Structure Theorems for Ordered Groupoids

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Abstract

The Ehresmann-Schein-Nambooripad theorem, which states that the category of inverse semigroups is isomorphic to the category of inductive groupoids, suggests a route for the generalisation of ideas from inverse semigroup theory to the more general setting of ordered groupoids. We use ordered groupoid analogues of the maximum group image and the E-unitary property – namely the level groupoid and incompressibility – to address structural questions about ordered groupoids. We extend the definition of the Margolis-Meakin graph expansion to an expansion of an ordered groupoid, and show that an ordered groupoid and its expansion have the same level groupoid and that the incompressibility of one determines the incompressibility of the other. We give a new proof of a P-theorem for incompressible ordered groupoids based on the Cayley graph of an ordered groupoid, and also use Ehresmann's Maximum Enlargement Theorem to prove a generalisation of the P-theorem for more general immersions of ordered groupoids. We then carry out an explicit comparison between the Gomes-Szendrei approach to idempotent pure maps of inverse semigroups and our construction derived from the Maximum Enlargement Theorem.

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Contents

Chapter 1

Introduction

The first developments in the theory of inverse semigroups and their connection with partial symmetry are due to Wagner [28] and Preston [22], independently in the early 1950s. Ehresmann's work on pseudogroups of transformations was phrased in terms of ordered groupoids, and according to [10], Ehresmann knew of the connection between ordered groupoids and inverse semigroups that was later codified as the Ehresmann-Schein-Nambooripad theorem (see[10]), stating that the category of inverse semigroups is isomorphic to the category of inductive groupoids. This theorem is an instance of the idea of regarding groupoids (or more generally, categories) as algebraic structures in their own right, a point of view advanced, with applications to combinatorial group theory, in [8]. It also suggests a route for the generalisation of ideas from inverse semigroup theory: convert them to the corresponding notion for inductive groupoids, and then try to generalise them to ordered groupoids. This route gives rise to most of the structure theorems in this thesis.

After dealing with some preliminaries in chapter 1, we turn in chapter 2 to a well-known structure theorem from inverse semigroup theory, the classification of all bisimple inverse ω-semigroups using Bruck-Reilly extensions of a group. The classification is based on ideas by Bruck [3], Reilly [23], and Munn [19]. As an introduction to the use of ordered groupoids, we prove this theorem by classifying the inductive groupoids that correspond to bisimple ω–semigroups. A groupoid corresponding to a bisimple inverse ω-semigroup under the Ehresmann-Schein-Nambooripad theorem has a transparent basic structure: to recover the semigroup from the groupoid we need to know the inductive structure. We proceed by constructing all possible inductive structures on the corresponding groupoid, and show that (up to isomorphism) each resulting inductive groupoid then corresponds to a Bruck-Reilly extension of a group.

An important concept for the remainder of this thesis is that of an E-unitary inverse semigroup. An inverse semigroup is E-unitary if any element lying above an idempotent, in the natural partial order, is also an idempotent. Equivalently, an inverse semigroup is E-unitary if the natural map to its maximum group image is idempotent pure. Saito, [24], was the first to introduce these semigroups in 1965 as *proper* inverse semigroups, and E-unitary inverse semigroups were then classified by McAlister in his celebrated P-theorem [16] in terms of a group action of the maximum group image on a certain partially ordered set. For ordered groupoids, we use the analogues developed in [5] of the maximum group image and the E-unitary property, namely the *level groupoid* of an ordered groupoid and the notion of an *incompressible* ordered groupoid. An ordered groupoid G is incompressible if the map from G to its level groupoid is star-injective (the analogue of being idempotent pure).

In chapters 3 and 4 we look at ideas based on Cayley graphs and their generalisations. A Cayley graph is a graphical representation of a presentation of a group, and we look at analogous constructions for inverse semigroups and ordered groupoids. For example, *Frucht's theorem* states that every finite group is isomorphic to the automorphism group of a Cayley graph. We prove similar results for inverse semigroups and ordered groupoids, realising them as partial symmetries of the appropriate analogue of the Cayley graph. Our main results in this direction concern the *Margolis-Meakin graph expansion* of a group presentation. The Margolis-Meakin graph expansion constructs, from the Cayley graph of a group with a given generating set Δ, an inverse semigroup $(G,\Delta)^{MM}$. Margolis and Meakin show that their expansion $(G,\Delta)^{MM}$ is Δ-generated as an inverse semigroup, has maximal group image G and is E-unitary. We use our definition of the Cayley graph of an ordered groupoid to extend the Margolis-Meakin graph expansion to an expansion of an ordered groupoid. We then show, for an ordered groupoid G generated by Δ, that $(G,\Delta)^{MM}$ is also an ordered groupoid generated by Δ, that the level groupoid of G is isomorphic to the level groupoid of $(G,\Delta)^{MM}$ and further, $(G,\Delta)^{MM}$ is incompressible if and only if G is incompressible. Thus the analogues of the key properties of the Margolis-Meakin expansion of a group hold for the Margolis-Meakin expansion of an ordered groupoid.

In chapter 5 we turn to McAlister's P-theorem, and its generalisation to the class of incompressible ordered groupoids proved by Gilbert [5]. Since McAlister published his P-theorem in 1974 there have been many alternative proofs given for the theorem. One of the versions is Steinberg's succinct proof based upon Schützenberger graphs, [26]. We give a new proof of Gilbert's P-theorem for ordered groupoids based on Steinberg's approach, using the Cayley graph of on ordered groupoid introduced in chapter 3. The P-theorem may also be viewed, as explained by Lawson in [10], as a consequence of Ehresmann's Maximum Enlargement Theorem, a very general structure theorem for star-injective maps between ordered groupoids. We give an account of this theorem in chapter 5, following Lawson's approach, and use it to prove a generalisation of the P-theorem that corresponds to O'Carroll's theorem [21] on idempotent pure extensions by inverse semigroups. O'Carroll's structure theorem is in turn a generalisation of McAlister's P-theorem for E-unitary inverse semigroups. Given a star-injective map $\nu : G \to T$ between ordered groupoids, we show how to reconstruct G from a so-called $\mathbb{L}$–system, involving the action of T on a certain poset.

In chapter 6 we look at some results of Gomes and Szendrei [7], who describe by means of category-like structures that they call *quivers*, the structure of regular semigroups that are idempotent pure regular extensions by inverse semigroups. There are close connections with the work of O'Carroll, and indeed the results of Gomes and Szendrei generalise those of O'Carroll. We carry out an explicit comparison of the Gomes-Szendrei quiver construction for an idempotent pure map of inverse semigroups, with the $\mathbb{L}$–system construction derived in chapter 5 via the Maximum Enlargement Theorem.

1.1 Inverse Semigroups

Symmetry groups are well known and it is the developement of the symmetry group of a geometry that led to 'inverse semigroups'. Inverse semigroups can be used to explain the 'partial' symmetry of a geometry. Wagner in 1952, [28], and Preston in 1954, [22], idependently constructed inverse semigroups. After introducing some semigroups definitions and properties we state the Wagner-Preston theorem which is a generalisation of Cayley's theorem for groups. As Cayley's theorem shows us that a group can always be described as a group of bijections, so the Wagner -Preston theorem shows us that an inverse semigroup can be described in terms of partial bijections.

One of the most studied types of inverse semigroup is the E-unitary inverse semigroup. In 1974 McAlister's P-theorem classified all E-unitary inverse semigroups, [16], and over the decades many have given alternative proofs to this theorem. We define the E-unitary property in this chapter and go on to discuss McAlister's P-theorem in more detail in chapter five.

1.1.1 Semigroup Definitions

A *semigroup* is a set with an associative binary operation. If a semigroup has an identity element it is called a *monoid*. A semigroup S is commutative if for all elements $s, t \in S$, $st = ts$. Given a semigroup S and element $a \in S$ then a is *regular* if there is another element $b \in S$ such that $a = aba$ and $b = bab$, in which case b is called an inverse of a. If every element of S is regular then S is called a *regular semigroup*. We denote the set of all inverses of an element a by $V(a)$. In order to show that a semigroup is regular we need only show that for an element $a \in S$ there is an element $b \in S$ such that $a = aba$. For then

$$a = aba = (aba)ba = a(bab)a$$

and

$$bab = b(aba)b = bab(aba)b = (bab)a(bab)$$

so bab is an inverse of a and S is regular.

An element e of a semigroup S is an *idempotent* if $e^2 = e$. Denote by $E(S)$ the set of all idempotents of S. Note that $e = eee$ so e is its own inverse.

An *inverse semigroup*, [10], is a semigroup S such that

1. S is regular, and
2. the idempotents of S commute.

Equivalently (see [10]), an inverse semigroup is a semigroup S such that

1. S is regular, and

2. every element of S has a unique inverse.

In an inverse semigroup the inverse of element s is usually denoted s^{-1}.

If S is a semigroup with no identity element then we can adjoin one. Denote by S^1 the set $S \cup \{1\}$ with binary operation extended as follows:

$$s1 = s = 1s \text{ for all } s \in S.$$

If S is an inverse monoid then a *unit* is an element $u \in S$ with an inverse v, say, such that $uv = 1 = vu$. Let $U(S)$ denote the set of all units of S . Then $U(S)$ forms a group.

We will look at a very important example of an inverse semigroup, the symmetric inverse monoid. For this we need first to consider partial functions. Let X and Y be sets. Then a *partial bijection* f is a bijection from a subset of X to a subset of Y, [10]. Denote the domain of f by $domf$ and the image of f by imf. Let X, Y and Z be sets. Let f be a partial bijection from X to Y and let g be a partial bijection from Y to Z. Denote by f^{-1} the partial bijection from Y to X that is the inverse of f. Then the composite of f followed by g is a partial function fg from X to Z, where

$$dom(fg) = (imf \cap domg)f^{-1}$$

and

$$im(fg) = (imf \cap domg)g.$$

The *partial identity* on the subset $A \subset X$ is the identity function on A, denoted 1_A. Also if 1_B is also a partial identity on X then

$$1_A 1_B = 1_{A\cap B} = 1_B 1_A$$

for if $x \in dom(1_A 1_B)$ then $x(1_A 1_B) = (x1_A)1_B = x1_B = x$. Thus $1_A 1_B$ is an partial identity on X. Also $dom(1_A 1_B) = A \cap B$ so $1_A 1_B = 1_{A \cap B}$. Similarly $1_B 1_A = 1_{A \cap B}$. Thus partial identities commute.

If $f : X \to Y$ is a partial bijection then $ff^{-1} = 1_{domf}$ and $f^{-1}f = 1_{imf}$. The inverse of fg is denoted $(fg)^{-1}$ and equals $g^{-1}f^{-1}$.

A *permutation* of a set X is a bijection from X to itself. The *permutation group* or *symmetric group* S_X consists of all permuations of the set X. In much the same way we have the *symmetric inverse monoid* $\mathcal{I}(X)$ which consists of the set of all partial bijections defined on subsets of X.

Theorem 1.1.1. $\mathcal{I}(X)$ *is an inverse monoid.*

Example 1.1.2. We give the particular examples S_X and $\mathcal{I}(X)$ where $X = \{1, 2, 3\}$.

S_X consists of the following six elements:

$$\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \}$$

The symmetric inverse monoid $\mathcal{I}(X)$ as a set is considerably larger than S_X; it has thirty four elements:

$$\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & * \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & * & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ * & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & * \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & * & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ * & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & * \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & * & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ * & 2 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & * \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & * & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ * & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & * \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & * & 2 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 \\ * & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & * \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & * & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ * & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & * & * \end{pmatrix},$$
$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & * & * \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & * & * \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ * & 1 & * \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ * & 2 & * \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ * & 3 & * \end{pmatrix},$$
$$\begin{pmatrix} 1 & 2 & 3 \\ * & * & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ * & * & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ * & * & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ * & * & * \end{pmatrix}\}$$

The element $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & * \end{pmatrix} \in \mathcal{I}(X)$, for example, tells us $1 \mapsto 1$, $2 \mapsto 3$ and the map is undefined for $3 \in X$. Hence this is a 'partial' bijection.

S_X corresponds to the symmetries of an equilateral triangle, where $1, 2$ and 3 label the corners of the triangle and elements of S_X are bijections that preserve the structure of the geometric figure. For example see Fig. 1.1.1. The partial bijection $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & * \end{pmatrix}$ could be considered then as a symmetry

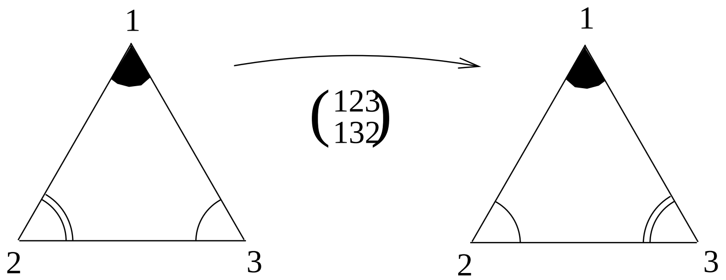

Figure 1.1.1: bijection.

of only part of the triangle. See Fig. 1.1.2. Here we know nothing of the corner 3, we have only part of the symmerty.

We note that the symmetric (or permutation) group S_X embeds into $\mathcal{I}(X)$.

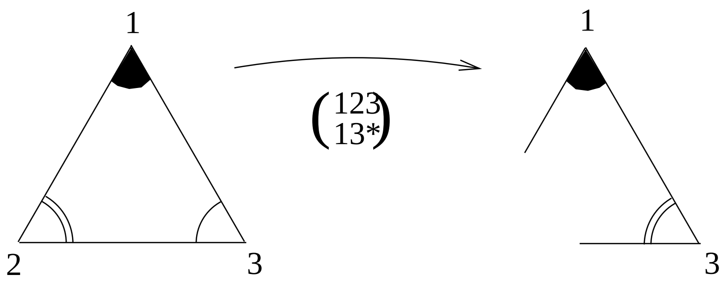

Figure 1.1.2: partial bijection.

1.1.2 Inverse Semigroup Properties

In this section we introduce some properties of inverse semigroups which we will use thoughout this thesis, often without reference. Most of this subsection, including the proofs, comes from Mark Lawson's book *Inverse Semigroups*, [10].

Idempotents

Proposition 1.1.3. *[10]. Let S be an inverse semigroup. Then the idempotents of S have the form ss^{-1} or $s^{-1}s$ for some $s \in S$.*

Further, the idempotents of S, denoted $E(S)$, forms an inverse semigroup.

Proposition 1.1.4. *[10]. Let S be an inverse semigroup. Then*

1. $(s^{-1})^{-1} = s$ *for every* $s \in S$, *and*
2. $(s_1 s_2 \dots s_n)^{-1} = s_n^{-1} \dots s_2^{-1} s_1^{-1}$ *for all* $s_i \in S$, $n \geqslant 2$.

For every element s in an inverse semigroup S we define the domain of s, $\mathbf{d}(s)$, and the range of s, $\mathbf{r}(s)$, as

$$\mathbf{d}(s) = ss^{-1} \text{ and } \mathbf{r}(s) = s^{-1}s.$$

The following result gives the property an inverse semigroup must have for it to be a group.

Proposition 1.1.5. *[10]. Groups are precisely the inverse semigroups with exactly one idempotent.*

Natural Partial Order

A relation $\leqslant$ is a *partial order* on a set X if the following axioms hold.

1. Reflexivity: $x \leqslant x$ for all $x \in X$.
2. Antisymmetry: $x \leqslant y$ and $y \leqslant x$ imply that $x = y$.
3. Transitivity: $x \leqslant y$ and $y \leqslant z$ imply that $x \leqslant z$.

In this case the set X with $\leqslant$ is called a *partially ordered set* or *poset*.

A relation $\leqslant$ on a set is a *preorder* or *quasiorder* if the relation is reflexive and transitive.

The *natural partial order* on an inverse semigroup S is given as follows.

$$s \leqslant t \Leftrightarrow s = te \text{ for some idempotent } e \in E(S).$$

If $s \leqslant t$ we say that s *lies beneath* t.

Lemma 1.1.6. *[10]. Let S be an inverse semigroup. Then the following are equivalent:*

1. $s \leqslant t$
2. $s = ft$ *for some idempotent* f

3. $s^{-1} \leqslant t^{-1}$

4. $s = ss^{-1}t$

5. $s = ts^{-1}s$

Proposition 1.1.7. *[10]. Let S be an inverse semigroup. Let $s, t, u, v \in S$.*

1. *If $s \leqslant t$ and $u \leqslant v$ then $su \leqslant tv$.*

2. *If $s \leqslant t$ then $s^{-1}s \leqslant t^{-1}t$ and $ss^{-1} \leqslant tt^{-1}$.*

3. *The relation $\leqslant$ is a partial order on S.*

On the idempotents of inverse semigroup S the natural partial order becomes

$$e \leqslant f \Leftrightarrow e = ef = fe.$$

Proposition 1.1.8. *[10]. Let S be an inverse semigroup. Then $\leqslant$ is a partial order on $E(S)$.*

Ideals and Meet Semilattices

Let S be a semigroup. A subset I of S is a *left ideal* if, for each $a \in I$ and $s \in S$, then $sa \in I$. Similarly I is a *right ideal* if $as \in I$. If subset I of S is both a left and right ideal it is called an *ideal*.

Let S be an inverse semigroup. The smallest ideal containing the element $s \in S$ is called the *principal left ideal* containing s and is $Ss = \{xs : x \in S\}$. In this case s is called a *generator* for the ideal. It is clear that $s \in Ss$ as $s = (ss^{-1})s$ and $ss^{-1} \in S$. Similarly the *principal right ideal* containing s is $sS = \{sx : x \in S\}$. The *principal (two-sided) ideal* containing s is $SsS = \{xsz : x, z \in S\}$.

We note here that the only ideal of a group is the group itself because $gG = G = Gg$ for all $g \in G$.

Lemma 1.1.9. *[10]. Let S be an inverse semigroup.*

1. *$aS = aa^{-1}S$ for all $a \in S$. Further aa^{-1} is the unique idempotent generator of aS.*

2. *$Sa = Sa^{-1}a$ for all $a \in S$. Further $a^{-1}a$ is the unique idempotent generator of Sa.*

3. *$eS \cap fS = efS$ where $e, f \in E(S)$.*

4. *$Se \cap Sf = Sef$ where $e, f \in E(S)$.*

Let $(P, \leqslant)$ be a poset. A subset Q of P is an *order ideal* if, for $x \in P$ and $y \in Q$, $x \leqslant y$ implies $x \in Q$. The smallest or *principal* order ideal of P containing an element x is the set $[x] = \{y \in P : y \leqslant x\}$.

If $x, y, z \in P$ and $z \leqslant x, y$ then z is a *lower bound* of x and y. If z lies above all other lower bounds of x and y it is the *greatest lower bound* and z is denoted by $x \wedge y$. If every pair of elements in P has a greatest lower bound then P is a *meet semilattice*.

Proposition 1.1.10. *[10]. Let S be an inverse semigroup. Then $E(S)$ is an order ideal of S. Further $E(S)$ is a meet semilattice.*

Proof. Let $x, y \in S$ and $y \in E(S)$ such that $x \leqslant y$. Then $x = ye$ where $e \in E(S)$. Then $ye \in E(S)$ so $x \in E(S)$. Therefore $E(S)$ is an order ideal.

Let $e, f \in E(S)$. Then $(ef)e = (fe)e = fe^2 = fe = ef$ so $ef \leqslant e$. Also $(ef)f = ef^2 = ef$ so $ef \leqslant f$. Thus ef is a lower bound for e and f.

Now let $z \in E(S)$ be another lower bound for e and f. So $z \leqslant e$ implying $z = ze = ez$ and $z \leqslant f$ implying $z = zf = fz$. Then $z(ef) = (ze)f = zf = z$

and $(ef)z = e(fz) = ez = z$, so $z = z(ef) = (ef)z$. Thus $z \leqslant ef$ and so ef is the greatest lower bound of e and f. Hence $e \wedge f = ef$.

It follows that $E(S)$ is a meet semilattice. □

Proposition 1.1.11. *[10]. Meet semilattices are the inverse semigroups in which every element is an idempotent.*

Compatibility Relation

Let S be an inverse semigroup. The *left compatibility relation* on S is defined, for all $s, t \in S$,

$$s \sim_l t \Leftrightarrow st^{-1} \in E(S).$$

Similarly the *right compatibility relation* is

$$s \sim_r t \Leftrightarrow s^{-1}t \in E(S).$$

The *compatibility relation* is then given by

$$s \sim t \Leftrightarrow st^{-1}, s^{-1}t \in E(S).$$

Lemma 1.1.12. *[10] Let S be an inverse semigroup.*

1. *$s \sim_l t$ if and only if $s \wedge t$ exists. Further*

$$s \wedge t = st^{-1}t = ts^{-1}t = ts^{-1}s = st^{-1}s.$$

2. *$s \sim_r t$ if and only if $s \wedge t$ exists. Further*

$$s \wedge t = ss^{-1}t = st^{-1}s = tt^{-1}s = ts^{-1}s.$$

3. *$s \sim t$ if and only if $s \wedge t$ exists. Further*

$$s \wedge t = st^{-1}t = ts^{-1}t = ts^{-1}s = st^{-1}s = ss^{-1}t = tt^{-1}s.$$

Inverse Semigroup Homomorphisms

Let S and T be semigroups. A semigroup homomorphism $\theta : S \to T$ is a function such that, for all $s, t \in S$,

$$(st)\theta = (s\theta)(t\theta).$$

An injective semigroup homomorphism is also called an *embedding*. An inverse semigroup homomorphism is just a semigroup homomorphism between inverse semigroups. If P and Q are posets then function $\theta : P \to Q$ is *order preserving* if $x \leqslant y$ in P then $x\theta \leqslant y\theta$ in Q. An *order isomorphism* is then a bijective order preserving homomorphism whose inverse is order preserving.

Proposition 1.1.13. *[10]. Let S, T be inverse semigroups and let $\theta : S \to T$ be a homomorphism.*

1. *$s^{-1}\theta = (s\theta)^{-1}$ for all $s \in S$.*
2. *If $e \in E(S)$ then $e\theta \in E(T)$.*
3. *If $s\theta \in E(T)$ then there exists $e \in E(S)$ such that $e\theta = s\theta$.*
4. *θ is order preserving.*
5. *Let $x, y \in S$ be such that $x\theta \leqslant y\theta$. Then there exists $z \in S$ such that $z \leqslant y$ and $z\theta = x\theta$.*
6. *$im\theta$ is an inverse subsemigroup of T.*
7. *If A is an inverse subsemigroup of T then $A\theta^{-1}$ is an inverse subsemigroup of S.*

The kernel of semigroup homomorphism $\theta : S \to T$ is defined as

$$ker\theta = \{(x, y) \in S \times S : x\theta = y\theta\}.$$

Let $\theta : S \to T$ be a homomorphism of inverse semigroups. Then θ induces a homomorphism $\theta|_{E(S)} : E(S) \to E(T)$, the restriction of θ to $E(S)$. If $\theta|_{E(S)}$ is injective θ is called an *idempotent separating* homomorphism.

An inverse semigroup homomorphism $\theta : S \to T$ is *idempotent pure* if when $s\theta \in E(T)$ then $s \in E(S)$.

Congruences

An *equivalence relation* ρ on a set X is a subset of $X \times X$ such that

1. ρ is reflexive: $(x, x) \in \rho$ for all $x \in X$;
2. ρ is symmetric: if $(x, y) \in \rho$ then $(y, x) \in \rho$;
3. ρ is transitive: if $(x, y), (y, z) \in \rho$ then $(x, z) \in \rho$.

We often denote $(x, y) \in \rho$ by $x\rho y$.

A quasiorder on a set induces a partial order via an equivalence relation. Given a quasiorder $\preccurlyeq$ on a set S we can construct an equivalence relation $\sim$ by defining

$$s \sim t \Leftrightarrow s \preccurlyeq t \text{ and } t \preccurlyeq s.$$

We check this is indeed an equivalence relation. First $s \preccurlyeq s$ so $s \sim s$. Secondly, $s \sim t$ implies $s \preccurlyeq t$ and $t \preccurlyeq s$ which implies $t \sim s$. Finally let $s \sim t$ and $t \sim u$. Then $s \preccurlyeq t$ and $t \preccurlyeq u$ so $s \preccurlyeq u$. Also $u \preccurlyeq t$ and $t \preccurlyeq s$ so $u \preccurlyeq s$. Then $s \sim u$ and $\sim$ is an equivalence.

The quasiorder induces a partial order on the set of equivalence classes $S/\sim$,

$$[x]_\sim \leqslant [y]_\sim \Leftrightarrow x \preccurlyeq y.$$

We show that $\leqslant$ is well-defined. If $[x]_\sim \leqslant [y]_\sim$ and $a \sim x$ and $b \sim y$ then $x \preccurlyeq y$ and $[a]_\sim = [x]_\sim$ and $[b]_\sim = [y]_\sim$ so $a \preccurlyeq x \preccurlyeq y \preccurlyeq b$.

Now we show $\leqslant$ is a partial order. As $\sim$ is an equivalence relation $x \sim x$ and so $x \preccurlyeq x$ which implies $[x]_\sim \leqslant [x]_\sim$ and the relation is reflexive. If $[x]_\sim \leqslant [y]_\sim$ and $[y]_\sim \leqslant [x]_\sim$ then $x \preccurlyeq y$ and $y \preccurlyeq x$. So $x \sim y$ and $[x]_\sim = [y]_\sim$. The relation is then antisymmetric. Now let us check transitivity so let $[x]_\sim \leqslant [y]_\sim$ and $[y]_\sim \leqslant [z]_\sim$. Then $x \preccurlyeq y$ and $y \preccurlyeq z$. As $\preccurlyeq$ is a quasiorder it is transitive so $x \preccurlyeq z$ and $[x]_\sim \leqslant [z]_\sim$.

A *congruence* on a semigroup S is an equivalence relation such that if $(u, v), (x, y) \in \rho$ then $(ux, vy) \in \rho$. A *left congruence* on a semigroup S is an equivalence relation such that if $(x, y) \in \rho$ then $(ux, uy) \in \rho$. Similarly we can define a *right congruence*. If ρ is both a left and right congruence then, if $(u, v), (x, y) \in \rho$ we have that $(ux, uy) \in \rho$ and $(uy, vy) \in \rho$. As ρ is an equivalence relation it is transitive, thus $(ux, vy) \in \rho$. So ρ is a congruence if it is both a left and a right congruence.

