Definition 3.3.2. *Define the* ***Sato Weights*** *for the curve variables and constants in the equation of a cyclic* (n, s) *curve to be*

$$\begin{aligned} \text{weight}[x] &= -n, & \text{weight}[\lambda_{s-1}] &= -n, \\ \text{weight}[y] &= -s, & \text{weight}[\lambda_{s-2}] &= -2n, \\ & & &\vdots \\ & & \text{weight}[\lambda_1] &= -n(s-1), \\ & & \text{weight}[\lambda_0] &= -ns. \end{aligned} \tag{3.41}$$

Define the Sato weight of all other constants to be zero and define the Sato weights of other objects to be those weights that follow logically from these.

If we had instead used equation (2.32) for a general (n, s) curve then we would have still had the same condition on the weights of x and y. Making the same choice would lead us to define the weights of the curve constant μ_i to be the negative of their subscripts. The subscripts were chosen in Definition 2.2.1 as a label to satisfy this property.

Example 3.3.3. In the cyclic (4,5)-case we have

	x	y	λ_4	λ_3	λ_2	λ_1	λ_0
Weight	-4	-5	-4	-8	-12	-16	-20

□

The next step is to derive the weights of the variables $\boldsymbol{u}$. They can be determined uniquely given Definition 3.3.2. Recall the definition of the local parameter for (x, y) as infinity.

$$\xi = x^{-\frac{1}{n}}.$$

Applying the mapping (3.40) to this equation,

$$t^{\alpha_\xi}\tilde{\xi} = \left(t^{\alpha_x}\tilde{x}\right)^{-\frac{1}{n}} = t^{-\frac{1}{n}(-n)}\tilde{x} = t\tilde{x}.$$

Hence the Sato weight of the parameter ξ is one for any (n, s)-curve.

Recall that series expansions in ξ can be derived for the basis of holomorphic differentials and hence the variables $\boldsymbol{u}$.

In the (4,5)-case these were given in equations (3.25) as

$$\begin{aligned} u_1 &= -\tfrac{1}{11}\xi^{11} + O(\xi^{15}), & u_3 &= -\tfrac{1}{6}\xi^{6} + O(\xi^{10}), & u_5 &= -\tfrac{1}{2}\xi^{2} + O(\xi^{6}), \\ u_2 &= -\tfrac{1}{7}\xi^{7} + O(\xi^{11}), & u_4 &= -\tfrac{1}{3}\xi^{3} + O(\xi^{7}), & u_6 &= -\xi + O(\xi^{5}). \end{aligned}$$

Applying the mapping (3.40) and considering the first term of the expansions would lead us to conclude that the weights of $\boldsymbol{u}$ were given by the leading powers of ξ in the expansions.

Note that the expansions rise in powers of ξ by $n = 4$, and that the coefficients of the higher order terms depend on the curve parameters. The weights of the curve parameters in each coefficient decrease by $n = 4$ ensuring that the expansion remains homogeneous and the $\boldsymbol{u}$ have definite Sato weight. (See the larger expansions for the (4,5)-curve given in Appendix C.1 to check this is the case.)

Theorem 3.3.4. *The variables $\boldsymbol{u}$ associated to an (n,s)-curve have definite weight that may be derived from the series expansions in ξ.*

Further these weights may be predicted as follows using the Weierstrass gap sequence $W_{n,s} = \{w_1, \dots, w_g\}$ defined in Definition 2.2.3.

$$\begin{aligned} \text{weight}[u_1] &= w_g = 2g-1 \\ \text{weight}[u_2] &= w_{g-1} \\ &\vdots \\ \text{weight}[u_{g-1}] &= w_2 && (3.42) \\ \text{weight}[u_g] &= w_1 = 1 && (3.43) \end{aligned}$$

Sketch of Proof. Since the weight of ξ is always one it is simple to choose weights of $\boldsymbol{u}$ to match the first term in the series expansions. The expansions in ξ will always increase in steps of n, with the coefficients in the curve parameters increasing in weight by n. This is because the infinite expansions came only from the variable y which was in each differential of the basis. Hence the variables $\boldsymbol{u}$ associated to an (n,s)-curve always have definite weight.

The dependency on the Weierstrass gap sequence is due to the fact that the standard basis of holomorphic differentials that we use was constructed from $W_{n,s}$. (See Proposition 2.2.4).

■

Example 3.3.5. In the (4,5)-case we assign the following weights to $\boldsymbol{u}$.

	u_1	u_2	u_3	u_4	u_5	u_6
Weight	+11	+7	+6	+3	+2	+1

□

Remark 3.3.6.

(i) The weights of the variables coincide with the order of their zero at ∞.

(ii) The weights of the entries of the standard standard basis of holomorphic differentials are one less than the corresponding variables.

$$\begin{aligned}
\text{weight}[du_1] &= w_g - 1 = 2g - 2 \\
\text{weight}[du_2] &= w_{g-1} - 1 \\
&\vdots \\
\text{weight}[du_g] &= w_2 - 1 \\
\text{weight}[du_g] &= w_1 - 1 = 0
\end{aligned} \tag{3.44}$$

This can be proved in an identical way using the series expansions for the differentials in ξ. In the (4,5)-case these were given by equations (3.24).

(iii) The weights of the basis of differentials of the second kind, $\boldsymbol{dr}$ may be established similarly.

Consider the definition of the σ-function as an infinite sum. Given that the variables and differentials have definite weight we can conclude that this is an infinite sum in which each term has the same weight. Hence the σ-function will have definite weight, which may be predicted using the following theorem.

Corollary 3.3.7. *The Sato weight of the σ-function is given by*

$$\text{weight}[\sigma(\boldsymbol{u})] = \frac{1}{24}(n^2-1)(s^2-1).$$

Proof. Recall Theorem 2.2.21 that established the first part of the series expansion of the σ-function about the origin. From equation (2.59) the σ-function will have the same weight as the corresponding Schur-Weierstrass polynomial. Next recall Remark 2.2.22(ii), that $SW_{n,s}$ will have, with respect to the variable u_g, a leading term given by a constant multiplied by

$$u_g^{\frac{1}{24}(n^2-1)(s^2-1)}.$$

Since the variable u_g always has weight one we can conclude that $SW_{n,s}$ and hence $\sigma(\boldsymbol{u})$ has the weight described.

■

Example 3.3.8. The σ-function associated to the (4,5)-curve has weight

$$\text{weight}\big[\sigma(\boldsymbol{u})\big] = \frac{1}{24}(4^2-1)(5^2-1) = 15.$$

□

The next task is to derive the weights of the Abelian functions we use. We start by establishing the weight of the σ-derivatives.

Lemma 3.3.9. *The weight of the σ-derivatives may be calculated from the weight of the σ-function as*

$$\text{weight}\big[\sigma_{i_1,i_2,\ldots,i_n}(\boldsymbol{u})\big] = \text{weight}[\sigma(\boldsymbol{u})] - \left(\sum_{k=1,\ldots,n} \text{weight}[u_{i_k}]\right). \tag{3.45}$$

Proof. We know that $\sigma(\boldsymbol{u})$ can be expressed as a Taylor series in $\boldsymbol{u}$. Since $\sigma(\boldsymbol{u})$ has definite weight, each term in that expansions must have the same weight. Consider just one of the terms, and denote the power of u_i in this term by a. Denote the remainder of the term by T.

$$\sigma(\boldsymbol{u}) = \cdots + u_i^a \cdot T(u_1, u_2, \ldots, u_{i-1}, u_{i+1}, \ldots, u_g) + \cdots$$

Then

$$\sigma_i(\mathbf{u}) = \cdots + a u_i^{a-1} \cdot T(u_1, u_2, \ldots, u_{i-1}, u_{i+1}, \ldots, u_g) + \cdots$$

So the weight of this term will have decreased by the weight of u_i. This will be true for all terms in the series and hence $\sigma_i(\boldsymbol{u})$ $\sigma_i(\boldsymbol{u})$ has definite weight given by the weight of $\sigma(\boldsymbol{u})$ minus the weight of u_i.

An identical argument may be applied to a k-index σ-derivative in order to establish the weight of a $(k+1)$-index σ-derivative. Hence the result may be established by induction.

■

Lemma 3.3.10. *The 2-index $\wp$-functions have a definite weight that is independent of the weight of $\sigma(\boldsymbol{u})$. It is given by*

$$\text{weight}\big[\wp_{ij}(\boldsymbol{u})\big] = -\big(\text{weight}[u_i] + \text{weight}[u_j]\big). \tag{3.46}$$

Proof. Recall equation (2.61) expressing the 2-index $\wp$-function as a ratio of σ-derivatives.

$$\wp_{ij}(\boldsymbol{u}) = \frac{\sigma(\boldsymbol{u})\sigma_{i_1,i_2}(\boldsymbol{u}) - \sigma_{i_1}(\boldsymbol{u})\sigma_{i_2}(\boldsymbol{u})}{\sigma(\boldsymbol{u})^2}.$$

Let α_χ be the weight of χ. We calculate the weights of the individual parts with the aid of Lemma 3.3.9.

$$\begin{aligned}
\text{weight}\big[\sigma(\boldsymbol{u})\sigma_{i_1,i_2}(\boldsymbol{u})\big] &= \text{weight}[\sigma(\boldsymbol{u})] + \text{weight}[\sigma_{i_1,i_2}(\boldsymbol{u})] \\
&= [\alpha_\sigma] + [\alpha_\sigma - \alpha_{u_{i_1}} - \alpha_{u_{i_2}}] = 2\alpha_\sigma - \alpha_{u_{i_1}} - \alpha_{u_{i_2}}, \\
\text{weight}\big[\sigma_{i_1}(\boldsymbol{u})\sigma_{i_2}(\boldsymbol{u})\big] &= 2\alpha_\sigma - \alpha_{u_{i_1}} - \alpha_{u_{i_2}}, \\
\text{weight}\big[\sigma(\boldsymbol{u})^2\big] &= 2\alpha_\sigma.
\end{aligned}$$

Now, perform the change of variables in Definition 3.3.1 on $\wp_{ij}$ to find that

$$\begin{aligned}\wp_{ij}(\boldsymbol{u})\Big|_{u_i=t^{\alpha_{u_i}}} &= \frac{t^{2\alpha_\sigma-\alpha_{u_{i_1}}-\alpha_{u_{i_2}}}}{t^{2\alpha_\sigma}}\frac{\tilde{\sigma}(\boldsymbol{u})\tilde{\sigma}_{i_1,i_2}(\boldsymbol{u})-\tilde{\sigma_{i_1}}(\boldsymbol{u})\tilde{\sigma}_{i_2}(\boldsymbol{u})}{\tilde{\sigma}(\boldsymbol{u})^2} \\ &= t^{-\alpha_{u_1}-\alpha_{u_2}}\tilde{\wp}_{ij}(\boldsymbol{u}).\end{aligned}$$

So after the change of variables we find that we can express $\wp_{ij}(\boldsymbol{u})$ as a power of t multiplied by a function not varying with t. Hence t has definite weight and we can see that weight is as given in the lemma. Note that the weight of the σ-function canceled in the calculation and does not have any influence of the weight of $\wp_{ij}(\boldsymbol{u})$.

■

Example 3.3.11. Given Example 3.3.5 specifying the weights of $\boldsymbol{u}$ in the (4,5)-case, we can conclude the weights of the associated $\wp_{ij}$. For example,

$$\text{wt}\big[\wp_{1,2}(\boldsymbol{u})\big] = -\text{wt}[u_1] - \text{wt}[u_2] = -(11) - (7) = -18.$$

We can calculate the weights of the other functions similarly.

$$\begin{array}{lll}
\text{weight}[\wp_{11}] = -22 & \text{weight}[\wp_{23}] = -13 & \text{weight}[\wp_{36}] = -7 \\
\text{weight}[\wp_{12}] = -18 & \text{weight}[\wp_{24}] = -10 & \text{weight}[\wp_{44}] = -6 \\
\text{weight}[\wp_{13}] = -17 & \text{weight}[\wp_{25}] = -9 & \text{weight}[\wp_{45}] = -5 \\
\text{weight}[\wp_{14}] = -14 & \text{weight}[\wp_{26}] = -8 & \text{weight}[\wp_{46}] = -4 \\
\text{weight}[\wp_{15}] = -13 & \text{weight}[\wp_{33}] = -12 & \text{weight}[\wp_{55}] = -4 \\
\text{weight}[\wp_{16}] = -12 & \text{weight}[\wp_{34}] = -9 & \text{weight}[\wp_{56}] = -3 \\
\text{weight}[\wp_{22}] = -14 & \text{weight}[\wp_{35}] = -8 & \text{weight}[\wp_{66}] = -2
\end{array}$$

□

Lemma 3.3.12. *The n-index $\wp$-functions have weight given by*

$$\text{weight}\big[\wp_{i_1,i_2,\dots,i_n}(\boldsymbol{u})\big] = -\sum_{k=1,\dots,n}\text{weight}[u_k]. \tag{3.47}$$

Proof. We already know the weight of the 2-index $\wp$-functions from Lemma 3.3.10. We can hence use a similar proof to that of Lemma 3.3.9 to conclude that taking derivatives reduces the weight by that of the variables that the differentiation is with respect to.

■

Example 3.3.13. In the (4,5)-case the highest weight 4-index $\wp$-function will be $\wp_{6666}$ with weight -4, while the lowest weight one will be $\wp_{1111}$ with weight -44.

□

Finally we derive the weight of the Q-functions.

Lemma 3.3.14. *The n-index Q-functions have weight*

$$\text{weight}\big[Q_{i_1,i_2,\ldots,i_n}(\boldsymbol{u})\big] = -\sum_{k=1,\ldots,n} \text{weight}[u_{i_k}]. \tag{3.48}$$

Proof. The lemma states that the weight of a Q-function is the same as the weight of a $\wp$-function with the same indices. The must clearly be the case for the 4-index and 6-index Q-functions by applying the mapping to equation (3.32) and Proposition (3.1.6).

To see the statement generally consider repeatedly applying the Hirota operator to $\sigma(\boldsymbol{u})\sigma(\boldsymbol{v})$.

$$\begin{aligned}
\Delta_j\big[\sigma(\boldsymbol{u})\sigma(\boldsymbol{v})\big] &= \tfrac{\partial}{\partial u_j}\left[\sigma(\boldsymbol{u})\sigma(\boldsymbol{v})\right] - \tfrac{\partial}{\partial v_j}\left[\sigma(\boldsymbol{u})\sigma(\boldsymbol{v})\right] \\
&= \sigma(\boldsymbol{v})\sigma_j(\boldsymbol{u}) - \sigma(\boldsymbol{u})\sigma_j(\boldsymbol{v}), \\
\Delta_i\Delta_j\big[\sigma(\boldsymbol{u})\sigma(\boldsymbol{v})\big] &= \sigma(\boldsymbol{v})\sigma_{ij}(\boldsymbol{u}) - \sigma_i(\boldsymbol{u})\sigma_j(\boldsymbol{v}) - \sigma_i(\boldsymbol{v})\sigma_j(\boldsymbol{u}) + \sigma(\boldsymbol{u})\sigma_{ij}(\boldsymbol{u}).
\end{aligned}$$

Each application of the Hirota operator results in a sum of terms, each the product of two σ-derivatives. Consider the indices present in each term as a whole. They will be the same indices as before the operator with one extra added (the index of the operator). Hence the weight of each term will be the weight before with the weight of the new index variable subtracted. Hence each application of the operator decreases the weight as expected. Setting $\boldsymbol{v} = \boldsymbol{u}$ does not change the weight and the division by $\sigma(\boldsymbol{u})^2$ removes the dependency on the weight of the σ-function.

■

So to summarise, the weights of the curve variables were determined up to a multiplicative constant. After choosing this constant in Definition 3.3.2 the weights of the curve constants were determined uniquely and the weights of the other constants were assumed to be zero. Unique weights for the differentials, variables, and σ-function also follow from this choice. The weights of the Abelian functions were derived from their definitions and shown to depend only on the weights of the variables indicated by their subscripts.

We can now conclude that any equation involving only objects with definite weight must also be homogeneous with respect to the Sato weights. This is a powerful tool that simplifies many calculations in the remainder of this document.

3.4 The σ-function expansion

In this section we establish some properties for the σ-function associated with the cyclic (4,5)-curve and then calculate the Taylor series expansion for $\sigma(\boldsymbol{u})$ about the origin. This expansion represents a significant computational challenge, but gives rise to a number of interesting results later in the document.

3.4.1 Properties of the σ-function

Theorem 3.4.1. *The Taylor series expansion of $\sigma(\boldsymbol{u})$ about the origin may be written as*

$$\sigma(\boldsymbol{u}) = K \cdot SW_{4,5}(\boldsymbol{u}) + \textit{ terms dependent on curve parameters}, \tag{3.49}$$

where K is a constant and $SW_{4,5}$ the Schur Weierstrass polynomial generated by $(4,5)$.

$$\begin{aligned} SW_{4,5} = {} & \tfrac{1}{8382528}u_6^{15} + \tfrac{1}{336}u_6^8u_5^2u_4 - \tfrac{1}{12}u_6^4u_1 - \tfrac{1}{126}u_6^7u_3u_5 - \tfrac{1}{6}u_4u_3u_5u_6^4 - {u_5}^2u_1 \\ & - \tfrac{1}{72}u_4^3u_6^6 - \tfrac{1}{33264}u_6^{11}u_5^2 + \tfrac{1}{27}u_5^6u_6^3 + \tfrac{2}{3}u_4u_5^3u_3 - 2u_4^2u_6u_3u_5 - u_2^2u_6 + u_4u_6u_1 \\ & - \tfrac{2}{9}u_5^3u_3u_6^3 - u_4u_3^2 + \tfrac{1}{12}u_4^4u_6^3 - \tfrac{1}{3024}u_6^9u_4^2 - \tfrac{1}{756}u_6^7u_5^4 + \tfrac{1}{1008}u_6^8u_2 - \tfrac{1}{36}\,{u_5}^4u_4{u_6}^4 \\ & + \tfrac{1}{3}u_5^4u_2 + \tfrac{1}{3}u_6^3u_3^2 - \tfrac{1}{9}u_4u_5^6 + \tfrac{1}{399168}u_6^{12}u_4 + u_4u_6u_5^2u_2 + \tfrac{1}{4}u_4^5 - \tfrac{1}{3}\,{u_4}^2u_6{u_5}^4 \\ & + 2\,u_5u_3u_2 + \tfrac{1}{6}\,{u_5}^2{u_6}^4u_2 + \tfrac{1}{12}\,{u_6}^5u_2u_4 - \tfrac{1}{2}\,{u_4}^2{u_6}^2u_2 + \tfrac{1}{2}\,{u_4}^3{u_6}^2{u_5}^2. \end{aligned} \tag{3.50}$$

(See Example A.5.21 in Appendix A.5 for the construction of this polynomial.)

Proof. This is an application of a result for the general case given in Theorem 2.2.21. The general result result was originally stated in [20]. A flaw in the proof have been discovered and there has recently been an alternative proof offered in [60]. This paper is currently only available as a preprint, so we present here a simple proof of the result that is specific to the (4,5)-case.

Consider $\boldsymbol{u} \in \Theta^{[5]}$ expressed as

$$\boldsymbol{u} = \int_\infty^{P_1} \boldsymbol{du} + \cdots + \int_\infty^{P_5} \boldsymbol{du},$$

where the P_k are points on C. Consider $\boldsymbol{u}$ close to the origin and use equations (3.24) to express the variables in $\boldsymbol{u}$ with five local parameters.

$$\begin{aligned} 0 &= u_1 + \tfrac{1}{11}\xi_1^{11} + \cdots + \tfrac{1}{11}\xi_5^{11} + O(\xi_1^{15}) + \cdots + O(\xi_5^{15}), \\ &\ \ \vdots \\ 0 &= u_6 + \xi_1 + \cdots + \xi_5 + O(\xi_1^5) + \cdots + O(\xi_5^5). \end{aligned} \tag{3.51}$$

Now consider the case when $\boldsymbol{\lambda} = \boldsymbol{0}$. Since the higher order terms in equations (3.24) all depend on curve parameters they simplify to

$$u_1 = -\tfrac{1}{11}\xi^{11},\ u_2 = -\tfrac{1}{7}\xi^7,\ u_3 = -\tfrac{1}{6}\xi^6,\ u_4 = -\tfrac{1}{3}\xi^3,\ u_5 = -\tfrac{1}{2}\xi^2,\ u_6 = -\xi,$$

and hence the higher order terms in equations (3.51) all reduce to zero. This leaves

$$\begin{aligned} 0 &= u_1 + \tfrac{1}{11}\xi_1^{11} + \cdots + \tfrac{1}{11}\xi_5^{11}, \\ &\vdots \\ 0 &= u_6 + \xi_1 + \cdots + \xi_5. \end{aligned}$$

Label these six equations $\mathrm{eq}_1, \ldots,\ \mathrm{eq}_6$ and recall the theory of resultants summarised in Appendix A.6. We can take pairs of equations and obtain a third equation, the resultant, which has one less variable and when satisfied implies the original equations are satisfied. Calculate five new polynomials by taking the resultant of eq_1 with each of the other four equations, eliminating the variable ξ_5 by choice.

$$\begin{aligned} &\mathrm{eq}_{12} = \mathrm{Res}(\mathrm{eq}_1, \mathrm{eq}_2, \xi_5), \quad \mathrm{eq}_{14} = \mathrm{Res}(\mathrm{eq}_1, \mathrm{eq}_4, \xi_5), \quad \mathrm{eq}_{16} = \mathrm{Res}(\mathrm{eq}_1, \mathrm{eq}_6, \xi_5). \\ &\mathrm{eq}_{13} = \mathrm{Res}(\mathrm{eq}_1, \mathrm{eq}_3, \xi_5), \quad \mathrm{eq}_{15} = \mathrm{Res}(\mathrm{eq}_1, \mathrm{eq}_5, \xi_5), \end{aligned}$$

Factor each of these polynomials and discard smaller factors. Next calculate four new polynomials by taking the resultant of eq_{12} with each of the other three equations, eliminating the variable ξ_4 each time.

$$\begin{aligned} &\mathrm{eq}_{123} = \mathrm{Res}(\mathrm{eq}_{12}, \mathrm{eq}_{13}, \xi_4), \qquad \mathrm{eq}_{125} = \mathrm{Res}(\mathrm{eq}_{12}, \mathrm{eq}_{15}, \xi_4), \\ &\mathrm{eq}_{124} = \mathrm{Res}(\mathrm{eq}_{12}, \mathrm{eq}_{14}, \xi_4), \qquad \mathrm{eq}_{126} = \mathrm{Res}(\mathrm{eq}_{12}, \mathrm{eq}_{16}, \xi_4). \end{aligned}$$

Continue in the way until only one equation remains and $\xi_1, \ldots, \xi_5$ have all been eliminated. (Alternatively, it should be possible to perform the calculation in one step using a multivariant resultant method, see for example [43] Chapter 13.)

From the theory of resultants we are left with a polynomial, unique up to multiplication by a non-vanishing holomorphic function, that must be zero for $\boldsymbol{u} \in \Theta^{[5]}$. By Lemma 2.2.19 we can conclude this polynomial to be a multiple of $\sigma(\boldsymbol{u})$.

Performing this calculation in Maple we find that the largest factor is indeed the Schur-Weierstrass polynomial (3.50).

■

Recall that by Remark 2.2.23 the constant in the definition of the σ-function was chosen so that $K = 1$ in Theorem 3.4.1.

Lemma 3.4.2. *The function $\sigma(\boldsymbol{u})$ associated with the (4,5)-curve is odd with respect to the change of variables $\boldsymbol{u} \mapsto [-1]\boldsymbol{u}$.*

Proof. By Lemma 2.2.20 the σ-function has definite parity and with $(n,s) = (4,5)$ we can conclude it to be odd. Note that this matches $SW_{4,5}$ above which is also odd under this change in variables.

∎

We now have enough information to derive a Taylor series expansion for $\sigma(\boldsymbol{u})$.

Theorem 3.4.3. *The function $\sigma(\boldsymbol{u})$ associated with the cyclic (4,5)-curve may be expanded about the origin as*

$$\sigma(\boldsymbol{u}) = \sigma(u_1,u_2,u_3,u_4,u_5,u_6) = C_{15}(\boldsymbol{u}) + C_{19}(\boldsymbol{u}) + \cdots + C_{15+4n}(\boldsymbol{u}) + \cdots$$

where each C_k is a finite, odd polynomial composed of products of monomials in $\boldsymbol{u} = (u_1,u_2,\dots,u_6)$ of weight $+k$ multiplied by monomials in $\boldsymbol{\lambda} = (\lambda_4,\lambda_3,\dots,\lambda_0)$ of weight $15-k$.

Proof. The theoretical part of the proof follows [30] and [11]. By Theorem 3(i) in [60] we know the expansion will be a sum of monomials in $\boldsymbol{u}$ and $\boldsymbol{\lambda}$ with rational coefficients, and by Lemma 3.4.2 we conclude that the expansion must be odd.

We also know that $\sigma(\boldsymbol{u})$ has definite weight, which we calculated in Example 3.3.8 to be $+15$. The rationale of the construction is that although the expansion is homogeneous of weight $+15$, it will contain both $\boldsymbol{u}$ (with positive weight) and $\boldsymbol{\lambda}$ (with negative weight). We hence split up the infinite expansion into finite polynomials whose terms share common weight ratios.

The first polynomial will be the terms with the lowest weight in $\boldsymbol{u}$. These must be the terms that do not vary with $\boldsymbol{\lambda}$ and have weight 15 in $\boldsymbol{u}$. The indices then increase by four since the weights of possible $\boldsymbol{\lambda}$-monomials decrease by four, (see Example 3.3.3).

∎

3.4.2 Constructing the expansion

We now describe how the σ-expansion was constructed using the framework established by Theorem 3.4.3. First, by Theorem 3.4.1 and the choice of c in Remark 2.2.23 we have $C_{15} = SW_{4,5}$ as given by equation (3.50). Using the computer algebra package Maple we calculate the other polynomials successively as follows.

1. Select the terms that could appear in C_k. These are a finite number of monomials formed by entries of $\boldsymbol{u}$ and $\boldsymbol{\lambda}$ with the appropriate weight ratio.

2. Construct $\hat{\sigma}(\boldsymbol{u})$ as the sum of C_k derived thus far. Then add to this each of the possible terms, each multiplied by an independent, unidentified constant coefficient.

3. Determine the unknown constants by ensuring $\hat{\sigma}(\boldsymbol{u})$ satisfies various properties of the σ-function. There are three methods that we use.

   (I) Ensuring the relations presented in Lemma 3.5.5 are satisfied. These are relations which express 4-index Q-functions as a linear combination of fundamental Abelian functions. All the functions are defined using the σ-function and so we ensure they are satisfied when $\hat{\sigma}(\boldsymbol{u})$ is used.

   (II) Ensuring the relations between the $\wp$-functions that were derived from the Kleinian formula in Section 3.2 are satisfied. These relations were denoted by $\mathcal{K}(\rho_{i,j}, n, \pm)$ as defined in Definition 3.2.3. We substitute the $\wp$-functions for their definition in $\sigma(\boldsymbol{u})$ to give conditions on the σ-function.

   (III) Ensuring $\sigma(\boldsymbol{u}) = 0$ for $\boldsymbol{u} \in \Theta^{[5]}$ as predicted by Lemma 2.2.19.

Step 1 is simple to complete using the in-built Maple command `partition`, part of the `combinat` package. This command will identify all the partitions of an integer. When identifying the terms in C_k we compute all the partitions of k and discard all those that contain integers that are not the weight of a variable in $\boldsymbol{u}$. (Discard any partitions that contain non-gaps.) We are left with partitions which we can easily identify with $\boldsymbol{u}$-monomials.

We can also discard all those which have a total degree in $\boldsymbol{u}$ even as we know that $\sigma(\boldsymbol{u})$ is an odd function.

We use the same command to identify those $\boldsymbol{\lambda}$-monomials of weight $15 - k$, this time discarding any partitions which contain integers that are not multiples of $n = 4$. The possible terms can all be written as a constant multiplied by one of the $\boldsymbol{u}$-monomials, multiplied by one of the $\boldsymbol{\lambda}$-monomials.

Hence Step 2 is to simply write $\hat{\sigma}(\boldsymbol{u})$ as the sum of those C_k already determined with each of the possible terms added, each multiplied by an unknown constant c_i. We then need to identify these constants using the methods in Step 3.

Notes on method (I)

It is computationally easiest to use method (I), however is not possible to use this method at the beginning stages of the expansion. The relations in Lemma 3.5.5 were themselves derived using the σ-expansion, which is why they have not been presented yet. However, as we discuss below, this is not a circular argument.

Note that each successive C_k contains terms with a weight in $\boldsymbol{\lambda}$ four less than the previous C_k. The relations in Lemma 3.5.5 are between Abelian functions and so by the weight properties discussed in Section 3.3 must be homogeneous. The relations were derived weight by weight starting with the highest. At each weight the $\boldsymbol{\lambda}$-monomials which may be present are either those monomials of that weight, or those of a higher weight, which are multiplied by an Abelian function to ensure the correct weight overall. Hence, in the calculation of that equation using the σ-expansion, it is only necessary to use the σ-expansion truncated after the corresponding C_k.

So the relations in Lemma 3.5.5 were derived in decreasing weight order, in tandem with the σ-expansion. I.e. once a new C_k was calculated, four more weight levels of the equations in Lemma 3.5.5 could be calculated. These could then be employed in the calculation of the next C_k.

Once a sufficient number of equations have been derived in Lemma 3.5.5 it is possible to use only method (I) when constructing the C_k. However in the (4,5)-case this only becomes possible once the σ-expansion has been calculated up to C_{39}. For the lower C_k a combination of all three methods was used.

We now discuss how to implement method (I). We take a relation from Lemma 3.5.5 and substitute the Abelian functions for their definition in terms of the σ-function. We evaluate these definitions so that we have a sum of rational function of σ-derivatives equal to zero. Since all the functions used in Lemma 3.5.5 have poles of order two, this sum can be factored so the denominator is $\sigma(\boldsymbol{u})^2$. Since this is not zero in general we can multiply up to leave the numerator. Note that the numerator will be a sum of products of pairs of σ-derivatives.

The next step is to substitute $\sigma(\boldsymbol{u})$ for the expansion $\hat{\sigma}(\boldsymbol{u})$, evaluating the σ-derivatives in the process. We now have a sum of products of expansions. We must expand these products to leave a polynomial with terms in the unknown constants, the $\boldsymbol{u}$ and the $\boldsymbol{\lambda}$. We will collect together terms with common $\boldsymbol{u\lambda}$-monomials and identify their coefficients as sums in the unknown constants, thus giving us conditions on the constants since each coefficient must equal zero. Note the following important points.

- Expanding these pairs of products will create terms with $\boldsymbol{\lambda}$-monomials of a lower weight than those in $\hat{\sigma}(\boldsymbol{u})$. These terms must be discarded since we would not have all such terms at this weight and hence any conditions gained from them would be invalid.

- The terms in $\hat{\sigma}(\boldsymbol{u})$ which had unknown constants were those terms with the lowest weight $\boldsymbol{\lambda}$-monomials. Since any multiples of such terms are discarded by the previous point, the conditions on the constants will be linear.

So we are left with linear equations in the constant coefficients. Since they are linear it is relatively easy for a computer package like Maple to solve these for the constants, (using the Maple command `solve`). However, as with everything in this section, it gets computationally more difficult for the higher C_k.

We may repeat this procedure on all the currently available equations from Lemma 3.5.5.

We now present a number of points about the implementation of method (1) for the higher C_k.

- The most difficult part of this method is expanding the products of expansions, especially for the later C_k for which $\hat{\sigma}(\boldsymbol{u})$ is very long. Using the standard Maple command `expand` is only acceptable for the first few C_k, after which it takes a great deal of CPU time and eventually more memory than a standard machine has available. This problem is overcome by implementing a new procedure, specifically designed to make use of the weight simplifications, which we describe now.

  Upon completion of the expansion we are left with only those terms that are of the $\boldsymbol{\lambda}$ weight of C_k. Those with lower weight were discarded for the reasons discussed above. Those with higher weight must, at the end of the expansion, cancel each other out since the relation we started with was true. (The only reason the end polynomial is not zero is because some of the coefficients were not yet determined.) Hence the computation may be simplified by missing the middle step and not considering terms that when multiplied together do not have $\boldsymbol{\lambda}$-weight $15 - k$.

  We achieve this by first cataloguing the terms in each of the expansions into sets of terms with the same $\boldsymbol{\lambda}$-weight. We then identify the pairs of weights which together add to $15 - k$. For example if we were working on C_{31} then we would be interested in $\boldsymbol{\lambda}$-monomials of weight -16. The possible pairs are

$$[0, 16], \quad [4, 12], \quad [8, 8], \quad [12, 4], \quad [16, 0].$$

  We would then form the end polynomial as a sum of pairs of products of terms. The first group of products would have been each term with weight 0 in $\boldsymbol{\lambda}$ in the first expansion, multiplied by each of terms with weight 16 in $\boldsymbol{\lambda}$ in the second. We then add to this products of the terms with weights 4 and 12 respectively and continue for each the possible pairs.

  The final polynomial is the same we would have achieved using the regular `expand` command and then discarding the terms with lower $\boldsymbol{\lambda}$-weight. However this method is considerable quicker, and more importantly, far less memory intensive. It is necessary to use such a procedure after the first few C_k to overcome memory limitations.

- Even using the improved procedure described above, the computation can still be time and memory intensive. This may be alleviated by using Distributed computing to run the computations in parallel on a cluster of machines. The problem is already prepared for parallelization since each product of expansions may be evaluated individually. It is logically simple to allow each of these expansions to be conducted on a different machine and then add them up at the end to achieve the final polynomial. We implemented this using the Distributed Maple package.

This is a free piece of software that opens Maple kernels on a cluster of machines and allows data and commands to be sent from a master kernel to the others. For more information see [72] and [65].

- The computation may be very memory intensive, even when using the above suggestions. The problem may be broken down into smaller problems by considering each possible $\boldsymbol{\lambda}$-monomial at weight $15-k$ separately. Such monomials never appear together (since we discard lower weight terms) and so are not interrelated. If there are n different $\boldsymbol{\lambda}$-monomial at weight $15-k$ then the problem may be split into n parts by considering each case separately. This is far simpler for some $\boldsymbol{\lambda}$-monomials than others and involves a repetition of work, increasing computation time. However, it is necessary in the later stages to make the computation feasible with the available memory allowances.

- For the higher C_k there are many relations in Lemma 3.5.5 that may be used. However, in practice we find that all the information may be gathered from a small group of these, if chosen carefully. This choice can be made by considering the special case with $\lambda_3 = \lambda_2 = \lambda_1 = \lambda_0 = 0$. There will hence be only one $\boldsymbol{\lambda}$ monomial, λ_4^n for some n. This case is very easy to implement (just substitute all the others to be zero) and so it is reasonably quick to check if a given relation is useful.

  Note also that while it may be possible to identify the coefficients with two or three high weight relations, it may be quicker to use a larger number of relations of a lower weight. This is because the relations at the higher weights involve lower index σ-derivatives and so the expansions that must be multiplied are longer. Again, experimentation with the case $\lambda_3 = \lambda_2 = \lambda_1 = \lambda_0 = 0$ can identify optimal choices.

Notes on method (II)

We may implement method (II) in the same way as method (I). Start by substituting the $\wp$-functions in $\mathcal{K}(\rho_{i,j}, n, \pm)$ for their definitions as σ-functions so that we have an equation involving σ-derivatives. Then take the numerator, substitute in $\hat{\sigma}(\boldsymbol{u})$, expand the products, discard the terms with $\boldsymbol{\lambda}$ weight lower than $\hat{\sigma}(\boldsymbol{u})$, collect terms with the same $\boldsymbol{u}\boldsymbol{\lambda}$-monomials and set their coefficients to zero. Finally solve the resulting linear equations in the unknown constants.

Note however that method (II) is computationally much more difficult than method (I). This is because the relations used in method (I) were linear in functions with poles of order two. Hence the numerator involved the multiplication of two large polynomials. The $\mathcal{K}(\rho_{i,j}, n, \pm)$ however involve products of functions with no limit on their poles. The simplest relation, $\mathcal{K}(\rho_{2,3}, 5, +)$, has poles of order eight and so using it involves multiplying eight expansions together. Hence method (II) should only be used if method (I) has been exhausted.

Method (II) was necessary in the derivation of all the C_k up to and including C_{39}, after which method (I) was sufficient. For the higher C_k it was not desirable or necessary to implement method (II) in the same way as method (I). The first reason for this is that it becomes very computationally difficult. The second is that it is not clear which of the $\mathcal{K}(\rho_{i,j}, n, \pm)$ will be useful. It is unlikely that the first few will be and just deriving the $\mathcal{K}(\rho_{i,j}, n, \pm)$ alone can be computationally intensive.

Instead note that we use method (II) only after method (I) has been exhausted. Usually by this stage all except a few of the constants have been determined. This means that we can simplify considerably by looking at specific values of $\boldsymbol{u}$ for the answer.

Start with the relations ρ_i defined in Section 3.2. Then at this stage perform the substitution for $\sigma(\boldsymbol{u})$, factor and take the numerator. Then instead of substituting for $\hat{\sigma}(\boldsymbol{u})$, we substitute for $\hat{\sigma}(\boldsymbol{u})$ evaluated at a specific value of $\boldsymbol{u}$, say $\boldsymbol{u} = (1, 1, 1, 1, 1, 1)$. First calculate the σ-derivatives at this special point (computationally easy) and then substitute for these in the relations derived from the ρ_i. We achieve a relation in the curve parameters and the variables (z, w). Then follow the same steps as in Section 3.2. Eliminate w by taking resultants and then reduce equations to degree five in z. If the resulting equation is not zero then use the coefficients of the $\boldsymbol{\lambda}$ to find the values of the constants.

Note that when searching for the equations that will assist we can simplify by setting $\lambda_3 = \lambda_2 = \lambda_1 = \lambda_0 = 0$ as this allows for speedy computations.

Notes on method (III)

Method (III) is to ensure that $\hat{\sigma}(\boldsymbol{u}) = 0$ for $\boldsymbol{u} \in \Theta^{[5]}$. We do this by substituting $\boldsymbol{u}$ in $\hat{\sigma}(\boldsymbol{u})$ for the series expansion in five local parameters, using the equations in Appendix C.1. We then take a multivariate series in $\xi_1, \ldots, \xi_5$ and identify the coefficients of ξ. These will be relations between the constants c_i and the curve parameters. We must discard those coefficients with $\boldsymbol{\lambda}$-monomials of weight lower than those in C_k. The remaining coefficients can be set to zero and used to identify the coefficients c_i.

The multivariate Taylor series can be taken using the Maple command `mtaylor`. This is simple for the first few C_k but quickly becomes very difficult. Since $\Theta^{[k]} \subset \Theta^{[5]}$ for $k < 5$ we can alternatively check that $\sigma(\boldsymbol{u}) = 0$ on a lower strata. This is computationally easier, but gives less conditions on the constants, (since there are less local parameters and hence less coefficients).

Example: Calculating C_{19}

We now describe how C_{19} was constructed, making use of all three methods to identify the coefficients.

The first step is to identify all the different monomials in $\boldsymbol{u}$ that have weight 19. We use the Maple procedure `partition` to find all the partitions of 19. We then discard those partitions that contain an integer not in $W_{4,5} = \{1, 2, 3, 6, 7, 11\}$. We relate the remaining partitions to the monomials in $\boldsymbol{u}$ they represent. For example the partition of 19 into 19 ones is related to u_6^{19}.

We find 115 possible monomials. However, we can discard all those that are not odd functions to leave 62.

The next step is to identify all those $\boldsymbol{\lambda}$-monomials that have weight $15 - 19 = -4$ in the curve parameters. The only such monomial is clearly λ_4. We hence form $\hat{\sigma}(\boldsymbol{u})$ using constants c_i as

$$\hat{\sigma}(\boldsymbol{u}) = SW_{4,5} + \lambda_4\big(c_1 u_6^{19} + c_2 u_6^{17} u_5 + \cdots\big)$$

We need to determine the 62 coefficients c_i. We start with method (I) by considering relations from Lemma 3.5.5. There is only one relations that may be derived here using $\sigma(\boldsymbol{u}) = SW_{4,5}(\boldsymbol{u})$ only. This is the relation for Q_{5666}, which has weight -5. (The relation for this function cannot involve any lower weight $\boldsymbol{\lambda}$-monomials and it cannot involve λ_4 since there is no Abelian function of weight -1 with which it could be combined. See Section 3.5.1 for full details.) We were able to use $\sigma = SW_{4,5}$ to find

$$Q_{5666} = -2\wp_{45}.$$

We substitute the Q and $\wp$-functions for their definitions in $\sigma(\boldsymbol{u})$to leave a rational function in σ-derivatives with denominator $\sigma(\boldsymbol{u})^2$. We multiply up to leave

$$\begin{aligned} 0 = &-\sigma_{5666}(\boldsymbol{u})\sigma(\boldsymbol{u}) + 3\sigma_{566}(\boldsymbol{u})\sigma_6(\boldsymbol{u}) - 3\sigma_{56}(\boldsymbol{u})\sigma_{66}(\boldsymbol{u}) \\ &+ \sigma_5(\boldsymbol{u})\sigma_{666}(\boldsymbol{u}) - 2\sigma_{45}(\boldsymbol{u})\sigma(\boldsymbol{u}) + 2\sigma_4\sigma_5. \end{aligned}$$

We now substitute $\sigma(\boldsymbol{u})$ for $\hat{\sigma}(\boldsymbol{u})$ and calculate the relevant derivatives. We then expand the products and remove the terms with higher weight $\boldsymbol{\lambda}$-monomials. We are left with a polynomial in $\boldsymbol{u}$ and λ_4. Collecting common $\boldsymbol{u}\boldsymbol{\lambda}$-monomials together gives us coefficients in the c_i. We set these to zero and solve for the unknowns using the Maple `solve` command. We find that we can express 49 of the 62 unknowns using the others.

We next use method (III) by ensuring that $\hat{\sigma}(\boldsymbol{u})$ vanishes for $\boldsymbol{u} \in \Theta^{[5]}$. We substitute $\boldsymbol{u}$ in $\hat{\sigma}(\boldsymbol{u})$ for the series expansion in five local parameters. We then take a multivariate Taylor series, with those terms containing $\boldsymbol{\lambda}$-monomials of weight lower than -4 discarded. We identify the coefficients of the remaining $\boldsymbol{\lambda}\xi_i$-monomials. These are relations between the constants c_i and the curve parameters. We substitute in the equations for the 49 coefficients derived using method (I). We then solve to find that 60 of the 62 coefficients can be expressed as a linear combination of the other two.

To find the final two c_i we use method (II). In this case we can use the simplest relation derived from the Kleinian equation, $\mathcal{K}(\rho_{2,3}, 5, +)$. Again we substitute the $\wp$-functions for σ-derivatives, factor, take the numerator and then substitute $\sigma(\boldsymbol{u})$ for $\hat{\sigma}(\boldsymbol{u})$. We evaluate the derivatives, expand the polynomial and collect the coefficients of the monomials in $\boldsymbol{u}$ and $\boldsymbol{\lambda}$. Then substitute in the conditions on c_i and we are able to assign a numerical value for the one c_i in which the others were evaluated. We hence gain numerical values for all the c_i. We find that

$$\begin{aligned}
C_{19} = \lambda_4 \cdot \big[&\tfrac{1}{13970880}\, u_6{}^{16}u_4 + \tfrac{2}{135}\, u_6{}^3u_5{}^8 - u_6{}^2u_5u_4{}^3u_3 - u_6{}^2u_4u_2{}^2 + \tfrac{4}{45}\, u_5{}^6u_2 \\
&- \tfrac{2}{45}\, u_6{}^3u_5{}^5u_3 - \tfrac{1}{5}\, u_6u_5{}^6u_4{}^2 + \tfrac{1}{90}\, u_6{}^4u_5{}^6u_4 - \tfrac{1}{30}\, u_6{}^5u_4u_1 + \tfrac{5}{3024}\, u_6{}^9u_5{}^2u_4{}^2 + \tfrac{1}{10}\, u_5{}^2u_4{}^5 \\
&- \tfrac{1}{1890}\, u_6{}^7u_5{}^6 + \tfrac{1}{30}\, u_6{}^5u_2{}^2 + \tfrac{1}{20956320}\, u_6{}^{15}u_5{}^2 - \tfrac{1}{3}\, u_5{}^4u_1 - \tfrac{2}{45}\, u_5{}^8u_4 - \tfrac{1}{83160}\, u_6{}^{11}u_5{}^4 \\
&- \tfrac{1}{630}\, u_6{}^7u_3{}^2 - \tfrac{1}{5040}\, u_6{}^{10}u_4{}^3 + \tfrac{1}{3024}\, u_6{}^9u_4u_2 + \tfrac{1}{2}\, u_6{}^2u_4{}^2u_1 + u_6{}^2u_5{}^2u_4{}^2u_2 - \tfrac{1}{120}\, u_6{}^7u_4{}^4 \\
&- \tfrac{1}{280}\, u_6{}^8u_5u_4u_3 + \tfrac{2}{15}\, u_6{}^5u_5{}^2u_4u_2 - \tfrac{2}{9}\, u_6{}^4u_5{}^3u_4u_3 - \tfrac{2}{3}\, u_6u_5{}^3u_4{}^2u_3 + \tfrac{1}{665280}\, u_6{}^{13}u_4{}^2 \\
&- \tfrac{1}{10}\, u_6{}^5u_5u_4{}^2u_3 + \tfrac{1}{3}\, u_6u_5{}^4u_4u_2 - u_6u_4{}^2u_3{}^2 + \tfrac{1}{20}\, u_6{}^4u_4{}^5 + \tfrac{1}{2520}\, u_6{}^8u_5{}^4u_4 + \tfrac{1}{5040}\, u_6{}^8u_1 \\
&+ \tfrac{1}{30}\, u_6{}^6u_4{}^2u_2 + \tfrac{1}{6}\, u_6{}^4u_4u_3{}^2 + \tfrac{4}{3}\, u_5{}^3u_3u_2 - \tfrac{4}{945}\, u_6{}^7u_5{}^3u_3 + \tfrac{1}{18}\, u_6{}^4u_5{}^4u_2 + \tfrac{1}{2}\, u_4{}^4u_2 \\
&- \tfrac{1}{180}\, u_6{}^6u_5{}^2u_4{}^3 - \tfrac{17}{997920}\, u_6{}^{12}u_5{}^2u_4 + \tfrac{1}{630}\, u_6{}^8u_5{}^2u_2 + \tfrac{3}{20}\, u_6u_4{}^6 + \tfrac{1}{997920}\, u_6{}^{12}u_2 \\
&+ \tfrac{2}{15}\, u_5{}^5u_4u_3 - \tfrac{1}{6}\, u_6{}^3u_4{}^3u_2 - \tfrac{1}{83160}\, u_6{}^{11}u_5u_3 - \tfrac{1}{60}\, u_6{}^5u_5{}^4u_4{}^2 + \tfrac{1}{3}\, u_6{}^3u_5{}^2u_4{}^4\big].
\end{aligned} \tag{3.52}$$

Summary

Using the techniques described above we have calculated the σ-expansion associated to the (4,5)-curve up to and including C_{59}. Appendix C.2 contains C_{23} while the rest of the expansion can be found in the extra Appendix of files or online at [38]. The Maple worksheets in which the C_k are derived can also be found in the extra Appendix of files. The polynomial get increasingly large as indicated by the table below displaying the number of non-zero terms in each.

C_{15}	C_{19}	C_{23}	C_{27}	C_{31}	C_{35}	C_{39}	C_{43}	C_{47}	C_{51}	C_{55}	C_{59}
32	50	176	386	1048	2193	4452	8463	16264	28359	49753	81832

So the final polynomial calculated had almost 82,000 terms. In fact, these were just the terms with non-zero coefficients; there were over 120,000 possible terms in C_{59}.

The later polynomials represent a significant amount of computation. Many of the calculations were run in parallel on a cluster of machines using the Distributed Maple package ([72]). This expansion is sufficient for any explicit calculations. However, it would be ideal to find a recursive construction of the expansion generalising equation (2.24) in the elliptic case, (see for example [32]).

3.5 Relations between the Abelian functions

In Section 3.3 we proved that the Abelian functions all have definite weight and hence any equations between them must be homogeneous. We will construct several important sets of such relations in this section. These include several classes of equations that generalise the theory from the elliptic case. They also include the results that allow us to find applications in the KP hierarchy in Section 3.7.

First we will introduce the following definition to classify the Abelian functions associated with C by their pole structure.

Definition 3.5.1. *Define*

$$\Gamma\big(J, \mathcal{O}(m\Theta^{[k]})\big)$$

as the vector space of Abelian functions defined upon J which have poles of order at most m, occurring only on the kth strata, $\Theta^{[k]}$.

The case where $k = g - 1$ is of interest to us because all the Abelian functions considered in this document vanish for $\boldsymbol{u} \in \Theta^{[g-1]}$. Note that the dimension of the space $\Gamma\big(J, \mathcal{O}(m\Theta^{[g-1]})\big)$ is m^g by the Riemann-Roch theorem for Abelian varieties. (See for example [53] page 99.)

Recall that the n-index $\wp$-functions have poles of order n (Lemma 2.2.28), while the n-index Q functions all have poles of order two (Lemma 3.1.3). In both cases these poles occurred if and only if $\sigma(\boldsymbol{u}) = 0$ which by Lemma 2.2.19 is when $\boldsymbol{u} \in \Theta^{[5]}$.

Therefore we classify the Abelian functions associated to the (4,5)-curve as follows.

$$\begin{aligned} \wp_{i_1,i_2,\dots,i_n} &\in \Gamma\big(J, \mathcal{O}(n\Theta^{[5]}\big), \\ Q_{i_1,i_2,\dots,i_n} &\in \Gamma\big(J, \mathcal{O}(2\Theta^{[5]}\big). \end{aligned} \tag{3.53}$$

Recall also that since holomorphic Abelian functions are constants and there exists no Abelian function with simple poles, those with poles of order two have the simplest pole structure. Hence we sometimes refer to those functions that belong to $\big(J, \mathcal{O}(2\Theta^{[5]}\big)$ as *fundamental Abelian functions.*

We will construct a basis for this space of fundamental Abelian function, and in doing so generate linear relations between such functions. First we must prove the following lemma which restricts the linear combinations of basis entries to those with polynomial coefficients in $\boldsymbol{\lambda}$.

Lemma 3.5.2. *Suppose we have a basis for the vector space $\Gamma\big(J, \mathcal{O}(m\Theta^{[k]})\big)$. Then an element of the space that is not contained in the basis can be expressed as a linear combination of the basis entries, with coefficients polynomial in $\boldsymbol{\lambda} = \{\lambda_4, \lambda_3, \lambda_2, \lambda_1, \lambda_0\}$.*

Proof. The significance of the lemma is that we need not consider the coefficients to be rational functions of $\boldsymbol{\lambda}$, as may be expected. We can modify the argument from Theorem 9.1 in [30] to prove this. Let X be an element of the vector space that is not in the basis. Then

$$X = \sum_j A_j(\boldsymbol{\lambda}) Y_j,$$

where the Y_j are elements of the basis and the $A_j(\boldsymbol{\lambda})$ are rational functions of $\boldsymbol{\lambda}$. Since X is an Abelian function it will have definite weight, and hence the terms $A_j(\boldsymbol{\lambda})Y_j$ must all be homogeneous with this weight.

Now, we have a set of polynomials, $A_j(\boldsymbol{\lambda})$, with finite indeterminates, $\boldsymbol{\lambda}$. This is a polynomial ring and hence a UFD, which means we can write each polynomial as a product of prime elements. Let us write each rational function $A_j(\boldsymbol{\lambda})$ in reduced fractional form,

$$X = \sum_j \frac{P_j(\boldsymbol{\lambda})}{Q_j(\boldsymbol{\lambda})} Y_j, \tag{3.54}$$

where P_j and Q_j are homogeneous polynomials that do not share prime elements. Now we will suppose for a contradiction that at least one of the $A_j(\boldsymbol{\lambda})$ is not polynomial.

Let B denote the least common multiple of the set $\{Q_j(\boldsymbol{\lambda})\}$, and multiply equation (3.54) by B. There will be specific values of $\boldsymbol{\lambda}$ that set $B = 0$ while leaving at least one of the $P_j(\boldsymbol{\lambda})$ non-zero, (Chapter 1 of [46]). In such a case we obtain an equation that contradicts the linear independence of the basis.

For example, suppose that $A_1(\boldsymbol{\lambda})$ is not polynomial. Then $Q_1(\boldsymbol{\lambda})$ will contain a non-trivial irreducible. Consider the zeros of this irreducible. At least one of the zeros must not be shared by $P_1(\boldsymbol{\lambda})$, or $P_1(\boldsymbol{\lambda})$ would also factor to give this irreducible. Therefore, when $\boldsymbol{\lambda}$ is fixed at this value we find that $Q_1(\boldsymbol{\lambda}) = B = 0$ but $P_1(\boldsymbol{\lambda}) \neq 0$.

Hence, in such a special case we would have contradicted the linear independence of the basis. Therefore we conclude that the $A_j(\boldsymbol{\lambda})$ must all be polynomial in $\boldsymbol{\lambda}$.

■

3.5.1 Basis for the fundamental Abelian functions

We now present the basis for the space of fundamental Abelian functions, followed by the details of how it was constructed.

Theorem 3.5.3. *A basis for* $\Gamma\big(J, \mathcal{O}(2\Theta^{[5]})\big)$ *is given by*

$$\begin{array}{ccccccccccccc}
 & \mathbb{C}1 & \oplus & \mathbb{C}\wp_{11} & \oplus & \mathbb{C}\wp_{12} & \oplus & \mathbb{C}\wp_{13} & \oplus & \mathbb{C}\wp_{14} & \oplus & \mathbb{C}\wp_{15} \\
\oplus & \mathbb{C}\wp_{16} & \oplus & \mathbb{C}\wp_{22} & \oplus & \mathbb{C}\wp_{23} & \oplus & \mathbb{C}\wp_{24} & \oplus & \mathbb{C}\wp_{25} & \oplus & \mathbb{C}\wp_{26} \\
\oplus & \mathbb{C}\wp_{33} & \oplus & \mathbb{C}\wp_{34} & \oplus & \mathbb{C}\wp_{35} & \oplus & \mathbb{C}\wp_{36} & \oplus & \mathbb{C}\wp_{44} & \oplus & \mathbb{C}\wp_{45} \\
\oplus & \mathbb{C}\wp_{46} & \oplus & \mathbb{C}\wp_{55} & \oplus & \mathbb{C}\wp_{56} & \oplus & \mathbb{C}\wp_{66} & \oplus & \mathbb{C}Q_{5566} & \oplus & \mathbb{C}Q_{4556}
\end{array}$$

$$\begin{aligned}
&\oplus\ \mathbb{C}Q_{4555} \oplus \mathbb{C}Q_{4455} \oplus \mathbb{C}Q_{3566} \oplus \mathbb{C}Q_{3556} \oplus \mathbb{C}Q_{2566} \\
&\oplus\ \mathbb{C}Q_{2556} \oplus \mathbb{C}Q_{3456} \oplus \mathbb{C}Q_{2456} \oplus \mathbb{C}Q_{3366} \oplus \mathbb{C}Q_{3445} \\
&\oplus\ \mathbb{C}Q_{2366} \oplus \mathbb{C}Q_{2445} \oplus \mathbb{C}Q_{1466} \oplus \mathbb{C}Q_{1556} \oplus \mathbb{C}Q_{2266} \\
&\oplus\ \mathbb{C}Q_{2356} \oplus \mathbb{C}Q_{2256} \oplus \mathbb{C}Q_{2346} \oplus \mathbb{C}Q_{1455} \oplus \mathbb{C}Q_{2345} \\
&\oplus\ \mathbb{C}Q_{3344} \oplus \mathbb{C}Q_{2245} \oplus \mathbb{C}Q_{2344} \oplus \mathbb{C}Q_{1266} \oplus \mathbb{C}Q_{1356} \\
&\oplus\ \mathbb{C}Q_{1444} \oplus \mathbb{C}Q_{1346} \oplus \mathbb{C}Q_{2236} \oplus \mathbb{C}Q_{2335} \oplus \mathbb{C}Q_{1246} \\
&\oplus\ \mathbb{C}Q_{1255} \oplus \mathbb{C}Q_{1245} \oplus \mathbb{C}Q_{1166} \oplus \mathbb{C}Q_{1244} \oplus \mathbb{C}Q_{1156} \\
&\oplus\ \mathbb{C}Q_{1146} \oplus \mathbb{C}Q_{1155} \oplus \mathbb{C}Q_{1145} \oplus \mathbb{C}Q_{1144} \oplus \mathbb{C}Q_{114466}.
\end{aligned} \tag{3.55}$$

Proof. The dimension of the space is $2^g = 2^6 = 64$ by the Riemann-Roch theorem for Abelian varieties. It was concluded above in equation (3.53) that all the selected elements do in fact belong to the space. All that remains is to prove their linear independence, which can be done explicitly using the σ-expansions in Maple.

■

The actual construction of the basis is as follows. We start by including all 21 of the 2-index $\wp$-functions in the basis, since they are all linearly independent. The next step is to decide which 4-index Q-functions to include. We do this by systematically considering weight levels, starting at -4 (the highest weight of any Q-function) and decreasing.

At each weight level we identify Q_{ijkl} to be added to the basis and derive equations for the non-basis entries as a linear combinations of basis entries. This is achieved using the following method, described for weight level $-k$, and implemented with Maple.

1. Start by forming a sum of existing basis entries, each multiplied by an undetermined constant coefficient, c_i. Note however that we need not include all the basis entries in this sum, since the sum must be homogeneous of weight $-k$.

   First include any basis entries at weight $-k$ that have already been determined, (the $\wp_{ij}$). Then include any elements with a higher weight, that may be multiplied by an appropriate $\boldsymbol{\lambda}$-monomial to balance the weight at $-k$ overall. (Note that all the possible elements of a higher weight have already been determined since we are working systematically in decreasing weights.) Note also that from Lemma 3.5.2 we need not consider basis entries multiplied by rational functions in the $\boldsymbol{\lambda}$.

   Note that we must account for the possibility of a constant term of weight $-k$. Such a term must be a multiple of a $\boldsymbol{\lambda}$-monomial at weight $-k$.

2. Add to this sum the Q_{ijkl} of weight $-k$, each multiplied by an undetermined constant coefficient, q_i.

3. Substitute the Abelian functions for their definitions as σ-derivatives. This will give a sum of rational function which may be factored to leave $\sigma(\boldsymbol{u})^2$ on the denominator. (This is because the sum is a linear combination of functions with poles of order two

when $\sigma(\boldsymbol{u}) = 0$.) Take the numerator, which should be a sum of products of pairs of σ-derivatives.

4. Substitute $\sigma(\boldsymbol{u})$ in this sum for the σ-expansion calculated in Section 3.4. Note that the sum contains $\boldsymbol{\lambda}$-monomials with weight no lower than $-k$. Hence we may truncate the σ-expansion after the polynomial which contains $\boldsymbol{\lambda}$-monomials of weight $-k$. Evaluate the σ-derivatives as derivatives of this expansion.

5. Expand the pairs of products to obtain a polynomial. Note that this will create terms with $\boldsymbol{\lambda}$-monomials of a lower weight than $-k$. These terms must be discarded since we would not have all such terms with this $\boldsymbol{\lambda}$-monomial, (because we truncated the expansion). Hence any information gained from them would be invalid.

6. Collect this polynomial into a sum of the various $\boldsymbol{u\lambda}$-monomials with coefficients in the unknown constants, $\{c_i, q_i\}$. Note that these coefficients will be linear in the unknowns, (since the unknowns were from the original sum, not the expansions).

7. Consider these coefficients as a series of linear equations in the unknowns set to zero. Since it is linear it is quite simple for a computer algebra package like Maple to solve.

   - If there is a unique solution for the c_i in terms of the q_i then each of the Q_{ijkl} at weight $-k$ may be expressed as a linear combination of existing basis entries. (To obtain the relations repeatedly set one of the q_i to one and the others to zero in the solution that was obtained.)

   - However, it will more often be the case that this is not possible, and hence some of the Q_{ijkl} must be selected as basis entries.

     Suppose that there are x unknowns and $(x - y)$ of the unknowns may be expressed using the other y. This means that specifying y of the q_i determines numerical values for all the c_i and the other q_i. Hence y of the Q_{ijkl} may be expressed as a linear combination of basis entries and the other Q_{ijkl}. These other Q_{ijkl} must be added to the basis.

     There is usually several choices for which Q_{ijkl} to add to the basis, (since the basis (3.55) is clearly not unique). However, we should check that the Q_{ijkl} chosen are acceptable as in some special cases the Q_{ijkl} at a weight level may be linear combinations of each other.

     To obtain the relations for the non-basis Q_{ijk} repeatedly set one of the corresponding q_i to one and the others to zero in the solution that was obtained.

   Hence we have identified the basis entries at weight $-k$ and obtained relations for all the non-basis entries as a linear combination of basis entries.

We give an example of this procedure for weight -8.

Example 3.5.4. Start by constructing the sum of existing basis entries. The entries at weight -8 are $\wp_{26}$ and $\wp_{35}$. We additionally take the entries at weight -4 and combine them with λ_4. Finally we may have a constant at weight -8 so we need to include λ_4^2 or λ_3. We then add to this sum the three Q-functions at weight 8 to end with the equation,

$$\begin{aligned} 0 = c_1\wp_{26} + c_2\wp_{35} + c_3\wp_{46}\lambda_4 + c_4\wp_{55}\lambda_4 + c_5\lambda_4^2 + c_6\lambda_3 \\ + q_1Q_{4466} + q_2Q_{4556} + q_3Q_{5555}. \end{aligned} \tag{3.56}$$

We substitute the Abelian functions for σ-derivatives and take the numerator. We then substitute in the σ-expansion truncated after C_{23}, since this was the polynomial that contained λ_4^2 and λ_3. Expanding the products of expansions will generate terms with $\{\lambda_3^2, \lambda_4^2, \lambda_3\lambda_4, \lambda_3\lambda_4^2\}$ which must all be discarded. The polynomial that is left is collected into $\boldsymbol{u\lambda}$-monomials with coefficients in the unknowns. Setting these to zero and using the Maple `solve` command we find a solution is given by

$$\begin{aligned} &h_1 = -72q_3 + 6h_4, \quad h_2 = -12q_3 + h_4, \quad h_3 = -64q_3 + 4h_4, \quad h_5 = 0, \\ &h_6 = -48q_3 + 4h_4, \quad q_1 = -16q_3 + h_4, \quad q_2 = -10q_3 + h_4, \end{aligned}$$

for any values of h_4 and q_3. Therefore once two of the unknowns are determined the others are established. This means that only two of the three Q_{ijkl} may be expressed as a linear combination of the basis entries and the other Q_{ijkl}. We choose to add Q_{5566} to the basis. To find the equations for the other two Q_{ijkl} first rearrange the equation for q_1 above so that it takes the role of h_4.

$$\begin{aligned} &h_1 = 6q_1 + 24q_3, \quad h_2 = q_1 + 4q_3, \quad h_3 = 4q_1, \quad h_4 = q_1 + 16q_3, \quad h_5 = 0, \\ &h_6 = 4q_1 + 16q_3, \quad q_2 = q_1 + 6q_3. \end{aligned}$$

Then set $q_1 = 1, q_3 = 0$, substitute in (3.56) and rearrange to find

$$Q_{4466} = 6\wp_{26} + \wp_{35} + 4\lambda_4\wp_{46} + \lambda_4\wp_{55} + 4\lambda_3 - Q_{4556}.$$

Similarly setting $q_1 = 0, q_3 = 1$ we find

$$Q_{5555} = 24\wp_{26} + 4\wp_{35} + 16\lambda_4\wp_{55} + 16\lambda_3 - 6Q_{4556}.$$

□

So we follow this procedure at successively lower weights, constructing the basis and equations as we proceed. Note that after every four weight levels an additional C_k of the σ-expansion is required. As discussed in Section 3.4 these relations were constructed in tandem with the σ-expansion. Once a new C_k was found four more weight levels could be examined. The relations obtained could then be used to construct the next C_k.

Note also that as the weight decreases, both the the possible number of terms in the original sum and the size of the σ-expansion increase. Hence the computations take more time and memory. The most computationally intensive part is Step 5 where the products of expansions must be calculated and the terms with weight in $\boldsymbol{\lambda}$ lower than $-k$ discarded.

- Since all the terms in the end polynomial contain an unknown, we should not use the procedure described in the last section to perform this calculation, or we would lose many constraints.
- However, we do find that the resulting system of linear equations for the unknowns is very overdetermined. Hence we can simplify by setting some of the $\boldsymbol{u}$ to specific values. We find that setting $\boldsymbol{u} = (u_1, u_2, 1, 1, 1, 1)$ achieves enough constraints to specify the unknowns, while being computationally far simpler.
- To further save computation time and memory distributed computing may be employed to expand the pairs of products in parallel. This was implemented using the Distributed Maple package, [72], discussed in Section 3.4.

Upon examining all the 4-index Q-functions, we find that 63 basis elements have been identified. To find the final basis element we repeat the procedure using the 6-index Q-functions. We find that all those of weight higher than -30 can be expressed as a linear combination of existing basis entries. However, at weight -30 one of the Q_{ijklmn} is required in the basis to express the others. The Maple worksheets in which these calculations were performed can be found in the extra Appendix of files.

3.5.2 Differential equations in the Abelian functions

We present a number of differential equations between the Abelian functions. The number in brackets on the left indicates the weight of each equation.

Lemma 3.5.5. *Those 4-index Q-functions not in the basis can be expressed as a linear combination of the basis elements.*

$$
\begin{aligned}
&(-4) \quad Q_{6666} = -3\wp_{55} + 4\wp_{46} \\
&(-5) \quad Q_{5666} = -2\wp_{45} \\
&(-6) \quad Q_{4666} = 6\lambda_4\wp_{66} - 2\wp_{44} - \tfrac{3}{2}Q_{5566} \\
&(-7) \quad Q_{4566} = 2\lambda_4\wp_{56} + 2\wp_{36} \\
&(-7) \quad Q_{5556} = 4\lambda_4\wp_{56} + 4\wp_{36} \\
&(-8) \quad Q_{4466} = 4\lambda_4\wp_{46} + \lambda_4\wp_{55} + \wp_{35} + 6\wp_{26} - Q_{4556} + 4\lambda_3 \\
&(-8) \quad Q_{5555} = 16\lambda_4\wp_{55} + 4\wp_{35} + 24\wp_{26} - 6Q_{4556} + 16\lambda_3 \\
&(-9) \quad Q_{4456} = \tfrac{8}{3}\lambda_4\wp_{45} + 2\wp_{25} - \tfrac{4}{3}\wp_{34} - \tfrac{1}{6}Q_{4555}
\end{aligned}
$$

$$
\begin{aligned}
&\mathbf{(-9)} \quad Q_{3666} = 2\lambda_4\wp_{45} - \tfrac{1}{2}Q_{4555}\\
&\mathbf{(-10)} \quad Q_{2666} = -\tfrac{1}{2}Q_{4455} - \tfrac{1}{2}Q_{3566} - \tfrac{1}{2}\lambda_4 Q_{5566} + 4\wp_{66}\lambda_3\\
&\mathbf{(-10)} \quad Q_{4446} = 6\lambda_4\wp_{44} - 2\wp_{24} - 2\lambda_4^2\wp_{66} - Q_{4455} + \tfrac{1}{2}\lambda_4 Q_{5566} - \tfrac{5}{2}Q_{3566} + 4\wp_{66}\lambda_3\\
&\mathbf{(-11)} \quad Q_{3466} = 4\wp_{36}\lambda_4 - \tfrac{1}{2}Q_{3566}\\
&\mathbf{(-11)} \quad Q_{4445} = 6\wp_{36}\lambda_4 - 2\wp_{56}\lambda_4^2 + 8\wp_{56}\lambda_3 - 3Q_{2566} - \tfrac{3}{2}Q_{3556}\\
&\mathbf{(-12)} \quad Q_{2466} = 8\wp_{16} - \wp_{33} + 2\lambda_2 - \tfrac{1}{2}Q_{2456} - Q_{3456} + 4\lambda_4\wp_{26} + 2\lambda_4\wp_{35}\\
&\mathbf{(-12)} \quad Q_{3555} = 24\wp_{16} - 8\wp_{33} + 16\lambda_4\wp_{35} - 6Q_{3456}\\
&\mathbf{(-12)} \quad Q_{4444} = -12\wp_{16} + 9\wp_{33} + 6\lambda_2 + 12Q_{3456} + 12\wp_{55}\lambda_3 + 4\wp_{46}\lambda_4^2\\
&\qquad\qquad - 16\wp_{46}\lambda_3 - 3\wp_{55}\lambda_4^2 + 12\lambda_4\wp_{26} - 18\lambda_4\wp_{35} - 6Q_{2556}\\
&\qquad\qquad \vdots
\end{aligned}
$$

The full list down to Q_{1111} can be found in the extra Appendix of files or online at [38].

There are similar equations for all the 6-index Q-functions, except Q_{114466} which is in the basis. Explicit relations have been calculated down to weight -39. The first few are given below while the full list can be found in the extra Appendix of files or online at [38].

$$
\begin{aligned}
&\mathbf{(-6)} \quad Q_{666666} = 40\wp_{44} + 15Q_{5566} - 24\wp_{66}\lambda_4\\
&\mathbf{(-7)} \quad Q_{566666} = 20\wp_{36} - 4\wp_{56}\lambda_4\\
&\mathbf{(-8)} \quad Q_{556666} = 24\wp_{26} - 12\wp_{35} - 2Q_{4556}\\
&\mathbf{(-8)} \quad Q_{466666} = -20\wp_{35} + 5Q_{4556} + 16\wp_{46}\lambda_4 - 20\wp_{55}\lambda_4 - 8\lambda_3\\
&\mathbf{(-9)} \quad Q_{456666} = -16\wp_{25} + \tfrac{8}{3}\wp_{34} + \tfrac{1}{3}Q_{4555} - \tfrac{16}{3}\wp_{45}\lambda_4\\
&\mathbf{(-9)} \quad Q_{555666} = -16\wp_{34} + 2Q_{4555} - 16\wp_{45}\lambda_4\\
&\mathbf{(-10)} \quad Q_{446666} = 8\wp_{24} + Q_{4455} - 2Q_{3566} - 24\wp_{44}\lambda_4 - 14Q_{5566}\lambda_4 + 40\wp_{66}\lambda_4^2\\
&\mathbf{(-10)} \quad Q_{455666} = -8\wp_{24} - 4Q_{3566} + 2Q_{5566}\lambda_4 + 8\wp_{66}\lambda_3\\
&\mathbf{(-10)} \quad Q_{555566} = -16\wp_{24} + 4Q_{4455} + 8Q_{3566} + 8Q_{5566}\lambda_4 - 16\wp_{66}\lambda_3\\
&\mathbf{(-11)} \quad Q_{366666} = -5Q_{3556} + 16\wp_{36}\lambda_4\\
&\mathbf{(-11)} \quad Q_{445666} = -6Q_{2566} + 3Q_{3556} + 12\wp_{36}\lambda_4 + 4\wp_{56}\lambda_4^2 + 8\wp_{56}\lambda_3\\
&\mathbf{(-11)} \quad Q_{455566} = -2Q_{3556} + 24\wp_{36}\lambda_4 + 8\wp_{56}\lambda_4^2\\
&\mathbf{(-11)} \quad Q_{555556} = -20Q_{3556} + 112\wp_{36}\lambda_4 + 16\wp_{56}\lambda_4^2 + 32\wp_{56}\lambda_3\\
&\qquad\qquad \vdots
\end{aligned}
$$

Proof. By Theorem 3.5.3 and Lemma 3.5.2 it is clear such relations must exist. The explicit differential equations were calculated in the construction of the basis, as discussed above. ∎

Corollary 3.5.6. *There are a set of differential equations that express 4-index $\wp$-functions as a polynomial of Abelian functions of total degree at most two.*

$$\begin{aligned}
&(-4) \quad \wp_{6666} = 6\wp_{66}^2 - 3\wp_{55} + 4\wp_{46} && (3.57)\\
&(-5) \quad \wp_{5666} = 6\wp_{56}\wp_{66} - 2\wp_{45} && (3.58)\\
&(-6) \quad \wp_{4666} = 6\wp_{46}\wp_{66} + 6\lambda_4\wp_{66} - 2\wp_{44} - \tfrac{3}{2}\wp_{5566} + 3\wp_{66}\wp_{55} + 6\wp_{56}^2\\
&(-7) \quad \wp_{4566} = 2\wp_{45}\wp_{66} + 4\wp_{46}\wp_{56} + 2\lambda_4\wp_{56} + 2\wp_{36}\\
&(-7) \quad \wp_{5556} = 6\wp_{55}\wp_{56} + 4\lambda_4\wp_{56} + 4\wp_{36}\\
&(-8) \quad \wp_{4466} = 2\wp_{44}\wp_{66} + 4\wp_{46}^2 + 4\lambda_4\wp_{46} + \lambda_4\wp_{55} + \wp_{35} + 6\wp_{26}\\
&\qquad\qquad - \wp_{4556} + 4\wp_{45}\wp_{56} + 2\wp_{46}\wp_{55} + 4\lambda_3\\
&(-8) \quad \wp_{5555} = 6\wp_{55}^2 + 16\lambda_4\wp_{55} + 4\wp_{35} + 24\wp_{26} - 6\wp_{4556}\\
&\qquad\qquad + 24\wp_{45}\wp_{56} + 12\wp_{46}\wp_{55} + 16\lambda_3\\
&(-9) \quad \wp_{4456} = 2\wp_{44}\wp_{56} + 4\wp_{45}\wp_{46} + \tfrac{8}{3}\lambda_4\wp_{45} + 2\wp_{25} - \tfrac{4}{3}\wp_{34} - \tfrac{1}{6}\wp_{4555} + \wp_{45}\wp_{55}\\
&(-9) \quad \wp_{3666} = 6\wp_{36}\wp_{66} + 2\lambda_4\wp_{45} - \tfrac{1}{2}\wp_{4555} + 3\wp_{45}\wp_{55}\\
&(-10) \quad \wp_{2666} = -\tfrac{1}{2}\wp_{4455} + \wp_{55}\wp_{44} + 2\lambda_4\wp_{56}^2 + \wp_{66}\lambda_4\wp_{55} + \wp_{66}\wp_{35}\\
&\qquad\qquad + 6\wp_{26}\wp_{66} + 2\wp_{45}^2 + 2\wp_{36}\wp_{56} - \tfrac{1}{2}\wp_{3566} - \tfrac{1}{2}\lambda_4\wp_{5566} + 4\wp_{66}\lambda_3\\
&(-10) \quad \wp_{4446} = 2\wp_{55}\wp_{44} - 2\wp_{24} - \wp_{4455} - 2\lambda_4^2\wp_{66} + 6\lambda_4\wp_{44} - 2\lambda_4\wp_{56}^2 - \tfrac{5}{2}\wp_{3566} + 4\wp_{45}^2\\
&\qquad\qquad - \wp_{66}\lambda_4\wp_{55} + 5\wp_{66}\wp_{35} + 10\wp_{36}\wp_{56} + 6\wp_{46}\wp_{44} + 4\wp_{66}\lambda_3 + \tfrac{1}{2}\lambda_4\wp_{5566}\\
&\qquad\qquad \vdots
\end{aligned}$$

Proof. Apply equation (3.32) to the first set of relations in Lemma 3.5.5 and rearrange. ■

The set of equations in Corollary 3.5.6 is of particular interest because it gives a generalisation of Corollary 2.1.15 from the elliptic case. This was the elliptic differential equation that allows higher order derivatives to be substituted out.

$$\wp''(u) = 6\wp(u)^2 - \tfrac{1}{2}g_2.$$

We discuss this generalisation more in Section 5.1 and present a full set of equations from $\wp_{6666}$ to to $\wp_{1111}$ in Appendix C.3. The other elliptic differential equation, which was derived in Theorem 2.1.14, is

$$[\wp'(u)]^2 = 4\wp(u)^3 - g_2\wp(u) - g_3.$$

A generalisation would be a set of equations that express the product of two 3-index $\wp$-functions using a polynomial of fundamental Abelian functions with total degree at most three. These are referred to as *quadratic 3-index relations*. For the cyclic-(4,5) curve these relations have been derived for the first few weights and are presented below.

Note that the Q-functions present here were those included in the basis (3.55).

$$
\begin{aligned}
&(-6) & \wp_{666}^2 &= 4\wp_{66}^3 + \wp_{56}^2 + 4\wp_{46}\wp_{66} - 4\wp_{55}\wp_{66} - 4\wp_{66}\lambda_4 + 4\wp_{44} + 2Q_{5566} \\
&(-7) & \wp_{566}\wp_{666} &= 4\wp_{66}^2\wp_{56} + 2\wp_{46}\wp_{56} - \wp_{55}\wp_{56} - 2\wp_{45}\wp_{66} + 2\wp_{36} \\
&(-8) & \wp_{566}^2 &= 8\wp_{26} + 4\wp_{55}\lambda_4 + 4\lambda_3 - 2Q_{4556} - 4\wp_{45}\wp_{56} + 4\wp_{56}^2\wp_{66} + \wp_{55}^2 \\
&(-8) & \wp_{556}\wp_{666} &= 2\wp_{56}^2\wp_{66} + 2\wp_{46}\wp_{55} - 2\wp_{55}^2 - 4\wp_{55}\lambda_4 + 2\wp_{45}\wp_{56} + \wp_{66}Q_{5566} \\
&&&\quad + 2\wp_{55}\wp_{66}^2 - 4\wp_{26} - 2\wp_{35} - 4\lambda_3 + 2Q_{4556} \\
&(-8) & \wp_{466}\wp_{666} &= 4\wp_{46}\wp_{66}^2 + 4\wp_{66}^2\lambda_4 + 2\wp_{46}^2 - 2\wp_{46}\wp_{55} - 2\wp_{55}\lambda_4 + \wp_{45}\wp_{56} \\
&&&\quad - 2\wp_{44}\wp_{66} - \wp_{66}Q_{5566} - 2\wp_{26} - 2\wp_{35} - 2\lambda_3 + Q_{4556} \\
&(-9) & \wp_{556}\wp_{566} &= \wp_{56}Q_{5566} + 2\wp_{66}\wp_{55}\wp_{56} + 2\wp_{56}^3 - \tfrac{4}{3}\wp_{45}\lambda_4 + \tfrac{1}{3}Q_{4555} - \tfrac{4}{3}\wp_{34} \\
&(-9) & \wp_{555}\wp_{666} &= -2\wp_{56}^3 + 6\wp_{66}\wp_{55}\wp_{56} + 4\wp_{45}\wp_{46} + \tfrac{8}{3}\wp_{45}\lambda_4 - 4\wp_{44}\wp_{56} \\
&&&\quad - 3\wp_{56}Q_{5566} + 8\wp_{36}\wp_{66} - \tfrac{4}{3}\wp_{34} - \tfrac{2}{3}Q_{4555} - 4\wp_{45}\wp_{55} + 8\wp_{56}\wp_{66}\lambda_4 \\
&(-9) & \wp_{466}\wp_{566} &= \tfrac{1}{3}Q_{4555} + \wp_{45}\wp_{55} - 2\wp_{25} + \tfrac{2}{3}\wp_{34} + 4\wp_{66}\wp_{46}\wp_{56} - \wp_{56}Q_{5566} \\
&&&\quad + 4\wp_{56}\wp_{66}\lambda_4 - 2\wp_{44}\wp_{56} - 2\wp_{45}\wp_{46} - \tfrac{4}{3}\wp_{45}\lambda_4 \\
&(-9) & \wp_{456}\wp_{666} &= 2\wp_{66}\wp_{46}\wp_{56} + 2\wp_{45}\wp_{66}^2 + 2\wp_{45}\wp_{46} + \tfrac{4}{3}\wp_{45}\lambda_4 + 2\wp_{44}\wp_{56} \\
&&&\quad + \tfrac{1}{2}\wp_{56}Q_{5566} + 2\wp_{36}\wp_{66} - \tfrac{2}{3}\wp_{34} - \tfrac{1}{3}Q_{4555} - 2\wp_{45}\wp_{55}
\end{aligned}
\tag{3.59}
$$

We do not give the details on the construction of these relations here, but instead refer the reader to Chapter 5. This chapter deals with new results for the first trigonal cases including a section on quadratic 3-index relations. The methods, problems and comments there are also relevant for the (4,5)-case.

Proposition 3.5.7. *There are a set of relations that are bi-linear in the 2-index and 3-index $\wp$-functions starting with the relations below.*

$$
\begin{aligned}
&(-6) & 0 &= -\wp_{555} + 2\wp_{456} + 2\wp_{566}\wp_{66} - 2\wp_{56}\wp_{666} \qquad (3.60) \\
&(-7) & 0 &= -2\wp_{446} + 2\wp_{455} - 2\wp_{466}\wp_{66} + 2\wp_{666}\lambda_4 + 2\wp_{46}\wp_{666} \\
&&&\quad - 2\wp_{556}\wp_{66} + \wp_{55}\wp_{666} + \wp_{566}\wp_{56} \\
&(-8) & 0 &= 2\wp_{46}\wp_{566} - 2\wp_{56}\wp_{466} + \wp_{555}\wp_{66} - 2\wp_{55}\wp_{566} + \wp_{556}\wp_{56} - 2\wp_{366} \\
&(-8) & 0 &= 2\wp_{456}\wp_{66} - \wp_{445} + \wp_{56}\wp_{466} - \wp_{366} - \wp_{566}\lambda_4 - \wp_{45}\wp_{666} - 2\wp_{46}\wp_{566} \\
&(-9) & 0 &= -2\wp_{455}\wp_{66} + 4\wp_{266} + 2\wp_{45}\wp_{566} + 2\wp_{466}\wp_{55} - 2\wp_{46}\wp_{556} \\
&&&\quad - \wp_{555}\wp_{56} + \wp_{556}\wp_{55} - 2\wp_{356} \\
&(-10) & 0 &= 2\wp_{256} - 4\wp_{346} - 2\wp_{446}\wp_{56} + 2\wp_{44}\wp_{566} + 2\wp_{45}\wp_{466} + \wp_{45}\wp_{556} \\
&&&\quad - 2\wp_{445}\wp_{66} - \wp_{455}\wp_{56} \\
&(-10) & 0 &= -2\wp_{366}\wp_{66} + 2\wp_{36}\wp_{666} + \wp_{45}\wp_{556} - 2\wp_{346} + \wp_{355} - \wp_{455}\wp_{56}
\end{aligned}
$$

$$\begin{aligned}
\textbf{(−10)}\quad 0 &= 6\wp_{256} + \wp_{355} - 2\wp_{456}\lambda_4 + \wp_{555}\lambda_4 + 2\wp_{44}\wp_{566} + 4\wp_{456}\wp_{46} - 2\wp_{455}\wp_{56} \\
&\quad + 4\wp_{45}\wp_{556} - 4\wp_{456}\wp_{55} + 2\wp_{46}\wp_{555} - 6\wp_{346} - 2\wp_{45}\wp_{466} - 4\wp_{446}\wp_{56} \\
\textbf{(−11)}\quad 0 &= \wp_{45}\wp_{555} - 4\wp_{246} - 2\wp_{445}\wp_{56} - 2\wp_{446}\wp_{55} + 2\wp_{46}\wp_{455} + 2\wp_{255} \\
&\quad - 2\wp_{345} + 2\wp_{44}\wp_{556} - \wp_{455}\wp_{55} \\
\textbf{(−11)}\quad 0 &= 8\wp_{666}\lambda_3 - 12\wp_{266}\wp_{66} + 12\wp_{26}\wp_{666} + 2\wp_{35}\wp_{666} - 12\wp_{246} + 6\wp_{255} \\
&\quad + 2\wp_{345} - 4\wp_{445}\wp_{56} + 4\wp_{45}\wp_{456} - 4\lambda_4\wp_{556}\wp_{66} + 2\lambda_4\wp_{55}\wp_{666} \\
&\quad + 2\lambda_4\wp_{566}\wp_{56} - 2\wp_{46}\wp_{455} + 2\wp_{44}\wp_{556} - 4\wp_{356}\wp_{66} + 6\wp_{36}\wp_{566} \\
&\quad - \wp_{455}\wp_{55} + \wp_{45}\wp_{555} - 4\wp_{56}\wp_{366}, \\
\textbf{(−12)}\quad 0 &= -6\wp_{344} - 10\wp_{445}\lambda_4 + 4\wp_{444}\wp_{56} - 8\wp_{44}\wp_{456} - 4\wp_{445}\wp_{46} + 8\wp_{45}\wp_{446} \\
&\quad + 10\wp_{356}\wp_{56} + 10\wp_{46}\wp_{366} + 5\wp_{55}\wp_{366} + 7\lambda_4\wp_{555}\wp_{66} - 14\lambda_4\wp_{55}\wp_{566} \\
&\quad + 7\lambda_4\wp_{556}\wp_{56} + 18\wp_{245} - 10\wp_{36}\wp_{466} - 2\wp_{445}\wp_{55} + \wp_{45}\wp_{455} \\
&\quad - 4\wp_{366}\lambda_4 - 10\wp_{566}\lambda_4^2 + 12\wp_{566}\lambda_3 + 30\wp_{26}\wp_{566} - 5\wp_{35}\wp_{566} \\
&\quad - 10\wp_{36}\wp_{556} + \wp_{44}\wp_{555} - 30\wp_{56}\wp_{266} \\
\textbf{(−13)}\quad 0 &= 2\wp_{34}\wp_{566} + 4\wp_{36}\wp_{456} - \wp_{355}\wp_{56} + 2\wp_{35}\wp_{556} - 2\wp_{356}\wp_{55} + \wp_{36}\wp_{555} \\
&\quad - 4\wp_{336} - 2\wp_{45}\wp_{366} - 4\wp_{346}\wp_{56}
\end{aligned}$$

Proof. The relations above can be calculated by cross differentiating suitable pairs of equations from Corollary 3.5.6. For example, equation (3.57) expresses $\wp_{6666}(\boldsymbol{u})$ while equation (3.58) expresses $\wp_{5666}(\boldsymbol{u})$. If we substitute for these equations into

$$\frac{\partial}{\partial u_5}\wp_{6666}(\boldsymbol{u}) - \frac{\partial}{\partial u_6}\wp_{5666}(\boldsymbol{u}) = 0,$$

then we find bilinear equation (3.60).

■

There is no analogue of Proposition 3.5.7 in the elliptic case, although similar relations have been derived using the cross differentiation method in the hyperelliptic and trigonal cases. Note that this method is much trickier in the (4,5)-case due to the large number of Q-functions in the basis. This means that often more than two pairs of equations from Corollary 3.5.6 need to be differentiated and examined. This gets increasing difficult at the higher weights as there are more interrelated Q-functions.

In Chapter 5 we present a new method of deriving every bilinear relation associated with a particular (n, s)-curve. This is used to complete the sets in the cyclic (3,4) and (3,5)-cases. It was applied also to the cyclic (4,5)-case with all the bilinear relations that have been derived available in the extra Appendix of files.

3.6 Addition formula

In this section we develop the two term addition formula for the σ-function. This generalises Theorem 2.1.26 in the elliptic case which was first generalised by Baker in equation (2.67) for a hyperelliptic function.

Theorem 3.6.1. *The σ-function associated to the cyclic (4,5)-curve satisfies a two term addition formula,*

$$-\frac{\sigma(\boldsymbol{u}+\boldsymbol{v})\sigma(\boldsymbol{u}-\boldsymbol{v})}{\sigma(\boldsymbol{u})^2\sigma(\boldsymbol{v})^2} = f(\boldsymbol{u},\boldsymbol{v}) - f(\boldsymbol{v},\boldsymbol{u}),$$

where $f(\boldsymbol{u},\boldsymbol{v})$ is a finite polynomial of Abelian functions associated to C. It may be written as

$$f(\boldsymbol{u},\boldsymbol{v}) = \big[P_{30}+P_{26}+P_{22}+P_{18}+P_{14}+P_{10}+P_6+P_2\big](\boldsymbol{u},\boldsymbol{v}),$$

where each $P_k(\boldsymbol{u},\boldsymbol{v})$ is a sum of terms with weight $-k$ in the Abelian functions and weight $k-30$ in $\boldsymbol{\lambda}$-monomials.

Proof. We seek to express the following ratio of sigma functions, labeled $\mathrm{LHS}(\boldsymbol{u},\boldsymbol{v})$, using a sum of Abelian functions.

$$\mathrm{LHS}(\boldsymbol{u},\boldsymbol{v}) = -\frac{\sigma(\boldsymbol{u}+\boldsymbol{v})\sigma(\boldsymbol{u}-\boldsymbol{v})}{\sigma(\boldsymbol{u})^2\sigma(\boldsymbol{v})^2}. \tag{3.61}$$

By Lemma 2.2.19 the σ-function has zeros of order one along $\Theta^{[5]}$ and no zeros anywhere else. This implies that $\mathrm{LHS}(\boldsymbol{u},\boldsymbol{v})$ has poles of order two along

$$(\Theta^{[5]}\times J)\cup(\Theta^{[5]}\times J)$$

but nowhere else. Hence we can express $\mathrm{LHS}(\boldsymbol{u},\boldsymbol{v})$ as

$$\mathrm{LHS}(\boldsymbol{u},\boldsymbol{v}) = \sum_j\Big(c_{1j}X_j(\boldsymbol{u})Y_j(\boldsymbol{v}) + c_{2j}X_j(\boldsymbol{v})Y_j(\boldsymbol{u})\Big), \tag{3.62}$$

where the X_j and Y_j are functions chosen from the basis in Theorem 3.5.3 and the $\{c_{1j},c_{2j}\}$ are constant coefficients. A simple modification of Lemma 3.5.2 will show that these constant coefficients must be polynomial functions of $\boldsymbol{\lambda}$.

Recall Lemma 3.4.2 which stated that $\sigma(\boldsymbol{u})$ is odd with respect to the change of variables $\boldsymbol{u}\mapsto[-1]\boldsymbol{u}$. Now consider the effect of $(\boldsymbol{u},\boldsymbol{v})\mapsto(\boldsymbol{v},\boldsymbol{u})$ on $\mathrm{LHS}(\boldsymbol{u},\boldsymbol{v})$.

$$\mathrm{LHS}(\boldsymbol{v},\boldsymbol{u}) = -\frac{\sigma(\boldsymbol{u}+\boldsymbol{v})\sigma\big([-1](\boldsymbol{u}-\boldsymbol{v})\big)}{\sigma(\boldsymbol{u})^2\sigma(\boldsymbol{v})^2} = -\mathrm{LHS}(\boldsymbol{u},\boldsymbol{v}).$$

So $\mathrm{LHS}(\boldsymbol{u},\boldsymbol{v})$ is antisymmetric, or odd with respect to the change $(\boldsymbol{u},\boldsymbol{v})\mapsto(\boldsymbol{v},\boldsymbol{u})$. Hence we must have $c_{2j}=-c_{1j}$ in equation (3.62) and equivalently $\mathrm{LHS}(\boldsymbol{u},\boldsymbol{v})$ can be written as $f(\boldsymbol{u},\boldsymbol{v})-f(\boldsymbol{v},\boldsymbol{u})$ for some function f.

Finally we use the fact that sigma has weight $+15$ to determine that the weight of LHS$(\boldsymbol{u}, \boldsymbol{v})$ is -30. Hence we need only consider those from the basis (3.62) that may give the correct overall weight. These will be the terms of weight -30 and the terms with higher weight that be combined with an appropriate $\boldsymbol{\lambda}$-monomial. This allows us to conclude that the function $f(\boldsymbol{u}, \boldsymbol{v})$ may be split up as indicated in the theorem.

■

The formula is derived explicitly below. This is just the first of a family of similar addition formula for the σ-function, related to the invariance expressed in equation (3.2). Work has been conducted on the extra addition formula in the trigonal cases in [30]. In [36] we see that this has inspired new results in the lower genus cases. Unfortunately such addition formula cannot currently be derived for the (4,5)-case using this approach as it requires the derivation of an additional basis and more polynomials in the σ-expansion, which is not currently computationally feasible.

Constructing $f(u, v)$

We now describe how Maple was used to explicitly find $f(\boldsymbol{u}, \boldsymbol{v})$. (The corresponding Maple worksheet is available in the extra Appendix of files.) First equation (3.62) is constructed with the undetermined constants. This contains 1348 terms, but only 647 undetermined coefficients due to the antisymmetry property. We may then determine the coefficients using the σ-expansion, in a similar method to that described in the previous section.

We must substitute the Abelian functions for their definitions to give a sum of rational function in σ-derivatives. We then take the numerator and substitute in the σ-expansion truncated after C_{43}. (The addition formula may contain $\boldsymbol{\lambda}$-monomials of weight -28, which were contained in C_{43} of the σ-expansion.) We must then multiply out the products, discard terms with weight in $\boldsymbol{\lambda}$ less than -28, collect the $\boldsymbol{uv\lambda}$-monomials, set the coefficients to zero and solve to find numerical values of the constants.

Note that the addition formula will have poles of order four and so we are required to multiply the product of four large expansions. This can be difficult and there are a number of techniques used for simplifying this problem.

- We design a procedure that is more efficient that the Maple `expand` command. We cannot take the same approach of Section 3.4 and consider only the terms at a single weight as there are unknowns at all weights. However, we can take a similar approach and only multiply the terms that will not be discarded. To achieve this we categorise the terms in the four expansions by their weights, and only consider the combinations of four entries that result in an acceptable $\boldsymbol{\lambda}$-weight.

- We may determine the conditions gradually by repeating the calculation, systematically adding C_k to the σ-expansion. For example, if we just use $\sigma(\boldsymbol{u}) = SW_{45}$ then we identify the coefficients in P_{30}. This causes much repetition of work, but can ease the later calculations as many coefficients are found to be zero at each stage.

- The system of equations is again over-determined so it may be possible to use specific values of $\boldsymbol{u}$ and $\boldsymbol{v}$ to ease computations.
- We again use distributed computing to expand the products in parallel. This was implemented using Distributed Maple, [72].

We find that polynomials are given as below.

$$\begin{aligned} P_2(\boldsymbol{u},\boldsymbol{v}) &= \tfrac{7}{3}\wp_{66}(\boldsymbol{u})\lambda_4\lambda_3\lambda_1 - \tfrac{3}{5}\wp_{66}(\boldsymbol{u})\lambda_2\lambda_1 - \tfrac{16}{3}\wp_{66}(\boldsymbol{u})\lambda_4^3\lambda_1 + \tfrac{4}{3}\wp_{66}(\boldsymbol{u})\lambda_4^2\lambda_0 \\ &\quad - \tfrac{2}{3}\wp_{66}(\boldsymbol{u})\lambda_3\lambda_0 \end{aligned}$$

$$\begin{aligned} P_6(\boldsymbol{u},\boldsymbol{v}) &= \tfrac{16}{3}\wp_{66}(\boldsymbol{u})\wp_{55}(\boldsymbol{v})\lambda_3\lambda_1 + \tfrac{7}{12}Q_{5566}(\boldsymbol{v})\lambda_4\lambda_0 - 4\wp_{46}(\boldsymbol{u})\wp_{66}(\boldsymbol{v})\lambda_4^2\lambda_1 \\ &\quad - 4\wp_{46}(\boldsymbol{v})\wp_{66}(\boldsymbol{u})\lambda_3\lambda_1 + \tfrac{5}{2}\wp_{55}(\boldsymbol{u})\wp_{66}(\boldsymbol{v})\lambda_4\lambda_0 - 6\wp_{55}(\boldsymbol{v})\wp_{66}(\boldsymbol{u})\lambda_4^2\lambda_1 + 3\wp_{44}(\boldsymbol{u})\lambda_3\lambda_1 \\ &\quad - \tfrac{3}{2}Q_{5566}(\boldsymbol{v})\lambda_3\lambda_1 + \tfrac{4}{3}\wp_{44}(\boldsymbol{v})\lambda_4\lambda_0 - \tfrac{16}{3}\wp_{44}(\boldsymbol{v})\lambda_4^2\lambda_1 - 2Q_{5566}(\boldsymbol{v})\lambda_4^2\lambda_1 \end{aligned}$$

$$\begin{aligned} P_{10}(\boldsymbol{u},\boldsymbol{v}) &= \big[5\wp_{36}(\boldsymbol{u})\wp_{56}(\boldsymbol{v}) - 2\wp_{24}(\boldsymbol{v}) - \tfrac{5}{3}Q_{3566}(\boldsymbol{v}) - \tfrac{5}{2}\wp_{44}(\boldsymbol{v})\wp_{55}(\boldsymbol{u}) - \tfrac{1}{24}Q_{4455}(\boldsymbol{v}) \\ &\quad - \tfrac{5}{2}\wp_{35}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) + \tfrac{5}{4}Q_{5566}(\boldsymbol{u})\wp_{55}(\boldsymbol{v})\big]\lambda_0 + \big[4\wp_{44}(\boldsymbol{v})\wp_{46}(\boldsymbol{u}) + \tfrac{8}{3}\wp_{24}(\boldsymbol{v}) \\ &\quad + \tfrac{18}{5}\wp_{36}(\boldsymbol{u})\wp_{56}(\boldsymbol{v}) + 6\wp_{26}(\boldsymbol{v})\wp_{66}(\boldsymbol{u}) - \tfrac{11}{2}\wp_{35}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) + 6\wp_{44}(\boldsymbol{v})\wp_{55}(\boldsymbol{u}) \\ &\quad - \tfrac{3}{2}Q_{5566}(\boldsymbol{u})\wp_{46}(\boldsymbol{v}) + \tfrac{7}{3}Q_{5566}(\boldsymbol{u})\wp_{55}(\boldsymbol{v}) - \tfrac{5}{3}Q_{3566}(\boldsymbol{u}) - \tfrac{2}{3}Q_{4455}(\boldsymbol{u})\big]\lambda_4\lambda_1 \end{aligned}$$

$$\begin{aligned} P_{14}(\boldsymbol{u},\boldsymbol{v}) &= \big[\wp_{25}(\boldsymbol{u})\wp_{45}(\boldsymbol{v}) + \tfrac{1}{3}Q_{3445}(\boldsymbol{u}) + \tfrac{4}{3}\wp_{14}(\boldsymbol{v}) + \tfrac{5}{12}Q_{3366}(\boldsymbol{v}) - \tfrac{11}{20}Q_{3556}(\boldsymbol{u})\wp_{56}(\boldsymbol{v}) \\ &\quad + \wp_{22}(\boldsymbol{u}) - 6\wp_{44}(\boldsymbol{u})\wp_{26}(\boldsymbol{v}) + 3Q_{5566}(\boldsymbol{v})\wp_{26}(\boldsymbol{u}) + \tfrac{18}{5}\wp_{16}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) - \tfrac{7}{3}\wp_{24}(\boldsymbol{u})\wp_{55}(\boldsymbol{v}) \\ &\quad + \tfrac{23}{12}Q_{3566}(\boldsymbol{v})\wp_{55}(\boldsymbol{u}) - \tfrac{1}{24}Q_{4555}(\boldsymbol{u})\wp_{45}(\boldsymbol{v}) - \tfrac{27}{10}\wp_{33}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) - \tfrac{1}{2}Q_{4455}(\boldsymbol{v})\wp_{46}(\boldsymbol{u}) \\ &\quad - \tfrac{3}{5}Q_{3456}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) - 2\wp_{24}(\boldsymbol{v})\wp_{46}(\boldsymbol{u}) + \tfrac{2}{3}Q_{4455}(\boldsymbol{v})\wp_{55}(\boldsymbol{u}) - 3\wp_{35}(\boldsymbol{u})Q_{5566}(\boldsymbol{v}) \\ &\quad - \tfrac{2}{3}\wp_{34}(\boldsymbol{u})\wp_{45}(\boldsymbol{v}) - \tfrac{11}{2}\wp_{35}(\boldsymbol{u})\wp_{44}(\boldsymbol{v}) + \tfrac{3}{10}Q_{2556}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) - \tfrac{3}{2}Q_{3566}(\boldsymbol{v})\wp_{46}(\boldsymbol{u}) \\ &\quad + \tfrac{1}{10}Q_{2566}(\boldsymbol{v})\wp_{56}(\boldsymbol{u})\big]\lambda_1 + \big[\tfrac{1}{24}Q_{3566}(\boldsymbol{v})\wp_{55}(\boldsymbol{u}) + \tfrac{1}{6}\wp_{25}(\boldsymbol{u})\wp_{45}(\boldsymbol{v}) - \tfrac{1}{10}Q_{3556}(\boldsymbol{u})\wp_{56}(\boldsymbol{v}) \\ &\quad + \tfrac{1}{6}\wp_{34}(\boldsymbol{u})\wp_{45}(\boldsymbol{v}) - \tfrac{1}{20}Q_{2566}(\boldsymbol{v})\wp_{56}(\boldsymbol{u}) - \tfrac{1}{30}Q_{3366}(\boldsymbol{v}) - \tfrac{1}{15}Q_{3445}(\boldsymbol{u}) + \tfrac{8}{15}\wp_{14}(\boldsymbol{v})\big]\lambda_4\lambda_2 \\ &\quad - \tfrac{64}{3}\wp_{16}(\boldsymbol{v})\wp_{66}(\boldsymbol{u})\lambda_4^4 + \tfrac{68}{3}\wp_{16}(\boldsymbol{v})\wp_{66}(\boldsymbol{u})\lambda_4^2\lambda_3 - \tfrac{32}{3}\wp_{16}(\boldsymbol{v})\wp_{66}(\boldsymbol{u})\lambda_3^2 \end{aligned}$$

$$\begin{aligned} P_{18}(\boldsymbol{u},\boldsymbol{v}) &= \big[\tfrac{3}{10}Q_{2566}(\boldsymbol{u})\wp_{36}(\boldsymbol{v}) - \wp_{23}(\boldsymbol{u})\wp_{45}(\boldsymbol{v}) - \tfrac{1}{5}Q_{2345}(\boldsymbol{v}) + \tfrac{3}{4}\wp_{56}(\boldsymbol{u})Q_{2366}(\boldsymbol{v}) \\ &\quad + \tfrac{2}{5}\wp_{12}(\boldsymbol{u}) + \tfrac{1}{24}Q_{4555}(\boldsymbol{u})\wp_{25}(\boldsymbol{v}) + \tfrac{4}{3}\wp_{34}(\boldsymbol{u})\wp_{25}(\boldsymbol{v}) - 2\wp_{45}(\boldsymbol{u})\wp_{15}(\boldsymbol{v}) + \tfrac{1}{3}\wp_{55}(\boldsymbol{u})\wp_{14}(\boldsymbol{v}) \\ &\quad - \tfrac{1}{24}\wp_{55}(\boldsymbol{u})Q_{3445}(\boldsymbol{v}) + \tfrac{1}{6}\wp_{55}(\boldsymbol{u})Q_{3366}(\boldsymbol{v}) - \tfrac{1}{24}\wp_{34}(\boldsymbol{v})Q_{4555}(\boldsymbol{u}) - \tfrac{3}{5}Q_{3556}(\boldsymbol{u})\wp_{36}(\boldsymbol{v}) \\ &\quad - \tfrac{1}{10}Q_{3344}(\boldsymbol{v})\big]\lambda_2 + \big[\tfrac{7}{20}Q_{2566}(\boldsymbol{u})\wp_{36}(\boldsymbol{v}) - \tfrac{3}{5}Q_{3556}(\boldsymbol{u})\wp_{36}(\boldsymbol{v}) - \tfrac{1}{4}Q_{3566}(\boldsymbol{u})\wp_{35}(\boldsymbol{v}) + \ldots \end{aligned}$$

$$\cdots + \tfrac{1}{3}\wp_{23}(\boldsymbol{u})\wp_{45}(\boldsymbol{v}) - \tfrac{2}{3}\wp_{16}(\boldsymbol{v})Q_{5566}(\boldsymbol{u}) - \tfrac{1}{6}\wp_{45}(\boldsymbol{u})\wp_{15}(\boldsymbol{v}) + 2\wp_{66}(\boldsymbol{u})Q_{1556}(\boldsymbol{v})$$
$$+ 2\wp_{66}(\boldsymbol{u})Q_{1466}(\boldsymbol{v}) + \tfrac{4}{3}\wp_{44}(\boldsymbol{v})\wp_{16}(\boldsymbol{u})\big]\lambda_4\lambda_3 + \big[4\wp_{66}(\boldsymbol{u})Q_{1556}(\boldsymbol{v})$$
$$+ \tfrac{1}{10}Q_{2566}(\boldsymbol{u})\wp_{36}(\boldsymbol{v}) + 8\wp_{16}(\boldsymbol{v})Q_{5566}(\boldsymbol{u}) + 4\wp_{66}(\boldsymbol{u})Q_{1466}(\boldsymbol{v}) - \tfrac{64}{3}\wp_{44}(\boldsymbol{v})\wp_{16}(\boldsymbol{u})\big]\lambda_4^3$$

$$P_{22}(\boldsymbol{u},\boldsymbol{v}) = \big[\wp_{35}(\boldsymbol{v})\wp_{14}(\boldsymbol{u}) + \tfrac{1}{2}\wp_{15}(\boldsymbol{v})\wp_{25}(\boldsymbol{u}) - \tfrac{58}{15}\wp_{16}(\boldsymbol{u})Q_{3566}(\boldsymbol{v}) - \tfrac{1}{2}Q_{3366}(\boldsymbol{v})\wp_{35}(\boldsymbol{u})$$
$$- \tfrac{1}{40}Q_{2566}(\boldsymbol{u})Q_{3556}(\boldsymbol{v}) - \tfrac{1}{5}Q_{3456}(\boldsymbol{u})Q_{3566}(\boldsymbol{v}) - 2Q_{1466}(\boldsymbol{u})\wp_{44}(\boldsymbol{v}) - Q_{1556}(\boldsymbol{u})Q_{5566}(\boldsymbol{v})$$
$$+ 2\wp_{44}(\boldsymbol{u})Q_{1556}(\boldsymbol{v}) - Q_{1466}(\boldsymbol{v})Q_{5566}(\boldsymbol{u}) - \tfrac{1}{24}\wp_{23}(\boldsymbol{u})Q_{4555}(\boldsymbol{v}) + \tfrac{2}{5}\wp_{33}(\boldsymbol{u})Q_{3566}(\boldsymbol{v})$$
$$+ \tfrac{8}{3}Q_{1266}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) + \tfrac{16}{3}Q_{1356}(\boldsymbol{v})\wp_{66}(\boldsymbol{u}) - \tfrac{4}{3}\wp_{23}(\boldsymbol{v})\wp_{34}(\boldsymbol{u}) + \tfrac{1}{10}Q_{2556}(\boldsymbol{u})Q_{3566}(\boldsymbol{v})$$
$$+ \tfrac{4}{3}Q_{1444}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) + \tfrac{1}{4}Q_{3445}(\boldsymbol{v})\wp_{35}(\boldsymbol{u}) - \tfrac{4}{3}\wp_{16}(\boldsymbol{u})Q_{4455}(\boldsymbol{v}) - \tfrac{1}{4}Q_{2366}(\boldsymbol{u})\wp_{36}(\boldsymbol{v})$$
$$- 4\wp_{16}(\boldsymbol{u})\wp_{24}(\boldsymbol{v}) - \tfrac{13}{6}\wp_{15}(\boldsymbol{u})\wp_{34}(\boldsymbol{v}) + \tfrac{1}{24}\wp_{15}(\boldsymbol{u})Q_{4555}(\boldsymbol{v}) - \wp_{23}(\boldsymbol{v})\wp_{25}(\boldsymbol{u})\big]\lambda_3$$
$$+ \big[3Q_{1266}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) - 6Q_{1356}(\boldsymbol{v})\wp_{66}(\boldsymbol{u}) - \tfrac{1}{15}Q_{3456}(\boldsymbol{u})Q_{3566}(\boldsymbol{v}) - 4\wp_{44}(\boldsymbol{u})Q_{1556}(\boldsymbol{v})$$
$$- \tfrac{3}{2}Q_{1466}(\boldsymbol{v})Q_{5566}(\boldsymbol{u}) - \tfrac{1}{10}Q_{2445}(\boldsymbol{u})\wp_{36}(\boldsymbol{v}) + 4Q_{1466}(\boldsymbol{u})\wp_{44}(\boldsymbol{v}) + \tfrac{1}{20}Q_{2566}(\boldsymbol{u})Q_{3556}(\boldsymbol{v})$$
$$- \tfrac{3}{10}\wp_{33}(\boldsymbol{u})Q_{3566}(\boldsymbol{v}) + \tfrac{1}{30}Q_{2556}(\boldsymbol{u})Q_{3566}(\boldsymbol{v}) - \tfrac{4}{3}Q_{1444}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) + \tfrac{3}{2}Q_{1556}(\boldsymbol{u})Q_{5566}(\boldsymbol{v})$$
$$+ \tfrac{8}{3}\wp_{16}(\boldsymbol{u})Q_{4455}(\boldsymbol{v}) + \tfrac{106}{15}\wp_{16}(\boldsymbol{u})Q_{3566}(\boldsymbol{v}) + \tfrac{1}{2}Q_{2366}(\boldsymbol{u})\wp_{36}(\boldsymbol{v}) + \tfrac{32}{3}\wp_{16}(\boldsymbol{u})\wp_{24}(\boldsymbol{v})\big]\lambda_4^2$$

$$P_{26}(\boldsymbol{u},\boldsymbol{v}) = \big[\tfrac{1}{24}Q_{3566}(\boldsymbol{u})Q_{2266}(\boldsymbol{v}) - \tfrac{1}{60}Q_{2556}(\boldsymbol{u})Q_{3366}(\boldsymbol{v}) - \tfrac{1}{2}Q_{1556}(\boldsymbol{u})Q_{4455}(\boldsymbol{v})$$
$$- Q_{1146}(\boldsymbol{u}) - \tfrac{7}{6}Q_{5566}(\boldsymbol{v})Q_{1266}(\boldsymbol{u}) + \tfrac{1}{2}Q_{1466}(\boldsymbol{v})Q_{4455}(\boldsymbol{u}) + \tfrac{1}{30}Q_{2556}(\boldsymbol{u})Q_{3445}(\boldsymbol{v})$$
$$- \tfrac{1}{15}Q_{3445}(\boldsymbol{v})Q_{3456}(\boldsymbol{u}) - \tfrac{7}{3}Q_{5566}(\boldsymbol{v})Q_{1356}(\boldsymbol{u}) + \tfrac{7}{6}Q_{3566}(\boldsymbol{u})Q_{1556}(\boldsymbol{v}) - \wp_{11}(\boldsymbol{u})\wp_{55}(\boldsymbol{v})$$
$$- \tfrac{1}{2}Q_{5566}(\boldsymbol{u})Q_{1444}(\boldsymbol{v}) + \tfrac{1}{20}Q_{3556}(\boldsymbol{v})Q_{2445}(\boldsymbol{u}) + \tfrac{1}{6}Q_{3566}(\boldsymbol{u})Q_{2356}(\boldsymbol{v}) + \tfrac{1}{4}Q_{1155}(\boldsymbol{u})$$
$$- \tfrac{1}{30}Q_{3456}(\boldsymbol{v})Q_{3366}(\boldsymbol{u}) + \tfrac{1}{40}Q_{2566}(\boldsymbol{u})Q_{2445}(\boldsymbol{v}) + \tfrac{7}{6}Q_{1466}(\boldsymbol{v})Q_{3566}(\boldsymbol{u})$$
$$+ 2Q_{1466}(\boldsymbol{v})\wp_{24}(\boldsymbol{u}) - \tfrac{3}{10}Q_{3445}(\boldsymbol{v})\wp_{33}(\boldsymbol{u}) + \tfrac{4}{15}Q_{2556}(\boldsymbol{u})\wp_{14}(\boldsymbol{v}) - \tfrac{1}{6}\wp_{45}(\boldsymbol{u})Q_{1346}(\boldsymbol{v})$$
$$- \tfrac{128}{15}\wp_{16}(\boldsymbol{v})\wp_{14}(\boldsymbol{u}) + \tfrac{2}{5}\wp_{33}(\boldsymbol{u})Q_{3366}(\boldsymbol{v}) - \tfrac{8}{15}Q_{3366}(\boldsymbol{v})\wp_{16}(\boldsymbol{u}) - \tfrac{1}{24}Q_{2335}(\boldsymbol{u})\wp_{45}(\boldsymbol{v})$$
$$+ 2\wp_{23}(\boldsymbol{u})\wp_{15}(\boldsymbol{v}) + 4\wp_{16}(\boldsymbol{v})\wp_{22}(\boldsymbol{u}) - \tfrac{12}{5}\wp_{33}(\boldsymbol{u})\wp_{14}(\boldsymbol{v}) - \tfrac{1}{5}\wp_{36}(\boldsymbol{u})Q_{2245}(\boldsymbol{v})$$
$$+ 3\wp_{44}(\boldsymbol{u})Q_{1266}(\boldsymbol{v}) + \tfrac{4}{3}\wp_{44}(\boldsymbol{v})Q_{1444}(\boldsymbol{u}) + \tfrac{3}{5}\wp_{36}(\boldsymbol{v})Q_{2344}(\boldsymbol{u}) - 2\wp_{25}(\boldsymbol{u})\wp_{13}(\boldsymbol{v})$$
$$- 2\wp_{35}(\boldsymbol{u})\wp_{12}(\boldsymbol{v}) + \tfrac{1}{24}\wp_{45}(\boldsymbol{u})Q_{2236}(\boldsymbol{v}) - \tfrac{8}{15}Q_{3456}(\boldsymbol{u})\wp_{14}(\boldsymbol{v}) - \wp_{66}(\boldsymbol{v})Q_{1244}(\boldsymbol{u})$$
$$+ \tfrac{14}{15}\wp_{16}(\boldsymbol{v})Q_{3445}(\boldsymbol{u}) + 6\wp_{44}(\boldsymbol{u})Q_{1356}(\boldsymbol{v}) - 2Q_{1556}(\boldsymbol{u})\wp_{24}(\boldsymbol{v})\big]\lambda_4$$

$$P_{30}(\boldsymbol{u},\boldsymbol{v}) = \tfrac{1}{4}Q_{114466}(\boldsymbol{u}) + \tfrac{5}{3}Q_{3566}(\boldsymbol{v})Q_{1356}(\boldsymbol{u}) - \tfrac{1}{3}\wp_{14}(\boldsymbol{v})Q_{2356}(\boldsymbol{u}) + \tfrac{1}{3}Q_{1556}(\boldsymbol{v})\wp_{14}(\boldsymbol{u})$$
$$+ \tfrac{1}{10}Q_{3344}(\boldsymbol{v})Q_{3456}(\boldsymbol{u}) - \tfrac{1}{2}\wp_{25}(\boldsymbol{v})Q_{1346}(\boldsymbol{u}) + \tfrac{1}{2}Q_{4556}(\boldsymbol{v})\wp_{11}(\boldsymbol{u}) - \tfrac{1}{4}Q_{2236}(\boldsymbol{u})\wp_{25}(\boldsymbol{v})$$
$$+ \tfrac{3}{5}Q_{3344}(\boldsymbol{u})\wp_{16}(\boldsymbol{v}) + \tfrac{1}{24}Q_{1346}(\boldsymbol{v})Q_{4555}(\boldsymbol{u}) + \tfrac{1}{2}Q_{1145}(\boldsymbol{v})\wp_{56}(\boldsymbol{u}) - \tfrac{2}{5}Q_{2345}(\boldsymbol{u})\wp_{33}(\boldsymbol{v})$$
$$- \tfrac{1}{2}Q_{1556}(\boldsymbol{v})\wp_{22}(\boldsymbol{u}) + \tfrac{1}{2}Q_{1155}(\boldsymbol{u})\wp_{46}(\boldsymbol{v}) + \tfrac{5}{6}Q_{3566}(\boldsymbol{v})Q_{1266}(\boldsymbol{u}) + \tfrac{1}{12}Q_{2236}(\boldsymbol{v})\wp_{34}(\boldsymbol{u})$$
$$+ \tfrac{7}{5}\wp_{12}(\boldsymbol{v})Q_{3456}(\boldsymbol{u}) - \tfrac{2}{3}Q_{1356}(\boldsymbol{v})Q_{4455}(\boldsymbol{u}) - \tfrac{1}{20}Q_{2245}(\boldsymbol{v})Q_{2566}(\boldsymbol{u}) + \ldots$$

$$
\begin{aligned}
&\cdots + \tfrac{1}{3}Q_{1466}(\boldsymbol{u})Q_{3445}(\boldsymbol{v}) - \tfrac{4}{3}Q_{1466}(\boldsymbol{u})\wp_{14}(\boldsymbol{v}) - \wp_{26}(\boldsymbol{v})\wp_{11}(\boldsymbol{u}) + \wp_{36}(\boldsymbol{v})Q_{1245}(\boldsymbol{u}) \\
&+ \tfrac{1}{10}Q_{2345}(\boldsymbol{u})Q_{2556}(\boldsymbol{v}) + \wp_{26}(\boldsymbol{v})Q_{1246}(\boldsymbol{u}) + Q_{2346}(\boldsymbol{v})\wp_{15}(\boldsymbol{u}) + \tfrac{1}{2}Q_{1444}(\boldsymbol{v})Q_{3566}(\boldsymbol{u}) \\
&+ \tfrac{1}{5}\wp_{16}(\boldsymbol{v})Q_{2345}(\boldsymbol{u}) - \tfrac{1}{12}Q_{3445}(\boldsymbol{v})Q_{2266}(\boldsymbol{u}) + \tfrac{1}{2}Q_{5566}(\boldsymbol{v})Q_{1244}(\boldsymbol{u}) - \tfrac{1}{5}\wp_{33}(\boldsymbol{v})Q_{3344}(\boldsymbol{u}) \\
&+ \tfrac{1}{6}Q_{1466}(\boldsymbol{v})Q_{3366}(\boldsymbol{u}) + \tfrac{1}{4}8Q_{4555}(\boldsymbol{v})Q_{2335}(\boldsymbol{u}) + \tfrac{1}{3}Q_{4455}(\boldsymbol{v})Q_{1266}(\boldsymbol{u}) + Q_{1246}(\boldsymbol{v})\wp_{35}(\boldsymbol{u}) \\
&- \tfrac{1}{6}Q_{4455}(\boldsymbol{v})Q_{1444}(\boldsymbol{u}) + \tfrac{1}{5}Q_{2345}(\boldsymbol{v})Q_{3456}(\boldsymbol{u}) + \tfrac{1}{2}Q_{1156}(\boldsymbol{v})\wp_{45}(\boldsymbol{u}) - \wp_{46}(\boldsymbol{v})Q_{1146}(\boldsymbol{u}) \\
&+ Q_{2256}(\boldsymbol{v})\wp_{15}(\boldsymbol{u}) - \tfrac{32}{5}\wp_{12}(\boldsymbol{v})\wp_{16}(\boldsymbol{u}) - \tfrac{7}{3}Q_{1356}(\boldsymbol{v})\wp_{24}(\boldsymbol{u}) - \tfrac{1}{2}\wp_{33}(\boldsymbol{v})Q_{1455}(\boldsymbol{u}) \\
&- \tfrac{1}{48}Q_{2236}(\boldsymbol{v})Q_{4555}(\boldsymbol{u}) - \tfrac{1}{20}Q_{3344}(\boldsymbol{v})Q_{2556}(\boldsymbol{u}) - \tfrac{1}{10}Q_{2566}(\boldsymbol{v})Q_{2344}(\boldsymbol{u}) \\
&- \tfrac{1}{10}Q_{3556}(\boldsymbol{v})Q_{2245}(\boldsymbol{u}) - \tfrac{1}{20}Q_{3556}(\boldsymbol{v})Q_{2344}(\boldsymbol{u}) + \tfrac{1}{8}Q_{2445}(\boldsymbol{v})Q_{2366}(\boldsymbol{u}) \\
&+ \tfrac{1}{6}Q_{1556}(\boldsymbol{v})Q_{3366}(\boldsymbol{u}) - \tfrac{1}{2}Q_{1255}(\boldsymbol{u})\wp_{26}(\boldsymbol{v}) - \tfrac{1}{2}Q_{1144}(\boldsymbol{v})\wp_{66}(\boldsymbol{u}) + \tfrac{1}{2}\wp_{44}(\boldsymbol{v})Q_{1166}(\boldsymbol{u}) \\
&- \tfrac{5}{12}Q_{2335}(\boldsymbol{v})\wp_{34}(\boldsymbol{u}) + \tfrac{1}{6}Q_{3366}(\boldsymbol{v})Q_{2266}(\boldsymbol{u}) + \tfrac{2}{3}\wp_{24}(\boldsymbol{v})Q_{1266}(\boldsymbol{u}) - Q_{1455}(\boldsymbol{v})\wp_{16}(\boldsymbol{u}) \\
&- \tfrac{1}{2}Q_{2256}(\boldsymbol{v})\wp_{23}(\boldsymbol{u}) - \wp_{22}(\boldsymbol{u})Q_{1466}(\boldsymbol{v}) + \tfrac{1}{2}\wp_{55}(\boldsymbol{v})Q_{1146}(\boldsymbol{u}) - Q_{2346}(\boldsymbol{v})\wp_{23}(\boldsymbol{u}) \\
&+ \tfrac{1}{6}Q_{1346}(\boldsymbol{u})\wp_{34}(\boldsymbol{v}) - \tfrac{1}{3}\wp_{14}(\boldsymbol{u})Q_{2266}(\boldsymbol{v}) + \wp_{13}(\boldsymbol{v})Q_{2456}(\boldsymbol{u}) + \tfrac{14}{5}\wp_{33}(\boldsymbol{u})\wp_{12}(\boldsymbol{v}) \\
&- \tfrac{1}{5}\wp_{12}(\boldsymbol{v})Q_{2556}(\boldsymbol{u}) - \tfrac{2}{3}\wp_{24}(\boldsymbol{v})Q_{1444}(\boldsymbol{u}) + \tfrac{1}{4}Q_{2335}(\boldsymbol{v})\wp_{25}(\boldsymbol{u}) + \wp_{44}(\boldsymbol{v})Q_{1244}(\boldsymbol{u}) \\
&- \tfrac{1}{3}Q_{2356}(\boldsymbol{v})Q_{3366}(\boldsymbol{u}) + \tfrac{1}{3}Q_{3445}(\boldsymbol{v})Q_{1556}(\boldsymbol{u}) - \tfrac{1}{6}Q_{3445}(\boldsymbol{v})Q_{2356}(\boldsymbol{u}) - \wp_{35}(\boldsymbol{v})\wp_{11}(\boldsymbol{u}).
\end{aligned}
$$

Note that the 6-index Q-function in basis (3.55) is present in P_{30} and was essential for the construction of the addition formula.

The two term addition formulae are the only results in this document that will change with a different choice of c in the definition of the σ-function, (see Remark 2.2.23). Using an alternative constant will change the polynomial $f(\boldsymbol{u}, \boldsymbol{v})$ by a multiplicative constant factor, (see Appendix B).

3.7 Applications in the KP hierarchy

The Weierstrass $\wp$-function had applications in many areas of mathematics, (see for example [3]). It may be used to construct solutions for problems such as the pendulum equation, and in particular the function,

$$W(x,t) = A\wp(x - ct) + B,$$

gives a traveling wave solution to the KdV equation,

$$W_t + 12WW_x + W_{xxx} = 0.$$

A similar application may be derived from Baker's results on the Kleinian $\wp$-functions associated to a (2,5)-curve. In [10] (page 47 of the reprinted edition) Baker derived the following equation between the hyperelliptic $\wp$-functions.

$$\wp_{2222} - 6\wp_{22}^2 = \frac{1}{2}\lambda_3 + \lambda_4\wp_{22} + 4\wp_{12}.$$

We make the change of variables $u_1 = T, u_2 = x$ to get

$$\wp_{xxxx} - 6\wp_{xx}^2 = \frac{1}{2}\lambda_3 + \lambda_4\wp_{xx} + 4\wp_{Tx}.$$

We then differentiate with respect to x.

$$\wp_{xxxxx} - 12\wp_{xx}\wp_{xxx} = \lambda_4\wp_{xxx} + 4\wp_{Txx}.$$

Now set $\lambda_4 = 0$, $t = -4T$ and define $W(x,t) = -\wp_{xx}(x,t)$. Then $W(x,t)$ is a solution of

$$0 = W_t + 12WW_x + W_{xxx}.$$

So Baker had constructed a 2-soliton solution to the periodic KdV equation.

More recently, in [21] and [35], the cyclic trigonal curves have been linked to the Boussinesq equation. In [11] the $\wp$-functions associated to the cyclic (3,5)-curve are shown to satisfy

$$\wp_{4444} = 6\wp_{44}^2 - 3\wp_{33}.$$

Differentiate twice with respect to u_4 to give the Boussinesq equation for $\wp_{44}$, with u_4 playing the role of the space variable and u_3 the time variable.

Given these results it is not surprising to see a similar result occurring in the cyclic (4,5)-case. This time a solution to the KP-equation can be constructed.

Recall equation (3.57) derived in Corollary 3.5.6.

$$\wp_{6666} = 6\wp_{66}^2 - 3\wp_{55} + 4\wp_{46}.$$

Differentiate twice with respect to u_6 to obtain

$$\wp_{666666} = 12\frac{\partial}{\partial u_6}\big(\wp_{66}\wp_{666}\big) - 3\wp_{5566} + 4\wp_{4666}. \tag{3.63}$$

This time we use the substitutions $u_6 = x, u_5 = y, u_4 = t$ and define the function

$$W(x,y,t) = \wp_{xx}(u_1,u_2,u_3,t,y,x).$$

Substituting into equation (3.63) and rearranging will show that $W(x,y,t)$ is a solution of to the following form of the KP-equation.

$$\big[W_{xxx} - 12WW_x - 4W_t\big]_x + 3W_{yy} = 0.$$

In fact this is just a special case of the following general result for Abelian functions associated with algebraic curves.

Theorem 3.7.1. *Let C be an (n,s)-curve with genus g as defined by equation (2.34). Define the σ-function and Abelian functions associated with C as normal. Finally define the function $W(\boldsymbol{u}) = \wp_{gg}(\boldsymbol{u})$, which we may denote $W(x,y,t)$ after applying the substitutions*

$$u_g = x, \qquad u_{g-1} = y, \qquad u_{g-2} = t.$$

Then, if $n \geq 4$, the function $W(x,y,t)$ will satisfy the following version of the KP-equation.

$$\big(W_{xxx} - 12WW_x - bW_t\big)_x - aW_{yy} = 0, \tag{3.64}$$

for some constants a, b.

Proof. Recall Definition 3.3.2 and Theorem 3.3.4 which gave the Sato weights for C. The weights of the variables $\boldsymbol{u}$ were the entries in the Weierstrass gap sequence generated by (n,s). These are the natural numbers not representable in the form $\alpha n + \beta s$ where $\alpha, \beta \in \mathbb{N}$.

Since $s > n > 4$ we know that $\{1,2,3\}$ cannot be represented in this form. Therefore we have

$$\text{weight}(u_g) = +1, \qquad \text{weight}(u_{g-1}) = +2, \qquad \text{weight}(u_{g-2}) = +3.$$

By Lemma 3.3.10 this implies the 2-index $\wp$-functions will have weights

$$\text{wt}(\wp_{g,g}) = -2, \;\; \text{wt}(\wp_{g,g-1}) = -3, \;\; \text{wt}(\wp_{g-1,g-1}) = -4, \;\; \text{wt}(\wp_{g-2,g}) = -4,$$

with all the other 2-index $\wp$-functions having a lower weight.

Next consider Q_{gggg}, which by Lemma 3.3.14 is the highest weight Q-function with weight -4. We noted in Section 3.5 that the Q-functions all belong to $\Gamma\big(J, \mathcal{O}(2\Theta^{[g-1]})\big)$, the space of Abelian functions associated to C with poles of at most order 2 when $\boldsymbol{u} \in \Theta^{[g-1]}$. By Lemma 3.5.2 we can express Q_{gggg} as

$$Q_{g,g,g,g} = a\wp_{g-1,g-1} + b\wp_{g-2,g} + c, \qquad (a, b, c \text{ constants}),$$

since these are the only Abelian functions of weight -4, and there is no function of lower weight that could be combined with a $\boldsymbol{\lambda}$-monomial to give the correct weight. We use Definition 3.1.2 to substitute Q_{gggg} for its expression in $\wp$-functions.

$$\wp_{g,g,g,g} - 6\wp_{g,g}^2 = a\wp_{g-1,g-1} + b\wp_{g-2,g} + c.$$

Then differentiate twice with respect to u_g to give

$$\wp_{g,g,g,g,g,g} \quad = 12\tfrac{\partial}{\partial u_g}\big(\wp_{g,g}\wp_{g,g,g}\big) + a\wp_{g-1,g-1,g,g} + b\wp_{g-2,g,g,g}.$$

Then make the substitutions suggested in the theorem to obtain equation (3.64).

■

Hence the Abelian functions associated to the cyclic (n, s)-curve with $n > 4$ may all be used to construct solutions to the same KP-equation as those functions associated to the cyclic tetragonal curve of genus six.

Chapter 4

Higher genus trigonal curves

The results of Chapter 3 required the development of new techniques, methods and the corresponding Maple programs. In this Chapter we have applied these to generate results for the higher genus trigonal curves. In most cases the same methods and programs could be used with only slight modifications and hence we do not repeat the discussion of these.

We start in Section 4.1 by investigating the Abelian functions associated to the cyclic (3,7)-curve. This curve has genus $g = 6$, the same as the cyclic (4,5)-curve investigated in Chapter 3. It will be interesting to see whether the theory is more closely related to that of the lower genus trigonal curves studied in [30] and [11] or to that of the (4,5)-curve with which there is a shared genus.

Next, in Section 4.2 we repeat the process for the cyclic (3,8)-curve which has genus $g = 7$ and is the highest genus curve to have been considered. The Maple worksheets that were used to derive results can be found in the extra Appendix of files.

The main motivation for the investigation of these two higher genus curves is to facilitate with the future construction of a general theory for the trigonal curve. However, they also have potential applications in both integrable systems and the theory of Weierstrass semigroups.

4.1 The cyclic trigonal curve of genus six

In this section we present results for the Abelian functions associated with the cyclic (3,7)-curve. This is the curve C given by

$$y^3 = x^7 + \lambda_6 x^6 + \lambda_5 x^5 \lambda_4 x^4 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0. \tag{4.1}$$

Using equation (2.33) we find this curve has genus $g = 6$. (Note this curve has the same genus as the cyclic (4,5)-curve investigated in Chapter 3.)

4.1.1 Differentials and functions

We need to start by investigating the differentials on C. Recalling Definition 2.2.3 we construct the Weierstrass gap sequence generated by $(n,s) = (3,7)$.

$$W_{3,7} = \{1,2,4,5,8,11\}, \qquad \overline{W}_{3,7} = \{3,6,7,9,10,12,13,\dots\}. \tag{4.2}$$

We follow Proposition 2.2.4 to construct the basis of holomorphic differentials upon C.

$$\begin{aligned} &\boldsymbol{du} = (du_1,\dots,du_6), \qquad \text{where} \quad du_i(x,y) = \frac{g_i(x,y)}{3y^2}dx, \\ &\text{with} \quad \begin{array}{lll} g_1(x,y) = 1, & g_2(x,y) = x, & g_3(x,y) = x^2, \\ g_4(x,y) = y, & g_5(x,y) = x^3, & g_6(x,y) = xy. \end{array} \end{aligned} \tag{4.3}$$

Any point $\boldsymbol{u} \in \mathbb{C}^6$ can be expressed as

$$\boldsymbol{u} = (u_1,u_2,u_3,u_4,u_5,u_6) = \sum_{i=1}^{6} \int_{\infty}^{P_i} \boldsymbol{du}, \tag{4.4}$$

where the P_i are six variable points upon C.

Next we must construct the fundamental differential of the second kind, (Definition 2.2.6). We follow Klein's explicit realisation set out in Proposition 2.2.8 and find that the fundamental differential may be expressed as

$$\Omega\big((x,y),(z,w)\big) = \frac{F\big((x,y),(z,w)\big)dxdz}{9(x-z)^2y^2w^2}, \tag{4.5}$$

where F is the following symmetric polynomial.

$$\begin{aligned} F\big((x,y),(z,w)\big) &= 2\lambda_1 yz + \lambda_2 x^2 w + 2\lambda_1 wx + 3w\lambda_0 + 3x\lambda_3 yz^2 - yz^4\lambda_4 \\ &+ 2x^2yz^3\lambda_5 + 4xyz^3\lambda_4 + yz^4x\lambda_5 + \lambda_2 yz^2 + 3yz^4x^2\lambda_6 + 3x^4w\lambda_6 z^2 + yz^4x^3 \\ &+ 2x^3w\lambda_5 z^2 + 2x^5wz^2 + 2x^2yz^5 + x^4wz^3 + 3y^2w^2 + \lambda_1 yx + \lambda_1 wz + 3y\lambda_0 \\ &+ \lambda_5 x^4 wz - \lambda_4 x^4 w + 2\lambda_2 xwz + 2\lambda_2 xyz + 3\lambda_3 x^2 wz + 4\lambda_4 x^3 wz. \end{aligned} \tag{4.6}$$

(See Section 3.1.1 for a detailed example of such a construction.)

In obtaining the realisation, an explicit basis for the differentials of the second kind associated with the cyclic (3,7)-curve was derived.

$$\boldsymbol{dr} = (dr_1, \dots, dr_6), \qquad \text{where} \quad dr_j(x,y) = \frac{h_j(x,y)}{3y^2}dx, \tag{4.7}$$

$$\text{with} \quad \begin{aligned} h_1(x,y) &= y\big(9x^4\lambda_6 + 5x^2\lambda_4 + 11x^5 + \lambda_2 + 3x\lambda_3 + 7x^3\lambda_5\big),\\ h_2(x,y) &= xy\big(4x\lambda_5 + 6x^2\lambda_6 + 8x^3 + 2\lambda_4\big),\\ h_3(x,y) &= y\big(5x^3 - \lambda_4 + x\lambda_5 + 3x^2\lambda_6\big),\\ h_4(x,y) &= x^3\big(3x\lambda_6 + 4x^2 + 2\lambda_5\big),\\ h_5(x,y) &= 2x^2y,\\ h_6(x,y) &= x^4. \end{aligned}$$

We can now proceed to define Abelian functions as in the general case. Start by choosing an appropriate basis of cycles upon the surface defined by C. (Ensure condition (2.45) on the intersection numbers is satisfied.) Then define the period matrices as in equations (2.46) and (2.52) by integrating the differentials (4.3) and (4.7) around these cycles.

Let Λ denote the lattice generated by the first pair of period matrices, (Definition 2.2.10), and define an Abelian function associated with C as a meromorphic function that is periodic over this lattice, (Definition 2.2.11). Define the Jacobian as the complex space modulo this lattice, (Definition 2.2.12) and recall Definition 2.2.13 for the Abel map.

$$\begin{aligned} &\mathfrak{A} : \mathrm{Sym}^k(C) \to J\\ &(P_1, \dots, P_k) \mapsto \left(\int_\infty^{P_1} \boldsymbol{du} + \dots + \int_\infty^{P_k} \boldsymbol{du}\right) \pmod{\Lambda}, \end{aligned} \tag{4.8}$$

where the P_i are points upon C. Finally by Definition 2.2.14 the strata of the Jacobian are the images of the Abel map.

When $k = 1$ the Abel map gives an embedding of the curve C upon which we define ξ as the local parameter at the origin, $\mathfrak{A}_1(\infty)$.

$$\xi = x^{-\frac{1}{3}}. \tag{4.9}$$

We can derive series near the origin in ξ for the key variables.

$$x = \frac{1}{\xi^3}, \qquad \frac{dx}{d\xi} = -\frac{3}{\xi^4},$$

$$y = \frac{1}{\xi^7} + \left(\frac{\lambda_6}{3}\right)\frac{1}{\xi^4} + \left(\frac{\lambda_5}{3} - \frac{\lambda_6^2}{9}\right)\frac{1}{\xi} + \left(\frac{\lambda_4}{3} - \frac{2\lambda_6\lambda_5}{9} + \frac{5\lambda_6^3}{81}\right)\xi^2 + O(\xi^5).$$

$$\begin{aligned} du_1 &= [-\xi^{10} + O(\xi^{13})]d\xi, & du_4 &= [-\xi^{3} + O(\xi^{6})]d\xi, \\ du_2 &= [-\xi^{7} + O(\xi^{10})]d\xi, & du_5 &= [-\xi^{1} + O(\xi^{4})]d\xi, \\ du_3 &= [-\xi^{4} + O(\xi^{7})]d\xi, & du_6 &= [-1 + O(\xi^{3})]d\xi. \end{aligned}$$

$$\begin{aligned} u_1 &= -\tfrac{1}{11}\xi^{11} + O(\xi^{14}), & u_3 &= -\tfrac{1}{5}\xi^{5} + O(\xi^{8}), & u_5 &= -\tfrac{1}{2}\xi^{2} + O(\xi^{5}), \\ u_2 &= -\tfrac{1}{8}\xi^{8} + O(\xi^{11}), & u_4 &= -\tfrac{1}{4}\xi^{4} + O(\xi^{7}), & u_6 &= -\xi + O(\xi^{4}). \end{aligned} \tag{4.10}$$

We will work with the Abelian functions derived from the Kleinian σ-function associated with the $(3,7)$-curve. This is defined as in Definition 2.2.15 and is here a function of $g = 6$ variables.

$$\sigma = \sigma(\boldsymbol{u}) = \sigma(u_1, u_2, u_3, u_4, u_5, u_6).$$

The Schur-Weierstrass polynomial can be constructed as in Example A.5.21.

$$\begin{aligned} SW_{3,7} &= \tfrac{1}{22528000}u_6^{16} + u_2^2 + \tfrac{1}{80}u_4u_6^6u_5^3 + \tfrac{1}{3200}u_4u_6^{10}u_5 + \tfrac{1}{8}u_6^2u_5^5u_4 - \tfrac{1}{2}u_3u_6^3u_4^2 \\ &+ \tfrac{1}{2}u_6^2u_5u_4^3 - \tfrac{1}{16}u_4^2u_6^4u_5^2 + u_6u_5^2u_1 - \tfrac{1}{2}u_2u_6^2u_5^3 - u_2u_5^2u_4 + \tfrac{1}{40}u_2u_6^6u_5 - \tfrac{1}{20}u_6^5u_1 \\ &- \tfrac{1}{80}u_6^7u_5^2u_3 + \tfrac{1}{2}u_3^2u_6^2u_5^2 + \tfrac{1}{8}u_3u_6^3u_5^4 + 2u_5u_4u_3^2 + \tfrac{3}{8}u_5^4u_4^2 - \tfrac{1}{640}u_4^2u_6^8 + \tfrac{1}{4}u_4u_6^4u_2 \\ &- u_3u_6u_5^3u_4 - \tfrac{1}{64}u_5^8 - \tfrac{1}{20}u_4u_6^5u_3u_5 + \tfrac{1}{2560}u_6^8u_5^4 - \tfrac{1}{4}u_4^4 - u_3u_1 - 2u_2u_3u_6u_5 \\ &- \tfrac{1}{35200}u_3u_6^{11} + u_6u_3^3 - \tfrac{3}{281600}u_6^{12}u_5^2 - \tfrac{1}{64}u_5^6u_6^4 + \tfrac{1}{40}u_3^2u_6^6 \end{aligned} \tag{4.11}$$

Recall that this polynomial is the canonical limit of the σ-function.

Define the n-index $\wp$-functions associated with the $(3,7)$-curve as in Definition 2.2.27. Since the genus is six there will be the same number of $\wp$-functions as in the (4,5)-case. We will also need to use the n-index Q-functions as defined in Section 3.1.3.

In Section 3.3 we proved that all the objects in the theory associated with any (n,s)-curve may be given a weight so that all equations are homogeneous. In the case of the cyclic (3,7)-curve these weights are as follows.

	x	y	λ_6	λ_5	λ_4	λ_3	λ_2	λ_1	λ_0
Weight	-3	-7	-3	-6	-9	-12	-15	-18	-21

	u_1	u_2	u_3	u_4	u_5	u_6	$\sigma(\boldsymbol{u})$
Weight	$+11$	$+8$	$+5$	$+4$	$+2$	$+1$	$+16$

Note that as always the local parameter ξ has weight one, while the weights of the $\wp$-functions and Q-functions may be derived from their indices as described in Lemmas 3.3.12 and 3.3.14.

Note that while we have the same $\wp$-functions as in the (4,5)-case, their weights are different.

$$\begin{array}{lll}
\text{weight}[\wp_{11}] = -22 & \text{weight}[\wp_{23}] = -13 & \text{weight}[\wp_{36}] = -6 \\
\text{weight}[\wp_{12}] = -19 & \text{weight}[\wp_{24}] = -12 & \text{weight}[\wp_{44}] = -8 \\
\text{weight}[\wp_{13}] = -16 & \text{weight}[\wp_{25}] = -10 & \text{weight}[\wp_{45}] = -6 \\
\text{weight}[\wp_{14}] = -15 & \text{weight}[\wp_{26}] = -9 & \text{weight}[\wp_{46}] = -5 \\
\text{weight}[\wp_{15}] = -13 & \text{weight}[\wp_{33}] = -10 & \text{weight}[\wp_{55}] = -4 \\
\text{weight}[\wp_{16}] = -12 & \text{weight}[\wp_{34}] = -9 & \text{weight}[\wp_{56}] = -3 \\
\text{weight}[\wp_{22}] = -16 & \text{weight}[\wp_{35}] = -7 & \text{weight}[\wp_{66}] = -2
\end{array}$$

(Compare with Example 3.3.11 for the (4,5)-case).

4.1.2 Expanding the Kleinian formula

We need to derive relations between the $\wp$-functions from the Kleinian formula, (Theorem 3.2.1). We discussed the procedure to do this in detail in Section 3.2. As before, we use the expansions of the variables in ξ to expand equation (3.33) as a series in ξ. The coefficients are polynomials in the variables (z, w) and the $\wp$-functions, starting with the five below.

$$0 = \rho_1 = -z^4 + \wp_{56}z^3 + \wp_{36}z^2 + \big(\wp_{66}w + \wp_{26}\big)z + \wp_{46}w + \wp_{16} \tag{4.12}$$

$$\begin{aligned}
0 = \rho_2 = {}& \big(\wp_{55} - \wp_{566}\big)z^3 + \big(\wp_{35} - 2w - \wp_{366}\big)z^2 + \big(\wp_{25} - \wp_{266} + \wp_{56}w \\
& - \wp_{666}w\big)z + \wp_{15} - \wp_{466}w - \wp_{166} + \wp_{45}w
\end{aligned} \tag{4.13}$$

$$\begin{aligned}
0 = \rho_3 = {}& \big(\tfrac{1}{2}\wp_{5666} - \tfrac{3}{2}\wp_{556}\big)z^3 + \big(\tfrac{1}{2}\wp_{3666} - \tfrac{3}{2}\wp_{356}\big)z^2 + \big(\tfrac{1}{2}\wp_{2666} - \tfrac{3}{2}\wp_{256} \\
& + (\tfrac{1}{2}\wp_{6666} - \tfrac{3}{2}\wp_{566})w\big)z - \tfrac{3}{2}\wp_{156} - 3w^2 + \tfrac{1}{2}\wp_{1666} + \big(-\tfrac{3}{2}\wp_{456} + \tfrac{1}{2}\wp_{4666}\big)w \\
0 = \rho_4 = {}& -2z^5 - \big(2\wp_{56} + \tfrac{10}{3}\lambda_6\big)z^4 + \big(\tfrac{1}{3}\wp_{56}\lambda_6 + \wp_{5566} - \tfrac{1}{6}\wp_{56666} - \tfrac{1}{2}\wp_{555} - 2\wp_{36} \\
& - 2\lambda_5 + \wp_{45}\big)z^3 + \big(\wp_{3566} - 2\wp_{26} - 2\wp_{66}w - \tfrac{1}{2}\wp_{355} + \wp_{34} + \tfrac{1}{3}\wp_{36}\lambda_6 - \tfrac{1}{6}\wp_{36666}\big)z^2 \\
& + \big(\wp_{2566} - \tfrac{1}{6}\wp_{26666} - 2\wp_{16} + \tfrac{1}{3}\wp_{26}\lambda_6 - \tfrac{1}{2}\wp_{255} + \big(\tfrac{1}{3}\wp_{66}\lambda_6 - \wp_{46} - \tfrac{1}{6}\wp_{66666} \\
& - \tfrac{1}{2}\wp_{556} + \wp_{5666}\big)w + \wp_{24}\big)z + \big(\wp_{4566} - \tfrac{1}{6}\wp_{46666} + \wp_{44} + \tfrac{1}{3}\wp_{46}\lambda_6 - \tfrac{1}{2}\wp_{455}\big)w \\
& + \tfrac{1}{3}\wp_{16}\lambda_6 + \wp_{14} - \tfrac{1}{2}\wp_{155} + \wp_{1566} - \tfrac{1}{6}\wp_{16666} \\
0 = \rho_5 = {}& \big(2\wp_{566} - 2\wp_{55}\big)z^4 + \big(\tfrac{5}{8}\wp_{5556} - \tfrac{5}{12}\wp_{55666} + 2\wp_{366} - \wp_{35} - \tfrac{5}{4}\wp_{456} - w \\
& + \tfrac{1}{24}\wp_{566666} - \tfrac{1}{4}\wp_{566}\lambda_6\big)z^3 + \big(\big(2\wp_{666} - 3\lambda_6 - 2\wp_{56}\big)w - \tfrac{5}{12}\wp_{35666} + 2\wp_{266} \\
& + \tfrac{1}{24}\wp_{366666} + \wp_{33} + \tfrac{5}{8}\wp_{3556} - \tfrac{1}{4}\wp_{366}\lambda_6 - 2\wp_{25} - \tfrac{5}{4}\wp_{346}\big)z^2 + \big(2\wp_{166} - \tfrac{5}{4}\wp_{246} \\
& - \tfrac{1}{4}\wp_{266}\lambda_6 + \wp_{23} - \tfrac{5}{12}\wp_{25666} - 2\wp_{15} + \tfrac{5}{8}\wp_{2556} + \big(\wp_{36} - 2\wp_{45} + \tfrac{5}{8}\wp_{5566} - \tfrac{5}{12}\wp_{56666} \\
& + \tfrac{3}{4}\wp_{466} - \lambda_5 + \tfrac{1}{24}\wp_{666666} - \tfrac{1}{4}\wp_{666}\lambda_6)w + \tfrac{1}{24}\wp_{266666}\big)z + \big(-\tfrac{1}{4}\wp_{466}\lambda_6 + \tfrac{5}{8}\wp_{4556} \\
& - \tfrac{5}{12}\wp_{45666} - \tfrac{5}{4}\wp_{446} + \tfrac{1}{24}\wp_{466666} + \lambda_4 + \wp_{34}\big)w - \tfrac{5}{12}\wp_{15666} + \tfrac{5}{8}\wp_{1556} - \tfrac{1}{4}\wp_{166}\lambda_6 \\
& + \wp_{13} + \tfrac{1}{24}\wp_{166666} - \tfrac{5}{4}\wp_{146}
\end{aligned}$$

They are valid for any $\boldsymbol{u} \in \mathbb{C}^6$, with (z,w) one of the point on C that are used in equation (4.4) to represent $\boldsymbol{u}$. They are presented in ascending order (as the coefficients of ξ) and have been calculated explicitly, (using Maple), up to ρ_{10}. They get increasingly larger in size and can be found in the extra Appendix of files.

As described in Section 3.2 we next take resultants of pairs of equations, eliminating the variable w by choice. (See Appendix A.6 for an introduction to resultants.) We denote the resultant of ρ_i and ρ_j by $\rho_{i,j}$. These equations are quite large and not presented here. Instead we present Table 4.1 detailing the number of terms they contain once expanded and their degree in z.

Table 4.1: The polynomials $\rho_{i,j}$

Res	# terms	degree	Res	# terms	degree
$\rho_{1,2}$	40	6	$\rho_{3,4}$	2025	10
$\rho_{1,3}$	79	8	$\rho_{3,5}$	4188	9
$\rho_{1,4}$	77	6	$\rho_{3,6}$	4333	8
$\rho_{1,5}$	154	7	$\rho_{3,7}$	19043	10
$\rho_{1,6}$	344	8	$\rho_{3,8}$	28422	10
$\rho_{1,7}$	290	7	$\rho_{3,9}$	44409	10
$\rho_{1,8}$	412	7	$\rho_{4,5}$	793	8
$\rho_{1,9}$	1055	8	$\rho_{4,6}$	8315	10
$\rho_{2,3}$	307	7	$\rho_{4,7}$	1183	8
$\rho_{2,4}$	219	7	$\rho_{4,8}$	2112	8
$\rho_{2,5}$	226	6	$\rho_{4,9}$	24569	10
$\rho_{2,6}$	1468	8	$\rho_{5,6}$	18356	10
$\rho_{2,7}$	712	7	$\rho_{5,7}$	2535	8
$\rho_{2,8}$	737	7	$\rho_{5,8}$	2316	8
$\rho_{2,9}$	4536	9	$\rho_{5,9}$	54384	11

In general, the higher the ρ_i involved in the resultant, the more terms the resultant would have. There were only two polynomials found with degree six in z, ρ_{12} and ρ_{14}, with all the others having higher degree. Recall that in the (4,5)-case the corresponding polynomials had much more terms and all had degree in z of at least seven.

With regards to the expansion of the Kleinian formula, the (3,7)-case has far more in common with the lower genus trigonal curves that the other higher genus curves that share the genus. This was partly predicted by the results of [21].

Next recall the result of Theorem 3.2.2 on equations between the $\wp$-functions and a point z on the curve from which they are defined. It stated that such an equation with degree $g-1=5$ in z must be identically zero. We hence aim to manipulate these polynomials to achieve this.