Let ρ be a congruence on a semigroup S. Denote by $a\rho$ the congruence class (or ρ-equivalence class) of $s \in S$. Denote the set of congruence classes by S/ρ. We say that S/ρ is the *quotient of S by ρ*. Define a binary operation on S/ρ by

$$(a\rho)(b\rho) = (ab)\rho.$$

The binary operation on S/ρ is clearly associative. Thus S/ρ is a semigroup.

The *associated natural homomorphism* $\rho^\natural : S \to S/\rho$ is defined by $s \mapsto s\rho$. A congruence is *idempotent separating* if its associated natural homomorphism is idempotent separating.

Let ρ be any relation on an inverse semigroup S, then the *congruence generated by ρ* is the intersection of all the congruences contatining ρ. We denote this congruence by $\rho^\sharp$.

Proposition 1.1.14. *[10]. Let ρ be a congruence on an inverse semigroup S.*

1. *If* $(s,t) \in \rho$ *then* $(s^{-1}, t^{-1}) \in \rho$, $(s^{-1}s, t^{-1}t) \in \rho$ *and* $(ss^{-1}, tt^{-1}) \in \rho$.

2. *If* $(s,e) \in \rho$ *with* $e \in E(S)$ *then* $(s, s^{-1}) \in \rho$, $(s, s^{-1}s) \in \rho$ *and* $(s, ss^{-1}) \in \rho$.

Let ρ be a congruence on an inverse semigroup S. The *Kernel of* ρ is the union of the ρ-classes containing idempotents and is denoted $\mathrm{Ker}\rho$. We distinguish this Kernel from the one previously defined by the use of the capital K.

Lemma 1.1.15. $\mathrm{Ker}\rho$ *is an inverse subsemigroup of* S

Let S be an inverse semigroup. A congruence ρ on S is *idempotent pure* if, for $s \in S$ and $e \in E(S)$, $s\rho e$ implies $s \in E(S)$.

Proposition 1.1.16. *[10]. Let* S *be an inverse semigroup. Then a congruence* ρ *is idempotent pure if and only if the compatibility relation contains* ρ.

Any ideal I of an inverse semigroup S determines a congruence ρ_I on S as follows, [10], for $s, t \in S$,

$$s\rho_I t \text{ if and only if } s, t \in I \text{ or } s = t.$$

The *Rees quotient* is defined as S/ρ_I.

Proposition 1.1.17. *[10]. Let* I *be an ideal of inverse semigroup* S*. Then the Rees quotient* S/ρ_I *is isomorphic as a semigroup to the set* $S \setminus I \cup \{0\}$ *with composition*

$$st = \left\{ \begin{array}{ll} st & \text{if } s, t \in S \setminus I \\ 0 & \text{otherwise} \end{array} \right\}$$

We now introduce an important congruence on an inverse semigroup that will be used frequently thoughout this thesis. The *minimum group congruence*, σ, on an inverse semigroup S is defined by

$$s\sigma t \text{ if and only if there exists } u \in S \text{ such that } u \leqslant s, t$$

for all $s, t \in S$.

Theorem 1.1.18. *[10]. Let S be an inverse semigroup.*

1. *σ is the smallest congruence on S containing the compatibility relation.*
2. *S/σ is a group.*
3. *If ρ is any congruence on S such that S/ρ is a group then $\sigma \subseteq \rho$.*

Proof, [10]. (1) We show that σ is an equivalence relation. As $s \leqslant s$ then $s\sigma s$ and σ is reflexive. Let $s\sigma t$ then there exists $u \in S$ such that $u \leqslant s, t$ so $t\sigma s$ and σ is symmetric. If $a\sigma b$ and $b\sigma c$ then there exists $u, v \in S$ such that $u \leqslant a, b$ and $v \leqslant b, c$. Then $v^{-1} \leqslant b^{-1}$ so $uv^{-1} \leqslant bb^{-1} \in E(S)$ and so $uv^{-1} \in E(S)$. Thus $u \sim_l v$. By lemma 1.1.12, $u \wedge v$ exists. Now $u \wedge v \leqslant u \leqslant a$ and $u \wedge v \leqslant v \leqslant c$ so $a\sigma c$.

We show that σ is a congruence. Let $a\sigma b$ and $c\sigma d$. Then there exists $u, v \in S$ such that $u \leqslant ab$ and $v \leqslant c, d$. Then $uv \leqslant ac$ and $uv \leqslant bd$ so $ac\sigma bd$ and σ is a congruence.

Now we show that $\sim \subseteq \sigma$. Let $s \sim t$. Then $s \wedge t$ exists by lemma 1.1.12. So $s \wedge t \leqslant s, t$ and $s\sigma t$.

Let ρ be any congruence containing $\sim$. Let $a\sigma b$. Then there exists $u \leqslant a, b$. So $u = ae = fa$ for some $e, f \in E(S)$. Then

$$au^{-1} = a(afa)^{-1} = aa^{-1}f \in E(S)$$

and

$$a^{-1}u = a^{-1}(ae) = a^{-1}ae \in E(S)$$

so $a \sim u$. Similarly $u \sim b$. By assumption $a\rho u$ and $u\rho b$. As ρ is an equivalence $a\rho b$. Thus $\sigma \subseteq \rho$.

(2) We show that all idempotents are contained in a single σ-class. Let $e, f \in E(S)$. Then $ef \leqslant e, f$ so $e\sigma f$. Thus $e\sigma = f\sigma$. Consequently, S/σ is an inverse semigroup with exactly one idempotent. By proposition 1.1.5, S/ρ is then a group.

(3) Let ρ be a congruence on S such that S/ρ is a group. Let $a\sigma b$ then there exists $u \leqslant a, b$. Then $u = ae$ so $u\rho = (ae)\rho = (a\rho)(e\rho)$ and by $e\rho \in E(S/\rho)$ so $u\rho \leqslant a\rho$. Similarly $u\rho \leqslant b\rho$. S/ρ is a group so there is only one idempotent, 1, so $u\rho = a\rho 1 = a\rho$ and $u\rho = b\rho 1 b\rho$. So $a\rho = b\rho$ hence $a\rho b$. Therefore $\sigma \subseteq \rho$. $\square$

The group S/σ is called the *maximal group image* of S.

1.1.3 Cayley's Theorem and the Wagner-Preston Theorem

We start this section with Cayley's theorem followed by the Wagner-Preston theorem. This gives an introduction to the notion of generalising a theorem for groups to one for inverse semigroups. This notion of generalising ideas to a wider class of structures is the main theme running through this thesis.

Theorem 1.1.19. Cayley's Theorem. *Let G be a group and S_G be the permutation group of the underlying set G. Then G is isomorphic to a subgroup of S_G.*

If G is of order n, then G is isomorphic to a subgroup of the permutation group S_n.

Now we give the Wagner-Preston theorem.

Theorem 1.1.20. *[10]. Let S be an inverse semigroup. Then there is an injective homomorphism $\theta : S \to \mathcal{I}(S)$ such that*

$$a \leqslant b \Leftrightarrow a\theta \leqslant b\theta.$$

Proof. For each element $a \in S$ define $\theta_a : Saa^{-1} \to Sa^{-1}a$ by $x \mapsto xa$. This is well-defined because $Sa = Saa^{-1}a \subseteq Sa^{-1}a \subseteq Sa$ so $Sa = Sa^{-1}a$.

Now $\theta_{a^{-1}} : Sa^{-1}a \to Saa^{-1}$ and $\theta_{a^{-1}}\theta_a$ is the identity on $Sa^{-1}a$ and $\theta_a\theta_{a^{-1}}$ is the identity on Saa^{-1}. Thus $\theta_a^{-1} = \theta_{a^{-1}}$ and θ_a is a bijection.

Define $\theta : S \to \mathcal{I}(S)$ by $a \mapsto \theta_a$. As θ_a is well-defined, θ is well-defined.

We show next that θ is a homomorphism, i.e. that $\theta_a\theta_b = \theta_{ab}$. Now

$$\theta_a : Saa^{-1} \to Sa^{-1}a, \quad \theta_b : Sbb^{-1} \to Sb^{-1}b$$

and

$$\theta_{ab} : S(ab)(ab)^{-1} \to S(ab)^{-1}(ab).$$

By lemma 1.1.9, and because $Sa = Sa^{-1}a$ for any $a \in S$,

$$\begin{aligned} dom(\theta_a\theta_b) &= (im\theta_a \cap dom\theta_b)\theta_a^{-1} \\ &= (Sa^{-1}a \cap Sbb^{-1})\theta_a^{-1} \\ &= (Sa^{-1}abb^{-1})\theta_{a^{-1}} \\ &= Sa^{-1}abb^{-1}a^{-1} \\ &= Sbb^{-1}a^{-1}aa^{-1} \\ &= Sbb^{-1}a^{-1} \\ &= S(bb^{-1}a^{-1})^{-1}(bb^{-1}a^{-1}) \\ &= Sabb^{-1}bb^{-1}a^{-1} \\ &= Sabb^{-1}a^{-1} \\ &= S(ab)(ab)^{-1} \\ &= dom\theta_{ab} \end{aligned}$$

If $x \in S(ab)(ab)^{-1}$, then $x\theta_a\theta_b = (xa)\theta_b = xab = x\theta_{ab}$. Therefore θ is a homomorphism.

Assume now that $a \leqslant b$, so $a = fb$ for some $f \in E(S)$. Then $aa^{-1} = fbb^{-1}$. Let $y \in Saa^{-1}$ then $y = saa^{-1}$ for some $s \in S$. Then $y = sfbb^{-1}$ thus $y \in Sbb^{-1}$. Therefore $Saa^{-1} \subseteq Sbb^{-1}$. Let $x \in Saa^{-1}$, then

$$\begin{aligned} x\theta_b &= xb \\ &= saa^{-1}b \\ &= saa^{-1}fb \\ &= xfb \\ &= xa \\ &= x\theta_a. \end{aligned}$$

So $\theta_a \leqslant \theta_b$.

Conversely, assume $\theta_a \leqslant \theta_b$, then $dom\theta_a = Saa^{-1} \subseteq Sbb^{-1} = dom\theta_b$. Now, $a^{-1} \in Ssa^{-1}$ since $Saa^{-1} = Sa^{-1}$ and $a^{-1} \in Sa^{-1}$, and so

$$a^{-1}\theta_a = a^{-1}\theta_b.$$

This implies $a^{-1}a = a^{-1}b$ which in turn implies $a = aa^{-1}a = aa^{-1}b$ so $a \leqslant b$.

Let $g, h \in S$ and assume $g\theta = h\theta$. Then $\theta_g = \theta_h$. So $\theta_g \leqslant \theta_h$ and $\theta_h \leqslant \theta_g$. Then, by the result above, $g \leqslant h$ and $h \leqslant g$. Then $g = h$ and θ is injective. $\square$

The restriction of the Wagner-Preston theorem to a group S gives us Cayley's theorem, so the Wagner-Preston theorem is a generalisation of Cayley's theorem. The only ideal in a group is the group itself, thus $\theta_a : Saa^{-1} \to Sa^{-1}a$ restricted from an inverse semigroup to a group S gives $\theta_a : S \to S$ and we find ourselves in the realm of Cayley's theorem. The main difference between the two theorems is that in Cayley's theorem θ_a has the group itself as the domain and codomain, whereas ideals must be used in the Wagner-Preston theorem to ensure that the maps θ_a are injective.

1.1.4 E-unitary Inverse Semigroups

The E-unitary property of an inverse semigroup is one that has been studied in some depth. McAlister's P-theorem classifies all E-unitary inverse semigroups. We discuss this theorem later in this thesis as well as a generalisation of this theorem and the E-unitary idea.

Let S be an inverse semigroup. A subset A of S is *left unitary* if, for $a \in A$ and $s \in S$, $as \in A$ implies $s \in A$. Similarly A is *right unitary* if $sa \in A$ implies $s \in A$. If a subset is both left and right unitary it is simply called *unitary*.

Proposition 1.1.21. *[10] Let S be an inverse semigroup. Then the following are equivalent:*

1. *$E(S)$ is left unitary.*
2. *$E(S)$ is right unitary.*
3. *If $e \in E(S)$, $s \in S$ and $e \leqslant s$ then $s \in E(S)$.*

An inverse semigroup S is E-unitary if, for $e \in E(S)$ and $s \in S$, $e \leqslant s$ then $s \in E(S)$.

We now introduce some properties of E-unitary inverse semigroups.

Theorem 1.1.22. *[10] and [14, lemma 1.1]. Let S be an inverse semigroup. Then the following are equivalent:*

1. *S is E-unitary.*
2. $\sim = \sigma$
3. *σ is idempotent pure.*
4. *$e\sigma = E(S)$ for any idempotent e.*
5. *If $st = s$ then $t \in E(S)$ for all $s, t \in S$.*

Proof. (1)⇒(2) Let S be E-unitary. By theorem 1.1.18, $\sim \subseteq \sigma$. Now let $s\sigma t$. Then there exists $u \leqslant s, t$. Then $u^{-1}u \leqslant s^{-1}t$ and $uu^{-1} \leqslant s \sim t$. Therefore $\sim = \sigma$.

(2)⇒(3) By proposition 1.1.16, $\sigma \subset \sim$ if and only if σ is idempotent pure.

(3)⇒(4) Follows directly from the definition of idempotent pure.

(4)$\Rightarrow$(5) Suppose $s, t \in S$ with $st = s$. Then $s^{-1}st = s^{-1}s$. So $s^{-1}s \leqslant t$ and $s^{-1}s \leqslant s^{-1}s$. Then $t\sigma s^{-1}s$. It follows that $t\sigma = (s^{-1}s)\sigma \in E(S)$, so $t \in E(S)$.

(5)$\Rightarrow$(1) Assume for all $s, t \in S$, $st = s$ implies $t \in E(S)$. Suppose now that $e \in E(S)$, $s \in S$ and $e \leqslant s$. Then $e = ee^{-1}s = ees = es$ so, by assumption, $s \in E(S)$. $\square$

Proposition 1.1.23. *Let S be an inverse semigroup. If S is E-unitary then any subgroup of S embeds in the maximal group image S/σ.*

Proof. Let G be a subgroup of an E-unitary inverse semigroup S. Then $\sigma^\natural$ restricted to G is $\sigma^\natural|_G : G \to S/\sigma$ defined as $g \mapsto g\sigma$.

$\sigma^\natural$ is well-defined for if $a\sigma c$ and $b\sigma d$ then $ab\sigma cd$ so $a\sigma b\sigma = (ab)\sigma = (cd)\sigma = c\sigma d\sigma$. Thus for $g, h \in G$,

$$g\sigma^\natural h\sigma^\natural = g\sigma h\sigma = (gh)\sigma = (gh)\sigma^\natural$$

and $\sigma^\natural$ is a homomorphism.

Let $g, h \in G \subseteq S$ and suppose $g\sigma^\natural = h\sigma^\natural$. So $g\sigma = h\sigma$ and $g\sigma h$. As S is E-unitary then, by theorem 1.1.22, $\sigma = \sim$ so $g \sim h$. Then $gh^{-1}, g^{-1}h \in E(S)$. Thus $gh^{-1}gh^{-1} = gh^{-1}$ in S, and so also in G. So, if e is the identity of G, $gh^{-1}gh^{-1} = gh^{-1}$ implies $gh^{-1} = e$, and so $g = h$.

Therefore $\sigma^\natural|_G$ is an embedding. $\square$

Let S be a semigroup. A *zero* element, $0 \in S$, is an element such that $s0 = 0 = 0s$ for all $s \in S$. A semigroup with no zero element can be converted into a semigroup with zero by adjoining a zero element and letting $s0 = 0 = 0s$ for all elements $s \in S \cup \{0\}$. We denote the semigroup with zero adjoined by S^0.

An inverse semigroup S with zero is *E^*-unitary*, [27], if, for $e \in E(S)$ and $s \in S$, $0 \neq e \leqslant s$ then $s \in E(S)$.

An inverse semigroup S with zero is *strongly E^*-unitary*, [2], if, for some group G, there is a function $\theta : S \to G^0$ such that

1. $x\theta = 0 \Leftrightarrow x = 0$;
2. $x\theta = 1 \Leftrightarrow x \in E(S)$;
3. if $xy \neq 0$ then $(xy)\theta = (x\theta)(y\theta)$.

Proposition 1.1.24. *If the inverse semigroup S with zero is strongly E^*-unitary then S is also E^*-unitary.*

Proof. Let $s \in S$ and $e \in E(S)$ be such that $0 \neq e \leqslant s$. Then $e = sf$ for some $f \in E(S)$. By axiom (3) above, $e\theta = (sf)\theta = (s\theta)(f\theta)$. By axiom (2) above, as $e, f \in E(S)$ then $e\theta = f\theta = 1$, so $1 = (s\theta)1$ which implies $1 = s\theta$. Again by axiom (2), $s \in E(S)$. Therefore S is E^*-unitary. □

1.1.5 Green's Relations

Green's Relations on Semigroups

If S is a semigroup then we define the Green's relations $\mathcal{R}$ and $\mathcal{L}$ by

$$a\mathcal{R}b \Leftrightarrow aS^1 = bS^1$$

$$a\mathcal{L}b \Leftrightarrow S^1a = S^1b$$

Equivalently $a\mathcal{R}b$ if and only if there exists $x, y \in S^1$ such that $a = bx$ and $b = ay$. Similarly $a\mathcal{L}b$ if and only if there exists $u, v \in S^1$ such that $a = ub$ and $b = va$. Both $\mathcal{R}$ and $\mathcal{L}$ are equivalence relations.

Green's equivalence relation $\mathcal{H}$ is defined as

$$\mathcal{H} = \mathcal{R} \cap \mathcal{L}.$$

Thus $a\mathcal{H}b$ if a and b are both $\mathcal{R}$ and $\mathcal{L}$ related.

Proposition 1.1.25. *[10]. Let S be a semigroup. If $a, b \in S$ then $a\mathcal{L}s\mathcal{R}b$ for some $s \in S$ if and only if $a\mathcal{R}t\mathcal{L}b$ for some $t \in S$.*

Green's equivalence relation $\mathcal{D}$ is defined as

$$\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}.$$

Equivalently $a\mathcal{D}b$ if and only if $a\mathcal{L}s\mathcal{R}b$ for some $s \in S$ and if and only if $a\mathcal{R}t\mathcal{L}b$ for some $t \in S$.

The Green's equivalence relation $\mathcal{J}$ is defined as

$$a\mathcal{J}b \Leftrightarrow S^1aS^1 = S^1bS^1.$$

Equivalently $a\mathcal{J}b$ if and only if $a = xby$ and $b = uav$ for some $x, y, u, v \in S^1$.

Proposition 1.1.26. *[10]. $\mathcal{D} \subseteq \mathcal{J}$*

If $\mathcal{K}$ is one of the Green's relations then we denote the $\mathcal{K}$-class of element $a \in S$ by K_a.

Green's Relations on Inverse Semigroups

For an inverse semigroup S the Green's equivalence relations $\mathcal{R}$ and $\mathcal{L}$ simplify to:

$$a\mathcal{R}b \Leftrightarrow aa^{-1} = bb^{-1}$$

$$a\mathcal{L}b \Leftrightarrow a^{-1}a = b^{-1}b$$

Proposition 1.1.27. *Every $\mathcal{R}$ and $\mathcal{L}$ class in S contains a unique idempotent.*

Proof. Let $a \in S$ and consider R_a. Then $aa^{-1} \in E(S)$ and $aa^{-1} = (aa^{-1})(aa^{-1}) = (aa^{-1})(aa^{-1})^{-1}$, so $a\mathcal{R}aa^{-1}$. Thus $aa^{-1} \in R_a$.

If $e \in E(S)$ and $e \in R_a$ then $ee^{-1} = aa^{-1}$ so $e^2 = aa^{-1}$ and so $e = aa^{-1}$. So aa^{-1} is the unique idempotent of R_a.

We can prove the result for $\mathcal{L}$ similarly. □

Lemma 1.1.28. *Let S be an inverse semigroup and $a, s \in S$. Then $s\mathcal{R}sa$ if and only if $s^{-1}s \leqslant aa^{-1}$.*

Proof.

$$\begin{aligned} s\mathcal{R}sa &\Leftrightarrow ss^{-1} = sa(sa)^{-1} \\ &\Leftrightarrow s^{-1}ss^{-1} = s^{-1}saa^{-1}s^{-1} \\ &\Leftrightarrow s^{-1}ss^{-1}s = s^{-1}saa^{-1}s^{-1}s \\ &\Leftrightarrow s^{-1}s = s^{-1}ss^{-1}saa^{-1} \\ &\Leftrightarrow s^{-1}s = (s^{-1}s)(aa^{-1}) \\ &\Leftrightarrow s^{-1}s \leqslant aa^{-1} \end{aligned}$$

□

Let $\theta : S \to T$ be an inverse semigroup homomorphism. We can restrict θ to the R-class of $E(S)$, $\theta|_{R_e} : R_e \to R_{e\theta}$. If this restricted function is injective we say θ is $\mathcal{R}$-injective. Similarly we can define $\mathcal{R}$-surjective and $\mathcal{R}$-bijective functions. Also the notions of $\mathcal{L}$-injective, $\mathcal{L}$-surjective and $\mathcal{L}$-bijective functions can similarly be defined.

Proposition 1.1.29. *[10]. Let $\theta : S \to T$ be a homomorphism between inverse semigroups. Then the following are equivalent:*

1. *θ is idempotent pure.*
2. *θ is $\mathcal{R}$-injective.*
3. *θ is $\mathcal{L}$-injective.*

Proof. (1)⇒(2) Let $s\mathcal{R}t$ and $s\theta = t\theta$. Then $s^{-1}\theta s\theta = s^{-1}\theta t\theta = (s^{-1}t)\theta \in E(T)$ and so, by assumption, $s^{-1}t \in E(S)$. Now $s^{-1}ts^{-1}t = s^{-1}t$. Then $s^{-1}ts^{-1}tt^{-1} = s^{-1}tt^{-1}$ but $s\mathcal{R}t$ so $s^{-1}ts^{-1}ss^{-1} = s^{-1}ss^{-1}$. Hence $s^{-1}ts^{-1} = s^{-1}$. By symmetry $t^{-1}st^{-1} = t^{-1}$. As inverses are unique in inverse semigroups $s^{-1} = t^{-1}$. Therefore $s = t$.

(2)⇒(3) Suppose $s\mathcal{L}t$ and $s\theta = t\theta$. Then $s^{-1}\mathcal{R}t^{-1}$ and $s^{-1}\theta = (s\theta)^{-1} = (t\theta)^{-1} = t^{-1}\theta$. Then, by assumption, $s^{-1} = t^{-1}$ and so $s = t$.

(3)⇒(1) Suppose $s\theta \in E(T)$. Then $(s^{-1}s)\theta = (s^{-1}\theta)(s\theta) = (s\theta)^{-1}(s\theta) = (s\theta)^2 = s\theta$ but $s^{-1}s\mathcal{R}s$ and so, by assumption, $s = s^{-1}s \in E(S)$. □

Proposition 1.1.30. *[10]. Let S be an inverse semigroup and let $e \in E(S)$.*

1. *eSe is an inverse monoid.*
2. *$H_e = U(eSe)$. In particular, H_e is a group.*
3. *Every subgroup of S is contained in an $\mathcal{H}$-class.*

Proposition 1.1.31. *[10]. Let S be an inverse semigroup and let $s, t \in S$.*

1. *If $s\mathcal{R}t$ and $s \leqslant t$ then $s = t$.*
2. *If $s\mathcal{L}t$ and $s \leqslant t$ then $s = t$.*

3. If $s\mathcal{H}t$ and $s \leqslant t$ then $s = t$.

Proposition 1.1.32. *Let S be an inverse semigroup.*

1. $a\mathcal{D}b \Leftrightarrow a^{-1}a\mathcal{D}b^{-1}b$

2. $a\mathcal{D}b \Leftrightarrow aa^{-1}\mathcal{D}bb^{-1}$

An inverse semigroup S is *bisimple* if S has a single $\mathcal{D}$-class. An inverse semigroup with zero is *0-bisimple* if it has two $\mathcal{D}$-classes. For if $a\mathcal{R}0$ then $aa^{-1} = 0$ and so $a = aa^{-1}a = aa^{-1}0 = 0$. Similarly if $a\mathcal{L}0$ then $a = 0$. So the zero element forms its own $\mathcal{D}$-class.

1.1.6 Extensions

For groups F and K a group G is an *extension of K by F* if there is an embedding $\iota : K \to G$ and a surjection $\omega : G \to F$ such that $(K)\iota = \ker\omega$. In [10, page 137], Lawson gives an analogous definition for inverse semigroups F and K and surjective homomorphism $\pi : K \to E(F)$. An inverse semigroup S is a *normal extension of K by F along* π if there is an embedding $\iota : K \to S$ and a surjection $\omega : S \to F$ such that $(K)\iota = \mathrm{Ker}\omega$ and $\iota\omega = \pi$. Refer to Fig. 1.1.3. We often just call S an *extension by F*. If π

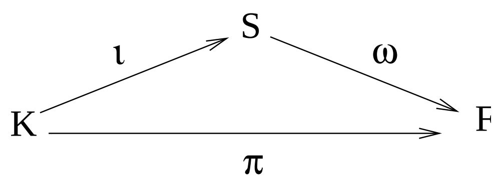

Figure 1.1.3: extension by F.

is an idempotent-separating homomorphism then the extension is called an *idempotent-seperating extension.* If K is a semilattice then the extension is an *idempotent pure extension.*

Proposition 1.1.33. *Let S be an inverse semigroup with maximal group image S/σ. Inverse semigroup S is E-unitary if and only if S is an idempotent pure extension of its semilattice $E(S)$ by its maximal group image S/σ.*

Given a regular semigroup S and an inverse semigroup T with epimorphism $\theta : S \to T$, take $K = E(T)\theta^{-1}$ and $\pi : K \to E(T)$ to be the restriction of θ to K. Now K is a regular subsemigroup of S. Then S is a *regular extension of K by T along π*. Again we often call S merely a *regular extension by T*, [7].