We select $\rho_{1,2}$ as it is the smallest polynomial of degree six, and rearrange it to give an equation for z^6.

$$\begin{aligned} z^6 = &-\tfrac{1}{2}z^5\wp_{666} - \tfrac{1}{2}\wp_{56}^2z^4 - \tfrac{1}{2}\wp_{16}\wp_{45} + \tfrac{1}{2}\wp_{16}\wp_{466} + \wp_{16}z^2 - \tfrac{1}{2}\wp_{166}\wp_{46} - \tfrac{1}{2}\wp_{166}z\wp_{66} \\ &- \tfrac{1}{2}z^2\wp_{26}\wp_{56} + \tfrac{1}{2}\wp_{15}\wp_{46} + \tfrac{3}{2}z^5\wp_{56} + \tfrac{1}{2}z^2\wp_{26}\wp_{666} - \tfrac{1}{2}z^4\wp_{466} + \tfrac{1}{2}z\wp_{16}\wp_{666} + z^3\wp_{26} \\ &+ \tfrac{1}{2}z^2\wp_{25}\wp_{66} + \tfrac{1}{2}z^3\wp_{35}\wp_{66} - \tfrac{1}{2}z^4\wp_{566}\wp_{66} - \tfrac{1}{2}z\wp_{266}\wp_{46} + \tfrac{1}{2}z^4\wp_{55}\wp_{66} - \tfrac{1}{2}z^3\wp_{566}\wp_{46} \\ &+ \tfrac{1}{2}\wp_{36}z^2\wp_{466} + \tfrac{1}{2}z\wp_{26}\wp_{466} - \tfrac{1}{2}z\wp_{26}\wp_{45} + \wp_{36}z^4 - \tfrac{1}{2}\wp_{16}z\wp_{56} + \tfrac{1}{2}z^4\wp_{45} + \tfrac{1}{2}z\wp_{25}\wp_{46} \\ &- \tfrac{1}{2}\wp_{36}z^3\wp_{56} + \tfrac{1}{2}\wp_{36}z^3\wp_{666} + \tfrac{1}{2}\wp_{56}z^4\wp_{666} + \tfrac{1}{2}\wp_{56}z^3\wp_{466} - \tfrac{1}{2}\wp_{56}z^3\wp_{45} - \tfrac{1}{2}\wp_{36}z^2\wp_{45} \\ &- \tfrac{1}{2}z^2\wp_{266}\wp_{66} + \tfrac{1}{2}z^3\wp_{55}\wp_{46} + \tfrac{1}{2}z^2\wp_{35}\wp_{46} + \tfrac{1}{2}\wp_{15}z\wp_{66} - \tfrac{1}{2}z^2\wp_{366}\wp_{46} - \tfrac{1}{2}z^3\wp_{366}\wp_{66}. \end{aligned}$$

We may now take any of the other $\rho_{i,j}$ and repeatedly substitute for z^6 until we have a polynomial of degree five in z. By Theorem 3.2.2 the coefficients, with respect to z, of such an equation must be zero. Further, since they are zero we can take the numerator of the coefficients leaving us with six polynomial equations between the $\wp$-functions. Finally, recalling Lemma 2.2.32 which stated that all the $\wp$-functions have definite parity, we can separate each of these six relations into their odd and even parts. We denote the polynomial equations between $\wp$-functions that we generate using this method by $\mathcal{K}(\rho_{i,j}, n, \pm)$ as defined in Definition 3.2.3.

In the (4,5)-case every polynomial needed at least two rounds of substitution, and the equations for z^6 and z^7 were far more complicated. In this case we need only substitute once into ρ_{14} and so the polynomials achieved here are far simpler.

$$\begin{aligned} \mathcal{K}(\rho_{14}, 5, +) &= \big(2\wp_{666} - 3\lambda_6 - 6\wp_{56}\big)\wp_{66} + \wp_{5666} - 3\wp_{46} \\ \mathcal{K}(\rho_{14}, 5, -) &= -\tfrac{1}{2}\wp_{556} - \tfrac{1}{6}\wp_{66666} \\ \mathcal{K}(\rho_{14}, 4, +) &= \big(2\wp_{566} - 2\wp_{55}\big)\wp_{66}^2 + \big(\wp_{5566} - \wp_{45} - 2\lambda_5 - 4\wp_{36} + 2\wp_{466} \\ &\quad - 2\wp_{56}\wp_{666} + 2\wp_{56}^2\big)\wp_{66} + \wp_{44} + \wp_{4566} - \wp_{56}\wp_{5666} - \wp_{56}\wp_{46} - 3\wp_{46}\lambda_6 \\ \mathcal{K}(\rho_{14}, 4, -) &= -\big(\tfrac{1}{6}\wp_{56666} + \tfrac{1}{2}\wp_{555}\big)\wp_{66} - \tfrac{1}{6}\wp_{46666} + \tfrac{1}{2}\wp_{56}\wp_{556} \\ &\quad + \tfrac{1}{6}\wp_{56}\wp_{66666} - \tfrac{1}{2}\wp_{455} \\ &\vdots \end{aligned}$$

However, the other relations all involve two rounds of substitution for z^6 and get increasingly long and complex, involving higher index $\wp$-functions.

Although the relations we achieve are far simpler than those in the (4,5)-case, there do not seem to be enough simple relations to manipulate to find the desired differential equations between the $\wp$-functions. Hence we proceed to use the tools developed for the (4,5)-curve to find these relations.

However, as predicted by [21], we may still use the relations from the Kleinian formula to find the solution of the Jacobi Inversion Problem.

Recall that the Jacobi Inversion Problem is, given a point $\boldsymbol{u} \in J$, to find the preimage of this point under the Abel map (4.8).

Theorem 4.1.1. *Suppose we are given $\{u_1, \dots, u_6\} = \boldsymbol{u} \in J$. Then we could solve the Jacobi Inversion Problem explicitly using the equations derived from the Kleinian formula.*

Proof. Consider the polynomial ρ_{12} which had degree six in z. Denote by $(z_1, \dots, z_6)$ the six zeros of the polynomial, which will be expressions in $\wp$-functions. Next consider equation (4.12) which is degree one in w. Substitute each z_i into equation (4.12) in turn and solve to find the corresponding w_i.

Therefore the set of points $\{(z_1, w_1), \dots, (z_6, w_6)\}$ on the curve C which are the Abel preimage of $\boldsymbol{u}$ have been identified.

■

4.1.3 The $\boldsymbol{\sigma}$-function expansion

We construct a σ-function expansion for the cyclic (3,7)-curve using the methods and techniques discussed in detail in Section 3.4. We start with a statement on the structure of the expansion.

Theorem 4.1.2. *The function $\sigma(\boldsymbol{u})$ associated with the cyclic (3,7)-curve may be expanded about the origin as*

$$\sigma(\boldsymbol{u}) = \sigma(u_1, u_2, u_3, u_4, u_5, u_6) = SW_{3,7}(\boldsymbol{u}) + C_{19}(\boldsymbol{u}) + C_{22} + \cdots + C_{16+3n}(\boldsymbol{u}) + \cdots$$

where each C_k is a finite, even polynomial composed of products of monomials in $\boldsymbol{u} = (u_1, u_2, \dots, u_6)$ of weight $+k$ multiplied by monomials in $\boldsymbol{\lambda} = (\lambda_6, \lambda_5, \dots, \lambda_0)$ of weight $16 - k$.

Proof. This is very similar to the proof of Theorem 3.4.3. First we note that the σ-function associated with the (3,7)-curve is even by Lemma 2.2.20. Then by Theorem 3(i) in [60] we know the expansion will be a sum of monomials in $\boldsymbol{u}$ and $\boldsymbol{\lambda}$ with rational coefficients.

We split the expansion into terms with common weight ratios. By Lemma 3.3.7 we know that $\sigma(\boldsymbol{u})$ has weight $+16$ and by Theorem 2.2.21 we know that the part of the expansion without $\boldsymbol{\lambda}$ will be $SW_{3,7}$. We then have a sum of other polynomials with increasing weight in $\boldsymbol{u}$. The subscripts increase by three since the possible weights of $\boldsymbol{\lambda}$-monomials decrease by three.

■

We presented $SW_{3,7}$ earlier in equation (4.11). We construct the other C_k successively following the steps set out in Section 3.4. The same Maple procedures could be easily adapted to perform the calculations.

Note that those polynomials up to and including C_{39} required the use of method (III) — ensuring polynomials derived from the expansions of the Kleinian formula in the last section were satisfied. After that it was possible to use only method (I) — ensuring the relations in Lemma 4.1.5 are satisfied.

As in the (4,5)-case we simplify calculations by rewriting procedures to take into account weight simplifications and by using parallel computing implemented with Distributed Maple, [72].

Example 4.1.3. We now describe how C_{19} was constructed, making use of all three methods to identify the coefficients.

The first step is to identify all the different monomials in $\boldsymbol{u}$ that have weight 19. We use the Maple procedure `partition` to find all the partitions of 19, discard those partitions that contain an integer not in $W_{3,7}$ and relate the remaining partitions to the monomials in $\boldsymbol{u}$ they represent.

We find 100 possible monomials. However, we can discard all those that are not even functions to leave 48.

The next step is to identify all those $\boldsymbol{\lambda}$-monomials that have weight $16 - 19 = -3$ in the curve parameters. The only such monomial is clearly λ_6. We hence form $\hat{\sigma}(\boldsymbol{u})$ using constants c_i as

$$\hat{\sigma}(\boldsymbol{u}) = SW_{3,7} + \lambda_6\big(c_1 u_6^{19} + c_2 u_6^{17} u_5 + \dots\big).$$

We need to determine the 48 coefficients c_i and we start by implementing method (I). The only equation that can already be derived in Lemma 4.1.5 is the one for Q_{6666}, which has weight -4. The relation for this function cannot involve any lower weight $\boldsymbol{\lambda}$-monomials and it cannot involve λ_6 since there is no Abelian function of weight -1 with which it could be combined. Hence we can derive it using only $\sigma = SW_{3,7}$, and find that

$$Q_{6666} = -3\wp_{55}.$$

We substitute the Q and $\wp$-function for their definitions in $\sigma(\boldsymbol{u})$. We then obtain a rational function in σ-derivatives with denominator $\sigma(\boldsymbol{u})^2$. We multiply up to leave

$$0 = -\sigma_{6666}(\boldsymbol{u})\sigma(\boldsymbol{u}) + 4\sigma_{666}(\boldsymbol{u})\sigma_6(\boldsymbol{u}) - 3\sigma_{66}(\boldsymbol{u})^2 - 3\sigma_{55}(\boldsymbol{u})\sigma(\boldsymbol{u}) + 3\sigma_5(\boldsymbol{u})^2.$$

We now substitute $\sigma(\boldsymbol{u})$ for $\hat{\sigma}(\boldsymbol{u})$ and calculate the relevant derivatives. We then expand the products and remove the terms with higher weight $\boldsymbol{\lambda}$-monomials. We are left with a polynomial in $\boldsymbol{u}$ and λ_6. Collecting common $\boldsymbol{u\lambda}$-monomials together gives us coefficients in the c_i.

Using the Maple `solve` command we find that 42 of the 48 coefficients can be expressed as a linear combination of the other six.

We next use method (III) and ensure that $\hat{\sigma}(\boldsymbol{u})$ vanishes for $\boldsymbol{u} \in \Theta^{[5]}$. We substitute $\boldsymbol{u}$ in $\hat{\sigma}(\boldsymbol{u})$ for the series expansion in five local parameters. We then take a multivariate Taylor series, with those terms containing $\boldsymbol{\lambda}$-monomials of weight lower than -3 discarded. We identify the coefficients of the remaining $\boldsymbol{\lambda}\xi_i$-monomials.

We substitute in the conditions we have already derived for the c_i. Those coefficients that are not zero may be used to achieve more conditions on the c_i. After this we find that 47 of the 48 coefficients may be described using the other one.

Finally we use method (II) by ensuring $\mathcal{K}(\rho_{2,3}, 5, +)$ was satisfied. Again we substitute the $\wp$-functions for σ-derivatives, factor, take the numerator and then substitute $\sigma(\boldsymbol{u})$ for $\hat{\sigma}(\boldsymbol{u})$. We evaluate the derivatives, expand the polynomial and collect the coefficients of the monomials in $\boldsymbol{u}$ and $\boldsymbol{\lambda}$. Then we substitute in the conditions on c_i already derived to find that we are able to find a numerical value for the one c_i in which the others were evaluated. We hence gain numerical values for all the c_i. We find that

$$\begin{aligned}
C_{19} = \lambda_6 \cdot \big[&\tfrac{17}{2240}u_6^7u_5^4u_4 - u_6u_5^3u_4u_2 - \tfrac{1}{40}u_6^6u_5^2u_4u_3 - \tfrac{1}{4}u_6^2u_5^4u_4u_3 - \tfrac{3}{8}u_6^4u_5u_4^2u_3 \\
&- \tfrac{3}{2}u_6^2u_5^2u_3u_2 + \tfrac{3}{20}u_6^5u_5u_4u_2 + \tfrac{1}{40}u_6^6u_3u_2 + u_6^2u_5u_3^3 + \tfrac{79}{492800}u_6^{11}u_5^2u_4 \\
&+ \tfrac{1}{4}u_6^3u_5^2u_4^3 - \tfrac{1}{40}u_6^6u_5u_1 + \tfrac{1}{2}u_6^3u_5^3u_3^2 + \tfrac{1}{2}u_5^3u_4^2u_3 - \tfrac{1}{40}u_6^5u_5^3u_4^2 + \tfrac{1}{32}u_6^4u_5^5u_3 \\
&+ \tfrac{1}{4}u_6u_5^5u_4^2 - \tfrac{3}{492800}u_6^13u_5^3 - \tfrac{1}{4}u_6^3u_5^4u_2 + \tfrac{1}{11200}u_6^10u_4u_3 - \tfrac{29}{4480}u_6^8u_5^3u_3 \\
&- \tfrac{1}{39424}u_6^{12}u_5u_3 + \tfrac{1}{2}u_6^2u_5^3u_1 + \tfrac{3}{280}u_6^7u_5^2u_2 + \tfrac{1}{280}u_6^7u_5u_3^2 - \tfrac{1}{2240}u_6^9u_5u_4^2 \\
&+ \tfrac{1}{39424000}u_6^{17}u_5 - \tfrac{1}{112}u_6^5u_5^7 - \tfrac{1}{560}u_6^7u_4^3 - \tfrac{1}{123200}u_6^{11}u_2 + \tfrac{1}{9856000}u_6^{15}u_4 \\
&- \tfrac{1}{112}u_6u_5^9 - \tfrac{3}{56}u_5^7u_3 + u_4u_3^3 + \tfrac{1}{16}u_6^3u_5^6u_4 + \tfrac{1}{4480}u_6^9u_5^5\big].
\end{aligned}$$

□

Using these techniques we have calculated the σ-expansion associated with the (3,7)-curve up to and including C_{49}. The polynomials can be found in the extra Appendix of files and the table below indicates the number of terms in each.

C_{16}	C_{19}	C_{22}	C_{25}	C_{28}	C_{31}	C_{34}	C_{37}	C_{40}	C_{43}	C_{46}	C_{49}
32	36	105	223	513	982	2098	3885	7037	12237	21237	21261

As in the (4,5)-case the later polynomials are extremely large and represent a significant amount of computation, with many calculations run in parallel on a cluster of machines using the Distributed Maple package, [72]. Although the genus is the same as the (4,5)-case, the computations are marginally more difficult. This is because the $\boldsymbol{\lambda}$ increase in steps of three instead of four, and so there are a greater number of possible $\boldsymbol{\lambda}$-monomials at each stage. Hence there are more polynomials C_k with more terms in each.

4.1.4 Relations between the Abelian functions

Following Section 3.5 we can derive sets of relations between the Abelian functions associated with the (3,7)-curve. We start with the construction of a basis for this space of fundamental Abelian functions, and in doing so generate linear relations between such functions.

Theorem 4.1.4. *A basis for* $\Gamma\big(J, \mathcal{O}(2\Theta^{[5]})\big)$ *is given by*

$$\begin{array}{cccccccccc}
 & \mathbb{C}1 & \oplus & \mathbb{C}\wp_{11} & \oplus & \mathbb{C}\wp_{12} & \oplus & \mathbb{C}\wp_{13} & \oplus & \mathbb{C}\wp_{14} \\
\oplus & \mathbb{C}\wp_{15} & \oplus & \mathbb{C}\wp_{16} & \oplus & \mathbb{C}\wp_{22} & \oplus & \mathbb{C}\wp_{23} & \oplus & \mathbb{C}\wp_{24} \\
\oplus & \mathbb{C}\wp_{25} & \oplus & \mathbb{C}\wp_{26} & \oplus & \mathbb{C}\wp_{33} & \oplus & \mathbb{C}\wp_{34} & \oplus & \mathbb{C}\wp_{35} \\
\oplus & \mathbb{C}\wp_{36} & \oplus & \mathbb{C}\wp_{44} & \oplus & \mathbb{C}\wp_{45} & \oplus & \mathbb{C}\wp_{46} & \oplus & \mathbb{C}\wp_{55} \\
\oplus & \mathbb{C}\wp_{56} & \oplus & \mathbb{C}\wp_{66} & \oplus & \mathbb{C}Q_{5556} & \oplus & \mathbb{C}Q_{5555} & \oplus & \mathbb{C}Q_{4466} \\
\oplus & \mathbb{C}Q_{3466} & \oplus & \mathbb{C}Q_{4456} & \oplus & \mathbb{C}Q_{4455} & \oplus & \mathbb{C}Q_{4446} & \oplus & \mathbb{C}Q_{3355} \\
\oplus & \mathbb{C}Q_{3446} & \oplus & \mathbb{C}Q_{4445} & \oplus & \mathbb{C}Q_{3346} & \oplus & \mathbb{C}Q_{3445} & \oplus & \mathbb{C}Q_{1556} \\
\oplus & \mathbb{C}Q_{2356} & \oplus & \mathbb{C}Q_{1555} & \oplus & \mathbb{C}Q_{2355} & \oplus & \mathbb{C}Q_{2446} & \oplus & \mathbb{C}Q_{2346} \\
\oplus & \mathbb{C}Q_{2445} & \oplus & \mathbb{C}Q_{3344} & \oplus & \mathbb{C}Q_{2345} & \oplus & \mathbb{C}Q_{3334} & \oplus & \mathbb{C}Q_{1355} \\
\oplus & \mathbb{C}Q_{1446} & \oplus & \mathbb{C}Q_{2255} & \oplus & \mathbb{C}Q_{2246} & \oplus & \mathbb{C}Q_{2344} & \oplus & \mathbb{C}Q_{2245} \\
\oplus & \mathbb{C}Q_{2334} & \oplus & \mathbb{C}Q_{1255} & \oplus & \mathbb{C}Q_{1335} & \oplus & \mathbb{C}Q_{1344} & \oplus & \mathbb{C}Q_{2244} \\
\oplus & \mathbb{C}Q_{2226} & \oplus & \mathbb{C}Q_{2234} & \oplus & \mathbb{C}Q_{1155} & \oplus & \mathbb{C}Q_{1235} & \oplus & \mathbb{C}Q_{1136} \\
\oplus & \mathbb{C}Q_{1155} & \oplus & \mathbb{C}Q_{1133} & \oplus & \mathbb{C}Q_{223466} & \oplus & \mathbb{C}Q_{113666}. & &
\end{array} \tag{4.14}$$

Proof. Recall that the dimension of the space is $2^g = 2^6 = 64$ by the Riemann-Roch theorem for Abelian varieties. It was concluded in equation (3.53) that all the selected elements do in fact belong to the space. All that remains is to prove their linear independence, which can be done explicitly using the σ-expansions in Maple.

■

To actually construct the basis we started by including all 21 of the $\wp_{ij}$ since they are all linearly independent. We then decided which Q_{ijkl} to include by testing at decreasing weight levels to see which could be written as a linear combination. This process was described in detail for the (4,5)-curve in Section 3.5.1 and so we do not repeat the details here. The same Maple procedures could be used to carry out the computations.

(Note that these computations were actually performed in tandem with the construction of the σ-expansion. Once a new C_k was found three more weight levels of the basis could be examined. The relations obtained could then be used to construct the next C_k in the expansion.)

Upon examining all the 4-index Q-functions, we find that 62 basis elements have been identified. To find the final two basis elements we repeat the procedure using the 6-index Q-functions. We find that all those of weight higher than -27 can be expressed as a linear combination of existing basis entries. However, at weight -27 one of the Q_{ijklmn} is required in the basis to express the others. This was the case also at weight -30. (Note that the basis used here is much more similar to the (4,5)-case than the lower genus trigonal cases.)

Lemma 4.1.5. *Those 4-index Q-functions not in the basis can be expressed as a linear combination of the basis elements.*

$$\begin{aligned}
&\mathbf{(-4)} \quad Q_{6666} = -3\wp_{55} \\
&\mathbf{(-5)} \quad Q_{5666} = 3\wp_{46} + 3\lambda_6\wp_{66} \\
&\mathbf{(-6)} \quad Q_{5566} = 4\wp_{36} - \wp_{45} + 3\lambda_6\wp_{56} + 2\lambda_5 \\
&\mathbf{(-7)} \quad Q_{4666} = 3\lambda_6\wp_{55} - Q_{5556} \\
&\mathbf{(-8)} \quad Q_{3666} = -\tfrac{1}{4}Q_{5555} - \tfrac{3}{4}\wp_{44} + \tfrac{3}{2}\lambda_6\wp_{46} - \tfrac{3}{4}\lambda_6^2\wp_{66} + 3\lambda_5\wp_{66} \\
&\mathbf{(-8)} \quad Q_{4566} = -\wp_{44} + 3\lambda_6\wp_{46} \\
&\mathbf{(-9)} \quad Q_{3566} = 4\wp_{26} - \wp_{34} + 3\lambda_6\wp_{36} \\
&\mathbf{(-9)} \quad Q_{4556} = 3\wp_{26} - 2\wp_{34} + 3\lambda_6\wp_{45} - 2\lambda_4 \\
&\qquad\qquad\vdots
\end{aligned}$$

The relations have been calculated down to weight -37 and are available in the extra Appendix of files.

There are similar equations for all the 6-index Q-functions, except Q_{223466} and Q_{113666} which were in the basis. Explicit relations have been calculated down to weight -32. The first few are given below with all available relations in the extra Appendix of files.

$$\begin{aligned}
&\mathbf{(-6)} \quad Q_{666666} = 36\wp_{36} - 45\wp_{45} - 9\lambda_6\wp_{56} - 6\lambda_5 \\
&\mathbf{(-7)} \quad Q_{566666} = -24\wp_{35} + 5Q_{5556} - 24\lambda_6\wp_{55} \\
&\mathbf{(-8)} \quad Q_{556666} = 12\wp_{44} + Q_{5555} + 24\lambda_6\wp_{46} + 12\lambda_6^2\wp_{66} \\
&\mathbf{(-9)} \quad Q_{555666} = 9\wp_{26} + 18\wp_{34} + 36\lambda_6\wp_{36} - 18\lambda_6\wp_{45} \\
&\qquad\qquad\qquad + 9\wp_{56}\lambda_6^2 + 6\lambda_6\lambda_5 + 42\lambda_4 \\
&\mathbf{(-9)} \quad Q_{466666} = 15\wp_{26} + 6\wp_{34} - 9\lambda_6\wp_{45} + 6\lambda_4 \\
&\qquad\qquad\vdots
\end{aligned}$$

Proof. By Theorem 4.1.4 and Lemma 3.5.2 it is clear that such relations must exist. The explicit differential equations were calculated in the construction of the basis.

■

Corollary 4.1.6. *There are a set of differential equations that express 4-index $\wp$-functions associated with the cyclic (3,7)-curve as a polynomial of Abelian functions of total degree* *at most two. The full set down to weight -37 can be found in the extra Appendix of files.*

$$
\begin{aligned}
&(-4) \quad \wp_{6666} = 6\wp_{66}^2 - 3\wp_{55} \\
&(-5) \quad \wp_{5666} = 6\wp_{56}\wp_{66} + 3\wp_{46} + 3\lambda_6\wp_{66} \\
&(-6) \quad \wp_{5566} = 4\wp_{36} - \wp_{45} + 3\lambda_6\wp_{56} + 2\lambda_5 + 2\wp_{55}\wp_{66} + 4\wp_{56}^2 \\
&(-7) \quad \wp_{4666} = 3\wp_{55}\lambda_6 + 6\wp_{46}\wp_{66} - \wp_{5556} + 6\wp_{55}\wp_{56} \\
&(-8) \quad \wp_{3666} = \tfrac{3}{2}\wp_{46}\lambda_6 - \tfrac{3}{4}\wp_{44} - \tfrac{3}{4}\wp_{66}\lambda_6^2 + 3\wp_{66}\lambda_5 + 6\wp_{36}\wp_{66} \\
&\qquad\qquad - \tfrac{1}{4}\wp_{5555} + \tfrac{3}{2}\wp_{55}^2 \\
&(-8) \quad \wp_{4566} = 3\wp_{46}\lambda_6 - \wp_{44} + 2\wp_{45}\wp_{66} + 4\wp_{46}\wp_{56} \\
&(-9) \quad \wp_{3566} = 4\wp_{26} - \wp_{34} + 3\wp_{36}\lambda_6 + 2\wp_{35}\wp_{66} + 4\wp_{36}\wp_{56} \\
&(-9) \quad \wp_{4556} = 3\wp_{26} - 2\wp_{34} + 3\wp_{45}\lambda_6 - 2\lambda_4 + 4\wp_{45}\wp_{56} + 2\wp_{46}\wp_{55} \\
&\qquad\qquad \vdots
\end{aligned}
$$

Proof. Apply equation (3.32) to the first set of relations in Lemma 4.1.5.

■

Remark 4.1.7. The first equation in Corollary 4.1.6 may be differentiated twice with respect to u_6 to give the Boussinesq equation for $\wp_{66}$ with u_6 playing the space variable and u_5 the time variable. The connection of the cyclic trigonal curves with the Boussinesq equation has been well established in [35], [21], [30] and [11].

Corollary 4.1.6 is of particular interest because it gives a generalisation of Corollary 2.1.15 from the elliptic case. Further discussion on the generalisation of the elliptic equations is given in the next Chapter. Details on the bilinear and quadratic 3-index relations for the (3,7)-curve are also included there.

4.1.5 Addition formula

In this section we develop the two term addition formula for the σ-function associated with the (3,7)-curve. This generalises Theorem 2.1.26 in the elliptic case and follows the approach taken in Section 3.6 for the (4,5)-curve.

Theorem 4.1.8. *The σ-function associated with the cyclic (3,7)-curve satisfies a two term addition formula*

$$
-\frac{\sigma(\boldsymbol{u}+\boldsymbol{v})\sigma(\boldsymbol{u}-\boldsymbol{v})}{\sigma(\boldsymbol{u})^2\sigma(\boldsymbol{v})^2} = f(\boldsymbol{u},\boldsymbol{v}) + f(\boldsymbol{v},\boldsymbol{u}),
$$

where $f(\boldsymbol{u},\boldsymbol{v})$ is a finite polynomial of Abelian functions associated with C. It may be written as

$$
f(\boldsymbol{u},\boldsymbol{v}) = \big[P_{32} + P_{29} + P_{26} + P_{23} + P_{20} + P_{17} + P_{14} + P_{11} + P_8 + P_5 + P_2\big](\boldsymbol{u},\boldsymbol{v})
$$

where each $P_k(\boldsymbol{u},\boldsymbol{v})$ is a sum of terms with weight $-k$ in the Abelian functions and weight $k-32$ in $\boldsymbol{\lambda}$-monomials.

Proof. We follow the proof of Theorem 4.1.8 for the (4,5)-case. This time the σ-function is even instead of odd so the addition formula becomes symmetric instead of anti-symmetric. The ratio of σ-derivatives has weight -32 and the polynomials have weights shifting by three since the weights of possible $\boldsymbol{\lambda}$-monomials are all multiples of three. ∎

The formula may be derived explicitly using the σ-function expansion. The same approach and Maple procedures can be used as in Section 3.6 for the (4,5)-case. Since $f(\boldsymbol{u},\boldsymbol{v})$ may contain $\boldsymbol{\lambda}$-monomials of weight -30, the σ-expansion should be truncated after C_{46}. We derive the polynomials P_k as below. Recall that the two term addition formulae are the only results in this document that will change with a different choice of c in the definition of the σ-function, (see Remark 2.2.23). Using an alternative constant will change the polynomial $f(\boldsymbol{u},\boldsymbol{v})$ by a multiplicative constant factor, (see Appendix B).

$$\begin{aligned}
P_2(\boldsymbol{u},\boldsymbol{v}) &= -\tfrac{4}{63}\wp_{66}(\boldsymbol{v})\lambda_6\lambda_5\lambda_4\lambda_3 - \tfrac{3}{2}\wp_{66}(\boldsymbol{v})\lambda_4\lambda_0 - \tfrac{1}{3}\wp_{66}(\boldsymbol{u})\lambda_5^2\lambda_1 - \tfrac{9}{4}\wp_{66}(\boldsymbol{v})\lambda_6^3\lambda_0 \\
&\quad + \tfrac{4}{3}\wp_{66}(\boldsymbol{u})\lambda_3\lambda_1 - 3\wp_{66}(\boldsymbol{v})\lambda_5\lambda_4\lambda_2 + \tfrac{3}{4}\wp_{66}(\boldsymbol{v})\lambda_6^2\lambda_4\lambda_2 + \tfrac{1}{3}\wp_{66}(\boldsymbol{u})\lambda_2^2 + 9\wp_{66}(\boldsymbol{v})\lambda_6\lambda_5\lambda_0 \\
&\quad + \tfrac{1}{12}\wp_{66}(\boldsymbol{u})\lambda_6^2\lambda_5\lambda_1 + \tfrac{7}{6}\wp_{66}(\boldsymbol{v})\lambda_6\lambda_4\lambda_1 - \tfrac{41}{21}\wp_{66}(\boldsymbol{v})\lambda_4^2\lambda_3 - \tfrac{25}{42}\wp_{66}(\boldsymbol{u})\lambda_6\lambda_4^3 \\
&\quad + \tfrac{5}{21}\wp_{66}(\boldsymbol{v})\lambda_5^2\lambda_4^2 + \tfrac{1}{63}\wp_{66}(\boldsymbol{u})\lambda_6\lambda_5^3\lambda_4 - \tfrac{1}{252}\wp_{66}(\boldsymbol{u})\lambda_6^3\lambda_5^2\lambda_4 - \tfrac{25}{252}\wp_{66}(\boldsymbol{u})\lambda_6^2\lambda_5\lambda_4^2
\end{aligned}$$

$$\begin{aligned}
P_5(\boldsymbol{u},\boldsymbol{v}) &= \big[\tfrac{25}{42}\wp_{56}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) - \tfrac{3}{2}\wp_{46}(\boldsymbol{v})\big]\lambda_6\lambda_4\lambda_2 + \big[\tfrac{20}{21}\wp_{66}(\boldsymbol{u})\wp_{56}(\boldsymbol{v}) - \wp_{46}(\boldsymbol{u})\big]\lambda_3\lambda_2 \\
&\quad + \tfrac{5}{84}\wp_{56}(\boldsymbol{u})\wp_{66}(\boldsymbol{v})\lambda_6^2\lambda_5\lambda_2 + \big[\tfrac{15}{2}\wp_{56}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) - 3\wp_{46}(\boldsymbol{u})\big]\lambda_5\lambda_0 + \tfrac{13}{6}\wp_{46}(\boldsymbol{v})\lambda_6\lambda_5\lambda_1 \\
&\quad + \big[\tfrac{5}{3}\wp_{56}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) + \tfrac{11}{3}\wp_{46}(\boldsymbol{v})\big]\lambda_4\lambda_1 + \big[\tfrac{9}{2}\wp_{46}(\boldsymbol{v}) - \tfrac{15}{8}\wp_{66}(\boldsymbol{u})\wp_{56}(\boldsymbol{v})\big]\lambda_6^2\lambda_0 \\
&\quad - \tfrac{2}{63}\wp_{46}(\boldsymbol{u})\lambda_6^2\lambda_5^2\lambda_4 - \tfrac{11}{42}\wp_{46}(\boldsymbol{v})\lambda_6\lambda_5\lambda_4^2 - \big[\tfrac{1}{2}\wp_{46}(\boldsymbol{v}) + \tfrac{5}{21}\wp_{56}(\boldsymbol{u})\wp_{66}(\boldsymbol{v})\big]\lambda_5^2\lambda_2 \\
&\quad - \tfrac{25}{14}\wp_{46}(\boldsymbol{v})\lambda_4^3 - \tfrac{5}{2}\wp_{46}(\boldsymbol{v})\lambda_5\lambda_4\lambda_3
\end{aligned}$$

$$\begin{aligned}
P_8(\boldsymbol{u},\boldsymbol{v}) &= -\big[\tfrac{221}{126}\wp_{45}(\boldsymbol{v})\wp_{66}(\boldsymbol{u}) + \tfrac{8}{63}\wp_{36}(\boldsymbol{u})\wp_{66}(\boldsymbol{v})\big]\lambda_6\lambda_4\lambda_3 + \tfrac{5}{7}\wp_{45}(\boldsymbol{u})\wp_{66}(\boldsymbol{v})\lambda_5^2\lambda_3 \\
&\quad + \big[\tfrac{1}{4}\wp_{45}(\boldsymbol{v})\wp_{66}(\boldsymbol{u}) + \tfrac{1}{4}Q_{5555}(\boldsymbol{u}) + \tfrac{3}{4}\wp_{44}(\boldsymbol{u}) + \tfrac{103}{42}\wp_{46}(\boldsymbol{u})\wp_{56}(\boldsymbol{v})\big]\lambda_4\lambda_2 + \big[\tfrac{5}{12}\wp_{44}(\boldsymbol{u}) \\
&\quad + 3\wp_{36}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) + \tfrac{1}{36}Q_{5555}(\boldsymbol{u}) - 4\wp_{45}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) + \tfrac{7}{3}\wp_{56}(\boldsymbol{v})\wp_{46}(\boldsymbol{u})\big]\lambda_5\lambda_1 \\
&\quad - \tfrac{5}{28}\wp_{66}(\boldsymbol{u})\wp_{45}(\boldsymbol{v})\lambda_6^2\lambda_5\lambda_3 + \big[\wp_{45}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) - \tfrac{3}{4}\wp_{36}(\boldsymbol{u})\wp_{66}(\boldsymbol{v})\big]\lambda_6^2\lambda_1 \\
&\quad + \big[\tfrac{2}{63}\wp_{36}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) - \tfrac{1}{756}Q_{5555}(\boldsymbol{u}) - \tfrac{1}{126}\wp_{45}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) - \tfrac{1}{84}\wp_{44}(\boldsymbol{v})\big]\lambda_6\lambda_5^2\lambda_4 \\
&\quad - \big[\tfrac{5}{28}\wp_{44}(\boldsymbol{v}) + \tfrac{5}{252}Q_{5555}(\boldsymbol{v})\big]\lambda_5\lambda_4^2 - \big[5\wp_{55}(\boldsymbol{u})\wp_{55}(\boldsymbol{v}) + \tfrac{33}{4}\wp_{44}(\boldsymbol{u})\big]\lambda_6\lambda_0 \\
&\quad + \big[\tfrac{5}{252}\wp_{66}(\boldsymbol{v})\wp_{45}(\boldsymbol{u}) - \tfrac{5}{63}\wp_{66}(\boldsymbol{u})\wp_{36}(\boldsymbol{v})\big]\lambda_6^2\lambda_4^2 + \tfrac{10}{21}\wp_{46}(\boldsymbol{u})\wp_{56}(\boldsymbol{v})\lambda_6\lambda_5\lambda_2 \\
&\quad + \big[\tfrac{1}{504}\wp_{66}(\boldsymbol{u})\wp_{45}(\boldsymbol{v}) - \tfrac{1}{126}\wp_{36}(\boldsymbol{u})\wp_{66}(\boldsymbol{v})\big]\lambda_6^3\lambda_5\lambda_4 - \tfrac{20}{7}\wp_{45}(\boldsymbol{u})\wp_{66}(\boldsymbol{v})\lambda_3^2 \\
&\quad + \big[\tfrac{15}{4}\wp_{56}(\boldsymbol{u})\wp_{46}(\boldsymbol{v}) - \tfrac{3}{4}Q_{5555}(\boldsymbol{v})\big]\lambda_6\lambda_0
\end{aligned}$$

$$
\begin{aligned}
P_{11}(\boldsymbol{u},\boldsymbol{v}) &= -\tfrac{1}{252}\big[\wp_{66}(\boldsymbol{u})\wp_{34}(\boldsymbol{v}) + \wp_{26}(\boldsymbol{u})\wp_{66}(\boldsymbol{v})\big]\lambda_5^2\lambda_6^3 + \big(\big[\tfrac{1}{63}Q_{3466}(\boldsymbol{u}) + \tfrac{1}{63}Q_{4456}(\boldsymbol{u}) \\
&\quad - \tfrac{4}{63}\wp_{36}(\boldsymbol{u})\wp_{46}(\boldsymbol{v}) - \tfrac{41}{504}\wp_{26}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) + \tfrac{1}{63}\wp_{46}(\boldsymbol{u})\wp_{45}(\boldsymbol{v}) - \tfrac{37}{252}\wp_{66}(\boldsymbol{u})\wp_{34}(\boldsymbol{v})\big]\lambda_4\lambda_5 \\
&\quad + \big[\tfrac{1}{2}\wp_{34}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) - \tfrac{1}{8}\wp_{66}(\boldsymbol{u})\wp_{26}(\boldsymbol{v})\big]\lambda_2\big)\lambda_6^2 + \big(\big[\tfrac{5}{21}Q_{3466}(\boldsymbol{v}) - \tfrac{5}{12}\wp_{66}(\boldsymbol{u})\wp_{26}(\boldsymbol{v}) \\
&\quad - \tfrac{3}{28}\wp_{45}(\boldsymbol{u})\wp_{46}(\boldsymbol{v}) + \tfrac{5}{21}Q_{4456}(\boldsymbol{v}) - \tfrac{15}{14}\wp_{66}(\boldsymbol{u})\wp_{34}(\boldsymbol{v}) + \tfrac{3}{7}\wp_{46}(\boldsymbol{u})\wp_{36}(\boldsymbol{v})\big]\lambda_4^2 \\
&\quad + \big[\tfrac{1}{63}\wp_{66}(\boldsymbol{u})\wp_{34}(\boldsymbol{v}) + \tfrac{1}{63}\wp_{26}(\boldsymbol{u})\wp_{66}(\boldsymbol{v})\big]\lambda_5^3 + \big[\tfrac{1}{12}\wp_{26}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) - \wp_{45}(\boldsymbol{u})\wp_{46}(\boldsymbol{v}) \\
&\quad - \tfrac{5}{6}Q_{3466}(\boldsymbol{u}) + 4\wp_{36}(\boldsymbol{u})\wp_{46}(\boldsymbol{v}) - 4\wp_{35}(\boldsymbol{u})\wp_{55}(\boldsymbol{v}) - \tfrac{2}{3}\wp_{34}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) - \tfrac{5}{6}Q_{4456}(\boldsymbol{u})\big]\lambda_1 \\
&\quad - \big[\tfrac{4}{63}\wp_{26}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) + \tfrac{4}{63}\wp_{34}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) + \tfrac{10}{7}\wp_{45}(\boldsymbol{u})\wp_{46}(\boldsymbol{v})\big]\lambda_3\lambda_5\big)\lambda_6 + \big[\tfrac{3}{2}Q_{4456}(\boldsymbol{u}) \\
&\quad - \tfrac{1}{2}\wp_{34}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) + 5\wp_{46}(\boldsymbol{u})\wp_{45}(\boldsymbol{v}) - \tfrac{5}{8}Q_{5555}(\boldsymbol{u})\wp_{56}(\boldsymbol{v}) - \tfrac{47}{8}\wp_{44}(\boldsymbol{u})\wp_{56}(\boldsymbol{v}) \\
&\quad + \tfrac{5}{3}Q_{5556}(\boldsymbol{u})\wp_{55}(\boldsymbol{v}) - 4\wp_{26}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) - \tfrac{5}{2}\wp_{35}(\boldsymbol{u})\wp_{55}(\boldsymbol{v}) + \tfrac{17}{2}\wp_{46}(\boldsymbol{u})\wp_{36}(\boldsymbol{v}) \\
&\quad + \tfrac{1}{2}Q_{3466}(\boldsymbol{u})\big]\lambda_0 + \big[\tfrac{1}{6}\wp_{26}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) + \tfrac{3}{7}\wp_{34}(\boldsymbol{u})\wp_{66}(\boldsymbol{v})\big]\lambda_4\lambda_5^2 + \big[\tfrac{1}{6}\wp_{36}(\boldsymbol{u})\wp_{46}(\boldsymbol{v}) \\
&\quad + \tfrac{5}{252}\wp_{56}(\boldsymbol{u})Q_{5555}(\boldsymbol{v}) + \tfrac{7}{12}\wp_{26}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) - \tfrac{5}{6}\wp_{45}(\boldsymbol{u})\wp_{46}(\boldsymbol{v}) + \tfrac{5}{28}\wp_{56}(\boldsymbol{u})\wp_{44}(\boldsymbol{v}) \\
&\quad - \tfrac{5}{3}\wp_{34}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) + \tfrac{1}{6}Q_{4456}(\boldsymbol{v}) + \tfrac{1}{6}Q_{3466}(\boldsymbol{v})\big]\lambda_2\lambda_5 + \big[\tfrac{1}{3}\wp_{66}(\boldsymbol{u})\wp_{26}(\boldsymbol{v}) + \tfrac{1}{2}Q_{4456}(\boldsymbol{v}) \\
&\quad - \wp_{36}(\boldsymbol{u})\wp_{46}(\boldsymbol{v}) - \tfrac{19}{7}\wp_{34}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) + \tfrac{1}{2}Q_{3466}(\boldsymbol{u}) - \tfrac{199}{28}\wp_{46}(\boldsymbol{u})\wp_{45}(\boldsymbol{v})\big]\lambda_3\lambda_4
\end{aligned}
$$

$$
\begin{aligned}
P_{14}(\boldsymbol{u},\boldsymbol{v}) &= \big[\tfrac{1}{84}\wp_{24}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) - \tfrac{1}{84}\wp_{16}(\boldsymbol{u})\wp_{66}(\boldsymbol{v})\big]\lambda_5\lambda_6^4 + \big[\tfrac{5}{42}\wp_{24}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) \\
&\quad - \tfrac{5}{42}\wp_{16}(\boldsymbol{v})\wp_{66}(\boldsymbol{u})\big]\lambda_4\lambda_6^3 + \big(\big[\tfrac{95}{168}\wp_{24}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) + \tfrac{47}{84}\wp_{16}(\boldsymbol{u})\wp_{66}(\boldsymbol{v})\big]\lambda_3 \\
&\quad - \big[\tfrac{1}{21}\wp_{16}(\boldsymbol{v})\wp_{66}(\boldsymbol{u}) + \tfrac{2}{63}\wp_{34}(\boldsymbol{v})\wp_{46}(\boldsymbol{u}) + \tfrac{5}{84}\wp_{24}(\boldsymbol{v})\wp_{66}(\boldsymbol{u}) + \tfrac{2}{63}\wp_{46}(\boldsymbol{v})\wp_{26}(\boldsymbol{u})\big]\lambda_5^2\big)\lambda_6^2 \\
&\quad + \big(\big[\tfrac{9}{4}\wp_{25}(\boldsymbol{u})\wp_{55}(\boldsymbol{v}) - \tfrac{5}{12}Q_{4466}(\boldsymbol{u})\wp_{55}(\boldsymbol{v}) - \tfrac{5}{21}\wp_{56}(\boldsymbol{u})Q_{4456}(\boldsymbol{v}) - \tfrac{5}{21}\wp_{56}(\boldsymbol{u})Q_{3466}(\boldsymbol{v}) \\
&\quad - \tfrac{1}{2}\wp_{24}(\boldsymbol{v})\wp_{66}(\boldsymbol{u}) + \tfrac{3}{2}\wp_{16}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) - \tfrac{1}{2}\wp_{46}(\boldsymbol{u})\wp_{26}(\boldsymbol{v}) - \wp_{55}(\boldsymbol{u})\wp_{33}(\boldsymbol{v}) \\
&\quad - \wp_{34}(\boldsymbol{v})\wp_{46}(\boldsymbol{u})\big]\lambda_2 + \big[\tfrac{1}{1512}\wp_{45}(\boldsymbol{u})Q_{5555}(\boldsymbol{v}) - \tfrac{5}{42}\wp_{46}(\boldsymbol{v})\wp_{26}(\boldsymbol{u}) - \tfrac{9}{14}\wp_{34}(\boldsymbol{v})\wp_{46}(\boldsymbol{u}) \\
&\quad - \tfrac{1}{21}Q_{3446}(\boldsymbol{v}) - \tfrac{1}{42}\wp_{36}(\boldsymbol{u})\wp_{44}(\boldsymbol{v}) + \tfrac{1}{27}Q_{4445}(\boldsymbol{u}) - \tfrac{1}{378}\wp_{36}(\boldsymbol{u})Q_{5555}(\boldsymbol{v}) + \tfrac{1}{126}Q_{3355}(\boldsymbol{v}) \\
&\quad - \tfrac{5}{42}\wp_{24}(\boldsymbol{v})\wp_{66}(\boldsymbol{u}) - \tfrac{20}{21}\wp_{16}(\boldsymbol{v})\wp_{66}(\boldsymbol{u}) + \tfrac{1}{168}\wp_{45}(\boldsymbol{u})\wp_{44}(\boldsymbol{v})\big]\lambda_4\lambda_5\big)\lambda_6 \\
&\quad + \big[\tfrac{8}{21}\wp_{16}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) + \tfrac{1}{21}\wp_{24}(\boldsymbol{u})\wp_{66}(\boldsymbol{v})\big]\lambda_5^3 + \big[\tfrac{2}{3}\wp_{16}(\boldsymbol{v})\wp_{66}(\boldsymbol{u}) - \tfrac{4}{9}Q_{4445}(\boldsymbol{u}) \\
&\quad + \tfrac{2}{7}Q_{3446}(\boldsymbol{u}) - \tfrac{107}{42}\wp_{34}(\boldsymbol{v})\wp_{46}(\boldsymbol{u}) - \tfrac{4}{3}\wp_{24}(\boldsymbol{v})\wp_{66}(\boldsymbol{u}) - \tfrac{25}{4}\wp_{46}(\boldsymbol{v})\wp_{26}(\boldsymbol{u}) \\
&\quad + \tfrac{5}{42}Q_{3355}(\boldsymbol{v})\big]\lambda_4^2 + \big[\tfrac{1}{4}\wp_{46}(\boldsymbol{v})\wp_{26}(\boldsymbol{u}) - \tfrac{95}{21}\wp_{16}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) - \tfrac{71}{42}\wp_{24}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) \\
&\quad - \tfrac{5}{2}\wp_{46}(\boldsymbol{v})\wp_{34}(\boldsymbol{u}) - \tfrac{5}{84}Q_{5555}(\boldsymbol{u})\wp_{45}(\boldsymbol{v}) - \tfrac{15}{28}\wp_{45}(\boldsymbol{u})\wp_{44}(\boldsymbol{v})\big]\lambda_3\lambda_5 + \big[\tfrac{5}{9}Q_{4445}(\boldsymbol{v}) \\
&\quad - \tfrac{1}{6}Q_{3355}(\boldsymbol{u}) + \tfrac{5}{4}\wp_{25}(\boldsymbol{v})\wp_{55}(\boldsymbol{u}) - \tfrac{1}{4}Q_{4466}(\boldsymbol{u})\wp_{55}(\boldsymbol{v}) - 3\wp_{35}(\boldsymbol{u})\wp_{35}(\boldsymbol{v}) - \tfrac{2}{3}Q_{3446}(\boldsymbol{u}) \\
&\quad - \tfrac{1}{3}\wp_{16}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) - \tfrac{8}{3}\wp_{46}(\boldsymbol{v})\wp_{34}(\boldsymbol{u}) - \tfrac{11}{12}\wp_{36}(\boldsymbol{v})\wp_{44}(\boldsymbol{u}) - \tfrac{1}{4}\wp_{36}(\boldsymbol{u})Q_{5555}(\boldsymbol{v}) \\
&\quad - \tfrac{5}{6}Q_{3466}(\boldsymbol{u})\wp_{56}(\boldsymbol{v}) + \tfrac{1}{12}Q_{4455}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) - \tfrac{2}{3}Q_{4456}(\boldsymbol{u})\wp_{56}(\boldsymbol{v}) - \wp_{46}(\boldsymbol{v})\wp_{26}(\boldsymbol{u}) \\
&\quad + \tfrac{4}{3}Q_{5556}(\boldsymbol{u})\wp_{35}(\boldsymbol{v}) + \tfrac{11}{3}\wp_{44}(\boldsymbol{u})\wp_{45}(\boldsymbol{v}) + \tfrac{11}{4}\wp_{24}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) + \tfrac{1}{3}Q_{5555}(\boldsymbol{u})\wp_{45}(\boldsymbol{v})\big]\lambda_1
\end{aligned}
$$

$$
\begin{aligned}
P_{17}(\boldsymbol{u},\boldsymbol{v}) &= \big[\tfrac{1}{504}Q_{3346}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) - \tfrac{2}{21}\wp_{16}(\boldsymbol{u})\wp_{46}(\boldsymbol{v}) - \tfrac{19}{252}\wp_{66}(\boldsymbol{v})\wp_{14}(\boldsymbol{u}) \\
&+ \tfrac{2}{21}\wp_{24}(\boldsymbol{v})\wp_{46}(\boldsymbol{u})\big]\lambda_5\lambda_6^3 + \big[\tfrac{5}{252}\wp_{66}(\boldsymbol{u})Q_{3346}(\boldsymbol{v}) - \tfrac{1}{126}Q_{3466}(\boldsymbol{u})\wp_{45}(\boldsymbol{v}) + \wp_{15}(\boldsymbol{u})\wp_{55}(\boldsymbol{v}) \\
&+ \tfrac{47}{63}\wp_{66}(\boldsymbol{v})\wp_{14}(\boldsymbol{u}) - \tfrac{9}{14}\wp_{24}(\boldsymbol{u})\wp_{46}(\boldsymbol{v}) + \tfrac{9}{14}\wp_{16}(\boldsymbol{u})\wp_{46}(\boldsymbol{v}) + \tfrac{2}{63}Q_{4456}(\boldsymbol{u})\wp_{36}(\boldsymbol{v}) \\
&- \tfrac{1}{126}Q_{4456}(\boldsymbol{u})\wp_{45}(\boldsymbol{v}) + \tfrac{2}{63}Q_{3466}(\boldsymbol{u})\wp_{36}(\boldsymbol{v})\big]\lambda_4\lambda_6^2 + \big(\big[\tfrac{19}{63}\wp_{66}(\boldsymbol{v})\wp_{14}(\boldsymbol{u}) \\
&- \tfrac{1}{126}Q_{3346}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) - \tfrac{1}{756}\wp_{34}(\boldsymbol{v})Q_{5555}(\boldsymbol{u}) - \tfrac{16}{21}\wp_{16}(\boldsymbol{u})\wp_{46}(\boldsymbol{v}) - \tfrac{1}{84}\wp_{34}(\boldsymbol{u})\wp_{44}(\boldsymbol{v}) \\
&- \tfrac{2}{21}\wp_{24}(\boldsymbol{u})\wp_{46}(\boldsymbol{v}) - \tfrac{1}{84}\wp_{44}(\boldsymbol{u})\wp_{26}(\boldsymbol{v}) - \tfrac{1}{756}\wp_{26}(\boldsymbol{u})Q_{5555}(\boldsymbol{v})\big]\lambda_5^2 + \big[\tfrac{1}{3}Q_{4446}(\boldsymbol{u})\wp_{55}(\boldsymbol{v}) \\
&+ \tfrac{2}{63}Q_{3346}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) - 6\wp_{16}(\boldsymbol{u})\wp_{46}(\boldsymbol{v}) - \tfrac{76}{63}\wp_{66}(\boldsymbol{v})\wp_{14}(\boldsymbol{u}) + \tfrac{5}{7}\wp_{45}(\boldsymbol{u})Q_{3466}(\boldsymbol{v}) \\
&+ \tfrac{3}{2}\wp_{23}(\boldsymbol{u})\wp_{55}(\boldsymbol{v}) + \tfrac{5}{7}Q_{4456}(\boldsymbol{u})\wp_{45}(\boldsymbol{v}) + \tfrac{3}{4}\wp_{24}(\boldsymbol{v})\wp_{46}(\boldsymbol{u}) + 2\wp_{15}(\boldsymbol{u})\wp_{55}(\boldsymbol{v})\big]\lambda_3\big)\lambda_6 \\
&+ \big[\tfrac{1}{3}Q_{2446}(\boldsymbol{u}) - \tfrac{1}{8}\wp_{44}(\boldsymbol{u})\wp_{26}(\boldsymbol{v}) - \tfrac{1}{28}\wp_{34}(\boldsymbol{v})Q_{5555}(\boldsymbol{u}) - \tfrac{1}{72}\wp_{26}(\boldsymbol{u})Q_{5555}(\boldsymbol{v}) \\
&- \tfrac{28}{3}\wp_{66}(\boldsymbol{u})\wp_{14}(\boldsymbol{v}) - \tfrac{9}{28}\wp_{34}(\boldsymbol{u})\wp_{44}(\boldsymbol{v}) - \tfrac{55}{7}\wp_{16}(\boldsymbol{u})\wp_{46}(\boldsymbol{v}) - \tfrac{11}{21}\wp_{24}(\boldsymbol{u})\wp_{46}(\boldsymbol{v}) \\
&- \tfrac{1}{9}Q_{1555}(\boldsymbol{v})\big]\lambda_4\lambda_5 + \big[\tfrac{1}{6}Q_{3466}(\boldsymbol{u})\wp_{36}(\boldsymbol{v}) + \tfrac{1}{3}\wp_{33}(\boldsymbol{u})Q_{5556}(\boldsymbol{v}) + \tfrac{1}{6}Q_{3466}(\boldsymbol{u})\wp_{45}(\boldsymbol{v}) \\
&- 2\wp_{15}(\boldsymbol{u})\wp_{55}(\boldsymbol{v}) + \tfrac{1}{6}\wp_{34}(\boldsymbol{u})Q_{5555}(\boldsymbol{v}) + \tfrac{9}{4}\wp_{25}(\boldsymbol{u})\wp_{35}(\boldsymbol{v}) - \tfrac{5}{12}Q_{4455}(\boldsymbol{u})\wp_{46}(\boldsymbol{v}) \\
&- \tfrac{3}{4}Q_{4466}(\boldsymbol{v})\wp_{35}(\boldsymbol{u}) - \wp_{33}(\boldsymbol{u})\wp_{35}(\boldsymbol{v}) - \tfrac{5}{2}\wp_{66}(\boldsymbol{v})\wp_{14}(\boldsymbol{u}) - \tfrac{1}{24}\wp_{26}(\boldsymbol{u})Q_{5555}(\boldsymbol{v}) \\
&+ \tfrac{1}{3}Q_{4446}(\boldsymbol{u})\wp_{55}(\boldsymbol{v}) - \tfrac{1}{8}\wp_{44}(\boldsymbol{v})\wp_{26}(\boldsymbol{u}) + \tfrac{7}{2}\wp_{16}(\boldsymbol{u})\wp_{46}(\boldsymbol{v}) + \tfrac{5}{18}Q_{4445}(\boldsymbol{u})\wp_{56}(\boldsymbol{v}) \\
&+ \tfrac{1}{4}\wp_{24}(\boldsymbol{u})\wp_{46}(\boldsymbol{v}) - \tfrac{1}{6}Q_{4456}(\boldsymbol{u})\wp_{36}(\boldsymbol{v}) - \tfrac{2}{7}Q_{3446}(\boldsymbol{u})\wp_{56}(\boldsymbol{v}) + \tfrac{1}{2}\wp_{34}(\boldsymbol{u})\wp_{44}(\boldsymbol{v}) \\
&+ \tfrac{5}{36}Q_{4466}(\boldsymbol{v})Q_{5556}(\boldsymbol{u}) - \tfrac{1}{6}Q_{3445}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) - \tfrac{5}{42}Q_{3355}(\boldsymbol{u})\wp_{56}(\boldsymbol{v}) + \tfrac{1}{3}Q_{2446}(\boldsymbol{u}) \\
&- \tfrac{1}{12}Q_{3346}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) + \tfrac{2}{3}Q_{4456}(\boldsymbol{u})\wp_{45}(\boldsymbol{v}) - \tfrac{3}{4}\wp_{25}(\boldsymbol{u})Q_{5556}(\boldsymbol{v})\big]\lambda_2
\end{aligned}
$$

$$
\begin{aligned}
P_{20}(\boldsymbol{u},\boldsymbol{v}) &= \big[\tfrac{1}{84}\wp_{66}(\boldsymbol{u})Q_{3344}(\boldsymbol{v}) + \tfrac{1}{28}\wp_{66}(\boldsymbol{u})Q_{2445}(\boldsymbol{v}) + \tfrac{1}{63}Q_{4456}(\boldsymbol{u})\wp_{34}(\boldsymbol{v}) \\
&- \tfrac{38}{63}\wp_{46}(\boldsymbol{u})\wp_{14}(\boldsymbol{v}) + \tfrac{1}{63}\wp_{46}(\boldsymbol{u})Q_{3346}(\boldsymbol{v}) + \tfrac{1}{63}Q_{3466}(\boldsymbol{u})\wp_{34}(\boldsymbol{v}) + \tfrac{1}{252}\wp_{24}(\boldsymbol{u})Q_{5555}(\boldsymbol{v}) \\
&+ \tfrac{1}{28}\wp_{24}(\boldsymbol{u})\wp_{44}(\boldsymbol{v}) + \tfrac{1}{63}Q_{3466}(\boldsymbol{u})\wp_{26}(\boldsymbol{v}) + \tfrac{1}{84}\wp_{66}(\boldsymbol{u})Q_{2346}(\boldsymbol{v}) - \tfrac{1}{252}\wp_{16}(\boldsymbol{v})Q_{5555}(\boldsymbol{u}) \\
&- \tfrac{1}{28}\wp_{16}(\boldsymbol{u})\wp_{44}(\boldsymbol{v}) + \tfrac{1}{63}Q_{4456}(\boldsymbol{u})\wp_{26}(\boldsymbol{v})\big]\lambda_5\lambda_6^2 + \big[\tfrac{1}{6}Q_{4456}(\boldsymbol{u})\wp_{26}(\boldsymbol{v}) - \tfrac{1}{3}Q_{1446}(\boldsymbol{v}) \\
&+ \tfrac{3}{7}Q_{3466}(\boldsymbol{u})\wp_{34}(\boldsymbol{v}) + \tfrac{5}{14}\wp_{66}(\boldsymbol{u})Q_{2445}(\boldsymbol{v}) + \tfrac{1}{42}Q_{3446}(\boldsymbol{v})\wp_{45}(\boldsymbol{u}) + \wp_{15}(\boldsymbol{v})\wp_{35}(\boldsymbol{u}) \\
&+ \tfrac{2}{27}Q_{4445}(\boldsymbol{u})\wp_{36}(\boldsymbol{v}) - \tfrac{2}{21}Q_{3446}(\boldsymbol{u})\wp_{36}(\boldsymbol{v}) - \tfrac{3}{28}Q_{3346}(\boldsymbol{u})\wp_{46}(\boldsymbol{v}) - \tfrac{139}{14}\wp_{14}(\boldsymbol{u})\wp_{46}(\boldsymbol{v}) \\
&- \tfrac{1}{3}\wp_{15}(\boldsymbol{u})Q_{5556}(\boldsymbol{v}) + \tfrac{5}{42}Q_{2346}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) + \tfrac{1}{6}\wp_{26}(\boldsymbol{u})Q_{3466}(\boldsymbol{v}) - \tfrac{1}{3}Q_{1556}(\boldsymbol{v})\wp_{55}(\boldsymbol{u}) \\
&- \tfrac{1}{252}Q_{3355}(\boldsymbol{u})\wp_{45}(\boldsymbol{v}) + \tfrac{1}{63}Q_{3355}(\boldsymbol{u})\wp_{36}(\boldsymbol{v}) + \tfrac{4}{3}\wp_{13}(\boldsymbol{u})\wp_{55}(\boldsymbol{v}) + \tfrac{3}{7}Q_{4456}(\boldsymbol{v})\wp_{34}(\boldsymbol{u}) \\
&+ \wp_{22}(\boldsymbol{u})\wp_{55}(\boldsymbol{v}) - \tfrac{1}{54}Q_{4445}(\boldsymbol{u})\wp_{45}(\boldsymbol{v}) + \tfrac{5}{42}\wp_{66}(\boldsymbol{u})Q_{3344}(\boldsymbol{v})\big]\lambda_4\lambda_6 - \big[\tfrac{11}{3}\wp_{14}(\boldsymbol{u})\wp_{46}(\boldsymbol{v}) \\
&+ \tfrac{1}{21}\wp_{66}(\boldsymbol{v})Q_{3344}(\boldsymbol{u}) + \tfrac{1}{252}\wp_{24}(\boldsymbol{v})Q_{5555}(\boldsymbol{u}) + \tfrac{2}{63}\wp_{16}(\boldsymbol{u})Q_{5555}(\boldsymbol{v}) + \tfrac{1}{21}Q_{2346}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) \\
&+ \tfrac{1}{28}\wp_{24}(\boldsymbol{u})\wp_{44}(\boldsymbol{v}) + \tfrac{1}{7}Q_{2445}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) + \tfrac{2}{7}\wp_{16}(\boldsymbol{u})\wp_{44}(\boldsymbol{v})\big]\lambda_5^2 + \cdots
\end{aligned}
$$

$$\cdots + \big[\tfrac{4}{21}\wp_{66}(\boldsymbol{u})Q_{2346}(\boldsymbol{v}) - \tfrac{3}{4}\wp_{25}(\boldsymbol{u})\wp_{25}(\boldsymbol{v}) + \tfrac{3}{2}\wp_{23}(\boldsymbol{u})\wp_{35}(\boldsymbol{v}) + \tfrac{3}{8}\wp_{24}(\boldsymbol{u})\wp_{44}(\boldsymbol{v})$$
$$+ \tfrac{5}{4}\wp_{16}(\boldsymbol{u})\wp_{44}(\boldsymbol{v}) + \tfrac{1}{8}\wp_{24}(\boldsymbol{u})Q_{5555}(\boldsymbol{v}) + \tfrac{1}{2}Q_{3466}(\boldsymbol{v})\wp_{34}(\boldsymbol{u}) - \tfrac{1}{2}Q_{3466}(\boldsymbol{u})\wp_{26}(\boldsymbol{v})$$
$$- \tfrac{1}{2}Q_{5556}(\boldsymbol{u})\wp_{23}(\boldsymbol{v}) + \tfrac{6}{7}Q_{3446}(\boldsymbol{u})\wp_{45}(\boldsymbol{v}) + Q_{4466}(\boldsymbol{u})\wp_{25}(\boldsymbol{v}) + \tfrac{5}{14}\wp_{45}(\boldsymbol{u})Q_{3355}(\boldsymbol{v})$$
$$+ \tfrac{4}{21}\wp_{66}(\boldsymbol{v})Q_{3344}(\boldsymbol{u}) - \tfrac{1}{6}\wp_{35}(\boldsymbol{v})Q_{4446}(\boldsymbol{u}) + 3\wp_{15}(\boldsymbol{u})\wp_{35}(\boldsymbol{v}) - \tfrac{1}{9}Q_{5556}(\boldsymbol{u})Q_{4446}(\boldsymbol{v})$$
$$+ \tfrac{1}{2}Q_{4456}(\boldsymbol{u})\wp_{34}(\boldsymbol{v}) - \tfrac{1}{12}Q_{4466}(\boldsymbol{u})Q_{4466}(\boldsymbol{v}) - \tfrac{13}{2}\wp_{46}(\boldsymbol{u})\wp_{14}(\boldsymbol{v}) - \tfrac{2}{3}\wp_{15}(\boldsymbol{u})Q_{5556}(\boldsymbol{v})$$
$$- \tfrac{5}{6}\wp_{45}(\boldsymbol{u})Q_{4445}(\boldsymbol{v}) + \tfrac{1}{4}\wp_{16}(\boldsymbol{u})Q_{5555}(\boldsymbol{v}) + \tfrac{1}{4}\wp_{46}(\boldsymbol{u})Q_{3346}(\boldsymbol{v}) + \tfrac{4}{7}\wp_{66}(\boldsymbol{u})Q_{2445}(\boldsymbol{v})\big]\lambda_3$$

$$P_{23}(\boldsymbol{u},\boldsymbol{v}) = \big[\tfrac{1}{21}\wp_{16}(\boldsymbol{u})Q_{4456}(\boldsymbol{v}) - \tfrac{1}{21}Q_{3466}(\boldsymbol{u})\wp_{24}(\boldsymbol{v}) - \tfrac{1}{21}Q_{4456}(\boldsymbol{u})\wp_{24}(\boldsymbol{v})$$
$$+ \tfrac{1}{21}Q_{3466}(\boldsymbol{u})\wp_{16}(\boldsymbol{v})\big]\lambda_6^3 + \big[\tfrac{1}{3}Q_{4466}(\boldsymbol{u})\wp_{15}(\boldsymbol{v}) + \tfrac{2}{21}\wp_{46}(\boldsymbol{u})Q_{3344}(\boldsymbol{v}) - \wp_{12}(\boldsymbol{u})\wp_{55}(\boldsymbol{v})$$
$$+ \tfrac{8}{21}\wp_{16}(\boldsymbol{u})Q_{3466}(\boldsymbol{v}) + \tfrac{1}{1512}Q_{3346}(\boldsymbol{u})Q_{5555}(\boldsymbol{v}) + \tfrac{1}{27}Q_{4445}(\boldsymbol{u})\wp_{26}(\boldsymbol{v}) + \wp_{25}(\boldsymbol{v})\wp_{15}(\boldsymbol{u})$$
$$- \tfrac{19}{84}\wp_{14}(\boldsymbol{u})\wp_{44}(\boldsymbol{v}) + \tfrac{1}{126}Q_{3355}(\boldsymbol{u})\wp_{26}(\boldsymbol{v}) + \tfrac{2}{21}\wp_{46}(\boldsymbol{v})Q_{2346}(\boldsymbol{u}) + \tfrac{1}{126}Q_{3355}(\boldsymbol{u})\wp_{34}(\boldsymbol{v})$$
$$- \tfrac{1}{21}Q_{3446}(\boldsymbol{u})\wp_{26}(\boldsymbol{v}) - \tfrac{1}{21}\wp_{34}(\boldsymbol{u})Q_{3446}(\boldsymbol{v}) + \tfrac{1}{27}\wp_{34}(\boldsymbol{v})Q_{4445}(\boldsymbol{u}) + \tfrac{2}{7}\wp_{46}(\boldsymbol{u})Q_{2445}(\boldsymbol{v})$$
$$+ \tfrac{8}{21}\wp_{16}(\boldsymbol{u})Q_{4456}(\boldsymbol{v}) - \tfrac{19}{756}\wp_{14}(\boldsymbol{u})Q_{5555}(\boldsymbol{v}) + \tfrac{1}{21}\wp_{24}(\boldsymbol{u})Q_{4456}(\boldsymbol{v})$$
$$+ \tfrac{1}{21}\wp_{24}(\boldsymbol{u})Q_{3466}(\boldsymbol{v}) + \tfrac{1}{168}Q_{3346}(\boldsymbol{u})\wp_{44}(\boldsymbol{v})\big]\lambda_5\lambda_6 + \big[\tfrac{8}{21}Q_{3446}(\boldsymbol{u})\wp_{34}(\boldsymbol{v}) + \tfrac{1}{3}Q_{1255}(\boldsymbol{v})$$
$$- \tfrac{2}{9}Q_{1555}(\boldsymbol{u})\wp_{36}(\boldsymbol{v}) + \wp_{35}(\boldsymbol{u})\wp_{22}(\boldsymbol{v}) - \tfrac{1}{3}Q_{4446}(\boldsymbol{v})\wp_{33}(\boldsymbol{u}) - \tfrac{1}{3}\wp_{66}(\boldsymbol{u})Q_{2246}(\boldsymbol{v})$$
$$+ \tfrac{1}{6}Q_{4466}(\boldsymbol{u})\wp_{15}(\boldsymbol{v}) + \tfrac{1}{12}Q_{3355}(\boldsymbol{u})\wp_{26}(\boldsymbol{v}) + \tfrac{2}{3}Q_{3446}(\boldsymbol{u})\wp_{26}(\boldsymbol{v}) + \tfrac{4}{3}\wp_{35}(\boldsymbol{u})\wp_{13}(\boldsymbol{v})$$
$$- \tfrac{1}{3}\wp_{35}(\boldsymbol{u})Q_{1556}(\boldsymbol{v}) - \tfrac{7}{9}\wp_{26}(\boldsymbol{u})Q_{4445}(\boldsymbol{v}) + \tfrac{1}{2}Q_{5555}(\boldsymbol{u})\wp_{14}(\boldsymbol{v}) + \tfrac{17}{6}\wp_{44}(\boldsymbol{u})\wp_{14}(\boldsymbol{v})$$
$$+ \tfrac{1}{18}Q_{1555}(\boldsymbol{u})\wp_{45}(\boldsymbol{v}) + \tfrac{1}{3}\wp_{66}(\boldsymbol{u})Q_{2344}(\boldsymbol{v}) - \tfrac{3}{2}Q_{4446}(\boldsymbol{v})\wp_{25}(\boldsymbol{u}) - \tfrac{3}{2}\wp_{23}(\boldsymbol{u})\wp_{25}(\boldsymbol{v})$$
$$+ \tfrac{3}{14}Q_{3355}(\boldsymbol{u})\wp_{34}(\boldsymbol{v}) + \tfrac{29}{14}\wp_{46}(\boldsymbol{v})Q_{2445}(\boldsymbol{u}) + \tfrac{2}{3}Q_{3466}(\boldsymbol{u})\wp_{24}(\boldsymbol{v}) + \tfrac{9}{2}\wp_{25}(\boldsymbol{v})\wp_{15}(\boldsymbol{u})$$
$$+ \tfrac{1}{3}Q_{4456}(\boldsymbol{u})\wp_{16}(\boldsymbol{v}) - \tfrac{2}{3}\wp_{34}(\boldsymbol{v})Q_{4445}(\boldsymbol{u}) + \tfrac{1}{2}Q_{4466}(\boldsymbol{u})\wp_{23}(\boldsymbol{v}) + 2\wp_{15}(\boldsymbol{u})\wp_{33}(\boldsymbol{v})$$
$$+ \tfrac{1}{9}Q_{4446}(\boldsymbol{v})Q_{4466}(\boldsymbol{u}) + \tfrac{71}{42}\wp_{46}(\boldsymbol{v})Q_{2346}(\boldsymbol{u}) - \tfrac{1}{3}\wp_{22}(\boldsymbol{u})Q_{5556}(\boldsymbol{v})$$
$$+ \tfrac{2}{3}Q_{4456}(\boldsymbol{u})\wp_{24}(\boldsymbol{v}) - \tfrac{1}{3}\wp_{16}(\boldsymbol{u})Q_{3466}(\boldsymbol{v}) + \tfrac{7}{12}\wp_{45}(\boldsymbol{u})Q_{2446}(\boldsymbol{v}) + \tfrac{4}{21}\wp_{46}(\boldsymbol{u})Q_{3344}(\boldsymbol{v})$$
$$+ \tfrac{1}{3}\wp_{36}(\boldsymbol{v})Q_{2446}(\boldsymbol{u}) - \tfrac{4}{9}Q_{5556}(\boldsymbol{u})\wp_{13}(\boldsymbol{v}) + \tfrac{1}{9}Q_{1556}(\boldsymbol{u})Q_{5556}(\boldsymbol{v}) + \tfrac{1}{3}Q_{1335}(\boldsymbol{u})\big]\lambda_4$$

$$P_{26}(\boldsymbol{u},\boldsymbol{v}) = \big[\tfrac{1}{2}\wp_{46}(\boldsymbol{u})Q_{2344}(\boldsymbol{v}) + \tfrac{1}{252}Q_{5555}(\boldsymbol{u})Q_{3344}(\boldsymbol{v}) + \tfrac{6}{7}\wp_{16}(\boldsymbol{u})Q_{3446}(\boldsymbol{v})$$
$$+ \tfrac{1}{3}Q_{5556}(\boldsymbol{u})\wp_{12}(\boldsymbol{v}) - \tfrac{1}{9}Q_{4466}(\boldsymbol{u})Q_{1556}(\boldsymbol{v}) + \tfrac{1}{42}\wp_{24}(\boldsymbol{u})Q_{3355}(\boldsymbol{v}) + \wp_{15}(\boldsymbol{u})\wp_{15}(\boldsymbol{v})$$
$$+ \tfrac{1}{9}\wp_{13}(\boldsymbol{u})Q_{4466}(\boldsymbol{v}) - \tfrac{1}{2}\wp_{46}(\boldsymbol{u})Q_{2246}(\boldsymbol{v}) - \tfrac{13}{42}\wp_{24}(\boldsymbol{u})Q_{3446}(\boldsymbol{v}) + \tfrac{1}{4}\wp_{22}(\boldsymbol{u})\wp_{25}(\boldsymbol{v})$$
$$- \tfrac{1}{9}\wp_{34}(\boldsymbol{u})Q_{1555}(\boldsymbol{v}) + \tfrac{1}{2}8\wp_{44}(\boldsymbol{u})Q_{2346}(\boldsymbol{v}) + \tfrac{3}{2}8\wp_{44}(\boldsymbol{u})Q_{2445}(\boldsymbol{v}) - \tfrac{5}{3}\wp_{25}(\boldsymbol{u})\wp_{13}(\boldsymbol{v})$$
$$+ \tfrac{5}{18}\wp_{24}(\boldsymbol{u})Q_{4445}(\boldsymbol{v}) - \tfrac{1}{9}Q_{4446}(\boldsymbol{u})Q_{4446}(\boldsymbol{v}) - \tfrac{1}{12}\wp_{22}(\boldsymbol{v})Q_{4466}(\boldsymbol{u}) - \wp_{35}(\boldsymbol{u})\wp_{12}(\boldsymbol{v})$$
$$- \tfrac{1}{9}\wp_{26}(\boldsymbol{u})Q_{1555}(\boldsymbol{v}) + \tfrac{1}{12}Q_{2446}(\boldsymbol{u})\wp_{26}(\boldsymbol{v}) + \tfrac{1}{3}\wp_{34}(\boldsymbol{u})Q_{2446}(\boldsymbol{v}) + \tfrac{1}{2}\wp_{66}(\boldsymbol{u})Q_{2244}(\boldsymbol{v})$$
$$+ \tfrac{1}{252}Q_{5555}(\boldsymbol{u})Q_{2346}(\boldsymbol{v}) + \wp_{66}(\boldsymbol{u})Q_{1344}(\boldsymbol{v}) + \tfrac{1}{28}\wp_{44}(\boldsymbol{u})Q_{3344}(\boldsymbol{v}) + \wp_{15}(\boldsymbol{u})\wp_{23}(\boldsymbol{v})$$
$$+ \tfrac{1}{84}Q_{5555}(\boldsymbol{u})Q_{2445}(\boldsymbol{v}) + \wp_{14}(\boldsymbol{u})Q_{4456}(\boldsymbol{v}) - \tfrac{1}{6}Q_{4446}(\boldsymbol{u})\wp_{23}(\boldsymbol{v}) + \cdots$$

$$\cdots - \tfrac{1}{3}Q_{1556}(\boldsymbol{v})\wp_{25}(\boldsymbol{u}) + \tfrac{5}{3}\wp_{14}(\boldsymbol{v})Q_{3466}(\boldsymbol{u}) - \tfrac{10}{9}\wp_{16}(\boldsymbol{u})Q_{4445}(\boldsymbol{v})$$
$$+ \tfrac{4}{21}\wp_{16}(\boldsymbol{u})Q_{3355}(\boldsymbol{v})\big]\lambda_5 + \big[\tfrac{1}{7}Q_{3446}(\boldsymbol{u})\wp_{24}(\boldsymbol{v}) - \tfrac{1}{42}\wp_{24}(\boldsymbol{v})Q_{3355}(\boldsymbol{u})$$
$$+ 6\wp_{15}(\boldsymbol{u})\wp_{15}(\boldsymbol{v}) - \tfrac{1}{126}Q_{3466}(\boldsymbol{u})Q_{3346}(\boldsymbol{v}) - \tfrac{4}{3}Q_{4446}(\boldsymbol{u})\wp_{15}(\boldsymbol{v}) - \tfrac{1}{7}Q_{3446}(\boldsymbol{u})\wp_{16}(\boldsymbol{v})$$
$$- \tfrac{1}{8}\wp_{66}(\boldsymbol{u})Q_{2244}(\boldsymbol{v}) + \tfrac{1}{9}\wp_{16}(\boldsymbol{u})Q_{4445}(\boldsymbol{v}) - \tfrac{1}{4}\wp_{66}(\boldsymbol{u})Q_{1344}(\boldsymbol{v})$$
$$+ \tfrac{19}{63}\wp_{14}(\boldsymbol{v})Q_{3466}(\boldsymbol{u}) - 3\wp_{55}(\boldsymbol{u})\wp_{11}(\boldsymbol{v}) + \tfrac{19}{63}Q_{4456}(\boldsymbol{u})\wp_{14}(\boldsymbol{v})$$
$$- \tfrac{1}{126}Q_{4456}(\boldsymbol{u})Q_{3346}(\boldsymbol{v}) - \tfrac{1}{9}\wp_{24}(\boldsymbol{u})Q_{4445}(\boldsymbol{v}) + \tfrac{1}{42}Q_{3355}(\boldsymbol{u})\wp_{16}(\boldsymbol{v})\big]\lambda_6^2$$

$$P_{29}(\boldsymbol{u},\boldsymbol{v}) = \big[\tfrac{1}{4}Q_{2244}(\boldsymbol{u})\wp_{46}(\boldsymbol{v}) - \tfrac{1}{54}Q_{3346}(\boldsymbol{v})Q_{4445}(\boldsymbol{u}) + \tfrac{1}{42}Q_{3446}(\boldsymbol{v})Q_{3346}(\boldsymbol{u})$$
$$+ \tfrac{19}{126}\wp_{14}(\boldsymbol{v})Q_{3355}(\boldsymbol{u}) + \tfrac{1}{3}\wp_{22}(\boldsymbol{v})Q_{4446}(\boldsymbol{u}) + \tfrac{5}{9}\wp_{13}(\boldsymbol{v})Q_{4446}(\boldsymbol{u}) - \tfrac{1}{2}\wp_{55}(\boldsymbol{v})Q_{2234}(\boldsymbol{u})$$
$$- \tfrac{1}{21}Q_{4456}(\boldsymbol{u})Q_{2346}(\boldsymbol{v}) - \tfrac{1}{21}Q_{3344}(\boldsymbol{v})Q_{4456}(\boldsymbol{u}) - \tfrac{1}{7}Q_{2445}(\boldsymbol{v})Q_{3466}(\boldsymbol{u})$$
$$- \tfrac{1}{252}Q_{3346}(\boldsymbol{v})Q_{3355}(\boldsymbol{u}) - \tfrac{1}{7}Q_{4456}(\boldsymbol{v})Q_{2445}(\boldsymbol{u}) - \tfrac{1}{21}Q_{2346}(\boldsymbol{v})Q_{3466}(\boldsymbol{u})$$
$$+ \tfrac{1}{3}\wp_{24}(\boldsymbol{v})Q_{1555}(\boldsymbol{u}) - \tfrac{8}{27}\wp_{14}(\boldsymbol{v})Q_{4445}(\boldsymbol{u}) - \tfrac{1}{2}Q_{2446}(\boldsymbol{v})\wp_{24}(\boldsymbol{u}) - \tfrac{1}{3}\wp_{34}(\boldsymbol{u})Q_{1446}(\boldsymbol{v})$$
$$- \tfrac{1}{3}Q_{1555}(\boldsymbol{v})\wp_{16}(\boldsymbol{u}) - \tfrac{1}{2}\wp_{12}(\boldsymbol{v})Q_{4466}(\boldsymbol{u}) - \wp_{15}(\boldsymbol{v})Q_{2356}(\boldsymbol{u}) + \tfrac{1}{6}\wp_{26}(\boldsymbol{v})Q_{1446}(\boldsymbol{u})$$
$$+ \tfrac{23}{21}Q_{3446}(\boldsymbol{v})\wp_{14}(\boldsymbol{u}) - \tfrac{4}{3}\wp_{15}(\boldsymbol{v})Q_{1556}(\boldsymbol{u}) + \tfrac{1}{2}Q_{1344}(\boldsymbol{u})\wp_{46}(\boldsymbol{v}) - 2\wp_{22}(\boldsymbol{u})\wp_{15}(\boldsymbol{v})$$
$$+ \tfrac{4}{9}Q_{4446}(\boldsymbol{u})Q_{1556}(\boldsymbol{v}) - \tfrac{1}{3}\wp_{55}(\boldsymbol{v})Q_{2226}(\boldsymbol{u}) + Q_{5556}(\boldsymbol{v})\wp_{11}(\boldsymbol{u}) + Q_{2446}(\boldsymbol{v})\wp_{16}(\boldsymbol{u})$$
$$+ \tfrac{3}{2}\wp_{12}(\boldsymbol{v})\wp_{25}(\boldsymbol{u}) - 3\wp_{35}(\boldsymbol{v})\wp_{11}(\boldsymbol{u}) - \tfrac{1}{21}Q_{3344}(\boldsymbol{v})Q_{3466}(\boldsymbol{u}) - \tfrac{8}{3}\wp_{15}(\boldsymbol{v})\wp_{13}(\boldsymbol{u})\big]\lambda_6$$

$$P_{32}(\boldsymbol{u},\boldsymbol{v}) = -\tfrac{1}{6}Q_{113666}(\boldsymbol{u})\wp_{66}(\boldsymbol{v}) + \tfrac{1}{6}Q_{223466}(\boldsymbol{v})\wp_{46}(\boldsymbol{u}) - \tfrac{5}{21}Q_{2445}(\boldsymbol{u})Q_{3446}(\boldsymbol{v})$$
$$+ \tfrac{4}{3}\wp_{12}(\boldsymbol{v})Q_{4446}(\boldsymbol{u}) - \tfrac{1}{9}Q_{1555}(\boldsymbol{v})\wp_{14}(\boldsymbol{u}) - \tfrac{1}{3}Q_{3334}(\boldsymbol{v})\wp_{15}(\boldsymbol{u}) + \tfrac{1}{3}Q_{1335}(\boldsymbol{u})\wp_{34}(\boldsymbol{v})$$
$$- \tfrac{1}{6}Q_{2446}(\boldsymbol{u})Q_{3445}(\boldsymbol{v}) + \tfrac{1}{18}Q_{3334}(\boldsymbol{u})Q_{4446}(\boldsymbol{v}) - \tfrac{1}{42}Q_{2346}(\boldsymbol{u})Q_{3355}(\boldsymbol{v})$$
$$- \tfrac{1}{2}\wp_{16}(\boldsymbol{u})Q_{2255}(\boldsymbol{v}) + \tfrac{1}{2}Q_{1135}(\boldsymbol{u})\wp_{56}(\boldsymbol{v}) - \tfrac{11}{3}\wp_{16}(\boldsymbol{u})Q_{1446}(\boldsymbol{v}) + \tfrac{3}{2}Q_{2446}(\boldsymbol{v})\wp_{14}(\boldsymbol{u})$$
$$+ \tfrac{1}{18}Q_{3344}(\boldsymbol{u})Q_{4445}(\boldsymbol{v}) + \tfrac{1}{18}Q_{1555}(\boldsymbol{u})Q_{3346}(\boldsymbol{v}) - \tfrac{1}{6}Q_{3466}(\boldsymbol{u})Q_{2344}(\boldsymbol{v})$$
$$- \tfrac{1}{6}Q_{2226}(\boldsymbol{u})\wp_{35}(\boldsymbol{v}) + \tfrac{1}{2}Q_{1136}(\boldsymbol{u})\wp_{55}(\boldsymbol{v}) + \tfrac{1}{4}\wp_{25}(\boldsymbol{u})Q_{2334}(\boldsymbol{v}) + \tfrac{2}{9}Q_{2346}(\boldsymbol{u})Q_{4445}(\boldsymbol{v})$$
$$- \tfrac{2}{9}\wp_{13}(\boldsymbol{u})Q_{1556}(\boldsymbol{v}) - \tfrac{1}{12}Q_{2446}(\boldsymbol{u})Q_{3346}(\boldsymbol{v}) - \tfrac{1}{6}Q_{4456}(\boldsymbol{u})Q_{2344}(\boldsymbol{v}) + \wp_{13}(\boldsymbol{v})\wp_{22}(\boldsymbol{u})$$
$$- \tfrac{2}{9}Q_{1556}(\boldsymbol{u})Q_{1556}(\boldsymbol{v}) + \tfrac{3}{2}Q_{1446}(\boldsymbol{u})\wp_{24}(\boldsymbol{v}) - \tfrac{1}{2}Q_{2234}(\boldsymbol{v})\wp_{35}(\boldsymbol{u}) + \tfrac{2}{3}Q_{2356}(\boldsymbol{v})\wp_{13}(\boldsymbol{u})$$
$$- \tfrac{1}{42}Q_{3344}(\boldsymbol{u})Q_{3446}(\boldsymbol{v}) - \tfrac{4}{21}Q_{2346}(\boldsymbol{u})Q_{3446}(\boldsymbol{v}) - \tfrac{1}{12}Q_{4466}(\boldsymbol{u})Q_{2334}(\boldsymbol{v}) + \tfrac{1}{2}Q_{1133}(\boldsymbol{v})$$
$$- \tfrac{1}{2}Q_{1155}(\boldsymbol{u})\wp_{36}(\boldsymbol{v}) - \tfrac{1}{24}Q_{2244}(\boldsymbol{u})Q_{5555}(\boldsymbol{v}) - \tfrac{1}{12}Q_{1344}(\boldsymbol{u})Q_{5555}(\boldsymbol{v}) + \wp_{33}(\boldsymbol{v})\wp_{11}(\boldsymbol{u})$$
$$+ \tfrac{1}{6}Q_{2445}(\boldsymbol{u})Q_{4445}(\boldsymbol{v}) - \tfrac{2}{3}Q_{4466}(\boldsymbol{u})\wp_{11}(\boldsymbol{v}) + \tfrac{1}{3}\wp_{26}(\boldsymbol{u})Q_{1255}(\boldsymbol{v}) + \tfrac{1}{3}Q_{1255}(\boldsymbol{u})\wp_{34}(\boldsymbol{v})$$
$$- \tfrac{7}{12}Q_{1344}(\boldsymbol{u})\wp_{44}(\boldsymbol{v}) + \tfrac{1}{6}Q_{1446}(\boldsymbol{u})Q_{4455}(\boldsymbol{v}) + \tfrac{1}{9}Q_{2226}(\boldsymbol{v})Q_{5556}(\boldsymbol{u}) - \tfrac{5}{9}\wp_{13}(\boldsymbol{u})\wp_{13}(\boldsymbol{v})$$
$$- \tfrac{1}{14}Q_{2445}(\boldsymbol{u})Q_{3355}(\boldsymbol{v}) - \tfrac{1}{6}Q_{1335}(\boldsymbol{u})\wp_{26}(\boldsymbol{v}) - Q_{2345}(\boldsymbol{u})\wp_{15}(\boldsymbol{v}) - Q_{1235}(\boldsymbol{u})\wp_{45}(\boldsymbol{v})$$
$$+ \tfrac{1}{6}Q_{2234}(\boldsymbol{u})Q_{5556}(\boldsymbol{v}) + \tfrac{1}{6}Q_{3466}(\boldsymbol{u})Q_{2246}(\boldsymbol{v}) - \tfrac{1}{6}Q_{4466}(\boldsymbol{u})Q_{2245}(\boldsymbol{v}) - \wp_{12}(\boldsymbol{v})\wp_{23}(\boldsymbol{u})$$
$$- \tfrac{1}{8}Q_{2244}(\boldsymbol{u})\wp_{44}(\boldsymbol{v}) + \tfrac{1}{3}Q_{2345}(\boldsymbol{u})Q_{4446}(\boldsymbol{v}) - \tfrac{1}{42}Q_{3344}(\boldsymbol{u})Q_{3355}(\boldsymbol{v}) - 4\wp_{12}(\boldsymbol{v})\wp_{15}(\boldsymbol{u})$$
$$+ \tfrac{1}{3}Q_{1556}(\boldsymbol{u})Q_{2356}(\boldsymbol{v}) - Q_{1355}(\boldsymbol{u})\wp_{24}(\boldsymbol{v}) + Q_{2355}(\boldsymbol{v})\wp_{14}(\boldsymbol{u}) + \wp_{16}(\boldsymbol{u})Q_{1355}(\boldsymbol{v})$$

4.2 The cyclic trigonal curve of genus seven

In this section we present results for the Abelian functions associated with the cyclic (3,8)-curve. This is the curve C given by

$$y^3 = x^8 + \lambda_7 x^7 + \lambda_6 x^6 + \lambda_5 x^5 \lambda_4 x^4 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0. \tag{4.15}$$

Using equation (2.33) we find this curve has genus $g = 7$. So this is the highest genus curve to have been considered.

4.2.1 Differentials and functions

We start by investigating the differentials on C. Recalling Definition 2.2.3 we construct the Weierstrass gap sequence generated by $(n, s) = (3, 8)$.

$$W_{3,8} = \{1, 2, 4, 5, 7, 10, 13\}, \qquad \overline{W}_{3,8} = \{3, 6, 8, 9, 11, 12, 14, \dots\}. \tag{4.16}$$

We now follow Proposition 2.2.4 to construct the standard basis of holomorphic differentials upon C.

$$\begin{aligned} &\boldsymbol{du} = (du_1, \dots, du_7), \qquad \text{where} \quad du_i(x,y) = \frac{g_i(x,y)}{3y^2}dx, \\ &\text{with} \quad \begin{array}{lll} g_1(x,y) = 1, & g_2(x,y) = x, & g_3(x,y) = x^2, \\ g_4(x,y) = y, & g_5(x,y) = x^3, & g_6(x,y) = xy, \\ g_7(x,y) = x^4. & & \end{array} \end{aligned} \tag{4.17}$$

Any point $\boldsymbol{u} \in \mathbb{C}^7$ can be expressed as

$$\boldsymbol{u} = (u_1, u_2, u_3, u_4, u_5, u_6, u_7) = \sum_{i=1}^{7} \int_{\infty}^{P_i} \boldsymbol{du}, \tag{4.18}$$

where the P_i are seven variable points upon C.

Next we must construct the fundamental differential of the second kind, (Definition 2.2.6). We follow Klein's explicit realisation set out in Proposition 2.2.8 and find that the fundamental differential may be expressed as

$$\Omega\big((x,y),(z,w)\big) = \frac{F\big((x,y),(z,w)\big)dxdz}{9(x-z)^2y^2w^2}, \tag{4.19}$$

where F is the following symmetric polynomial

$$
\begin{aligned}
F\big((x,y),(z,w)\big) &= 3y\lambda_0 + \lambda_1 yx + \lambda_1 wz + \lambda_2 x^2 w + 2\lambda_1 yz - \lambda_4 x^4 w + 2\lambda_1 wx \\
&+ 3y^2w^2 + \lambda_2 yz^2 - yz^4\lambda_4 + wx^6z^2 + 2yx^3z^5 + 2x^5z^3w + z^6yx^2 + 2\lambda_2 xwz \\
&+ 2\lambda_2 xyz + 3\lambda_3 x^2wz + 4\lambda_4 x^3wz + \lambda_5 x^4wz + 3\lambda_3 yxz^2 + 4xyz^3\lambda_4 \\
&+ 2x^2yz^3\lambda_5 + yz^4\lambda_5 x + 3yz^4\lambda_6 x^2 + yz^4\lambda_7 x^3 + 2wx^3\lambda_5 z^2 \\
&+ 3wx^4\lambda_6 z^2 + 2wx^5\lambda_7 z^2 + 2yx^2z^5\lambda_7 + x^4z^3w\lambda_7 + 3w\lambda_0 \qquad (4.20)
\end{aligned}
$$

(See Section 3.1.1 for a detailed example of such a construction.)

In obtaining the realisation an explicit basis for the differentials of the second kind associated with the cyclic (3,8)-curve was derived.

$$
\boldsymbol{dr} = (dr_1, \dots, dr_7), \qquad \text{where} \quad dr_j(x,y) = \frac{h_j(x,y)}{3y^2}dx, \qquad (4.21)
$$

$$
\text{with} \quad
\begin{aligned}
h_1(x,y) &= y\big(\lambda_2 + 13x^6 + 3\lambda_3 x + 5\lambda_4 x^2 + 7\lambda_5 x^3 + 9\lambda_6 x^4 + 11\lambda_7 x^5\big),\\
h_2(x,y) &= 2xy\big(\lambda_4 + 2\lambda_5 x + 3\lambda_6 x^2 + 4\lambda_7 x^3 + 5x^4\big),\\
h_3(x,y) &= y\big(\lambda_5 x + 3\lambda_6 x^2 + 5\lambda_7 x^3 + 7x^4 - \lambda_4\big),\\
h_4(x,y) &= x^3\big(5x^3 + 2\lambda_5 + 3\lambda_6 x + 4\lambda_7 x^2\big),\\
h_5(x,y) &= 2x^2y\big(\lambda_7 + 2x\big),\\
h_6(x,y) &= x^4\big(\lambda_7 + 2x\big),\\
h_7(x,y) &= x^2y.
\end{aligned}
$$

We can now proceed to define the period matrices, period lattice, Jacobian and Abel map as in the general case.

The local parameter at the origin, $\mathfrak{A}_1(\infty)$ is again

$$
\xi = x^{-\frac{1}{3}} \qquad (4.22)
$$

The key variables may be expressed near this point as

$$
x = \frac{1}{\xi^3}, \qquad \frac{dx}{d\xi} = -\frac{3}{\xi^4},
$$

$$
y = \frac{1}{\xi^8} + \left(\frac{\lambda_7}{3}\right)\frac{1}{\xi^5} + \left(\frac{\lambda_6}{3} - \frac{\lambda_7^2}{9}\right)\frac{1}{\xi^2} + \left(\frac{\lambda_5}{3} - \frac{2\lambda_7\lambda_6}{9} + \frac{5\lambda_7^3}{81}\right)\xi + O\big(\xi^4\big).
$$

$$
\begin{aligned}
du_1 &= [-\xi^{12} + O(\xi^{15})]d\xi, & du_5 &= [-\xi^3 + O(\xi^6)]d\xi,\\
du_2 &= [-\xi^{9} + O(\xi^{12})]d\xi, & du_6 &= [-\xi^1 + O(\xi^4)]d\xi,\\
du_3 &= [-\xi^{6} + O(\xi^{9})]d\xi, & du_7 &= [-1 + O(\xi^3)]d\xi.\\
du_4 &= [-\xi^{4} + O(\xi^{7})]d\xi, & &
\end{aligned} \qquad (4.23)
$$

$$
\begin{aligned}
&u_1 = -\tfrac{1}{13}\xi^{13} + O(\xi^{16}), \quad u_4 = -\tfrac{1}{5}\xi^{5} + O(\xi^{8}), \quad u_6 = -\tfrac{1}{2}\xi^{2} + O(\xi^{5}),\\
&u_2 = -\tfrac{1}{10}\xi^{10} + O(\xi^{13}), \quad u_5 = -\tfrac{1}{4}\xi^{4} + O(\xi^{7}), \quad u_7 = -\xi + O(\xi^{4}).\\
&u_3 = -\tfrac{1}{7}\xi^{7} + O(\xi^{10}),
\end{aligned} \tag{4.24}
$$

Define the Kleinian σ-function associated with the $(3,8)$-curve as in Definition 2.2.15 and note that it is a function of $g = 7$ variables.

$$\sigma = \sigma(\boldsymbol{u}) = \sigma(u_1, u_2, u_3, u_4, u_5, u_6, u_7).$$

The Schur-Weierstrass polynomial can be constructed as in Example A.5.21.

$$
\begin{aligned}
SW_{3,8} = {}& \tfrac{1}{45660160000}u_7^{21} + u_3^3 + u_2^2u_7 - u_5^4u_4 + \tfrac{1}{5}u_7^2u_6^5u_5u_4 - \tfrac{1}{10}u_7^5u_6u_5u_4^2 - \tfrac{1}{80}u_6^8u_4\\
& - \tfrac{1}{70}u_7^7u_5u_2 - \tfrac{2}{u\,5}u_3u_2 + \tfrac{1}{10}u_6^5u_2u_7 - \tfrac{1}{1120}u_7^9u_2u_6 + \tfrac{1}{20}u_6^3u_7^5u_2 + u_3^2u_7^2u_4 + \tfrac{1}{1120}u_7^8u_5^2u_4\\
& - \tfrac{1}{8}u_3u_7^4u_4^2 + \tfrac{2}{5}u_6^5u_5u_3 - \tfrac{1}{8}u_6^2u_7^4u_1 - u_3u_7u_1 + u_6^3u_4u_2 + \tfrac{1}{4}u_6^2u_7^4u_5^2u_4 + \tfrac{1}{2}u_5^2u_4^2u_7^3\\
& + u_3u_7u_5^2u_4 - \tfrac{3}{326144000}u_7^{17}u_6^2 + \tfrac{1}{1164800}u_7^{13}u_6^4 + \tfrac{1}{400}u_6^{10}u_7 + \tfrac{1}{280}u_7^8u_6u_5u_3\\
& - \tfrac{1}{1600}u_6^8u_7^5 - u_7^2u_6^2u_5^2u_3 - \tfrac{3}{2240}u_7^9u_4u_3 + 2u_7u_6u_5u_3^2 - \tfrac{3}{2}u_3u_6^2u_4^2 + \tfrac{1}{20}u_3u_7^6u_5^2\\
& + \tfrac{3}{11200}u_7^{10}u_6^2u_3 - u_7^3u_6u_5u_4u_3 - \tfrac{1}{22400}u_7^9u_6^6 - \tfrac{1}{40}u_6^2u_7^5u_4u_3 - \tfrac{1}{20}u_3u_7^2u_6^6\\
& - \tfrac{1}{4}u_3u_7u_6^4u_4 + \tfrac{1}{2}u_6^2u_7^3u_3^2 - u_3u_7^2u_2u_6 - u_7^2u_5u_4u_2 - 2u_5^2u_2u_7u_6 - \tfrac{1}{20}u_5^4u_7^5\\
& - \tfrac{1}{291200}u_7^{13}u_5^2 + \tfrac{1}{4}u_7^4u_6u_4u_2 + u_7^2u_6u_5u_1 + \tfrac{1}{2548000}u_7^{15}u_6u_5 + \tfrac{1}{350}u_7^7u_6^5u_5\\
& - \tfrac{1}{40}u_6^4u_7^5u_5^2 - \tfrac{1}{10}u_5^2u_7u_6^6 + u_5^4u_7u_6^2 - \tfrac{1}{1120}u_7^9u_6^2u_5^2 - \tfrac{1}{2}u_5^2u_6^4u_4 + u_4^3u_6u_5\\
& - \tfrac{1}{80}u_6^4u_7^6u_3 + \tfrac{1}{2800}u_7^{10}u_6u_5u_4 + \tfrac{1}{448}u_7^8u_1 - \tfrac{1}{4}u_6^4u_1 + u_5^2u_1 - \tfrac{1}{280}u_7^7u_3^2\\
& - \tfrac{1}{25088000}u_7^{16}u_4 - \tfrac{1}{4}u_7u_4^4 + \tfrac{1}{80}u_7^6u_4^3 + \tfrac{1}{4076800}u_7^{14}u_3 + \tfrac{1}{4}u_7^2u_6^2u_4^3 - \tfrac{1}{8}u_6^4u_7^3u_4^2\\
& + \tfrac{1}{80}u_6^6u_7^4u_4 - \tfrac{1}{4480}u_7^8u_6^4u_4 - \tfrac{1}{44800}u_7^{12}u_6^2u_4 + \tfrac{1}{560}u_7^7u_6^2u_4^2 + \tfrac{1}{22400}u_7^{11}u_4^2
\end{aligned} \tag{4.25}
$$

Recall that this polynomial is the canonical limit of the σ-function.

Define the n-index $\wp$-functions and n-index Q-functions associated with the $(3,8)$-curve as in Definitions 2.2.27 and 3.1.2. Note these are all now functions of $g = 7$ variables, and so there are more functions than in the (3,7) and (4,5)-cases.

The weights for the theory of the cyclic (3,8)-curve are given below. All equations must be homogeneous with respect to these weights.

	x	y	λ_7	λ_6	λ_5	λ_4	λ_3	λ_2	λ_1	λ_0
Weight	-3	-8	-3	-6	-9	-12	-15	-18	-21	-24

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	$\sigma(\boldsymbol{u})$
Weight	$+13$	$+10$	$+7$	$+5$	$+4$	$+2$	$+1$	$+21$

4.2.2 Expanding the Kleinian formula

We need to derive relations between the $\wp$-functions from the Kleinian formula, (Theorem 3.2.1). We again follow the procedure in Section 3.2. We use the expansions of the variables in ξ to obtain a series for equation (3.33) in ξ. The coefficients are polynomials in the variables $(z, w)$and the $\wp$-functions, starting with the three below.

$$0 = \rho_1 = \wp_{77}z^4 + \wp_{57}z^3 + (\wp_{37} - w)z^2 + (\wp_{67}w + \wp_{27})z + \wp_{47}w + \wp_{17} \tag{4.26}$$

$$\begin{aligned} 0 = \rho_2 = &-2z^5 + (\wp_{67} - \wp_{777} - \lambda_7)z^4 + (\wp_{56} - \wp_{577})z^3 + (\wp_{36} - \wp_{377})z^2 \\ &+ \big((\wp_{66} - \wp_{677})w - \wp_{277} + \wp_{26}\big)z + (\wp_{46} - \wp_{477})w - \wp_{177} + \wp_{16} \end{aligned} \tag{4.27}$$

$$\begin{aligned} 0 = \rho_3 = &(\tfrac{1}{2}\wp_{7777} - \tfrac{3}{2}\wp_{677})z^4 + (\tfrac{1}{2}\wp_{5777} - \tfrac{3}{2}\wp_{567})z^3 + (\tfrac{1}{2}\wp_{3777} - \tfrac{3}{2}\wp_{367})z^2 + (\tfrac{1}{2}\wp_{2777} \\ &- \tfrac{3}{2}\wp_{267} + (\tfrac{1}{2}\wp_{6777} - \tfrac{3}{2}\wp_{667})w)z - 3w^2 + (\tfrac{1}{2}\wp_{4777} - \tfrac{3}{2}\wp_{467})w - \tfrac{3}{2}\wp_{167} + \tfrac{1}{2}\wp_{1777} \end{aligned}$$

They are valid for any $\boldsymbol{u} \in \mathbb{C}^7$, with (z, w) one of the point on C that are used in equation (4.18) to represent $\boldsymbol{u}$. They are presented in ascending order (as the coefficients of ξ). The polynomials have been calculated explicitly, (using Maple), up to ρ_8. They get increasingly large in size and can be found in the extra Appendix of files.

Following Section 3.2, we next take resultants of pairs of equations, eliminating the variable w by choice. We denote the resultant of ρ_i and ρ_j by $\rho_{i,j}$ and present Table 4.2 detailing the number of terms they contain once expanded and the degree in z.

Table 4.2: The polynomials $\rho_{i,j}$

Res	# terms	degree	Res	# terms	degree
$\rho_{1,2}$	55	7	$\rho_{3,4}$	2151	10
$\rho_{1,3}$	135	8	$\rho_{3,5}$	5752	12
$\rho_{1,4}$	109	7	$\rho_{3,6}$	5915	10
$\rho_{1,5}$	228	8	$\rho_{3,7}$	20375	12
$\rho_{1,6}$	648	9	$\rho_{3,8}$	54807	12
$\rho_{1,7}$	404	8	$\rho_{4,5}$	952	9
$\rho_{1,8}$	730	8	$\rho_{4,6}$	10093	11
$\rho_{2,3}$	370	10	$\rho_{4,7}$	1323	9
$\rho_{2,4}$	233	8	$\rho_{4,8}$	3054	9
$\rho_{2,5}$	255	7	$\rho_{5,6}$	23087	12
$\rho_{2,6}$	1596	10	$\rho_{5,7}$	2905	9
$\rho_{2,7}$	740	8	$\rho_{5,8}$	3324	9
$\rho_{2,8}$	1002	8			

Although the genus is higher, these polynomials are not as complex as the corresponding ones in the (4,5)-case. Also, there are two polynomials present with degree $g = 7$ in z, ρ_{12} and ρ_{14} with all the others having higher degree. Recall that in the (4,5)-case there was no polynomial with degree $g - 1$ while in the (3,7)-case there were again these two.

We need to manipulate these polynomials to obtain a polynomial with degree $g-1=6$. We select $\rho_{1,2}$ and rearrange it to give an equation for z^7.

We may now take any of the other $\rho_{i,j}$ and repeatedly substitute for z^7 until we have a polynomial of degree six in z. By Theorem 3.2.2 the coefficients with respect to z of such an equation must be zero. Further, since they are zero we can take the numerator of the coefficients leaving us with seven polynomial equations between the $\wp$-functions. Finally, we separate each of these seven relations into their odd and even parts. We denote the polynomial equations between $\wp$-functions that we generate using this method as $\mathcal{K}(\rho_{i,j}, n, \pm)$ as defined in Definition 3.2.3.

We need only substitute once into ρ_{14} and so the polynomials achieved here are far simpler than the corresponding polynomials in the (4,5)-case. They start with

$$\mathcal{K}(\rho_{1,4}, 5, +) = \big(6\wp_{67} - 2\wp_{777}\big)\wp_{77} - \wp_{6777},$$
$$\mathcal{K}(\rho_{1,4}, 5, -) = \tfrac{1}{6}\wp_{77777} + \tfrac{1}{2}\wp_{667}.$$

However, the other relations require two rounds of substitution get increasingly long and complex, involving higher index $\wp$-functions. There are not enough simple relations to manipulate to find the desired differential equations between the $\wp$-functions, but we may still use these to solve the Jacobi Inversion Problem.

Theorem 4.2.1. *Suppose we are given $\{u_1, \ldots, u_7\} = \boldsymbol{u} \in J$. Then we could solve the Jacobi Inversion Problem explicitly using the equations derived from the Kleinian formula.*

Proof. The polynomial ρ_{12} has degree seven in z and so we denote by $(z_1, \ldots, z_7)$ the zeros of the polynomial, (expressions in $\wp$-functions). Next consider equation (4.26) which is degree one in w. Substitute each z_i into equation (4.26) in turn and solve to find the corresponding w_i. Therefore the set of points $\{(z_1, w_1), \ldots, (z_7, w_7)\}$ on the curve C which are the Abel preimage of $\boldsymbol{u}$ have been identified.

■

4.2.3 The σ-function expansion

We construct a σ-function expansion for the cyclic (3,8)-curve using the methods and techniques discussed in detail in Section 3.4. We start with a statement on the structure of the expansion.

Theorem 4.2.2. *The function $\sigma(\boldsymbol{u})$ associated with the cyclic (3,8)-curve may be expanded about the origin as*

$$\sigma(\boldsymbol{u}) = \sigma(u_1, u_2, u_3, u_4, u_5, u_6, u_7) = SW_{3,8}(\boldsymbol{u}) + C_{24}(\boldsymbol{u}) + C_{27} + \cdots + C_{21+3n}(\boldsymbol{u}) + \cdots$$

where each C_k is a finite, odd polynomial composed of products of monomials in $\boldsymbol{u} = (u_1, u_2 \dots, u_7)$ of weight $+k$ multiplied by monomials in $\boldsymbol{\lambda} = (\lambda_7, \lambda_6, \dots, \lambda_0)$ of weight $21 - k$.

Proof. This again follows the proof of Theorem 3.4.3. This time the σ-function is odd by Lemma 2.2.20 and by Lemma 3.3.7 has weight $+21$.

∎

We presented $SW_{3,7}$ earlier in equation (4.11). We construct the other C_k successively following the steps set out in Section 3.4. The same Maple procedures could be easily adapted to perform the calculations. Using these techniques we have calculated the σ-expansion associated with the (3,8)-curve up to and including C_{42}. The expansion may be found in the extra Appendix of files while the table below indicates the number of terms in each polynomial. Note that all these polynomials required the use of method (III) — using equations derived from the Kleinian expansion.

C_{21}	C_{24}	C_{27}	C_{30}	C_{33}	C_{36}	C_{39}	C_{42}
71	100	267	521	1121	2282	4780	8791

Note that these polynomials are rising in size much quicker than in the (4,5) and (3,7)-cases. Derivation of further polynomials is currently computationally unfeasible.

4.2.4 Relations between the Abelian functions

The basis for $\Gamma\big(J, \mathcal{O}(2\Theta^{[5]})\big)$ has not been completed for the cyclic (3,8)-curve yet. The dimension of the space is $2^g = 2^7 = 128$ by the Riemann-Roch theorem which is considerably larger than in the previous cases. So far the 4-index Q-functions down to weight -22 have been examined and the following 70 basis elements have been identified.

$$\begin{array}{cccccccccc}
 & \mathbb{C}1 & \oplus & \mathbb{C}\wp_{11} & \oplus & \mathbb{C}\wp_{12} & \oplus & \mathbb{C}\wp_{13} & \oplus & \mathbb{C}\wp_{14} \\
\oplus & \mathbb{C}\wp_{15} & \oplus & \mathbb{C}\wp_{16} & \oplus & \mathbb{C}\wp_{17} & \oplus & \mathbb{C}\wp_{22} & \oplus & \mathbb{C}\wp_{23} \\
\oplus & \mathbb{C}\wp_{24} & \oplus & \mathbb{C}\wp_{25} & \oplus & \mathbb{C}\wp_{26} & \oplus & \mathbb{C}\wp_{27} & \oplus & \mathbb{C}\wp_{33} \\
\oplus & \mathbb{C}\wp_{34} & \oplus & \mathbb{C}\wp_{35} & \oplus & \mathbb{C}\wp_{36} & \oplus & \mathbb{C}\wp_{37} & \oplus & \mathbb{C}\wp_{44} \\
\oplus & \mathbb{C}\wp_{45} & \oplus & \mathbb{C}\wp_{46} & \oplus & \mathbb{C}\wp_{47} & \oplus & \mathbb{C}\wp_{55} & \oplus & \mathbb{C}\wp_{56} \\
\oplus & \mathbb{C}\wp_{57} & \oplus & \mathbb{C}\wp_{66} & \oplus & \mathbb{C}\wp_{67} & \oplus & \mathbb{C}\wp_{77} & \oplus & \mathbb{C}Q_{6667} \\
\oplus & \mathbb{C}Q_{4777} & \oplus & \mathbb{C}Q_{4667} & \oplus & \mathbb{C}Q_{5577} & \oplus & \mathbb{C}Q_{5567} & \oplus & \mathbb{C}Q_{3667} \\
\oplus & \mathbb{C}Q_{4477} & \oplus & \mathbb{C}Q_{4467} & \oplus & \mathbb{C}Q_{4566} & \oplus & \mathbb{C}Q_{5557} & \oplus & \mathbb{C}Q_{3477} \\
\oplus & \mathbb{C}Q_{3567} & \oplus & \mathbb{C}Q_{2667} & \oplus & \mathbb{C}Q_{3467} & \oplus & \mathbb{C}Q_{3466} & \oplus & \mathbb{C}Q_{3557} \\
\oplus & \mathbb{C}Q_{4447} & \oplus & \mathbb{C}Q_{4456} & \oplus & \mathbb{C}Q_{5555} & \oplus & \mathbb{C}Q_{2567} & \oplus & \mathbb{C}Q_{3367} \\
\oplus & \mathbb{C}Q_{4555} & \oplus & \mathbb{C}Q_{1667} & \oplus & \mathbb{C}Q_{2467} & \oplus & \mathbb{C}Q_{2566} & \oplus & \mathbb{C}Q_{3366} \\
\oplus & \mathbb{C}Q_{2377} & \oplus & \mathbb{C}Q_{2466} & \oplus & \mathbb{C}Q_{2557} & \oplus & \mathbb{C}Q_{3555} & \oplus & \mathbb{C}Q_{4445}
\end{array}$$

$$\begin{array}{llllllllll} \oplus & \mathbb{C}Q_{1477} & \oplus & \mathbb{C}Q_{1567} & \oplus & \mathbb{C}Q_{2367} & \oplus & \mathbb{C}Q_{2457} & \oplus & \mathbb{C}Q_{1377} \\ \oplus & \mathbb{C}Q_{1466} & \oplus & \mathbb{C}Q_{1557} & \oplus & \mathbb{C}Q_{2277} & \oplus & \mathbb{C}Q_{2357} & \oplus & \mathbb{C}Q_{2555} \\ & & & & & \vdots & & & & \end{array} \qquad (4.28)$$

In the construction of this basis we obtain equations for those Q-functions not in the basis as a linear combination of basis entries.

$$\begin{array}{ll}
\mathbf{(-4)} & Q_{7777} = -3\wp_{66} \\
\mathbf{(-5)} & Q_{6777} = 3\wp_{57} \\
\mathbf{(-6)} & Q_{6677} = 4\wp_{47} - \wp_{56} + 2\lambda_7\wp_{67} \\
\mathbf{(-7)} & Q_{5777} = -Q_{6667} \\
\mathbf{(-8)} & Q_{6666} = 12\wp_{37} - 3\wp_{55} - 4Q_{4777} \\
\mathbf{(-8)} & Q_{5677} = 4\wp_{37} - \wp_{55} + 2\lambda_7\wp_{57} \\
\mathbf{(-9)} & Q_{5667} = 2\wp_{36} - 2\wp_{45} + 3\lambda_7\wp_{47} + 3\lambda_6\wp_{67} + \lambda_5 \\
\mathbf{(-9)} & Q_{4677} = -\wp_{45} + 2\wp_{47}\lambda_7 \\
 & \vdots
\end{array}$$

The relations have been calculated down to weight -20 and can be found in the extra Appendix of files.

Applying equation (3.32) to these relations give us the set between the $\wp$-functions which generalise Corollary 2.1.15.

$$\begin{array}{ll}
\mathbf{(-4)} & \wp_{7777} = 6\wp_{77}^2 - 3\wp_{66} \\
\mathbf{(-5)} & \wp_{6777} = 3\wp_{57} + 6\wp_{67}\wp_{77} \\
\mathbf{(-6)} & \wp_{6677} = 4\wp_{47} - \wp_{56} + 2\wp_{67}\lambda_7 + 2\wp_{66}\wp_{77} + 4\wp_{67}^2 \\
\mathbf{(-7)} & \wp_{5777} = -\wp_{6667} + 6\wp_{66}\wp_{67} + 6\wp_{57}\wp_{77} \\
\mathbf{(-8)} & \wp_{6666} = 12\wp_{37} - 3\wp_{55} - 4\wp_{4777} + 24\wp_{47}\wp_{77} + 6\wp_{66}^2 \\
\mathbf{(-8)} & \wp_{5677} = 4\wp_{37} - \wp_{55} + 2\wp_{57}\lambda_7 + 2\wp_{56}\wp_{77} + 4\wp_{57}\wp_{67} \\
\mathbf{(-9)} & \wp_{5667} = 2\wp_{36} - 2\wp_{45} + 3\wp_{47}\lambda_7 + 3\wp_{67}\lambda_6 + \lambda_5 + 4\wp_{56}\wp_{67} + 2\wp_{57}\wp_{66} \\
\mathbf{(-9)} & \wp_{4677} = -\wp_{45} + 2\wp_{47}\lambda_7 + 2\wp_{46}\wp_{77} + 4\wp_{47}\wp_{67}
\end{array}$$

Once again, the first equation may be differentiated twice with respect to u_7 to give the Boussinesq equation for $\wp_{77}$ with u_7 playing the space variable and u_6 the time variable.

An addition formula for this curve has not been constructed as it requires the basis to be completed and more of the σ-expansion to be derived. Further progress here is limited by the computational time and memory required for calculations.

Chapter 5

New approaches for Abelian functions associated to trigonal curves

In this chapter we discuss some new approaches, inspired by the techniques developed in the investigation of the cyclic (4,5)-curve in Chapter 3. The techniques presented here will be valid for any (n,s)-curve and we have used them to generate new results for the two canonical trigonal curves. These results will be presented in full with some additional details for the higher genus curves included.

5.1 Introduction

Our main aim in this chapter is to generalise the classic differential equations for the Weierstrass $\wp$-function, presented earlier in Theorem 2.1.14 and Corollary 2.1.15.

$$[\wp'(u)]^2 = 4\wp(u)^3 - g_2\wp(u) - g_3.$$
$$\wp''(u) = 6\wp(u)^2 - \tfrac{1}{2}g_2.$$

In Corollary 3.5.6 we presented a generalisation of the second equation for the cyclic (4,5)-curve. This was a set of partial differential equations that expressed 4-index $\wp$-functions using a polynomial of $\wp$-functions. Such relations have also been presented for lower genus generalisations. For example Baker derived equations (2.66) for the (2,5)-case, while [30] and [11] contain relations for the general (3,4) and cyclic (3,5)-curves respectively.

Baker's hyperelliptic generalisations expressed each of the different $\wp_{ijkl}$ using a polynomial of 2-index $\wp$-functions with degree two. (This is the logical generalisation given $\wp(u)$ is a 2-index $\wp$-function in the new notation and $\wp''(u)$ a 4-index $\wp$-function.) However, the trigonal and tetragonal generalisations all contain at least one equation with multiple 4-index $\wp$-functions which cannot be separated out.

We explain this by examining the method used in Chapter 3 to derive the relations for the (4,5)-case. Here we attained them through the derivation of a basis for $\Gamma\big(J, \mathcal{O}(2\Theta^{[g-1]})\big)$, the space of Abelian functions with poles of order at most two. The 2-index $\wp$-functions were not sufficient to construct the basis and so Q-functions were used as well. Hence the relations that express the non-basis entries as linear combinations of basis entries had multiple Q-functions in them, and so our generalisation had equations with multiple $\wp_{ijkl}$.

Note that this problem still occurs if the relations are derived from alternative methods. For example, in [30] the relation for $\wp_{2222}$ also contained a $\wp_{1333}$ term. Here the relations were not constructed using the σ-expansion and the basis, but rather using the expansion of the Kleinian formula and algebraic manipulation of the formulae. However, in the construction of the basis for $\Gamma\big(J, \mathcal{O}(2\Theta^{[3]})\big)$, we find that either Q_{2222} or Q_{1333} must be included.

In [9] Baker considers a genus three hyperelliptic curve and defines a function

$$\Delta = \wp_{12}\wp_{23} - \wp_{23}\wp_{22} + \wp_{13}^2 - \wp_{11}\wp_{33}.$$

Each of the terms in Δ has poles of order four and in general this function will have poles of order three. (This may be checked by substituting the $\wp$-functions for their definitions in σ-derivatives.) However, Baker showed that in the special case of the (2,7)-curve Δ has poles of order two. Hence it may be used in the basis of $\Gamma\big(J, \mathcal{O}(2\Theta^{[g-1]})\big)$ avoiding the need for any Q-functions. This would ensure that independent relations for each of the 4-index $\wp$-functions may be derived.

A function similar to Δ, (quadratic in 2-index $\wp$-functions with poles of order two) does not exist in any of the trigonal or tetragonal cases that have been tested. Hence, since the method in Chapter 3 of using Q-functions can be used for any (n,s)-curve it is reasonable to focus on this. Recall that in general the number of 2-index $\wp$-functions is

$$\frac{(g+r-1)!}{r!(g-1)!},$$

while the dimension of $\Gamma\big(J, \mathcal{O}(2\Theta^{[g-1]})\big)$ is 2^g. This means that the dimension of the basis grows faster than the number of 2-index $\wp$-functions and so an increasing number of Q-functions will be required. Hence the relations for 4-index $\wp$-functions will become increasingly interrelated. A more appropriate definition for the generalisation of the second elliptic differential equation may be to put emphasis on fundamental Abelian functions (those with poles of order two) instead of the 2-index $\wp$-functions.

For example, we could instead look for *a set of equations for all the second derivatives of fundamental Abelian functions, in which they are expressed using a polynomial of fundamental Abelian functions of degree at most two*. However, this would imply an infinite set of equations since there are an infinite number of fundamental Abelian functions. (For example, the n-index Q-functions for progressively higher n.)

The approach that we use instead is to find expressions for all the 4-index $\wp$-functions using a polynomial of the fundamental Abelian functions with degree two.

Definition 5.1.1. *We will define a* ***generalisation of the second elliptic differential equation****, (Corollary 2.1.15) as a set of equations that express all the 4-index $\wp$-functions using a polynomial of degree two in those fundamental Abelian functions that give a basis of the space $\Gamma\big(J, \mathcal{O}(2\Theta^{[g-1]})\big)$.*

Hence we will not substitute those Q-functions in the basis for their definition in $\wp$-functions, but rather to treat them as *extra* 2-index $\wp$-functions. We can easily extract the desired generalisation from the existing equations that have been derived by substituting those 4-index $\wp$-functions that share their indices with basis Q-functions for these functions. Using equation (3.32),

$$\wp_{ijk\ell} = -Q_{ijk\ell} + 2\wp_{ij}\wp_{k\ell} + 2\wp_{ik}\wp_{j\ell} + 2\wp_{i\ell}\wp_{jk}.$$

This also trivially gives us the equations in the generalisation that express the 4-index $\wp$-functions with these indices. In Appendix C.3 we have presented the full generalisation, as defined in Definition 5.1.1, for the cyclic (4,5)-curve. This contains relations for all 126 of the 4-index $\wp$-functions from $\wp_{6666}$ to $\wp_{1111}$.

Note that we may now achieve the generalisation given in Definition 5.1.1 for any (n, s)-curve using the methods set out in Chapter 3.

The next aim will be to find generalisations of the first elliptic differential equation,

$$[\wp'(u)]^2 = 4\wp(u)^3 - g_2\wp(u) - g_3.$$

Definition 5.1.2. *We will define a* ***generalisation of the first elliptic differential equation****, (Theorem 2.1.14) as a set of equations for each of the possible products of two 3-index $\wp$-functions using a polynomial of fundamental Abelian function of degree at most three.*

We often refer to such equations as ***quadratic 3-index relations****, or just* ***quadratic relations*** *for brevity.*

In equation (2.65) we presented Baker's generalisation of the first differential equation for the functions associated to the (2,5)-curve, which matches Definition 5.1.2. In [30] (Lemma 6.7) and [11] (Proposition 7.6) generalisations are presented for the general (3,4) and cyclic (3,5)-curves. These contain 4-index $\wp$-functions with indices matching the Q-functions in the basis, and when these are substituted back to Q-functions the relations match Definition 5.1.2. However, both these sets of relations are incomplete, with the lower weight relations missing. We aim to complete these sets in this chapter.

In Section 5.3.1 we discuss the methods that we use to derive the quadratic relations. In the remainder of Section 5.3 we present generalisations for the specific cases. The most efficient methods relies on the following set of equations associated with (n, s)-curves.

Definition 5.1.3. *We define the* ***bilinear relations*** *associated with an* (n, s)*-curve as the set of equations that are bilinear in the 2 and 3-index* $\wp$*-functions.*

These do not exist in the elliptic case but have been found in the lower genus generalisations. For example, in [11] the authors derived the following relations for the $\wp$-functions associated to the cyclic (3,5)-curve.

$$\begin{aligned}
\wp_{333} &= 2\wp_{44}\wp_{344} - 2\wp_{34}\wp_{444} - \wp_{244} \\
\wp_{234} &= \tfrac{1}{2}\wp_{34}\wp_{344} - \wp_{334}\wp_{44} + \tfrac{1}{2}\wp_{33}\wp_{444} + \tfrac{1}{2}\lambda_4\wp_{344} \\
\wp_{233} &= -\wp_{33}\wp_{344} - \tfrac{3}{2}\wp_{444}\wp_{24} + \tfrac{1}{2}\wp_{334}\wp_{34} + \tfrac{3}{2}\wp_{244}\wp_{44} + \tfrac{1}{2}\lambda_4\wp_{334} + \tfrac{1}{2}\wp_{333}\wp_{44} \\
\wp_{144} &= -\tfrac{1}{2}\wp_{334}\wp_{33} + \tfrac{1}{2}\wp_{333}\wp_{34} + \wp_{344}\wp_{24} - \tfrac{1}{2}\wp_{34}\wp_{244} \\
\wp_{134} &= \wp_{234}\wp_{34} - \wp_{24}\wp_{334} + \tfrac{1}{2}\wp_{33}\wp_{244} - \tfrac{1}{2}\wp_{344}\wp_{23} \\
\wp_{133} &= \tfrac{1}{2}\wp_{333}\wp_{24} - \wp_{33}\wp_{234} - \tfrac{1}{2}\wp_{23}\wp_{334} + \wp_{34}\wp_{233} - 3\wp_{444}\wp_{14} + 3\wp_{144}\wp_{44} \\
\wp_{124} &= -\wp_{134}\wp_{44} - \tfrac{1}{2}\wp_{144}\wp_{34} + \wp_{14}\wp_{344} + \tfrac{1}{2}\wp_{13}\wp_{444} + \tfrac{1}{2}\lambda_4\wp_{144} \\
\wp_{134}\wp_{34} &= -\tfrac{1}{2}\wp_{33}\wp_{144} + \tfrac{1}{2}\wp_{344}\wp_{13} + \wp_{334}\wp_{14} \\
\wp_{114} &= -\tfrac{1}{2}\wp_{144}\wp_{23} - \wp_{134}\wp_{24} + \wp_{234}\wp_{14} + \tfrac{1}{2}\wp_{244}\wp_{13} \\
\wp_{111} &= \tfrac{2}{3}\wp_{22}\wp_{123} + \tfrac{1}{3}\wp_{23}\wp_{122} + \lambda_3\wp_{114} - \lambda_1\wp_{144} - \tfrac{1}{3}\wp_{13}\wp_{222} - \tfrac{2}{3}\wp_{223}\wp_{12} \\
&\quad - \tfrac{2}{3}\lambda_2\wp_{124} + \tfrac{1}{3}\lambda_1\wp_{224} + \lambda_0\wp_{244} + \tfrac{1}{3}\lambda_4\wp_{112}
\end{aligned} \tag{5.1}$$

These were derived directly from the set of relations that generalises the second elliptic differential equation by using cross-differentiation on selected pairs from the set. For example, substituting the relations for $\wp_{4444}$ and $\wp_{3444}$ into

$$\frac{\partial}{\partial u_3}\big(\wp_{4444}\big) - \frac{\partial}{\partial u_4}\big(\wp_{3444}\big) = 0$$

would result in the first relation of (5.1). The same technique was used to derive the relations in Proposition 3.5.7 for the (4,5)-case.

We note the following problems with this approach to the derivation of the bilinear relations.

- It is not possible to know for sure if all such bilinear relations have been calculated.
- It is not known if the relations are given in the simplest form. (For example, it may be possible to separate the relations to give two independent equations.)
- Using the approach on higher genus cases can become quite complicated. This is because the higher genus curve require more Q-functions in the basis for $\big(J, \mathcal{O}(2\Theta^{[k]})\big)$

and hence there are more 4-index $\wp$-functions in each relation. Then the cross differentiation approach may require triplets or quadruplets instead of pairs of equations for example.

It becomes more difficult to derive the equations and to ensure all the bilinear relations are found.

In Section 5.2 of this chapter we present a new method to derive the full set of bilinear relations associated to an (n,s) curve. This method employs the σ-function expansion at the origin, discussed throughout this document and derived for the (4,5)-curve in Section 3.4.

While interesting on their own merit, these bilinear relations are motivated by their use in the computationally efficient method to derive the quadratic 3-index relations, which is discussed in Section 5.3. We also develop another method to find quadratic equations using pole cancellations and the σ-expansion, for use once the efficient method is exhausted.

Throughout these two sections we present the full results of the calculations for the cyclic (3,4) and (3,5)-cases, giving also some details for the higher genus curves. Finally in Section 5.4 we consider the problem of constructing a basis for $\big(J, \mathcal{O}(3\Theta^{[3]})\big)$ in the (3,5)-case. This problem was solved in [30] for the (3,4)-curve but the corresponding results for the (3,5)-case has yet to be considered. In Section 5.4 we use ideas from the previous two sections to identify elements in this basis.

5.2 Bilinear relations and B-functions

In this section we introduce a new set of odd functions that belong to $\Gamma\big(J, \mathcal{O}(3\Theta^{[g-1]})\big)$ — the space of Abelian functions associated to an (n,s)-curve with poles of order at most three. The new functions may be expressed as a sum of products of 2-index and 3-index $\wp$-functions in which the poles of order five and four cancel.

We use the new functions, along with the 3-index $\wp$-functions to construct the odd part of the basis for $\Gamma\big(J, \mathcal{O}(3\Theta^{[g-1]})\big)$. Those functions that are not in the basis are expressed as a linear combination of basis entries, which generate the bilinear relations discussed in the introduction above.

The new functions were constructed to aid with the derivation of bilinear relations, however they may also be used in other areas of the theory where a basis for $\Gamma\big(J, \mathcal{O}(3\Theta^{[g-1]})\big)$ is required.

5.2.1 Defining the B-functions

Consider a 2-index $\wp$-function multiplied by a 3-index $\wp$-function. This has poles of order five when $\sigma(\boldsymbol{u}) = 0$. We seek to find the combinations of such products that cancel higher order poles. We do this using the definition of the functions as σ-derivatives and so these calculations are valid for functions associated to any (n,s)-curve. We start by forming the polynomial $\mathcal{P}(\boldsymbol{u})$ of all the possible combinations, (assuming five distinct indices).

$$\begin{aligned}\mathcal{P}(\boldsymbol{u}) = \big[&c_1\wp_{ij}\wp_{klm} + c_2\wp_{ik}\wp_{jlm} + c_3\wp_{il}\wp_{jkm} + c_4\wp_{im}\wp_{jkl} + c_5\wp_{jk}\wp_{ilm} \\ &+ c_6\wp_{jl}\wp_{ikm} + c_7\wp_{jm}\wp_{ikl} + c_8\wp_{kl}\wp_{ijm} + c_9\wp_{km}\wp_{ijl} + c_{10}\wp_{lm}\wp_{ijk}\big](\boldsymbol{u}),\end{aligned} \tag{5.2}$$

for constants $c_1, \dots c_{10}$. We then substitute in the definition of the $\wp$-functions to obtain an expression involving σ-derivatives. For example,

$$\begin{aligned}\wp_{ij}(\boldsymbol{u})\wp_{klm}(\boldsymbol{u}) = &\frac{\sigma_{ij}(\boldsymbol{u})\sigma_{klm}(\boldsymbol{u})}{\sigma(\boldsymbol{u})^2} - \frac{\sigma_i(\boldsymbol{u})\sigma_j(\boldsymbol{u})\sigma_{klm}(\boldsymbol{u})}{\sigma(\boldsymbol{u})^3} \\ &- \frac{\sigma_{ij}(\boldsymbol{u})\sigma_{lm}(\boldsymbol{u})\sigma_k(\boldsymbol{u})}{\sigma(\boldsymbol{u})^3} - \frac{\sigma_{ij}(\boldsymbol{u})\sigma_{km}(\boldsymbol{u})\sigma_l(\boldsymbol{u})}{\sigma(\boldsymbol{u})^3} - \frac{\sigma_{ij}(\boldsymbol{u})\sigma_{kl}(\boldsymbol{u})\sigma_m(\boldsymbol{u})}{\sigma(\boldsymbol{u})^3} \\ &+ \frac{2\sigma_{ij}(\boldsymbol{u})\sigma_k(\boldsymbol{u})\sigma_l(\boldsymbol{u})\sigma_m(\boldsymbol{u})}{\sigma(\boldsymbol{u})^4} + \frac{\sigma_i(\boldsymbol{u})\sigma_j(\boldsymbol{u})\sigma_k(\boldsymbol{u})\sigma_{lm}(\boldsymbol{u})}{\sigma(\boldsymbol{u})^4} \\ &+ \frac{\sigma_i(\boldsymbol{u})\sigma_j(\boldsymbol{u})\sigma_l(\boldsymbol{u})\sigma_{km}(\boldsymbol{u})}{\sigma(\boldsymbol{u})^4} + \frac{\sigma_i(\boldsymbol{u})\sigma_j(\boldsymbol{u})\sigma_m(\boldsymbol{u})\sigma_{kl}(\boldsymbol{u})}{\sigma(\boldsymbol{u})^4} \\ &- \frac{2\sigma_i(\boldsymbol{u})\sigma_j(\boldsymbol{u})\sigma_k(\boldsymbol{u})\sigma_l(\boldsymbol{u})\sigma_m(\boldsymbol{u})}{\sigma(\boldsymbol{u})^5}\end{aligned} \tag{5.3}$$

So $\mathcal{P}(\boldsymbol{u})$ is a sum of rational functions of σ-derivatives, with the denominators all $\sigma(\boldsymbol{u})^k$ where $k = 2, 3, 4$ or 5. Each of the 10 original products contributes 10 fractions, as in equation (5.3). Therefore $\mathcal{P}(\boldsymbol{u})$ has 100 terms in total — 10 with poles of order five, 40 with poles of order four, 40 with poles of order three and 10 with poles of order two.

We identify the numerator of those parts of the sum with denominator $\sigma(\boldsymbol{u})^k$ by $\mathcal{N}_k(\boldsymbol{u})$.

$$\mathcal{P}(\boldsymbol{u}) = \frac{\mathcal{N}_5(\boldsymbol{u})}{\sigma(\boldsymbol{u})^5} + \frac{\mathcal{N}_4(\boldsymbol{u})}{\sigma(\boldsymbol{u})^4} + \frac{\mathcal{N}_3(\boldsymbol{u})}{\sigma(\boldsymbol{u})^3} + \frac{\mathcal{N}_2(\boldsymbol{u})}{\sigma(\boldsymbol{u})^2}.$$

We start by considering the parts which contribute to the poles of highest order. We find this can be factored simply to give

$$\mathcal{N}_5(\boldsymbol{u}) = -2\sigma_i(\boldsymbol{u})\sigma_j(\boldsymbol{u})\sigma_k(\boldsymbol{u})\sigma_l(\boldsymbol{u})\sigma_m(\boldsymbol{u})\big[c_1 + c_2 + \cdots + c_{10}\big].$$

Hence only one condition of the coefficients is required to ensure the poles do not have order five.

$$c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7 + c_8 + c_9 + c_{10} = 0.$$

However, we find that the other parts do not factor as simply. We write a Maple procedure to collect the parts together into terms with the same σ-derivatives. We find

$$\begin{aligned}
\mathcal{N}_4(\boldsymbol{u}) = \big[&c_1 + c_3 + c_6 + 2c_9\big]\sigma_i(\boldsymbol{u})\sigma_j(\boldsymbol{u})\sigma_{km}(\boldsymbol{u})\sigma_l(\boldsymbol{u}) \\
&+ \big[c_2 + c_3 + c_8 + 2c_7\big]\sigma_i(\boldsymbol{u})\sigma_k(\boldsymbol{u})\sigma_{jm}(\boldsymbol{u})\sigma_l(\boldsymbol{u}) \\
&+ \big[c_2 + c_4 + c_9 + 2c_6\big]\sigma_i(\boldsymbol{u})\sigma_k(\boldsymbol{u})\sigma_{jl}(\boldsymbol{u})\sigma_m(\boldsymbol{u}) \\
&+ \big[c_1 + c_2 + c_5 + 2c_{10}\big]\sigma_i(\boldsymbol{u})\sigma_j(\boldsymbol{u})\sigma_k(\boldsymbol{u})\sigma_{lm}(\boldsymbol{u}) \\
&+ \big[c_1 + c_4 + c_7 + 2c_8\big]\sigma_i(\boldsymbol{u})\sigma_j(\boldsymbol{u})\sigma_{kl}(\boldsymbol{u})\sigma_m(\boldsymbol{u}) \\
&+ \big[c_3 + c_4 + c_{10} + 2c_5\big]\sigma_i(\boldsymbol{u})\sigma_l(\boldsymbol{u})\sigma_{jk}(\boldsymbol{u})\sigma_m(\boldsymbol{u}) \\
&+ \big[c_5 + c_7 + c_9 + 2c_3\big]\sigma_{il}(\boldsymbol{u})\sigma_j(\boldsymbol{u})\sigma_k(\boldsymbol{u})\sigma_m(\boldsymbol{u}) \\
&+ \big[c_5 + c_6 + c_8 + 2c_4\big]\sigma_{im}(\boldsymbol{u})\sigma_j(\boldsymbol{u})\sigma_k(\boldsymbol{u})\sigma_l(\boldsymbol{u}) \\
&+ \big[c_6 + c_7 + c_{10} + 2c_2\big]\sigma_{ik}(\boldsymbol{u})\sigma_j(\boldsymbol{u})\sigma_l(\boldsymbol{u})\sigma_m(\boldsymbol{u}) \\
&+ \big[c_8 + c_9 + c_{10} + 2c_1\big]\sigma_{ij}(\boldsymbol{u})\sigma_k(\boldsymbol{u})\sigma_l(\boldsymbol{u})\sigma_m(\boldsymbol{u}),
\end{aligned}$$

and hence we have a further 10 conditions of the constants, necessary to ensure $\mathcal{P}(\boldsymbol{u})$ has poles of order at most three. We find that all 11 of the conditions may be satisfied if

$$\begin{aligned}
c_1 &= 2c_4 + 3c_5 - c_2 + 2c_3, & c_8 &= -2c_5 - c_3 + c_2 - 2c_4, \\
c_6 &= c_3 + c_5 - c_2, & c_9 &= -2c_5 - c_4 + c_2 - 2c_3, \\
c_7 &= c_4 + c_5 - c_2, & c_{10} &= -c_3 - c_4 - 2c_5,
\end{aligned} \tag{5.4}$$

where c_2, c_3, c_4, c_5 may take any value.

If we repeat this process on $\mathcal{N}_3(\boldsymbol{u})$ we see that the 40 terms may be collected into 20 terms with each coefficient involving two of the constants. Hence we have a further 20 conditions to be satisfied if $\mathcal{P}(\boldsymbol{u})$ is to have poles of order at most two. We find that the only simultaneous solution to these 20 conditions is to have all constants set to zero. Hence it is not possible to remove from $\mathcal{P}(\boldsymbol{u})$ the terms contributing poles of order three and leave a non-zero function.

However, we do have a non-zero solution in equation (5.4), that restricts $\mathcal{P}(\boldsymbol{u})$ to poles of order at most three. There are four free parameters so we find that this may be separated to give four different solutions. (To achieve this we let each of the free parameters be one in turn, setting the others to zero.) Substituting these four solutions into equation (5.2) identifies the four independent combinations that have poles of order at most three. We define these combinations to be the B-functions.

Definition 5.2.1. *Define the four **B-functions** as follows.*

$$\begin{aligned}
B^A_{ijklm} &= \wp_{ij}\wp_{klm} + \tfrac{1}{3}\wp_{jk}\wp_{ilm} + \tfrac{1}{3}\wp_{jl}\wp_{ikm} + \tfrac{1}{3}\wp_{jm}\wp_{ikl} - \tfrac{2}{3}\wp_{kl}\wp_{ijm} \\
&\quad - \tfrac{2}{3}\wp_{km}\wp_{ijl} - \tfrac{2}{3}\wp_{lm}\wp_{ijk}. \\
B^B_{ijklm} &= \wp_{ik}\wp_{jlm} + \tfrac{1}{3}\wp_{jk}\wp_{ilm} - \tfrac{2}{3}\wp_{jl}\wp_{ikm} - \tfrac{2}{3}\wp_{jm}\wp_{ikl} + \tfrac{1}{3}\wp_{kl}\wp_{ijm} \\
&\quad + \tfrac{1}{3}\wp_{km}\wp_{ijl} - \tfrac{2}{3}\wp_{lm}\wp_{ijk}. \\
B^C_{ijklm} &= \wp_{il}\wp_{jkm} - \tfrac{2}{3}\wp_{jk}\wp_{ilm} + \tfrac{1}{3}\wp_{jl}\wp_{ikm} - \tfrac{2}{3}\wp_{jm}\wp_{ikl} + \tfrac{1}{3}\wp_{kl}\wp_{ijm} \\
&\quad - \tfrac{2}{3}\wp_{km}\wp_{ijl} + \tfrac{1}{3}\wp_{lm}\wp_{ijk}. \\
B^D_{ijklm} &= \wp_{im}\wp_{jkl} - \tfrac{2}{3}\wp_{jk}\wp_{ilm} - \tfrac{2}{3}\wp_{jl}\wp_{ikm} + \tfrac{1}{3}\wp_{jm}\wp_{ikl} - \tfrac{2}{3}\wp_{kl}\wp_{ijm} \\
&\quad + \tfrac{1}{3}\wp_{km}\wp_{ijl} + \tfrac{1}{3}\wp_{lm}\wp_{ijk}.
\end{aligned} \tag{5.5}$$

Note that this derivation and definition is for functions associated to an arbitrary (n, s) curve and not specific to the cyclic case. (The Maple worksheet in which the derivation was conducted can be found in the extra Appendix of files.)

Remark 5.2.2.

(i) The B-functions are defined as a sum of Abelian functions and hence are Abelian functions themselves. From the derivation above we know they have poles of order at most three and hence belong to the vector space $\Gamma\big(J, \mathcal{O}(3\Theta^{[g-1]})\big)$.

(ii) By Lemma 2.2.32 the 2-index $\wp$-functions are even with respect to the change of variables $\boldsymbol{u} \to [-1]\boldsymbol{u}$ while the 3-index $\wp$-functions are odd. Hence the B-functions are all odd functions since each term is the product of an odd and even function.

Remark 5.2.3. When the indices are not distinct the B-functions specify to simpler formulae. In particular, if the indices are all equal then all four B-functions are zero.

$$B^A_{iiiii} = B^B_{iiiii} = B^C_{iiiii} = B^D_{iiiii} = 0.$$

Hence, no B-functions exist in the genus one case, which is related to the fact that there are no bilinear relations associated with the elliptic curve.

Note also that when the indices are specified the B-functions often equal each other, or are linearly dependent on each other. For example,

$$B^A_{iiiij} = B^B_{iiiij} = B^C_{iiiij} = \wp_{ii}\wp_{iij} - \wp_{ij}\wp_{iii}.$$
$$B^D_{iiiij} = 2\wp_{ij}\wp_{iii} - 2\wp_{ii}\wp_{iij} = -2B^A_{iiiij}.$$

5.2.2 Deriving bilinear relations

We will derive bilinear relations systematically through the construction of a basis for the odd functions in $\Gamma\big(J, \mathcal{O}(3\Theta^{[g-1]})\big)$. This mimics the derivation of the relations between Q-functions in Lemma 3.5.5 through the construction of the basis $\Gamma\big(J, \mathcal{O}(2\Theta^{[g-1]})\big)$ in Theorem 3.5.3.

Recall that the space $\Gamma\big(J, \mathcal{O}(3\Theta^{[g-1]})\big)$ will have dimension 3^g by the Riemann-Roch theorem. We may assign to this basis the 2^g elements already identified in $\Gamma\big(J, \mathcal{O}(2\Theta^{[g-1]})\big)$ and then search for the remaining elements amongst those functions with poles of order three.

We start by including all the 3-index $\wp$-functions and then search systematically among the B-functions at decreasing weight levels using techniques similar to those discussed in Section 3.5. Since these functions are odd we may ignore the elements in the smaller basis when constructing the linear polynomials, (since these were all even). We identify those elements to add to the basis and obtain expressions for the non-basis entries as linear combinations of basis entries. Substituting for the B-functions in these will leave us with bilinear relations.

We use the following method, described for weight level $-k$, and implemented with Maple.

1. Start by forming a sum of existing odd basis entries, each multiplied by an undetermined constant coefficient, c_i. (These will be the 3-index $\wp$-functions and B-functions of a higher weight). Include only those which may be combined with an appropriate $\boldsymbol{\lambda}$-monomial to give weight $-k$ overall. (Note that all the possible elements of a higher weight have already been determined since we are working systematically in decreasing weights.)

   From a simple extension of Lemma 3.5.2 we need not consider basis entries multiplied by rational functions in the $\boldsymbol{\lambda}$. We also do not need to consider any constant term since this would not be odd.

2. Add to this sum the B-functions of weight $-k$, each multiplied by an undetermined constant coefficient, q_i. (Note that often the different B-functions may be equal or linearly dependent when the indices have been specified specified. If this is the case then we need only include a minimal set of them.)

3. Substitute the functions for their definitions as σ-derivatives. This will give a sum of rational functions which may be factored to leave $\sigma(\boldsymbol{u})^3$ on the denominator. Take the numerator, which should be a sum of products of triplets of σ-derivatives.

4. Substitute $\sigma(\boldsymbol{u})$ in this sum for the σ-expansion at the origin. Note that the sum contains $\boldsymbol{\lambda}$-monomials with weight no lower than $-k$. Hence we may truncate the σ-expansion after the polynomial which contains $\boldsymbol{\lambda}$-monomials of weight $-k$. Evaluate the σ-derivatives as derivatives of this expansion.

5. Expand the products to obtain a polynomial. Note that this will create terms with $\boldsymbol{\lambda}$-monomials of a lower weight than $-k$. These terms must be discarded since we would not have all such terms with this $\boldsymbol{\lambda}$-monomial, (because we truncated the expansion). Hence any information gained from them would be invalid.

6. Collect this polynomial into a sum of the various $\boldsymbol{u\lambda}$-monomials with coefficients in the unknown constants $\{c_i, q_i\}$. Note that these coefficients will be linear in the unknowns, (since the unknowns were from the original sum, not the expansions that were multiplied).

7. Consider these coefficients as a series of linear equations in the unknowns that must equal zero. Since it is linear it is relatively simple for a computer algebra package like Maple to solve.

   - If there is a unique solution for the c_i in terms of the q_i then each of the functions at weight $-k$ may be expressed as a linear combination of existing basis entries. (To obtain the relations repeatedly set one of the q_i to one and the others to zero in the solution that was obtained.)
   - However, it will more often be the case that this is not possible, and hence some of the functions must be selected as basis entries.

     Suppose that there are x unknowns and $(x - y)$ of the unknowns may be expressed using the other y. This means that specifying y of the q_i determines numerical values for all the c_i and the other q_i. Hence y of the functions may be expressed as a linear combination of basis entries and the others, which are added to the basis.

     To obtain the relations for the non-basis functions repeatedly set one of the corresponding q_i to one and the others to zero in the solution that was obtained.

Hence we have identified the basis entries at weight $-k$ and obtained relations for all the non-basis entries as a linear combination of basis entries. Substituting the B-functions for their definitions into these will produce the bilinear relations.

Example 5.2.4 below demonstrates the use of this procedure. We follow it at successively lower weights, constructing the basis and equations as we proceed. Note that as the weight decreases, both the the possible number of terms in the original sum and the size of the σ-expansion increase. Hence the computations take more time and memory. The most computationally intensive part is Step 5 where the triplets of expansions must be calculated and the terms with weight in $\boldsymbol{\lambda}$ lower than $-k$ discarded.

- Since all the terms in the end polynomial contain an unknown, we should not use the new procedure described in Section 3.4. However, we can implement a procedure similar to the one discussed in Section 3.6 for the addition formula.

  This procedure only multiplies those terms that will not be discarded. To achieve this we categorise the terms in the three expansions by their weights, and only consider the combinations of three entries that result in an acceptable $\boldsymbol{\lambda}$-weight.

- To further save computation time and memory distributed computing may be employed to expand the triplets of products in parallel.

5.2.3 The cyclic (3,4)-case

We have derived a complete set of bilinear relations for the cyclic (3,4)-case. We start by giving an example of the procedure above by discussing the steps taken when the first weight level was examined.

Example 5.2.4. The B-functions have five indices and when these are all the same the functions reduce to zero. Hence in the (3,4)-case the lowest weight functions will be at weight -6 with indices $\{2,3,3,3,3\}$.

Start by constructing the sum of existing basis entries. The entries at weight -6 are $\wp_{222}$ and $\lambda_3\wp_{333}$, (since we do not consider the even or constant functions).

We additionally add the B-functions at weight -6, multiplied by an unknown constant. Since they are all linearly dependent we include only B^A_{23333}. We are left with the equation

$$0 = c_1\wp_{222} + c_2\lambda_3\wp_{333} + q_1 B^A_{23333}.$$

We substitute the Abelian functions for σ-derivatives and take the numerator.

$$\begin{aligned} 0 = c_1\big(3\sigma_{22}\sigma_2\sigma - \sigma_{222}\sigma^2 - 2\sigma_2^3\big) + \lambda_3 c_2\big(3\sigma_{33}\sigma_3\sigma - \sigma_{333}\sigma^2 - 2\sigma_3^3\big) \\ + q_1\big(\sigma_{23}\sigma\sigma_{333} - \sigma_{23}\sigma_{33}\sigma_3 - \sigma_2\sigma_3\sigma_{333} - \sigma_{33}\sigma\sigma_{233} + \sigma_{33}^2\sigma_2 + \sigma_3^2\sigma_{233}\big). \end{aligned}$$

We then substitute in the σ-expansion truncated after C_{11}, since this was the first polynomial to contain λ_3. Expanding the products of expansions will generate terms with λ_3^2 and λ_3^2 which must be discarded.

The polynomial that is left is collected into $\boldsymbol{u\lambda}$-monomials with coefficients in the unknowns. Setting these to zero we see that there is a solution for any value of q_1. Setting $q_1 = 1$ we find

$$B^A_{23333} = -\tfrac{1}{2}\wp_{222} - \tfrac{1}{2}\lambda_3\wp_{333}.$$

Hence no B-functions at weight -6 need to be added to the basis. Substituting for the B-function using equation (5.5) we have

$$0 = \wp_{222} + \lambda_3\wp_{333} + 2\wp_{23}\wp_{333} - 2\wp_{33}\wp_{233},$$

which is the first of the bilinear relations associated with this curve.

□

Following this procedure at successively lower weights allows us to examine all the B-functions. We find that three need to be included in the basis, one each at weight -9, -10 and -13. We choose B^A_{13333}, B^B_{12333} and B^A_{11333} respectively. Then all the other B-functions may be expressed as a linear combination of these and the $\wp_{ijk}$. (The Maple worksheet in which these calculations are performed can be found in the extra Appendix of files.) We obtain the following set of relations.

$$
\begin{aligned}
&\textbf{(−6)} && B^A_{23333} = -\tfrac{1}{2}\wp_{222} - \tfrac{1}{2}\lambda_3\wp_{333} \\
&\textbf{(−7)} && B^A_{22333} = -2\wp_{133} \\
&\textbf{(−8)} && B^A_{22233} = \tfrac{4}{3}\wp_{123} \\
&\textbf{(−9)} && B^A_{22223} = -2\wp_{122} + \tfrac{1}{2}\lambda_3\wp_{222} + \tfrac{1}{2}{\lambda_3}^2\wp_{333} - 2\lambda_2\wp_{333} - 4B^A_{13333} \\
&\textbf{(−10)} && B^A_{12333} = -2B^B_{12333} \\
&\textbf{(−11)} && B^A_{12233} = \tfrac{4}{3}\wp_{113} \\
&\textbf{(−11)} && B^C_{12233} = \tfrac{1}{3}\wp_{113} \\
&\textbf{(−12)} && B^A_{12223} = -\lambda_1\wp_{333} - \lambda_3 B^A_{13333} \\
&\textbf{(−12)} && B^D_{12223} = \wp_{112} + 2\lambda_1\wp_{333} + 2\lambda_3 B^A_{13333} \\
&\textbf{(−13)} && B^A_{12222} = \tfrac{2}{3}\lambda_2\wp_{133} - \tfrac{1}{3}\lambda_1\wp_{233} - \tfrac{4}{3}B^A_{11333} - \lambda_3 B^C_{12333} \\
&\textbf{(−14)} && B^A_{11233} = \tfrac{3}{2}\lambda_3\wp_{113} - \lambda_2\wp_{123} + \tfrac{1}{2}\lambda_1\wp_{223} \\
&\textbf{(−14)} && B^C_{11233} = 0 \\
&\textbf{(−15)} && B^A_{11223} = \tfrac{3}{2}\lambda_3\wp_{112} - \lambda_2\wp_{122} - \wp_{111} + \tfrac{1}{3}\lambda_1\wp_{222} - \tfrac{1}{6}\lambda_3\lambda_1\wp_{333} \\
&&& \qquad - \tfrac{4}{3}\lambda_0\wp_{333} - \tfrac{2}{3}\lambda_2 B^A_{13333} \\
&\textbf{(−15)} && B^B_{11223} = -\tfrac{1}{6}\lambda_1\wp_{222} - \tfrac{1}{6}\lambda_3\lambda_1\wp_{333} - \tfrac{4}{3}\lambda_0\wp_{333} - \tfrac{2}{3}\lambda_2 B^A_{13333} \\
&\textbf{(−16)} && B^A_{11222} = 3\lambda_1\wp_{133} - 4\lambda_0\wp_{233} - B^A_{11333}\lambda_3 - 2\lambda_2 B^B_{12333} \\
&\textbf{(−17)} && B^A_{11133} = \tfrac{2}{3}\lambda_2\wp_{113} - \tfrac{4}{3}\lambda_1\wp_{123} + 2\lambda_0\wp_{223}
\end{aligned}
$$

$$
\begin{aligned}
&\mathbf{(-18)} \quad B^A_{11123} = -\lambda_3\lambda_0\wp_{333} + \lambda_0\wp_{222} + \tfrac{2}{3}\lambda_2\wp_{112} - \tfrac{4}{3}\lambda_1\wp_{122} - \lambda_1 B^A_{13333} \\
&\mathbf{(-18)} \quad B^C_{11123} = -\lambda_3\lambda_0\wp_{333} - 2\lambda_0\wp_{222} - \tfrac{1}{3}\lambda_2\wp_{112} + \tfrac{2}{3}\lambda_1\wp_{122} - \lambda_1 B^A_{13333} \\
&\mathbf{(-19)} \quad B^A_{11122} = \tfrac{8}{3}\lambda_0\wp_{133} - \tfrac{4}{9}{\lambda_2}^2\wp_{133} + \tfrac{2}{9}\lambda_2\lambda_1\wp_{233} + \tfrac{4}{3}\lambda_3\lambda_1\wp_{133} - 2\lambda_3\lambda_0\wp_{233} \\
&\qquad\qquad - 2B^B_{12333}\lambda_1 - \tfrac{4}{9}\lambda_2 B^A_{11333} \\
&\mathbf{(-21)} \quad B^A_{11113} = -2\lambda_2\lambda_0\wp_{333} - 3\lambda_0\wp_{122} + \tfrac{1}{2}{\lambda_1}^2\wp_{333} + \tfrac{1}{2}\lambda_1\wp_{112} - 4\lambda_0 B^A_{13333} \\
&\mathbf{(-22)} \quad B^A_{11112} = -\tfrac{1}{3}\lambda_2\lambda_1\wp_{133} + \tfrac{2}{3}{\lambda_1}^2\wp_{233} + 3\lambda_3\lambda_0\wp_{133} - 2\lambda_2\lambda_0\wp_{233} \\
&\qquad\qquad - 4\lambda_0 B^B_{12333} - \tfrac{1}{3}\lambda_1 B^A_{11333}
\end{aligned} \tag{5.6}
$$

These calculations required the σ-function expansion up to C_{26}. We can now make the following statement.

Theorem 5.2.5. *Every bilinear relation associated to the cyclic (3,4)-curve may be given as a linear combination of the set below.*

$$
\begin{aligned}
&\mathbf{(-6)} \quad 0 = \wp_{222} + \lambda_3\wp_{333} + 2\wp_{23}\wp_{333} - 2\wp_{33}\wp_{233} \\
&\mathbf{(-7)} \quad 0 = 2\wp_{133} + \wp_{22}\wp_{333} + \wp_{23}\wp_{233} - 2\wp_{33}\wp_{223} \\
&\mathbf{(-8)} \quad 0 = -4\wp_{123} + 4\wp_{22}\wp_{233} - 2\wp_{23}\wp_{223} - 2\wp_{33}\wp_{222} \\
&\mathbf{(-9)} \quad 0 = 4\wp_{122} - \lambda_3\wp_{222} - {\lambda_3}^2\wp_{333} + 4\lambda_2\wp_{333} - 2\wp_{23}\wp_{222} + 2\wp_{22}\wp_{223} \\
&\qquad\qquad + 8\wp_{13}\wp_{333} - 8\wp_{33}\wp_{133} \\
&\mathbf{(-10)} \quad 0 = -\wp_{12}\wp_{333} + \wp_{23}\wp_{133} + 2\wp_{33}\wp_{123} - 2\wp_{13}\wp_{233} \\
&\mathbf{(-11)} \quad 0 = 4\wp_{113} - 3\wp_{12}\wp_{233} - \wp_{22}\wp_{133} + 2\wp_{23}\wp_{123} + 2\wp_{33}\wp_{122} \\
&\mathbf{(-11)} \quad 0 = \wp_{113} - 3\wp_{13}\wp_{223} + 2\wp_{22}\wp_{133} + 2\wp_{23}\wp_{123} - \wp_{33}\wp_{122} \\
&\mathbf{(-12)} \quad 0 = \lambda_3\big(\wp_{13}\wp_{333} - \wp_{33}\wp_{133}\big) + \lambda_1\wp_{333} + \wp_{12}\wp_{223} - \wp_{23}\wp_{122} \\
&\mathbf{(-12)} \quad 0 = \wp_{13}\wp_{222} - \wp_{112} - 2\lambda_3\big(\wp_{13}\wp_{333} - \wp_{33}\wp_{133}\big) - 2\lambda_1\wp_{333} \\
&\qquad\qquad - 2\wp_{22}\wp_{123} + \wp_{23}\wp_{122} \\
&\mathbf{(-13)} \quad 0 = 3\lambda_3\big(\wp_{13}\wp_{233} - \wp_{23}\wp_{133}\big) - 2\lambda_2\wp_{133} + \lambda_1\wp_{233} + 3\wp_{12}\wp_{222} \\
&\qquad\qquad - 3\wp_{22}\wp_{122} + 4\wp_{11}\wp_{333} + 4\wp_{13}\wp_{133} - 8\wp_{33}\wp_{113} \\
&\mathbf{(-14)} \quad 0 = -4\wp_{12}\wp_{133} + 4\wp_{13}\wp_{123} - 2\wp_{23}\wp_{113} + 2\wp_{33}\wp_{112} \\
&\mathbf{(-14)} \quad 0 = \tfrac{9}{2}\lambda_3\wp_{113} - 3\lambda_2\wp_{123} + \tfrac{3}{2}\lambda_1\wp_{223} - 3\wp_{11}\wp_{233} - \wp_{12}\wp_{133} - 2\wp_{13}\wp_{123} \\
&\qquad\qquad + 4\wp_{23}\wp_{113} + 2\wp_{33}\wp_{112} \\
&\mathbf{(-15)} \quad 0 = -\wp_{11}\wp_{223} - \tfrac{2}{3}\wp_{12}\wp_{123} - \tfrac{1}{3}\wp_{13}\wp_{122} + \tfrac{2}{3}\wp_{22}\wp_{113} + \tfrac{4}{3}\wp_{23}\wp_{112} \\
&\qquad\qquad + \tfrac{3}{2}\lambda_3\wp_{112} - \tfrac{2}{3}\lambda_2\big(\wp_{13}\wp_{333} - \wp_{33}\wp_{133}\big) - \lambda_2\wp_{122} - \wp_{111} \\
&\qquad\qquad + \tfrac{1}{3}\lambda_1\wp_{222} - \tfrac{1}{6}\lambda_3\lambda_1\wp_{333} - \tfrac{4}{3}\lambda_0\wp_{333}
\end{aligned}
$$

$$
\begin{aligned}
\textbf{(−15)}\quad 0 &= -\tfrac{2}{3}\wp_{12}\wp_{123} + \tfrac{2}{3}\wp_{13}\wp_{122} - \tfrac{1}{3}\wp_{22}\wp_{113} + \tfrac{1}{3}\wp_{23}\wp_{112} \\
&\quad - \tfrac{2}{3}\lambda_2\big(\wp_{13}\wp_{333} - \wp_{33}\wp_{133}\big) - \tfrac{1}{6}\lambda_1\wp_{222} - \tfrac{1}{6}\lambda_3\lambda_1\wp_{333} - \tfrac{4}{3}\lambda_0\wp_{333} \\
\textbf{(−16)}\quad 0 &= -\lambda_3\big(\wp_{11}\wp_{333} + \wp_{13}\wp_{133} - 2\wp_{33}\wp_{113}\big) - 2\lambda_2\big(\wp_{13}\wp_{233} - \wp_{23}\wp_{133}\big) \\
&\quad + 3\lambda_1\wp_{133} - 4\lambda_0\wp_{233} - \wp_{11}\wp_{222} - \wp_{12}\wp_{122} + 2\wp_{22}\wp_{112} \\
\textbf{(−17)}\quad 0 &= \tfrac{2}{3}\lambda_2\wp_{113} - \tfrac{4}{3}\lambda_1\wp_{123} + 2\lambda_0\wp_{223} - \tfrac{4}{3}\wp_{11}\wp_{133} + \tfrac{2}{3}\wp_{13}\wp_{113} + \tfrac{2}{3}\wp_{33}\wp_{111} \\
\textbf{(−18)}\quad 0 &= -\lambda_3\lambda_0\wp_{333} + \lambda_0\wp_{222} + \tfrac{2}{3}\lambda_2\wp_{112} - \tfrac{4}{3}\wp_{11}\wp_{123} + \tfrac{1}{3}\wp_{12}\wp_{113} \\
&\quad + \tfrac{1}{3}\wp_{13}\wp_{112} + \tfrac{2}{3}\wp_{23}\wp_{111} - \tfrac{4}{3}\lambda_1\wp_{122} - \lambda_1\big(\wp_{13}\wp_{333} - \wp_{33}\wp_{133}\big) \\
\textbf{(−18)}\quad 0 &= -\lambda_3\lambda_0\wp_{333} - 2\lambda_0\wp_{222} - \tfrac{1}{3}\lambda_2\wp_{112} - \tfrac{5}{3}\wp_{12}\wp_{113} + \tfrac{2}{3}\wp_{11}\wp_{123} \\
&\quad + \tfrac{4}{3}\wp_{13}\wp_{112} - \tfrac{1}{3}\wp_{23}\wp_{111} + \tfrac{2}{3}\lambda_1\wp_{122} - \lambda_1\big(\wp_{13}\wp_{333} - \wp_{33}\wp_{133}\big) \\
\textbf{(−19)}\quad 0 &= \tfrac{8}{3}\lambda_0\wp_{133} - \tfrac{4}{9}{\lambda_2}^2\wp_{133} - 2\lambda_1\big(\wp_{13}\wp_{233} - \wp_{23}\wp_{133}\big) \\
&\quad - \tfrac{4}{9}\lambda_2\big(\wp_{11}\wp_{333} + \wp_{13}\wp_{133} - 2\wp_{33}\wp_{113}\big) + \tfrac{2}{9}\lambda_2\lambda_1\wp_{233} \\
&\quad + \tfrac{4}{3}\lambda_3\lambda_1\wp_{133} - 2\lambda_3\lambda_0\wp_{233} - \tfrac{4}{3}\wp_{11}\wp_{122} + \tfrac{2}{3}\wp_{12}\wp_{112} + \tfrac{2}{3}\wp_{22}\wp_{111} \\
\textbf{(−21)}\quad 0 &= -2\lambda_2\lambda_0\wp_{333} - 3\lambda_0\wp_{122} - 4\lambda_0\big(\wp_{13}\wp_{333} - \wp_{33}\wp_{133}\big) + \tfrac{1}{2}{\lambda_1}^2\wp_{333} \\
&\quad + \tfrac{1}{2}\lambda_1\wp_{112} + \wp_{13}\wp_{111} - \wp_{11}\wp_{113} \\
\textbf{(−22)}\quad 0 &= -\tfrac{1}{3}\lambda_2\lambda_1\wp_{133} - \wp_{11}\wp_{112} + \wp_{12}\wp_{111} - 4\lambda_0\big(\wp_{13}\wp_{233} - \wp_{23}\wp_{133}\big) \\
&\quad - \tfrac{1}{3}\lambda_1\big(\wp_{11}\wp_{333} + \wp_{13}\wp_{133} - 2\wp_{33}\wp_{113}\big) + \tfrac{2}{3}{\lambda_1}^2\wp_{233} \\
&\quad + 3\lambda_3\lambda_0\wp_{133} - 2\lambda_2\lambda_0\wp_{233}
\end{aligned}
$$

Proof. First we note that these identities are derived simply from equations (5.6) by substituting for the definition of the B-functions.

Now, consider a relation that is bilinear in 2 and 3-index $\wp$-functions. Due to the parity properties of the $\wp$-functions we know that the relations must be odd or even. If the relation were even then it could not contain any $\wp_{ijk}$ and so would only be trivially bilinear. Hence the relations we consider are odd, and therefore cannot contain a constant term or terms with a single 2-index $\wp$-function. Each term must either be a constant multiplied by a $\wp_{ijk}$ or a constant multiplied by a $\wp_{ij}\wp_{ijk}$.

Further, the terms with $\wp_{ij}\wp_{ijk}$ must together cancel the poles of order four and five, since $\wp_{ijk}$ has only poles of order three. We have established that all such combinations are given by B-functions as defined in Definition 5.2.1.

Next we note that due to the results of Section 3.3 all such relations must be homogeneous in the Sato weights. Now, in the derivation of the basis described above we found all the relations between the $\wp_{ijk}$ and B-functions at each weight, (up to linear dependence).

The relations in the theorem were derived from these by substituting the B-functions for their definitions. (Note that we exclude relations for those B-functions that are linearly dependent on the ones used, since these would have resulted in linearly dependent bilinear relations.) Hence the set presented above is sufficient to generate all such relations.

■

Remark 5.2.6.

(i) In Lemma 6.5 of [30] bilinear relations associated to the general (3,4)-curve are presented. These were derived using the method of cross-differentiation discussed above. If we make the following change of variables for the curve coefficients then we will move from the general to the cyclic case.

$$\begin{array}{lllll} \mu_1 = 0, & \mu_4 = 0, & \mu_2 = 0, & \mu_5 = 0, & \mu_8 = 0, \\ \mu_3 = \lambda_3, & \mu_6 = \lambda_2, & \mu_9 = \lambda_9, & \mu_{12} = \lambda_0. & \end{array}$$

Substituting these into the relations derived in [30] will transform them into relations from Theorem 5.2.5. (Or a linear combination of the relations from Theorem 5.2.5.) Note however, that not all the relations were present in [30] and this new method has allowed the complete set to be derived.

(ii) In Lemma 8.1 of [30] a full basis for $\Gamma\big(J, \mathcal{O}(3\Theta^{[2]})\big)$ was derived. The odd Abelian functions included alongside the $\wp_{ijk}$ here were derivatives of Q-functions. These are not identical to B-functions in general, however we note that the three entries selected are at the same weights as the B-functions selected above, and so play the same role in the basis.

5.2.4 The cyclic (3,5)-case

This calculation was repeated for the cyclic (3,5)-case. The following nineteen B-functions were used as a basis at the weight levels indicated in brackets.

$$\begin{array}{llll} (\mathbf{-8}) & B^B_{33344} & (\mathbf{-16}) & B^B_{12334} \\ (\mathbf{-9}) & B^B_{23444} & (\mathbf{-17}) & B^B_{12333},\ B^C_{12244},\ B^A_{11444} \\ (\mathbf{-11}) & B^D_{23334},\ B^A_{22444} & (\mathbf{-18}) & B^B_{11344} \\ (\mathbf{-12}) & B^B_{13444} & (\mathbf{-19}) & B^C_{12233} \\ (\mathbf{-13}) & B^C_{13344} & (\mathbf{-20}) & B^D_{12224},\ B^B_{11244} \\ (\mathbf{-14}) & B^A_{22244},\ B^D_{13334},\ B^B_{12444} & (\mathbf{-23}) & B^A_{11144} \\ (\mathbf{-15}) & B^C_{12344} & & \end{array} \tag{5.7}$$

At each weight level for which B-functions exist we obtained a set of relations expressing the linearly independent B-functions as a linear combination of basis entries. (This required the σ-function expansion up to C_{38}.)

$$\begin{aligned}
&\textbf{(−6)} \quad B^A_{34444} = -\tfrac{1}{2}\wp_{244} - \tfrac{1}{2}\wp_{333} \\
&\textbf{(−7)} \quad B^A_{33444} = 2\wp_{234} - \lambda_4\wp_{344} \\
&\textbf{(−8)} \quad B^A_{24444} = -\tfrac{2}{3}\wp_{233} + \tfrac{1}{3}\lambda_4\wp_{334} - \tfrac{1}{2}B^A_{33344} \\
&\textbf{(−9)} \quad B^A_{33334} = -2\wp_{144} + B^B_{23444} \\
&\textbf{(−9)} \quad B^A_{23444} = -2\wp_{144} + 2\wp_{224} - \lambda_4\wp_{244} - 2B^B_{23444} \\
&\qquad\vdots \\
&\textbf{(−32)} \quad B^A_{11112} = -\tfrac{2}{7}\lambda_2\lambda_1\wp_{133} - 2\lambda_2\lambda_0\wp_{233} + \tfrac{5}{7}\lambda_3\lambda_0\wp_{133} - \tfrac{1}{2}\lambda_4{\lambda_1}^2\wp_{334} + {\lambda_1}^2\wp_{233} \\
&\quad - \tfrac{1}{21}{\lambda_4}^2\lambda_2\lambda_1\wp_{334} - \tfrac{1}{7}\lambda_4\lambda_3\lambda_0\wp_{233} - \tfrac{6}{7}{\lambda_4}^2\lambda_3\lambda_0\wp_{334} + \tfrac{1}{42}\lambda_4\lambda_2\lambda_1\wp_{233} + 3\lambda_4\lambda_2\lambda_0\wp_{334} \\
&\quad + \tfrac{8}{21}{\lambda_4}^4\lambda_0\wp_{334} - \tfrac{6}{7}{\lambda_3}^2\lambda_0\wp_{334} - \tfrac{4}{21}{\lambda_4}^3\lambda_0\wp_{233} + \lambda_1\lambda_0\wp_{334} + \tfrac{1}{7}\lambda_3\lambda_2\lambda_1\wp_{334} \\
&\quad - 6\lambda_0 B^C_{12244} + \left(\tfrac{3}{2}\lambda_3\lambda_1 - 8\lambda_4\lambda_0\right)B^B_{12444} + (6\lambda_4\lambda_0 - \lambda_3\lambda_1)B^D_{13334} \\
&\quad + \left(\tfrac{3}{14}\lambda_2\lambda_1 + \tfrac{2}{7}{\lambda_4}^2\lambda_0 - \tfrac{9}{7}\lambda_3\lambda_0\right)B^D_{23334} + \tfrac{2}{7}{\lambda_4}^2\lambda_0\wp_{133} + \tfrac{1}{2}\lambda_1 B^D_{12224} \\
&\quad + \left(\tfrac{15}{14}\lambda_4\lambda_3\lambda_0 - \tfrac{3}{4}{\lambda_1}^2 + \tfrac{1}{14}\lambda_4\lambda_2\lambda_1 - \tfrac{4}{7}{\lambda_4}^3\lambda_0\right)B^B_{33344} + 2\lambda_0 B^A_{11444} \\
&\quad + \left(\tfrac{1}{14}\lambda_2\lambda_1 - \tfrac{4}{7}{\lambda_4}^2\lambda_0 - \tfrac{3}{7}\lambda_3\lambda_0\right)B^A_{22444} + \tfrac{3}{2}\lambda_4\lambda_0 B^A_{22244} - \tfrac{3}{4}\lambda_1 B^B_{11244}
\end{aligned} \tag{5.8}$$

The full list can be found in Appendix D.1 and the corresponding Maple worksheet in the extra Appendix of files.

Theorem 5.2.7. *Every bilinear relation associated to the cyclic (3,5)-curve may be given as a linear combination of the set obtained from equations (5.8) by substituting for the B-functions. The full list is available in the extra Appendix of files.*

$$\begin{aligned}
&\textbf{(−6)} \quad 0 = -\tfrac{1}{2}\wp_{244} - \tfrac{1}{2}\wp_{333} - \wp_{34}\wp_{444} + \wp_{44}\wp_{344} \\
&\textbf{(−7)} \quad 0 = 2\wp_{234} - \lambda_4\wp_{344} - \wp_{33}\wp_{444} - \wp_{34}\wp_{344} + 2\wp_{44}\wp_{334} \\
&\textbf{(−8)} \quad 0 = \tfrac{1}{3}\lambda_4\wp_{334} - \tfrac{2}{3}\wp_{233} - \tfrac{2}{3}\wp_{33}\wp_{344} + \tfrac{1}{3}\wp_{34}\wp_{334} + \tfrac{1}{3}\wp_{44}\wp_{333} - \wp_{24}\wp_{444} + \wp_{44}\wp_{244} \\
&\textbf{(−9)} \quad 0 = -2\wp_{144} - \wp_{33}\wp_{334} + \wp_{34}\wp_{333} + \wp_{24}\wp_{344} - \wp_{34}\wp_{244} \\
&\textbf{(−9)} \quad 0 = 2\wp_{224} - 2\wp_{144} - \lambda_4\wp_{244} - \wp_{23}\wp_{444} + \wp_{34}\wp_{244} + 2\wp_{44}\wp_{234} - 2\wp_{24}\wp_{344} \\
&\qquad\vdots \\
&\textbf{(−32)} \quad 0 = \tfrac{1}{7}\lambda_4\lambda_3\lambda_0\wp_{233} - \tfrac{15}{14}\lambda_4\lambda_3\lambda_0\left(\tfrac{4}{3}\wp_{33}\wp_{344} - \tfrac{2}{3}\wp_{34}\wp_{334} - \tfrac{2}{3}\wp_{44}\wp_{333}\right) - \tfrac{5}{7}\lambda_3\lambda_0\wp_{133} \\
&\quad + \lambda_3\lambda_1\left(\wp_{14}\wp_{333} - 2\wp_{33}\wp_{134} + \wp_{34}\wp_{133}\right) - \tfrac{1}{42}\lambda_4\lambda_2\lambda_1\wp_{233} + \tfrac{4}{21}\lambda_4^3\lambda_0\wp_{233} - \lambda_1\lambda_0\wp_{334} \\
&\quad - \tfrac{3}{2}\lambda_4\lambda_0\left(\tfrac{4}{3}\wp_{22}\wp_{244} - \tfrac{2}{3}\wp_{24}\wp_{224} - \tfrac{2}{3}\wp_{44}\wp_{222}\right) + 8\lambda_4\lambda_0\left(\wp_{14}\wp_{244} - \wp_{24}\wp_{144}\right) - \lambda_1^2\wp_{233} \\
&\quad - \tfrac{1}{14}\lambda_2\lambda_1\left(\wp_{22}\wp_{444} + \wp_{24}\wp_{244} - 2\wp_{44}\wp_{224}\right) - 6\lambda_4\lambda_0\left(\wp_{14}\wp_{333} - 2\wp_{33}\wp_{134} + \wp_{34}\wp_{133}\right) \\
&\quad - \tfrac{3}{14}\lambda_2\lambda_1\left(\wp_{24}\wp_{333} - 2\wp_{33}\wp_{234} + \wp_{34}\wp_{233}\right) - \tfrac{3}{2}\lambda_3\lambda_1\left(\wp_{14}\wp_{244} - \wp_{24}\wp_{144}\right) + \wp_{11}\wp_{112} \\
&\quad - \tfrac{2}{7}\lambda_4^2\lambda_0\left(\wp_{24}\wp_{333} - 2\wp_{33}\wp_{234} + \wp_{34}\wp_{233}\right) + \tfrac{4}{7}\lambda_4^2\lambda_0\left(\wp_{22}\wp_{444} + \wp_{24}\wp_{244} - 2\wp_{44}\wp_{224}\right) \\
&\quad + \tfrac{6}{7}\lambda_4^2\lambda_3\lambda_0\wp_{334} - \tfrac{1}{14}\lambda_4\lambda_2\lambda_1\left(\tfrac{4}{3}\wp_{33}\wp_{344} - \tfrac{2}{3}\wp_{34}\wp_{334} - \tfrac{2}{3}\wp_{44}\wp_{333}\right) + \tfrac{6}{7}\lambda_3^2\lambda_0\wp_{334} + \ldots
\end{aligned}$$

$$
\begin{aligned}
&\cdots + \tfrac{4}{7}\lambda_4^3\lambda_0\big(\tfrac{4}{3}\wp_{33}\wp_{344} - \tfrac{2}{3}\wp_{34}\wp_{334} - \tfrac{2}{3}\wp_{44}\wp_{333}\big) - 3\lambda_4\lambda_2\lambda_0\wp_{334} + \tfrac{1}{21}\lambda_4^2\lambda_2\lambda_1\wp_{334} \\
&+ \tfrac{9}{7}\lambda_3\lambda_0\big(\wp_{24}\wp_{333} - 2\wp_{33}\wp_{234} + \wp_{34}\wp_{233}\big) + \tfrac{3}{7}\lambda_3\lambda_0\big(\wp_{22}\wp_{444} + \wp_{24}\wp_{244} - 2\wp_{44}\wp_{224}\big) \\
&+ \tfrac{3}{4}\lambda_1^2\big(\tfrac{4}{3}\wp_{33}\wp_{344} - \tfrac{2}{3}\wp_{34}\wp_{334} - \tfrac{2}{3}\wp_{44}\wp_{333}\big) + \tfrac{1}{2}\lambda_4\lambda_1^2\wp_{334} - \tfrac{8}{21}\lambda_4^4\lambda_0\wp_{334} + \tfrac{2}{7}\lambda_2\lambda_1\wp_{133} \\
&- \tfrac{1}{2}\lambda_1\big(\wp_{14}\wp_{222} - 2\wp_{22}\wp_{124} + \wp_{24}\wp_{122}\big) - 2\lambda_0\big(\wp_{11}\wp_{444} + \wp_{14}\wp_{144} - 2\wp_{44}\wp_{114}\big) \\
&+ \tfrac{3}{4}\lambda_1\big(\tfrac{4}{3}\wp_{12}\wp_{144} - \tfrac{4}{3}\wp_{14}\wp_{124} + \tfrac{2}{3}\wp_{24}\wp_{114} - \tfrac{2}{3}\wp_{44}\wp_{112}\big) + 2\lambda_2\lambda_0\wp_{233} - \tfrac{2}{7}\lambda_4^2\lambda_0\wp_{133} \\
&+ 6\lambda_0\big(\wp_{14}\wp_{224} - \tfrac{2}{3}\wp_{22}\wp_{144} - \tfrac{2}{3}\wp_{24}\wp_{124} + \tfrac{1}{3}\wp_{44}\wp_{122}\big) - \wp_{12}\wp_{111} - \tfrac{1}{7}\lambda_3\lambda_2\lambda_1\wp_{334}
\end{aligned}
$$

Proof. We follow the arguments in the proof of Theorem 5.2.5. ■

Remark 5.2.8. In Proposition 7.5 of [11] the authors presented a set of bilinear relations associated to the cyclic (3,5)-curve that was recalled in equations (5.1) during the introduction to this chapter. These relations were derived using the method of cross-differentiation of relations from Definition 5.1.1. We can see that the first four relations match ours, (differing by an overall multiplicative constant). However, note that the second relation at weight -9 had not been picked up by the method of cross-differentiation. Additionally, the new approach has found extra relations at some other weight levels considered in [11] and relations at all the higher weights that were not considered.

In [11] the authors did not present a basis for $\Gamma\big(J, \mathcal{O}(3\Theta^{[2]})\big)$. The B-functions we have identified in (5.7) may be used but this will not be enough to complete the basis. We consider this problem is Section 5.4.

5.2.5 Higher genus curves

The procedure has been applied to the cyclic (4,5)-curve down to weight -35, (which uses the σ-expansion up to C_{47}). The first few bilinear relations were presented earlier in Proposition 3.5.7, with all the available relations in the extra Appendix of files. The lowest weight relation will involve B_{11112} at weight -51 but these lower weight equations have not been derived due to computational limitations.

The first few bilinear identities for the cyclic (3,7)-case are presented below.

$$
\begin{aligned}
&\mathbf{(-6)} \quad 0 = -\tfrac{1}{2}\wp_{466} - \tfrac{1}{2}\wp_{555} - \tfrac{1}{2}\lambda_6\wp_{666} - \wp_{56}\wp_{666} + \wp_{66}\wp_{566} \\
&\mathbf{(-7)} \quad 0 = -2\wp_{366} + 2\wp_{456} - \wp_{55}\wp_{666} - \wp_{56}\wp_{566} + 2\wp_{66}\wp_{556} \\
&\mathbf{(-8)} \quad 0 = \tfrac{2}{3}\wp_{356} - \tfrac{2}{3}\wp_{455} - \tfrac{2}{3}\wp_{55}\wp_{566} + \tfrac{1}{3}\wp_{56}\wp_{556} + \tfrac{1}{3}\wp_{66}\wp_{555} - \wp_{46}\wp_{666} + \wp_{66}\wp_{466} \\
&\mathbf{(-9)} \quad 0 = -2\wp_{355} - \tfrac{3}{2}\lambda_6\wp_{466} + \tfrac{1}{2}\lambda_6\wp_{555} + \tfrac{1}{2}\lambda_6^2\wp_{666} - 2\lambda_5\wp_{666} \\
&\qquad - \wp_{55}\wp_{556} + \wp_{56}\wp_{555} + \wp_{46}\wp_{566} - \wp_{56}\wp_{466} - 4\wp_{36}\wp_{666} + 4\wp_{66}\wp_{366} \\
&\mathbf{(-9)} \quad 0 = 2\wp_{446} - \wp_{45}\wp_{666} + \wp_{56}\wp_{466} + 2\wp_{66}\wp_{456} - 2\wp_{46}\wp_{566}
\end{aligned}
$$

5.3 Quadratic relations

In this section we aim to find generalisations of the classic differential equation for the Weierstrass $\wp$-function,

$$[\wp'(u)]^2 = 4\wp(u)^3 - g_2\wp(u) - g_3.$$

In Definition 5.1.2 we defined these as quadratic relations — a set of equations for each of the possible products of two 3-index $\wp$-functions using a polynomial of fundamental Abelian function of degree at most three.

We start by discussing the various methods used to obtain the relations and present a procedure to use for any (n,s)-curve. We then discuss the results for specific cases including the canonical trigonal curves.

5.3.1 Deriving quadratic relations

Deriving quadratic relations from bilinear relations

Consider the bilinear relations discussed in the previous section. An arbitrary relation will be a sum of terms containing only a 3-index $\wp$-function, plus a sum of terms containing the product of a 3-index and a 2-index $\wp$-function.

$$0 = \sum_i \big(c_i \wp_{i_1 i_2 i_3}\big) + \sum_j \big(c_j \wp_{j_1 j_2} \wp_{j_3 j_4 j_5}\big). \tag{5.9}$$

We can derive equations with quadratic terms in 3-index $\wp$-functions from the bilinear relations using the following two methods.

- *Differentiating the bilinear relations:*
  For example, differentiating equation (5.9) with respect to u_k gives

  $$0 = \sum_i \big(c_i \wp_{i_1 i_2 i_3 k}\big) + \sum_j \big(c_j \wp_{j_1 j_2 k} \wp_{j_3 j_4 j_5} + \wp_{j_1 j_2} \wp_{j_3 j_4 j_5 k}\big).$$

  We then need to substitute for the 4-index $\wp$-functions using equations in Definition 5.1.1. Since these were of total degree two, the total degree of the equation we are left with will be three. (Note that this method assumes the relations for the 4-index $\wp$-functions at this weight have already been derived. This may be achieved for any (n,s)-curve using the methods of Chapter 3, as discussed in the introduction to this Chapter.)

  So this leaves us with a set of equations, each containing a sum of quadratic 3-index terms and a degree three polynomial in the 2-index $\wp$-functions.

- *Multiply bilinear equations by a 3-index $\wp$-function:*
  For example, multiplying equation (5.9) by $\wp_{klm}$ gives

$$0 = \sum_i \big(c_i \wp_{i_1 i_2 i_3} \wp_{klm}\big) + \sum_j \big(c_j \wp_{j_1 j_2} \wp_{j_3 j_4 j_5} \wp_{klm}\big).$$

  Due to the results of Section 3.3 we know such an equation must have homogeneous weight. Hence the weight of the quadratic 3-index terms that are multiplied by 2-index $\wp$-functions are higher than those that are not. We will search for relations at successively lower weights and so assume that the relations for these higher weight quadratic terms are known. Substituting for these, we are left with a sum of quadratic 3-index terms and a polynomial in the 2-index $\wp$-functions of degree at most four.

Remark 5.3.1. Those equations derived from the second method with degree four terms should be combined to eliminate these terms. If it is not possible to reduce them all them we may discard some of the equations.

The degree four polynomials in the 2-index $\wp$-functions that are generated as a side product are known as *Kummer relations* and may have geometric significant in another area of the theory. We do not comment further, but note that it is an ongoing area of research.

Once we have found all the relations at a specific weight using these methods, we need to manipulate the equations to find separate relations for each of the quadratic 3-index terms. We achieve this by treating these terms as variables and solving simultaneously. If there is enough relations then the generalisation can be completed at that weight. If not then we should be able to express most of the quadratic 3-index terms as a polynomial involving the others.

These two methods were developed in [30] for the functions associated to the (3,4)-curve. At each weight level the methods were supplemented by relations obtained from the expansion of the Kleinian formula, (see Section 3.2). As discussed elsewhere in this document, the expansion of this formula does not yield useful relations for (n, s)-curves with $n > 4$. Also, the sets of relations for the trigonal cases have yet to be completed using only these approaches and it is questionable whether this is possible.

Deriving quadratic relations from the σ-expansion

An alternative way to derive quadratic 3-index relations would be to use the σ-function expansion about the origin which may be derived using the methods discussed in Section 3.4. We can determine the possible terms in the expression for a particular quadratic 3-index product as those which are of degree three or less in the fundamental Abelian functions and have the correct weight. We then form the polynomial with unidentified constant coefficients, substitute in the σ-expansion and set the coefficients of the $\boldsymbol{\lambda u}$-monomials to zero to determine the constants.

Example 5.3.2. Consider $\wp_{333}^2$ associated to the cyclic (3,4)-curve. We will derive an expression for it using the σ-expansion. Start by considering the possible terms in the expression, which will be those of degree at most three and weight -6.

$$\wp_{333}^2 = g_1\wp_{33}^3 + g_2\wp_{23}^2 + g_3\wp_{22}\wp_{33} + g_4\wp_{13} + g_5\lambda_3\wp_{23} + g_6\lambda_3^2 + g_7\lambda_2,$$

for some constants $g_1,\dots,g_7$. We substitute the $\wp$-functions for their definitions as σ-derivatives and take the numerator to find

$$\begin{aligned}0 = &-\sigma_{333}^2\sigma^4 + 6\sigma_{333}\sigma^3\sigma_{33}\sigma_3 - 4\sigma_{333}\sigma^2\sigma_3^3 - 9\sigma_{33}^2\sigma_3^2\sigma^2 + 12\sigma_{33}\sigma_3^4\sigma - 4\sigma_3^6\\ &+ g_1(3\sigma_{33}^2\sigma_3^2\sigma^2 - \sigma_{33}^3\sigma^3 - 3\sigma_{33}\sigma_3^4\sigma + \sigma_3^6) + g_2(\sigma^4\sigma_{23}^2 - 2\sigma^3\sigma_{23}\sigma_2\sigma_3 + \sigma^2\sigma_2^2\sigma_3^2)\\ &+ g_3(\sigma^4\sigma_{22}\sigma_{33} - \sigma^3\sigma_{22}\sigma_3^2 - \sigma^3\sigma_2^2\sigma_{33} + \sigma^2\sigma_2^2\sigma_3^2) + g_4(\sigma^4\sigma_1\sigma_3 - \sigma^5\sigma_{13})\\ &+ g_5(\lambda_3\sigma^4\sigma_2\sigma_3 - \lambda_3\sigma^5\sigma_{23}) + g_6\lambda_3^2\sigma^6 + g_7\lambda_2\sigma^6. \end{aligned} \tag{5.10}$$

We then substitute in the σ-expansion truncated after C_{11}, the first polynomial that contains λ_2. We expand the products and discard terms with lower weights in $\boldsymbol{\lambda}$. Setting the coefficients of the $\boldsymbol{\lambda u}$-monomials to zero we find

$$\wp_{333}^2 = 4\wp_{33}^3 + 4\wp_{13} + \wp_{23}^2 - 4\wp_{22}\wp_{33}.$$

□

Since these relations have poles of order six this procedure involves the multiplication of six large expansions. (See equation (5.10) which in which each term was a multiple of six σ-derivatives.) This can be computationally difficult even at the high weights and is not easily applicable for generating the complete set of relations.

Canceling the higher order poles

The process of deriving quadratic equations from the σ-expansion may be simplified by the following result, valid for the $\wp$-functions associated to any (n,s)-curve.

Theorem 5.3.3. *For arbitrary constants h_1,h_2,h_3,h_4,h_5 the following polynomial has poles of order four and no higher.*

$$\begin{aligned}&\wp_{ijk}\wp_{lmn} - (h_1+h_2+h_3+h_4+h_5-2)\wp_{in}\wp_{jl}\wp_{km}\\ &- (h_4+h_5)\wp_{ij}\wp_{kl}\wp_{mn} + (h_1+h_3+h_4+h_5-2)\wp_{ij}\wp_{km}\wp_{ln}\\ &- (h_1+h_3)\wp_{ij}\wp_{kn}\wp_{lm} + (h_1+h_2+h_4+h_5-2)\wp_{ik}\wp_{jl}\wp_{mn}\\ &- (h_1+h_5)\wp_{ik}\wp_{jm}\wp_{ln} + (h_1+h_2+h_3+h_4-2)\wp_{in}\wp_{jk}\wp_{lm}\\ &- (h_1+h_2)\wp_{il}\wp_{jk}\wp_{mn} + h_1\wp_{il}\wp_{jm}\wp_{kn} + h_2\wp_{il}\wp_{jn}\wp_{km} - (h_3+h_4)\wp_{im}\wp_{jk}\wp_{ln}\\ &+ h_3\wp_{im}\wp_{jl}\wp_{kn} + h_4\wp_{im}\wp_{jn}\wp_{kl} - (h_2+h_4)\wp_{ik}\wp_{jn}\wp_{lm} + h_5\wp_{in}\wp_{jm}\wp_{kl}.\end{aligned}$$

Proof. Substitute the $\wp$-functions for their definitions as σ-derivatives. Each term becomes a sum of rational functions of σ-derivatives with the denominator $\sigma(\boldsymbol{u})^k$ for $k \leq 6$. We find that all those terms with $k = 5, 6$ cancel. Hence, since $\sigma(\boldsymbol{u})$ is an entire function, the polynomial has poles of order at most four, occurring when $\sigma(\boldsymbol{u}) = 0$.

■

This result was constructed after noting the similarities in the cubic terms of the quadratic equations already discovered. It was derived by considering the general sum of terms cubic in the 2-index $\wp$-functions, with each term multiplied by an undetermined constant. Then the constants were set to ensure the poles of order five and six canceled with those in $\wp_{ijk}\wp_{lmn}$. (A similar approach to the construction of the B-functions in the last section.) Note that there are no values of $h_1, \ldots, h_5$ that cancel the poles of order four.

This same approach was used with additional terms in the polynomial to make the following remarks.

Remark 5.3.4. We have considered $\wp_{ijk}\wp_{klm}$ added to increasing wider classes of polynomials in an attempt to cancel poles using only the definition of the Abelian functions as σ-derivatives.

1. Using a polynomial of $\wp_{ij}$ in which each term has degree three — the poles of order five and six may be canceled. (This is Theorem 5.3.3 above.)
2. Include also terms in $\wp_{ij}$ with degree less than three — the poles of order five and six may be canceled.
3. Include also linear terms in the 4-index Q-functions — the poles of order five and six may be canceled.
4. Include also terms with a product of a 2-index $\wp$-function and a 4-index Q-function — the poles of order four, five and six may be canceled.
5. Include also terms with a product of two 4-index Q-functions — the poles of order four, five and six may be canceled.

(The Maple worksheet in which the cancellations were investigated is available in the extra Appendix of files.)

Thus, when deriving quadratic 3-index relations using the σ-expansion, we may start by identifying the cubic terms using Theorem 5.3.3. We then need to identify the correct values of $h_1, \ldots, h_5$ along with the coefficients of the degree two terms in the fundamental Abelian functions.

This reduces the polynomial to one with poles of degree four, and so four expansions must be multiplied instead of six. This is much easier computationally, but still far more difficult that the methods using bilinear relations and so should only be used when this is exhausted.

It is accepted that the Q-functions in the basis need to be present. However, it is an open question as to whether they need only appear linearly, or whether quadratic terms or terms with a Q-function multiplied by a $\wp$-function should be included. There are two schools of thought here.

1. When the Q-functions are substituted for $\wp$-functions, it introduces terms quadratic in the 2-index $\wp$-functions. Hence the traditional argument is that it is acceptable to multiply a Q-function by a 2-index $\wp$-function as the resulting expression in still at most cubic in the 2-index $\wp$-functions.

2. A counter argument is that the basis Q-functions should be treated as if they were 2-index $\wp$-functions and so terms quadratic, or even cubic, in the Q-functions should be allowed. (Note that we need not actually consider terms cubic in the Q-functions since the poles of order five and six may already be canceled using the results of Theorem 5.3.3).

Definition 5.1.2 follows the second school of thought however, in keeping with tradition, we will first attempt to construct the relations without quadratic Q-functions. We now present the procedure used to derive the quadratic 3-index relations.

Procedure for finding quadratic relations

We will derive quadratic 3-index relations at successively lower weights using the procedure presented here for weight level $-k$.

1. Derive a set of relations at weight $-k$ using the methods involving bilinear relations that were discussed above.

    - Differentiating bilinear relations.
    - Multiplying bilinear relations by a 3-index $\wp$-function.

    We obtain an exhaustive set by considering all the possible options that produce the desired weight. Note that for lowers weights this may be quite a large set. (A complete set of bilinear relations may be calculated using cross-differentiation or the σ-expansion method set out in Section 5.2.)

2. Substitute for the 4-index $\wp$-functions using the relations that generalise the second elliptic differential equation. (These may be derived for any (n,s)-curve through the construction of a basis for $\Gamma\big(J, \mathcal{O}(2\Theta^{[g-1]})\big)$ as discussed above). Substitute also for those quadratic 3-index relations of weight higher than $-k$.

3. There may be terms in 2-index $\wp$-functions with degree four. Eliminate these if possible by treating them as variables and solving simultaneously. Discard any equations where the terms of degree four cannot be eliminated.

4. Identify the quadratic 3-index terms as variables in the remaining set of equations and solve simultaneously. If a solution can be found that expresses each quadratic 3-index term as a polynomial in fundamental Abelian functions then we are finished. However, it is often the case that several of the quadratic 3-index terms are used in the expressions for the others.

5. Using the σ-expansion method, identify expressions for those quadratic 3-index terms which are used to express the others.

   (i) First use Theorem 5.3.3 to identify the cubic terms, (with unknown constants $h_1, \dots, h_5$).

   (ii) Then form the general quadratic polynomial of 2-index $\wp$-functions at this weight, (with each term multiplied by an unknown constant). Add to this terms involving the basis Q-functions that are either linear or multiplied by a 2-index $\wp$-function. (Note that if we cannot find a solution in the next step we may return to this step and also include terms which are a multiple of two Q-functions.)

   (iii) Identify the unknown constants using the σ-expansion as normal — substitute the Abelian functions for σ-derivatives, take the numerator, substitute in the σ-expansion truncated at the appropriate weight, expand the products, discard lower weight $\boldsymbol{\lambda}$ terms, collect the coefficients and solve for the unknown constants.

6. Substitute these expressions for the quadratic terms identified by the σ-expansions into the set from Step 4. We obtain similar expressions for all the quadratic 3-index terms at this weight.

Remark 5.3.5.

(i) In theory this procedure may miss quadratic terms that are not introduced by the methods involving bilinear relations. In this case the expressions for these functions should also be derived by the σ-expansion. However, in practice, all the quadratic terms are introduced this way.

(ii) The σ-expansion method is the computationally difficult part of this procedure. We may simplify by writing a new Maple procedure to multiply four expansions and discard the lower weight terms. This was discussed in Section 3.6 for the addition formula construction. This procedure essentially only multiplies those terms that will not be discarded at the next stage

(iii) Time and memory constraints may be eased further by using parallel computing.

5.3.2 The cyclic (3,4)-case

We used the procedure presented above to derive a complete set of relations for the cyclic (3,4)-case. (The corresponding Maple worksheet is available in the extra Appendix of files.)

Theorem 5.3.6. *The complete set of quadratic 3-index relations associated to the cyclic (3,4)-curve is given below. These match our generalisation of the classic elliptic differential equation, (Definition 5.1.2).*

Note that Q_{1333} was the sole Q-function used in the basis of fundamental Abelian functions and that it only appears here as a linear term, or multiplied by a 2-index $\wp$-function.

The relations are presented in decreasing weight order as indicated by the number in brackets. Readers who wish to skip them should proceed to page 164.

$$\begin{aligned}
&\mathbf{(-6)} & \wp_{333}^2 &= 4\wp_{33}^3 + 4\wp_{13} + \wp_{23}^2 - 4\wp_{22}\wp_{33} \\
&\mathbf{(-7)} & \wp_{233}\wp_{333} &= 4\wp_{23}\wp_{33}^2 - 2\wp_{12} - \wp_{22}\wp_{23} + 2\lambda_3\wp_{33}^2 \\
&\mathbf{(-8)} & \wp_{233}^2 &= 4\wp_{23}^2\wp_{33} + \wp_{22}^2 + 4\lambda_3\wp_{23}\wp_{33} + 4\wp_{33}\lambda_2 - \tfrac{4}{3}Q_{1333} \\
&\mathbf{(-8)} & \wp_{223}\wp_{333} &= 2\wp_{23}^2\wp_{33} + 2\wp_{22}\wp_{33}^2 - 2\wp_{22}^2 + \lambda_3\wp_{23}\wp_{33} + 4\wp_{13}\wp_{33} + \tfrac{2}{3}Q_{1333} \\
&\mathbf{(-9)} & \wp_{223}\wp_{233} &= 2\wp_{13}\lambda_3 + 2\lambda_1 + 4\wp_{13}\wp_{23} + 2\wp_{23}\wp_{22}\wp_{33} + 2\wp_{23}^3 \\
&&&\quad + \wp_{22}\wp_{33}\lambda_3 + 2\wp_{23}^2\lambda_3 + 2\wp_{23}\lambda_2 \\
&\mathbf{(-9)} & \wp_{222}\wp_{333} &= 6\wp_{23}\wp_{22}\wp_{33} - 4\wp_{12}\wp_{33} - 2\wp_{23}^3 + 4\wp_{22}\wp_{33}\lambda_3 - 8\wp_{13}\wp_{23} \\
&&&\quad - \wp_{23}^2\lambda_3 - 4\wp_{13}\lambda_3 \\
&\mathbf{(-10)} & \wp_{223}^2 &= 4\wp_{23}^2\wp_{22} + 4\wp_{11} - 4\wp_{12}\wp_{23} + 4\wp_{13}\wp_{22} - 4\wp_{12}\lambda_3 \\
&&&\quad + 4\lambda_3\wp_{22}\wp_{23} + \lambda_3^2\wp_{33}^2 + 4\lambda_2\wp_{22} - 4\lambda_2\wp_{33}^2 + \tfrac{4}{3}\wp_{33}Q_{1333} \\
&\mathbf{(-10)} & \wp_{222}\wp_{233} &= 2\wp_{23}^2\wp_{22} + 2\wp_{22}^2\wp_{33} + 4\wp_{12}\wp_{23} + 2\wp_{12}\lambda_3 + \lambda_3\wp_{22}\wp_{23} \\
&&&\quad - 2\lambda_3^2\wp_{33}^2 + 8\lambda_2\wp_{33}^2 - \tfrac{8}{3}\wp_{33}Q_{1333} \\
&\mathbf{(-10)} & \wp_{133}\wp_{333} &= \wp_{12}\wp_{23} - 2\wp_{13}\wp_{22} + 4\wp_{13}\wp_{33}^2 + \tfrac{2}{3}\wp_{33}Q_{1333} \\
&\mathbf{(-11)} & \wp_{222}\wp_{223} &= 2\wp_{22}^2\lambda_3 - \lambda_3^2\wp_{23}\wp_{33} + 4\wp_{22}^2\wp_{23} + 4\lambda_2\wp_{23}\wp_{33} + 4\wp_{33}\lambda_1 \\
&&&\quad - \tfrac{2}{3}\lambda_3 Q_{1333} - \tfrac{4}{3}\wp_{23}Q_{1333} \\
&\mathbf{(-11)} & \wp_{133}\wp_{233} &= 2\lambda_3\wp_{13}\wp_{33} + 2\wp_{33}\lambda_1 + \wp_{12}\wp_{22} + 4\wp_{33}\wp_{23}\wp_{13} + \tfrac{2}{3}\wp_{23}Q_{1333} \\
&\mathbf{(-11)} & \wp_{123}\wp_{333} &= 2\wp_{12}\wp_{33}^2 - 2\wp_{12}\wp_{22} + 2\wp_{33}\wp_{23}\wp_{13} + 2\lambda_3\wp_{13}\wp_{33} - \tfrac{1}{3}\wp_{23}Q_{1333} \\
&\mathbf{(-12)} & \wp_{222}^2 &= 4\wp_{22}^3 + 8\wp_{11}\wp_{33} - 8\wp_{13}^2 - 4\wp_{33}\wp_{12}\lambda_3 + 4\wp_{23}\wp_{13}\lambda_3 + 4\wp_{13}\lambda_3^2 \\
&&&\quad - 4\wp_{22}\wp_{33}\lambda_3^2 + \wp_{23}^2\lambda_3^2 - 8\wp_{13}\lambda_2 + 16\wp_{33}\lambda_2\wp_{22} - 4\wp_{23}^2\lambda_2 \\
&&&\quad - 4\wp_{23}\lambda_1 - 8\lambda_0 - 4\wp_{22}Q_{1333} \\
&\mathbf{(-12)} & \wp_{133}\wp_{223} &= 2\wp_{13}\wp_{33}\wp_{22} + 2\wp_{13}\wp_{23}^2 + 2\wp_{11}\wp_{33} + 2\wp_{13}^2 - \wp_{33}\wp_{12}\lambda_3 \\
&&&\quad + 2\wp_{13}\lambda_2 + 2\wp_{23}\wp_{13}\lambda_3 + 2\lambda_0 + \tfrac{2}{3}\wp_{22}Q_{1333} \\
&\mathbf{(-12)} & \wp_{123}\wp_{233} &= 2\wp_{13}\lambda_2 + 2\wp_{13}^2 + 2\wp_{33}\wp_{12}\lambda_3 + 2\wp_{23}\wp_{33}\wp_{12} + 2\wp_{13}\wp_{23}^2 \\
&&&\quad - 2\wp_{11}\wp_{33} + 2\wp_{23}\wp_{13}\lambda_3 + 2\lambda_0 - \tfrac{1}{3}\wp_{22}Q_{1333}
\end{aligned}$$

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\begin{aligned}
\textbf{(−12)}\quad \wp_{122}\wp_{333} &= 2\wp_{13}\wp_{33}\wp_{22} - 2\wp_{13}\wp_{23}^2 + 4\wp_{23}\wp_{33}\wp_{12} - 2\wp_{11}\wp_{33} - 6\wp_{13}^2 + 2\lambda_0 \\
&\quad + 2\wp_{33}\wp_{12}\lambda_3 - \wp_{23}\wp_{13}\lambda_3 - 2\wp_{13}\lambda_2 + \wp_{23}\lambda_1 + \tfrac{2}{3}\wp_{22}Q_{1333} \\
\textbf{(−13)}\quad \wp_{133}\wp_{222} &= -2\wp_{12}\wp_{23}^2 + 4\wp_{23}\wp_{13}\wp_{22} - \wp_{23}\wp_{12}\lambda_3 + 2\wp_{22}\wp_{13}\lambda_3 \\
&\quad + 2\wp_{33}\wp_{12}\wp_{22} + 4\wp_{33}^2\lambda_1 - \tfrac{2}{3}\wp_{33}\lambda_3Q_{1333} \\
\textbf{(−13)}\quad \wp_{123}\wp_{223} &= 2\wp_{22}\lambda_1 + 2\wp_{22}\wp_{13}\lambda_3 - 2\wp_{33}^2\lambda_1 + 2\wp_{12}\wp_{23}^2 + 2\wp_{23}\wp_{13}\wp_{22} \\
&\quad - 2\wp_{11}\wp_{23} + 2\wp_{23}\wp_{12}\lambda_3 + \tfrac{1}{3}\wp_{33}\lambda_3Q_{1333} \\
\textbf{(−13)}\quad \wp_{122}\wp_{233} &= -\wp_{22}\wp_{13}\lambda_3 + 4\wp_{12}\wp_{13} + 2\wp_{12}\wp_{23}^2 + 2\wp_{12}\lambda_2 - \wp_{22}\lambda_1 \\
&\quad + 2\wp_{23}\wp_{12}\lambda_3 + 4\wp_{33}^2\lambda_1 + 2\wp_{33}\wp_{12}\wp_{22} - \tfrac{2}{3}\wp_{33}\lambda_3Q_{1333} \\
\textbf{(−14)}\quad \wp_{133}^2 &= \wp_{12}^2 + 4\wp_{33}\lambda_0 + 4\wp_{33}\wp_{13}^2 + \tfrac{4}{3}\wp_{13}Q_{1333} \\
\textbf{(−14)}\quad \wp_{123}\wp_{222} &= 2\wp_{22}\wp_{23}\wp_{12} - 2\wp_{12}^2 + 2\wp_{13}\wp_{22}^2 - 2\wp_{23}\wp_{33}\lambda_1 - 2\lambda_3^2\wp_{13}\wp_{33} \\
&\quad + 2\wp_{22}\wp_{12}\lambda_3 + 8\wp_{33}\wp_{13}\lambda_2 - \tfrac{8}{3}\wp_{13}Q_{1333} + \tfrac{1}{3}\wp_{23}\lambda_3Q_{1333} \\
\textbf{(−14)}\quad \wp_{122}\wp_{223} &= 4\wp_{22}\wp_{23}\wp_{12} - 2\wp_{11}\wp_{22} + 2\wp_{12}^2 + 2\wp_{22}\wp_{12}\lambda_3 + \lambda_3^2\wp_{13}\wp_{33} \\
&\quad + 4\wp_{23}\wp_{33}\lambda_1 + \wp_{33}\lambda_3\lambda_1 + 8\wp_{33}\lambda_0 + \tfrac{4}{3}\wp_{13}Q_{1333} \\
&\quad - \tfrac{2}{3}\wp_{23}\lambda_3Q_{1333} - 4\wp_{33}\wp_{13}\lambda_2 - \tfrac{2}{3}\lambda_2Q_{1333} \\
\textbf{(−14)}\quad \wp_{113}\wp_{333} &= 2\wp_{33}\wp_{13}^2 + 2\wp_{33}^2\wp_{11} - 2\wp_{12}^2 + 2\wp_{33}\wp_{13}\lambda_2 - \wp_{23}\wp_{33}\lambda_1 \\
&\quad - 2\wp_{33}\lambda_0 - \tfrac{2}{3}\wp_{13}Q_{1333} \\
\textbf{(−15)}\quad \wp_{123}\wp_{133} &= 2\wp_{23}\wp_{13}^2 + 2\wp_{13}\lambda_1 + 2\wp_{13}^2\lambda_3 - 2\wp_{23}\lambda_0 + 2\wp_{33}\wp_{12}\wp_{13} + \tfrac{1}{3}\wp_{12}Q_{1333} \\
\textbf{(−15)}\quad \wp_{122}\wp_{222} &= 6\wp_{33}\wp_{22}\lambda_1 - \wp_{23}\lambda_3\lambda_1 - 2\wp_{33}\wp_{12}\lambda_3^2 - 2\wp_{23}^2\lambda_1 + 2\lambda_3\wp_{13}\lambda_2 \\
&\quad + 2\lambda_3\wp_{11}\wp_{33} + \wp_{23}\wp_{13}\lambda_3^2 - 2\wp_{13}^2\lambda_3 - \tfrac{2}{3}\wp_{22}\lambda_3Q_{1333} \\
&\quad + 4\wp_{33}\wp_{12}\lambda_2 - 2\lambda_3\lambda_0 + 4\wp_{22}^2\wp_{12} - \tfrac{4}{3}\wp_{12}Q_{1333} - 8\wp_{13}\lambda_1 \\
\textbf{(−15)}\quad \wp_{113}\wp_{233} &= 2\wp_{13}^2\lambda_3 + 2\wp_{23}\wp_{13}^2 + 2\wp_{33}\wp_{12}\lambda_2 + 2\wp_{13}\lambda_1 + 2\wp_{33}\wp_{23}\wp_{11} \\
&\quad - 2\wp_{23}\lambda_0 - \wp_{33}\wp_{22}\lambda_1 - \tfrac{2}{3}\wp_{12}Q_{1333} \\
\textbf{(−15)}\quad \wp_{112}\wp_{333} &= 2\wp_{23}\wp_{13}\lambda_2 - \wp_{23}^2\lambda_1 - 4\wp_{13}^2\lambda_3 + 4\wp_{33}\wp_{12}\wp_{13} - 2\wp_{23}\lambda_0 \\
&\quad + 2\wp_{33}\wp_{23}\wp_{11} - 2\wp_{23}\wp_{13}^2 + \tfrac{4}{3}\wp_{12}Q_{1333} \\
\textbf{(−16)}\quad \wp_{123}^2 &= \tfrac{1}{3}\wp_{23}\wp_{12}\lambda_2 - \tfrac{8}{3}\wp_{11}\wp_{13} + \tfrac{10}{3}\wp_{13}\wp_{23}\wp_{12} - \tfrac{1}{3}\wp_{33}\wp_{11}\wp_{22} + \tfrac{2}{9}\wp_{33}\lambda_2Q_{1333} \\
&\quad - \tfrac{1}{3}\wp_{23}\wp_{22}\lambda_1 + \tfrac{1}{3}\wp_{22}\wp_{13}\lambda_2 + 3\wp_{22}\lambda_0 + \tfrac{1}{3}\wp_{12}\lambda_1 + \tfrac{1}{3}\wp_{23}^2\wp_{11} \\
&\quad + \tfrac{1}{3}\wp_{13}^2\wp_{22} - \tfrac{8}{3}\wp_{33}^2\lambda_0 + \tfrac{1}{3}\wp_{33}\wp_{12}^2 + 3\wp_{12}\wp_{13}\lambda_3 - \tfrac{1}{3}\wp_{33}^2\lambda_3\lambda_1 \\
\textbf{(−16)}\quad \wp_{122}\wp_{133} &= -\tfrac{8}{3}\wp_{11}\wp_{13} + \tfrac{4}{3}\wp_{13}\wp_{23}\wp_{12} + \tfrac{2}{3}\wp_{33}\wp_{11}\wp_{22} - \tfrac{2}{3}\wp_{23}^2\wp_{11} \\
&\quad + \tfrac{4}{3}\wp_{13}^2\wp_{22} + \tfrac{4}{3}\wp_{33}\wp_{12}^2 + \tfrac{2}{3}\wp_{23}\wp_{22}\lambda_1 - \tfrac{2}{3}\wp_{22}\wp_{13}\lambda_2 - \tfrac{2}{3}\wp_{23}\wp_{12}\lambda_2 \\
&\quad + 3\wp_{12}\wp_{13}\lambda_3 + \tfrac{2}{3}\wp_{33}^2\lambda_3\lambda_1 + \tfrac{1}{3}\wp_{12}\lambda_1 + \tfrac{16}{3}\wp_{33}^2\lambda_0 - \tfrac{4}{9}\wp_{33}\lambda_2Q_{1333}
\end{aligned}
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\begin{aligned}
\textbf{(−16)}\quad \wp_{113}\wp_{223} &= \tfrac{4}{3}\wp_{11}\wp_{13} + \tfrac{4}{3}\wp_{13}\wp_{23}\wp_{12} + \tfrac{2}{3}\wp_{33}\wp_{11}\wp_{22} + \tfrac{4}{3}\wp_{23}^2\wp_{11} + \tfrac{4}{3}\wp_{23}\wp_{12}\lambda_2 \\
&\quad + \tfrac{4}{3}\wp_{13}^2\wp_{22} - \tfrac{2}{3}\wp_{33}\wp_{12}^2 - \tfrac{4}{3}\wp_{23}\wp_{22}\lambda_1 + \tfrac{4}{3}\wp_{22}\wp_{13}\lambda_2 \\
&\quad - \tfrac{1}{3}\wp_{33}^2\lambda_3\lambda_1 + \tfrac{4}{3}\wp_{12}\lambda_1 - \tfrac{8}{3}\wp_{33}^2\lambda_0 + \tfrac{2}{9}\wp_{33}\lambda_2 Q_{1333} \\
\textbf{(−16)}\quad \wp_{112}\wp_{233} &= \tfrac{16}{3}\wp_{11}\wp_{13} + \tfrac{4}{3}\wp_{13}\wp_{23}\wp_{12} + \tfrac{2}{3}\wp_{33}\wp_{11}\wp_{22} + \tfrac{4}{3}\wp_{23}^2\wp_{11} - \tfrac{4}{9}\wp_{33}\lambda_2 Q_{1333} \\
&\quad - \tfrac{2}{3}\wp_{13}^2\wp_{22} + \tfrac{4}{3}\wp_{33}\wp_{12}^2 - \tfrac{1}{3}\wp_{23}\wp_{22}\lambda_1 - \tfrac{2}{3}\wp_{22}\wp_{13}\lambda_2 - 6\wp_{22}\lambda_0 \\
&\quad + \tfrac{4}{3}\wp_{23}\wp_{12}\lambda_2 + \tfrac{2}{3}\wp_{33}^2\lambda_3\lambda_1 + \tfrac{10}{3}\wp_{12}\lambda_1 + \tfrac{16}{3}\wp_{33}^2\lambda_0 \\
\textbf{(−17)}\quad \wp_{122}\wp_{123} &= 2\wp_{33}\lambda_3\lambda_0 + 2\wp_{12}\wp_{13}\wp_{22} - 2\wp_{11}\wp_{12} + 2\lambda_1\wp_{13}\wp_{33} + 2\lambda_3\wp_{12}^2 \\
&\quad + 2\wp_{23}\wp_{12}^2 - \tfrac{1}{3}\lambda_1 Q_{1333} - \tfrac{1}{3}\wp_{13}\lambda_3 Q_{1333} \\
\textbf{(−17)}\quad \wp_{113}\wp_{222} &= -8\wp_{23}\wp_{33}\lambda_0 - 4\wp_{33}\lambda_3\lambda_0 + 8\lambda_1\wp_{13}\wp_{33} + 4\wp_{22}\wp_{12}\lambda_2 + \tfrac{2}{3}\lambda_1 Q_{1333} \\
&\quad + 4\wp_{12}\wp_{13}\wp_{22} + 4\wp_{11}\wp_{12} - \lambda_3\wp_{23}\wp_{33}\lambda_1 - 2\wp_{23}\wp_{12}^2 + 2\wp_{22}\wp_{23}\wp_{11} \\
&\quad - 4\lambda_3\wp_{12}^2 - 2\wp_{22}^2\lambda_1 + \tfrac{2}{3}\wp_{23}\lambda_2 Q_{1333} - \tfrac{4}{3}\wp_{13}\lambda_3 Q_{1333} \\
\textbf{(−17)}\quad \wp_{112}\wp_{223} &= 8\wp_{33}\lambda_3\lambda_0 + \lambda_3\wp_{23}\wp_{33}\lambda_1 - 4\lambda_1\wp_{13}\wp_{33} + 2\wp_{23}\wp_{12}^2 + \tfrac{2}{3}\wp_{13}\lambda_3 Q_{1333} \\
&\quad + 2\wp_{22}\wp_{23}\wp_{11} + 8\wp_{23}\wp_{33}\lambda_0 + 2\lambda_3\wp_{12}^2 - \tfrac{2}{3}\wp_{23}\lambda_2 Q_{1333} - \tfrac{4}{3}\lambda_1 Q_{1333} \\
\textbf{(−18)}\quad \wp_{122}^2 &= -4\lambda_2\wp_{13}^2 - 4\lambda_2\lambda_0 + \lambda_1^2 - 16\lambda_0\wp_{13} - 8\lambda_0\wp_{23}^2 + 4\wp_{22}\wp_{12}^2 \\
&\quad + 2\lambda_3\wp_{13}\lambda_1 + \wp_{13}^2\lambda_3^2 + 4\lambda_1\wp_{13}\wp_{23} + 4\lambda_1\wp_{12}\wp_{33} \\
&\quad + 8\wp_{33}\wp_{22}\lambda_0 - \tfrac{4}{3}\wp_{12}\lambda_3 Q_{1333} + \tfrac{4}{3}\wp_{11} Q_{1333} - 4\lambda_3\wp_{23}\lambda_0 \\
\textbf{(−18)}\quad \wp_{113}\wp_{133} &= 2\wp_{33}\wp_{13}\wp_{11} + 2\lambda_2\wp_{13}^2 + 2\lambda_0\wp_{13} + 2\lambda_0\wp_{23}^2 + 2\wp_{13}^3 \\
&\quad + \lambda_1\wp_{12}\wp_{33} - 2\wp_{33}\wp_{22}\lambda_0 - 2\lambda_1\wp_{13}\wp_{23} \\
\textbf{(−18)}\quad \wp_{111}\wp_{333} &= -9\lambda_3\wp_{23}\lambda_0 - 2\wp_{13}\lambda_2^2 + \lambda_2\wp_{23}\lambda_1 - 2\lambda_2\wp_{11}\wp_{33} + 2\wp_{11} Q_{1333} \\
&\quad - 10\lambda_0\wp_{23}^2 + 6\wp_{33}\wp_{13}\wp_{11} - 10\lambda_0\wp_{13} - 2\wp_{13}^3 + 7\lambda_1\wp_{13}\wp_{23} \\
&\quad + 2\lambda_1\wp_{12}\wp_{33} - 2\wp_{33}\wp_{22}\lambda_0 + 6\lambda_3\wp_{13}\lambda_1 + 2\lambda_1^2 - 6\lambda_2\lambda_0 - 4\lambda_2\wp_{13}^2 \\
\textbf{(−18)}\quad \wp_{112}\wp_{222} &= 4\lambda_2\wp_{13}^2 + 2\wp_{23}\lambda_3\wp_{13}\lambda_2 - 2\lambda_2\wp_{23}\lambda_1 + 4\lambda_2\wp_{11}\wp_{33} - \wp_{23}^2\lambda_3\lambda_1 \\
&\quad + 4\lambda_2\lambda_0 - 2\lambda_1^2 + 8\lambda_0\wp_{23}^2 + 2\wp_{11}\wp_{22}^2 + 2\wp_{22}\wp_{12}^2 + \tfrac{2}{3}\wp_{12}\lambda_3 Q_{1333} \\
&\quad - 2\lambda_3\wp_{33}\wp_{12}\lambda_2 + 2\lambda_3\wp_{33}\wp_{22}\lambda_1 + 4\wp_{13}\lambda_2^2 - 8\lambda_3\wp_{13}\lambda_1 - \tfrac{8}{3}\wp_{11} Q_{1333} \\
&\quad - 8\lambda_1\wp_{13}\wp_{23} + 4\lambda_1\wp_{12}\wp_{33} + 8\lambda_3\wp_{23}\lambda_0 - \tfrac{2}{3}\lambda_2\wp_{22} Q_{1333} - 2\wp_{13}^2\lambda_3^2 \\
\textbf{(−19)}\quad \wp_{113}\wp_{123} &= -2\wp_{33}^2\lambda_3\lambda_0 + 2\lambda_2\wp_{12}\wp_{13} + 2\wp_{12}\wp_{13}^2 + 2\wp_{12}\lambda_0 - 2\wp_{23}\wp_{22}\lambda_0 \\
&\quad + 2\wp_{13}\wp_{23}\wp_{11} + \tfrac{1}{3}\wp_{33}\lambda_1 Q_{1333} \\
\textbf{(−19)}\quad \wp_{111}\wp_{233} &= 2\wp_{12}\wp_{33}\wp_{11} - 2\wp_{12}\wp_{13}^2 + 6\lambda_1\wp_{12}\lambda_3 + 6\lambda_1\wp_{12}\wp_{23} \\
&\quad + 4\wp_{13}\wp_{23}\wp_{11} - 2\wp_{12}\lambda_2^2 - 2\wp_{11}\lambda_1 - 4\lambda_2\wp_{12}\wp_{13} + \lambda_2\wp_{22}\lambda_1 \\
&\quad - 2\lambda_2\wp_{23}\wp_{11} - 8\wp_{23}\wp_{22}\lambda_0 + \wp_{13}\wp_{22}\lambda_1 + 4\wp_{33}^2\lambda_3\lambda_0 \\
&\quad + 6\wp_{11}\wp_{13}\lambda_3 - 9\lambda_3\wp_{22}\lambda_0 + 2\wp_{12}\lambda_0 - \tfrac{2}{3}\wp_{33}\lambda_1 Q_{1333}
\end{aligned}
$$

$$\begin{aligned}
\textbf{(−19)}\quad \wp_{112}\wp_{133} &= 4\wp_{33}^2\lambda_3\lambda_0 + 2\wp_{12}\wp_{33}\wp_{11} + 2\lambda_2\wp_{12}\wp_{13} - 2\wp_{13}\wp_{22}\lambda_1 + 2\wp_{12}\lambda_0 \\
&\quad + 2\wp_{12}\wp_{13}^2 + 4\wp_{23}\wp_{22}\lambda_0 - \tfrac{2}{3}\wp_{33}\lambda_1 Q_{1333} - \lambda_1\wp_{12}\wp_{23} \\
\textbf{(−20)}\quad \wp_{113}\wp_{122} &= 2\wp_{22}\wp_{13}\wp_{11} - 4\wp_{33}\lambda_2\lambda_0 + 2\wp_{12}^2\wp_{13} - 4\wp_{22}^2\lambda_0 + \wp_{33}\lambda_1^2 \\
&\quad + 2\wp_{12}\wp_{22}\lambda_1 - 4\wp_{23}\wp_{33}\lambda_3\lambda_0 + \lambda_1\lambda_3\wp_{13}\wp_{33} - \tfrac{2}{3}\wp_{13}\lambda_2 Q_{1333} \\
&\quad + 8\wp_{13}\wp_{33}\lambda_0 + \tfrac{4}{3}\lambda_0 Q_{1333} + \tfrac{2}{3}\wp_{23}\lambda_1 Q_{1333} \\
\textbf{(−20)}\quad \wp_{112}\wp_{123} &= 8\wp_{33}\lambda_2\lambda_0 + 2\wp_{12}^2\wp_{13} + 2\wp_{12}\wp_{23}\wp_{11} + 2\wp_{22}^2\lambda_0 - 2\wp_{33}\lambda_1^2 \\
&\quad + 2\lambda_2\wp_{12}^2 - 2\wp_{12}\wp_{22}\lambda_1 + 2\wp_{23}\wp_{33}\lambda_3\lambda_0 - \tfrac{8}{3}\lambda_0 Q_{1333} - \tfrac{1}{3}\wp_{23}\lambda_1 Q_{1333} \\
\textbf{(−20)}\quad \wp_{111}\wp_{223} &= -12\wp_{33}\lambda_2\lambda_0 + 2\wp_{22}\wp_{13}\wp_{11} + 6\wp_{11}\lambda_3\wp_{12} - 4\wp_{11}^2 + 4\wp_{33}\lambda_1^2 \\
&\quad - 4\wp_{22}^2\lambda_0 - 2\lambda_2\wp_{11}\wp_{22} - 8\wp_{13}\wp_{33}\lambda_0 - \tfrac{4}{3}\wp_{23}\lambda_1 Q_{1333} \\
&\quad - \lambda_2\wp_{33}\lambda_3\lambda_1 + 8\wp_{23}\wp_{33}\lambda_3\lambda_0 + \tfrac{2}{3}\wp_{13}\lambda_2 Q_{1333} + \tfrac{2}{3}\lambda_2^2 Q_{1333} \\
&\quad - 2\lambda_2\wp_{12}^2 - 2\wp_{12}^2\wp_{13} + 4\wp_{12}\wp_{23}\wp_{11} - \lambda_1\lambda_3\wp_{13}\wp_{33} \\
&\quad - 2\lambda_3\lambda_1 Q_{1333} + 4\wp_{12}\wp_{22}\lambda_1 + \tfrac{4}{3}\lambda_0 Q_{1333} + 9\wp_{33}\lambda_3^2\lambda_0 \\
\textbf{(−21)}\quad \wp_{111}\wp_{222} &= 6\lambda_3\lambda_2\lambda_0 - 6\lambda_1\lambda_0 - 2\lambda_3\lambda_1^2 - 2\wp_{22}\lambda_1 Q_{1333} - 6\lambda_3^2\wp_{13}\lambda_1 - 8\lambda_3\lambda_0\wp_{13} \\
&\quad - 2\wp_{12}^3 + 2\wp_{12}\lambda_2 Q_{1333} + 10\lambda_3\lambda_0\wp_{23}^2 + 4\wp_{23}\lambda_2^2\wp_{13} - 2\lambda_3\wp_{11}Q_{1333} \\
&\quad + 6\wp_{11}\wp_{22}\wp_{12} - 4\wp_{33}\wp_{12}\lambda_2^2 - 16\lambda_0\wp_{13}\wp_{23} + 9\lambda_3^2\wp_{23}\lambda_0 - 4\wp_{23}\lambda_1^2 \\
&\quad - 2\lambda_3\lambda_2\wp_{13}^2 + 2\lambda_3\wp_{13}\lambda_2^2 + 6\lambda_2\wp_{13}\lambda_1 - 8\lambda_0\wp_{12}\wp_{33} + 10\lambda_1\wp_{13}^2 \\
&\quad + 2\lambda_3\lambda_2\wp_{11}\wp_{33} - 2\wp_{23}^2\lambda_2\lambda_1 + 4\lambda_3\lambda_1\wp_{12}\wp_{33} + 2\lambda_2\wp_{33}\wp_{22}\lambda_1 \\
&\quad - \lambda_3\lambda_2\wp_{23}\lambda_1 + 2\wp_{11}\wp_{33}\lambda_1 - 7\lambda_3\lambda_1\wp_{13}\wp_{23} + 2\lambda_3\lambda_0\wp_{22}\wp_{33} \\
\textbf{(−21)}\quad \wp_{112}\wp_{122} &= 2\wp_{11}\wp_{22}\wp_{12} + 2\wp_{11}\wp_{33}\lambda_1 - 2\lambda_1\lambda_0 - 8\lambda_3\lambda_0\wp_{13} + 2\wp_{12}^3 \\
&\quad - 2\lambda_3\lambda_0\wp_{23}^2 + 8\lambda_0\wp_{12}\wp_{33} + 2\lambda_3\lambda_0\wp_{22}\wp_{33} + 2\lambda_2\wp_{13}\lambda_1 \\
&\quad + \lambda_3\lambda_1\wp_{13}\wp_{23} - \wp_{23}\lambda_1^2 - 2\lambda_1\wp_{13}^2 - \tfrac{2}{3}\wp_{12}\lambda_2 Q_{1333} \\
\textbf{(−22)}\quad \wp_{113}^2 &= 4\wp_{13}^2\wp_{11} - 4\lambda_0\wp_{13}\wp_{22} - 4\lambda_0\wp_{12}\wp_{23} - 4\wp_{33}^2\lambda_2\lambda_0 + \wp_{33}^2\lambda_1^2 \\
&\quad + 4\lambda_1\wp_{12}\wp_{13} + \tfrac{4}{3}\wp_{33}\lambda_0 Q_{1333} \\
\textbf{(−22)}\quad \wp_{111}\wp_{133} &= 8\lambda_0\wp_{12}\wp_{23} - \lambda_1\wp_{12}\lambda_2 - 6\lambda_2\wp_{22}\lambda_0 + 2\lambda_2\wp_{11}\wp_{13} + 2\wp_{22}\lambda_1^2 \\
&\quad - 2\wp_{33}^2\lambda_1^2 - 6\lambda_0\wp_{11} + 2\wp_{33}\wp_{11}^2 + 9\lambda_0\wp_{12}\lambda_3 - \tfrac{8}{3}\wp_{33}\lambda_0 Q_{1333} \\
&\quad - \lambda_1\wp_{12}\wp_{13} - 2\lambda_1\wp_{11}\wp_{23} + 2\wp_{13}^2\wp_{11} - 4\lambda_0\wp_{13}\wp_{22} + 8\wp_{33}^2\lambda_2\lambda_0 \\
\textbf{(−23)}\quad \wp_{112}\wp_{113} &= -4\wp_{23}\wp_{33}\lambda_2\lambda_0 + 2\lambda_1\wp_{12}^2 + \wp_{23}\wp_{33}\lambda_1^2 - 4\lambda_0\wp_{12}\wp_{22} \\
&\quad 4\wp_{12}\wp_{13}\wp_{11} + \tfrac{4}{3}\wp_{23}\lambda_0 Q_{1333} - \tfrac{2}{3}\wp_{13}\lambda_1 Q_{1333} + 4\lambda_0\lambda_3\wp_{13}\wp_{33} \\
\textbf{(−23)}\quad \wp_{111}\wp_{123} &= 2\wp_{12}\wp_{13}\wp_{11} - 2\wp_{33}\lambda_3\lambda_1^2 + 2\lambda_1\wp_{33}\lambda_0 - 2\lambda_0\lambda_3\wp_{13}\wp_{33} \\
&\quad + 2\wp_{11}\wp_{12}\lambda_2 - 2\wp_{11}\wp_{22}\lambda_1 + 6\lambda_2\wp_{33}\lambda_3\lambda_0 + 8\wp_{23}\wp_{33}\lambda_2\lambda_0 \\
&\quad + 2\lambda_0\wp_{12}\wp_{22} - 2\wp_{23}\wp_{33}\lambda_1^2 + \tfrac{1}{3}\lambda_2\lambda_1 Q_{1333} - 3\lambda_3\lambda_0 Q_{1333} \\
&\quad - \tfrac{8}{3}\wp_{23}\lambda_0 Q_{1333} + \tfrac{1}{3}\wp_{13}\lambda_1 Q_{1333} + 2\wp_{23}\wp_{11}^2
\end{aligned}$$

$$
\begin{aligned}
\mathbf{(-24)} \quad \wp_{111}\wp_{122} &= -3\lambda_3\lambda_1\lambda_0 - \lambda_2\lambda_1^2 + 4\lambda_2^2\lambda_0 - 8\lambda_0^2 - \lambda_1\wp_{13}^2\lambda_3 - 8\lambda_1\lambda_0\wp_{23} - 2\wp_{23}^2\lambda_1^2 \\
&+ 4\lambda_2\lambda_0\wp_{23}^2 - \tfrac{8}{3}\wp_{22}\lambda_0 Q_{1333} + \tfrac{2}{3}\wp_{12}\lambda_1 Q_{1333} - \tfrac{2}{3}\lambda_2\wp_{11}Q_{1333} + \lambda_1^2\wp_{13} \\
&+ 8\lambda_2\lambda_0\wp_{13} - 9\lambda_3^2\lambda_0\wp_{13} - 2\lambda_3\lambda_1^2\wp_{23} + 2\wp_{11}^2\wp_{22} - 2\lambda_2\lambda_1\wp_{12}\wp_{33} \\
&+ 2\wp_{11}\wp_{12}^2 + 8\lambda_0\wp_{13}^2 + \lambda_2\lambda_3\wp_{13}\lambda_1 + 4\lambda_3\lambda_0\wp_{12}\wp_{33} + 2\lambda_2\lambda_1\wp_{13}\wp_{23} \\
&- 8\lambda_3\lambda_0\wp_{13}\wp_{23} + 4\lambda_2\wp_{33}\wp_{22}\lambda_0 + 2\lambda_3\lambda_1\wp_{11}\wp_{33} + 6\lambda_2\lambda_3\wp_{23}\lambda_0 \\
\mathbf{(-24)} \quad \wp_{112}^2 &= -8\lambda_0\wp_{13}^2 + 8\lambda_0\wp_{11}\wp_{33} - 4\lambda_1\lambda_0\wp_{23} - 4\lambda_2\lambda_0\wp_{23}^2 - 8\lambda_2\lambda_0\wp_{13} \\
&+ 4\wp_{11}\wp_{12}^2 - 8\lambda_0^2 + 4\lambda_3\lambda_0\wp_{13}\wp_{23} + 4\lambda_3\lambda_0\wp_{12}\wp_{33} + \wp_{23}^2\lambda_1^2 \\
&+ 4\lambda_1^2\wp_{13} + \tfrac{4}{3}\wp_{22}\lambda_0 Q_{1333} - \tfrac{4}{3}\wp_{12}\lambda_1 Q_{1333} \\
\mathbf{(-26)} \quad \wp_{111}\wp_{113} &= 4\lambda_0\wp_{33}\wp_{13}\lambda_2 + 4\wp_{11}^2\wp_{13} - \wp_{33}\lambda_2\lambda_1^2 + 4\wp_{33}\lambda_2^2\lambda_0 - \lambda_1^2\wp_{13}\wp_{33} \\
&- 3\wp_{33}\lambda_3\lambda_1\lambda_0 + 2\wp_{12}\wp_{11}\lambda_1 - 6\wp_{22}\lambda_0\wp_{11} + 2\lambda_0\wp_{12}^2 \\
&- 2\lambda_2\lambda_0 Q_{1333} - \tfrac{4}{3}\lambda_0\wp_{13}Q_{1333} + \tfrac{2}{3}\lambda_1^2 Q_{1333} + 8\wp_{33}\lambda_0^2 \\
\mathbf{(-27)} \quad \wp_{111}\wp_{112} &= -\tfrac{4}{3}\wp_{12}\lambda_0 Q_{1333} + 2\lambda_3\wp_{13}\lambda_1^2 + 8\lambda_2\lambda_1\lambda_0 + \lambda_1^2\wp_{13}\wp_{23} - \tfrac{2}{3}\wp_{11}\lambda_1 Q_{1333} \\
&+ 6\lambda_3\wp_{33}\lambda_0\wp_{11} + 4\wp_{11}^2\wp_{12} + 4\wp_{23}\lambda_2^2\lambda_0 - 18\lambda_3\lambda_0^2 - 2\lambda_1^3 - 16\wp_{23}\lambda_0^2 \\
&- 2\lambda_3\wp_{13}^2\lambda_0 - 6\lambda_3\wp_{13}\lambda_2\lambda_0 - 3\lambda_1\lambda_3\wp_{23}\lambda_0 + 2\lambda_1\wp_{33}\wp_{22}\lambda_0 \\
&- \lambda_2\wp_{23}\lambda_1^2 - 2\lambda_1^2\wp_{12}\wp_{33} - 2\lambda_1\lambda_0\wp_{23}^2 + 8\lambda_1\lambda_0\wp_{13} + 4\lambda_0\wp_{33}\wp_{12}\lambda_2 \\
\mathbf{(-30)} \quad \wp_{111}^2 &= -4\wp_{23}\lambda_1^3 - 8\wp_{23}^2\lambda_0^2 - 8\wp_{13}\lambda_2^2\lambda_0 - 4\lambda_2\lambda_0\wp_{13}^2 + 16\wp_{23}\lambda_2\lambda_1\lambda_0 \\
&+ 4\wp_{23}\lambda_1\lambda_0\wp_{13} + 6\lambda_3\lambda_1\lambda_0\wp_{13} - 4\lambda_0\wp_{11}Q_{1333} - 36\lambda_3\wp_{23}\lambda_0^2 \\
&+ 18\lambda_3\lambda_2\lambda_1\lambda_0 + \lambda_2^2\lambda_1^2 - 4\lambda_3\lambda_1^3 - 4\lambda_2^3\lambda_0 - 4\lambda_1^2\lambda_0 - 27\lambda_3^2\lambda_0^2 \\
&+ 12\lambda_2\lambda_0^2 + 2\wp_{13}\lambda_2\lambda_1^2 + 16\wp_{13}\lambda_0^2 + \lambda_1^2\wp_{13}^2 - 4\wp_{12}\lambda_1\wp_{33}\lambda_0 \\
&- 4\wp_{11}\wp_{33}\lambda_1^2 + 16\wp_{11}\wp_{33}\lambda_2\lambda_0 + 4\wp_{11}^3 + 8\wp_{33}\wp_{22}\lambda_0^2
\end{aligned}
$$

These calculations required the σ-function expansion up to C_{35}.

Remark 5.3.7. In [30] the quadratic 3-index relations associated to the general (3,4)-curve were presented down to weight -15. Make a change of variables on the curve coefficients to move from the general case to the cyclic case. Substituting

$$
\begin{aligned}
&\mu_1 = 0, \quad \mu_4 = 0, \quad \mu_2 = 0, \quad \mu_5 = 0, \quad \mu_8 = 0, \quad \mu_3 = \lambda_3, \\
&\mu_6 = \lambda_2, \quad \mu_9 = \lambda_9, \quad \mu_{12} = \lambda_0, \quad \wp_{1333} = Q_{1333} + 6\wp_{13}\wp_{33},
\end{aligned}
$$

will change the equations derived in [30] to equations from Theorem 5.3.6.

The equations in [30] were derived using the bilinear relation methods, supplemented by equations obtained from the expansion of the Kleinian formula. The authors found all the relations down to weight -23 this way, (with the relations from -16 to -23 presented on-line at [29]).

5.3.3 The cyclic (3,5)-curve

We have also used the procedure to derive the relations associated to the cyclic (3,5)-curve. All those from weight -6 to weight -19 were derived as normal, starting with the relations below.

$$
\begin{aligned}
&\mathbf{(-6)} & \wp_{444}^2 &= 4\wp_{44}^3 - 4\wp_{23} - 4\wp_{33}\wp_{44} + \wp_{34}^2 + 2\wp_{34}\lambda_4 + \lambda_4^2 - 4\lambda_3 \\
&\mathbf{(-7)} & \wp_{344}\wp_{444} &= 4\wp_{34}\wp_{44}^2 + 2\wp_{24}\wp_{44} - \wp_{33}\wp_{34} - \lambda_4\wp_{33} - \tfrac{2}{3}Q_{2444} \\
&\mathbf{(-8)} & \wp_{344}^2 &= 4\wp_{34}^2\wp_{44} + 4\wp_{14} + 4\wp_{24}\wp_{34} + \wp_{33}^2 \\
&\mathbf{(-8)} & \wp_{334}\wp_{444} &= 2\wp_{34}^2\wp_{44} + 2\wp_{33}\wp_{44}^2 - 4\wp_{14} + 2\wp_{22} - 2\wp_{23}\wp_{44} - \wp_{24}\wp_{34} \\
& & &\quad - 2\wp_{33}^2 - \wp_{24}\lambda_4 + 2\lambda_4\wp_{34}\wp_{44} \\
&\mathbf{(-9)} & \wp_{334}\wp_{344} &= -2\wp_{13} + 2\wp_{44}\wp_{33}\wp_{34} - 2\wp_{23}\wp_{34} + \wp_{24}\wp_{33} + 2\wp_{34}^3 + 2\lambda_4\wp_{34}^2 \\
&\mathbf{(-9)} & \wp_{333}\wp_{444} &= 6\wp_{44}\wp_{33}\wp_{34} - 4\lambda_4\wp_{34}^2 + 2\wp_{13} + 7\wp_{23}\wp_{34} + 2\wp_{24}\wp_{33} + \lambda_4\wp_{23} \\
& & &\quad - 2\lambda_4\wp_{33}\wp_{44} - 2\wp_{34}^3 - 2\wp_{34}\lambda_4^2 + 6\wp_{34}\lambda_3 + 2\lambda_2 - 2\wp_{44}Q_{2444} \\
&\mathbf{(-9)} & \wp_{244}\wp_{444} &= -2\wp_{13} + \wp_{23}\wp_{34} - 2\wp_{24}\wp_{33} - \lambda_4\wp_{23} + 2\wp_{34}\lambda_3 - 2\lambda_2 \\
& & &\quad + 4\wp_{24}\wp_{44}^2 + \tfrac{2}{3}\wp_{44}Q_{2444}
\end{aligned}
$$

The full set of relations to weight -19 can be found in Appendix D.2. Note that Q_{2444} was one of the five Q-functions used in the basis of fundamental Abelian functions for the (3,5)-case. These functions only appear linearly or multiplied by a 2-index $\wp$-function in the relations down to weight -19.

At weight -20 there are 11 quadratic 3-index terms,

$$
\begin{aligned}
\{\wp_{ijk}\wp_{lmn}\} =\{&\wp_{223}^2, \wp_{222}\wp_{233}, \wp_{134}\wp_{223}, \wp_{134}^2, \wp_{133}\wp_{224}, \wp_{133}\wp_{144}, \\
&\wp_{124}\wp_{233}, \wp_{123}\wp_{234}, \wp_{122}\wp_{334}, \wp_{114}\wp_{334}, \wp_{113}\wp_{344}\}.
\end{aligned}
$$

Following Steps 1-5 of the procedure we find that $\{\wp_{12}\wp_{334}, \wp_{114}\wp_{334}, \wp_{113}\wp_{344}\}$ may be used to give separate expressions for the other eight quadratic 3-index products. We hence try to determine expressions for these three pairs using the σ-expansion. We find this is not possible when only allowing the basis Q-functions to appear linearly. (There is no solution to the equations in the constants obtained as the coefficients in the expansion).

We repeat the calculation, allowing quadratic terms in the basis Q-functions. This time there is a solution, however it is not unique. We find that we can separate out two polynomials from the expression such that adding any multiple of these polynomials does not effect the remaining equation. Hence these two polynomials must be zero. (This has been double checked using the σ-expansion on the polynomials themselves).

So we have the following two relations, cubic in the fundamental Abelian functions, including quadratic terms in the basis Q-functions.

$$\begin{aligned}
0 &= Q_{2333}^2 - 12Q_{2444}Q_{1244} - 4\lambda_4 Q_{2444}Q_{2333} - 6\wp_{22}Q_{2233} - 6\wp_{33}\lambda_3 Q_{2333} \\
&+ 12\wp_{33}\lambda_4\lambda_3 Q_{2444} + 24\wp_{44}\wp_{13}\lambda_2 - 36\wp_{33}^2\lambda_1 + 24\wp_{14}\wp_{23}^2 + 12\wp_{33}\wp_{1224} \\
&+ 9\wp_{33}^2\lambda_3^2 + 24\wp_{22}\wp_{13}\lambda_4 - 36\wp_{24}\lambda_0 - 12\wp_{11}\wp_{23} - 24\wp_{24}\wp_{13}\wp_{23} \\
&- 12\wp_{12}\lambda_2 - 24\wp_{24}\wp_{12}\wp_{33} + 24\wp_{22}\wp_{13}\wp_{34} - 12\wp_{44}\wp_{23}\lambda_1 - 36\wp_{34}\wp_{12}\lambda_3 \\
&- 72\wp_{44}\wp_{34}\lambda_0 + 24\lambda_4\wp_{14}\lambda_2 - 24\wp_{34}\wp_{12}\wp_{23} - 36\wp_{24}\wp_{13}\lambda_3 + 18\wp_{22}\wp_{23}\lambda_3 \\
&+ 36\wp_{14}\wp_{23}\lambda_3 + 24\lambda_4\wp_{11}\wp_{34} + 12\wp_{22}\lambda_4\lambda_2 + 24\wp_{22}\wp_{34}\lambda_2 - 12\lambda_4\wp_{24}\lambda_1 \\
&- 48\wp_{22}\wp_{14}\wp_{33}.
\end{aligned}$$

$$\begin{aligned}
0 &= Q_{2333}^2 - 4Q_{2444}Q_{1244} - \tfrac{4}{3}\lambda_4 Q_{2444}Q_{2333} - 2\wp_{22}Q_{2233} - 10\wp_{33}\lambda_3 Q_{2333} \\
&+ 4\wp_{33}\lambda_4\lambda_3 Q_{2444} - 8\wp_{14}Q_{2233} + 8\wp_{22}\wp_{13}\lambda_4 + 8\wp_{34}\wp_{24}\lambda_1 + 16\wp_{44}\wp_{11}\wp_{33} \\
&+ 6\wp_{22}\wp_{23}\lambda_3 - 4\wp_{12}\lambda_2 - 36\wp_{24}\lambda_0 - 4\wp_{11}\wp_{23} + 8\wp_{22}\wp_{14}\wp_{33} - 4\wp_{44}\wp_{23}\lambda_1 \\
&- 12\wp_{34}\wp_{12}\lambda_3 - 72\wp_{44}\wp_{34}\lambda_0 - 16\wp_{14}^2\wp_{33} + 24\lambda_4\wp_{14}\lambda_2 - 12\wp_{24}\wp_{13}\lambda_3 \\
&+ 16\wp_{24}\wp_{12}\wp_{33} + 36\wp_{14}\wp_{23}\lambda_3 + 8\lambda_4\wp_{11}\wp_{34} + 4\wp_{22}\lambda_4\lambda_2 + 8\wp_{22}\wp_{34}\lambda_2 \\
&- 4\lambda_4\wp_{24}\lambda_1 + 8\wp_{44}\wp_{13}\lambda_2 + 32\wp_{34}\wp_{13}\wp_{14} - 16\wp_{11}\wp_{34}^2 + 16\wp_{34}\wp_{14}\lambda_2 \\
&- 16\wp_{44}\wp_{13}^2 + 21\wp_{33}^2\lambda_3^2 + 12\wp_{33}^2\lambda_1 - 4\wp_{33}\wp_{1224} + 8\wp_{12}\wp_{13} + 16\lambda_4\wp_{13}\wp_{14}.
\end{aligned}$$

Such equations have not been derived in lower genus cases and their significance has yet to be fully understood. It is likely that they relate to the Kummer surface of the curve, (see for example [19]).

We factor out these polynomials to obtain expressions for the three quadratic 3-index functions. We then substitute back to obtain expressions for the other eight quadratic 3-index functions. For example we have

$$\begin{aligned}
\wp_{134}\wp_{223} &= -\tfrac{1}{3}\wp_{33}\lambda_4\lambda_3 Q_{2444} - \wp_{11}\wp_{23} - \wp_{12}\wp_{13} - \wp_{12}\lambda_2 - \tfrac{8}{3}\wp_{14}\lambda_1 + \wp_{14}\wp_{23}^2 \\
&+ \tfrac{4}{3}\lambda_4\wp_{14}\lambda_2 + \tfrac{1}{2}\wp_{33}^2\lambda_3^2 + \boldsymbol{\tfrac{1}{3}Q_{2444}Q_{1244} + \tfrac{1}{9}\lambda_4 Q_{2444}Q_{2333}} - \wp_{34}\wp_{24}\lambda_1 \\
&+ 2\wp_{33}^2\lambda_1 + \tfrac{2}{3}\lambda_4\wp_{13}\wp_{14} + \wp_{34}\wp_{12}\lambda_3 + \tfrac{2}{3}\wp_{34}\wp_{14}\lambda_2 + \tfrac{2}{3}\wp_{22}\wp_{13}\lambda_4 + \tfrac{1}{3}\wp_{22}\lambda_4\lambda_2 \\
&+ \tfrac{1}{2}\wp_{22}\wp_{23}\lambda_3 + \tfrac{2}{3}\wp_{22}\wp_{34}\lambda_2 + \wp_{24}\wp_{13}\lambda_3 + \wp_{22}\wp_{14}\wp_{33} - \wp_{24}\wp_{12}\wp_{33} + \tfrac{4}{3}\wp_{22}\lambda_1 \\
&- \wp_{44}\wp_{23}\lambda_1 - \lambda_4\lambda_1\wp_{24} - \wp_{33}Q_{1224} + \wp_{34}\wp_{12}\wp_{23} + \wp_{22}\wp_{13}\wp_{34} + \wp_{24}\wp_{13}\wp_{23} \\
&- \tfrac{1}{6}\wp_{33}\lambda_3 Q_{2333} - 2\lambda_4\wp_{44}\lambda_0 + \tfrac{1}{3}\wp_{14}Q_{2233} - \tfrac{1}{6}\wp_{22}Q_{2233} - 2\wp_{44}\wp_{34}\lambda_0.
\end{aligned}$$

The other ten relations at weight -20 can be found in Appendix D.2.

At weights -21, -22, and -23 the relations could be constructed without quadratic Q-terms as normal. These relations can be found in Appendix D.2. However, at weight -24 we again find this is not possible. This time allowing quadratic basis Q-function terms is not effective in finding a solution and at the time of writing no quadratic 3-index relations have been identified at weight -24.

We have proceeded to look at the lower weights and have been able to calculate the remaining relations as normal, without using quadratic Q-terms. They have all been calculated, down to $\wp_{111}^2$ at weight -42.

$$\begin{aligned}\wp_{111}^2 &= 8\lambda_0^2\wp_{13}\wp_{34} - 42\lambda_3\lambda_0^2\wp_{23} - 4\lambda_4\lambda_1^2\lambda_0 + 8\lambda_4\lambda_2\lambda_0^2 + 18\lambda_3\lambda_2\lambda_1\lambda_0 \\ &+ 2\lambda_2\lambda_1^2\wp_{13} - 8\lambda_2^2\lambda_0\wp_{13} + 8\lambda_4\lambda_0^2\wp_{13} - 8\lambda_0^2\wp_{14}\wp_{33} - 4\lambda_2\lambda_0\wp_{13}^2 + 8\lambda_0^2\wp_{22}\wp_{33} \\ &- 4\lambda_3\lambda_1^3 - 4\lambda_2^3\lambda_0 - 27\lambda_3^2\lambda_0^2 + 4\lambda_1\lambda_0^2 + 6\lambda_3\lambda_1\lambda_0\wp_{13} + 12\lambda_4\lambda_3\lambda_0\wp_{11}\wp_{33} \\ &- 4\lambda_1\lambda_0\wp_{12}\wp_{33} + 4\lambda_1\lambda_0\wp_{13}\wp_{23} + 8\lambda_2\lambda_0^2\wp_{34} - 4\lambda_1^2\wp_{11}\wp_{33} - 4\lambda_1^2\lambda_0\wp_{34} \\ &- 8\lambda_0^2\wp_{23}^2 - 4\lambda_1^3\wp_{23} + \lambda_1^2\wp_{13}^2 + 16\lambda_2\lambda_0\wp_{11}\wp_{33} + 16\lambda_2\lambda_1\lambda_0\wp_{23} + 4\wp_{11}^3 \\ &- 4\lambda_0\wp_{11}\lambda_4 Q_{2333} + 2\lambda_0^2 Q_{2233} - 12\wp_{11}\lambda_0 Q_{1244} + \lambda_2^2\lambda_1^2.\end{aligned}$$

The full set of relations descending to this from weight -25 can be found in the extra Appendix of files. These calculations required the σ-function expansion up to C_{50}. (Note that when deriving these relations the quadratic 3-index terms at weight -24 were sometimes present in the set of relations obtained from the bilinear identities. Relations with these terms were discarded and all the relations presented in the Appendix are independent of them).

The Maple worksheet in which these calculations were performed is available in the extra Appendix of files.

Remark 5.3.8. The fact that quadratic relations at weight -24 are not present in the desired form does not contradict any mathematical theory.

Our generalisation of the second elliptic differential equation must exist as a corollary from the construction of the basis for these functions. However, there is no known mathematical proof why the quadratic relations defined in Definition 5.1.1 must exist. We have searched for them without a proof for their existence.

Nevertheless, since many such relations have been derived it is probable that a subtle change in Definition 5.1.1 is required to give a generalisation that must exist mathematically. This is clearly an important topic for future study.

5.3.4 Higher genus curves

The problems at weight -24 in the (3,5)-case propagate with the higher genus curves. In equations (3.59) we presented the quadratic identities associated with the cyclic (4,5)-curve from weight -6 to weight -9. At weight -10 we have been unable to identify expressions, even when allowing quadratic terms in the basis Q-functions.

When the cyclic (3,7)-curve was considered, we were able to derive relations from weight -6 to weight -11 which are presented below. At lower weights we were unable to derive expressions that match Definition 5.1.2. It appears that the problem here is related to the higher genus rather than the classification of the curve, (value of n).

$$\begin{aligned}
&\textbf{(−6)} && \wp_{666}^2 = 4\wp_{66}^3 + \wp_{56}^2 - 4\wp_{55}\wp_{66} + 4\wp_{36} - 4\wp_{45} \\
&\textbf{(−7)} && \wp_{566}\wp_{666} = 4\wp_{66}^2\wp_{56} + 2\wp_{66}^2\lambda_6 - \wp_{55}\wp_{56} - 2\wp_{55}\lambda_6 + 2\wp_{46}\wp_{66} - 2\wp_{35} + \tfrac{2}{3}Q_{5556} \\
&\textbf{(−8)} && \wp_{566}^2 = \wp_{44} + 2\wp_{46}\lambda_6 + \wp_{66}\lambda_6^2 + \tfrac{1}{3}Q_{5555} + \wp_{55}^2 + 4\wp_{46}\wp_{56} \\
&&& \quad + 4\wp_{56}^2\wp_{66} + 4\lambda_6\wp_{56}\wp_{66} \\
&\textbf{(−8)} && \wp_{556}\wp_{666} = 2\wp_{56}^2\wp_{66} + \lambda_6\wp_{56}\wp_{66} - \tfrac{1}{2}\wp_{66}\lambda_6^2 + 2\wp_{55}\wp_{66}^2 - 2\wp_{55}^2 - \wp_{46}\wp_{56} \\
&&& \quad - \lambda_6\wp_{46} + 4\wp_{36}\wp_{66} - 2\wp_{45}\wp_{66} + 2\wp_{66}\lambda_5 + \tfrac{3}{2}\wp_{44} - \tfrac{1}{6}Q_{5555} \\
&\textbf{(−9)} && \wp_{556}\wp_{566} = 2\wp_{34} + 2\wp_{55}\wp_{56}\wp_{66} + 2\wp_{56}\lambda_5 + \wp_{55}\wp_{66}\lambda_6 + 2\wp_{56}^3 + 4\wp_{36}\wp_{56} \\
&&& \quad - 2\wp_{45}\wp_{56} + \wp_{46}\wp_{55} + 2\wp_{56}^2\lambda_6 - 2\wp_{45}\lambda_6 + 2\wp_{36}\lambda_6 + 4\lambda_4 \\
&\textbf{(−9)} && \wp_{555}\wp_{666} = 6\wp_{55}\wp_{56}\wp_{66} - 8\wp_{36}\wp_{56} - 4\wp_{35}\wp_{66} - \wp_{56}^2\lambda_6 - 2\wp_{26} + 2\wp_{66}Q_{5556} \\
&&& \quad + 7\wp_{45}\wp_{56} - 4\wp_{36}\lambda_6 - 2\wp_{56}^3 + 4\wp_{45}\lambda_6 - 2\wp_{55}\wp_{66}\lambda_6 + 2\wp_{46}\wp_{55} \\
&\textbf{(−9)} && \wp_{466}\wp_{666} = 2\wp_{55}\wp_{66}\lambda_6 + 4\wp_{46}\wp_{66}^2 - 2\wp_{46}\wp_{55} + \wp_{45}\wp_{56} - \tfrac{2}{3}\wp_{66}Q_{5556} + 2\wp_{26} \\
&\textbf{(−10)} && \wp_{556}^2 = \wp_{46}^2 - 6\wp_{25} - 4\wp_{35}\wp_{56} + 4\wp_{55}\wp_{56}^2 + \tfrac{4}{3}\wp_{56}Q_{5556} - \wp_{44}\wp_{66} \\
&&& \quad + 2Q_{4466} - \tfrac{1}{3}\wp_{66}Q_{5555} + 4\wp_{33} - 4\wp_{35}\lambda_6 + 4\wp_{36}\wp_{55} + 4\wp_{55}\lambda_5 \\
&&& \quad - 4\wp_{55}\lambda_6^2 + \tfrac{4}{3}\lambda_6 Q_{5556} \\
&\textbf{(−10)} && \wp_{466}\wp_{566} = 2\wp_{46}^2 - \wp_{25} + 2\lambda_6\wp_{55}\wp_{56} + 4\wp_{66}\wp_{46}\wp_{56} + \wp_{45}\wp_{55} + Q_{4466} \\
&&& \quad + 2\wp_{66}\wp_{46}\lambda_6 - \tfrac{2}{3}\wp_{56}Q_{5556} \\
&\textbf{(−10)} && \wp_{555}\wp_{566} = -2\wp_{46}^2 + \wp_{25} + 3\lambda_6\wp_{55}\wp_{56} + 4\wp_{35}\wp_{56} + 2\wp_{55}\wp_{56}^2 + 2\wp_{55}\lambda_6^2 \\
&&& \quad + 2\wp_{44}\wp_{66} - \wp_{45}\wp_{55} - Q_{4466} + 2\wp_{55}^2\wp_{66} - \tfrac{2}{3}\wp_{56}Q_{5556} \\
&&& \quad - \tfrac{2}{3}\lambda_6 Q_{5556} + 2\wp_{35}\lambda_6 + \tfrac{2}{3}\wp_{66}Q_{5555} \\
&\textbf{(−10)} && \wp_{456}\wp_{666} = -Q_{4466} - 2\wp_{44}\wp_{66} - \lambda_6\wp_{55}\wp_{56} - 2\wp_{45}\wp_{55} + 2\wp_{66}\wp_{46}\wp_{56} \\
&&& \quad + 2\wp_{45}\wp_{66}^2 + \tfrac{1}{3}\wp_{56}Q_{5556} - \wp_{25} + 2\wp_{66}\wp_{46}\lambda_6 \\
&\textbf{(−10)} && \wp_{366}\wp_{666} = -\wp_{25} + \wp_{35}\wp_{56} - \tfrac{1}{2}\wp_{44}\wp_{66} - Q_{4466} - \tfrac{1}{6}\wp_{66}Q_{5555} - 2\wp_{36}\wp_{55} \\
&&& \quad + \wp_{66}\wp_{46}\lambda_6 + 4\wp_{36}\wp_{66}^2 - \tfrac{1}{2}\wp_{66}^2\lambda_6^2 + 2\wp_{66}^2\lambda_5
\end{aligned}$$

$$\begin{aligned}
\textbf{(−11)} \qquad \wp_{555}\wp_{556} &= -\tfrac{2}{3}Q_{3466} + 2\wp_{34}\wp_{66} - 2\wp_{36}\wp_{46} + 4\wp_{55}^2\wp_{56} + \tfrac{2}{3}\wp_{55}Q_{5556} \\
&\quad + \tfrac{2}{3}Q_{4456} - \wp_{44}\wp_{56} + \wp_{45}\wp_{46} + \tfrac{1}{3}\wp_{56}Q_{5555} + \tfrac{1}{2}\wp_{66}\lambda_6^3 \\
&\quad + \tfrac{1}{6}\lambda_6 Q_{5555} - \tfrac{2}{3}\wp_{46}\lambda_5 + 2\tfrac{2}{3}\wp_{66}\lambda_4 + \wp_{56}\wp_{46}\lambda_6 - \wp_{66}\wp_{45}\lambda_6 \\
&\quad + \wp_{46}\lambda_6^2 - \tfrac{3}{2}\wp_{44}\lambda_6 - 2\wp_{66}\lambda_6\lambda_5 \\
\textbf{(−11)} \qquad \wp_{466}\wp_{556} &= \tfrac{2}{3}Q_{3466} + 2\wp_{34}\wp_{66} + 2\wp_{36}\wp_{46} - \tfrac{2}{3}Q_{4456} - 2\wp_{44}\wp_{56} - \wp_{45}\wp_{46} \\
&\quad + \tfrac{2}{3}\wp_{46}\lambda_5 + \tfrac{2}{3}\wp_{66}\lambda_4 + 2\wp_{66}\wp_{46}\wp_{55} + 2\lambda_6\wp_{55}^2 - \tfrac{2}{3}\wp_{55}Q_{5556} \\
&\quad + 2\wp_{46}\wp_{56}^2 - \wp_{66}\wp_{45}\lambda_6 + 2\wp_{56}\wp_{46}\lambda_6 \\
\textbf{(−11)} \qquad \wp_{456}\wp_{566} &= -\tfrac{4}{3}Q_{3466} - 2\wp_{34}\wp_{66} + 2\wp_{36}\wp_{46} + \tfrac{1}{3}\wp_{55}Q_{5556} + 2\wp_{66}\wp_{45}\wp_{56} \\
&\quad - \tfrac{2}{3}Q_{4456} + 2\wp_{46}\wp_{56}^2 - \lambda_6\wp_{55}^2 + \tfrac{14}{3}\wp_{46}\lambda_5 + \tfrac{2}{3}\wp_{66}\lambda_4 \\
&\quad + 2\wp_{56}\wp_{46}\lambda_6 + 2\wp_{66}\wp_{45}\lambda_6 \\
\textbf{(−11)} \qquad \wp_{455}\wp_{666} &= \tfrac{2}{3}Q_{3466} - 2\wp_{34}\wp_{66} - 6\wp_{36}\wp_{46} - \wp_{56}\wp_{46}\lambda_6 - \tfrac{10}{3}\wp_{46}\lambda_5 \\
&\quad + \wp_{44}\wp_{56} + 6\wp_{45}\wp_{46} + 6\wp_{26}\wp_{66} + 4\wp_{66}\wp_{45}\wp_{56} \\
&\quad - 2\wp_{46}\wp_{56}^2 + 2\lambda_6\wp_{55}^2 - \tfrac{10}{3}\wp_{66}\lambda_4 + 2\wp_{66}\wp_{45}\lambda_6 \\
&\quad + \tfrac{4}{3}Q_{4456} - \tfrac{2}{3}\wp_{55}Q_{5556} + 2\wp_{66}\wp_{46}\wp_{55} \\
\textbf{(−11)} \qquad \wp_{366}\wp_{566} &= -\tfrac{4}{3}Q_{3466} + 2\wp_{36}\wp_{46} - \tfrac{2}{3}Q_{4456} - \tfrac{1}{2}\wp_{44}\wp_{56} + \wp_{35}\wp_{55} \\
&\quad + 4\wp_{66}\wp_{36}\wp_{56} + 1\tfrac{4}{3}\wp_{46}\lambda_5 + \tfrac{14}{3}\wp_{66}\lambda_4 - \tfrac{1}{2}\wp_{56}\wp_{66}\lambda_6^2 \\
&\quad + 2\wp_{66}\wp_{36}\lambda_6 + 2\wp_{56}\wp_{66}\lambda_5 - \tfrac{1}{6}\wp_{56}Q_{5555} + \wp_{56}\wp_{46}\lambda_6 \\
\textbf{(−11)} \qquad \wp_{356}\wp_{666} &= \tfrac{2}{3}Q_{3466} - 2\wp_{34}\wp_{66} + \tfrac{4}{3}Q_{4456} + \tfrac{1}{4}\wp_{44}\wp_{56} - 2\wp_{35}\wp_{55} \\
&\quad + \tfrac{1}{12}\wp_{56}Q_{5555} + 2\wp_{66}\wp_{36}\wp_{56} + 2\wp_{35}\wp_{66}^2 - \tfrac{10}{3}\wp_{46}\lambda_5 \\
&\quad + \tfrac{1}{4}\wp_{56}\wp_{66}\lambda_6^2 - \tfrac{1}{2}\wp_{56}\wp_{46}\lambda_6 + 2\wp_{66}\wp_{36}\lambda_6 - \wp_{56}\wp_{66}\lambda_5 \\
&\quad - \tfrac{10}{3}\wp_{66}\lambda_4 + 4\wp_{26}\wp_{66}
\end{aligned}$$

The corresponding Maple worksheets for the cyclic (4,5) and (3,7)-curves are available in the extra Appendix of files.

5.4 Calculating the second basis in the (3,5)-case

Recall Definition 3.5.1 for the vector spaces of Abelian functions associated to (n,s)-curves. The simplest such space is $\Gamma\big(J, \mathcal{O}(2\Theta^{[g-1]})\big)$ which contains those function with poles of order at most two. Using the method set out in Chapter 3 for the (4,5)-case, we can identify a basis for this space using the 2-index $\wp$-functions and the Q-functions of increasing index. The next vector space to consider is $\Gamma\big(J, \mathcal{O}(3\Theta^{[g-1]})\big)$ for the functions with poles of order at most three. The basis for this space has been determined for the (3,4)-case in [30] but has yet to be considered for the (3,5)-case.

We can automatically include all those functions from the basis for $\Gamma\big(J, \mathcal{O}(2\Theta^{[g-1]})\big)$ since they trivially satisfy the conditions. We then need to look for the remaining basis functions from those with poles of order three. The 3-index $\wp$-functions may all be included, but this is not enough to complete the basis in general. (Recall that the dimension if $\Gamma\big(J, \mathcal{O}(3\Theta^{[g-1]})\big)$ will be 3^g from the Riemann-Roch theorem.) The aim of this section is to identify the remaining basis entries.

5.4.1 Possible functions for inclusion in the basis

We start by identifying sets of functions in the space $\Gamma\big(J, \mathcal{O}(3\Theta^{[g-1]})\big)$ which may therefore be used to construct the basis. We will recall the results for the (3,4)-curve to motivate our choice of functions. Lemma 8.1 in [30] identified the basis for $\Gamma\big(J, \mathcal{O}(2\Theta^{[2]})\big)$ as

$$\mathbb{C}1 \oplus \mathbb{C}\wp_{11} \oplus \mathbb{C}\wp_{12} \oplus \mathbb{C}\wp_{13} \oplus \mathbb{C}\wp_{22} \oplus \mathbb{C}\wp_{23} \oplus \mathbb{C}\wp_{33} \oplus \mathbb{C}Q_{1333}$$

and the basis for $\Gamma\big(J, \mathcal{O}(3\Theta^{[2]})\big)$ as

$$\begin{array}{cccccccccc}
 & \mathbb{C}1 & \oplus & \mathbb{C}\wp_{11} & \oplus & \mathbb{C}\wp_{12} & \oplus & \mathbb{C}\wp_{13} & \oplus & \mathbb{C}\wp_{22} \\
\oplus & \mathbb{C}\wp_{23} & \oplus & \mathbb{C}\wp_{33} & \oplus & \mathbb{C}Q_{1333} & \oplus & \mathbb{C}\wp_{111} & \oplus & \mathbb{C}\wp_{112} \\
\oplus & \mathbb{C}\wp_{113} & \oplus & \mathbb{C}\wp_{122} & \oplus & \mathbb{C}\wp_{123} & \oplus & \mathbb{C}\wp_{133} & \oplus & \mathbb{C}\wp_{222} \\
\oplus & \mathbb{C}\wp_{223} & \oplus & \mathbb{C}\wp_{233} & \oplus & \mathbb{C}\wp_{333} & \oplus & \mathbb{C}\wp^{[11]} & \oplus & \mathbb{C}\wp^{[12]} \\
\oplus & \mathbb{C}\wp^{[13]} & \oplus & \mathbb{C}\wp^{[22]} & \oplus & \mathbb{C}\wp^{[23]} & \oplus & \mathbb{C}\wp^{[33]} & & \\
\oplus & \mathbb{C}\partial_1 Q_{1333} & \oplus & \mathbb{C}\partial_2 Q_{1333} & \oplus & \mathbb{C}\partial_3 Q_{1333}. & & & &
\end{array} \tag{5.11}$$

Here, alongside the 3-index $\wp$-functions, the authors had also considered two other classes of functions with poles of order three. The first are the derivatives of the basis Q-functions, which we have denoted by

$$\partial_m Q_{ijkl} = \frac{\partial}{\partial u_m} Q_{ijkl}$$

for brevity. The derivatives of the basis Q-functions may be considered similarly for the (3,5)-case. (Note that we do not need to consider derivatives of the non-basis Q-functions. Since these Q-functions can be written as a linear combination of 2-index $\wp$-functions and basis Q-functions it follows that their derivatives may be written as a linear combination of 3-index $\wp$-functions and derivatives of the basis Q-functions.)

The second set of functions the authors of [30] considered were cross products of 2-index $\wp$-functions. Consider the matrix

$$\big[\wp_{ij}\big]_{3\times 3} = \begin{bmatrix} \wp_{11} & \wp_{12} & \wp_{13} \\ \wp_{21} & \wp_{22} & \wp_{23} \\ \wp_{31} & \wp_{32} & \wp_{33} \end{bmatrix} = \begin{bmatrix} \wp_{11} & \wp_{12} & \wp_{13} \\ \wp_{12} & \wp_{22} & \wp_{23} \\ \wp_{13} & \wp_{23} & \wp_{33} \end{bmatrix}.$$

Then the function $\wp^{[ij]}$ is defined to be the (i,j) minor of $[\wp_{ij}]_{3\times 3}$. (The determinant of the matrix obtained from $[\wp_{ij}]_{3\times 3}$ by eliminating row i and column j.) For example,

$$\wp^{[12]} = \begin{vmatrix} \wp_{12} & \wp_{23} \\ \wp_{13} & \wp_{33} \end{vmatrix} = \wp_{12}\wp_{33} - \wp_{23}\wp_{13}.$$

Each of these functions will be a sum of two pairs of products. While each term has poles of order four we find, (using the definition of the $\wp$-functions as σ-derivatives to check), that these always cancel to leave poles of order three.

We aim to redefine these functions for the (3,5)-case, which has genus $g = 4$. Define the matrix

$$\big[\wp_{ij}\big]_{4\times 4} = \begin{bmatrix} \wp_{11} & \wp_{12} & \wp_{13} & \wp_{14} \\ \wp_{12} & \wp_{22} & \wp_{23} & \wp_{24} \\ \wp_{13} & \wp_{23} & \wp_{33} & \wp_{34} \\ \wp_{14} & \wp_{24} & \wp_{34} & \wp_{44} \end{bmatrix}.$$

This time the minors would be cubic polynomials in the 2-index $\wp$-functions, which contain poles of order greater than three. However, we can generalise the concept by considering double minors of the matrix.

Definition 5.4.1. *Define the* ***double minor functions,*** $\wp^{[(\boldsymbol{ij}),(\boldsymbol{kl})]}$ *to be the determinant of the* 2×2 *submatrix of* $[\wp_{ij}]_{4\times 4}$ *formed from rows* i *and* j *and columns* k *and* l.

For example,

$$\wp^{[(12),(34)]} = \begin{vmatrix} \wp_{13} & \wp_{14} \\ \wp_{23} & \wp_{24} \end{vmatrix} = \wp_{13}\wp_{24} - \wp_{14}\wp_{23}.$$

We can easily check, by substituting the $\wp$-functions for σ-derivatives, that all the double minor functions associated to a genus four curve have poles of order three and no higher. Note that we need to impose the conditions

$$i \in \{1,\dots,3\}, \quad j \in \{i+1,\dots,4\}, \quad k \in \{1,\dots,3\}, \quad l \in \{k+1,\dots,4\},$$

to ensure we always use 2×2 determinants.

So we have identified the functions that correspond to those used to find a basis for $\Gamma\big(J, \mathcal{O}(3\Theta^{[g-1]})\big)$ in the (3,4)-case. However, we find these are not sufficient to complete the basis in the (3,5)-case and hence we incorporate some new functions, inspired by the earlier results of this section.

We may use the B-functions defined in Definition 5.2.1 since in Remark 5.2.2(i) we concluded that these belonged to $\Gamma\big(J, \mathcal{O}(3\Theta^{[g-1]})\big)$. In fact, we have already identified in equation (5.7) which B-functions are required to express all the others as a linear combination of these and the 3-index $\wp$-functions. Hence we may consider only this set of 19 functions in the construction of the basis for the (3,5)-case.

Next recall Remark 5.3.4 where we stated that the poles of order greater than three in $\wp_{ijk}\wp_{lmn}$ may be canceled using a polynomial involving the Q-functions. We specify this result in the theorem below.

Note that this theorem holds for any (n,s)-curve. The Maple worksheet where it was derived can be found in the extra Appendix of files.

Theorem 5.4.2. *The function $\mathcal{T}_{ijklmn}$ defined below has poles of order at most three, occurring when $\sigma(\boldsymbol{u}) = 0$.*

$$\begin{aligned}
\mathcal{T}_{ijklmn} &= \wp_{ijk}\wp_{lmn} - \tfrac{2}{3}\wp_{ij}\wp_{kl}\wp_{mn} - \tfrac{2}{3}\wp_{ij}\wp_{km}\wp_{ln} - \tfrac{2}{3}\wp_{ij}\wp_{kn}\wp_{lm} - \tfrac{2}{3}\wp_{ik}\wp_{jl}\wp_{mn} \\
&\quad - \tfrac{2}{3}\wp_{ik}\wp_{jm}\wp_{ln} - \tfrac{2}{3}\wp_{ik}\wp_{jn}\wp_{lm} - \tfrac{2}{3}\wp_{il}\wp_{jk}\wp_{mn} + \tfrac{1}{3}\wp_{il}\wp_{jm}\wp_{kn} + \tfrac{1}{3}\wp_{il}\wp_{jn}\wp_{km} \\
&\quad - \tfrac{2}{3}\wp_{im}\wp_{jk}\wp_{ln} + \tfrac{1}{3}\wp_{im}\wp_{jl}\wp_{kn} + \tfrac{1}{3}\wp_{im}\wp_{jn}\wp_{kl} - \tfrac{2}{3}\wp_{in}\wp_{jk}\wp_{lm} + \tfrac{1}{3}\wp_{in}\wp_{jl}\wp_{km} \\
&\quad + \tfrac{1}{3}\wp_{in}\wp_{jm}\wp_{kl} - \tfrac{2}{3}Q_{ijkl}\wp_{mn} - \tfrac{2}{3}Q_{ijkm}\wp_{ln} - \tfrac{2}{3}Q_{ijkn}\wp_{lm} + \tfrac{1}{3}Q_{ijlm}\wp_{kn} \\
&\quad + \tfrac{1}{3}Q_{ijln}\wp_{km} + \tfrac{1}{3}Q_{ijmn}\wp_{kl} + \tfrac{1}{3}Q_{iklm}\wp_{jn} + \tfrac{1}{3}Q_{ikln}\wp_{jm} + \tfrac{1}{3}Q_{ikmn}\wp_{jl} \\
&\quad - \tfrac{2}{3}Q_{ilmn}\wp_{jk} + \tfrac{1}{3}Q_{jklm}\wp_{in} + \tfrac{1}{3}Q_{jkln}\wp_{im} + \tfrac{1}{3}Q_{jkmn}\wp_{il} - \tfrac{2}{3}Q_{jlmn}\wp_{ik} - \tfrac{2}{3}Q_{klmn}\wp_{ij}
\end{aligned}$$

Proof. Substitute the Q and $\wp$-functions for their definitions in σ-derivatives. We find that $\mathcal{T}_{ijklmn}$ may be expressed as a sum of rational functions in σ-derivatives, with denominators $\sigma(\boldsymbol{u})^k$ where $k \leq 3$.

■

Hence we also consider the functions $\mathcal{T}_{ijklmn}$ for inclusion in the basis. To avoid repetition we set $i \leq j \leq k \leq l \leq m \leq n$.

5.4.2 Deriving basis entries

We derive a basis for $\Gamma\big(J, \mathcal{O}(3\Theta^{[3]})\big)$ associated with the cyclic (3,5)-curve as follows. We start by including the $2^g = 16$ functions with poles of order two that were identified as a basis for $\Gamma\big(J, \mathcal{O}(2\Theta^{[g-1]})\big)$. This basis was presented in [11] as

$$\mathbb{C}\wp_{11} \oplus \mathbb{C}\wp_{12} \oplus \mathbb{C}\wp_{13} \oplus \mathbb{C}\wp_{14}\mathbb{C}\wp_{22} \oplus \mathbb{C}\wp_{23} \oplus \mathbb{C}\wp_{24} \oplus \mathbb{C}\wp_{33} \oplus \mathbb{C}\wp_{34} \oplus \mathbb{C}\wp_{44}$$
$$\oplus \mathbb{C}Q_{2444} \oplus \mathbb{C}Q_{1444} \oplus \mathbb{C}Q_{2233} \oplus \mathbb{C}Q_{1244} \oplus \mathbb{C}Q_{1144} \oplus \mathbb{C}1.$$

We then identify the elements with poles of order three by considering successive weight levels in decreasing weight order. We use the σ-expansion to see if the possible functions can be written as a linear combination of existing basis entries, or if new entries need to be added. We use the following procedure, presented for weight level $-k$.

1. Start by forming a sum of existing basis entries, each multiplied by an undetermined constant coefficient, c_i. Include only those which may be combined with an appropriate $\boldsymbol{\lambda}$-monomial to give weight $-k$ overall. (Note that all the possible elements of a higher weight have already been determined since we are working systematically in decreasing weights.)

   From a simple extension of Lemma 3.5.2 we need not consider basis entries multiplied by rational functions in the $\boldsymbol{\lambda}$.

2. Add to this a sum of functions with poles of order three at weight $-k$, each multiplied by an undetermined constant coefficient, q_i. We choose from the following:

   - The 3-index $\wp$-functions.
   - The derivatives of the basis Q functions.
   - The double minor functions $\wp^{[(ij),(kl)]}$ given in Definition 5.4.1.
   - The set of 19 B-functions given in equation (5.7) and defined in Definition 5.2.1.
   - The functions $\mathcal{T}_{ijklmn}$ defined above in Theorem 5.4.2.

3. Substitute the functions for their definitions as σ-derivatives. This will give a sum of rational function which may be factored to leave $\sigma(\boldsymbol{u})^3$ on the denominator. Take the numerator, which should be a sum of products of triplets of σ-derivatives.

4. Substitute $\sigma(\boldsymbol{u})$ in this sum for the σ-expansion about the origin. Note that the sum contains $\boldsymbol{\lambda}$-monomials with weight no lower than $-k$. Hence we may truncate the σ-expansion after the polynomial which contains $\boldsymbol{\lambda}$-monomials of weight $-k$. Evaluate the σ-derivatives as derivatives of this expansion.

5. Expand the products to obtain a polynomial. Note that this will create terms with $\boldsymbol{\lambda}$-monomials of a lower weight than $-k$, which must be discarded.

   As before, this step is the most computationally difficult. We may simplify using the procedure for multiplying triplets discussed in the derivation of the bilinear equations above.

To save further computation time and memory distributed computing may be employed to expand the products in parallel.

6. Collect this polynomial into a sum of the various $\boldsymbol{u\lambda}$-monomials with coefficients linear in the unknown constants $\{c_i, q_i\}$.
7. Consider these coefficients as a series of linear equations in the unknowns that must equal zero and solve for the unknowns.
   - If there is a unique solution for the c_i in terms of the q_i then each of the functions at weight $-k$ may be expressed as a linear combination of existing basis entries.
   - If this is not possible then add the necessary functions to the basis.

We have implemented this procedure in Maple and the corresponding Maple worksheet may be found in the extra Appendix of files. It has allowed us to identify 80 of the $3^4 = 81$ entries in the following basis.

$$
\begin{array}{cccccccccc}
 & \mathbb{C}1 & \oplus & \mathbb{C}\wp_{11} & \oplus & \mathbb{C}\wp_{12} & \oplus & \mathbb{C}\wp_{13} & \oplus & \mathbb{C}\wp_{14} \\
\oplus & \mathbb{C}\wp_{22} & \oplus & \mathbb{C}\wp_{23} & \oplus & \mathbb{C}\wp_{24} & \oplus & \mathbb{C}\wp_{33} & \oplus & \mathbb{C}\wp_{34} \\
\oplus & \mathbb{C}\wp_{44} & \oplus & \mathbb{C}Q_{2444} & \oplus & \mathbb{C}Q_{1444} & \oplus & \mathbb{C}Q_{2233} & \oplus & \mathbb{C}Q_{1244} \\
\oplus & \mathbb{C}Q_{1144} & \oplus & \mathbb{C}\wp_{111} & \oplus & \mathbb{C}\wp_{112} & \oplus & \mathbb{C}\wp_{113} & \oplus & \mathbb{C}\wp_{114} \\
\oplus & \mathbb{C}\wp_{122} & \oplus & \mathbb{C}\wp_{123} & \oplus & \mathbb{C}\wp_{124} & \oplus & \mathbb{C}\wp_{133} & \oplus & \mathbb{C}\wp_{134} \\
\oplus & \mathbb{C}\wp_{144} & \oplus & \mathbb{C}\wp_{222} & \oplus & \mathbb{C}\wp_{223} & \oplus & \mathbb{C}\wp_{224} & \oplus & \mathbb{C}\wp_{233} \\
\oplus & \mathbb{C}\wp_{234} & \oplus & \mathbb{C}\wp_{244} & \oplus & \mathbb{C}\wp_{333} & \oplus & \mathbb{C}\wp_{334} & \oplus & \mathbb{C}\wp_{344} \\
\oplus & \mathbb{C}\wp_{444} & \oplus & \mathbb{C}\partial_4 Q_{2444} & \oplus & \mathbb{C}\partial_3 Q_{2444} & \oplus & \mathbb{C}\partial_4 Q_{1444} & \oplus & \mathbb{C}\partial_2 Q_{2444} \\
\oplus & \mathbb{C}\partial_3 Q_{1444} & \oplus & \mathbb{C}\partial_4 Q_{2233} & \oplus & \mathbb{C}\partial_4 Q_{1244} & \oplus & \mathbb{C}\partial_4 Q_{1144} & \oplus & \mathbb{C}\partial_1 Q_{1444} \\
\oplus & \mathbb{C}\partial_2 Q_{1244} & \oplus & \mathbb{C}\partial_3 Q_{1244} & \oplus & \mathbb{C}\partial_2 Q_{2233} & \oplus & \mathbb{C}\partial_3 Q_{1144} & \oplus & \mathbb{C}\partial_1 Q_{2233} \\
\oplus & \mathbb{C}\partial_1 Q_{1244} & \oplus & \mathbb{C}\partial_2 Q_{1144} & \oplus & \mathbb{C}\partial_1 Q_{1144} & \oplus & \mathbb{C}\wp^{[(34),(34)]} & \oplus & \mathbb{C}\wp^{[(24),(34)]} \\
\oplus & \mathbb{C}\wp^{[(23),(34)]} & \oplus & \mathbb{C}\wp^{[(24),(24)]} & \oplus & \mathbb{C}\wp^{[(23),(24)]} & \oplus & \mathbb{C}\wp^{[(14),(34)]} & \oplus & \mathbb{C}\wp^{[(13),(34)]} \\
\oplus & \mathbb{C}\wp^{[(23),(23)]} & \oplus & \mathbb{C}\wp^{[(14),(24)]} & \oplus & \mathbb{C}\wp^{[(13),(24)]} & \oplus & \mathbb{C}\wp^{[(14),(23)]} & \oplus & \mathbb{C}\wp^{[(13),(23)]} \\
\oplus & \mathbb{C}\wp^{[(14),(14)]} & \oplus & \mathbb{C}\wp^{[(12),(24)]} & \oplus & \mathbb{C}\wp^{[(13),(14)]} & \oplus & \mathbb{C}\wp^{[(12),(23)]} & \oplus & \mathbb{C}\wp^{[(13),(13)]} \\
\oplus & \mathbb{C}\wp^{[(12),(14)]} & \oplus & \mathbb{C}\wp^{[(12),(13)]} & \oplus & \mathbb{C}\wp^{[(12),(12)]} & \oplus & \mathbb{C}B^{A}_{22244} & \oplus & \mathbb{C}B^{B}_{12444} \\
\oplus & \mathbb{C}\mathcal{T}_{133344} & \oplus & \mathbb{C}\mathcal{T}_{114444} & \oplus & \mathbb{C}\mathcal{T}_{222233} & \oplus & \mathbb{C}\mathcal{T}_{222224} & \oplus & \mathbb{C}\mathcal{T}_{222222} \\
\oplus & ? & & & & & & & &
\end{array}
$$

Note that the derivatives of basis Q-functions have been included instead of many of the B-functions used in Section 5.2.4. However, they occur at the same weight levels and so play the same role. The basis currently includes 41 even functions and 39 odd functions, but it is not known what parity the final function should have. An additional class of functions will be required to identify the final basis element. It is possible that the identification of such a function may aid the derivation of the missing quadratic 3-index relations at weight -24. Clearly, this is an important problem for further research.

Chapter 6

Reductions of the Benney equations

6.1 Introduction

This Chapter develops an application of the theory of Abelian functions to a problem in non-linear differential equations. An explicit example is presented which uses the results of Chapter 3 for the multiply periodic functions associated with the cyclic (4,5)-curve.

The material in this chapter was recently summarised and published in [40] which was co authored by Dr. John Gibbons. Some of the more lengthly results discussed here may be found in the extra Appendix of files, or online at [37]. The Maple worksheets in which the results were calculated are also available in the extra Appendix of files.

In [48] and [49] it was shown that Benney's moment equations

$$A^n_t = A^{n+1}_x + nA^{n-1}A^0_x, \qquad n \geq 0,$$

admit *reductions* in which only finitely many N of the moments A^n are independent. Further, it was shown that a large class of such reductions may be parametrised by conformal maps from an upper half plane to a slit domain — an upper half plane cut along N non intersecting curves with one fixed end point on the real line and the other *free end* a Riemann invariant of the reduced equations.

A natural subclass of these occur where the curves of the slit domain are straight lines, leading to a polygonal domain and an N-parameter Schwartz-Christoffel map. An important and tractable subfamily of these is the case in which the angles are all rational multiples of π. In [73] it was shown that this subclass leads to a mapping given by an integral of a second kind Abelian differential on an algebraic curve. Such examples have been worked out explicitly, in [73], [12], [13] and [14]. In all these examples the underlying curves have been from the class of (n, s)-curves, with the reductions constructed explicitly by evaluating both the integrand and its integral using quotients of σ-derivatives associated with the respective curves.

These papers have considered elliptic and hyperelliptic curves as well as a cyclic trigonal example. In this chapter we generalise the approach to the tetragonal curve that was discussed in Chapter 3. We develop new methods which should now be easily applied to reductions related to any (n, s)-curve.

This chapter is organised as follows. First background information of the Benney equations and the reductions is presented in Section 6.2. Then, in Section 6.3, the particular reduction associated with the tetragonal curve is developed, with the necessary theory introduced. In Section 6.4 we derive relations between the σ-derivatives that hold on the various strata of the Jacobian, with the relations themselves available in Appendix E. These relations allows us to evaluate the mapping integrand in Section 6.5 and obtain an explicit formula in Section 6.6.

6.2 Benney's equations

In 1973 Benney published [17] which considered an approximation for the two-dimensional equations of motion of an incompressible perfect fluid under a gravitational force. He showed that if moments are defined by

$$A_n(x,t) = \int_0^h v^n \, \mathrm{d}y, \tag{6.1}$$

where $v(x,y,t)$ is the horizontal fluid velocity and $h(x,t)$ the height of the free surface, then the moments $A_n(x,t)$ satisfy an infinite set of hydrodynamic type equations, now called the Benney moment equations.

Definition 6.2.1. *The* ***Benney moment equations*** *are the following infinite set of partial differential equations.*

$$\frac{\partial A_n}{\partial t} + \frac{\partial A_{n+1}}{\partial x} + n\, A_{n-1}\, \frac{\partial A_0}{\partial x} = 0 \qquad\qquad n = 1, 2, \ldots \tag{6.2}$$

Identical moment equations were alternatively derived from a Vlasov equation in [44], [74]. In this case the moments were defined by

$$A_n = \int_{-\infty}^{\infty} p^n f \, dp \tag{6.3}$$

where $f = f(x,q,t)$ is a distribution function.

Benney showed in [17] that system (6.2) has infinitely many conserved densities, polynomial in the A_n. One of the most direct ways to calculate these is to use generating functions as described in [52]. Let $q(x,p,t)$, be the series in p which acts as a generating function of the moments,

$$q(x,p,t) = p + \sum_{n=0}^{\infty} \frac{A_n}{p^{n+1}}, \tag{6.4}$$

and let $p(x,q,t)$ be the inverse series,

$$p(x,q,t) = q - \sum_{m=0}^{\infty} \frac{H_m}{q^{m+1}}.$$

We note here that if equation (6.3) is substituted into equation (6.4), then we obtain the asymptotic series, as $p \to \infty$, of an integral,

$$q = p + \int_{-\infty}^{\infty} \frac{f(x,p',t)}{(p-p')} \, dp'. \tag{6.5}$$

Here p' runs along the real axis, and we take $\mathrm{Im}(p) > 0$. It follows that $q(p)$ is holomorphic in its domain of definition.

Comparing the first derivatives of $q(x,p,t)$, we obtain the differential equation

$$\frac{\partial q}{\partial t} + p\,\frac{\partial q}{\partial x} = \frac{\partial q}{\partial p}\left(\frac{\partial p}{\partial t} + p\,\frac{\partial p}{\partial x} + \frac{\partial A_0}{\partial x}\right). \tag{6.6}$$

If we now hold p constant, we obtain the same form of Vlasov equation from which the moments were defined. Hence any function of q and f must satisfy the same equation.

Alternatively, if we hold q constant in equation (6.6), then we obtain the conservation equation,

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}p^2 + A_0\right) = 0. \tag{6.7}$$

Substituting the formal series of $p(x,q,t)$ into equation (6.7), we see that each H_n is polynomial in the A_n and is a conserved density.

6.2.1 Reductions of the moment equations

Suppose that for some family of points, $p = \hat{p}_i(x,t),\ q(\hat{p}_i) = \hat{q}_i(x,t)$, we have

$$\left.\frac{\partial q}{\partial p}\right|_{p=\hat{p}_i} = 0.$$

Then equation (6.6) reduces to

$$\frac{\partial \hat{q}_i}{\partial t} + \hat{p}_i\,\frac{\partial \hat{q}_i}{\partial x} = 0,$$

where

$$\frac{\partial \hat{q}_i}{\partial t} = \left.\frac{\partial q}{\partial t}\right|_{p=\hat{p}_i} \qquad \text{and} \qquad \frac{\partial \hat{q}_i}{\partial x} = \left.\frac{\partial q}{\partial x}\right|_{p=\hat{p}_i}.$$

We say that $\hat{q}_i$ is a *Riemann invariant* with characteristic speed $\hat{p}_i$. We will see that there are families of functions $q(p)$ which are invariant under the Benney dynamics, and are parametrised by N Riemann invariants q_i.

A hydrodynamic type system with $N \geq 3$ independent variables can not in general be expressed in terms of Riemann invariants. If such a system does have N Riemann invariants, it is called *diagonalisable*. Using results from Tsarev in [66] we can conclude that any system of this type can be solved in principle using the method of a *hodograph transformation*.

This construction cannot easily be applied to the Benney equations however, as these have infinitely many dependent variables. Instead we must consider families of distribution functions f, which are parameterised by finitely many, N, Riemann invariants $\hat{q}_i(x,t)$.

Definition 6.2.2. *Consider the case where the function $q(p,x,t)$ is such that only N of the moments are independent. In [48] and [49] we find this implies N characteristic speeds, assumed real and distinct, and N corresponding Riemann invariants $(\hat{p}_i, \hat{q}_i)$. In this case Benney's equations reduce to a diagonal system of hydrodynamic type with finitely many* *dependent variables $\hat{q}_i$.*

$$\frac{\partial \hat{q}_i}{\partial t} + \hat{p}_i(\hat{q})\frac{\partial \hat{q}_i}{\partial x} = 0, \qquad i = 1, 2, \ldots, N. \tag{6.8}$$

*Such a system is called a **reduction of Benney's equations**.*

The construction of a general family of solutions for equations of this type was outlined in [48] and [49].

6.2.2 Schwartz-Christoffel reductions

Recall that q must map from the upper half p-plane to an upper half plane. Explicit solutions have been constructed in the case where the mapping function maps to a polygonal N-slit domain.

Definition 6.2.3. *We define **polygonal N-slit domain** as follows. Suppose the real p-axis has M vertices marked on it and the preimage in the p-plane of each slit runs from a vertex $\hat{p}_j$ to a point $\hat{v}_i$, the preimage of the end of the slit, and then to another vertex $\hat{p}_{j+1}$. The angle π in the p-plane at $\hat{p}_j$ is mapped to an angle $\alpha_j\pi$ at the image point. The internal angle at the end of each slit is 2π.*

In this case the mapping function is shown to be given, up to a constant of integration, by

$$q = \int^p \left[\frac{\prod_{i=1}^{N}(p - \hat{v}_i)}{\prod_{j=1}^{2N}(p - \hat{p}_j)^{1-\alpha_j}}\right] dp.$$

If the integrand is to converge to one as $p \to \infty$, we require

$$\sum_{j=1}^{2N} \alpha_j = N.$$

To avoid a logarithmic singularity, we further impose,

$$\sum_{j=1}^{M} \alpha_j \hat{p}_j = \sum_{i=1}^{N} \hat{v}_i.$$

We may then define q more precisely as follows.

Definition 6.2.4. *Define q to be a **Schwartz-Christoffel mapping** if it is of the form*

$$q = p + \int_{-\infty}^{p} \left[\frac{\prod_{i=1}^{N}(p - \hat{v}_i)}{\prod_{j=1}^{M}(p - \hat{p}_j)^{1-\alpha_j}} - 1\right] dp.$$

Other constraints are imposed by requiring the vertices $\hat{p}_j$ to map to points q_i^0, the fixed base points of the slits. There remain N independent parameters, which can be taken to be the movable end points of the slits $q_i(x,t)$. These satisfy the equations of motion

$$\frac{\partial q_i}{\partial t} + \hat{v}_i \frac{\partial q_i}{\partial x} = 0.$$

Hence, to use the general solution to these equations we must evaluate the Schwartz-Christoffel integral explicitly.

Example 6.2.5. An elementary example is the case where the map takes the upper half p-plane to the upper half q-plane with a vertical slit, as shown in Figure 6.1.

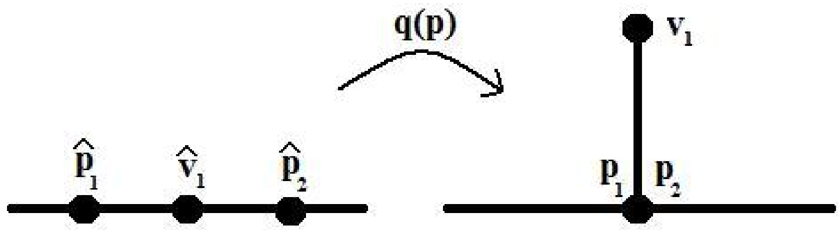

Figure 6.1: The domain and codomain of the map in Example 6.2.5

This relates to the following Schwarz-Christoffel map.

$$q(p) = p + \int_{\infty}^{p} \frac{p' - \hat{v}_1}{\sqrt{(p' - \hat{p}_1)(p' - \hat{p}_2)}} \, \mathrm{d}p'.$$

If the residue at infinity is set to be zero, then this imposes the condition $\hat{v}_1 = \frac{1}{2}(\hat{p}_1 + \hat{p}_2)$ and we get the solution

$$\begin{aligned} q(p) &= \hat{v}_1 + \sqrt{p^2 - (\hat{p}_1 + \hat{p}_2)p + \hat{p}_1\hat{p}_2} \\ &= \hat{v}_1 + \sqrt{(p - \hat{v}_1)^2 + 2A_0}, \end{aligned}$$

(from the expansion as $p \to \infty$). The two parameters $\hat{p}_1$ and $\hat{p}_2$ are not independent, as for consistency their sum must be a constant. Hence only the end point of the slit in the q-plane is variable. This is the Riemann invariant.

□

The most tractable cases are where all the α_j are rational, so that the integrand becomes a meromorphic second kind differential on some algebraic curve. In this case the only singularity is as $p \to \infty$, where the integrand has a double pole with no residue, and the integral thus has a simple pole.

The integral has been calculated explicitly in [73], [12], [13] and [14], in each case using the theory of (n,s)-curves. These papers have considered elliptic and hyperelliptic curves as well as a cyclic trigonal example. In each case the mappings were found as rational functions of derivatives of the Kleinian σ-function of the associated curve.

6.3 A tetragonal reduction

We will consider reductions that allow us to work on a tetragonal surface, which has not been considered before in the context of this application. The first step is to specify the domain and codomain of the mapping q for this case. For the reduction to involve a tetragonal surface we will require two or more sets of straight slits, making angles of $\pi/4, \pi/2,$ and $3\pi/4$ to the horizontal.

Define $\mathcal{P}$ to be the upper half p-plane with 14 points marked on the real axis, as in Figure 6.2.

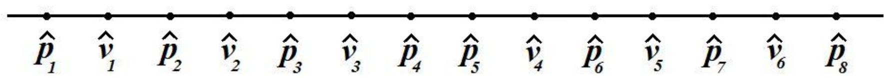

Figure 6.2: The domain $\mathcal{P}$ within the p-plane.

These points satisfy

$$\hat{p}_1 < \hat{v}_1 < \hat{p}_2 < \hat{v}_2 < \hat{p}_3 < \hat{v}_3 < \hat{p}_4 < \hat{p}_5 < \hat{v}_4 < \hat{p}_6 < \hat{v}_5 < \hat{p}_7 < \hat{v}_6 < \hat{p}_8.$$

Then define the codomain $\mathcal{Q}'$ as the upper-half q-plane with two triplets of slits, as described above. We let the first trio of slits radiate from the fixed point p_1, with the end points of these three slits labeled v_1, v_2 and v_3 respectively. Similarly, let the second trio of slits radiate from p_5 and have end points v_4, v_5, v_6. Finally impose the conditions that

$$\begin{aligned} q(\hat{p}_i) &= p_i, \qquad p_1 = p_2 = p_3 = p_4, \\ q(\hat{v}_i) &= v_i, \qquad p_5 = p_6 = p_7 = p_8. \end{aligned}$$

We then see that $\mathcal{Q}'$ is the slit domain as shown in Figure 6.3, and the mapping $q : \mathcal{P} \to \mathcal{Q}'$ can be given in Schwartz Christoffel form by

$$q(p) = p + \int_{\infty}^{p} \left[\varphi(p') - 1\right] dp' \tag{6.9}$$

with

$$\varphi(p) = \frac{\prod_{i=1}^{6}(p - \hat{v}_i)}{\left[\prod_{i-1}^{8}(p - \hat{p}_i)\right]^{\frac{3}{4}}} = \frac{\prod_{i=1}^{6}(p - \hat{v}_i)}{Y^3}, \tag{6.10}$$

where

$$Y^4 = \prod_{i=1}^{8}(p - \hat{p}_i). \tag{6.11}$$

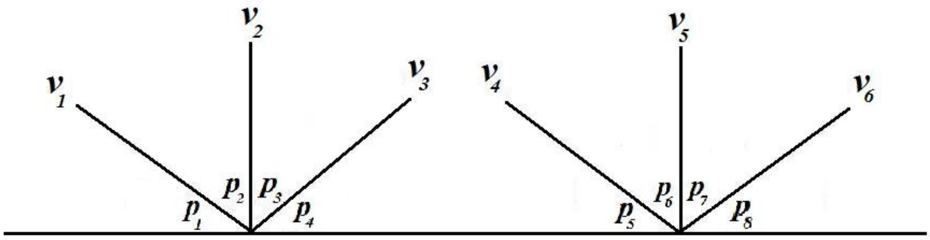

Figure 6.3: The domain $\mathcal{Q}'$ within the q-plane.

Note that we require the following zero residue property.

$$\lim_{p\to\infty} \varphi(p) \sim 1 + O\left(\frac{1}{p^2}\right). \tag{6.12}$$

This mapping would lead us to consider the Riemann surface given by points (p, Y) that satisfy equation (6.11). However, we wish to consider the simplest possible tetragonal surface (one with only six branch points) and so we collapse two of the slits, (the final two by choice). This simplifies our q-plane to $\mathcal{Q}$, given in Figure 6.4. Our problem will be to determine an explicit formula for the mapping $q(p)$ in this case.

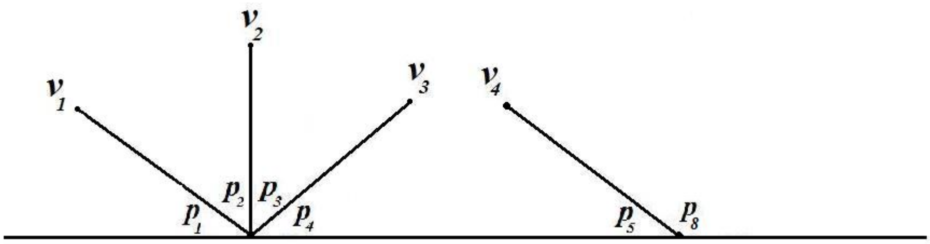

Figure 6.4: The domain $\mathcal{Q}$ within the q-plane.

As in the trigonal case, the analysis of this surface is eased if we put it into canonical form, by mapping one of the branch points, (p_8 by choice), to infinity.

We use the following invertible rational map to perform these simplifications on our curve and integrand.

$$\begin{aligned} &\hat{p}_6 = \hat{p}_8, \quad \hat{p}_7 = \hat{p}_8, \quad \hat{v}_5 = \hat{p}_8, \quad \hat{v}_6 = \hat{p}_8, \\ &p = \hat{p}_8 - \left(\frac{1}{x}\right), \qquad \hat{p}_i = \hat{p}_8 - \frac{1}{T_i}, \quad i = 1, \dots, 5, \\ &Y = \frac{yk}{x^2} \quad \text{where} \quad k^4 = -\prod_{i=1}^{5}(\hat{p}_8 - \hat{p}_i) = -\prod_{i=1}^{5}\frac{1}{T_i}. \end{aligned} \tag{6.13}$$

If we perform the mapping (6.13) on the curve (6.11) we obtain

$$\begin{aligned}\frac{y^4k^4}{x^8} &= \left[\prod_{i=1}^{5}\left(\hat{p}_8 - \frac{1}{x} - \hat{p}_8 + \frac{1}{T_i}\right)\right]\left(\hat{p}_8 - \frac{1}{x} - \hat{p}_8\right)^3 = -\frac{1}{x^3}\prod_{i=1}^{5}\left(\frac{1}{T_i} - \frac{1}{x}\right)\\ &= -\frac{1}{x^3}\left[\prod_{i=1}^{5}(x - T_i)\frac{1}{xT_i}\right] = \left[\prod_{i=1}^{5}(x - T_i)\right]\cdot\left(\frac{1}{x^8}\right)\cdot(-1)\left[\prod_{i=1}^{5}\frac{1}{T_i}\right].\end{aligned}$$

This simplifies to give the following canonical form of the curve (6.11).

$$\begin{aligned} y^4 &= \prod_{i=1}^{5}(x - T_i)\\ &= x^5 + \lambda_4x^4 + \lambda_3x^3 + \lambda_2x^2 + \lambda_1x + \lambda_0, \end{aligned} \tag{6.14}$$

for constants $\lambda_0, \ldots, \lambda_4$. Let C denote the Riemann surface defined by equation (6.14).