1.2 Graphs

We introduce the algebraic structures "categories" and "groupoids" over the next few sections of chapter one. In order to visually depict these algebraic structures we require "graphs". The focus of chapter three is two specific types of graphs, Cayley graphs and Schützenberger graphs. This section gives the graph concepts we will use throughout this thesis. Most of the definitions from this section can be found in [1].

A *graph* Γ consists of a non-empty set V of vertices and a collection E (permitting repetitions) of two-element subsets of V callled edges. The edge $\{u, v\} \in E$ connects vertices $u \in V$ and $v \in V$. There may be many edges connecting vertices u and v, such edges are called *multiple edges*.

Example 1.2.1. The graph shown in Fig. 1.2.1 has vertex set $V = \{a, b, c, d\}$ and edge set $E = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\}$. This graph has no multiple edges.

The graph Δ with vertex set W and edge set F is a *subgraph* of Γ if $W \subseteq V$ and the edge set F joining the vertices of W is a subset of E.

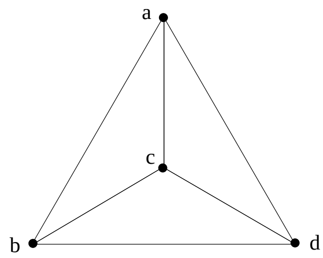

Figure 1.2.1: graph example.

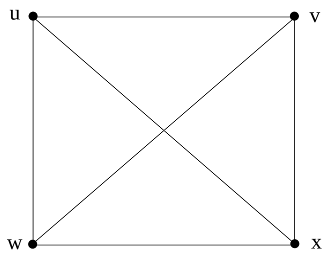

Figure 1.2.2: isomorphic graph.

Let Γ be a graph with vertex set V and edge set E and let Γ' be another graph with vertex set V' and edge set E'. A *graph map* from Γ to Γ' is a map $\alpha : V \to V'$ such that if $\{u, v\} \in E$ then $\{u\alpha, v\alpha\} \in E'$. Graph Γ is *isomorphic* to Γ' if there is a bijection $\alpha : V \to V'$ such that $\{u\alpha, v\alpha\} \in E'$ if and only if $\{u, v\} \in E$.

Example 1.2.2. The graph shown in Fig. 1.2.2 is isomorphic to the graph given in example 1.2.1. The isomorphism is given by $\alpha : a \mapsto u, b \mapsto w, c \mapsto v, d \mapsto x$.

A *directed graph* is a graph with a direction on each edge. So an edge is now an ordered pair (u, v) with initial vertex u and terminal vertex v.

Example 1.2.3. The graph shown in Fig. 1.2.3 is a directed graph.

For a graph Γ, the *anti-isomorphic graph* has the same set of vertices and edes as Γ but the edges have the opposite direction.

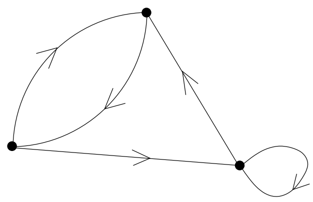

Figure 1.2.3: directed graph.

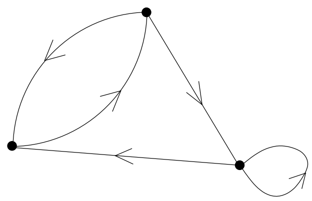

Figure 1.2.4: anti-isomorphic graph.

Example 1.2.4. The graph shown in Fig. 1.2.4 is anti-isomorphic to the graph in example 1.2.3.

A *walk* in a graph Γ is a sequence of vertices $(v_1, v_2, \ldots, v_n)$ such that there exists edges (v_i, v_{i+1}) for all $1 \leqslant i \leqslant n-1$. If the vertices are distinct we call the walk a *path*. A *component* of a graph is a subgraph that consists of all the vertices one can walk to from a given vertex. The graph in Fig. 1.2.5 consists of three components. A graph is *connected* if it has only one component.

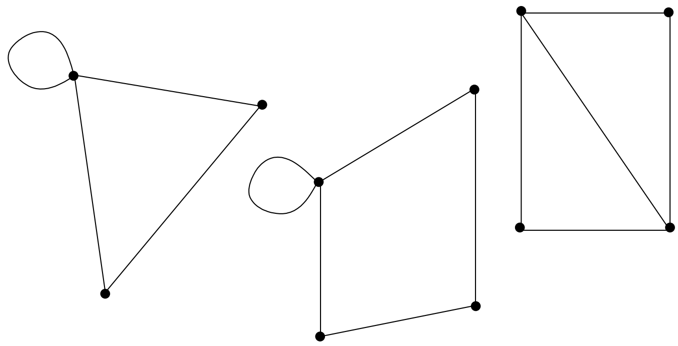

Figure 1.2.5: components.

1.3 Categories

Categories are a way of studying the relationship between different structures but we may also think of categories as algebraic structures with a partially defined binary operation. The focus of Higgins' book [8] is to consider categories as algebraic structures in their own right. We present some of the features of categories in this section.

1.3.1 Introducing Categories

A *category*, [8], consisits of a set of *objects* $\{A, B, C, \ldots\}$ and *morphisms* between the objects such that:

- For any morphism $\alpha : A \to B$ and $\beta : B \to C$, the morphism $\alpha\beta : A \to C$ exists, otherwise $\alpha\beta$ is not defined.

- Composition is associative: if $\alpha : A \to B$, $\beta : B \to C$, $\gamma : C \to D$ are morphisms, $(\alpha\beta)\gamma = \alpha(\beta\gamma)$.

- There is an identity morphism e_A associated to each object A such that for morphism $\alpha : A \to B$, $e_A\alpha = \alpha = \alpha e_B$.

Higgins, [8], discusses two types of categories, "Categories" and "categories". The first is considered as a way of studying the connection between structures. The objects are algebraic structures and the morphisms are functors between said structures. Examples include objects consisting of groups with morphisms being group homomorphisms, objects consisting metric spaces with morphisms being isometries and objects consisting of topological spaces with morphisms being continuous maps. The latter type of category is considered as an abstract algebraic structure in its own right. We can visualise this category by means of a directed graph. The objects are represented by vertices and the morphisms by arrows. Throughout this thesis we represent categories by graphs, often referring to a morphism of a category as an arrow.

If $\mathcal{C}$ and $\mathcal{D}$ are categories then a *functor* $F : \mathcal{C} \to \mathcal{D}$ is a structure preserving map such that if $\alpha : A \to B$ is a morphism in $\mathcal{C}$ then $(\alpha)F : (A)F \to (B)F$ is a morphism of $\mathcal{D}$. The functor preserves identities: if e_A is an identity of $\mathcal{C}$ then $(e_A)F$ is an identity of $\mathcal{D}$. Composition is also preserved: if α and β compose in $\mathcal{C}$ then $(\alpha)F$ and $(\beta)F$ compose in $\mathcal{D}$ and $(\alpha)F(\beta)F = (\alpha\beta)F$.

A *category-with-involution* is a category in which for every morphism $\alpha : A \to B$ there exists another morphism $\beta : B \to A$.

We give an example of a category.

Example 1.3.1. Take Γ to be a directed graph with vertex set $V = \{v_1, v_2, \dots\}$. Take p to be the directed path shown in Fig. 1.3.1. Denote

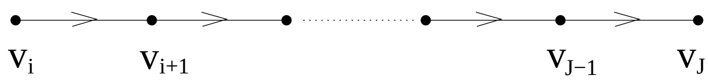

Figure 1.3.1: directed path.

this path by $(v_i, v_{i+1}, \dots, v_j)$. Let q be the path $(v_j, v_{j+1}, \dots, v_k)$. Then pq is the path $(v_i, v_{i+1}, \dots, v_j, \dots, v_k)$. Denote by P_{ij} *all* directed paths from v_i

to v_j. Denote by $P(\Gamma)$ the category whose objects are the vertices of Γ and whose morphisms are the sets of directed paths of the form P_{ij}. Composition is given by $P_{ij} \times P_{jk} = P_{ik}$ and is associative. The identities are the paths of zero length.

Consider a category $\mathcal{C}$ thought of as a graph. Then the *opposite category of* $\mathcal{C}$, $\mathcal{C}^{\mathrm{OP}}$, is the category represented by the anti-isomorphic graph of $\mathcal{C}$. So $\mathcal{C}^{\mathrm{OP}}$ consists of the same object set as $\mathcal{C}$ but $\hat{\alpha} : B \to A$ is a morphism of $\mathcal{C}^{\mathrm{OP}}$ only if $\alpha : A \to B$ is a morphism of $\mathcal{C}$. Composition in $\mathcal{C}^{\mathrm{OP}}$ is given by $\hat{\alpha}\hat{\beta} = \widehat{\beta\alpha}$. If $\mathcal{A}$ and $\mathcal{B}$ are categories then a functor from $\mathcal{A}^{\mathrm{OP}}$ to $\mathcal{B}$ is called a *contravariant* functor from $\mathcal{A}$ to $\mathcal{B}$. These functors are thought of as functors from $\mathcal{A}$ to $\mathcal{B}$ that reverse the direction of the arrows.

1.3.2 Equivalent Categories

Let $\mathcal{A}$, $\mathcal{B}$ be categories and let F_1, F_2 be functors from $\mathcal{A}$ to $\mathcal{B}$. A *natural transformation* τ [8], is a family of morphisms in $\mathcal{B}$ such that for each object a_1 in $\mathcal{A}$, there exists a morphism $\tau_{a_1} : (a_1)F_1 \to (a_1)F_2$ in $\mathcal{B}$ and for every morphism $\alpha : a_1 \to a_2$ in $\mathcal{A}$, the diagram in Fig.1.3.2 commutes. If τ is a

$$
\begin{tikzcd}
(a_1)F_1 \arrow[r, "(\alpha)F_1"] \arrow[d, "\tau_{a_1}"'] & (a_2)F_1 \arrow[d, "\tau_{a_2}"] \\
(a_1)F_2 \arrow[r, "(\alpha)F_2"'] & (a_2)F_2
\end{tikzcd}
$$

Figure 1.3.2: natural transformation.

family of isomorphisms in $\mathcal{B}$ then τ is called a *natural equivalence*. Further we denote this natural equivalence between F_1 and F_2 by $F_1 \simeq F_2$.

Suppose we are given two categories $\mathcal{A}$ and $\mathcal{B}$ and functors $F_A : \mathcal{A} \to \mathcal{B}$ and $F_B : \mathcal{B} \to \mathcal{A}$. Then categories $\mathcal{A}$ and $\mathcal{B}$ are *isomorphic* if $F_A F_B = 1_A$, the identity functor on $\mathcal{A}$, and $F_B F_A = 1_B$, the identity functor on $\mathcal{B}$.

Categories $\mathcal{A}$ and $\mathcal{B}$ are *equivalent* if $F_A F_B \simeq 1_A$ and $F_A F_B \simeq 1_B$. In detail then, categories $\mathcal{A}$ and $\mathcal{B}$ are *equivalent* if there is a natural equivalence τ such that for every morphism $\alpha : a_1 \to a_2$ in $\mathcal{A}$ the diagram shown in Fig. 1.3.3 commutes, and another natural equivalence $\tilde{\tau}$ such that for every morphism $\beta : b_1 \to b_2$ in $\mathcal{B}$, the diagram shown in Fig. 1.3.4 commutes.

$$\begin{tikzcd}
(a_1)F_A F_B \arrow[r, "(\alpha)F_A F_B"] \arrow[d, "\tau_{a_1}"'] & (a_2)F_A F_B \arrow[d, "\tau_{a_2}"] \\
(a_1)1_A \arrow[r, "(\alpha)1_A"'] & (a_2)1_A
\end{tikzcd}$$

Figure 1.3.3: equivalent categories.

$$\begin{tikzcd}
(b_1)F_B F_A \arrow[r, "(\beta)F_B F_A"] \arrow[d, "\tilde{\tau}_{b_1}"'] & (b_2)F_B F_A \arrow[d, "\tilde{\tau}_{b_2}"] \\
(b_1)1_B \arrow[r, "(\beta)1_B"'] & (b_2)1_B
\end{tikzcd}$$

Figure 1.3.4: equivalent categories.

1.3.3 Cones and Presheaves

Pullbacks

Let $\alpha : A \to C$ and $\beta : B \to C$ be morphims. A *cone over α and β* is a commutative square shown in Fig. 1.3.5. We call the object X the *vertex* of

$$\begin{tikzcd} X \arrow[r] \arrow[d] & B \arrow[d, "\beta"] \\ A \arrow[r, "\alpha"'] & C \end{tikzcd}$$

Figure 1.3.5: cone.

the cone. The *limiting cone over α and β* or *pullback* is the unique cone over α and β through which every other cone factors uniquely.

Cones and Limits

Define a *diagram* to be a set of objects with arrows between these objects. Given a diagram Γ a *cone* is a family of morphisms ρ from an object X to the vertices of Γ. Denote by ρ_i the *cone map* from X to the vertex d_i of Γ. If δ is an arrow of Γ from d_i to d_j then $\rho_i\delta = \rho_j$ for all arrows of Γ. See Fig, 1.3.6. We denote the cone ρ from X by (X, ρ).

A *cone limit* of Γ is a cone (Z, ϕ) of Γ such that for any other cone (X, ρ) of Γ there is a unique morphism $\psi : X \to Z$ such that for all cone maps ρ_i and ϕ_i we have that $\psi\phi_i = \rho_i$. See Fig. 1.3.7.

Dually we can define a *co-cone* (ρ, Y) of a diagram Γ as a family of morphisms ρ from Γ to Y such that, for arrow δ in Γ from d_i to d_j and cone maps ρ_i

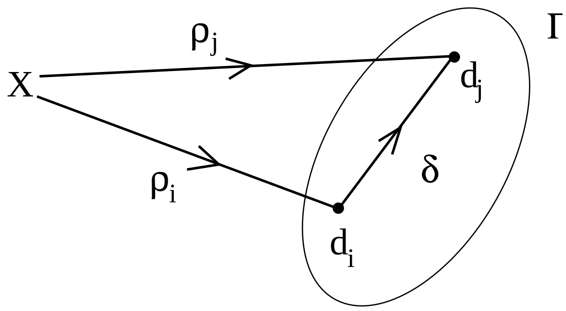

Figure 1.3.6: cone.

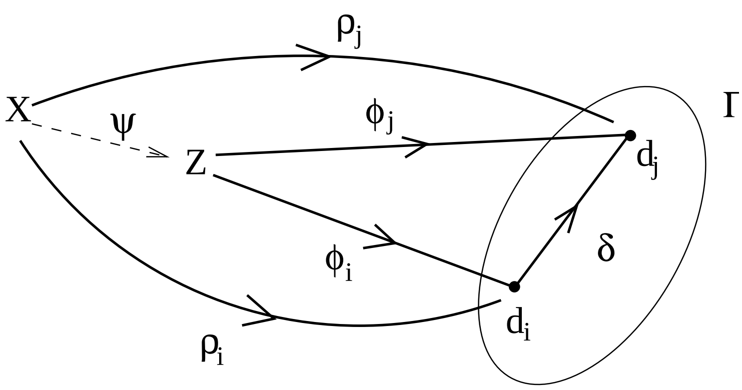

Figure 1.3.7: cone limit.

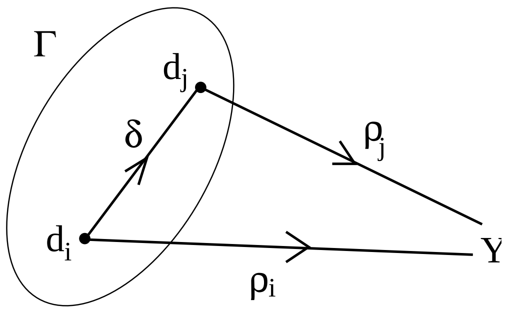

Figure 1.3.8: co-cone.

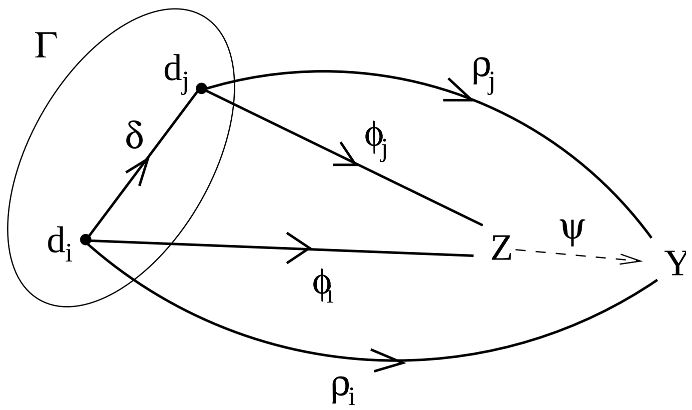

Figure 1.3.9: co-cone limit.

and ρ_j, $\rho_i = \delta\rho_j$. See Fig. 1.3.8.

Also a *co-cone limit*, or *co-limit*, is a unique co-cone (ϕ, Z) such that there exists a unique morphism $\psi : Z \to Y$ such that $\phi_i\psi = \rho_i$ for any co-cone (ρ, Y). See Fig. 1.3.9.

If our diagram Γ is indexed by directed sets the colimit can be called a *direct limit*.

Presheaves

Let E be a partially ordered set and take **Grph** to be the category of graphs. The objects of this category are graphs and the morphisms are graph maps. We consider E as a category with object set E and a unique morphism from e to f whenever $e \leqslant f$ in E. A *presheaf of graphs over* E is a contravariant functor from E to **Grph**.

1.4 Ordered Groupoids

Groupoids are categories in which every morphism is invertible and a groupoid with a partial ordering on the morphisms that conforms to certain axioms is an ordered groupoid. We describe these axioms and give properties of both groupoids and ordered groupoids. We also describe maps of ordered groupoids, namely immersions and coverings.

1.4.1 Groupoids

In this section we introduce an important type of category, the "groupoid". We will give a simple example of such a category and some basic defintions. Next we state the relationship between groupoids and groups. Following this we discuss "connected" groupoids before ending this subsection with a description of groupoid quotients.

Introducing Groupoids

A *groupoid* is a category in which for every morphism $\alpha : A \to B$ there exists an inverse morphism $\alpha^{-1} : B \to A$ with $\alpha\alpha^{-1} = e_A$ and $\alpha^{-1}\alpha = e_B$. Consider a morphism $\alpha : A \to B$ represented by an arrow of the associated graph from vertex A to B, then $\alpha^{-1} : B \to A$ is thought of as travelling backwards along the arrow α from vertex B to vertex A.

Example 1.4.1. The *simplicial* groupoid of the set X, has object set X. Denote by $\mathbb{I}^n$ the simplicial groupoid of the set with $n+1$ identities.

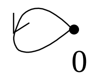

Figure 1.4.1: $\mathbb{I}^0$.

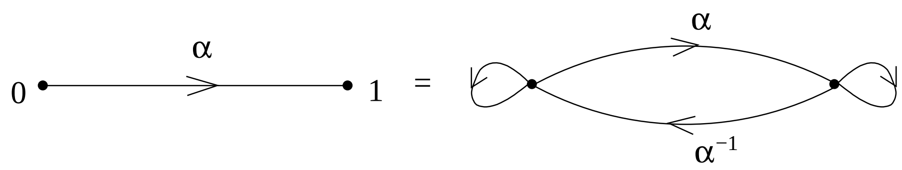

Figure 1.4.2: $\mathbb{I}^1$.

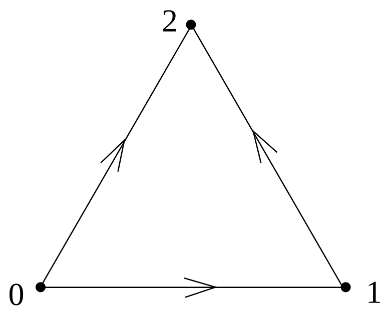

Figure 1.4.3: $\mathbb{I}^2$.

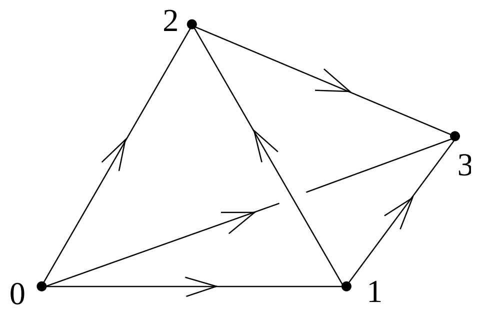

Figure 1.4.4: $\mathbb{I}^3$.

We denote the set of identities of the groupoid G by $E(G)$. If g is a morphism of the groupoid G from object e to object f then we write $g \in G(e, f)$. The *source map* maps a morphism $g \in G(e, f)$ to the object e and the *target map* maps g to the object f. We shall usually identify an object with the identity morphism at that object, and so identify the source and target maps with the domain and range maps given by $\mathbf{d}(g) = gg^{-1}$ and $\mathbf{r}(g) = g^{-1}g$. This corresponds with the usage for inverse semigroups, as in [10].

A *subgroupoid* of G consists of a subset I of the objects of G and a subset of the morphisms of G between the objects of I such that these subsets form a groupoid.

Proposition 1.4.2. *G is a group if and only if G is a groupoid with only one object.*

Connected Groupoids

A *connected* groupoid is one in which any two objects are connected by at least one morphism. Then the graph associated to this groupoid is connected.

Proposition 1.4.3. *Let A be a set and G be a group. The set $A \times G \times A$ is a groupoid with source map $(a, g, b) \mapsto a$ and target map $(a, g, b) \mapsto b$.*

Composition is

$$(a, g, b)(c, h, d) = (a, gh, d) \text{ provided } b = c.$$

The inverse of (a, g, b) *is* (b, g^{-1}, a) *and so the domain is* $\mathbf{d}(a, g, b) = (a, 1, a)$ *and the range is* $\mathbf{d}(a, g, b) = (b, 1, b)$.

The groupoid $A \times G \times A$ *is connected.*

Proof. The element $(b, 1, c)$ exists in $A \times G \times A$ for every two elements $(a, g, b), (c, h, d) \in A \times G \times A$ so the groupoid is connected. □

Consider a connected groupoid $\mathcal{G}$ with vertex set A. Let $u \in A$ and let G_u be the local group at u:

$$G_u = \{g \in \mathcal{G} : \mathbf{d}(g) = u = \mathbf{r}(g)\}.$$

Here $\mathbf{d}(x) = xx^{-1}$ and $\mathbf{r}(x) = x^{-1}x$ are the source and target maps respectively.

Proposition 1.4.4. *With the above notation,* $\mathcal{G}$ *is isomorphic to* $A \times G_u \times A$.

Proof. For each $a \in A$ choose a morphism α_a from u to a. Such a morphism will exist as $\mathcal{G}$ is connected. The required isomorphism is then $\theta : \mathcal{G} \to A \times G_u \times A$ with $g\theta = (\mathbf{d}(g), \alpha_{\mathbf{d}(g)} g {\alpha_{\mathbf{r}(g)}}^{-1}, \mathbf{r}(g))$. □

It makes no difference which u we choose to fix in A.

Proposition 1.4.5. *The local groups at any two vertices of a connected group are isomorphic.*

Proof. Let $u, v \in A$ and $g \in G_v$. Then the map $\psi : g \mapsto \alpha_v g {\alpha_v}^{-1}$ is an isomorphism between G_v and G_u. □

Groupoid Quotients and the Universal Groupoid

In a groupoid G we denote the set of all arrows from i to j by G_{ij}. A subgroupoid N of G is *normal* if

(i) N contains all the identities of G and

(ii) $x \in N_{ii}$ and $g \in G_{ij}$ implies $g^{-1}xg \in N_{jj}$.

We describe now groupoid quotients, [8]. Let N be a normal subgroupoid of groupoid G. The components of N define a partition on the objects of G. We denote the class containing object i as $[i]$ and the set of all classes by I. On the arrows of G, N defines an equivalence relation as follows: $a \equiv_m b$ if and only if $a = xby$ for some $x, y \in N$. Equivalent arrows of G must have their domains in the same component of N, similarly with their ranges. Hence each class $[a]$ of arrows can be assigned a unique domain and range in I and this assignment produces a graph G/N. The map $\lambda : G \to G/N$ is given by $g \mapsto [g]$ on the arrows of G and $i \mapsto [i]$ on the objects of G and is surjective.

Composition on the edges of G/N is as follows: $[a][b]$ is defined provided there exists $a_1 \in [a]$, $b_1 \in [b]$ such that a_1b_1 is defined in G, in which case $[a][b] = [a_1b_1]$. To show this composition is well-defined suppose $a_2 \in [a]$, $b_2 \in [b]$ and a_2b_2 is defined in G. Then $a_2 = xa_1y$, $b_2 = pbq$ with $x, y, p, q \in N$ and $a_2b_2 = xa_1ypb_1q \in G$. We know a_1b_1 is defined in G so $\mathbf{d}(yp) = \mathbf{r}(yp) = \mathbf{d}(b_1)$ and so $w = b_1^{-1}ypb_1$ is defined and lies in N. Hence $a_2b_2 = xa_1b_1wq \equiv_m a_1b_1$. We note now that if $[a]$ has domain $[i]$ and range $[j]$ and $[b]$ has domain $[k]$ and range $[l]$ then $[a][b]$ is defined if and only if axb is defined for some $x \in N$, i.e. if and only if $[j] = [k]$. So $[a][b] = [axb]$ has domain $[i]$ and range $[l]$. Moreover, if $([a][b])[c]$ is defined then $axbyc$ is defined for some $x, y \in N$. So $[a]([b][c])$ is defined and equals $([a][b])[c]$. Thus composition is associative when defined.

We now have a category G/N and as $\lambda : a^{-1} \mapsto [a^{-1}] = [a]^{-1}$ we see G/N is a groupoid. G/N is called the *quotient groupoid* and $\lambda : G \to G/N$ the *quotient map of groupoids*.

Note that if $n \in N$ then $\mathbf{r}(n) \in N$ and $n = n\mathbf{r}(n)\mathbf{r}(n)$ so $n \equiv_m \mathbf{r}(n)$ and so $[n]$ is an identity of G/N.

Given a groupoid G and a function $\sigma : E(G) \to V$ Higgins [8] constructs the "universal groupoid" $U_\sigma(G)$ as follows. We define a graph G^σ with vertex set V and edge set the non-identity arrows of G. Domain and range of arrows g are $(\mathbf{d}(g))\sigma$ and $(\mathbf{r}(g))\sigma$ respectively. In G^σ let $p = g_1 g_2 \ldots g_n$ be a path of length $n > 0$. In our path p we can replace $g_j g_{j+1}$ by a if $a = g_j g_{j+1}$ in G. We can also delete g_j if it is an identity of G. These two modifications of p are *elementary reductions*. The elementary reductions generate an equivalence relation $\simeq$. Modulo this equivalence relation the path category $P(G^\sigma)$ becomes a groupoid $U_\sigma(G) = P(G^\sigma)/\simeq$. We call this groupoid the *universal groupoid*.

1.4.2 Ordered and Inductive Groupoids

Definitions and Properties

Let G be a groupoid and let $\leqslant$ be a partial order on G. Then $(G, \leqslant)$ is an *ordered groupoid*, [10], if the following axioms hold:

(OG1) For all $g, h \in G$, $g \leqslant h$ implies $g^{-1} \leqslant h^{-1}$.

(OG2) For $g, h, u, v \in G$ such that gu and hv exist, if $g \leqslant h$ and $u \leqslant v$ then $gu \leqslant hv$.

(OG3) Let $g \in G$ and e be an identity of G such that $e \leqslant \mathbf{d}(g)$. Then there exists a unique element $(e|g)$, called the *restriction of g to e*, such that

$(e|g) \leqslant g$ and $\mathbf{d}(e|g) = e$.

Proposition 1.4.6. *If $(G, \leqslant)$ is an ordered groupoid then for $g \in G$ and $e \in E(G)$ such that $e \leqslant \mathbf{r}(g)$. Then there exists a unique element $(g|e)$, called the* co-restriction of g to e *, such that $(g|e) \leqslant g$ and $\mathbf{r}(g|e) = e$.*

Proof [10]. Suppose $(G, \leqslant)$ is an ordered groupoid so axioms (OG1) and (OG3) hold. Let $e \leqslant \mathbf{r}(g)$, then $e \leqslant \mathbf{d}(g^{-1})$ so, by axiom (OG3), the restriction $(e|g^{-1})$ exists. Define $(g|e) = (e|g^{-1})^{-1}$. Then $(e|g^{-1}) \leqslant g^{-1}$ so $(g|e) \leqslant g$ by axiom (OG1). Also, $\mathbf{r}(g|e) = \mathbf{d}(e|g^{-1}) = e$. We check uniqeness. Suppose $y \leqslant g$ and $\mathbf{r}(y) = e$. Then $y^{-1} \leqslant g^{-1}$ by axiom (OG1). Also $\mathbf{d}(y^{-1}) = e$. Thus by uniqueness of restriction $y^{-1} = (e|g^{-1})$ and so $y = (g|e)$ by (OG1). $\square$

We introduce some properties of ordered groupoids, especially the restriction operation, the proofs of which can be found in [10].

Proposition 1.4.7. *[10]. Let $(G, \leqslant)$ be an ordered groupoid. Suppose $x, y, z \in G$.*

1. *If $x \leqslant y$ then $\mathbf{d}(x) \leqslant \mathbf{d}(y)$ and $\mathbf{r}(x) \leqslant \mathbf{r}(y)$.*
2. *If $x \leqslant y$, $\mathbf{d}(x) = \mathbf{d}(y)$ and $\mathbf{r}(x) = \mathbf{r}(y)$ then $x = y$.*

Proposition 1.4.8. *[10]. Let $(G, \leqslant)$ be an ordered groupoid. If the product xy exists in G and $e \in E(G)$ is such that $e \leqslant \mathbf{d}(xy)$ then the restriction of xy to e equals $(e|x)(\mathbf{r}(e|x)|y)$. A similar result holds for the corestriction of a product: $(xy|e) = (x|\mathbf{d}(y|e))(y|e)$.*

Further, if $z \leqslant xy$ then there exists elements x' and y' such that the product $x'y'$ exists and $x' \leqslant x$, $y \leqslant y$ and $z = x'y'$.

Proposition 1.4.9. *[10]. Let $(G, \leqslant)$ be an ordered groupoid. Suppose $x, y \in G$ and $e, f \in E(G)$.*

1. *If* $f \leqslant e \leqslant \mathbf{d}(x)$ *then* $(f|x) \leqslant (e|x) \leqslant x$. *A similar result holds for corestriction.*

2. *If* $x \leqslant y$ *and* $f \leqslant e$ *with* $f \leqslant \mathbf{d}(x)$ *and* $e \leqslant \mathbf{d}(y)$ *then* $(f|x) \leqslant (e|y)$.

Proposition 1.4.10. *[10]. Let* $(G, \leqslant)$ *be an ordered groupid. Then* $E(G)$ *is an order ideal of* G.

An ordered groupoid is *inductive* if the partially ordered set of identities forms a meet-semilattice. A functor between ordered groupoids that is order-preserving is called an *ordered functor*. An ordered functor between inductive groupoids that preserves the meet operation on the set of identities is called *inductive*.

Proposition 1.4.11. *[10]. Let* G *and* H *be ordered groupoids and let* $\theta : G \to H$ *be an ordered functor. If* $(e|x)$ *is defined in* G *then* $(e\theta|x\theta)$ *is defined in* H *and* $(e|x)\theta = (e\theta|x\theta)$. *A similar result holds for corestriction.*

Example 1.4.12. Denote by $\mathbb{I}$ the ordered groupoid that consists of two copies of the interval groupoid $\mathbb{I}^1$ as shown in Fig. 1.4.5. Ordering is given

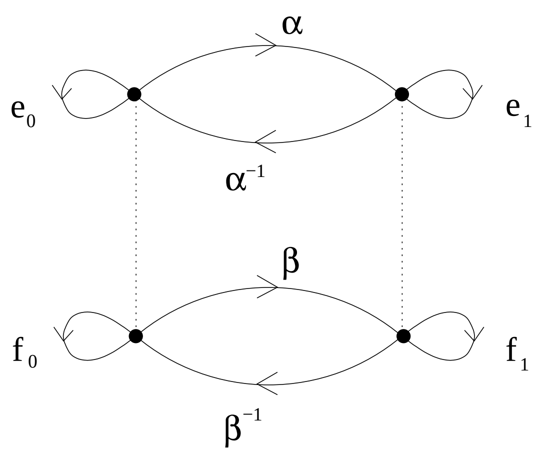

Figure 1.4.5: groupoid $\mathbb{I}$.

by $f_0 \leqslant e_0$, $f_1 \leqslant e_1$, $\beta \leqslant \alpha$ and $\beta^{-1} \leqslant \alpha^{-1}$. In the graph we represent the ordering by a dotted line. Ordered groupoid $\mathbb{I}$ is not inductive for identities e_0 and e_1 have no greatest lower bound.

Let $(G, \leqslant)$ be an ordered groupoid and let $x, y \in G$. Suppose that $\mathbf{r}(x)$ and $\mathbf{d}(y)$ have a greatest lower bound $e = \mathbf{r}(x) \wedge \mathbf{d}(y)$. Then the *pseudoproduct* of x and y is defined as follows:

$$x * y = (x|e)(e|y).$$

The pseudoproduct in an inductive groupoid is everywhere defined because the identites of an inductive groupoid form a meet semilattice.

Inductors

Given a groupoid G we can define an ordered groupoid structure on G as follows:

- Define a partial order on the identities of G.
- Define a restriction operation: for $g \in G$ and $e \in E(G)$ with $e \leqslant \mathbf{d}(g)$ define a unique arrow $\bar{g}$ with $\mathbf{d}(\bar{g}) = e$.
- Define a relation on the arrows of G by

$$g \leqslant h \Leftrightarrow \mathbf{d}(g) \leqslant \mathbf{d}(h) \text{ and } g = \bar{h}.$$

- Check that $\leqslant$ is a partial order and that $(OG1)$ and $(OG2)$ hold.

If, in addition, the set of identites of G forms a meet semilattice we have defined an inductive groupoid. We then call the restriction operation an *inductor*.

Stars and Coverings

Let G be a groupoid and let $e \in E(G)$. Define the *star of* e *in* G to be the set

$$\mathrm{star}_G(e) = \{g \in G : \mathbf{d}(g) = e\}.$$

Similarly we can define the *costar of* e *in* G to be the set

$$\mathrm{costar}_G(e) = \{g \in G : \mathbf{r}(g) = e\}.$$

Take G and H to be groupoids and let $F : G \to H$ be a functor. If, for each $e \in E(G)$, the restriction $F : \mathrm{star}_G(e) \to \mathrm{star}_H(eF)$ is injective then the functor F is called *star injective* or an *immersion*. Similarly we can define *star surjective* and *star bijective*. A star bijection is called a *covering*.

An *ordered covering* is an order preserving covering between ordered groupoids. Given ordered groupoids A, B, C and D, two ordered coverings $\alpha : A \to B$ and $\beta : C \to D$ are isomorphic if there are two isomorphisms $\phi_1 : A \to C$ and $\phi_2 : B \to D$ such that $\alpha\phi_2 = \phi_1\beta$.

Green's Relation on Groupoids

The Green's relations on ordered groupoid G are as follows:

$$a\mathcal{R}b \Leftrightarrow \mathbf{d}(a) = \mathbf{d}(b)$$

$$a\mathcal{L}b \Leftrightarrow \mathbf{r}(a) = \mathbf{r}(b)$$

$$\mathcal{H} = \mathcal{R} \cap \mathcal{L}$$

$$a\mathcal{D}b \Leftrightarrow a \text{ is connected to } b$$

$$a\mathcal{J}b \Leftrightarrow a\mathcal{D}b' \leqslant b \text{ and } b\mathcal{D}a' \leqslant a \text{ for some } a', b' \in G$$

So the $\mathcal{D}$-classes of a groupoid are the connected components of G. An ordered groupoid is then *bisimple* if G is connected.

Proposition 1.4.13. *[10]. Let G be an ordered groupoid.*

1. *Let $e, f \in E(G)$. Then $e\mathcal{D}f$ if and only if there is an element $a \in G$ such that $aa^{-1} = e$ and $a^{-1}a = f$.*

2. *Let $s, t \in G$. Then $s\mathcal{D}t$ if and only if there exists elements $a, b \in G$ such that $\mathbf{d}(a) = \mathbf{r}(s)$, $\mathbf{r}(a) = \mathbf{r}(t)$, $\mathbf{d}(b) = \mathbf{d}(s)$, $\mathbf{r}(b) = \mathbf{d}(t)$ and $s = bta^{-1}$.*

3. *If $s = a_1 a_2 \ldots a_n$ then $s\mathcal{D}a_i$ for each $i = 1, 2, \ldots, n$.*

Lemma 1.4.14. *[10]. Let G be an ordered groupoid. If $x \leqslant y\mathcal{D}v$ then there exists $u \in G$ such that $u \leqslant v$ and $u\mathcal{D}x$.*

1.5 Semigroups and Groupoids

In this final section of chapter one we explain the relation between inverse semigroups and inductive groupoids. By means of the 'restricted product' we can convert an inverse semigroup into an inductive groupoid. Further any inductive groupoid can be converted into an inverse semigroup via the pseudoproduct. It transpires that the category of inverse semigroups and homomorphisms is isomorphic to the category of inductive groupoids and inductive functors. This result is attributed in [10] to Ehresmann, Schein and Nambooripad. We use the Ehresmann-Schein-Nambooripad theorem to convert inverse semigroup concepts into inductive groupoid terminology. Once ideas have been transferred into inductive groupoid terms we may try to generalise these ideas for ordered groupoids. This is the way most of the structure theorems in this thesis have arisen.

Let S be an inverse semigroup. If $s, t \in S$ then the *restricted product* $s \cdot t$ exists if $s^{-1}s = tt^{-1}$ in which case $s \cdot t = st$. In other words, the restricted product exists if $\mathbf{r}(s) = \mathbf{d}(t)$.

Proposition 1.5.1. *[10] Every inverse semigroup S is an inductive groupoid with respect to the restricted product $\cdot$ and the natural partial order $\leqslant$.*

This groupoid is denoted $\mathbb{G}(S)$ and is called the *associated groupoid* of S. The inductive groupoid $\mathbb{G}(S)$ is constructed as follows. The object set of $\mathbb{G}(S)$ is $E(G)$. Each element $s \in S$ is a morphism of $\mathbb{G}(S)$ from ss^{-1} to $s^{-1}s$. If $s, t \in S$ then $s \cdot t$ is defined in $\mathbb{G}(S)$ only if $s^{-1}s = tt^{-1}$, i.e. only if the arrows match up, in which case $s \cdot t = st$. The ordering comes from the natural partial order on S.

Proposition 1.5.2. *[10]. Every inductive groupoid G is an inverse semigroup $\mathbb{S}(G)$ with respect to the pseudoproduct.*

Theorem 1.5.3. *[10]. Let G be an inductive groupoid and let S be an inverse semigroup.*

1. $\mathbb{G}(\mathbb{S}(G)) = G$.
2. $\mathbb{S}(\mathbb{G}(S)) = S$.

We now give the Ehresmann-Schein-Nambooripad theorem, referring to [10] for the details of the proof.

Theorem 1.5.4. *The category of inverse semigroups and homomorphisms is isomorphic to the category of inductive groupoids and inductive functors.*

This result means that any property of an inverse semigroup may be interpreted in terms of inductive groupoids and vice-versa. So any property that holds in an inverse semigroup can be transferred into inductive groupoid terms for $\mathbb{G}(S)$ and this then gives us the possibility of generalising this property to ordered groupoids. We focus on this idea throught the thesis and in doing so construct some interesting structure theorems for ordered groupoids.

Chapter 2

A Classification of Bisimple Inductive ω-Groupoids

In this chapter we consider Reilly's classification of bisimple inverse ω-semigroups using Bruck-Reilly extensions of a group. Beginning with a bisimple inverse ω-semigroup S, we construct its associated inductive groupoid $\mathbb{G}(S)$. By classifying the inductive groupoid structures that can arise, we show that each one corresponds (up to isomorphism) with an inductive groupoid obtained from a Bruck-Reilly extension.

2.1 A Structure Theorem for Bisimple Inverse ω-Semigroups

In this section we describe a Bruck-Reilly extension for a group. We then state the theorem that classifies bisimple inverse ω-semigroups as given in [9, section 5.6]. This classification is based on ideas by Bruck [3], Reilly [23] and Munn [19].

Proposition 2.1.1. *Let G be a group and let $\theta : G \to G$ be a homomorphism. Then the set $\mathbb{N} \times G \times \mathbb{N}$, with compostion*

$$(m, a, n)(p, b, q) = (m - n + t, (a\theta^{t-n})(b\theta^{t-p}), q - p + t)$$

where $t = \max\{n, p\}$ and $\theta^0$is the identity map of G, is a semigroup.

The semigroup $\mathbb{N} \times G \times \mathbb{N}$ described above is called the *Bruck-Reilly extension of G determined by θ* and is denoted by $\mathrm{BR}(G, \theta)$.

The *ω-ordering* on $\mathbb{N}$ is the ordering that reverses the natural order, i.e.

$$0 \geqslant 1 \geqslant 2 \geqslant 3 \geqslant \dots .$$

An *inverse ω-semigroup* is a semigroup whose semilattice of idempotents is isomorphic to $\mathbb{N}$ with the ω-ordering.

The following theorem is due to Reilly [23], see also [9]. Recall that an inverse semigroup is bisimple if it has a single $\mathcal{D}$-class.

Theorem 2.1.2. *Let G be a group and $\theta : G \to G$ an endomorphism. Then $\mathrm{BR}(G, \theta)$ is a bisimple inverse ω-semigroup.*

Proof. By the previous proposition $\mathbb{N} \times G \times \mathbb{N}$ is a semigroup. The following calculation shows that $\mathrm{BR}(G, \theta)$ is a regular semigroup:

$$\begin{aligned}
&(m,a,n)(n,a^{-1},m)(m,a,n)\\
&\qquad\qquad = (m-n+t,(a\theta^{t-n})(a^{-1}\theta^{t-n}),m-n+t)(m,a,n)\\
&\qquad\qquad = (m-n+n,,(a\theta^{n-n})(a^{-1}\theta^{n-n}),m-n+n)(m,a,n)\\
&\qquad\qquad = (m,aa^{-1},m)(m,a,n)\\
&\qquad\qquad = ((m-m+\bar{t},(aa^{-1}\theta^{\bar{t}-m})(a\theta^{\bar{t}-m}),n-m+\bar{t})\\
&\qquad\qquad = (m-m+m,(aa^{-1})\theta^{m-m}(a\theta^{m-m}),n-m+m)\\
&\qquad\qquad = (m,aa^{-1}a,n)\\
&\qquad\qquad = (m,a,n).
\end{aligned}$$

If $(m,a,n)(m,a,n) = (m,a,n)$, then for $t = \max\{m,n\}$,

$$(m-n+t,(a\theta^{t-n})(a\theta^{t-m}),n-m+t) = (m,a,n).$$

So $m = m-n+t$ implies that $n = t$ and $n = n-m+t$ implies that $m = t$, and so $n = m$. Then $(a\theta^{t-n})(a\theta^{t-m}) = a$ becomes $a\theta^0 a\theta^0 = a^2 = a$. As G is a group we deduce that $a = 1_G$, the identity element of G. Therefore the idempotents of $\mathrm{BR}(G,\theta)$ have the form $(m,1_G,m)$.

We show that idempotents commute: let $t = \max\{m,n\}$,

$$\begin{aligned}
(m,1_G,m)(n,1_G,n) &= (m-m+t,(1_G\theta^{t-m})(1_G\theta^{t-n}),n-n+t)\\
&= (t,1_G,t)\\
&= (n-n+t,(1_G\theta^{t-n})(1_G\theta^{t-m}),m-m+t)\\
&= (n,1_G,n)(m,1_G,m).
\end{aligned}$$

Therefore $\mathrm{BR}(G,\theta)$ is an inverse semigroup.

The natural partial order on the idempotents is now given by

$$(m, 1_G, m) \leqslant (n, 1_G, n) \Leftrightarrow (m, 1_G, m) = (m, 1_G, m)(n, 1_G, n).$$

So $(m, 1_G, m) = (t, 1_G, t)$ where $t = \max\{m, n\}$ thus $m = \max\{m, n\}$ and so $m \geqslant n$ in $\mathbb{N}$. Therefore $(m, 1_G, m) \leqslant (n, 1_G, n)$ in the natural partial order if and only if $m \geqslant n$, i.e.

$$(0, 1_G, 0) \geqslant (1, 1_G, 1) \geqslant (2, 1_G, 2) \geqslant (3, 1_G, 3) \geqslant \dots.$$

This is the ω ordering so $\mathrm{BR}(G, \theta)$ is an inverse ω-semigroup.

To show that $\mathrm{BR}(G, \theta)$ is bisimple, let $(m, a, n), (p, b, q) \in \mathrm{BR}(G, \theta)$. Since $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$ we want an $(s, c, l) \in \mathbb{N} \times G \times \mathbb{N}$ such that $(m, a, n)\mathcal{R}(s, c, l)\mathcal{L}(p, b, q)$. Now, for $t = \max\{n, n\} = n$,

$$\begin{aligned}
(m, a, n)(m, a, n)^{-1} &= (m, a, n)(n, a^{-1}, m) \\
&= (m - n + t, (a\theta^{t-n})(a^{-1}\theta^{t-n}), m - n + t) \\
&= (m - n + n, (a\theta^{n-n})(a^{-1}\theta^{n-n}), m - n + n) \\
&= (m, aa^{-1}, m) \\
&= (m, 1_G, m)
\end{aligned}$$

and, for $\bar{t} = \max\{m, m\} = m$

$$\begin{aligned}
(m, a, n)^{-1}(m, a, n) &= (n, a^{-1}, m)(m, a, n) \\
&= (mn - m + \bar{t}, (a^{-1}\theta^{\bar{t}-m})(a\theta^{\bar{t}-m}), n - m + \bar{t}) \\
&= (n - m + m, (a^{-1}\theta^{m-m})(a\theta^{m-m}), n - m + m) \\
&= (n, a^{-1}a, n) \\
&= (n, 1_G, n)
\end{aligned}$$

So we require an $(s,c,l) \in \mathrm{BR}(G,\theta)$ such that $(m,1_G,m) = (s,1_G,s)$ and $(l,1_G,l) = (q,1_G,q)$. Now $(m,1_G,q)$ is such an element. An element such as this exists for every $(m,a,n),(p,b,q) \in \mathrm{BR}(G,\theta)$. Therefore $(m,a,n)\mathcal{D}(p,b,q)$ for every two elements in $\mathrm{BR}(G,\theta)$, and so there exists only one $\mathcal{D}$-class. Thus $\mathrm{BR}(G,\theta)$ is bisimple. □

Lemma 2.1.3. *The natural partial order on* $\mathrm{BR}(G,\theta)$ *is given by*

$$(m,a,n) \leqslant (p,b,q) \text{ if and only if } m-n = p-q \text{ and } a = b\theta^{n-q}.$$

Proof. In $\mathrm{BR}(G,\theta)$, $(m,a,n) \leqslant (p,b,q)$ if and only if $(m,a,n) = (p,b,q)(r,1_G,r)$ for some $(r,1_G,r) \in E(\mathrm{BR}(G,\theta))$ and

$$(p,b,q)(r,1_G,r) = (p-q+t,(b\theta^{t-q})(1_G\theta^{t-r}),r-r+t)$$

where $t = \max\{q,r\}$. Thus $m = p-q+t$ and $r-r+t = n$ so $t = n$ and $m = p-q+n$ which implies $m-n = p-q$. Also, as θ is a homomorphism $a = (b\theta^{n-q})(1_G\theta^{n-r}) = (b\theta^{n-q})1_G = b\theta^{n-q}$. □

We can now classify bisimple inverse ω-semigroups as follows.

Theorem 2.1.4. *Every bisimple inverse ω-semigroup is isomorphic to some Bruck-Reilly extension of a group G determined by an endomorphism of G.*

This classification of bisimple inverse ω-semigroups using Bruck-Reilly extensions is due to Reilly and is given in [9, pg 174].

2.2 Classifying Inductors for Groupoid Corresponding to $\mathrm{BR}(G,\theta)$

The aim of this subsection is to determine the structure of the inductive groupoids that correspond to bisimple inverse ω-semigroups. We construct *all* possible inductors on an ω–ordered, bisimple groupoid and so classify all possible bisimple inductive ω-groupoids.

Let S be a bisimple inverse ω-semigroup. The groupoid $\mathcal{G}$ associated to S is given by theorem 1.4.3 and proposition 1.4.4 where the set A is taken to be $\mathbb{N}$.

Theorem 2.2.1. *Given a group G then $\mathcal{G} = \mathbb{N} \times G \times \mathbb{N}$, with composition $(a, g, b)(c, h, d) = (a, gh, d)$ if and only if $b = c$, is a groupoid.*

The identities of $\mathcal{G}$ are of the form $(a, 1, a)$ and we have the ω ordering on the identities of $\mathcal{G}$, so

$$(0, 1, 0) \geqslant (1, 1, 1) \geqslant (2, 1, 2) \geqslant (3, 1, 3) \geqslant \dots .$$

Note that the ω-ordering on $\mathcal{G}$ gives us a semilattice structure on the identities of $\mathcal{G}$. As an element $(m, g, n) \in \mathcal{G}$ has domain $(m, 1, m)$ which is completely determined by $m \in \mathbb{N}$, we say $\mathbf{d}(m, g, n) = m$. Similarly $\mathbf{r}(m, g, n) = n$.

Recall that the element $(b, 1, c)$ exists in $\mathcal{G}$ for every two elements $(a, g, b), (c, h, d) \in \mathcal{G}$ so the groupoid is connected.

We now have a connected groupoid $\mathcal{G} = \mathbb{N} \times G \times \mathbb{N}$ with the ω-ordering on the identities of $\mathcal{G}$ and we wish to classify all possible inductors on $\mathcal{G}$. We denote the restriction of (m, g, n) to $(r, 1, r)$ for some $(r, 1, r) \leqslant (m, g, m)$ in the ordering of $\mathcal{G}$ (i.e. for some $r \geqslant m$) by $(r|(m, g, n))$. We consider initially the $\mathbb{N}$ components of $\mathcal{G}$ with respect to inductors.

Lemma 2.2.2. *For $r \geqslant m$, the restriction of (m, g, n) in $\mathcal{G}$ to r has the form*

$$(r|(m, g, n)) = (r, g', r + n - m)$$

for some $g' \in G$.

Proof. First we show that $(m+1|(m,g,m+1)) = (m+1,h,m+2)$ where h is some element of G which is yet to be determined.

Let $\gamma = (m,g,m+1)$, $\alpha = (m+1|\gamma)$ and $\mathbf{r}(\alpha) = p$. Suppose $p \neq m+2$ and assume $p > m+2$. Let $\beta = (\gamma|m+2)$ and $\mathbf{d}(\beta) = q$. Then $\beta = (q|\gamma)$. (So we have that $q \geqslant m$.) See Fig. 2.2.1. If $q = m$ then by uniqueness of restriction

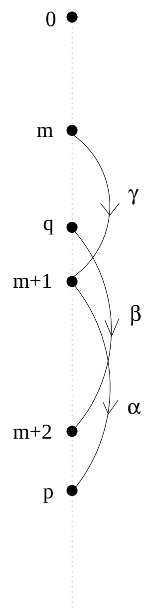

Figure 2.2.1: restriction.

$\beta = (m|\gamma) = \gamma$ but by assumption $\mathbf{r}(\beta) = m+2$ and $\mathbf{r}(\gamma) = m+1$, so β can not equal γ, and so $q \neq m$. If instead $q = m+1$ then $\beta = (m+1|\gamma) = \alpha$ but $\mathbf{r}(\beta) = m+2$ and $\mathbf{r}(\alpha) = p \neq m+2$, so β can not equal α, and so $q \neq m+1$. Hence $q \geqslant m+2$. See Fig. 2.2.2. Then $\mathbf{d}(\beta) = q$ and $q \geqslant m+2$, $\mathbf{d}(\alpha) = m+1$ and $\mathbf{d}(\gamma) = m$. Also $\beta = (q|\gamma)$ and $\alpha = (m+1|\gamma)$ so $\mathbf{d}(\beta) < \mathbf{d}(\alpha) < \mathbf{d}(\gamma)$, hence $\beta < \alpha < \gamma$. This implies $\mathbf{r}(\beta) < \mathbf{r}(\alpha)$. So $p \leqslant m+2$. We reach a contradiction, therefore $p = m+2$. See Fig. 2.2.3. So for some $h \in G$ yet to be determined

$$(m+1|(m,g,m+1)) = (m+1,h,m+2) \tag{2.2.1}$$

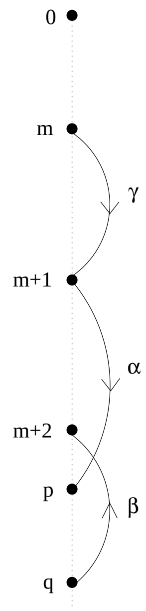

Figure 2.2.2: restriction.