We now consider $q(p)$ as mapping $\mathcal{P} \to \mathcal{Q}$ by performing (6.13) on the integrand (6.10).

$$\begin{aligned}\varphi(p)dp &= \left(\frac{x^6}{y^3k^3}\right)\left[\prod_{i=1}^{4}\left(\hat{p}_8 - \frac{1}{x} - \hat{v}_i\right)\right]\left(\hat{p}_8 - \frac{1}{x} - \hat{p}_8\right)^2\left(-\frac{1}{x^2}dx\right)\\ &= \left(\frac{x^6}{y^3k^3}\right)\left[\prod_{i=1}^{4}((\hat{p}_8 - \hat{v}_i)x - 1)\right]\left[\frac{1}{x^4}\right]\left(\frac{1}{x}\right)^2\left(-\frac{1}{x^2}dx\right)\\ &= K\left[A_4x^4 + A_3x^3 + A_2x^2 + A_1x + 1\right]\frac{1}{x^2}\frac{dx}{4y^3} \equiv \varphi(x)dx, \end{aligned} \tag{6.15}$$

where $K = -4/k^3$ and $A_1, \ldots, A_4$ are constants. We consider this transformed integrand as varying with x instead of p and so denote it $\varphi(x)dx$. We will evaluate this integrand using Kleinian functions defined upon C.

The tetragonal surface C

Equation (6.14) is the cyclic tetragonal curve of genus six that was studied in detail in Chapter 3. The surface is constructed from four sheets of the complex plane, with branch points of order four at

$$T_1, T_2, T_3, T_4, T_5, T_6 = \infty,$$

and branch cuts along the intervals

$$[T_1, T_2], \quad [T_2, T_3], \quad [T_3, T_4], \quad [T_5, \infty].$$

In Section 3.1 we constructed the standard basis of holomorphic differentials upon C, denoted $\boldsymbol{du}$ and given in equation (3.4).

Any point $\boldsymbol{u} \in \mathbb{C}^6$ can be expressed as

$$\boldsymbol{u} = (u_1, u_2, u_3, u_4, u_5, u_6) = \sum_{i=1}^{6} \int_{\infty}^{P_i} \boldsymbol{du},$$

where the P_i are six variable points upon C.

In Chapter 3 we used the Kleinian σ-function defined in association with C to construct and investigate the Abelian functions associated with C. In this chapter we will use the σ-function to evaluate the integral $q(x)$.

Recall the Abel map, defined in Definition 2.2.13, that mapped points on the curve to the Jacobian. We denoted the image of the kth Abel map by $W^{[k]}$ and in Definition 2.2.14 defined strata of the Jacobian as

$$\Theta^{[k]} = W^{[k]} \cup [-1]W^{[k]}.$$

In fact, since this (n, s)-curve has $n = 4$ which is even the process is simplified and we have

$$W^{[k]} = [-1]W^{[k]} = \Theta^{[k]}.$$

When $k = 1$ the Abel map gives a one dimensional image of the curve C. Since our mapping was given by a single integral with respect to one parameter, it will make sense to rewrite this as an integral on the one-dimensional strata, $\Theta^{[1]}$. In addition to [12], [13] and [14], similar problems of inverting meromorphic differentials on lower dimensional strata of the Jacobian have been studied, in the case of hyperelliptic surfaces, in [41], [47], [1] and [2] for example.

6.4 Relations between the σ-derivatives

In Section 6.5 we will evaluate the integrand (6.15) as a function of σ-derivatives restricted to $\Theta^{[1]}$. In order to achieve this we will first need to derive equations that hold between the various σ-derivatives when $\boldsymbol{u} \in \Theta^{[1]}$. In Chapter 3 sets of relations between the Abelian functions associated with C were calculated. However, these were the relations that held everywhere, and do not give us sufficient information for the behaviour of $\sigma(\boldsymbol{u})$ on the various strata.

We start in Section 6.4.1 by deriving those relations that define the various strata. Then in Section 6.4.2 we obtain further relations from these through a technique we have developed known as *descending*. (Essentially since each relation on Θ^k is also valid on Θ^{k-1} we can consider the effect on it as $\boldsymbol{u}$ descends the strata.)

6.4.1 The defining strata relations

Recall from Lemma 2.2.19 that $\Theta^{[g-1]}$ may be defined as those places where $\sigma(\boldsymbol{u}) = 0$.

$$\Theta^{[5]} = \{\boldsymbol{u} \mid \sigma(\boldsymbol{u}) = 0\}.$$

We may derive similar defining relations for the other strata using the following theorem of Jorgenson, (from [50]).

Theorem 6.4.1. *Let $\boldsymbol{u} \in \Theta^{[k]}$ for some $k < g$. Then for a set of k points $P_i = (x_i, y_i)$ on C we have*

$$\boldsymbol{u} = \sum_{i=1}^{k} \int_{\infty}^{P_i} \boldsymbol{du},$$

and the following statement holds for vectors $\boldsymbol{a}, \boldsymbol{b}$ of arbitrary constants.

$$\frac{\sum_{j=1}^{g} a_j \sigma_j(\boldsymbol{u})}{\sum_{j=1}^{g} b_j \sigma_j(\boldsymbol{u})} = \frac{\det \left[\boldsymbol{a}\middle|\boldsymbol{du}(P_1)\middle|\cdots\middle|\boldsymbol{du}(P_k)\middle|\boldsymbol{du}(P_k)^{(g-k-1)}\middle|\cdots\middle|\boldsymbol{du}(P_k)^{(1)}\right]}{\det \left[\boldsymbol{b}\middle|\boldsymbol{du}(P_1)\middle|\cdots\middle|\boldsymbol{du}(P_k)\middle|\boldsymbol{du}(P_k)^{(g-k-1)}\middle|\cdots\middle|\boldsymbol{du}(P_k)^{(1)}\right]}.$$

Here, $\boldsymbol{du}^{(i)}$ denotes the column of ith derivatives of the holomorphic differentials $\boldsymbol{du}$, and should be ignored if $i < 1$.

Start by considering Theorem 6.4.1 in the case when $k = 5$. Then for arbitrary $\boldsymbol{a}, \boldsymbol{b}$

$$\frac{\sum_{j=1}^{6} a_j \sigma_j(\boldsymbol{u})}{\sum_{j=1}^{6} b_j \sigma_j(\boldsymbol{u})} = \frac{\det \left[\boldsymbol{a}\middle|\boldsymbol{du}(P_1)\middle|\cdots\middle|\boldsymbol{du}(P_5)\right]}{\det \left[\boldsymbol{b}\middle|\boldsymbol{du}(P_1)\middle|\cdots\middle|\boldsymbol{du}(P_5)\right]}.$$

Now, as $\boldsymbol{u} \in \Theta^{[5]}$ approaches $\Theta^{[4]}$ we must have one of the points P_i approaching ∞. Let $P_5 = (x_5, y_5)$ approach ∞ and use the local coordinate ξ to replace the final column of the determinants by the expansions for $\boldsymbol{du}$ in ξ. (These expansions were derived in Chapter 3 in equation (3.24) and are summarised below. However, for these calculations we will need more terms and so refer to the longer expansions presented in Appendix C.1).

$$\begin{aligned} du_1 &= [-\xi^{10} + O(\xi^{14})]d\xi, & du_4 &= [-\xi^2 + \tfrac{3}{4}\lambda_4\xi^6 + O(\xi^{10})]d\xi, \\ du_2 &= [-\xi^6 + O(\xi^{10})]d\xi, & du_5 &= [-\xi^1 + \tfrac{1}{2}\lambda_4\xi^5 + O(\xi^9)]d\xi, \\ du_3 &= [-\xi^5 + O(\xi^9)]d\xi, & du_6 &= [-1 + \tfrac{1}{4}\lambda_4\xi^4 + O(\xi^8)]d\xi. \end{aligned} \tag{6.16}$$

When $\boldsymbol{u}$ arrives at $\Theta^{[4]}$ we have $\xi = 0$ and hence the determinant in the numerator becomes

$$\begin{vmatrix} a_1 & \frac{dx_1}{4y_1^3} & \cdots & \frac{dx_4}{4y_4^3} & 0 \\ a_2 & \frac{x_1dx_1}{4y_1^3} & \cdots & \frac{x_4dx_4}{4y_4^3} & 0 \\ a_3 & \frac{y_1dx_1}{4y_1^3} & \cdots & \frac{y_4dx_4}{4y_4^3} & 0 \\ a_4 & \frac{x_1^2dx_1}{4y_1^3} & \cdots & \frac{x_4^2dx_4}{4y_4^3} & 0 \\ a_5 & \frac{x_1y_1dx_1}{4y_1^3} & \cdots & \frac{x_4y_4dx_4}{4y_4^3} & 0 \\ a_6 & \frac{y_1^2dx_1}{4y_1^3} & \cdots & \frac{y_4^2dx_4}{4y_4^3} & -1 \end{vmatrix} = \left(\frac{dx_1}{4y_1^3}\cdots\frac{dx_4}{4y_4^3}\right) \times \begin{vmatrix} a_1 & 1 & \cdots & 1 & 0 \\ a_2 & x_1 & \cdots & x_4 & 0 \\ a_3 & y_1 & \cdots & y_4 & 0 \\ a_4 & x_1^2 & \cdots & x_4^2 & 0 \\ a_5 & x_1y_1 & \cdots & x_4y_4 & 0 \\ a_6 & y_1^2 & \cdots & y_4^2 & -1 \end{vmatrix}.$$

The determinant in the denominator will be identical, except with the entries of $\boldsymbol{a}$ replaced by the entries of $\boldsymbol{b}$. Hence the factored terms will cancel, leaving us with the simpler determinants. It is clear from the final column, that when we expand the determinants the resulting quotient of polynomials will not vary with the arbitrary constant a_6.

$$\frac{\sum_{j=1}^{g} a_j\sigma_j(\boldsymbol{u})}{\sum_{j=1}^{g} b_j\sigma_j(\boldsymbol{u})} = \text{function that does not vary with } a_6.$$

Since a_6 can be any constant we must conclude that for $\boldsymbol{u} \in \Theta^{[4]}$, the function $\sigma_6(\boldsymbol{u}) = 0$. (Note the same conclusion could have been drawn from considering the constant b_6.)

$$\Theta^{[4]} = \{\boldsymbol{u} \mid \sigma(\boldsymbol{u}) = \sigma_6(\boldsymbol{u}) = 0\}.$$

We repeat this process by considering Theorem 6.4.1 in the case when $k = 4$.

$$\frac{\sum_{j=1}^{6} a_j\sigma_j(\boldsymbol{u})}{\sum_{j=1}^{6} b_j\sigma_j(\boldsymbol{u})} = \frac{\det\left[\boldsymbol{a}\middle|\boldsymbol{du}(P_1)\middle|\cdots\middle|\boldsymbol{du}(P_4)\middle|\boldsymbol{du}(P_4)^{(1)}\right]}{\det\left[\boldsymbol{b}\middle|\boldsymbol{du}(P_1)\middle|\cdots\middle|\boldsymbol{du}(P_4)\middle|\boldsymbol{du}(P_4)^{(1)}\right]}.$$

This time we consider $\boldsymbol{u}$ descending to $\Theta^{[3]}$, by letting the fourth point move towards infinity. The penultimate column in each determinant can be given with the expansions (6.16) as before. For the final column we will need to determine the derivative of these expansions.

$$\frac{d^2u_1}{d\xi^2} = -10\xi^9 + O(\xi^{13}), \qquad \frac{d^2u_4}{d\xi} = -2\xi + \tfrac{9}{2}\lambda_4\xi^5 + O(\xi^9),$$

$$\frac{d^2u_2}{d\xi^2} = -6\xi^5 + O(\xi^9), \qquad \frac{d^2u_5}{d\xi^2} = -1 + \tfrac{5}{2}\lambda_4\xi^4 + O(\xi^8),$$
$$\frac{d^2u_3}{d\xi^2} = -5\xi^4 + O(\xi^8), \qquad \frac{d^2u_6}{d\xi^2} = \lambda_4\xi^3 + O(\xi^7). \tag{6.17}$$

When $\boldsymbol{u}$ arrives at $\Theta^{[3]}$ we will have $\xi = 0$. Our determinants will again factor and cancel to leave the numerator as

$$\begin{vmatrix} a_1 & 1 & \cdots & 1 & 0 & 0 \\ a_2 & x_1 & \cdots & x_3 & 0 & 0 \\ a_3 & y_1 & \cdots & y_3 & 0 & 0 \\ a_4 & x_1^2 & \cdots & x_3^2 & 0 & 0 \\ a_5 & x_1y_1 & \cdots & x_3y_3 & 0 & -1 \\ a_6 & y_1^2 & \cdots & y_3^2 & -1 & 0 \end{vmatrix}$$

with the denominator identical except for $\boldsymbol{b}$ instead of $\boldsymbol{a}$. From the final two columns it is clear that the resulting quotient of polynomials will not vary with the arbitrary constants a_6 and a_5. Hence we must conclude that

$$\Theta^{[3]} = \{\boldsymbol{u} \mid \sigma(\boldsymbol{u}) = \sigma_6(\boldsymbol{u}) = \sigma_5(\boldsymbol{u}) = 0\}.$$

We repeat the procedure once more for $k = 3$.

$$\frac{\sum_{j=1}^6 a_j\sigma_j(\boldsymbol{u})}{\sum_{j=1}^6 b_j\sigma_j(\boldsymbol{u})} = \frac{\det\left[\boldsymbol{a}\middle|\boldsymbol{du}(P_1)\middle|\cdots\middle|\boldsymbol{du}(P_3)\middle|\boldsymbol{du}(P_3)^{(1)}\middle|\boldsymbol{du}(P_3)^{(2)}\right]}{\det\left[\boldsymbol{b}\middle|\boldsymbol{du}(P_1)\middle|\cdots\middle|\boldsymbol{du}(P_3)\middle|\boldsymbol{du}(P_3)^{(1)}\middle|\boldsymbol{du}(P_3)^{(2)}\right]}.$$

We let $\boldsymbol{u}$ descend to $\Theta^{[2]}$ and use the expansions (6.16) and (6.17) for the fourth and fifth columns. The final column will require the derivatives of the expansions in (6.17).

$$\frac{d^3u_1}{d\xi^3} = -90\xi^8 + O(\xi^{12}), \qquad \frac{d^3u_4^3}{d\xi} = -2 + \tfrac{45}{2}\lambda_4\xi^4 + O(\xi^8),$$
$$\frac{d^3u_2}{d\xi^3} = -30\xi^4 + O(\xi^8), \qquad \frac{d^3u_5}{d\xi^3} = 10\lambda_4\xi^3 + O(\xi^7), \tag{6.18}$$
$$\frac{d^3u_3}{d\xi^3} = -20\xi^3 + O(\xi^7), \qquad \frac{d^3u_6}{d\xi^3} = 3\lambda_4\xi^2 + O(\xi^6).$$

We let $\xi = 0$ and cancel the common factors to leave the numerator as

$$\begin{vmatrix} a_1 & 1 & \cdots & 1 & 0 & 0 & 0 \\ a_2 & x_1 & \cdots & x_2 & 0 & 0 & 0 \\ a_3 & y_1 & \cdots & y_2 & 0 & 0 & -2 \\ a_4 & x_1^2 & \cdots & x_2^2 & 0 & 0 & 0 \\ a_5 & x_1y_1 & \cdots & x_2y_2 & 0 & -1 & 0 \\ a_6 & y_1^2 & \cdots & y_2^2 & -1 & 0 & 0 \end{vmatrix}$$

and similarly for the denominator. Expand the determinants and the statement reduces to

$$\frac{\sum_{j=1}^{6} a_j\sigma_j(\boldsymbol{u})}{\sum_{j=1}^{6} b_j\sigma_j(\boldsymbol{u})} = \frac{a_1x_1y_2 - a_1y_1x_2 + a_2y_1 - a_2y_2 - a_3x_1 + a_3x_2}{b_1x_1y_2 - b_1y_1x_2 + b_2y_1 - b_2y_2 - b_3x_1 + b_3x_2} \tag{6.19}$$

for $\boldsymbol{u} \in \Theta^{[2]}$. The right hand side is the same for all values of a_6, a_5, a_4 and hence we conclude that

$$\Theta^{[2]} = \{\boldsymbol{u} \mid \sigma(\boldsymbol{u}) = \sigma_6(\boldsymbol{u}) = \sigma_5(\boldsymbol{u}) = \sigma_4(\boldsymbol{u}) = 0\}.$$

Finally we consider Theorem 6.4.1 in the case when $k = 2$. Here, when we let $\boldsymbol{u}$ descend to $\Theta^{[1]}$, we find that the statement of the Theorem involves singular matrices. (Consider a further derivative of the series in (6.18). All these series would vanish when ξ is set to zero.) Hence this time the theorem will give us no information.

Instead we can consider equation (6.19) which held for $\boldsymbol{u} \in \Theta^{[2]}$. Let $\boldsymbol{u}$ descend to $\Theta^{[1]}$ here, by replacing (x_2, y_2) by their expansions in ξ, given in equations (3.22) and (3.23). We find that

$$\frac{\sum_{j=1}^{6} a_j\sigma_j(\boldsymbol{u})}{\sum_{j=1}^{6} b_j\sigma_j(\boldsymbol{u})} = \frac{a_1x_1 - a_2}{b_1x_1 - b_2} + O(\xi). \tag{6.20}$$

When $\boldsymbol{u} \in \Theta^{[1]}$ we set $\xi = 0$ and conclude that the right hand side has no dependence on a_3, a_4, a_5 or a_6. Hence

$$\Theta^{[1]} = \{\boldsymbol{u} \mid \sigma(\boldsymbol{u}) = \sigma_6(\boldsymbol{u}) = \sigma_5(\boldsymbol{u}) = \sigma_4(\boldsymbol{u}) = \sigma_3(\boldsymbol{u}) = 0\}.$$

Note that there is a Maple worksheet available in the extra Appendix of files in which these calculations are performed.

Summary

The strata of C can be defined by the zeros of the σ-function and its derivatives as follows. The definition of $\Theta^{[5]}$ is a classical result (Lemma 2.2.19), while the others were derived from Theorem 6.4.1.

$$\begin{aligned}
\Theta^{[5]} &= \{\boldsymbol{u} \mid \sigma(\boldsymbol{u}) = 0\} \\
\Theta^{[4]} &= \{\boldsymbol{u} \mid \sigma(\boldsymbol{u}) = \sigma_6(\boldsymbol{u}) = 0\} \\
\Theta^{[3]} &= \{\boldsymbol{u} \mid \sigma(\boldsymbol{u}) = \sigma_6(\boldsymbol{u}) = \sigma_5(\boldsymbol{u}) = 0\} \\
\Theta^{[2]} &= \{\boldsymbol{u} \mid \sigma(\boldsymbol{u}) = \sigma_6(\boldsymbol{u}) = \sigma_5(\boldsymbol{u}) = \sigma_4(\boldsymbol{u}) = 0\} \\
\Theta^{[1]} &= \{\boldsymbol{u} \mid \sigma(\boldsymbol{u}) = \sigma_6(\boldsymbol{u}) = \sigma_5(\boldsymbol{u}) = \sigma_4(\boldsymbol{u}) = \sigma_3(\boldsymbol{u}) = 0\}
\end{aligned} \tag{6.21}$$

6.4.2 Further relations

We next use the defining strata relations (6.21) to generate further relations between the σ-derivatives, holding on each strata. We use a systematic method, implemented in Maple, to achieve this.

Start with the relation $\sigma(\boldsymbol{u}) = 0$, valid for $\boldsymbol{u} \in \Theta^{[5]}$. Consider $\boldsymbol{u} \in \Theta^{[5]}$ as it descends to $\Theta^{[4]}$. We write $\boldsymbol{u}$ as $\boldsymbol{u} = \hat{\boldsymbol{u}} + \boldsymbol{u}_{\boldsymbol{\xi}}$ where $\hat{\boldsymbol{u}}$ is an arbitrary point on $\Theta^{[4]}$ and $\boldsymbol{u}_{\boldsymbol{\xi}}$ is a vector containing the series expansions of $\boldsymbol{u}$. (These may be obtained by integrating (6.16) or refer to the larger expansions are presented in Appendix C.1.) We calculate the series expansion of $\sigma(\hat{\boldsymbol{u}} + \boldsymbol{u}_{\boldsymbol{\xi}}) = 0$ in ξ as

$$\begin{aligned} 0 = \sigma(\hat{\boldsymbol{u}} + \boldsymbol{u}_{\boldsymbol{\xi}}) &= \sigma(\hat{\boldsymbol{u}}) - \sigma_6(\hat{\boldsymbol{u}})\xi + \tfrac{1}{2}\big[\sigma_{66}(\hat{\boldsymbol{u}}) - \sigma_5(\hat{\boldsymbol{u}})\big]\xi^2 + \big[\tfrac{1}{2}\sigma_{56}(\hat{\boldsymbol{u}}) - \tfrac{1}{6}\sigma_{666}(\hat{\boldsymbol{u}}) \\ &- \tfrac{1}{3}\sigma_4(\hat{\boldsymbol{u}})\big]\xi^3 + \big[\tfrac{1}{8}\sigma_{55}(\hat{\boldsymbol{u}}) + \tfrac{1}{3}\sigma_{46}(\hat{\boldsymbol{u}}) - \tfrac{1}{4}\sigma_{566}(\hat{\boldsymbol{u}}) + \tfrac{1}{24}\sigma_{6666}(\hat{\boldsymbol{u}})\big]\xi^4 + \big[-\tfrac{1}{120}\sigma_{66666}(\hat{\boldsymbol{u}}) \\ &+ \tfrac{1}{12}\sigma_{5666}(\hat{\boldsymbol{u}}) + \tfrac{1}{6}\sigma_{45}(\hat{\boldsymbol{u}}) - \tfrac{1}{6}\sigma_{466}(\hat{\boldsymbol{u}}) - \tfrac{1}{8}\sigma_{556}(\hat{\boldsymbol{u}}) + \tfrac{1}{20}\sigma_6(\hat{\boldsymbol{u}})\lambda_4\big]\xi^5 + O\big(\xi^6\big). \end{aligned} \tag{6.22}$$

Setting the coefficients of ξ to zero gives us a set of relations for $\boldsymbol{u} \in \Theta^{[4]}$.

$$\begin{aligned} \sigma_6(\boldsymbol{u}) &= 0 \\ \sigma_{66}(\boldsymbol{u}) &= \sigma_5(\boldsymbol{u}) \\ \sigma_{666}(\boldsymbol{u}) &= 3\sigma_{56}(\boldsymbol{u}) - 2\sigma_4(\boldsymbol{u}) \\ \sigma_{6666}(\boldsymbol{u}) &= 6\sigma_{566}(\boldsymbol{u}) - 8\sigma_{46}(\boldsymbol{u}) - 3\sigma_{55}(\boldsymbol{u}) \\ \sigma_{66666}(\boldsymbol{u}) &= 10\sigma_{5666}(\boldsymbol{u}) - 20\sigma_{466}(\boldsymbol{u}) - 15\sigma_{556}(\boldsymbol{u}) + 20\sigma_{45}(\boldsymbol{u}) + 6\lambda_4\sigma_6(\boldsymbol{u}) \\ &\vdots \end{aligned} \tag{6.23}$$

If we calculate the expansion to a higher order of ξ then infinitely more relations can be obtained, involving progressively higher index σ-derivatives. Note however, that the expansions for $\boldsymbol{u}_{\boldsymbol{\xi}}$ must first be calculated to a sufficiently high order first. Using the Maple `series` command we find that the series in equation (6.22) continues with

$$\begin{aligned} \ldots &+ \big[\tfrac{1}{720}\sigma_{666666} - \tfrac{1}{48}\sigma_{56666} - \tfrac{1}{48}\sigma_{555} + \tfrac{1}{16}\sigma_{5566} - \tfrac{1}{6}\sigma_3 + \tfrac{1}{18}\sigma_{44} + \tfrac{1}{12}\sigma_5\lambda_4 \\ &+ \tfrac{1}{18}\sigma_{4666} - \tfrac{1}{6}\sigma_{456} - \tfrac{1}{20}\sigma_{66}\lambda_4\big]\xi^6 + \big[\tfrac{1}{48}\sigma_{5556} + \tfrac{1}{240}\sigma_{566666} - \tfrac{1}{24}\sigma_{455} - \tfrac{1}{18}\sigma_{446} \\ &+ \tfrac{3}{28}\sigma_4\lambda_4 + \tfrac{1}{12}\sigma_{4566} - \tfrac{1}{7}\sigma_2 - \tfrac{1}{72}\sigma_{46666} - \tfrac{1}{5040}\sigma_{6666666} - \tfrac{1}{48}\sigma_{55666} + \tfrac{1}{6}\sigma_{36} \\ &+ \tfrac{1}{40}\sigma_{666}\lambda_4 - \tfrac{13}{120}\sigma_{56}\lambda_4\big]\xi^7 + \big[\tfrac{1}{360}\sigma_{466666} + \tfrac{1}{192}\sigma_{556666} - \tfrac{1}{96}\sigma_{55566} + \tfrac{1}{36}\sigma_{4466} \\ &+ \tfrac{1}{12}\sigma_{35} + \tfrac{1}{7}\sigma_{26} + \tfrac{1}{40320}\sigma_{66666666} - \tfrac{13}{105}\sigma_{46}\lambda_4 - \tfrac{1}{36}\sigma_{445} + \tfrac{1}{384}\sigma_{5555} + \tfrac{1}{24}\sigma_{4556} \\ &- \tfrac{1}{1440}\sigma_{5666666} - \tfrac{1}{12}\sigma_{366} - \tfrac{1}{120}\sigma_{6666}\lambda_4 - \tfrac{1}{24}\sigma_{55}\lambda_4 - \tfrac{1}{36}\sigma_{45666} + \tfrac{1}{15}\sigma_{566}\lambda_4\big]\xi^8 \\ &+ \big[\tfrac{1}{144}\sigma_{4555} - \tfrac{1}{12}\sigma_{356} + \tfrac{1}{36}\sigma_{4456} - \tfrac{1}{162}\sigma_{444} - \tfrac{1}{14}\sigma_{266} - \tfrac{1}{384}\sigma_{55556} + \tfrac{59}{840}\sigma_{466}\lambda_4 \\ &- \tfrac{19}{720}\sigma_{5666}\lambda_4 + \tfrac{1}{36}\sigma_6\lambda_3 - \tfrac{5}{288}\sigma_6{\lambda_4}^2 + \tfrac{1}{480}\sigma_{66666}\lambda_4 - \tfrac{1}{362880}\sigma_{666666666} + \tfrac{1}{18}\sigma_{34} + \ldots \end{aligned}$$

$$\cdots + \tfrac{1}{14}\sigma_{25} + \tfrac{1}{144}\sigma_{456666} + \tfrac{1}{288}\sigma_{555666} - \tfrac{1}{48}\sigma_{45566} - \tfrac{1}{2160}\sigma_{4666666} - \tfrac{1}{960}\sigma_{5566666}$$
$$+ \tfrac{1}{10080}\sigma_{56666666} + \tfrac{23}{480}\lambda_4\sigma_{556} - \tfrac{1}{108}\sigma_{44666} - \tfrac{41}{504}\lambda_4\sigma_{45} + \tfrac{1}{36}\sigma_{3666}\big]\xi^9 + O\big(\xi^{10}\big).$$

We have calculated an expansion for $\sigma(\hat{\boldsymbol{u}}+\boldsymbol{u}_{\boldsymbol{\xi}})$ up to $O(\xi^{29})$, generating a set of 28 relations for the σ-derivatives valid for $\boldsymbol{u} \in \Theta^{[4]}$. Note that the standard Maple `series` command was not able to derive an expansion further than $O(\xi^{14})$ due to memory constraints. Instead an alternative procedure was implemented, making use of the weight properties described in Section 3.3.

Recall that the σ-function has a definite Sato weight, which is $+15$ in the case of the cyclic (4,5)-curve. Hence this expansion in ξ must also be homogeneous with this weight, (check series (6.22) to see this is the case). Since ξ has weight one we know the coefficient of ξ^n must be a sum of terms with weight $15 - n$. Hence we can determine which σ-derivatives may be present, (those that can be combined with a $\boldsymbol{\lambda}$-monomial to give the correct weight).

We write a procedure that calculates each coefficient of ξ successively. Recall that the expansions of $\boldsymbol{u}$ in ξ also involve terms with decreasing $\boldsymbol{\lambda}$ weight. At each stage we truncate these expansions at the appropriate weight necessary to calculate the next coefficient. We then calculate that coefficient only, (using the general equation for an unknown function of six variables).

The remainder of the expansion and the full set of relations can be found in the extra Appendix of files or online at [37]. The Maple worksheet in which the series is derived using the new procedure can also be found in the extra Appendix of files.

The next step in this process will be to find the relations valid for $\boldsymbol{u} \in \Theta^{[3]}$. Since $\Theta^{[3]} \subset \Theta^{[4]}$ we can conclude that the relations (6.23) for $\boldsymbol{u} \in \Theta^{[4]}$ are valid here also. However, we can derive a larger set of relations for $\boldsymbol{u} \in \Theta^{[3]}$ by repeating the descent procedure for all those relations valid for $\boldsymbol{u} \in \Theta^{[4]}$.

We do not need to consider the relation $\sigma(\boldsymbol{u}) = 0$ since that will only give us the same relations as above. Instead choose the second defining relation $\sigma_6(\boldsymbol{u}) = 0$. We again write $\boldsymbol{u} = \hat{\boldsymbol{u}} + \boldsymbol{u}_{\boldsymbol{\xi}}$ where $\boldsymbol{u}_{\boldsymbol{\xi}}$ is the vector of expansions as before and $\hat{\boldsymbol{u}}$ is now an arbitrary point on $\Theta^{[3]}$. Note that calculating the series expansion in ξ for $\sigma_6(\hat{\boldsymbol{u}} + \boldsymbol{u}_{\boldsymbol{\xi}})$ is not as arduous as above. We can easily derive it from the expansion (6.22) for $\sigma(\hat{\boldsymbol{u}} + \boldsymbol{u}_{\boldsymbol{\xi}})$ by adding a six to each index.

$$0 = \big(\sigma_6 - \sigma_{66}\xi + \tfrac{1}{2}\big[\sigma_{666} - \sigma_{56}\big]\xi^2 + \big[\tfrac{1}{2}\sigma_{566} - \tfrac{1}{3}\sigma_{46} - \tfrac{1}{6}\sigma_{6666}\big]\xi^3 + \cdots\big)(\hat{\boldsymbol{u}})$$

Setting these coefficients of ξ to zero gives us more relations valid for $\boldsymbol{u} \in \Theta^{[3]}$, starting with $\sigma_{66}(\boldsymbol{u}) = 0$. We can obtain further relations for $\boldsymbol{u} \in \Theta^{[3]}$ by descending all of the 28 relations in (6.23) for $\boldsymbol{u} \in \Theta^{[4]}$. After this we find we have obtained 238 relations.

We need to organise these relations by choosing a ranking system to decide which σ-

derivatives should be expressed in terms of the others. It is logical to remove the higher index σ-derivatives and so we always solve for those with the greater number of indices first. (Note this was the approach that was taken with equations (6.23)). When we have the choice between σ-derivatives with the same number of indices we choose to solve for those with the highest numerical indices first.

So we take the 238 equations valid for $\boldsymbol{u} \in \Theta^{[3]}$, substitute in the equations for $\boldsymbol{u} \in \Theta^{[4]}$ and solve according to these rules to give a set of equations in the σ-derivatives for $\boldsymbol{u} \in \Theta^{[3]}$.

The next step is to descend the relations valid for $\boldsymbol{u} \in \Theta^{[3]}$ to $\Theta^{[2]}$. We automate the process of descending from $\Theta^{[k]}$ to $\Theta^{[k-1]}$ in Maple as follows.

1. Take a relation valid for $\boldsymbol{u} \in \Theta^{[k]}$ and expand as a series in ξ. To do this we replace each σ-derivative by the series expansion for $\sigma(\hat{\boldsymbol{u}} + \boldsymbol{u}_{\boldsymbol{\xi}})$, adding the relevant index to each σ-derivative in the expansion.

2. Set each coefficient with respect to ξ to zero, and save the resulting equations, valid for $\boldsymbol{u} \in \Theta^{[k-1]}$.

3. Repeat Steps 1 and 2 for all known relations valid for $\boldsymbol{u} \in \Theta^{[k]}$ to obtain a large set of equations between σ-derivatives, valid for $\boldsymbol{u} \in \Theta^{[k-1]}$.

4. Substitute the existing equations for $\boldsymbol{u} \in \Theta^{[k]}$ into the set. Then rearrange to obtain additional equations for $\boldsymbol{u} \in \Theta^{[k-1]}$ by first solving for the higher index functions, and then for those with the greater numerical indices.

So we use this process to obtain first a set of relations for $\boldsymbol{u} \in \Theta^{[2]}$ and then a final set for $\boldsymbol{u} \in \Theta^{[1]}$. (The corresponding Maple worksheet can be found in the extra Appendix of files.) Appendix E.1 contains all the equations valid for $\boldsymbol{u} \in \Theta^{[1]}$ that express k-index σ-derivatives with $k \leq 4$. The full set of relations that have been derived for $\Theta^{[1]}$, along with the sets of relations for higher strata, can be found in the extra Appendix of files or online at [37]. The most interesting result of these calculations was to find that on $\Theta^{[1]}$ we have

$$\sigma_1(\boldsymbol{u}) = \sigma_2(\boldsymbol{u}) = 0.$$

Together with equation (6.21) this shows that all the first derivatives of $\sigma(\boldsymbol{u})$ are zero on $\Theta^{[1]}$. (This may be double checked using the σ-expansion as derived in Section 3.4, and then substituting in the expansions in ξ.)

This result is very surprising as it was not the case for the lower genus examples that were considered. It also causes extra complications in the next Section.

Note that the calculations in this Section were computationally much more difficult that those in [14]. The latter stages were performed in parallel on a small cluster of machines, using Distributed Maple, (see [72]).

6.5 Evaluating the integrand

Recall our integrand as given by equation (6.15),

$$\varphi(x)dx = K\left(A_4x^2 + A_3x + A_2 + \frac{A_1}{x} + \frac{1}{x^2}\right)\left(\frac{dx}{4y^3}\right).$$

Now, the map $q(x)$ was given by a single integral with respect to one parameter, the point (x, y) on C. So we will rewrite this as an integral on the one-dimensional strata $\Theta^{[1]}$ of J. We will then evaluate it using σ-derivatives restricted to $\Theta^{[1]}$. In [14] Jorgenson's Theorem was used to express x in terms of σ-derivatives. However, if we followed this approach and solved equation (6.20) for $\boldsymbol{u} \in \Theta^{[1]}$ we would find

$$x = -\frac{\sigma_1(\boldsymbol{u})}{\sigma_2(\boldsymbol{u})},$$

which makes no sense given that $\sigma_1(\boldsymbol{u}) = 0$ for $\boldsymbol{u} \in \Theta^{[1]}$. Instead let us take equation (6.19) which was also derived from Jorgenson's Theorem and which holds for $\boldsymbol{u}$ in $\Theta^{[2]}$. We consider what happens to this as $\boldsymbol{u}$ descends to $\Theta^{[1]}$. We replace (x_2, y_2) with the expansions in the parameter ξ and replace the σ-derivatives by their Taylor series in ξ. (The expansions for x and y were given in equations (3.22) and (3.23) while the series of the σ-derivatives is derived from equation (6.22) as discussed in the last section.) If we then take a series expansion of this in ξ and set $\xi = 0$ we find that for $\boldsymbol{u} \in \Theta^{[1]}$ we have

$$\frac{a_1\sigma_{23}(\boldsymbol{u}) + a_2\sigma_{34}(\boldsymbol{u})}{b_1\sigma_{23}(\boldsymbol{u}) + b_2\sigma_{34}(\boldsymbol{u})} = \frac{a_1x - a_2}{b_1x - b_2}.$$

Solving this for x gives,

$$x = -\frac{\sigma_{23}(\boldsymbol{u})}{\sigma_{34}(\boldsymbol{u})}, \tag{6.24}$$

for $\boldsymbol{u} \in \Theta^{[1]}$. We must verify that $\sigma_{34}(\boldsymbol{u})$ is not identically zero on $\Theta^{[1]}$. Consider the expansion about the origin that we derived in Chapter 3. Using the Schur-Weierstrass polynomial (3.50) and the substitutions for $\boldsymbol{u}$ in ξ we see that

$$\sigma_{34}(\boldsymbol{u}) = \xi^6 + O(\xi^7),$$

and so $\sigma_{34}(\boldsymbol{u})$ is not identically zero. Further, this technique will allow us to specify all the zeros of the function. First note that $\sigma_{34}(\boldsymbol{u})$ has exactly six zeros on $\Theta^{[1]}$. (The Riemann vanishing theorem stated that the θ-function, and hence the σ-function, is either identically zero or has exactly $g = 6$ zeros on the curve. The σ-derivatives may be defined using the same θ-function, and while $\sigma(\boldsymbol{u})$ is identically zero on $\Theta^{[1]}$, not all the σ-derivatives are.) Since the expansion above has a zero of order six at the origin, we can hence conclude that $\sigma_{34}(\boldsymbol{u})$ restricted to $\Theta^{[1]}$ cannot vanish for $\boldsymbol{u} \neq 0 \bmod \Lambda$.

Similarly, $\sigma_{23}(\boldsymbol{u})$ also has six zeros on $\Theta^{[1]}$. However,

$$\sigma_{23}(\boldsymbol{u}) = \xi^2 + O(\xi^3),$$

and hence there is a double zero at the origin and four other zeros corresponding to the points $x = 0$ on each of the four sheets of C.

So we may evaluate x as the ratio of σ-derivatives in equation (6.24). Using this and the standard basis of holomorphic differentials (3.4), we can rewrite our integrand as

$$\varphi(x)dx = K[\varphi_1(x)dx + \varphi_2(x)dx] \tag{6.25}$$

where

$$\varphi_1(x)dx = A_2du_1 + A_3du_2 + A_4du_4, \tag{6.26}$$

$$\varphi_2(x)dx = \left(\left(\frac{\sigma_{34}(\boldsymbol{u})}{\sigma_{23}(\boldsymbol{u})}\right)^2 - A_1\frac{\sigma_{34}(\boldsymbol{u})}{\sigma_{23}(\boldsymbol{u})}\right)du_1 \equiv \varphi_2(\boldsymbol{u})du_1. \tag{6.27}$$

Thus φ_1 is a sum of holomorphic differentials on C, and φ_2 is a second kind meromorphic differential. As in the lower genus examples, we will need to find a suitable function $\Psi(\boldsymbol{u})$ such that

$$\frac{d}{du_1}\Psi(\boldsymbol{u}) = \varphi_2(\boldsymbol{u}), \qquad \boldsymbol{u} \in \Theta^{[1]}. \tag{6.28}$$

We will identify $\Psi(\boldsymbol{u})$ and evaluate $\varphi_2(\boldsymbol{u})$ as follows. First we must derive the expansions for $\varphi_2(\boldsymbol{u})$ at its poles. We will then find a function $\Psi(\boldsymbol{u})$, which has simple poles at the same points as the double poles of $\varphi_2(\boldsymbol{u})$, and which varies by at worst an additive constant over the period lattice Λ.

The function $\Psi(\boldsymbol{u})$ will be chosen so that $\frac{d}{du_1}\Psi(\boldsymbol{u})$ has the same expansion at the poles as φ_2 and is regular elsewhere. It then follows that the difference $\frac{d}{du_1}\Psi(\boldsymbol{u}) - \varphi_2(\boldsymbol{u})$ is both holomorphic and Abelian. By the generalisation of Liouville's theorem we can conclude that this difference is a constant, which will be evaluated at a convenient point.

6.5.1 The expansion of $\varphi_2(\boldsymbol{u})$ at the poles

Recall that $\sigma(\boldsymbol{u})$ was an entire function, and so $\varphi_2(\boldsymbol{u})$ will have poles at those points where $\sigma_{23}(\boldsymbol{u}) = 0$. As discussed above, we know $\sigma_{23}(\boldsymbol{u})$ has six zeros — a double zero at the origin and zeros at the four points, one on each sheet, where $x = 0$. We will first investigate what happens at these four points and consider the origin later.

Recall the invariance condition on the cyclic (n, s)-curve given in equation (2.35). In the (4,5)-case the cyclic symmetry is [i], the imaginary unit. This relates the four different sheets and acts on (x, y) by

$$[\mathrm{i}](x, y) \mapsto (x, \mathrm{i}y).$$

Hence it will act on $\boldsymbol{u}$ as follows.

$$\begin{array}{lll} u_1 \mapsto \mathrm{i}u_1 & u_2 \mapsto \mathrm{i}u_2 & u_3 \mapsto -u_3 \\ u_4 \mapsto \mathrm{i}u_4 & u_5 \mapsto -u_5 & u_6 \mapsto -\mathrm{i}u_6. \end{array} \tag{6.29}$$

Let $\boldsymbol{u}_0$ be the Abel image of the point on the principal sheet where $\sigma_{23}(\boldsymbol{u}) = 0$. This is the point where $x = 0$ and $y = (\lambda_0)^{1/4}$. Then the four zeros of $\sigma_{23}(\boldsymbol{u})$ away from the origin are given by

$$\boldsymbol{u}_{0,N} = [\mathrm{i}]^N \boldsymbol{u}_0, \qquad N = 0, 1, 2, 3.$$

We will require the poles to match at all four of these points.

We need to find an expansion for $\wp_2(\boldsymbol{u})$ at the points $\boldsymbol{u}_{0,N}$. To start, we consider $\boldsymbol{u} \in \Theta^{[1]}$ and calculate the Taylor series of $\sigma(\boldsymbol{u})$ around the point

$$\boldsymbol{u} = \boldsymbol{u}_0 = (u_{0,1}, u_{0,2}, u_{0,3}, u_{0,4}, u_{0,5}, u_{0,6}).$$

Writing $w_i = (u_i - u_{0,i})$ we have,

$$\begin{aligned} \sigma(\boldsymbol{u}) = \sigma(\boldsymbol{u}_0) &+ \big[\sigma_1(\boldsymbol{u}_0)w_1 + \sigma_2(\boldsymbol{u}_0)w_2 + \sigma_3 w_3(\boldsymbol{u}_0) + \sigma_4(\boldsymbol{u}_0)w_4 \\ &+ \sigma_5(\boldsymbol{u}_0)w_5 + \sigma_6(\boldsymbol{u}_0)w_6\big] + \big[\tfrac{1}{2}\sigma_{11}(\boldsymbol{u}_0)w_1^2 + \sigma_{12}(\boldsymbol{u}_0)w_1w_2 + \sigma_{13}(\boldsymbol{u}_0)w_1w_3 \\ &+ \sigma_{14}(\boldsymbol{u}_0)w_1w_4 + \sigma_{15}(\boldsymbol{u}_0)w_1w_5 + \sigma_{16}(\boldsymbol{u}_0)w_1w_6 + \tfrac{1}{2}\sigma_{22}(\boldsymbol{u}_0)w_2^2 + \ldots \end{aligned}$$

We will have similar expansions around the other $\boldsymbol{u}_{0,N}$. Note we can use this expansion to easily compute the expansions for the σ-derivatives, (by simply adding the relevant indices). Now, since $\boldsymbol{u}_{0,N}$ are the points where $t = 0$, we can write their components as

$$\big(\boldsymbol{u}_{0,N}\big)_i = \int_\infty^0 du_i, \qquad i = 1, \ldots, 6,$$

evaluated on the sheet where $s = [i]^N(\lambda_0)^{1/4}$. Therefore we can denote

$$w_{i,N} = \big(u_i - \big(\boldsymbol{u}_{0,N}\big)_i\big) = \int_\infty^x du_i - \int_\infty^0 du_i = \int_0^x du_i,$$

evaluated on the Nth sheet. Using our basis of holomorphic differentials (3.4), we can find expansions for $w_{1,N}, \ldots w_{6,N}$ in the parameter x.

$$\begin{aligned} w_{1,N} &= \frac{1}{4}\frac{\mathrm{i}^N}{{\lambda_0}^{3/4}}x - \frac{3}{32}\frac{\mathrm{i}^{3N}\lambda_1}{{\lambda_0}^{7/4}}x^2 - \frac{\mathrm{i}^{3N}}{128}\frac{8\lambda_2\lambda_0 - 7\lambda_1^2}{{\lambda_0}^{11/4}}x^3 \\ &\quad - \frac{\mathrm{i}^N}{2048}\frac{(96\lambda_3{\lambda_0}^2 - 168\lambda_1\lambda_2\lambda_0 + 77{\lambda_1}^3)}{{\lambda_0}^{15/4}}x^4 + O\left(x^5\right) \\ w_{2,N} &= \frac{1}{8}\frac{\mathrm{i}^N}{{\lambda_0}^{3/4}}x^2 - \frac{1}{16}\frac{\mathrm{i}^N\lambda_1}{{\lambda_0}^{7/4}}x^3 - \frac{3\mathrm{i}^N}{512}\frac{8\lambda_2\lambda_0 - 7{\lambda_1}^2}{{\lambda_0}^{11/4}}x^4 + O\left(x^5\right) \end{aligned} \tag{6.30}$$

$$
\begin{aligned}
w_{3,N} &= \frac{1}{4}\frac{\mathrm{i}^{2N}}{\lambda_0^{1/2}}x - \frac{1}{16}\frac{\mathrm{i}^{2N}\lambda_1}{{\lambda_0}^{3/2}}x^2 - \frac{\mathrm{i}^{2N}}{96}\frac{4\lambda_2\lambda_0 - 3{\lambda_1}^2}{{\lambda_0}^{5/2}}x^3 \\
&\quad - \frac{\mathrm{i}^{2N}}{256}\frac{8\lambda_3{\lambda_0}^2 - 12\lambda_1\lambda_2\lambda_0 + 5{\lambda_1}^3}{{\lambda_0}^{7/2}}x^4 + O\left(x^5\right) \\
w_{4,N} &= \frac{1}{12}\frac{\mathrm{i}^{N}}{{\lambda_0}^{3/4}}x^3 - \frac{3}{64}\frac{\mathrm{i}^{N}\lambda_1}{{\lambda_0}^{7/4}}x^4 + O\left(x^5\right) \\
w_{5,N} &= \frac{1}{8}\frac{\mathrm{i}^{2N}}{\lambda_0^{1/2}}x^2 - \frac{1}{24}\frac{\mathrm{i}^{2N}\lambda_1}{{\lambda_0}^{3/2}}x^3 - \frac{\mathrm{i}^{2N}}{128}\frac{4\lambda_2\lambda_0 - 3{\lambda_1}^2}{{\lambda_0}^{5/2}}x^4 + O\left(x^5\right) \\
w_{6,N} &= \frac{1}{4}\frac{\mathrm{i}^{3N}}{\lambda_0^{1/4}}x - \frac{1}{32}\frac{\mathrm{i}^{3N}\lambda_1}{{\lambda_0}^{5/4}}x^2 - \frac{\mathrm{i}^{3N}}{384}\frac{8\lambda_2\lambda_0 - 5{\lambda_1}^2}{{\lambda_0}^{9/4}}x^3 \\
&\quad - \frac{\mathrm{i}^{3N}}{2048}\frac{(32\lambda_3{\lambda_0}^2 - 40\lambda_1\lambda_2\lambda_0 + 15{\lambda_1}^3)}{\lambda_0^{13/4}}x^4 + O\left(x^5\right)
\end{aligned}
$$

Note that all these expansions are given for the general sheet, since we need to check the behaviour at all the poles. The quantity $\lambda_0^{1/4}$, and its positive integer powers, are defined on the principal sheet as before. We can move between the sheets by selecting the appropriate value of N.

We can invert equation (6.30) on the N-th sheet to give an expansion for x in $w_{1,N}$.

$$
\begin{aligned}
x &= 4\,\mathrm{i}^{3N}{\lambda_0}^{3/4}w_{1,N} + 6\lambda_1\mathrm{i}^{6N}\lambda_0^{1/2}w_{1,N}^2 \\
&\quad + 4\mathrm{i}^{N}\lambda_0^{1/4}(\lambda_1^2 + 4\lambda_2\lambda_0)w_{1,N}^3 + O\left(w_{1,N}^4\right).
\end{aligned}
$$

We can hence use $w_{1,N}$ as a local parameter near $\boldsymbol{u}_{0,N}$. We start by substituting for x to give the expansions of $w_{2,N}, \ldots, w_{6,N}$ with respect to $w_{1,N}$.

$$
\begin{aligned}
w_{2,N} &= 2\mathrm{i}^{3N}{\lambda_0}^{3/4}w_{1,N}^2 + 2\mathrm{i}^{2N}\lambda_1\lambda_0^{1/2}w_{1,N}^3 + O\left(w_{1,N}^4\right). \\
w_{3,N} &= \mathrm{i}^{N}\lambda_0^{1/4}w_{1,N} + \tfrac{1}{2}\lambda_1 w_{1,N}^2 + \tfrac{4}{3}\mathrm{i}^{3N}\lambda_2\lambda_0^{3/4}w_{1,N}^3 + O\left(w_{1,N}^4\right). \\
w_{4,N} &= \tfrac{16}{3}\mathrm{i}^{2N}\lambda_0^{3/2}w_{1,N}^3 + O\left(w_{1,N}^4\right). \\
w_{5,N} &= 2\lambda_0 w_{1,N}^2 + \tfrac{10}{3}\mathrm{i}^{3N}\lambda_0^{3/4}\lambda_1 w_{1,N}^3 + O\left(w_{1,N}^4\right). \\
w_{6,N} &= \mathrm{i}^{2N}\lambda_0^{1/2}w_{1,N} + \lambda_0^{1/4}\lambda_1\mathrm{i}^{N}w_{1,N}^2 + \tfrac{1}{3}(\lambda_1^2 + 8\lambda_2\lambda_0)w_{1,N}^3 + O\left(w_{1,N}^4\right).
\end{aligned}
\tag{6.31}
$$

We use these in turn to give the σ-derivative expansions at $\boldsymbol{u}_{0,N}$ as series in $w_{1,N}$. (Take the Taylor series at $\boldsymbol{u}_{0,N}$, substitute in equation (6.31) and find the series in $w_{1,N}$.) For example we have,

$$
\begin{aligned}
\sigma_{34}(\boldsymbol{u}) &= \sigma_{34}(\boldsymbol{u}_{0,N}) + \Big[\sigma_{346}(\boldsymbol{u}_{0,N})\mathrm{i}^{2N}\lambda_0^{1/2} + \sigma_{334}(\boldsymbol{u}_{0,N})\mathrm{i}^{N}\lambda_0^{1/4} \\
&\qquad + \sigma_{134}(\boldsymbol{u}_{0,N})\Big]w_{1,N} + O\left(w_{1,N}^2\right), \\
\sigma_{23}(\boldsymbol{u}) &= \Big[\mathrm{i}^{2N}\lambda_0^{\frac{1}{2}}\sigma_{236}(\boldsymbol{u}_{0,N}) + \sigma_{123}(\boldsymbol{u}_{0,N}) + \mathrm{i}^{N}\lambda_0^{\frac{1}{4}}\sigma_{233}(\boldsymbol{u}_{0,N})\Big]\, w_{1,N}, +O\left(w_{1,N}^2\right).
\end{aligned}
$$

(In the second example we recall that $\sigma_{23}(\boldsymbol{u}_{0,N}) = 0$ by the definition of $\boldsymbol{u}_{0,N}$.)

We substitute these into equation (6.27) to obtain an expansion for the meromorphic part of the integrand, $\varphi_2(\boldsymbol{u})$, at $\boldsymbol{u} = \boldsymbol{u}_{0,N}$, as a series in $w_{1,N}$.

$$\varphi_2(\boldsymbol{u}) = \frac{1}{w_{1,N}^2} \left(\frac{\sigma_{34}}{\mathrm{i}^N \lambda_0^{1/4} \sigma_{233} + \mathrm{i}^{2N} \lambda_0^{1/2} \sigma_{236} + \sigma_{123}} \right)^2 (\boldsymbol{u}_{0,N}) + \frac{\mathcal{C}(\boldsymbol{u}_{0,N})}{w_{1,N}} + O(w_{1,N}^0), \tag{6.32}$$

where $\mathcal{C}(\boldsymbol{u}_{0,N})$ is the following polynomial in the σ-derivatives. We need to evaluate this polynomial to ensure that $\varphi_2 dx$ has zero residue as required.

$$\begin{aligned}
\mathcal{C}(\boldsymbol{u_0}) = &- \left(\frac{\sigma_{34}}{\left(\mathrm{i}^N \lambda_0^{1/4} \sigma_{233} + \mathrm{i}^{2N} \lambda_0^{1/2} \sigma_{236} + \sigma_{123}\right)^3} \right) (\boldsymbol{u_0}) \times \Big[2i^N \sqrt[4]{\lambda_0} \sigma_{34} \sigma_{1233} \\
&- 2i^N \sqrt[4]{\lambda_0} \sigma_{334} \sigma_{123} - 2i^{2N} \sqrt{\lambda_0} \sigma_{346} \sigma_{123} + 2i^{5N} \sqrt[4]{\lambda_0} \lambda_1 \sigma_{34} \sigma_{236} \\
&- 2i^N \sqrt[4]{\lambda_0} \sigma_{134} \sigma_{233} + 2A_1 i^N \sqrt[4]{\lambda_0} \sigma_{233} \sigma_{123} + \sigma_{34} \sigma_{1123} - 2i^{2N} \sqrt{\lambda_0} \sigma_{334} \sigma_{233} \\
&+ i^{4N} \lambda_1 \sigma_{34} \sigma_{233} + i^{4N} \lambda_0 \sigma_{34} \sigma_{2366} + 4i^{4N} \lambda_0 \sigma_{34} \sigma_{235} - 2\sigma_{134} \sigma_{123} \\
&+ 2i^{2N} \sqrt{\lambda_0} \sigma_{34} \sigma_{1236} - 2i^{3N} {\lambda_0}^{3/4} \sigma_{346} \sigma_{233} - 2i^{3N} {\lambda_0}^{3/4} \sigma_{334} \sigma_{236} \\
&+ 2A_1 i^{3N} {\lambda_0}^{3/4} \sigma_{233} \sigma_{236} + A_1 i^{4N} \lambda_0 {\sigma_{236}}^2 + A_1 i^{2N} \sqrt{\lambda_0} {\sigma_{233}}^2 \\
&- 2i^{4N} \lambda_0 \sigma_{346} \sigma_{236} + 2A_1 i^{2N} \sqrt{\lambda_0} \sigma_{236} \sigma_{123} + A_1 {\sigma_{123}}^2 + i^{2N} \sqrt{\lambda_0} \sigma_{34} \sigma_{2333} \\
&+ 2i^{3N} {\lambda_0}^{3/4} \sigma_{34} \sigma_{2336} + 4i^{3N} {\lambda_0}^{3/4} \sigma_{34} \sigma_{223} - 2i^{2N} \sqrt{\lambda_0} \sigma_{134} \sigma_{236} \Big] (\boldsymbol{u_0}).
\end{aligned}$$

In the previous section we derived a set of relations for $\boldsymbol{u} \in \Theta^{[1]}$, but these are not sufficient to simplify $\mathcal{C}(\boldsymbol{u}_{0,N})$. We need to generate a further set of relations which are valid only at $\boldsymbol{u} = \boldsymbol{u}_{0,N}$.

We will use a similar approach to that used in Section 6.4.2 to descend relations down the strata. First take a relation valid on $\Theta^{[1]}$ and calculate its expansion around $\boldsymbol{u} = \boldsymbol{u}_{0,N}$ and then obtain a series in $w_{1,N}$ using the substitutions (6.31). We then set to zero the coefficients of $w_{1,N}$ to obtain relations. We do this for each relation valid on $\Theta^{[1]}$ and obtain a set of equations between σ-derivatives valid only at the points $\boldsymbol{u} = \boldsymbol{u}_{0,N}$. Many of these are presented in Appendix E.2, with the full set that was derived available on-line at [37].

If we substitute the equations of Appendix E.2 into equation (6.32), the expansion of $\varphi_2(\boldsymbol{u})$, we see $\mathcal{C}(\boldsymbol{u}_{0,N})$ simplifies considerably and we obtain

$$\varphi_2(\boldsymbol{u}) = \left[\frac{\mathrm{i}^{2N}}{16} \frac{1}{\lambda_0^{3/2}} \right] \frac{1}{w_{1,N}^2} + \left[\frac{\mathrm{i}^N}{16} (4\lambda_0 A_1 - 3\lambda_1) \right] \frac{1}{w_{1,N}} + O(w_{1,N}^0). \tag{6.33}$$

Recall equation (6.12) which stated that $\varphi(p)$ has zero residue at $p = \infty$ on all sheets. Since residues are invariant under conformal maps, we can conclude that $\varphi(x)$ must also have zero residue at the poles.

Since $\varphi_1 dx$ was holomorphic we see that $\varphi_2(\boldsymbol{u})$ must have zero residue, and so the constant A_1 must be given by

$$A_1 = \frac{3}{4}\frac{\lambda_1}{\lambda_0}. \tag{6.34}$$

Hence

$$\varphi_2(\boldsymbol{u}) = \left[\frac{\mathrm{i}^{2N}}{16}\frac{1}{\lambda_0^{3/2}}\right]\frac{1}{w_{1,N}^2} + O(w_{1,N}^0). \tag{6.35}$$

(The corresponding Maple worksheet in which these results was derived can be found in the extra Appendix of files).

6.5.2 Finding a suitable function $\Psi(\boldsymbol{u})$

We need to derive a function $\Psi(\boldsymbol{u})$ such that the Laurent expansion of $\frac{\mathrm{d}}{\mathrm{d}u_1}\Psi(\boldsymbol{u})$ has the same principal part at the poles as $\varphi_2(\boldsymbol{u})$. Hence we will restrict our search to linear expressions in σ-derivatives, divided by $\sigma_{23}(\boldsymbol{u})$. For these functions we will derive expansions in $w_{1,N}$ at $\boldsymbol{u} = \boldsymbol{u}_{0,N}$ using the techniques described above. Let us take the function

$$\Psi(\boldsymbol{u}) = \sum_{1\le i\le 6} \eta_i \frac{\sigma_i(\boldsymbol{u})}{\sigma_{23}(\boldsymbol{u})} + \sum_{\substack{1\le i\le j\le 6\\ [i,j]\neq[2,3]}} \eta_{ij}\frac{\sigma_{ij}(\boldsymbol{u})}{\sigma_{23}(\boldsymbol{u})} + \sum_{1\le i\le j\le k\le 6} \eta_{ijk}\frac{\sigma_{ijk}(\boldsymbol{u})}{\sigma_{23}(\boldsymbol{u})},$$

where the η_{ij} and η_{ijk} are undetermined constants. We do not include any higher index σ-derivatives in $\Psi(\boldsymbol{u})$ since it should be possible to evaluate them all at the poles as a linear combination of lower index functions.

Now, since we are working on $\Theta^{[1]}$ we will find that many of these σ-derivatives are equal to zero, or can be expressed as a linear combination of other such functions using the equations in Appendix E.1. Let us set the coefficients of these functions to zero, leaving

$$\begin{aligned}\Psi(\boldsymbol{u}) = \big[&\eta_{22}\sigma_{22} + \eta_{34}\sigma_{34} + \eta_{111}\sigma_{111} + \eta_{122}\sigma_{122} + \eta_{123}\sigma_{123} + \eta_{134}\sigma_{134} + \eta_{222}\sigma_{222}\\ &+ \eta_{223}\sigma_{223} + \eta_{224}\sigma_{224} + \eta_{225}\sigma_{225} + \eta_{226}\sigma_{226} + \eta_{233}\sigma_{233} + \eta_{234}\sigma_{234} + \eta_{235}\sigma_{235}\\ &+ \eta_{236}\sigma_{236} + \eta_{334}\sigma_{334} + \eta_{344}\sigma_{344} + \eta_{345}\sigma_{345} + \eta_{346}\sigma_{346}\big](\boldsymbol{u}) \cdot \frac{1}{\sigma_{23}(\boldsymbol{u})}.\end{aligned}$$

We emphasise that we need to work with the *total*, not the partial, derivative of $\Psi(\boldsymbol{u})$ with respect to u_1. In practice the other u_i are expressed in terms of $w_{1,N}$ in the vicinity of $\boldsymbol{u}_{0,N}$ so there is no ambiguity. Note from equation (3.4) that

$$\frac{\partial}{\partial u_2} = x\frac{\partial}{\partial u_1}, \qquad \frac{\partial}{\partial u_3} = y\frac{\partial}{\partial u_1}, \qquad \frac{\partial}{\partial u_4} = x^2\frac{\partial}{\partial u_1},$$
$$\frac{\partial}{\partial u_5} = xy\frac{\partial}{\partial u_1}, \qquad \frac{\partial}{\partial u_6} = y^2\frac{\partial}{\partial u_1}.$$

Let D_1 be the operator of differentiation with respect to u_1 on $\Theta^{[1]}$.

$$D_1 = \frac{d}{du_1}\Big|_{\Theta^{[1]}} = \frac{\partial}{\partial u_1} + x\frac{\partial}{\partial u_2} + y\frac{\partial}{\partial u_3} + x^2\frac{\partial}{\partial u_4} + yx\frac{\partial}{\partial u_5} + y^2\frac{\partial}{\partial u_6}$$
$$= \frac{\partial}{\partial u_1} - \left(\frac{\sigma_{23}(\boldsymbol{u})}{\sigma_{34}(\boldsymbol{u})}\right)\frac{\partial}{\partial u_2} + y\frac{\partial}{\partial u_3} + \left(\frac{\sigma_{23}(\boldsymbol{u})}{\sigma_{34}(\boldsymbol{u})}\right)^2\frac{\partial}{\partial u_4} - y\frac{\sigma_{23}}{\sigma_{34}}\frac{\partial}{\partial u_5} + y^2\frac{\partial}{\partial u_6}.$$

We can now evaluate $\frac{d}{du_1}\Psi(\boldsymbol{u})$ as a sum of quotients of σ-derivatives. For example, the term with σ_{236} in the numerator may be differentiated to give

$$D_1\left(\frac{\sigma_{236}}{\sigma_{23}}\right) = \left[\frac{\sigma_{1236}}{\sigma_{23}} - \frac{\sigma_{236}\sigma_{123}}{\sigma_{23}^2} - \frac{\sigma_{2236}}{\sigma_{34}} + \frac{\sigma_{236}\sigma_{223}}{\sigma_{23}\sigma_{34}} + y\frac{\sigma_{2336}}{\sigma_{23}} - y\frac{\sigma_{236}\sigma_{233}}{\sigma_{23}^2}\right.$$
$$\left. + \frac{\sigma_{23}\sigma_{2346}}{\sigma_{34}^2} - \frac{\sigma_{236}\sigma_{234}}{\sigma_{34}^2} - y\frac{\sigma_{2356}}{\sigma_{34}^2} + y\frac{\sigma_{236}\sigma_{235}}{\sigma_{23}\sigma_{34}} + y^2\frac{\sigma_{2366}}{\sigma_{23}} - y^2\frac{\sigma_{236}^2}{\sigma_{23}^2}\right].$$

Now let us consider the expansion of $D_1(\Psi(\boldsymbol{u}))$ at $\boldsymbol{u} = \boldsymbol{u}_{0,N}$. We generate series expansions in $w_{1,N}$ for the relevant σ-derivatives using the method described in the previous subsection. We can use the relations in Appendix E.1 and E.2 to simplify these expansions, and so obtain a series in $w_{1,N}$ for $D_1(\Psi(\boldsymbol{u}))$. We find,

$$\frac{d}{du_1}\Psi(\boldsymbol{u}) \quad \Big|_{\boldsymbol{u}=\boldsymbol{u}_{0,N}} = \frac{\mathcal{L}(\boldsymbol{u}_{0,N})}{\sigma_{22}(\boldsymbol{u}_{0,N})}\left[\frac{1}{w_{1,N}^2}\right] + O(w_{1,N}^0),$$

where $\mathcal{L}(\boldsymbol{u}_{0,N})$ is a linear expression in $\{\sigma_{22}, \sigma_{122}, \sigma_{222}, \sigma_{223}, \sigma_{224}, \sigma_{225}, \sigma_{226}\}$. This set of σ-derivatives can be used to express all other 2 and 3-index σ-derivatives when $\boldsymbol{u} = \boldsymbol{u}_{0,N}$ (as in Appendix E.2).

We find the coefficients of $\mathcal{L}(\boldsymbol{u}_{0,N})$ with respect to each of these seven σ-derivatives and determine conditions on the constants η_{ij}, η_{ijk} that set all the coefficients, except that of σ_{22}, to zero. We then obtain further conditions by ensuring the expansion we are left with (now independent of any σ-derivatives) matches expansion (6.33) on the four sheets.

Imposing these ten conditions on $\Psi(\boldsymbol{u})$ leaves us with

$$\Psi(\boldsymbol{u}) = \left[\eta_{22}\frac{\sigma_{22}}{\sigma_{23}} + \eta_{111}\frac{\sigma_{111}}{\sigma_{23}} + 2\eta_{334}\frac{\sigma_{235}}{\sigma_{23}} + \frac{8\eta_{22}\lambda_0 - 1}{4\lambda_0}\frac{\sigma_{236}}{\sigma_{23}} + \eta_{334}\frac{\sigma_{334}}{\sigma_{23}}\right](\boldsymbol{u}).$$

Note from Appendix E.2 that the terms containing $\sigma_{111}, \sigma_{235}, \sigma_{334}$ all vanish at the points $\boldsymbol{u} = \boldsymbol{u}_{0,N}$ and so have no effect on the expansion here. Set η_{111} and η_{334} to zero to leave,

$$\Psi(\boldsymbol{u}) = \eta_{22}\frac{\sigma_{22}(\boldsymbol{u})}{\sigma_{23}(\boldsymbol{u})} + \frac{1}{4}\frac{(8\eta_{22}\lambda_0 - 1)}{\lambda_0}\frac{\sigma_{236}(\boldsymbol{u})}{\sigma_{23}(\boldsymbol{u})}.$$

We now have two functions, $\varphi_2(\boldsymbol{u})$ and $D_1(\Psi(\boldsymbol{u}))$, which both have poles at $\boldsymbol{u}_{0,N}$. We have derived expansions at these points, given in the local parameter $w_{1,N}$, and ensured that they match. We need to also explicitly check what happens at the point $\boldsymbol{u} = \boldsymbol{0}$, where both the functions have a double pole. Note that $w_{1,N}$ is not a suitable local parameter here.

We can instead use the Taylor series expansion of $\sigma(\boldsymbol{u})$ about the origin derived in Section 3.4 of this document. We differentiate this to give expansions for the σ-derivatives and then, since we are at the origin, replace the variables $u_1, \ldots, u_6$ with their expansions, in the local parameter ξ, (found in Appendix C.1).

Now, the σ-expansion was given as a sum of polynomials, C_k, with increasing weight in $\boldsymbol{u}$ and hence the expansions will have increasing order of ξ. Since the functions we consider all contain ratios of σ-derivatives, we will only need the leading terms from each expansion in order to check regularity. Hence we only require a minimum amount of the sigma expansion, sufficient to give non-zero expansions for the derivatives we consider. We find that for the functions used here, we can truncate the σ-expansion after C_{35}.

Substituting these expansions into $\wp_2(\boldsymbol{u})$, we find

$$\lim_{\boldsymbol{u}\to\boldsymbol{0}} \wp_2(\boldsymbol{u}) = \lim_{\xi\to 0}\left[\frac{3}{4}\frac{\lambda_1}{\lambda_0}\xi^4 + \xi^8 + O(\xi^{36})\right] = 0. \tag{6.36}$$

So $\wp_2(\boldsymbol{u})$ is regular at the origin, and hence we must ensure that $\Psi(\boldsymbol{u})$ is as well if they are to match at all the poles. Upon substitution into $\Psi(\boldsymbol{u})$, we find that we must set $\eta_{22} = 0$ for the expansion to be regular. This leaves

$$\Psi(\boldsymbol{u}) = -\frac{1}{4}\frac{1}{\lambda_0}\frac{\sigma_{236}(\boldsymbol{u})}{\sigma_{23}(\boldsymbol{u})}, \qquad \text{with}$$
$$\lim_{\boldsymbol{u}\to\boldsymbol{0}} \Psi(\boldsymbol{u}) = \lim_{\xi\to 0}\left[-\frac{1}{28}\frac{\lambda_3}{\lambda_0}\xi^7 + \frac{1}{176}\frac{(-8\lambda_2 + 3\lambda_4\lambda_3)}{\lambda_0}\xi^{11} + O(\xi^{15})\right] = 0. \tag{6.37}$$

So the two functions match at all the poles. Now we need to check the periodicity properties of the functions. Recall equation (2.56) which gave the quasi-periodicity property of $\sigma(\boldsymbol{u})$. In equation (2.64) we derived the corresponding quasi-periodicity properties of the first and second σ-derivatives.

Since we are working on $\Theta^{[1]}$ we know that $\sigma(\boldsymbol{u})$ and all its first derivatives are zero which simplifies the quasi-periodicity conditions. For any $\boldsymbol{\ell} \in \Lambda$, the lattice of periods,

$$\begin{aligned} \sigma_{23}(\boldsymbol{u}+\boldsymbol{\ell}) &= \chi(\boldsymbol{\ell})\exp(L(\boldsymbol{u}+\tfrac{\boldsymbol{\ell}}{2},\boldsymbol{\ell}))\sigma_{23}(\boldsymbol{u}),\\ \sigma_{34}(\boldsymbol{u}+\boldsymbol{\ell}) &= \chi(\boldsymbol{\ell})\exp(L(\boldsymbol{u}+\tfrac{\boldsymbol{\ell}}{2},\boldsymbol{\ell}))\sigma_{34}(\boldsymbol{u}). \end{aligned}$$

(The functions χ and L were defined in Lemma 2.2.18). Therefore, the ratio of these functions will be periodic. Substituting into equation (6.27) we see that

$$\wp_2(\boldsymbol{u}+\boldsymbol{\ell}) = \wp_2(\boldsymbol{u}),$$

and so $\wp_2(\boldsymbol{u})$ is an Abelian function associated with the cyclic $(4,5)$-curve. We now need to check the periodicity of $\Psi(\boldsymbol{u})$. We can calculate the quasi-periodicity property of a 3-

index σ-derivative by differentiating equation (2.64). We find that when restricted to $\Theta^{[1]}$ this simplifies considerably. Specifically,

$$\sigma_{236}(\boldsymbol{u}+\boldsymbol{\ell}) = \chi(\boldsymbol{\ell})\exp(L(\boldsymbol{u}+\tfrac{\boldsymbol{\ell}}{2},\boldsymbol{\ell}))\cdot\left[\tfrac{\partial}{du_6}L(\boldsymbol{u}+\tfrac{\boldsymbol{\ell}}{2},\boldsymbol{\ell})\cdot\sigma_{23}+\sigma_{236}\right]. \tag{6.38}$$

It then follows that

$$\Psi(\boldsymbol{u}+\boldsymbol{\ell}) = -\frac{1}{4}\frac{1}{\lambda_0}\frac{\sigma_{236}(\boldsymbol{u}+\boldsymbol{\ell})}{\sigma_{23}(\boldsymbol{u}+\boldsymbol{\ell})} = \Psi(\boldsymbol{u}) - \frac{1}{4}\frac{1}{\lambda_0}\frac{\partial}{du_6}\big(L(\boldsymbol{u}+\tfrac{\boldsymbol{\ell}}{2},\boldsymbol{\ell})\big).$$

Since the first derivatives of $L(\boldsymbol{u}+\frac{\boldsymbol{\ell}}{2},\boldsymbol{\ell})$ are constant we know that $D_1(\Psi(\boldsymbol{u}))$ is an Abelian function, although $\Psi(\boldsymbol{u})$ itself is not.

Summary

We have shown that the two functions $\wp_2(\boldsymbol{u})$ and $D_1(\Psi(\boldsymbol{u}))$ share the same poles and have matching expansions at each. Further, they are both Abelian functions, and hence the difference between them is a holomorphic periodic function. Therefore by by the generalisation of Liouville's theorem we can conclude that this difference is a constant.

We may therefore write our integrand $\varphi(x)dx$ as

$$\wp_2(\boldsymbol{u})du_1 + A_2du_1 + A_3du_2 + A_4du_4 = D_1(\Psi(\boldsymbol{u})) + \boldsymbol{B}^T\boldsymbol{du}, \tag{6.39}$$

for some vector of constants $\boldsymbol{B}^T = (B_1, B_2, B_3, B_4, B_5, B_6)$.

6.5.3 Evaluating the vector $\boldsymbol{B}$

We can evaluate the vector $\boldsymbol{B}$ by considering the integral of equation (6.39) at the point $\boldsymbol{u} = \boldsymbol{0}$. Using equation (6.16), the expansions for $\boldsymbol{du}$ in ξ and equations (6.36) and (6.37) for the functions we obtain

$$\begin{aligned}0 &= B_6\xi + \tfrac{1}{2}B_5\xi^2 + \tfrac{1}{3}\left(-A_4+B_4\right)\xi^3 - \tfrac{1}{20}B_6\lambda_4\xi^5 + \tfrac{1}{12}\left(-B_5\lambda_4+2B_3\right)\xi^6 \\ &+ \tfrac{1}{28}\tfrac{1}{\lambda_0}\big(3\lambda_4\lambda_0A_4 - \lambda_3 - 3\lambda_4\lambda_0B_4 - 4A_3\lambda_0 + 4B_2\lambda_0\big)\xi^7 \\ &+ \tfrac{1}{288}B_6(5\lambda_4^2 - 8\lambda_3)\xi^9 + \left(-\tfrac{1}{20}B_3\lambda_4 - \tfrac{1}{20}B_5\lambda_3 + \tfrac{3}{80}B_5\lambda_4^2\right)\xi^{10} \\ &- \tfrac{1}{352}\tfrac{1}{\lambda_0}\big(32A_2\lambda_0 - 32B_1\lambda_0 + 16\lambda_2 - 6\lambda_4\lambda_3 - 24\lambda_0A_4\lambda_3 + 21\lambda_0A_4\lambda_4^2 \\ &+ 24\lambda_0B_4\lambda_3 - 21\lambda_0B_4\lambda_4^2 - 24\lambda_4\lambda_0A_3 + 24\lambda_4\lambda_0B_2\big)\xi^{11} + O(\xi^{13}).\end{aligned}$$

Setting each coefficient of ξ to zero, we find

$$\begin{aligned}B_1 &= \frac{1}{2}\frac{(\lambda_2+2A_2\lambda_0)}{\lambda_0}, & B_2 &= \frac{1}{4}\frac{(4A_3\lambda_0+\lambda_3)}{\lambda_0}, & B_3 &= 0, \\ B_4 &= A_4, & B_5 &= 0, & B_6 &= 0.\end{aligned}$$

6.6 An explicit formula for the mapping

We now use the results of Section 6.5 to derive an explicit formula for the mapping $q(p)$. Start by applying the change of coordinates given in equation (6.13) to $q(p)$ as given in equation (6.9).

$$\begin{aligned} q(p) = p + \int_{\infty}^{p} [\varphi(p') - 1] dp' &= \left(\hat{p}_8 - \frac{1}{x}\right) + \int_0^{\frac{1}{\hat{p}_8 - p}} \left(\varphi(x) - \frac{1}{x^2}\right) dx \\ &= \left(\hat{p}_8 - \frac{1}{x}\right) - \int_0^{\frac{1}{\hat{p}_8 - p}} \left[\frac{1}{x^2}\right] dx \\ &\quad + \int_0^{\frac{1}{\hat{p}_8 - p}} \left(K\left[A_4 x^4 + A_3 x^3 + A_2 x^2 + A_1 x + 1\right] \frac{1}{x^2} \frac{dx}{4y^3}\right), \end{aligned}$$

where the constants A_1, A_2, A_3, A_4 and K were defined by equation (6.15). Note from equation (6.24) that

$$p = \hat{p}_8 + \frac{\sigma_{34}(\boldsymbol{u})}{\sigma_{23}(\boldsymbol{u})}, \qquad \boldsymbol{u} \in \Theta^{[1]}. \tag{6.40}$$

So let us take $\boldsymbol{u} \in \Theta^{[1]}$, and use equation (6.40) and the evaluation of $\varphi(x)dx$ from the previous section to write $q(p)$ as,

$$\begin{aligned} q(p) = &\left(\hat{p}_8 + \frac{\sigma_{34}(\boldsymbol{u})}{\sigma_{23}(\boldsymbol{u})}\right) - \int_0^{\frac{1}{\hat{p}_8 - p}} \left[\frac{1}{x^2}\right] dx + K \int_0^{\frac{1}{\hat{p}_8 - p}} \left[\frac{1}{2}\frac{(\lambda_2 + 2A_2\lambda_0)}{\lambda_0} du_1 \right.\\ &\left. + \frac{1}{4}\frac{(4A_3\lambda_0 + \lambda_3)}{\lambda_0} du_2 + A_4 du_4\right] + K \int_0^{\frac{1}{\hat{p}_8 - p}} \left[\frac{d}{du_1}\left(-\frac{1}{4}\frac{1}{\lambda_0}\frac{\sigma_{236}(\boldsymbol{u})}{\sigma_{23}(\boldsymbol{u})}\right)\right] du_1. \end{aligned}$$

Integrating, gives

$$\begin{aligned} q(p) = &\left(\hat{p}_8 + \frac{\sigma_{34}(\boldsymbol{u})}{\sigma_{23}(\boldsymbol{u})}\right) - \left[\frac{\sigma_{34}(\boldsymbol{u})}{\sigma_{23}(\boldsymbol{u})}\right] + K\left[\frac{1}{2}\frac{(\lambda_2 + 2A_2\lambda_0)}{\lambda_0} u_1 \right.\\ &\left. + \frac{1}{4}\frac{(4A_3\lambda_0 + \lambda_3)}{\lambda_0} u_2 + A_4 u_4\right] + K\left[-\frac{1}{4}\frac{1}{\lambda_0}\frac{\sigma_{236}(\boldsymbol{u})}{\sigma_{23}(\boldsymbol{u})}\right] + \hat{C} \\ = &\ \hat{p}_8 + K\left[-\frac{1}{4\lambda_0}\frac{\sigma_{236}(\boldsymbol{u})}{\sigma_{23}(\boldsymbol{u})} + \frac{\lambda_2 + 2A_2\lambda_0}{2\lambda_0} u_1 + \frac{4A_3\lambda_0 + \lambda_3}{4\lambda_0} u_2 + A_4 u_4\right] + \hat{C}, \end{aligned}$$

for some constant $\hat{C}$. We can determine $\hat{C}$ by ensuring that the following condition on the mapping is satisfied.

$$\lim_{p \to \infty} q(p) = p + O\left(\frac{1}{p}\right)$$

Note from equation (6.40) that $p \to \infty$ implies $\sigma_{23}(\boldsymbol{u}) \to 0$ and therefore $\boldsymbol{u} \to \boldsymbol{u}_{0,N}$.

$$\lim_{p\to\infty}[q(p)-p] = \lim_{\boldsymbol{u}\to\boldsymbol{u}_{0,N}}\Bigg[\hat{C} - \frac{\sigma_{34}(\boldsymbol{u})}{\sigma_{23}(\boldsymbol{u})} + K\Bigg(- \frac{1}{4}\frac{1}{\lambda_0}\frac{\sigma_{236}(\boldsymbol{u})}{\sigma_{23}(\boldsymbol{u})} + \frac{\lambda_2 + 2A_2\lambda_0}{2\lambda_0}u_1 + \frac{4A_3\lambda_0+\lambda_3}{4\lambda_0}u_2 + A_4u_4\Bigg)\Bigg]$$

Let us ensure that the condition is met on the sheet of the surface C associated with $\lim_{t\to 0}(s) = \lambda_0^{1/4}$. We can write this as a series expansion in the local parameter w_1, (as described in Section 6.5.1). Recall that $u_i = w_i + u_{0,i}$, and use the expansions (6.31) and the existing expansions for the σ-derivatives to obtain

$$\lim_{p\to\infty}[q(p)-p] = \left[\frac{1}{4}\frac{1}{\lambda_0^{\frac{3}{4}}} - \frac{K}{16}\frac{1}{\lambda_0^{\frac{3}{2}}}\right]\frac{1}{w_1} + \Bigg[\hat{C} - \frac{3}{8}\frac{\lambda_1}{\lambda_0} + K\Bigg(- \frac{1}{4}\frac{1}{\lambda_0}\frac{\sigma_{226}(\boldsymbol{u_0})}{\sigma_{22}(\boldsymbol{u_0})} + \frac{1}{32}\frac{\lambda_1}{\lambda_0^{7/4}} + \frac{(\lambda_2+2A_2\lambda_0)}{2\lambda_0}u_{0,1} + \frac{(4A_3\lambda_0+\lambda_3)}{4\lambda_0}u_{0,2} + A_4u_{0,4}\Bigg)\Bigg] + O(w_1).$$

Therefore, we must set the constants of integration, $\hat{C}$, to be

$$\hat{C} = \frac{3}{8}\frac{\lambda_1}{\lambda_0} + K\Bigg(+ \frac{1}{4}\frac{1}{\lambda_0}\frac{\sigma_{226}(\boldsymbol{u_0})}{\sigma_{22}(\boldsymbol{u_0})} - \frac{1}{32}\frac{\lambda_1}{\lambda_0^{7/4}} - \frac{1}{2}\frac{(\lambda_2+2A_2\lambda_0)}{\lambda_0}u_{0,1} - \frac{1}{4}\frac{(4A_3\lambda_0+\lambda_3)}{\lambda_0}u_{0,2} - A_4u_{0,4}\Bigg).$$

This gives us the following explicit formula for the mapping $q(p)$.

$$q(p) = \hat{p}_8 + \frac{3}{8}\frac{\lambda_1}{\lambda_0} + K\Bigg[- \frac{1}{4}\frac{1}{\lambda_0}\left[\frac{\sigma_{236}(\boldsymbol{u})}{\sigma_{23}(\boldsymbol{u})} - \frac{\sigma_{226}(\boldsymbol{u_0})}{\sigma_{22}(\boldsymbol{u_0})}\right] - \frac{1}{32}\frac{\lambda_1}{\lambda_0^{7/4}} + \frac{\lambda_2+2A_2\lambda_0}{2\lambda_0}(u_1-u_{1,0}) + \frac{4A_3\lambda_0+\lambda_3}{4\lambda_0}(u_2-u_{2,0}) + A_4(u_4-u_{4,0})\Bigg],$$

where $\boldsymbol{u} \in \Theta^{[1]}$ and $\boldsymbol{u_0}$ is the point on the principal sheet of the surface C where $t = 0$.

Appendices

Appendix A

Background Mathematics

A.1 Properties of elliptic functions

Theorem A.1.1. *We now present a number of properties for elliptic functions.*

(i) The number of poles in a cell is finite.

Proof. Suppose for a contradiction that there is an infinite sequence of poles, bounded by the cell. By the BolzanoWeierstrass theorem this would contain a convergent subsequence. The limit point to which it converges would be a singularity of the function that is not isolated. Hence by Definition 2.1.2 it is an essential singularity and so the function is not elliptic, which is a contradiction.

■

(ii) The number of zeros in a cell is finite.

Proof. If not then the inverse function would have an infinite number of poles, and therefore an essential singularity. This would then be an essential singularity of the function itself and hence the function would not be elliptic which is a contradiction.

■

(iii) The sum of residues of poles in a cell is zero.

Proof. The residue theorem from complex analysis tells us that the sum of residues of the function $f(u)$ around the domain P is equal to

$$\frac{1}{2\pi i}\int_{\delta P} f(u)du,$$

where δP is the boundary of P taken counterclockwise. For our case it is intuitive that the integrals over opposite sides of the parallelogram will cancel each other out. To see this more clearly let C be the contour formed around the edges of the cell and let the corners of the cell be given by $t, t+\omega_1, t+\omega_2, t+\omega_1+\omega_2$. Then the sum of the residues of an elliptic function $f(u)$ at its poles inside C is given by

$$\frac{1}{2\pi i}\int_C f(u)du = \frac{1}{2\pi i}\left[\int_t^{t+\omega_1} + \int_{t+\omega_1}^{t+\omega_1+\omega_2} + \int_{t+\omega_1+\omega_2}^{t+\omega_2} + \int_{t+\omega_2}^{t}\right] f(u)du.$$

Then use the substitutions $u' = u - \omega_1$ and $u'' = u - \omega_2$ in the second and third intervals respectively. We obtain

$$\begin{aligned}
&= \frac{1}{2\pi i}\left[\int_t^{t+\omega_1} f(u)du + \int_t^{t+\omega_2} f(u')du' + \int_{t+\omega_1}^{t} f(u'')du'' + \int_{t+\omega_2}^{t} f(u)du\right] \\
&= \frac{1}{2\pi i}\left[\int_t^{t+\omega_1} f(u)du + \int_t^{t+\omega_2} f(u')du' - \int_t^{t+\omega_1} f(u'')du'' - \int_t^{t+\omega_2} f(u)du\right] \\
&= \frac{1}{2\pi i}\left[\int_t^{t+\omega_1} \{f(u) - f(u-\omega_2)\}du\right] + \frac{1}{2\pi i}\left[\int_t^{t+\omega_2} \{f(u-\omega_1) - f(u)\}du\right] \\
&= 0
\end{aligned}$$

The final step is down to the periodicity properties of $f(u)$ causing each integrand to be zero.

■

(iv) There does not exist an elliptic function with a single simple pole.

Proof. This is an obvious corollary to the previous property.

■

(v) An elliptic function with no poles in a cell is a constant.

Proof. If $f(u)$ is an elliptic function with no poles then it is holomorphic. It is therefore bounded inside and on the boundary of the cell and hence a constant by Liouville's Theorem below.

■

Theorem A.1.2 (Liouville's theorem). *For all values of u let $f(u)$ be holomorphic and satisfy $|f(u)| < K$ for K constant. Then $f(u)$ is a constant.*

Proof. $f : \mathbb{C} \to \mathbb{C}$ is a bounded, entire function and so by Taylor's theorem,

$$f(u) = \sum_{n=0}^{\infty} c_n u^n \qquad \text{where} \qquad c_n = \frac{1}{2\pi i}\int_{\Gamma_r} \frac{f(w)}{w^{n+1}}dw$$

and Γ_r is the circle of radius r about 0, for $r > 0$. Now consider c_n.

$$|c_n| \leq \frac{1}{2\pi}\,\mathrm{length}(\Gamma_r)\cdot \sup\left\{\left|\frac{f(w)}{w^{n+1}}\right| : w \in \Gamma_r\right\} = \frac{1}{2\pi}2\pi r\frac{M_r}{r^{n+1}} = \frac{M_r}{r^n}$$

where $M_r = \sup\{|f(w)| : w \in \Gamma_r\}$.

But f is bounded, so there is M such that $M_r \leq M$ for all r. Then

$$|c_n| \leq \frac{M}{r^n} \qquad \text{for all n and all } r > 0.$$

Since r is arbitrary, this gives $c_n = 0$ whenever $n > 0$. So $f(u) = c_0$ for all u, giving $f(u)$ constant, as required.

■

Theorem A.1.3. *A non-constant elliptic function $f(u)$ has exactly as many poles as zeros (counting multiplicities)*

Proof. Both the poles and zeros of $f(u)$ will be simple poles of the function $f'(u)/f(u)$. Suppose $f(u)$ has a pole of order n at $u = 0$ and a zero of order m at $u = c$. Then

$$f(u) = \frac{1}{u^n} g(u) \qquad \Longrightarrow \qquad f'(u) = \frac{-n}{u^{n+1}} g(u) + \frac{1}{u^n} g'(u)$$
$$\Longrightarrow \quad \frac{f'(u)}{f(u)} = \frac{-ng(u)}{u^{n+1}} \cdot \frac{u^n}{g(u)} + \frac{g'(u)}{u^n} \cdot \frac{u^n}{g(u)} = \frac{-n}{u} + \frac{g'(u)}{g(u)}$$

$$f(u) = (u-c)^m h(u) \quad \Longrightarrow \quad f'(u) = m(u-c)^{m-1} h(u) + (u-c)^m h'(u)$$
$$\Longrightarrow \quad \frac{f'(u)}{f(u)} = \frac{m(u-c)^{m-1} h(u)}{(u-c)^m h(u)} + \frac{(u-c)^m h'(u)}{((u-c)^m h(u)} = \frac{m}{u-c} + \frac{h'(u)}{h(u)}$$

So we see that $f'(u)/f(u)$ has poles at both $u = 0$ and $u = c$ as expected.

Also note that the multiplicities will be the residues of $f'(u)/f(u)$ counted positive for zeros and negative for poles. Therefore the sum of the residues for $f'(u)/f(u)$ will be the sum of the poles and zeros of $f(u)$ counted according to multiplicities.

Now it is clear that $f'(u)/f(u)$ will be an elliptic function given that $f(u)$ is elliptic. Therefore, by Theorem A.1.1(iii) this sum must be zero and so the number of poles must equal the number of zeros.

■

Corollary A.1.4. *If $f(u)$ is an elliptic function and c a constant then the number of roots of $f(u) = c$ in any cell is the number of poles of $f(u)$ in a cell. We define this number as the **order** of the elliptic function.*

Proof. The number of roots of $f(u) = c$ is equal to the number of zeros of the function

$$F(u) = f(u) - c.$$

Now every pole of $F(u)$ will be a pole of $f(u)$ and conversely. Hence $F(u)$ has the same number of poles as $f(u)$.

Also, $F(u)$ is an elliptic function so by Theorem A.1.3 the number of zeros it has is equal the number of poles it has. Hence the number of roots of $f(u) - c$ is equal to the number of poles of $f(u)$.

■

Theorem A.1.5. *Let $a_1, \dots, a_n$ denote the zeros and $b_1, \dots, b_n$ the poles of an elliptic function (counting multiplicities). Then*

$$a_1 + \cdots + a_n \equiv b_1 + \cdots + b_n \pmod{\omega_1 + \omega_2}. \tag{A.1}$$

Proof. By Cauchy's argument principle, a meromorphic function $f(u)$ inside and on some closed contour C, with no zeros or poles on C, satisfies the following formula.

$$\int_C uf'(u)f(u)du = \sum_i a_i - \sum_j b_j$$

where a_i and b_j denote respectively the zeros and poles of $f(u)$ inside the contour. We can apply this here with C the boundary of the parallelogram. (Note that we can shift the parallelogram if necessary to satisfy the restrictions). We obtain

$$\begin{aligned}&(a_1 + \cdots + a_n) - (b_1 + \cdots + b_n)\\ &= \frac{1}{2\pi i}\left[\int_t^{t+\omega_1} + \int_{t+\omega_1}^{t+\omega_1+\omega_2} + \int_{t+\omega_1+\omega_2}^{t+\omega_2} + \int_{t+\omega_2}^{t}\right]\frac{uf'(u)}{f(u)}du.\end{aligned}$$

Using the same substitutions as in the proof of Theorem A.1.1 (iii) will give

$$\begin{aligned}(a_1 + \cdots + a_n) - (b_1 + \cdots + b_n) = \ & \frac{1}{2\pi i}\int_t^{t+\omega_1}\left[\frac{uf'(u)}{f(u)} - \frac{(u+\omega_2)f'(u+\omega_2)}{f(u+\omega_2)}\right]du\\ & -\frac{1}{2\pi i}\int_t^{t+\omega_2}\left[\frac{uf'(u)}{f(u)} - \frac{(u+\omega_1)f'(u+\omega_1)}{f(u+\omega_1)}\right]du.\end{aligned}$$

Then applying the periodicity properties of $f(u)$ gives

$$\begin{aligned}(a_1 + \cdots + a_n) - (b_1 + \cdots + b_n) &= \frac{1}{2\pi i}\left[-\omega_2\int_t^{t+2\omega_1}\frac{f'(u)}{f(u)}du + \omega_1\int_t^{t+2\omega_2}\frac{f'(u)}{f(u)}du\right]\\ &= \frac{1}{2\pi i}\left[-\omega_2\big[\log f(u)\big]_t^{t+2\omega_1} + \omega_1\big[\log f(u)\big]_t^{t+2\omega_2}\right].\end{aligned}$$

Now $f(u)$ has the same value at the points $t + \omega_1, t + \omega_2$ as at t so the values of $\log f(u)$ can only differ by integer multiples of $2\pi i$ say $-2n\pi i$ and $2m\pi i$. Then we have that

$$(a_1 + \cdots + a_n) - (b_1 + \cdots + b_n) = m\omega_1 = n\omega_2,$$

and therefore $a_1 + \cdots + a_n \equiv b_1 + \cdots + b_n \pmod{\omega_1 + \omega_2}$.

■

A.2 Jacobi θ-functions

The θ-functions were first studied by Jacobi as part of his research on elliptic functions. They can be used to move between the Jacobi and Weierstrass theory of elliptic functions. They are also the most efficient way to perform numerical computations with elliptic functions. In this appendix we give the definitions, some of the key properties and the formulae that link them to elliptic functions. For a more detailed introduction to these functions see Chapter 21 of [70] and for full details and proofs see [54].

The **Jacobi θ-function** is defined as

$$\theta(u;q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2inu}. \tag{A.2}$$

Here u is a complex variable and q is a complex number labeled the **nome**, which should satisfy $|q| < 1$. Alternatively, q may be replaced by $q = e^{i\pi\tau}$ where τ has positive imaginary part. The θ-function is periodic with period π,

$$\theta(u+\pi;q) = \theta(u;q), \tag{A.3}$$

and is quasi-periodic with respect to $\pi\tau$,

$$\theta(u+\pi\tau;q) = -q^{-1}e^{-2iu}\theta(u;q). \tag{A.4}$$

It is customary to define three further θ-functions and relabel the Jacobi θ-function as $\theta_4(u)$. Note that it is possible to express them as either infinite sums or infinite products.

$$\begin{aligned}
\theta_1(u;q) &= -i \sum_{n=-\infty}^{\infty} (-1)^n q^{\left(n+\frac{1}{2}\right)^2} e^{i(2n+1)u} && \text{(A.5)}\\
&= 2q^{\frac{1}{4}} \sin(u) \prod_{n=1}^{\infty} (1-q^{2n})(1-2q^{2n}\cos(2u)+q^{4n}).\\
\theta_2(u;q) &= \sum_{n=-\infty}^{\infty} q^{\left(n+\frac{1}{2}\right)^2} e^{i(2n+1)u} && \text{(A.6)}\\
&= 2q^{\frac{1}{4}} \cos(u) \prod_{n=1}^{\infty} (1-q^{2n})(1+2q^{2n}\cos(2u)+q^{4n}).\\
\theta_3(u;q) &= \sum_{n=-\infty}^{\infty} q^{n^2} e^{2inu} && \text{(A.7)}\\
&= \prod_{n=1}^{\infty} (1-q^{2n})(1+2q^{2n}\cos(2u)+q^{4n-2}).
\end{aligned}$$

$$\begin{aligned}\theta_4(u;q) &= \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} e^{2inu} \\ &= \prod_{n=1}^{\infty}(1-q^{2n})(1-2q^{2n}\cos(2u)+q^{4n-2}).\end{aligned} \tag{A.8}$$

These are all entire functions of u. Often it is only the dependence on u that is considered and in these cases $\theta_i(u)$ is used. The function $\theta_1(u)$ is an odd function of u, while the other three are all even functions. All four functions are periodic in the real direction.

$$\begin{aligned}\theta_1(u+\pi) = -\theta_1(u) &\implies \theta_1(u+2\pi) = \theta_1(u), \\ \theta_2(u+\pi) = -\theta_2(u) &\implies \theta_2(u+2\pi) = \theta_2(u), \\ \theta_3(u+\pi) = +\theta_3(u), & \\ \theta_4(u+\pi) = +\theta_4(u). &\end{aligned} \tag{A.9}$$

Similarly, all four functions satisfy a second quasi-periodicity condition in the complex direction.

$$\begin{aligned}\theta_1(u+\pi\tau) &= -q^{-1}e^{-2iu}\theta_1(u), \qquad \theta_3(u+\pi\tau) = +q^{-1}e^{-2iu}\theta_3(u), \\ \theta_2(u+\pi\tau) &= +q^{-1}e^{-2iu}\theta_2(u), \qquad \theta_4(u+\pi\tau) = -q^{-1}e^{-2iu}\theta_4(u).\end{aligned} \tag{A.10}$$

It is natural to consider a cell formed by these two periods; say the the cell with corners $t, t+\pi, t+\pi+\pi\tau, t+\pi\tau$. We can show that any θ-function will have one simple zero in such a cell. Further each of the four functions has its zero on the respective corner of the fundamental period parallelogram.

$$\begin{aligned}\theta_1(u) = 0 \quad &\text{where} \quad u = m\pi + n\pi\tau, \\ \theta_2(u) = 0 \quad &\text{where} \quad u = \left(m+\tfrac{1}{2}\right)\pi + n\pi\tau, \\ \theta_3(u) = 0 \quad &\text{where} \quad u = \left(m+\tfrac{1}{2}\right)\pi + (n+\tfrac{1}{2})\pi\tau, \\ \theta_4(u) = 0 \quad &\text{where} \quad u = m\pi + (n+\tfrac{1}{2})\pi\tau,\end{aligned} \tag{A.11}$$

where $m, n \in \mathbb{Z}$.

There is a wealth of identities and relations for the θ-functions that are not presented here, (see for example [54] and [15]). We also note that they have several applications outside of elliptic function theory.

We now proceed to demonstrate how the Weierstrass functions may be expressed using the θ-function.

We can express the Weierstrass functions with periods ω_1, ω_2 using θ-functions with parameter τ where $\tau = \omega_2/\omega_1$. This ratio of the periods must have non-zero imaginary part by definition and we label the periods here so that this imaginary part is positive, satisfying the assumption on τ. The Weierstrass σ-function can then be expressed as,

$$\sigma(u,\omega_1,\omega_2) = \frac{\omega_1}{\pi\theta_1'(0)}\exp\left(-\frac{\pi^2}{6\omega_1^2}\frac{\theta_1'''(0)}{\theta_1'(0)}u^2\right)\theta_1\left(\frac{\pi u}{\omega_1};\frac{\omega_2}{\omega_1}\right), \tag{A.12}$$

where the prime notation indicates derivatives with respect to the variable u. Similarly the Weierstrass $\wp$-function can be expressed as,

$$\wp(u) = \left(\frac{\pi}{\omega_1}\right)^2\left[\frac{1}{3}\frac{\theta_1'''(0)}{\theta_1'(0)} - \frac{d^2}{dz^2}\Big(\log\left[\theta_1(z;\tau)\right]\Big)\right], \tag{A.13}$$

where $z = \pi u/\omega_1$ and $\tau = \omega_2/\omega_1$.

A.3 Multivariate θ-functions

The multivariate θ-functions are generalisations of the Jacobi θ-functions discussed in Appendix A.2. They were first studied by Riemann and are used in this document to realise the Kleinian σ-function. We give here the definitions and essential information but refer the reader to [59], [42] and [53] for a detailed study. Note that these functions have applications outside of Abelian function theory, see for example [16].

Let $\boldsymbol{u} \in \mathbb{C}^g$ be a vector of variables, $\boldsymbol{m}$ an arbitrary integer column vector and τ a symmetric matrix with positive definite imaginary part. (This plays a similar role to the complex number τ in the case of Jacobi θ-functions.)

Definition A.3.1. *The **canonical θ-function** is*

$$\theta(\boldsymbol{u};\tau) = \sum_{\boldsymbol{m}\in\mathbb{Z}^g}\exp\Big[2\pi i\left(\tfrac{1}{2}\boldsymbol{m}^T\tau\boldsymbol{m} + \boldsymbol{m}^T\boldsymbol{u}\right)\Big]. \tag{A.14}$$

The condition on τ ensures this series converges. As in the classic case this is an even function of $\boldsymbol{u}$ which satisfies a periodicity condition in the real direction and a quasi-periodicity condition in the complex direction.

$$\begin{aligned}\theta(\boldsymbol{u}+\boldsymbol{n};\tau) &= \theta(\boldsymbol{u};\tau) && \forall\boldsymbol{n}\in\mathbb{Z}^g.\\ \theta(\boldsymbol{u}+\tau\boldsymbol{n};\tau) &= \exp\big[-2\pi i\{\tfrac{1}{2}\boldsymbol{n}^T\tau\boldsymbol{n}+\boldsymbol{u}^T\boldsymbol{n}\}\big]\theta(\boldsymbol{u};\tau) && \forall\boldsymbol{n}\in\mathbb{Z}^g.\end{aligned} \tag{A.15}$$

As in the classic case we can define a set of additional θ-functions.

Definition A.3.2. *Define a **characteristic** $[\epsilon]$ as a column vector*

$$[\epsilon] = \begin{bmatrix}\boldsymbol{\epsilon}'\\ \boldsymbol{\epsilon}''\end{bmatrix} \in \mathbb{R}^{2g},$$

which we split into the two column vectors of length g denoted by $\boldsymbol{\epsilon}'$ and $\boldsymbol{\epsilon}''$.

*Then the **θ-function with characteristic** $[\boldsymbol{\epsilon}]$ is given by*

$$\theta[\epsilon](\boldsymbol{u};\tau) = \sum_{\boldsymbol{m}\in\mathbb{Z}^g} \exp\left[2\pi i \left\{\tfrac{1}{2}(\boldsymbol{m}+\boldsymbol{\epsilon'})^T\tau(\boldsymbol{m}+\boldsymbol{\epsilon'}) + (\boldsymbol{m}+\boldsymbol{\epsilon'})^T(\boldsymbol{u}+\boldsymbol{\epsilon''})\right\}\right]. \quad \text{(A.16)}$$

The θ-functions with characteristic can be expressed in terms of the canonical θ-function,

$$\theta[\epsilon](\boldsymbol{u};\tau) = \exp\left[2\pi i(\boldsymbol{\epsilon'})^T(\tfrac{1}{2}\tau\boldsymbol{\epsilon'} + \boldsymbol{u} + \boldsymbol{\epsilon''})\right]\theta(\boldsymbol{u} + \boldsymbol{\epsilon''} + \tau\boldsymbol{\epsilon'};\tau),$$

and satisfies similar periodicity properties.

$$\begin{aligned}\theta[\epsilon](\boldsymbol{u}+\boldsymbol{n_1}+\boldsymbol{n_2}\tau;\tau) = \exp\big[-2\pi i\big\{&\boldsymbol{n_2}^T\boldsymbol{u} + \tfrac{1}{2}\boldsymbol{n_2}^T\tau\boldsymbol{n_2} \\ &- \boldsymbol{n_1}^T\boldsymbol{\epsilon'} + \boldsymbol{n_2}^T\boldsymbol{\epsilon''}\big\}\big]\cdot\theta(\boldsymbol{u};\tau), \qquad \boldsymbol{n_1},\boldsymbol{n_2}\in\mathbb{Z}^g.\end{aligned}$$

It is often assumed that the characteristic $[\epsilon]$ is a vector of half integers. Given this property the θ-functions with characteristic have definite parity.

$$\theta[\epsilon](-\boldsymbol{u};\tau) = \begin{cases} +\theta[\epsilon](\boldsymbol{u};\tau) & \text{if} \quad 4(\boldsymbol{\epsilon''})^T\boldsymbol{\epsilon'} \equiv 0 \pmod 2 \\ -\theta[\epsilon](\boldsymbol{u};\tau) & \text{if} \quad 4(\boldsymbol{\epsilon''})^T\boldsymbol{\epsilon'} \equiv 1 \pmod 2 \end{cases}$$

We now consider those θ-functions connected to the (n,s)-curve C by setting the matrix $\tau = (\omega')^{-1}\omega''$ where ω', ω'' are the period matrices defined in equation (2.46). Earlier we noted that $(\omega')^{-1}\omega''$ does satisfy the condition on τ.

Let t_a be some base point on the curve. The branching points of C are connected with the half-integer characteristics. For a branching point a_j the vector,

$$\int_{t_a}^{a_j} \boldsymbol{dv} = \boldsymbol{\epsilon''} + \tau\boldsymbol{\epsilon'},$$

where $\boldsymbol{\epsilon'}, \boldsymbol{\epsilon''}$ constitute a half-integer characteristic $[\epsilon]$.

We now discuss how to define a special θ-function that vanishes on a particular subset of the Jacobian of C. It is this function that is used to construct the Kleinian σ-function.

Definition A.3.3. *Define*

$$\boldsymbol{dv} = (\omega')^{-1}\boldsymbol{du} \quad \text{(A.17)}$$

*to be the **basis of normalised holomorphic differentials**.*

Definition A.3.4. *Denote the **vector of Riemann constants with base point*** $\boldsymbol{t_a}$ *by* Δ_{t_a} *and define it as below.*

$$\left(\Delta_{t_a}\right)_i = \frac{1}{2}(1+\tau_{ii}) - \sum_{\substack{j=1\\ j\neq i}}^{g}\left(\oint_{\alpha_j}\boldsymbol{dv}_j\int_{t_a}^{t}\boldsymbol{dv}_i\right), \qquad i = 1,\ldots,g. \quad \text{(A.18)}$$

Here the subscripts indicate the components of the vector and matrices. The vector of Riemann constants may be expressed as,

$$\Delta_{t_a} = \sum_{k=1}^{a_i} \int_{t_a}^{a_i} \boldsymbol{dv},$$

where the a_i are branching point of the curve. Hence the vector of Riemann constants is connected to a particular θ-function characteristic $[\epsilon]$ by

$$\Delta_{t_a} = \omega' \boldsymbol{\epsilon''} + \omega'' \boldsymbol{\epsilon'}.$$

Definition A.3.5. *Define the* ***Riemann θ-function*** *using the canonical θ-function as*

$$R(t) = \theta\left(\int_{t_a}^{t} \boldsymbol{dv} - \boldsymbol{p} - \Delta_{t_a}; \tau\right), \tag{A.19}$$

where t is a point on the curve C and $\boldsymbol{p}$ is a point on the Jacobian J.

Theorem A.3.6. ***[Riemann's vanishing theorem]***
The Riemann θ-function defined above either vanishes identically or has exactly g zeros $t_1, \dots, t_g \in C$ such that

$$\sum_{i=1}^{g} \int_{t_a}^{t_i} dv_j = p_j, \qquad j = 1 \dots g, \tag{A.20}$$

where the p_j are the components of the point $\boldsymbol{p}$. The points t_i are defined uniquely up to permutation and are not congruent on the lattice, Λ.

Corollary A.3.7. *Define the* ***Theta divisor*** *as the set*

$$\overline{\Theta} = \left\{ \boldsymbol{u} \in J : \quad \boldsymbol{u} = \sum_{i=1}^{g-1} \int_{t_a}^{t_i} \boldsymbol{du} \right\}. \tag{A.21}$$

Then

$$\theta\big((\omega)^{-1}\boldsymbol{u} - \Delta_{t_a}\big) = 0 \qquad \forall \boldsymbol{u} \in \overline{\Theta}. \tag{A.22}$$

It is beneficial to set the base point t_a to be the point ∞ on the curve C. Note that the Theta divisor then becomes the strata $\Theta^{[g-1]}$ from Definition 2.2.14. Denote by Δ_∞ the vector of Riemann constants with base point ∞ and let

$$[\delta] = \begin{bmatrix} \boldsymbol{\delta'} \\ \boldsymbol{\delta''} \end{bmatrix}$$

be the θ-function characteristic that is related to Δ_∞.

A.4 The Weierstrass Gap Sequence

The Weierstrass gap sequence was presented in Definition 2.2.3 as follows.

Definition. *Let (n, s) be a pair of coprime integers such that $s > n \geq 2$. Then the natural numbers not representable in the form*

$$an + bs \qquad \text{where} \qquad a, b \in \mathbb{N} = \{0, 1, 2, \dots\} \tag{A.23}$$

*form a **Weierstrass Sequence**, $W_{n,s}$. The numbers in the sequence are the **gaps**, with the numbers representable in the form of equation (A.23) the **non-gaps**.*

In this Appendix we prove some of the properties of the Weierstrass gap sequence.

Lemma A.4.1. *Any element of the Weierstrass sequence $w \in W_{n,s}$ can be represented in the form*

$$w = -\alpha n + \beta s \qquad \text{where} \qquad \alpha, \beta \in \mathbb{Z}, \qquad \alpha > 0, \quad 0 < \beta < n. \tag{A.24}$$

The integers α, β are determined uniquely.

Proof. Bézout's identity tells us that for two non-zero integers, a, b there will exist $x, y \in \mathbb{Z}$ such that $xa + yb = \gcd(a, b)$. Therefore here we have $xn + ys = \gcd(n, s) = 1$ for some $x, y \in \mathbb{Z}$. Let w be an element of $W_{n,s}$. Then

$$\begin{aligned} w &= (wx)n + (wy)s \\ &= (wx + \kappa s)n + (wy - \kappa n)s \end{aligned}$$

where κ can be any integer. Choose κ such that $(wy - \kappa n) \in (0, n)$ — there will be one unique choice of κ that achieves this. Since this chosen κ ensures the coefficient of s is positive, the coefficient of n must be negative (or else $w \notin W_{n,s}$ by definition). Also, since the choice of κ was unique the coefficient of n is also unique.

■

Lemma A.4.2. *A Weierstrass sequence $W_{n,s} = \{w_1, w_2, \dots, w_g\}$ has the following properties*

(1) The length of the sequence is $g = \frac{1}{2}(n-1)(s-1)$.

(2) The maximal element w_g is equal to $2g - 1$.

(3) If $w \in W_{n,s}$ then $(w_g - w) \notin W_{n,s}$.

(4) If $w > \overline{w}$ where $w \in W_{n,s}$ and $\overline{w} \notin W_{n,s}$, then $(w - \overline{w}) \in W_{n,s}$.

(5) Each element in the gap sequence satisfies $i \leq w_i \leq 2i - 1$.

We prove this lemma below. First note that we usually denote the sequence of non-gaps by $\overline{W}_{n,s}$ and write both sequences in ascending order.

$$W_{n,s} = \{w_1, w_2, \ldots, w_g\} \qquad \overline{W}_{n,s} = \{\overline{w}_1, \overline{w}_2, \ldots\}$$

Example A.4.3. to calculate $W_{3,4}$ we take each natural number in turn and see if it can be represented as $3a + 4b$ for $a, b \in \mathbb{N}$. If so it is a nongap and we overline it. We stop once we have found $g = \frac{1}{2}(n-1)(s-1) = 3$ gaps.

$$\overline{0}, 1, 2, \overline{3, 4}, 5, \overline{6, 7, 8, 9, 10, \ldots}$$

So $W_{3,4} = \{1, 2, 5\}$.

□

Proof of Lemma A.4.2

(1) We want to find how many integers there are of the form (A.24). Now, we know that $\beta \in \{1, \ldots, n-1\}$, $\alpha > 0$ and $-\alpha n + \beta s = w > 0$. If we set $\beta = l$ then we know that $\alpha \leq \lfloor ls/n \rfloor$ since if it were any greater we would have $w < 0$. So α ranges over $\{1, \ldots, \lfloor ls/n \rfloor\}$ meaning for each possible β there are $\lfloor ls/n \rfloor$ possible α. Hence the length of the sequence will be

$$g = \sum_{l=1}^{n-1} \left\lfloor \frac{ls}{n} \right\rfloor$$

We can then use Lemma A.4.4 below to conclude $g = \frac{1}{2}(n-1)(s-1)$.

■

Lemma A.4.4. *Let p, q be two coprime integers. Then*

$$\sum_{i=1}^{q-1} \left\lfloor \frac{ip}{q} \right\rfloor = \left\lfloor \frac{p}{q} \right\rfloor + \left\lfloor \frac{2p}{q} \right\rfloor + \cdots + \left\lfloor \frac{(q-1)p}{q} \right\rfloor = \frac{(p-1)(q-1)}{2}$$

Proof. Consider a system of Cartesian coordinates. Draw the line connecting the origin to the point (p, q). Note that since (p, q) are coprime this line contains no other points with integer coordinates.

The number of points inside the rectangle $[(0,0) \to (0,p) \to (p,q) \to (q,0) \to (0,0)]$ is clearly $(p-1)(q-1)$. The equation of the diagonal line will be $y = (p/q)x$. Figure A.1 below gives an example for $p = 7, q = 16$.

Now consider an integer m. The point mp/q will be the point on the diagonal line where $x = m$. The value of $\lfloor mp/q \rfloor$ will be the number of integer points below that point. For example, in the diagram we see that when $m = 8$ there are three such points; (8,1), (8,2) and (8,3). This corresponds to $\lfloor 8 \cdot 7/16 \rfloor = \lfloor 3.5 \rfloor = 3$.

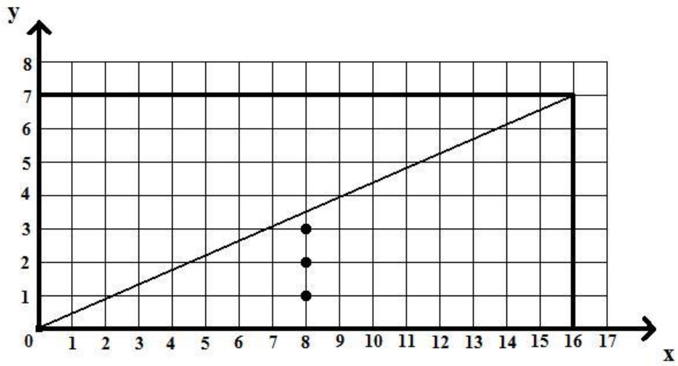

Figure A.1: Example of Lemma A.4.4 with $p = 7, q = 16$.

Therefore the left hand side of the equality above will be the number of points below the diagonal line. Now since no integer points lay on the diagonal line, and the total number of points is $(p-1)(q-1)$ we have that exactly half lie below, and so the left hand side is equal to $(p-1)(q-1)/2$.

■

(2) First note that

$$\begin{aligned} 2g - 1 &= (n-1)(s-1) - 1 = ns - n - s \\ &= s(n-1) - n. \end{aligned}$$

So by Lemma A.4.1 we know that $2g - 1$ is in the gap sequence. Suppose for a contradiction that there exists another number w of the form (A.24) that is greater than $2g-1$. Then we have α, β such that

$$-\alpha n + \beta s > -n + s(n-1) \quad \Longrightarrow \quad \alpha < 1 - sn(n-1-\beta).$$

We know that $\alpha \geq 1$, which means that $-sn(n-1-\beta) > 0$. Since n and s are both positive we must have,

$$n - 1 - \beta < 0 \quad \Longrightarrow \quad \beta > n - 1 \quad \Longrightarrow \quad \beta \geq n,$$

which is a contradiction.

■

(3) From part (2) we have that $w_g = 2g - 1 = -n + s(n-1)$. Therefore, for some α, β

$$w_g - w = -n + s(n-1) - (-\alpha n + \beta s) = (\alpha - 1)n + (n - 1 - \beta)s.$$

Therefore $(w_g - w)$ can be expressed in the form of (A.23) and hence $(w_g - w) \notin W_{n,s}$.

■

(4) Let $w \in W_{n,s}$ and $\overline{w} \notin W_{n,s}$. Therefore

$$w = -\alpha n + \beta s, \qquad \overline{w} = an + bs.$$

Then

$$w - \overline{w} = -\alpha n + \beta s - (an + bs) = -(\alpha + a)n + (\beta - b)s > 0.$$

(The > 0 was by assumption in the statement.) Now we can check that $(w - \overline{w}) \in W_{n,s}$. Clearly $(\alpha + a) > 0$ and since $(w - \overline{w}) > 0$ we have $(\beta - b) > 0$. Finally, since $\beta < n$ we also have $(\beta - b) < n$ which shows $(w - \overline{w})$ to be of the form (A.24).

■

(5) Consider the set,

$$\tilde{W}_{n,s} = \{2g - 1 - w_{g+1-i}\}_{i=1,\ldots,g} = \{(w_g - w_g), (w_g - w_{g-1}), \ldots, (w_g - w_1)\},$$

recalling that $w_g = 2g - 1$ from part (2). Next, by part (3), we know that $\tilde{W}_{n,s} \cap W_{n,s} = \{\phi\}$. Then, since both $\tilde{W}_{n,s}$ and $W_{n,s}$ contain g elements, we have that $W_{n,s} \cup \tilde{W}_{n,s}$ contains $2g$ different non-negative integers, each of which is less than $2g - 1$. Hence $\tilde{W}_{n,s} \cup W_{n,s}$ is a permutation of

$$\{0, 1, 2, \ldots, (2g - 1)\}.$$

Now let $N(L)$ be the number of integers of the form (A.23) that are less than L. Then

$$\{\tilde{w}_1, \ldots, \tilde{w}_{N(w_i)}\} \subset \tilde{W}_{n,s}, \qquad \{w_1, \ldots, w_i\} \subset W_{n,s}$$

and the intersection $\{\tilde{w}_1, \ldots, \tilde{w}_{N(w_i)}\} \cap \{w_1, \ldots, w_i\}$ is therefore empty. Similarly, the union $\{\tilde{w}_1, \ldots, \tilde{w}_{N(w_i)}\} \cup \{w_1, \ldots, w_i\}$ will be permutation of the set $\{0, 1, 2, \ldots, w_i\}$ for any $w_i \in W_{n,s}$. Now, in this case $2g = i + N(w_i)$ and hence $w_i = i + N(w_i) - 1$.

Next note that $\tilde{w}_0 = 0 < w_i$ and so $N(w_i) \geq 1$. Then by the definition of $N(w_i)$ we see that $w_i \geq i$ for all $w_i \in W_{n,s}$.

Next, by part (4) we have that

$$\{(w_i - \tilde{w}_1), (w_2 - \tilde{w}_2) \ldots, (w_i - \tilde{w}_{N(w_i)})\} \subset \{w_1, \ldots, w_i\}$$

and so $N(w_i) \leq i$. Thus

$$w_i = i + N(w_i) - 1 \leq 2i - 1.$$

So, as required, we have $i \leq w_i \leq 2i - 1$ for all $i = 1, \ldots, g$.

■

A.5 The Schur-Weierstrass polynomial

In this Appendix we define and construct Schur Weierstrass polynomials. The material here loosely follows [20] with extra material cited independently when used. These polynomials will be defined by a pair of coprime integers (n, s) and use results on the Weierstrass gap sequence that were discussed in the previous Appendix. Schur-Weierstrass polynomials are important in this document as they form the first part of the series expansion about the origin of the Kleinian σ-function associated with the corresponding (n, s)-curve.

A.5.1 Weierstrass Partitions

Definition A.5.1. *A **partition**, π of length m is a non-increasing set of m positive integers π_i written $\pi = \{\pi_1, \dots, \pi_m\}$.*

For example, $\{3, 2, 2, 2, 1\}$ is a partition of length 5. Let Par_m denote the set of all partitions of length m and let the symbol $\#$ denote the operation of taking the cardinality (number of elements) of a set.

Definition A.5.2. *We define the following **conjugation operation** on the set of partitions and denote it using the dash symbol.*

$$\pi' = \{\pi_1, \dots, \pi_m\}' \stackrel{def}{=} \{\pi_1', \dots, \pi_{m'}'\} \qquad \textit{where} \quad \pi_i' = \#\{j : \pi_j \geq i\} \quad \textit{and} \quad m' \ = \pi_1.$$

Note that $\pi'' = \pi$.

Example A.5.3. We apply the operation on the partition $\pi = \{3, 2, 2, 2, 1\}$. In this case $m' = \pi_1 = 3$. The elements of the new set will be

$$\pi_1' = \#\{j : \pi_j \geq 1\} = \#\pi = 5, \quad \pi_2' \ = \#\{j : \pi_j \geq 2\} = 4, \quad \pi_3' = \#\{j : \pi_j \geq 3\} = 1.$$

So $\pi' = \{5, 4, 1\}$. Similarly we can check that $\pi'' = \pi$.

□

Let WS_g denote the set of all Weierstrass gap sequences of length g.

Lemma A.5.4. *Let $W_{n,s} = \{w_1, \dots, w_g\}$ be a Weierstrass gap sequence. Then the map*

$$\chi(W_{n,s}) = \pi \qquad \textit{where} \quad \pi_k = w_{g-k+1} + k - g, \qquad k = 1, \dots, g \tag{A.25}$$

defines an embedding $\chi : WS_g \to Par_g$.

Proof. We need to show that $\chi(W_{n,s})$ is a partition as defined above. (I.e. show that the π_i are all positive, non increasing integers.) Clearly all the π_j are integers. We have that

$w_i - 1 \geq 0$ for all $i = 1, \ldots, g$ from Lemma A.4.2(5). Now, for $j = 1, \ldots, g$ we find that $i = g + 1 - j$ is an integer within $[1, \ldots, g]$. Therefore

$$\begin{aligned} w_{g+1-j} - g - 1 + j &\geq 0 \\ w_{g+1-j} - g + j &> 0 \qquad \Longrightarrow \qquad \pi_j > 0 \qquad j = 1, \ldots, g \end{aligned}$$

It remains to show that the π_k are non increasing. Consider

$$\pi_k - \pi_{k+1} = [w_{g-k+1} + k - g] - [w_{g-(k+1)+1} + (k+1) - g] = w_{g-k+1} - w_{g-k} - 1$$

Now $W_{n,s}$ is a strictly increasing sequence by definition so $\pi_k - \pi_{k+1} \geq 1 - 1 = 0$, and hence the the π_k are non increasing as required.

■

Definition A.5.5. *A partition π that is the image of a Weierstrass gap sequence under the mapping χ is defined to be a **Weierstrass Partition**, denoted by*

$$\pi_{n,s} = \chi(W_{n,s}).$$

Example A.5.6. In equation (3.3) we derived $W_{4,5} = \{1, 2, 3, 6, 7, 11\}$. Hence

$$\pi_{4,5} = \chi(W_{4,5}) = \{w_6 - 5, w_5 - 4, w_4 - 3, w_3 - 2, w_2 - 1, w_1\} = \{6, 3, 3, 1, 1, 1\}$$

□

Lemma A.5.7. *Weierstrass Partitions have the following properties.*

(1) $\pi'_{n,s} = \pi_{n,s}$.

(2) $\pi_1 = g, \pi_g = 1$ *and* $\pi_{n,s} \subset \{g, g-1, \ldots, 1\}$.

Example A.5.8. In Example A.5.6 we derived $\pi_{4,5}$. A quick check will show that, as predicted by Lemma A.5.7, we have $\pi'_{4,5} = \{6, 3, 3, 1, 1, 1\} = \pi_{4,5}$. Also we can check that $\pi_1 = g = 6$ and $\pi_g = \pi_6 = 1$.

□

For the proof of Lemma A.5.7 we require the following theorem.

Theorem A.5.9. *For any partition μ $\in$Par$_n$ such that $\mu_1 = m$, the set consisting of the $m + n$ numbers,*

$$\mu_i + n - 1 \quad (1 \leq i \leq n) \qquad\qquad n - 1 + j - \mu'_j \quad (1 \leq j \leq m),$$

is a permutation of the set $\{0, 1, 2, \ldots, (m + n - 1)\}$.

Proof. (We follow [56] page 3.) Let $\mu \in \mathrm{Par}_n$ and $\mu_1 = m$. Note that we have $\mu'_1 = n$. First, draw the Young diagram induced by the partition. This will consist of rows of cubes, with the first row containing μ_1 cubes, the second μ_2 etc. Figure A.2 contains the Young diagram for the partition $\{5, 4, 4, 1\}$.

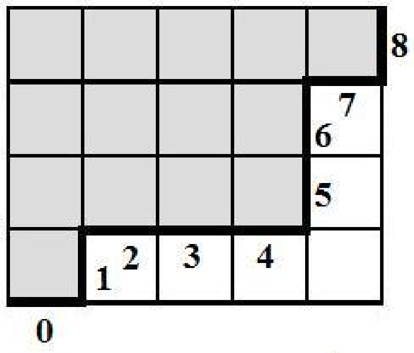

Figure A.2: The Young diagram for the partition $\{5, 4, 4, 1\}$ is given by the shaded squares. We have $n = 4$ and $m = 5$.

Such a diagram will be contained within a rectangle of $m \times n$ cubes as in Figure A.2. We separate the diagram from its complement inside the rectangle, by highlighting the line segments between the two sets. There will clearly be $(n + m)$ line segments that separate the diagram of μ from its complement within the rectangle. Label these line segments $\{0, 1, 2, \ldots, (m + n - 1)\}$, starting at the bottom left hand corner. (See Figure A.2 for example). We find that the numbers attached to the vertical line segments are, starting from the top left, given by

$$\mu_j + n - i \qquad (1 \leq i \leq n).$$

This is because at each vertical segment you will have already traveled μ_i segments across and $(n - i)$ segments up. We then see the numbers attached to the horizontal segments are, starting from the bottom right,

$$(m + n - 1) - (\mu'_j + m - j) = n - 1 + j - \mu'_j \qquad (1 \leq j \leq m).$$

This is because, before you travel along a horizontal segment, you will have traveled the total $(m + n - 1)$, minus the number of rows left (μ'_j) and minus the number of columns not yet considered, $(m - j)$.

So, we have shown that the $n + m$ numbers described by the two equations above will correspond to the $n + m$ line segments in the diagram. Hence these are a permutation of the set $\{0, 1, 2, \ldots, (m + n - 1)\}$.

■

Example A.5.10. We demonstrate the main argument in the proof of Theorem A.5.9 for the partition $\{5, 4, 4, 1\}$, with Young diagram given in Figure A.2. Here

$$\begin{array}{llllll} \mu_1 = 5, & \mu_2 = 4, & \mu_3 = 4, & \mu_4 = 1, & & m = 5, \\ \mu'_1 = 4, & \mu'_2 = 3, & \mu'_3 = 3, & \mu'_4 = 3, & \mu'_5 = 1 & n = 4. \end{array}$$

So the vertical line segments should be given (from top to bottom) by

$$\begin{array}{ccccccc} & \mu_j + n - i & (1 \le i \le n) & \Longrightarrow & \mu_j + 4 - i & (1 \le i \le 4). \\ i = 1 & \Longrightarrow & 5 + 4 - 1 = 8, & & i = 3 & \Longrightarrow & 4 + 4 - 3 = 5, \\ i = 2 & \Longrightarrow & 4 + 4 - 2 = 6, & & i = 4 & \Longrightarrow & 1 + 4 - 4 = 1. \end{array}$$

While the horizontal line segments should be given (from bottom to top) by

$$\begin{array}{ccccccc} & n - 1 + j - \mu'_j & (1 \le j \le m) & \Longrightarrow & 4 - 1 + j - \mu'_j & (1 \le j \le 5). \\ j = 1 & \Longrightarrow & 3 + 1 - 4 = 0, & & j = 4 & \Longrightarrow & 3 + 4 - 3 = 4, \\ j = 2 & \Longrightarrow & 3 + 2 - 3 = 2, & & j = 5 & \Longrightarrow & 3 + 5 - 1 = 7. \\ j = 3 & \Longrightarrow & 3 + 3 - 3 = 3, & & & & \end{array}$$

As expected, these match the labels in Figure A.2.

□

We can now prove Lemma A.5.7.

Proof. [Lemma A.5.7] We will apply Theorem A.5.9 to $\pi_{n,s}$. It has length g, and, from equation (A.25) we see

$$\pi_1 = w_g + 1 - g = (2g - 1) + 1 - g = g.$$

Therefore, the set of $2g$ numbers

$$\begin{array}{rcll} \pi_i + g - 1 & = & w_{g+1-i} & i = 1, \ldots, g \\ g - 1 + j - \pi'_j & = & (2g - 1) - (\pi'_j + g - j) & j = 1, \ldots, g \end{array}$$

is a permutation of the set $\{0, 1, 2, \ldots, (2g - 1)\}$. This implies that the set $\{2g - 1 - (\pi'_j + g - 1)\}_{j=1,\ldots,g}$ is the complement of the sequence $W_{n,s}$ within the set $\{0, 1, 2, \ldots, (2g-1)\}$. However, from Lemma A.4.2(3) we know that this complement can be expressed as

$$\{w_g - w_i\}_{i=1,\ldots,g} = \{w_g - w_{g+1-j}\}_{j=1,\ldots,g} = \{(2g - 1) - w_{g+1-j}\}.$$

It then follows that

$$\pi'_j = w_{g+1-j} + j - g = \pi_j \qquad (1 \le j \le g),$$

which completes the proof of Lemma A.5.7(1).

To prove part (2) we need to show that $\pi_k \in \{1, \ldots, g\}$ for $k = 1, \ldots, g$. So we must prove that, for $k = 1, \ldots, g$

$$\begin{array}{ccccc} 1 & \le & \pi_k & \le & g \\ 1 & \le & w_{g-k+1} + k - g & \le & g \end{array}$$

$$\begin{aligned} g-k+1 &\leq w_{g-k+1} \leq 2g-1 \\ g-k+1 &\leq w_{g-k+1} \leq g+(g-k+1)-1 \end{aligned}$$

Now we make the substitution $j = (g-k+1) \implies g = j+k-1$ into the above inequality (j will run from g to 1). So now we must show that

$$\begin{aligned} j &\leq w_j \leq (j+k-1)+(j)-1 \\ j &\leq w_j \leq (2j-1)+(k-1) \end{aligned}$$

Finally, we note that $(k-1) \geq 0$ so it would satisfy to show that

$$j \leq w_j \leq 2j-1 \qquad j = 1, \ldots, g.$$

This is the case by Lemma A.4.2 (5).

■

A.5.2 Symmetric polynomials

We now need some preliminary results on symmetric polynomials. For more detailed information on these polynomials see [56] Section 1.2 for example.

Definition A.5.11. *A* ***symmetric polynomial*** *in n variables $\{x_1, \ldots, x_n\}$ is a function that is unchanged by any permutation of its variables. Therefore symmetric polynomials satisfy*

$$f(y_1, y_2, \ldots, y_n) = f(x_1, x_2, \ldots, x_n),$$

where $y_i = x_{\pi_i}$ and $\pi = \{\pi_i\}_{i=1,\ldots,n}$ is an arbitrary permutation of the indices $1, 2, \ldots, n$.

We now define a class of symmetric polynomials from which all other symmetric polynomials may be defined

Definition A.5.12. *The* ***elementary symmetric polynomials*** *in m variables are defined by*

$$e_r(x_1, x_2, \ldots, x_m) = \sum_{1 \leq j_1 < j_2 < \cdots < j_r \leq m} x_{j_1} \ldots x_{j_r} \qquad r = 0, 1, 2, \ldots, m.$$

So we have

$$\begin{aligned} e_0(x_1, x_2, \ldots, x_m) &= 1 \\ e_1(x_1, x_2, \ldots, x_m) &= \sum_{1 \leq j \leq m} x_j \\ e_2(x_1, x_2, \ldots, x_m) &= \sum_{1 \leq j_1 < j_2 \leq m} x_{j_1} x_{j_2} \\ &\vdots \\ e_m(x_1, x_2, \ldots, x_m) &= x_1 x_2 \cdots x_m \end{aligned}$$

These can be introduced by means of the generating function below. Here the coefficient of t^r in the expansion on the right hand side will be the rth elementary symmetric polynomial in m variables.

$$E_m(t) = \sum_{r=0}^{m} e_r t^r = \prod_{i=1}^{m} (1 + x_i t).$$

The elementary symmetric polynomials form a basis for the space of all symmetric polynomials.

The elementary Newton polynomials

Definition A.5.13. *The **elementary Newton polynomials** in m variables are*

$$p_r(x_1, \dots, x_n) = x_1^r + \dots + x_m^r = \sum_{i=1}^{m} x_i^r.$$

These can be introduced by means of the generating function below.

$$P(t) = \sum_{r \geq 1} p_r t^{r-1} = \sum_{i \geq 1} \frac{x_i}{(1 - x_i t)}. \tag{A.26}$$

The functions p_r are algebraically independent over $\mathbb{Q}$, the field of rational numbers.

Theorem A.5.14. *The elementary symmetric polynomials may be expressed using elementary Newton polynomials using the following determinant expression.*

$$e_k = \frac{1}{k!} \begin{vmatrix} p_1 & 1 & 0 & \dots & 0 & 0 & 0 \\ p_2 & p_1 & 2 & \dots & 0 & 0 & 0 \\ p_3 & p_2 & p_1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ p_{k-2} & p_{k-3} & p_{k-4} & \dots & p_1 & k-2 & 0 \\ p_{k-1} & p_{k-2} & p_{k-3} & \dots & p_2 & p_1 & k-1 \\ p_k & p_{k-1} & p_{k-2} & \dots & p_3 & p_2 & p_1 \end{vmatrix}. \tag{A.27}$$

The generating functions are related by the formula $P(-t)E(t) = E'(t)$.

Proof. See [56] page 28.

■

Example A.5.15. We calculate e_3 in $m = 3$ variables using equation (A.27).

$$e_3 = \frac{1}{6} \begin{vmatrix} p_1 & 1 & 0 \\ p_2 & p_1 & 2 \\ p_3 & p_2 & p_1 \end{vmatrix}$$

$$6e_3 = p_1(p_1^2 - 2p_2) - (p_1 p_2 - 2p_3) = p_1^3 - 3p_1 p_2 + 2p_3$$

Now, in 3 variables the elementary Newton polynomials are

$$p_1 = x_1 + x_2 + x_3, \qquad p_2 = x_1^2 + x_2^2 + x_3^2, \qquad p_3 = x_1^3 + x_2^3 + x_3^3.$$

So, multiplying out and canceling we find

$$6e_3 = p_1^3 - 3p_1p_2 + 2p_3 = 6x_1x_2x_3$$

So, with 3 variables, $e_3 = x_1x_2x_3$, as expected.

□

A.5.3 Schur-Weierstrass Polynomials

Schur polynomials are defined using a partition as follows.

Definition A.5.16. *Given a partition of length* n*,*

$$\pi = \{\pi_1, \pi_2, \cdots, \pi_n\}$$

the corresponding ***Schur polynomial*** *of* n *variables is*

$$s_\pi(x_1, x_2, \ldots x_n) = \frac{\det \begin{bmatrix} x_1^{\pi_1+n-1} & x_2^{\pi_1+n-1} & \ldots & x_n^{\pi_1+n-1} \\ x_1^{\pi_2+n-2} & x_2^{\pi_2+n-2} & \ldots & x_n^{\pi_2+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\pi_n} & x_2^{\pi_n} & \ldots & x_n^{\pi_n} \end{bmatrix}}{\det \begin{bmatrix} x_1^{n-1} & x_2^{n-1} & \ldots & x_n^{n-1} \\ x_1^{n-2} & x_2^{n-2} & \ldots & x_n^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 \end{bmatrix}}$$

$$= \frac{\det(x_j^{\pi_i+n-i})_{1\le i,j\le n}}{\det(x_j^{n-1})_{1\le i,j\le n}}.$$

The denominator is a Vandermonte determinant and so

$$\det \begin{bmatrix} x_1^{n-1} & x_2^{n-1} & \ldots & x_n^{n-1} \\ x_1^{n-2} & x_2^{n-2} & \ldots & x_n^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 \end{bmatrix} = \prod_{1\le j<k\le n} (x_j - x_k).$$

The numerator will be divisible by each factor $(x_j - x_k)$ and so s_π is a polynomial. Further, since both the numerator and denominator are determinants, both will change sign under any transposition of the variables. These sign changes will cancel each other out and hence the Schur polynomial itself is a symmetric polynomial.

Theorem A.5.17. *A Schur polynomial, s_π corresponding to an arbitrary partition π of length m can be represented using elementary symmetric polynomials as*

$$s_\pi = \det\left((e_{\pi'_i - i + j})_{1\le i,j\le m}\right). \tag{A.28}$$

Proof. See [56] page 40.

■

Definition A.5.18. *Denote by $S_{n,s}$ the Schur polynomial corresponding to a Weierstrass partition $\pi_{n,s}$.*

Given such a Schur polynomial we have case $m = g$ and $\pi'_i = \pi_i$ and so equation (A.28) becomes

$$S_{n,s} = \det\left((e_{\pi_i - i + j})_{1\le i,j\le g}\right)$$

$$= \begin{vmatrix} e_{\pi_1} & e_{\pi_1+1} & e_{\pi_1+2} & \cdots & e_{\pi_1+\lambda-1} & \cdots & e_{\pi_1+g-1} \\ e_{\pi_2-1} & e_{\pi_2} & e_{\pi_2+1} & \cdots & e_{\pi_2+\lambda-2} & \cdots & e_{\pi_2+g-2} \\ e_{\pi_3-2} & e_{\pi_3-1} & e_{\pi_3} & \cdots & e_{\pi_3+\lambda-3} & \cdots & e_{\pi_3+g-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots \\ e_{\pi_\lambda-\lambda+1} & e_{\pi_\lambda-\lambda+2} & e_{\pi_\lambda-\lambda+3} & \cdots & e_{\pi_\lambda} & \cdots & e_{\pi_\lambda+g-\lambda} \\ \vdots & \vdots & \vdots & & \vdots & \ddots & \vdots \\ e_{\pi_g-g+1} & e_{\pi_g-g+2} & e_{\pi_g-g+3} & \cdots & e_{\pi_g+-g+\lambda} & \cdots & e_{\pi_g} \end{vmatrix}. \tag{A.29}$$

Now recall that $\pi_{n,s} \subset \{g, \dots, 1\}$ with $\pi_1 = g$ and $\pi_g = 1$. Therefore, the largest subscript will be $\pi_1 + g - 1 = 2g - 1$. The smallest subscript will actually be $\pi_g - g + 1 = -g$ however, by convention we set e_k to 0 for $k < 0$. Hence

$$S_{n,s} = S_{n,s}(e_1, \dots, e_{2g-1}).$$

Theorem A.5.19. *Using equation (A.27) we can represent $S_{n,s}$ via the elementary Newton polynomials p_i. When using this representation $S_{n,s}$ will be a polynomial in g variables $\{p_{w_1}, \dots, p_{w_g}\}$ where $\{w_1, \dots, w_g\} = W_{n,s}$.*

Proof. See [20] Theorem 4.1.

■

Therefore, given a polynomial $S_{n,s}$, it may be represented using Theorem A.5.19 as $S_{n,s}(p_{w_1}, \dots, p_{w_g})$.

Definition A.5.20. *We define the **Schur-Weierstrass polynomial** associated with (n, s) as follows. First form the Schur polynomial generated by (n, s) as a function of elementary Newton polynomials,*

$$S_{n,s}(p_{w_1}, \ldots, p_{w_g}).$$

We then find the Schur-Weierstrass polynomial by applying the change of variables

$$p_{w_i} = w_i u_{g+1-i}, \qquad i = 1, \ldots, g$$

and denote this polynomial by $SW_{n,s}$.

$$SW_{n,s} = S_{n,s}(w_g u_g, \ldots, w_1 u_1).$$

Summary

To calculate the Schur-Weierstrass polynomial generated by (n, s):

(1) Calculate the Weierstrass Sequence $W_{n,s}$ of length $g = \frac{1}{2}(n-1)(s-1)$.

(2) Calculate the corresponding Weierstrass partition $\pi_{n,s}$.

(3) Define the elementary symmetric polynomials

$$e_k, \qquad k = 1, \ldots, (2g-1)$$

in terms of the elementary Newton polynomials p_i using equation (A.27).

(4) Calculate $S_{n,s}$ as the determinant in equation (A.29). By Theorem A.5.19 this is a polynomial in g variables, $\{p_{w_1}, \ldots, p_{w_g}\}$.

(5) Find the Schur-Weierstrass polynomial by making the change of variables

$$p_{w_i} = w_i u_{g+1-i} \qquad i = 1, \ldots, g.$$

Example A.5.21. We now construct the Schur-Weierstrass polynomial associated with the (4,5)-curve. In equation (3.3) we calculated the gap sequence, $SW_{4,5} = \{1, 2, 3, 6, 7, 11\}$. Then in Example A.5.6 the associated Weierstrass partition was calculated as

$$\pi_{4,5} = \{6, 3, 3, 1, 1, 1\}.$$

Substitute the values of $\pi_{4,5}$ into equation (A.29). Recall that $e_0 = 1$ and that e_k is zero for negative k to find

$$S_{4,5} = \begin{vmatrix} e_6 & e_7 & e_8 & e_9 & e_{10} & e_{11} \\ e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\ e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ 0 & 0 & 1 & e_1 & 1 & 0 \\ 0 & 0 & 0 & 0 & e_1 & 1 \\ 0 & 0 & 0 & 0 & 1 & e_1 \end{vmatrix}.$$

Now use equation (A.27) to describe the elementary symmetric polynomials using Newton polynomials.

$$e_1 = \frac{1}{1!}|1| = 1, \qquad e_2 = \frac{1}{2!}\begin{vmatrix} p_1 & 1 \\ p_2 & p_1 \end{vmatrix} = \frac{p_1^2 - p_2}{2},$$

$$e_3 = \frac{1}{3!}\begin{vmatrix} p_1 & 1 & 0 \\ p_2 & p_1 & 2 \\ p_3 & p_2 & p_1 \end{vmatrix} = \frac{1}{6}\Big(p_1(p_1^2 - 2p_2) - (p_1p_2 - 2p_3)\Big) = \frac{p_1^3}{6} - \frac{p_1p_2}{2} + \frac{p_3}{3}.$$

Using Maple, we find similarly that

$$\begin{aligned}
e_4 &= \tfrac{1}{24}p_1^4 - \tfrac{1}{4}p_2p_1^2 + \tfrac{1}{3}p_3p_1 + \tfrac{1}{8}p_2^2 - \tfrac{1}{4}p_4, \\
e_5 &= \tfrac{1}{120}p_1^5 - \tfrac{1}{12}p_2p_1^3 + \tfrac{1}{6}p_3p_1^2 + \tfrac{1}{8}p_1p_2^2 - \tfrac{1}{4}p_4p_1 - \tfrac{1}{6}p_2p_3 + \tfrac{1}{5}p_5, \\
e_6 &= \tfrac{1}{720}{p_1}^6 - \tfrac{1}{48}p_2{p_1}^4 + \tfrac{1}{18}p_3{p_1}^3 + \tfrac{11}{48}16{p_1}^2{p_2}^2 - \tfrac{1}{8}p_4{p_1}^2 - \tfrac{1}{6}p_3p_1p_2 \\
&\quad - \tfrac{11}{48}48{p_2}^3 + \tfrac{1}{8}p_4p_2 + \tfrac{1}{18}{p_3}^2 + \tfrac{1}{5}p_5p_1 - \tfrac{1}{6}p_6, \\
e_7 &= -\tfrac{1}{12}{p_1}^2p_2p_3 + \tfrac{1}{8}p_4p_1p_2 + \tfrac{1}{24}{p_2}^2p_3 - \tfrac{1}{10}p_5p_2 - \tfrac{1}{12}p_3p_4 + \tfrac{1}{5040}{p_1}^7 + \tfrac{1}{7}p_7 - \tfrac{1}{6}p_6p_1 \\
&\quad - \tfrac{1}{240}{p_1}^5p_2 + \tfrac{1}{72}{p_1}^4p_3 + \tfrac{1}{48}{p_1}^3{p_2}^2 - \tfrac{1}{24}p_4{p_1}^3 + \tfrac{1}{10}p_5{p_1}^2 - \tfrac{1}{48}p_1{p_2}^3 + \tfrac{1}{18}{p_3}^2p_1, \\
e_8 &= \tfrac{1}{16}{p_1}^2p_2p_4 - \tfrac{1}{36}{p_1}^3p_2p_3 - \tfrac{1}{36}p_2{p_3}^2 - \tfrac{1}{32}{p_2}^2p_4 - \tfrac{1}{10}p_5p_1p_2 + \tfrac{1}{24}p_1{p_2}^2p_3 + \tfrac{11}{127}p_7p_1 \\
&\quad + \tfrac{1}{15}p_3p_5 - \tfrac{1}{12}p_3p_1p_4 + \tfrac{1}{12}p_6p_2 + \tfrac{1}{32}{p_4}^2 + \tfrac{1}{384}{p_2}^4 - \tfrac{1}{12}p_6{p_1}^2 + \tfrac{1}{36}{p_1}^2{p_3}^2 - \tfrac{1}{8}p_8 \\
&\quad - \tfrac{1}{96}{p_1}^2{p_2}^3 + \tfrac{1}{30}p_5{p_1}^3 - \tfrac{1}{96}p_4{p_1}^4 + \tfrac{1}{192}{p_2}^2{p_1}^4 + \tfrac{1}{360}p_3{p_1}^5 - \tfrac{1}{1440}{p_1}^6p_2 + \tfrac{1}{40320}{p_1}^8, \\
e_9 &= -\tfrac{1}{24}{p_1}^2p_3p_4 + \tfrac{1}{9}p_9 + \tfrac{1}{24}p_2p_3p_4 - \tfrac{1}{36}p_6{p_1}^3 + \tfrac{1}{14}p_7{p_1}^2 - \tfrac{1}{14}p_7p_2 - \tfrac{1}{8}p_8p_1 \\
&\quad + \tfrac{1}{12}p_6p_1p_2 + \tfrac{1}{48}{p_1}^3p_2p_4 - \tfrac{1}{20}{p_1}^2p_2p_5 - \tfrac{1}{144}{p_2}^3p_3 + \tfrac{1}{40}{p_2}^2p_5 + \tfrac{1}{2160}{p_1}^6p_3 \\
&\quad - \tfrac{1}{10080}{p_1}^7p_2 + \tfrac{1}{384}p_1{p_2}^4 - \tfrac{1}{480}{p_1}^5p_4 + \tfrac{1}{162}{p_3}^3 + \tfrac{1}{32}p_1{p_4}^2 + \tfrac{1}{48}{p_1}^2{p_2}^2p_3 - \tfrac{1}{18}p_3p_6 \\
&\quad + \tfrac{1}{362880}{p_1}^9 + \tfrac{1}{960}{p_1}^5{p_2}^2 + \tfrac{1}{108}{p_1}^3{p_3}^2 - \tfrac{1}{288}{p_1}^3{p_2}^3 + \tfrac{1}{120}{p_1}^4p_5 - \tfrac{1}{32}p_1{p_2}^2p_4 \\
&\quad - \tfrac{1}{36}p_2p_1{p_3}^2 - \tfrac{1}{20}p_4p_5 - \tfrac{1}{144}{p_1}^4p_2p_3 + \tfrac{1}{15}p_3p_1p_5, \\
e_{10} &= -\tfrac{1}{720}p_2p_3{p_1}^5 + \tfrac{1}{15120}{p_1}^7p_3 + \tfrac{1}{9}p_9p_1 - \tfrac{1}{2880}{p_1}^6p_4 + \tfrac{1}{162}p_1{p_3}^3 - \tfrac{1}{10}p_{10} \\
&\quad - \tfrac{1}{64}{p_1}^2{p_2}^2p_4 + \tfrac{1}{5760}{p_1}^6{p_2}^2 + \tfrac{1}{4}2p_7{p_1}^3 - \tfrac{1}{16}p_8{p_1}^2 + \tfrac{1}{16}p_8p_2 - \tfrac{1}{14}p_7p_1p_2 \\
&\quad + \tfrac{1}{600}{p_1}^5p_5 + \tfrac{1}{50}{p_5}^2 - \tfrac{1}{144}{p_1}^4p_6 + \tfrac{1}{432}{p_1}^4{p_3}^2 - \tfrac{1}{1152}{p_1}^4{p_2}^3 + \tfrac{1}{768}{p_1}^2{p_2}^4 + \cdots
\end{aligned}$$

$$\begin{aligned}
&\cdots + \tfrac{1}{24}{p_1}^2p_2p_6 - \tfrac{1}{20}p_1p_4p_5 + \tfrac{1}{24}p_4p_6 - \tfrac{1}{72}{p_1}^2p_2{p_3}^2 + \tfrac{1}{30}{p_1}^2p_3p_5 - \tfrac{1}{80640}{p_1}^8p_2 \\
&+ \tfrac{1}{64}{p_1}^2{p_4}^2 - \tfrac{1}{3840}{p_2}^5 + \tfrac{1}{40}p_1{p_2}^2p_5 - \tfrac{1}{144}p_1{p_2}^3p_3 + \tfrac{1}{192}{p_2}^3p_4 - \tfrac{1}{48}{p_2}^2p_6 \\
&+ \tfrac{1}{144}{p_2}^2{p_3}^2 - \tfrac{1}{64}p_2{p_4}^2 + \tfrac{1}{192}{p_1}^4p_2p_4 - \tfrac{1}{60}{p_1}^3p_2p_5 + \tfrac{1}{144}{p_1}^3{p_2}^2p_3 - \tfrac{1}{72}{p_3}^2p_4 \\
&- \tfrac{1}{18}p_3p_1p_6 + \tfrac{1}{21}p_3p_7 - \tfrac{1}{72}{p_1}^3p_3p_4 + \tfrac{1}{3628800}{p_1}^{10} - \tfrac{1}{30}p_2p_3p_5 + \tfrac{1}{24}p_2p_1p_3p_4,
\end{aligned}$$

$$\begin{aligned}
e_{11} = {} & \tfrac{1}{90}p_5p_3{p_1}^3 - \tfrac{1}{240}p_5p_2{p_1}^4 - \tfrac{1}{288}p_4{p_1}^4p_3 + \tfrac{1}{960}p_4{p_1}^5p_2 - \tfrac{1}{20160}p_4{p_1}^7 + \tfrac{1}{1152}{p_2}^4p_3 \\
&+ \tfrac{1}{3600}p_5{p_1}^6 + \tfrac{1}{48}{p_1}^2p_2p_3p_4 + \tfrac{1}{40}p_2p_4p_5 - \tfrac{1}{30}p_2p_1p_3p_5 + \tfrac{1}{36}p_2p_3p_6 - \tfrac{1}{30}p_5p_6 \\
&- \tfrac{1}{96}{p_2}^2p_3p_4 + \tfrac{1}{39916800}{p_1}^{11} - \tfrac{1}{10}p_{10}p_1 - \tfrac{1}{48}p_8{p_1}^3 + \tfrac{1}{18}p_9{p_1}^2 - \tfrac{1}{18}p_9p_2 + \tfrac{1}{144}p_1{p_2}^2{p_3}^2 \\
&+ \tfrac{1}{324}{p_1}^2{p_3}^3 - \tfrac{1}{720}{p_1}^5p_6 + \tfrac{1}{2160}{p_1}^5{p_3}^2 - \tfrac{1}{5760}{p_1}^5{p_2}^3 + \tfrac{1}{40320}{p_1}^7{p_2}^2 + \tfrac{1}{11}p_{11} \\
&+ \tfrac{1}{120960}{p_1}^8p_3 - \tfrac{1}{725760}{p_1}^9p_2 + \tfrac{1}{168}{p_1}^4p_7 + \tfrac{1}{2304}{p_1}^3{p_2}^4 + \tfrac{1}{192}{p_1}^3{p_4}^2 - \tfrac{1}{240}{p_2}^3p_5 \\
&- \tfrac{1}{3840}p_1{p_2}^5 + \tfrac{1}{50}p_1{p_5}^2 + \tfrac{1}{16}p_8p_1p_2 - \tfrac{1}{324}p_2{p_3}^3 - \tfrac{1}{72}p_1{p_3}^2p_4 + \tfrac{1}{90}{p_3}^2p_5 + \tfrac{1}{56}{p_2}^2p_7 \\
&+ \tfrac{1}{96}p_3{p_4}^2 - \tfrac{1}{24}p_3p_8 + \tfrac{1}{21}p_3p_1p_7 + \tfrac{1}{24}p_1p_4p_6 - \tfrac{1}{2}8p_4p_7 - \tfrac{1}{64}p_2p_1{p_4}^2 + \tfrac{1}{192}p_1{p_2}^3p_4 \\
&+ \tfrac{1}{72}{p_1}^3p_2p_6 - \tfrac{1}{40}{p_1}^2p_4p_5 - \tfrac{1}{216}{p_1}^3p_2{p_3}^2 - \tfrac{1}{4320}{p_1}^6p_2p_3 - \tfrac{1}{192}{p_1}^3{p_2}^2p_4 \\
&- \tfrac{1}{28}{p_1}^2p_2p_7 + \tfrac{1}{80}{p_1}^2{p_2}^2p_5 - \tfrac{1}{288}{p_1}^2{p_2}^3p_3 + \tfrac{1}{576}{p_1}^4{p_2}^2p_3 - \tfrac{1}{36}{p_1}^2p_3p_6 - \tfrac{1}{48}p_1{p_2}^2p_6.
\end{aligned}$$

The next step is to substitute these into $S_{4,5}$ and evaluating the determinant. Using Maple we find that many terms cancel to leave

$$\begin{aligned}
S_{4,5} = {} & \tfrac{1}{4032}{p_2}^2{p_1}^8p_3 - \tfrac{1}{1512}p_2{p_1}^7p_6 + \tfrac{1}{7056}p_7{p_1}^8 + \tfrac{1}{252}p_7p_3{p_1}^5 + \tfrac{1}{168}p_7{p_2}^2{p_1}^4 \\
&- \tfrac{1}{1944}{p_3}^3{p_1}^6 - \tfrac{1}{27216}{p_1}^9{p_3}^2 + \tfrac{1}{1197504}p_3{p_1}^{12} - \tfrac{1}{12096}{p_2}^4{p_1}^7 - \tfrac{1}{133056}{p_2}^2{p_1}^{11} \\
&- \tfrac{1}{44}{p_2}^2p_{11} - \tfrac{1}{126}{p_3}^2{p_1}^2p_7 + \tfrac{1}{216}{p_3}^3{p_1}^2{p_2}^2 + \tfrac{1}{33}p_3p_1p_{11} - \tfrac{1}{108}p_3{p_6}^2 \\
&- \tfrac{1}{1728}p_3{p_2}^6 - \tfrac{1}{432}{p_3}^2p_1{p_2}^4 + \tfrac{1}{336}{p_2}^4p_7 + \tfrac{1}{42}p_6p_2p_7 + \tfrac{1}{84}p_3p_1p_7{p_2}^2 + \tfrac{1}{108}{p_1}^3{p_6}^2 \\
&- \tfrac{1}{54}{p_3}^2p_1p_6p_2 + \tfrac{1}{216}p_3{p_2}^3p_6 - \tfrac{1}{216}p_3p_6p_2{p_1}^4 + \tfrac{1}{972}{p_3}^5 - \tfrac{1}{49}{p_7}^2p_1 + \tfrac{1}{972}{p_3}^4{p_1}^3 \\
&- \tfrac{1}{132}{p_1}^4p_{11} + \tfrac{1}{1728}{p_2}^6{p_1}^3 - \tfrac{1}{216}{p_2}^3p_6{p_1}^3 - \tfrac{1}{1728}{p_2}^4p_3{p_1}^4 + \tfrac{1}{8382528}{p_1}^{15}.
\end{aligned}$$

Note that as predicted by Theorem A.5.19, this is a function of g variables, $\{p_{w_1}, \ldots, p_{w_g}\}$. Finally, by Definition A.5.20, we make the change of variables to give,

$$\begin{aligned}
SW_{4,5} = {} & \tfrac{1}{8382528}u_6^{15} + \tfrac{1}{336}u_6^8u_5^2u_4 - \tfrac{1}{12}u_6^4u_1 - \tfrac{1}{126}u_6^7u_3u_5 - \tfrac{1}{6}u_4u_3u_5u_6^4 - {u_5}^2u_1 \\
&- \tfrac{1}{72}u_4^3u_6^6 - \tfrac{1}{33264}u_6^{11}u_5^2 + \tfrac{1}{27}u_5^6u_6^3 + \tfrac{2}{3}u_4u_5^3u_3 - 2u_4^2u_6u_3u_5 - u_2^2u_6 + u_4u_6u_1 \\
&- \tfrac{2}{9}u_5^3u_3u_6^3 - u_4u_3^2 + \tfrac{1}{12}u_4^4u_6^3 - \tfrac{1}{3024}u_6^9u_4^2 - \tfrac{1}{756}u_6^7u_5^4 + \tfrac{1}{1008}u_6^8u_2 - \tfrac{1}{3}\,{u_4}^2u_6{u_5}^4 \\
&+ \tfrac{1}{3}u_5^4u_2 + \tfrac{1}{3}u_6^3u_3^2 - \tfrac{1}{9}u_4u_5^6 + \tfrac{1}{399168}u_6^{12}u_4 + u_4u_6u_5^2u_2 + \tfrac{1}{4}u_4^5 - \tfrac{1}{36}\,{u_5}^4u_4{u_6}^4 \\
&+ 2\,u_5u_3u_2 + \tfrac{1}{6}\,{u_5}^2{u_6}^4u_2 + \tfrac{1}{12}\,{u_6}^5u_2u_4 - \tfrac{1}{2}\,{u_4}^2{u_6}^2u_2 + \tfrac{1}{2}\,{u_4}^3{u_6}^2{u_5}^2.
\end{aligned}$$

□

A.6 Resultants

This Appendix will give a short overview of the theory of resultants. For more information see Chapter 12 of [43]. We use resultants in several parts of this document as a tool to eliminate variables from equations. In these computations it was implemented using the standard `resultant` command in the computer algebra package Maple.