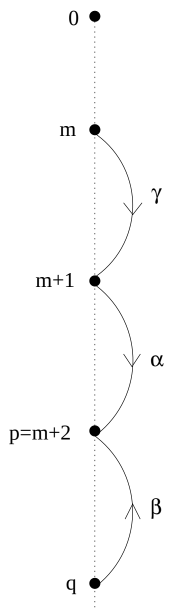

Figure 2.2.3: restriction

We now find $(r|(m,g,m+1))$ for any given r by induction. Equation (2.2.1) gives us our basis step. We then assume for some $r \in \mathbb{N}$,

$$(r|(m,g,m+1) = (r,g',r+1).$$

Then by the assumption we have

$$\begin{aligned}(r+1|(m,g,m+1)) &= (r+1|(r|(m,g,m+1))) \\ &= (r+1|(r,g',r+1))\end{aligned}$$

By equation (2.2.1) we get

$$\begin{aligned}(r+1|(m,g,m+1)) &= (r+1|(r,g',r+1)) \\ &= (r+1,\tilde{h},r+2).\end{aligned}$$

Hence for any $r \in \mathbb{N}$ we have

$$(r|(m,g,m+1)) = (r,g',r+1) \tag{2.2.2}$$

We now determine $(r|(m,g,n))$ by induction on n. By equation (2.2.2), when $n = m+1$

$$\begin{aligned}(r|(m,g,n)) &= (r,g',r+1) \\ &= (r,g',r+(m+1)-m)\end{aligned}$$

and the basis step is complete. Assume then that for some $n \geqslant m+1$ $(r|(m,g,n)) = (r,g',r+n-m)$. Then by assumption and equation (2.2.2) we have

$$\begin{aligned}(r|(m,g,n+1)) &= (r|(m,g,n)(n,1,n+1))\\ &= (r,g',r+n-m)(r+n-m|(n,l,n+1))\\ &= (r,g',r+n-m)(r+n-m,l,r+n-m+1)\\ &= (r,g'l,r+n-m+1)\\ &= (r,k,r+(n+1)-m)\end{aligned}$$

Therefore for all $r \geqslant m$

$$(r|(m,g,n)) = (r,g',r+n-m)$$

for some $g' \in G$. $\square$

This complete, we now require the G components of $\mathcal{G} = \mathbb{N} \times G \times \mathbb{N}$. We begin with restricting (m,g,m) and $(m,1,m+1)$ to $m+1$. Let

$$(m+1|(m,g,m)) = (m+1,g\theta_m,m+1) \tag{2.2.3}$$

and

$$(m+1|(m,1,m+1)) = (m+1,a_{m+1},m+2). \tag{2.2.4}$$

Since

$$\begin{aligned}&(m+1|(m,g,m)(m,h,m))\\ &= (m+1,g\theta_m,m+1)(m+1,h\theta_m,m+1)\\ &= (m+1,g\theta_m h\theta_m,m+1)\end{aligned}$$

and

$$\begin{aligned}(m+1|(m,g,m)(m,h,m)) &= (m+1|(m,gh,m))\\ &= (m+1,(gh)\theta_m,m+1)\end{aligned}$$

then

$$(gh)\theta_m = g\theta_m h\theta_m.$$

Therefore $\theta_m : G \to G$ is a homomorphism of groups.

Now $(m,g,m+1) = (m,g,m)(m,1,m+1) = (m,1,m+1)(m+1,g,m+1)$ and so we can restrict $(m,g,m+1)$ to $m+1$ in two possible ways;

$$\begin{aligned}(m+1|(m,g,m)(m,1,m+1)) &= (m+1,g\theta_m,m+1)(m+1,a_{m+1},m+2)\\ &= (m+1,g\theta_m a_{m+1},m+2)\end{aligned}$$

and

$$\begin{aligned}&(m+1|(m,1,m+1)(m+1,g,m+1))\\ &=(m+1,a_{m+1},m+2)(m+2,g\theta_{m+1},m+2)\\ &=(m+1,a_{m+1}g\theta_{m+1},m+2).\end{aligned}$$

Thus for all m,

$$g\theta_m a_{m+1} = a_{m+1}(g\theta_{m+1}).$$

Rewriting gives us

$$g\theta_{m+1} = a_{m+1}^{-1}(g\theta_m)a_{m+1}. \tag{2.2.5}$$

Hence, writing $\theta = \theta_0$, we have

$$g\theta_m = a_m^{-1}\dots a_1^{-1}(g\theta)a_1\dots a_m. \tag{2.2.6}$$

Set $u_m = a_1 \dots a_m$ with $u_0 = 1$ and

$$w_{m,k} = u_m\theta^k u_{m+1}\theta^{k-1} \dots u_{m+k-1}\theta^1 u_{m+k}. \qquad (2.2.7)$$

Then

$$\begin{aligned}(w_{m,k-1})\theta u_{m+k} &= (u_m\theta^{k-1}u_{m+1}\theta^{k-2} \dots u_{m+k-2}\theta^1 u_{m+k-1})\theta u_{m+k} \\ &= (u_m\theta^{k-1})\theta(u_{m+1}\theta^{k-2})\theta \dots (u_{m+k-2}\theta)\theta(u_{m+k-1})\theta u_{m+k} \\ &= u_m\theta^k u_{m+1}\theta^{k-1} \dots u_{m+k-2}\theta^2 u_{m+k-1}\theta^1 u_{m+k} \\ &= w_{m,k}\end{aligned}$$

so we have

$$w_{m,k} = (w_{m,k-1})\theta u_{m+k}. \qquad (2.2.8)$$

Note that $w_{m,0} = u_m$ and equation (2.2.6) then becomes

$$g\theta_m = u_m^{-1}(g\theta)u_m. \qquad (2.2.9)$$

Lemma 2.2.3. *For $r > m$,*

$$(r|(m, g, m)) = (r, w_{m,r-m-1}^{-1}(g\theta^{r-m})w_{m,r-m-1}, r).$$

Proof. We prove the lemma by induction on $r - m$. If $r - m = 1$ (i.e. $r = m + 1$) then

$$\begin{aligned}&(m + 1|(m, g, m)) \\ &= (m + 1, g\theta_m, m + 1) && \text{by equation (2.2.3)} \\ &= (m + 1, u_m^{-1}(g\theta)u_m, m + 1) && \text{by equation (2.2.9)} \\ &= (m + 1, w_{m,0}^{-1}(g\theta)w_{m,0}, m + 1) && \text{as } w_{m,0} = u_m\end{aligned}$$

which confirms the basis case.

Assume for all $r - m = k$ that

$$(m+k|(m,g,m)) = (m+k, w_{m,k-1}^{-1}(g\theta^k)w_{m,k-1}, m+k).$$

Now if $r - m = k+1$ we have

$$\begin{aligned}
&(m+k+1|(m,g,m)) \\
&= (m+k+1|(m+k|(m,g,m))) \\
&= (m+k+1|(m+k, w_{m,k-1}^{-1}(g\theta^k)w_{m,k-1}, m+k)) && \text{by assumption} \\
&= (m+k+1, (w_{m,k-1}^{-1}(g\theta^k)w_{m,k-1})\theta_{m+k}, m+k+1) && \text{using (2.2.3)} \\
&= (m+k+1, u_{m+k}^{-1}(w_{m,k-1}^{-1}(g\theta^k)w_{m,k-1})\theta u_{m+k}, m+k+1) && \text{using (2.2.9)} \\
&= (m+k+1, w_{m,k}^{-1}(g\theta^{k+1})w_{m,k}, m+k+1) && \text{using (2.2.8)}
\end{aligned}$$

and so the lemma is true for all $r - m \geqslant 1$. $\square$

Lemma 2.2.4. *For $r - m \geqslant 2$,*

$$(r|(m,1,m+1)) = (r, w_{m+1,r-m-2}^{-1}(a_{m+1}\theta^{r-m-1})w_{m+2,r-m-2}, r+1).$$

Proof. We again use proof by induction on $r-m$. If $r-m=2$ then

$$\begin{aligned}
&(m+2|(m,1,m+1))\\
&=(m+2|(m+1|(m,1,m+1)))\\
&=(m+2|(m+1|(m,1,m)(m,1,m+1)))\\
&=(m+2|(m+1,1\theta_m,m+1)(m+1,a_{m+1},m+2)) && \text{by (2.2.3) and (2.2.4)}\\
&=(m+2|(m+1,1,m+1)(m+1,a_{m+1},m+2)) && \text{as } 1\theta_m=1\\
&=(m+2|(m+1,a_{m+1},m+2))\\
&=(m+2|(m+1,a_{m+1},m+1)(m+1,1,m+2))\\
&=(m+2,a_{m+1}\theta_{m+1},m+2)(m+2,a_{m+2},m+3) && \text{by (2.2.3) and (2.2.4)}\\
&=(m+2,a_{m+1}\theta_{m+1}a_{m+2},m+3)\\
&=(m+2,u_{m+1}^{-1}(a_{m+1}\theta)u_{m+1}a_{m+2},m+3) && \text{by (2.2.9)}\\
&=(m+2,u_{m+1}^{-1}(a_{m+1}\theta)u_{m+2},m+3) && \text{by definition of } u_{m+2}\\
&=(m+2,w_{m+1,0}^{-1}(a_{m+1}\theta)w_{m+2,0},m+3) && \text{as } w_{m,0}=u_m
\end{aligned}$$

and so the basis step holds true.

If $r-m=k\geqslant 2$ assume that

$$(m+k|(m,1,m+1))=(m+k,w_{m+1,k-2}^{-1}(a_{m+1}\theta^{k-1})w_{m+2,k-2},m+k+1).$$

Now if $r - m = k + 1 > 2$ then

$$
\begin{aligned}
&(m+k+1|(m,1,m+1)) \\
&= (m+k+1|(m+k|(m,1,m+1))) \\
&= (m+k+1|(m+k, w_{m+1,k-2}^{-1}(a_{m+1}\theta^{k-1})w_{m+2,k-2}, m+k+1)) \\
&\qquad\qquad\qquad\qquad \text{by assumption} \\
&= (m+k+1|(m+k, w_{m+1,k-2}^{-1}(a_{m+1}\theta^{k-1})w_{m+2,k-2}, m+k) \\
&\quad (m+k,1,m+k+1)) \\
&= (m+k+1, (w_{m+1,k-2}^{-1}(a_{m+1}\theta^{k-1})w_{m+2,k-2})\theta_{m+k}a_{m+k+1}, \\
&\quad m+k+2) \\
&\qquad\qquad\qquad\qquad \text{by equations (2.2.3) and (2.2.4)} \\
&= (m+k+1, u_{m+k}^{-1}(w_{m+1,k-2}^{-1}(a_{m+1}\theta^{k-1})w_{m+2,k-2})\theta u_{m+k}a_{m+k+1}, \\
&\quad m+k+2) \\
&\qquad\qquad\qquad\qquad \text{by equation (2.2.9)} \\
&= (m+k+1, w_{m+1,k-1}^{-1}(a_{m+1}\theta^{k})w_{m+2,k-1}, m+k+2) \\
&\qquad\qquad\qquad\qquad \text{by equation (2.2.8)}
\end{aligned}
$$

Therefore the lemma is true for all $r - m \geqslant 2$. $\square$

Theorem 2.2.5. *An inductor on $\mathcal{G} = \mathbb{N} \times G \times \mathbb{N}$ arises from any choice of homomorphism $\theta : G \to G$ and sequence $\mathbf{a} = (a_n)_{n \geqslant 1}$ of elements of G. It is given by the restriction operation (for $r > m$)*

$$(r|(m,g,n)) = (r, w_{m,r-m-1}^{-1}(g\theta^{r-m})w_{n,r-m-1}, r+n-m)$$

where $u_m = a_1 \ldots a_m$ and $w_{m,k} = u_m\theta^k u_{m+1}\theta^{k-1} \ldots u_{m+k}$ and every inductor on $\mathcal{G}$ arises in this way.

Proof. Any inductor on $\mathcal{G}$ must satisfy lemmas 2.2.3 and 2.2.4. Then

$$\begin{aligned}
&(r|(m,g,n)) \\
&\quad = (r|(m,g,m)(m,1,n)) \\
&\quad = (r|(m,g,m))(r|(m,1,m+1))(r+1|(m+1,1,m+2))\dots \\
&\quad\ \dots(r+n-m-1|(n-1,1,n)) \\
&\quad = (r,g',r+n-m)
\end{aligned}$$

where

$$\begin{aligned}
g' = w^{-1}_{m,r-m-1}(g\theta^{r-m})w_{m,r-m-1}&[w^{-1}_{m+1,r-m-2}(a_{m+1}\theta^{r-m-1})w_{m+2,r-m-2}] \\
&[w^{-1}_{m+2,r-m-2}(a_{m+2}\theta^{r-m-1})w_{m+3,r-m-2}]\dots \\
&\dots[w^{-1}_{n,r-m-2}(a_n\theta^{r-m-1})w_{n+1,r-m-2}]
\end{aligned}$$

Now the terms $w_{m+k,r-m-2}w^{-1}_{m+k,r-m-2}$ cancel out, whilst

$$w_{m,r-m-1}w^{-1}_{m+1,r-m-2} = u_m\theta^{r-m-1}.$$

Hence

$$\begin{aligned}
g' &= w^{-1}_{m,r-m-1}(g\theta^{r-m})u_m\theta^{r-m-1}(a_{m+1}\theta^{r-m-1})(a_{m+2}\theta^{r-m-1})\cdots \\
&\cdots(a_n\theta^{r-m-1})w_{n+1,r-m-2} \\
&= w^{-1}_{m,r-m-1}(g\theta^{r-m})(u_m a_{m+1}a_{m+2}\dots a_n)\theta^{r-m-1}w_{n+1,r-m-2} \\
&= w^{-1}_{m,r-m-1}(g\theta^{r-m})u_n\theta^{r-m-1}w_{n+1,r-m-2} \\
&= w^{-1}_{m,r-m-1}(g\theta^{r-m})w_{n,r-m-1}
\end{aligned}$$

as claimed. Hence every restriction has the form given.

We now define $(r,h,s) \preccurlyeq (m,g,n)$ if and only if $r \geqslant m$, $s = r+n-m$ and $h = g'$ as given in the theorem. We show that $\preccurlyeq$ is an inductor on $\mathcal{G}$.

We begin by showing we have defined a partial order and then show that (OG1) and (OG2) hold. First note that if $r = m$ then $(r|(m,g,n)) = (m,g,n)$ so $g' = g$.

Clearly $(r,h,s) \preccurlyeq (r,h,s)$ so $\preccurlyeq$ is reflexive. Now if $(r,h,s) \preccurlyeq (m,g,n)$ and $(m,g,n) \preccurlyeq (r,h,s)$ then $r \geqslant m$ and $m \geqslant r$ imply $r = m$ and so $s = r+n-m = r+n-r = n$. Also $h = g' = g$ so $(r,h,s) = (m,g,n)$ and $\preccurlyeq$ is antisymmetric. Our last partial order check is for transitivity so let $(r,h,s) \preccurlyeq (m,g,n)$ and $(m,g,n) \preccurlyeq (p,k,q)$. Then $r \geqslant m$ and $m \geqslant p$ so $r \geqslant p$. Now $n = m+q-p$ so $s = r+n-m$ becomes $s = r+m+q-p-m = r+q-p$. Now,

$$h = w^{-1}_{m,r-m-1}(g\theta^{r-m})w_{n,r-m-1}$$

and

$$g = w^{-1}_{p,m-p-1}(k\theta^{m-p})w_{q,m-p-1}.$$

We have that

$$\begin{aligned}
& w_{p,m-p-1} = u_p\theta^{m-p-1}u_{p+1}\theta^{m-p-2}\dots u_{p+m-p-2}\theta u_{p+m-p-1} \\
\Rightarrow\ & w^{-1}_{p,m-p-1} = u^{-1}_{m-1}u^{-1}_{m-2}\theta\dots u^{-1}_{p+1}\theta^{m-p-2}u^{-1}_{p}\theta^{m-p-1} \\
\Rightarrow\ & w^{-1}_{p,m-p-1}\theta^{r-m} = u^{-1}_{m-1}\theta^{r-m}u^{-1}_{m-2}\theta^{r-m+1}\dots u^{-1}_{p+1}\theta^{r-p-2}u^{-1}_{p}\theta^{r-p-1}
\end{aligned}$$

and

$$\begin{aligned}
w^{-1}_{m,r-m-1} &= u^{-1}_{m+r-m-1}u^{-1}_{m+r-m-2}\theta\dots u^{-1}_{m+1}\theta^{r-m-2}u^{-1}_{m}\theta^{r-m-1} \\
&= u^{-1}_{r-1}u^{-1}_{r-2}\theta\dots u^{-1}_{m+1}\theta^{r-m-2}u^{-1}_{m}\theta^{r-m-1}
\end{aligned}$$

so

$$w^{-1}_{m,r-m-1}w^{-1}_{p,m-p-1}\theta^{r-m} = w^{-1}_{p,r-p-1}.$$

Also,

$$w_{q,m-p-1}\theta^{r-m} = u_q\theta^{r-p-1}u_{q+1}\theta^{r-p-2}\dots u_{q+m-p-2}\theta^{r-m-1}u_{q+m-p-1}\theta^{r-m}$$

and as $n = q+m-p$,

$$w_{n,r-m-1} = u_{q+m-p}\theta^{r-m-1}\dots u_{q+r-p-1}$$

so,

$$w_{q,m-p-1}\theta^{r-m}w_{n,r-m-1} = w_{q,r-p-1}.$$

Giving us

$$h = w_{p,r-p-1}^{-1}(k\theta^{r-p})w_{q,r-p-1}.$$

Therefore $(r, h, s) \preccurlyeq (p, k, q)$.

We show we have an ordered groupoid structure by checking that (OG1) and (OG2) hold.

(OG1). Let $(r, h, s) \preccurlyeq (m, g, n)$. We require $(r, h, s)^{-1} = (s, h^{-1}, r) \preccurlyeq (n, g^{-1}, m) = (m, g, n)^{-1}$. By definition $s = r + n - m$ so $r = s + m - n$, and $r \geqslant m$ which implies $s \geqslant n$. Also $s = r + n - m$ implies $s - n = r - m$. As $h = g' = w_{m,r-m-1}^{-1}(g\theta^{r-m})w_{n,r-m-1}$, we have

$$\begin{aligned} h^{-1} &= (g')^{-1} \\ &= w_{n,r-m-1}^{-1}(g\theta^{r-m})^{-1}w_{m,r-m-1} \\ &= w_{n,s-n-1}^{-1}(g^{-1}\theta^{s-n})w_{m,s-n-1} \\ &= (g^{-1})'. \end{aligned}$$

Hence $(s, h^{-1}, r) \preccurlyeq (n, g^{-1}, m)$. Therefore (OG1) holds.

(OG2). Now let $(r, h, s) \preccurlyeq (m, g, n)$ and $(s, f, t) \preccurlyeq (n, j, p)$. We require $(r, h, s)(s, f, t) = (r, hf, t) \preccurlyeq (m, gj, p) = (m, g, n)(n, j, p)$. By definition, $r \geqslant m$ and $t = s + p - n = (r + n - m) + p - n = r + p - m$. Also,

$s = r + n - m$ implies $s - n = r - m$. Now,

$$\begin{aligned}
hf &= g'j' \\
&= (w_{m,r-m-1}^{-1}(g\theta^{r-m})w_{n,r-m-1})(w_{n,s-n-1}^{-1}(j\theta^{s-n})w_{p,s-n-1}) \\
&= (w_{m,r-m-1}^{-1}(g\theta^{r-m})w_{n,r-m-1})(w_{n,r-m-1}^{-1}(j\theta^{r-m})w_{p,r-m-1}) \\
&= w_{m,r-m-1}^{-1}(g\theta^{r-m})(j\theta^{r-m})w_{p,r-m-1}) \\
&= w_{m,r-m-1}^{-1}((gj)\theta^{r-m})w_{p,r-m-1}) \\
&= (gj)'
\end{aligned}$$

Hence $(r, hf, t) \preccurlyeq (m, gj, p)$. Therefore (OG2) holds.

As $\mathbb{N}$ forms a meet semilattice under ω then the set of identities of $\mathcal{G}$ form a meet semilattice under $\preccurlyeq$. Therefore $\mathcal{G}$ is an inductive groupid. $\square$

As seen, an inductor arises from any choice of homomorphism $\theta : G \to G$ and sequence $\mathbf{a} = (a_n)_{n \geqslant 1}$ of elements of G. We denote the corresponding inductive groupoid by $\mathcal{G}(\theta, \mathbf{a})$. If $\mathbf{a}$ is the constant sequence at $1 \in G$ then we write $\mathcal{G}(\theta, \mathbf{a})$ simply as $\mathcal{G}(\theta)$. Note that in $\mathcal{G}(\theta)$ the restriction operation is given by

$$(r|(m, g, n)) = (r, g\theta^{r-m}, r + n - m).$$

It is then easy to see that the inverse semigroup associated to $\mathcal{G}(\theta)$ is the Bruck-Reilly extension $\mathrm{BR}(G, \theta)$ described in proposition 2.1.1.

Theorem 2.2.6. *For any choice of sequence* $\mathbf{a}$ *the inductive groupoids* $\mathcal{G}(\theta, \mathbf{a})$ *and* $\mathcal{G}(\theta)$ *are isomorphic.*

Proof. We shall construct an isomorphism of groupoids $\alpha : \mathcal{G}(\theta, \mathbf{a}) \to \mathcal{G}(\theta)$ such that

$$(m, g, m) \mapsto (m, g\alpha_m, m)$$

and

$$(m, 1, m+1) \mapsto (m, h_m, m+1)$$

where $\alpha_m : G \to G$ is an isomorphism and $h_m \in G$. We shall eventually determine a choice of these $h_m \in G$ that will make α an isomorphism of inductive groupoids.

We begin by applying α to $(m, g, m+1)$ in the two possible ways. Apply first to $(m, g, m)(m, 1, m+1)$,

$$\begin{aligned}(m, g, m+1)\alpha &= [(m, g, m)(m, 1, m+1)]\alpha \\ &= (m, g\alpha_m, m)(m, h_m, m+1) \\ &= (m, (g\alpha_m)h_m, m+1).\end{aligned}$$

Apply now to $(m, 1, m+1)(m+1, g, m+1)$,

$$\begin{aligned}(m, g, m+1)\alpha &= [(m, 1, m+1)(m+1, g, m+1)]\alpha \\ &= (m, h_m, m+1)(m+1, g\alpha_{m+1}, m+1) \\ &= (m, h_m(g\alpha_{m+1}), m+1).\end{aligned}$$

Comparing G components gives, for all m,

$$g\alpha_{m+1} = h_m^{-1}(g\alpha_m)h_m.$$

Assume α_0 is the identity function and $h_0 = 1$, the identity element of G, then we see that

$$g\alpha_1 = h_0^{-1}(g\alpha_0)h_0 = 1g1 = g$$

so α_1 is also the identity function and

$$g\alpha_m = v_m^{-1} g v_m \tag{2.2.10}$$

where $v_m = h_1h_2h_3 \ldots h_{m-1}$ and we take $v_0 = 1$.

Since

$$\begin{aligned}
&(m, g, n)\alpha \\
&= [(m, g, m)(m, 1, m+1) \ldots (n-1, 1, n)]\alpha \\
&= (m, g\alpha_m h_m h_{m+1} \ldots h_{n-1}, n) \\
&= (m, v_m^{-1} g v_m h_m h_{m+1} \ldots h_{n-1}, n) && \text{by equation (2.2.10)} \\
&= (m, v_m^{-1} g v_n, n)
\end{aligned}$$

we define α to be

$$\alpha : (m, g, n) \mapsto (m, v_m^{-1} g v_n, n).$$

Now let $m \leqslant n \leqslant l$ and $k, g \in G$. Then using (2.2.10),

$$\begin{aligned}
&(m, g, n)\alpha(n, k, l)\alpha \\
&= (m, v_m^{-1} g v_n, n)(n, v_n^{-1} k v_l, l) \\
&= (m, v_m^{-1} g v_n v_n^{-1} k v_l, l) \\
&= (m, v_m^{-1} g k v_l, l) \\
&= (m, gk, l)\alpha \\
&= [(m, g, n)(n, k, l)]\alpha
\end{aligned}$$

Therefore $(m, g, n)\alpha(n, k, l)\alpha = [(m, g, n)(n, k, l)]\alpha$ so α is a homomorphism. All the α_m are isomorphisms so they are all bijective hence α is bijective and α is an isomorphism of groupoids.

We now need to check that α is an isomorphism of ordered groupoids. As the ordering on the elements of an ordered groupoids is equivalent to the restriction operation we need to check that applying α to an element in

$\mathcal{G}(\theta, \mathbf{a})$ then restricting it results in the same element of $\mathcal{G}(\theta)$ if we first restrict in $\mathcal{G}(\theta, \mathbf{a})$ and then apply α.

Let us look initially at the restriction of (m, g, m) to $m+1$. We start with applying α,

$$(m, g, m)\alpha = (m, v_m^{-1} g v_m, m)$$

now we restrict to $m+1$ in $\mathcal{G}(\theta)$,

$$\begin{aligned}(m+1|(m, v_m^{-1} g v_m, m)) &= (m+1, (v_m^{-1} g v_m)\theta, m+1) \\ &= (m+1, (v_m^{-1})\theta(g\theta)(v_m)\theta, m+1).\end{aligned}$$

On the other hand first restricting (m, g, m) to $m+1$ in $\mathcal{G}(\theta, \mathbf{a})$ gives

$$\begin{aligned}&(m+1|(m, g, m)) \\ &= (m+1, w_{m,0}^{-1}(g\theta)w_{m,0}, m+1) && \text{by theorem 2.2.5} \\ &= (m+1, u_m^{-1}(g\theta)u_m, m+1) && \text{as } w_{m,0} = u_m\end{aligned}$$

and then applying α results in

$$(m+1, u_m^{-1}(g\theta)u_m, m+1)\alpha = (m+1, v_{m+1}^{-1} u_m^{-1}(g\theta)u_m v_{m+1}, m+1).$$

So we require

$$(v_m^{-1}\theta)(g\theta)(v_m\theta) = v_{m+1}^{-1} u_m^{-1}(g\theta)u_m v_{m+1}.$$

We define the v_i (and hence the h_i) inductively by setting $v_0 = 1 = v_1$ and then

$$v_{m+1} = u_m^{-1}(v_m\theta)$$

Moreover, for $k \geqslant 2$,

$$v_k = w_{1,k-2}^{-1} \tag{2.2.11}$$

and $h_m = v_m^{-1} v_{m+1}$, so

$$h_m = w_{1,m-2} w_{1,m-1}^{-1}.$$

We can now refine our definition of α for $m, n \geqslant 2$. We deal with the cases where m or/and n is less than 2 separately in the appendix.

$$\begin{aligned}
&(m, g, n)\alpha \\
&= (m, v_m^{-1} g v_n, n) \\
&= (m, w_{1,m-2} g w_{1,n-2}^{-1}, n) \qquad\qquad \text{by equation (2.2.11)}
\end{aligned}$$

so

$$(m, g, n)\alpha = (m, w_{1,m-2} g w_{1,m-2}^{-1}, n) \tag{2.2.12}$$

Restricting to $r > m$ in $\mathcal{G}(\theta)$ gives

$$(r|(m, w_{1,m-2} g w_{1,n-2}^{-1}, n)) = (r, (w_{1,m-2}\theta^{r-m})(g\theta^{r-m})(w_{1,n-2}^{-1}\theta^{r-m}, r+n-m).$$

On the other hand, restricting in $\mathcal{G}(\theta, \mathbf{a})$,

$$(r|(m, g, n)) = (r, w_{m,r-m-1}^{-1}(g\theta^{r-m}) w_{n,r-m-1}, r+n-m).$$

Applying α using (2.2.12) gives

$$\begin{aligned}
&(r, w_{m,r-m-1}^{-1}(g\theta^{r-m}) w_{n,r-m-1}, r+n-m)\alpha \\
&\qquad = (r, w_{1,r-2} w_{m,r-m-1}^{-1}(g\theta^{r-m}) w_{n,r-m-1} w_{1,r+n-m-2}^{-1}, r+n-m).
\end{aligned}$$

Whence for α to be an isomorphism of ordered groupoid we require

$$\begin{aligned}
&(w_{1,m-2}\theta^{r-m})(g\theta^{r-m})(w_{1,n-2}^{-1}\theta^{r-m}) \\
&\qquad = w_{1,r-2} w_{m,r-m-1}^{-1}(g\theta^{r-m}) w_{n,r-m-1} w_{1,r+n-m-2}^{-1}.
\end{aligned}$$

Now,

$$\begin{aligned}
&w_{1,r-2}w_{m,r-m-1}^{-1}\\
&= [u_1\theta^{r-2}u_2\theta^{r-3}\dots u_{r-1}][u_m\theta^{r-m-1}u_{m+1}\theta^{r-m-2}\dots u_{r-1}]^{-1} \quad \text{using (2.2.7)}\\
&= u_1\theta^{r-2}u_2\theta^{r-3}\dots u_{r-1}u_{r-1}^{-1}\dots u_{m+1}^{-1}\theta^{r-m-2}u_m^{-1}\theta^{r-m-1}\\
&= u_1\theta^{r-2}u_2\theta^{r-3}\dots u_{m-1}\theta^{r-m}\\
&= [u_1\theta^{m-2}u_2\theta^{m-3}\dots u_{m-1}]\theta^{r-m}\\
&= w_{1,m-2}\theta^{r-m} \quad \text{using (2.2.7)}
\end{aligned}$$

also,

$$\begin{aligned}
&w_{n,r-m-1}w_{1,r+n-m-2}^{-1}\\
&= [u_n\theta^{r-m-1}u_{n+1}\theta^{r-m-2}\dots u_{r+n-m-1}][u_1\theta^{r+n-m-2}\\
&\qquad u_2\theta^{r+n-m-1}\dots u_{r+n-m-1}]^{-1} \quad \text{using (2.2.7)}\\
&= [u_1\theta^{r+n-m-2}u_2\theta^{r+n-m-1}\dots u_{n-1}\theta^{r-m}]^{-1}\\
&= [u_1\theta^{n-2}u_2\theta^{n-1}\dots u_{n-1}]^{-1}\theta^{r-m}\\
&= w_{1,n-2}^{-1}\theta^{r-m} \quad \text{using (2.2.7).}
\end{aligned}$$

As required we have

$$\begin{aligned}
&(w_{1,m-2}\theta^{r-m})(g\theta^{r-m})(w_{1,n-2}^{-1}\theta^{r-m})\\
&\qquad = w_{1,r-2}w_{m,r-m-1}^{-1}(g\theta^{r-m})w_{n,r-m-1}w_{1,r+n-m-2}^{-1}.
\end{aligned}$$

Therefore, for $m, n \geqslant 2$

$$(r|[(m,g,n)\alpha]) = [(r|(m,g,n))]\alpha.$$

Together with the extra calculations collected in the appendix, this completes the proof. $\square$

Theorems 2.2.5 and 2.2.6 give us the inductive groupoids that correspond to bisimple inverse ω-semigroups. As a consequence, every inductive groupoid associated to a bisimple inverse ω-semigroup is isomorphic to one associated to a Bruck-Reilly extension.

Chapter 3

Cayley Graphs and Frucht's Theorem

We can visually represent a group by a graph known as a *Cayley graph*. In this chapter we give the Cayley graph of a group and also introduce Cayley graphs for inverse semigroups, groupoids and ordered groupoids. Frucht's theorem states that every finite group is isomorphic to the automorphism group of a graph. Following Sieben [25], we give generalisations of Frucht's theorem for inverse semigroups and groupoids. The Cayley graph of a group and the Cayley graph of an ordered groupoid we discuss in this chapter will be used throughout this thesis. As well as being of vital importance to chapter four's discussion of the Margolis-Meakin graph expansion [14], the Cayley graph of an ordered groupoid also becomes important in chapter five when we discuss a significant structure theorem, McAlister's P-theorem [16].

3.1 Group Presentations and Cayley Graphs

Given a set X and a group G, suppose we have a map $f : X \to G$ such that every element of G can be expressed as a product of elements of the image of f and their inverses. We then say that (X, f) generates G. Usually we will suppress mention of f. If f is injective we can identify X as a subset of G.

Let $F(X)$ denote the free group on the set X. If (X, f) generates G then f induces an epimorphism $\phi : F(X) \to G$. Given a subset R of $F(X)$, the set of all products of conjugates of elements of R and their inverses forms a normal subgroup of $F(X)$, which we denote by $\ll R \gg$, that is the smallest normal subgroup of $F(X)$ that contains R. A subset R of $F(X)$ is called a *set of defining relations for* G if $\ll R \gg = \ker(\phi)$. Further, $F(X)/ \ll R \gg$ is isomorphic to G and $\langle X : R\rangle$ is called a *presentation of* G, [6].

A *Cayley graph* or *Cayley colour graph*, [6] is a way of visually representing the information given about a group by a group presentation. The Cayley colour graph for a group G with presentation $\langle X : R\rangle$ is a directed graph. The vertex set $V = G$ and the edge set is

$$E = \{(g, x, gx) : x \in X,\ g, gx \in G\}.$$

Each generator assigns a unique colour to the edge it labels.

Example 3.1.1. The dihedral group D_3 has six elements and is the symmetry group of an equilateral triangle. We give two presentations for this group, [6, pg132], and then the Cayley graphs for each presentation;

(i) $D_3 = \langle x, y : x^3 = y^2 = (xy)^2 = 1\rangle$;

(ii) $D_3 = \langle a, b : a^2 = b^2 = (ab)^3 = 1\rangle$.

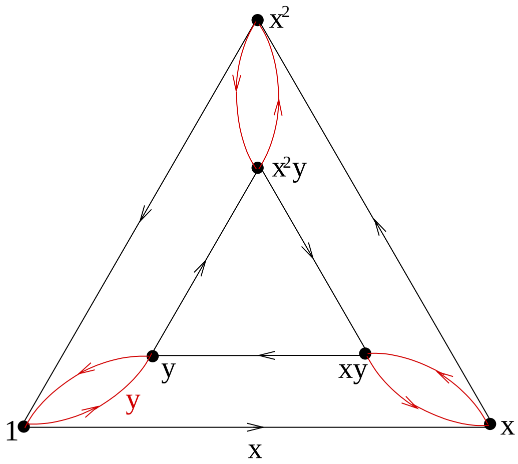

Figure 3.1.1: Cayley graph for presentation (i).

The Cayley graph for each presentation will have six vertices, one vertex for each element in the group. Other than this, the Cayley graphs of each are quite different in appearance. The Cayley graph for presentation (i) is given in Fig. 3.1.1. From Fig. 3.1.1 we see generator x is labelled by a black directed edge and y by a red directed edge. The relation $(xy)^2 = 1$ can be traced in the graph: starting at vertex 1 we follow edge x to vertex x, then edge y to vertex xy, followed by edge x to vertex y, then edge y bringing us back to vertex 1 again. So $xyxy = 1$.

The Cayley graph of D_3 given by presentation (ii) is given in Fig. 3.1.2.

3.2 Cayley Graphs for Inverse Semigroups

Let X be a non-empty set. An inverse semigroup F equipped with a function $\iota : X \to F$ is a *free inverse semigroup on* X if for every inverse semigroup S and function $\theta : X \to S$ there exists a unique homomorphism $k : F \to S$ such that $\iota\theta = k$. Let $FIS(X)$ denote the free inverse semigroup on X. If ρ

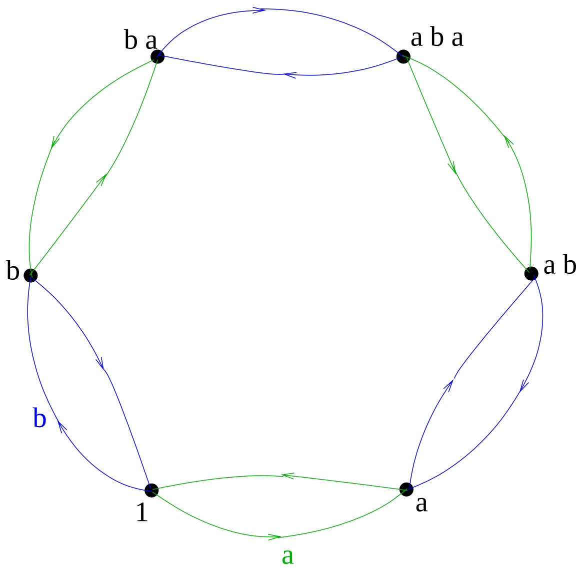

Figure 3.1.2: Cayley graph for presentation (ii).

is a relation on $FIS(X)$ then denote by $\rho^\natural$ the intersection of all congruences containing ρ. Then if S is isomorphic to the inverse semigroup $FIS(X)/\rho^\natural$ we say that $\langle X : \rho \rangle$ is an *inverse semigroup presentation* for S, [10].

Nándor Sieben considers a generalisation of the Cayley graph for a group to a Cayley graph for an inverse semigroup [25]. Let Δ be a subset of S that generates S, so that every element of S can be written as a finite product of elements of Δ. As with the Cayley graph of a group, the Cayley graph of the inverse semigroup S with generating set Δ is a directed graph, which we denote by $\Gamma(S, \Delta)$. The vertex set $V = S$ and edge set $E = \{(s, z, sz) : z \in \Delta, s \in S\}$. Each generator in Δ will assign a unique colour to the edge it labels.

Example 3.2.1. The bicyclic monoid can be given by the inverse semigroup presentation $\langle s, t : ts = 1 \rangle$. Now $sts = s1 = s$ and $tst = 1t = t$ so t is the inverse of s. The Cayley graph of the bicyclic monoid is given in Fig. 3.2.1.

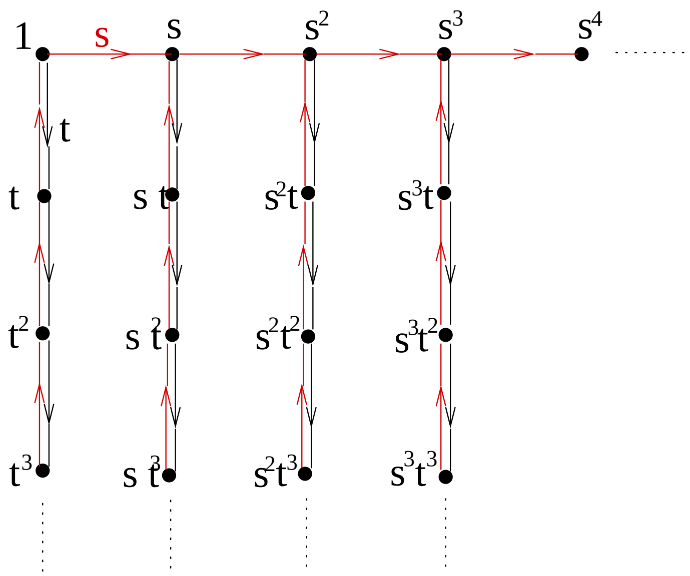

Figure 3.2.1: Cayley graph of the bicyclic monoid.

Note that $ts = 1$ but $st \neq 1$. All the elements of the bicyclic monoid can be written in the form $s^m t^n$ where $m, n \geqslant 0$. Thus the underlying set of the bicyclic monoid is bijective with the set $\mathbb{N} \times \mathbb{N}$.

Example 3.2.2. Meakin [18] gives the following presentation for the symmetric inverse monoid of degree 3.

$$\langle a, b, e : a^2 = b^2 = (ab)^3 = 1, e^2 = e, ea = ae, ebe = bebe \rangle$$

Here a corresponds to $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$, b corresponds to $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ and e to the idempotent $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & * \end{pmatrix}$. We use the above presentation to create the Cayley colour graph shown in Fig. 3.2.2.

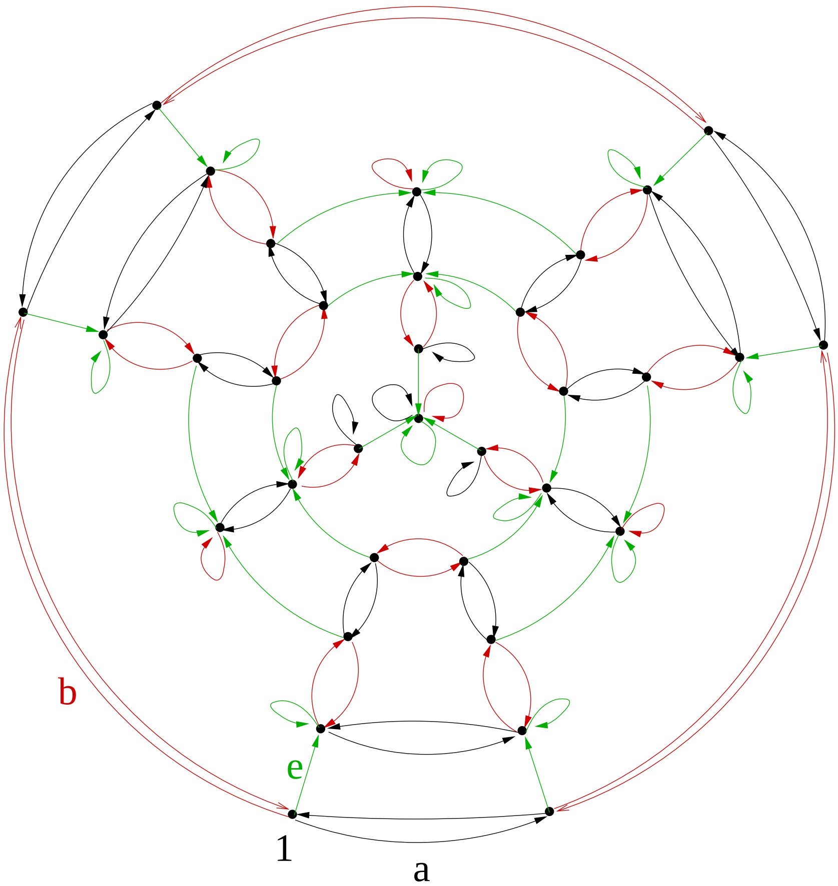

Figure 3.2.2: Cayley graph of the symmetric inverse monoid.

3.3 Generalisation of Frucht's Theorem

We introduce Frucht's theorem, a proof of which can be found in [29]. The rest of this chapter concerns the generalisation of this theorem.

Theorem 3.3.1. *Every finite group G is isomorphic to the group of colour preserving automorphisms of a Cayley graph of G*

Sieben's paper [25] extends the result of Frucht's theorem for groups to inverse semigroups and groupoids. Before giving the result for an inverse semigroup we need some definitions, and note that for this section it makes sense to write maps on the left.

Sieben defines the *tail* of a vertex $s \in \Gamma(S, \Delta)$ to be the set consisting of all vertices that can be reached by a finite directed walk from s. So

$$\mathrm{tail}(s) = \{sr : r \in S\}$$

since $r = g_1 g_2 \ldots g_n$ for some $g_1, g_2, \ldots, g_n \in \Delta$. Vertex s is called the *head* of its tail. The head is not unique but there does exist a unique idempotent head for every tail.

Lemma 3.3.2. *[25, Lemma 3.8]. For the tail of $s \in S$ the idempotent head ss^{-1} is unique.*

$$\mathrm{tail}(s) = \mathrm{tail}(ss^{-1})$$

Proof. If $t \in \mathrm{tail}(s)$ then $t = sr = ss^{-1}sr = ss^{-1}(sr)$ so $t \in \mathrm{tail}(ss^{-1})$. If $t \in \mathrm{tail}(ss^{-1})$ then $t = ss^{-1}r' = s(s^{-1}r')$ so $t \in \mathrm{tail}(s)$.

If e and f are both idempotent heads of $\mathrm{tail}(s)$ then $e \in \mathrm{tail}(s) = \mathrm{tail}(f)$ so $e = fr$ for some $r \in S$. Then $fe = ffr = fr = e$ implies $e \leqslant f$. Similarly $f \leqslant e$. Therefore $e = f$. □

We note here that $\mathrm{tail}(s) = ss^{-1}S$ and so the partial automorphism between Sieben's "tails" are nothing more than the partial automorphisms between ideals of S, like those used in the Wagner-Preston representation theorem 1.1.20. So the lemma above gives us the same information as lemma 1.1.9. The tail definition relates closely to the Cayley graph and gives us a definition that is easier to visualise.

A *partial automorphism* of $\Gamma(S, \Delta)$ is a bijection α between two tails of $\Gamma(S, \Delta)$ such that, for all $g \in \Delta$, $s \in dom(\alpha)$,

$$\alpha(sg) = \alpha(s)g.$$

Since every element of S can be considered as a string of generators it is clear that, for all $r \in S$, $s \in dom(\alpha)$,

$$\alpha(sr) = \alpha(s)r.$$

Lemma 3.3.3. *[25, Lemma 3.10]. $\alpha_s : \mathrm{tail}(s^{-1}) \to \mathrm{tail}(s)$ given by $x \mapsto sx$ is a partial automorphism of $\Gamma(S, \Delta)$.*

Proof. If $p \in \mathrm{tail}(s)$ then $p = sr' = ss^{-1}sr'$ for some $r' \in S$. Let $x = s^{-1}sr'$, then $x \in \mathrm{tail}(s^{-1})$ and $\alpha_s(x) = p$. So α_s is surjective.

Now let $s^{-1}t, s^{-1}r \in \mathrm{tail}(s^{-1})$ and assume $\alpha_s(s^{-1}t) = \alpha_s(s^{-1}r)$. Then $s^{-1}\alpha_s(s^{-1}t) = s^{-1}\alpha_s(s^{-1}r)$, so $s^{-1}ss^{-1}t = s^{-1}ss^{-1}r$ which implies that $s^{-1}t = s^{-1}r$, and so α_s is injective and therefore a bijection.

Also, let $g \in \Delta$ and $t \in \mathrm{tail}(s^{-1})$, then $\alpha_s(tg) = stg = \alpha_s(t)g$. □

Lemma 3.3.4. *[25, Lemmas 3.9 and 3.10]. The inverse of a partial automorphism is a partial automorphism. Furthermore, $\alpha_{s^{-1}} = {\alpha_s}^{-1}$.*

Proposition 3.3.5. *[25, Prop 3.11]. Every partial automorphism of $\Gamma(S, \Delta)$ is of the form α_s for some $s \in S$.*

Proof. Let e be the unique idempotent head of the domain of α and let $s = \alpha(e)$ (i.e. $e = \alpha^{-1}(s)$). The domain of α_s is tail(s^{-1}) = tail$(s^{-1}s)$. Now,

$$e(s^{-1}s) = \alpha^{-1}(s)s^{-1}s = \alpha^{-1}(ss^{-1}s) = \alpha^{-1}(s) = e$$

so $e \leqslant s^{-1}s$, and

$$s^{-1}se = s^{-1}\alpha(e)e = s^{-1}\alpha(ee) = s^{-1}\alpha(e) = s^{-1}s$$

so $s^{-1}s \leqslant e$ and $s^{-1}s = e$. Thus the domain of α_s is tail$(s^{-1}s)$ = tail(e), which is the domain of α. Let $t \in dom(\alpha) = \text{tail}(e)$, then $t = er$ for some $r \in S$ and

$$\alpha(t) = \alpha(er) = \alpha(eer) = \alpha(e)er = st = \alpha_s(t).$$

□

Theorem 3.3.6. *[25, Theorem 3.12]. If S is an inverse semigroup with generating set Δ then the set of partial automorphisms of $\Gamma(S,\Delta)$, denoted* Part.Aut.$(\Gamma(S,\Delta))$, *is an inverse semigroup isomorphic to S.*

Proof. Let $s \in S$, then $\alpha_s : \text{tail}(s^{-1}) \to \text{tail}(s)$ and $\alpha_s{\alpha_s}^{-1}\alpha_s : \text{tail}(s^{-1}) \to \text{tail}(s)$.

$$\alpha_s{\alpha_s}^{-1}\alpha_s(x) = ss^{-1}sx = sx = \alpha_s(x).$$

Similarly, $\alpha_{s^{-1}} = \alpha_{s^{-1}}\alpha_s\alpha_{s^{-1}}$. As α_s is an automorphism, $\alpha_{s^{-1}}$ is the unique inverse of α_s. Hence Part.Aut.$(\Gamma(S,\Delta))$ is an inverse semigroup.

Next, let $\phi : S \to \text{Part.Aut.}(\Gamma(S,\Delta))$ be given by $s \mapsto \alpha_s$. By proposition 3.3.5, ϕ is a surjective mapping. Now let $s, t \in S$ and assume $\phi(s) = \phi(t)$. Then $\alpha_s = \alpha_t$ and so α_s and α_t must have the same domain, i.e. tail(s) = tail(t). As each tail has a unique idempotent head $s^{-1}s = t^{-1}t$ and so

$$s = ss^{-1}s = \alpha_s(s^{-1}s) = \alpha_t(s^{-1}s) = \alpha_t(t^{-1}t) = tt^{-1}t = t.$$

Lastly, if $a, b \in S$ and $x \in \text{tail}(a)$ then $\phi(a)\phi(b) = \alpha_a\alpha_b$ and $\phi(ab) = \alpha_{ab}$. Now,

$$\alpha_{ab}(x) = abx = \alpha_a(bx) = \alpha_a\alpha_b(x)$$

so we have an isomorphism. □

3.4 Cayley Graphs for Groupoids

Sieben [25] also defines a Cayley graph of a groupoid $\mathcal{G}$ with generating set Δ, denoted $\Gamma(\mathcal{G}, \Delta)$. Here Δ generates $\mathcal{G}$ if every morphism in $\mathcal{G}$ is a finite composition of morphisms in Δ. As before, this is a directed graph with vertices labelled by elements of $\mathcal{G}$ and edge set

$$E = \{(x, z, xz) : x \in \mathcal{G}, z \in \Delta, \mathbf{r}(x) = \mathbf{d}(z)\}.$$

The edge (x, z, xz) starts at vertex x and ends at vertex xz.

Example 3.4.1. Suppose $\mathcal{G} = A \times \mathbb{Z}_2 \times A$ where $A = \{a, b\}$, [25, Example 4]. Given generating set $\Delta = \{(b, 0, a), (a, 1, b)\}$ we can create the Cayley colour graph $\Gamma(\mathcal{G}, \Delta)$, see Fig. 3.4.1.

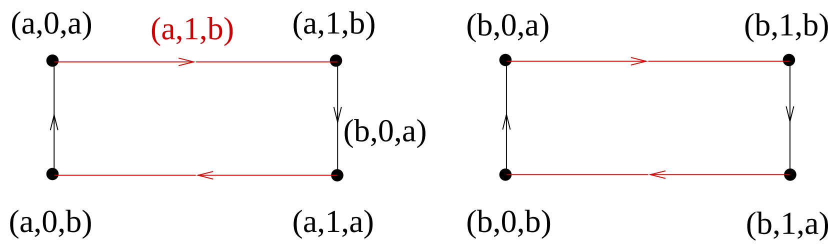

Figure 3.4.1: $\Gamma(\mathcal{G}, \Delta)$.

The tail of a vertex in $\Gamma(\mathcal{G}, \Delta)$ is defined as before,

$$\mathrm{tail}(x) = \{xr : r \in \mathcal{G}\}.$$

A partial automorphism of $\Gamma(\mathcal{G}, \Delta)$ is again a bijection α between two tails of $\Gamma(\mathcal{G}, \Delta)$ such that, for $z \in \Delta$,

$$\alpha(xz) = \alpha(x)z$$

for all composable $x, z \in \mathcal{G}$. Sieben shows that any partial automorphism of $\Gamma(\mathcal{G}, \Delta)$ can be written in the form $\alpha_x : \mathrm{tail}(x^{-1}) \to \mathrm{tail}(x)$ given by $y \mapsto xy$.