Definition A.6.1. *Let $p(x)$ be a polynomial of degree n,*

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \tag{A.30}$$

and denote the roots of $p(x)$ by α_i where $i = 1, \ldots, n$. Similarly, let $q(x)$ be a polynomial of degree m,

$$q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0, \tag{A.31}$$

*and denote the roots of $q(x)$ by β_j where $j = 1, \ldots, m$. Then the **resultant** of the polynomials is defined as*

$$\textit{Res}(p, q) = a_n^m b_m^n \prod_{i=1}^{n} \prod_{j=1}^{m} \left[\alpha_i - \beta_j\right].$$

So the resultant of the polynomials is zero if and only if they share a common root. The resultant may be realised as the determinant of the corresponding Sylvester matrix.

Definition A.6.2. *Consider the two polynomials of degrees n and m in equations (A.30) and (A.31). The corresponding **Sylvester matrix** is an $(m + n) \times (m + n)$ matrix formed as follows.*

Start at the upper left corner and fill the top row with the coefficients of $p(x)$. Repeat this on the second row, shifting each entry to the right. Continue this process until the coefficients hit the right hand side. Then repeat the process again using the coefficients of $q(x)$, starting on the next row down. All other matrix entries are set to zero.

For example, if $m = 3$ and $n = 2$ then

$$\text{Res}(p, q) = \begin{vmatrix} a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & a_3 & a_2 & a_1 & a_0 \\ b_2 & b_1 & b_0 & 0 & 0 \\ 0 & b_2 & b_1 & b_0 & 0 \\ 0 & 0 & b_2 & b_1 & b_0 \end{vmatrix}.$$

Note that the resultant does not depend on the variable x.

The resultant can be used to search for solutions to the equations

$$p(x) = 0, \qquad q(x) = 0. \tag{A.32}$$

This is because when the resultant is zero, the polynomials have a common root and hence equations (A.32) are satisfied.

More generally it may be used to eliminate variables between equations, (the use it is put to in this document). To do this we write two equations as polynomials that must be zero, and take the resultant with respect to the variable that we want eliminated. We obtain a polynomial that does not contain that variable and that when equal to zero implies the original equations are satisfied.

We demonstrate this with a simple example

Example A.6.3. Suppose we wish to eliminate the variable x_1 from the equations

$$\begin{aligned} 4x_1 + x_1^2 + 9x_2 &= 0 \\ -x_1^2 + 10x_2 + 4x_2^3 &= 0. \end{aligned}$$

The resultant of these equations with respect to x_1 is

$$\begin{aligned} \text{Res} &= \begin{vmatrix} 1 & 4 & 9x_2 & 0 \\ 0 & 1 & 4 & 9x_2 \\ -1 & 0 & 10x_2 + 4x_2^3 & 0 \\ 0 & -1 & 0 & 10x_2 + 4x_2^3 \end{vmatrix} \\ &= -160x_2 - 64x_2^3 + 361x_2^2 + 152x_2^4 + 16x_2^6. \end{aligned}$$

Hence we have eliminated x_1.

□

This was a simple example to demonstrate the concept. The resultant approach may be used in far more complicated circumstance, with polynomials of many variables and a high degree. In these cases it is an efficient method for eliminating variables. Note that the Sylvester matrix is not the only method of determining the resultant. If $m = n$ then the resultants may be computed using a Bezout matrix, (see [43] Chapter 12).

It is possible to define the multivariant result of a group of polynomials in analogy to Definition A.6.1, however we do not use these in this document. For more information see [43] Chapter 13.

Appendix B

Independence from the constant c

Recall Definition 2.2.15 which defined the Kleinian σ-function using the period matrices and θ-function characteristic $[\delta]$ as

$$\sigma(\boldsymbol{u}) = c\exp\left(\tfrac{1}{2}\boldsymbol{u}\eta'(\omega')^{-1}\boldsymbol{u}^T\right) \times \sum_{\boldsymbol{m}\in\mathbb{Z}^g} \exp\Bigg[2\pi i\big\{$$
$$\tfrac{1}{2}(\boldsymbol{m}+\boldsymbol{\delta'})^T(\omega')^{-1}\omega''(\boldsymbol{m}+\boldsymbol{\delta'}) + (\boldsymbol{m}+\boldsymbol{\delta'})^T((\omega')^{-1}\boldsymbol{u}^T+\boldsymbol{\delta''})\big\}\Bigg].$$

Here c was a constant that depends on the curve parameters and was fixed in Remark 2.2.23 so that $K = 1$ in Theorem 2.2.21. It was remarked at the time that this choice of c differs from the choice of some other authors working in this area, and that these choice are not equivalent in general. However, all the Abelian functions and almost all the results in this document are independent of the choice of c. This Appendix is dedicated to proving this fact.

Notation

Note that the constant c does not vary with $\boldsymbol{u}$ and hence may be taken outside of the differentiation when considering σ-derivatives. Hence both the σ-function all the σ-derivatives may be factored to give c multiplied by an infinite sum that does not vary with c. In this section, when a function may be factored in this way we denote the part of a function that does not vary with c with the bar notation. Hence

$$\sigma(\boldsymbol{u}) = c\overline{\sigma}(\boldsymbol{u})$$
$$\sigma_{i_1,i_2,\dots,i_n}(\boldsymbol{u}) = c\overline{\sigma}_{i_1,i_2,\dots,i_n}(\boldsymbol{u})$$

If a function $f(\boldsymbol{u})$ is independent of c then $f(\boldsymbol{u}) = \overline{f}(\boldsymbol{u})$.

Kleinian $\wp$-functions

Now we will prove that all the Kleinian $\wp$-functions are independent of c.

Proposition B.0.4. *The 2-index $\wp$-functions are independent of the constant c.*

Proof. Recall Definition 2.2.24 for the 2-index $\wp$-functions. Take the derivatives to obtain

$$\begin{aligned}\wp_{ij}(\boldsymbol{u}) &= \frac{\sigma_i(\boldsymbol{u})\sigma_j(\boldsymbol{u}) - \sigma(\boldsymbol{u})\sigma_{ij}(\boldsymbol{u})}{\sigma(\boldsymbol{u})^2} \qquad \text{(B.1)}\\ &= \frac{c\overline{\sigma}_i(\boldsymbol{u})c\overline{\sigma_j}(\boldsymbol{u}) - c\overline{\sigma}(\boldsymbol{u})c\overline{\sigma}_{ij}(\boldsymbol{u})}{c^2\overline{\sigma}(\boldsymbol{u})^2} = \frac{\overline{\sigma}_i(\boldsymbol{u})\overline{\sigma_j}(\boldsymbol{u}) - \overline{\sigma}(\boldsymbol{u})\overline{\sigma}_{ij}(\boldsymbol{u})}{\overline{\sigma}(\boldsymbol{u})^2}\\ &= \overline{\wp}_{ij}(\boldsymbol{u}) \qquad \text{(B.2)}\end{aligned}$$

Once the definition was expanded it was possible to factor out the constant c to obtain something independent of c which we label $\overline{\wp}_{ij}(\boldsymbol{u})$. Both the numerator and denominator contained c^2 which canceled so we can conclude $\wp_{ij}(\boldsymbol{u})$ independent of the constant c.

■

Such a cancellation occurs for all the n-index $\wp$-functions. We can prove this using the following lemma.

Lemma B.0.5. *An n-index $\wp$-function can be expressed as*

$$\wp_{i_1,i_2,\dots,i_n}(\boldsymbol{u}) = \frac{h(\boldsymbol{u})}{\sigma(\boldsymbol{u})^n},$$

where $h(\boldsymbol{u})$ is a finite polynomial in which each term is a product of n functions. These functions are all either $\sigma(\boldsymbol{u})$ or σ-derivatives

$$\sigma_{i_1,i_2,\dots,i_m}(\boldsymbol{u}), \qquad \textit{where } m \le n.$$

Proof. We will prove this by induction. The statement is clearly true for $n = 2$ from equation (B.1). Now assume that the statement holds for a k-index $\wp$-function:

$$\wp_{i_1,i_2,\dots,i_k}(\boldsymbol{u}) = \frac{h(\boldsymbol{u})}{\sigma(\boldsymbol{u})^k}$$

where $h(\boldsymbol{u})$ is a sum of terms, each a product of k m-index σ-derivatives with $m \le k$. Now differentiate this fraction with respect to u_i to obtain a $(k+1)$-index $\wp$-function.

$$\wp_{i_1,i_2,\dots,i_k,i_{k+1}}(\boldsymbol{u}) = \frac{\left[\sigma(\boldsymbol{u})^k\right]\left[\dfrac{\partial}{\partial u_i}h(\boldsymbol{u})\right] - \left[h(\boldsymbol{u})\right]\left[k\sigma(\boldsymbol{u})^{k-1}\sigma_i(\boldsymbol{u})\right]}{\sigma(\boldsymbol{u})^{2k}}.$$

We can factor out $\sigma(\boldsymbol{u})^{k-1}$ from the numerator and cancel to leave $2k - (k-1) = k+1$

powers of $\sigma(\boldsymbol{u})$ in the denominator.

$$\wp_{i_1,i_2,\ldots,i_k,i_{k+1}}(\boldsymbol{u}) = \frac{\sigma(\boldsymbol{u})\left[\dfrac{\partial}{\partial u_i}h(\boldsymbol{u})\right] - kh(\boldsymbol{u})\sigma_i(\boldsymbol{u})}{\sigma(\boldsymbol{u})^{k+1}}. \tag{B.3}$$

The denominator of equation (B.3) is already in the correct form. Due to the assumption on $h(\boldsymbol{u})$ so is the second term in the numerator.

Consider the first term in the numerator of equation (B.3). By assumption $h(\boldsymbol{u})$ is sum of terms, each a product of k functions. Consider one such product, $f_1 \cdot f_2 \cdots f_k$. (Note that all the functions f_i are m-index σ-derivatives with $m \leq k$.) The derivative of this term will be

$$\begin{aligned}
\frac{\partial}{\partial u_i}(f_1 f_2 \cdots f_k) &= f_1 \frac{\partial}{\partial u_i}(f_2 \cdots f_k) + \left(\frac{\partial}{\partial u_i}f_1\right) f_2 \cdots f_k \\
&= f_1\left[f_2\frac{\partial}{\partial u_i}(f_3 \cdots f_k) + f_3 \cdots f_k\left(\frac{\partial}{\partial u_i}f_2\right)\right] + \left(\frac{\partial}{\partial u_i}f_1\right) f_2 \cdots f_k \\
&= f_1 f_2 \frac{\partial}{\partial u_i}(f_3 \cdots f_k) + \left[\left(\frac{\partial}{\partial u_i}f_1\right) f_2 \cdots f_k + f_1\left(\frac{\partial}{\partial u_i}f_2\right) f_3 \cdots f_k\right].
\end{aligned}$$

We can continue to take derivatives using the product rule,

$$\begin{aligned}
&\frac{\partial}{\partial u_i}(f_1 f_2 \cdots f_k) \\
&\qquad = f_1 f_2\left[f_3\frac{\partial}{\partial u_i}(f_4 \cdots f_k) + f_4 \cdots f_k\left(\frac{\partial}{\partial u_i}f_3\right)\right] \\
&\qquad\quad + \left[\left(\frac{\partial}{\partial u_i}f_1\right) f_2 \cdots f_k + f_1\left(\frac{\partial}{\partial u_i}f_2\right) f_3 \cdots f_k\right] \\
&\qquad = f_1 f_2 f_3 \frac{\partial}{\partial u_i}(f_4 \cdots f_k) + \left[\left(\frac{\partial}{\partial u_i}f_1\right) f_2 \cdots f_k\right. \\
&\qquad\quad \left. + f_1\left(\frac{\partial}{\partial u_i}f_2\right) f_3 \cdots f_k + f_1 f_2\left(\frac{\partial}{\partial u_i}f_3\right) f_4 \cdots f_k\right],
\end{aligned}$$

until eventually we have

$$\begin{aligned}
\frac{\partial}{\partial u_i}(f_1 f_2 \cdots f_k) = \Bigg[&\left(\frac{\partial}{\partial u_i}f_1\right) f_2 \cdots f_k + f_1\left(\frac{\partial}{\partial u_i}f_2\right) f_3 \cdots f_k \\
&+ f_1 f_2\left(\frac{\partial}{\partial u_i}f_3\right) f_4 \cdots f_k + \cdots + f_1 f_2 \cdots f_{t-1}\left(\frac{\partial}{\partial u_i}f_k\right)\Bigg].
\end{aligned}$$

Therefore the derivative of the term gives a polynomial of new terms, each of which is a product of k functions. Due to the assumption on $h(\boldsymbol{u})$ each of the f_i was a m-index σ-derivative with $m < k$. Therefore each of the functions used in the derivative of the term are m-index σ-derivative with $m < k+1$.

This will also be true for all the terms in $h(\boldsymbol{u})$. Therefore, using term-by-term differentiation, we know that $\frac{\partial}{\partial u_i}h(\boldsymbol{u})$ is a polynomial of terms, each a product of k m-index

σ-derivatives with $m < k+1$.

Hence the first product in the numerator of equation (B.3) will be equal to this polynomial, with each term multiplied by $\sigma(\boldsymbol{u})$. Therefore each term will be a product of $k+1$ of the desired functions.

So we have shown that the numerator and both parts of the numerator are of the desired form. Hence, if the statement holds for $n=k$, it will hold for $n=k+1$.

■

Corollary B.0.6. *Any n-index Kleinian $\wp$-function is independent of the constant c.*

Proof. From Lemma B.0.5 we have

$$\wp_{i_1,i_2,\ldots,i_n}(\boldsymbol{u}) = \frac{h(\boldsymbol{u})}{\sigma(\boldsymbol{u})^{2^{n-1}}} = \frac{c^n\overline{h}(\boldsymbol{u})}{c^n\overline{\sigma}(\boldsymbol{u})^{2^{n-1}}} = \overline{\wp}_{i_1,i_2,\ldots,i_n}(\boldsymbol{u}).$$

Due to the assumption on $h(\boldsymbol{u})$ the numerator could factor to give c^n multiplied by a function not dependent on c. This is true also for the denominator and so the powers of c cancel allowing us to conclude that an n-index $\wp$-function is independent of c. ■

The Q-functions

We shortly prove that the Q-functions are independent of c. These functions were introduced in Definition 3.1.2 using the Hirota operator given in Definition 3.1.1. The proof is based on the following lemma.

Lemma B.0.7. *Consider the polynomial obtained by applying the Hirota operator n times to $\sigma(\boldsymbol{u})\sigma(\boldsymbol{v})$, (where n is even). This will be a finite polynomial, in which each term is the product of two functions. One function will vary with $\boldsymbol{u} = (u_1,\ldots,u_g)$ while the other will vary with $\boldsymbol{v} = (v_1,\ldots,v_g)$.*

$$\Delta_{i_n}\ldots\Delta_{i_2}\Delta_{i_1}\sigma(\boldsymbol{u})\sigma(\boldsymbol{v}) = f_{11}(\boldsymbol{u})f_{12}(\boldsymbol{v}) + f_{21}(\boldsymbol{u})f_{22}(\boldsymbol{v}) + f_{31}(\boldsymbol{u})f_{32}(\boldsymbol{v}) + \ldots$$

Further, these functions will either be the σ-function or a σ-derivative,

$$\sigma_{i_1,i_2,\ldots,i_m}, \qquad \textit{where } m \leq n.$$

Proof. We will prove this by induction, starting with the case $n=2$. After the Q-functions were defined we checked that applying the Hirota operator twice gave

$$\Delta_j\Delta_i\sigma(\boldsymbol{u})\sigma(\boldsymbol{v}) = \sigma_{ij}(\boldsymbol{u})\sigma(\boldsymbol{v}) - \sigma_i(\boldsymbol{v})\sigma_j(\boldsymbol{u}) - \sigma_i(\boldsymbol{u})\sigma_j(\boldsymbol{v}) + \sigma_{ij}(\boldsymbol{v})\sigma(\boldsymbol{u}),$$

which is a polynomial of the desired form.

Now suppose that the statement holds for an even number $n = k$.

$$\Delta_{i_k} \dots \Delta_{i_2} \Delta_{i_1} \sigma(\boldsymbol{u}) \sigma(\boldsymbol{v}) = f_{11}(\boldsymbol{u}) f_{12}(\boldsymbol{v}) + f_{21}(\boldsymbol{u}) f_{22}(\boldsymbol{v}) + \dots \tag{B.4}$$

We need to consider the result of applying two Hirota operators to equation (B.4). This could be done term by term, so just consider the affect on the first term of equation (B.4).

$$\begin{aligned}
&\Delta_{i_{k+2}} \Delta_{i_{k+1}} f_{11}(\boldsymbol{u}) f_{12}(\boldsymbol{v}) = \\
&\quad \Delta_{i_{k+2}} \left[\left(\frac{\partial}{\partial u_{i_{k+1}}} f_{11}(\boldsymbol{u}) \right) f_{12}(\boldsymbol{v}) - f_{11}(\boldsymbol{u}) \left(\frac{\partial}{\partial v_{i_{k+1}}} f_{12}(\boldsymbol{v}) \right) \right] \\
&\quad = \Delta_{i_{k+2}} \left[\left(\frac{\partial}{\partial u_{i_{k+1}}} f_{11}(\boldsymbol{u}) \right) f_{12}(\boldsymbol{v}) \right] - \Delta_{i_{k+2}} \left[f_{11}(\boldsymbol{u}) \left(\frac{\partial}{\partial v_{i_{k+1}}} f_{12}(\boldsymbol{v}) \right) \right] \\
&\quad = \left[\left(\frac{\partial^2}{\partial u_{i_{k+1}} \partial u_{i_{k+2}}} f_{11}(\boldsymbol{u}) \right) f_{12}(\boldsymbol{v}) - \left(\frac{\partial}{\partial u_{i_{k+1}}} f_{11}(\boldsymbol{u}) \right) \left(\frac{\partial}{\partial v_{i_{k+2}}} f_{12}(\boldsymbol{v}) \right) \right] \\
&\qquad - \left[\left(\frac{\partial}{\partial u_{i_{k+2}}} f_{11}(\boldsymbol{u}) \right) \left(\frac{\partial}{\partial v_{i_{k+1}}} f_{12}(\boldsymbol{v}) \right) - f_{11}(\boldsymbol{u}) \left(\frac{\partial^2}{\partial v_{i_{k+1}} \partial v_{i_{k+2}}} f_{12}(\boldsymbol{v}) \right) \right].
\end{aligned}$$

Consider this final polynomial. Each term is a product of two functions, one which varies with $\boldsymbol{u}$ and one which varies with $\boldsymbol{v}$. Also, it is clear that all the functions are σ-derivatives by the assumption. This conclusion would be drawn in the same way for all the terms of equation (B.4), and therefore it is true for their sum. So we have shown that if the statement is true for $n = k$ it is true for $n = k + 2$ and hence by induction the statement is true.

■

Corollary B.0.8. *Any n-index Q-function is independent of the constant c.*

Proof. Such a function can be expressed as

$$Q_{i_1, i_2, \dots, i_n}(\boldsymbol{u}) = \frac{(-1)}{2\sigma(\boldsymbol{u})^2} g(\boldsymbol{u})$$

where, by Lemma B.0.7, $g(\boldsymbol{u})$ is a polynomial in which each term is the product of two σ-derivatives. Hence

$$Q_{i_1, i_2, \dots, i_n}(\boldsymbol{u}) = \frac{(-1)}{2c^2 \overline{\sigma}(\boldsymbol{u})^2} c^2 \overline{g}(\boldsymbol{u}) = \overline{Q}_{i_1, i_2, \dots, i_n}(\boldsymbol{u}).$$

The numerator and denominator can both be factored to give c^2 multiplied by something that does not vary with c. The c^2 parts will cancel and so the corollary is true. ■

Conclusions

We have shown that all the Abelian functions we consider in this document are independent of the constant c. Hence any function, relation or equation defined between these functions is also independent of the constant c.

The only equations that may not be independent are those that contain the σ-function or its derivatives. In fact, most of these are still independent. For example, the equations in Chapter 6 are all either linear in σ-derivatives, (in which case c may be factored out), or contain ratios of σ-derivatives which have the same number of σ-derivatives in the numerator as in the denominator. Either way the parts of c may be canceled out to leave the theory independent.

The only results in this document that are not independent of the constant c are the two term addition formulae for the σ-function. This was presented for the cyclic (4,5)-curve in Section 3.6 and for the cyclic (3,7)-curve in Section 4.1.5.

In each case we proved that

$$-\frac{\sigma(\boldsymbol{u}+\boldsymbol{v})\sigma(\boldsymbol{u}-\boldsymbol{v})}{\sigma(\boldsymbol{u})^2\sigma(\boldsymbol{v})^2} = f(\boldsymbol{u},\boldsymbol{v}) \pm f(\boldsymbol{v},\boldsymbol{u}),$$

where $f(\boldsymbol{u},\boldsymbol{v})$ was a sum of products of pairs of Abelian functions. The left hand side was antisymmetric in the (4,5)-case (minus sign) and symmetric in the (3,7)-case (plus sign).

Let c represent the choice we use as given in Remark 2.2.23 and let $\hat{c}$ be an alternative acceptable choice. Then

$$-\frac{\sigma(\boldsymbol{u}+\boldsymbol{v},\hat{c})\sigma(\boldsymbol{u}-\boldsymbol{v},\hat{c})}{\sigma(\boldsymbol{u},\hat{\boldsymbol{c}})^2\sigma(\boldsymbol{v},\hat{\boldsymbol{c}})^2} = \frac{1}{\hat{c}^2}\frac{\overline{s}(\boldsymbol{u}+\boldsymbol{v})\overline{s}(\boldsymbol{u}-\boldsymbol{v})}{\overline{s}(\boldsymbol{u})^2\overline{s}(\boldsymbol{v})^2},$$

with a similar equation for c. Hence changing to the alternative choice of constant would change the addition formulae as follows.

$$-\frac{\sigma(\boldsymbol{u}+\boldsymbol{v},\hat{c})\sigma(\boldsymbol{u}-\boldsymbol{v},\hat{c})}{\sigma(\boldsymbol{u},\hat{\boldsymbol{c}})^2\sigma(\boldsymbol{v},\hat{\boldsymbol{c}})^2} = \left[\frac{c^2}{\hat{c}^2}\right]\big[f(\boldsymbol{u},\boldsymbol{v}) \pm f(\boldsymbol{v},\boldsymbol{u})\big].$$

Therefore the polynomial $f(\boldsymbol{u},\boldsymbol{v})$ is changed only by a multiplicative constant.

Appendix C

Results for the cyclic tetragonal curve of genus six

C.1 Expansions in the local parameter

Recall the local parameter at $(x,y)=\infty$, (where $\boldsymbol{u}=\boldsymbol{0}$), is given by $\xi = x^{-\frac{1}{4}}$. Hence

$$x = \frac{1}{\xi^4}, \qquad \Longrightarrow \qquad \frac{dx}{d\xi} = -\frac{4}{\xi^5}.$$

We can then obtain a series expansion for $y(\xi)$.

$$\begin{aligned}
y = \frac{1}{\xi^{-5}} &+ \left(\frac{\lambda_4}{4}\right)\frac{1}{\xi} + \left(\frac{\lambda_3}{4} - \frac{3\lambda_4^2}{32}\right)\xi^3 + \left(\frac{\lambda_2}{4} - \frac{3\lambda_4\lambda_3}{16} + \frac{7\lambda_4^3}{128}\right)\xi^7 \\
&+ \left(\frac{\lambda_1}{4} - \frac{3\lambda_4\lambda_3}{16} - \frac{3\lambda_3^2}{32} + \frac{21\lambda_4^2\lambda_3}{128} - \frac{77\lambda_4^4}{2048}\right)\xi^{11} \\
&+ \left(\frac{\lambda_0}{4} + \frac{21}{64}\,{\lambda_3}^2\lambda_4 - \frac{3}{8}\,\lambda_4\lambda_1 - \frac{77}{64}\,{\lambda_4}^3\lambda_3 + \frac{231}{256}\,{\lambda_4}^5\right)\xi^{15} \\
&+ \Bigg(\frac{21}{32}\,{\lambda_4}^2\lambda_1 + \frac{7\lambda_3^3}{128} - \frac{3}{8}\,\lambda_4\lambda_0 - \frac{3}{16}\,\lambda_3\lambda_1 - \frac{231}{256}\,{\lambda_3}^2{\lambda_4}^2 \\
&\qquad + \frac{1155}{512}\,{\lambda_4}^4\lambda_3 - \frac{1463}{1024}\,{\lambda_4}^6\Bigg)\xi^{19} + O(\xi^{23}).
\end{aligned}$$

We substitute these into the basis of holomorphic differential (3.4) to obtain series expansions for $\boldsymbol{du}$. Integrating gives the following expansions for the variables $\boldsymbol{u}$.

$$\begin{aligned}
u_1 &= -\tfrac{1}{11}\xi^{11} + \tfrac{1}{20}\,\lambda_4\xi^{15} + \big(-\tfrac{21}{608}\,{\lambda_4}^2 + \tfrac{3}{76}\,\lambda_3\big)\xi^{19} + O(\xi^{23}) \\
u_2 &= -\tfrac{1}{7}\xi^7 + \tfrac{3}{44}\,\lambda_4\xi^{11} + \big(-\tfrac{7}{160}\,{\lambda_4}^2 + \tfrac{1}{20}\,\lambda_3\big)\xi^{15} + \big(\tfrac{1}{19}\,\lambda_4\big(-\tfrac{1}{4}\,\lambda_3 + \tfrac{5}{32}\,{\lambda_4}^2\big) \\
&\quad + \tfrac{45}{2432}\,{\lambda_4}^3 + \tfrac{3}{76}\,\lambda_2 - \tfrac{15}{304}\,\lambda_4\lambda_3 + \tfrac{1}{76}\,\lambda_4\big(3/8\,{\lambda_4}^2 - 1/2\,\lambda_3\big)\big)\xi^{19} + O(\xi^{23}) \\
u_3 &= -\tfrac{1}{6}\xi^6 + \tfrac{1}{20}\,\lambda_4\xi^{10} + \big(-\tfrac{3}{112}\,{\lambda_4}^2 + \tfrac{1}{28}\,\lambda_3\big)\xi^{14} + \big(\tfrac{1}{36}\,\lambda_4\big(-\tfrac{1}{4}\,\lambda_3 + \tfrac{5}{32}\,{\lambda_4}^2\big) \\
&\quad + \tfrac{5}{384}\,{\lambda_4}^3 + \tfrac{1}{36}\,\lambda_2 - \tfrac{5}{144}\,\lambda_4\lambda_3\big)\xi^{18} + O(\xi^{22})
\end{aligned}$$

$$\begin{aligned}
u_4 &= -\tfrac{1}{3}\xi^3 + \tfrac{3}{28}\,\lambda_4\xi^7 + \big(-\tfrac{21}{352}\,{\lambda_4}^2 + \tfrac{3}{44}\,\lambda_3\big)\xi^{11} + \big(\tfrac{1}{15}\,\lambda_4\big(-\tfrac{1}{4}\,\lambda_3 + \tfrac{5}{32}\,{\lambda_4}^2\big) \\
&\quad + \tfrac{3}{128}\,{\lambda_4}^3 + \tfrac{1}{20}\,\lambda_2 - \tfrac{1}{16}\,\lambda_4\lambda_3 + \tfrac{1}{60}\,\lambda_4\big(3/8\,{\lambda_4}^2 - 1/2\,\lambda_3\big)\big)\xi^{15} + O(\xi^{19}) \\
u_5 &= -\tfrac{1}{2}\xi^2 + \tfrac{1}{12}\,\lambda_4\xi^6 + \big(-\tfrac{3}{80}\,{\lambda_4}^2 + \tfrac{1}{20}\,\lambda_3\big)\xi^{10} + \big(\tfrac{1}{28}\,\lambda_4\big(-\tfrac{1}{4}\,\lambda_3 + \tfrac{5}{32}\,{\lambda_4}^2\big) \\
&\quad + \tfrac{15}{896}\,{\lambda_4}^3 + \tfrac{1}{28}\,\lambda_2 - \tfrac{5}{112}\,\lambda_4\lambda_3\big)\xi^{14} + \big(-\tfrac{5}{288}\,{\lambda_3}^2 + \tfrac{5}{128}\,{\lambda_4}^2\lambda_3 - \tfrac{65}{6144}\,{\lambda_4}^4 \\
&\quad - \tfrac{5}{144}\,\lambda_4\lambda_2 + \tfrac{1}{36}\,\lambda_1 + \tfrac{1}{36}\,\lambda_4\big(-\tfrac{15}{128}\,{\lambda_4}^3 - \tfrac{1}{4}\,\lambda_2 + \tfrac{5}{16}\,\lambda_4\lambda_3\big) - \tfrac{1}{18}\,\big(-\tfrac{1}{4}\,\lambda_3 \\
&\quad + \tfrac{5}{32}\,{\lambda_4}^2\big)^2\big)\xi^{18} + O(\xi^{22}) \\
u_6 &= -\xi + \tfrac{1}{20}\,\lambda_4\xi^5 + \big(\tfrac{1}{36}\,\lambda_3 - \tfrac{5}{288}\,{\lambda_4}^2\big)\xi^9 + \big(\tfrac{7}{1664}\,{\lambda_4}^3 + \tfrac{1}{52}\,\lambda_2 - \tfrac{3}{208}\,\lambda_4\lambda_3 \\
&\quad - \tfrac{1}{52}\,\lambda_4\big(\tfrac{1}{4}\,\lambda_3 - \tfrac{3}{32}\,{\lambda_4}^2\big) - \tfrac{1}{52}\,\big(\tfrac{1}{4}\,\lambda_3 - \tfrac{5}{32}\,{\lambda_4}^2\big)\lambda_4\big)\xi^{13} + \big(-\tfrac{1}{17}\,\big(\tfrac{1}{4}\,\lambda_3 \\
&\quad - \tfrac{5}{32}\,{\lambda_4}^2\big)\big(\tfrac{1}{4}\,\lambda_3 - \tfrac{3}{32}\,{\lambda_4}^2\big) - \tfrac{77}{34816}\,{\lambda_4}^4 - \tfrac{3}{272}\,\lambda_4\lambda_2 + \tfrac{21}{2176}\,{\lambda_4}^2\lambda_3 - \tfrac{3}{544}\,{\lambda_3}^2 \\
&\quad + \tfrac{1}{68}\,\lambda_1 - \tfrac{1}{68}\,\lambda_4\big(\tfrac{7}{128}\,{\lambda_4}^3 + \tfrac{1}{4}\,\lambda_2 - 3/16\,\lambda_4\lambda_3\big) - \tfrac{1}{68}\,\big(\tfrac{15}{128}\,{\lambda_4}^3 + \tfrac{1}{4}\,\lambda_2 \\
&\quad - \tfrac{5}{16}\,\lambda_4\lambda_3\big)\lambda_4\big)\xi^{17} + O(\xi^{21})
\end{aligned}$$

Note that the series expansions of the differentials $\boldsymbol{du}$ are the same with each power of ξ reduced by one.

C.2 The σ-function expansion

We defined the σ-function expansion as the following infinite sum of finite polynomials.

$$\sigma(\mathbf{u}) = C_{15} + C_{19} + C_{23} + C_{27} + C_{31} + C_{35} + \cdots$$

We know that C_{15} is equal to the Schur-Weierstrass polynomial as given in equation (3.50), and we constructed C_{19} in equation (3.52). The other polynomials were calculated in turn using the method described in Section 3.4. The next polynomial, C_{23}, is given below, while the rest of the expansion can be found in the extra Appendix of files or online at [38].

$$\begin{aligned}
C_{23}(\boldsymbol{u}) &= \lambda_4^2 \cdot \big[\tfrac{1}{9}\,{u_6}^3{u_4}^3u_1 - \tfrac{1}{180}\,{u_6}^6{u_4}^2u_1 + \tfrac{1}{16200}\,{u_6}^{10}{u_4}^2u_2 - \tfrac{13}{2041200}\,{u_6}^{12}{u_5}^4u_4 \\
&\quad + \tfrac{1}{2993760}\,{u_6}^{13}u_4u_2 + \tfrac{4}{315}\,{u_5}^7u_4u_3 + \tfrac{1}{22680}\,{u_6}^9u_4u_1 + \tfrac{4}{15}\,{u_5}^5u_3u_2 + \tfrac{1}{467775}\,{u_6}^{12}{u_5}^2u_2 \\
&\quad - \tfrac{1}{3}\,{u_6}^3{u_4}^2{u_2}^2 - \tfrac{17}{315}\,u_6{u_5}^8{u_4}^2 + \tfrac{5}{756}\,{u_6}^4{u_5}^8u_4 - \tfrac{1}{14175}\,{u_6}^8{u_5}^6u_4 + \tfrac{1}{15}\,u_6{u_5}^2{u_4}^6 \\
&\quad + \tfrac{13}{120}\,{u_6}^4{u_5}^2{u_4}^5 + \tfrac{7}{108}\,{u_6}^3{u_5}^4{u_4}^4 - \tfrac{1}{280}\,{u_6}^7{u_5}^2{u_4}^4 - \tfrac{1}{30}\,u_5{u_4}^5u_3 - \tfrac{1}{162}\,{u_6}^6{u_5}^4{u_4}^3 \\
&\quad + \tfrac{101}{226800}\,{u_6}^{10}{u_5}^2{u_4}^3 - \tfrac{4}{467775}\,{u_6}^{11}{u_5}^3u_3 - \tfrac{8}{135}\,{u_6}^2{u_5}^6{u_4}^3 - \tfrac{4}{4725}\,{u_6}^7{u_5}^5u_3 + \tfrac{1}{3}\,{u_4}^3{u_2}^2 \\
&\quad + \tfrac{17}{838252800}\,{u_6}^{17}{u_4}^2 + \tfrac{1}{105}\,{u_5}^8u_2 + \tfrac{1}{30}\,{u_6}^5{u_4}^2{u_3}^2 - \tfrac{1}{2520}\,{u_6}^8u_4{u_3}^2 + \tfrac{17}{60}\,u_6{u_4}^5u_2 \\
&\quad - \tfrac{2}{45}\,{u_5}^6u_1 - \tfrac{1}{22680}\,{u_6}^9{u_2}^2 - \tfrac{1}{28668245760}\,{u_6}^{19}{u_5}^2 - \tfrac{1}{14968800}\,{u_6}^{12}u_1 - \tfrac{1}{72}\,{u_6}^4{u_4}^4u_2 \\
&\quad + \tfrac{23}{3780}\,{u_6}^7{u_4}^3u_2 - \tfrac{4}{945}\,{u_6}^3{u_5}^7u_3 + \tfrac{1}{6}\,{u_5}^2{u_4}^4u_2 + \tfrac{1}{135}\,{u_6}^4{u_5}^6u_2 - \tfrac{1}{3}\,{u_6}^2{u_4}^3{u_3}^2 + \cdots
\end{aligned}$$

$$\cdots + \tfrac{2}{2835}\,u_6{}^9u_5{}^2u_4u_2 + \tfrac{1}{573364915200}\,u_6{}^{20}u_4 - \tfrac{4}{45}\,u_6{}^5u_5{}^3u_4{}^2u_3 - \tfrac{1}{199584}\,u_6{}^{13}u_5{}^2u_4{}^2$$
$$+ \tfrac{41}{14175}\,u_6{}^3u_5{}^{10} - \tfrac{4}{9}\,u_6{}^2u_5{}^3u_4{}^3u_3 - \tfrac{2}{15}\,u_6u_5{}^5u_4{}^2u_3 - \tfrac{31}{7484400}\,u_6{}^{12}u_5u_4u_3 + \tfrac{1}{24}\,u_6{}^2u_4{}^7$$
$$- \tfrac{43}{396900}\,u_6{}^7u_5{}^8 - \tfrac{1}{1350}\,u_6{}^5u_5{}^6u_4{}^2 + \tfrac{29}{68040}\,u_6{}^9u_5{}^4u_4{}^2 - \tfrac{247}{100800}\,u_6{}^8u_4{}^5 + \tfrac{1}{36}\,u_5{}^4u_4{}^5$$
$$- \tfrac{593}{9979200}\,u_6{}^{11}u_4{}^4 + \tfrac{1}{67359600}\,u_6{}^{15}u_5{}^4 + \tfrac{1}{1247400}\,u_6{}^{11}u_3{}^2 + \tfrac{1}{2514758400}\,u_6{}^{16}u_2 - \tfrac{1}{12}\,u_4{}^4u_1$$
$$+ \tfrac{19}{41912640}\,u_6{}^{14}u_4{}^3 + \tfrac{1}{90}\,u_6{}^6u_4u_2{}^2 - \tfrac{5}{18}\,u_6{}^3u_5u_4{}^4u_3 - \tfrac{2}{945}\,u_6{}^8u_5{}^3u_4u_3 + \tfrac{2}{45}\,u_6u_5{}^6u_4u_2$$
$$+ \tfrac{13}{900}\,u_6{}^5u_4{}^6 + \tfrac{2}{45}\,u_6{}^5u_5{}^4u_4u_2 + \tfrac{2}{45}\,u_6{}^6u_5{}^2u_4{}^2u_2 - \tfrac{7}{270}\,u_6{}^6u_5u_4{}^3u_3 + \tfrac{4}{9}\,u_6{}^3u_5{}^2u_4{}^3u_2$$
$$- \tfrac{2}{45}\,u_6{}^4u_5{}^5u_4u_3 + \tfrac{1}{3}\,u_6{}^2u_5{}^4u_4{}^2u_2 - \tfrac{17}{22680}\,u_6{}^9u_5u_4{}^2u_3 - \tfrac{1}{561330}\,u_6{}^{11}u_5{}^6 - \tfrac{41}{4725}\,u_5{}^{10}u_4$$
$$+ \tfrac{1}{1890}\,u_6{}^8u_5{}^4u_2 + \tfrac{1}{39916800}\,u_6{}^{16}u_5{}^2u_4 - \tfrac{1}{157172400}\,u_6{}^{15}u_5u_3 - \tfrac{1}{26489459082240}\,u_6{}^{23}\big]$$

$$+ \lambda_3 \cdot \big[u_5{}^2u_3{}^2u_2 - \tfrac{1}{360}\,u_6{}^6u_4{}^2u_1 + \tfrac{1}{6}\,u_5{}^4u_4u_3{}^2 - \tfrac{1}{18}\,u_6{}^3u_5{}^4u_3{}^2 - \tfrac{1}{252}\,u_6{}^7u_5{}^2u_3{}^2$$
$$+ \tfrac{1}{72}\,u_6{}^4u_5{}^4u_1 + \tfrac{1}{6}\,u_6u_5{}^4u_2{}^2 + \tfrac{1}{12}\,u_6{}^5u_5{}^2u_2{}^2 + \tfrac{1}{12}\,u_6{}^4u_3{}^2u_2 + \tfrac{1}{2016}\,u_6{}^8u_5{}^2u_1$$
$$+ \tfrac{1}{111767040}\,u_6{}^{16}u_5{}^2u_4 + \tfrac{17}{11975040}\,u_6{}^{13}u_4u_2 - \tfrac{17}{315}\,u_5{}^7u_4u_3 - \tfrac{1}{30}\,u_6{}^5u_5u_4u_3u_2$$
$$+ \tfrac{1}{18}\,u_6{}^3u_4{}^3u_1 + \tfrac{1}{5}\,u_5{}^5u_3u_2 - \tfrac{1}{74844}\,u_6{}^{12}u_5{}^2u_2 - \tfrac{1}{6}\,u_6{}^3u_4{}^2u_2{}^2 + \tfrac{5}{144}\,u_6{}^4u_4{}^4u_2$$
$$+ \tfrac{1}{630}\,u_6u_5{}^8u_4{}^2 + \tfrac{1}{7560}\,u_6{}^4u_5{}^8u_4 - \tfrac{1}{45360}\,u_6{}^8u_5{}^6u_4 - \tfrac{1}{60}\,u_6u_5{}^2u_4{}^6 - \tfrac{29}{17962560}\,u_6{}^{12}u_5{}^4u_4$$
$$+ \tfrac{1}{240}\,u_6{}^4u_5{}^2u_4{}^5 + \tfrac{1}{216}\,u_6{}^3u_5{}^4u_4{}^4 + \tfrac{1}{1680}\,u_6{}^7u_5{}^2u_4{}^4 + \tfrac{2}{15}\,u_5u_4{}^5u_3 + \tfrac{1}{6480}\,u_6{}^6u_5{}^4u_4{}^3$$
$$+ \tfrac{1}{36288}\,u_6{}^{10}u_5{}^2u_4{}^3 + \tfrac{1}{299376}\,u_6{}^{11}u_5{}^3u_3 - \tfrac{1}{2}\,u_6{}^2u_2{}^3 - \tfrac{1}{540}\,u_6{}^2u_5{}^6u_4{}^3 + \tfrac{1}{25147584}\,u_6{}^{15}u_5u_3$$
$$- \tfrac{1}{1260}\,u_6{}^7u_5{}^5u_3 + \tfrac{1}{335301120}\,u_6{}^{17}u_4{}^2 - \tfrac{1}{630}\,u_5{}^8u_2 - \tfrac{1}{1890}\,u_5{}^{10}u_4 + \tfrac{1}{60}\,u_6{}^5u_4{}^2u_3{}^2$$
$$+ \tfrac{1}{6}\,u_4{}^3u_2{}^2 - \tfrac{11}{10080}\,u_6{}^8u_4u_3{}^2 + \tfrac{7}{60}\,u_6u_4{}^5u_2 - \tfrac{37}{181440}\,u_6{}^{10}u_4{}^2u_2 - \tfrac{13}{1890}\,u_6{}^7u_4{}^3u_2$$
$$+ \tfrac{17}{945}\,u_6{}^3u_5{}^7u_3 + \tfrac{1}{12}\,u_5{}^2u_4{}^4u_2 + \tfrac{2}{135}\,u_6{}^4u_5{}^6u_2 - \tfrac{1}{3024}\,u_6{}^8u_5{}^4u_2 + \tfrac{1}{158760}\,u_6{}^7u_5{}^8$$
$$- \tfrac{1}{6}\,u_6u_5{}^4u_4u_1 + \tfrac{1}{1995840}\,u_6{}^{13}u_5{}^2u_4{}^2 + \tfrac{1}{120}\,u_6{}^2u_4{}^7 + \tfrac{5}{18144}\,u_6{}^9u_2{}^2 + \tfrac{1}{11467298304}\,u_6{}^{19}u_5{}^2$$
$$+ \tfrac{1}{5670}\,u_6{}^3u_5{}^{10} - \tfrac{1}{18}\,u_6{}^4u_5{}^3u_3u_2 + \tfrac{1}{9}\,u_6{}^2u_5{}^3u_4{}^3u_3 - \tfrac{1}{5}\,u_6u_5{}^5u_4{}^2u_3 + \tfrac{2}{3}\,u_6u_5{}^3u_4u_3u_2$$
$$- \tfrac{43}{5987520}\,u_6{}^{12}u_5u_4u_3 + \tfrac{113}{90720}\,u_6{}^9u_5{}^2u_4u_2 - \tfrac{1}{229345966080}\,u_6{}^{20}u_4 + \tfrac{1}{2395008}\,u_6{}^{12}u_1$$
$$- \tfrac{1}{90}\,u_6{}^5u_5{}^3u_4{}^2u_3 + \tfrac{1}{540}\,u_6{}^5u_5{}^6u_4{}^2 + \tfrac{1}{2}\,u_6{}^2u_5{}^2u_4u_2{}^2 + \tfrac{1}{17010}\,u_6{}^9u_5{}^4u_4{}^2 - \tfrac{1}{8064}\,u_6{}^8u_4{}^5$$
$$- \tfrac{7}{360}\,u_5{}^4u_4{}^5 - \tfrac{1}{199584}\,u_6{}^{11}u_3{}^2 + \tfrac{1}{28740096}\,u_6{}^{16}u_2 + \tfrac{1}{720}\,u_6{}^5u_4{}^6 - \tfrac{1}{167650560}\,u_6{}^{14}u_4{}^3$$
$$- \tfrac{1}{3991680}\,u_6{}^{11}u_4{}^4 - \tfrac{1}{12}\,u_6{}^4u_5{}^2u_4u_3{}^2 - \tfrac{1}{1008}\,u_6{}^8u_5u_3u_2 + \tfrac{1}{2}\,u_6{}^2u_5{}^2u_4{}^2u_1 + \tfrac{11}{90720}\,u_6{}^9u_4u_1$$
$$- \tfrac{1}{75442752}\,u_6{}^{15}u_5{}^4 + \tfrac{11}{360}\,u_6{}^6u_4u_2{}^2 - u_6{}^2u_5u_4{}^2u_3u_2 - u_6u_4u_3{}^2u_2 - u_6u_5{}^2u_4{}^2u_3{}^2$$
$$+ \tfrac{1}{9}\,u_6{}^3u_5u_4{}^4u_3 + \tfrac{11}{15120}\,u_6{}^8u_5{}^3u_4u_3 + \tfrac{1}{60}\,u_6{}^5u_5{}^2u_4u_1 - \tfrac{7}{90}\,u_6u_5{}^6u_4u_2 + \tfrac{2}{9}\,u_6{}^3u_5{}^2u_4{}^3u_2$$
$$- \tfrac{7}{360}\,u_6{}^5u_5{}^4u_4u_2 - \tfrac{1}{360}\,u_6{}^6u_5{}^2u_4{}^2u_2 - \tfrac{7}{540}\,u_6{}^6u_5u_4{}^3u_3 - \tfrac{1}{30}\,u_5{}^6u_1 - \tfrac{1}{6}\,u_6{}^2u_4{}^3u_3{}^2$$
$$- \tfrac{1}{60}\,u_6{}^4u_5{}^5u_4u_3 - \tfrac{1}{12}\,u_6{}^2u_5{}^4u_4{}^2u_2 + \tfrac{1}{45360}\,u_6{}^9u_5u_4{}^2u_3 + \tfrac{1}{3}\,u_4{}^4u_1 + \tfrac{1}{10595783632896}\,u_6{}^{23}$$
$$- \tfrac{1}{641520}\,u_6{}^{11}u_5{}^6\big]$$

C.3 The 4-index ℘-function relations

This Appendix contains a list of relations expressing 4-index ℘-functions as a polynomial of fundamental Abelian functions of degree at most two.

The fundamental Abelian functions used are those in the basis (3.55), which consisted of 2-index ℘-functions and selected Q-functions. (Note that we only use 4-index Q-functions here and they only appear as linear terms).

These relations are a generalisation, as defined in Definition 5.1.1, of the second differential equation from the elliptic case.

$$\wp''(u) = 6\wp(u)^2 - \tfrac{1}{2}g_2.$$

There are 126 relations, one for each 4-index ℘-functions. They run from $\wp_{6666}$ to $\wp_{1111}$ in decreasing weight order, as indicated by the the number in brackets. Those readers who wish to skip the equations should proceed to page 248.

(−4) $\wp_{6666} = 6\wp_{66}^2 - 3\wp_{55} + 4\wp_{46}$

(−5) $\wp_{5666} = 6\wp_{56}\wp_{66} - 2\wp_{45}$

(−6) $\wp_{4666} = 6\wp_{46}\wp_{66} + 6\lambda_4\wp_{66} - 2\wp_{44} - \tfrac{3}{2}Q_{5566}$

(−6) $\wp_{5566} = Q_{5566} + 2\wp_{55}\wp_{66} + 4\wp_{56}^2$

(−7) $\wp_{4566} = 2\wp_{45}\wp_{66} + 4\wp_{46}\wp_{56} + 2\lambda_4\wp_{56} + 2\wp_{36}$

(−7) $\wp_{5556} = 6\wp_{55}\wp_{56} + 4\lambda_4\wp_{56} + 4\wp_{36}$

(−8) $\wp_{4466} = 2\wp_{44}\wp_{66} + 4\wp_{46}^2 + 4\lambda_4\wp_{46} + \lambda_4\wp_{55} + \wp_{35} + 6\wp_{26} - Q_{4556} + 4\lambda_3$

(−8) $\wp_{5555} = 6\wp_{55}^2 + 16\lambda_4\wp_{55} + 4\wp_{35} + 24\wp_{26} - 6Q_{4556} + 16\lambda_3$

(−8) $\wp_{4556} = Q_{4556} + 4\wp_{45}\wp_{56} + 2\wp_{46}\wp_{55}$

(−9) $\wp_{4456} = 2\wp_{44}\wp_{56} + 4\wp_{45}\wp_{46} + \tfrac{8}{3}\lambda_4\wp_{45} + 2\wp_{25} - \tfrac{4}{3}\wp_{34} - \tfrac{1}{6}Q_{4555}$

(−9) $\wp_{3666} = 6\wp_{36}\wp_{66} + 2\lambda_4\wp_{45} - \tfrac{1}{2}Q_{4555}$

(−9) $\wp_{4555} = Q_{4555} + 6\wp_{45}\wp_{55}$

(−10) $\wp_{2666} = 6\wp_{26}\wp_{66} - \tfrac{1}{2}Q_{4455} - \tfrac{1}{2}Q_{3566} - \tfrac{1}{2}\lambda_4 Q_{5566} + 4\lambda_3\wp_{66}$

(−10) $\wp_{4446} = 6\wp_{44}\wp_{46} + 6\lambda_4\wp_{44} - 2\wp_{24} - 2\lambda_4^2\wp_{66} - Q_{4455}$
$+ \tfrac{1}{2}\lambda_4 Q_{5566} - \tfrac{5}{2}Q_{3566} + 4\lambda_3\wp_{66}$

(−10) $\wp_{4455} = Q_{4455} + 2\wp_{44}\wp_{55} + 4\wp_{45}^2$

(−10) $\wp_{3566} = Q_{3566} + 2\wp_{35}\wp_{66} + 4\wp_{36}\wp_{56}$

(−11) $\wp_{3466} = 2\wp_{34}\wp_{66} + 4\wp_{36}\wp_{46} + 4\lambda_4\wp_{36} - \tfrac{1}{2}Q_{3566}$

(−11) $\wp_{4445} = 6\wp_{44}\wp_{45} + 6\lambda_4\wp_{36} - 2\lambda_4^2\wp_{56} + 8\lambda_3\wp_{56} - 3Q_{2566} - \tfrac{3}{2}Q_{3556}$

(−11) $\wp_{3556} = Q_{3556} + 4\wp_{35}\wp_{56} + 2\wp_{36}\wp_{55}$

(−11) $\wp_{2566} = Q_{2566} + 2\wp_{25}\wp_{66} + 4\wp_{26}\wp_{56}$

(−12) $\wp_{2466} = 2\wp_{24}\wp_{66} + 4\wp_{26}\wp_{46} + 8\wp_{16} - \wp_{33} + 2\lambda_2 - \frac{1}{2}Q_{2456} - Q_{3456}$
$+ 4\lambda_4\wp_{26} + 2\lambda_4\wp_{35}$

(−12) $\wp_{3555} = 6\wp_{35}\wp_{55} + 24\wp_{16} - 8\wp_{33} + 16\lambda_4\wp_{35} - 6Q_{3456}$

(−12) $\wp_{4444} = 6\wp_{44}^2 - 12\wp_{16} + 9\wp_{33} + 6\lambda_2 + 12Q_{3456} + 12\wp_{55}\lambda_3 + 4\wp_{46}\lambda_4^2$
$- 16\wp_{46}\lambda_3 - 3\wp_{55}\lambda_4^2 + 12\lambda_4\wp_{26} - 18\lambda_4\wp_{35} - 6Q_{2556}$

(−12) $\wp_{2556} = Q_{2556} + 4\wp_{25}\wp_{56} + 2\wp_{26}\wp_{55}$

(−12) $\wp_{3456} = Q_{3456} + 2\wp_{34}\wp_{56} + 2\wp_{35}\wp_{46} + 2\wp_{36}\wp_{45}$

(−13) $\wp_{2555} = 6\wp_{25}\wp_{55} + 2\wp_{15} - 8\wp_{23} + 16\wp_{25}\lambda_4 - 6Q_{2456}$

(−13) $\wp_{3446} = 4\wp_{34}\wp_{46} + 2\wp_{36}\wp_{44} + 6\wp_{15} - 4\wp_{23} + 4\wp_{25}\lambda_4 + \frac{8}{3}\wp_{34}\lambda_4$
$+ \frac{2}{3}\wp_{45}\lambda_4^2 - \frac{1}{6}Q_{4555}\lambda_4 - 2Q_{2456}$

(−13) $\wp_{3455} = 2\wp_{34}\wp_{55} + 4\wp_{35}\wp_{45} - 4\wp_{25}\lambda_4 + \frac{8}{3}\wp_{34}\lambda_4 - \frac{4}{3}\wp_{45}\lambda_4^2$
$+ \frac{1}{3}Q_{4555}\lambda_4 + 2Q_{2456}$

(−13) $\wp_{2456} = Q_{2456} + 2\wp_{24}\wp_{56} + 2\wp_{25}\wp_{46} + 2\wp_{26}\wp_{45}$

(−14) $\wp_{2446} = 4\wp_{24}\wp_{46} + 2\wp_{26}\wp_{44} + \frac{8}{3}\wp_{14} - 2\wp_{22} + \frac{8}{3}\wp_{24}\lambda_4 - \frac{2}{3}\wp_{66}\lambda_4\lambda_3 + \frac{5}{6}Q_{3366}$
$- \frac{1}{6}Q_{5566}\lambda_4^2 + 2\wp_{44}\lambda_3 - \frac{1}{6}\lambda_4 Q_{4455} - \frac{4}{3}\lambda_4 Q_{3566} - \frac{7}{6}Q_{3445} + Q_{5566}\lambda_3$

(−14) $\wp_{2455} = 2\wp_{24}\wp_{55} + 4\wp_{25}\wp_{45} + \frac{8}{3}\wp_{14} - 2\wp_{22} + \frac{8}{3}\wp_{24}\lambda_4 - \frac{2}{3}\wp_{66}\lambda_4\lambda_3 - \frac{5}{3}Q_{3366}$
$+ \frac{1}{3}Q_{5566}\lambda_4^2 + 6\wp_{66}\lambda_2 - 2\wp_{44}\lambda_3 + \frac{1}{3}\lambda_4 Q_{4455} + \frac{2}{3}\lambda_4 Q_{3566} - Q_{5566}\lambda_3 + \frac{1}{3}Q_{3445}$

(−14) $\wp_{1666} = 6\wp_{16}\wp_{66} - \frac{1}{2}\lambda_4 Q_{3566} + Q_{3366} - \frac{1}{2}Q_{3445}$

(−14) $\wp_{3366} = Q_{3366} + 2\wp_{33}\wp_{66} + 4\wp_{36}^2$

(−14) $\wp_{3445} = Q_{3445} + 4\wp_{34}\wp_{45} + 2\wp_{35}\wp_{44}$

(−15) $\wp_{1566} = 2\wp_{15}\wp_{66} + 4\wp_{16}\wp_{56} - \frac{1}{4}Q_{2445} - \frac{1}{4}Q_{2455}\lambda_4 + \frac{3}{4}Q_{2366}$
$- \frac{1}{2}\wp_{36}\lambda_3 + \frac{3}{2}\wp_{56}\lambda_2$

(−15) $\wp_{3356} = 2\wp_{33}\wp_{56} + 4\wp_{35}\wp_{36} - \frac{1}{2}Q_{2445} + 3\wp_{36}\lambda_3 + 3\wp_{56}\lambda_2 - \frac{1}{2}Q_{2566}\lambda_4 - \frac{1}{2}Q_{2366}$

(−15) $\wp_{3444} = 6\wp_{34}\wp_{44} - \frac{3}{4}Q_{2445} + 4\wp_{36}\lambda_4^2 - \frac{3}{2}Q_{3556}\lambda_4 + \frac{13}{2}\wp_{36}\lambda_3 + \frac{9}{2}\wp_{56}\lambda_2$
$- \frac{3}{4}Q_{2566}\lambda_4 - \frac{15}{4}Q_{2366}$

(−15) $\wp_{2366} = Q_{2366} + 2\wp_{23}\wp_{66} + 4\wp_{26}\wp_{36}$

(−15) $\wp_{2445} = Q_{2445} + 4\wp_{24}\wp_{45} + 2\wp_{25}\wp_{44}$

(−16) $\wp_{2444} = 6\wp_{24}\wp_{44} + 2\lambda_4\lambda_2 + 10\lambda_1 + 36\wp_{16}\lambda_4 + \frac{3}{2}\wp_{35}\lambda_3 + \frac{15}{2}\wp_{55}\lambda_2 - Q_{3456}\lambda_4$
$- \frac{3}{2}Q_{2556}\lambda_4 - 7\wp_{26}\lambda_3 - 6\wp_{46}\lambda_2 + 3\wp_{33}\lambda_4 + 6\lambda_4\wp_{3456} - 9Q_{1556} + \frac{3}{2}Q_{2266} + 3Q_{2356}$
$- 12\lambda_4\wp_{34}\wp_{56} - 12\lambda_4\wp_{35}\wp_{46} - 12\lambda_4\wp_{36}\wp_{45} + 4\wp_{26}\lambda_4^2 - 6\wp_{35}\lambda_4^2 - 9Q_{1466}$

$$\mathbf{(-16)} \quad \wp_{3346} = 2\wp_{33}\wp_{46} + 4\wp_{34}\wp_{36} - 2Q_{1466} + 4\lambda_1 - Q_{1556} - Q_{2266} + 16\wp_{16}\lambda_4 + 3\wp_{35}\lambda_3 - \wp_{55}\lambda_2 + 2\wp_{26}\lambda_3 - 2Q_{2356}$$

$$\mathbf{(-16)} \quad \wp_{3355} = 2\wp_{33}\wp_{55} + 4\wp_{35}^2 + 8Q_{1466} - 8\lambda_1 + 6Q_{1556} - 32\wp_{16}\lambda_4 + 4\wp_{35}\lambda_3 - 2Q_{2356}$$

$$\mathbf{(-16)} \quad \wp_{1466} = Q_{1466} + 2\wp_{14}\wp_{66} + 4\wp_{16}\wp_{46}$$

$$\mathbf{(-16)} \quad \wp_{1556} = Q_{1556} + 4\wp_{15}\wp_{56} + 2\wp_{16}\wp_{55}$$

$$\mathbf{(-16)} \quad \wp_{2266} = Q_{2266} + 2\wp_{22}\wp_{66} + 4\wp_{26}^2$$

$$\mathbf{(-16)} \quad \wp_{2356} = Q_{2356} + 2\wp_{23}\wp_{56} + 2\wp_{25}\wp_{36} + 2\wp_{26}\wp_{35}$$

$$\mathbf{(-17)} \quad \wp_{1456} = 2\wp_{14}\wp_{56} + 2\wp_{15}\wp_{46} + 2\wp_{16}\wp_{45} - 2\wp_{13} - \wp_{45}\lambda_2 + 2\wp_{25}\wp_{26} + \wp_{22}\wp_{56} + 4\wp_{15}\lambda_4 - Q_{2346} - \tfrac{1}{12}Q_{4555}\lambda_3 - \tfrac{1}{2}\wp_{2256} + \wp_{25}\lambda_3 + \tfrac{1}{3}\wp_{45}\lambda_4\lambda_3 + \tfrac{4}{3}\wp_{34}\lambda_3$$

$$\mathbf{(-17)} \quad \wp_{1555} = 6\wp_{15}\wp_{55} + 4\wp_{13} + 6\wp_{45}\lambda_2 + 3Q_{2256} - 8\wp_{15}\lambda_4 + 6Q_{2346} + \tfrac{1}{2}Q_{4555}\lambda_3 - 6\wp_{25}\lambda_3 - 2\wp_{45}\lambda_4\lambda_3 - 8\wp_{34}\lambda_3$$

$$\mathbf{(-17)} \quad \wp_{2355} = 2\wp_{23}\wp_{55} + 4\wp_{25}\wp_{35} - 4\wp_{13} - 4\wp_{45}\lambda_2 - 2Q_{2256} + 8\wp_{15}\lambda_4 - 2Q_{2346} + 4\wp_{25}\lambda_3 + 4\wp_{34}\lambda_3$$

$$\mathbf{(-17)} \quad \wp_{3345} = 2\wp_{33}\wp_{45} + 4\wp_{34}\wp_{35} - 2\wp_{25}\lambda_3 + \tfrac{4}{3}\wp_{34}\lambda_3 - \tfrac{2}{3}\wp_{45}\lambda_4\lambda_3 + 2\wp_{45}\lambda_2 + Q_{2256} + \tfrac{1}{6}Q_{4555}\lambda_3$$

$$\mathbf{(-17)} \quad \wp_{2256} = Q_{2256} + 2\wp_{22}\wp_{56} + 4\wp_{25}\wp_{26}$$

$$\mathbf{(-17)} \quad \wp_{2346} = Q_{2346} + 2\wp_{23}\wp_{46} + 2\wp_{24}\wp_{36} + 2\wp_{26}\wp_{34}$$

$$\mathbf{(-18)} \quad \wp_{1446} = 4\wp_{14}\wp_{46} + 2\wp_{16}\wp_{44} - 6\wp_{12} + 6\wp_{66}\lambda_1 + \tfrac{1}{2}Q_{3566}\lambda_4^2 - Q_{1455} + 8\wp_{14}\lambda_4 - Q_{3566}\lambda_3 - Q_{2345} - \tfrac{1}{2}Q_{3344} - \tfrac{1}{2}Q_{3366}\lambda_4 + \tfrac{1}{2}Q_{3445}\lambda_4$$

$$\mathbf{(-18)} \quad \wp_{2246} = 2\wp_{22}\wp_{46} + 4\wp_{24}\wp_{26} + 2\wp_{44}\lambda_2 - 4\wp_{66}\lambda_1 - 2\wp_{66}\lambda_4\lambda_2 + \tfrac{1}{2}Q_{3566}\lambda_3 + Q_{3366}\lambda_4 - Q_{3445}\lambda_4 - Q_{3566}\lambda_4^2 + 2\wp_{24}\lambda_3 + \tfrac{3}{2}Q_{5566}\lambda_2 + Q_{1455}$$

$$\mathbf{(-18)} \quad \wp_{2255} = 2\wp_{22}\wp_{55} + 4\wp_{25}^2 - 8\wp_{12} - 8\wp_{44}\lambda_2 + 16\wp_{66}\lambda_1 + \tfrac{32}{3}\wp_{14}\lambda_4 + 8\wp_{66}\lambda_4\lambda_2 - \tfrac{2}{3}Q_{3566}\lambda_3 - \tfrac{8}{3}Q_{3366}\lambda_4 + \tfrac{4}{3}Q_{3445}\lambda_4 + \tfrac{4}{3}Q_{3566}\lambda_4^2 + \tfrac{8}{3}\wp_{24}\lambda_3 - 2Q_{1455} + 4\wp_{23}\wp_{45} - \tfrac{2}{3}\lambda_4\lambda_3\wp_{55}\wp_{66} - \tfrac{4}{3}\lambda_4\lambda_3\wp_{56}^2 - \tfrac{8}{3}\wp_{66}\lambda_3^2 - 2\wp_{2345} + \tfrac{1}{3}Q_{4455}\lambda_3 + \tfrac{1}{3}\lambda_4\lambda_3\wp_{5566} + 4\wp_{24}\wp_{35} + 4\wp_{25}\wp_{34} - 3Q_{5566}\lambda_2$$

$$\mathbf{(-18)} \quad \wp_{1455} = Q_{1455} + 2\wp_{14}\wp_{55} + 4\wp_{15}\wp_{45}$$

$$\mathbf{(-18)} \quad \wp_{2345} = Q_{2345} + 2\wp_{23}\wp_{45} + 2\wp_{24}\wp_{35} + 2\wp_{25}\wp_{34}$$

$$\mathbf{(-18)} \quad \wp_{3344} = Q_{3344} + 2\wp_{33}\wp_{44} + 4\wp_{34}^2$$

$$\mathbf{(-19)} \quad \wp_{1366} = 2\wp_{13}\wp_{66} + 4\wp_{16}\wp_{36} - \tfrac{3}{10}Q_{2566}\lambda_3 - \tfrac{1}{4}Q_{2366}\lambda_4 + \tfrac{3}{20}Q_{2455}\lambda_4 - \tfrac{1}{4}Q_{3556}\lambda_3 - \tfrac{3}{10}Q_{2245} - \tfrac{2}{5}Q_{2344} + \tfrac{3}{20}Q_{2566}\lambda_4^2 + \tfrac{12}{5}\wp_{56}\lambda_1 - \tfrac{3}{10}\wp_{56}\lambda_4\lambda_2 + \tfrac{7}{10}\wp_{36}\lambda_4\lambda_3 + \tfrac{9}{5}\wp_{36}\lambda_2$$

$$\mathbf{(-19)} \quad \wp_{2245} = Q_{2245} + 2\wp_{22}\wp_{45} + 4\wp_{24}\wp_{25}$$

$$(\mathbf{-19})\quad \wp_{3336} = 6\wp_{33}\wp_{36} - \tfrac{3}{5}Q_{2245} + \tfrac{6}{5}Q_{2344} - \tfrac{3}{5}Q_{2566}\lambda_3 + \tfrac{3}{2}Q_{2366}\lambda_4 + \tfrac{3}{10}Q_{2455}\lambda_4$$
$$+ \tfrac{3}{10}Q_{2566}\lambda_4^2 + \tfrac{38}{5}\wp_{36}\lambda_2 - \tfrac{16}{5}\wp_{56}\lambda_1 - \tfrac{3}{5}\wp_{56}\lambda_4\lambda_2 - \tfrac{33}{5}\wp_{36}\lambda_4\lambda_3 + \tfrac{3}{2}Q_{3556}\lambda_3$$

$$(\mathbf{-19})\quad \wp_{1445} = 4\wp_{14}\wp_{45} + 2\wp_{15}\wp_{44} + 8\wp_{56}\lambda_1 - \tfrac{1}{2}Q_{2245} - Q_{2344} - \tfrac{1}{2}Q_{2566}\lambda_3$$
$$- \tfrac{3}{2}Q_{2366}\lambda_4 + \tfrac{1}{2}Q_{2566}\lambda_4^2 + \tfrac{1}{2}Q_{2455}\lambda_4 - 2\wp_{56}\lambda_4\lambda_2 + 3\wp_{36}\lambda_4\lambda_3 + 3\wp_{36}\lambda_2 - \tfrac{3}{4}Q_{3556}\lambda_3$$

$$(\mathbf{-19})\quad \wp_{2344} = Q_{2344} + 2\wp_{23}\wp_{44} + 4\wp_{24}\wp_{34}$$

$$(\mathbf{-20})\quad \wp_{2244} = 2\wp_{22}\wp_{44} + 4\wp_{24}^2 + 2\lambda_3\lambda_2 + 28\lambda_4\lambda_1 + 2\wp_{55}\lambda_4\lambda_2 - 6\wp_{35}\lambda_4\lambda_3 + 3\wp_{33}\lambda_3$$
$$- 24\wp_{46}\lambda_1 + 112\wp_{16}{\lambda_4}^2 - 44\wp_{16}\lambda_3 + 28\wp_{55}\lambda_1 + 9\wp_{35}\lambda_2 - 22Q_{1556}\lambda_4 - Q_{2556}\lambda_3$$
$$+ 20Q_{1356} - 6Q_{1444} + 10Q_{1266} - 22Q_{1466}\lambda_4 + Q_{2266}\lambda_4 + 2Q_{2356}\lambda_4 + 2Q_{3456}\lambda_3$$
$$- 6\wp_{26}\lambda_2$$

$$(\mathbf{-20})\quad \wp_{2336} = 4\wp_{23}\wp_{36} + 2\wp_{26}\wp_{33} + 8\wp_{16}\lambda_3 - 2\wp_{55}\lambda_1 + 2\wp_{35}\lambda_2 - 2Q_{1356} - 2Q_{1266}$$

$$(\mathbf{-20})\quad \wp_{3335} = 6\wp_{33}\wp_{35} - 32\lambda_4\lambda_1 + 24\wp_{46}\lambda_1 - 128\wp_{16}{\lambda_4}^2 + 56\wp_{16}\lambda_3 - 32\wp_{55}\lambda_1$$
$$- 30Q_{1356} + 8Q_{1444} - 12Q_{1266} + 24Q_{1466}\lambda_4 + 24Q_{1556}\lambda_4 + 4\wp_{35}\lambda_2$$

$$(\mathbf{-20})\quad \wp_{1266} = Q_{1266} + 2\wp_{12}\wp_{66} + 4\wp_{16}\wp_{26}$$

$$(\mathbf{-20})\quad \wp_{1356} = Q_{1356} + 2\wp_{13}\wp_{56} + 2\wp_{15}\wp_{36} + 2\wp_{16}\wp_{35}$$

$$(\mathbf{-20})\quad \wp_{1444} = Q_{1444} + 6\wp_{14}\wp_{44}$$

$$(\mathbf{-21})\quad \wp_{1256} = 2\wp_{12}\wp_{56} + 2\wp_{15}\wp_{26} + 2\wp_{16}\wp_{25} - \tfrac{1}{2}Q_{2236} - Q_{1346} - \tfrac{1}{12}Q_{4555}\lambda_2$$
$$+ \tfrac{4}{3}\wp_{34}\lambda_2 + \tfrac{1}{3}\wp_{45}\lambda_4\lambda_2 + 3\wp_{15}\lambda_3 - 2\wp_{45}\lambda_1$$

$$(\mathbf{-21})\quad \wp_{1355} = 2\wp_{13}\wp_{55} + 4\wp_{15}\wp_{35} + \tfrac{1}{2}Q_{2236} + Q_{1346} + \tfrac{1}{4}Q_{4555}\lambda_2 - \tfrac{1}{2}Q_{2335}$$
$$- \wp_{45}\lambda_4\lambda_2 + \wp_{15}\lambda_3$$

$$(\mathbf{-21})\quad \wp_{3334} = 6\wp_{33}\wp_{34} - \tfrac{3}{2}Q_{2236} - 3Q_{1346} + \tfrac{1}{4}Q_{4555}\lambda_2 + \tfrac{3}{2}Q_{2335} - \wp_{45}\lambda_4\lambda_2$$
$$+ 9\wp_{15}\lambda_3 + 12\wp_{34}\lambda_2 - 8\wp_{45}\lambda_1$$

$$(\mathbf{-21})\quad \wp_{1346} = Q_{1346} + 2\wp_{13}\wp_{46} + 2\wp_{14}\wp_{36} + 2\wp_{16}\wp_{34}$$

$$(\mathbf{-21})\quad \wp_{2236} = Q_{2236} + 2\wp_{22}\wp_{36} + 4\wp_{23}\wp_{26}$$

$$(\mathbf{-21})\quad \wp_{2335} = Q_{2335} + 4\wp_{23}\wp_{35} + 2\wp_{25}\wp_{33}$$

$$(\mathbf{-22})\quad \wp_{1345} = 2\wp_{13}\wp_{45} + 2\wp_{14}\wp_{35} + 2\wp_{15}\wp_{34} + 5\wp_{66}\lambda_0 - \wp_{11} + \tfrac{16}{15}\wp_{14}\lambda_4^2 + \tfrac{11}{5}\wp_{66}\lambda_4\lambda_1$$
$$- \tfrac{7}{30}Q_{3566}\lambda_4\lambda_3 - \tfrac{1}{5}Q_{3344}\lambda_4 - \tfrac{1}{2}Q_{1255} + \tfrac{10}{3}\wp_{14}\lambda_3 - \wp_{44}\lambda_1 - Q_{1246} + \tfrac{2}{15}Q_{3445}{\lambda_4}^2$$
$$+ \tfrac{2}{15}Q_{3566}{\lambda_4}^3 - \tfrac{1}{15}Q_{3366}{\lambda_4}^2 - \tfrac{1}{3}Q_{3366}\lambda_3 + \tfrac{1}{6}Q_{3445}\lambda_3 - \tfrac{2}{5}Q_{2345}\lambda_4 - \tfrac{4}{5}\wp_{12}\lambda_4$$

$$(\mathbf{-22})\quad \wp_{2226} = 6\wp_{22}\wp_{26} - 6\wp_{11} + \tfrac{7}{2}Q_{3566}\lambda_2 + \tfrac{1}{2}Q_{4455}\lambda_2 + 6Q_{5566}\lambda_1 + \tfrac{32}{5}\wp_{14}{\lambda_4}^2$$
$$- \tfrac{14}{5}\wp_{66}\lambda_4\lambda_1 - \tfrac{39}{10}Q_{3566}\lambda_4\lambda_3 - \tfrac{24}{5}\wp_{12}\lambda_4 + 4\wp_{24}\lambda_2 + \tfrac{4}{5}Q_{3566}{\lambda_4}^3$$
$$+ 12\wp_{14}\lambda_3 + 6\wp_{44}\lambda_1 - 2\wp_{66}\lambda_3\lambda_2 - 6Q_{1246} + \tfrac{4}{5}Q_{3445}{\lambda_4}^2 + \tfrac{1}{2}Q_{5566}\lambda_4\lambda_2$$
$$- \tfrac{6}{5}Q_{3344}\lambda_4 - \tfrac{2}{5}Q_{3366}{\lambda_4}^2 + \tfrac{3}{2}Q_{3366}\lambda_3 - \tfrac{3}{2}Q_{3445}\lambda_3 - 10\wp_{66}\lambda_0 - \tfrac{12}{5}Q_{2345}\lambda_4$$

$$(\mathbf{-22})\quad \wp_{1246} = Q_{1246} + 2\wp_{12}\wp_{46} + 2\wp_{14}\wp_{26} + 2\wp_{16}\wp_{24}$$

$$\textbf{(−22)}\quad \wp_{2235} = 2\wp_{22}\wp_{35} + 4\wp_{23}\wp_{25} - 2\wp_{11} - Q_{3566}\lambda_2 + 2Q_{1246} - 4Q_{5566}\lambda_1 + \tfrac{22}{15}Q_{3566}\lambda_4\lambda_3 + \tfrac{8}{5}\wp_{12}\lambda_4 + 10\wp_{66}\lambda_0 - Q_{1255} + \tfrac{2}{15}Q_{3366}{\lambda_4}^2 - \tfrac{4}{3}Q_{3366}\lambda_3 - 10\wp_{44}\lambda_1 - \tfrac{4}{15}Q_{3566}{\lambda_4}^3 - \tfrac{4}{15}Q_{3445}{\lambda_4}^2 + \tfrac{2}{5}Q_{3344}\lambda_4 + \tfrac{4}{3}\wp_{14}\lambda_3 + \tfrac{2}{3}Q_{3445}\lambda_3 - \tfrac{32}{15}\wp_{14}{\lambda_4}^2 + \tfrac{4}{5}Q_{2345}\lambda_4 + \tfrac{38}{5}\wp_{66}\lambda_4\lambda_1$$

$$\textbf{(−22)}\quad \wp_{2334} = 4\wp_{23}\wp_{34} + 2\wp_{24}\wp_{33} - 4\wp_{11} - Q_{3566}\lambda_2 + 2Q_{5566}\lambda_1 - \tfrac{32}{15}\wp_{14}{\lambda_4}^2 + \tfrac{7}{15}Q_{3566}\lambda_4\lambda_3 + \tfrac{8}{5}\wp_{12}\lambda_4 + Q_{1255} + \tfrac{4}{3}\wp_{14}\lambda_3 + 4\wp_{44}\lambda_1 - \tfrac{4}{15}Q_{3566}{\lambda_4}^3 - \tfrac{32}{5}\wp_{66}\lambda_4\lambda_1 - \tfrac{4}{15}Q_{3445}{\lambda_4}^2 + \tfrac{2}{5}Q_{3344}\lambda_4 + \tfrac{2}{15}Q_{3366}{\lambda_4}^2 + \tfrac{2}{3}Q_{3366}\lambda_3 - \tfrac{1}{3}Q_{3445}\lambda_3 + \tfrac{4}{5}Q_{2345}\lambda_4$$

$$\textbf{(−22)}\quad \wp_{1255} = Q_{1255} + 2\wp_{12}\wp_{55} + 4\wp_{15}\wp_{25}$$

$$\textbf{(−23)}\quad \wp_{1344} = 2\wp_{13}\wp_{44} + 4\wp_{14}\wp_{34} + 6\wp_{36}\lambda_1 + \tfrac{3}{5}\wp_{36}\lambda_4\lambda_2 - \tfrac{1}{10}\wp_{56}{\lambda_4}^2\lambda_2 + \tfrac{29}{10}\wp_{36}{\lambda_4}^2\lambda_3 + \wp_{36}{\lambda_3}^2 + \tfrac{2}{5}Q_{2566}\lambda_4\lambda_3 - \tfrac{3}{4}Q_{3556}\lambda_4\lambda_3 - 2Q_{1246} - 3\wp_{56}\lambda_3\lambda_2 + 10\wp_{56}\lambda_0 + \tfrac{1}{20}Q_{2566}{\lambda_4}^3 + \tfrac{1}{2}Q_{2445}\lambda_3 - \tfrac{3}{2}Q_{2366}\lambda_3 - \tfrac{4}{5}Q_{2344}\lambda_4 - \tfrac{1}{10}Q_{2245}\lambda_4 - \tfrac{3}{4}Q_{2366}{\lambda_4}^2 + \tfrac{1}{20}Q_{2445}{\lambda_4}^2 + \tfrac{24}{5}\wp_{56}\lambda_4\lambda_1$$

$$\textbf{(−23)}\quad \wp_{2225} = 6\wp_{22}\wp_{25} + 24\wp_{36}\lambda_1 - \tfrac{36}{5}\wp_{36}\lambda_4\lambda_2 + \tfrac{6}{5}\wp_{56}{\lambda_4}^2\lambda_2 + \tfrac{52}{5}\wp_{56}\lambda_4\lambda_1 - \tfrac{3}{5}Q_{2445}{\lambda_4}^2 - 3Q_{2366}{\lambda_4}^2 - \tfrac{3}{2}Q_{2366}\lambda_3 - \tfrac{12}{5}Q_{2344}\lambda_4 - 3Q_{2566}\lambda_2 + \tfrac{6}{5}Q_{2245}\lambda_4 - 7\wp_{56}\lambda_3\lambda_2 + 3Q_{3556}\lambda_2 - 3\wp_{36}{\lambda_3}^2 + \tfrac{27}{10}Q_{2566}\lambda_4\lambda_3 - 3Q_{3556}\lambda_4\lambda_3 + \tfrac{66}{5}\wp_{36}{\lambda_4}^2\lambda_3 - 6Q_{1246} + 20\wp_{56}\lambda_0 - \tfrac{3}{5}Q_{2566}{\lambda_4}^3 + \tfrac{3}{2}Q_{2445}\lambda_3$$

$$\textbf{(−23)}\quad \wp_{2234} = 2\wp_{22}\wp_{34} + 4\wp_{23}\wp_{24} - 4\wp_{36}\lambda_1 + \tfrac{14}{5}\wp_{36}\lambda_4\lambda_2 + \tfrac{8}{5}\wp_{56}{\lambda_4}^2\lambda_2 - \tfrac{28}{5}\wp_{56}\lambda_4\lambda_1 - \tfrac{29}{5}\wp_{36}{\lambda_4}^2\lambda_3 - \tfrac{1}{2}\wp_{36}{\lambda_3}^2 - \tfrac{1}{20}Q_{2566}\lambda_4\lambda_3 + \tfrac{3}{2}Q_{3556}\lambda_4\lambda_3 + \tfrac{3}{2}\wp_{56}\lambda_3\lambda_2 + \tfrac{8}{5}Q_{2344}\lambda_4 + 2Q_{1246} - \tfrac{1}{10}Q_{2566}{\lambda_4}^3 - \tfrac{1}{4}Q_{2445}\lambda_3 + \tfrac{3}{4}Q_{2366}\lambda_3 - \tfrac{1}{10}Q_{2445}{\lambda_4}^2 + \tfrac{3}{2}Q_{2366}{\lambda_4}^2 + \tfrac{8}{5}Q_{2245}\lambda_4 - \tfrac{3}{2}Q_{3556}\lambda_2$$

$$\textbf{(−23)}\quad \wp_{1245} = Q_{1245} + 2\wp_{12}\wp_{45} + 2\wp_{14}\wp_{25} + 2\wp_{15}\wp_{24}$$

$$\textbf{(−24)}\quad \wp_{1336} = 4\wp_{13}\wp_{36} + 2\wp_{16}\wp_{33} + \wp_{35}\lambda_1 - 2\wp_{26}\lambda_1 + 4\wp_{46}\lambda_0 - 3\wp_{55}\lambda_0 + 4\wp_{16}\lambda_2 - Q_{1166}$$

$$\textbf{(−24)}\quad \wp_{3333} = 6\wp_{33}^2 - 256\wp_{16}{\lambda_4}^3 - 50\wp_{35}\lambda_1 + 16\wp_{16}\lambda_4\lambda_3 - 72\wp_{55}\lambda_4\lambda_1 - 32\lambda_3\lambda_1 + 4\wp_{33}\lambda_2 + 48\wp_{46}\lambda_4\lambda_1 - 64{\lambda_4}^2\lambda_1 + 6Q_{1166} - 36Q_{1266}\lambda_4 + 72\wp_{46}\lambda_0 - 72Q_{1356}\lambda_4 - 54\wp_{55}\lambda_0 + 16Q_{1444}\lambda_4 + 48Q_{1556}{\lambda_4}^2 + 24Q_{1466}\lambda_3 + 12Q_{1244} + 36\wp_{26}\lambda_1 + 48Q_{1466}{\lambda_4}^2 + 24Q_{1556}\lambda_3$$

$$\textbf{(−24)}\quad \wp_{2224} = 6\wp_{22}\wp_{24} + 6{\lambda_2}^2 - 6Q_{1244} - 3\wp_{26}{\lambda_3}^2 - 40\wp_{46}\lambda_0 + 9\wp_{33}\lambda_2 + 18\lambda_3\lambda_1 + 48{\lambda_4}^2\lambda_1 + 192\wp_{16}{\lambda_4}^3 - \tfrac{9}{2}\wp_{35}{\lambda_3}^2 + 3Q_{2356}\lambda_3 + 24Q_{1266}\lambda_4 - 24\wp_{26}\lambda_1 + 48Q_{1356}\lambda_4 - 12Q_{1444}\lambda_4 - 12\wp_{16}\lambda_2 - 36Q_{1556}{\lambda_4}^2 - 40\wp_{46}\lambda_4\lambda_1 - 15Q_{1466}\lambda_3 + 36\wp_{35}\lambda_1 - 36Q_{1466}{\lambda_4}^2 + 3Q_{3456}\lambda_2 + \tfrac{3}{2}Q_{2266}\lambda_3 + \tfrac{3}{2}\wp_{55}\lambda_3\lambda_2 - 15Q_{1556}\lambda_3 + \tfrac{3}{2}Q_{2556}\lambda_2 + 30\wp_{55}\lambda_0 + 54\wp_{55}\lambda_4\lambda_1 - 24\wp_{16}\lambda_4\lambda_3 + 2\wp_{46}\lambda_3\lambda_2$$

$$\textbf{(−24)}\quad \wp_{1166} = Q_{1166} + 2\wp_{11}\wp_{66} + 4\wp_{16}^2$$

$$\textbf{(−24)}\quad \wp_{1244} = Q_{1244} + 2\wp_{12}\wp_{44} + 4\wp_{14}\wp_{24}$$

$$\textbf{(−25)}\quad \wp_{1236} = 2\wp_{12}\wp_{36} + 2\wp_{13}\wp_{26} + 2\wp_{16}\wp_{23} + 2\wp_{15}\lambda_2 + \tfrac{4}{3}\wp_{34}\lambda_1 - 3\wp_{45}\lambda_0 - \wp_{25}\lambda_1 + \tfrac{1}{3}\wp_{45}\lambda_4\lambda_1 - \tfrac{1}{2}Q_{1156} - \tfrac{1}{12}Q_{4555}\lambda_1$$

$$\textbf{(−25)}\quad \wp_{1335} = 4\wp_{13}\wp_{35} + 2\wp_{15}\wp_{33} + 4\wp_{15}\lambda_2 + \tfrac{4}{3}\wp_{34}\lambda_1 - 2\wp_{45}\lambda_0 - 2\wp_{25}\lambda_1 - \tfrac{2}{3}\wp_{45}\lambda_4\lambda_1 - Q_{1156} + \tfrac{1}{6}Q_{4555}\lambda_1$$

$$\textbf{(−25)}\quad \wp_{2333} = 6\wp_{23}\wp_{33} + 4\wp_{34}\lambda_1 + 4\wp_{23}\lambda_2 - 6\wp_{45}\lambda_0 - 2\wp_{25}\lambda_1 - 2\wp_{45}\lambda_4\lambda_1 + 3Q_{1156} + \tfrac{1}{2}Q_{4555}\lambda_1$$

$$\textbf{(−25)}\quad \wp_{1156} = Q_{1156} + 2\wp_{11}\wp_{56} + 4\wp_{15}\wp_{16}$$

$$\textbf{(−26)}\quad \wp_{1235} = 2\wp_{12}\wp_{35} + 2\wp_{13}\wp_{25} + 2\wp_{15}\wp_{23} + \tfrac{4}{3}\wp_{66}\lambda_3\lambda_1 + 8\wp_{66}\lambda_4\lambda_0 - \tfrac{4}{3}\wp_{24}\lambda_1 - 8\wp_{44}\lambda_0 - \tfrac{1}{2}Q_{1155} - \tfrac{1}{6}Q_{5566}\lambda_4\lambda_1 + \tfrac{1}{3}Q_{3566}\lambda_4\lambda_2 - \tfrac{5}{2}Q_{5566}\lambda_0 + \tfrac{8}{3}\wp_{14}\lambda_2 - \tfrac{2}{3}Q_{3366}\lambda_2 - \tfrac{2}{3}Q_{3566}\lambda_1 + \tfrac{1}{3}Q_{3445}\lambda_2 - \tfrac{1}{6}Q_{4455}\lambda_1$$

$$\textbf{(−26)}\quad \wp_{1226} = 4\wp_{12}\wp_{26} + 2\wp_{16}\wp_{22} + \tfrac{8}{5}\wp_{14}\lambda_4\lambda_3 + \tfrac{2}{15}\wp_{66}\lambda_3\lambda_1 - 6\wp_{66}\lambda_4\lambda_0 + \tfrac{2}{3}\wp_{24}\lambda_1 + 2\wp_{44}\lambda_0 + \tfrac{1}{3}Q_{5566}\lambda_4\lambda_1 - \tfrac{1}{10}Q_{3366}\lambda_4\lambda_3 + \tfrac{1}{5}Q_{3445}\lambda_4\lambda_3 + \tfrac{1}{5}Q_{3566}{\lambda_4}^2\lambda_3 - \tfrac{1}{6}Q_{3566}\lambda_4\lambda_2 + \tfrac{5}{2}Q_{5566}\lambda_0 + \tfrac{8}{3}\wp_{14}\lambda_2 - \tfrac{6}{5}\wp_{12}\lambda_3 - \tfrac{3}{10}Q_{3344}\lambda_3 - \tfrac{3}{5}Q_{2345}\lambda_3 + \tfrac{1}{3}Q_{4455}\lambda_1 + \tfrac{1}{3}Q_{3366}\lambda_2 + \tfrac{11}{6}Q_{3566}\lambda_1 - \tfrac{1}{6}Q_{3445}\lambda_2 - Q_{1146} - \tfrac{3}{5}Q_{3566}{\lambda_3}^2$$

$$\textbf{(−26)}\quad \wp_{2233} = 2\wp_{22}\wp_{33} + 4\wp_{23}^2 - \tfrac{16}{5}\wp_{14}\lambda_4\lambda_3 - \tfrac{64}{15}\wp_{66}\lambda_3\lambda_1 + 4\wp_{66}\lambda_4\lambda_0 - \tfrac{4}{3}\wp_{24}\lambda_1 - 4\wp_{44}\lambda_0 + Q_{1155} + \tfrac{1}{3}Q_{5566}\lambda_4\lambda_1 + \tfrac{1}{5}Q_{3366}\lambda_4\lambda_3 - \tfrac{2}{5}Q_{3445}\lambda_4\lambda_3 + 2Q_{1146} - \tfrac{2}{5}Q_{3566}{\lambda_4}^2\lambda_3 - \tfrac{2}{3}Q_{3566}\lambda_4\lambda_2 + \tfrac{8}{3}\wp_{14}\lambda_2 - 8\wp_{11}\lambda_4 + \tfrac{32}{5}\wp_{12}\lambda_3 + \tfrac{6}{5}Q_{2345}\lambda_3 + \tfrac{3}{5}Q_{3344}\lambda_3 + \tfrac{1}{3}Q_{4455}\lambda_1 + \tfrac{1}{3}Q_{3366}\lambda_2 - \tfrac{5}{3}Q_{3566}\lambda_1 - \tfrac{2}{3}Q_{3445}\lambda_2 + \tfrac{6}{5}Q_{3566}{\lambda_3}^2$$

$$\textbf{(−26)}\quad \wp_{1334} = 4\wp_{13}\wp_{34} + 2\wp_{14}\wp_{33} - 2\wp_{66}\lambda_4\lambda_0 - 2\wp_{24}\lambda_1 + 6\wp_{44}\lambda_0 + 4\wp_{14}\lambda_2 - \tfrac{1}{2}Q_{3566}\lambda_1 + \tfrac{5}{2}Q_{5566}\lambda_0 - Q_{1146}$$

$$\textbf{(−26)}\quad \wp_{1146} = Q_{1146} + 2\wp_{11}\wp_{46} + 4\wp_{14}\wp_{16}$$

$$\textbf{(−26)}\quad \wp_{1155} = Q_{1155} + 2\wp_{11}\wp_{55} + 4\wp_{15}^2$$

$$\textbf{(−27)}\quad \wp_{1225} = 4\wp_{12}\wp_{25} + 2\wp_{15}\wp_{22} - \tfrac{13}{10}\wp_{36}\lambda_3\lambda_2 + \tfrac{33}{10}\wp_{36}\lambda_4{\lambda_3}^2 + 2\wp_{56}\lambda_4\lambda_0 - \tfrac{3}{2}\wp_{56}{\lambda_2}^2 + \tfrac{1}{4}Q_{2566}\lambda_4\lambda_2 - \tfrac{3}{20}Q_{2445}\lambda_4\lambda_3 - \tfrac{3}{20}Q_{2566}{\lambda_4}^2\lambda_3 - \tfrac{3}{4}Q_{2366}\lambda_4\lambda_3 - Q_{1145} + \tfrac{3}{2}Q_{3556}\lambda_1 - \tfrac{3}{4}Q_{2366}\lambda_2 - \tfrac{3}{5}Q_{2344}\lambda_3 + \tfrac{1}{4}Q_{2445}\lambda_2 + \tfrac{8}{5}\wp_{56}\lambda_3\lambda_1 + \tfrac{3}{10}Q_{2245}\lambda_3 - \tfrac{3}{4}Q_{3556}{\lambda_3}^2 - Q_{2566}\lambda_1 + \tfrac{3}{10}Q_{2566}{\lambda_3}^2 + \tfrac{3}{10}\wp_{56}\lambda_4\lambda_3\lambda_2 + 10\wp_{36}\lambda_0$$

$$\textbf{(−27)}\quad \wp_{2223} = 6\wp_{22}\wp_{23} - \tfrac{38}{5}\wp_{36}\lambda_3\lambda_2 - \tfrac{27}{5}\wp_{36}\lambda_4{\lambda_3}^2 + 6\wp_{56}\lambda_4\lambda_0 - \tfrac{24}{5}\wp_{56}\lambda_3\lambda_1 + \tfrac{3}{5}\wp_{56}\lambda_4\lambda_3\lambda_2 - \tfrac{3}{10}Q_{2445}\lambda_4\lambda_3 - \tfrac{3}{10}Q_{2566}{\lambda_4}^2\lambda_3 + \tfrac{3}{2}Q_{2366}\lambda_4\lambda_3 + 8\wp_{36}\lambda_4\lambda_1 + 3Q_{1145} - \tfrac{3}{2}Q_{3556}\lambda_1 + 3Q_{2366}\lambda_2 + \tfrac{9}{5}Q_{2344}\lambda_3 + \tfrac{3}{5}Q_{2566}{\lambda_3}^2 - 3Q_{2566}\lambda_1 - 10\wp_{36}\lambda_0 + \tfrac{3}{5}Q_{2245}\lambda_3 + \tfrac{3}{2}Q_{3556}{\lambda_3}^2$$

$$(\mathbf{-27})\quad \wp_{1234} = 2\wp_{12}\wp_{34} + 2\wp_{13}\wp_{24} + 2\wp_{14}\wp_{23} + \tfrac{1}{2}\wp_{36}\lambda_3\lambda_2 + 3\wp_{56}\lambda_4\lambda_0 + \tfrac{1}{4}Q_{2566}\lambda_4\lambda_2 - \tfrac{1}{2}Q_{1145} - \tfrac{3}{4}Q_{3556}\lambda_1 + 5\wp_{36}\lambda_0 - \tfrac{3}{4}Q_{2366}\lambda_2 + \tfrac{1}{4}Q_{2445}\lambda_2 + 2\wp_{36}\lambda_4\lambda_1 + \tfrac{1}{2}Q_{2566}\lambda_1 - \tfrac{3}{2}\wp_{56}{\lambda_2}^2$$

$$(\mathbf{-27})\quad \wp_{1145} = Q_{1145} + 2\wp_{11}\wp_{45} + 4\wp_{14}\wp_{15}$$

$$(\mathbf{-28})\quad \wp_{1144} = Q_{1144} + 2\wp_{11}\wp_{44} + 4\wp_{14}^2$$

$$(\mathbf{-28})\quad \wp_{2222} = 6\wp_{22}^2 + 6Q_{2266}\lambda_2 + 16\lambda_4\lambda_3\lambda_1 + 42\wp_{35}\lambda_4\lambda_1 + 12\wp_{33}\lambda_4\lambda_2 + 4Q_{1444}\lambda_3 - 16\wp_{26}\lambda_3\lambda_2 - 28\wp_{26}\lambda_4\lambda_1 - 12Q_{1466}\lambda_4\lambda_3 - 48\wp_{46}{\lambda_4}^2\lambda_1 - 30\wp_{35}\lambda_0 + 12Q_{3456}\lambda_1 - 12\wp_{35}\lambda_3\lambda_2 - 80\wp_{16}{\lambda_4}^2\lambda_3 - 12Q_{1244}\lambda_4 + 6Q_{1144} - 3\wp_{33}{\lambda_3}^2 - 48\wp_{16}\lambda_4\lambda_2 + 64{\lambda_4}^3\lambda_1 + 256\wp_{16}{\lambda_4}^4 + 3\wp_{55}{\lambda_2}^2 + 8\lambda_4{\lambda_2}^2 + 36Q_{1266}{\lambda_4}^2 + 72\wp_{55}{\lambda_4}^2\lambda_1 - 12\wp_{55}\lambda_3\lambda_1 + 12Q_{2356}\lambda_2 - 48Q_{1466}{\lambda_4}^3 - 16Q_{1444}{\lambda_4}^2 - 12Q_{1556}\lambda_4\lambda_3 - 24\wp_{16}\lambda_1 + 72Q_{1356}{\lambda_4}^2 - 48Q_{1556}{\lambda_4}^3 - 6Q_{2556}\lambda_1 + 12\wp_{46}{\lambda_2}^2 - 2{\lambda_3}^2\lambda_2 - 72\wp_{46}\lambda_4\lambda_0 - 12Q_{1266}\lambda_3 + 4\lambda_2\lambda_1 + 52\wp_{16}{\lambda_3}^2 + 18\wp_{33}\lambda_1 + 54\wp_{55}\lambda_4\lambda_0 - 24Q_{1356}\lambda_3 + 20\wp_{26}\lambda_0$$

$$(\mathbf{-28})\quad \wp_{1224} = 4\wp_{12}\wp_{24} + 2\wp_{14}\wp_{22} + 12\lambda_4\lambda_3\lambda_1 - 3Q_{1444}\lambda_3 - 8\wp_{46}\lambda_3\lambda_1 + 48\wp_{16}{\lambda_4}^2\lambda_3 + 3\wp_{35}\lambda_4\lambda_1 - 2\wp_{26}\lambda_4\lambda_1 + 3\wp_{55}\lambda_4\lambda_0 - 4\wp_{46}\lambda_4\lambda_0 + 12\wp_{55}\lambda_3\lambda_1 + 16\wp_{16}\lambda_4\lambda_2 - 3Q_{1466}\lambda_2 + 6Q_{1266}\lambda_3 + 6\lambda_2\lambda_1 - 9Q_{1556}\lambda_4\lambda_3 - 9Q_{1466}\lambda_4\lambda_3 - 3Q_{1556}\lambda_2 - 24\wp_{16}{\lambda_3}^2 + 3\wp_{33}\lambda_1 + 15\wp_{35}\lambda_0 - Q_{1155} + 12Q_{1356}\lambda_3 - 10\wp_{26}\lambda_0 - 4\wp_{16}\lambda_1$$

$$(\mathbf{-29})\quad \wp_{1136} = 2\wp_{11}\wp_{36} + 4\wp_{13}\wp_{16} - 4\wp_{25}\lambda_0 + \tfrac{2}{3}\wp_{45}\lambda_4\lambda_0 + 2\wp_{15}\lambda_1 + \tfrac{8}{3}\wp_{34}\lambda_0 - \tfrac{1}{6}Q_{4555}\lambda_0$$

$$(\mathbf{-29})\quad \wp_{1333} = 6\wp_{13}\wp_{33} + 4\wp_{13}\lambda_2 - 12\wp_{25}\lambda_0 - 4\wp_{45}\lambda_4\lambda_0 + 4\wp_{15}\lambda_1 + 8\wp_{34}\lambda_0 + Q_{4555}\lambda_0$$

$$(\mathbf{-30})\quad \wp_{1126} = 2\wp_{11}\wp_{26} + 4\wp_{12}\wp_{16} + \tfrac{16}{15}\wp_{14}\lambda_4\lambda_2 + \tfrac{6}{5}\wp_{66}\lambda_2\lambda_1 - 4\wp_{66}\lambda_3\lambda_0 + \tfrac{1}{6}Q_{3566}\lambda_4\lambda_1 + \tfrac{2}{15}Q_{3445}\lambda_4\lambda_2 + \tfrac{2}{15}Q_{3566}{\lambda_4}^2\lambda_2 - \tfrac{2}{5}Q_{3566}\lambda_3\lambda_2 - \tfrac{2}{5}Q_{2345}\lambda_2 + \tfrac{1}{6}Q_{3445}\lambda_1 + \tfrac{4}{3}\wp_{14}\lambda_1 - \tfrac{4}{5}\wp_{12}\lambda_2 + \tfrac{1}{2}Q_{4455}\lambda_0 - \tfrac{1}{5}Q_{3344}\lambda_2 - \tfrac{3}{5}Q_{3566}\lambda_0 + \tfrac{1}{6}Q_{3366}\lambda_1 - \tfrac{1}{15}Q_{3366}\lambda_4\lambda_2 + \tfrac{1}{2}Q_{5566}\lambda_4\lambda_0$$

$$(\mathbf{-30})\quad \wp_{1135} = 2\wp_{11}\wp_{35} + 4\wp_{13}\wp_{15} + \tfrac{16}{3}\wp_{66}\lambda_3\lambda_0 - \tfrac{16}{3}\wp_{24}\lambda_0 - \tfrac{2}{3}Q_{5566}\lambda_4\lambda_0 + \tfrac{8}{3}\wp_{14}\lambda_1 + \tfrac{1}{3}Q_{3445}\lambda_1 - \tfrac{2}{3}Q_{4455}\lambda_0 + \tfrac{1}{3}Q_{3566}\lambda_4\lambda_1 - \tfrac{5}{3}Q_{3566}\lambda_0 - \tfrac{2}{3}Q_{3366}\lambda_1$$

$$(\mathbf{-30})\quad \wp_{1233} = 2\wp_{12}\wp_{33} + 4\wp_{13}\wp_{23} - \tfrac{16}{15}\wp_{14}\lambda_4\lambda_2 - \tfrac{6}{5}\wp_{66}\lambda_2\lambda_1 + \tfrac{10}{3}\wp_{66}\lambda_3\lambda_0 - \tfrac{16}{3}\wp_{24}\lambda_0 + \tfrac{1}{15}Q_{3366}\lambda_4\lambda_2 - \tfrac{1}{3}Q_{3566}\lambda_4\lambda_1 + \tfrac{16}{3}\wp_{14}\lambda_1 - \tfrac{2}{15}Q_{3445}\lambda_4\lambda_2 - \tfrac{2}{15}Q_{3566}{\lambda_4}^2\lambda_2 + \tfrac{2}{5}Q_{2345}\lambda_2 - 2\wp_{11}\lambda_3 + \tfrac{24}{5}\wp_{12}\lambda_2 - 2\wp_{22}\lambda_1 - \tfrac{1}{3}Q_{3445}\lambda_1 - \tfrac{1}{3}Q_{3366}\lambda_1 + \tfrac{2}{5}Q_{3566}\lambda_3\lambda_2 + \tfrac{1}{3}Q_{4455}\lambda_0 + \tfrac{1}{5}Q_{3344}\lambda_2 - \tfrac{5}{3}Q_{3566}\lambda_0 + \tfrac{1}{3}Q_{5566}\lambda_4\lambda_0$$

$$(\mathbf{-31})\quad \wp_{1134} = 2\wp_{11}\wp_{34} + 4\wp_{13}\wp_{14} + \tfrac{1}{4}\lambda_4\lambda_1\wp_{2566} + 4\wp_{36}\lambda_4\lambda_0 - \tfrac{3}{4}Q_{2366}\lambda_1 - \tfrac{3}{2}\wp_{56}\lambda_2\lambda_1 + \tfrac{1}{2}\wp_{36}\lambda_3\lambda_1 - \tfrac{1}{2}\lambda_4\lambda_1\wp_{25}\wp_{66} + \tfrac{1}{4}Q_{2445}\lambda_1 - \tfrac{3}{2}Q_{3556}\lambda_0 + 2Q_{2566}\lambda_0 - \lambda_4\lambda_1\wp_{26}\wp_{56}$$

$$(-31)\quad \wp_{1125} = 2\wp_{11}\wp_{25} + 4\wp_{12}\wp_{15} + 2\wp_{36}\lambda_3\lambda_1 + \tfrac{11}{5}\wp_{36}\lambda_4\lambda_3\lambda_2 - \tfrac{1}{2}Q_{2366}\lambda_4\lambda_2 - \tfrac{1}{2}Q_{3556}\lambda_3\lambda_2 - \tfrac{1}{10}Q_{2566}{\lambda_4}^2\lambda_2 - \tfrac{6}{5}\wp_{36}{\lambda_2}^2 - \tfrac{1}{10}Q_{2445}\lambda_4\lambda_2 + 2Q_{3556}\lambda_0 + \tfrac{2}{5}\wp_{56}\lambda_2\lambda_1 + \tfrac{1}{5}Q_{2245}\lambda_2 + \tfrac{1}{5}\wp_{56}\lambda_4{\lambda_2}^2 - Q_{2366}\lambda_1 - Q_{2566}\lambda_0 - \tfrac{2}{5}Q_{2344}\lambda_2 + \tfrac{1}{5}Q_{2566}\lambda_3\lambda_2$$

$$(-31)\quad \wp_{1223} = 4\wp_{12}\wp_{23} + 2\wp_{13}\wp_{22} + \tfrac{1}{2}Q_{2445}\lambda_1 - \tfrac{9}{5}\wp_{36}{\lambda_2}^2 - \tfrac{3}{20}Q_{2566}{\lambda_4}^2\lambda_2 + \tfrac{1}{4}Q_{2366}\lambda_4\lambda_2 + \tfrac{1}{4}Q_{3556}\lambda_3\lambda_2 + \tfrac{3}{10}Q_{2566}\lambda_3\lambda_2 - \tfrac{7}{10}\wp_{36}\lambda_4\lambda_3\lambda_2 + \tfrac{3}{10}Q_{2245}\lambda_2 + \tfrac{1}{2}Q_{2366}\lambda_1 + 6\wp_{56}\lambda_3\lambda_0 - Q_{3556}\lambda_0 - \tfrac{3}{20}Q_{2445}\lambda_4\lambda_2 + \tfrac{3}{10}\wp_{56}\lambda_4{\lambda_2}^2 - \wp_{36}\lambda_3\lambda_1 + \tfrac{1}{2}Q_{2566}\lambda_4\lambda_1 - 2Q_{2566}\lambda_0 + \tfrac{2}{5}Q_{2344}\lambda_2 - \tfrac{27}{5}\wp_{56}\lambda_2\lambda_1 + 8\wp_{36}\lambda_4\lambda_0$$

$$(-32)\quad \wp_{1124} = 2\wp_{11}\wp_{24} + 4\wp_{12}\wp_{14} + \wp_{26}\lambda_3\lambda_1 - 6Q_{1466}\lambda_4\lambda_2 - 6Q_{1556}\lambda_4\lambda_2 - \tfrac{1}{2}Q_{2266}\lambda_1 + 6\wp_{35}\lambda_4\lambda_0 + \tfrac{15}{2}\wp_{55}\lambda_2\lambda_1 + 8\lambda_4\lambda_2\lambda_1 - 16\wp_{16}\lambda_3\lambda_2 + 32\wp_{16}{\lambda_4}^2\lambda_2 - 4\wp_{16}\lambda_0 + 2{\lambda_1}^2 + \tfrac{3}{2}\wp_{35}\lambda_3\lambda_1 - 6\wp_{46}\lambda_2\lambda_1 - 4\wp_{26}\lambda_4\lambda_0 + \tfrac{1}{2}Q_{2556}\lambda_0 + 8\wp_{16}\lambda_4\lambda_1 + 3\wp_{33}\lambda_0 - Q_{1466}\lambda_1 - Q_{1556}\lambda_1 - Q_{2356}\lambda_1 - Q_{3456}\lambda_0 + 2\lambda_2\lambda_0 + 4Q_{1266}\lambda_2 - 2Q_{1444}\lambda_2 + 8Q_{1356}\lambda_2$$

$$(-32)\quad \wp_{1222} = 6\wp_{12}\wp_{22} + 4{\lambda_1}^2 - 18\wp_{26}\lambda_3\lambda_1 - 6Q_{1466}\lambda_4\lambda_2 - 6Q_{1556}\lambda_4\lambda_2 + 12\lambda_4\lambda_2\lambda_1 - 12\wp_{16}\lambda_3\lambda_2 + 64\wp_{16}{\lambda_4}^3\lambda_3 - 4\wp_{16}\lambda_4{\lambda_3}^2 + 18Q_{1356}\lambda_4\lambda_3 + 9Q_{1266}\lambda_4\lambda_3 + 32\wp_{16}{\lambda_4}^2\lambda_2 + 8\wp_{16}\lambda_4\lambda_1 - 24\wp_{46}\lambda_3\lambda_0 + 18\wp_{55}\lambda_3\lambda_0 + 16{\lambda_4}^2\lambda_3\lambda_1 - 4Q_{1444}\lambda_4\lambda_3 - 12Q_{1466}{\lambda_4}^2\lambda_3 - 12Q_{1556}{\lambda_4}^2\lambda_3 + 9\wp_{55}\lambda_2\lambda_1 + 6\wp_{33}\lambda_4\lambda_1 - 12\wp_{46}\lambda_4\lambda_3\lambda_1 + 18\wp_{55}\lambda_4\lambda_3\lambda_1 - 3Q_{2556}\lambda_0 - 16\wp_{16}\lambda_0 + 8{\lambda_3}^2\lambda_1 - 6Q_{1556}{\lambda_3}^2 + 9\wp_{35}\lambda_3\lambda_1 - 3Q_{1244}\lambda_3 - 6Q_{1466}{\lambda_3}^2 - 6Q_{1556}\lambda_1 + 6Q_{2356}\lambda_1 + 6Q_{3456}\lambda_0 - 6Q_{1466}\lambda_1 + 8\lambda_2\lambda_0 + 3Q_{1266}\lambda_2 - 2Q_{1444}\lambda_2 + 6Q_{1356}\lambda_2 + 3Q_{2266}\lambda_1 + 12\wp_{33}\lambda_0$$

$$(-34)\quad \wp_{1116} = 6\wp_{11}\wp_{16} - 4\wp_{66}\lambda_2\lambda_0 - \tfrac{3}{5}Q_{3566}\lambda_3\lambda_1 + \tfrac{3}{2}Q_{3566}\lambda_4\lambda_0 - \tfrac{1}{10}Q_{3366}\lambda_4\lambda_1 + \tfrac{1}{5}Q_{3445}\lambda_4\lambda_1 + \tfrac{1}{5}Q_{3566}{\lambda_4}^2\lambda_1 - \tfrac{3}{5}Q_{2345}\lambda_1 + \tfrac{14}{5}\wp_{66}{\lambda_1}^2 - \tfrac{3}{10}Q_{3344}\lambda_1 - \tfrac{6}{5}\wp_{12}\lambda_1 + \tfrac{3}{2}Q_{3445}\lambda_0 + \tfrac{8}{5}\wp_{14}\lambda_4\lambda_1$$

$$(-34)\quad \wp_{1133} = 2\wp_{11}\wp_{33} + 4\wp_{13}^2 + 8\wp_{66}\lambda_2\lambda_0 + \tfrac{2}{5}Q_{3566}\lambda_3\lambda_1 - \tfrac{2}{3}Q_{3566}\lambda_4\lambda_0 - \tfrac{2}{15}Q_{3566}{\lambda_4}^2\lambda_1 + \tfrac{2}{5}Q_{2345}\lambda_1 - \tfrac{2}{15}Q_{3445}\lambda_4\lambda_1 + \tfrac{32}{3}\wp_{14}\lambda_0 - 8\wp_{22}\lambda_0 - \tfrac{5}{3}Q_{3366}\lambda_0 - \tfrac{16}{5}\wp_{66}{\lambda_1}^2 - \tfrac{16}{15}\wp_{14}\lambda_4\lambda_1 - \tfrac{2}{3}Q_{3445}\lambda_0 + \tfrac{1}{5}Q_{3344}\lambda_1 + \tfrac{24}{5}\wp_{12}\lambda_1 + \tfrac{1}{15}Q_{3366}\lambda_4\lambda_1$$

$$(-35)\quad \wp_{1123} = 2\wp_{11}\wp_{23} + 4\wp_{12}\wp_{13} + Q_{2445}\lambda_0 - \tfrac{12}{5}\wp_{36}\lambda_2\lambda_1 + \tfrac{2}{5}\wp_{36}\lambda_4\lambda_3\lambda_1 + \tfrac{3}{10}Q_{2245}\lambda_1 + \tfrac{2}{5}\wp_{56}\lambda_4\lambda_2\lambda_1 - \tfrac{1}{5}Q_{2566}{\lambda_4}^2\lambda_1 - \tfrac{1}{5}Q_{2445}\lambda_4\lambda_1 + Q_{2566}\lambda_4\lambda_0 + 2\wp_{36}\lambda_3\lambda_0 - 2\wp_{56}\lambda_2\lambda_0 + \tfrac{1}{5}Q_{2344}\lambda_1 + \tfrac{2}{5}Q_{2245}\lambda_1 - \tfrac{16}{5}\wp_{56}{\lambda_1}^2 + \tfrac{2}{5}Q_{2566}\lambda_3\lambda_1$$

$$(-35)\quad \wp_{1115} = 6\wp_{11}\wp_{15} - \tfrac{3}{4}Q_{2445}\lambda_0 + \tfrac{33}{10}\wp_{36}\lambda_4\lambda_3\lambda_1 + \tfrac{3}{10}\wp_{56}\lambda_4\lambda_2\lambda_1 + \tfrac{3}{10}Q_{2566}\lambda_3\lambda_1 + \tfrac{3}{4}Q_{2566}\lambda_4\lambda_0 - \tfrac{3}{20}Q_{2445}\lambda_4\lambda_1 - \tfrac{3}{4}Q_{2366}\lambda_4\lambda_1 - \tfrac{3}{4}Q_{3556}\lambda_3\lambda_1 + \tfrac{8}{5}\wp_{56}{\lambda_1}^2 - \tfrac{9}{5}\wp_{36}\lambda_2\lambda_1 - \tfrac{3}{20}Q_{2566}{\lambda_4}^2\lambda_1 + \tfrac{21}{2}\wp_{36}\lambda_3\lambda_0 + \tfrac{1}{2}\wp_{56}\lambda_2\lambda_0 - \tfrac{3}{5}Q_{2344}\lambda_1 - \tfrac{15}{4}Q_{2366}\lambda_0 + \tfrac{3}{10}Q_{2245}\lambda_1$$

$$\begin{aligned}
\mathbf{(-36)}\quad \wp_{1114} &= 6\wp_{11}\wp_{14} + 6\wp_{26}\lambda_3\lambda_0 - 3\wp_{55}\lambda_2\lambda_0 - 4\wp_{46}\lambda_2\lambda_0 + 48\wp_{16}{\lambda_4}^2\lambda_1\\
&+ 3Q_{2266}\lambda_0 + 12Q_{1356}\lambda_1 + 9\wp_{35}\lambda_3\lambda_0 - 6\lambda_0\wp_{2266} + 3Q_{1466}\lambda_0 + 3Q_{1556}\lambda_0\\
&+ 6Q_{1266}\lambda_1 - 3Q_{1444}\lambda_1 - 6Q_{2356}\lambda_0 - 9Q_{1466}\lambda_4\lambda_1 + 24\lambda_0{\wp_{26}}^2 - 9Q_{1556}\lambda_4\lambda_1\\
&+ 12\lambda_4{\lambda_1}^2 - 8\wp_{46}{\lambda_1}^2 + 12\wp_{55}{\lambda_1}^2 + 12\lambda_0\wp_{22}\wp_{66} - 24\wp_{16}\lambda_3\lambda_1
\end{aligned}$$

$$\begin{aligned}
\mathbf{(-36)}\quad Q_{1122} &= 2\wp_{11}\wp_{22} + 4\wp_{12}^2 + \tfrac{40}{3}\lambda_4{\lambda_1}^2 - 2Q_{1244}\lambda_2 - 8Q_{1556}\lambda_0 + 3Q_{2266}\lambda_0\\
&- 8\wp_{46}{\lambda_1}^2 - 8Q_{1466}\lambda_0 + 12\wp_{55}{\lambda_1}^2 + 6\lambda_3\lambda_2\lambda_1 + 9\wp_{35}\lambda_2\lambda_1 - \tfrac{10}{3}Q_{1444}\lambda_1 + 6Q_{1266}\lambda_1\\
&+ 6Q_{2356}\lambda_0 + \wp_{33}\lambda_3\lambda_1 + \tfrac{20}{3}\lambda_1\lambda_0 - \tfrac{8}{3}\wp_{16}\lambda_4\lambda_3\lambda_2 + \tfrac{32}{3}{\lambda_4}^2\lambda_2\lambda_1 - \tfrac{8}{3}Q_{1444}\lambda_4\lambda_2\\
&+ 8\wp_{33}\lambda_4\lambda_0 - 8Q_{1466}{\lambda_4}^2\lambda_2 + 6Q_{1266}\lambda_4\lambda_2 - 4Q_{1466}\lambda_3\lambda_2 + 12Q_{1356}\lambda_4\lambda_2\\
&- 8Q_{1556}{\lambda_4}^2\lambda_2 - 4Q_{1556}\lambda_3\lambda_2 - 8\wp_{46}\lambda_4\lambda_2\lambda_1 + 12\wp_{55}\lambda_4\lambda_2\lambda_1 - 10Q_{1556}\lambda_4\lambda_1\\
&- 6\wp_{26}\lambda_2\lambda_1 - 10Q_{1466}\lambda_4\lambda_1 - \tfrac{76}{3}\wp_{16}\lambda_3\lambda_1 - 8\wp_{46}\lambda_2\lambda_0 + \tfrac{160}{3}\wp_{16}{\lambda_4}^2\lambda_1\\
&+ \tfrac{128}{3}\wp_{16}{\lambda_4}^3\lambda_2 - 12\wp_{26}\lambda_3\lambda_0 + 12\wp_{55}\lambda_2\lambda_0 + 16\wp_{16}\lambda_4\lambda_0 + 12Q_{1356}\lambda_1 + \tfrac{16}{3}\lambda_4\lambda_2\lambda_0
\end{aligned}$$

$$\begin{aligned}
\mathbf{(-39)}\quad \wp_{1113} &= 6\wp_{11}\wp_{12} - 2\wp_{36}{\lambda_1}^2 - \wp_{36}\lambda_2\lambda_0 - \tfrac{3}{4}Q_{3556}\lambda_3\lambda_0 + \tfrac{3}{2}Q_{2566}\lambda_3\lambda_0\\
&- \tfrac{3}{4}Q_{2366}\lambda_4\lambda_0 - \tfrac{3}{4}Q_{2445}\lambda_4\lambda_0 - 6\wp_{56}\lambda_1\lambda_0 - \tfrac{3}{4}Q_{2566}{\lambda_4}^2\lambda_0 + \tfrac{3}{2}Q_{2245}\lambda_0\\
&+ \tfrac{3}{2}\wp_{56}\lambda_4\lambda_2\lambda_0 + \tfrac{9}{2}\wp_{36}\lambda_4\lambda_3\lambda_0
\end{aligned}$$

$$\begin{aligned}
\mathbf{(-40)}\quad \wp_{1112} &= 6\wp_{11}\wp_{12} + 8\lambda_3{\lambda_1}^2 + 4\lambda_3\lambda_2\lambda_0 + 32\lambda_4\lambda_1\lambda_0 + 6\wp_{33}\lambda_3\lambda_0 - 68\wp_{16}\lambda_3\lambda_0\\
&+ 6\wp_{35}\lambda_2\lambda_0 + 42\wp_{55}\lambda_1\lambda_0 - 36\wp_{46}\lambda_1\lambda_0 + 18\wp_{55}\lambda_4{\lambda_1}^2 - 4\wp_{26}\lambda_2\lambda_0\\
&+ 64\wp_{16}{\lambda_4}^3\lambda_1 + 18Q_{1356}\lambda_1\lambda_4 - 4Q_{1444}\lambda_1\lambda_4 - 12Q_{1556}{\lambda_4}^2\lambda_1 - 6Q_{1466}\lambda_3\lambda_1\\
&+ 128\wp_{16}{\lambda_4}^2\lambda_0 - 24Q_{1556}\lambda_0\lambda_4 - 24Q_{1466}\lambda_0\lambda_4 + 9Q_{1266}\lambda_1\lambda_4 - 6Q_{1556}\lambda_3\lambda_1\\
&- 12\wp_{46}\lambda_4{\lambda_1}^2 - 12Q_{1466}{\lambda_4}^2\lambda_1 - 4\wp_{16}\lambda_4\lambda_3\lambda_1 + 16{\lambda_4}^2{\lambda_1}^2 + 30Q_{1356}\lambda_0\\
&- 8Q_{1444}\lambda_0 + 15Q_{1266}\lambda_0 - 3Q_{1244}\lambda_1 - 8\wp_{26}{\lambda_1}^2 + 12\wp_{35}{\lambda_1}^2
\end{aligned}$$

$$\begin{aligned}
\mathbf{(-44)}\quad \wp_{1111} &= 6\wp_{11}^2 + 64{\lambda_4}^2\lambda_1\lambda_0 + 32\lambda_3\lambda_1\lambda_0 + 8{\lambda_2}^2\lambda_0 + 12Q_{1244}\lambda_0 - 2\lambda_2{\lambda_1}^2\\
&+ 16Q_{1444}\lambda_4\lambda_0 - 48Q_{1466}{\lambda_4}^2\lambda_0 - 24Q_{1556}\lambda_3\lambda_0 - 48Q_{1556}{\lambda_4}^2\lambda_0 + 4\wp_{16}{\lambda_1}^2\\
&- 24Q_{1466}\lambda_3\lambda_0 - 16\wp_{16}\lambda_4\lambda_3\lambda_0 + 72\wp_{55}\lambda_4\lambda_1\lambda_0 + 54\wp_{35}\lambda_1\lambda_0 - 16\wp_{16}\lambda_2\lambda_0\\
&+ 72Q_{1356}\lambda_4\lambda_0 + 12\wp_{33}\lambda_2\lambda_0 - 36\wp_{26}\lambda_1\lambda_0 + 256\wp_{16}{\lambda_4}^3\lambda_0 - 60\wp_{46}{\lambda_0}^2\\
&+ 36Q_{1266}\lambda_4\lambda_0 + 45\wp_{55}{\lambda_0}^2 - 3\wp_{33}{\lambda_1}^2 - 48\wp_{46}\lambda_4\lambda_1\lambda_0
\end{aligned}$$

Appendix D

New results for the cyclic trigonal curve of genus four

D.1 Expressions for the B-functions

In Section 5.2.4 we derived relations between the B-functions associated to the cyclic (3,5)-curve. We discussed how we would need to add nineteen B-functions to the basis for $\Gamma\big(J, \mathcal{O}(3\Theta^{[3]})\big)$ in order to express the others as a linear combination of these and the 3-index $\wp$-functions. We have derived equations at each weight for those B-functions which are linearly independent and present the complete set below in decreasing weight order.

Every bilinear relation associated to the cyclic (3,5)-curve may be given as a linear combination of the set obtained by substituting for the B-functions in the equations, (Theorem 5.2.7). Those readers who wish to skip the equations should proceed to page 254.

$$(\mathbf{-6})\quad B^A_{34444} = -\tfrac{1}{2}\wp_{244} - \tfrac{1}{2}\wp_{333}$$

$$(\mathbf{-7})\quad B^A_{33444} = 2\wp_{234} - \lambda_4\wp_{344}$$

$$(\mathbf{-8})\quad B^A_{24444} = -\tfrac{2}{3}\wp_{233} + \tfrac{1}{3}\lambda_4\wp_{334} - \tfrac{1}{2}B^A_{33344}$$

$$(\mathbf{-9})\quad B^A_{33334} = -2\wp_{144} + B^B_{23444}$$

$$(\mathbf{-9})\quad B^A_{23444} = -2\wp_{144} + 2\wp_{224} - \lambda_4\wp_{244} - 2B^B_{23444}$$

$$(\mathbf{-10})\quad B^A_{23344} = -\tfrac{1}{3}\wp_{223} + \tfrac{2}{3}\lambda_4\wp_{234} - \lambda_3\wp_{344}$$

$$(\mathbf{-10})\quad B^C_{23344} = -\wp_{134} + \tfrac{1}{6}\wp_{223} - \tfrac{1}{3}\lambda_4\wp_{234} + \tfrac{1}{2}\lambda_3\wp_{344}$$

$$(\mathbf{-11})\quad B^A_{23334} = -\tfrac{2}{7}\wp_{133} - \tfrac{1}{7}\lambda_4\wp_{233} + \tfrac{2}{7}{\lambda_4}^2\wp_{334} - \tfrac{6}{7}\lambda_3\wp_{334} - \tfrac{3}{7}\lambda_4 B^B_{33344} - \tfrac{2}{7}B^D_{23334} - \tfrac{3}{7}B^A_{22444}$$

$$(\mathbf{-11})\quad B^A_{14444} = -\tfrac{2}{7}\wp_{133} - \tfrac{1}{21}{\lambda_4}^2\wp_{334} + \tfrac{1}{7}\lambda_3\wp_{334} + \tfrac{3}{14}B^D_{23334} + \tfrac{1}{42}\lambda_4\wp_{233} + \tfrac{1}{14}\lambda_4 B^B_{33344} + \tfrac{1}{14}B^A_{22444}$$

$$(\mathbf{-12})\quad B^A_{22344} = \tfrac{2}{3}\lambda_4 B^B_{23444} + \tfrac{7}{3}\wp_{124} - \tfrac{1}{2}\wp_{222} - \tfrac{4}{3}\lambda_4\wp_{144} + \tfrac{2}{3}\lambda_4\wp_{224} - \lambda_3\wp_{244} - \tfrac{1}{6}\lambda_3\wp_{333} + \tfrac{2}{3}\lambda_2\wp_{444} + \tfrac{2}{3}B^B_{13444}$$

$$\mathbf{(-12)} \quad B^{A}_{23333} = -\tfrac{1}{2}\lambda_3\wp_{333} - \tfrac{1}{2}\wp_{222} + 3\wp_{124} - 2B^{B}_{13444}$$

$$\mathbf{(-12)} \quad B^{B}_{22344} = \tfrac{1}{3}\lambda_3\wp_{333} + \tfrac{4}{3}\wp_{124} + \tfrac{2}{3}\lambda_2\wp_{444} - \tfrac{4}{3}\lambda_4\wp_{144} + \tfrac{2}{3}\lambda_4\wp_{224} - \lambda_3\wp_{244} - \tfrac{4}{3}B^{B}_{13444} - \tfrac{4}{3}\lambda_4 B^{B}_{23444}$$

$$\mathbf{(-12)} \quad B^{A}_{13444} = 2\wp_{124} - \lambda_4\wp_{144} - 2B^{B}_{13444}$$

$$\mathbf{(-13)} \quad B^{A}_{22334} = \tfrac{4}{3}\wp_{123} - \tfrac{4}{3}\lambda_4\wp_{134} + 2B^{C}_{13344}$$

$$\mathbf{(-13)} \quad B^{B}_{22334} = -\tfrac{2}{3}\wp_{123} + \tfrac{2}{3}\lambda_4\wp_{134} - \lambda_2\wp_{344} + 2B^{C}_{13344}$$

$$\mathbf{(-13)} \quad B^{A}_{13344} = -2B^{C}_{13344}$$

$$\mathbf{(-14)} \quad B^{A}_{22333} = 12B^{D}_{13334} - \tfrac{24}{7}\lambda_4\lambda_3\wp_{334} + 7\lambda_2\wp_{334} - \tfrac{4}{7}{\lambda_4}^2\wp_{233} - \tfrac{8}{7}\lambda_4\wp_{133} + \tfrac{8}{7}{\lambda_4}^3\wp_{334} + \tfrac{9}{2}B^{A}_{22244} + \tfrac{6}{7}\lambda_4 B^{D}_{23334} - \tfrac{12}{7}\lambda_4 B^{A}_{22444} - 30B^{B}_{12444} + \tfrac{9}{2}\lambda_3 B^{B}_{33344} - \tfrac{12}{7}{\lambda_4}^2 B^{B}_{33344}$$

$$\mathbf{(-14)} \quad B^{A}_{13334} = B^{D}_{13334} - 3B^{B}_{12444}$$

$$\mathbf{(-14)} \quad B^{A}_{12444} = \tfrac{10}{7}\lambda_4\lambda_3\wp_{334} - 3\lambda_2\wp_{334} + \tfrac{5}{21}{\lambda_4}^2\wp_{233} + \tfrac{5}{7}{\lambda_4}^2 B^{B}_{33344} - \tfrac{10}{21}{\lambda_4}^3\wp_{334} - \tfrac{3}{2}B^{A}_{22244} + \tfrac{1}{7}\lambda_4 B^{D}_{23334} + \tfrac{5}{7}\lambda_4 B^{A}_{22444} - 5B^{D}_{13334} + 11B^{B}_{12444} - \tfrac{3}{2}\lambda_3 B^{B}_{33344} + \tfrac{1}{7}\lambda_4\wp_{133}$$

$$\mathbf{(-15)} \quad B^{A}_{22234} = 4\wp_{114} - \tfrac{2}{3}\wp_{122} + 2\lambda_3\wp_{144} + \lambda_3 B^{B}_{23444} - \tfrac{2}{3}\lambda_2\wp_{244} - \lambda_2\wp_{333} - \tfrac{2}{3}\lambda_4\wp_{124} + 4\lambda_4 B^{B}_{13444} - 6B^{C}_{12344}$$

$$\mathbf{(-15)} \quad B^{C}_{22234} = -5\wp_{114} + \tfrac{1}{3}\wp_{122} - 7\lambda_3\wp_{144} - 2\lambda_3 B^{B}_{23444} + \tfrac{1}{3}\lambda_2\wp_{244} + 3\lambda_1\wp_{444} + 2\lambda_2\wp_{333} + \tfrac{10}{3}\lambda_4\wp_{124} - 8\lambda_4 B^{B}_{13444} + 12B^{C}_{12344}$$

$$\mathbf{(-15)} \quad B^{A}_{13333} = \lambda_4\wp_{124} - \tfrac{3}{2}\lambda_3\wp_{144} + \tfrac{1}{2}\lambda_1\wp_{444} + \tfrac{1}{2}\wp_{114} - \tfrac{1}{2}\wp_{122} + 3B^{C}_{12344} - 2\lambda_4 B^{B}_{13444}$$

$$\mathbf{(-15)} \quad B^{A}_{12344} = -\tfrac{3}{2}\lambda_3\wp_{144} + \tfrac{1}{2}\lambda_1\wp_{444} + \lambda_4\wp_{124} + \tfrac{1}{2}\wp_{114} - \tfrac{1}{2}\wp_{122} + B^{C}_{12344}$$

$$\mathbf{(-15)} \quad B^{B}_{12344} = 2\wp_{114} - 2B^{C}_{12344}$$

$$\mathbf{(-16)} \quad B^{A}_{22233} = -2\lambda_1\wp_{344} + \tfrac{4}{3}\lambda_2\wp_{234} - 4\lambda_3\wp_{134} + \tfrac{4}{3}\lambda_4\wp_{123} - \tfrac{2}{3}\wp_{113} + 4B^{D}_{12334}$$

$$\mathbf{(-16)} \quad B^{A}_{12334} = 2\lambda_1\wp_{344} + 2\wp_{113} - 2B^{D}_{12334}$$

$$\mathbf{(-16)} \quad B^{B}_{12334} = -\tfrac{1}{2}\wp_{113} - \tfrac{3}{2}\lambda_1\wp_{344} + B^{D}_{12334}$$

$$\mathbf{(-17)} \quad B^{A}_{22224} = -\tfrac{1}{21}\lambda_4\lambda_3\wp_{233} - \tfrac{10}{7}{\lambda_4}^2\lambda_3\wp_{334} + \tfrac{10}{3}\lambda_4\lambda_2\wp_{334} - \tfrac{1}{7}\lambda_3 B^{D}_{22444} - 2\lambda_2 B^{B}_{33344} + \tfrac{8}{3}B^{A}_{11444} + \tfrac{1}{3}\lambda_2\wp_{233} + \tfrac{10}{3}B^{B}_{12333} - 2B^{C}_{12244} - \tfrac{16}{63}{\lambda_4}^3\wp_{233} + \tfrac{32}{63}{\lambda_4}^4\wp_{334} + 4\lambda_4 B^{D}_{13334} - \tfrac{20}{3}\lambda_4 B^{B}_{12444} - \tfrac{16}{21}{\lambda_4}^3 B^{B}_{33344} - \tfrac{10}{7}\lambda_3 B^{D}_{23334} + \tfrac{13}{7}\lambda_4\lambda_3 B^{B}_{33344} - \tfrac{16}{21}{\lambda_4}^2 B^{A}_{22444} + \tfrac{8}{21}{\lambda_4}^2 B^{D}_{23334} + \tfrac{8}{21}{\lambda_4}^2\wp_{133} + 2\lambda_4 B^{B}_{22244} - \tfrac{58}{21}\lambda_3\wp_{133} - \tfrac{2}{7}{\lambda_3}^2\wp_{334}$$

$$\mathbf{(-17)} \quad B^{A}_{12333} = 2B^{A}_{11444} - \tfrac{1}{7}\lambda_4\lambda_3\wp_{233} - \tfrac{6}{7}{\lambda_4}^2\lambda_3\wp_{334} + 3\lambda_4\lambda_2\wp_{334} - \tfrac{3}{7}\lambda_3 B^{D}_{22444} + 2\lambda_1\wp_{334} + 2B^{B}_{12333} - 6B^{C}_{12244} - \tfrac{4}{21}{\lambda_4}^3\wp_{233} + \tfrac{8}{21}{\lambda_4}^4\wp_{334} + 6\lambda_4 B^{D}_{13334} - 8\lambda_4 B^{B}_{12444} - \tfrac{4}{7}{\lambda_4}^3 B^{B}_{33344} - \tfrac{9}{7}\lambda_3 B^{D}_{23334} - \tfrac{4}{7}{\lambda_4}^2 B^{A}_{22444} + \tfrac{2}{7}{\lambda_4}^2 B^{D}_{23334} + \tfrac{2}{7}{\lambda_4}^2\wp_{133} + \tfrac{3}{2}\lambda_4 B^{B}_{22244} - \tfrac{16}{7}\lambda_3\wp_{133} - \tfrac{6}{7}{\lambda_3}^2\wp_{334} + \tfrac{15}{14}\lambda_4\lambda_3 B^{B}_{33344}$$

$$(\mathbf{-17})\quad B^A_{12244} = \tfrac{2}{21}\lambda_4\lambda_3\wp_{233} + \tfrac{4}{7}{\lambda_4}^2\lambda_3\wp_{334} - 2\lambda_4\lambda_2\wp_{334} + \tfrac{2}{7}\lambda_3 B^D_{22444}$$
$$- 2\lambda_1\wp_{334} - \tfrac{5}{3}B^B_{12333} + 2B^C_{12244} + \tfrac{8}{63}{\lambda_4}^3\wp_{233} - \tfrac{16}{63}{\lambda_4}^4\wp_{334} - 4\lambda_4 B^D_{13334}$$
$$+ \tfrac{22}{3}\lambda_4 B^B_{12444} + \tfrac{8}{21}{\lambda_4}^3 B^B_{33344} + \tfrac{6}{7}\lambda_3 B^D_{23334} + \tfrac{8}{21}{\lambda_4}^2 B^A_{22444} - \tfrac{4}{21}{\lambda_4}^2 B^D_{23334}$$
$$- \tfrac{4}{21}{\lambda_4}^2\wp_{133} - \lambda_4 B^B_{22244} + \tfrac{32}{21}\lambda_3\wp_{133} + \tfrac{4}{7}{\lambda_3}^2\wp_{334} - \tfrac{5}{7}\lambda_4\lambda_3 B^B_{33344} - \tfrac{4}{3}B^A_{11444}$$

$$(\mathbf{-18})\quad B^A_{22223} = -4{\lambda_4}^2\wp_{124} + \tfrac{1}{2}\lambda_3\wp_{222} - 2\lambda_2\wp_{224} + \tfrac{1}{2}{\lambda_3}^2\wp_{333} + 3\lambda_3\wp_{124}$$
$$- \wp_{112} - 2\lambda_4\lambda_1\wp_{444} + 8{\lambda_4}^2 B^B_{13444} + 6\lambda_4\lambda_3\wp_{144} - 2\lambda_4\lambda_2\wp_{333} + 6\lambda_4\wp_{114}$$
$$+ 4\lambda_2 B^B_{23444} - 2\lambda_2\wp_{144} - \lambda_1\wp_{333} - 3B^B_{11344} - 12\lambda_4 B^C_{12344} - 4\lambda_3 B^B_{13444}$$

$$(\mathbf{-18})\quad B^A_{12234} = \lambda_0\wp_{444} - \tfrac{1}{2}\wp_{112} + \lambda_4\wp_{114} + \lambda_3 B^B_{13444} - \tfrac{2}{3}\lambda_2\wp_{144} - \tfrac{1}{2}\lambda_1\wp_{333} + \tfrac{1}{2}B^B_{11344}$$

$$(\mathbf{-18})\quad B^C_{12234} = \lambda_0\wp_{444} + \lambda_1\wp_{244} + \lambda_4\wp_{114} - 2\lambda_3 B^B_{13444} - \tfrac{5}{3}\lambda_2\wp_{144} + \lambda_1\wp_{333} - B^B_{11344}$$

$$(\mathbf{-18})\quad B^D_{12234} = -2\lambda_0\wp_{444} - \tfrac{1}{2}\lambda_1\wp_{244} + \lambda_3 B^B_{13444} + \tfrac{4}{3}\lambda_2\wp_{144} - \tfrac{1}{2}\lambda_1\wp_{333} + \tfrac{1}{2}B^B_{11344}$$

$$(\mathbf{-18})\quad B^A_{11344} = -\wp_{11}\wp_{344} - \wp_{13}\wp_{144} + \wp_{34}\wp_{114} + \wp_{44}\wp_{113} + \lambda_4\wp_{114}$$
$$+ \tfrac{1}{2}\lambda_1\wp_{244} - \tfrac{1}{2}\wp_{112} - \tfrac{1}{2}B^B_{11344} - \lambda_2\wp_{144}$$

$$(\mathbf{-19})\quad B^A_{12233} = -2\lambda_2\wp_{134} + \lambda_4\wp_{113} + 3\lambda_0\wp_{344} + 2\lambda_1\wp_{234} + B^C_{12233}$$

$$(\mathbf{-19})\quad B^A_{11334} = \lambda_2\wp_{134} + \tfrac{1}{2}\lambda_4\wp_{113} - \tfrac{5}{2}\lambda_0\wp_{344} - \lambda_1\wp_{234} - \tfrac{3}{2}B^C_{12233}$$

$$(\mathbf{-19})\quad B^A_{11334} = -\lambda_0\wp_{344} + \wp_{14}\wp_{133} - \wp_{33}\wp_{114} - \wp_{34}\wp_{113} + \wp_{11}\wp_{334}$$

$$(\mathbf{-20})\quad B^A_{11333} = \tfrac{1}{7}{\lambda_4}^2\lambda_2\wp_{334} - \tfrac{3}{7}\lambda_3\lambda_2\wp_{334} + \tfrac{3}{2}\lambda_4\lambda_1\wp_{334} - \tfrac{1}{14}\lambda_4\lambda_2\wp_{233}$$
$$+ \tfrac{9}{4}\lambda_1 B^B_{33344} - \tfrac{3}{14}\lambda_2 B^B_{22444} - \tfrac{9}{2}\lambda_3 B^B_{12444} - \tfrac{9}{14}\lambda_2 B^D_{23334} - \tfrac{3}{2}B^D_{12224} + \tfrac{9}{4}B^B_{11244}$$
$$+ 3\lambda_3 B^D_{13334} - \tfrac{3}{14}\lambda_4\lambda_2 B^B_{33344} - \tfrac{1}{7}\lambda_2\wp_{133} - \lambda_1\wp_{233} + 3\lambda_0\wp_{334}$$

$$(\mathbf{-20})\quad B^A_{11244} = \tfrac{2}{7}{\lambda_4}^3\lambda_3\wp_{334} + \tfrac{1}{21}{\lambda_4}^2\lambda_3\wp_{233} - \tfrac{5}{6}\lambda_4\lambda_1\wp_{334} + \tfrac{1}{42}\lambda_4\lambda_2\wp_{233}$$
$$+ \tfrac{2}{7}\lambda_4{\lambda_3}^2\wp_{334} + \tfrac{1}{7}\lambda_3\lambda_2\wp_{334} + \tfrac{1}{4}B^B_{11244} - \tfrac{1}{2}B^D_{12224} + \tfrac{5}{7}\lambda_2\wp_{133} - \tfrac{22}{21}{\lambda_4}^2\lambda_2\wp_{334}$$
$$- \tfrac{8}{63}{\lambda_4}^5\wp_{334} + \tfrac{4}{63}{\lambda_4}^4\wp_{233} - \tfrac{2}{21}{\lambda_4}^3\wp_{133} - \tfrac{5}{14}{\lambda_4}^2\lambda_3 B^B_{33344} + \tfrac{3}{7}\lambda_4\lambda_3 B^D_{23334}$$
$$+ \tfrac{1}{7}\lambda_4\lambda_3 B^B_{22444} + \tfrac{1}{14}\lambda_4\lambda_2 B^B_{33344} + \tfrac{5}{4}\lambda_1 B^B_{33344} + \tfrac{1}{14}\lambda_2 B^B_{22444} - \tfrac{3}{2}\lambda_3 B^B_{12444} - \tfrac{1}{3}\lambda_1\wp_{233}$$
$$+ \tfrac{3}{14}\lambda_2 B^D_{23334} + \lambda_3 B^D_{13334} - \tfrac{2}{3}\lambda_4 B^A_{11444} - 2{\lambda_4}^2 B^D_{13334} + \tfrac{16}{21}\lambda_4\lambda_3\wp_{133} + \tfrac{4}{21}{\lambda_4}^3 B^B_{22444}$$
$$- \tfrac{2}{21}{\lambda_4}^3 B^D_{23334} - 2{\lambda_4}^2 B^B_{12444} - \tfrac{4}{3}\lambda_4 B^B_{12333} - \tfrac{4}{21}{\lambda_4}^4 B^B_{33344} + 2\lambda_4 B^C_{12244} - \tfrac{1}{2}{\lambda_4}^2 B^A_{22244}$$

$$(\mathbf{-20})\quad B^A_{12224} = -\tfrac{2}{21}{\lambda_4}^2\lambda_2\wp_{334} + \tfrac{2}{7}\lambda_3\lambda_2\wp_{334} - \tfrac{4}{3}\lambda_4\lambda_1\wp_{334} + \tfrac{1}{21}\lambda_4\lambda_2\wp_{233} + \tfrac{2}{3}\lambda_1\wp_{233}$$
$$- \tfrac{5}{2}\lambda_1 B^B_{33344} + \tfrac{1}{7}\lambda_2 B^B_{22444} + 2\wp_{14}\wp_{124} - 2\wp_{12}\wp_{144} + \wp_{44}\wp_{112} - \wp_{24}\wp_{114} - 5\lambda_0\wp_{334}$$
$$+ 6\lambda_3 B^B_{12444} + \tfrac{3}{7}\lambda_2 B^D_{23334} + B^D_{12224} - 3\lambda_3 B^D_{13334} + \tfrac{1}{7}\lambda_4\lambda_2 B^B_{33344} + \tfrac{3}{7}\lambda_2\wp_{133}$$

$$(\mathbf{-21})\quad B^A_{12223} = -\lambda_0\wp_{333} - \lambda_1\wp_{144} + \tfrac{1}{2}\lambda_3\wp_{122} - \lambda_1\wp_{224} + \tfrac{5}{2}\lambda_3\wp_{114} - \lambda_4\lambda_1\wp_{333}$$
$$- \tfrac{1}{2}\lambda_3\lambda_1\wp_{444} - \lambda_4\lambda_3\wp_{124} - \wp_{111} + \tfrac{3}{2}{\lambda_3}^2\wp_{144} - 3\lambda_3 B^C_{12344} + 2\lambda_4\lambda_3 B^B_{13444} + 2\lambda_1 B^B_{23444}$$

$$(\mathbf{-21})\quad B^D_{12223} = 6\lambda_3 B^C_{12344} + 2\lambda_4\lambda_1\wp_{333} + \lambda_3\lambda_1\wp_{444} + 2\lambda_4\lambda_3\wp_{124} + \lambda_4\wp_{112}$$
$$+ 3\lambda_0\wp_{244} - \lambda_3\wp_{122} - 3{\lambda_3}^2\wp_{144} - 2\lambda_2\wp_{124} - \lambda_1\wp_{144} - 4\lambda_1 B^B_{23444}$$
$$- 4\lambda_4\lambda_3 B^B_{13444} + 2\lambda_0\wp_{333} + 3\lambda_1\wp_{224} - 2\lambda_3\wp_{114} - \wp_{111}$$

$$(\mathbf{-21})\quad B^A_{11234} = \lambda_0\wp_{244} - \tfrac{1}{2}\wp_{111} - \lambda_2\wp_{124} - \tfrac{7}{6}\lambda_1\wp_{144} - \tfrac{1}{3}\lambda_1 B^B_{23444} - \tfrac{2}{3}\lambda_0\wp_{333} + \tfrac{1}{2}\lambda_1\wp_{224} + \tfrac{3}{2}\lambda_3\wp_{114} + \tfrac{2}{3}\lambda_2 B^B_{13444}$$

$$(\mathbf{-21})\quad B^B_{11234} = -\tfrac{2}{3}\lambda_0\wp_{333} - \tfrac{2}{3}\lambda_1\wp_{144} + \lambda_0\wp_{244} + \tfrac{2}{3}\lambda_2 B^B_{13444} - \tfrac{1}{3}\lambda_1 B^B_{23444}$$

$$(\mathbf{-21})\quad B^C_{11234} = \tfrac{4}{3}\lambda_0\wp_{333} - \tfrac{2}{3}\lambda_1\wp_{144} + \lambda_0\wp_{244} + \tfrac{2}{3}\lambda_1 B^B_{23444} - \tfrac{4}{3}\lambda_2 B^B_{13444}$$

$$(\mathbf{-22})\quad B^A_{11233} = -\lambda_2\wp_{123} + \tfrac{3}{2}\lambda_3\wp_{113} + \tfrac{1}{2}\lambda_1\wp_{223} - \tfrac{1}{3}\lambda_1\wp_{134} + \lambda_0\wp_{234}$$