Theorem 3.4.2. *[25, Theorem 4.10]. A groupoid $\mathcal{G}$ is isomorphic to the groupoid of partial automorphisms of $\Gamma(\mathcal{G}, \Delta)$.*

Proof. $\phi : \mathcal{G} \to \mathrm{Part.Aut.}(\Gamma(\mathcal{G}, \Delta))$ defined by $g \mapsto \alpha_g$ is the required isomorphism. □

3.5 Schützenberger Graphs and Presheaves

Let S be an inverse semigroup generated by a set Δ. For each idempotent $e \in S$, or equivalently for each $\mathcal{R}$-class in S, we define the Schützenberger graph $\mathrm{Sch}(S, \Delta, e)$ of (S, Δ) at e as folows. The vertex set of $\mathrm{Sch}(S, \Delta, e)$ is the $\mathcal{R}$-class R_e of e, i.e. $\{s \in S : ss^{-1} = e\}$. If $s \in S$ and $a \in \Delta$ such that $s\mathcal{R}sa$ (or equivalently, by lemma 1.1.28, $s^{-1}s \leqslant aa^{-1}$ in the natural partial order on S), then there exists a directed edge (s, a, sa) in $\mathrm{Sch}(S, \Delta, e)$ starting at s and ending at sa coloured by a. So the edge set is

$$E = \{(s, a, sa) : s \in S, a \in \Delta, s\mathcal{R}sa\}.$$

The Cayley graph $\Gamma(S, \Delta)$ we discussed in section 3.2 contains each Schützenberger graph $\mathrm{Sch}(S, \Delta, e)$ as a subgraph induced on the vertex set R_e.

Example 3.5.1. We recall the symmetric inverse monoid example 3.2.2 and give the Schützenberger graphs of the symmetric inverse monoid of degree 3 in Fig. 3.5.1. It is obvious these Schützenberger graphs embed into the Cayley graph given in Fig. 3.2.2.

A directed path $p = (s, a_1, sa_1)(s, a_2, sa_2) \dots (s, a_k, sa_k)$ in $\mathrm{Sch}(S, \Delta, e)$ has *label* $\lambda(p) = a_1 \dots a_k$. $\mathcal{R}$ is a left congruence so elements of S act on the left of paths in $\mathrm{Sch}(S, \Delta, e)$: if q is a path from s to t with label u (so that $s\mathcal{R}t$ and $t = su$) and $r \in S$ then rq is a path from rs to rt with label u.

Lemma 3.5.2. *Let $s, t \in S$ and $s\mathcal{R}st$. Then there exists a directed path p in $\mathrm{Sch}(S, \Delta, e)$ from s to st with label t.*

Proof. We write t in terms of generators, $t = a_1 a_2 \cdots a_m\,, (a_j \in \Delta)$ and proceed by induction on m.

Let $m = 1$, then $t = a \in \Delta$ and p is the edge (s, a, sa).

Now let $m > 1$ and let $t_1 = a_2 \cdots a_m$ so that $t = a_1 t_1$. By assumption $s\mathcal{R}st$, so we have, by lemma 1.1.28,

$$s^{-1}s \leqslant tt^{-1} = (a_1 t_1)(a_1 t_1)^{-1} = a_1 t_1 t_1^{-1} a_1^{-1}$$

and so

$$(sa_1)^{-1}(sa_1) = a_1^{-1} s^{-1} s a_1 \leqslant a_1^{-1} a_1 t_1 t_1^{-1} a_1^{-1} a_1 = a_1^{-1} a_1 t_1 t_1^{-1} \leqslant t_1 t_1^{-1}.$$

Also,

$$sa_1 \mathcal{R} sa_1 t \Leftrightarrow (sa_1)^{-1}(sa_1) \leqslant t_1 t_1^{-1}.$$

Thus, by lemma 1.1.28, $sa_1 \mathcal{R} sa_1 t_1 = st$.

Inductively we may assume that there exists a path p_1 in $\mathrm{Sch}(S, \Delta, ss^{-1})$ from sa_1 to st with label t_1, i.e. $sa_1 \mathcal{R} st$ and $st = sa_1 t_1$. We want a path p

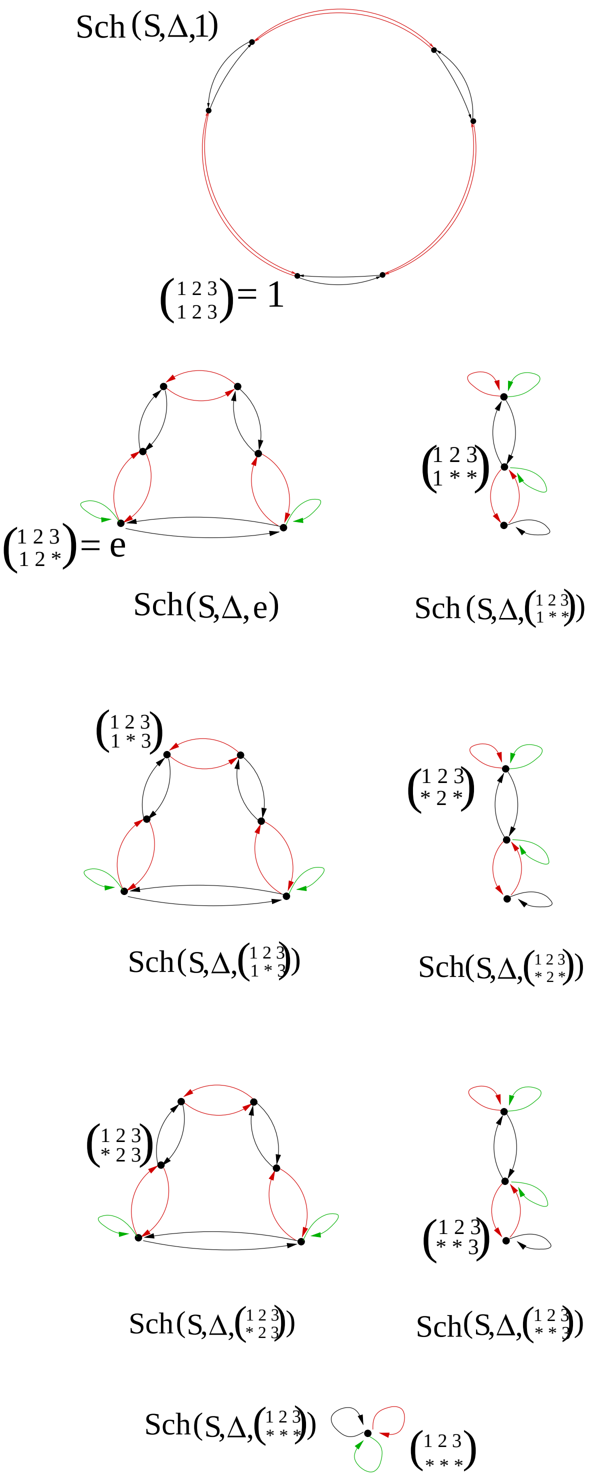

Figure 3.5.1: Schützenberger graphs.

in $\mathrm{Sch}(S, \Delta, ss^{-1})$ from s to st with label $t = a_1 t_1$. Now,

$$ss^{-1} \leqslant a_1 t_1 t_1^{-1} a_1^{-1} = (a_1 a_1^{-1})(a_1 t_1 t_1^{-1} a_1^{-1}) = (a_1 a_1^{-1})(tt^{-1}) \leqslant a_1 a_1^{-1}.$$

Hence $s\mathcal{R}sa_1$ and so (s, a_1, sa_1) is an edge in $\mathrm{Sch}(S, \Delta, ss^{-1})$ from s to sa_1. Edge (s, a_1, sa_1) may be prefixed to p_1 to give the required path p. $\square$

If $e, f \in E(S)$ with $e \leqslant f$ then left translation by e induces a graph map

$$\rho_e^f : \mathrm{Sch}(S, \Delta, f) \to \mathrm{Sch}(S, \Delta, e)$$

given by $s \mapsto es$ on vertices and $(s, a, sa) \mapsto (es, a, esa)$ on edges. Take ζ to be the functor from the semilattice $E(S)$, thought of as a category with unique morphism $e \to f$ whenever $e \leqslant f$, to the category of graphs **Grph** that maps objects $e \mapsto \mathrm{Sch}(S, \Delta, e)$ and morphisms $(e \to f) \mapsto \rho_e^f$. Then ζ is a contravariant functor. Therefore we have a presheaf of graphs over the semilattice $E(S)$, called the *Schützenberger presheaf* of (S, Δ), denoted $\mathrm{Sch}(S, \Delta)$. As Steinberg remarks in [26], the direct limit is the Cayley graph of the maximum group homomorphic image $\hat{S}$ of S with generating set Δ,

$$\lim_{e \in E(S)} \mathrm{Sch}(S, \Delta, e) = \Gamma(\hat{S}, \Delta)$$

with the cone maps

$$\rho_e : \mathrm{Sch}(S, \Delta, e) \to \Gamma(\hat{S}, \Delta) = \mathrm{Sch}(\hat{S}, \Delta, 1)$$

induced by the natural map $S \to \hat{S}$ so $s \mapsto s\sigma$ and $(s, a, sa) \mapsto (s\sigma, a\sigma, s\sigma a\sigma)$. As noted in [15], S is E–unitary if and only if ρ_e is injective for all e.

3.6 Another Generalisation of Frucht's Theorem

The generalisation of Frucht's theorem in this section derives from another remark of Steinberg in [26], that the left Wagner-Preston representation of

S induces a left action of S on the Schützenberger presheaf by partial automorphisms. As our actions are on the left, we write our maps in this section on the left.

A *partial automorphism* α of the presheaf $\mathrm{Sch}(S,\Delta)$ is a colour-preserving graph isomorphism between two subgraphs K and L of the total graph

$$\mathcal{T} = \bigcup_{e \in E(S)} \mathrm{Sch}(S,\Delta,e)$$

such that α respects the presheaf structure: that is, for any $e, f \in E(S)$ with $e \leqslant f$, we have a commutative square

$$\begin{array}{ccc} K \cap \mathrm{Sch}(S,\Delta,f) & \xrightarrow[\alpha]{} & L \cap \mathrm{Sch}(S,\Delta,(\alpha(f))(\alpha(f))^{-1}) \\ {\scriptstyle \rho^f_e}\big\downarrow & & \big\downarrow{\scriptstyle \rho^{(\alpha(f))(\alpha(f))^{-1}}_{(\alpha(e))(\alpha(e))^{-1}}} \\ K \cap \mathrm{Sch}(S,\Delta,e) & \xrightarrow[\alpha]{} & L \cap \mathrm{Sch}(S,\Delta,(\alpha(e))(\alpha(e))^{-1}) \end{array}$$

Note that since α is colour-preserving it is completely determined by its action on the vertex set of K.

Lemma 3.6.1. *For any partial automorphism α of $\mathrm{Sch}(S,\Delta)$ we have $\alpha(st) = \alpha(s)t$ whenever $s\mathcal{R}st$.*

Proof. By lemma 3.5.2, there exists a path in $\mathrm{Sch}(S,\Delta,ss^{-1})$ from s to st labelled t. Hence there is a path from $\alpha(s)$ to $\alpha(st)$ labelled t. Therefore $\alpha(s)t = \alpha(st)$. $\square$

A *principal* partial automorphism of $\mathrm{Sch}(S,\Delta)$ is a colour-preserving graph isomorphism $\alpha : K \to L$ where K and L are induced subgraphs of $\mathcal{T}$ on principal right ideals of S. Since the principal right ideal sS has a unique idempotent generator ss^{-1}, we may assume that on the vertex sets of K and L we have $\alpha : eS \to fS$ for some $e, f \in E(S)$.

Lemma 3.6.2. *Any principal ideal eS is a union of $\mathcal{R}$–classes of idempotents $k \in E(S)$ with $k \leqslant e$:*

$$eS = \bigcup_{k \leqslant e} R_k.$$

Proof. Let $s \in eS$, so that $s = es'$ for some $s' \in S$. Then $ss^{-1} = (es')(es')^{-1} = es'(s')^{-1}e = es'(s')^{-1} \leqslant e$. Let $k = ss^{-1}$ then $s\mathcal{R}k$ and $k \leqslant e$ so $s \in \bigcup_{k \leqslant e} R_k$. For the other inclusion let $s \in R_k$ for some $k \leqslant e$. Then $k = ek = ke$ so $s = ss^{-1}s = ks = eks = e(ks) \in eS$. □

Theorem 3.6.3. *The set of principal partial automorphisms of the Schützenberger presheaf* Sch(S, Δ) *is an inverse semigroup isomorphic to S.*

Proof. Let α be a partial automorphism of Sch(S, Δ) given by a bijection $eS \to fS$. Set $u = \alpha(e)$. We show that α is determined by the Wagner-Preston representation of u on the principal ideal $u^{-1}uS$. Suppose that $t = es \in eS$. Then $t \in R_k$ for some $k \leqslant e$, and so by lemma 3.6.1 we have

$$\alpha(t) = \alpha(kt) = \alpha(k)t$$

and hence $\alpha(t) \in \alpha(k)S = \alpha(k)\alpha(k)^{-1}S$. In particular, when $k = e$ we have $\alpha(t) \in uu^{-1}S$. Now applying lemma 3.6.1 to α^{-1}, and since $u = \alpha(e)$ implies $\alpha^{-1}(u) = e$, we have

$$eu^{-1}u = \alpha^{-1}(u)u^{-1}u = \alpha^{-1}(uu^{-1}u) = \alpha^{-1}(u) = e$$

and therefore $e \leqslant u^{-1}u$. Also,

$$u^{-1}ue = u^{-1}\alpha(e)e = u^{-1}\alpha(e^2) = u^{-1}\alpha(e) = u^{-1}u,$$

so $u^{-1}u \leqslant e$. Therefore $e = u^{-1}u$. Similarly, for any $k \leqslant e$ we have

$k = (\alpha(k))^{-1}\alpha(k)$. Since α preserves the presheaf structure,

$$\begin{aligned}\alpha(k) &= \alpha(\rho_k^e(e)) \\ &= \rho^{(\alpha(e))(\alpha(e))^{-1}}_{(\alpha(k))(\alpha(k))^{-1}}(\alpha(e)) \\ &= \rho^{uu^{-1}}_{(\alpha(k))(\alpha(k))^{-1}}(u) \\ &= (\alpha(k))(\alpha(k))^{-1}u\end{aligned}$$

so $\alpha(k) \leqslant u$. Then, by lemma 1.1.6, $\alpha(k) = u(\alpha(k))^{-1}\alpha(k) = uk$.

If $t \in eS$ then $tt^{-1} \leqslant e$. Therefore, for all $t \in eS$, we have

$$\alpha(t) = \alpha(tt^{-1}t) = \alpha(tt^{-1})t = (utt^{-1})t = ut.$$

So $\alpha : eS \to fS$ is the principal partial automorphism

$$\alpha_u : u^{-1}uS \to uu^{-1}S$$

given by left translation by u. In other words α is determined by the Wagner-Preston representation of $u \in S$. $\square$

3.7 Cayley Graph for Ordered Groupoids

In this section we give the definition of the Cayley graph of an ordered groupoid. This is the definition we will be using throughout this thesis from hereon.

Let G be an ordered groupoid, and let Δ be a set with a given function $\gamma : \Delta \to G$. We say that (Δ, γ) *generates* G if, for each arrow $g \in G$, there exists a sequence $(a_1, a_2, \cdots, a_m)$ with $m \geqslant 1$ such that:

- for each j, $(1 \leqslant j \leqslant m)$ either $a_j \in \Delta\gamma$ or $a_j^{-1} \in \Delta\gamma$,

- if $m \geqslant 2$, the pseudoproducts

$$a_1 * a_2, (a_1 * a_2) * a_3, \ldots, ((\cdots((a_1 * a_2) * a_3)\cdots) * a_{m-1}) * a_m$$

all exist,

- $g = ((\cdots((a_1 * a_2) * a_3)\cdots) * a_{m-1}) * a_m$.

In short, (Δ, γ) generates G if every arrow in G is expressible as a left-normed pseudoproduct of elements of $\Delta\gamma$ and their inverses. When the mapping γ is understood, we shall just say that Δ generates G, and we shall occasionally suppress mention of γ without comment.

We give some useful results for the generating elements of an ordered groupoid. The following lemma is a direct result of proposition 1.4.8.

Lemma 3.7.1. *If, for some bracketing of the terms, the pseudoproduct $a_1 * a_2 * \cdots * a_n$ exists in G, then it is equal in G to some composition of arrows $b_1 b_2 \cdots b_n$ with $b_j \leqslant a_j$ for $j = 1, \ldots, n$.*

Lemma 3.7.2. *Let $h = b_1 * b_2 * \cdots * b_m$ be a left-normed pseudoproduct in G.*

1. *If $g \in G$ and $\mathbf{r}(g) \leqslant \mathbf{d}(h)$ then $g * b_1 * \cdots * b_m$ is also a left-normed pseudoproduct, and is equal to $g * h$ in G.*
2. *$b_1 * b_2 * \cdots * b_m * b_m^{-1} * b_{m-1}^{-1} * \cdots * b_1^{-1}$ is a left-normed pseudoproduct, and is equal to $\mathbf{d}(h)$ in G.*

Proof. We prove part (1) by induction on m: the case $m = 1$ is trivial. If $m > 1$, let $h' = b_1 * \cdots * b_{m-1}$. This expression is a left-normed pseudoproduct, and $h = h' * b_m$. It follows that $\mathbf{d}(h) \leqslant \mathbf{d}(h')$ and so $\mathbf{r}(g) \leqslant \mathbf{d}(h')$: by induction, the pseudoproduct $g * h'$ exists and is equal to the left-normed pseudoproduct $g*b_1*\cdots*b_{m-1}$. By lemma 3.7.1, for each j with $1 \leqslant j \leqslant m-1$, there exist b'_j, b''_j and $\tilde{b}_j$ with $\tilde{b}_j \leqslant b''_j \leqslant b'_j \leqslant b_j$, and $b''_m \leqslant b_m$, such that

$$\begin{aligned} h' &= b'_1 b'_2 \cdots b'_{m-1}, \\ h &= h' * b_m = b''_1 b''_2 \cdots b''_m, \end{aligned}$$

and

$$g * h' = g\tilde{b}_1\tilde{b}_2 \cdots \tilde{b}_{m-1}$$

are compositions of arrows defined in G. Now $\mathbf{r}(g*h') = \mathbf{r}(\tilde{b}_{m-1}) \leqslant \mathbf{r}(b''_{m-1}) = \mathbf{d}(b''_m) \leqslant \mathbf{d}(b_m)$. Hence the left-normed pseudoproduct $(g*h')*b_m = g*b_1*\cdots*b_{m-1}*b_m$ exists, and is equal to the composition $g\tilde{b}_1\tilde{b}_2\cdots\tilde{b}_{m-1}\tilde{b}_m$ for some arrow $\tilde{b}_m$ with $\tilde{b}_m \leqslant b''_m \leqslant b_m$. Now $\tilde{b}_1\tilde{b}_2\cdots\tilde{b}_{m-1}\tilde{b}_m \leqslant h$ in G and so is equal to the restriction $(\mathbf{r}(g)|h)$. But we have $g(\mathbf{r}(g)|h) = g*h$. This proves part (1).

For part (2), we again proceed by induction, and again the case $m = 1$ is trivial. If $m > 1$ we write $h = b''_1 b''_2 \cdots b''_m$. Now $\mathbf{r}(h) = \mathbf{r}(b''_m) \leqslant \mathbf{r}(b_m) = \mathbf{d}(b_m^{-1})$ and so $h * b_m^{-1}$ exists, and we have

$$\begin{aligned} h * b_m^{-1} &= b''_1 b''_2 \cdots b''_m (b''_m)^{-1} = b''_1 b''_2 \cdots b''_{m-1} \\ &\leqslant b''_1 b''_2 \cdots b''_{m-1} = b_1 * \cdots * b_{m-1} = h'. \end{aligned}$$

By induction, the left-normed pseudoproduct $h' * b_{m-1}^{-1} * \cdots * b_1^{-1}$ exists and is equal to $\mathbf{d}(h')$. Therefore the left-normed pseudoproduct $h*b_m^{-1}*b_{m-1}^{-1}*\cdots*b_1^{-1}$ exists, and is equal in G to the composition

$$b''_1 b''_2 \cdots b''_m (b''_m)^{-1} \cdots (b''_1)^{-1} = \mathbf{d}(h).$$

□

The Cayley graph $\Gamma(G, \Delta)$ of the ordered groupoid G with generating set (Δ, γ) has vertex set $V = G$ and edge set

$$E = \{(g, a, g(\mathbf{r}(g)|a\gamma)) : g \in G, a \in \Delta, \mathbf{r}(g) \leqslant \mathbf{d}(a\gamma)\}.$$

Lemma 3.7.3. *Let G be an ordered groupoid generated by (Δ, γ). Suppose that $g \in G$ and that for some $x_1, x_2, \ldots, x_m \in \Delta$, g is equal to the left-normed pseudoproduct $(x_1\gamma)^{\varepsilon_1} * (x_2\gamma)^{\varepsilon_2} * \cdots * (x_m\gamma)^{\varepsilon_m}$. Then there exists a path $\mathbf{x}$ in $\Gamma(G, \Delta)$ from $\mathbf{d}(g)$ to g with label $(x_1^{\varepsilon_1}, \ldots, x_m^{\varepsilon_m})$. Conversely, any path in $\Gamma(G, \Delta)$ from $\mathbf{d}(g)$ to g yields a left-normed pseudoproduct $\mathbf{d}(g)*y_1^{\varepsilon_1}*\cdots*y_m^{\varepsilon_m}$ (with $y_j \in \Delta\gamma$) that is equal to g.*

Proof. We proceed by induction on m. If $m = 1$ then $g = (x\gamma)^{\varepsilon}$ for some $x \in \Delta$ and $\varepsilon = \pm 1$, and $\mathbf{x}$ consists of a single edge in the Cayley graph $\Gamma(G, \Delta)$. Now if $m > 1$ let $g' = (x_1\gamma)^{\varepsilon_1}*\cdots*(x_{m-1}\gamma)^{\varepsilon_{m-1}}$. By our inductive assumption, there exists a path $\mathbf{x}'$ from $\mathbf{d}(g')$ to g' with label $(x_1^{\varepsilon_1}, \ldots, x_{m-1}^{\varepsilon_{m-1}})$. Now $g = g' * (x_m\gamma)^{\varepsilon_m}$, and it follows that $\mathbf{d}(g) \leqslant \mathbf{d}(g')$. We define $g_0 = \mathbf{d}(g)$, $g_1 = (\mathbf{d}(g)|x_1^{\varepsilon_1})$, and then iteratively define $g_k = (\mathbf{r}(g_{k-1})|x_k^{\varepsilon_k})$ for each k with $1 < k < m$. Since $\mathbf{r}(g_{k-1}) \leqslant \mathbf{d}(x_k^{\varepsilon_k})$ the edges $(g_{k-1}, x_k^{\varepsilon_k}, g_{k-1}(\mathbf{r}(g_{k-1})|x_k\gamma^{\varepsilon_k}))$ exist in $\Gamma(G, \Delta)$ and form a path from $\mathbf{d}(g)$ to g with label $(x_1^{\varepsilon_1}, \ldots, x_{m-1}^{\varepsilon_{m-1}}, x_m^{\varepsilon_m})$.

The converse is straightforward: each edge $(h, y_k^{\varepsilon_k}, h(\mathbf{r}(h)|y_k^{\varepsilon_k})$ of the path extends the left-normed pseudoproduct by one factor. □

Like the collection of all Schützenberger graphs of an inverse semigroup, the Cayley graph of an ordered groupoid consists of several components, one component for every identity. If $e \in E(G)$ then the component of $\Gamma(G, \Delta)$ containing the unique identity e will have vertex set the star of e in G. Denote this component by Γ_e.

If Γ_f is a component of $\Gamma(G, \Delta)$ and $e \leqslant f$ in $E(G)$, then define $(e|\Gamma_f)$ as the graph that has vertex set $V = \{(e|g) : g \in \Gamma_f\}$ and edge set

$$E = \{((e|g), a, (e|g)(\mathbf{r}(e|g)|a\gamma)) : (g, a, g(\mathbf{r}(g)|a\gamma)) \in \Gamma_f\}.$$

Define a graph map $\bar{\rho}_e^f : \Gamma_f \to (e|\Gamma_f)$ to map vertices $g \mapsto (e|g)$ and edges $(g, a, g(\mathbf{r}(g)|a\gamma)) \mapsto ((e|g), a, (e|g)(\mathbf{r}(e|g)|a\gamma))$. The vertices of $(e|\Gamma_f)$ are the elements of the star of G at f restricted to e. So the vertex set of $(e|\Gamma_f)$ is contained in the star of e. Hence the graph $(e|\Gamma_f)$ is contained in the component Γ_e. Therefore $\bar{\rho}_e^f$ is a graph map from Γ_f to Γ_e. We note that if $e \leqslant f \leqslant k$ in $E(G)$ then $(e|(f|\Gamma_k)) = (e|\Gamma_k)$ so $\bar{\rho}_e^k = \bar{\rho}_f^k\bar{\rho}_e^f$. Further $(e|\Gamma_e) = \Gamma_e$ so $\bar{\rho}_e^e$ is the identity map on Γ_e.

Letting $\bar{\zeta} : E(G) \to \mathbf{Grph}$ be the contravariant functor that maps objects $e \mapsto \Gamma_e$ and morphisms $(e \to f) \mapsto (\bar{\rho} : \Gamma_f \to \Gamma_e)$, we have a presheaf of graphs over the poset $E(G)$. We call this presheaf the *Cayley presheaf.*

If S is an inverse semigroup generated by Δ then recall from theorem 1.5.1 the associated inductive groupoid $\mathbb{G}(S)$.

Proposition 3.7.4. *Let S be an inverse semigroup with generating set $\Delta \subseteq S$. The Schützenberger presheaf of S is isomorphic to the Cayley presheaf of $\mathbb{G}(S)$.*

Proof. Let $\mathrm{Sch}(S, \Delta, e)$ be a Schützenberger graph of the Schützenberger presheaf. We show that this is isomorphic to the component Γ_e of the Cayley presheaf. The isomorphism relies heavily on the isomorphism between S and its associated groupoid $\mathbb{G}(S)$ as described in chapter 1.