$$(\mathbf{-22})\quad B^B_{11233} = -\tfrac{4}{3}\lambda_1\wp_{134} + 4\lambda_0\wp_{234}$$

$$(\mathbf{-23})\quad B^A_{12222} = 2\lambda_3\lambda_1\wp_{334} + \tfrac{2}{7}{\lambda_4}^2\lambda_2\wp_{233} - \tfrac{4}{7}{\lambda_4}^3\lambda_2\wp_{334} + \tfrac{15}{14}\lambda_4{\lambda_3}^2 B^B_{33344} - \tfrac{4}{7}{\lambda_4}^3\lambda_3 B^B_{33344} + \tfrac{2}{3}\lambda_4\lambda_1\wp_{233} + \tfrac{8}{21}{\lambda_4}^4\lambda_3\wp_{334} - 4\lambda_4\lambda_1 B^B_{33344} + 2\lambda_4\lambda_3 B^D_{13334} - \tfrac{4}{21}{\lambda_4}^3\lambda_3\wp_{233} + \tfrac{3}{2}\lambda_4\lambda_3 B^A_{22244} - \tfrac{4}{3}{\lambda_4}^2\lambda_1\wp_{334} + \tfrac{6}{7}\lambda_4\lambda_2 B^A_{22444} + \tfrac{4}{7}\lambda_4\lambda_2\wp_{133} - \lambda_1 B^D_{22444} - \tfrac{1}{7}\lambda_4{\lambda_3}^2\wp_{233} + \tfrac{6}{7}{\lambda_4}^2\lambda_2 B^B_{33344} - \tfrac{6}{7}{\lambda_4}^2{\lambda_3}^2\wp_{334} + \tfrac{2}{7}{\lambda_4}^2\lambda_3\wp_{133} - 4\lambda_4\lambda_0\wp_{334} + \tfrac{33}{7}\lambda_4\lambda_3\lambda_2\wp_{334} - 3\lambda_0\wp_{233} - 3{\lambda_2}^2\wp_{334} - \tfrac{6}{7}{\lambda_3}^3\wp_{334} - \tfrac{4}{7}{\lambda_4}^2\lambda_3 B^A_{22444} - \tfrac{16}{7}{\lambda_3}^2\wp_{133} - \lambda_1 B^D_{23334} - 2\lambda_4\lambda_3 B^B_{12444} - \tfrac{3}{2}\lambda_3\lambda_2 B^B_{33344} + \tfrac{2}{7}{\lambda_4}^2\lambda_3 B^D_{23334} + \tfrac{4}{7}\lambda_4\lambda_2 B^D_{23334} - \tfrac{9}{7}{\lambda_3}^2 B^D_{23334} - 6\lambda_3 B^C_{12244} - 6\lambda_2 B^D_{13334} + 2\lambda_3 B^A_{11444} + 2\lambda_4 B^D_{12224} - \tfrac{3}{7}{\lambda_3}^2 B^A_{22444} + 3\lambda_3 B^B_{12333} + 16\lambda_2 B^B_{12444} - 3\lambda_4 B^B_{11244} - 6\lambda_0 B^B_{33344} - \tfrac{3}{2}\lambda_2 B^A_{22244}$$

$$(\mathbf{-23})\quad B^A_{11224} = \tfrac{4}{3}\lambda_2 B^D_{13334} + \tfrac{4}{3}\lambda_4\lambda_0\wp_{334} + \tfrac{2}{63}\lambda_4\lambda_1\wp_{233} - \tfrac{4}{63}{\lambda_4}^2\lambda_1\wp_{334} + \tfrac{4}{21}\lambda_3\lambda_1\wp_{334} + \tfrac{2}{21}\lambda_4\lambda_1 B^B_{33344} - \tfrac{8}{3}\lambda_2 B^B_{12444} - \tfrac{8}{21}\lambda_1 B^D_{23334} + \wp_{11}\wp_{224} + 2\wp_{12}\wp_{124} + \tfrac{2}{21}\lambda_1 B^D_{22444} - \wp_{14}\wp_{122} - 2\wp_{24}\wp_{112} - \tfrac{2}{3}\lambda_0\wp_{233} + 2\lambda_0 B^B_{33344} - \tfrac{8}{21}\lambda_1\wp_{133}$$

$$(\mathbf{-23})\quad B^B_{11224} = -\tfrac{2}{3}\lambda_2 B^D_{13334} - \tfrac{2}{3}\lambda_4\lambda_0\wp_{334} - \tfrac{1}{63}\lambda_4\lambda_1\wp_{233} + \tfrac{2}{63}{\lambda_4}^2\lambda_1\wp_{334} - \tfrac{1}{21}\lambda_1 B^D_{22444} - \tfrac{1}{21}\lambda_4\lambda_1 B^B_{33344} + \tfrac{4}{3}\lambda_2 B^B_{12444} + \tfrac{4}{21}\lambda_1 B^D_{23334} + \tfrac{1}{3}\lambda_0\wp_{233} - \lambda_0 B^B_{33344} + \tfrac{4}{21}\lambda_1\wp_{133} - \tfrac{2}{21}\lambda_3\lambda_1\wp_{334}$$

$$(\mathbf{-24})\quad B^A_{11223} = \tfrac{3}{2}\lambda_3\wp_{112} - \lambda_4\wp_{111} - \tfrac{1}{6}\lambda_3\lambda_1\wp_{333} - \tfrac{1}{3}\lambda_2\lambda_1\wp_{444} - \tfrac{2}{3}\lambda_4\lambda_2\wp_{124} - \tfrac{4}{3}\lambda_4\lambda_0\wp_{333} + \lambda_0\wp_{224} + \lambda_3\lambda_2\wp_{144} - \tfrac{2}{3}\lambda_2\wp_{122} + 3\lambda_2\wp_{114} + 2\lambda_0 B^B_{23444} - 3\lambda_0\wp_{144} + \tfrac{1}{3}\lambda_1\wp_{222} + \tfrac{2}{3}\lambda_1 B^B_{13444} - \tfrac{8}{3}\lambda_1\wp_{124} - \tfrac{4}{3}\lambda_4\lambda_2 B^B_{13444} - 2\lambda_2 B^C_{12344}$$

$$(\mathbf{-24})\quad B^B_{11223} = -\tfrac{1}{6}\lambda_3\lambda_1\wp_{333} - \tfrac{1}{3}\lambda_2\lambda_1\wp_{444} - \tfrac{2}{3}\lambda_4\lambda_2\wp_{124} - \tfrac{4}{3}\lambda_4\lambda_0\wp_{333} + \lambda_3\lambda_2\wp_{144} + \tfrac{1}{3}\lambda_2\wp_{122} + \lambda_2\wp_{114} + 2\lambda_0 B^B_{23444} - 2\lambda_2 B^C_{12344} - \tfrac{1}{6}\lambda_1\wp_{222} + \tfrac{1}{3}\lambda_1\wp_{124} - \tfrac{4}{3}\lambda_4\lambda_2 B^B_{13444} + \tfrac{2}{3}\lambda_1 B^B_{13444} - 2\lambda_0\wp_{224}$$

$$(\mathbf{-24})\quad B^A_{11134} = 2\lambda_0\wp_{224} - \tfrac{4}{3}\lambda_1\wp_{124} + \tfrac{2}{3}\lambda_2\wp_{114} - \lambda_0\wp_{144} - 2\lambda_0 B^B_{23444} + \lambda_1 B^B_{13444}$$

$$(\mathbf{-24})\quad B^C_{11134} = \tfrac{2}{3}\lambda_1\wp_{124} - \lambda_0\wp_{224} - \tfrac{1}{3}\lambda_2\wp_{114} - \lambda_0\wp_{144} + 4\lambda_0 B^B_{23444} - 2\lambda_1 B^B_{13444}$$

$$(\mathbf{-24})\quad B^D_{11134} = \tfrac{2}{3}\lambda_1\wp_{124} - \lambda_0\wp_{224} - \tfrac{1}{3}\lambda_2\wp_{114} + 2\lambda_0\wp_{144} - 2\lambda_0 B^B_{23444} + \lambda_1 B^B_{13444}$$

$$(\mathbf{-25})\quad B^A_{11133} = -\tfrac{4}{3}\lambda_1\wp_{123} + 2\lambda_0\wp_{223} + \tfrac{2}{3}\lambda_2\wp_{113}$$

$$(-26)\quad B^A_{11222} = -\tfrac{8}{21}{\lambda_4}^3\lambda_1\wp_{334} - \tfrac{29}{21}\lambda_3\lambda_2\wp_{133} - \tfrac{45}{7}\lambda_3\lambda_0\wp_{334} + 2\lambda_4{\lambda_2}^2\wp_{334} + \tfrac{4}{3}\lambda_2 B^A_{11444}$$
$$+ \tfrac{4}{21}{\lambda_4}^2\lambda_2\wp_{133} + \tfrac{4}{21}{\lambda_4}^2\lambda_1\wp_{233} + \tfrac{8}{7}{\lambda_4}^2\lambda_0\wp_{334} + \tfrac{19}{7}\lambda_4\lambda_1\wp_{133} - 3{\lambda_3}^2 B^D_{13334} + \tfrac{6}{7}\lambda_0 B^D_{23334}$$
$$- \tfrac{1}{7}{\lambda_3}^2\lambda_2\wp_{334} + 4\lambda_2\lambda_4 B^D_{13334} - \tfrac{8}{63}{\lambda_4}^3\lambda_2\wp_{233} - \tfrac{9}{4}\lambda_3 B^B_{11244} - \tfrac{32}{7}\lambda_4\lambda_0\wp_{233} - \tfrac{3}{2}\lambda_1 B^A_{22244}$$
$$+ \tfrac{16}{63}{\lambda_4}^4\lambda_2\wp_{334} - \lambda_2\lambda_1\wp_{334} + \lambda_3\lambda_1\wp_{233} - \tfrac{5}{7}{\lambda_4}^2\lambda_3\lambda_2\wp_{334} - \tfrac{1}{42}\lambda_4\lambda_3\lambda_2\wp_{233} + \tfrac{3}{2}\lambda_3 B^D_{12224}$$
$$- \tfrac{5}{14}\lambda_4\lambda_3\lambda_1\wp_{334} + \lambda_2\lambda_4 B^A_{22244} + \tfrac{13}{14}\lambda_2\lambda_3\lambda_4 B^B_{33344} - 4\lambda_2 B^C_{12244} + \tfrac{4}{21}\lambda_2{\lambda_4}^2 B^D_{23334}$$
$$+ \tfrac{9}{2}{\lambda_3}^2 B^B_{12444} - \tfrac{12}{7}\lambda_0 B^A_{22444} + \tfrac{27}{7}\lambda_0\wp_{133} + 18\lambda_1 B^B_{12444} + \tfrac{2}{3}\lambda_2 B^B_{12333} - \tfrac{15}{4}\lambda_1\lambda_3 B^B_{33344}$$
$$- \tfrac{8}{21}\lambda_2{\lambda_4}^3 B^B_{33344} - \tfrac{12}{7}\lambda_0\lambda_4 B^B_{33344} - \tfrac{8}{21}\lambda_2{\lambda_4}^2 B^A_{22444} - \tfrac{2}{7}\lambda_4\lambda_1 B^D_{23334} - 6\lambda_1 B^D_{13334}$$
$$- \tfrac{1}{14}\lambda_3\lambda_2 B^A_{22444} - \tfrac{3}{14}\lambda_3\lambda_2 B^D_{23334} + \tfrac{4}{7}\lambda_4\lambda_1 B^A_{22444} - \tfrac{16}{3}\lambda_2\lambda_4 B^B_{12444} + \tfrac{4}{7}\lambda_1{\lambda_4}^2 B^B_{33344}$$

$$(-26)\quad B^A_{11124} = \tfrac{16}{63}{\lambda_4}^3\lambda_1\wp_{334} + \tfrac{32}{63}\lambda_3\lambda_2\wp_{133} - \tfrac{4}{7}\lambda_3\lambda_0\wp_{334} + \tfrac{5}{21}\lambda_4\lambda_0\wp_{233} - \tfrac{2}{3}\lambda_4{\lambda_2}^2\wp_{334}$$
$$- \tfrac{4}{63}{\lambda_4}^2\lambda_2\wp_{133} - \tfrac{8}{63}{\lambda_4}^2\lambda_1\wp_{233} - \tfrac{10}{21}{\lambda_4}^2\lambda_0\wp_{334} + \tfrac{4}{21}\lambda_4\lambda_1\wp_{133} + \tfrac{4}{21}{\lambda_3}^2\lambda_2\wp_{334}$$
$$- \tfrac{16}{189}{\lambda_4}^4\lambda_2\wp_{334} + \tfrac{4}{3}\lambda_2 B^C_{12244} - \tfrac{8}{9}\lambda_2 B^B_{12333} + \lambda_1 B^A_{22244} + \tfrac{1}{7}\lambda_0 B^D_{23334} - \tfrac{4}{9}\lambda_2 B^A_{11444}$$
$$+ \tfrac{4}{3}\lambda_2\lambda_1\wp_{334} + \tfrac{5}{7}\lambda_0 B^A_{22444} - \tfrac{20}{3}\lambda_1 B^B_{12444} + \lambda_1\lambda_3 B^B_{33344} + \tfrac{31}{21}\lambda_0\wp_{133} - \tfrac{4}{63}\lambda_2{\lambda_4}^2 B^D_{23334}$$
$$+ \tfrac{2}{7}\lambda_3\lambda_2 B^D_{23334} - \tfrac{8}{21}\lambda_4\lambda_1 B^A_{22444} + \tfrac{16}{9}\lambda_2\lambda_4 B^B_{12444} - \tfrac{8}{21}\lambda_1{\lambda_4}^2 B^B_{33344} + \tfrac{4}{21}{\lambda_4}^2\lambda_3\lambda_2\wp_{334}$$
$$+ \tfrac{5}{7}\lambda_0\lambda_4 B^B_{33344} + \tfrac{8}{63}\lambda_2{\lambda_4}^2 B^A_{22444} + \tfrac{4}{21}\lambda_4\lambda_1 B^D_{23334} - \tfrac{4}{3}\lambda_2\lambda_4 B^B_{13334} + \tfrac{8}{63}\lambda_2{\lambda_4}^3 B^B_{33344}$$
$$+ \tfrac{2}{63}\lambda_4\lambda_3\lambda_2\wp_{233} - \tfrac{16}{21}\lambda_4\lambda_3\lambda_1\wp_{334} + \tfrac{8}{189}{\lambda_4}^3\lambda_2\wp_{233} - \tfrac{5}{21}\lambda_2\lambda_3\lambda_4 B^B_{33344} - \tfrac{1}{3}\lambda_2\lambda_4 B^A_{22244}$$
$$+ 3\lambda_1 B^D_{13334} + \tfrac{2}{21}\lambda_3\lambda_2 B^A_{22444}$$

$$(-26)\quad B^C_{11124} = -\tfrac{8}{63}{\lambda_4}^3\lambda_1\wp_{334} - \tfrac{16}{63}\lambda_3\lambda_2\wp_{133} - \lambda_3\lambda_0\wp_{334} + \tfrac{1}{3}\lambda_4{\lambda_2}^2\wp_{334}$$
$$+ \tfrac{2}{63}{\lambda_4}^2\lambda_2\wp_{133} + \tfrac{4}{63}{\lambda_4}^2\lambda_1\wp_{233} + \tfrac{2}{3}{\lambda_4}^2\lambda_0\wp_{334} - \tfrac{2}{21}\lambda_4\lambda_1\wp_{133} - \tfrac{2}{21}{\lambda_3}^2\lambda_2\wp_{334}$$
$$+ \tfrac{2}{3}\lambda_2\lambda_4 B^D_{13334} - \tfrac{4}{189}{\lambda_4}^3\lambda_2\wp_{233} - \tfrac{1}{3}\lambda_4\lambda_0\wp_{233} + \tfrac{8}{189}{\lambda_4}^4\lambda_2\wp_{334} - \tfrac{2}{3}\lambda_2\lambda_1\wp_{334}$$
$$- \tfrac{2}{21}{\lambda_4}^2\lambda_3\lambda_2\wp_{334} - \tfrac{1}{63}\lambda_4\lambda_3\lambda_2\wp_{233} + \tfrac{8}{21}\lambda_4\lambda_3\lambda_1\wp_{334} + \tfrac{1}{3}\lambda_0\wp_{133} + \tfrac{2}{63}\lambda_2{\lambda_4}^2 B^D_{23334}$$
$$+ \tfrac{5}{42}\lambda_2\lambda_3\lambda_4 B^B_{33344} - \tfrac{2}{3}\lambda_2 B^C_{12244} + \tfrac{4}{9}\lambda_2 B^B_{12333} - \tfrac{1}{2}\lambda_1 B^A_{22244} + \lambda_0 B^D_{23334}$$
$$- 3\lambda_1 B^D_{13334} - \lambda_0 B^A_{22444} + \tfrac{19}{3}\lambda_1 B^B_{12444} - \tfrac{1}{2}\lambda_1\lambda_3 B^B_{33344} + \tfrac{1}{6}\lambda_2\lambda_4 B^A_{22244}$$
$$- \tfrac{1}{21}\lambda_3\lambda_2 B^A_{22444} - \tfrac{1}{7}\lambda_3\lambda_2 B^D_{23334} + \tfrac{4}{21}\lambda_4\lambda_1 B^A_{22444} - \tfrac{8}{9}\lambda_2\lambda_4 B^B_{12444} + \tfrac{4}{21}\lambda_1{\lambda_4}^2 B^B_{33344}$$
$$- \tfrac{4}{63}\lambda_2{\lambda_4}^3 B^B_{33344} - \lambda_0\lambda_4 B^B_{33344} - \tfrac{4}{63}\lambda_2{\lambda_4}^2 B^A_{22444} - \tfrac{2}{21}\lambda_4\lambda_1 B^D_{23334} + \tfrac{2}{9}\lambda_2 B^A_{11444}$$

$$(-27)\quad B^A_{11123} = \tfrac{2}{3}\lambda_2\wp_{112} - 4\lambda_0\wp_{124} + \tfrac{17}{6}\lambda_1\wp_{114} - \tfrac{5}{6}\lambda_1\wp_{122} + \lambda_0\wp_{222} - \tfrac{1}{2}{\lambda_1}^2\wp_{444}$$
$$- \lambda_4\lambda_1\wp_{124} + \tfrac{3}{2}\lambda_3\lambda_1\wp_{144} - \lambda_3\lambda_0\wp_{333} + (2\lambda_0 + 2\lambda_4\lambda_1)\, B^B_{13444} - 3\lambda_1 B^C_{12344}$$

$$(-27)\quad B^C_{11123} = -\tfrac{1}{3}\lambda_2\wp_{112} + 2\lambda_0\wp_{124} + \tfrac{5}{6}\lambda_1\wp_{114} + \tfrac{7}{6}\lambda_1\wp_{122} - 2\lambda_0\wp_{222} - 3\lambda_1 B^C_{12344}$$
$$- \tfrac{1}{2}{\lambda_1}^2\wp_{444} - \lambda_4\lambda_1\wp_{124} + \tfrac{3}{2}\lambda_3\lambda_1\wp_{144} - \lambda_3\lambda_0\wp_{333} + (2\lambda_0 + 2\lambda_4\lambda_1)\, B^B_{13444}$$

$$\mathbf{(-29)} \quad B^A_{11122} = -\tfrac{8}{3}\lambda_2\lambda_0\wp_{334} - 2\lambda_3\lambda_0\wp_{233} - \tfrac{16}{189}\lambda_4{}^3\lambda_1\wp_{233} + \tfrac{2}{63}\lambda_4\lambda_2{}^2\wp_{233}$$
$$+ \left(\tfrac{2}{7}\lambda_2{}^2 - \tfrac{4}{7}\lambda_3\lambda_1 + \tfrac{8}{63}\lambda_4{}^2\lambda_1\right) B^D_{23334} + \left(2\lambda_3\lambda_2 - \tfrac{32}{9}\lambda_4\lambda_1 + 4\lambda_0\right) B^B_{12444}$$
$$+ \left(-\tfrac{16}{63}\lambda_4{}^3\lambda_1 + \tfrac{2}{21}\lambda_4\lambda_2{}^2 - \lambda_2\lambda_1 + \tfrac{10}{21}\lambda_4\lambda_3\lambda_1\right) B^B_{33344} - \tfrac{8}{21}\lambda_4{}^2\lambda_3\lambda_1\wp_{334} - \lambda_2 B^B_{11244}$$
$$+ \left(\tfrac{2}{21}\lambda_2{}^2 - \tfrac{16}{63}\lambda_4{}^2\lambda_1 - \tfrac{4}{21}\lambda_3\lambda_1\right) B^A_{22444} + \left(\tfrac{8}{3}\lambda_4\lambda_1 - \tfrac{4}{3}\lambda_3\lambda_2\right) B^D_{13334} - \tfrac{8}{21}\lambda_2{}^2\wp_{133}$$
$$- \tfrac{2}{9}\lambda_1 B^B_{12333} + \tfrac{2}{3}\lambda_2 B^D_{12224} - \tfrac{8}{3}\lambda_1 B^C_{12244} + \tfrac{8}{9}\lambda_1 B^A_{11444} + \tfrac{2}{3}\lambda_2\lambda_1\wp_{233} + \tfrac{4}{3}\lambda_1{}^2\wp_{334}$$
$$- \tfrac{8}{21}\lambda_3{}^2\lambda_1\wp_{334} - \tfrac{4}{63}\lambda_4{}^2\lambda_2{}^2\wp_{334} + \tfrac{4}{21}\lambda_3\lambda_2{}^2\wp_{334} + \tfrac{8}{3}\lambda_4\lambda_0\wp_{133} + \tfrac{2}{3}\lambda_4\lambda_1 B^A_{22244}$$
$$+ \tfrac{20}{63}\lambda_3\lambda_1\wp_{133} + \tfrac{32}{189}\lambda_4{}^4\lambda_1\wp_{334} + \tfrac{8}{63}\lambda_4{}^2\lambda_1\wp_{133} + \tfrac{2}{3}\lambda_4\lambda_2\lambda_1\wp_{334} - \tfrac{4}{63}\lambda_4\lambda_3\lambda_1\wp_{233}$$

$$\mathbf{(-29)} \quad B^A_{11114} = \tfrac{8}{21}\lambda_3\lambda_1\wp_{133} + \tfrac{1}{7}\lambda_3{}^2\lambda_1\wp_{334} + \tfrac{3}{7}\lambda_4\lambda_0\wp_{133} - \tfrac{1}{21}\lambda_4{}^2\lambda_1\wp_{133}$$
$$+ \tfrac{2}{63}\lambda_4{}^3\lambda_1\wp_{233} - \tfrac{4}{63}\lambda_4{}^4\lambda_1\wp_{334} + \tfrac{5}{2}\lambda_2\lambda_0\wp_{334} + \left(\tfrac{9}{4}\lambda_0 - \tfrac{1}{4}\lambda_4\lambda_1\right) B^A_{22244}$$
$$+ \left(\tfrac{2}{21}\lambda_4{}^3\lambda_1 - \tfrac{5}{28}\lambda_4\lambda_3\lambda_1 + \tfrac{9}{4}\lambda_3\lambda_0 - \tfrac{6}{7}\lambda_4{}^2\lambda_0\right) B^B_{33344} + \tfrac{1}{42}\lambda_4\lambda_3\lambda_1\wp_{233}$$
$$+ \left(\tfrac{2}{21}\lambda_4{}^2\lambda_1 + \tfrac{1}{14}\lambda_3\lambda_1 - \tfrac{6}{7}\lambda_4\lambda_0\right) B^A_{22444} + \left(5\lambda_0 - \lambda_4\lambda_1\right) B^D_{13334} - \tfrac{2}{7}\lambda_4{}^2\lambda_0\wp_{233}$$
$$+ \left(\tfrac{3}{14}\lambda_3\lambda_1 - \tfrac{1}{21}\lambda_4{}^2\lambda_1 + \tfrac{3}{7}\lambda_4\lambda_0\right) B^D_{23334} - \tfrac{12}{7}\lambda_4\lambda_3\lambda_0\wp_{334} - \tfrac{2}{3}\lambda_1 B^B_{12333} + \tfrac{4}{7}\lambda_4{}^3\lambda_0\wp_{334}$$
$$+ \lambda_1 B^C_{12244} - \tfrac{1}{3}\lambda_1 B^A_{11444} + \tfrac{1}{7}\lambda_4{}^2\lambda_3\lambda_1\wp_{334} - \tfrac{1}{2}\lambda_4\lambda_2\lambda_1\wp_{334} + \left(\tfrac{4}{3}\lambda_4\lambda_1 - 12\lambda_0\right) B^B_{12444}$$

$$\mathbf{(-30)} \quad B^A_{11113} = \tfrac{1}{2}\lambda_1\wp_{112} + \tfrac{1}{2}\lambda_1{}^2\wp_{333} + 7\lambda_0\wp_{114} - 2\lambda_1\lambda_0\wp_{444} - 4\lambda_4\lambda_0\wp_{124}$$
$$- \lambda_0\wp_{122} + 6\lambda_3\lambda_0\wp_{144} - 2\lambda_2\lambda_0\wp_{333} - 12\lambda_0 B^C_{12344} + 8\lambda_4\lambda_0 B^B_{13444}$$

$$\mathbf{(-32)} \quad B^A_{11112} = -\tfrac{2}{7}\lambda_2\lambda_1\wp_{133} - 2\lambda_2\lambda_0\wp_{233} + \tfrac{5}{7}\lambda_3\lambda_0\wp_{133} - \tfrac{6}{7}\lambda_4{}^2\lambda_3\lambda_0\wp_{334} - \tfrac{1}{2}\lambda_4\lambda_1{}^2\wp_{334}$$
$$+ \tfrac{8}{21}\lambda_4{}^4\lambda_0\wp_{334} - \tfrac{6}{7}\lambda_3{}^2\lambda_0\wp_{334} - \tfrac{4}{21}\lambda_4{}^3\lambda_0\wp_{233} + \lambda_1\lambda_0\wp_{334} + \lambda_1{}^2\wp_{233} - \tfrac{1}{7}\lambda_4\lambda_3\lambda_0\wp_{233}$$
$$+ \tfrac{1}{42}\lambda_4\lambda_2\lambda_1\wp_{233} + \left(\tfrac{3}{2}\lambda_3\lambda_1 - 8\lambda_4\lambda_0\right) B^B_{12444} + \left(\tfrac{3}{14}\lambda_2\lambda_1 + \tfrac{2}{7}\lambda_4{}^2\lambda_0 - \tfrac{9}{7}\lambda_3\lambda_0\right) B^D_{23334}$$
$$- \tfrac{1}{21}\lambda_4{}^2\lambda_2\lambda_1\wp_{334} + \left(\tfrac{15}{14}\lambda_4\lambda_3\lambda_0 - \tfrac{3}{4}\lambda_1{}^2 + \tfrac{1}{14}\lambda_4\lambda_2\lambda_1 - \tfrac{4}{7}\lambda_4{}^3\lambda_0\right) B^B_{33344} + \tfrac{1}{7}\lambda_3\lambda_2\lambda_1\wp_{334}$$
$$+ \left(6\lambda_4\lambda_0 - \lambda_3\lambda_1\right) B^D_{13334} + \left(\tfrac{1}{14}\lambda_2\lambda_1 - \tfrac{4}{7}\lambda_4{}^2\lambda_0 - \tfrac{3}{7}\lambda_3\lambda_0\right) B^A_{22444} + \tfrac{2}{7}\lambda_4{}^2\lambda_0\wp_{133}$$
$$+ 2\lambda_0 B^A_{11444} + \tfrac{1}{2}\lambda_1 B^D_{12224} - \tfrac{3}{4}\lambda_1 B^B_{11244} - 6\lambda_0 B^C_{12244} + \tfrac{3}{2}\lambda_4\lambda_0 B^A_{22244} + 3\lambda_4\lambda_2\lambda_0\wp_{334}$$

D.2 Quadratic 3-index relations

This Appendix contains quadratic 3-index relations associated to the cyclic (3,5)-curve. Each equation expresses the product of two 3-index $\wp$-functions as a degree two polynomial in the 2-index $\wp$-functions and the five Q-functions that were used in the basis for the space of fundamental Abelian functions. The derivation was discussed in Section 5.3.3. Those readers who wish to skip this Appendix should proceed to page 289.

We start by presenting the complete set from weight -6 to weight -19. Note that the Q-functions appear as either linear terms, or multiplied by a 2-index $\wp$-function. To skip to the weight -20 relations proceed to page 264.

$$\mathbf{(-6)} \qquad \wp_{444}^2 = 4\wp_{44}^3 - 4\wp_{23} - 4\wp_{33}\wp_{44} + \wp_{34}^2 + 2\wp_{34}\lambda_4 + \lambda_4^2 - 4\lambda_3$$

$$\mathbf{(-7)} \qquad \wp_{344}\wp_{444} = 4\wp_{34}\wp_{44}^2 + 2\wp_{24}\wp_{44} - \wp_{33}\wp_{34} - \lambda_4\wp_{33} - \tfrac{2}{3}Q_{2444}$$

$$\mathbf{(-8)} \qquad \wp_{344}^2 = 4\wp_{34}^2\wp_{44} + 4\wp_{14} + 4\wp_{24}\wp_{34} + \wp_{33}^2$$

$$\begin{aligned}\mathbf{(-8)} \qquad \wp_{334}\wp_{444} &= 2\wp_{34}^2\wp_{44} + 2\wp_{33}\wp_{44}^2 - 4\wp_{14} + 2\wp_{22} - 2\wp_{23}\wp_{44} - \wp_{24}\wp_{34} \\ &\quad - 2\wp_{33}^2 - \wp_{24}\lambda_4 + 2\lambda_4\wp_{34}\wp_{44}\end{aligned}$$

$$\mathbf{(-9)} \qquad \wp_{334}\wp_{344} = -2\wp_{13} + 2\wp_{44}\wp_{33}\wp_{34} - 2\wp_{23}\wp_{34} + \wp_{24}\wp_{33} + 2\wp_{34}^3 + 2\lambda_4\wp_{34}^2$$

$$\begin{aligned}\mathbf{(-9)} \qquad \wp_{333}\wp_{444} &= 6\wp_{44}\wp_{33}\wp_{34} - 4\lambda_4\wp_{34}^2 + 2\wp_{13} + 7\wp_{23}\wp_{34} + 2\wp_{24}\wp_{33} + \lambda_4\wp_{23} \\ &\quad - 2\lambda_4\wp_{33}\wp_{44} - 2\wp_{34}^3 - 2\wp_{34}\lambda_4^2 + 6\wp_{34}\lambda_3 + 2\lambda_2 - 2\wp_{44}Q_{2444}\end{aligned}$$

$$\begin{aligned}\mathbf{(-9)} \qquad \wp_{244}\wp_{444} &= -2\wp_{13} + \wp_{23}\wp_{34} - 2\wp_{24}\wp_{33} - \lambda_4\wp_{23} + 2\wp_{34}\lambda_3 - 2\lambda_2 \\ &\quad + 4\wp_{24}\wp_{44}^2 + \tfrac{2}{3}\wp_{44}Q_{2444}\end{aligned}$$

$$\mathbf{(-10)} \qquad \wp_{334}^2 = 4\wp_{33}\wp_{34}^2 - 4\wp_{14}\wp_{44} + \wp_{24}^2 - 2\wp_{33}\lambda_3 - \tfrac{4}{3}\wp_{34}Q_{2444} + \tfrac{2}{3}Q_{2333}$$

$$\begin{aligned}\mathbf{(-10)} \qquad \wp_{333}\wp_{344} &= 2\wp_{33}^2\wp_{44} + 2\wp_{33}\wp_{34}^2 + 8\wp_{14}\wp_{44} - 2\wp_{24}^2 - \wp_{23}\wp_{33} \\ &\quad + 2\lambda_4\wp_{33}\wp_{34} + \wp_{33}\lambda_3 + \tfrac{2}{3}\wp_{34}Q_{2444} - \tfrac{1}{3}Q_{2333}\end{aligned}$$

$$\mathbf{(-10)} \qquad \wp_{244}\wp_{344} = 2\wp_{24}^2 + \wp_{23}\wp_{33} - \wp_{33}\lambda_3 + 4\wp_{34}\wp_{24}\wp_{44} + \tfrac{2}{3}\wp_{34}Q_{2444} + \tfrac{1}{3}Q_{2333}$$

$$\begin{aligned}\mathbf{(-10)} \qquad \wp_{234}\wp_{444} &= 2\wp_{34}\wp_{24}\wp_{44} + 2\wp_{23}\wp_{44}^2 + 4\wp_{14}\wp_{44} - 2\wp_{22}\wp_{44} - 2\wp_{23}\wp_{33} \\ &\quad + 2\lambda_4\wp_{24}\wp_{44} - 2\wp_{33}\lambda_3 - \tfrac{1}{3}\wp_{34}Q_{2444} - \tfrac{1}{3}\lambda_4Q_{2444}\end{aligned}$$

$$\begin{aligned}\mathbf{(-11)} \qquad \wp_{333}\wp_{334} &= -2\wp_{12} + 2\wp_{14}\lambda_4 - 2\wp_{13}\wp_{44} + 6\wp_{14}\wp_{34} - 2\wp_{22}\wp_{34} + \wp_{23}\wp_{24} \\ &\quad + 4\wp_{33}^2\wp_{34} - \tfrac{2}{3}\wp_{33}Q_{2444}\end{aligned}$$

$$\begin{aligned}\mathbf{(-11)} \qquad \wp_{244}\wp_{334} &= 2\wp_{12} - 2\wp_{14}\lambda_4 - 2\wp_{13}\wp_{44} + 2\lambda_4\wp_{24}\wp_{34} + 2\wp_{14}\wp_{34} \\ &\quad + 2\wp_{24}\wp_{34}^2 + 2\wp_{44}\wp_{24}\wp_{33} - 2\wp_{22}\wp_{34} - \wp_{23}\wp_{24} + \tfrac{2}{3}\wp_{33}Q_{2444}\end{aligned}$$

$$\begin{aligned}\mathbf{(-11)} \qquad \wp_{233}\wp_{444} &= 4\wp_{44}\wp_{23}\wp_{34} - 2\wp_{24}\lambda_4^2 - 2\wp_{24}\wp_{34}^2 + \lambda_4\wp_{22} - 2\wp_{14}\wp_{34} \\ &\quad - 2\lambda_4\wp_{23}\wp_{44} - 4\lambda_4\wp_{24}\wp_{34} + 2\wp_{44}\wp_{24}\wp_{33} - 2\wp_{13}\wp_{44} + \tfrac{2}{3}\wp_{33}Q_{2444} \\ &\quad + 6\wp_{44}\wp_{34}\lambda_3 - 2\wp_{14}\lambda_4 + 6\wp_{24}\lambda_3 - 2\wp_{44}\lambda_2 + \wp_{22}\wp_{34} + 6\wp_{23}\wp_{24}\end{aligned}$$

$$\begin{aligned}
\textbf{(−11)}\quad \wp_{234}\wp_{344} &= 2\wp_{14}\lambda_4 + 2\lambda_4\wp_{24}\wp_{34} + 2\wp_{13}\wp_{44} + 2\wp_{14}\wp_{34} + 2\wp_{24}\wp_{34}^2 \\
&\quad + 2\wp_{44}\wp_{23}\wp_{34} - \tfrac{1}{3}\wp_{33}Q_{2444} \\
\textbf{(−12)}\quad \wp_{333}^2 &= -4\wp_{22}\wp_{33} + \wp_{23}^2 - 4\wp_{13}\wp_{34} + 16\wp_{14}\wp_{33} + 4\wp_{33}^3 + 4\lambda_1 \\
&\quad - 4\lambda_4\lambda_2 - 4\wp_{13}\lambda_4 - 6\wp_{23}\lambda_3 - 4\wp_{34}\lambda_2 + 2Q_{2233} \\
\textbf{(−12)}\quad \wp_{244}\wp_{333} &= 2\wp_{33}\wp_{23}\wp_{44} - \wp_{23}^2 + 4\wp_{13}\wp_{34} + \tfrac{8}{3}\wp_{13}\lambda_4 + 4\wp_{33}\wp_{24}\wp_{34} - \tfrac{2}{3}Q_{2233} \\
&\quad - 2\wp_{23}\wp_{34}^2 - 2\wp_{44}\wp_{33}\lambda_3 + 2\lambda_4\wp_{23}\wp_{34} + \tfrac{4}{3}\lambda_1 + \tfrac{4}{3}\lambda_4\lambda_2 \\
&\quad + \tfrac{8}{3}\wp_{34}\lambda_2 + 2\wp_{23}\lambda_3 - 4\wp_{34}^2\lambda_3 - \tfrac{4}{3}\wp_{24}Q_{2444} + \tfrac{2}{3}\wp_{44}Q_{2333} \\
\textbf{(−12)}\quad \wp_{244}^2 &= \wp_{23}^2 - \tfrac{8}{3}\wp_{13}\lambda_4 + 4\wp_{44}\wp_{24}^2 - \tfrac{4}{3}\lambda_4\lambda_2 + \tfrac{4}{3}\wp_{34}\lambda_2 - 2\wp_{23}\lambda_3 \\
&\quad - \tfrac{4}{3}\lambda_1 + \tfrac{4}{3}\wp_{24}Q_{2444} + \tfrac{2}{3}Q_{2233} \\
\textbf{(−12)}\quad \wp_{234}\wp_{334} &= 2\wp_{14}\wp_{33} - \tfrac{4}{3}\wp_{13}\lambda_4 + 2\wp_{33}\wp_{24}\wp_{34} + 2\wp_{23}\wp_{34}^2 + \wp_{44}\wp_{33}\lambda_3 \\
&\quad - \tfrac{2}{3}\lambda_4\lambda_2 + \tfrac{2}{3}\wp_{34}\lambda_2 - \wp_{23}\lambda_3 + \tfrac{4}{3}\lambda_1 + 2\wp_{34}^2\lambda_3 - \tfrac{1}{3}\wp_{24}Q_{2444} \\
&\quad + \tfrac{1}{3}Q_{2233} - \tfrac{1}{3}\wp_{44}Q_{2333} \\
\textbf{(−12)}\quad \wp_{233}\wp_{344} &= 2\wp_{33}\wp_{23}\wp_{44} - \wp_{22}\wp_{33} - \tfrac{4}{3}\wp_{13}\lambda_4 + 2\wp_{23}\wp_{34}^2 + 2\wp_{33}\wp_{24}\lambda_4 \\
&\quad - 2\wp_{44}\wp_{33}\lambda_3 - \tfrac{2}{3}\lambda_4\lambda_2 + \tfrac{2}{3}\wp_{34}\lambda_2 - \wp_{23}\lambda_3 + \tfrac{4}{3}\lambda_1 + 2\wp_{34}^2\lambda_3 \\
&\quad + \tfrac{2}{3}\wp_{24}Q_{2444} + \tfrac{2}{3}\wp_{44}Q_{2333} + \tfrac{1}{3}Q_{2233} \\
\textbf{(−12)}\quad \wp_{224}\wp_{444} &= 2\lambda_4\wp_{23}\wp_{34} + 3\wp_{44}\wp_{33}\lambda_3 - 2\wp_{23}^2 + 2\wp_{13}\wp_{34} - 2\wp_{14}\wp_{33} \\
&\quad - \wp_{34}^2\lambda_3 + \lambda_4\wp_{34}\lambda_3 + 2\wp_{22}\wp_{44}^2 + 2\wp_{44}\wp_{24}^2 - 2\wp_{33}\wp_{24}\lambda_4 \\
&\quad - \tfrac{4}{3}\lambda_1 - \tfrac{1}{3}\lambda_4\lambda_2 - \tfrac{2}{3}\wp_{13}\lambda_4 - \wp_{23}\lambda_3 + \tfrac{7}{3}\wp_{34}\lambda_2 + \tfrac{2}{3}\wp_{44}\lambda_4Q_{2444} \\
&\quad - \tfrac{2}{3}\wp_{24}Q_{2444} - \tfrac{1}{3}\wp_{44}Q_{2333} - \tfrac{1}{3}Q_{2233} \\
\textbf{(−12)}\quad \wp_{144}\wp_{444} &= \wp_{44}\wp_{33}\lambda_3 + \wp_{13}\wp_{34} - 2\wp_{14}\wp_{33} + 4\wp_{14}\wp_{44}^2 - \tfrac{4}{3}\lambda_1 + \tfrac{2}{3}\lambda_4\lambda_2 \\
&\quad + \tfrac{1}{3}\wp_{13}\lambda_4 + \wp_{23}\lambda_3 + \tfrac{4}{3}\wp_{34}\lambda_2 - \tfrac{1}{3}\wp_{44}Q_{2333} - \tfrac{1}{3}Q_{2233}
\end{aligned}$$

The following relations are at weight -13.

$$\begin{aligned}
\wp_{134}\wp_{444} &= \tfrac{3}{2}\wp_{33}\lambda_4\lambda_3 - 2\wp_{12}\wp_{44} - 2\wp_{13}\wp_{33} - \tfrac{1}{2}\wp_{34}\wp_{33}\lambda_3 + 2\wp_{34}\wp_{14}\wp_{44} \\
&\quad + 2\wp_{44}\wp_{14}\lambda_4 + 2\wp_{44}^2\wp_{13} - \tfrac{1}{2}\lambda_4Q_{2333} + \tfrac{1}{6}\wp_{34}Q_{2333} - 2Q_{1244} \\
\wp_{234}\wp_{244} &= 4\wp_{14}\wp_{24} - 2\wp_{22}\wp_{24} + 2\wp_{34}\wp_{24}^2 - 2\wp_{33}\lambda_2 + 2\wp_{24}^2\lambda_4 - 2\wp_{33}\lambda_4\lambda_3 \\
&\quad + 2\wp_{24}\wp_{23}\wp_{44} + \tfrac{1}{3}\wp_{23}Q_{2444} + \tfrac{2}{3}\lambda_4Q_{2333} + 2Q_{1244} \\
\wp_{233}\wp_{334} &= -2\wp_{12}\wp_{44} - 2\wp_{14}\wp_{24} + \wp_{22}\wp_{24} + 2\wp_{33}\lambda_2 + 2\wp_{33}\lambda_4\lambda_3 - \tfrac{2}{3}\wp_{23}Q_{2444} \\
&\quad + 4\wp_{34}\wp_{23}\wp_{33} + \tfrac{2}{3}\wp_{34}Q_{2333} - \tfrac{2}{3}\lambda_4Q_{2333} - 4Q_{1244} \\
\wp_{234}\wp_{333} &= 2\wp_{34}\wp_{23}\wp_{33} + 3\wp_{34}\wp_{33}\lambda_3 + 4\wp_{14}\wp_{24} + 2\wp_{13}\wp_{33} - 2\wp_{22}\wp_{24} + 2Q_{1244} \\
&\quad - \wp_{33}\lambda_4\lambda_3 + 2\wp_{24}\wp_{33}^2 + \tfrac{1}{3}\wp_{23}Q_{2444} - \tfrac{1}{3}\wp_{34}Q_{2333} + \tfrac{1}{3}\lambda_4Q_{2333} + 4\wp_{12}\wp_{44}
\end{aligned}$$

$$\wp_{224}\wp_{344} = 2\wp_{34}\wp_{33}\lambda_3 - 2\wp_{12}\wp_{44} + 2\wp_{14}\wp_{24} + 2\wp_{22}\wp_{24} + 2\wp_{34}\wp_{24}^2 + \wp_{33}\lambda_4\lambda_3 - 2Q_{1244}$$
$$+ \wp_{33}\lambda_2 + 2\wp_{34}\wp_{22}\wp_{44} - \tfrac{2}{3}\wp_{23}Q_{2444} + \tfrac{2}{3}\wp_{34}\lambda_4 Q_{2444} - \tfrac{1}{3}\wp_{34}Q_{2333} - \tfrac{1}{3}\lambda_4 Q_{2333}$$
$$\wp_{144}\wp_{344} = \wp_{13}\wp_{33} + \wp_{34}\wp_{33}\lambda_3 + 4\wp_{34}\wp_{14}\wp_{44} + 2\wp_{14}\wp_{24} - \tfrac{1}{3}\wp_{34}Q_{2333}$$
$$\wp_{223}\wp_{444} = \tfrac{4}{3}\wp_{23}Q_{2444} + \tfrac{1}{3}\wp_{34}Q_{2333} + 4\wp_{24}\wp_{23}\wp_{44} - 4\wp_{44}^2\lambda_2 - 2\wp_{34}\wp_{33}\lambda_3$$
$$- 4\wp_{12}\wp_{44} - 8\wp_{14}\wp_{24} - 4Q_{1244} + 4\wp_{22}\wp_{24} + 6\lambda_3\wp_{24}\wp_{44} - \lambda_4 Q_{2333}$$
$$- 2\wp_{34}\wp_{24}^2 + 2\wp_{33}\lambda_4\lambda_3 + 4\wp_{33}\lambda_2 + 8\wp_{44}\wp_{14}\lambda_4 - 2\wp_{24}^2\lambda_4 - \tfrac{2}{3}\lambda_4^2 Q_{2444}$$
$$+ 2\wp_{34}\wp_{22}\wp_{44} - \tfrac{2}{3}\wp_{34}\lambda_4 Q_{2444} + 2\lambda_3 Q_{2444} - 2\lambda_4\wp_{22}\wp_{44}$$

The following relations are at weight -14.

$$\wp_{234}^2 = -\tfrac{4}{3}\wp_{44}\wp_{34}\lambda_2 + \tfrac{8}{3}\wp_{14}\wp_{23} + \tfrac{1}{3}\wp_{13}\wp_{24} - \tfrac{1}{3}\wp_{12}\wp_{34} + \tfrac{1}{3}\lambda_4\wp_{22}\wp_{34} + 2\wp_{44}\wp_{13}\lambda_4$$
$$+ \tfrac{1}{3}\wp_{22}\wp_{34}^2 + \wp_{44}\wp_{23}\lambda_3 + \tfrac{2}{3}\wp_{44}\lambda_4\lambda_2 + \tfrac{1}{3}\wp_{23}\wp_{24}\lambda_4 + 3\lambda_3\wp_{24}\wp_{34} + \tfrac{1}{9}\wp_{33}Q_{2333} + \tfrac{2}{3}\wp_{11}$$
$$- 2\wp_{44}\lambda_1 + 2\wp_{14}\lambda_3 - \tfrac{1}{3}\wp_{22}\wp_{23} + \tfrac{1}{3}\wp_{24}\lambda_2 + \tfrac{1}{3}\wp_{33}\wp_{24}^2 - \tfrac{1}{3}\wp_{44}Q_{2233} - \tfrac{2}{9}\lambda_4\wp_{33}Q_{2444}$$
$$- \tfrac{2}{3}\wp_{33}^2\lambda_3 + \tfrac{1}{3}\wp_{44}\wp_{23}^2 - \tfrac{1}{3}\lambda_4\wp_{12} + \tfrac{2}{3}\wp_{14}\lambda_4^2 + \tfrac{10}{3}\wp_{23}\wp_{24}\wp_{34} - \tfrac{1}{3}\wp_{44}\wp_{22}\wp_{33} + \tfrac{2}{3}\wp_{34}\wp_{14}\lambda_4$$
$$\wp_{233}\wp_{333} = 2\wp_{11} - \wp_{22}\wp_{23} - 2\wp_{12}\wp_{34} - 2\wp_{13}\wp_{24} + 8\wp_{14}\wp_{23} - 2\wp_{24}\lambda_2$$
$$+ 2\wp_{44}\lambda_1 - 2\lambda_4\wp_{12} + 4\wp_{23}\wp_{33}^2 + \tfrac{2}{3}\wp_{33}Q_{2333} + 6\wp_{14}\lambda_3$$
$$\wp_{233}\wp_{244} = \tfrac{8}{3}\wp_{44}\wp_{34}\lambda_2 + \tfrac{8}{3}\wp_{14}\wp_{23} + \tfrac{10}{3}\wp_{13}\wp_{24} + \tfrac{2}{3}\wp_{12}\wp_{34} + \tfrac{2}{3}\wp_{44}\wp_{22}\wp_{33} - \tfrac{4}{3}\wp_{34}\wp_{14}\lambda_4$$
$$- \tfrac{2}{3}\lambda_4\wp_{22}\wp_{34} + \tfrac{2}{3}\wp_{11} - \tfrac{2}{3}\wp_{22}\wp_{34}^2 - 2\wp_{44}\wp_{23}\lambda_3 - \tfrac{4}{3}\wp_{44}\lambda_4\lambda_2 + \tfrac{4}{3}\wp_{23}\wp_{24}\lambda_4 + \tfrac{2}{3}\wp_{44}Q_{2233}$$
$$+ \tfrac{4}{9}\lambda_4\wp_{33}Q_{2444} - 2\wp_{44}\lambda_1 + 2\wp_{14}\lambda_3 - \tfrac{1}{3}\wp_{22}\wp_{23} + \tfrac{10}{3}\wp_{24}\lambda_2 + \tfrac{4}{3}\wp_{33}\wp_{24}^2 - 4\wp_{44}\wp_{13}\lambda_4$$
$$- \tfrac{2}{9}\wp_{33}Q_{2333} + \tfrac{4}{3}\wp_{33}^2\lambda_3 + \tfrac{4}{3}\wp_{44}\wp_{23}^2 + \tfrac{2}{3}\lambda_4\wp_{12} - \tfrac{4}{3}\wp_{14}\lambda_4^2 + \tfrac{4}{3}\wp_{23}\wp_{24}\wp_{34}$$
$$\wp_{134}\wp_{344} = -\tfrac{1}{2}\wp_{33}^2\lambda_3 - 2\wp_{14}\wp_{23} + 2\wp_{13}\wp_{24} + 2\wp_{14}\wp_{34}^2 + 2\wp_{34}\wp_{14}\lambda_4$$
$$+ 2\wp_{44}\wp_{13}\wp_{34} + \tfrac{1}{6}\wp_{33}Q_{2333}$$
$$\wp_{144}\wp_{334} = \wp_{33}^2\lambda_3 + 2\wp_{14}\wp_{34}^2 - 2\wp_{12}\wp_{34} + 2\wp_{34}\wp_{14}\lambda_4 - \wp_{13}\wp_{24} - \tfrac{1}{3}\wp_{33}Q_{2333}$$
$$+ 2\wp_{33}\wp_{14}\wp_{44}$$
$$\wp_{224}\wp_{334} = -\tfrac{4}{3}\wp_{44}\wp_{34}\lambda_2 + \tfrac{2}{3}\wp_{14}\wp_{23} - \tfrac{8}{3}\wp_{13}\wp_{24} - \tfrac{10}{3}\wp_{12}\wp_{34} - \tfrac{2}{3}\lambda_4\wp_{22}\wp_{34} + \tfrac{2}{3}\wp_{11}$$
$$+ \tfrac{4}{3}\wp_{22}\wp_{34}^2 + \wp_{44}\wp_{23}\lambda_3 + \tfrac{2}{3}\wp_{44}\lambda_4\lambda_2 - \tfrac{2}{3}\wp_{23}\wp_{24}\lambda_4 + 3\lambda_3\wp_{24}\wp_{34} + \tfrac{4}{9}\lambda_4\wp_{33}Q_{2444}$$
$$- 2\wp_{44}\lambda_1 + 2\wp_{14}\lambda_3 + \tfrac{2}{3}\wp_{22}\wp_{23} + \tfrac{1}{3}\wp_{24}\lambda_2 + \tfrac{4}{3}\wp_{33}\wp_{24}^2 - \tfrac{1}{3}\wp_{44}Q_{2233} - \tfrac{2}{9}\wp_{33}Q_{2333}$$
$$+ \tfrac{4}{3}\wp_{33}^2\lambda_3 - \tfrac{2}{3}\wp_{44}\wp_{23}^2 + \tfrac{2}{3}\lambda_4\wp_{12} - \tfrac{4}{3}\wp_{14}\lambda_4^2 + \tfrac{4}{3}\wp_{23}\wp_{24}\wp_{34} + \tfrac{2}{3}\wp_{44}\wp_{22}\wp_{33} + \tfrac{8}{3}\wp_{34}\wp_{14}\lambda_4$$
$$\wp_{223}\wp_{344} = \tfrac{8}{3}\wp_{11} - \tfrac{4}{3}\wp_{22}\wp_{23} + \tfrac{4}{3}\lambda_4\wp_{22}\wp_{34} - \tfrac{4}{3}\wp_{12}\wp_{34} + \tfrac{4}{3}\wp_{13}\wp_{24} - \tfrac{4}{3}\wp_{14}\wp_{23}$$
$$+ \tfrac{4}{3}\wp_{22}\wp_{34}^2 + \tfrac{4}{3}\wp_{24}\lambda_2 - 4\wp_{14}\lambda_3 - \tfrac{2}{3}\wp_{33}\wp_{24}^2 + \tfrac{4}{3}\wp_{23}\wp_{24}\lambda_4 + \tfrac{4}{3}\wp_{23}\wp_{24}\wp_{34}$$
$$+ \tfrac{2}{3}\wp_{44}Q_{2233} + \tfrac{1}{9}\wp_{33}Q_{2333} - \tfrac{2}{9}\lambda_4\wp_{33}Q_{2444} - \tfrac{2}{3}\wp_{33}^2\lambda_3 + \tfrac{4}{3}\wp_{44}\wp_{23}^2 - \tfrac{4}{3}\wp_{44}\lambda_4\lambda_2$$
$$- 2\wp_{44}\wp_{23}\lambda_3 - \tfrac{4}{3}\wp_{44}\wp_{34}\lambda_2 + \tfrac{2}{3}\wp_{44}\wp_{22}\wp_{33} + \tfrac{8}{3}\wp_{34}\wp_{14}\lambda_4 - \tfrac{4}{3}\lambda_4\wp_{12} + \tfrac{8}{3}\wp_{14}\lambda_4^2$$

$$\begin{aligned}\wp_{133}\wp_{444} &= -2\wp_{11} + \wp_{12}\wp_{34} + 6\wp_{14}\wp_{23} - 2\wp_{44}\lambda_1 + 6\wp_{14}\lambda_3 + \wp_{33}^2\lambda_3 - 2\wp_{14}\lambda_4^2 \\ &\quad + \lambda_4\wp_{12} + 2\wp_{44}\wp_{13}\lambda_4 - 4\wp_{34}\wp_{14}\lambda_4 + 3\wp_{44}\wp_{23}\lambda_3 + 4\wp_{44}\wp_{34}\lambda_2 + 2\wp_{44}\lambda_4\lambda_2 \\ &\quad + 2\wp_{33}\wp_{14}\wp_{44} + 4\wp_{44}\wp_{13}\wp_{34} - 2\wp_{14}\wp_{34}^2 - \tfrac{1}{3}\wp_{33}Q_{2333} - \wp_{44}Q_{2233}\end{aligned}$$

The following relations are at weight -15.

$$\begin{aligned}\wp_{233}\wp_{234} &= \tfrac{2}{3}\wp_{34}^2\lambda_2 + 2\wp_{24}\wp_{33}\wp_{23} + \tfrac{4}{3}\lambda_4\lambda_1 + \tfrac{1}{3}\wp_{34}Q_{2233} - \tfrac{2}{3}\lambda_4^2\lambda_2 + 2\wp_{13}\wp_{23} \\ &\quad - \tfrac{2}{3}\wp_{14}Q_{2444} + 2\wp_{13}\lambda_3 + \tfrac{4}{3}\wp_{34}\lambda_1 - 2\lambda_4\wp_{44}\wp_{33}\lambda_3 + 2\wp_{34}\wp_{23}^2 + 2\wp_{44}Q_{1244} \\ &\quad - \tfrac{4}{3}\wp_{34}\lambda_4\wp_{13} + \wp_{34}\wp_{23}\lambda_3 - \lambda_4\wp_{23}\lambda_3 - 2\wp_{33}\wp_{44}\lambda_2 + 2\wp_{24}\wp_{33}\lambda_3 + \tfrac{1}{3}\lambda_4Q_{2233} \\ &\quad - \tfrac{4}{3}\lambda_4^2\wp_{13} + \tfrac{2}{3}\wp_{44}\lambda_4Q_{2333} + \tfrac{1}{3}\wp_{22}Q_{2444}\end{aligned}$$

$$\begin{aligned}\wp_{224}\wp_{333} &= -\tfrac{2}{3}\lambda_4Q_{2233} + \tfrac{20}{3}\wp_{34}\lambda_4\wp_{13} + \tfrac{2}{3}\wp_{44}\lambda_4Q_{2333} + \wp_{24}Q_{2333} + 2\wp_{34}\wp_{33}\wp_{22} - \tfrac{2}{3}\lambda_4\lambda_1 \\ &\quad + 2\wp_{34}\lambda_4\lambda_2 - 2\lambda_4\wp_{44}\wp_{33}\lambda_3 + \tfrac{4}{3}\lambda_4^2\lambda_2 - 2\wp_{12}\wp_{33} + 6\lambda_0 + \wp_{34}\wp_{23}\lambda_3 + 2\wp_{44}Q_{1244} \\ &\quad + 2\lambda_4\wp_{23}\lambda_3 + 4\wp_{24}\wp_{33}\wp_{23} - \wp_{24}\wp_{33}\lambda_3 - \tfrac{10}{3}\wp_{34}^2\lambda_2 - 6\wp_{13}\wp_{23} - \tfrac{2}{3}\wp_{22}Q_{2444} + \wp_{23}\lambda_2 \\ &\quad - \tfrac{2}{3}\wp_{14}Q_{2444} - 4\wp_{13}\lambda_3 - \tfrac{8}{3}\wp_{34}\lambda_1 - 2\wp_{34}\wp_{23}^2 + \tfrac{8}{3}\lambda_4^2\wp_{13} - \tfrac{2}{3}\wp_{34}Q_{2233} - 2\wp_{33}\wp_{44}\lambda_2\end{aligned}$$

$$\begin{aligned}\wp_{222}\wp_{444} &= -8\wp_{33}\wp_{14}\lambda_4 - 2\wp_{24}^3 - 6\wp_{34}\lambda_3^2 - 2\lambda_4\wp_{23}^2 + 6\wp_{44}\wp_{24}\wp_{22} - \wp_{24}Q_{2333} \\ &\quad + 2\wp_{34}\lambda_4\lambda_2 + 2\lambda_4\wp_{44}\wp_{33}\lambda_3 + 2\lambda_4^2\wp_{34}\lambda_3 - 2\lambda_3\lambda_2 + 4\lambda_4^2\wp_{23}\wp_{34} + 2\wp_{12}\wp_{33} \\ &\quad - 4\lambda_0 + 2\wp_{33}\wp_{44}\lambda_2 - \lambda_4\wp_{23}\lambda_3 - 2\lambda_4\wp_{34}^2\lambda_3 + 7\wp_{24}\wp_{33}\lambda_3 + 2\wp_{34}^2\lambda_2 - 4\wp_{23}\lambda_2 \\ &\quad + 2\wp_{22}Q_{2444} - 4\wp_{14}Q_{2444} - 2\wp_{13}\lambda_3 + 4\wp_{34}\lambda_1 + 2\wp_{34}Q_{2233} - 2\wp_{13}\wp_{23} \\ &\quad - 4\lambda_4^2\wp_{24}\wp_{33} - 2\wp_{24}\lambda_4Q_{2444} + 2\wp_{44}\lambda_3Q_{2444} + 2\lambda_4\wp_{33}\wp_{22} - 13\wp_{34}\wp_{23}\lambda_3\end{aligned}$$

$$\begin{aligned}\wp_{224}\wp_{244} &= 2\wp_{24}^3 - \tfrac{2}{3}\wp_{34}^2\lambda_2 - \tfrac{2}{3}\lambda_4\lambda_1 - \tfrac{2}{3}\lambda_4^2\lambda_2 + \tfrac{2}{3}\wp_{14}Q_{2444} - 2\wp_{44}Q_{1244} - \tfrac{1}{3}\wp_{34}Q_{2233} \\ &\quad + 2\lambda_4\wp_{44}\wp_{33}\lambda_3 - \tfrac{4}{3}\lambda_4^2\wp_{13} + \tfrac{4}{3}\wp_{34}\lambda_4\wp_{13} + 2\wp_{34}\wp_{23}\lambda_3 + \tfrac{4}{3}\wp_{34}\lambda_4\lambda_2 - \lambda_4\wp_{23}\lambda_3 + \tfrac{8}{3}\wp_{34}\lambda_1 \\ &\quad + 2\wp_{33}\wp_{44}\lambda_2 + 2\wp_{44}\wp_{24}\wp_{22} - \wp_{23}\lambda_2 + \tfrac{2}{3}\wp_{24}\lambda_4Q_{2444} - \tfrac{2}{3}\wp_{44}\lambda_4Q_{2333} + \tfrac{1}{3}\lambda_4Q_{2233} - 2\lambda_0\end{aligned}$$

$$\begin{aligned}\wp_{223}\wp_{334} &= 2\wp_{34}\wp_{33}\wp_{22} + \tfrac{2}{3}\wp_{34}^2\lambda_2 + \tfrac{4}{3}\lambda_4\lambda_1 - \tfrac{2}{3}\lambda_4^2\lambda_2 - 2\wp_{12}\wp_{33} + 2\wp_{13}\wp_{23} \\ &\quad - \tfrac{2}{3}\wp_{22}Q_{2444} + \tfrac{4}{3}\wp_{14}Q_{2444} + 2\wp_{13}\lambda_3 + \tfrac{4}{3}\wp_{34}\lambda_1 + 4\lambda_4\wp_{44}\wp_{33}\lambda_3 + 2\wp_{34}\wp_{23}^2 \\ &\quad - \tfrac{4}{3}\lambda_4^2\wp_{13} + 4\wp_{33}\wp_{14}\lambda_4 - \tfrac{4}{3}\wp_{34}\lambda_4\wp_{13} + \wp_{34}\wp_{23}\lambda_3 - \lambda_4\wp_{23}\lambda_3 + 2\wp_{33}\wp_{44}\lambda_2 \\ &\quad - \wp_{24}\wp_{33}\lambda_3 - \tfrac{4}{3}\wp_{44}\lambda_4Q_{2333} + \tfrac{1}{3}\wp_{34}Q_{2233} + \tfrac{1}{3}\lambda_4Q_{2233} - 4\wp_{44}Q_{1244}\end{aligned}$$

$$\begin{aligned}\wp_{144}\wp_{333} &= 2\lambda_0 - \tfrac{8}{3}\wp_{34}^2\lambda_2 - \wp_{13}\wp_{23} + 2\wp_{33}\wp_{44}\wp_{13} + 4\wp_{33}\wp_{14}\wp_{34} + \tfrac{1}{3}\wp_{24}Q_{2333} \\ &\quad - 2\wp_{34}\wp_{23}\lambda_3 - \wp_{24}\wp_{33}\lambda_3 - 2\wp_{34}^2\wp_{13} - \tfrac{4}{3}\wp_{34}\lambda_4\lambda_2 + \tfrac{2}{3}\wp_{34}\lambda_1 - \tfrac{2}{3}\wp_{14}Q_{2444} \\ &\quad + \tfrac{2}{3}\wp_{34}Q_{2233} - \tfrac{2}{3}\wp_{34}\lambda_4\wp_{13}\end{aligned}$$

$$\wp_{144}\wp_{244} = \wp_{13}\wp_{23} - 2\lambda_0 + \wp_{24}\wp_{33}\lambda_3 + 2\wp_{34}\lambda_1 + 4\wp_{44}\wp_{14}\wp_{24} + \tfrac{2}{3}\wp_{14}Q_{2444} - \tfrac{1}{3}\wp_{24}Q_{2333}$$

$$\begin{aligned}\wp_{134}\wp_{334} &= 2\wp_{34}^2\wp_{13} + 2\lambda_0 + 2\wp_{33}\wp_{14}\wp_{34} + \tfrac{4}{3}\wp_{34}\lambda_4\wp_{13} + \wp_{34}\wp_{23}\lambda_3 + \tfrac{1}{2}\wp_{24}\wp_{33}\lambda_3 \\ &\quad + \tfrac{4}{3}\wp_{34}^2\lambda_2 + \tfrac{2}{3}\wp_{34}\lambda_4\lambda_2 + \tfrac{2}{3}\wp_{34}\lambda_1 - \tfrac{2}{3}\wp_{14}Q_{2444} - \tfrac{1}{6}\wp_{24}Q_{2333} - \tfrac{1}{3}\wp_{34}Q_{2233}\end{aligned}$$

$$\wp_{133}\wp_{344} = 2\wp_{34}^2\wp_{13} + 2\lambda_0 - \wp_{12}\wp_{33} + 2\wp_{33}\wp_{44}\wp_{13} - \wp_{24}\wp_{33}\lambda_3 + \tfrac{4}{3}\wp_{34}\lambda_4\wp_{13} + \wp_{34}\wp_{23}\lambda_3$$
$$+ 2\wp_{33}\wp_{14}\lambda_4 + \tfrac{4}{3}\wp_{34}^2\lambda_2 + \tfrac{2}{3}\wp_{34}\lambda_4\lambda_2 + \tfrac{2}{3}\wp_{34}\lambda_1 + \tfrac{4}{3}\wp_{14}Q_{2444} + \tfrac{1}{3}\wp_{24}Q_{2333} - \tfrac{1}{3}\wp_{34}Q_{2233}$$
$$\wp_{124}\wp_{444} = \tfrac{1}{3}\lambda_4^2\lambda_2 - \tfrac{2}{3}\wp_{34}^2\lambda_2 - 2\wp_{13}\lambda_3 - 2\wp_{13}\wp_{23} - \tfrac{2}{3}\wp_{14}Q_{2444} + 2\wp_{44}Q_{1244} - \tfrac{1}{6}\lambda_4Q_{2233}$$
$$- \lambda_4\wp_{44}\wp_{33}\lambda_3 + 2\wp_{12}\wp_{44}^2 + \tfrac{2}{3}\lambda_4^2\wp_{13} + 2\wp_{44}\wp_{14}\wp_{24} - 2\wp_{33}\wp_{14}\lambda_4 + \tfrac{2}{3}\wp_{34}\lambda_1$$
$$- \tfrac{1}{2}\wp_{34}\wp_{23}\lambda_3 + 2\wp_{12}\wp_{44}^2 + \tfrac{1}{3}\wp_{34}\lambda_4\lambda_2 + \tfrac{1}{2}\lambda_4\wp_{23}\lambda_3 + \tfrac{1}{3}\wp_{44}\lambda_4Q_{2333} - \tfrac{2}{3}\lambda_4\lambda_1$$
$$+ \tfrac{1}{6}\wp_{34}Q_{2233} + \tfrac{4}{3}\wp_{34}\lambda_4\wp_{13}$$

The following relations are at weight -16.

$$\wp_{233}^2 = 4\wp_{33}\lambda_4^2\lambda_3 + 20\wp_{33}Q_{2233} - 36\wp_{33}\lambda_4\lambda_2 + 87\wp_{33}\lambda_3^2 + 120\wp_{34}Q_{1244}$$
$$- 120\wp_{34}\wp_{33}\lambda_4\lambda_3 + 120\wp_{11}\wp_{44} - 120\wp_{14}^2 - 124\wp_{12}\wp_{24} + 120\wp_{14}\wp_{22} + \wp_{22}^2$$
$$- 62Q_{1224} + 204\wp_{33}\lambda_1 - 80\wp_{33}\wp_{13}\lambda_4 - 80\wp_{33}\wp_{34}\lambda_2 + 4\wp_{33}\wp_{23}^2 - 40\wp_{13}Q_{2444}$$
$$+ \tfrac{4}{3}\lambda_2Q_{2444} - 4\lambda_4Q_{1244} + 40\wp_{34}\lambda_4Q_{2333} - \tfrac{56}{3}\wp_{23}Q_{2333} - 29\lambda_3Q_{2333} - \tfrac{4}{3}\lambda_4^2Q_{2333}$$
$$\wp_{224}\wp_{234} = 2\wp_{23}\wp_{24}^2 - 9\tfrac{5}{3}\lambda_3Q_{2333} + \tfrac{1}{3}\lambda_2Q_{2444} + 2\lambda_3\wp_{24}^2 - 84\wp_{33}\wp_{13}\lambda_4 - 2\wp_{44}\wp_{14}\lambda_3$$
$$- 42\wp_{33}\lambda_4\lambda_2 - 124\wp_{34}\wp_{33}\lambda_4\lambda_3 + 21\wp_{33}Q_{2233} + 95\wp_{33}\lambda_3^2 - 124\wp_{14}^2$$
$$+ 124\wp_{34}Q_{1244} + 126\wp_{11}\wp_{44} - 128\wp_{12}\wp_{24} + 126\wp_{14}\wp_{22} + 212\wp_{33}\lambda_1$$
$$- 42\wp_{13}Q_{2444} - 82\wp_{33}\wp_{34}\lambda_2 + 2\wp_{44}^2\lambda_1 - 21\wp_{23}Q_{2333} + \tfrac{1}{3}\wp_{34}\lambda_3Q_{2444}$$
$$- 64Q_{1224} + 2\wp_{22}\wp_{24}\wp_{34} - 2\lambda_2\wp_{24}\wp_{44} + \tfrac{124}{3}\wp_{34}\lambda_4Q_{2333} + 4\lambda_4\wp_{14}\wp_{24}$$
$$\wp_{223}\wp_{333} = -2\wp_{33}\lambda_4^2\lambda_3 - 3\wp_{33}\lambda_3^2 - 6\wp_{34}Q_{1244} + 6\wp_{34}\wp_{33}\lambda_4\lambda_3 + 8\wp_{12}\wp_{24} + \lambda_3Q_{2333}$$
$$- 2\wp_{22}^2 - 12\wp_{33}\lambda_1 + 4\wp_{33}\wp_{13}\lambda_4 + 4\wp_{33}\wp_{34}\lambda_2 + 2\wp_{33}\wp_{23}^2 + 2\wp_{22}\wp_{33}^2 + \tfrac{2}{3}\lambda_4^2Q_{2333}$$
$$+ 2\wp_{13}Q_{2444} - \tfrac{2}{3}\lambda_2Q_{2444} + 2\lambda_4Q_{1244} - 2\wp_{34}\lambda_4Q_{2333} + \tfrac{1}{3}\wp_{23}Q_{2333} + 4Q_{1224}$$
$$\wp_{223}\wp_{244} = 2\lambda_3\wp_{23}\wp_{33} + 2\wp_{33}\lambda_3^2 + 4\wp_{34}Q_{1244} + 2\lambda_3\wp_{24}^2 - 4\wp_{34}\wp_{33}\lambda_4\lambda_3 - 4\wp_{14}^2$$
$$- 4\wp_{12}\wp_{24} + 4\wp_{14}\wp_{22} - 2Q_{1224} + 8\wp_{33}\lambda_1 - 4\wp_{33}\wp_{34}\lambda_2 + 2\wp_{22}\wp_{23}\wp_{44}$$
$$+ 4\wp_{44}\wp_{14}\lambda_3 - 4\wp_{44}^2\lambda_1 + 2\wp_{23}\wp_{24}^2 - \tfrac{2}{3}\wp_{34}\lambda_3Q_{2444} + \tfrac{4}{3}\lambda_2Q_{2444}$$
$$+ \tfrac{4}{3}\wp_{34}\lambda_4Q_{2333} - \tfrac{1}{3}\wp_{23}Q_{2333} + \tfrac{2}{3}\lambda_4\wp_{23}Q_{2444} - \tfrac{2}{3}\lambda_3Q_{2333}$$
$$\wp_{222}\wp_{344} = -\tfrac{2}{3}\lambda_4\wp_{23}Q_{2444} + 4\lambda_4\wp_{22}\wp_{24} - 2\wp_{23}\wp_{24}^2 - 9\tfrac{2}{3}\lambda_3Q_{2333}$$
$$- \tfrac{2}{3}\lambda_2Q_{2444} - 4\lambda_3\wp_{24}^2 + 2\wp_{22}\wp_{23}\wp_{44} - 84\wp_{33}\wp_{13}\lambda_4 - 118\wp_{34}\wp_{33}\lambda_4\lambda_3 + 4\wp_{44}\wp_{14}\lambda_3$$
$$- 42\wp_{33}\lambda_4\lambda_2 + 21\wp_{33}Q_{2233} + 92\wp_{33}\lambda_3^2 - 124\wp_{14}^2 - 4\lambda_4\wp_{12}\wp_{44} + 118\wp_{34}Q_{1244}$$
$$+ \tfrac{118}{3}\wp_{34}\lambda_4Q_{2333} - 116\wp_{12}\wp_{24} + 128\wp_{14}\wp_{22} + 200\wp_{33}\lambda_1 - 42\wp_{13}Q_{2444}$$
$$- 2\lambda_3\wp_{23}\wp_{33} - 76\wp_{33}\wp_{34}\lambda_2 - 4\wp_{44}^2\lambda_1 - 2\wp_{22}^2 - 6\tfrac{2}{3}\wp_{23}Q_{2333} + \tfrac{4}{3}\wp_{34}\lambda_3Q_{2444}$$
$$- 58Q_{1224} + 4\wp_{22}\wp_{24}\wp_{34} + 4\lambda_2\wp_{24}\wp_{44} - 4\lambda_4\wp_{14}\wp_{24} + 120\wp_{11}\wp_{44} - 2\lambda_4Q_{1244}$$

$$
\begin{aligned}
\wp_{133}\wp_{334} &= 11\wp_{33}Q_{2233} - 22\wp_{33}\lambda_4\lambda_2 + 48\wp_{33}\lambda_3^2 - 68\wp_{34}\wp_{33}\lambda_4\lambda_3 + 4\wp_{34}\wp_{13}\wp_{33} \\
&\quad + 64\wp_{11}\wp_{44} - 68\wp_{14}^2 - 65\wp_{12}\wp_{24} + 66\wp_{14}\wp_{22} + 108\wp_{33}\lambda_1 - 44\wp_{33}\wp_{13}\lambda_4 + 68\wp_{34}Q_{1244} \\
&\quad - 44\wp_{33}\wp_{34}\lambda_2 - 6\tfrac{8}{3}\wp_{13}Q_{2444} + 6\tfrac{8}{3}\wp_{34}\lambda_4Q_{2333} - 11\wp_{23}Q_{2333} - 16\lambda_3Q_{2333} - 32Q_{1224} \\
\wp_{144}\wp_{234} &= \lambda_3\wp_{23}\wp_{33} + 21\wp_{33}Q_{2233} - 42\wp_{33}\lambda_4\lambda_2 + 96\wp_{33}\lambda_3^2 + 126\wp_{34}Q_{1244} \\
&\quad + 2\wp_{34}\wp_{14}\wp_{24} + 128\wp_{11}\wp_{44} - 124\wp_{14}^2 - 128\wp_{12}\wp_{24} + 126\wp_{14}\wp_{22} - 64Q_{1224} \\
&\quad + 212\wp_{33}\lambda_1 - 126\wp_{34}\wp_{33}\lambda_4\lambda_3 - 84\wp_{33}\wp_{13}\lambda_4 - 84\wp_{33}\wp_{34}\lambda_2 + 2\lambda_4\wp_{14}\wp_{24} \\
&\quad + 2\wp_{44}\wp_{14}\wp_{23} - \tfrac{127}{3}\wp_{13}Q_{2444} + 42\wp_{34}\lambda_4Q_{2333} - 6\tfrac{4}{3}\wp_{23}Q_{2333} - 32\lambda_3Q_{2333} \\
\wp_{134}\wp_{244} &= -\tfrac{1}{2}\lambda_3\wp_{23}\wp_{33} + 21\wp_{33}Q_{2233} - 42\wp_{33}\lambda_4\lambda_2 + 93\wp_{33}\lambda_3^2 + 126\wp_{34}Q_{1244} \\
&\quad + 2\wp_{34}\wp_{14}\wp_{24} + 124\wp_{11}\wp_{44} - 124\wp_{14}^2 - 126\wp_{12}\wp_{24} + 124\wp_{14}\wp_{22} - 62Q_{1224} \\
&\quad + 206\wp_{33}\lambda_1 - 126\wp_{34}\wp_{33}\lambda_4\lambda_3 - 84\wp_{33}\wp_{13}\lambda_4 - 84\wp_{33}\wp_{34}\lambda_2 + 2\lambda_4\wp_{14}\wp_{24} \\
&\quad + 2\wp_{24}\wp_{13}\wp_{44} - \tfrac{124}{3}\wp_{13}Q_{2444} + 42\wp_{34}\lambda_4Q_{2333} - 1\tfrac{25}{6}\wp_{23}Q_{2333} - 31\lambda_3Q_{2333} \\
\wp_{134}\wp_{333} &= \tfrac{3}{2}\lambda_3\wp_{23}\wp_{33} + 6\wp_{33}\lambda_1 + 4\wp_{11}\wp_{44} - 2\wp_{12}\wp_{24} + 3\wp_{33}\lambda_3^2 + 2\wp_{34}\wp_{13}\wp_{33} \\
&\quad + 2\wp_{14}\wp_{33}^2 - 2\wp_{34}\wp_{33}\lambda_4\lambda_3 + 4\wp_{14}^2 - \tfrac{2}{3}\wp_{13}Q_{2444} - \lambda_3Q_{2333} - \tfrac{1}{2}\wp_{23}Q_{2333} \\
&\quad + 2\wp_{34}Q_{1244} - 2Q_{1224} + \tfrac{2}{3}\wp_{34}\lambda_4Q_{2333} \\
\wp_{124}\wp_{344} &= -\tfrac{1}{2}\lambda_3\wp_{23}\wp_{33} + 2\wp_{34}\wp_{12}\wp_{44} + 6\tfrac{3}{2}\wp_{33}Q_{2233} - 63\wp_{33}\lambda_4\lambda_2 + 141\wp_{33}\lambda_3^2 \\
&\quad - 189\wp_{34}\wp_{33}\lambda_4\lambda_3 + 188\wp_{11}\wp_{44} - 188\wp_{14}^2 - 188\wp_{12}\wp_{24} + 190\wp_{14}\wp_{22} \\
&\quad + 2\wp_{34}\wp_{14}\wp_{24} - 126\wp_{33}\wp_{13}\lambda_4 - 126\wp_{33}\wp_{34}\lambda_2 - \tfrac{190}{3}\wp_{13}Q_{2444} + 63\wp_{34}\lambda_4Q_{2333} \\
&\quad - \tfrac{9}{3}\wp_{23}Q_{2333} - 47\lambda_3Q_{2333} + 190\wp_{34}Q_{1244} - 94Q_{1224} + 314\wp_{33}\lambda_1 \\
\wp_{123}\wp_{444} &= 2\wp_{44}\wp_{14}\wp_{23} - 2\lambda_4\wp_{14}\wp_{24} + 2\wp_{34}\wp_{12}\wp_{44} + \lambda_3\wp_{23}\wp_{33} + 6\wp_{44}\wp_{14}\lambda_3 \\
&\quad - 4\wp_{33}\lambda_1 - 2\wp_{44}^2\lambda_1 + 4\wp_{12}\wp_{24} - 2\wp_{11}\wp_{44} - \wp_{34}Q_{1244} + 2\wp_{24}\wp_{13}\wp_{44} - 4\wp_{14}^2 \\
&\quad - \tfrac{1}{3}\wp_{23}Q_{2333} - 2\lambda_4\wp_{12}\wp_{44} - \lambda_4Q_{1244} + 2Q_{1224} + \tfrac{2}{3}\wp_{13}Q_{2444} - 2\wp_{34}\wp_{14}\wp_{24}
\end{aligned}
$$

The following relations are at weight -17.

$$
\begin{aligned}
\wp_{224}\wp_{233} &= 8\wp_{13}\wp_{14} - 4\wp_{11}\wp_{34} + 2\wp_{12}\wp_{23} - 2\wp_{13}\wp_{22} + 2\wp_{24}\wp_{23}^2 + 4\wp_{24}\lambda_1 \\
&\quad + 6\wp_{14}\lambda_2 + 2\wp_{12}\lambda_3 - 6\wp_{44}\lambda_0 - \wp_{24}\wp_{23}\lambda_3 + 2\wp_{24}\wp_{22}\wp_{33} - \wp_{22}\lambda_3\wp_{34} \\
&\quad + 4\wp_{44}\wp_{34}\lambda_1 - 2\wp_{44}\lambda_4\lambda_1 + \tfrac{2}{3}\wp_{33}\lambda_3Q_{2444} + \wp_{24}Q_{2233} - \tfrac{4}{3}\lambda_4\wp_{33}Q_{2333} \\
&\quad - 4\wp_{33}Q_{1244} - \wp_{22}\lambda_2 + 4\wp_{33}^2\lambda_2 - 2\lambda_3\wp_{14}\lambda_4 - 4\lambda_3\wp_{13}\wp_{44} + 4\wp_{33}^2\lambda_4\lambda_3 \\
\wp_{223}\wp_{234} &= -4\wp_{13}\wp_{14} + 2\wp_{11}\wp_{34} - 4\wp_{12}\wp_{23} + 2\wp_{13}\wp_{22} + 2\wp_{24}\wp_{23}^2 - 4\wp_{14}\lambda_2 \\
&\quad + 2\wp_{11}\lambda_4 - 4\wp_{12}\lambda_3 + 2\wp_{24}\wp_{23}\lambda_3 + 2\wp_{22}\lambda_3\wp_{34} - 2\wp_{44}\wp_{34}\lambda_1 - 2\wp_{44}\lambda_4\lambda_1 \\
&\quad - \tfrac{1}{3}\wp_{33}\lambda_3Q_{2444} + \tfrac{2}{3}\lambda_4\wp_{33}Q_{2333} + 2\wp_{33}Q_{1244} + 2\wp_{22}\lambda_2 - 2\wp_{33}^2\lambda_2 + 4\lambda_3\wp_{14}\lambda_4 \\
&\quad + 2\lambda_3\wp_{13}\wp_{44} + 4\wp_{23}\wp_{14}\lambda_4 - 2\wp_{23}\wp_{44}\lambda_2 + 2\wp_{23}\wp_{22}\wp_{34} - 2\wp_{33}^2\lambda_4\lambda_3
\end{aligned}
$$

$$\begin{aligned}
\wp_{222}\wp_{334} &= 2\wp_{24}\wp_{23}\lambda_3 + 4\wp_{24}\wp_{34}\lambda_2 + 2\lambda_3\wp_{13}\wp_{44} - 4\lambda_4\wp_{13}\wp_{24} - \tfrac{4}{3}\lambda_4\wp_{33}Q_{2333} \\
&+ 4\wp_{33}^2\lambda_4\lambda_3 + 2\wp_{22}\lambda_3\wp_{34} + 2\wp_{24}\wp_{22}\wp_{33} + \tfrac{2}{3}\wp_{33}\lambda_3 Q_{2444} - 4\lambda_4\wp_{12}\wp_{34} \\
&- 8\wp_{44}\wp_{34}\lambda_1 + 6\lambda_3\wp_{14}\wp_{34} - 4\wp_{11}\wp_{34} - 4\wp_{13}\wp_{14} + 4\wp_{12}\wp_{23} + 4\wp_{13}\wp_{22} \\
&- 2\wp_{24}\wp_{23}^2 - 4\wp_{24}\lambda_1 + 8\wp_{14}\lambda_2 + 2\wp_{12}\lambda_3 - 12\wp_{44}\lambda_0 - \wp_{24}Q_{2233} - 4\wp_{33}Q_{1244} \\
&+ 4\wp_{33}^2\lambda_2 - 2\lambda_3\wp_{14}\lambda_4 + 4\wp_{23}\wp_{22}\wp_{34} + 2\wp_{24}\lambda_4\lambda_2 \\
\wp_{144}\wp_{233} &= \tfrac{8}{3}\wp_{23}\wp_{14}\wp_{34} + 2\wp_{13}\wp_{14} - \tfrac{2}{3}\wp_{11}\wp_{34} - \tfrac{1}{3}\wp_{13}\wp_{22} + \tfrac{4}{3}\wp_{24}\lambda_1 + \tfrac{4}{3}\wp_{14}\lambda_2 \\
&- 2\wp_{44}\lambda_0 - \wp_{24}\wp_{23}\lambda_3 - \tfrac{4}{3}\wp_{24}\wp_{34}\lambda_2 + \tfrac{4}{3}\wp_{24}\wp_{14}\wp_{33} + \tfrac{8}{3}\wp_{44}\wp_{34}\lambda_1 - \tfrac{2}{3}\wp_{12}\wp_{34}^2 \\
&+ \tfrac{1}{3}\wp_{24}Q_{2233} + \tfrac{2}{3}\wp_{33}Q_{1244} - \tfrac{2}{3}\lambda_4\wp_{13}\wp_{24} - \tfrac{4}{3}\wp_{34}\wp_{13}\wp_{24} + 2\lambda_3\wp_{14}\wp_{34} \\
&- \tfrac{2}{3}\wp_{24}\lambda_4\lambda_2 - \tfrac{2}{3}\lambda_4\wp_{12}\wp_{34} + \tfrac{2}{3}\wp_{23}\wp_{14}\lambda_4 + \tfrac{4}{3}\wp_{23}\wp_{13}\wp_{44} + \tfrac{2}{3}\wp_{44}\wp_{12}\wp_{33} \\
\wp_{134}\wp_{234} &= \tfrac{5}{3}\wp_{23}\wp_{14}\wp_{34} + 2\wp_{13}\wp_{14} - \tfrac{2}{3}\wp_{11}\wp_{34} - \tfrac{1}{3}\wp_{13}\wp_{22} - \tfrac{2}{3}\wp_{24}\lambda_1 + \tfrac{4}{3}\wp_{14}\lambda_2 \\
&- 2\wp_{44}\lambda_0 + \tfrac{1}{2}\wp_{24}\wp_{23}\lambda_3 + \tfrac{2}{3}\wp_{24}\wp_{34}\lambda_2 + \tfrac{1}{3}\wp_{24}\wp_{14}\wp_{33} - \tfrac{4}{3}\wp_{44}\wp_{34}\lambda_1 + \tfrac{1}{3}\wp_{12}\wp_{34}^2 \\
&- \tfrac{1}{6}\wp_{24}Q_{2233} - \tfrac{1}{3}\wp_{33}Q_{1244} + \tfrac{4}{3}\lambda_4\wp_{13}\wp_{24} + \tfrac{5}{3}\wp_{34}\wp_{13}\wp_{24} + 2\lambda_3\wp_{14}\wp_{34} \\
&+ \tfrac{1}{3}\wp_{24}\lambda_4\lambda_2 + \tfrac{1}{3}\lambda_4\wp_{12}\wp_{34} - \tfrac{1}{3}\wp_{23}\wp_{14}\lambda_4 + \tfrac{1}{3}\wp_{23}\wp_{13}\wp_{44} - \tfrac{1}{3}\wp_{44}\wp_{12}\wp_{33} \\
\wp_{133}\wp_{244} &= -\tfrac{4}{3}\wp_{23}\wp_{14}\wp_{34} + 2\wp_{13}\wp_{14} + \tfrac{10}{3}\wp_{11}\wp_{34} - \wp_{12}\wp_{23} + \tfrac{2}{3}\wp_{13}\wp_{22} - \tfrac{2}{3}\wp_{24}\lambda_1 \\
&+ \tfrac{10}{3}\wp_{14}\lambda_2 - 2\wp_{44}\lambda_0 + 2\wp_{24}\wp_{23}\lambda_3 + \tfrac{8}{3}\wp_{24}\wp_{34}\lambda_2 + \tfrac{4}{3}\wp_{24}\wp_{14}\wp_{33} + \tfrac{8}{3}\wp_{44}\wp_{34}\lambda_1 \\
&- \tfrac{2}{3}\wp_{24}Q_{2233} + \tfrac{2}{3}\wp_{33}Q_{1244} + \tfrac{4}{3}\lambda_4\wp_{13}\wp_{24} + \tfrac{4}{3}\wp_{23}\wp_{13}\wp_{44} + \tfrac{8}{3}\wp_{34}\wp_{13}\wp_{24} + \tfrac{2}{3}\wp_{44}\wp_{12}\wp_{33} \\
&+ \tfrac{4}{3}\wp_{24}\lambda_4\lambda_2 - \tfrac{2}{3}\lambda_4\wp_{12}\wp_{34} + \tfrac{8}{3}\wp_{23}\wp_{14}\lambda_4 - 4\lambda_3\wp_{14}\wp_{34} - \tfrac{2}{3}\wp_{12}\wp_{34}^2 \\
\wp_{133}\wp_{333} &= 6\wp_{13}\wp_{14} - 2\wp_{11}\wp_{34} + \wp_{12}\wp_{23} - 2\wp_{13}\wp_{22} - 2\wp_{24}\lambda_1 + 2\wp_{14}\lambda_2 \\
&+ 6\wp_{44}\lambda_0 + 4\wp_{13}\wp_{33}^2 - 2\wp_{33}^2\lambda_4\lambda_3 + \tfrac{2}{3}\lambda_4\wp_{33}Q_{2333} + 2\wp_{33}Q_{1244} \\
\wp_{124}\wp_{334} &= \tfrac{2}{3}\wp_{23}\wp_{14}\wp_{34} - 2\wp_{13}\wp_{14} - \tfrac{2}{3}\wp_{11}\wp_{34} + \tfrac{2}{3}\wp_{13}\wp_{22} - \tfrac{2}{3}\wp_{24}\lambda_1 + \tfrac{4}{3}\wp_{14}\lambda_2 \\
&- 2\wp_{44}\lambda_0 + \tfrac{1}{2}\wp_{24}\wp_{23}\lambda_3 + \tfrac{2}{3}\wp_{24}\wp_{34}\lambda_2 + \tfrac{4}{3}\wp_{24}\wp_{14}\wp_{33} - \tfrac{4}{3}\wp_{44}\wp_{34}\lambda_1 + \tfrac{4}{3}\wp_{12}\wp_{34}^2 \\
&+ \tfrac{2}{3}\wp_{44}\wp_{12}\wp_{33} - \tfrac{1}{6}\wp_{24}Q_{2233} + \tfrac{2}{3}\wp_{33}Q_{1244} - \tfrac{2}{3}\lambda_4\wp_{13}\wp_{24} + \tfrac{2}{3}\wp_{34}\wp_{13}\wp_{24} \\
&+ 2\lambda_3\wp_{14}\wp_{34} + \tfrac{1}{3}\wp_{24}\lambda_4\lambda_2 - \tfrac{2}{3}\lambda_4\wp_{12}\wp_{34} + \tfrac{2}{3}\wp_{23}\wp_{14}\lambda_4 - \tfrac{2}{3}\wp_{23}\wp_{13}\wp_{44} \\
\wp_{123}\wp_{344} &= \tfrac{2}{3}\wp_{23}\wp_{14}\wp_{34} + 2\wp_{13}\wp_{14} - \tfrac{8}{3}\wp_{11}\wp_{34} - \tfrac{4}{3}\wp_{13}\wp_{22} + \tfrac{4}{3}\wp_{24}\lambda_1 - \tfrac{2}{3}\wp_{14}\lambda_2 \\
&- 2\wp_{44}\lambda_0 - \wp_{24}\wp_{23}\lambda_3 - \tfrac{4}{3}\wp_{24}\wp_{34}\lambda_2 - \tfrac{2}{3}\wp_{24}\wp_{14}\wp_{33} + \tfrac{2}{3}\wp_{44}\wp_{34}\lambda_1 + \tfrac{4}{3}\wp_{12}\wp_{34}^2 \\
&+ \tfrac{2}{3}\wp_{44}\wp_{12}\wp_{33} + \tfrac{1}{3}\wp_{24}Q_{2233} - \tfrac{1}{3}\wp_{33}Q_{1244} + \tfrac{4}{3}\lambda_4\wp_{13}\wp_{24} + \tfrac{2}{3}\wp_{34}\wp_{13}\wp_{24} \\
&+ 2\lambda_3\wp_{14}\wp_{34} - \tfrac{2}{3}\wp_{24}\lambda_4\lambda_2 + \tfrac{4}{3}\lambda_4\wp_{12}\wp_{34} - \tfrac{4}{3}\wp_{23}\wp_{14}\lambda_4 + \tfrac{4}{3}\wp_{23}\wp_{13}\wp_{44}
\end{aligned}$$

The following relations are at weight -18.

$$\wp_{144}^2 = 4\wp_{34}\lambda_0 + \wp_{13}^2 + 2\wp_{14}\wp_{33}\lambda_3 + 4\wp_{44}\wp_{14}^2 - \tfrac{2}{3}\wp_{14}Q_{2333}$$

$$\begin{aligned}
\wp_{224}^2 &= -\tfrac{4}{3}\lambda_4\wp_{13}\wp_{23} + \tfrac{10}{3}\wp_{34}\lambda_4\lambda_1 - \tfrac{4}{3}\lambda_4\lambda_3\lambda_2 + \tfrac{8}{3}\wp_{23}\wp_{34}\lambda_2 + \tfrac{2}{3}\lambda_3 Q_{2233} \\
&- \wp_{44}\wp_{33}\lambda_3^2 + \tfrac{8}{3}\wp_{34}\lambda_4^2\wp_{13} + 4\lambda_4\wp_{24}\wp_{33}\lambda_3 - \tfrac{4}{3}\wp_{24}\lambda_4 Q_{2333} + \tfrac{1}{3}\lambda_3\wp_{44}Q_{2333} \\
&+ 2\wp_{14}\wp_{33}\lambda_3 + \tfrac{4}{3}\wp_{14}\lambda_4 Q_{2444} + 4\lambda_2\wp_{24}\wp_{33} - 6\wp_{33}\wp_{44}\lambda_1 - 2\lambda_3\wp_{13}\wp_{34} - 2\lambda_4\lambda_0 \\
&+ 2\lambda_4\wp_{34}\wp_{23}\lambda_3 - \tfrac{2}{3}\lambda_3\wp_{34}\lambda_2 - \tfrac{2}{3}\wp_{34}\lambda_4 Q_{2233} + \tfrac{4}{3}\wp_{34}\lambda_4^2\lambda_2 - \tfrac{2}{3}\wp_{23}\lambda_4\lambda_2 - \tfrac{4}{3}\lambda_3\lambda_1 \\
&- \wp_{23}^2\lambda_3 - \tfrac{2}{3}\wp_{12}Q_{2444} - 2\wp_{34}^2\lambda_1 + \wp_{34}^2\lambda_3^2 + \lambda_2^2 + 2\wp_{13}^2 - 2\wp_{23}\lambda_3^2 + 4\wp_{22}\wp_{24}^2 \\
&+ 2\wp_{44}Q_{1224} - 6\wp_{24}Q_{1244} - 2\wp_{11}\wp_{33} + \tfrac{1}{3}\wp_{23}Q_{2233} + 2\wp_{13}\lambda_2 + 6\wp_{34}\lambda_0 - \tfrac{8}{3}\wp_{23}\lambda_1 \\
&- \tfrac{1}{3}\wp_{22}Q_{2333} - \tfrac{8}{3}\lambda_3\wp_{13}\lambda_4 - \tfrac{4}{3}\lambda_4\wp_{34}^2\lambda_2 + \tfrac{4}{3}\wp_{24}\lambda_3 Q_{2444} + \wp_{22}\wp_{33}\lambda_3 - \tfrac{4}{3}\wp_{44}\lambda_2 Q_{2444} \\
\wp_{223}\wp_{233} &= 2\lambda_4^2\lambda_1 - 2\lambda_4\lambda_3\lambda_2 + 2\wp_{22}\wp_{23}\wp_{33} + \wp_{22}\wp_{33}\lambda_3 - 2\wp_{11}\wp_{33} - 2\lambda_3\wp_{13}\lambda_4 \\
&+ \lambda_3 Q_{2233} + 4\wp_{13}\lambda_2 - 2\wp_{33}\wp_{44}\lambda_1 + 4\wp_{34}\lambda_4\lambda_1 - 4\lambda_3\lambda_1 - 4\wp_{23}\lambda_1 + 2\lambda_2^2 + 2\wp_{14}\wp_{33}\lambda_3 \\
&+ 2\wp_{34}^2\lambda_1 - 3\wp_{23}\lambda_3^2 + 2\wp_{23}^3 + \wp_{23}Q_{2233} - \wp_{23}^2\lambda_3 - 2\lambda_3\wp_{34}\lambda_2 - 2\lambda_3\wp_{13}\wp_{34} + 2\wp_{13}^2 \\
\wp_{222}\wp_{333} &= -6\wp_{34}\lambda_4\lambda_1 + 4\lambda_4\lambda_3\lambda_2 - 2\lambda_3 Q_{2233} - 4\lambda_4\wp_{12}\wp_{33} - 6\lambda_4\wp_{24}\wp_{33}\lambda_3 \\
&+ 2\wp_{24}\lambda_4 Q_{2333} + 6\wp_{22}\wp_{23}\wp_{33} + 2\wp_{14}\wp_{33}\lambda_3 - 4\lambda_2\wp_{24}\wp_{33} - 2\wp_{33}\wp_{44}\lambda_1 \\
&+ 10\lambda_3\wp_{13}\wp_{34} + 4\lambda_3\wp_{34}\lambda_2 + 4\wp_{23}\lambda_4\lambda_2 + 5\wp_{23}^2\lambda_3 - 2\wp_{12}Q_{2444} - 10\wp_{34}^2\lambda_1 \\
&- 10\wp_{13}^2 + 6\wp_{23}\lambda_3^2 + 6\wp_{24}Q_{1244} - 2\wp_{11}\wp_{33} - 2\wp_{23}Q_{2233} - 6\wp_{13}\lambda_2 - 6\wp_{34}\lambda_0 \\
&+ \wp_{22}Q_{2333} - 2\wp_{23}^3 + 4\lambda_3\wp_{13}\lambda_4 + \wp_{22}\wp_{33}\lambda_3 + 6\lambda_4\lambda_0 - 4\lambda_3\lambda_1 \\
\wp_{222}\wp_{244} &= \tfrac{8}{3}\lambda_4\wp_{13}\wp_{23} + \tfrac{20}{3}\wp_{34}\lambda_4\lambda_1 + \tfrac{8}{3}\lambda_4\lambda_3\lambda_2 - \tfrac{16}{3}\wp_{23}\wp_{34}\lambda_2 + 2\wp_{22}^2\wp_{44} \\
&+ 4\lambda_4^2\wp_{44}\wp_{33}\lambda_3 - \tfrac{4}{3}\lambda_3 Q_{2233} + \tfrac{2}{3}\lambda_4^2 Q_{2233} + 2\wp_{44}\wp_{33}\lambda_3^2 - \tfrac{8}{3}\wp_{34}\lambda_4^2\wp_{13} \\
&- \tfrac{2}{3}\wp_{24}\lambda_4 Q_{2333} - \tfrac{2}{3}\lambda_3\wp_{44}Q_{2333} - 6\wp_{14}\wp_{33}\lambda_3 - \tfrac{4}{3}\wp_{14}\lambda_4 Q_{2444} + 4\lambda_2\wp_{24}\wp_{33} \\
&+ 14\wp_{33}\wp_{44}\lambda_1 + \tfrac{2}{3}\lambda_4^2\lambda_1 + 6\lambda_3\wp_{13}\wp_{34} - \tfrac{8}{3}\lambda_3\wp_{34}\lambda_2 + \tfrac{2}{3}\wp_{34}\lambda_4 Q_{2233} - \tfrac{2}{3}\wp_{23}\lambda_4\lambda_2 \\
&+ \wp_{23}^2\lambda_3 + \tfrac{4}{3}\wp_{12}Q_{2444} + 2\wp_{34}^2\lambda_1 - 2\wp_{34}^2\lambda_3^2 - 2\lambda_2^2 - 6\wp_{13}^2 + 4\wp_{23}\lambda_3^2 + 2\wp_{22}\wp_{24}^2 \\
&- 4\wp_{44}Q_{1224} + 6\wp_{11}\wp_{33} - \tfrac{2}{3}\wp_{23}Q_{2233} - 8\wp_{13}\lambda_2 - \tfrac{8}{3}\wp_{23}\lambda_1 - \tfrac{4}{3}\lambda_4^3\lambda_2 \\
&- \tfrac{4}{3}\lambda_4^2\wp_{44}Q_{2333} + \tfrac{1}{3}\wp_{22}Q_{2333} - 2\lambda_4^2\wp_{23}\lambda_3 + 4\lambda_4\wp_{33}\wp_{44}\lambda_2 - \tfrac{8}{3}\lambda_4^3\wp_{13} \\
&+ \tfrac{22}{3}\lambda_3\wp_{13}\lambda_4 + \tfrac{4}{3}\lambda_4\wp_{34}^2\lambda_2 - \tfrac{2}{3}\wp_{24}\lambda_3 Q_{2444} + \wp_{22}\wp_{33}\lambda_3 + \tfrac{8}{3}\wp_{44}\lambda_2 Q_{2444} \\
&- 4\lambda_4\lambda_0 - \tfrac{4}{3}\lambda_3\lambda_1 + \tfrac{2}{3}\lambda_4\wp_{22}Q_{2444} - 4\lambda_4\wp_{44}Q_{1244} \\
\wp_{134}\wp_{233} &= \tfrac{2}{3}\lambda_4\wp_{13}\wp_{23} + \wp_{34}\lambda_4\lambda_1 + \tfrac{2}{3}\wp_{23}\wp_{34}\lambda_2 - \lambda_4\wp_{24}\wp_{33}\lambda_3 + \tfrac{1}{3}\wp_{24}\lambda_4 Q_{2333} \\
&- \wp_{14}\wp_{33}\lambda_3 - \wp_{33}\wp_{44}\lambda_1 + \lambda_3\wp_{13}\wp_{34} + \tfrac{1}{3}\wp_{23}\lambda_4\lambda_2 + \tfrac{1}{2}\wp_{23}^2\lambda_3 - \tfrac{1}{3}\wp_{12}Q_{2444} \\
&+ \wp_{13}^2 + \wp_{24}Q_{1244} - \wp_{11}\wp_{33} - \tfrac{1}{6}\wp_{23}Q_{2233} + \wp_{13}\lambda_2 + \wp_{34}\lambda_0 - \tfrac{2}{3}\wp_{23}\lambda_1 + \lambda_4\lambda_0 \\
&- \tfrac{1}{3}\wp_{22}Q_{2333} + \tfrac{2}{3}\wp_{14}Q_{2333} + 2\wp_{23}\wp_{14}\wp_{33} + 2\wp_{23}\wp_{13}\wp_{34} + \wp_{22}\wp_{33}\lambda_3 + \wp_{34}^2\lambda_1 \\
\wp_{144}\wp_{224} &= \tfrac{1}{2}\wp_{22}\wp_{33}\lambda_3 + 2\wp_{22}\wp_{14}\wp_{44} - \wp_{11}\wp_{33} + \wp_{13}^2 - \tfrac{1}{6}\wp_{22}Q_{2333} + 5\wp_{34}\lambda_0 \\
&- \wp_{34}^2\lambda_1 - \tfrac{1}{6}\wp_{23}Q_{2233} + \tfrac{1}{2}\wp_{23}^2\lambda_3 + \lambda_4\wp_{24}\wp_{33}\lambda_3 - \wp_{24}Q_{1244} + \tfrac{2}{3}\lambda_4\wp_{13}\wp_{23} \\
&+ 2\wp_{14}\wp_{33}\lambda_3 - \tfrac{1}{3}\wp_{14}Q_{2333} - \tfrac{1}{3}\wp_{12}Q_{2444} + \tfrac{2}{3}\wp_{14}\lambda_4 Q_{2444} - \tfrac{1}{3}\wp_{24}\lambda_4 Q_{2333} \\
&+ \wp_{33}\wp_{44}\lambda_1 + \wp_{34}\lambda_4\lambda_1 - \lambda_4\lambda_0 - \tfrac{2}{3}\wp_{23}\lambda_1 + 2\wp_{14}\wp_{24}^2 + \tfrac{1}{3}\wp_{23}\lambda_4\lambda_2 + \tfrac{2}{3}\wp_{23}\wp_{34}\lambda_2
\end{aligned}$$

$$
\begin{aligned}
\wp_{133}\wp_{234} &= \tfrac{2}{3}\lambda_4\wp_{13}\wp_{23} + \wp_{34}\lambda_4\lambda_1 + \tfrac{2}{3}\wp_{23}\wp_{34}\lambda_2 + 2\wp_{33}\wp_{13}\wp_{24} - \lambda_4\wp_{24}\wp_{33}\lambda_3 \\
&\quad + \tfrac{1}{3}\wp_{24}\lambda_4 Q_{2333} + 2\wp_{14}\wp_{33}\lambda_3 - \wp_{33}\wp_{44}\lambda_1 + \lambda_3\wp_{13}\wp_{34} + \tfrac{1}{3}\wp_{23}\lambda_4\lambda_2 + \tfrac{1}{2}\wp_{23}^2\lambda_3 \\
&\quad + \tfrac{2}{3}\wp_{12}Q_{2444} + \wp_{34}^2\lambda_1 + \wp_{13}^2 + \wp_{24}Q_{1244} + \wp_{11}\wp_{33} - \tfrac{1}{6}\wp_{23}Q_{2233} + \wp_{13}\lambda_2 \\
&\quad - \tfrac{2}{3}\wp_{23}\lambda_1 + \tfrac{1}{6}\wp_{22}Q_{2333} - \tfrac{1}{3}\wp_{14}Q_{2333} + 2\wp_{23}\wp_{13}\wp_{34} - \tfrac{1}{2}\wp_{22}\wp_{33}\lambda_3 + \lambda_4\lambda_0 + \wp_{34}\lambda_0 \\
\wp_{124}\wp_{333} &= -\tfrac{4}{3}\lambda_4\wp_{13}\wp_{23} - \wp_{34}\lambda_4\lambda_1 - \tfrac{4}{3}\wp_{23}\wp_{34}\lambda_2 + 2\wp_{33}\wp_{13}\wp_{24} - \lambda_4\wp_{24}\wp_{33}\lambda_3 \\
&\quad + \tfrac{1}{3}\wp_{24}\lambda_4 Q_{2333} - \wp_{14}\wp_{33}\lambda_3 - \wp_{33}\wp_{44}\lambda_1 + \lambda_3\wp_{13}\wp_{34} - \tfrac{2}{3}\wp_{23}\lambda_4\lambda_2 - \wp_{23}^2\lambda_3 \\
&\quad - \tfrac{1}{3}\wp_{12}Q_{2444} - 3\wp_{34}^2\lambda_1 - 3\wp_{13}^2 + \wp_{24}Q_{1244} - \wp_{11}\wp_{33} + \tfrac{1}{3}\wp_{23}Q_{2233} - \wp_{13}\lambda_2 \\
&\quad - 5\wp_{34}\lambda_0 + \tfrac{4}{3}\wp_{23}\lambda_1 + \tfrac{1}{6}\wp_{22}Q_{2333} + 2\wp_{33}\wp_{12}\wp_{34} + \tfrac{2}{3}\wp_{14}Q_{2333} + 2\wp_{23}\wp_{14}\wp_{33} \\
&\quad - 2\wp_{23}\wp_{13}\wp_{34} - \tfrac{1}{2}\wp_{22}\wp_{33}\lambda_3 + \lambda_4\lambda_0 \\
\wp_{123}\wp_{334} &= \tfrac{2}{3}\lambda_4\wp_{13}\wp_{23} + \wp_{34}\lambda_4\lambda_1 + \tfrac{2}{3}\wp_{23}\wp_{34}\lambda_2 + 2\lambda_4\wp_{24}\wp_{33}\lambda_3 - \tfrac{2}{3}\wp_{24}\lambda_4 Q_{2333} \\
&\quad + 2\wp_{14}\wp_{33}\lambda_3 + \wp_{33}\wp_{44}\lambda_1 + \lambda_3\wp_{13}\wp_{34} + \tfrac{1}{3}\wp_{23}\lambda_4\lambda_2 + \tfrac{1}{2}\wp_{23}^2\lambda_3 - \tfrac{1}{3}\wp_{12}Q_{2444} \\
&\quad + \wp_{13}^2 - 2\wp_{24}Q_{1244} - \wp_{11}\wp_{33} - \tfrac{1}{6}\wp_{23}Q_{2233} + \wp_{13}\lambda_2 + \wp_{34}\lambda_0 - \tfrac{2}{3}\wp_{23}\lambda_1 + \lambda_4\lambda_0 \\
&\quad + \tfrac{1}{6}\wp_{22}Q_{2333} + 2\wp_{33}\wp_{12}\wp_{34} - \tfrac{1}{3}\wp_{14}Q_{2333} + 2\wp_{23}\wp_{13}\wp_{34} - \tfrac{1}{2}\wp_{22}\wp_{33}\lambda_3 + \wp_{34}^2\lambda_1 \\
\wp_{122}\wp_{444} &= 4\wp_{24}\wp_{12}\wp_{44} - 4\lambda_4\wp_{13}\wp_{23} + \tfrac{10}{3}\wp_{34}\lambda_4\lambda_1 - 2\lambda_4\lambda_3\lambda_2 - 2\wp_{23}\wp_{34}\lambda_2 \\
&\quad - 2\lambda_4^2\wp_{44}\wp_{33}\lambda_3 - \tfrac{1}{3}\lambda_4^2 Q_{2233} + 2\lambda_4\wp_{12}\wp_{33} + \tfrac{8}{3}\wp_{34}\lambda_4^2\wp_{13} - 2\lambda_4\wp_{24}\wp_{33}\lambda_3 \\
&\quad - \lambda_4\wp_{34}\wp_{23}\lambda_3 - 4\lambda_3\wp_{34}\lambda_2 + \tfrac{1}{3}\wp_{34}\lambda_4 Q_{2233} + \tfrac{2}{3}\wp_{34}\lambda_4^2\lambda_2 - \wp_{23}\lambda_4\lambda_2 - \tfrac{3}{2}\wp_{23}^2\lambda_3 \\
&\quad + \tfrac{1}{3}\wp_{12}Q_{2444} + 5\wp_{34}^2\lambda_1 + 3\wp_{13}^2 - 3\wp_{23}\lambda_3^2 + 2\wp_{44}Q_{1224} + \wp_{24}Q_{1244} - 3\wp_{11}\wp_{33} \\
&\quad + \tfrac{1}{2}\wp_{23}Q_{2233} + \wp_{13}\lambda_2 - 2\wp_{23}\lambda_1 + \tfrac{2}{3}\lambda_4^3\lambda_2 + 2\lambda_4\wp_{44}Q_{1244} + \tfrac{2}{3}\lambda_4^2\wp_{44}Q_{2333} \\
&\quad - \tfrac{5}{6}\wp_{22}Q_{2333} + \lambda_4^2\wp_{23}\lambda_3 - 4\wp_{33}\wp_{14}\lambda_4^2 + \tfrac{1}{3}\wp_{14}Q_{2333} + 2\wp_{22}\wp_{14}\wp_{44} + \tfrac{4}{3}\lambda_4^3\wp_{13} \\
&\quad - 2\wp_{14}\wp_{24}^2 - 5\lambda_3\wp_{13}\lambda_4 - \tfrac{4}{3}\lambda_4\wp_{34}^2\lambda_2 + \tfrac{5}{2}\wp_{22}\wp_{33}\lambda_3 + 4\wp_{34}\lambda_0 + \lambda_3 Q_{2233} \\
&\quad + \tfrac{2}{3}\wp_{24}\lambda_4 Q_{2333} + 2\wp_{14}\wp_{33}\lambda_3 - \tfrac{4}{3}\wp_{14}\lambda_4 Q_{2444} - 5\wp_{33}\wp_{44}\lambda_1 - \tfrac{1}{3}\lambda_4^2\lambda_1 - 4\lambda_3\wp_{13}\wp_{34} \\
\wp_{124}\wp_{244} &= -\wp_{13}\lambda_2 - \wp_{34}^2\lambda_1 - \lambda_4\lambda_0 + \wp_{34}\lambda_0 + \wp_{11}\wp_{33} - \wp_{13}^2 + 2\wp_{14}\wp_{24}^2 \\
&\quad + \wp_{34}\lambda_4\lambda_1 + \wp_{33}\wp_{44}\lambda_1 - \tfrac{1}{2}\wp_{22}\wp_{33}\lambda_3 + \lambda_3\wp_{13}\wp_{34} + \tfrac{1}{3}\wp_{12}Q_{2444} + \tfrac{1}{6}\wp_{22}Q_{2333} \\
&\quad + 2\wp_{24}\wp_{12}\wp_{44} + \wp_{24}Q_{1244} - \wp_{14}\wp_{33}\lambda_3 \\
\wp_{114}\wp_{444} &= -\tfrac{1}{2}\lambda_3\wp_{44}Q_{2333} + \tfrac{1}{2}\wp_{22}\wp_{33}\lambda_3 - \wp_{11}\wp_{33} - \wp_{13}^2 - \tfrac{1}{6}\wp_{22}Q_{2333} - \wp_{13}\lambda_2 \\
&\quad + 3\wp_{33}\wp_{44}\lambda_1 + \tfrac{2}{3}\wp_{23}\lambda_1 - \tfrac{1}{3}\wp_{23}\lambda_4\lambda_2 - \tfrac{2}{3}\wp_{23}\wp_{34}\lambda_2 + \tfrac{1}{6}\wp_{23}Q_{2233} - \tfrac{1}{2}\wp_{23}^2\lambda_3 \\
&\quad - 2\wp_{14}\wp_{33}\lambda_3 + 2\wp_{44}\wp_{14}^2 + \tfrac{3}{2}\wp_{44}\wp_{33}\lambda_3^2 - \lambda_4\wp_{24}\wp_{33}\lambda_3 + \wp_{24}Q_{1244} + \lambda_3\wp_{13}\wp_{34} \\
&\quad - \tfrac{2}{3}\lambda_4\wp_{13}\wp_{23} - \wp_{44}Q_{1224} + 2\wp_{11}\wp_{44}^2 + \tfrac{1}{3}\wp_{14}Q_{2333} - \tfrac{1}{3}\wp_{12}Q_{2444} + \tfrac{1}{3}\wp_{24}\lambda_4 Q_{2333}
\end{aligned}
$$

The following relations are at weight -19.

$$
\begin{aligned}
\wp_{134}\wp_{144} &= 2\wp_{14}\wp_{13}\wp_{44} - 2\wp_{12}\wp_{14} - 2\wp_{33}\lambda_0 + \tfrac{1}{2}\wp_{33}\wp_{13}\lambda_3 + 2\wp_{14}^2\wp_{34} \\
&\quad - \tfrac{1}{6}\wp_{13}Q_{2333} + 2\wp_{14}^2\lambda_4
\end{aligned}
$$

$$
\begin{aligned}
\wp_{134}\wp_{224} &= \tfrac{187}{2}\wp_{33}\lambda_4\lambda_3^2 - \tfrac{10}{3}\wp_{12}\wp_{14} + \tfrac{2}{9}\lambda_1 Q_{2444} + \tfrac{1}{18}\lambda_2 Q_{2333} + \tfrac{376}{3}\lambda_4\wp_{14}\wp_{22} \\
&\quad - \tfrac{187}{3}\lambda_4 Q_{1224} - 18\tfrac{7}{6}\lambda_4\lambda_3 Q_{2333} + 126\wp_{34}\lambda_4 Q_{1244} + \tfrac{374}{3}\lambda_4\wp_{11}\wp_{44} + 2\wp_{44}^2\lambda_0 \\
&\quad + \tfrac{2}{3}\wp_{22}\wp_{13}\wp_{44} + \tfrac{2}{3}\wp_{23}\wp_{14}\wp_{24} + \tfrac{2}{3}\wp_{34}\wp_{12}\wp_{24} + \tfrac{4}{3}\wp_{33}\wp_{13}\lambda_3 - \tfrac{2}{9}\wp_{13}Q_{2333} \\
&\quad + \tfrac{4}{3}\wp_{13}\wp_{24}^2 - 36\tfrac{8}{3}\wp_{14}^2\lambda_4 - \tfrac{374}{9}\wp_{13}\lambda_4 Q_{2444} + 21\lambda_4\wp_{33}Q_{2233} - 126\wp_{34}\wp_{33}\lambda_4^2\lambda_3 \\
&\quad + 2\lambda_3\wp_{14}\wp_{24} - 21\lambda_4\wp_{23}Q_{2333} - 42\wp_{33}\lambda_4^2\lambda_2 - \tfrac{1}{6}\wp_{33}\lambda_3\lambda_2 - \tfrac{376}{3}\lambda_4\wp_{12}\wp_{24} \\
&\quad - 84\lambda_4\wp_{33}\wp_{34}\lambda_2 - 84\wp_{33}\wp_{13}\lambda_4^2 + 208\wp_{33}\lambda_4\lambda_1 + \tfrac{1}{3}\wp_{34}Q_{1224} - \tfrac{2}{3}\wp_{23}Q_{1244} \\
&\quad + \tfrac{4}{3}\wp_{34}\wp_{14}\wp_{22} - \tfrac{8}{3}\wp_{33}\lambda_0 - \tfrac{2}{3}\wp_{11}\wp_{24} + 42\wp_{34}\lambda_4^2 Q_{2333} - \tfrac{2}{3}\wp_{23}\wp_{12}\wp_{44} - 2\lambda_2\wp_{14}\wp_{44} \\
\wp_{222}\wp_{234} &= \wp_{33}\lambda_4\lambda_3^2 + 8\wp_{12}\wp_{14} - 2\lambda_2\wp_{24}^2 + \tfrac{4}{3}\lambda_1 Q_{2444} - \tfrac{1}{3}\lambda_2 Q_{2333} - \tfrac{1}{3}\lambda_4\lambda_3 Q_{2333} \\
&\quad - 2\lambda_3 Q_{1244} - 2\lambda_2\wp_{23}\wp_{33} - 8\wp_{34}\lambda_4 Q_{1244} + 8\wp_{34}\wp_{33}\lambda_4^2\lambda_3 + 2\lambda_3\wp_{14}\wp_{24} \\
&\quad - 2\wp_{12}\wp_{22} + \tfrac{2}{3}\lambda_4\wp_{23}Q_{2333} + 2\lambda_1\wp_{24}\wp_{44} + \wp_{33}\lambda_3\lambda_2 + 4\lambda_4\wp_{12}\wp_{24} + 6\wp_{33}\lambda_0 \\
&\quad + 2\wp_{23}\wp_{22}\wp_{24} - \tfrac{1}{3}\wp_{23}\lambda_3 Q_{2444} - \tfrac{2}{3}\wp_{34}\lambda_3 Q_{2333} + 2\wp_{22}^2\wp_{34} + 2\lambda_3\wp_{22}\wp_{24} \\
&\quad - \tfrac{8}{3}\wp_{34}\lambda_4^2 Q_{2333} - 2\wp_{23}\wp_{33}\lambda_4\lambda_3 + 8\lambda_4\wp_{33}\wp_{34}\lambda_2 + 2\wp_{33}\lambda_4\lambda_1 + 16\wp_{33}\wp_{34}\lambda_1 \\
&\quad - 4\lambda_3\wp_{12}\wp_{44} + \tfrac{8}{3}\wp_{34}\lambda_2 Q_{2444} - 4\wp_{34}Q_{1224} + 2\wp_{23}Q_{1244} - 6\wp_{11}\wp_{24} \\
\wp_{144}\wp_{223} &= 2\wp_{33}\lambda_4\lambda_3^2 - \tfrac{4}{3}\wp_{12}\wp_{14} + \tfrac{8}{9}\lambda_1 Q_{2444} + \tfrac{2}{9}\lambda_2 Q_{2333} - \tfrac{2}{3}\lambda_4\lambda_3 Q_{2333} \\
&\quad + \tfrac{8}{3}\lambda_4\wp_{11}\wp_{44} + \tfrac{2}{3}\wp_{22}\wp_{13}\wp_{44} + \tfrac{8}{3}\wp_{23}\wp_{14}\wp_{24} - \tfrac{4}{3}\wp_{34}\wp_{12}\wp_{24} - \tfrac{2}{3}\wp_{33}\wp_{13}\lambda_3 \\
&\quad + \tfrac{1}{9}\wp_{13}Q_{2333} - \tfrac{2}{3}\wp_{13}\wp_{24}^2 + \tfrac{4}{3}\wp_{14}^2\lambda_4 - \tfrac{2}{9}\wp_{13}\lambda_4 Q_{2444} + \tfrac{4}{3}\lambda_4\wp_{14}\wp_{22} \\
&\quad - \tfrac{2}{3}\wp_{33}\lambda_3\lambda_2 - \tfrac{4}{3}\lambda_4\wp_{12}\wp_{24} + \tfrac{4}{3}\wp_{34}\wp_{14}\wp_{22} + \tfrac{4}{3}\wp_{33}\lambda_0 - \tfrac{8}{3}\wp_{11}\wp_{24} + \tfrac{4}{3}\wp_{23}\wp_{12}\wp_{44} \\
&\quad + 4\wp_{33}\lambda_4\lambda_1 - \tfrac{2}{3}\wp_{34}Q_{1224} + \tfrac{4}{3}\wp_{23}Q_{1244} - \tfrac{4}{3}\lambda_4 Q_{1224} - 4\wp_{44}^2\lambda_0 + 2\lambda_3\wp_{14}\wp_{24} \\
\wp_{124}\wp_{234} &= 95\wp_{33}\lambda_4\lambda_3^2 + \tfrac{2}{3}\wp_{12}\wp_{14} + \tfrac{2}{9}\lambda_1 Q_{2444} + \tfrac{1}{18}\lambda_2 Q_{2333} - \tfrac{190}{3}\lambda_4 Q_{1224} \\
&\quad - \tfrac{95}{3}\lambda_4\lambda_3 Q_{2333} + 126\wp_{34}\lambda_4 Q_{1244} + \tfrac{380}{3}\lambda_4\wp_{11}\wp_{44} - 126\wp_{34}\wp_{33}\lambda_4^2\lambda_3 + \tfrac{1}{3}\wp_{33}\lambda_0 \\
&\quad + \tfrac{5}{3}\wp_{23}\wp_{14}\wp_{24} + \tfrac{5}{3}\wp_{34}\wp_{12}\wp_{24} - \tfrac{2}{3}\wp_{33}\wp_{13}\lambda_3 + \tfrac{1}{9}\wp_{13}Q_{2333} + 2\wp_{44}^2\lambda_0 + \tfrac{1}{3}\wp_{13}\wp_{24}^2 \\
&\quad - \tfrac{374}{3}\wp_{14}^2\lambda_4 - \tfrac{380}{9}\wp_{13}\lambda_4 Q_{2444} + 21\lambda_4\wp_{33}Q_{2233} + \tfrac{379}{3}\lambda_4\wp_{14}\wp_{22} + 2\lambda_3\wp_{14}\wp_{24} \\
&\quad - 21\lambda_4\wp_{23}Q_{2333} - 42\wp_{33}\lambda_4^2\lambda_2 - \tfrac{1}{6}\wp_{33}\lambda_3\lambda_2 - \tfrac{379}{3}\lambda_4\wp_{12}\wp_{24} + \tfrac{1}{3}\wp_{34}\wp_{14}\wp_{22} \\
&\quad - \tfrac{2}{3}\wp_{11}\wp_{24} + 42\wp_{34}\lambda_4^2 Q_{2333} + \tfrac{1}{3}\wp_{23}\wp_{12}\wp_{44} - 2\lambda_2\wp_{14}\wp_{44} + \tfrac{1}{3}\wp_{23}Q_{1244} \\
&\quad - 84\lambda_4\wp_{33}\wp_{34}\lambda_2 - 84\wp_{33}\wp_{13}\lambda_4^2 + 211\wp_{33}\lambda_4\lambda_1 + \tfrac{1}{3}\wp_{34}Q_{1224} - \tfrac{1}{3}\wp_{22}\wp_{13}\wp_{44} \\
\wp_{122}\wp_{344} &= 92\wp_{33}\lambda_4\lambda_3^2 + \tfrac{14}{3}\wp_{12}\wp_{14} - \tfrac{10}{9}\lambda_1 Q_{2444} - \tfrac{7}{9}\lambda_2 Q_{2333} - 18\tfrac{4}{3}\lambda_4 Q_{1224} + \tfrac{2}{3}\wp_{22}\wp_{13}\wp_{44} \\
&\quad - \tfrac{92}{3}\lambda_4\lambda_3 Q_{2333} + 126\wp_{34}\lambda_4 Q_{1244} + 36\tfrac{8}{3}\lambda_4\wp_{11}\wp_{44} - 126\wp_{34}\wp_{33}\lambda_4^2\lambda_3 \\
&\quad - \tfrac{4}{3}\wp_{23}\wp_{14}\wp_{24} + \tfrac{8}{3}\wp_{34}\wp_{12}\wp_{24} - \tfrac{2}{3}\wp_{33}\wp_{13}\lambda_3 - 2\wp_{12}\wp_{22} + \tfrac{1}{9}\wp_{13}Q_{2333} - 4\wp_{44}^2\lambda_0 \\
&\quad + 4\lambda_2\wp_{14}\wp_{44} - \tfrac{380}{3}\wp_{14}^2\lambda_4 - \tfrac{380}{9}\wp_{13}\lambda_4 Q_{2444} + 21\lambda_4\wp_{33}Q_{2233} + \tfrac{382}{3}\lambda_4\wp_{14}\wp_{22} \\
&\quad - 4\lambda_3\wp_{14}\wp_{24} - 21\lambda_4\wp_{23}Q_{2333} - 42\wp_{33}\lambda_4^2\lambda_2 + \tfrac{7}{3}\wp_{33}\lambda_3\lambda_2 - \tfrac{370}{3}\lambda_4\wp_{12}\wp_{24} \\
&\quad + \tfrac{4}{3}\wp_{34}\wp_{14}\wp_{22} + \tfrac{13}{3}\wp_{33}\lambda_0 + \tfrac{10}{3}\wp_{11}\wp_{24} + 42\wp_{34}\lambda_4^2 Q_{2333} + \tfrac{4}{3}\wp_{23}\wp_{12}\wp_{44} - \tfrac{2}{3}\wp_{13}\wp_{24}^2 \\
&\quad - 84\lambda_4\wp_{33}\wp_{34}\lambda_2 - 84\wp_{33}\wp_{13}\lambda_4^2 + 205\wp_{33}\lambda_4\lambda_1 + \tfrac{4}{3}\wp_{34}Q_{1224} - \tfrac{2}{3}\wp_{23}Q_{1244}
\end{aligned}
$$

$$\begin{aligned}
\wp_{123}\wp_{333} &= 2\wp_{33}\wp_{13}\wp_{23} + 2\wp_{12}\wp_{33}^2 - 2\wp_{33}\lambda_0 + 4\wp_{11}\wp_{24} + 4\wp_{12}\wp_{14} - 2\wp_{12}\wp_{22} \\
&\quad + \wp_{33}\lambda_3\lambda_2 + \wp_{23}\wp_{33}\lambda_4\lambda_3 + 3\wp_{33}\wp_{13}\lambda_3 + 2\wp_{33}\wp_{34}\lambda_1 - \tfrac{2}{3}\lambda_1 Q_{2444} - \tfrac{1}{3}\lambda_4\wp_{23}Q_{2333} \\
&\quad - \tfrac{1}{3}\wp_{13}Q_{2333} - \tfrac{1}{3}\lambda_2 Q_{2333} - \wp_{23}Q_{1244} \\
\wp_{223}\wp_{224} &= 4\wp_{33}\lambda_4\lambda_3^2 - 4\wp_{12}\wp_{14} + 2\lambda_2\wp_{24}^2 + \tfrac{4}{3}\lambda_1 Q_{2444} + \tfrac{1}{3}\lambda_2 Q_{2333} - 2\lambda_4 Q_{1224} \\
&\quad - \tfrac{4}{3}\lambda_4\lambda_3 Q_{2333} + 4\lambda_2\wp_{23}\wp_{33} + \tfrac{2}{3}\lambda_4\lambda_2 Q_{2444} + 4\wp_{34}\lambda_4 Q_{1244} - 4\wp_{34}\wp_{33}\lambda_4^2\lambda_3 \\
&\quad - 2\lambda_3 Q_{1244} + 4\lambda_4\wp_{14}\wp_{22} + 2\lambda_3\wp_{14}\wp_{24} - \tfrac{4}{3}\lambda_4\wp_{23}Q_{2333} - 4\lambda_1\wp_{24}\wp_{44} \\
&\quad + 2\lambda_3\wp_{22}\wp_{24} + 4\wp_{23}\wp_{22}\wp_{24} + \tfrac{2}{3}\wp_{23}\lambda_3 Q_{2444} + \tfrac{1}{3}\wp_{34}\lambda_3 Q_{2333} + \tfrac{4}{3}\wp_{34}\lambda_4^2 Q_{2333} \\
&\quad - 2\lambda_2\wp_{22}\wp_{44} + 4\wp_{23}\wp_{33}\lambda_4\lambda_3 - 4\lambda_4\wp_{33}\wp_{34}\lambda_2 + 8\wp_{33}\lambda_4\lambda_1 - 8\wp_{33}\wp_{34}\lambda_1 \\
&\quad + 2\lambda_3\wp_{12}\wp_{44} - \tfrac{4}{3}\wp_{34}\lambda_2 Q_{2444} + 2\wp_{34}Q_{1224} - 4\wp_{23}Q_{1244} - 4\lambda_4\wp_{12}\wp_{24} \\
\wp_{133}\wp_{233} &= 2\wp_{23}Q_{1244} - 2\wp_{11}\wp_{24} - 2\wp_{12}\wp_{14} + \wp_{12}\wp_{22} - 2\wp_{33}\lambda_0 - 2\wp_{33}\lambda_3\lambda_2 + \tfrac{2}{3}\wp_{13}Q_{2333} \\
&\quad + 2\wp_{33}\lambda_4\lambda_1 + \tfrac{4}{3}\lambda_1 Q_{2444} + \tfrac{2}{3}\lambda_4\wp_{23}Q_{2333} + \tfrac{2}{3}\lambda_2 Q_{2333} + 4\wp_{33}\wp_{13}\wp_{23} - 2\wp_{23}\wp_{33}\lambda_4\lambda_3 \\
\wp_{123}\wp_{244} &= -\wp_{33}\lambda_4\lambda_3^2 + \tfrac{8}{3}\wp_{12}\wp_{14} + \tfrac{2}{9}\lambda_1 Q_{2444} - \tfrac{4}{9}\lambda_2 Q_{2333} + \tfrac{2}{3}\lambda_4 Q_{1224} + \tfrac{1}{3}\lambda_4\lambda_3 Q_{2333} \\
&\quad - \tfrac{4}{3}\lambda_4\wp_{11}\wp_{44} + \tfrac{2}{3}\wp_{22}\wp_{13}\wp_{44} + \tfrac{2}{3}\wp_{23}\wp_{14}\wp_{24} + \tfrac{2}{3}\wp_{34}\wp_{12}\wp_{24} + \tfrac{4}{3}\wp_{33}\wp_{13}\lambda_3 \\
&\quad - \tfrac{2}{9}\wp_{13}Q_{2333} + \tfrac{4}{3}\wp_{13}\wp_{24}^2 - \tfrac{8}{3}\wp_{14}^2\lambda_4 + \tfrac{4}{9}\wp_{13}\lambda_4 Q_{2444} - \tfrac{2}{3}\lambda_4\wp_{14}\wp_{22} + 2\lambda_3\wp_{14}\wp_{24} \\
&\quad - 2\lambda_1\wp_{24}\wp_{44} + \tfrac{4}{3}\wp_{33}\lambda_3\lambda_2 + \tfrac{2}{3}\lambda_4\wp_{12}\wp_{24} - \tfrac{2}{3}\wp_{34}\wp_{14}\wp_{22} + \tfrac{10}{3}\wp_{33}\lambda_0 - \tfrac{2}{3}\wp_{11}\wp_{24} \\
&\quad + \tfrac{4}{3}\wp_{23}\wp_{12}\wp_{44} + 4\lambda_2\wp_{14}\wp_{44} - 2\wp_{33}\lambda_4\lambda_1 - \tfrac{2}{3}\wp_{34}Q_{1224} + \tfrac{1}{3}\wp_{23}Q_{1244} - 4\wp_{44}^2\lambda_0 \\
\wp_{113}\wp_{444} &= -\tfrac{3}{2}\wp_{33}\lambda_4\lambda_3^2 - 3\wp_{33}\lambda_4\lambda_1 + 4\wp_{11}\wp_{24} + \wp_{34}Q_{1224} + 2\wp_{33}\lambda_3\lambda_2 \\
&\quad + 4\lambda_2\wp_{14}\wp_{44} + \tfrac{1}{2}\lambda_4\lambda_3 Q_{2333} + \tfrac{1}{2}\wp_{34}\lambda_3 Q_{2333} + 4\wp_{14}\wp_{13}\wp_{44} - \tfrac{2}{3}\lambda_2 Q_{2333} \\
&\quad - \tfrac{2}{3}\wp_{13}Q_{2333} - \tfrac{2}{3}\lambda_1 Q_{2444} - \tfrac{3}{2}\wp_{34}\wp_{33}\lambda_3^2 + 2\wp_{34}\wp_{11}\wp_{44} + 2\wp_{33}\wp_{13}\lambda_3 \\
&\quad + \lambda_4 Q_{1224} - 2\wp_{14}^2\wp_{34} - 2\wp_{14}^2\lambda_4 - 2\lambda_4\wp_{11}\wp_{44} - 2\lambda_1\wp_{24}\wp_{44} - 3\wp_{33}\wp_{34}\lambda_1 \\
\wp_{114}\wp_{344} &= 2\wp_{12}\wp_{14} + \wp_{33}\lambda_0 + \tfrac{3}{2}\wp_{34}\wp_{33}\lambda_3^2 + 3\wp_{33}\wp_{34}\lambda_1 - \wp_{33}\wp_{13}\lambda_3 \\
&\quad + 2\wp_{34}\wp_{11}\wp_{44} + 2\wp_{14}^2\wp_{34} - \tfrac{1}{2}\wp_{34}\lambda_3 Q_{2333} + \tfrac{1}{3}\wp_{13}Q_{2333} - \wp_{34}Q_{1224}
\end{aligned}$$

We present the relations at weight -20 below. As discussed in Section 5.3.3, these contain terms that are quadratic in the basis Q-functions. To skip to the next weight level proceed to page 267.

$$\begin{aligned}
\wp_{134}\wp_{223} &= -\tfrac{1}{3}\wp_{33}\lambda_4\lambda_3 Q_{2444} - \wp_{11}\wp_{23} - \wp_{12}\wp_{13} - \wp_{12}\lambda_2 - \tfrac{8}{3}\wp_{14}\lambda_1 + \wp_{14}\wp_{23}^2 \\
&\quad + \tfrac{4}{3}\lambda_4\wp_{14}\lambda_2 + \tfrac{1}{2}\wp_{33}^2\lambda_3^2 + \tfrac{1}{3}Q_{2444}Q_{1244} + \tfrac{1}{9}\lambda_4 Q_{2444}Q_{2333} - \wp_{34}\wp_{24}\lambda_1 \\
&\quad + 2\wp_{33}^2\lambda_1 + \tfrac{2}{3}\lambda_4\wp_{13}\wp_{14} + \wp_{34}\wp_{12}\lambda_3 + \tfrac{2}{3}\wp_{34}\wp_{14}\lambda_2 + \tfrac{2}{3}\wp_{22}\wp_{13}\lambda_4 + \tfrac{1}{3}\wp_{22}\lambda_4\lambda_2 \\
&\quad + \tfrac{1}{2}\wp_{22}\wp_{23}\lambda_3 + \tfrac{2}{3}\wp_{22}\wp_{34}\lambda_2 + \wp_{24}\wp_{13}\lambda_3 + \wp_{22}\wp_{14}\wp_{33} - \wp_{24}\wp_{12}\wp_{33} + \tfrac{4}{3}\wp_{22}\lambda_1 \\
&\quad - \wp_{44}\wp_{23}\lambda_1 - \lambda_4\lambda_1\wp_{24} - \wp_{33}Q_{1224} + \wp_{34}\wp_{12}\wp_{23} + \wp_{22}\wp_{13}\wp_{34} + \wp_{24}\wp_{13}\wp_{23} \\
&\quad - \tfrac{1}{6}\wp_{33}\lambda_3 Q_{2333} - 2\lambda_4\wp_{44}\lambda_0 + \tfrac{1}{3}\wp_{14}Q_{2233} - \tfrac{1}{6}\wp_{22}Q_{2233} - 2\wp_{44}\wp_{34}\lambda_0
\end{aligned}$$

$$\begin{aligned}
\wp_{223}^2 &= -8\lambda_3\wp_{44}\lambda_1 + 4\wp_{22}\wp_{13}\lambda_4 - 4\wp_{44}\wp_{11}\wp_{33} - 8\lambda_4\wp_{13}\wp_{14} - 8\wp_{34}\wp_{14}\lambda_2 \\
&\quad - 8\wp_{11}\wp_{23} - 4\wp_{24}\wp_{13}\wp_{23} - 4\wp_{22}\wp_{14}\wp_{33} + 4\wp_{24}\wp_{12}\wp_{33} + 4\wp_{22}\wp_{13}\wp_{34} \\
&\quad - 8\wp_{44}\wp_{23}\lambda_1 - 4\wp_{34}\wp_{12}\lambda_3 + 4\wp_{14}^2\wp_{33} - 8\lambda_4\wp_{14}\lambda_2 - 4\wp_{34}\wp_{12}\wp_{23} - 4\wp_{24}\wp_{13}\lambda_3 \\
&\quad + 4\wp_{22}\wp_{23}\lambda_3 + 16\wp_{14}\wp_{23}\lambda_3 + 8\lambda_4\wp_{11}\wp_{34} + 4\wp_{22}\lambda_4\lambda_2 + 4\wp_{22}\wp_{34}\lambda_2 \\
&\quad + 8\wp_{44}\wp_{13}\lambda_2 - 8\wp_{34}\wp_{13}\wp_{14} + 4\wp_{11}\wp_{34}^2 + 4\wp_{44}\wp_{13}^2 + 4\wp_{22}\wp_{23}^2 + 4\wp_{14}\wp_{23}^2 \\
&\quad - 8\wp_{33}^2\lambda_1 - 4\lambda_2\wp_{23}\wp_{24} - \tfrac{4}{3}\wp_{33}\lambda_2 Q_{2444} + \tfrac{4}{3}\wp_{33}\lambda_4^2 Q_{2333} - 4\wp_{33}^2\lambda_4^2\lambda_3 \\
&\quad - 4\lambda_4\wp_{12}\wp_{23} - 4\lambda_4\wp_{12}\lambda_3 + \tfrac{1}{3}\wp_{33}\lambda_3 Q_{2333} - 4\lambda_2\wp_{24}\lambda_3 + 2\wp_{33}Q_{1224} \\
&\quad + 12\wp_{14}\lambda_3^2 + 4\lambda_4\wp_{33}Q_{1244} + 4\wp_{11}\lambda_4^2 + 4\wp_{44}\lambda_2^2 - 8\lambda_3\wp_{11} - 4\wp_{33}^2\lambda_4\lambda_2
\end{aligned}$$

$$\begin{aligned}
\wp_{222}\wp_{233} &= -2\lambda_3\wp_{44}\lambda_1 + \tfrac{4}{3}\wp_{22}\wp_{13}\lambda_4 + \tfrac{4}{3}\wp_{34}\wp_{24}\lambda_1 - \tfrac{4}{3}\wp_{44}\wp_{11}\wp_{33} + 4\lambda_4\wp_{14}\lambda_2 \\
&\quad + \tfrac{8}{3}\lambda_4\wp_{13}\wp_{14} + \tfrac{8}{3}\wp_{34}\wp_{14}\lambda_2 + \tfrac{4}{3}\wp_{12}\lambda_2 - \tfrac{8}{3}\wp_{11}\wp_{23} - 4\wp_{24}\wp_{13}\wp_{23} - 4\wp_{22}\wp_{14}\wp_{33} \\
&\quad + 4\wp_{24}\wp_{12}\wp_{33} + 4\wp_{22}\wp_{13}\wp_{34} - \tfrac{8}{3}\wp_{44}\wp_{23}\lambda_1 - 6\wp_{34}\wp_{12}\lambda_3 + \tfrac{4}{3}\wp_{14}^2\wp_{33} \\
&\quad - 4\wp_{34}\wp_{12}\wp_{23} - 6\wp_{24}\wp_{13}\lambda_3 + 2\wp_{22}\wp_{23}\lambda_3 - 8\wp_{14}\wp_{23}\lambda_3 - \tfrac{8}{3}\lambda_4\wp_{11}\wp_{34} \\
&\quad + \tfrac{4}{3}\wp_{22}\wp_{34}\lambda_2 + \tfrac{4}{3}\lambda_4\wp_{24}\lambda_1 - \tfrac{8}{3}\wp_{44}\wp_{13}\lambda_2 - \tfrac{8}{3}\wp_{34}\wp_{13}\wp_{14} - \tfrac{8}{9}\lambda_4 Q_{2444}Q_{2333} \\
&\quad + \tfrac{4}{3}\wp_{11}\wp_{34}^2 + \tfrac{4}{3}\wp_{44}\wp_{13}^2 + 2\wp_{22}\wp_{23}^2 + 2\wp_{22}^2\wp_{33} - 4\wp_{33}^2\lambda_3^2 + 4\wp_{14}\wp_{23}^2 \\
&\quad + \tfrac{8}{3}\wp_{33}\lambda_2 Q_{2444} + \tfrac{4}{3}\wp_{12}\wp_{13} - \tfrac{8}{3}\wp_{33}\lambda_4^2 Q_{2333} + 8\wp_{33}^2\lambda_4^2\lambda_3 + 4\lambda_4\wp_{12}\wp_{23} \\
&\quad + \tfrac{2}{3}\wp_{33}\lambda_3 Q_{2333} + 2\lambda_2\wp_{24}\lambda_3 + \tfrac{4}{3}\wp_{33}Q_{1224} - 8\lambda_4\wp_{33}Q_{1244} - 6\wp_{14}\lambda_3^2 - 2\lambda_3\wp_{11} \\
&\quad - 12\lambda_4\wp_{44}\lambda_0 + 16\wp_{14}\lambda_1 + 8\wp_{33}^2\lambda_4\lambda_2 + \tfrac{8}{3}\wp_{33}\lambda_4\lambda_3 Q_{2444} - \tfrac{1}{3}\wp_{22}Q_{2233} \\
&\quad - \tfrac{8}{3}Q_{2444}Q_{1244} + \tfrac{8}{3}\wp_{14}Q_{2233} + 2\lambda_4\wp_{12}\lambda_3 + \tfrac{2}{3}\wp_{22}\lambda_4\lambda_2 + 4\lambda_2\wp_{23}\wp_{24} - 4\wp_{22}\lambda_1
\end{aligned}$$

$$\begin{aligned}
\wp_{133}\wp_{224} &= \tfrac{2}{3}\wp_{22}\wp_{13}\lambda_4 + 5\wp_{34}\wp_{24}\lambda_1 + \tfrac{14}{3}\lambda_4\wp_{13}\wp_{14} + \tfrac{1}{9}\lambda_4 Q_{2444}Q_{2333} - \wp_{11}\wp_{23} \\
&\quad + 3\wp_{24}\wp_{13}\wp_{23} + \wp_{22}\wp_{14}\wp_{33} + \wp_{24}\wp_{12}\wp_{33} + \wp_{22}\wp_{13}\wp_{34} + \wp_{44}\wp_{23}\lambda_1 \\
&\quad + 4\wp_{44}\wp_{34}\lambda_0 + \tfrac{4}{3}\lambda_4\wp_{14}\lambda_2 - \wp_{34}\wp_{12}\wp_{23} + \wp_{24}\wp_{13}\lambda_3 + \tfrac{1}{2}\wp_{22}\wp_{23}\lambda_3 - \tfrac{10}{3}\wp_{34}\wp_{14}\lambda_2 \\
&\quad + 2\lambda_4\wp_{11}\wp_{34} + \tfrac{1}{3}\wp_{22}\lambda_4\lambda_2 + \tfrac{2}{3}\wp_{22}\wp_{34}\lambda_2 + \lambda_4\wp_{24}\lambda_1 - 2\wp_{44}\wp_{13}\lambda_2 + 3\wp_{14}\wp_{23}\lambda_3 \\
&\quad + \tfrac{1}{2}\wp_{33}^2\lambda_3^2 + 2\wp_{33}^2\lambda_1 - \wp_{14}\wp_{23}^2 - \wp_{12}\wp_{13} - \tfrac{1}{6}\wp_{33}\lambda_3 Q_{2333} - 2\lambda_4\wp_{44}\lambda_0 + \tfrac{16}{3}\wp_{14}\lambda_1 \\
&\quad - \tfrac{2}{3}\wp_{22}\lambda_1 - \tfrac{1}{3}\wp_{33}\lambda_4\lambda_3 Q_{2444} - \tfrac{1}{6}\wp_{22}Q_{2233} + \tfrac{1}{3}Q_{2444}Q_{1244} - \tfrac{2}{3}\wp_{14}Q_{2233} - 2\wp_{34}\wp_{12}\lambda_3
\end{aligned}$$

$$\begin{aligned}
\wp_{124}\wp_{233} &= \wp_{34}\wp_{24}\lambda_1 - \tfrac{4}{3}\wp_{22}\wp_{13}\lambda_4 + \tfrac{2}{3}\lambda_4\wp_{13}\wp_{14} + \tfrac{2}{3}\wp_{34}\wp_{14}\lambda_2 + \wp_{12}\lambda_2 + \wp_{11}\wp_{23} \\
&\quad + \wp_{24}\wp_{13}\wp_{23} + \wp_{22}\wp_{14}\wp_{33} + \wp_{24}\wp_{12}\wp_{33} - \wp_{22}\wp_{13}\wp_{34} + \wp_{44}\wp_{23}\lambda_1 + \wp_{34}\wp_{12}\lambda_3 \\
&\quad + 4\wp_{44}\wp_{34}\lambda_0 - \tfrac{2}{3}\lambda_4\wp_{14}\lambda_2 + \wp_{34}\wp_{12}\wp_{23} + \wp_{24}\wp_{13}\lambda_3 - \wp_{22}\wp_{23}\lambda_3 - 2\lambda_4\wp_{11}\wp_{34} \\
&\quad + 2\wp_{33}^2\lambda_1 + \wp_{14}\wp_{23}^2 + \wp_{12}\wp_{13} - 2\lambda_4\wp_{44}\lambda_0 + \tfrac{1}{3}\wp_{14}Q_{2233} + \tfrac{16}{3}\wp_{14}\lambda_1 - \tfrac{2}{3}\wp_{22}\lambda_1 \\
&\quad - \tfrac{1}{3}\wp_{33}\lambda_4\lambda_3 Q_{2444} + \tfrac{1}{3}\wp_{22}Q_{2233} + \tfrac{1}{2}\wp_{33}^2\lambda_3^2 + \tfrac{1}{3}Q_{2444}Q_{1244} - \tfrac{1}{6}\wp_{33}\lambda_3 Q_{2333} \\
&\quad - \tfrac{2}{3}\wp_{22}\lambda_4\lambda_2 - \tfrac{4}{3}\wp_{22}\wp_{34}\lambda_2 + \lambda_4\wp_{24}\lambda_1 - 2\wp_{44}\wp_{13}\lambda_2 + \tfrac{1}{9}\lambda_4 Q_{2444}Q_{2333}
\end{aligned}$$

$$\begin{aligned}
\wp_{134}^2 &= \tfrac{20}{9}\lambda_4\wp_{13}\wp_{14} - \tfrac{2}{9}\wp_{22}\wp_{13}\lambda_4 - \tfrac{2}{9}\wp_{34}\wp_{24}\lambda_1 - \tfrac{4}{9}\wp_{44}\wp_{11}\wp_{33} + \tfrac{20}{9}\wp_{34}\wp_{14}\lambda_2 \\
&+ \tfrac{1}{9}\wp_{12}\lambda_2 - 3\wp_{24}\lambda_0 + \tfrac{1}{9}\wp_{11}\wp_{23} + \tfrac{1}{9}\wp_{44}\wp_{23}\lambda_1 + \tfrac{1}{3}\wp_{34}\wp_{12}\lambda_3 - 2\wp_{44}\wp_{34}\lambda_0 \\
&+ \tfrac{4}{9}\wp_{14}^2\wp_{33} + \tfrac{2}{3}\lambda_4\wp_{14}\lambda_2 + \tfrac{1}{3}\wp_{24}\wp_{13}\lambda_3 - \tfrac{1}{6}\wp_{22}\wp_{23}\lambda_3 + \wp_{14}\wp_{23}\lambda_3 - \tfrac{2}{9}\lambda_4\wp_{11}\wp_{34} \\
&- \tfrac{1}{9}\wp_{22}\lambda_4\lambda_2 - \tfrac{2}{9}\wp_{22}\wp_{34}\lambda_2 + \tfrac{1}{9}\lambda_4\wp_{24}\lambda_1 - \tfrac{2}{9}\wp_{44}\wp_{13}\lambda_2 + \tfrac{28}{9}\wp_{34}\wp_{13}\wp_{14} - \tfrac{4}{9}\wp_{14}Q_{2233} \\
&+ \tfrac{1}{27}\lambda_4 Q_{2444}Q_{2333} + \tfrac{4}{9}\wp_{44}\wp_{13}^2 - \tfrac{1}{3}\wp_{33}^2\lambda_3^2 - \tfrac{1}{3}\wp_{33}^2\lambda_1 - \tfrac{2}{9}\wp_{12}\wp_{13} + \tfrac{1}{9}\wp_{33}\lambda_3 Q_{2333} \\
&+ \tfrac{1}{9}\wp_{33}Q_{1224} + \tfrac{4}{3}\wp_{14}\lambda_1 - \tfrac{1}{9}\wp_{33}\lambda_4\lambda_3 Q_{2444} + \tfrac{1}{18}\wp_{22}Q_{2233} + \tfrac{1}{9}Q_{2444}Q_{1244} + \tfrac{4}{9}\wp_{11}\wp_{34}^2
\end{aligned}$$

$$\begin{aligned}
\wp_{133}\wp_{144} &= \tfrac{4}{9}\wp_{22}\wp_{13}\lambda_4 + \tfrac{4}{9}\wp_{34}\wp_{24}\lambda_1 + \tfrac{8}{9}\wp_{44}\wp_{11}\wp_{33} + \tfrac{14}{9}\lambda_4\wp_{13}\wp_{14} - \tfrac{4}{9}\wp_{34}\wp_{14}\lambda_2 \\
&- \tfrac{2}{9}\wp_{12}\lambda_2 - \tfrac{2}{9}\wp_{11}\wp_{23} - \tfrac{2}{9}\wp_{44}\wp_{23}\lambda_1 - \tfrac{2}{3}\wp_{34}\wp_{12}\lambda_3 + 4\wp_{44}\wp_{34}\lambda_0 + \tfrac{10}{9}\wp_{14}^2\wp_{33} \\
&- \tfrac{2}{9}\wp_{33}\lambda_3 Q_{2333} + \tfrac{2}{9}\wp_{22}\lambda_4\lambda_2 + \tfrac{1}{3}\wp_{22}\wp_{23}\lambda_3 + \wp_{14}\wp_{23}\lambda_3 + \tfrac{2}{9}\wp_{33}\lambda_4\lambda_3 Q_{2444} \\
&+ \tfrac{4}{9}\lambda_4\wp_{11}\wp_{34} - \tfrac{2}{3}\wp_{24}\wp_{13}\lambda_3 + \tfrac{4}{9}\wp_{22}\wp_{34}\lambda_2 - \tfrac{2}{9}\lambda_4\wp_{24}\lambda_1 + \tfrac{4}{9}\wp_{44}\wp_{13}\lambda_2 - \tfrac{5}{9}\wp_{12}\wp_{13} \\
&- \tfrac{8}{9}\wp_{11}\wp_{34}^2 - \tfrac{2}{27}\lambda_4 Q_{2444}Q_{2333} + \tfrac{10}{9}\wp_{44}\wp_{13}^2 + \tfrac{2}{3}\wp_{33}^2\lambda_3^2 + \tfrac{2}{3}\wp_{33}^2\lambda_1 + \tfrac{16}{9}\wp_{34}\wp_{13}\wp_{14} \\
&+ \tfrac{2}{3}\lambda_4\wp_{14}\lambda_2 - \tfrac{2}{9}\wp_{33}Q_{1224} + \tfrac{4}{3}\wp_{14}\lambda_1 - \tfrac{2}{9}Q_{2444}Q_{1244} - \tfrac{1}{9}\wp_{22}Q_{2233} - \tfrac{1}{9}\wp_{14}Q_{2233}
\end{aligned}$$

$$\begin{aligned}
\wp_{123}\wp_{234} &= -\tfrac{2}{3}\wp_{34}\wp_{24}\lambda_1 + \tfrac{2}{3}\wp_{44}\wp_{11}\wp_{33} + \tfrac{4}{3}\lambda_4\wp_{13}\wp_{14} + \tfrac{4}{3}\wp_{34}\wp_{14}\lambda_2 - \tfrac{2}{3}\wp_{12}\lambda_2 \\
&- \tfrac{2}{3}\wp_{11}\wp_{23} + 2\wp_{24}\wp_{13}\wp_{23} - \tfrac{2}{3}\wp_{44}\wp_{23}\lambda_1 + 2\wp_{34}\wp_{12}\lambda_3 - 2\wp_{44}\wp_{34}\lambda_0 \\
&+ \tfrac{4}{3}\lambda_4\wp_{14}\lambda_2 + 2\wp_{34}\wp_{12}\wp_{23} + 2\wp_{24}\wp_{13}\lambda_3 - \tfrac{2}{3}\lambda_4\wp_{11}\wp_{34} - \tfrac{2}{3}\lambda_4\wp_{24}\lambda_1 \\
&+ \tfrac{4}{3}\wp_{34}\wp_{13}\wp_{14} - \tfrac{2}{3}\wp_{11}\wp_{34}^2 - \tfrac{1}{9}\lambda_4 Q_{2444}Q_{2333} - \tfrac{2}{3}\wp_{44}\wp_{13}^2 - \tfrac{1}{2}\wp_{33}^2\lambda_3^2 - 2\wp_{33}^2\lambda_1 \\
&- \tfrac{2}{3}\wp_{12}\wp_{13} + \tfrac{1}{6}\wp_{33}\lambda_3 Q_{2333} + \tfrac{1}{3}\wp_{33}Q_{1224} - 2\lambda_4\wp_{44}\lambda_0 - \tfrac{8}{3}\wp_{14}\lambda_1 + \tfrac{4}{3}\wp_{22}\lambda_1 \\
&+ \tfrac{1}{3}\wp_{33}\lambda_4\lambda_3 Q_{2444} - \tfrac{1}{3}Q_{2444}Q_{1244} - \tfrac{2}{3}\wp_{44}\wp_{13}\lambda_2 - \tfrac{2}{3}\wp_{14}^2\wp_{33}
\end{aligned}$$

$$\begin{aligned}
\wp_{122}\wp_{334} &= \tfrac{2}{3}\wp_{22}\wp_{13}\lambda_4 - \wp_{34}\wp_{24}\lambda_1 - \tfrac{10}{3}\lambda_4\wp_{13}\wp_{14} + \tfrac{14}{3}\wp_{34}\wp_{14}\lambda_2 + \wp_{12}\lambda_2 - 9\wp_{24}\lambda_0 \\
&+ \wp_{11}\wp_{23} - \wp_{24}\wp_{13}\wp_{23} + \wp_{22}\wp_{14}\wp_{33} + \wp_{24}\wp_{12}\wp_{33} + \wp_{22}\wp_{13}\wp_{34} - \wp_{44}\wp_{23}\lambda_1 \\
&+ \wp_{34}\wp_{12}\lambda_3 - 8\wp_{44}\wp_{34}\lambda_0 + \tfrac{4}{3}\lambda_4\wp_{14}\lambda_2 + 3\wp_{34}\wp_{12}\wp_{23} - 2\wp_{24}\wp_{13}\lambda_3 + \tfrac{1}{2}\wp_{22}\wp_{23}\lambda_3 \\
&+ 3\wp_{14}\wp_{23}\lambda_3 - 2\lambda_4\wp_{11}\wp_{34} + \tfrac{1}{3}\wp_{22}\lambda_4\lambda_2 + \tfrac{2}{3}\wp_{22}\wp_{34}\lambda_2 + 2\wp_{44}\wp_{13}\lambda_2 + \tfrac{1}{2}\wp_{33}^2\lambda_3^2 \\
&+ \tfrac{1}{9}\lambda_4 Q_{2444}Q_{2333} + 2\wp_{33}^2\lambda_1 - \wp_{14}\wp_{23}^2 + 5\wp_{12}\wp_{13} - \tfrac{1}{6}\wp_{33}\lambda_3 Q_{2333} - 2\lambda_4\wp_{44}\lambda_0 \\
&+ \tfrac{16}{3}\wp_{14}\lambda_1 - \tfrac{2}{3}\wp_{22}\lambda_1 - \tfrac{1}{3}\wp_{33}\lambda_4\lambda_3 Q_{2444} - \tfrac{1}{6}\wp_{22}Q_{2233} + \tfrac{1}{3}Q_{2444}Q_{1244} - \tfrac{2}{3}\wp_{14}Q_{2233}
\end{aligned}$$

$$\begin{aligned}
\wp_{114}\wp_{334} &= \tfrac{4}{9}\wp_{22}\wp_{13}\lambda_4 - \tfrac{5}{9}\wp_{34}\wp_{24}\lambda_1 + \tfrac{8}{9}\wp_{44}\wp_{11}\wp_{33} - \tfrac{4}{9}\lambda_4\wp_{13}\wp_{14} + \tfrac{14}{9}\wp_{34}\wp_{14}\lambda_2 \\
&- \tfrac{2}{9}\wp_{12}\lambda_2 - 3\wp_{24}\lambda_0 - \tfrac{2}{9}\wp_{11}\wp_{23} - \tfrac{2}{9}\wp_{44}\wp_{23}\lambda_1 - \tfrac{2}{3}\wp_{34}\wp_{12}\lambda_3 - 2\wp_{44}\wp_{34}\lambda_0 - \tfrac{1}{9}\wp_{14}Q_{2233} \\
&+ \tfrac{10}{9}\wp_{14}^2\wp_{33} + \tfrac{2}{3}\lambda_4\wp_{14}\lambda_2 - \tfrac{2}{3}\wp_{24}\wp_{13}\lambda_3 + \tfrac{1}{3}\wp_{22}\wp_{23}\lambda_3 + \wp_{14}\wp_{23}\lambda_3 + \tfrac{4}{9}\lambda_4\wp_{11}\wp_{34} \\
&+ \tfrac{2}{9}\wp_{22}\lambda_4\lambda_2 + \tfrac{4}{9}\wp_{22}\wp_{34}\lambda_2 - \tfrac{2}{9}\lambda_4\wp_{24}\lambda_1 + \tfrac{4}{9}\wp_{44}\wp_{13}\lambda_2 + \tfrac{16}{9}\wp_{34}\wp_{13}\wp_{14} - \tfrac{2}{9}Q_{2444}Q_{1244} \\
&+ \tfrac{10}{9}\wp_{11}\wp_{34}^2 - \tfrac{2}{27}\lambda_4 Q_{2444}Q_{2333} - \tfrac{8}{9}\wp_{44}\wp_{13}^2 + \tfrac{2}{3}\wp_{33}^2\lambda_3^2 + \tfrac{2}{3}\wp_{33}^2\lambda_1 + \tfrac{4}{9}\wp_{12}\wp_{13} \\
&- \tfrac{2}{9}\wp_{33}\lambda_3 Q_{2333} - \tfrac{2}{9}\wp_{33}Q_{1224} + \tfrac{4}{3}\wp_{14}\lambda_1 + \tfrac{2}{9}\wp_{33}\lambda_4\lambda_3 Q_{2444} - \tfrac{1}{9}\wp_{22}Q_{2233}
\end{aligned}$$

$$\begin{aligned}
\wp_{113}\wp_{344} &= -\tfrac{8}{9}\wp_{22}\wp_{13}\lambda_4 - \tfrac{8}{9}\wp_{34}\wp_{24}\lambda_1 + \tfrac{2}{9}\wp_{44}\wp_{11}\wp_{33} + \tfrac{8}{9}\lambda_4\wp_{13}\wp_{14} + \tfrac{8}{9}\wp_{34}\wp_{14}\lambda_2 \\
&+ \tfrac{4}{9}\wp_{12}\lambda_2 + \tfrac{4}{9}\wp_{11}\wp_{23} + \tfrac{4}{9}\wp_{44}\wp_{23}\lambda_1 + \tfrac{4}{3}\wp_{34}\wp_{12}\lambda_3 + 4\wp_{44}\wp_{34}\lambda_0 - \tfrac{2}{9}\wp_{14}^2\wp_{33} \\
&- \tfrac{4}{3}\lambda_4\wp_{14}\lambda_2 + \tfrac{4}{3}\wp_{24}\wp_{13}\lambda_3 - \tfrac{2}{3}\wp_{22}\wp_{23}\lambda_3 - 2\wp_{14}\wp_{23}\lambda_3 - \tfrac{8}{9}\lambda_4\wp_{11}\wp_{34} - \tfrac{4}{9}\wp_{22}\lambda_4\lambda_2 \\
&- \tfrac{8}{9}\wp_{44}\wp_{13}\lambda_2 + \tfrac{4}{9}\wp_{34}\wp_{13}\wp_{14} + \tfrac{16}{9}\wp_{11}\wp_{34}^2 + \tfrac{4}{27}\lambda_4 Q_{2444}Q_{2333} + \tfrac{16}{9}\wp_{44}\wp_{13}^2 + \tfrac{4}{9}\lambda_4\wp_{24}\lambda_1 \\
&+ \tfrac{1}{6}\wp_{33}^2\lambda_3^2 + \tfrac{5}{3}\wp_{33}^2\lambda_1 - \tfrac{8}{9}\wp_{12}\wp_{13} - \tfrac{1}{18}\wp_{33}\lambda_3 Q_{2333} - \tfrac{5}{9}\wp_{33}Q_{1224} + \tfrac{4}{3}\wp_{14}\lambda_1 \\
&- \tfrac{4}{9}\wp_{33}\lambda_4\lambda_3 Q_{2444} + \tfrac{2}{9}\wp_{22}Q_{2233} + \tfrac{4}{9}Q_{2444}Q_{1244} + \tfrac{2}{9}\wp_{14}Q_{2233} - \tfrac{8}{9}\wp_{22}\wp_{34}\lambda_2
\end{aligned}$$

The relations at weights -21, -22 and -23 could be derived normally, without terms quadratic in the Q-functions. To skip these proceed to page 273.

The following relations are at weight -21.

$$\begin{aligned}
\wp_{222}\wp_{224} &= \tfrac{2}{3}\lambda_4\lambda_3\lambda_1 - \tfrac{4}{3}\lambda_4^2\lambda_3\lambda_2 + \tfrac{10}{3}\lambda_4\lambda_2^2 - \tfrac{20}{3}\lambda_2\lambda_1 - 4\lambda_4^2\lambda_0 + \tfrac{2}{3}\wp_{12}Q_{2333} \\
&+ 6\lambda_3\lambda_0 - 4\wp_{14}Q_{1244} + \tfrac{4}{3}\wp_{11}Q_{2444} - \tfrac{2}{3}\wp_{13}Q_{2233} - 4\wp_{22}Q_{1244} - 2\lambda_2\wp_{23}^2 \\
&+ 4\wp_{13}\lambda_3^2 + 4\wp_{44}\wp_{33}\lambda_0 + 8\wp_{24}\wp_{33}\lambda_1 - \tfrac{2}{3}\lambda_3\wp_{34}^2\lambda_2 + 4\lambda_3\wp_{13}\wp_{23} - 2\wp_{24}\wp_{33}\lambda_3^2 \\
&+ 2\wp_{34}\wp_{23}\lambda_3^2 + \tfrac{8}{3}\lambda_2\wp_{13}\wp_{34} - 4\lambda_4\wp_{23}\lambda_1 + 4\lambda_2\wp_{23}\lambda_3 - 4\lambda_3\wp_{12}\wp_{33} + 4\lambda_4\wp_{13}\lambda_2 \\
&+ \tfrac{8}{3}\lambda_3\wp_{34}\lambda_1 + 4\lambda_4\wp_{11}\wp_{33} + 4\wp_{34}\lambda_4^2\lambda_1 + 4\lambda_4\wp_{34}\lambda_0 + 4\lambda_4\wp_{14}\wp_{33}\lambda_3 + \tfrac{2}{3}\lambda_4\lambda_3 Q_{2233} \\
&+ \tfrac{4}{3}\lambda_4\lambda_3\wp_{13}\wp_{34} + 4\lambda_4\wp_{22}\wp_{33}\lambda_3 - \tfrac{32}{3}\lambda_1\wp_{13} + 4\wp_{23}\lambda_0 - 4\wp_{34}^2\lambda_0 - \tfrac{4}{3}\lambda_4\wp_{13}^2 \\
&- \tfrac{4}{3}\wp_{34}\lambda_2^2 + 4\lambda_4^2\wp_{24}\wp_{33}\lambda_3 - \tfrac{4}{3}\wp_{14}\lambda_4 Q_{2333} + 6\lambda_2\wp_{22}\wp_{33} + 4\wp_{33}\wp_{24}\lambda_4\lambda_2 \\
&+ 4\wp_{24}\wp_{22}^2 - \tfrac{4}{3}\wp_{24}\lambda_4^2 Q_{2333} - \tfrac{8}{3}\lambda_3\wp_{13}\lambda_4^2 - 2\lambda_4\wp_{23}\lambda_3^2 + \tfrac{4}{3}\wp_{24}\lambda_2 Q_{2444} - \tfrac{5}{3}\lambda_2 Q_{2233} \\
&- \tfrac{4}{3}\lambda_4\wp_{22}Q_{2333} + \tfrac{2}{3}\wp_{14}\lambda_3 Q_{2444} + \tfrac{2}{3}\wp_{22}\lambda_3 Q_{2444} - \tfrac{1}{3}\wp_{34}\lambda_3 Q_{2233} - 2\wp_{24}Q_{1224} \\
&+ 2\wp_{44}\wp_{33}\lambda_4\lambda_3^2 + 2\wp_{44}\wp_{33}\lambda_3\lambda_2 - \tfrac{2}{3}\lambda_4\lambda_3\wp_{44}Q_{2333} - 4\wp_{24}\lambda_4 Q_{1244} - 2\wp_{44}\lambda_3 Q_{1244} \\
\wp_{133}\wp_{223} &= -\tfrac{2}{3}\lambda_4\lambda_2^2 + \tfrac{4}{3}\lambda_2\lambda_1 + 2\lambda_4^2\lambda_0 - 6\lambda_3\lambda_0 - 4\wp_{14}Q_{1244} + \tfrac{4}{3}\wp_{11}Q_{2444} - 6\wp_{23}\lambda_0 \\
&+ \tfrac{1}{3}\wp_{13}Q_{2233} + 2\wp_{22}Q_{1244} - 2\wp_{44}\wp_{33}\lambda_0 - 2\wp_{24}\wp_{33}\lambda_1 + \lambda_3\wp_{13}\wp_{23} - \lambda_2\wp_{23}\lambda_3 \\
&- \tfrac{4}{3}\lambda_2\wp_{13}\wp_{34} - \lambda_3\wp_{12}\wp_{33} + 2\lambda_3\wp_{34}\lambda_1 + 2\lambda_4\wp_{11}\wp_{33} + 2\wp_{22}\wp_{13}\wp_{33} + 4\lambda_4\wp_{34}\lambda_0 \\
&+ 2\wp_{23}\wp_{34}\lambda_1 + 4\lambda_2\wp_{14}\wp_{33} + 4\lambda_4\wp_{14}\wp_{33}\lambda_3 - 2\lambda_4\wp_{22}\wp_{33}\lambda_3 + \tfrac{4}{3}\lambda_1\wp_{13} + \tfrac{1}{3}\lambda_2 Q_{2233} \\
&+ 2\wp_{34}^2\lambda_0 + \tfrac{2}{3}\lambda_4\wp_{13}^2 - \tfrac{4}{3}\wp_{34}\lambda_2^2 + 2\wp_{13}\wp_{23}^2 - \tfrac{4}{3}\wp_{14}\lambda_4 Q_{2333} + \tfrac{2}{3}\lambda_4\wp_{22}Q_{2333} \\
\wp_{124}\wp_{144} &= 2\wp_{14}^2\wp_{24} + \tfrac{2}{3}\lambda_2\wp_{13}\wp_{34} + 2\lambda_4\wp_{34}\lambda_0 + 2\wp_{44}\wp_{33}\lambda_0 - \tfrac{1}{3}\wp_{14}\lambda_4 Q_{2333} + 2\wp_{14}\wp_{12}\wp_{44} \\
&- 2\wp_{34}^2\lambda_0 + \tfrac{1}{2}\lambda_3\wp_{13}\wp_{23} + \tfrac{1}{3}\lambda_4\wp_{13}\lambda_2 - \tfrac{2}{3}\lambda_1\wp_{13} - \tfrac{1}{6}\wp_{13}Q_{2233} + \lambda_4\wp_{14}\wp_{33}\lambda_3 + \tfrac{2}{3}\lambda_4\wp_{13}^2 \\
\wp_{123}\wp_{233} &= -\tfrac{2}{3}\lambda_4\lambda_2^2 + \tfrac{4}{3}\lambda_2\lambda_1 + 2\lambda_4^2\lambda_0 - 6\lambda_3\lambda_0 + 2\wp_{14}Q_{1244} - \tfrac{2}{3}\wp_{11}Q_{2444} \\
&+ \tfrac{1}{3}\wp_{13}Q_{2233} - \wp_{22}Q_{1244} - 2\wp_{44}\wp_{33}\lambda_0 + \lambda_3\wp_{13}\wp_{23} - \tfrac{4}{3}\lambda_2\wp_{13}\wp_{34} - \lambda_2\wp_{23}\lambda_3 \\
&+ 2\lambda_3\wp_{12}\wp_{33} + 2\lambda_3\wp_{34}\lambda_1 - 2\lambda_4\wp_{11}\wp_{33} + 4\lambda_4\wp_{34}\lambda_0 + 2\wp_{23}\wp_{12}\wp_{33} + \tfrac{2}{3}\lambda_4\wp_{13}^2 \\
&- 2\lambda_4\wp_{14}\wp_{33}\lambda_3 + \lambda_4\wp_{22}\wp_{33}\lambda_3 + \tfrac{4}{3}\lambda_1\wp_{13} - 6\wp_{23}\lambda_0 + 2\wp_{34}^2\lambda_0 + 2\wp_{23}\wp_{34}\lambda_1 \\
&- \tfrac{4}{3}\wp_{34}\lambda_2^2 + 2\wp_{13}\wp_{23}^2 + \tfrac{2}{3}\wp_{14}\lambda_4 Q_{2333} - \tfrac{1}{3}\lambda_4\wp_{22}Q_{2333} + \tfrac{1}{3}\lambda_2 Q_{2233}
\end{aligned}$$

$$\begin{aligned}
\wp_{133}\wp_{134} &= \tfrac{2}{3}\wp_{14}\lambda_4 Q_{2333} + \tfrac{4}{3}\lambda_2\wp_{13}\wp_{34} - 2\lambda_4\wp_{14}\wp_{33}\lambda_3 + 2\wp_{34}\wp_{13}^2 + \lambda_3\wp_{13}\wp_{23} \\
&\quad + \tfrac{4}{3}\lambda_4\wp_{13}^2 - 2\wp_{44}\wp_{33}\lambda_0 + 2\lambda_4\wp_{34}\lambda_0 + 2\wp_{34}^2\lambda_0 - 2\wp_{23}\lambda_0 + 2\wp_{14}\wp_{13}\wp_{33} \\
&\quad + \tfrac{2}{3}\lambda_4\wp_{13}\lambda_2 + \tfrac{2}{3}\lambda_1\wp_{13} + \tfrac{1}{2}\lambda_3\wp_{12}\wp_{33} - \tfrac{1}{3}\wp_{13}Q_{2233} - \tfrac{1}{6}\wp_{12}Q_{2333} + 2\wp_{14}Q_{1244} \\
\wp_{144}\wp_{222} &= \tfrac{2}{3}\lambda_4\lambda_2^2 - \tfrac{4}{3}\lambda_2\lambda_1 - 2\lambda_3\lambda_0 + \tfrac{1}{3}\wp_{12}Q_{2333} - 8\wp_{14}Q_{1244} + 2\wp_{22}Q_{1244} \\
&\quad + 6\wp_{44}\wp_{33}\lambda_0 + 6\wp_{24}\wp_{33}\lambda_1 - \tfrac{4}{3}\lambda_3\wp_{34}^2\lambda_2 - \lambda_3\wp_{13}\wp_{23} + \wp_{24}\wp_{33}\lambda_3^2 - \wp_{34}\wp_{23}\lambda_3^2 \\
&\quad + 4\lambda_2\wp_{13}\wp_{34} - \tfrac{4}{3}\lambda_4\wp_{23}\lambda_1 + \lambda_2\wp_{23}\lambda_3 - \lambda_3\wp_{12}\wp_{33} + \tfrac{4}{3}\lambda_4\wp_{13}\lambda_2 - \tfrac{14}{3}\lambda_3\wp_{34}\lambda_1 \\
&\quad + 2\wp_{34}\lambda_4^2\lambda_1 + 6\lambda_4\wp_{34}\lambda_0 + 2\wp_{22}\wp_{12}\wp_{44} - 6\wp_{23}\wp_{34}\lambda_1 + 4\lambda_2\wp_{14}\wp_{33} \\
&\quad + 8\lambda_4\wp_{14}\wp_{33}\lambda_3 - \tfrac{4}{3}\lambda_4\lambda_3\wp_{13}\wp_{34} - \lambda_4\wp_{22}\wp_{33}\lambda_3 + 2\lambda_4\wp_{33}\wp_{44}\lambda_1 - \tfrac{2}{3}\lambda_4\lambda_3\wp_{34}\lambda_2 \\
&\quad - 4\lambda_1\wp_{13} - 2\wp_{23}\lambda_0 - 2\wp_{34}^2\lambda_0 + \tfrac{4}{3}\wp_{34}\lambda_2^2 + 2\lambda_4^2\wp_{24}\wp_{33}\lambda_3 - \tfrac{8}{3}\wp_{14}\lambda_4 Q_{2333} \\
&\quad - 2\wp_{12}\wp_{24}^2 - \tfrac{2}{3}\wp_{24}\lambda_4^2 Q_{2333} + \tfrac{1}{3}\lambda_4\wp_{22}Q_{2333} + \tfrac{2}{3}\wp_{14}\lambda_3 Q_{2444} + \tfrac{1}{3}\wp_{34}\lambda_3 Q_{2233} \\
&\quad - 2\wp_{24}Q_{1224} - 2\wp_{24}\lambda_4 Q_{1244} - \tfrac{1}{3}\lambda_4\wp_{23}Q_{2233} - \tfrac{1}{3}\wp_{24}\lambda_3 Q_{2333} + \tfrac{4}{3}\lambda_4^2\wp_{13}\wp_{23} \\
&\quad + \tfrac{2}{3}\wp_{23}\lambda_4^2\lambda_2 + \lambda_4\wp_{23}^2\lambda_3 + \tfrac{4}{3}\lambda_4\wp_{23}\wp_{34}\lambda_2 + 4\wp_{22}\wp_{14}\wp_{24} + \tfrac{4}{3}\wp_{44}\lambda_1 Q_{2444} \\
&\quad - \tfrac{1}{3}\lambda_2 Q_{2233} + 2\lambda_4\wp_{34}^2\lambda_1 \\
\wp_{124}\wp_{224} &= -\tfrac{1}{3}\lambda_4\lambda_2^2 + \tfrac{2}{3}\lambda_2\lambda_1 - 2\lambda_3\lambda_0 - 2\wp_{14}Q_{1244} - 2\wp_{24}\wp_{33}\lambda_1 - \tfrac{2}{3}\wp_{44}\lambda_1 Q_{2444} \\
&\quad - \tfrac{1}{2}\wp_{24}\wp_{33}\lambda_3^2 + \tfrac{1}{2}\wp_{34}\wp_{23}\lambda_3^2 - \tfrac{1}{2}\lambda_2\wp_{23}\lambda_3 - \tfrac{2}{3}\lambda_4\wp_{13}\lambda_2 + \tfrac{4}{3}\lambda_3\wp_{34}\lambda_1 + 4\lambda_4\wp_{34}\lambda_0 \\
&\quad + 2\wp_{23}\wp_{34}\lambda_1 + 2\lambda_2\wp_{14}\wp_{33} - 2\lambda_4\wp_{34}^2\lambda_1 + 2\lambda_4\wp_{14}\wp_{33}\lambda_3 + \tfrac{2}{3}\lambda_4\lambda_3\wp_{13}\wp_{34} + \tfrac{1}{6}\lambda_2 Q_{2233} \\
&\quad + \tfrac{1}{3}\lambda_4\lambda_3\wp_{34}\lambda_2 - 2\wp_{23}\lambda_0 - 2\wp_{34}^2\lambda_0 - \tfrac{2}{3}\wp_{34}\lambda_2^2 + 2\wp_{12}\wp_{24}^2 - \tfrac{2}{3}\wp_{14}\lambda_4 Q_{2333} + \tfrac{2}{3}\lambda_3\wp_{34}^2\lambda_2 \\
&\quad + \tfrac{2}{3}\wp_{14}\lambda_3 Q_{2444} - \tfrac{1}{6}\wp_{34}\lambda_3 Q_{2233} + \wp_{24}Q_{1224} + \tfrac{1}{6}\wp_{24}\lambda_3 Q_{2333} + 2\wp_{22}\wp_{14}\wp_{24} \\
\wp_{122}\wp_{333} &= \tfrac{4}{3}\lambda_4\lambda_2^2 - \tfrac{2}{3}\lambda_2\lambda_1 + 2\lambda_4^2\lambda_0 + \wp_{12}Q_{2333} + 2\wp_{14}Q_{1244} - \tfrac{2}{3}\wp_{11}Q_{2444} \\
&\quad - 6\lambda_3\lambda_0 - \tfrac{2}{3}\wp_{13}Q_{2233} + 2\wp_{22}Q_{1244} - 2\wp_{44}\wp_{33}\lambda_0 - 2\wp_{24}\wp_{33}\lambda_1 + \lambda_3\wp_{13}\wp_{23} \\
&\quad + \tfrac{20}{3}\lambda_2\wp_{13}\wp_{34} + \lambda_4\wp_{23}\lambda_1 + 2\lambda_2\wp_{23}\lambda_3 - \lambda_3\wp_{12}\wp_{33} + 2\lambda_4\wp_{13}\lambda_2 - 4\lambda_3\wp_{34}\lambda_1 \\
&\quad - 2\lambda_4\wp_{11}\wp_{33} - 8\lambda_4\wp_{34}\lambda_0 + 2\wp_{22}\wp_{13}\wp_{33} + 4\wp_{23}\wp_{12}\wp_{33} - 6\wp_{23}\wp_{34}\lambda_1 \\
&\quad - 2\lambda_4\wp_{14}\wp_{33}\lambda_3 - 2\lambda_4\wp_{22}\wp_{33}\lambda_3 - \tfrac{8}{3}\lambda_1\wp_{13} - 3\wp_{23}\lambda_0 - 10\wp_{34}^2\lambda_0 - \tfrac{10}{3}\lambda_4\wp_{13}^2 \\
&\quad + \tfrac{8}{3}\wp_{34}\lambda_2^2 - 2\wp_{13}\wp_{23}^2 + \tfrac{2}{3}\wp_{14}\lambda_4 Q_{2333} + \tfrac{2}{3}\lambda_4\wp_{22}Q_{2333} - \tfrac{2}{3}\lambda_2 Q_{2233} \\
\wp_{122}\wp_{244} &= -\tfrac{2}{3}\lambda_4\lambda_2^2 - \tfrac{2}{3}\lambda_2\lambda_1 - 2\lambda_3\lambda_0 - \tfrac{2}{3}\wp_{12}Q_{2333} + 6\wp_{14}Q_{1244} - \tfrac{2}{3}\wp_{11}Q_{2444} \\
&\quad + \tfrac{1}{3}\wp_{13}Q_{2233} - 2\wp_{44}\wp_{33}\lambda_0 + 4\wp_{24}\wp_{33}\lambda_1 - \tfrac{4}{3}\lambda_3\wp_{34}^2\lambda_2 - 2\lambda_3\wp_{13}\wp_{23} \\
&\quad - \wp_{34}\wp_{23}\lambda_3^2 - \tfrac{4}{3}\lambda_2\wp_{13}\wp_{34} - \lambda_4\wp_{23}\lambda_1 - \lambda_2\wp_{23}\lambda_3 + 4\lambda_3\wp_{12}\wp_{33} - 2\lambda_4\wp_{13}\lambda_2 \\
&\quad - \tfrac{8}{3}\lambda_3\wp_{34}\lambda_1 + 2\wp_{34}\lambda_4^2\lambda_1 + 6\lambda_4\wp_{34}\lambda_0 + 2\wp_{22}\wp_{12}\wp_{44} - 4\wp_{23}\wp_{34}\lambda_1 \\
&\quad + 2\lambda_4\wp_{34}^2\lambda_1 - 8\lambda_4\wp_{14}\wp_{33}\lambda_3 + \tfrac{2}{3}\lambda_4\lambda_3\wp_{13}\wp_{34} + 2\lambda_4\wp_{33}\wp_{44}\lambda_1 - \tfrac{2}{3}\lambda_4\lambda_3\wp_{34}\lambda_2 \\
&\quad - \tfrac{8}{3}\lambda_1\wp_{13} - 3\wp_{23}\lambda_0 + 6\wp_{34}^2\lambda_0 - \tfrac{4}{3}\lambda_4\wp_{13}^2 - \tfrac{4}{3}\wp_{34}\lambda_2^2 + 2\wp_{12}\wp_{24}^2 + 2\wp_{14}\lambda_4 Q_{2333} \\
&\quad - \tfrac{4}{3}\wp_{14}\lambda_3 Q_{2444} + \tfrac{1}{3}\wp_{34}\lambda_3 Q_{2233} - \tfrac{1}{3}\wp_{24}\lambda_3 Q_{2333} + \tfrac{4}{3}\wp_{44}\lambda_1 Q_{2444} \\
&\quad + \tfrac{1}{3}\lambda_2 Q_{2233} + \tfrac{2}{3}\lambda_4\wp_{12}Q_{2444} - 4\lambda_2\wp_{14}\wp_{33} + \wp_{24}\wp_{33}\lambda_3^2
\end{aligned}$$

$$\wp_{114}\wp_{333} = \tfrac{2}{3}\wp_{14}\lambda_4 Q_{2333} + 2\wp_{34}\wp_{11}\wp_{33} - \tfrac{2}{3}\lambda_2\wp_{13}\wp_{34} - 2\lambda_4\wp_{14}\wp_{33}\lambda_3 - 2\wp_{34}\wp_{13}^2 - 2\lambda_3\wp_{13}\wp_{23} - \tfrac{8}{3}\lambda_4\wp_{13}^2 - 2\wp_{44}\wp_{33}\lambda_0 - 4\lambda_4\wp_{34}\lambda_0 - \wp_{23}\wp_{34}\lambda_1 - 4\wp_{34}^2\lambda_0 + \tfrac{2}{3}\wp_{13}Q_{2233} + \wp_{23}\lambda_0 + 4\wp_{14}\wp_{13}\wp_{33} - \tfrac{4}{3}\lambda_4\wp_{13}\lambda_2 + \tfrac{2}{3}\lambda_1\wp_{13} - \lambda_3\wp_{12}\wp_{33} + 2\wp_{14}Q_{1244} + \tfrac{1}{3}\wp_{12}Q_{2333}$$

$$\wp_{114}\wp_{244} = \tfrac{2}{3}\lambda_2\wp_{13}\wp_{34} - \tfrac{1}{2}\wp_{24}\lambda_3 Q_{2333} + \tfrac{2}{3}\wp_{14}\lambda_4 Q_{2333} - 2\lambda_2\wp_{14}\wp_{33} - 2\lambda_4\wp_{14}\wp_{33}\lambda_3 + \tfrac{3}{2}\wp_{24}\wp_{33}\lambda_3^2 - \lambda_3\wp_{13}\wp_{23} - \tfrac{4}{3}\lambda_4\wp_{13}^2 + 2\wp_{14}^2\wp_{24} + 4\wp_{24}\wp_{33}\lambda_1 + 2\wp_{24}\wp_{11}\wp_{44} - \wp_{23}\wp_{34}\lambda_1 + \wp_{23}\lambda_0 - \tfrac{2}{3}\lambda_4\wp_{13}\lambda_2 - \tfrac{2}{3}\lambda_1\wp_{13} + 2\wp_{14}Q_{1244} + \tfrac{1}{3}\wp_{13}Q_{2233} - \wp_{24}Q_{1224}$$

$$\wp_{113}\wp_{334} = 2\wp_{34}\wp_{11}\wp_{33} - \tfrac{4}{3}\wp_{14}\lambda_4 Q_{2333} + \tfrac{4}{3}\lambda_2\wp_{13}\wp_{34} + 2\lambda_2\wp_{14}\wp_{33} + 4\lambda_4\wp_{14}\wp_{33}\lambda_3 + 2\wp_{34}\wp_{13}^2 + \lambda_3\wp_{13}\wp_{23} + \tfrac{4}{3}\lambda_4\wp_{13}^2 - \wp_{24}\wp_{33}\lambda_1 + 4\wp_{44}\wp_{33}\lambda_0 + 2\lambda_4\wp_{34}\lambda_0 + \tfrac{1}{3}\wp_{12}Q_{2333} - 2\wp_{23}\lambda_0 + \tfrac{2}{3}\lambda_4\wp_{13}\lambda_2 + \tfrac{2}{3}\lambda_1\wp_{13} - \lambda_3\wp_{12}\wp_{33} - 4\wp_{14}Q_{1244} - \tfrac{1}{3}\wp_{13}Q_{2233} + 2\wp_{34}^2\lambda_0$$

$$\wp_{112}\wp_{444} = -\tfrac{4}{3}\lambda_4\lambda_2^2 + \tfrac{2}{3}\lambda_2\lambda_1 - 2\wp_{14}^2\wp_{24} - \tfrac{1}{3}\wp_{12}Q_{2333} - 4\wp_{14}Q_{1244} + \tfrac{4}{3}\wp_{11}Q_{2444} - \tfrac{1}{3}\wp_{13}Q_{2233} + 4\wp_{44}\wp_{33}\lambda_0 - 6\wp_{24}\wp_{33}\lambda_1 + \lambda_3\wp_{13}\wp_{23} - \tfrac{3}{2}\wp_{24}\wp_{33}\lambda_3^2 - \tfrac{14}{3}\lambda_2\wp_{13}\wp_{34} + \tfrac{1}{3}\lambda_4\wp_{23}\lambda_1 - 2\lambda_2\wp_{23}\lambda_3 + \lambda_3\wp_{12}\wp_{33} - 2\lambda_4\wp_{13}\lambda_2 + 2\lambda_3\wp_{34}\lambda_1 + 4\lambda_4\wp_{34}\lambda_0 + 3\wp_{23}\wp_{34}\lambda_1 + 6\lambda_2\wp_{14}\wp_{33} + 2\lambda_4\lambda_3\wp_{13}\wp_{34} + \lambda_4\wp_{22}\wp_{33}\lambda_3 + \tfrac{2}{3}\lambda_1\wp_{13} + \tfrac{2}{3}\lambda_2 Q_{2233} + \tfrac{4}{3}\lambda_4\wp_{13}^2 - \tfrac{8}{3}\wp_{34}\lambda_2^2 - 2\lambda_4^2\wp_{24}\wp_{33}\lambda_3 - \tfrac{2}{3}\wp_{14}\lambda_4 Q_{2333} + \tfrac{2}{3}\wp_{24}\lambda_4^2 Q_{2333} - 2\wp_{23}\lambda_0 - \tfrac{1}{3}\lambda_4\wp_{22}Q_{2333} + \wp_{24}Q_{1224} + 2\wp_{44}\wp_{33}\lambda_3\lambda_2 + 2\wp_{24}\lambda_4 Q_{1244} + \tfrac{1}{3}\lambda_4\wp_{23}Q_{2233} + \tfrac{1}{2}\lambda_3\wp_{24}Q_{2333} - \tfrac{4}{3}\lambda_4^2\wp_{13}\wp_{23} - \tfrac{2}{3}\wp_{23}\lambda_4^2\lambda_2 - \lambda_4\wp_{23}^2\lambda_3 - \tfrac{4}{3}\lambda_4\wp_{23}\wp_{34}\lambda_2 + 4\wp_{34}^2\lambda_0 - \tfrac{2}{3}\wp_{44}\lambda_1 Q_{2444} + 4\wp_{14}\wp_{12}\wp_{44} - \tfrac{2}{3}\lambda_4\wp_{12}Q_{2444} + 2\wp_{24}\wp_{11}\wp_{44} - \tfrac{2}{3}\wp_{44}\lambda_2 Q_{2333}$$

The following relations are at weight -22.

$$\wp_{133}^2 = 21\lambda_3 Q_{1224} + 14\wp_{33}\lambda_4\lambda_3\lambda_2 + 42\lambda_3\wp_{14}^2 - 4\wp_{11}\wp_{14} + 4\wp_{13}Q_{1244} - 42\lambda_3\wp_{11}\wp_{44} + 24\lambda_3\wp_{33}\wp_{13}\lambda_4 + 14\wp_{13}\lambda_3 Q_{2444} + \tfrac{4}{3}\wp_{13}\lambda_4 Q_{2333} - 7\wp_{33}\lambda_3 Q_{2233} - 14\lambda_4\wp_{34}\lambda_3 Q_{2333} + 4\wp_{33}\wp_{13}^2 + \wp_{12}^2 + 4\wp_{33}\lambda_4\lambda_0 - 72\wp_{33}\lambda_3\lambda_1 - 42\lambda_3\wp_{14}\wp_{22} + 42\lambda_3\wp_{12}\wp_{24} + \tfrac{21}{2}\lambda_3^2 Q_{2333} + \tfrac{2}{3}\lambda_1 Q_{2333} + 28\lambda_3\wp_{33}\wp_{34}\lambda_2 + 7\wp_{23}\lambda_3 Q_{2333} + 42\wp_{34}\wp_{33}\lambda_4\lambda_3^2 - \tfrac{63}{2}\wp_{33}\lambda_3^3 + 4\lambda_0 Q_{2444} - 42\wp_{34}\lambda_3 Q_{1244}$$

$$\wp_{134}\wp_{222} = -\lambda_2 Q_{1244} - \tfrac{41}{2}\lambda_3 Q_{1224} - \wp_{23}Q_{1224} - 14\wp_{33}\lambda_4\lambda_3\lambda_2 - 37\lambda_3\wp_{14}^2 - 4\wp_{11}\wp_{14} - 4\wp_{13}Q_{1244} - \wp_{12}^2 + \wp_{44}\wp_{22}\lambda_1 + 2\wp_{23}\wp_{14}\wp_{22} - 2\wp_{23}\wp_{12}\wp_{24} - 2\lambda_2\wp_{12}\wp_{44} + 71\wp_{33}\lambda_3\lambda_1 + 6\lambda_0\wp_{24}\wp_{44} + 42\lambda_3\wp_{14}\wp_{22} - 40\lambda_3\wp_{12}\wp_{24} + \tfrac{1}{3}\lambda_4\lambda_1 Q_{2444} - \tfrac{41}{4}\lambda_3^2 Q_{2333} - \tfrac{2}{3}\lambda_1 Q_{2333} - 28\lambda_3\wp_{33}\wp_{34}\lambda_2 - 7\wp_{23}\lambda_3 Q_{2333} - 2\wp_{24}\wp_{11}\lambda_4 - 40\wp_{34}\wp_{33}\lambda_4\lambda_3^2 - \wp_{24}^2\lambda_1 + \wp_{11}\wp_{22} - 4\wp_{44}\wp_{14}\lambda_1 + 2\wp_{34}\wp_{12}\wp_{22} + 4\wp_{34}\wp_{33}\lambda_0 - 24\lambda_3\wp_{33}\wp_{13}\lambda_4 + 4\lambda_4\wp_{33}\wp_{34}\lambda_1 + 2\wp_{24}\wp_{13}\wp_{22} + \tfrac{4}{3}\wp_{34}\lambda_1 Q_{2444} - \tfrac{40}{3}\wp_{13}\lambda_3 Q_{2444} - \tfrac{4}{3}\wp_{13}\lambda_4 Q_{2333} + 7\wp_{33}\lambda_3 Q_{2233} + \tfrac{40}{3}\lambda_4\wp_{34}\lambda_3 Q_{2333} + 4\lambda_2\wp_{13}\wp_{33} + 41\lambda_3\wp_{11}\wp_{44} + 40\wp_{34}\lambda_3 Q_{1244} + \tfrac{123}{4}\wp_{33}\lambda_3^3 + \lambda_0 Q_{2444}$$

$$\wp_{222}\wp_{223} = 4\lambda_4\lambda_2\wp_{23}\wp_{33} + 4\lambda_4^2\lambda_3\wp_{23}\wp_{33} - 6\lambda_2 Q_{1244} + 6\wp_{33}\lambda_4\lambda_3\lambda_2 + 6\lambda_0 Q_{2444}$$
$$- 4\lambda_3 Q_{1224} - 2\wp_{23}Q_{1224} + 2\wp_{12}^2 - 6\wp_{44}\wp_{22}\lambda_1 + 12\wp_{33}\lambda_4\lambda_0 + 2\lambda_4^2\wp_{33}\lambda_3^2$$
$$+ 4\lambda_4^2\wp_{33}\lambda_1 + 4\wp_{33}\lambda_3\lambda_1 + 6\lambda_3\wp_{14}\wp_{22} - 2\lambda_3\wp_{12}\wp_{24} + 2\lambda_4\lambda_1 Q_{2444} + 3\wp_{33}\lambda_3^3$$
$$+ \tfrac{2}{3}\lambda_3\lambda_2 Q_{2444} - 2\lambda_4\lambda_2 Q_{2333} - \tfrac{2}{3}\lambda_4^2\lambda_3 Q_{2333} - 2\lambda_4\lambda_3 Q_{1244} - \lambda_3^2 Q_{2333} + 2\lambda_3\wp_{22}^2$$
$$+ \tfrac{4}{3}\lambda_1 Q_{2333} + \tfrac{4}{3}\wp_{23}\lambda_2 Q_{2444} - \tfrac{4}{3}\lambda_4^2\wp_{23}Q_{2333} + 8\wp_{23}\wp_{33}\lambda_1 - \tfrac{1}{3}\wp_{23}\lambda_3 Q_{2333}$$
$$- 4\lambda_4\wp_{23}Q_{1244} + 4\wp_{23}\wp_{22}^2 + 4\wp_{33}\lambda_2^2 + 2\wp_{24}^2\lambda_1 - 6\wp_{11}\wp_{22}$$

$$\wp_{124}\wp_{223} = -\lambda_2 Q_{1244} + \tfrac{19}{2}\lambda_3 Q_{1224} + \wp_{23}Q_{1224} + 8\wp_{33}\lambda_4\lambda_3\lambda_2 + 23\lambda_3\wp_{14}^2 - \tfrac{57}{4}\wp_{33}\lambda_3^3$$
$$- 4\wp_{11}\wp_{14} + 2\wp_{13}Q_{1244} + \wp_{12}^2 - \wp_{44}\wp_{22}\lambda_1 + 4\wp_{33}\lambda_4\lambda_0 + 2\wp_{23}\wp_{14}\wp_{22} + 3\lambda_0 Q_{2444}$$
$$+ 2\wp_{23}\wp_{12}\wp_{24} - 33\wp_{33}\lambda_3\lambda_1 - 18\lambda_3\wp_{14}\wp_{22} + 20\lambda_3\wp_{12}\wp_{24} + \tfrac{1}{3}\lambda_4\lambda_1 Q_{2444}$$
$$- \tfrac{1}{3}\lambda_4\lambda_2 Q_{2333} + \tfrac{19}{4}\lambda_3^2 Q_{2333} + \tfrac{2}{3}\lambda_1 Q_{2333} + 14\lambda_3\wp_{33}\wp_{34}\lambda_2 + \tfrac{7}{2}\wp_{23}\lambda_3 Q_{2333}$$
$$+ 20\wp_{34}\wp_{33}\lambda_4\lambda_3^2 - \wp_{11}\wp_{22} - 4\wp_{44}\wp_{14}\lambda_1 - 2\wp_{34}\wp_{33}\lambda_0 + 12\lambda_3\wp_{33}\wp_{13}\lambda_4$$
$$+ \wp_{24}^2\lambda_1 - 2\lambda_4\wp_{33}\wp_{34}\lambda_1 - \tfrac{2}{3}\wp_{34}\lambda_1 Q_{2444} + \tfrac{20}{3}\wp_{13}\lambda_3 Q_{2444} + \tfrac{2}{3}\wp_{13}\lambda_4 Q_{2333}$$
$$- \tfrac{7}{2}\wp_{33}\lambda_3 Q_{2233} - \tfrac{20}{3}\lambda_4\wp_{34}\lambda_3 Q_{2333} - 2\lambda_2\wp_{13}\wp_{33} - 19\lambda_3\wp_{11}\wp_{44} - 20\wp_{34}\lambda_3 Q_{1244}$$

$$\wp_{124}\wp_{134} = \tfrac{1}{6}\lambda_2 Q_{1244} - \tfrac{27}{4}\lambda_3 Q_{1224} + \tfrac{1}{3}\wp_{23}Q_{1224} - \tfrac{14}{3}\wp_{33}\lambda_4\lambda_3\lambda_2 - \tfrac{23}{2}\lambda_3\wp_{14}^2 + \tfrac{1}{3}\wp_{24}\wp_{11}\wp_{34}$$
$$- 2\wp_{11}\wp_{14} + \tfrac{1}{3}\wp_{13}Q_{1244} + \tfrac{1}{3}\wp_{23}\wp_{14}^2 + \tfrac{1}{6}\wp_{12}^2 - \tfrac{1}{6}\wp_{44}\wp_{22}\lambda_1 - \tfrac{1}{3}\lambda_2\wp_{14}\wp_{24} - \tfrac{1}{3}\lambda_4\wp_{12}\wp_{14}$$
$$+ \tfrac{1}{3}\lambda_2\wp_{12}\wp_{44} + \tfrac{133}{6}\wp_{33}\lambda_3\lambda_1 + 2\lambda_0\wp_{24}\wp_{44} - \tfrac{1}{2}\lambda_3^2\wp_{23}\wp_{33} + 14\lambda_3\wp_{14}\wp_{22} + \tfrac{81}{8}\wp_{33}\lambda_3^3$$
$$- 14\lambda_3\wp_{12}\wp_{24} - \tfrac{1}{18}\lambda_4\lambda_1 Q_{2444} - \tfrac{27}{8}\lambda_3^2 Q_{2333} + \tfrac{1}{9}\lambda_1 Q_{2333} - 9\lambda_3\wp_{33}\wp_{34}\lambda_2 + \tfrac{5}{6}\lambda_0 Q_{2444}$$
$$- \wp_{23}\wp_{33}\lambda_1 - \tfrac{13}{6}\wp_{23}\lambda_3 Q_{2333} + \tfrac{1}{3}\wp_{24}\wp_{11}\lambda_4 - 14\wp_{34}\wp_{33}\lambda_4\lambda_3^2 + \tfrac{1}{6}\wp_{24}^2\lambda_1$$
$$- \tfrac{1}{6}\wp_{11}\wp_{22} - 2\wp_{44}\wp_{14}\lambda_1 + \tfrac{4}{3}\wp_{34}\wp_{33}\lambda_0 - \tfrac{28}{3}\lambda_3\wp_{33}\wp_{13}\lambda_4 - \tfrac{1}{9}\wp_{34}\lambda_1 Q_{2444}$$
$$- \tfrac{14}{3}\wp_{13}\lambda_3 Q_{2444} + \tfrac{7}{3}\wp_{33}\lambda_3 Q_{2233} + \tfrac{14}{3}\lambda_4\wp_{34}\lambda_3 Q_{2333} + \tfrac{27}{2}\lambda_3\wp_{11}\wp_{44} + \tfrac{1}{3}\wp_{44}\wp_{12}\wp_{13}$$
$$+ \tfrac{5}{3}\wp_{24}\wp_{13}\wp_{14} + \tfrac{5}{3}\wp_{34}\wp_{12}\wp_{14} - \tfrac{1}{3}\wp_{23}\wp_{11}\wp_{44} + 14\wp_{34}\lambda_3 Q_{1244} - \tfrac{1}{9}\wp_{34}\lambda_2 Q_{2333}$$

$$\wp_{123}\wp_{144} = -\tfrac{1}{3}\lambda_2 Q_{1244} + \tfrac{69}{2}\lambda_3 Q_{1224} - \tfrac{2}{3}\wp_{23}Q_{1224} + \tfrac{70}{3}\wp_{33}\lambda_4\lambda_3\lambda_2 + 71\lambda_3\wp_{14}^2$$
$$- 2\wp_{11}\wp_{14} + \tfrac{1}{3}\wp_{13}Q_{1244} + \tfrac{4}{3}\wp_{23}\wp_{14}^2 - \tfrac{1}{3}\wp_{12}^2 + \tfrac{1}{3}\wp_{44}\wp_{22}\lambda_1 + \tfrac{2}{3}\lambda_2\wp_{14}\wp_{24}$$
$$- \tfrac{2}{3}\lambda_2\wp_{12}\wp_{44} - \tfrac{343}{3}\wp_{33}\lambda_3\lambda_1 - 4\lambda_0\wp_{24}\wp_{44} + \lambda_3^2\wp_{23}\wp_{33} - 70\lambda_3\wp_{14}\wp_{22}$$
$$+ 70\lambda_3\wp_{12}\wp_{24} + \tfrac{1}{9}\lambda_4\lambda_1 Q_{2444} + \tfrac{69}{4}\lambda_3^2 Q_{2333} - \tfrac{2}{9}\lambda_1 Q_{2333} + 46\lambda_3\wp_{33}\wp_{34}\lambda_2$$
$$+ 2\wp_{23}\wp_{33}\lambda_1 + \tfrac{34}{3}\wp_{23}\lambda_3 Q_{2333} - \tfrac{2}{3}\wp_{24}\wp_{11}\lambda_4 + 70\wp_{34}\wp_{33}\lambda_4\lambda_3^2 - \tfrac{1}{3}\wp_{24}^2\lambda_1$$
$$+ \tfrac{1}{3}\wp_{11}\wp_{22} + 2\wp_{44}\wp_{14}\lambda_1 - \tfrac{8}{3}\wp_{34}\wp_{33}\lambda_0 + \tfrac{140}{3}\lambda_3\wp_{33}\wp_{13}\lambda_4 + \tfrac{2}{9}\wp_{34}\lambda_1 Q_{2444}$$
$$+ \tfrac{70}{3}\wp_{13}\lambda_3 Q_{2444} - \tfrac{35}{3}\wp_{33}\lambda_3 Q_{2233} - \tfrac{70}{3}\lambda_4\wp_{34}\lambda_3 Q_{2333} - 69\lambda_3\wp_{11}\wp_{44}$$
$$+ \tfrac{2}{3}\wp_{24}\wp_{13}\wp_{14} + \tfrac{2}{3}\wp_{34}\wp_{12}\wp_{14} + \tfrac{2}{3}\wp_{23}\wp_{11}\wp_{44} - 70\wp_{34}\lambda_3 Q_{1244} + \tfrac{2}{9}\wp_{34}\lambda_2 Q_{2333}$$
$$+ \tfrac{4}{3}\wp_{44}\wp_{12}\wp_{13} - \tfrac{2}{3}\wp_{24}\wp_{11}\wp_{34} + \tfrac{2}{3}\lambda_4\wp_{12}\wp_{14} - 20\tfrac{7}{4}\wp_{33}\lambda_3^3 + \tfrac{1}{3}\lambda_0 Q_{2444}$$

$$\begin{aligned}
\wp_{123}\wp_{224} &= \tfrac{1}{2}\lambda_3 Q_{1224} - \lambda_3\wp_{14}^2 + 2\wp_{11}\wp_{14} - 4\wp_{13}Q_{1244} - \wp_{12}^2 - \wp_{44}\wp_{22}\lambda_1 + \wp_{34}\lambda_3 Q_{1244} \\
&+ 2\wp_{23}\wp_{12}\wp_{24} + 4\lambda_2\wp_{14}\wp_{24} + \wp_{33}\lambda_3\lambda_1 - 6\lambda_0\wp_{24}\wp_{44} + 2\lambda_3\wp_{12}\wp_{24} - \wp_{24}^2\lambda_1 - \lambda_3\wp_{11}\wp_{44} \\
&+ \tfrac{1}{3}\lambda_4\lambda_1 Q_{2444} + \tfrac{1}{4}\lambda_3^2 Q_{2333} - \tfrac{2}{3}\lambda_1 Q_{2333} - 2\wp_{24}\wp_{11}\lambda_4 - \wp_{34}\wp_{33}\lambda_4\lambda_3^2 + 4\lambda_2\wp_{13}\wp_{33} \\
&+ \wp_{11}\wp_{22} + 2\wp_{44}\wp_{14}\lambda_1 - 2\wp_{34}\wp_{33}\lambda_0 + 4\lambda_3\wp_{33}\wp_{13}\lambda_4 - 2\lambda_4\wp_{33}\wp_{34}\lambda_1 + \tfrac{1}{3}\lambda_4\wp_{34}\lambda_3 Q_{2333} \\
&+ 2\wp_{24}\wp_{13}\wp_{22} - \tfrac{2}{3}\wp_{34}\lambda_1 Q_{2444} + \tfrac{2}{3}\wp_{13}\lambda_3 Q_{2444} - \tfrac{4}{3}\wp_{13}\lambda_4 Q_{2333} - \lambda_0 Q_{2444} - \tfrac{3}{4}\wp_{33}\lambda_3^3
\end{aligned}$$

$$\begin{aligned}
\wp_{122}\wp_{234} &= 2\wp_{13}Q_{1244} + \wp_{33}\lambda_4\lambda_3\lambda_2 - \lambda_3\wp_{14}^2 + 2\wp_{11}\wp_{14} - \lambda_2 Q_{1244} + \tfrac{1}{2}\lambda_3 Q_{1224} - \tfrac{3}{4}\wp_{33}\lambda_3^3 \\
&- \wp_{12}^2 + \wp_{44}\wp_{22}\lambda_1 + 4\wp_{33}\lambda_4\lambda_0 + 2\wp_{23}\wp_{12}\wp_{24} - 2\lambda_2\wp_{12}\wp_{44} + \wp_{33}\lambda_3\lambda_1 + 4\lambda_4\wp_{12}\wp_{14} \\
&+ 2\lambda_3\wp_{12}\wp_{24} - \tfrac{1}{3}\lambda_4\lambda_2 Q_{2333} + \tfrac{1}{4}\lambda_3^2 Q_{2333} - \tfrac{2}{3}\lambda_1 Q_{2333} + 2\wp_{34}\wp_{33}\lambda_4\lambda_3^2 - \wp_{24}^2\lambda_1 \\
&+ 2\wp_{44}\wp_{14}\lambda_1 + 2\wp_{34}\wp_{12}\wp_{22} + 4\wp_{34}\wp_{33}\lambda_0 - 2\lambda_3\wp_{33}\wp_{13}\lambda_4 + 4\lambda_4\wp_{33}\wp_{34}\lambda_1 \\
&- \wp_{11}\wp_{22} + \tfrac{4}{3}\wp_{34}\lambda_1 Q_{2444} - \tfrac{1}{3}\wp_{13}\lambda_3 Q_{2444} + \tfrac{2}{3}\wp_{13}\lambda_4 Q_{2333} - \tfrac{2}{3}\lambda_4\wp_{34}\lambda_3 Q_{2333} \\
&- \lambda_3\wp_{11}\wp_{44} - 2\wp_{34}\lambda_3 Q_{1244} - 2\lambda_2\wp_{13}\wp_{33}
\end{aligned}$$

$$\begin{aligned}
\wp_{114}\wp_{234} &= -\tfrac{1}{3}\lambda_2 Q_{1244} - \tfrac{1}{2}\lambda_3 Q_{1224} - \tfrac{2}{3}\wp_{23}Q_{1224} + \lambda_3\wp_{14}^2 + 2\wp_{11}\wp_{14} + \tfrac{4}{3}\wp_{24}\wp_{11}\wp_{34} \\
&- \tfrac{2}{3}\wp_{13}Q_{1244} + \tfrac{4}{3}\wp_{23}\wp_{14}^2 - \tfrac{1}{3}\wp_{12}^2 + \tfrac{1}{3}\wp_{44}\wp_{22}\lambda_1 + \tfrac{2}{3}\lambda_2\wp_{14}\wp_{24} - \tfrac{2}{3}\lambda_2\wp_{12}\wp_{44} + \tfrac{3}{4}\wp_{33}\lambda_3^3 \\
&+ \tfrac{7}{3}\wp_{33}\lambda_3\lambda_1 + 2\lambda_0\wp_{24}\wp_{44} + \lambda_3^2\wp_{23}\wp_{33} + \tfrac{1}{9}\lambda_4\lambda_1 Q_{2444} - \tfrac{1}{4}\lambda_3^2 Q_{2333} + \tfrac{2}{3}\lambda_4\wp_{12}\wp_{14} \\
&- \tfrac{2}{9}\lambda_1 Q_{2333} + \tfrac{1}{3}\lambda_3\wp_{33}\wp_{34}\lambda_2 + 2\wp_{23}\wp_{33}\lambda_1 - \tfrac{1}{3}\wp_{23}\lambda_3 Q_{2333} - \tfrac{2}{3}\wp_{24}\wp_{11}\lambda_4 - \tfrac{2}{3}\lambda_0 Q_{2444} \\
&- \tfrac{1}{3}\wp_{24}^2\lambda_1 + \tfrac{1}{3}\wp_{11}\wp_{22} - 2\wp_{44}\wp_{14}\lambda_1 + \tfrac{4}{3}\wp_{34}\wp_{33}\lambda_0 - \tfrac{1}{9}\wp_{34}\lambda_1 Q_{2444} + \lambda_3\wp_{11}\wp_{44} \\
&+ \tfrac{2}{3}\wp_{24}\wp_{13}\wp_{14} + \tfrac{2}{3}\wp_{34}\wp_{12}\wp_{14} + \tfrac{2}{3}\wp_{23}\wp_{11}\wp_{44} - \tfrac{1}{9}\wp_{34}\lambda_2 Q_{2333} - \tfrac{2}{3}\wp_{44}\wp_{12}\wp_{13}
\end{aligned}$$

$$\begin{aligned}
\wp_{113}\wp_{244} &= \tfrac{2}{3}\lambda_2 Q_{1244} + 22\lambda_3 Q_{1224} + \tfrac{1}{3}\wp_{23}Q_{1224} + 14\wp_{33}\lambda_4\lambda_3\lambda_2 + 40\lambda_3\wp_{14}^2 \\
&+ 4\wp_{11}\wp_{14} + \tfrac{4}{3}\lambda_2\wp_{12}\wp_{44} - \tfrac{215}{3}\wp_{33}\lambda_3\lambda_1 - 4\lambda_0\wp_{24}\wp_{44} - \tfrac{1}{2}\lambda_3^2\wp_{23}\wp_{33} - 42\lambda_3\wp_{14}\wp_{22} \\
&+ \tfrac{4}{3}\wp_{13}Q_{1244} + 42\lambda_3\wp_{12}\wp_{24} - \tfrac{2}{9}\lambda_4\lambda_1 Q_{2444} + 11\lambda_3^2 Q_{2333} - \tfrac{5}{9}\lambda_1 Q_{2333} + \tfrac{82}{3}\lambda_3\wp_{33}\wp_{34}\lambda_2 \\
&- \tfrac{2}{3}\wp_{23}\wp_{14}^2 - \wp_{23}\wp_{33}\lambda_1 + \tfrac{43}{6}\wp_{23}\lambda_3 Q_{2333} + \tfrac{4}{3}\wp_{24}\wp_{11}\lambda_4 + 42\wp_{34}\wp_{33}\lambda_4\lambda_3^2 - \tfrac{4}{3}\wp_{24}^2\lambda_1 \\
&+ \tfrac{2}{3}\wp_{12}^2 - \tfrac{2}{3}\wp_{11}\wp_{22} + 4\wp_{44}\wp_{14}\lambda_1 - \tfrac{8}{3}\wp_{34}\wp_{33}\lambda_0 + 28\lambda_3\wp_{33}\wp_{13}\lambda_4 + \tfrac{2}{9}\wp_{34}\lambda_1 Q_{2444} \\
&- \tfrac{2}{3}\wp_{44}\wp_{22}\lambda_1 + 14\wp_{13}\lambda_3 Q_{2444} - 7\wp_{33}\lambda_3 Q_{2233} - 14\lambda_4\wp_{34}\lambda_3 Q_{2333} - 44\lambda_3\wp_{11}\wp_{44} \\
&+ \tfrac{8}{3}\lambda_2\wp_{14}\wp_{24} + \tfrac{8}{3}\wp_{24}\wp_{13}\wp_{14} - \tfrac{4}{3}\wp_{34}\wp_{12}\wp_{14} + \tfrac{2}{3}\wp_{23}\wp_{11}\wp_{44} - 42\wp_{34}\lambda_3 Q_{1244} \\
&+ \tfrac{2}{9}\wp_{34}\lambda_2 Q_{2333} + \tfrac{4}{3}\wp_{44}\wp_{12}\wp_{13} + \tfrac{4}{3}\wp_{24}\wp_{11}\wp_{34} - \tfrac{4}{3}\lambda_4\wp_{12}\wp_{14} - 33\wp_{33}\lambda_3^3 - \tfrac{2}{3}\lambda_0 Q_{2444}
\end{aligned}$$

$$\begin{aligned}
\wp_{112}\wp_{344} &= \tfrac{2}{3}\lambda_2 Q_{1244} + \lambda_3 Q_{1224} + \tfrac{1}{3}\wp_{23}Q_{1224} - 2\lambda_3\wp_{14}^2 + 4\wp_{11}\wp_{14} - \tfrac{2}{3}\wp_{13}Q_{1244} \\
&- \tfrac{2}{3}\wp_{23}\wp_{14}^2 - \tfrac{4}{3}\wp_{12}^2 - \tfrac{2}{3}\wp_{44}\wp_{22}\lambda_1 - \tfrac{4}{3}\lambda_2\wp_{14}\wp_{24} + \tfrac{4}{3}\lambda_2\wp_{12}\wp_{44} - \tfrac{5}{3}\wp_{33}\lambda_3\lambda_1 \\
&- 4\lambda_0\wp_{24}\wp_{44} - \tfrac{1}{2}\lambda_3^2\wp_{23}\wp_{33} - \tfrac{2}{9}\lambda_4\lambda_1 Q_{2444} + \tfrac{1}{2}\lambda_3^2 Q_{2333} - \tfrac{5}{9}\lambda_1 Q_{2333} \\
&+ \tfrac{4}{3}\lambda_3\wp_{33}\wp_{34}\lambda_2 - \wp_{23}\wp_{33}\lambda_1 + \tfrac{1}{6}\wp_{23}\lambda_3 Q_{2333} + \tfrac{4}{3}\wp_{24}\wp_{11}\lambda_4 + \tfrac{2}{3}\wp_{24}^2\lambda_1 \\
&- \tfrac{2}{3}\wp_{11}\wp_{22} + 4\wp_{44}\wp_{14}\lambda_1 + \tfrac{16}{3}\wp_{34}\wp_{33}\lambda_0 - \tfrac{4}{9}\wp_{34}\lambda_1 Q_{2444} - 2\lambda_3\wp_{11}\wp_{44} \\
&- \tfrac{4}{3}\wp_{24}\wp_{13}\wp_{14} + \tfrac{8}{3}\wp_{34}\wp_{12}\wp_{14} + \tfrac{2}{3}\wp_{23}\wp_{11}\wp_{44} - \tfrac{4}{9}\wp_{34}\lambda_2 Q_{2333} + \tfrac{4}{3}\wp_{44}\wp_{12}\wp_{13} \\
&+ \tfrac{4}{3}\wp_{24}\wp_{11}\wp_{34} + \tfrac{8}{3}\lambda_4\wp_{12}\wp_{14} - \tfrac{3}{2}\wp_{33}\lambda_3^3 - \tfrac{8}{3}\lambda_0 Q_{2444}
\end{aligned}$$

$$\begin{aligned}\wp_{113}\wp_{333} &= 2\lambda_3\wp_{33}\wp_{13}\lambda_4 + \wp_{33}\lambda_3\lambda_1 - 2\wp_{33}\lambda_4\lambda_0 - 2\wp_{12}^2 - \wp_{23}\wp_{33}\lambda_1 - \tfrac{1}{3}\lambda_1 Q_{2333} \\ &+ 8\wp_{11}\wp_{14} + 6\wp_{34}\wp_{33}\lambda_0 + 2\lambda_2\wp_{13}\wp_{33} + 2\wp_{11}\wp_{33}^2 + 2\wp_{33}\wp_{13}^2 - 2\lambda_0 Q_{2444} \\ &- \tfrac{2}{3}\wp_{13}\lambda_4 Q_{2333} - 2\wp_{13}Q_{1244}\end{aligned}$$

The following relations are at weight -23.

$$\begin{aligned}\wp_{133}\wp_{222} &= 12\wp_{14}\lambda_0 - 8\wp_{11}\wp_{13} + 4\wp_{33}^2\lambda_0 - 4\wp_{12}\lambda_1 - \wp_{12}Q_{2233} + 2\wp_{33}^2\lambda_4\lambda_3^2 \\ &- 2\wp_{33}\lambda_3 Q_{1244} + 4\wp_{23}\wp_{13}\wp_{22} + 6\wp_{14}\wp_{13}\lambda_3 - 4\wp_{34}\wp_{12}\lambda_2 + 8\lambda_1\wp_{14}\lambda_4 - 2\wp_{12}\wp_{23}^2 \\ &+ 4\wp_{22}\wp_{34}\lambda_1 + 4\lambda_4\wp_{12}\wp_{13} - 4\lambda_1\wp_{14}\wp_{34} - 4\lambda_1\wp_{13}\wp_{44} + 4\wp_{24}\wp_{23}\lambda_1 + 2\lambda_3\wp_{11}\wp_{34} \\ &+ 12\wp_{24}\wp_{34}\lambda_0 - 4\wp_{24}\wp_{13}\lambda_2 + 2\lambda_4\wp_{12}\lambda_2 + 2\wp_{23}\wp_{12}\lambda_3 + 2\wp_{33}\wp_{12}\wp_{22} - 6\wp_{44}\lambda_3\lambda_0 \\ &+ 2\wp_{22}\wp_{13}\lambda_3 + 2\lambda_3\wp_{24}\lambda_1 - 2\lambda_3\wp_{14}\lambda_2 + 4\wp_{33}^2\lambda_4\lambda_1 + \tfrac{4}{3}\wp_{33}\lambda_1 Q_{2444} - \tfrac{2}{3}\wp_{33}\lambda_4\lambda_3 Q_{2333} \\ \wp_{124}\wp_{133} &= \tfrac{8}{3}\wp_{33}^2\lambda_0 - \tfrac{4}{3}\wp_{11}\wp_{13} - \tfrac{2}{3}\wp_{12}\lambda_1 + 6\wp_{14}\lambda_0 + \tfrac{4}{3}\wp_{24}\wp_{13}^2 - \tfrac{1}{6}\wp_{12}Q_{2233} \\ &+ 2\wp_{14}\wp_{13}\lambda_3 + \tfrac{2}{3}\wp_{34}\wp_{12}\wp_{13} - \tfrac{2}{3}\wp_{34}\wp_{12}\lambda_2 - \tfrac{2}{3}\wp_{23}\wp_{11}\wp_{34} + \tfrac{4}{3}\lambda_1\wp_{14}\lambda_4 \\ &+ \tfrac{2}{3}\wp_{14}\wp_{13}\wp_{23} + 2\wp_{23}\wp_{44}\lambda_0 + \tfrac{2}{3}\wp_{22}\wp_{34}\lambda_1 + \tfrac{2}{3}\lambda_4\wp_{12}\wp_{13} - 2\lambda_1\wp_{14}\wp_{34} \\ &- \tfrac{2}{3}\lambda_1\wp_{13}\wp_{44} + 6\wp_{24}\wp_{34}\lambda_0 - \tfrac{2}{3}\wp_{24}\wp_{13}\lambda_2 + \tfrac{2}{3}\wp_{33}^2\lambda_3\lambda_2 + \tfrac{1}{3}\lambda_4\wp_{12}\lambda_2 - \tfrac{2}{9}\wp_{33}\lambda_2 Q_{2333} \\ &+ \tfrac{4}{3}\wp_{14}\wp_{12}\wp_{33} + \tfrac{1}{2}\wp_{23}\wp_{12}\lambda_3 + \tfrac{2}{3}\wp_{23}\wp_{14}\lambda_2 + \tfrac{2}{3}\wp_{24}\wp_{11}\wp_{33} - \tfrac{2}{9}\wp_{33}\lambda_1 Q_{2444} \\ \wp_{123}\wp_{223} &= \wp_{33}\lambda_3 Q_{1244} - 2\wp_{11}\wp_{13} - 2\wp_{33}^2\lambda_0 - 2\lambda_2\wp_{11} - \wp_{33}^2\lambda_4\lambda_3^2 + 2\wp_{23}\wp_{13}\wp_{22} \\ &- 2\wp_{23}\wp_{11}\lambda_4 - 6\wp_{23}\wp_{44}\lambda_0 + 2\wp_{22}\wp_{34}\lambda_1 - 4\lambda_1\wp_{14}\wp_{34} + 2\lambda_1\wp_{13}\wp_{44} \\ &- 4\wp_{24}\wp_{23}\lambda_1 + 2\wp_{23}\wp_{12}\lambda_3 + 4\wp_{23}\wp_{14}\lambda_2 + 2\wp_{22}\wp_{13}\lambda_3 + 4\lambda_3\wp_{14}\lambda_2 - 4\lambda_1\wp_{14}\lambda_4 \\ &- 2\wp_{33}^2\lambda_4\lambda_1 - \tfrac{2}{3}\wp_{33}\lambda_1 Q_{2444} + 2\lambda_3\wp_{11}\wp_{34} - 6\wp_{44}\lambda_3\lambda_0 + \tfrac{1}{3}\wp_{33}\lambda_4\lambda_3 Q_{2333} \\ &+ 2\lambda_2\wp_{44}\lambda_1 + 2\wp_{22}\lambda_4\lambda_1 + 2\wp_{12}\wp_{23}^2 - 4\lambda_3\wp_{24}\lambda_1 \\ \wp_{123}\wp_{134} &= -\tfrac{4}{3}\wp_{11}\wp_{13} - \tfrac{4}{3}\wp_{33}^2\lambda_0 - \tfrac{2}{3}\wp_{12}\lambda_1 - 6\wp_{14}\lambda_0 + \tfrac{1}{3}\wp_{24}\wp_{13}^2 - \tfrac{1}{6}\wp_{12}Q_{2233} \\ &+ 2\wp_{14}\wp_{13}\lambda_3 + \tfrac{5}{3}\wp_{34}\wp_{12}\wp_{13} + \tfrac{4}{3}\wp_{34}\wp_{12}\lambda_2 + \tfrac{1}{3}\wp_{23}\wp_{11}\wp_{34} + \tfrac{4}{3}\lambda_1\wp_{14}\lambda_4 \\ &+ \tfrac{5}{3}\wp_{14}\wp_{13}\wp_{23} - \wp_{23}\wp_{44}\lambda_0 - \tfrac{1}{3}\wp_{22}\wp_{34}\lambda_1 + \tfrac{2}{3}\lambda_4\wp_{12}\wp_{13} + 2\lambda_1\wp_{14}\wp_{34} + \tfrac{1}{9}\wp_{33}\lambda_1 Q_{2444} \\ &- \tfrac{2}{3}\lambda_1\wp_{13}\wp_{44} - 3\wp_{24}\wp_{34}\lambda_0 + \tfrac{1}{3}\wp_{24}\wp_{13}\lambda_2 - 3\lambda_4\wp_{24}\lambda_0 - \tfrac{1}{3}\wp_{33}^2\lambda_3\lambda_2 + \tfrac{1}{9}\wp_{33}\lambda_2 Q_{2333} \\ &+ \tfrac{1}{3}\lambda_4\wp_{12}\lambda_2 + \tfrac{1}{3}\wp_{14}\wp_{12}\wp_{33} + \tfrac{1}{2}\wp_{23}\wp_{12}\lambda_3 - \tfrac{1}{3}\wp_{23}\wp_{14}\lambda_2 - \tfrac{1}{3}\wp_{24}\wp_{11}\wp_{33} + 3\wp_{22}\lambda_0 \\ \wp_{114}\wp_{233} &= \tfrac{8}{3}\wp_{11}\wp_{13} + \tfrac{8}{3}\wp_{33}^2\lambda_0 + \tfrac{4}{3}\wp_{12}\lambda_1 + 6\wp_{14}\lambda_0 - \tfrac{2}{3}\wp_{24}\wp_{13}^2 + \tfrac{1}{3}\wp_{12}Q_{2233} \\ &+ 2\wp_{14}\wp_{13}\lambda_3 - \tfrac{4}{3}\wp_{34}\wp_{12}\wp_{13} - \tfrac{2}{3}\wp_{34}\wp_{12}\lambda_2 + \tfrac{4}{3}\wp_{23}\wp_{11}\wp_{34} + \tfrac{4}{3}\lambda_1\wp_{14}\lambda_4 \\ &+ \tfrac{8}{3}\wp_{14}\wp_{13}\wp_{23} + 2\wp_{23}\wp_{44}\lambda_0 - \tfrac{1}{3}\wp_{22}\wp_{34}\lambda_1 - \tfrac{4}{3}\lambda_4\wp_{12}\wp_{13} + 2\lambda_1\wp_{14}\wp_{34} \\ &- \tfrac{2}{3}\lambda_1\wp_{13}\wp_{44} - \tfrac{2}{3}\wp_{24}\wp_{13}\lambda_2 + \tfrac{2}{3}\wp_{33}^2\lambda_3\lambda_2 - \tfrac{2}{3}\lambda_4\wp_{12}\lambda_2 + \tfrac{4}{3}\wp_{14}\wp_{12}\wp_{33} - 3\wp_{22}\lambda_0 \\ &+ \tfrac{2}{3}\wp_{23}\wp_{14}\lambda_2 + \tfrac{2}{3}\wp_{24}\wp_{11}\wp_{33} - \tfrac{2}{9}\wp_{33}\lambda_1 Q_{2444} - \tfrac{2}{9}\wp_{33}\lambda_2 Q_{2333} - \wp_{23}\wp_{12}\lambda_3\end{aligned}$$

$$\begin{aligned}
\wp_{122}\wp_{233} &= 4\wp_{11}\wp_{13} + 4\wp_{33}^2\lambda_0 + 4\wp_{12}\lambda_1 + 18\wp_{14}\lambda_0 + \wp_{12}Q_{2233} - 2\lambda_2\wp_{11} + 2\wp_{12}\wp_{23}^2 \\
&+ 2\wp_{33}^2\lambda_4\lambda_3^2 - 2\wp_{33}\lambda_3 Q_{1244} + 6\lambda_1\wp_{14}\lambda_4 - 2\wp_{22}\wp_{34}\lambda_1 + 8\lambda_1\wp_{14}\wp_{34} - \tfrac{2}{3}\wp_{33}\lambda_4\lambda_3 Q_{2333} \\
&- 4\lambda_1\wp_{13}\wp_{44} + 2\wp_{24}\wp_{23}\lambda_1 - \wp_{23}\wp_{12}\lambda_3 + 2\wp_{33}\wp_{12}\wp_{22} - \wp_{22}\wp_{13}\lambda_3 - \wp_{22}\lambda_4\lambda_1 - 9\wp_{22}\lambda_0 \\
&- 2\lambda_3\wp_{14}\lambda_2 + 4\wp_{33}^2\lambda_4\lambda_1 + \tfrac{4}{3}\wp_{33}\lambda_1 Q_{2444} - 4\lambda_3\wp_{11}\wp_{34} - 6\wp_{44}\lambda_3\lambda_0 + 2\lambda_3\wp_{24}\lambda_1 \\
\wp_{113}\wp_{234} &= \tfrac{2}{3}\wp_{11}\wp_{13} - \tfrac{4}{3}\wp_{33}^2\lambda_0 + \tfrac{4}{3}\wp_{12}\lambda_1 + \tfrac{4}{3}\wp_{24}\wp_{13}^2 + \tfrac{1}{3}\wp_{12}Q_{2233} + 2\wp_{14}\wp_{13}\lambda_3 \\
&+ \tfrac{2}{3}\wp_{34}\wp_{12}\wp_{13} + \tfrac{4}{3}\wp_{34}\wp_{12}\lambda_2 + \tfrac{4}{3}\wp_{23}\wp_{11}\wp_{34} - \tfrac{2}{3}\lambda_1\wp_{14}\lambda_4 + \tfrac{2}{3}\wp_{14}\wp_{13}\wp_{23} \\
&+ 2\wp_{23}\wp_{44}\lambda_0 - \tfrac{4}{3}\wp_{22}\wp_{34}\lambda_1 - \tfrac{4}{3}\lambda_4\wp_{12}\wp_{13} + 2\lambda_1\wp_{14}\wp_{34} - \tfrac{8}{3}\lambda_1\wp_{13}\wp_{44} \\
&+ \tfrac{4}{3}\wp_{24}\wp_{13}\lambda_2 - \tfrac{1}{3}\wp_{33}^2\lambda_3\lambda_2 - \tfrac{2}{3}\lambda_4\wp_{12}\lambda_2 - \tfrac{2}{3}\wp_{14}\wp_{12}\wp_{33} - \wp_{23}\wp_{12}\lambda_3 \\
&+ \tfrac{2}{3}\wp_{24}\wp_{11}\wp_{33} + \tfrac{1}{9}\wp_{33}\lambda_1 Q_{2444} + \tfrac{1}{9}\wp_{33}\lambda_2 Q_{2333} - \tfrac{4}{3}\wp_{23}\wp_{14}\lambda_2 \\
\wp_{112}\wp_{334} &= \tfrac{8}{3}\wp_{11}\wp_{13} + \tfrac{8}{3}\wp_{33}^2\lambda_0 - \tfrac{2}{3}\wp_{12}\lambda_1 - \tfrac{2}{3}\wp_{24}\wp_{13}^2 - \tfrac{2}{3}\wp_{12}Q_{2233} - 4\wp_{14}\wp_{13}\lambda_3 \\
&+ \tfrac{8}{3}\wp_{34}\wp_{12}\wp_{13} + \tfrac{4}{3}\wp_{34}\wp_{12}\lambda_2 + \tfrac{4}{3}\wp_{23}\wp_{11}\wp_{34} + \tfrac{10}{3}\lambda_1\wp_{14}\lambda_4 - \tfrac{4}{3}\wp_{14}\wp_{13}\wp_{23} \\
&- 4\wp_{23}\wp_{44}\lambda_0 + \tfrac{2}{3}\wp_{22}\wp_{34}\lambda_1 + \tfrac{8}{3}\lambda_4\wp_{12}\wp_{13} + 2\lambda_1\wp_{14}\wp_{34} + \tfrac{10}{3}\lambda_1\wp_{13}\wp_{44} + \tfrac{4}{3}\wp_{14}\wp_{12}\wp_{33} \\
&- 6\wp_{24}\wp_{34}\lambda_0 - \tfrac{2}{3}\wp_{24}\wp_{13}\lambda_2 - 6\lambda_4\wp_{24}\lambda_0 + \tfrac{2}{3}\wp_{33}^2\lambda_3\lambda_2 + \tfrac{4}{3}\lambda_4\wp_{12}\lambda_2 - \wp_{24}\wp_{23}\lambda_1 \\
&+ 2\wp_{23}\wp_{12}\lambda_3 + \tfrac{8}{3}\wp_{23}\wp_{14}\lambda_2 + \tfrac{2}{3}\wp_{24}\wp_{11}\wp_{33} - \tfrac{2}{9}\wp_{33}\lambda_1 Q_{2444} - \tfrac{2}{9}\wp_{33}\lambda_2 Q_{2333}
\end{aligned}$$

The relations at weight -24 have not been derived, (see Section 5.3.3 for more details). The remaining relations from weight -25 to weight -42 could be calculated normally without any quadratic Q-terms. To skip the remainder of this Appendix proceed to page 289. The following relations are at weight -25.

$$\begin{aligned}
\wp_{222}\wp_{224} &= \tfrac{2}{3}\lambda_4\lambda_3\lambda_1 - \tfrac{4}{3}\lambda_4^2\lambda_3\lambda_2 + \tfrac{10}{3}\lambda_4\lambda_2^2 - \tfrac{20}{3}\lambda_2\lambda_1 - 4\lambda_4^2\lambda_0 + \tfrac{4}{3}\lambda_4\lambda_3\wp_{13}\wp_{34} \\
&+ \tfrac{2}{3}\wp_{12}Q_{2333} - 4\wp_{14}Q_{1244} + \tfrac{4}{3}\wp_{11}Q_{2444} - \tfrac{2}{3}\wp_{13}Q_{2233} - 4\wp_{22}Q_{1244} - 2\lambda_2\wp_{23}^2 \\
&+ 4\wp_{13}\lambda_3^2 + 4\wp_{44}\wp_{33}\lambda_0 + 8\wp_{24}\wp_{33}\lambda_1 - \tfrac{2}{3}\lambda_3\wp_{34}^2\lambda_2 + 4\lambda_3\wp_{13}\wp_{23} - 2\wp_{24}\wp_{33}\lambda_3^2 \\
&+ 2\wp_{34}\wp_{23}\lambda_3^2 + \tfrac{8}{3}\lambda_2\wp_{13}\wp_{34} - 4\lambda_4\wp_{23}\lambda_1 + 4\lambda_2\wp_{23}\lambda_3 - 4\lambda_3\wp_{12}\wp_{33} + 4\lambda_4\wp_{13}\lambda_2 \\
&+ \tfrac{8}{3}\lambda_3\wp_{34}\lambda_1 + 4\lambda_4\wp_{11}\wp_{33} + 4\wp_{34}\lambda_4^2\lambda_1 + 4\lambda_4\wp_{34}\lambda_0 + 4\lambda_4\wp_{14}\wp_{33}\lambda_3 + 6\lambda_3\lambda_0 \\
&+ 4\lambda_4\wp_{22}\wp_{33}\lambda_3 - \tfrac{4}{3}\wp_{14}\lambda_4 Q_{2333} + 6\lambda_2\wp_{22}\wp_{33} - \tfrac{4}{3}\lambda_4\wp_{13}^2 - \tfrac{4}{3}\wp_{34}\lambda_2^2 - \tfrac{5}{3}\lambda_2 Q_{2233} \\
&+ 4\lambda_4^2\wp_{24}\wp_{33}\lambda_3 - \tfrac{32}{3}\lambda_1\wp_{13} + 4\wp_{23}\lambda_0 - 4\wp_{34}^2\lambda_0 + 4\wp_{24}\wp_{22}^2 + 4\wp_{33}\wp_{24}\lambda_4\lambda_2 \\
&- \tfrac{2}{3}\lambda_4\lambda_3\wp_{44}Q_{2333} - \tfrac{8}{3}\lambda_3\wp_{13}\lambda_4^2 - 2\lambda_4\wp_{23}\lambda_3^2 + \tfrac{4}{3}\wp_{24}\lambda_2 Q_{2444} - \tfrac{4}{3}\lambda_4\wp_{22}Q_{2333} \\
&+ \tfrac{2}{3}\wp_{14}\lambda_3 Q_{2444} + \tfrac{2}{3}\wp_{22}\lambda_3 Q_{2444} - \tfrac{1}{3}\wp_{34}\lambda_3 Q_{2233} - 2\wp_{24}Q_{1224} + 2\wp_{44}\wp_{33}\lambda_4\lambda_3^2 \\
&- \tfrac{4}{3}\wp_{24}\lambda_4^2 Q_{2333} - 2\wp_{44}\lambda_3 Q_{1244} + 2\wp_{44}\wp_{33}\lambda_3\lambda_2 - 4\wp_{24}\lambda_4 Q_{1244} + \tfrac{2}{3}\lambda_4\lambda_3 Q_{2233} \\
\wp_{133}\wp_{134} &= \tfrac{2}{3}\wp_{14}\lambda_4 Q_{2333} + \tfrac{4}{3}\lambda_2\wp_{13}\wp_{34} - 2\lambda_4\wp_{14}\wp_{33}\lambda_3 + 2\wp_{34}\wp_{13}^2 + \lambda_3\wp_{13}\wp_{23} \\
&+ \tfrac{4}{3}\lambda_4\wp_{13}^2 - 2\wp_{44}\wp_{33}\lambda_0 + 2\lambda_4\wp_{34}\lambda_0 + 2\wp_{34}^2\lambda_0 - 2\wp_{23}\lambda_0 + 2\wp_{14}\wp_{13}\wp_{33} \\
&+ \tfrac{2}{3}\lambda_1\wp_{13} + \tfrac{1}{2}\lambda_3\wp_{12}\wp_{33} - \tfrac{1}{3}\wp_{13}Q_{2233} - \tfrac{1}{6}\wp_{12}Q_{2333} + 2\wp_{14}Q_{1244} + \tfrac{2}{3}\lambda_4\wp_{13}\lambda_2
\end{aligned}$$

$$\wp_{144}\wp_{222} = \tfrac{2}{3}\lambda_4\lambda_2^2 + 2\lambda_4\wp_{34}^2\lambda_1 - 2\lambda_3\lambda_0 + \tfrac{1}{3}\wp_{12}Q_{2333} - 8\wp_{14}Q_{1244} + 2\wp_{22}Q_{1244}$$
$$+ 6\wp_{44}\wp_{33}\lambda_0 + 6\wp_{24}\wp_{33}\lambda_1 - \tfrac{4}{3}\lambda_3\wp_{34}^2\lambda_2 - \lambda_3\wp_{13}\wp_{23} + \wp_{24}\wp_{33}\lambda_3^2 - \wp_{34}\wp_{23}\lambda_3^2$$
$$+ 4\lambda_2\wp_{13}\wp_{34} - \tfrac{4}{3}\lambda_4\wp_{23}\lambda_1 + \lambda_2\wp_{23}\lambda_3 - \lambda_3\wp_{12}\wp_{33} + \tfrac{4}{3}\lambda_4\wp_{13}\lambda_2 - \tfrac{14}{3}\lambda_3\wp_{34}\lambda_1$$
$$+ 2\wp_{34}\lambda_4^2\lambda_1 + 6\lambda_4\wp_{34}\lambda_0 + 2\wp_{22}\wp_{12}\wp_{44} - 6\wp_{23}\wp_{34}\lambda_1 + 4\lambda_2\wp_{14}\wp_{33} - \tfrac{4}{3}\lambda_2\lambda_1$$
$$+ 8\lambda_4\wp_{14}\wp_{33}\lambda_3 - \tfrac{4}{3}\lambda_4\lambda_3\wp_{13}\wp_{34} - \lambda_4\wp_{22}\wp_{33}\lambda_3 + 2\lambda_4\wp_{33}\wp_{44}\lambda_1 - \tfrac{2}{3}\lambda_4\lambda_3\wp_{34}\lambda_2$$
$$+ \tfrac{4}{3}\lambda_4\wp_{23}\wp_{34}\lambda_2 - 2\wp_{23}\lambda_0 - 2\wp_{34}^2\lambda_0 + \tfrac{4}{3}\wp_{34}\lambda_2^2 + \tfrac{2}{3}\wp_{14}\lambda_3Q_{2444} + \tfrac{1}{3}\wp_{34}\lambda_3Q_{2233}$$
$$- \tfrac{8}{3}\wp_{14}\lambda_4Q_{2333} - \tfrac{2}{3}\wp_{24}\lambda_4^2Q_{2333} + \tfrac{1}{3}\lambda_4\wp_{22}Q_{2333} - 2\wp_{12}\wp_{24}^2 + 2\lambda_4^2\wp_{24}\wp_{33}\lambda_3$$
$$- 2\wp_{24}Q_{1224} - 2\wp_{24}\lambda_4Q_{1244} - \tfrac{1}{3}\lambda_4\wp_{23}Q_{2233} - \tfrac{1}{3}\wp_{24}\lambda_3Q_{2333} + \tfrac{4}{3}\lambda_4^2\wp_{13}\wp_{23}$$
$$+ \tfrac{2}{3}\wp_{23}\lambda_4^2\lambda_2 + \lambda_4\wp_{23}^2\lambda_3 - 4\lambda_1\wp_{13} + \tfrac{4}{3}\wp_{44}\lambda_1Q_{2444} + 4\wp_{22}\wp_{14}\wp_{24} - \tfrac{1}{3}\lambda_2Q_{2233}$$

$$\wp_{133}\wp_{223} = -\tfrac{2}{3}\lambda_4\lambda_2^2 + \tfrac{4}{3}\lambda_2\lambda_1 + 2\lambda_4^2\lambda_0 - 6\lambda_3\lambda_0 - 4\wp_{14}Q_{1244} + \tfrac{4}{3}\wp_{11}Q_{2444} - 6\wp_{23}\lambda_0$$
$$+ \tfrac{1}{3}\wp_{13}Q_{2233} + 2\wp_{22}Q_{1244} - 2\wp_{44}\wp_{33}\lambda_0 - 2\wp_{24}\wp_{33}\lambda_1 + \lambda_3\wp_{13}\wp_{23} - \tfrac{4}{3}\lambda_2\wp_{13}\wp_{34}$$
$$- \lambda_2\wp_{23}\lambda_3 - \lambda_3\wp_{12}\wp_{33} + 2\lambda_3\wp_{34}\lambda_1 + 2\lambda_4\wp_{11}\wp_{33} + 4\lambda_4\wp_{34}\lambda_0 + 2\wp_{22}\wp_{13}\wp_{33}$$
$$+ 2\wp_{23}\wp_{34}\lambda_1 + 4\lambda_2\wp_{14}\wp_{33} + 4\lambda_4\wp_{14}\wp_{33}\lambda_3 - 2\lambda_4\wp_{22}\wp_{33}\lambda_3 + \tfrac{4}{3}\lambda_1\wp_{13} + \tfrac{1}{3}\lambda_2Q_{2233}$$
$$+ 2\wp_{34}^2\lambda_0 + \tfrac{2}{3}\lambda_4\wp_{13}^2 - \tfrac{4}{3}\wp_{34}\lambda_2^2 + 2\wp_{13}\wp_{23}^2 - \tfrac{4}{3}\wp_{14}\lambda_4Q_{2333} + \tfrac{2}{3}\lambda_4\wp_{22}Q_{2333}$$

$$\wp_{124}\wp_{224} = -\tfrac{1}{3}\lambda_4\lambda_2^2 + \tfrac{2}{3}\lambda_2\lambda_1 - 2\lambda_3\lambda_0 - 2\wp_{14}Q_{1244} - 2\wp_{24}\wp_{33}\lambda_1 + \tfrac{2}{3}\lambda_3\wp_{34}^2\lambda_2$$
$$- \tfrac{1}{2}\wp_{24}\wp_{33}\lambda_3^2 + \tfrac{1}{2}\wp_{34}\wp_{23}\lambda_3^2 - \tfrac{1}{2}\lambda_2\wp_{23}\lambda_3 - \tfrac{2}{3}\lambda_4\wp_{13}\lambda_2 + \tfrac{4}{3}\lambda_3\wp_{34}\lambda_1 - \tfrac{2}{3}\wp_{44}\lambda_1Q_{2444}$$
$$+ 2\wp_{23}\wp_{34}\lambda_1 + 2\lambda_2\wp_{14}\wp_{33} - 2\lambda_4\wp_{34}^2\lambda_1 + 2\lambda_4\wp_{14}\wp_{33}\lambda_3 + \tfrac{2}{3}\lambda_4\lambda_3\wp_{13}\wp_{34} + \tfrac{1}{6}\lambda_2Q_{2233}$$
$$+ \tfrac{1}{3}\lambda_4\lambda_3\wp_{34}\lambda_2 - 2\wp_{23}\lambda_0 - 2\wp_{34}^2\lambda_0 - \tfrac{2}{3}\wp_{34}\lambda_2^2 + 2\wp_{12}\wp_{24}^2 - \tfrac{2}{3}\wp_{14}\lambda_4Q_{2333} + 4\lambda_4\wp_{34}\lambda_0$$
$$+ \tfrac{2}{3}\wp_{14}\lambda_3Q_{2444} - \tfrac{1}{6}\wp_{34}\lambda_3Q_{2233} + \wp_{24}Q_{1224} + \tfrac{1}{6}\wp_{24}\lambda_3Q_{2333} + 2\wp_{22}\wp_{14}\wp_{24}$$

$$\wp_{123}\wp_{233} = -\tfrac{2}{3}\lambda_4\lambda_2^2 + \tfrac{4}{3}\lambda_2\lambda_1 + 2\lambda_4^2\lambda_0 + \tfrac{1}{3}\lambda_2Q_{2233} + 2\wp_{14}Q_{1244} - \tfrac{2}{3}\wp_{11}Q_{2444}$$
$$+ \tfrac{1}{3}\wp_{13}Q_{2233} - \wp_{22}Q_{1244} - 2\wp_{44}\wp_{33}\lambda_0 + \lambda_3\wp_{13}\wp_{23} - \tfrac{4}{3}\lambda_2\wp_{13}\wp_{34} - \lambda_2\wp_{23}\lambda_3$$
$$+ 2\lambda_3\wp_{12}\wp_{33} + 2\lambda_3\wp_{34}\lambda_1 - 2\lambda_4\wp_{11}\wp_{33} + 4\lambda_4\wp_{34}\lambda_0 + 2\wp_{23}\wp_{12}\wp_{33}$$
$$- 2\lambda_4\wp_{14}\wp_{33}\lambda_3 + \lambda_4\wp_{22}\wp_{33}\lambda_3 + \tfrac{4}{3}\lambda_1\wp_{13} - 6\wp_{23}\lambda_0 + 2\wp_{34}^2\lambda_0 + \tfrac{2}{3}\lambda_4\wp_{13}^2$$
$$- \tfrac{4}{3}\wp_{34}\lambda_2^2 + 2\wp_{13}\wp_{23}^2 + \tfrac{2}{3}\wp_{14}\lambda_4Q_{2333} - \tfrac{1}{3}\lambda_4\wp_{22}Q_{2333} + 2\wp_{23}\wp_{34}\lambda_1 - 6\lambda_3\lambda_0$$

$$\wp_{124}\wp_{144} = \lambda_4\wp_{14}\wp_{33}\lambda_3 - 2\wp_{34}^2\lambda_0 + \tfrac{2}{3}\lambda_2\wp_{13}\wp_{34} + 2\lambda_4\wp_{34}\lambda_0 + \tfrac{2}{3}\lambda_4\wp_{13}^2 - \tfrac{1}{6}\wp_{13}Q_{2233}$$
$$+ 2\wp_{14}^2\wp_{24} + \tfrac{1}{2}\lambda_3\wp_{13}\wp_{23} + \tfrac{1}{3}\lambda_4\wp_{13}\lambda_2 - \tfrac{2}{3}\lambda_1\wp_{13} + 2\wp_{14}\wp_{12}\wp_{44} + 2\wp_{44}\wp_{33}\lambda_0$$
$$- \tfrac{1}{3}\wp_{14}\lambda_4Q_{2333}$$

$$\wp_{114}\wp_{333} = \tfrac{2}{3}\wp_{14}\lambda_4Q_{2333} + 2\wp_{34}\wp_{11}\wp_{33} - \tfrac{2}{3}\lambda_2\wp_{13}\wp_{34} - 2\lambda_4\wp_{14}\wp_{33}\lambda_3 - 2\wp_{34}\wp_{13}^2$$
$$- 2\lambda_3\wp_{13}\wp_{23} - \tfrac{8}{3}\lambda_4\wp_{13}^2 - 2\wp_{44}\wp_{33}\lambda_0 - 4\lambda_4\wp_{34}\lambda_0 - \wp_{23}\wp_{34}\lambda_1 - 4\wp_{34}^2\lambda_0$$
$$+ 4\wp_{14}\wp_{13}\wp_{33} - \tfrac{4}{3}\lambda_4\wp_{13}\lambda_2 + \tfrac{2}{3}\lambda_1\wp_{13} - \lambda_3\wp_{12}\wp_{33} + 2\wp_{14}Q_{1244} + \tfrac{2}{3}\wp_{13}Q_{2233}$$
$$+ \tfrac{1}{3}\wp_{12}Q_{2333} + \wp_{23}\lambda_0$$

$$\wp_{122}\wp_{333} = \lambda_3\wp_{13}\wp_{23} - \tfrac{2}{3}\lambda_2\lambda_1 + 2\lambda_4^2\lambda_0 - 6\lambda_3\lambda_0 + \wp_{12}Q_{2333} + 2\wp_{14}Q_{1244} - \tfrac{2}{3}\lambda_2 Q_{2233}$$
$$- \tfrac{2}{3}\wp_{11}Q_{2444} - \tfrac{2}{3}\wp_{13}Q_{2233} + 2\wp_{22}Q_{1244} - 2\wp_{44}\wp_{33}\lambda_0 - 2\wp_{24}\wp_{33}\lambda_1 + \tfrac{4}{3}\lambda_4\lambda_2^2$$
$$+ \tfrac{20}{3}\lambda_2\wp_{13}\wp_{34} + \lambda_4\wp_{23}\lambda_1 + 2\lambda_2\wp_{23}\lambda_3 - \lambda_3\wp_{12}\wp_{33} - 2\wp_{13}\wp_{23}^2 + 2\lambda_4\wp_{13}\lambda_2 - 4\lambda_3\wp_{34}\lambda_1$$
$$- 2\lambda_4\wp_{11}\wp_{33} - 8\lambda_4\wp_{34}\lambda_0 + 2\wp_{22}\wp_{13}\wp_{33} + 4\wp_{23}\wp_{12}\wp_{33} - 6\wp_{23}\wp_{34}\lambda_1 + \tfrac{2}{3}\wp_{14}\lambda_4 Q_{2333}$$
$$- 2\lambda_4\wp_{14}\wp_{33}\lambda_3 - 2\lambda_4\wp_{22}\wp_{33}\lambda_3 - \tfrac{8}{3}\lambda_1\wp_{13} - 3\wp_{23}\lambda_0 - 10\wp_{34}^2\lambda_0 - \tfrac{10}{3}\lambda_4\wp_{13}^2 + \tfrac{8}{3}\wp_{34}\lambda_2^2$$
$$+ \tfrac{2}{3}\lambda_4\wp_{22}Q_{2333}$$

$$\wp_{122}\wp_{244} = -\tfrac{2}{3}\lambda_4\lambda_2^2 - \tfrac{2}{3}\lambda_2\lambda_1 - 2\lambda_3\lambda_0 - \tfrac{2}{3}\wp_{12}Q_{2333} + 6\wp_{14}Q_{1244} - \tfrac{2}{3}\wp_{11}Q_{2444} - \tfrac{8}{3}\lambda_1\wp_{13}$$
$$+ \tfrac{1}{3}\wp_{13}Q_{2233} - 2\wp_{44}\wp_{33}\lambda_0 + 4\wp_{24}\wp_{33}\lambda_1 - \tfrac{4}{3}\lambda_3\wp_{34}^2\lambda_2 - 2\lambda_3\wp_{13}\wp_{23} - \tfrac{8}{3}\lambda_3\wp_{34}\lambda_1$$
$$- \wp_{34}\wp_{23}\lambda_3^2 - \tfrac{4}{3}\lambda_2\wp_{13}\wp_{34} - \lambda_4\wp_{23}\lambda_1 - \lambda_2\wp_{23}\lambda_3 + 4\lambda_3\wp_{12}\wp_{33} - 2\lambda_4\wp_{13}\lambda_2$$
$$+ 2\wp_{34}\lambda_4^2\lambda_1 + 6\lambda_4\wp_{34}\lambda_0 + 2\wp_{22}\wp_{12}\wp_{44} - 4\wp_{23}\wp_{34}\lambda_1 - 4\lambda_2\wp_{14}\wp_{33} - \tfrac{4}{3}\wp_{14}\lambda_3 Q_{2444}$$
$$+ 2\lambda_4\wp_{34}^2\lambda_1 - 8\lambda_4\wp_{14}\wp_{33}\lambda_3 + \tfrac{2}{3}\lambda_4\lambda_3\wp_{13}\wp_{34} + 2\lambda_4\wp_{33}\wp_{44}\lambda_1 - \tfrac{2}{3}\lambda_4\lambda_3\wp_{34}\lambda_2$$
$$- 3\wp_{23}\lambda_0 + 6\wp_{34}^2\lambda_0 - \tfrac{4}{3}\lambda_4\wp_{13}^2 - \tfrac{4}{3}\wp_{34}\lambda_2^2 + 2\wp_{12}\wp_{24}^2 + 2\wp_{14}\lambda_4 Q_{2333} + \wp_{24}\wp_{33}\lambda_3^2$$
$$+ \tfrac{1}{3}\wp_{34}\lambda_3 Q_{2233} - \tfrac{1}{3}\wp_{24}\lambda_3 Q_{2333} + \tfrac{4}{3}\wp_{44}\lambda_1 Q_{2444} + \tfrac{2}{3}\lambda_4\wp_{12}Q_{2444} + \tfrac{1}{3}\lambda_2 Q_{2233}$$

$$\wp_{114}\wp_{244} = -\tfrac{1}{2}\wp_{24}\lambda_3 Q_{2333} + \tfrac{2}{3}\wp_{14}\lambda_4 Q_{2333} - \wp_{24}Q_{1224} - 2\lambda_2\wp_{14}\wp_{33} - 2\lambda_4\wp_{14}\wp_{33}\lambda_3$$
$$+ \tfrac{3}{2}\wp_{24}\wp_{33}\lambda_3^2 - \lambda_3\wp_{13}\wp_{23} - \tfrac{4}{3}\lambda_4\wp_{13}^2 + 2\wp_{14}^2\wp_{24} + 4\wp_{24}\wp_{33}\lambda_1 + 2\wp_{24}\wp_{11}\wp_{44}$$
$$- \wp_{23}\wp_{34}\lambda_1 + \wp_{23}\lambda_0 - \tfrac{2}{3}\lambda_4\wp_{13}\lambda_2 - \tfrac{2}{3}\lambda_1\wp_{13} + 2\wp_{14}Q_{1244} + \tfrac{1}{3}\wp_{13}Q_{2233} + \tfrac{2}{3}\lambda_2\wp_{13}\wp_{34}$$

$$\wp_{112}\wp_{444} = -\tfrac{4}{3}\lambda_4\lambda_2^2 + \tfrac{2}{3}\lambda_2\lambda_1 - 2\wp_{14}^2\wp_{24} - \tfrac{1}{3}\wp_{12}Q_{2333} - 4\wp_{14}Q_{1244} + \tfrac{4}{3}\wp_{11}Q_{2444}$$
$$- \tfrac{1}{3}\wp_{13}Q_{2233} + 4\wp_{44}\wp_{33}\lambda_0 - 6\wp_{24}\wp_{33}\lambda_1 + \lambda_3\wp_{13}\wp_{23} - \tfrac{3}{2}\wp_{24}\wp_{33}\lambda_3^2 + \tfrac{2}{3}\lambda_2 Q_{2233}$$
$$- \tfrac{14}{3}\lambda_2\wp_{13}\wp_{34} - 2\lambda_2\wp_{23}\lambda_3 + \lambda_3\wp_{12}\wp_{33} - 2\lambda_4\wp_{13}\lambda_2 + 2\lambda_3\wp_{34}\lambda_1 + 4\lambda_4\wp_{34}\lambda_0$$
$$+ 3\wp_{23}\wp_{34}\lambda_1 + 6\lambda_2\wp_{14}\wp_{33} + 2\lambda_4\lambda_3\wp_{13}\wp_{34} + \lambda_4\wp_{22}\wp_{33}\lambda_3 + \tfrac{2}{3}\lambda_1\wp_{13} - \tfrac{8}{3}\wp_{34}\lambda_2^2$$
$$+ 4\wp_{34}^2\lambda_0 + \tfrac{4}{3}\lambda_4\wp_{13}^2 - 2\lambda_4^2\wp_{24}\wp_{33}\lambda_3 + \tfrac{1}{3}\lambda_4\wp_{23}\lambda_1 - \tfrac{2}{3}\wp_{14}\lambda_4 Q_{2333} + \tfrac{2}{3}\wp_{24}\lambda_4^2 Q_{2333}$$
$$- \tfrac{1}{3}\lambda_4\wp_{22}Q_{2333} + \wp_{24}Q_{1224} + 2\wp_{44}\wp_{33}\lambda_3\lambda_2 + 2\wp_{24}\lambda_4 Q_{1244} + \tfrac{1}{3}\lambda_4\wp_{23}Q_{2233}$$
$$+ \tfrac{1}{2}\wp_{24}\lambda_3 Q_{2333} - \tfrac{4}{3}\lambda_4^2\wp_{13}\wp_{23} - \tfrac{2}{3}\wp_{23}\lambda_4^2\lambda_2 - \tfrac{4}{3}\lambda_4\wp_{23}\wp_{34}\lambda_2 - \tfrac{2}{3}\wp_{44}\lambda_1 Q_{2444}$$
$$+ 4\wp_{14}\wp_{12}\wp_{44} - \tfrac{2}{3}\lambda_4\wp_{12}Q_{2444} + 2\wp_{24}\wp_{11}\wp_{44} - \tfrac{2}{3}\wp_{44}\lambda_2 Q_{2333} - \lambda_4\wp_{23}^2\lambda_3 - 2\wp_{23}\lambda_0$$

$$\wp_{113}\wp_{334} = -\tfrac{4}{3}\wp_{14}\lambda_4 Q_{2333} + 2\wp_{34}\wp_{11}\wp_{33} + 2\lambda_2\wp_{14}\wp_{33} + 4\lambda_4\wp_{14}\wp_{33}\lambda_3 + \tfrac{1}{3}\wp_{12}Q_{2333}$$
$$+ 2\wp_{34}\wp_{13}^2 + \lambda_3\wp_{13}\wp_{23} + \tfrac{4}{3}\lambda_4\wp_{13}^2 - \wp_{24}\wp_{33}\lambda_1 + 4\wp_{44}\wp_{33}\lambda_0 + 2\lambda_4\wp_{34}\lambda_0 + 2\wp_{34}^2\lambda_0$$
$$- 2\wp_{23}\lambda_0 + \tfrac{2}{3}\lambda_4\wp_{13}\lambda_2 + \tfrac{2}{3}\lambda_1\wp_{13} - \lambda_3\wp_{12}\wp_{33} - 4\wp_{14}Q_{1244} - \tfrac{1}{3}\wp_{13}Q_{2233} + \tfrac{4}{3}\lambda_2\wp_{13}\wp_{34}$$

$$\wp_{114}\wp_{134} = \tfrac{1}{2}\wp_{33}\lambda_3\lambda_0 + \tfrac{1}{2}\lambda_3\wp_{33}\wp_{34}\lambda_1 - 2\lambda_1\wp_{14}\wp_{24} + 2\wp_{13}\wp_{14}^2 - \tfrac{1}{6}\wp_{34}\lambda_1 Q_{2333}$$
$$- 2\wp_{44}\wp_{14}\lambda_0 + 2\wp_{34}\wp_{11}\wp_{14} - \tfrac{2}{3}\wp_{34}\lambda_0 Q_{2444} - \tfrac{1}{6}\lambda_0 Q_{2333} + 2\wp_{24}^2\lambda_0 + 2\lambda_2\wp_{14}^2$$

The following relations are at weight -26.

$$\begin{aligned}
\wp_{123}^2 &= \tfrac{1}{3}\wp_{23}\wp_{12}\lambda_2 + \tfrac{1}{3}\wp_{22}\wp_{13}\lambda_2 - \tfrac{1}{3}\wp_{33}\wp_{11}\wp_{22} + 3\wp_{22}\wp_{34}\lambda_0 + \tfrac{1}{3}\lambda_1\wp_{12}\wp_{34} \\
&- \tfrac{8}{3}\wp_{33}^2\lambda_4\lambda_0 + 3\lambda_4\wp_{22}\lambda_0 + \tfrac{10}{3}\wp_{23}\wp_{12}\wp_{13} + 3\wp_{13}\wp_{12}\lambda_3 - \tfrac{2}{3}\wp_{33}\lambda_0 Q_{2444} - 6\lambda_0\wp_{14}\wp_{34} \\
&+ \tfrac{1}{9}\wp_{33}\lambda_1 Q_{2333} + \tfrac{2}{3}\wp_{33}\lambda_2 Q_{1244} + 2\lambda_1\wp_{14}\lambda_3 + \tfrac{1}{3}\wp_{11}\wp_{23}^2 + \tfrac{2}{3}\wp_{44}\lambda_1^2 + \tfrac{2}{3}\wp_{14}\lambda_2^2 \\
&- \tfrac{1}{3}\lambda_1\wp_{13}\wp_{24} - \tfrac{2}{3}\wp_{33}^2\lambda_4\lambda_3\lambda_2 + \tfrac{2}{9}\wp_{33}\lambda_4\lambda_2 Q_{2333} + \tfrac{1}{3}\wp_{12}^2\wp_{33} - \tfrac{1}{3}\wp_{11}Q_{2233} - \tfrac{4}{3}\lambda_4\wp_{11}\wp_{13} \\
&- 2\lambda_1\wp_{11} + 2\lambda_0\wp_{44}\lambda_2 - 9\wp_{23}\wp_{24}\lambda_0 + \wp_{11}\wp_{23}\lambda_3 + 2\lambda_0\wp_{13}\wp_{44} + 2\lambda_2\wp_{11}\wp_{34} \\
&- \tfrac{1}{3}\lambda_2\wp_{24}\lambda_1 + \tfrac{2}{3}\lambda_2\wp_{13}\wp_{14} - \tfrac{2}{3}\wp_{33}^2\lambda_3\lambda_1 - 9\lambda_3\wp_{24}\lambda_0 - \tfrac{1}{3}\wp_{22}\wp_{23}\lambda_1 + \tfrac{1}{3}\wp_{12}\lambda_4\lambda_1 \\
&- 6\lambda_0\wp_{14}\lambda_4 + \tfrac{8}{3}\lambda_1\wp_{14}\wp_{23} + \tfrac{1}{3}\wp_{22}\wp_{13}^2 + \tfrac{2}{3}\wp_{11}\lambda_4\lambda_2 \\
\wp_{122}\wp_{133} &= -\tfrac{2}{3}\wp_{23}\wp_{12}\lambda_2 - \tfrac{2}{3}\wp_{22}\wp_{13}\lambda_2 + \tfrac{2}{3}\wp_{22}\wp_{23}\lambda_1 + \tfrac{1}{3}\wp_{12}\lambda_4\lambda_1 - \tfrac{8}{3}\lambda_1\wp_{12}\wp_{34} - \tfrac{4}{3}\wp_{14}\lambda_2^2 \\
&+ \tfrac{16}{3}\wp_{33}^2\lambda_4\lambda_0 + \tfrac{4}{3}\wp_{23}\wp_{12}\wp_{13} + 3\wp_{13}\wp_{12}\lambda_3 + \tfrac{4}{3}\wp_{33}\lambda_0 Q_{2444} + \tfrac{2}{3}\lambda_1\wp_{14}\wp_{23} + \tfrac{2}{3}\lambda_2\wp_{24}\lambda_1 \\
&- \tfrac{4}{3}\wp_{33}\lambda_2 Q_{1244} + 2\lambda_1\wp_{14}\lambda_3 + \tfrac{4}{3}\wp_{12}^2\wp_{33} + \tfrac{4}{3}\wp_{22}\wp_{13}^2 - \tfrac{1}{3}\wp_{11}Q_{2233} - \tfrac{2}{9}\wp_{33}\lambda_1 Q_{2333} \\
&+ \tfrac{4}{3}\wp_{33}^2\lambda_4\lambda_3\lambda_2 - \tfrac{4}{9}\wp_{33}\lambda_4\lambda_2 Q_{2333} - 3\lambda_0\wp_{12} - \tfrac{2}{3}\wp_{11}\wp_{23}^2 + \tfrac{2}{3}\wp_{44}\lambda_1^2 + \tfrac{2}{3}\wp_{11}\lambda_4\lambda_2 \\
&- 2\lambda_1\wp_{11} - 4\lambda_0\wp_{44}\lambda_2 + 6\wp_{23}\wp_{24}\lambda_0 + \wp_{11}\wp_{23}\lambda_3 - 4\lambda_0\wp_{13}\wp_{44} - \tfrac{10}{3}\lambda_1\wp_{13}\wp_{24} \\
&+ \tfrac{8}{3}\lambda_2\wp_{13}\wp_{14} + \tfrac{4}{3}\wp_{33}^2\lambda_3\lambda_1 + \tfrac{2}{3}\wp_{33}\wp_{11}\wp_{22} + 6\wp_{22}\wp_{34}\lambda_0 + 12\lambda_0\wp_{14}\lambda_4 - \tfrac{4}{3}\lambda_4\wp_{11}\wp_{13} \\
\wp_{114}\wp_{133} &= 2\wp_{14}\wp_{13}^2 + 2\wp_{14}\wp_{11}\wp_{33} + 2\lambda_2\wp_{13}\wp_{14} + \wp_{33}^2\lambda_3\lambda_1 - \lambda_1\wp_{12}\wp_{34} - \tfrac{1}{3}\wp_{33}\lambda_1 Q_{2333} \\
&- 2\lambda_1\wp_{13}\wp_{24} + 2\wp_{23}\wp_{24}\lambda_0 + 2\lambda_0\wp_{13}\wp_{44} + 2\wp_{22}\wp_{34}\lambda_0 - \tfrac{4}{3}\wp_{33}\lambda_0 Q_{2444} - \lambda_0\wp_{12} \\
&+ 4\lambda_0\wp_{14}\lambda_4 \\
\wp_{113}\wp_{223} &= \tfrac{4}{3}\wp_{23}\wp_{12}\lambda_2 - \tfrac{4}{3}\wp_{22}\wp_{23}\lambda_1 + \tfrac{4}{3}\wp_{22}\wp_{13}^2 + \tfrac{4}{3}\wp_{12}\lambda_4\lambda_1 + \tfrac{4}{3}\lambda_1\wp_{12}\wp_{34} - \tfrac{4}{3}\lambda_1\wp_{14}\wp_{23} \\
&- \tfrac{8}{3}\wp_{33}^2\lambda_4\lambda_0 - \tfrac{2}{3}\wp_{33}\lambda_0 Q_{2444} + \tfrac{4}{3}\wp_{11}\wp_{23}^2 + \tfrac{1}{9}\wp_{33}\lambda_1 Q_{2333} + \tfrac{2}{3}\wp_{33}\lambda_2 Q_{1244} - \tfrac{4}{3}\wp_{11}\lambda_4\lambda_2 \\
&- 4\lambda_1\wp_{14}\lambda_3 - \tfrac{2}{3}\wp_{12}^2\wp_{33} + \tfrac{4}{3}\wp_{22}\wp_{13}\lambda_2 + \tfrac{2}{3}\wp_{11}Q_{2233} - \tfrac{4}{3}\lambda_2\wp_{24}\lambda_1 - \tfrac{2}{3}\wp_{33}^2\lambda_4\lambda_3\lambda_2 \\
&+ \tfrac{2}{9}\wp_{33}\lambda_4\lambda_2 Q_{2333} + \tfrac{4}{3}\wp_{23}\wp_{12}\wp_{13} + \tfrac{8}{3}\wp_{44}\lambda_1^2 + \tfrac{8}{3}\wp_{14}\lambda_2^2 - 4\lambda_0\wp_{44}\lambda_2 - 2\wp_{11}\wp_{23}\lambda_3 \\
&- 4\lambda_0\wp_{13}\wp_{44} - \tfrac{4}{3}\lambda_1\wp_{13}\wp_{24} + \tfrac{8}{3}\lambda_2\wp_{13}\wp_{14} - \tfrac{2}{3}\wp_{33}^2\lambda_3\lambda_1 + \tfrac{2}{3}\wp_{33}\wp_{11}\wp_{22} - \tfrac{4}{3}\lambda_4\wp_{11}\wp_{13} \\
\wp_{113}\wp_{134} &= 2\lambda_0\wp_{12} + 2\wp_{11}\wp_{13}\wp_{34} + 2\wp_{14}\wp_{13}^2 + 2\lambda_2\wp_{13}\wp_{14} - 2\lambda_1\wp_{14}\wp_{23} \\
&+ 2\lambda_1\wp_{12}\wp_{34} + 2\wp_{23}\wp_{24}\lambda_0 - 4\lambda_0\wp_{13}\wp_{44} + 6\lambda_0\wp_{14}\wp_{34} - 4\wp_{22}\wp_{34}\lambda_0 \\
&+ \tfrac{2}{3}\wp_{33}\lambda_0 Q_{2444} + \tfrac{1}{6}\wp_{33}\lambda_1 Q_{2333} - \tfrac{1}{2}\wp_{33}^2\lambda_3\lambda_1 - 2\lambda_0\wp_{14}\lambda_4 \\
\wp_{112}\wp_{233} &= \tfrac{4}{3}\wp_{23}\wp_{12}\lambda_2 - \tfrac{2}{3}\wp_{22}\wp_{13}\lambda_2 - \tfrac{1}{3}\wp_{22}\wp_{23}\lambda_1 + \tfrac{10}{3}\wp_{12}\lambda_4\lambda_1 + \tfrac{10}{3}\lambda_1\wp_{12}\wp_{34} - 2\lambda_1\wp_{11} \\
&+ \tfrac{16}{3}\wp_{33}^2\lambda_4\lambda_0 - 6\lambda_4\wp_{22}\lambda_0 + \tfrac{4}{3}\wp_{23}\wp_{12}\wp_{13} + \tfrac{4}{3}\wp_{33}\lambda_0 Q_{2444} - \tfrac{2}{9}\wp_{33}\lambda_1 Q_{2333} - \tfrac{4}{3}\lambda_2\wp_{13}\wp_{14} \\
&- \tfrac{4}{3}\wp_{33}\lambda_2 Q_{1244} + 2\lambda_1\wp_{14}\lambda_3 + \tfrac{4}{3}\wp_{12}^2\wp_{33} - \tfrac{2}{3}\wp_{22}\wp_{13}^2 + \tfrac{2}{3}\wp_{11}Q_{2233} + \tfrac{2}{3}\lambda_2\wp_{24}\lambda_1 \\
&+ \tfrac{4}{3}\wp_{33}^2\lambda_4\lambda_3\lambda_2 - \tfrac{4}{9}\wp_{33}\lambda_4\lambda_2 Q_{2333} + \tfrac{4}{3}\wp_{11}\wp_{23}^2 + \tfrac{2}{3}\wp_{44}\lambda_1^2 - \tfrac{4}{3}\wp_{14}\lambda_2^2 - \tfrac{4}{3}\wp_{11}\lambda_4\lambda_2 \\
&- 4\lambda_0\wp_{44}\lambda_2 - 2\wp_{11}\wp_{23}\lambda_3 - 4\lambda_0\wp_{13}\wp_{44} - 4\lambda_2\wp_{11}\wp_{34} + \tfrac{2}{3}\lambda_1\wp_{13}\wp_{24} + \tfrac{8}{3}\lambda_1\wp_{14}\wp_{23} \\
&+ \tfrac{4}{3}\wp_{33}^2\lambda_3\lambda_1 + \tfrac{2}{3}\wp_{33}\wp_{11}\wp_{22} - 6\wp_{22}\wp_{34}\lambda_0 + 12\lambda_0\wp_{14}\lambda_4 + 12\lambda_0\wp_{14}\wp_{34} + \tfrac{8}{3}\lambda_4\wp_{11}\wp_{13}
\end{aligned}$$

$$\begin{aligned}
\wp_{111}\wp_{334} &= -\tfrac{4}{3}\wp_{33}\lambda_0 Q_{2444} - \tfrac{1}{3}\wp_{33}\lambda_1 Q_{2333} + 2\wp_{14}\wp_{11}\wp_{33} + 6\lambda_1\wp_{14}\lambda_3 \\
&+ \lambda_2\wp_{24}\lambda_1 + 2\lambda_0\wp_{12} - 2\wp_{14}\wp_{13}^2 - 2\wp_{44}\lambda_1^2 - 2\wp_{14}\lambda_2^2 - 2\lambda_1\wp_{11} + 6\lambda_0\wp_{44}\lambda_2 \\
&- 10\wp_{23}\wp_{24}\lambda_0 + 3\wp_{11}\wp_{23}\lambda_3 + 8\lambda_0\wp_{13}\wp_{44} + 2\lambda_2\wp_{11}\wp_{34} + 4\wp_{11}\wp_{13}\wp_{34} \\
&- 4\lambda_2\wp_{13}\wp_{14} + \wp_{33}^2\lambda_3\lambda_1 - 9\lambda_3\wp_{24}\lambda_0 + 2\wp_{22}\wp_{34}\lambda_0 - 2\lambda_0\wp_{14}\lambda_4 - 6\lambda_0\wp_{14}\wp_{34} \\
&+ 6\lambda_1\wp_{14}\wp_{23} + 4\lambda_4\wp_{11}\wp_{13} + 2\wp_{11}\lambda_4\lambda_2 + \lambda_1\wp_{13}\wp_{24} - \wp_{11}Q_{2233}
\end{aligned}$$

The following relations are at weight -27.

$$\begin{aligned}
\wp_{122}\wp_{222} &= 4\wp_{44}\lambda_1\wp_{33}\lambda_4\lambda_3 - \tfrac{32}{3}\wp_{34}\lambda_1^2 + 4\wp_{13}\lambda_2^2 - 2\lambda_0\wp_{23}^2 - \tfrac{4}{3}\lambda_1\wp_{34}^2\lambda_2 - \tfrac{4}{3}\lambda_4^2\lambda_1\wp_{13} \\
&- 2\lambda_3\wp_{34}^2\lambda_0 - \tfrac{2}{3}\lambda_3\lambda_4\wp_{13}^2 + \tfrac{8}{3}\lambda_3\lambda_1\wp_{13} - 12\lambda_0\wp_{13}\lambda_4 + 8\lambda_1\wp_{12}\wp_{33} + 6\wp_{22}\wp_{33}\lambda_0 \\
&+ 12\lambda_3\wp_{23}\lambda_0 + 2\lambda_3^2\wp_{13}\wp_{23} - 2\lambda_3^2\wp_{12}\wp_{33} - 4\lambda_2\wp_{23}\lambda_1 - \tfrac{4}{3}\lambda_4\lambda_3\lambda_2^2 + 4\lambda_4\lambda_1\wp_{23}\lambda_3 \\
&- 2\lambda_4^2\lambda_3\lambda_0 + \tfrac{2}{3}\lambda_3\lambda_2\lambda_1 + \tfrac{10}{3}\lambda_4^2\lambda_2\lambda_1 - \tfrac{20}{3}\lambda_4\lambda_1^2 + 6\lambda_3^2\lambda_0 - 20\lambda_1\lambda_0 + 4\lambda_1\wp_{34}\wp_{23}\lambda_3 \\
&- \tfrac{5}{3}\lambda_4\lambda_1 Q_{2233} + \tfrac{2}{3}\lambda_3\lambda_2 Q_{2233} - \tfrac{8}{3}\lambda_3\wp_{34}\lambda_2^2 - 4\lambda_4\wp_{12}Q_{1244} + \tfrac{2}{3}\wp_{24}\lambda_1 Q_{2333} - 3\lambda_0 Q_{2233} \\
&- 2\wp_{22}\lambda_3 Q_{1244} + \tfrac{2}{3}\lambda_3\wp_{11}Q_{2444} - 2\wp_{14}\lambda_3 Q_{1244} - 4\wp_{44}\lambda_1 Q_{1244} - 2\lambda_4\lambda_1\wp_{23}^2 + 6\lambda_4\lambda_2\lambda_0 \\
&- \tfrac{4}{3}\lambda_4^2\wp_{12}Q_{2333} - \tfrac{2}{3}\wp_{34}\lambda_1 Q_{2233} + \tfrac{4}{3}\wp_{22}\lambda_1 Q_{2444} - \tfrac{1}{3}\wp_{13}\lambda_3 Q_{2233} + \tfrac{4}{3}\wp_{12}\lambda_2 Q_{2444} \\
&+ \tfrac{4}{3}\wp_{14}\lambda_1 Q_{2444} + 4\lambda_3^2\wp_{34}\lambda_1 - 2\lambda_2\wp_{23}\lambda_3^2 + 4\wp_{12}\wp_{22}^2 + 4\lambda_4^2\lambda_3\wp_{12}\wp_{33} + 4\lambda_4\lambda_2\wp_{12}\wp_{33} \\
&+ 4\lambda_2\wp_{33}\wp_{44}\lambda_1 + 2\lambda_4\lambda_3^2\wp_{22}\wp_{33} - 2\wp_{12}Q_{1224} - \tfrac{4}{3}\wp_{44}\lambda_4\lambda_1 Q_{2333} + 6\lambda_4\lambda_1\wp_{22}\wp_{33} \\
&- \tfrac{2}{3}\lambda_4\lambda_3\wp_{22}Q_{2333} - \tfrac{2}{3}\lambda_3\wp_{14}\lambda_4 Q_{2333} + 2\lambda_4\wp_{14}\wp_{33}\lambda_3^2 + \tfrac{4}{3}\lambda_3\lambda_2\wp_{13}\wp_{34} + 8\lambda_3\lambda_4\wp_{34}\lambda_0 \\
&+ \tfrac{8}{3}\lambda_1\wp_{34}\lambda_4\wp_{13} + 4\lambda_4\lambda_2\wp_{34}\lambda_1 + 2\lambda_3\wp_{44}\wp_{33}\lambda_0 + 2\wp_{11}\wp_{33}\lambda_4\lambda_3 - 4\lambda_1\wp_{24}\wp_{33}\lambda_3 \\
\wp_{122}\wp_{124} &= 2\wp_{12}^2\wp_{24} - 2\lambda_2\wp_{13}^2 - 4\lambda_0\wp_{23}^2 - \tfrac{2}{3}\lambda_4^2\lambda_1\wp_{13} - 2\lambda_3\wp_{34}^2\lambda_0 + \wp_{12}Q_{1224} + \tfrac{2}{3}\lambda_3\lambda_4\wp_{13}^2 \\
&+ 4\lambda_0\wp_{13}\wp_{34} - 2\wp_{14}\wp_{33}\lambda_0 + \tfrac{4}{3}\lambda_3\lambda_1\wp_{13} + 2\wp_{12}\wp_{14}\wp_{22} - 2\lambda_0\wp_{13}\lambda_4 - 2\lambda_1\wp_{12}\wp_{33} \\
&+ 4\wp_{22}\wp_{33}\lambda_0 - 2\lambda_2\wp_{34}\lambda_0 + \tfrac{1}{2}\lambda_3\wp_{23}\lambda_0 + 2\lambda_1\wp_{13}\wp_{23} + \tfrac{1}{2}\lambda_3^2\wp_{13}\wp_{23} - \tfrac{1}{2}\lambda_3^2\wp_{12}\wp_{33} \\
&+ \lambda_4\lambda_2\lambda_0 - \tfrac{1}{3}\lambda_4^2\lambda_2\lambda_1 + \tfrac{2}{3}\lambda_4\lambda_1^2 - 6\lambda_1\lambda_0 - \tfrac{3}{2}\lambda_0 Q_{2233} + \tfrac{1}{6}\lambda_4\lambda_1 Q_{2233} - 2\wp_{14}\lambda_3 Q_{1244} \\
&- \tfrac{1}{6}\wp_{13}\lambda_3 Q_{2233} + \tfrac{2}{3}\wp_{14}\lambda_1 Q_{2444} + \tfrac{2}{3}\wp_{11}\lambda_4 Q_{2333} + \tfrac{1}{6}\wp_{12}\lambda_3 Q_{2333} - \tfrac{1}{2}\lambda_4\lambda_1\wp_{23}\lambda_3 \\
&- \tfrac{2}{3}\lambda_3\wp_{14}\lambda_4 Q_{2333} + 2\wp_{11}Q_{1244} + 2\lambda_4\wp_{14}\wp_{33}\lambda_3^2 + \tfrac{2}{3}\lambda_3\lambda_2\wp_{13}\wp_{34} + 4\lambda_3\lambda_4\wp_{34}\lambda_0 \\
&+ 2\lambda_1\wp_{33}\wp_{14}\lambda_4 - \tfrac{2}{3}\lambda_4\lambda_2\wp_{34}\lambda_1 + 2\lambda_3\wp_{44}\wp_{33}\lambda_0 + \tfrac{1}{3}\lambda_3\lambda_4\wp_{13}\lambda_2 - 2\wp_{11}\wp_{33}\lambda_4\lambda_3 \\
\wp_{114}\wp_{124} &= \tfrac{1}{3}\wp_{44}\lambda_0 Q_{2333} - \tfrac{1}{3}\wp_{14}\lambda_2 Q_{2333} + \tfrac{1}{6}\lambda_0 Q_{2233} - \tfrac{2}{3}\wp_{24}\lambda_0 Q_{2444} - 2\lambda_3\wp_{34}^2\lambda_0 \\
&+ \lambda_2\wp_{14}\wp_{33}\lambda_3 - \tfrac{1}{2}\lambda_3\wp_{23}\lambda_0 + 2\wp_{14}\wp_{11}\wp_{24} - \tfrac{1}{3}\lambda_4\lambda_2\lambda_0 + \tfrac{4}{3}\lambda_2\wp_{34}\lambda_0 - \lambda_3\wp_{44}\wp_{33}\lambda_0 \\
&+ \tfrac{2}{3}\lambda_1\lambda_0 - \tfrac{2}{3}\wp_{34}\lambda_1^2 + \tfrac{2}{3}\lambda_1\wp_{34}^2\lambda_2 - \tfrac{2}{3}\lambda_0\wp_{13}\lambda_4 + 2\lambda_0\wp_{13}\wp_{34} + \tfrac{1}{3}\lambda_4\lambda_2\wp_{34}\lambda_1 \\
&+ \tfrac{1}{2}\lambda_1\wp_{34}\wp_{23}\lambda_3 + \tfrac{2}{3}\lambda_1\wp_{34}\lambda_4\wp_{13} + 2\wp_{14}\wp_{33}\lambda_0 + 2\wp_{12}\wp_{14}^2 - \tfrac{1}{6}\wp_{34}\lambda_1 Q_{2233} \\
\wp_{113}\wp_{133} &= -2\wp_{22}\wp_{33}\lambda_0 + 2\lambda_0\wp_{23}^2 + 2\wp_{13}^3 - 2\lambda_4\lambda_2\lambda_0 + 2\lambda_1\lambda_0 + 2\lambda_2\wp_{13}^2 - 2\wp_{34}\lambda_1^2 \\
&+ 2\wp_{11}\wp_{13}\wp_{33} - 2\lambda_1\wp_{13}\wp_{23} - 3\lambda_3\wp_{23}\lambda_0 + 4\lambda_2\wp_{34}\lambda_0 + \lambda_1\wp_{12}\wp_{33} - 2\lambda_0\wp_{13}\lambda_4 \\
&+ 4\lambda_0\wp_{13}\wp_{34} + 2\wp_{14}\wp_{33}\lambda_0 + \lambda_0 Q_{2233}
\end{aligned}$$

$$
\begin{aligned}
\wp_{114}\wp_{222} &= -2\wp_{12}^2\wp_{24} - 4\wp_{34}\lambda_1^2 + 2\wp_{13}\lambda_2^2 + 2\lambda_2\wp_{13}^2 + 6\lambda_0\wp_{23}^2 + 2\wp_{22}\wp_{11}\wp_{24} \\
&+ \tfrac{4}{3}\wp_{23}\wp_{34}\lambda_2^2 + \tfrac{2}{3}\wp_{23}\lambda_4\lambda_2^2 + 4\wp_{12}\wp_{14}\wp_{22} + \tfrac{4}{3}\lambda_4^2\lambda_1\wp_{13} - 2\lambda_3\wp_{34}^2\lambda_0 - \tfrac{4}{3}\lambda_3\lambda_4\wp_{13}^2 \\
&+ 4\wp_{14}\wp_{33}\lambda_0 - \tfrac{14}{3}\lambda_3\lambda_1\wp_{13} + 4\lambda_0\wp_{13}\wp_{34} - 4\lambda_0\wp_{13}\lambda_4 + 6\lambda_1\wp_{12}\wp_{33} + 2\wp_{22}\wp_{33}\lambda_0 \\
&+ 6\lambda_2\wp_{34}\lambda_0 + 2\lambda_2\wp_{11}\wp_{33} + 2\lambda_3\wp_{23}\lambda_0 - 6\lambda_1\wp_{13}\wp_{23} - \lambda_3^2\wp_{13}\wp_{23} + \lambda_3^2\wp_{12}\wp_{33} \\
&+ \lambda_2\wp_{23}^2\lambda_3 - \tfrac{4}{3}\lambda_2\wp_{23}\lambda_1 + \tfrac{2}{3}\lambda_4^2\lambda_2\lambda_1 - \tfrac{4}{3}\lambda_4\lambda_1^2 - \tfrac{1}{3}\lambda_4\lambda_1 Q_{2233} + \tfrac{1}{3}\wp_{24}\lambda_1 Q_{2333} \\
&+ \lambda_0 Q_{2233} - 2\wp_{14}\lambda_3 Q_{1244} - \tfrac{2}{3}\wp_{22}\lambda_1 Q_{2444} + \tfrac{1}{3}\wp_{13}\lambda_3 Q_{2233} + \tfrac{2}{3}\wp_{12}\lambda_2 Q_{2444} \\
&+ \tfrac{8}{3}\wp_{14}\lambda_1 Q_{2444} - \tfrac{4}{3}\wp_{11}\lambda_4 Q_{2333} - 2\wp_{24}\lambda_0 Q_{2444} - \tfrac{1}{3}\wp_{12}\lambda_3 Q_{2333} - 2\wp_{12} Q_{1224} \\
&- \tfrac{2}{3}\lambda_3\wp_{14}\lambda_4 Q_{2333} - 4\wp_{11} Q_{1244} - \tfrac{1}{3}\wp_{23}\lambda_2 Q_{2233} - \tfrac{1}{3}\wp_{22}\lambda_2 Q_{2333} + \tfrac{4}{3}\lambda_2\lambda_4\wp_{13}\wp_{23} \\
&+ 2\lambda_4\wp_{14}\wp_{33}\lambda_3^2 - \tfrac{4}{3}\lambda_3\lambda_2\wp_{13}\wp_{34} - 2\lambda_3\lambda_4\wp_{34}\lambda_0 - 4\lambda_4\wp_{23}\wp_{34}\lambda_0 + 4\lambda_1\wp_{33}\wp_{14}\lambda_4 \\
&- 4\wp_{24}\wp_{33}\lambda_4\lambda_0 + 4\lambda_1\wp_{34}\lambda_4\wp_{13} + \tfrac{4}{3}\lambda_4\lambda_2\wp_{34}\lambda_1 + 2\lambda_3\wp_{44}\wp_{33}\lambda_0 - \tfrac{2}{3}\lambda_3\lambda_4\wp_{13}\lambda_2 \\
&+ 4\wp_{11}\wp_{33}\lambda_4\lambda_3 - \lambda_1\wp_{24}\wp_{33}\lambda_3 + \lambda_2\wp_{22}\wp_{33}\lambda_3 - \lambda_1\wp_{34}\wp_{23}\lambda_3 + \lambda_4\lambda_1\wp_{23}\lambda_3 \\
\wp_{112}\wp_{144} &= -\tfrac{2}{3}\wp_{34}\lambda_1^2 + 2\wp_{12}\wp_{14}^2 + 2\lambda_0\wp_{23}^2 + 2\wp_{12}\wp_{11}\wp_{44} - \tfrac{4}{3}\lambda_1\wp_{34}^2\lambda_2 + 4\lambda_3\wp_{34}^2\lambda_0 \\
&+ 6\wp_{14}\wp_{33}\lambda_0 - 6\lambda_0\wp_{13}\wp_{34} - \tfrac{20}{3}\lambda_0\wp_{13}\lambda_4 + 4\lambda_1\wp_{12}\wp_{33} - 2\wp_{22}\wp_{33}\lambda_0 - \tfrac{8}{3}\lambda_2\wp_{34}\lambda_0 \\
&- 5\lambda_3\wp_{23}\lambda_0 - \lambda_1\wp_{13}\wp_{23} + \tfrac{3}{2}\lambda_3^2\wp_{12}\wp_{33} - \tfrac{10}{3}\lambda_4\lambda_2\lambda_0 + \tfrac{2}{3}\lambda_1\lambda_0 + \tfrac{5}{3}\lambda_0 Q_{2233} \\
&+ \tfrac{1}{3}\wp_{34}\lambda_1 Q_{2233} - \tfrac{2}{3}\wp_{14}\lambda_1 Q_{2444} + \tfrac{4}{3}\wp_{24}\lambda_0 Q_{2444} - \tfrac{2}{3}\wp_{44}\lambda_0 Q_{2333} - \tfrac{1}{2}\wp_{12}\lambda_3 Q_{2333} \\
&- \wp_{12} Q_{1224} - 4\lambda_4\wp_{23}\wp_{34}\lambda_0 - 2\lambda_1\wp_{33}\wp_{14}\lambda_4 + 4\wp_{24}\wp_{33}\lambda_4\lambda_0 + \tfrac{2}{3}\lambda_1\wp_{34}\lambda_4\wp_{13} \\
&- \tfrac{2}{3}\lambda_4\lambda_2\wp_{34}\lambda_1 + 2\lambda_3\wp_{44}\wp_{33}\lambda_0 - \lambda_1\wp_{34}\wp_{23}\lambda_3 \\
\wp_{112}\wp_{224} &= 2\lambda_4\lambda_2\wp_{24}\wp_{33}\lambda_3 - 2\lambda_2\wp_{14}\wp_{33}\lambda_3 - 2\wp_{44}\lambda_1\wp_{33}\lambda_4\lambda_3 + 2\wp_{12}^2\wp_{24} - \tfrac{8}{3}\wp_{34}\lambda_1^2 \\
&+ 2\wp_{13}\lambda_2^2 + 2\lambda_2\wp_{13}^2 + 4\lambda_0\wp_{23}^2 - \tfrac{4}{3}\lambda_1\wp_{34}^2\lambda_2 + 2\wp_{22}\wp_{11}\wp_{24} - \tfrac{4}{3}\lambda_4^2\lambda_1\wp_{13} + 4\lambda_3\wp_{34}^2\lambda_0 \\
&- \tfrac{4}{3}\lambda_3\lambda_4\wp_{13}^2 + 4\wp_{14}\wp_{33}\lambda_0 - \tfrac{8}{3}\lambda_3\lambda_1\wp_{13} - 8\lambda_0\wp_{13}\wp_{34} - 8\lambda_0\wp_{13}\lambda_4 + 4\lambda_1\wp_{12}\wp_{33} \\
&- 4\wp_{22}\wp_{33}\lambda_0 - 2\lambda_2\wp_{34}\lambda_0 + 2\lambda_2\wp_{11}\wp_{33} - 4\lambda_3\wp_{23}\lambda_0 - 4\lambda_1\wp_{13}\wp_{23} - \lambda_3^2\wp_{13}\wp_{23} \\
&+ \lambda_3^2\wp_{12}\wp_{33} - \lambda_2\wp_{23}\lambda_1 - 2\lambda_4\lambda_2\lambda_0 - \tfrac{2}{3}\lambda_4^2\lambda_2\lambda_1 - \tfrac{2}{3}\lambda_4\lambda_1^2 - 2\lambda_1\lambda_0 + 2\lambda_0 Q_{2233} \\
&+ \tfrac{1}{3}\lambda_4\lambda_1 Q_{2233} - \tfrac{2}{3}\wp_{24}\lambda_1 Q_{2333} + 4\wp_{14}\lambda_3 Q_{1244} + 2\wp_{44}\lambda_1 Q_{1244} + \tfrac{1}{3}\wp_{34}\lambda_1 Q_{2233} \\
&+ \tfrac{1}{3}\wp_{13}\lambda_3 Q_{2233} - 2\wp_{14}\lambda_1 Q_{2444} - \tfrac{4}{3}\wp_{11}\lambda_4 Q_{2333} + 2\wp_{24}\lambda_0 Q_{2444} - 2\wp_{24}\lambda_2 Q_{1244} \\
&- \tfrac{1}{3}\wp_{12}\lambda_3 Q_{2333} + \tfrac{2}{3}\wp_{44}\lambda_4\lambda_1 Q_{2333} + \tfrac{4}{3}\lambda_3\wp_{14}\lambda_4 Q_{2333} - 4\wp_{11} Q_{1244} - \tfrac{2}{3}\wp_{24}\lambda_4\lambda_2 Q_{2333} \\
&- 4\lambda_4\wp_{14}\wp_{33}\lambda_3^2 + \tfrac{2}{3}\lambda_3\lambda_2\wp_{13}\wp_{34} + 4\lambda_3\lambda_4\wp_{34}\lambda_0 - 4\lambda_1\wp_{33}\wp_{14}\lambda_4 + 8\wp_{24}\wp_{33}\lambda_4\lambda_0 \\
&- \tfrac{4}{3}\lambda_1\wp_{34}\lambda_4\wp_{13} - 2\lambda_4\lambda_2\wp_{34}\lambda_1 - 4\lambda_3\wp_{44}\wp_{33}\lambda_0 - \tfrac{2}{3}\lambda_3\lambda_4\wp_{13}\lambda_2 + 4\wp_{11}\wp_{33}\lambda_4\lambda_3 \\
&+ 4\lambda_1\wp_{24}\wp_{33}\lambda_3 - 2\lambda_1\wp_{34}\wp_{23}\lambda_3 - \lambda_4\lambda_1\wp_{23}\lambda_3 \\
\wp_{111}\wp_{333} &= -2\lambda_0 Q_{2233} + 6\wp_{11}\wp_{13}\wp_{33} + 7\lambda_1\wp_{13}\wp_{23} - 6\wp_{11}\wp_{33}\lambda_4\lambda_3 - 3\lambda_3\wp_{23}\lambda_0 \\
&- 2\lambda_4\lambda_2\lambda_0 - 2\lambda_2\wp_{11}\wp_{33} + \lambda_2\wp_{23}\lambda_1 - 2\lambda_2\wp_{34}\lambda_0 - 2\wp_{22}\wp_{33}\lambda_0 + 2\lambda_4\lambda_1^2 - 4\lambda_2\wp_{13}^2 \\
&- 2\wp_{13}\lambda_2^2 - 2\wp_{13}^3 - 4\lambda_1\lambda_0 + 2\lambda_1\wp_{12}\wp_{33} + 2\wp_{34}\lambda_1^2 - 10\lambda_0\wp_{23}^2 - 2\lambda_0\wp_{13}\lambda_4 \\
&- 2\lambda_0\wp_{13}\wp_{34} + 6\wp_{11} Q_{1244} + 6\lambda_3\lambda_1\wp_{13} - 4\wp_{14}\wp_{33}\lambda_0 + 2\wp_{11}\lambda_4 Q_{2333}
\end{aligned}
$$

$$\begin{aligned}
\wp_{111}\wp_{244} &= -2\lambda_4\lambda_2\wp_{24}\wp_{33}\lambda_3 - 4\lambda_2\wp_{14}\wp_{33}\lambda_3 + 4\wp_{44}\lambda_1\wp_{33}\lambda_4\lambda_3 + \tfrac{2}{3}\wp_{34}\lambda_1^2 - 2\wp_{12}\wp_{14}^2 \\
&\quad - 4\lambda_0\wp_{23}^2 + 4\wp_{14}\wp_{11}\wp_{24} + 2\wp_{12}\wp_{11}\wp_{44} + \tfrac{4}{3}\lambda_1\wp_{34}^2\lambda_2 - \tfrac{4}{3}\wp_{23}\wp_{34}\lambda_2^2 - \tfrac{2}{3}\wp_{23}\lambda_4\lambda_2^2 \\
&\quad - \tfrac{8}{3}\lambda_4^2\lambda_1\wp_{13} - 2\lambda_3\wp_{34}^2\lambda_0 - 6\wp_{14}\wp_{33}\lambda_0 + 2\lambda_3\lambda_1\wp_{13} - 2\lambda_0\wp_{13}\wp_{34} + \tfrac{4}{3}\lambda_0\wp_{13}\lambda_4 \\
&\quad - 6\lambda_1\wp_{12}\wp_{33} + 4\wp_{22}\wp_{33}\lambda_0 - \tfrac{8}{3}\lambda_2\wp_{34}\lambda_0 - 2\lambda_3\wp_{23}\lambda_0 + 3\lambda_1\wp_{13}\wp_{23} - \tfrac{3}{2}\lambda_3^2\wp_{12}\wp_{33} \\
&\quad - \lambda_2\wp_{23}^2\lambda_3 + \tfrac{1}{3}\lambda_2\wp_{23}\lambda_1 + \tfrac{2}{3}\lambda_4\lambda_2\lambda_0 - \tfrac{4}{3}\lambda_4^2\lambda_2\lambda_1 + \tfrac{2}{3}\lambda_4\lambda_1^2 - \tfrac{4}{3}\lambda_1\lambda_0 - \tfrac{1}{3}\lambda_0 Q_{2233} \\
&\quad + \tfrac{2}{3}\lambda_4\lambda_1 Q_{2233} - \tfrac{1}{3}\wp_{24}\lambda_1 Q_{2333} - 4\wp_{44}\lambda_1 Q_{1244} - \tfrac{1}{3}\wp_{34}\lambda_1 Q_{2233} - \tfrac{2}{3}\wp_{12}\lambda_2 Q_{2444} \\
&\quad + \tfrac{4}{3}\wp_{14}\lambda_1 Q_{2444} - \tfrac{8}{3}\wp_{24}\lambda_0 Q_{2444} + \tfrac{2}{3}\wp_{14}\lambda_2 Q_{2333} - \tfrac{2}{3}\wp_{44}\lambda_0 Q_{2333} + 2\wp_{24}\lambda_2 Q_{1244} \\
&\quad + \tfrac{1}{2}\wp_{12}\lambda_3 Q_{2333} + \wp_{12} Q_{1224} - \tfrac{4}{3}\wp_{44}\lambda_4\lambda_1 Q_{2333} + 2\wp_{11} Q_{1244} + \tfrac{2}{3}\wp_{24}\lambda_4\lambda_2 Q_{2333} \\
&\quad + \tfrac{1}{3}\wp_{23}\lambda_2 Q_{2233} - \tfrac{1}{3}\wp_{22}\lambda_2 Q_{2333} - \tfrac{4}{3}\lambda_2\lambda_4\wp_{13}\wp_{23} + 2\lambda_3\lambda_2\wp_{13}\wp_{34} + 6\lambda_3\lambda_4\wp_{34}\lambda_0 \\
&\quad + 8\lambda_4\wp_{23}\wp_{34}\lambda_0 + 6\lambda_1\wp_{33}\wp_{14}\lambda_4 - 8\wp_{24}\wp_{33}\lambda_4\lambda_0 - \tfrac{14}{3}\lambda_1\wp_{34}\lambda_4\wp_{13} - 2\lambda_4\lambda_2\wp_{34}\lambda_1 \\
&\quad + 8\lambda_3\wp_{44}\wp_{33}\lambda_0 + \lambda_1\wp_{24}\wp_{33}\lambda_3 + \lambda_2\wp_{22}\wp_{33}\lambda_3 + \lambda_1\wp_{34}\wp_{23}\lambda_3 - 2\lambda_4\lambda_1\wp_{23}\lambda_3
\end{aligned}$$

The following relations are at weight -28.

$$\begin{aligned}
\wp_{114}\wp_{123} &= \lambda_3\wp_{23}\wp_{33}\lambda_1 - 3\lambda_3\wp_{34}\wp_{33}\lambda_0 - \lambda_4\lambda_3\wp_{33}\wp_{34}\lambda_1 + 4\wp_{33}\lambda_4\lambda_3\lambda_0 \\
&\quad + \tfrac{1}{3}\wp_{34}\lambda_4\lambda_1 Q_{2333} + 2\wp_{33}\lambda_1^2 + 2\lambda_1\wp_{14}^2 + \tfrac{1}{3}\wp_{34}\lambda_0 Q_{2333} - \tfrac{4}{3}\wp_{23}\lambda_0 Q_{2444} \\
&\quad - 4\lambda_0\wp_{14}\wp_{24} + 2\wp_{13}\wp_{12}\wp_{14} + 2\wp_{11}\wp_{14}\wp_{23} + 2\wp_{22}\wp_{24}\lambda_0 + \wp_{34}\lambda_1 Q_{1244} \\
&\quad - \tfrac{1}{3}\wp_{23}\lambda_1 Q_{2333} - 4\lambda_0\wp_{12}\wp_{44} + \tfrac{3}{2}\wp_{33}\lambda_3^2\lambda_1 - 2\lambda_1\wp_{12}\wp_{24} + 2\lambda_1\wp_{11}\wp_{44} \\
&\quad + 2\wp_{12}\wp_{14}\lambda_2 + 4\wp_{33}\lambda_2\lambda_0 - \lambda_1 Q_{1224} - 5\lambda_0 Q_{1244} - \tfrac{1}{2}\lambda_3\lambda_1 Q_{2333} - \tfrac{4}{3}\lambda_4\lambda_0 Q_{2333} \\
\wp_{113}\wp_{222} &= -2\lambda_4\lambda_3\lambda_2\wp_{23}\wp_{33} + \tfrac{380}{3}\lambda_1\wp_{33}\lambda_4\lambda_2 - 2\lambda_3\wp_{23}\wp_{33}\lambda_1 + \tfrac{2}{3}\wp_{23}\lambda_4\lambda_2 Q_{2333} \\
&\quad - 8\lambda_0\lambda_4\wp_{23}\wp_{33} - 2\lambda_4^2\lambda_1\wp_{33}\lambda_3 - 84\lambda_3^2\wp_{33}\wp_{34}\lambda_2 - 42\wp_{33}\lambda_4\lambda_3^2\lambda_2 + \tfrac{784}{3}\lambda_1\wp_{33}\wp_{13}\lambda_4 \\
&\quad + \tfrac{760}{3}\lambda_1\wp_{33}\wp_{34}\lambda_2 - 4\wp_{11}^2 - 126\lambda_3^2\wp_{14}^2 + 380\lambda_4\lambda_3\wp_{33}\wp_{34}\lambda_1 + 42\lambda_4\lambda_3^2\wp_{34} Q_{2333} \\
&\quad - \tfrac{4}{3}\lambda_3\wp_{13}\lambda_4 Q_{2333} - 10\wp_{33}\lambda_4\lambda_3\lambda_0 - \tfrac{380}{3}\wp_{34}\lambda_4\lambda_1 Q_{2333} - 2\wp_{23}\wp_{12}^2 - \tfrac{1930}{3}\wp_{33}\lambda_1^2 \\
&\quad + 382\lambda_1\wp_{14}^2 + \tfrac{189}{2}\wp_{33}\lambda_3^4 - 4\lambda_3\wp_{12}^2 + 2\wp_{23}\lambda_2 Q_{1244} - 2\wp_{33}\lambda_3\lambda_2^2 - 4\lambda_2\wp_{11}\wp_{24} \\
&\quad + 4\wp_{22}\wp_{12}\lambda_2 + 2\wp_{23}\wp_{11}\wp_{22} - 2\wp_{23}\lambda_0 Q_{2444} + 12\lambda_0\wp_{14}\wp_{24} - 12\wp_{22}\wp_{24}\lambda_0 \\
&\quad + 126\wp_{34}\lambda_3^2 Q_{1244} - 4\wp_{13}\lambda_3 Q_{1244} - 380\wp_{34}\lambda_1 Q_{1244} + 21\wp_{33}\lambda_3^2 Q_{2233} + 8\lambda_0\wp_{13}\wp_{33} \\
&\quad - \tfrac{190}{3}\wp_{33}\lambda_1 Q_{2233} - 21\wp_{23}\lambda_3^2 Q_{2333} + \tfrac{191}{3}\wp_{23}\lambda_1 Q_{2333} - 42\wp_{13}\lambda_3^2 Q_{2444} + 2\lambda_4\lambda_1 Q_{1244} \\
&\quad + \tfrac{388}{3}\lambda_1\wp_{13} Q_{2444} + 126\lambda_3^2\wp_{11}\wp_{44} + 126\lambda_3^2\wp_{14}\wp_{22} - 126\lambda_3^2\wp_{12}\wp_{24} - \tfrac{155}{2}\wp_{33}\lambda_3^2\lambda_1 \\
&\quad + 4\lambda_3\wp_{11}\wp_{14} - 2\wp_{22}^2\lambda_1 - 378\lambda_1\wp_{14}\wp_{22} + 390\lambda_1\wp_{12}\wp_{24} - 386\lambda_1\wp_{11}\wp_{44} + \tfrac{2}{3}\lambda_2\lambda_1 Q_{2444} \\
&\quad + 4\wp_{12}\wp_{11}\lambda_4 + 4\wp_{12}\wp_{13}\wp_{22} - 4\wp_{12}\wp_{14}\lambda_2 - 12\wp_{33}\lambda_2\lambda_0 - 126\lambda_4\lambda_3^3\wp_{34}\wp_{33} \\
&\quad + 195\lambda_1 Q_{1224} + \tfrac{2}{3}\lambda_2^2 Q_{2333} - 80\lambda_4\lambda_3^2\wp_{33}\wp_{13} - \tfrac{63}{2}\lambda_3^3 Q_{2333} - 63\lambda_3^2 Q_{1224} \\
&\quad + 12\lambda_0 Q_{1244} - 4\lambda_3\lambda_0 Q_{2444} + \tfrac{2}{3}\lambda_4^2\lambda_1 Q_{2333} + 5\tfrac{75}{6}\lambda_3\lambda_1 Q_{2333} + 2\lambda_4\lambda_0 Q_{2333}
\end{aligned}$$

$$
\begin{aligned}
\wp_{122}\wp_{123} &= \tfrac{128}{3}\lambda_1\wp_{33}\lambda_4\lambda_2 + \lambda_4^2\lambda_1\wp_{33}\lambda_3 - 28\lambda_3^2\wp_{33}\wp_{34}\lambda_2 - 14\wp_{33}\lambda_4\lambda_3^2\lambda_2 - 3\lambda_4\lambda_0 Q_{2333}\\
&+ \tfrac{262}{3}\lambda_1\wp_{33}\wp_{13}\lambda_4 + \tfrac{256}{3}\lambda_1\wp_{33}\wp_{34}\lambda_2 + 2\wp_{11}^2 - 42\lambda_3^2\wp_{14}^2 + 128\lambda_4\lambda_3\wp_{33}\wp_{34}\lambda_1\\
&- \lambda_4\lambda_1 Q_{1244} + 14\lambda_4\lambda_3^2\wp_{34}Q_{2333} - \tfrac{1}{3}\lambda_3\wp_{13}\lambda_4 Q_{2333} + 11\wp_{33}\lambda_4\lambda_3\lambda_0 - \tfrac{128}{3}\wp_{34}\lambda_4\lambda_1 Q_{2333}\\
&+ 2\wp_{23}\wp_{12}^2 - \tfrac{634}{3}\wp_{33}\lambda_1^2 + 130\lambda_1\wp_{14}^2 + \tfrac{63}{2}\wp_{33}\lambda_3^4 + 2\lambda_3\wp_{12}^2 + 42\wp_{34}\lambda_3^2 Q_{1244}\\
&- \wp_{13}\lambda_3 Q_{1244} - 128\wp_{34}\lambda_1 Q_{1244} + 7\wp_{33}\lambda_3^2 Q_{2233} - \tfrac{64}{3}\wp_{33}\lambda_1 Q_{2233} - 7\wp_{23}\lambda_3^2 Q_{2333}\\
&+ \tfrac{64}{3}\wp_{23}\lambda_1 Q_{2333} + 2\lambda_0\wp_{13}\wp_{33} - 14\wp_{13}\lambda_3^2 Q_{2444} + \tfrac{130}{3}\lambda_1\wp_{13}Q_{2444} - 6\lambda_0\wp_{12}\wp_{44}\\
&+ 42\lambda_3^2\wp_{11}\wp_{44} + 42\lambda_3^2\wp_{14}\wp_{22} - 42\lambda_3^2\wp_{12}\wp_{24} - \tfrac{51}{2}\wp_{33}\lambda_3^2\lambda_1 - 2\lambda_3\wp_{11}\wp_{14} + \tfrac{191}{6}\lambda_3\lambda_1 Q_{2333}\\
&- 128\lambda_1\wp_{14}\wp_{22} + 126\lambda_1\wp_{12}\wp_{24} - 128\lambda_1\wp_{11}\wp_{44} - 2\wp_{12}\wp_{11}\lambda_4 + 2\wp_{12}\wp_{13}\wp_{22}\\
&+ 4\wp_{12}\wp_{14}\lambda_2 + 8\wp_{33}\lambda_2\lambda_0 - 42\lambda_4\lambda_3^3\wp_{34}\wp_{33} + 63\lambda_1 Q_{1224} - 27\lambda_4\lambda_3^2\wp_{33}\wp_{13}\\
&- \tfrac{21}{2}\lambda_3^3 Q_{2333} - 21\lambda_3^2 Q_{1224} - 9\lambda_0 Q_{1244} + 2\lambda_3\lambda_0 Q_{2444} - \tfrac{1}{3}\lambda_4^2\lambda_1 Q_{2333}\\
\wp_{113}\wp_{124} &= \lambda_2\wp_{33}\wp_{13}\lambda_3 - 21\lambda_1\wp_{33}\lambda_4\lambda_2 - \tfrac{1}{2}\lambda_3\wp_{23}\wp_{33}\lambda_1 - 3\lambda_3\wp_{34}\wp_{33}\lambda_0\\
&- 42\lambda_1\wp_{33}\wp_{13}\lambda_4 - 42\lambda_1\wp_{33}\wp_{34}\lambda_2 - 64\lambda_4\lambda_3\wp_{33}\wp_{34}\lambda_1 - 2\wp_{33}\lambda_4\lambda_3\lambda_0\\
&+ \tfrac{64}{3}\wp_{34}\lambda_4\lambda_1 Q_{2333} + 104\wp_{33}\lambda_1^2 - 62\lambda_1\wp_{14}^2 - \tfrac{1}{3}\wp_{13}\lambda_2 Q_{2333} + 2\wp_{13}\wp_{11}\wp_{24}\\
&+ \tfrac{1}{3}\wp_{34}\lambda_0 Q_{2333} + \tfrac{2}{3}\wp_{23}\lambda_0 Q_{2444} + 2\lambda_0\wp_{14}\wp_{24} + 2\wp_{13}\wp_{12}\wp_{14} - 4\wp_{22}\wp_{24}\lambda_0\\
&+ 64\wp_{34}\lambda_1 Q_{1244} + \tfrac{21}{2}\wp_{33}\lambda_1 Q_{2233} - \tfrac{31}{3}\wp_{23}\lambda_1 Q_{2333} + 4\lambda_0\wp_{13}\wp_{33} - \tfrac{64}{3}\lambda_1\wp_{13}Q_{2444}\\
&+ 2\lambda_0\wp_{12}\wp_{44} + \tfrac{93}{2}\wp_{33}\lambda_3^2\lambda_1 + 64\lambda_1\wp_{14}\wp_{22} - 62\lambda_1\wp_{12}\wp_{24} + 62\lambda_1\wp_{11}\wp_{44}\\
&- 2\wp_{33}\lambda_2\lambda_0 - 31\lambda_1 Q_{1224} + 4\lambda_0 Q_{1244} - \tfrac{31}{2}\lambda_3\lambda_1 Q_{2333} + \tfrac{2}{3}\lambda_4\lambda_0 Q_{2333}\\
\wp_{112}\wp_{223} &= 2\lambda_4\lambda_3\lambda_2\wp_{23}\wp_{33} + \tfrac{2}{3}\lambda_1\wp_{33}\lambda_4\lambda_2 + 2\lambda_3\wp_{23}\wp_{33}\lambda_1 - \tfrac{2}{3}\wp_{23}\lambda_4\lambda_2 Q_{2333}\\
&+ 8\lambda_0\lambda_4\wp_{23}\wp_{33} + 4\lambda_4^2\lambda_1\wp_{33}\lambda_3 - \tfrac{8}{3}\lambda_1\wp_{33}\wp_{13}\lambda_4 + \tfrac{4}{3}\lambda_1\wp_{33}\wp_{34}\lambda_2 - 4\lambda_4\lambda_1 Q_{1244}\\
&+ 2\lambda_4\lambda_3\wp_{33}\wp_{34}\lambda_1 + \tfrac{2}{3}\lambda_3\wp_{13}\lambda_4 Q_{2333} + 8\wp_{33}\lambda_4\lambda_3\lambda_0 - \tfrac{2}{3}\wp_{34}\lambda_4\lambda_1 Q_{2333} + 2\wp_{23}\wp_{12}^2\\
&+ \tfrac{14}{3}\wp_{33}\lambda_1^2 - 2\lambda_1\wp_{14}^2 + 2\lambda_3\wp_{12}^2 - 2\wp_{23}\lambda_2 Q_{1244} + 2\wp_{23}\wp_{11}\wp_{22} + 2\wp_{23}\lambda_0 Q_{2444}\\
&+ 2\wp_{13}\lambda_3 Q_{1244} - 2\wp_{34}\lambda_1 Q_{1244} - \tfrac{1}{3}\wp_{33}\lambda_1 Q_{2233} - 4\lambda_0\wp_{13}\wp_{33} - \tfrac{2}{3}\lambda_1\wp_{13}Q_{2444}\\
&+ \tfrac{1}{2}\wp_{33}\lambda_3^2\lambda_1 + 4\lambda_3\wp_{11}\wp_{14} + 2\lambda_1\wp_{14}\wp_{22} - 2\lambda_1\wp_{12}\wp_{24} - 2\lambda_1\wp_{11}\wp_{44} - 4\wp_{33}\lambda_2\lambda_0\\
&- \lambda_1 Q_{1224} - 2\lambda_4\lambda_3^2\wp_{33}\wp_{13} + 2\lambda_3\lambda_0 Q_{2444} - \tfrac{4}{3}\lambda_4^2\lambda_1 Q_{2333} - \tfrac{1}{6}\lambda_3\lambda_1 Q_{2333} - 4\wp_{11}^2\\
\wp_{111}\wp_{234} &= -\lambda_2\wp_{33}\wp_{13}\lambda_3 + \lambda_3\wp_{23}\wp_{33}\lambda_1 + 6\lambda_3\wp_{34}\wp_{33}\lambda_0 - 2\wp_{11}^2 + 2\lambda_4\lambda_3\wp_{33}\wp_{34}\lambda_1\\
&- 2\wp_{33}\lambda_4\lambda_3\lambda_0 - \tfrac{2}{3}\wp_{34}\lambda_4\lambda_1 Q_{2333} - 4\wp_{33}\lambda_1^2 - 4\lambda_1\wp_{14}^2 + 2\wp_{11}\wp_{12}\wp_{34}\\
&+ \tfrac{1}{3}\wp_{13}\lambda_2 Q_{2333} + 2\wp_{13}\wp_{11}\wp_{24} - \wp_{33}\lambda_3\lambda_2^2 - 2\lambda_2\wp_{11}\wp_{24} - \tfrac{2}{3}\wp_{34}\lambda_0 Q_{2333}\\
&- \tfrac{4}{3}\wp_{23}\lambda_0 Q_{2444} + 8\lambda_0\wp_{14}\wp_{24} - 2\wp_{13}\wp_{12}\wp_{14} + 2\wp_{11}\wp_{14}\wp_{23} - 4\wp_{22}\wp_{24}\lambda_0\\
&- 2\wp_{34}\lambda_1 Q_{1244} - \tfrac{1}{3}\wp_{23}\lambda_1 Q_{2333} - 4\lambda_0\wp_{13}\wp_{33} + \tfrac{1}{3}\lambda_1\wp_{13}Q_{2444} + 2\lambda_0\wp_{12}\wp_{44}\\
&+ 6\lambda_3\wp_{11}\wp_{14} + 4\lambda_1\wp_{12}\wp_{24} - 2\lambda_1\wp_{11}\wp_{44} - 2\wp_{12}\wp_{14}\lambda_2 - 6\wp_{33}\lambda_2\lambda_0 + 2\lambda_1 Q_{1224}\\
&+ \tfrac{1}{3}\lambda_2^2 Q_{2333} + 4\lambda_0 Q_{1244} - 3\lambda_3\lambda_0 Q_{2444} + \tfrac{2}{3}\lambda_4\lambda_0 Q_{2333} + \tfrac{1}{3}\lambda_2\lambda_1 Q_{2444}
\end{aligned}
$$

$$\begin{aligned}
\wp_{112}\wp_{134} &= -\tfrac{1}{2}\lambda_3\wp_{23}\wp_{33}\lambda_1 + 6\lambda_3\wp_{34}\wp_{33}\lambda_0 + 2\lambda_4\lambda_3\wp_{33}\wp_{34}\lambda_1 + 4\wp_{33}\lambda_4\lambda_3\lambda_0 \\
&- \tfrac{2}{3}\wp_{34}\lambda_4\lambda_1 Q_{2333} - 4\wp_{33}\lambda_1^2 + 2\lambda_1\wp_{14}^2 + 2\wp_{11}\wp_{12}\wp_{34} - \tfrac{2}{3}\wp_{34}\lambda_0 Q_{2333} \\
&+ \tfrac{2}{3}\wp_{23}\lambda_0 Q_{2444} - 4\lambda_0\wp_{14}\wp_{24} + 2\wp_{13}\wp_{12}\wp_{14} + 2\wp_{22}\wp_{24}\lambda_0 - 2\wp_{34}\lambda_1 Q_{1244} \\
&+ \tfrac{1}{6}\wp_{23}\lambda_1 Q_{2333} + 2\lambda_0\wp_{12}\wp_{44} - \tfrac{3}{2}\wp_{33}\lambda_3^2\lambda_1 - 2\lambda_1\wp_{14}\wp_{22} - 2\lambda_1\wp_{11}\wp_{44} + 2\wp_{12}\wp_{14}\lambda_2 \\
&+ 4\wp_{33}\lambda_2\lambda_0 + \lambda_1 Q_{1224} - 2\lambda_0 Q_{1244} + \tfrac{1}{2}\lambda_3\lambda_1 Q_{2333} - \tfrac{4}{3}\lambda_4\lambda_0 Q_{2333}
\end{aligned}$$

The following relations are at weight -29.

$$\begin{aligned}
\wp_{113}\wp_{123} &= 2\lambda_1\wp_{13}\wp_{14} - \wp_{33}^2\lambda_4\lambda_3\lambda_1 + \tfrac{1}{3}\wp_{33}\lambda_4\lambda_1 Q_{2333} + \wp_{33}\lambda_1 Q_{1244} - 4\lambda_0\wp_{13}\wp_{24} \\
&+ \tfrac{1}{3}\wp_{33}\lambda_0 Q_{2333} + 2\lambda_4\lambda_0\wp_{12} + 2\lambda_0\wp_{12}\wp_{34} - 4\lambda_2\wp_{24}\lambda_0 - 6\lambda_0\wp_{14}\lambda_3 - 2\lambda_0\wp_{14}\wp_{23} \\
&+ 2\wp_{13}\wp_{12}\lambda_2 + 2\wp_{12}\wp_{13}^2 + 2\wp_{11}\wp_{34}\lambda_1 + 2\wp_{11}\wp_{13}\wp_{23} + 2\lambda_1\wp_{14}\lambda_2 + 4\lambda_1\wp_{44}\lambda_0 \\
&- 2\wp_{23}\wp_{22}\lambda_0 - 2\wp_{11}\lambda_0 - 3\wp_{33}^2\lambda_3\lambda_0 \\
\wp_{112}\wp_{133} &= 2\lambda_1\wp_{13}\wp_{14} + 2\wp_{33}^2\lambda_4\lambda_3\lambda_1 - \tfrac{2}{3}\wp_{33}\lambda_4\lambda_1 Q_{2333} - 2\wp_{33}\lambda_1 Q_{1244} - 4\lambda_0\wp_{13}\wp_{24} \\
&- \tfrac{2}{3}\wp_{33}\lambda_0 Q_{2333} + 2\lambda_4\lambda_0\wp_{12} + 2\lambda_0\wp_{12}\wp_{34} - 4\lambda_2\wp_{24}\lambda_0 + 12\lambda_0\wp_{14}\lambda_3 + 2\wp_{24}\lambda_1^2 \\
&+ 4\lambda_0\wp_{14}\wp_{23} + 2\wp_{13}\wp_{12}\lambda_2 + 2\wp_{12}\wp_{13}^2 - 2\wp_{11}\wp_{34}\lambda_1 + 2\wp_{11}\wp_{12}\wp_{33} - 2\lambda_1\wp_{14}\lambda_2 \\
&- 2\lambda_1\wp_{44}\lambda_0 - \lambda_1\wp_{12}\wp_{23} - 2\lambda_1\wp_{13}\wp_{22} + 4\wp_{23}\wp_{22}\lambda_0 - 8\wp_{11}\lambda_0 + 6\wp_{33}^2\lambda_3\lambda_0 \\
\wp_{111}\wp_{233} &= -2\wp_{12}\wp_{13}^2 - 2\wp_{11}\lambda_0 - 2\wp_{12}\lambda_2^2 - \tfrac{2}{3}\wp_{33}\lambda_4\lambda_1 Q_{2333} - 9\lambda_3\wp_{22}\lambda_0 + 2\lambda_0\wp_{12}\wp_{34} \\
&+ 6\lambda_1\wp_{12}\lambda_3 - 2\wp_{11}\lambda_4\lambda_1 + 6\wp_{11}\wp_{13}\lambda_3 - 2\lambda_2\wp_{11}\wp_{23} + \lambda_2\wp_{22}\lambda_1 + 4\wp_{11}\wp_{13}\wp_{23} \\
&- \tfrac{2}{3}\wp_{33}\lambda_0 Q_{2333} - 2\wp_{33}\lambda_1 Q_{1244} + 2\wp_{33}^2\lambda_4\lambda_3\lambda_1 - 2\lambda_1\wp_{14}\lambda_2 + 12\lambda_0\wp_{14}\lambda_3 - 2\lambda_1\wp_{44}\lambda_0 \\
&+ \lambda_1\wp_{13}\wp_{22} + 2\lambda_4\lambda_0\wp_{12} + 6\lambda_1\wp_{12}\wp_{23} - 2\wp_{11}\wp_{34}\lambda_1 + 10\lambda_0\wp_{14}\wp_{23} + 6\wp_{33}^2\lambda_3\lambda_0 \\
&+ 2\lambda_0\wp_{13}\wp_{24} + 2\lambda_2\wp_{24}\lambda_0 - 4\wp_{13}\wp_{12}\lambda_2 + 2\wp_{11}\wp_{12}\wp_{33} - 2\lambda_1\wp_{13}\wp_{14} - 8\wp_{23}\wp_{22}\lambda_0
\end{aligned}$$

The following relations are at weight -30.

$$\begin{aligned}
\wp_{114}\wp_{122} &= 2\lambda_2\lambda_4\wp_{14}\wp_{33}\lambda_3 + 2\lambda_4\lambda_3\wp_{44}\wp_{33}\lambda_0 + \tfrac{1}{3}\wp_{12}\lambda_1 Q_{2444} + \tfrac{8}{3}\wp_{14}\lambda_0 Q_{2444} \\
&- 2\lambda_0\wp_{13}\wp_{23} + \tfrac{1}{2}\lambda_1\wp_{23}^2\lambda_3 - \tfrac{4}{3}\wp_{22}\lambda_0 Q_{2444} + \tfrac{4}{3}\lambda_0\wp_{13}\lambda_4^2 + 2\wp_{11}\wp_{14}\wp_{22} \\
&- \tfrac{1}{3}\wp_{14}\lambda_1 Q_{2333} + 4\lambda_2\wp_{44}\wp_{33}\lambda_0 + \tfrac{2}{3}\lambda_2\wp_{23}\wp_{34}\lambda_1 + 4\lambda_4\wp_{14}\wp_{33}\lambda_0 - \tfrac{1}{6}\wp_{23}\lambda_1 Q_{2233} \\
&- \lambda_1\lambda_4\wp_{24}\wp_{33}\lambda_3 + \lambda_2\lambda_1\wp_{13} - \wp_{33}\wp_{44}\lambda_1^2 + \lambda_1\wp_{11}\wp_{33} - \tfrac{1}{6}\wp_{22}\lambda_1 Q_{2333} - \tfrac{1}{3}\lambda_4\lambda_0 Q_{2233} \\
&- \tfrac{1}{3}\wp_{34}\lambda_0 Q_{2233} + \lambda_4\lambda_3\wp_{23}\lambda_0 + \tfrac{2}{3}\lambda_1\lambda_4\wp_{13}\wp_{23} + \tfrac{1}{3}\lambda_4\lambda_2\wp_{23}\lambda_1 + \tfrac{1}{2}\lambda_1\wp_{22}\wp_{33}\lambda_3 \\
&+ \tfrac{1}{3}\lambda_4\wp_{24}\lambda_1 Q_{2333} - \tfrac{2}{3}\lambda_4\wp_{14}\lambda_2 Q_{2333} - \tfrac{2}{3}\lambda_4\wp_{44}\lambda_0 Q_{2333} - \lambda_1\wp_{13}^2 - 5\lambda_0\wp_{24}\wp_{33}\lambda_3 \\
&- \tfrac{2}{3}\wp_{23}\lambda_1^2 - \tfrac{10}{3}\lambda_1\wp_{34}\lambda_0 - 5\lambda_0\wp_{13}\lambda_3 + 2\lambda_0\wp_{12}\wp_{33} - 2\wp_{44}\lambda_0 Q_{1244} + \wp_{24}\lambda_0 Q_{2333} \\
&+ \wp_{24}\lambda_1 Q_{1244} + 2\wp_{12}^2\wp_{14} - 2\wp_{14}\lambda_2 Q_{1244} - \tfrac{8}{3}\lambda_2\wp_{34}^2\lambda_0 + \wp_{34}^2\lambda_1^2 - \lambda_0\wp_{34}\wp_{23}\lambda_3 \\
&+ \tfrac{16}{3}\lambda_0\wp_{34}\lambda_4\wp_{13} + 2\lambda_1\wp_{14}\wp_{33}\lambda_3 + 2\lambda_2\lambda_4\wp_{34}\lambda_0 + \tfrac{2}{3}\lambda_4^2\lambda_2\lambda_0 - \tfrac{4}{3}\lambda_4\lambda_1\lambda_0 - 3\lambda_0^2
\end{aligned}$$

$$\begin{aligned}
\wp_{114}^2 &= -\tfrac{4}{3}\lambda_2\wp_{34}^2\lambda_0 + \wp_{34}^2\lambda_1^2 + \lambda_0^2 + 4\wp_{11}\wp_{14}^2 + 2\lambda_0\wp_{34}\wp_{23}\lambda_3 + \tfrac{8}{3}\lambda_0\wp_{34}\lambda_4\wp_{13} \\
&+ 2\lambda_1\wp_{14}\wp_{33}\lambda_3 - 2\lambda_0\wp_{24}\wp_{33}\lambda_3 + \tfrac{4}{3}\lambda_2\lambda_4\wp_{34}\lambda_0 - \tfrac{2}{3}\lambda_1\wp_{34}\lambda_0 - \tfrac{4}{3}\wp_{14}\lambda_0 Q_{2444} \\
&- \tfrac{2}{3}\wp_{14}\lambda_1 Q_{2333} + \tfrac{2}{3}\wp_{24}\lambda_0 Q_{2333} - \tfrac{2}{3}\wp_{34}\lambda_0 Q_{2233}
\end{aligned}$$

$$\begin{aligned}
\wp_{112}\wp_{222} &= 2\wp_{11}\wp_{22}^2 - 4\lambda_2\lambda_4\wp_{14}\wp_{33}\lambda_3 - 12\lambda_4\lambda_3\wp_{44}\wp_{33}\lambda_0 + 2\lambda_4\lambda_2\wp_{22}\wp_{33}\lambda_3 \\
&- 4\wp_{14}\lambda_0 Q_{2444} - 8\lambda_0\wp_{13}\wp_{23} + \lambda_1\wp_{23}^2\lambda_3 + \tfrac{4}{3}\lambda_2\lambda_4\wp_{13}^2 + \tfrac{2}{3}\wp_{13}\lambda_2 Q_{2233} + 4\lambda_3^2\wp_{23}\lambda_1 \\
&- 6\lambda_3^2\wp_{34}\lambda_0 + 2\wp_{12}\lambda_3 Q_{1244} - \tfrac{8}{3}\wp_{11}\lambda_4^2 Q_{2333} + 2\lambda_3^2\wp_{11}\wp_{33} + 2\wp_{22}\lambda_0 Q_{2444} + \tfrac{2}{3}\lambda_2^2 Q_{2233} \\
&+ 8\lambda_4\lambda_0\wp_{23}^2 - 8\lambda_0\wp_{13}\lambda_4^2 + \tfrac{4}{3}\lambda_2\wp_{11}Q_{2444} - 2\lambda_2^2\wp_{23}\lambda_3 - 4\lambda_2\wp_{23}\lambda_0 - 2\wp_{22}\lambda_2 Q_{1244} \\
&- 16\lambda_2\wp_{44}\wp_{33}\lambda_0 + \tfrac{8}{3}\lambda_2\wp_{23}\wp_{34}\lambda_1 + 6\lambda_1\lambda_3\wp_{13}\wp_{34} + \tfrac{22}{3}\lambda_3\lambda_2\wp_{34}\lambda_1 - \tfrac{8}{3}\lambda_4\lambda_3\lambda_1\wp_{13} \\
&+ 8\wp_{11}\wp_{33}\lambda_4^2\lambda_3 + \tfrac{2}{3}\lambda_4\wp_{12}\lambda_3 Q_{2333} + 4\lambda_4\lambda_1\wp_{12}\wp_{33} - \tfrac{2}{3}\wp_{22}\lambda_4\lambda_2 Q_{2333} + 16\lambda_0\wp_{24}\wp_{33}\lambda_3 \\
&- 2\lambda_4\lambda_3^2\wp_{12}\wp_{33} + 4\lambda_4\lambda_2\wp_{11}\wp_{33} - \tfrac{2}{3}\wp_{23}\lambda_1 Q_{2233} + 4\lambda_1\lambda_4\wp_{24}\wp_{33}\lambda_3 + \tfrac{20}{3}\lambda_2\lambda_1\wp_{13} \\
&+ 6\wp_{33}\wp_{44}\lambda_1^2 + 14\lambda_1\wp_{11}\wp_{33} + \tfrac{1}{3}\wp_{22}\lambda_1 Q_{2333} + 2\wp_{34}\lambda_0 Q_{2233} + 2\lambda_4\lambda_3\wp_{23}\lambda_0 \\
&- \tfrac{16}{3}\lambda_1\lambda_4\wp_{13}\wp_{23} - \tfrac{2}{3}\lambda_4\lambda_2\wp_{23}\lambda_1 + \lambda_1\wp_{22}\wp_{33}\lambda_3 - \tfrac{4}{3}\lambda_4\wp_{24}\lambda_1 Q_{2333} - 12\lambda_1\wp_{34}\lambda_0 \\
&+ \tfrac{4}{3}\lambda_4\wp_{14}\lambda_2 Q_{2333} + 4\lambda_4\wp_{44}\lambda_0 Q_{2333} + 2\lambda_1\wp_{13}^2 - 4\wp_{11}Q_{1224} - \tfrac{2}{3}\lambda_3\wp_{11}Q_{2333} \\
&- \tfrac{8}{3}\wp_{23}\lambda_1^2 - \tfrac{8}{3}\lambda_2^2\wp_{13}\wp_{34} - 8\wp_{34}\lambda_4\lambda_1^2 - 10\lambda_0\wp_{13}\lambda_3 + 4\lambda_0\wp_{12}\wp_{33} - 8\wp_{11}\lambda_4 Q_{1244} \\
&+ 12\wp_{44}\lambda_0 Q_{1244} - 2\wp_{24}\lambda_0 Q_{2333} - 4\wp_{24}\lambda_1 Q_{1244} - 2\lambda_3^2\wp_{13}^2 + 2\wp_{12}^2\wp_{22} - \tfrac{8}{3}\wp_{34}\lambda_2^3 \\
&- \tfrac{2}{3}\wp_{12}\lambda_2 Q_{2333} + 4\wp_{14}\lambda_2 Q_{1244} + 8\lambda_2\wp_{34}^2\lambda_0 - 16\lambda_0\wp_{34}\wp_{23}\lambda_3 - 8\lambda_0\wp_{34}\lambda_4\wp_{13} \\
&- 6\wp_{34}^2\lambda_1^2 - 6\lambda_1\wp_{14}\wp_{33}\lambda_3 + 8\lambda_2\lambda_4\wp_{34}\lambda_0 - \tfrac{4}{3}\lambda_3\lambda_1 Q_{2233} + 2\lambda_3\lambda_2\lambda_0 - \tfrac{4}{3}\lambda_4\lambda_2^3 \\
&+ \tfrac{2}{3}\lambda_2^2\lambda_1 - 2\lambda_4^2\lambda_1^2 - \tfrac{4}{3}\lambda_3\lambda_1^2 - 8\lambda_4\lambda_1\lambda_0 - 12\lambda_0^2 + \tfrac{8}{3}\lambda_4\lambda_3\lambda_2\lambda_1 + 2\lambda_4\lambda_0 Q_{2233}
\end{aligned}$$

$$\begin{aligned}
\wp_{111}\wp_{224} &= -\lambda_2\lambda_3\wp_{12}\wp_{33} - 4\lambda_2\lambda_4\wp_{14}\wp_{33}\lambda_3 + 8\lambda_4\lambda_3\wp_{44}\wp_{33}\lambda_0 - \tfrac{1}{3}\wp_{12}\lambda_1 Q_{2444} \\
&+ \tfrac{2}{3}\wp_{14}\lambda_0 Q_{2444} - 4\lambda_0\wp_{13}\wp_{23} - \tfrac{3}{2}\lambda_1\wp_{23}^2\lambda_3 + \tfrac{2}{3}\lambda_4\wp_{13}\lambda_2^2 + \tfrac{1}{3}\wp_{13}\lambda_2 Q_{2233} \\
&- \tfrac{4}{3}\lambda_2\lambda_4\wp_{13}^2 - 3\lambda_3^2\wp_{23}\lambda_1 + 9\lambda_3^2\wp_{34}\lambda_0 - \tfrac{4}{3}\wp_{22}\lambda_0 Q_{2444} - \tfrac{8}{3}\lambda_0\wp_{13}\lambda_4^2 + 2\wp_{11}\wp_{14}\wp_{22} \\
&+ \tfrac{1}{3}\wp_{14}\lambda_1 Q_{2333} - \tfrac{2}{3}\lambda_2\wp_{11}Q_{2444} + \lambda_2^2\wp_{23}\lambda_3 + 2\lambda_2\wp_{23}\lambda_0 + 4\wp_{11}\wp_{12}\wp_{24} - 4\lambda_2^2\wp_{14}\wp_{33} \\
&+ 8\lambda_2\wp_{44}\wp_{33}\lambda_0 - 4\lambda_2\wp_{23}\wp_{34}\lambda_1 - 4\lambda_1\lambda_3\wp_{13}\wp_{34} - 5\lambda_3\lambda_2\wp_{34}\lambda_1 - 4\lambda_4\lambda_3\lambda_1\wp_{13} \\
&+ 2\lambda_2\wp_{24}\wp_{33}\lambda_1 + \tfrac{1}{2}\wp_{23}\lambda_1 Q_{2233} + \lambda_1\lambda_4\wp_{24}\wp_{33}\lambda_3 + \tfrac{10}{3}\lambda_2\lambda_1\wp_{13} - 3\wp_{33}\wp_{44}\lambda_1^2 \\
&- 5\lambda_1\wp_{11}\wp_{33} - \tfrac{5}{6}\wp_{22}\lambda_1 Q_{2333} - \tfrac{4}{3}\wp_{34}\lambda_0 Q_{2233} - \lambda_2\lambda_3\wp_{13}\wp_{23} - 2\lambda_4\lambda_3\wp_{23}\lambda_0 \\
&- 2\lambda_1\lambda_4\wp_{13}\wp_{23} - \lambda_4\lambda_2\wp_{23}\lambda_1 + \tfrac{5}{2}\lambda_1\wp_{22}\wp_{33}\lambda_3 - \tfrac{1}{3}\lambda_4\wp_{24}\lambda_1 Q_{2333} + \tfrac{4}{3}\lambda_4\wp_{14}\lambda_2 Q_{2333} \\
&- \tfrac{8}{3}\lambda_4\wp_{44}\lambda_0 Q_{2333} + 5\lambda_1\wp_{13}^2 - 2\wp_{23}\lambda_1^2 + 2\wp_{11}Q_{1224} - \tfrac{1}{3}\lambda_1\wp_{34}\lambda_0 + \tfrac{8}{3}\lambda_2^2\wp_{13}\wp_{34} \\
&+ \wp_{34}\lambda_4\lambda_1^2 - 2\lambda_0\wp_{13}\lambda_3 - 4\lambda_0\wp_{12}\wp_{33} - 8\wp_{44}\lambda_0 Q_{1244} - \wp_{24}\lambda_1 Q_{1244} - 2\lambda_0\wp_{24}\wp_{33}\lambda_3 \\
&+ \tfrac{4}{3}\wp_{34}\lambda_2^3 + \tfrac{1}{3}\wp_{12}\lambda_2 Q_{2333} + 4\wp_{14}\lambda_2 Q_{1244} - \tfrac{8}{3}\lambda_2\wp_{34}^2\lambda_0 + 3\wp_{34}^2\lambda_1^2 + 14\lambda_0\wp_{34}\wp_{23}\lambda_3 \\
&- \tfrac{8}{3}\lambda_0\wp_{34}\lambda_4\wp_{13} + 2\lambda_1\wp_{14}\wp_{33}\lambda_3 - 4\lambda_2\lambda_4\wp_{34}\lambda_0 - \tfrac{4}{3}\lambda_4^2\lambda_2\lambda_0 + \lambda_3\lambda_1 Q_{2233} + 3\lambda_3\lambda_2\lambda_0 \\
&+ \tfrac{2}{3}\lambda_4\lambda_2^3 - \tfrac{1}{3}\lambda_2^2\lambda_1 - \tfrac{7}{3}\lambda_4\lambda_1\lambda_0 - 2\lambda_4\lambda_3\lambda_2\lambda_1 + \tfrac{2}{3}\lambda_4\lambda_0 Q_{2233} - \tfrac{1}{3}\lambda_2^2 Q_{2233} - 2\wp_{12}^2\wp_{14}
\end{aligned}$$

$$\begin{aligned}
\wp_{122}^2 &= -2\lambda_2\lambda_3\wp_{12}\wp_{33} + 4\lambda_2\lambda_4\wp_{14}\wp_{33}\lambda_3 + 12\lambda_4\lambda_3\wp_{44}\wp_{33}\lambda_0 + 2\wp_{12}\lambda_1 Q_{2444} \\
&+ 4\wp_{14}\lambda_0 Q_{2444} + 4\lambda_0\wp_{13}\wp_{23} - \lambda_1\wp_{23}^2\lambda_3 + \tfrac{4}{3}\lambda_4\wp_{13}\lambda_2^2 - \tfrac{4}{3}\lambda_2\lambda_4\wp_{13}^2 - \tfrac{2}{3}\wp_{13}\lambda_2 Q_{2233} \\
&- 2\lambda_3^2\wp_{23}\lambda_1 + 12\lambda_3^2\wp_{34}\lambda_0 - 4\wp_{12}\lambda_3 Q_{1244} + \tfrac{4}{3}\wp_{11}\lambda_4^2 Q_{2333} - \lambda_3^2\wp_{11}\wp_{33} - 8\lambda_4\lambda_0\wp_{23}^2 \\
&+ 16\lambda_2\wp_{44}\wp_{33}\lambda_0 - \tfrac{4}{3}\lambda_2\wp_{23}\wp_{34}\lambda_1 - 2\lambda_1\lambda_3\wp_{13}\wp_{34} - \tfrac{8}{3}\lambda_3\lambda_2\wp_{34}\lambda_1 - \tfrac{2}{3}\lambda_4\lambda_3\lambda_1\wp_{13} \\
&- 4\wp_{11}\wp_{33}\lambda_4^2\lambda_3 - \tfrac{4}{3}\lambda_4\wp_{12}\lambda_3 Q_{2333} + 4\lambda_4\lambda_1\wp_{12}\wp_{33} + 4\lambda_4\lambda_3^2\wp_{12}\wp_{33} + 8\lambda_4\lambda_0\wp_{22}\wp_{33} \\
&+ \tfrac{1}{3}\wp_{23}\lambda_1 Q_{2233} - 2\lambda_1\lambda_4\wp_{24}\wp_{33}\lambda_3 + \tfrac{10}{3}\lambda_2\lambda_1\wp_{13} - 2\wp_{33}\wp_{44}\lambda_1^2 - 4\lambda_4\wp_{44}\lambda_0 Q_{2333} \\
&- \tfrac{1}{3}\wp_{22}\lambda_1 Q_{2333} - 2\wp_{34}\lambda_0 Q_{2233} + 2\lambda_2\lambda_3\wp_{13}\wp_{23} + 8\lambda_4\lambda_3\wp_{23}\lambda_0 + \tfrac{8}{3}\lambda_1\lambda_4\wp_{13}\wp_{23} \\
&- \tfrac{2}{3}\lambda_4\lambda_2\wp_{23}\lambda_1 + \lambda_1\wp_{22}\wp_{33}\lambda_3 + \tfrac{2}{3}\lambda_4\wp_{24}\lambda_1 Q_{2333} - \tfrac{4}{3}\lambda_4\wp_{14}\lambda_2 Q_{2333} - 6\lambda_1\wp_{11}\wp_{33} \\
&- 2\lambda_1\wp_{13}^2 - \tfrac{8}{3}\wp_{23}\lambda_1^2 + 2\wp_{11}Q_{1224} - 22\lambda_1\wp_{34}\lambda_0 + \tfrac{1}{3}\lambda_3\wp_{11}Q_{2333} + \tfrac{8}{3}\lambda_2^2\wp_{13}\wp_{34} \\
&+ 2\wp_{34}\lambda_4\lambda_1^2 + 2\lambda_0\wp_{13}\lambda_3 + 4\lambda_0\wp_{12}\wp_{33} + 4\wp_{11}\lambda_4 Q_{1244} - 12\wp_{44}\lambda_0 Q_{1244} \\
&+ 2\wp_{24}\lambda_0 Q_{2333} + 2\wp_{24}\lambda_1 Q_{1244} + 4\wp_{12}^2\wp_{22} + \tfrac{2}{3}\wp_{12}\lambda_2 Q_{2333} - 4\wp_{14}\lambda_2 Q_{1244} \\
&- 8\lambda_2\wp_{34}^2\lambda_0 + 2\wp_{34}^2\lambda_1^2 + 14\lambda_0\wp_{34}\wp_{23}\lambda_3 + 8\lambda_0\wp_{34}\lambda_4\wp_{13} + 2\lambda_1\wp_{14}\wp_{33}\lambda_3 \\
&+ 4\lambda_2\lambda_4\wp_{34}\lambda_0 + 4\lambda_4^2\lambda_2\lambda_0 + \tfrac{2}{3}\lambda_3\lambda_1 Q_{2233} + 8\lambda_3\lambda_2\lambda_0 - \tfrac{4}{3}\lambda_3\lambda_1^2 - 16\lambda_4\lambda_1\lambda_0 \\
&- 15\lambda_0^2 - \tfrac{4}{3}\lambda_4\lambda_3\lambda_2\lambda_1 - 4\lambda_4\lambda_0 Q_{2233} - 14\lambda_0\wp_{24}\wp_{33}\lambda_3 + \lambda_4^2\lambda_1^2 + \lambda_3^2\wp_{13}^2 \\
\wp_{112}\wp_{124} &= \lambda_2\lambda_3\wp_{12}\wp_{33} - 4\lambda_4\lambda_3\wp_{44}\wp_{33}\lambda_0 - \tfrac{1}{3}\wp_{12}\lambda_1 Q_{2444} - \tfrac{4}{3}\wp_{14}\lambda_0 Q_{2444} \\
&+ \tfrac{2}{3}\wp_{22}\lambda_0 Q_{2444} - 2\lambda_2\wp_{23}\lambda_0 + 2\wp_{11}\wp_{12}\wp_{24} - 4\lambda_2\wp_{44}\wp_{33}\lambda_0 + \lambda_1\lambda_3\wp_{13}\wp_{34} \\
&+ \lambda_1\lambda_4\wp_{24}\wp_{33}\lambda_3 - \tfrac{8}{3}\lambda_0\wp_{13}\lambda_4^2 + \lambda_2\lambda_1\wp_{13} + \wp_{33}\wp_{44}\lambda_1^2 + \lambda_1\wp_{11}\wp_{33} \\
&+ \tfrac{2}{3}\wp_{34}\lambda_0 Q_{2233} - 2\lambda_4\lambda_3\wp_{23}\lambda_0 - \tfrac{1}{2}\lambda_1\wp_{22}\wp_{33}\lambda_3 - \tfrac{1}{3}\lambda_4\wp_{24}\lambda_1 Q_{2333} \\
&- \lambda_1\wp_{13}^2 - \tfrac{7}{3}\lambda_1\wp_{34}\lambda_0 - \wp_{34}\lambda_4\lambda_1^2 - 2\lambda_0\wp_{13}\lambda_3 + 4\lambda_0\wp_{12}\wp_{33} + 4\wp_{44}\lambda_0 Q_{1244} \\
&- \wp_{24}\lambda_0 Q_{2333} - \wp_{24}\lambda_1 Q_{1244} + 2\wp_{12}^2\wp_{14} - \tfrac{1}{3}\wp_{12}\lambda_2 Q_{2333} + \tfrac{4}{3}\lambda_2\wp_{34}^2\lambda_0 - \wp_{34}^2\lambda_1^2 \\
&- \tfrac{8}{3}\lambda_0\wp_{34}\lambda_4\wp_{13} - \lambda_1\wp_{14}\wp_{33}\lambda_3 - \tfrac{4}{3}\lambda_4^2\lambda_2\lambda_0 - \tfrac{1}{3}\lambda_4\lambda_1\lambda_0 - 6\lambda_0^2 + \tfrac{2}{3}\lambda_4\lambda_0 Q_{2233} \\
&- 4\lambda_0\wp_{34}\wp_{23}\lambda_3 + 7\lambda_0\wp_{24}\wp_{33}\lambda_3 + \tfrac{1}{6}\wp_{22}\lambda_1 Q_{2333} + \tfrac{4}{3}\lambda_4\wp_{44}\lambda_0 Q_{2333} \\
\wp_{111}\wp_{144} &= 2\wp_{11}^2\wp_{44} + 6\lambda_4\lambda_3\wp_{44}\wp_{33}\lambda_0 - \tfrac{1}{3}\wp_{12}\lambda_1 Q_{2444} + \tfrac{2}{3}\wp_{14}\lambda_0 Q_{2444} + 2\lambda_0\wp_{13}\wp_{23} \\
&- \tfrac{1}{2}\lambda_1\wp_{23}^2\lambda_3 + \tfrac{3}{2}\lambda_3^2\wp_{11}\wp_{33} - 4\lambda_0\wp_{13}\lambda_4^2 + \tfrac{1}{3}\wp_{14}\lambda_1 Q_{2333} - 2\lambda_2\wp_{23}\lambda_0 + 4\lambda_2\wp_{44}\wp_{33}\lambda_0 \\
&- \tfrac{2}{3}\lambda_2\wp_{23}\wp_{34}\lambda_1 + \lambda_1\lambda_3\wp_{13}\wp_{34} + 4\lambda_4\wp_{14}\wp_{33}\lambda_0 + \tfrac{1}{6}\wp_{23}\lambda_1 Q_{2233} - \lambda_1\lambda_4\wp_{24}\wp_{33}\lambda_3 \\
&- \wp_{33}\wp_{44}\lambda_1^2 + 3\lambda_1\wp_{11}\wp_{33} - \tfrac{1}{6}\wp_{22}\lambda_1 Q_{2333} + \tfrac{1}{3}\wp_{34}\lambda_0 Q_{2233} - 3\lambda_4\lambda_3\wp_{23}\lambda_0 \\
&- \tfrac{1}{3}\lambda_4\lambda_2\wp_{23}\lambda_1 + \tfrac{1}{2}\lambda_1\wp_{22}\wp_{33}\lambda_3 + \tfrac{1}{3}\lambda_4\wp_{24}\lambda_1 Q_{2333} - 2\lambda_4\wp_{44}\lambda_0 Q_{2333} \\
&- \wp_{11}Q_{1224} + \tfrac{1}{3}\lambda_1\wp_{34}\lambda_0 - \tfrac{1}{2}\lambda_3\wp_{11}Q_{2333} - \wp_{34}\lambda_4\lambda_1^2 + 3\lambda_0\wp_{13}\lambda_3 - 2\lambda_0\wp_{12}\wp_{33} \\
&+ \tfrac{2}{3}\wp_{23}\lambda_1^2 - 6\wp_{44}\lambda_0 Q_{1244} - \tfrac{1}{3}\wp_{24}\lambda_0 Q_{2333} + \wp_{24}\lambda_1 Q_{1244} + 2\wp_{11}\wp_{14}^2 + \tfrac{8}{3}\lambda_2\wp_{34}^2\lambda_0 \\
&- \wp_{34}^2\lambda_1^2 - \lambda_0\wp_{34}\wp_{23}\lambda_3 - \tfrac{16}{3}\lambda_0\wp_{34}\lambda_4\wp_{13} - 2\lambda_1\wp_{14}\wp_{33}\lambda_3 - \tfrac{2}{3}\lambda_2\lambda_4\wp_{34}\lambda_0 \\
&+ \lambda_4\lambda_1\lambda_0 - 2\lambda_0^2 + \lambda_4\lambda_0 Q_{2233} + \lambda_0\wp_{24}\wp_{33}\lambda_3 - \tfrac{2}{3}\lambda_1\lambda_4\wp_{13}\wp_{23} - 2\lambda_4^2\lambda_2\lambda_0
\end{aligned}$$

The following relations are at weight -31.

$$
\begin{aligned}
\wp_{111}\wp_{134} &= 2\lambda_2\wp_{11}\wp_{14} - 195\lambda_0\wp_{14}\wp_{22} - \tfrac{63}{2}\wp_{33}\lambda_0 Q_{2233} - 197\wp_{34}\lambda_0 Q_{1244} \\
&\quad + \tfrac{199}{2}\lambda_0 Q_{1224} - 2\wp_{11}\wp_{24}\lambda_1 + 126\lambda_0\wp_{33}\wp_{13}\lambda_4 + 63\lambda_4\lambda_2\wp_{33}\lambda_0 - \tfrac{1}{2}\lambda_2\wp_{33}\lambda_3\lambda_1 \\
&\quad - \tfrac{197}{3}\lambda_4\wp_{34}\lambda_0 Q_{2333} + \tfrac{191}{3}\lambda_0\wp_{13}Q_{2444} + \tfrac{1}{6}\lambda_1\wp_{13}Q_{2333} + \tfrac{63}{2}\wp_{23}\lambda_0 Q_{2333} + \tfrac{2}{3}\lambda_1^2 Q_{2444} \\
&\quad + 2\wp_{11}^2\wp_{34} + \tfrac{1}{6}\lambda_2\lambda_1 Q_{2333} + \tfrac{193}{4}\lambda_3\lambda_0 Q_{2333} - 2\lambda_2\lambda_0 Q_{2444} - \tfrac{1}{2}\lambda_1\wp_{33}\wp_{13}\lambda_3 \\
&\quad + 2\wp_{11}\wp_{13}\wp_{14} - 2\wp_{33}\wp_{34}\lambda_1^2 - 329\lambda_1\wp_{33}\lambda_0 - 193\lambda_0\wp_{11}\wp_{44} + 197\lambda_0\wp_{34}\wp_{33}\lambda_4\lambda_3 \\
&\quad + 134\lambda_2\wp_{34}\wp_{33}\lambda_0 - 5\tfrac{79}{4}\lambda_0\wp_{33}\lambda_3^2 + 191\lambda_0\wp_{14}^2 + 197\lambda_0\wp_{12}\wp_{24}
\end{aligned}
$$

$$
\begin{aligned}
\wp_{113}\wp_{114} &= \tfrac{3}{2}\lambda_0\wp_{33}\lambda_3^2 - \tfrac{1}{2}\lambda_3\lambda_0 Q_{2333} - \tfrac{4}{3}\lambda_0\wp_{13}Q_{2444} - 4\lambda_0\wp_{34}\wp_{33}\lambda_4\lambda_3 - 4\lambda_0\wp_{12}\wp_{24} \\
&\quad + 2\lambda_0\wp_{11}\wp_{44} - 4\lambda_2\wp_{34}\wp_{33}\lambda_0 + \wp_{33}\wp_{34}\lambda_1^2 + 4\wp_{11}\wp_{13}\wp_{14} + 3\lambda_1\wp_{33}\lambda_0 + \lambda_1\wp_{33}\wp_{13}\lambda_3 \\
&\quad + 2\lambda_1\wp_{12}\wp_{14} - \lambda_0 Q_{1224} + \tfrac{4}{3}\lambda_4\wp_{34}\lambda_0 Q_{2333} - \tfrac{1}{3}\lambda_1\wp_{13}Q_{2333} + 4\wp_{34}\lambda_0 Q_{1244} + 2\lambda_0\wp_{14}^2
\end{aligned}
$$

$$
\begin{aligned}
\wp_{113}\wp_{122} &= 2\lambda_4\lambda_2\wp_{33}\wp_{13}\lambda_3 - 4\lambda_4^2\lambda_1\wp_{34}\wp_{33}\lambda_3 + 5\lambda_0 Q_{1224} - 2\wp_{11}\wp_{24}\lambda_1 - 4\lambda_1\wp_{14}^2\lambda_4 \\
&\quad - 2\lambda_3\lambda_1\lambda_4\wp_{23}\wp_{33} + \tfrac{4}{3}\wp_{34}\lambda_4^2\lambda_1 Q_{2333} - \tfrac{2}{3}\lambda_4\wp_{13}\lambda_2 Q_{2333} + \tfrac{2}{3}\wp_{33}\lambda_4\lambda_1 Q_{2233} + 4\lambda_0\wp_{12}\wp_{24} \\
&\quad + 4\wp_{34}\lambda_4\lambda_1 Q_{1244} - 2\wp_{13}\lambda_2 Q_{1244} + 8\lambda_0\wp_{33}\wp_{13}\lambda_4 + 3\lambda_3^2\lambda_1\wp_{33}\lambda_4 - 4\lambda_1\lambda_4\wp_{12}\wp_{24} \\
&\quad + 4\lambda_1\lambda_4\wp_{11}\wp_{44} + 4\lambda_1\lambda_4\wp_{14}\wp_{22} - \tfrac{8}{3}\lambda_1\wp_{33}\wp_{13}\lambda_4^2 - 4\lambda_4^2\lambda_0\wp_{33}\lambda_3 - 6\lambda_0\lambda_3\wp_{23}\wp_{33} \\
&\quad - \tfrac{4}{3}\lambda_4^2\lambda_1\wp_{33}\lambda_2 - 4\lambda_4\lambda_2\wp_{33}\lambda_0 - \lambda_2\wp_{33}\lambda_3\lambda_1 + 2\wp_{11}\wp_{13}\wp_{22} + 2\lambda_1\wp_{12}\wp_{22} - \tfrac{3}{2}\lambda_0\wp_{33}\lambda_3^2 \\
&\quad + \tfrac{23}{3}\wp_{33}\lambda_4\lambda_1^2 + 2\lambda_0\wp_{13}Q_{2444} + 2\wp_{23}\lambda_1 Q_{1244} - \tfrac{1}{3}\lambda_1\wp_{13}Q_{2333} + \tfrac{2}{3}\wp_{23}\lambda_0 Q_{2333} \\
&\quad - \tfrac{4}{3}\lambda_1\wp_{13}\lambda_4 Q_{2444} - \lambda_4\lambda_3\lambda_1 Q_{2333} + \tfrac{2}{3}\lambda_1^2 Q_{2444} - 2\lambda_4\lambda_1 Q_{1224} + 4\lambda_4\lambda_0 Q_{1244} \\
&\quad - 4\wp_{22}^2\lambda_0 + 2\wp_{13}\wp_{12}^2 + \tfrac{4}{3}\lambda_4^2\lambda_0 Q_{2333} + \tfrac{1}{3}\lambda_2\lambda_1 Q_{2333} + \tfrac{1}{2}\lambda_3\lambda_0 Q_{2333} - \tfrac{4}{3}\lambda_2\lambda_0 Q_{2444} \\
&\quad - \tfrac{8}{3}\lambda_4\lambda_2\wp_{33}\wp_{34}\lambda_1 + 2\lambda_1\wp_{33}\wp_{13}\lambda_3 + 2\lambda_1\wp_{12}\wp_{14} - 11\lambda_1\wp_{33}\lambda_0 - 6\lambda_0\wp_{11}\wp_{44} + 6\lambda_0\wp_{14}^2
\end{aligned}
$$

$$
\begin{aligned}
\wp_{112}\wp_{123} &= 56\lambda_4\lambda_2\wp_{33}\wp_{13}\lambda_3 - 4\lambda_4^2\lambda_1\wp_{34}\wp_{33}\lambda_3 - 28\lambda_4\lambda_3\wp_{34}\lambda_2 Q_{2333} - \wp_{23}\lambda_1 Q_{1244} \\
&\quad + 84\lambda_2\wp_{34}\wp_{33}\lambda_4\lambda_3^2 - 193\lambda_0 Q_{1224} + 84\lambda_2\lambda_3\wp_{14}^2 - 63\lambda_2\wp_{33}\lambda_3^3 + 2\wp_{11}\wp_{12}\wp_{23} \\
&\quad + 378\lambda_0\wp_{14}\wp_{22} + 63\wp_{33}\lambda_0 Q_{2233} + 378\wp_{34}\lambda_0 Q_{1244} - 4\lambda_1\wp_{14}^2\lambda_4 + \lambda_3\lambda_1\lambda_4\wp_{23}\wp_{33} \\
&\quad + \tfrac{4}{3}\wp_{34}\lambda_4^2\lambda_1 Q_{2333} + \tfrac{2}{3}\wp_{33}\lambda_4\lambda_1 Q_{2233} + 4\wp_{34}\lambda_4\lambda_1 Q_{1244} - 252\lambda_0\wp_{33}\wp_{13}\lambda_4 - 378\lambda_0\wp_{14}^2 \\
&\quad + 3\lambda_3^2\lambda_1\wp_{33}\lambda_4 - 4\lambda_1\lambda_4\wp_{12}\wp_{24} + 4\lambda_1\lambda_4\wp_{11}\wp_{44} + 4\lambda_1\lambda_4\wp_{14}\wp_{22} - \tfrac{8}{3}\lambda_1\wp_{33}\wp_{13}\lambda_4^2 \\
&\quad + 8\lambda_4^2\lambda_0\wp_{33}\lambda_3 + 3\lambda_0\lambda_3\wp_{23}\wp_{33} - \tfrac{4}{3}\lambda_4^2\lambda_1\wp_{33}\lambda_2 - 118\lambda_4\lambda_2\wp_{33}\lambda_0 - 140\lambda_2\wp_{33}\lambda_3\lambda_1 \\
&\quad + 2\lambda_2\wp_{12}^2 + 126\lambda_4\wp_{34}\lambda_0 Q_{2333} - \lambda_4\wp_{23}\lambda_1 Q_{2333} + 14\lambda_2\wp_{23}\lambda_3 Q_{2333} - 14\wp_{33}\lambda_3\lambda_2 Q_{2233} \\
&\quad - 84\lambda_3\wp_{34}\lambda_2 Q_{1244} + 28\lambda_2\wp_{13}\lambda_3 Q_{2444} + 28\wp_{33}\lambda_4\lambda_3\lambda_2^2 + 56\lambda_3\wp_{33}\wp_{34}\lambda_2^2 - 386\lambda_0\wp_{12}\wp_{24} \\
&\quad - 84\lambda_2\lambda_3\wp_{14}\wp_{22} - 84\lambda_2\lambda_3\wp_{11}\wp_{44} + 84\lambda_2\lambda_3\wp_{12}\wp_{24} - 2\lambda_1\wp_{12}\wp_{22} + \tfrac{14}{3}\wp_{33}\lambda_4\lambda_1^2 \\
&\quad - 126\lambda_0\wp_{13}Q_{2444} - \tfrac{190}{3}\wp_{23}\lambda_0 Q_{2333} - \tfrac{4}{3}\lambda_1\wp_{13}\lambda_4 Q_{2444} - \lambda_4\lambda_3\lambda_1 Q_{2333} + 2\wp_{13}\wp_{12}^2 \\
&\quad + 42\lambda_3\lambda_2 Q_{1224} + 21\lambda_3^2\lambda_2 Q_{2333} - \tfrac{2}{3}\lambda_1^2 Q_{2444} - 2\lambda_4\lambda_1 Q_{1224} - 8\lambda_4\lambda_0 Q_{1244} + 2\wp_{22}^2\lambda_0 \\
&\quad - \tfrac{8}{3}\lambda_4^2\lambda_0 Q_{2333} - \tfrac{191}{2}\lambda_3\lambda_0 Q_{2333} + \tfrac{8}{3}\lambda_2\lambda_0 Q_{2444} - \tfrac{8}{3}\lambda_4\lambda_2\wp_{33}\wp_{34}\lambda_1 + \tfrac{573}{2}\lambda_0\wp_{33}\lambda_3^2 \\
&\quad + 4\lambda_1\wp_{12}\wp_{14} + 644\lambda_1\wp_{33}\lambda_0 + 378\lambda_0\wp_{11}\wp_{44} - 378\lambda_0\wp_{34}\wp_{33}\lambda_4\lambda_3 - 252\lambda_2\wp_{34}\wp_{33}\lambda_0
\end{aligned}
$$

$$\wp_{111}\wp_{223} = 54\lambda_4\lambda_2\wp_{33}\wp_{13}\lambda_3 - 4\lambda_4^2\lambda_1\wp_{34}\wp_{33}\lambda_3 - 28\lambda_4\lambda_3\wp_{34}\lambda_2 Q_{2333} - 4\lambda_1\wp_{14}^2\lambda_4$$
$$+ 84\lambda_2\wp_{34}\wp_{33}\lambda_4\lambda_3^2 + 2\lambda_2^2 Q_{1244} - 187\lambda_0 Q_{1224} + 8\lambda_2\wp_{11}\wp_{14} - 2\lambda_2\wp_{11}\wp_{22}$$
$$+ 6\wp_{11}\wp_{12}\lambda_3 + 84\lambda_2\lambda_3\wp_{14}^2 - 63\lambda_2\wp_{33}\lambda_3^3 + 4\wp_{11}\wp_{12}\wp_{23} + 4\lambda_3\lambda_1\lambda_4\wp_{23}\wp_{33}$$
$$+ 63\wp_{33}\lambda_0 Q_{2233} + 378\wp_{34}\lambda_0 Q_{1244} + 390\lambda_0\wp_{14}\wp_{22} + \tfrac{4}{3}\wp_{34}\lambda_4^2\lambda_1 Q_{2333} - 4\wp_{11}\wp_{24}\lambda_1$$
$$+ \tfrac{2}{3}\lambda_4\wp_{13}\lambda_2 Q_{2333} + \tfrac{2}{3}\wp_{33}\lambda_4\lambda_1 Q_{2233} + 4\wp_{34}\lambda_4\lambda_1 Q_{1244} + 2\wp_{13}\lambda_2 Q_{1244} - 374\lambda_0\wp_{12}\wp_{24}$$
$$- 260\lambda_0\wp_{33}\wp_{13}\lambda_4 + 9\lambda_3^2\lambda_1\wp_{33}\lambda_4 - 4\lambda_1\lambda_4\wp_{12}\wp_{24} + 4\lambda_1\lambda_4\wp_{11}\wp_{44} + 4\lambda_1\lambda_4\wp_{14}\wp_{22}$$
$$- \tfrac{8}{3}\lambda_1\wp_{33}\wp_{13}\lambda_4^2 - 4\lambda_4^2\lambda_0\wp_{33}\lambda_3 + 12\lambda_0\lambda_3\wp_{23}\wp_{33} - \tfrac{4}{3}\lambda_4^2\lambda_1\wp_{33}\lambda_2 - 138\lambda_4\lambda_2\wp_{33}\lambda_0$$
$$- 142\lambda_2\wp_{33}\lambda_3\lambda_1 + 126\lambda_4\wp_{34}\lambda_0 Q_{2333} - 2\lambda_4\wp_{23}\lambda_1 Q_{2333} + 14\lambda_2\wp_{23}\lambda_3 Q_{2333}$$
$$- 14\wp_{33}\lambda_3\lambda_2 Q_{2233} - 84\lambda_3\wp_{34}\lambda_2 Q_{1244} + 28\lambda_2\wp_{13}\lambda_3 Q_{2444} + 26\wp_{33}\lambda_4\lambda_3\lambda_2^2$$
$$+ 56\lambda_3\wp_{33}\wp_{34}\lambda_2^2 - 84\lambda_2\lambda_3\wp_{14}\wp_{22} - 84\lambda_2\lambda_3\wp_{11}\wp_{44} + 84\lambda_2\lambda_3\wp_{12}\wp_{24} + 2\wp_{11}\wp_{13}\wp_{22}$$
$$+ 4\lambda_1\wp_{12}\wp_{22} + \tfrac{32}{3}\wp_{33}\lambda_4\lambda_1^2 - 128\lambda_0\wp_{13}Q_{2444} - 4\wp_{23}\lambda_1 Q_{1244} + \tfrac{1}{3}\lambda_1\wp_{13}Q_{2333}$$
$$- \tfrac{193}{3}\wp_{23}\lambda_0 Q_{2333} - 6\lambda_3\lambda_1 Q_{1244} + \tfrac{2}{3}\lambda_4\lambda_2^2 Q_{2333} - \tfrac{4}{3}\lambda_1\wp_{13}\lambda_4 Q_{2444} - 3\lambda_4\lambda_3\lambda_1 Q_{2333}$$
$$+ 42\lambda_3\lambda_2 Q_{1224} + 21\lambda_3^2\lambda_2 Q_{2333} + \tfrac{4}{3}\lambda_1^2 Q_{2444} - 2\lambda_4\lambda_1 Q_{1224} + 4\lambda_4\lambda_0 Q_{1244} - 4\wp_{22}^2\lambda_0$$
$$- 2\wp_{13}\wp_{12}^2 - 2\lambda_2\wp_{12}^2 - 4\wp_{11}^2\lambda_4 + \tfrac{4}{3}\lambda_4^2\lambda_0 Q_{2333} + \tfrac{1}{3}\lambda_2\lambda_1 Q_{2333} - \tfrac{191}{2}\lambda_3\lambda_0 Q_{2333}$$
$$- \tfrac{10}{3}\lambda_2\lambda_0 Q_{2444} - \tfrac{8}{3}\lambda_4\lambda_2\wp_{33}\wp_{34}\lambda_1 - 2\lambda_1\wp_{33}\wp_{13}\lambda_3 - 8\lambda_1\wp_{12}\wp_{14} + 626\lambda_1\wp_{33}\lambda_0$$
$$+ 378\lambda_0\wp_{11}\wp_{44} - 378\lambda_0\wp_{34}\wp_{33}\lambda_4\lambda_3 - 252\lambda_2\wp_{34}\wp_{33}\lambda_0 + 5\tfrac{91}{2}\lambda_0\wp_{33}\lambda_3^2 - 390\lambda_0\wp_{14}^2$$

The following relations are at weight -32.

$$\wp_{113}^2 = -4\lambda_2\lambda_0\wp_{33}^2 + \lambda_1^2\wp_{33}^2 + 4\wp_{11}\wp_{13}^2 - 4\wp_{23}\lambda_0\wp_{12} + 4\wp_{14}\lambda_1^2 + 8\lambda_0\wp_{11}\wp_{34}$$
$$- 4\lambda_0\wp_{13}\wp_{22} - 4\lambda_0\wp_{24}\lambda_1 - 8\lambda_0\wp_{14}\lambda_2 + 12\wp_{44}\lambda_0^2 + 4\lambda_1\wp_{12}\wp_{13}$$
$$- 4\lambda_4\lambda_0\wp_{33}^2\lambda_3 + \tfrac{4}{3}\wp_{33}\lambda_4\lambda_0 Q_{2333} + 4\wp_{33}\lambda_0 Q_{1244}$$
$$\wp_{111}\wp_{133} = -\tfrac{8}{3}\wp_{33}\lambda_4\lambda_0 Q_{2333} + 6\lambda_0\wp_{13}\wp_{14} + 8\lambda_2\lambda_0\wp_{33}^2 - 2\lambda_1^2\wp_{33}^2 + 2\wp_{11}^2\wp_{33}$$
$$+ 2\wp_{11}\wp_{13}^2 + 8\wp_{23}\lambda_0\wp_{12} + 9\lambda_3\lambda_0\wp_{12} - 4\wp_{14}\lambda_1^2 + 2\wp_{22}\lambda_1^2 - 4\lambda_0\wp_{11}\wp_{34} - 4\lambda_0\wp_{13}\wp_{22}$$
$$+ 2\lambda_0\wp_{24}\lambda_1 + 10\lambda_0\wp_{14}\lambda_2 - 6\wp_{44}\lambda_0^2 - 6\lambda_0\wp_{11}\lambda_4 + 2\wp_{11}\wp_{13}\lambda_2 - \lambda_1\wp_{12}\lambda_2$$
$$- 2\lambda_1\wp_{11}\wp_{23} - \lambda_1\wp_{12}\wp_{13} - 6\lambda_2\lambda_0\wp_{22} + 8\lambda_4\lambda_0\wp_{33}^2\lambda_3 - 8\wp_{33}\lambda_0 Q_{1244}$$

The following relations are at weight -33.

$$\wp_{112}\wp_{114} = 2\lambda_0\lambda_3\wp_{13}\wp_{34} - 4\lambda_0\wp_{13}^2 + 2\lambda_1^2\wp_{13} + 2\lambda_1\lambda_4\wp_{14}\wp_{33}\lambda_3 - 2\lambda_4\lambda_0\wp_{24}\wp_{33}\lambda_3$$
$$- \tfrac{7}{3}\lambda_1\wp_{23}\lambda_0 + \lambda_0\wp_{23}^2\lambda_3 - 4\lambda_0\wp_{13}\lambda_2 + 4\lambda_0\wp_{11}\wp_{33} + \wp_{23}\wp_{34}\lambda_1^2 + 4\wp_{11}\wp_{12}\wp_{14} - 4\wp_{34}\lambda_0^2$$
$$- \tfrac{2}{3}\wp_{14}\lambda_0 Q_{2333} + 2\wp_{24}\lambda_0 Q_{1244} - \tfrac{1}{3}\wp_{12}\lambda_1 Q_{2333} + \tfrac{1}{3}\wp_{22}\lambda_0 Q_{2333} - \tfrac{1}{3}\wp_{23}\lambda_0 Q_{2233}$$
$$- 2\wp_{14}\lambda_1 Q_{1244} - \tfrac{2}{3}\lambda_0\wp_{12}Q_{2444} + \tfrac{2}{3}\lambda_4\wp_{24}\lambda_0 Q_{2333} - \tfrac{2}{3}\lambda_1\wp_{14}\lambda_4 Q_{2333} - \wp_{22}\lambda_3\wp_{33}\lambda_0$$
$$- \tfrac{8}{3}\lambda_0\wp_{23}\wp_{34}\lambda_2 + \tfrac{2}{3}\lambda_4\lambda_2\wp_{23}\lambda_0 + \tfrac{4}{3}\lambda_0\lambda_4\wp_{13}\wp_{23} + \lambda_1\lambda_3\wp_{12}\wp_{33} + 4\lambda_3\wp_{14}\wp_{33}\lambda_0 - 4\lambda_4\lambda_0^2$$

$$
\begin{aligned}
\wp_{112}\wp_{122} &= -18\wp_{34}\lambda_0^2 - 2\lambda_0\wp_{13}^2 + \tfrac{8}{3}\lambda_1^2\wp_{13} + 2\lambda_1\lambda_4\wp_{14}\wp_{33}\lambda_3 + 2\lambda_4\lambda_2\lambda_3\wp_{12}\wp_{33} \\
&+ \tfrac{1}{3}\lambda_2\lambda_1 Q_{2233} + 2\wp_{12}\wp_{11}\wp_{22} + 3\lambda_3^2\wp_{23}\lambda_0 - 7\lambda_1\wp_{23}\lambda_0 - 4\lambda_1\wp_{34}^2\lambda_0 - \lambda_4\wp_{23}\lambda_1^2 \\
&- 2\lambda_0\wp_{23}^2\lambda_3 + 2\lambda_0\wp_{13}\lambda_2 + 2\lambda_0\wp_{11}\wp_{33} - \tfrac{2}{3}\lambda_4\lambda_1\wp_{13}^2 - \tfrac{4}{3}\lambda_1\wp_{34}\lambda_2^2 + \tfrac{2}{3}\lambda_1\wp_{11}Q_{2444} \\
&- \tfrac{1}{3}\lambda_1\wp_{13}Q_{2233} - 2\wp_{14}\lambda_1 Q_{1244} - 2\wp_{12}\lambda_2 Q_{1244} + 2\lambda_0\wp_{12}Q_{2444} - \tfrac{2}{3}\lambda_4\wp_{12}\lambda_2 Q_{2333} \\
&- \tfrac{2}{3}\lambda_1\wp_{14}\lambda_4 Q_{2333} + \tfrac{4}{3}\lambda_1\lambda_2\wp_{13}\wp_{34} - 4\lambda_3\lambda_0\wp_{13}\lambda_4 + 8\lambda_4\lambda_0\wp_{12}\wp_{33} + 2\lambda_4\lambda_1\wp_{11}\wp_{33} \\
&+ 8\lambda_3\lambda_2\wp_{34}\lambda_0 + 4\lambda_1\wp_{44}\wp_{33}\lambda_0 - 6\lambda_1\lambda_4\wp_{34}\lambda_0 + \tfrac{4}{3}\lambda_1\lambda_4\wp_{13}\lambda_2 + 2\lambda_1\lambda_3\wp_{13}\wp_{23} \\
&- 2\lambda_3\wp_{14}\wp_{33}\lambda_0 + 2\lambda_0\lambda_3\wp_{13}\wp_{34} + 2\wp_{22}\lambda_3\wp_{33}\lambda_0 - \lambda_1\lambda_2\wp_{23}\lambda_3 - \tfrac{2}{3}\lambda_2\lambda_1^2 + 4\lambda_2^2\lambda_0 \\
&- 18\lambda_4\lambda_0^2 - \tfrac{2}{3}\lambda_4\lambda_2^2\lambda_1 - 2\lambda_4^2\lambda_1\lambda_0 - 2\lambda_3\lambda_1\lambda_0 - \lambda_3\lambda_0 Q_{2233} + 2\lambda_4\lambda_3\lambda_2\lambda_0 + 2\wp_{12}^3
\end{aligned}
$$

$$
\begin{aligned}
\wp_{111}\wp_{222} &= 6\wp_{34}\lambda_0^2 - 6\lambda_4^2\lambda_1\lambda_0 + 10\lambda_0\wp_{13}^2 + 4\lambda_1^2\wp_{13} - 12\lambda_1\lambda_4\wp_{14}\wp_{33}\lambda_3 - 6\lambda_4\lambda_2\lambda_3\wp_{12}\wp_{33} \\
&- 2\wp_{12}^3 + 6\lambda_4\lambda_0\wp_{24}\wp_{33}\lambda_3 + 6\lambda_1\lambda_4\wp_{22}\wp_{33}\lambda_3 - 4\lambda_2^2\wp_{12}\wp_{33} + 4\lambda_2^2\wp_{23}\wp_{13} + 4\lambda_3\lambda_1\lambda_0 \\
&+ 6\wp_{12}\wp_{11}\wp_{22} + 3\lambda_3^2\wp_{23}\lambda_0 + 2\wp_{24}\wp_{33}\lambda_1^2 - 6\lambda_1\wp_{23}\lambda_0 + 4\lambda_1\wp_{34}^2\lambda_0 - 2\lambda_3\wp_{34}\lambda_1^2 \\
&- 2\lambda_1\lambda_2\wp_{23}^2 - 4\lambda_4\wp_{23}\lambda_1^2 + 13\lambda_0\wp_{23}^2\lambda_3 + 6\lambda_0\wp_{13}\lambda_2 + 2\lambda_0\wp_{11}\wp_{33} + 2\lambda_4\lambda_3\lambda_2\lambda_0 \\
&- 2\wp_{23}\wp_{34}\lambda_1^2 - 6\lambda_3^2\lambda_1\wp_{13} + 2\lambda_3\lambda_2^2\wp_{13} - 2\lambda_3\lambda_2\wp_{13}^2 - 6\wp_{22}\lambda_1 Q_{1244} - 6\lambda_3\wp_{11}Q_{1244} \\
&- 6\wp_{24}\lambda_0 Q_{1244} - \wp_{12}\lambda_1 Q_{2333} - \wp_{22}\lambda_0 Q_{2333} - \wp_{23}\lambda_0 Q_{2233} + 2\lambda_1\wp_{13}Q_{2233} + 2\lambda_4\lambda_1\wp_{13}^2 \\
&+ 12\wp_{14}\lambda_1 Q_{1244} + 6\wp_{12}\lambda_2 Q_{1244} - 4\lambda_0\wp_{12}Q_{2444} - 2\lambda_3\wp_{11}\lambda_4 Q_{2333} - 2\wp_{22}\lambda_4\lambda_1 Q_{2333} \\
&- 2\lambda_4\wp_{24}\lambda_0 Q_{2333} + 2\lambda_4\wp_{12}\lambda_2 Q_{2333} + 4\lambda_1\wp_{14}\lambda_4 Q_{2333} + 4\lambda_2\wp_{24}\wp_{33}\lambda_0 + 2\lambda_3\lambda_0 Q_{2233} \\
&+ 6\wp_{11}\wp_{33}\lambda_4\lambda_3^2 + 2\lambda_3\lambda_2\wp_{11}\wp_{33} - 8\lambda_1\lambda_2\wp_{14}\wp_{33} + 2\lambda_1\lambda_2\wp_{22}\wp_{33} + 2\lambda_4\lambda_2\wp_{23}\lambda_0 \\
&- 12\lambda_0\lambda_4\wp_{13}\wp_{23} + 7\lambda_1\lambda_3\wp_{12}\wp_{33} - 16\lambda_3\lambda_0\wp_{13}\lambda_4 - 8\lambda_4\lambda_0\wp_{12}\wp_{33} + 2\lambda_4\lambda_1\wp_{11}\wp_{33} \\
&+ 2\lambda_3\lambda_2\wp_{34}\lambda_0 - 4\lambda_1\wp_{44}\wp_{33}\lambda_0 - 6\lambda_1\lambda_4\wp_{34}\lambda_0 + 2\lambda_1\lambda_4\wp_{13}\lambda_2 - 13\lambda_1\lambda_3\wp_{13}\wp_{23} \\
&+ 4\lambda_3\wp_{14}\wp_{33}\lambda_0 - 4\lambda_0\lambda_3\wp_{13}\wp_{34} + 5\wp_{22}\lambda_3\wp_{33}\lambda_0 - \lambda_1\lambda_2\wp_{23}\lambda_3 - 6\lambda_4\lambda_0^2 - 2\lambda_4\lambda_3\lambda_1^2
\end{aligned}
$$

$$
\begin{aligned}
\wp_{111}\wp_{124} &= 2\wp_{34}\lambda_0^2 + 4\lambda_0\wp_{13}^2 + \tfrac{2}{3}\lambda_1^2\wp_{13} - 2\lambda_1\lambda_4\wp_{14}\wp_{33}\lambda_3 + 4\lambda_4\lambda_0\wp_{24}\wp_{33}\lambda_3 \\
&+ 2\wp_{11}^2\wp_{24} - \tfrac{9}{2}\lambda_3^2\wp_{23}\lambda_0 - \tfrac{10}{3}\lambda_1\wp_{23}\lambda_0 + 2\lambda_1\wp_{34}^2\lambda_0 - 2\lambda_3\wp_{34}\lambda_1^2 - 2\lambda_0\wp_{23}^2\lambda_3 \\
&+ 4\lambda_0\wp_{13}\lambda_2 - \tfrac{2}{3}\lambda_4\lambda_1\wp_{13}^2 - 2\wp_{23}\wp_{34}\lambda_1^2 + \tfrac{2}{3}\lambda_1\wp_{34}\lambda_2^2 + 2\wp_{11}\wp_{12}\wp_{14} - \tfrac{1}{3}\lambda_2\wp_{11}Q_{2333} \\
&+ \tfrac{1}{3}\wp_{14}\lambda_0 Q_{2333} - 4\wp_{24}\lambda_0 Q_{1244} - \tfrac{2}{3}\wp_{22}\lambda_0 Q_{2333} + \tfrac{2}{3}\wp_{23}\lambda_0 Q_{2233} - \tfrac{2}{3}\lambda_1\wp_{11}Q_{2444} \\
&+ \tfrac{1}{6}\lambda_1\wp_{13}Q_{2233} + 2\wp_{14}\lambda_1 Q_{1244} - \tfrac{2}{3}\lambda_0\wp_{12}Q_{2444} - \tfrac{4}{3}\lambda_4\wp_{24}\lambda_0 Q_{2333} + \tfrac{2}{3}\lambda_1\wp_{14}\lambda_4 Q_{2333} \\
&+ \tfrac{4}{3}\lambda_0\wp_{23}\wp_{34}\lambda_2 + \tfrac{4}{3}\lambda_1\lambda_2\wp_{13}\wp_{34} + 4\lambda_2\wp_{24}\wp_{33}\lambda_0 + \lambda_3\lambda_2\wp_{11}\wp_{33} - 2\lambda_1\lambda_2\wp_{14}\wp_{33} \\
&- \tfrac{4}{3}\lambda_4\lambda_2\wp_{23}\lambda_0 - \tfrac{8}{3}\lambda_0\lambda_4\wp_{13}\wp_{23} - 6\lambda_3\lambda_0\wp_{13}\lambda_4 - 2\lambda_1\wp_{44}\wp_{33}\lambda_0 + \tfrac{1}{3}\lambda_1\lambda_4\wp_{13}\lambda_2 \\
&- \tfrac{1}{2}\lambda_1\lambda_3\wp_{13}\wp_{23} + \lambda_3\wp_{14}\wp_{33}\lambda_0 - 4\lambda_0\lambda_3\wp_{13}\wp_{34} + 2\wp_{22}\lambda_3\wp_{33}\lambda_0 + \tfrac{1}{2}\lambda_1\lambda_2\wp_{23}\lambda_3 \\
&- \tfrac{2}{3}\lambda_2\lambda_1^2 + 2\lambda_2^2\lambda_0 - 4\lambda_4\lambda_0^2 + \tfrac{1}{3}\lambda_4\lambda_2^2\lambda_1 + \tfrac{3}{2}\lambda_3\lambda_0 Q_{2233} - 3\lambda_4\lambda_3\lambda_2\lambda_0 - \tfrac{1}{6}\lambda_2\lambda_1 Q_{2233}
\end{aligned}
$$

The following relations are at weight -34.

$$\begin{aligned}
\wp_{112}\wp_{113} &= 4\wp_{11}\wp_{12}\wp_{13} - 4\wp_{11}\wp_{24}\lambda_0 + 4\wp_{23}\lambda_0 Q_{1244} - 2\lambda_1\wp_{13}Q_{1244} - \tfrac{2}{3}\lambda_2\lambda_0 Q_{2333} \\
&\quad - \tfrac{2}{3}\lambda_0\wp_{13}Q_{2333} + \tfrac{2}{3}\lambda_1\lambda_0 Q_{2444} + 2\lambda_2\wp_{33}\lambda_3\lambda_0 - 4\wp_{23}\wp_{33}\lambda_4\lambda_3\lambda_0 + 6\lambda_0\wp_{33}\wp_{13}\lambda_3 \\
&\quad - 4\wp_{23}\wp_{33}\lambda_2\lambda_0 - 4\wp_{12}\wp_{22}\lambda_0 + 8\wp_{33}\lambda_0^2 + \tfrac{1}{3}\lambda_1^2 Q_{2333} + \lambda_1^2\wp_{23}\wp_{33} - \wp_{33}\lambda_3\lambda_1^2 \\
&\quad + 2\lambda_1\wp_{12}^2 + 2\lambda_4\lambda_1\wp_{33}\wp_{13}\lambda_3 + \tfrac{4}{3}\wp_{23}\lambda_4\lambda_0 Q_{2333} - \tfrac{2}{3}\lambda_4\lambda_1\wp_{13}Q_{2333} + 8\lambda_0\wp_{12}\wp_{14} \\
\wp_{111}\wp_{123} &= 8\wp_{23}\wp_{33}\lambda_4\lambda_3\lambda_0 - 2\lambda_1^2\wp_{23}\wp_{33} - 3\lambda_0\wp_{33}\wp_{13}\lambda_3 + 8\wp_{23}\wp_{33}\lambda_2\lambda_0 \\
&\quad + 5\lambda_2\wp_{33}\lambda_3\lambda_0 + 2\wp_{12}\wp_{22}\lambda_0 - 4\lambda_0\wp_{12}\wp_{14} + 2\wp_{33}\lambda_0^2 - 2\lambda_1\wp_{11}\wp_{22} + 4\lambda_1\wp_{11}\wp_{14} \\
&\quad + 63\lambda_1\lambda_3\wp_{14}^2 - \tfrac{189}{4}\lambda_1\wp_{33}\lambda_3^3 + 2\wp_{11}\wp_{12}\lambda_2 + \tfrac{1}{3}\lambda_0\wp_{13}Q_{2333} - 8\wp_{23}\lambda_0 Q_{1244} \\
&\quad + \lambda_1\wp_{13}Q_{1244} + \tfrac{63}{2}\lambda_3\lambda_1 Q_{1224} - 9\lambda_3\lambda_0 Q_{1244} + \lambda_2\lambda_1 Q_{1244} + \tfrac{63}{4}\lambda_3^2\lambda_1 Q_{2333} \\
&\quad + \tfrac{1}{3}\lambda_2\lambda_0 Q_{2333} + \tfrac{2}{3}\lambda_1\lambda_0 Q_{2444} - 107\wp_{33}\lambda_3\lambda_1^2 + 2\wp_{11}\wp_{12}\wp_{13} + 41\lambda_4\lambda_1\wp_{33}\wp_{13}\lambda_3 \\
&\quad + 2\wp_{11}^2\wp_{23} - 21\lambda_4\lambda_1\wp_{34}\lambda_3 Q_{2333} + 63\lambda_4\lambda_1\wp_{34}\wp_{33}\lambda_3^2 + 20\lambda_1\wp_{33}\lambda_4\lambda_3\lambda_2 \\
&\quad + 42\lambda_1\lambda_3\wp_{33}\wp_{34}\lambda_2 + \tfrac{1}{3}\lambda_4\lambda_2\lambda_1 Q_{2333} - 3\lambda_4\lambda_3\lambda_0 Q_{2333} + \tfrac{1}{3}\lambda_4\lambda_1\wp_{13}Q_{2333} \\
&\quad - \tfrac{8}{3}\wp_{23}\lambda_4\lambda_0 Q_{2333} - 63\lambda_1\lambda_3\wp_{14}\wp_{22} + 9\lambda_3^2\lambda_0\wp_{33}\lambda_4 - 63\lambda_3\wp_{34}\lambda_1 Q_{1244} \\
&\quad + 2\lambda_4\lambda_1\wp_{33}\lambda_0 - \tfrac{21}{2}\wp_{33}\lambda_3\lambda_1 Q_{2233} + \tfrac{21}{2}\lambda_1\wp_{23}\lambda_3 Q_{2333} + 21\lambda_3\lambda_1\wp_{13}Q_{2444} \\
&\quad - 63\lambda_1\lambda_3\wp_{11}\wp_{44} - 4\wp_{11}\wp_{24}\lambda_0 + 63\lambda_1\lambda_3\wp_{12}\wp_{24}
\end{aligned}$$

The following relations are at weight -36.

$$\begin{aligned}
\wp_{112}^2 &= 4\lambda_3\lambda_0\wp_{12}\wp_{33} + 8\lambda_0\lambda_3\wp_{13}\wp_{23} + 4\lambda_3\lambda_2\wp_{23}\lambda_0 + \tfrac{16}{3}\lambda_0\lambda_2\wp_{13}\wp_{34} - \tfrac{4}{3}\lambda_2\lambda_0 Q_{2233} \\
&\quad + 4\lambda_3\lambda_1\wp_{34}\lambda_0 - 4\lambda_4\lambda_1\wp_{23}\lambda_0 - 16\lambda_4\wp_{34}\lambda_0^2 - \tfrac{8}{3}\lambda_2\wp_{34}\lambda_1^2 + \tfrac{4}{3}\lambda_4\lambda_1^2\wp_{13} - 2\lambda_3\lambda_1^2\wp_{23} \\
&\quad - \tfrac{8}{3}\lambda_0\lambda_4\wp_{13}^2 + \tfrac{20}{3}\lambda_1\wp_{13}\lambda_0 + \tfrac{16}{3}\lambda_2^2\wp_{34}\lambda_0 - 4\lambda_2\lambda_0\wp_{23}^2 + 8\wp_{44}\wp_{33}\lambda_0^2 + 4\wp_{22}\lambda_0 Q_{1244} \\
&\quad + 4\lambda_4\lambda_3\lambda_1\wp_{12}\wp_{33} + 8\lambda_0\lambda_4\wp_{14}\wp_{33}\lambda_3 - 4\wp_{22}\wp_{33}\lambda_4\lambda_3\lambda_0 - \tfrac{8}{3}\lambda_0\wp_{14}\lambda_4 Q_{2333} \\
&\quad + 8\wp_{11}\wp_{33}\lambda_4\lambda_0 - \tfrac{4}{3}\wp_{12}\lambda_4\lambda_1 Q_{2333} + \tfrac{4}{3}\lambda_4\wp_{22}\lambda_0 Q_{2333} - 8\wp_{34}^2\lambda_0^2 + 4\wp_{11}\wp_{12}^2 \\
&\quad - 12\wp_{23}\lambda_0^2 + \lambda_1^2\wp_{23}^2 + \tfrac{2}{3}\lambda_1^2 Q_{2233} - \tfrac{4}{3}\lambda_4\lambda_2\lambda_1^2 + \tfrac{8}{3}\lambda_4\lambda_2^2\lambda_0 + \tfrac{20}{3}\lambda_2\lambda_1\lambda_0 - \tfrac{4}{3}\lambda_1^3 \\
&\quad - 8\lambda_4^2\lambda_0^2 - 12\lambda_3\lambda_0^2 + \tfrac{8}{3}\wp_{11}\lambda_0 Q_{2444} - 4\wp_{12}\lambda_1 Q_{1244} - \tfrac{4}{3}\lambda_0\wp_{13}Q_{2233} - 8\wp_{14}\lambda_0 Q_{1244} \\
\wp_{111}\wp_{114} &= \lambda_3\lambda_0\wp_{12}\wp_{33} - \lambda_0\lambda_3\wp_{13}\wp_{23} - 3\lambda_3\lambda_2\wp_{23}\lambda_0 - \tfrac{4}{3}\lambda_0\lambda_2\wp_{13}\wp_{34} \\
&\quad - 3\lambda_3\lambda_1\wp_{34}\lambda_0 + \lambda_3\lambda_1\wp_{11}\wp_{33} + 2\lambda_0\wp_{24}\wp_{33}\lambda_1 - 2\lambda_0\wp_{23}\wp_{34}\lambda_1 + 4\lambda_2\wp_{14}\wp_{33}\lambda_0 \\
&\quad - \tfrac{14}{3}\lambda_2\lambda_0\wp_{13}\lambda_4 + 4\lambda_4\wp_{34}\lambda_0^2 + \tfrac{1}{3}\lambda_2\wp_{34}\lambda_1^2 + \tfrac{4}{3}\lambda_4\lambda_1^2\wp_{13} + \lambda_3\lambda_1^2\wp_{23} - \tfrac{4}{3}\lambda_0\lambda_4\wp_{13}^2 \\
&\quad + \tfrac{13}{3}\lambda_1\wp_{13}\lambda_0 - 2\lambda_1^2\wp_{14}\wp_{33} + \lambda_1^2\wp_{13}\wp_{34} - 4\wp_{44}\wp_{33}\lambda_0^2 + 2\lambda_0\lambda_4\wp_{14}\wp_{33}\lambda_3 \\
&\quad - \tfrac{2}{3}\lambda_0\wp_{14}\lambda_4 Q_{2333} + 4\wp_{34}^2\lambda_0^2 - 10\wp_{23}\lambda_0^2 - \tfrac{1}{3}\lambda_1^2 Q_{2233} + \tfrac{2}{3}\lambda_4\lambda_2\lambda_1^2 - 2\lambda_4\lambda_2^2\lambda_0 \\
&\quad + 5\lambda_2\lambda_1\lambda_0 - \tfrac{4}{3}\lambda_1^3 - 9\lambda_3\lambda_0^2 - 2\wp_{11}\lambda_0 Q_{2444} - \tfrac{1}{3}\lambda_1\wp_{11}Q_{2333} - \tfrac{1}{3}\lambda_0\wp_{12}Q_{2333} \\
&\quad + \tfrac{1}{3}\lambda_0\wp_{13}Q_{2233} - 2\wp_{14}\lambda_0 Q_{1244} + \lambda_2\lambda_0 Q_{2233} + 4\wp_{14}\wp_{11}^2 \\
\wp_{111}\wp_{122} &= 7\lambda_3\lambda_0\wp_{12}\wp_{33} - 13\lambda_0\lambda_3\wp_{13}\wp_{23} + \lambda_3\lambda_2\wp_{23}\lambda_0 - \tfrac{8}{3}\lambda_0\lambda_2\wp_{13}\wp_{34} \\
&\quad - 4\lambda_4\wp_{34}\lambda_0^2 - \tfrac{2}{3}\lambda_2\wp_{34}\lambda_1^2 + \tfrac{7}{3}\lambda_4\lambda_1^2\wp_{13} - \lambda_3\lambda_1^2\wp_{23} + \tfrac{4}{3}\lambda_0\lambda_4\wp_{13}^2 + \tfrac{23}{3}\lambda_1\wp_{13}\lambda_0 + \cdots
\end{aligned}$$

$$\cdots + \tfrac{4}{3}\lambda_2^2\wp_{34}\lambda_0 + 4\lambda_2\lambda_0\wp_{23}^2 - 2\lambda_1^2\wp_{14}\wp_{33} + 2\lambda_1^2\wp_{13}\wp_{34} - \lambda_3\lambda_1\wp_{13}^2 - 9\lambda_3^2\lambda_0\wp_{13}$$
$$- 2\lambda_3\lambda_1\wp_{34}\lambda_0 - 8\lambda_4\lambda_1\wp_{23}\lambda_0 + 3\lambda_3\lambda_1\wp_{11}\wp_{33} + 4\lambda_0\wp_{24}\wp_{33}\lambda_1 - 4\lambda_0\wp_{23}\wp_{34}\lambda_1$$
$$- 4\lambda_2\wp_{14}\wp_{33}\lambda_0 - 2\lambda_2\lambda_0\wp_{13}\lambda_4 - 2\lambda_2\lambda_1\wp_{12}\wp_{33} + 4\lambda_2\lambda_0\wp_{22}\wp_{33} + 2\lambda_2\lambda_1\wp_{13}\wp_{23}$$
$$- 8\wp_{22}\lambda_0 Q_{1244} - 2\lambda_4\lambda_3\lambda_1\wp_{12}\wp_{33} - 10\lambda_0\lambda_4\wp_{14}\wp_{33}\lambda_3 + 8\wp_{22}\wp_{33}\lambda_4\lambda_3\lambda_0$$
$$- 4\wp_{44}\wp_{33}\lambda_0^2 + 2\lambda_2\wp_{11}\wp_{33}\lambda_4\lambda_3 + \lambda_3\lambda_1\wp_{13}\lambda_2 + \tfrac{10}{3}\lambda_0\wp_{14}\lambda_4 Q_{2333} + \tfrac{2}{3}\wp_{12}\lambda_4\lambda_1 Q_{2333}$$
$$- \tfrac{8}{3}\lambda_4\wp_{22}\lambda_0 Q_{2333} - \tfrac{2}{3}\wp_{11}\lambda_4\lambda_2 Q_{2333} + 4\wp_{34}^2\lambda_0^2 + 2\wp_{11}\wp_{12}^2 - 6\wp_{23}\lambda_0^2 - 2\lambda_1^2\wp_{23}^2$$
$$+ 2\wp_{11}^2\wp_{22} - \tfrac{1}{3}\lambda_1^2 Q_{2233} - 3\lambda_4\lambda_3\lambda_1\lambda_0 - \tfrac{1}{3}\lambda_4\lambda_2\lambda_1^2 + \tfrac{2}{3}\lambda_4\lambda_2^2\lambda_0 + \tfrac{17}{3}\lambda_2\lambda_1\lambda_0 - \tfrac{4}{3}\lambda_1^3$$
$$- 8\lambda_4^2\lambda_0^2 - 3\lambda_3\lambda_0^2 - \tfrac{4}{3}\wp_{11}\lambda_0 Q_{2444} + 2\wp_{12}\lambda_1 Q_{1244} - \tfrac{1}{3}\lambda_1\wp_{11} Q_{2333} - \lambda_0\wp_{12} Q_{2333}$$
$$+ \tfrac{5}{3}\lambda_0\wp_{13} Q_{2233} + 10\wp_{14}\lambda_0 Q_{1244} + \tfrac{5}{3}\lambda_2\lambda_0 Q_{2233} - 2\lambda_2\wp_{11} Q_{1244}$$

The final three relations are below with the weights indicated by the number in brackets.

$$\mathbf{(-37)} \quad \wp_{111}\wp_{113} = 2\lambda_3\lambda_4\wp_{34}\lambda_0 Q_{2333} - 4\lambda_0\wp_{13} Q_{1244} + 4\lambda_0\wp_{33}\lambda_4\lambda_3\lambda_2 + 2\lambda_0\wp_{12}^2$$
$$- 4\lambda_0\lambda_3\wp_{33}\wp_{34}\lambda_2 - 6\lambda_4\lambda_0\wp_{34}\wp_{33}\lambda_3^2 - 6\lambda_0\lambda_3\wp_{12}\wp_{24} - 6\lambda_0\lambda_3\wp_{14}^2 + 4\wp_{13}\wp_{11}^2$$
$$+ 2\lambda_1^2 Q_{1244} + 2\lambda_0^2 Q_{2444} - 2\lambda_0\wp_{13}\lambda_3 Q_{2444} - \lambda_3\wp_{23}\lambda_0 Q_{2333} - \tfrac{4}{3}\lambda_0\wp_{13}\lambda_4 Q_{2333}$$
$$+ \wp_{33}\lambda_3\lambda_0 Q_{2233} - 2\lambda_4\lambda_2\lambda_0 Q_{2333} + 6\lambda_0\wp_{34}\lambda_3 Q_{1244} - 3\lambda_3\lambda_0 Q_{1224} - \tfrac{3}{2}\lambda_3^2\lambda_0 Q_{2333}$$
$$+ \tfrac{1}{3}\lambda_1\lambda_0 Q_{2333} - 6\lambda_2\lambda_0 Q_{1244} + \tfrac{2}{3}\lambda_4\lambda_1^2 Q_{2333} + 4\wp_{33}\lambda_2^2\lambda_0 + 8\lambda_4\lambda_0^2\wp_{33} + 4\lambda_0\wp_{11}\wp_{14}$$
$$- 6\lambda_0\wp_{11}\wp_{22} + 2\lambda_1\wp_{11}\wp_{12} + 4\wp_{33}\lambda_0\wp_{13}\lambda_2 - \lambda_2\wp_{33}\lambda_1^2 + 6\lambda_0\wp_{33}\lambda_3\lambda_1$$
$$- 2\wp_{33}\lambda_4\lambda_3\lambda_1^2 + 6\lambda_0\lambda_3\wp_{11}\wp_{44} + 6\lambda_0\lambda_3\wp_{14}\wp_{22} + \tfrac{9}{2}\lambda_0\wp_{33}\lambda_3^3 - \wp_{33}\lambda_1^2\wp_{13}$$

$$\mathbf{(-39)} \quad \wp_{111}\wp_{112} = 4\wp_{11}^2\wp_{12} - 2\lambda_1^3\wp_{34} + 8\lambda_0^2\wp_{13} + \lambda_1^2\wp_{13}\wp_{23} - 2\lambda_1^2\wp_{12}\wp_{33}$$
$$- 2\lambda_3\lambda_0\wp_{13}^2 + 4\lambda_2^2\lambda_0\wp_{23} - 2\lambda_1\lambda_0\wp_{23}^2 - 16\lambda_4\lambda_0^2\wp_{23} + 2\lambda_3\lambda_1^2\wp_{13} - 4\lambda_0^2\wp_{23}\wp_{34}$$
$$+ 4\lambda_0^2\wp_{24}\wp_{33} - \lambda_2\lambda_1^2\wp_{23} - 6\lambda_3\lambda_0^2\wp_{34} + \tfrac{2}{3}\lambda_1\lambda_0 Q_{2233} - 4\lambda_0\wp_{12} Q_{1244} - 2\lambda_1\wp_{11} Q_{1244}$$
$$- \tfrac{2}{3}\wp_{11}\lambda_0 Q_{2333} + 4\lambda_4\lambda_3\lambda_0\wp_{12}\wp_{33} + 2\lambda_4\lambda_3\lambda_1\wp_{11}\wp_{33} - 18\lambda_4\lambda_3\lambda_0^2 - 2\lambda_4\lambda_1^3 - \tfrac{4}{3}\lambda_1^2\lambda_0$$
$$+ 8\lambda_2\lambda_0^2 + \tfrac{20}{3}\lambda_4\lambda_2\lambda_1\lambda_0 + \tfrac{16}{3}\lambda_2\lambda_1\lambda_0\wp_{34} + 2\lambda_1\lambda_0\wp_{22}\wp_{33} - 5\lambda_3\lambda_1\lambda_0\wp_{23}$$
$$+ 8\lambda_3\lambda_0\wp_{11}\wp_{33} - 6\lambda_3\lambda_2\lambda_0\wp_{13} + 4\lambda_1\lambda_0\wp_{13}\wp_{34} + \tfrac{16}{3}\lambda_4\lambda_1\lambda_0\wp_{13} - 4\lambda_1\lambda_0\wp_{14}\wp_{33}$$
$$- \tfrac{2}{3}\wp_{11}\lambda_4\lambda_1 Q_{2333} + 4\lambda_2\lambda_0\wp_{12}\wp_{33} - \tfrac{4}{3}\lambda_4\lambda_0\wp_{12} Q_{2333}$$

$$\mathbf{(-42)} \quad \wp_{111}^2 = 8\lambda_0^2\wp_{13}\wp_{34} - 42\lambda_3\lambda_0^2\wp_{23} + 8\lambda_4\lambda_2\lambda_0^2 + 18\lambda_3\lambda_2\lambda_1\lambda_0$$
$$+ 2\lambda_2\lambda_1^2\wp_{13} - 8\lambda_2^2\lambda_0\wp_{13} + 8\lambda_4\lambda_0^2\wp_{13} - 8\lambda_0^2\wp_{14}\wp_{33} - 4\lambda_2\lambda_0\wp_{13}^2 + 8\lambda_0^2\wp_{22}\wp_{33}$$
$$- 4\lambda_3\lambda_1^3 - 4\lambda_2^3\lambda_0 - 27\lambda_3^2\lambda_0^2 + 4\lambda_1\lambda_0^2 + 6\lambda_3\lambda_1\lambda_0\wp_{13} + 12\lambda_4\lambda_3\lambda_0\wp_{11}\wp_{33}$$
$$- 4\lambda_1\lambda_0\wp_{12}\wp_{33} + 4\lambda_1\lambda_0\wp_{13}\wp_{23} + 8\lambda_2\lambda_0^2\wp_{34} - 4\lambda_1^2\wp_{11}\wp_{33} - 4\lambda_1^2\lambda_0\wp_{34}$$
$$- 8\lambda_0^2\wp_{23}^2 - 4\lambda_1^3\wp_{23} + \lambda_1^2\wp_{13}^2 + 16\lambda_2\lambda_0\wp_{11}\wp_{33} + 16\lambda_2\lambda_1\lambda_0\wp_{23} + 4\wp_{11}^3$$
$$- 4\lambda_0\wp_{11}\lambda_4 Q_{2333} + 2\lambda_0^2 Q_{2233} - 12\wp_{11}\lambda_0 Q_{1244} + \lambda_2^2\lambda_1^2 - 4\lambda_4\lambda_1^2\lambda_0$$

Appendix E

Strata relations for the cyclic tetragonal curve of genus six

E.1 Relations for $\boldsymbol{u} \in \Theta^{[1]}$

The equations is this Appendix are all valid for the σ-derivatives associated with the cyclic (4,5)-curve, when $\boldsymbol{u} \in \Theta^{[1]}$.

We start by noting that for $\boldsymbol{u} \in \Theta^{[1]}$ we have both the σ-function and all its first derivatives equal to zero.

$$\begin{aligned}
&\sigma(\boldsymbol{u}) = 0 && && \\
&\sigma_1(\boldsymbol{u}) = 0 && \sigma_2(\boldsymbol{u}) = 0 && \sigma_3(\boldsymbol{u}) = 0 \\
&\sigma_4(\boldsymbol{u}) = 0 && \sigma_5(\boldsymbol{u}) = 0 && \sigma_6(\boldsymbol{u}) = 0
\end{aligned}$$

We now present all those relations that express the second derivatives of $\sigma(\boldsymbol{u})$. Note that they may all be expressed as a linear combination of $\{\sigma_{22}, \sigma_{23}, \sigma_{34}\}$.

$$\begin{aligned}
&\sigma_{11} = 0, && \sigma_{24} = 0 && \sigma_{44} = 0 \\
&\sigma_{12} = 0, && \sigma_{25} = -\sigma_{34} && \sigma_{45} = 0 \\
&\sigma_{13} = 0, && \sigma_{26} = 0 && \sigma_{46} = 0 \\
&\sigma_{14} = -\tfrac{1}{2}\sigma_{22}, && \sigma_{33} = 0 && \sigma_{55} = 0 \\
&\sigma_{15} = -\sigma_{23}, && \sigma_{35} = 0 && \sigma_{56} = 0 \\
&\sigma_{16} = 0, && \sigma_{36} = 0 && \sigma_{66} = 0
\end{aligned}$$

We now give all the relations that express the 3-index σ-derivatives.

$$\sigma_{112} = \lambda_0\sigma_{34} + \lambda_1\sigma_{23}$$
$$\sigma_{113} = 0$$
$$\sigma_{114} = -\sigma_{122} + \lambda_1\sigma_{34} + \lambda_2\sigma_{23}$$
$$\sigma_{115} = -2\sigma_{123}$$
$$\sigma_{116} = 0$$
$$\sigma_{124} = -\tfrac{1}{6}\sigma_{222} + \tfrac{1}{2}\lambda_2\sigma_{34} + \tfrac{1}{2}\lambda_3\sigma_{23}$$
$$\sigma_{125} = -\tfrac{1}{2}\sigma_{223} - \sigma_{134}$$
$$\sigma_{126} = 0$$
$$\sigma_{133} = 0$$
$$\sigma_{135} = -\tfrac{1}{2}\sigma_{233}$$
$$\sigma_{136} = 0$$
$$\sigma_{144} = -\sigma_{224} + \lambda_4\sigma_{23} + \lambda_3\sigma_{34}$$
$$\sigma_{145} = -\sigma_{234} - \tfrac{1}{2}\sigma_{225}$$
$$\sigma_{146} = -\tfrac{1}{2}\sigma_{226}$$
$$\sigma_{155} = -\sigma_{334} - 2\sigma_{235}$$
$$\sigma_{156} = -\sigma_{236}$$
$$\sigma_{166} = \sigma_{23}$$
$$\sigma_{244} = \sigma_{23} + \lambda_4\sigma_{34}$$
$$\sigma_{245} = -\tfrac{1}{2}\sigma_{344}$$
$$\sigma_{246} = 0$$
$$\sigma_{255} = -2\sigma_{345}$$
$$\sigma_{256} = -\sigma_{346}$$
$$\sigma_{266} = \sigma_{34}$$
$$\sigma_{333} = 0$$
$$\sigma_{335} = 0$$
$$\sigma_{336} = -2\sigma_{23}$$
$$\sigma_{355} = 0$$
$$\sigma_{356} = -\sigma_{34}$$
$$\sigma_{366} = 0$$
$$\sigma_{444} = 3\sigma_{34}$$
$$\sigma_{445} = 0$$
$$\sigma_{446} = 0$$
$$\sigma_{455} = 0$$
$$\sigma_{456} = 0$$
$$\sigma_{466} = 0$$
$$\sigma_{555} = 0$$
$$\sigma_{556} = 0$$
$$\sigma_{566} = 0$$
$$\sigma_{666} = 0$$

We finish by giving all the relations that express the 4-index σ-derivatives. Note that the relations for higher index σ-derivatives evaluated for $\boldsymbol{u} \in \Theta^{[1]}$ are available in the extra Appendix of files and online at [37], (along with the sets of relations valid on the higher strata).

$$\sigma_{1136} = -\lambda_0\sigma_{34}$$
$$\begin{aligned}\sigma_{1144} = {} & \lambda_3\sigma_{223} - 2\sigma_{1224} - \tfrac{1}{6}\sigma_{2222} + 2\lambda_4\sigma_{123} + 2\lambda_2\sigma_{234} \\ & + 2\lambda_3\sigma_{134} + \lambda_1\sigma_{344}\end{aligned}$$
$$\sigma_{1145} = \lambda_2\sigma_{235} - 2\sigma_{1234} - \sigma_{1225} - \tfrac{1}{3}\sigma_{2223} + \tfrac{1}{2}\lambda_3\sigma_{233} + \lambda_2\sigma_{334} + \lambda_1\sigma_{345}$$
$$\sigma_{1146} = -\sigma_{1226} + \lambda_2\sigma_{22} + \lambda_2\sigma_{236} + \lambda_1\sigma_{346}$$
$$\sigma_{1155} = -2\sigma_{1334} - 4\sigma_{1235} - \sigma_{2233} + 2\lambda_2\sigma_{22}$$
$$\sigma_{1156} = -\lambda_1\sigma_{34} - 2\sigma_{1236}$$
$$\sigma_{1166} = 2\sigma_{123}$$
$$\sigma_{1244} = -\tfrac{1}{3}\sigma_{2224} + \sigma_{123} + \tfrac{1}{2}\lambda_4\sigma_{223} + \lambda_4\sigma_{134} + \tfrac{1}{2}\lambda_2\sigma_{344} + \lambda_3\sigma_{234}$$
$$\begin{aligned}\sigma_{1245} = {} & \tfrac{1}{4}\lambda_4\sigma_{233} - \tfrac{1}{6}\sigma_{2225} - \tfrac{1}{2}\sigma_{2234} - \tfrac{1}{2}\sigma_{1344} + \tfrac{1}{2}\lambda_3\sigma_{334} \\ & + \tfrac{1}{2}\lambda_2\sigma_{345} + \tfrac{1}{2}\lambda_3\sigma_{235}\end{aligned}$$
$$\sigma_{1246} = \tfrac{1}{2}\lambda_3\sigma_{22} - \tfrac{1}{6}\sigma_{2226} + \tfrac{1}{2}\lambda_3\sigma_{236} + \tfrac{1}{2}\lambda_2\sigma_{346}$$
$$\sigma_{1255} = \lambda_3\sigma_{22} - \sigma_{2235} - 2\sigma_{1345} - \sigma_{2334}$$
$$\sigma_{1256} = -\tfrac{1}{2}\sigma_{2236} - \tfrac{1}{2}\lambda_2\sigma_{34} - \sigma_{1346}$$

$$
\begin{aligned}
\sigma_{1266} &= \tfrac{1}{2}\sigma_{223} + \sigma_{134} \\
\sigma_{1333} &= -6\lambda_0\sigma_{34} + 2\lambda_1\sigma_{23} \\
\sigma_{1335} &= \tfrac{4}{3}\lambda_2\sigma_{23} - \tfrac{4}{3}\lambda_1\sigma_{34} - \tfrac{1}{3}\sigma_{2333} \\
\sigma_{1336} &= -2\sigma_{123} \\
\sigma_{1355} &= -\sigma_{2335} - \tfrac{1}{3}\sigma_{3334} - \tfrac{2}{3}\lambda_2\sigma_{34} + 2\lambda_3\sigma_{23} \\
\sigma_{1356} &= -\tfrac{1}{2}\sigma_{223} - \sigma_{134} - \tfrac{1}{2}\sigma_{2336} \\
\sigma_{1366} &= \tfrac{1}{2}\sigma_{233} \\
\sigma_{1444} &= \tfrac{3}{2}\sigma_{223} + 3\sigma_{134} - \tfrac{3}{2}\sigma_{2244} + 3\lambda_4\sigma_{234} + \tfrac{3}{2}\lambda_3\sigma_{344} \\
\sigma_{1445} &= \lambda_3\sigma_{345} + \tfrac{1}{2}\sigma_{233} - \sigma_{2245} - \sigma_{2344} + \lambda_4\sigma_{334} + \lambda_4\sigma_{235} \\
\sigma_{1446} &= \lambda_4\sigma_{22} - \sigma_{2246} + \lambda_3\sigma_{346} + \lambda_4\sigma_{236} \\
\sigma_{1455} &= -\tfrac{1}{2}\sigma_{3344} - 2\sigma_{2345} - \tfrac{1}{2}\sigma_{2255} + \lambda_4\sigma_{22} \\
\sigma_{1456} &= -\tfrac{1}{2}\sigma_{2256} - \tfrac{1}{2}\lambda_3\sigma_{34} - \sigma_{2346} \\
\sigma_{1466} &= \sigma_{234} - \tfrac{1}{2}\sigma_{2266} \\
\sigma_{1555} &= -3\sigma_{2355} + 8\lambda_4\sigma_{23} - 3\sigma_{3345} \\
\sigma_{1556} &= -2\sigma_{234} - \sigma_{3346} - 2\sigma_{2356} \\
\sigma_{1566} &= \sigma_{334} - \sigma_{2366} + \sigma_{235}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{1666} &= \sigma_{22} + 3\sigma_{236} & \sigma_{4446} &= 3\sigma_{346} \\
\sigma_{2444} &= 3\sigma_{234} + \tfrac{3}{2}\lambda_4\sigma_{344} & \sigma_{4455} &= 0 \\
\sigma_{2445} &= \sigma_{334} + \sigma_{235} - \tfrac{1}{3}\sigma_{3444} + \lambda_4\sigma_{345} & \sigma_{4456} &= -\sigma_{34} \\
\sigma_{2446} &= \sigma_{22} + \sigma_{236} + \lambda_4\sigma_{346} & \sigma_{4466} &= 0 \\
\sigma_{2455} &= \sigma_{22} - \sigma_{3445} & \sigma_{4556} &= 0 \\
\sigma_{2456} &= -\tfrac{1}{2}\lambda_4\sigma_{34} - \tfrac{1}{2}\sigma_{3446} & \sigma_{4566} &= 0 \\
\sigma_{2466} &= \tfrac{1}{2}\sigma_{344} & \sigma_{4666} &= 0 \\
\sigma_{2555} &= 10\sigma_{23} + 2\lambda_4\sigma_{34} - 3\sigma_{3455} & \sigma_{5555} &= 0 \\
\sigma_{2556} &= -\sigma_{344} - 2\sigma_{3456} & \sigma_{5556} &= 0 \\
\sigma_{2666} &= 3\sigma_{346} & \sigma_{3556} &= -2\sigma_{345} \\
\sigma_{3333} &= 0 & \sigma_{3566} &= -2\sigma_{346} \\
\sigma_{3335} &= 0 & \sigma_{3666} &= \sigma_{34} \\
\sigma_{3336} &= -3\sigma_{233} & \sigma_{4444} &= 6\sigma_{344} \\
\sigma_{3355} &= 0 & \sigma_{4445} &= 3\sigma_{345} \\
\sigma_{3356} &= -2\sigma_{334} - 2\sigma_{235} & \sigma_{4555} &= 4\sigma_{34} \\
\sigma_{3366} &= -2\sigma_{22} - 4\sigma_{236} & \sigma_{5566} &= 0 \\
\sigma_{3466} &= \sigma_{345} - \sigma_{2566} & \sigma_{5666} &= 0 \\
\sigma_{3555} &= 0 & \sigma_{6666} &= 0
\end{aligned}
$$

E.2 Relations for $\boldsymbol{u} = \boldsymbol{u}_{0,N}$

The following list of equations are valid only at the four points in $\Theta^{[1]}$ where $\sigma_{23}(\boldsymbol{u}) = 0$. These points were labeled $\boldsymbol{u}_{0,N}$ for $N = 1, 2, 3, 4$ in Section 6.5. To move between the four points substitute the appropriate value of N into the equations. (Note that all the equations in Appendix E.1 are still valid at these points and are not repeated here.)

This Appendix contains all those relations we have obtained that express n-index σ-functions for $n \leq 4$. A larger set that includes relations for $n > 4$ is available online at [37] or in the extra Appendix of files.

$$\sigma_{34} = \frac{1}{2}\frac{\sigma_{22}}{\mathrm{i}^N \lambda_0^{1/4}} \qquad \sigma_{334} = +\frac{\sigma_{223}}{\mathrm{i}^N \lambda_0^{1/4}}$$

$$\sigma_{111} = 0 \qquad \sigma_{344} = -\frac{1}{2}\frac{\sigma_{22}\lambda_3}{\mathrm{i}^{2N}\sqrt{\lambda_0}} + \frac{\sigma_{224}}{\mathrm{i}^N \lambda_0^{1/4}}$$

$$\sigma_{112} = \frac{1}{2}\sigma_{22}\mathrm{i}^{3N}{\lambda_0}^{3/4} \qquad \sigma_{345} = \frac{\mathrm{i}^{3N}\sigma_{225}}{2\lambda_0^{1/4}} + \frac{\mathrm{i}^{2N}\sigma_{222}}{6\sqrt{\lambda_0}} - \frac{\mathrm{i}^N\lambda_2\sigma_{22}}{4{\lambda_0}^{3/4}}$$

$$\sigma_{113} = 0 \qquad \sigma_{346} = \frac{1}{2}\frac{\sigma_{226}}{\mathrm{i}^N \lambda_0^{1/4}}$$

$$\sigma_{123} = -\frac{1}{2}\mathrm{i}^{2N}\sqrt{\lambda_0}\sigma_{22}$$

$$\sigma_{134} = \frac{\mathrm{i}^{3N}\sigma_{122}}{2\lambda_0^{1/4}} - \frac{\mathrm{i}^{2N}\sigma_{22}\lambda_1}{2\sqrt{\lambda_0}} - \frac{\sigma_{223}}{2}$$

$$\sigma_{233} = -\mathrm{i}^N\lambda_0^{1/4}\sigma_{22}$$

$$\sigma_{234} = \frac{1}{6}\frac{\sigma_{222}}{\mathrm{i}^N\lambda_0^{1/4}} - \frac{1}{4}\frac{\lambda_2\sigma_{22}}{\mathrm{i}^{2N}\sqrt{\lambda_0}}$$

$$\sigma_{235} = -\frac{1}{2}\frac{\sigma_{223}}{\mathrm{i}^N\lambda_0^{1/4}}$$

$$\sigma_{236} = -\frac{1}{2}\sigma_{22}$$

Note that these can all be expressed as a linear combination of

$$\{\sigma_{22}, \sigma_{122}, \sigma_{222}, \sigma_{223}, \sigma_{224}, \sigma_{225}, \sigma_{226}\}.$$

$$\sigma_{1111} = -6{\lambda_0}^{\frac{3}{2}}\sigma_{22}\mathrm{i}^{2N}$$

$$\sigma_{1112} = -3\lambda_1\mathrm{i}^{2N}\sqrt{\lambda_0}\sigma_{22} + \tfrac{3}{2}{\lambda_0}^{\frac{3}{4}}\sigma_{122}\mathrm{i}^{3N}$$

$$\sigma_{1113} = -\tfrac{3}{2}{\lambda_0}^{5/4}\sigma_{22}\mathrm{i}^N$$

$$\sigma_{1114} = -\frac{9}{4}\mathrm{i}^{2N}\sqrt{\lambda_0}\lambda_2\sigma_{22} - \frac{3}{2}\frac{\mathrm{i}^{2N}{\lambda_1}^2\sigma_{22}}{\sqrt{\lambda_0}} - \frac{3}{2}\sigma_{1122} + \frac{3}{2}\frac{\lambda_1\sigma_{122}}{\mathrm{i}^N\lambda_0^{1/4}} + \frac{1}{2}{\lambda_0}^{\frac{3}{4}}\sigma_{222}\mathrm{i}^{3N}$$

$$\sigma_{1115} = -\tfrac{3}{2}\mathrm{i}^{N}\sigma_{22}\lambda_1\lambda_0^{\frac{1}{4}} + 3\sqrt{\lambda_0}\sigma_{122}\mathrm{i}^{2N}$$

$$\sigma_{1116} = +\tfrac{3}{2}\sigma_{22}\lambda_0$$

$$\sigma_{1123} = +\tfrac{1}{2}\mathrm{i}^{3N}\lambda_0{}^{\frac{3}{4}}\sigma_{223} - \sqrt{\lambda_0}\mathrm{i}^{2N}\sigma_{122}$$

$$\sigma_{1124} = -\frac{3}{4}\frac{\mathrm{i}^{2N}\lambda_2\sigma_{22}\lambda_1}{\sqrt{\lambda_0}} - \frac{3}{4}\mathrm{i}^{2N}\sqrt{\lambda_0}\sigma_{22}\lambda_3 - \frac{1}{3}\sigma_{1222} + \frac{1}{2}\lambda_0{}^{\frac{3}{4}}\sigma_{224}\mathrm{i}^{3N} + \frac{1}{6}\frac{\lambda_1\sigma_{222}}{\mathrm{i}^{N}\lambda_0^{1/4}} + \frac{1}{2}\frac{\lambda_2\sigma_{122}}{\mathrm{i}^{N}\lambda_0^{1/4}}$$

$$\sigma_{1125} = -\frac{1}{2}\lambda_2\sigma_{22}\mathrm{i}^{N}\lambda_0^{\frac{1}{4}} - \frac{\mathrm{i}^{N}{\lambda_1}^{2}\sigma_{22}}{\lambda_0{}^{\frac{3}{4}}} - \frac{1}{2}\frac{\sigma_{1122}}{\mathrm{i}^{N}\lambda_0^{1/4}} + \frac{1}{2}\lambda_0{}^{\frac{3}{4}}\sigma_{225}\mathrm{i}^{3N} + \frac{\lambda_1\sigma_{122}}{\mathrm{i}^{2N}\sqrt{\lambda_0}} + \frac{2}{3}\frac{\sqrt{\lambda_0}\sigma_{222}}{\mathrm{i}^{2N}}$$

$$\sigma_{1126} = +\tfrac{1}{2}\lambda_1\sigma_{22} + \tfrac{1}{2}\sigma_{226}\mathrm{i}^{3N}\lambda_0{}^{\frac{3}{4}}$$

$$\sigma_{1133} = +2\sigma_{22}\lambda_0$$

$$\sigma_{1134} = +\frac{\mathrm{i}^{N}{\lambda_1}^{2}\sigma_{22}}{\lambda_0{}^{\frac{3}{4}}} - \frac{1}{4}\lambda_2\sigma_{22}\mathrm{i}^{N}\lambda_0^{\frac{1}{4}} - \sigma_{1223} + \frac{1}{2}\frac{\sigma_{223}\lambda_1}{\mathrm{i}^{N}\lambda_0^{1/4}} + \frac{1}{2}\frac{\sigma_{1122}}{\mathrm{i}^{N}\lambda_0^{1/4}} - \frac{1}{2}\frac{\sqrt{\lambda_0}\sigma_{222}}{\mathrm{i}^{2N}} - \frac{\lambda_1\sigma_{122}}{\mathrm{i}^{2N}\sqrt{\lambda_0}}$$

$$\sigma_{1135} = +\mathrm{i}^{2N}\sqrt{\lambda_0}\sigma_{223} + \mathrm{i}^{N}\lambda_0^{\frac{1}{4}}\sigma_{122}$$

$$\sigma_{1233} = +\sigma_{22}\lambda_1 - \frac{\sqrt{\lambda_0}\sigma_{223}}{\mathrm{i}^{2N}} - \frac{\sigma_{122}\lambda_0^{\frac{1}{4}}}{\mathrm{i}^{3N}}$$

$$\sigma_{1234} = -\frac{1}{2}\mathrm{i}^{N}\lambda_0^{\frac{1}{4}}\sigma_{22}\lambda_3 + \frac{1}{2}\frac{\lambda_2\sigma_{22}\mathrm{i}^{N}\lambda_1}{\lambda_0{}^{\frac{3}{4}}} - \frac{1}{6}\sigma_{2223} + \frac{1}{4}\frac{\sigma_{223}\lambda_2}{\mathrm{i}^{N}\lambda_0^{1/4}} + \frac{1}{6}\frac{\sigma_{1222}}{\mathrm{i}^{N}\lambda_0^{1/4}} - \frac{1}{2}\frac{\sqrt{\lambda_0}\sigma_{224}}{\mathrm{i}^{2N}} - \frac{1}{6}\frac{\lambda_1\sigma_{222}}{\mathrm{i}^{2N}\sqrt{\lambda_0}} - \frac{1}{4}\frac{\lambda_2\sigma_{122}}{\mathrm{i}^{2N}\sqrt{\lambda_0}}$$

$$\sigma_{1235} = \frac{1}{2}\frac{\mathrm{i}^{2N}\sigma_{223}\lambda_1}{\sqrt{\lambda_0}} - \frac{1}{2}\sqrt{\lambda_0}\mathrm{i}^{2N}\sigma_{225} - \frac{1}{6}\lambda_0^{\frac{1}{4}}\mathrm{i}^{N}\sigma_{222} + \frac{1}{2}\lambda_2\sigma_{22} - \frac{1}{2}\frac{\sigma_{1223}}{\mathrm{i}^{N}\lambda_0^{1/4}}$$

$$\sigma_{1236} = -\frac{1}{2}\mathrm{i}^{2N}\sqrt{\lambda_0}\sigma_{226} - \frac{1}{2}\sigma_{122}$$

$$\sigma_{1334} = -\frac{\mathrm{i}^{2N}\sigma_{223}\lambda_1}{\sqrt{\lambda_0}} + \frac{1}{2}\lambda_2\sigma_{22} - \frac{1}{2}\sigma_{2233} + \frac{\sigma_{1223}}{\mathrm{i}^{N}\lambda_0^{1/4}}$$

$$\sigma_{1344} = \frac{\mathrm{i}^{N}\lambda_3\sigma_{22}\lambda_1}{\lambda_0{}^{\frac{3}{4}}} - \frac{3}{2}\mathrm{i}^{N}\lambda_0^{\frac{1}{4}}\sigma_{22}\lambda_4 + \frac{1}{2}\frac{\mathrm{i}^{N}{\lambda_2}^{2}\sigma_{22}}{\lambda_0{}^{\frac{3}{4}}} - \sigma_{2234} + \frac{1}{6}\frac{\sigma_{2222}}{\mathrm{i}^{N}\lambda_0^{1/4}} + \frac{1}{2}\frac{\sigma_{223}\lambda_3}{\mathrm{i}^{N}\lambda_0^{1/4}} + \frac{\sigma_{1224}}{\mathrm{i}^{N}\lambda_0^{1/4}} - \frac{1}{2}\frac{\lambda_3\sigma_{122}}{\mathrm{i}^{2N}\sqrt{\lambda_0}} - \frac{\lambda_1\sigma_{224}}{\mathrm{i}^{2N}\sqrt{\lambda_0}} - \frac{1}{3}\frac{\lambda_2\sigma_{222}}{\mathrm{i}^{2N}\sqrt{\lambda_0}}$$

$$\sigma_{1345} = -\frac{1}{2}\sigma_{2235} - \frac{1}{4}\sigma_{22}\lambda_3 + \frac{3}{4}\frac{\lambda_1\lambda_2\sigma_{22}}{\lambda_0} + \frac{1}{2}\frac{\sigma_{1225}}{\mathrm{i}^{N}\lambda_0^{1/4}} - \frac{1}{6}\frac{\sigma_{2223}}{\mathrm{i}^{N}\lambda_0^{1/4}} + \frac{1}{2}\frac{\lambda_0^{\frac{1}{4}}\sigma_{224}}{\mathrm{i}^{3N}} - \frac{1}{4}\frac{\lambda_2\sigma_{122}}{\mathrm{i}^{3N}\lambda_0{}^{3/4}} - \frac{1}{2}\frac{\lambda_1\sigma_{225}}{\mathrm{i}^{2N}\sqrt{\lambda_0}} + \frac{1}{4}\frac{\sigma_{223}\lambda_2}{\mathrm{i}^{2N}\sqrt{\lambda_0}} + \frac{1}{6}\frac{\sigma_{1222}}{\mathrm{i}^{2N}\sqrt{\lambda_0}} - \frac{1}{3}\frac{\lambda_1\sigma_{222}}{\mathrm{i}^{3N}\lambda_0{}^{3/4}}$$

$$\sigma_{1346} = -\frac{1}{6}\sigma_{222} - \frac{1}{2}\sigma_{2236} - \frac{1}{4}\frac{\lambda_2\sigma_{22}}{\mathrm{i}^N\lambda_0^{1/4}} + \frac{1}{2}\frac{\sigma_{1226}}{\mathrm{i}^N\lambda_0^{1/4}} - \frac{1}{2}\frac{\lambda_1\sigma_{226}}{\mathrm{i}^{2N}\sqrt{\lambda_0}}$$

$$\sigma_{2333} = -2\frac{\lambda_1\sigma_{22}\mathrm{i}^{3N}}{\lambda_0^{1/4}} - 3\mathrm{i}^N\lambda_0^{1/4}\sigma_{223}$$

$$\sigma_{2334} = \frac{1}{2}\sigma_{22}\lambda_3 + \frac{1}{3}\frac{\sigma_{2223}}{\mathrm{i}^N\lambda_0^{1/4}} - \frac{1}{2}\frac{\sigma_{223}\lambda_2}{\mathrm{i}^{2N}\sqrt{\lambda_0}} - \frac{\lambda_0^{1/4}\sigma_{224}}{\mathrm{i}^{3N}}$$

$$\sigma_{2335} = -\mathrm{i}^N\lambda_0^{1/4}\sigma_{225} - \frac{2}{3}\sigma_{222} - \frac{1}{2}\frac{\sigma_{2233}}{\mathrm{i}^N\lambda_0^{1/4}} + \frac{\lambda_2\sigma_{22}}{\mathrm{i}^N\lambda_0^{1/4}}$$

$$\sigma_{2336} = -\mathrm{i}^N\lambda_0^{1/4}\sigma_{226} - \sigma_{223}$$

$$\sigma_{2344} = \frac{1}{2}\frac{\lambda_2\sigma_{22}\mathrm{i}^N\lambda_3}{\lambda_0{}^{3/4}} - 2\mathrm{i}^N\lambda_0^{1/4}\sigma_{22} + \frac{1}{2}\frac{\sigma_{223}\lambda_4}{\mathrm{i}^N\lambda_0^{1/4}} + \frac{1}{3}\frac{\sigma_{2224}}{\mathrm{i}^N\lambda_0^{1/4}} - \frac{1}{6}\frac{\sigma_{222}\lambda_3}{\mathrm{i}^{2N}\sqrt{\lambda_0}} - \frac{1}{2}\frac{\sigma_{224}\lambda_2}{\mathrm{i}^{2N}\sqrt{\lambda_0}}$$

$$\sigma_{2345} = -\frac{1}{2}\sigma_{22}\lambda_4 + \frac{3}{8}\frac{\lambda_2{}^2\sigma_{22}}{\lambda_0} + \frac{1}{6}\frac{\sigma_{2225}}{\mathrm{i}^N\lambda_0^{1/4}} - \frac{1}{2}\frac{\sigma_{2234}}{\mathrm{i}^N\lambda_0^{1/4}} + \frac{1}{12}\frac{\sigma_{2222}}{\mathrm{i}^{2N}\sqrt{\lambda_0}} - \frac{1}{4}\frac{\sigma_{225}\lambda_2}{\mathrm{i}^{2N}\sqrt{\lambda_0}} + \frac{1}{4}\frac{\sigma_{223}\lambda_3}{\mathrm{i}^{2N}\sqrt{\lambda_0}} - \frac{1}{4}\frac{\lambda_2\sigma_{222}}{\mathrm{i}^{3N}\lambda_0{}^{3/4}}$$

$$\sigma_{2346} = -\frac{1}{2}\sigma_{224} + \frac{1}{6}\frac{\sigma_{2226}}{\mathrm{i}^N\lambda_0^{1/4}} - \frac{1}{4}\frac{\sigma_{226}\lambda_2}{\mathrm{i}^{2N}\sqrt{\lambda_0}}$$

$$\sigma_{2355} = -\frac{\sigma_{2235}}{\mathrm{i}^N\lambda_0^{1/4}} + \frac{\sigma_{22}\lambda_3}{\mathrm{i}^N\lambda_0^{1/4}} - \frac{1}{3}\frac{\sigma_{2223}}{\mathrm{i}^{2N}\sqrt{\lambda_0}} + \frac{1}{2}\frac{\sigma_{223}\lambda_2}{\mathrm{i}^{3N}\lambda_0{}^{3/4}}$$

$$\sigma_{2356} = -\frac{1}{2}\sigma_{225} - \frac{1}{2}\frac{\sigma_{222}}{\mathrm{i}^N\lambda_0^{1/4}} - \frac{1}{2}\frac{\sigma_{2236}}{\mathrm{i}^N\lambda_0^{1/4}} + \frac{1}{2}\frac{\lambda_2\sigma_{22}}{\mathrm{i}^{2N}\sqrt{\lambda_0}}$$

$$\sigma_{2366} = -\sigma_{226} + \frac{1}{2}\frac{\sigma_{223}}{\mathrm{i}^N\lambda_0^{1/4}}$$

$$\sigma_{2566} = -\frac{1}{2}\frac{\lambda_2\sigma_{22}\mathrm{i}^N}{\lambda_0{}^{3/4}} + \frac{1}{2}\frac{\sigma_{225}}{\mathrm{i}^N\lambda_0^{1/4}} - \frac{1}{2}\frac{\sigma_{2266}}{\mathrm{i}^N\lambda_0^{1/4}} + \frac{1}{3}\frac{\sigma_{222}}{\mathrm{i}^{2N}\sqrt{\lambda_0}}$$

$$\sigma_{3334} = \sigma_{222} + \frac{3}{2}\frac{\sigma_{2233}}{\mathrm{i}^N\lambda_0^{1/4}} - \frac{5}{2}\frac{\lambda_2\sigma_{22}}{\mathrm{i}^N\lambda_0^{1/4}}$$

$$\sigma_{3344} = 2\sigma_{22}\lambda_4 - \frac{\lambda_2{}^2\sigma_{22}}{2\lambda_0} + 2\frac{\sigma_{2234}}{\mathrm{i}^N\lambda_0^{1/4}} - \frac{\sigma_{223}\lambda_3}{\mathrm{i}^{2N}\sqrt{\lambda_0}} - \frac{1}{6}\frac{\sigma_{2222}}{\mathrm{i}^{2N}\sqrt{\lambda_0}} + \frac{1}{3}\frac{\lambda_2\sigma_{222}}{\mathrm{i}^{3N}\lambda_0{}^{3/4}}$$

$$\sigma_{3345} = \frac{\sigma_{2235}}{\mathrm{i}^N\lambda_0^{1/4}} - \frac{\sigma_{22}\lambda_3}{\mathrm{i}^N\lambda_0^{1/4}} + \frac{1}{3}\frac{\sigma_{2223}}{\mathrm{i}^{2N}\sqrt{\lambda_0}} - \frac{1}{2}\frac{\sigma_{223}\lambda_2}{\mathrm{i}^{3N}\lambda_0{}^{3/4}}$$

$$\sigma_{3346} = \frac{1}{3}\frac{\sigma_{222}}{\mathrm{i}^N\lambda_0^{1/4}} + \frac{\sigma_{2236}}{\mathrm{i}^N\lambda_0^{1/4}}$$

$$\sigma_{3444} = \frac{3}{4}\frac{\mathrm{i}^N\sigma_{22}\lambda_3{}^2}{\lambda_0{}^{3/4}} + 3/4\frac{\mathrm{i}^N\lambda_4\lambda_2\sigma_{22}}{\lambda_0{}^{3/4}} + \frac{3}{2}\frac{\sigma_{223}}{\mathrm{i}^N\lambda_0^{1/4}} + \frac{3}{2}\frac{\sigma_{2244}}{\mathrm{i}^N\lambda_0^{1/4}} - \frac{3}{2}\frac{\sigma_{224}\lambda_3}{\mathrm{i}^{2N}\sqrt{\lambda_0}} - \frac{1}{2}\frac{\sigma_{222}\lambda_4}{\mathrm{i}^{2N}\sqrt{\lambda_0}}$$

$$\begin{aligned}
\sigma_{3445} &= -\sigma_{22} + \frac{3}{4}\frac{\lambda_2\sigma_{22}\lambda_3}{\lambda_0} + \frac{\sigma_{2245}}{\mathrm{i}^N\lambda_0^{1/4}} - \frac{1}{2}\frac{\sigma_{225}\lambda_3}{\mathrm{i}^{2N}\sqrt{\lambda_0}} + \frac{1}{3}\frac{\sigma_{2224}}{\mathrm{i}^{2N}\sqrt{\lambda_0}} \\
&\quad - \frac{1}{3}\frac{\sigma_{222}\lambda_3}{\mathrm{i}^{3N}{\lambda_0}^{3/4}} - \frac{1}{2}\frac{\sigma_{224}\lambda_2}{\mathrm{i}^{3N}{\lambda_0}^{3/4}} \\
\sigma_{3446} &= -\frac{1}{2}\frac{\sigma_{22}\lambda_4}{\mathrm{i}^N\lambda_0^{1/4}} + \frac{\sigma_{2246}}{\mathrm{i}^N\lambda_0^{1/4}} - \frac{1}{2}\frac{\sigma_{226}\lambda_3}{\mathrm{i}^{2N}\sqrt{\lambda_0}} \\
\sigma_{3455} &= -\frac{\sigma_{22}\lambda_4}{\mathrm{i}^N\lambda_0^{1/4}} + \frac{1}{2}\frac{\sigma_{2255}}{\mathrm{i}^N\lambda_0^{1/4}} + \frac{1}{2}\frac{{\lambda_2}^2\sigma_{22}}{\mathrm{i}^N{\lambda_0}^{5/4}} + \frac{1}{3}\frac{\sigma_{2225}}{\mathrm{i}^{2N}\sqrt{\lambda_0}} - \frac{1}{2}\frac{\sigma_{225}\lambda_2}{\mathrm{i}^{3N}{\lambda_0}^{3/4}} \\
&\quad + \frac{1}{12}\frac{\sigma_{2222}}{\mathrm{i}^{3N}{\lambda_0}^{3/4}} - \frac{1}{3}\frac{\lambda_2\sigma_{222}}{\lambda_0} \\
\sigma_{3456} &= -\frac{1}{2}\frac{\sigma_{224}}{\mathrm{i}^N\lambda_0^{1/4}} + \frac{1}{2}\frac{\sigma_{2256}}{\mathrm{i}^N\lambda_0^{1/4}} + \frac{1}{6}\frac{\sigma_{2226}}{\mathrm{i}^{2N}\sqrt{\lambda_0}} + \frac{1}{4}\frac{\sigma_{22}\lambda_3}{\mathrm{i}^{2N}\sqrt{\lambda_0}} - \frac{1}{4}\frac{\sigma_{226}\lambda_2}{\mathrm{i}^{3N}{\lambda_0}^{3/4}}
\end{aligned}$$

Bibliography

[1] S. Abenda and Yu. N. Fedorov. On the weak Kowalevski-Painleve property for hyperelliptic seperable systems. *Acta Applicandae Mathematicae*, 60:138–178, 2000.

[2] M.S. Alber and Yu. N. Fedorov. Wave solutions of evolution equations and Hamiltonian flows on nonlinear subvarieties of generalised Jacobians. *Journal of Physics A: Mathematical and Theoretical*, 33:8409–8425, 2000.

[3] J.V. Armitage and W.F. Eberlein. *Elliptic functions — LMS student texts 67*. Cambridge, 2006.

[4] C. Athorne. Identities for hyperelliptic $\wp$-functions of genus one, two and three in covariant form. *Journal of Physics A: Mathematical & Theoretical*, 41:415202, 2008.

[5] C. Athorne, J.C. Eilbeck, and V.Z. Enolskii. Identities for the classical genus two $\wp$-function. *Journal of Geometry and Physics*, 48(2-3): 354–368, 2003.

[6] C. Athorne, J.C. Eilbeck, and V.Z. Enolskii. A $sl(2)$ covariant theory of genus 2 hyperelliptic functions. *Mathematical Proceedings of the Cambridge Philosophical Society*, 136(2): 269–286, 2004.

[7] H.F. Baker. *Abelian functions: Abel's theorem and the allied theory of theta functions*. Cambridge University Press, 1897 (Reprinted in 1995).

[8] H.F. Baker. On the hyperelliptic sigma functions. *American Journal of Mathematics*, 20:301–384, 1898.

[9] H.F. Baker. On a system of differential equations leading to periodic functions. *Acta Mathematica*, 27:135–156, 1903.

[10] H.F. Baker. *Multiply periodic functions*. Cambridge University Press, Cambridge, 1907 (Reprinted in 2007 by Merchant Books. ISBN 193399880).

[11] S. Baldwin, J.C. Eilbeck, J. Gibbons, and Y. Ônishi. Abelian functions for cyclic trigonal curves of genus four. *Journal of Geometry and Physics*, 58:450–467, 2008.

[12] S. Baldwin and J. Gibbons. Hyperelliptic reduction of the Benney moment equations. *Journal of Physics A: Mathematical and Theoretical*, 36:8393, 2003.

[13] S. Baldwin and J. Gibbons. Higher genus hyperelliptic reductions of the Benney equations. *Journal of Physics A: Mathematical and Theoretical*, 37:5341–5354, 2004.

[14] S. Baldwin and J. Gibbons. Genus 4 trigonal reduction of the Benny equations. *Journal of Physics A: Mathematical and Theoretical*, 39:3607–3639, 2006.

[15] E. Bellman. *A brief introduction to theta functions*. Holt, Rinehart and Winston, 1961.

[16] E.D. Belokolos, A.I. Bobenko, V.Z. Enolskii, A.R. Its, and V.B. Matveev. *Algebro-geometric approach to nonlinear integrable equations*. Springer Verlag, 1994.

[17] D.J. Benney. Some properties of long nonlinear waves. *Studies in Applied Mathematics*, 52:45, 1973.

[18] O. Bolza. On the first and second derivatives of hyperelliptic σ-functions. *Mathematische Annalen*, 17:11–36, 1895.

[19] V.M. Buchstaber, V.Z. Enolskii, and D.V. Leykin. Kleinian functions, hyperelliptic Jacobians and applications. *Reviews in Math. and Math. Physics*, 10:1–125, 1997.

[20] V.M. Buchstaber, V.Z. Enolskii, and D.V. Leykin. Rational analogs of Abelian functions. *Functional Analysis and its Applications*, 33:83–94, 1999.

[21] V.M. Buchstaber, V.Z. Enolskii, and D.V. Leykin. Uniformization of Jacobi varieties of trigonal curves and nonlinear equations. *Functional Analysis and its Applications*, 34:159–171, 2000.

[22] V.M. Bukhshtaber, D.V. Leikin, and V.Z. Enol'skii. σ-functions of (n, s)-curves. *Communications of the Moscow Mathematical Society*, 54:No. 3, 1999.

[23] J.L. Burchnall and T.W. Chaundy. Commutative ordinary differential operators. *Proceedings of the London Mathematical Society*, 118:420–440, 1923.

[24] H. Burkhardt. Beiträge zur theorie der hyperelliptische sigmafunctionen. *Mathematische Annalen*, 32:381–442, 1888.

[25] J.W.S. Cassels and E.V. Flynn. *Prolegomena to a middlebrow arithmetic of curves of genus 2 — LMS Lecture note series 230*. Cambridge, 1996.

[26] K. Chandrasekharan. *Elliptic functions*. Springer Verlag, 1985.

[27] H. Cohen and G. Frey. *Handbook of elliptic and hyperelliptic curve cryptography*. Chapman and Hall / CRC, 2006.

[28] P. Du Val. *Elliptic functions and elliptic curves — LMS Lecture note series 9*. Cambridge University Press, 1973.

[29] J.C. Eilbeck. `http://www.ma.hw.ac.uk/Weierstrass/Trig34/`.

[30] J.C. Eilbeck, V.Z. Enolski, S. Matsutani, Y. Ônishi, and E. Previato. Abelian functions for trigonal curves of genus three. *International Mathematics Research Notices*, page Art.ID: rnm140 (38 pages), 2007.

[31] J.C. Eilbeck, V.Z. Enolski, S. Matsutani, Y. Ônishi, and E. Previato. Addition formulae over the Jacobian pre-image of hyperelliptic wirtinger varieties. *Journal fr die reine und angewandte Mathematik (Crelle's Journal)*, 618:37–48, 2008.

[32] J.C. Eilbeck and V.Z. Enolskii. Bilinear operators and the power series for the Weierstrass σ function. *Journal of Physics A: Mathematical and Theoretical*, 33:791–794, 2000.

[33] J.C. Eilbeck, V.Z. Enolskii, and E.Previato. Varieties of elliptic solitons. *Journal of Physics A: Mathematical and Theoretical*, 34:2215–2227, 2001.

[34] J.C. Eilbeck, V.Z. Enolskii, and E.Previato. On a generalized Frobenius–Stickelberger addition formula. *Letters in Mathematical Physics*, 65:5–17, 2003.

[35] J.C. Eilbeck, V.Z. Enolskii, and D.V. Leykin. On the Kleinian construction of Abelian functions of canonical algebraic curves. In D Levi and O Ragnisco, editors, *Proceedings of the 1998 SIDE III Conference, 1998: Symmetries of Integrable Differences Equations*, volume CRMP/25 of *CRM Proceedings and Lecture Notes*, pages 121–138, 2000.

[36] J.C. Eilbeck, S. Matsutani, and Y. Ônishi. Some addition formula for Abelian functions for elliptic and hyperelliptic curves of cyclotomic type. *preprint,arXiv:0803.3899v1*, 2008.

[37] M. England. `http://www.ma.hw.ac.uk/~matte/BenneyReduction/`.

[38] M. England. `http://www.ma.hw.ac.uk/Weierstrass/45Results/`.

[39] M. England and J.C. Eilbeck. Abelian functions associated with a cyclic tetragonal curve of genus six. *Journal of Physics A: Mathematical and Theoretical*, 42:095210 (27pp), 2009.

[40] M. England and J. Gibbons. A genus six cyclic tetragonal reduction of the Benney equations. *Journal of Physics A: Mathematical and Theoretical*, 42:375202 (27pp), 2009.

[41] V.Z. Enolskii, M. Pronine, and P.H. Richter. Double pendulum and θ-divisor. *Journal of Nonlinear Science*, 13:157–174, 2003.

[42] H.M. Farkas and I. Kra. *Riemann surfaces — Second edition.* Springer, 1992.

[43] I.M. Gelfand, M.M. Kapranov, and A.V. Zelevinsky. *Discriminants, resultants and multidimensional determinants*. Birkhäuser, 1994.

[44] J. Gibbons. Collisionless boltzmann equations and integrable moment equations. *Physica D*, 3:503, 1981.

[45] G. Göppel. Theoriae transcendentium Abelianarum primi ordinis adumbrato levis. *Journ.reine angew. Math*, 35:277, 1847.

[46] P. Griffiths and J. Harris. *Principles Of Algebraic Geometry*. Wiley-Interscience, 1978.

[47] E. Hackmann and C. Lammerzähl. Complete analytic solution of the geodesic equation in Schwarzschild-(Anti-)de Sitter spacetimes. *Physical Review Letters*, 100:171101, 2008.

[48] S.P. Tsarev J. Gibbons. Reductions of the Benney equations. *Physics Letters A*, 211:19, 1996.

[49] S.P. Tsarev J. Gibbons. Conformal maps and reductions of the Benney equations. *Physics Letters A*, 258, 1999.

[50] J. Jorgenson. On directional derivatives of the theta function along its divisor. *Israel Journal of Mathematics*, 77:273–284, 1992.

[51] F. Klein. Über hyperelliptische sigmafunctionen. *Mathematische Annalen*, 32:351–380, 1888.

[52] B.A. Kupershmidt and Yu.I. Manin. Long-wave equation with free boundaries. *Functional Analysis and its Applications*, 11:188, 1977.

[53] S. Lang. *Introduction to algebraic functions and Abelian functions*. Number 89 in Graduate Texts in Mathematics. Springer-Verlag, 2nd. edition, 1982.

[54] D. Lawden. *Elliptic Functions and Applications*. Springer Verlag, 1980.

[55] H. Ludmark. Complex analysis: Elliptic functions `http://www.mai.liu.se/ halun/complex/elliptic/`.

[56] I.G. Macdonald. *Symmetric Functions and Hall polynomials*. Clarendon Press, Oxford, 1995.

[57] H. McKean and V. Moll. *Elliptic curves*. Cambridge, 1997.

[58] R. Miranda. *Algebraic curves and Riemann surfaces*. AMS, 1995.

[59] D. Mumford. *Tata lectures on Theta I*, volume 28 of *Progress in Mathematics*. Birkhäuser, Boston, 1983.

[60] A. Nakayashiki. On algebraic expressions of sigma functions for (n, s)-curves. *preprint, arXiv:0803.2083v1*, 2008.

[61] Proceedings of symposia in pure mathematics. *Theta functions: Bowdoin: Volume 49, Part 1 & 2*. AMS, 1987.

[62] Y. Ônishi. Determinant expressions for hyperelliptic functions. *Proceedings of the Edinburgh Mathematical Society*, 48(3):705–742, 2005. (With an appendix by Shigeki Matsutani).

[63] Y. Ônishi. Abelian functions for trigonal curves of degree four and determinantal formulae in purely trigonal case. *International Journal of Mathematics*, 20(4):427–441, 2009.

[64] G. Rosenhain. Abhandlung uber die functionen zweier variabler mit vier perioden. *Mem. pre l'Acad de Sci. de France des savants*, IX:361–455, 1851.

[65] W. Schreiner, C. Mittermaier, and K. Bosa. Distributed Maple: Parallel computer algebra in networked environments. *Journal of Symbolic Computation*, 35:305–347, 2003.

[66] S.P. Tsarev. The geometry of Hamiltonian systems of hydrodynamic type. The generalized hodograph method. *Izvestiya: Mathematics*, 37:397, 1991.

[67] L.C. Washington. *Elliptic curves: Number theory and cryptography*. Chapman and Hall / CRC, 2003.

[68] K. Weierstrass. Beitrag zur theorie der Abel'schen integrale. *Jahreber, Königl. Katolischen Gymnasium zu Braunsberg in dem Schuljahre*, pages 3–23, 1848/1849.

[69] K. Weierstrass. Zur theorie der Abelschen functionen. *Journ.reine angew. Math*, 47:289–306, 1854.

[70] E.T. Whittaker and G.N. Watson. *A course of modern analysis*. Cambridge, 1947.

[71] E. Wiltheiss. Ueber die potenzreihen der hyperelliptischen thetafunctionen. *Mem. pre l'Acad de Sci. de France des savants*, 31:410–423, 1888.

[72] W.Schreiner. `www.risc.uni-linz.ac.at/software/distmaple/`.

[73] L. Yu and J. Gibbons. The initial value problem for reductions of the Benney equations. *Inverse Problems*, 16:605–618, 2000.

[74] V.E. Zakharov. On the Benney equations. *Physica D*, 3:193, 1981.