$\mathrm{Sch}(S, \Delta, e)$ has vertex set

$$\begin{aligned} V &= R_e \\ &= \{s \in S : s\mathcal{R}e\} \\ &= \{s \in S : ss^{-1} = e\} \\ &= \{s \in S : \mathbf{d}(s) = e\} \\ &= \mathrm{star}_S(e). \end{aligned}$$

Let $s \in S$ be a vertex of $\mathrm{Sch}(S, \Delta, e)$. Let ϕ_e map s to $(e|s) \in \mathbb{G}(S)$. As $s \in \mathrm{star}_S(e)$ then $\mathbf{d}(s) = e$ in S so $(e|s) = s$ in $\mathbb{G}(S)$. Whence ϕ_e maps

$\text{star}_S(e)$ to $\text{star}_{\mathbb{G}(S)}(e)$ which is exactly the vertex set of Γ_e. Clearly ϕ_e is a bijection.

The edge set of the Schützenberger graph of S at e is

$$E = \{(s, a, sa) : s \in S, a \in \Delta, s\mathcal{R}sa\}.$$

Now by lemma 1.1.28,

$$s\mathcal{R}sa \Leftrightarrow s^{-1}s \leqslant aa^{-1} \Leftrightarrow \mathbf{r}(s) \leqslant \mathbf{d}(a)$$

so

$$E = \{(s, a, sa) : s \in S, a \in \Delta, \mathbf{r}(s) \leqslant \mathbf{d}(a)\}.$$

Define ϕ_e on the edge set of $\text{Sch}(S, \Delta, e)$ to be

$$\phi_e : (es, a, esa) \mapsto ((e|s), a, (e|s)(\mathbf{d}(e|s)|a)).$$

This is well-defined because $(es, a, esa) = (s, a, sa)$ for all edges of $\text{Sch}(S, \Delta, e)$ and since $e = \mathbf{d}(s)$ in S we have that $e = \mathbf{d}(s)$ in $\mathbb{G}(S)$ so $(e|s) = s$ and $((e|s), a, (e|s)(\mathbf{d}(e|s)|a)) = (s, a, s(\mathbf{d}(s)|a))$ which is an edge of Γ_e. Further ϕ_e is clearly a bijection on the edge sets. Therefore $\text{Sch}(S, \Delta, e)$ is isomorphic to Γ_e.

Recall that $\rho_e^f : \text{Sch}(S, \Delta, f) \to \text{Sch}(S, \Delta, e)$ and $\bar{\rho}_e^f : \Gamma_f \to \Gamma_e$. Let $\phi_f : \text{Sch}(S, \Delta, f) \to \Gamma_f$ and $\phi_e : \text{Sch}(S, \Delta, e) \to \Gamma_e$ be defined as above. Then $\rho_e^f \phi_e = \phi_f \bar{\rho}_e^f$

Thus the contravariant functor ζ that maps $(e \to f)$ to ρ_e^f is isomorphic to the contravariant functor $\bar{\zeta}$ that maps $(e \to f)$ to $\bar{\rho}_e^f$. Hence the two presheafs are isomorphic. $\square$

Chapter 4

Graph Expansions

In this section we consider the Margolis-Meakin graph expansion for an ordered groupoid, introduced in [4]. In [14], Margolis and Meakin defined the graph expansion of a group – or more precisely, of a group with a given generating set – and established some of its key properties. We consider analogous properties for our extended version of the Margolis-Meakin expansion. One such property is that of being E-unitary. The corresponding notion for ordered groupoids is incompressibility [5], and we determine when the Margolis-Meakin expansion of an ordered groupoid is incompressible.

4.1 Extending the Margolis-Meakin Expansion

Given a group G generated by (Δ, γ) we denote the Cayley graph of (G, Δ) by $\Gamma(G, \Delta)$. The Margolis-Meakin expansion of (G, Δ), as defined in [14], is the set

$$(G, \Delta)^{MM} = \{(L, g) : L \text{ is a finite connected subgraph of } \Gamma(G, \Delta) \text{ and } 1, g \in L\}.$$

A connected subgraph P of $\Gamma(G, \Delta)$ containing the identity vertex is called a *pattern*, [4]. A *specified pattern* consists of a pattern and a specific vertex $g \in P$.

Margolis and Meakin construct the following binary operation. Given $(L, g), (K, h) \in (G, \Delta)^{MM}$, we define

$$(L, g)(K, h) = (L \cup g \cdot K, gh).$$

If a is an edge of K from h to ha then a is an edge in $g \cdot K$ from gh to gha.

Theorem 4.1.1. *[14, Theorem 2.1]. With the above binary operation, $(G, \Delta)^{MM}$ is an E-unitary inverse monoid.*

Theorem 4.1.2. *[14, Theorem 2.1]. Let $\delta \in \Delta$ and L_δ be the subgraph of $\Gamma(G, \Delta)$ consisting of the edge labelled δ from vertex 1 to vertex δ. The set $\{(L_\delta, \delta) : \delta \in \Delta\}$ generates $(G, \Delta)^{MM}$ as an inverse semigroup.*

Furthermore, G is the maximal group image of $(G, \Delta)^{MM}$.

Margolis and Meakin consider only the finite subgraphs of $\Gamma(G, \Delta)$. We extend the Margolis-Meakin expansion for a group from a construction with only *finite* connected subgraphs to one which includes *infinite* connected subgraphs. So we define the following expanded expansion:

$$(G, \Delta)^{MM}_{\infty} = \{(L, g) : L \text{ is a connected subgraph of } \Gamma(G, \Delta) \text{ and}, 1, g \in L\}$$

We use the binary operation above [14], letting $(L, g), (K, h) \in (G, \Delta)^{MM}_{\infty}$,

$$(L, g)(K, h) = (L \cup g \cdot K, gh).$$

Theorem 4.1.3. *$(G, \Delta)^{MM}_{\infty}$ with the above binary operation is an inverse monoid.*

Proof, based on [14, Theorem 2.1]. We show the binary operation is associative. Let $(P, g), (Q, h), (R, k) \in (G, \Delta)^{MM}_{\infty}$, then

$$[(P, g)(Q, h)](R, k) = (P \cup g \cdot Q, gh)(R, k) = (P \cup g \cdot Q \cup (gh) \cdot R, ghk)$$

and

$$(P, g)[(Q, h)(R, k)] = (P, g)(Q \cup h \cdot R, hk) = (P \cup g \cdot Q \cup (gh) \cdot R, ghk)$$

Hence $(G, \Delta)^{MM}_{\infty}$ is a semigroup.

For each element $(P, g) \in (G, \Delta)^{MM}_{\infty}$ the element $(g^{-1} \cdot P, g^{-1}) \in (G, \Delta)^{MM}_{\infty}$ is an inverse, since $(P, g) = (P, g)(g^{-1} \cdot P, g^{-1})(P, g)$ and $(g^{-1} \cdot P, g^{-1}) = (g^{-1} \cdot P, g^{-1})(P, g)(g^{-1} \cdot P, g^{-1})$. So we have a regular semigroup.

If $(P, g) = (P, g)(P, g) = (P \cup g \cdot P, g^2) = (P, g^2)$ then $g^2 = g$ so $g = 1$. Hence all idempotents are of the form $(P, 1)$. Now

$$(P, 1)(Q, 1) = (P \cup 1 \cdot Q, 1) = (P \cup Q, 1) = (Q \cup P, 1) = (Q, 1)(P, 1).$$

Since the idempotents in $(G, \Delta)^{MM}_{\infty}$ commute, we have an inverse semigroup.

Let ι be the subgraph of $\Gamma(G, \Delta)$ that contains only the vertex labelled by the identity of G. Then $(\iota, 1)(P, g) = (\iota \cup 1 \cdot P, 1g) = (P, g)$ and similarly $(P, g)(\iota, 1) = (P, g)$, giving us an identity element $(\iota, 1)$ and so an inverse monoid. □

We remark that the ordering on $(G, \Delta)^{MM}_{\infty}$ is given as follows: let $(P, g), (Q, h) \in (G, \Delta)^{MM}_{\infty}$ then

$$(P, g) \leqslant (Q, h) \Leftrightarrow P \supseteq Q \text{ and } g = h.$$

Lemma 4.1.4. *$(G, \Delta)^{MM}_{\infty}$ is E-unitary.*

Proof. Let $(P,g),(Q,h) \in (G,\Delta)^{MM}_{\infty}$. Suppose $(P,g)(Q,h) = (P,g)$. Then $(P \cup g \cdot Q, gh) = (P,g)$. So $P \cup g \cdot Q = P$ and $gh = h$. As G is a group $h = 1$ and $(Q,h) = (Q,1)$ is an idempotent. By lemma 1.1.22, $(G,\Delta)^{MM}_{\infty}$ is E-unitary. □

Lemma 4.1.5. *Denote the Cayley graph* $\Gamma(G,\Delta)$ *by* Γ. *Then* $K = \{(\Gamma,g) : g \in G\}$ *is a minimal ideal of* $(G,\Delta)^{MM}_{\infty}$.

Furthermore, K *is isomorphic to* G.

Proof. We note that $g \cdot \Gamma = \Gamma$ for all $g \in G$.

We show first that K is an ideal of $(G,\Delta)^{MM}_{\infty}$. Let $(\Gamma,g) \in K$ and let $(P,h) \in (G,\Delta)^{MM}_{\infty}$. Then

$$(\Gamma,g)(P,h) = (\Gamma \cup g \cdot P, gh) = (\Gamma, gh) \in K$$

and

$$(P,h)(\Gamma,g) = (P \cup h \cdot \Gamma, hg) = (\Gamma, hg) \in K.$$

Therefore K is an ideal of $(G,\Delta)^{MM}_{\infty}$.

We show that K is a group. Let $(\Gamma,g),(\Gamma,h) \in K$. Then

$$(\Gamma,g)(\Gamma,h) = (\Gamma \cup g \cdot \Gamma, gh) = (\Gamma, gh) \in K$$

and composition within K is always defined. Further, for $(\Gamma,g),(\Gamma,h),(\Gamma,j) \in K$, we have

$$\begin{aligned}
[(\Gamma,g)(\Gamma,h)](\Gamma,j) &= (\Gamma,gh)(\Gamma,j) \\
&= (\Gamma,ghj) \\
&= (\Gamma,g)(\Gamma,hj) \\
&= (\Gamma,g)[(\Gamma,h)(\Gamma,j)]
\end{aligned}$$

so composition is associative. The identity of K is $(\Gamma, 1)$, The inverse of $(\Gamma, g) \in K$ is $(\Gamma, g^{-1}) \in K$. Thus K is a group.

The only ideal of a group is the group itself. Hence, if an ideal of a semigroup is a group then this ideal must be minimal. Therefore K is a minimal ideal.

Furthermore, $\phi : K \to G$ defined by $(\Gamma, g) \mapsto g$ is an isomorphism. □

Proposition 4.1.6. *H is isomorphic to a subgroup of $(G, \Delta)^{MM}_{\infty}$ if and only if H is isomorphic to a subgroup of G.*

Proof. Let H be a subgroup of $(G, \Delta)^{MM}_{\infty}$. Now $(G, \Delta)^{MM}_{\infty}$ is E-unitary so, by proposition 1.1.23 each subgroup of $(G, \Delta)^{MM}_{\infty}$ embeds into the maximal group image G of $(G, \Delta)^{MM}_{\infty}$. Therefore every subgroup of $(G, \Delta)^{MM}_{\infty}$ is isomorphic to a subgroup of G.

By lemma 4.1.5, $(G, \Delta)^{MM}_{\infty}$ contains a copy of G as a minimal ideal and so the converse is immediate. □

We consider now the Margolis-Meakin expansion of a group G, with generating set Δ, which contains only those subgraphs of $\Gamma(G, \Delta)$ which are *infinite* connected subgraphs. We will denote this expansion by $(G, \Delta)^{MM}_{I}$.

$$(G, \Delta)^{MM}_{I} = \{(L, g) : L \,\text{is an infinite connected subgraph of}\, \Gamma(G, \Delta) \\ \text{and}\, 1, g \in L\}.$$

It is clear that $(G, \Delta)^{MM}_{I}$ is a subsemigroup of $(G, \Delta)^{MM}_{\infty}$. Like $(G, \Delta)^{MM}_{\infty}$, $(G, \Delta)^{MM}_{I}$ is an E-unitary inverse semigroup, but unlike $(G, \Delta)^{MM}_{\infty}$, it is not a monoid. We give an example of $(G, \Delta)^{MM}_{\infty}$ and show it is not a monoid.

Example 4.1.7. Consider the group $G = \mathbb{Z}$ with generating set $\Delta = \{1\}$ and identity 0. Then $\Gamma(\mathbb{Z}, \Delta)$ is given in Fig. 4.1.1. There are three types of

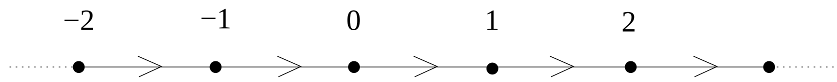

Figure 4.1.1: $\Gamma(\mathbb{Z}, \Delta)$.

infinite connected subgraphs of $\Gamma(\mathbb{Z}, \Delta)$.
1. A subgraph of $\Gamma(\mathbb{Z}, \Delta)$ that is infinite in the positive direction, see Fig. 4.1.2.

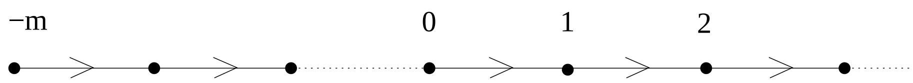

Figure 4.1.2: infinite in positive direction.

2. A subgraph of $\Gamma(\mathbb{Z}, \Delta)$ that is infinite in the negative direction, see Fig. 4.1.3.
3. The whole Cayley graph $\Gamma(\mathbb{Z}, \Delta)$.

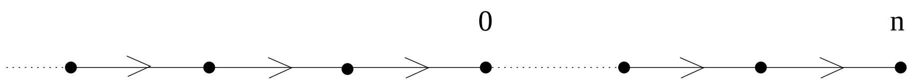

Figure 4.1.3: infinite in negative direction.

No identity element exisits so $(G, \Delta)_I^{MM}$ is not a monoid.

Proposition 4.1.8. *$(G, \Delta)_I^{MM}$ is an ideal of $(G, \Delta)_\infty^{MM}$.*

Proof. Let $(\Gamma, h) \in (G, \Delta)_I^{MM}$ and $(P, g) \in (G, \Delta)_\infty^{MM}$. Now $(\Gamma, h)(P, g) = (\Gamma \cup h \cdot P, hg)$ and since Γ is infinite so is $\Gamma \cup h \cdot P$. Also

$1 \in \Gamma$ and $g \in P$ so $hg \in h \cdot P$. Thus $(\Gamma, h)(P, g) \in (G, \Delta)_I^{MM}$. Again $(P, g)(\Gamma, h) = (P \cup g \cdot \Gamma, gh)$ and $g \cdot \Gamma$ is infinite because Γ is. Also $1 \in P$ and $h \in \Gamma$ so $gh \in g \cdot \Gamma$. Hence $(P, g)(\Gamma, h) \in (G, \Delta)_I^{MM}$ and so $(G, \Delta)_I^{MM}$ is an ideal of $(G, \Delta)_\infty^{MM}$. $\square$

We have that $(G, \Delta)^{MM} = (G, \Delta)_\infty^{MM} \setminus (G, \Delta)_I^{MM}$. Recall that an ideal determines a congruence ρ_I on an inverse semigroup. Two elements correspond under ρ_I if either the elements are equal or the elements both belong to the ideal. So by propositions 1.1.17 and 4.1.8, the Rees quotient $(G, \Delta)_\infty^{MM}/\rho_I$ is isomorphic as a semigroup to $(G, \Delta)^{MM} \cup \{0\}$ with composition

$$(L, g)(K, h) = \left\{ \begin{array}{cl} (L \cup g \cdot K, gh) & \text{if } (L, g), (K, h) \in (G, \Delta)^{MM} \\ 0 & \text{otherwise} \end{array} \right\}$$

4.2 Margolis-Meakin Expansion of an Ordered Groupoid

In this section we look at a construction of the Margolis-Meakin expansion for ordered groupoids [4], which is based on Lawson, Margolis and Steinberg's version of the Margolis-Meakin expansion for inverse semigroups [13].

Let G be an ordered groupoid with generating set Δ. In other words, every element of G can be expressed as a left-normed pseudoproduct of elements of $\Delta \cup \Delta^{-1}$, as shown in section 3.7. Denote by $\Gamma(G, \Delta)$ the Cayley graph for an ordered groupoid that we described in section 3.7.

A path in $\Gamma(G, \Delta)$ consists of a number of consecutive edges. The *label* of the path p is $(a_1^{\epsilon_1}, a_2^{\epsilon_2}, \ldots, a_n^{\epsilon_n})$ where each a_i is the generator involved in the corresponding edge in p and $\epsilon_i = \pm 1$ indicates the orientation of the corresponding edge in the path. This definition of label is a generalisation of the notion of label we introduced for paths in Schützenberger graphs for inverse semigroups in section 3.5.

A *pattern* P is a finite connected subgraph of $\Gamma(G, \Delta)$ that contains an identity vertex. A *marked pattern* is a pattern P and a specified vertex $g \in P$ such that P contains a path p from the identity vertex $e \in P$ to g derived from a representation of g as a left-normed pseudoproduct. We denote this marked pattern by (P, g).

The Margolis-Meakin expansion of an ordered groupoid is then defined as follows

$$(G, \Delta)^{MM} = \{(P, g) : P \,\text{is a marked pattern of}\, \Gamma(G, \Delta)\}.$$

We now define an ordered groupoid structure for $(G, \Delta)^{MM}$. Define composition to be $(P, g)(R, h) = (P, gh)$ whenever $g^{-1} \cdot P = R$ and $\mathbf{r}(g) = \mathbf{d}(h)$. An arrow (P, g) of $(G, \Delta)^{MM}$ has domain $(P, \mathbf{d}(g))$ and range $(g^{-1} \cdot P, \mathbf{r}(g))$. The inverse of (P, g) is $(g^{-1} \cdot P, g^{-1})$.

If (Q, h) is a marked pattern and $e \in E(G)$ with $e \leqslant \mathbf{d}(h)$, then $(e|Q)$ is the pattern with vertex set $V = \{(e|g) : g \in Q\}$ and edge set

$$E = \{((e|g), a, (e|g)(\mathbf{r}(e|g)|a\gamma)) : (g, a, g(\mathbf{r}(g)|a\gamma)) \in Q\}.$$

Further, for $(Q, h) \in (G, \Delta)^{MM}$ and $e \in E(G)$ with $e \leqslant \mathbf{r}(g)$ then $(Q|e)$ is the pattern with vertex set $V = \{(g|e) : g \in Q\}$ and edge set

$$\begin{aligned} E &= \{((g|e), a, (g|e)(\mathbf{r}(g|e)|a\gamma)) : (g, a, g(\mathbf{r}(g)|a\gamma)) \in Q\} \\ &= \{((g|e), a, (g|e)(e|a\gamma)) : (g, a, g(\mathbf{r}(g)|a\gamma)) \in Q\}. \end{aligned}$$

The ordering is given by $(R, g) \leqslant (Q, h)$ if and only if $g \leqslant h$ and $(\mathbf{d}(g)|Q) \subseteq R$.

We give the restriction and co-restriction operations explicitly. If $(P, g) \in (G, \Delta)^{MM}$ and (Q, e) is an identity of $(G, \Delta)^{MM}$ such that $(Q, e) \leqslant \mathbf{d}(P, g)$ then

$$((Q, e)|(P, g)) = (Q, (e|g)).$$

If instead, $(P, g) \in (G, \Delta)^{MM}$ and (Q, e) is an identity of $(G, \Delta)^{MM}$ such that $(Q, e) \leqslant \mathbf{r}(P, g)$ then

$$((P, g)|(Q, e)) = ((g|e) \cdot Q, (g|e)).$$

Theorem 4.2.1. *[4, Theorem 5.1]. $(G, \Delta)^{MM}$ with the above composition and ordering is an ordered groupoid. If G is inductive then so is $(G, \Delta)^{MM}$.*

4.3 Incompressibility and the Margolis-Meakin Expansion

The Margolis-Meakin expansion of a group is always E-unitary, as is the expanded Margolis-Meakin expansion $(G, \Delta)^{MM}_{\infty}$. We wish to determine when the Margolis-Meakin expansion of an ordered groupoid is E-unitary. Of course, for this to make sense, we need an appropriate version of the E–unitary property for ordered groupoids. Our first attempt at a definition simply rewrites the definition of E–unitary in the langauge of ordered groupoids. However, this does not give us a useful generalisation of the related properties of being E^*-unitary or strongly E^*-unitary. Instead, we adopt Gilbert's definition of incompressibility [4], which gives us a generalisation of all these properties of interest.

4.3.1 E-unitary Expansions

Let us now give our first attempt at defining the E-unitary property for an ordered groupoid. We define an ordered groupoid G as being *E-unitary* if whenever $g \in G$ and $e \in E(G)$ such that $e \leqslant \mathbf{d}(g)$ and $(e|g) \in E(G)$, then $g \in E(G)$. We note that Gilbert calls this property *filtered* in [4].

Theorem 4.3.1. *Let G be an ordered groupoid generated by Δ with Margolis-Meakin expansion $(G, \Delta)^{MM}$. Then G is E-unitary if and only if $(G, \Delta)^{MM}$ is E-unitary.*

Proof. Let G be an E-unitary ordered groupoid. Let $(P, g) \in (G, \Delta)^{MM}$, let (Q, f) be an identity of $(G, \Delta)^{MM}$ such that $(Q, f) \leqslant (P, \mathbf{d}(g))$ and let $((Q, f)|(P, g))$ be an identity. Then $((Q, f)|(P, g)) = (Q, (f|g))$ where $(f|g)$ is an identity of G. So we have $g \in G$, $f \in E(G)$, $f \leqslant \mathbf{d}(g)$ and $(f|g) \in E(G)$ which implies $g \in E(G)$ because G is E-unitary. Thus (P, g) is an identity of $(G, \Delta)^{MM}$.

Suppose now that G is not E-unitary. We take $g \in G, e \in E(G)$ such that $e \leqslant \mathbf{d}(g)$ and $(e|g) \in E(G)$ but g is not an identity of G.

Let $(P, g) \in (G, \Delta)^{MM}$ where P is a marked pattern with vertices the vertex set of a path in $\Gamma(G, \Delta)$ from $\mathbf{d}(g)$ to g. Take identity (Q, e) where $Q = (e|P)$. Then $(Q, e) \leqslant (P, \mathbf{d}(g))$. Restricting (P, g) to (Q, e) gives $((Q, e)|(P, g)) = (Q, (e|g))$ which is an identity of $(G, \Delta)^{MM}$ as $(e|g) \in E(G)$, but g is not an identity of G so (P, g) is not an identity. Therefore $(G, \Delta)^{MM}$ is not E-unitary. $\square$

4.3.2 Incompressibility

Gilbert defines incompressible ordered groupoids which generalises E-unitary inverse semigroups and strongly E^*-unitary inverse semigroups. In [5], Gilbert constructs a structure theorem for all incompressible ordered groupoids.

Given ordered groupoids G and H, the functor $\theta : G \to H$ is *levelling* if $g \leqslant h$ in G implies $g\theta = h\theta$ in H. Gilbert [5] constructs a groupoid $G_{\updownarrow}$ that is universal for levelling functors from G.

Let $\updownarrow$ be the equivalence relation on G generated by the partial order. Restrict $\updownarrow$ to the set of identities of G. Let $\lambda' : E(G) \to E(G)/\updownarrow$ be the quotient map and construct the universal groupoid $U_{\lambda'}(G)$, as defined in section 1.4.1. Now let N be the normal subgroupoid of $U_{\lambda'}(G)$ generated by the elements of the form $(a * b)b^{-1}a^{-1}$. Then $G_{\updownarrow}$ is defined as the quotient $U_{\lambda'}(G)/N$ with quotient map $\lambda : U_{\lambda'}(G) \to U_{\lambda'}(G)/N$. The groupoid $G_{\updownarrow}$ is called the *level groupoid of* G.

Note that $(a*b)b^{-1}a^{-1} \in N$ so $[(a*b)b^{-1}a^{-1}]$ is an identity of $U_{\lambda'}(G)/N = G_{\updownarrow}$, i.e. $((a * b)b^{-1}a^{-1})\lambda = e$ for some $e \in E(G_{\updownarrow})$, then $(a * b)\lambda(b^{-1}a^{-1})\lambda = e$ implies $(a * b)\lambda = (b^{-1}a^{-1})\lambda^{-1} = (ab)\lambda = (a\lambda)(b\lambda)$.

Gilbert shows the universality of $G_{\updownarrow}$ with the following lemma, [5, Lemma 2.1].

Lemma 4.3.2. *Let G, H be ordered groupoids.*
(i) Then $\lambda : G \to G_{\updownarrow}$ is a levelling functor.
(ii) If $\mu : G \to H$ is levelling, there exists a unique functor $\mu_{\updownarrow} : G_{\updownarrow} \to H$ such that $\mu = \lambda\mu_{\updownarrow}$.

Proof. (i) Suppose that $g \leqslant h$ in G. Then $\mathbf{d}(g) \leqslant \mathbf{d}(h)$ and $g^{-1} * h = g^{-1}(\mathbf{d}(g)|h) = g^{-1}g = \mathbf{r}(g)$. Hence in $G_{\updownarrow}$, $(g^{-1}\lambda)(h\lambda) = (\mathbf{r}(g))\lambda$. Therefore $g\lambda = h\lambda$.

(ii) Let $e, f \in E(G)$. If $e \updownarrow f$ then, since μ is levelling, $e\mu = f\mu$. It follows that μ induces a functor $\mu' : U_{\updownarrow}(G) \to H$. Now suppose that $a, b \in G$ have pseudoproduct $a * b = \tilde{a}\tilde{b}$ in G where $\tilde{a} = (a|\mathbf{r}(a)\mathbf{d}(b))$ and $\tilde{b} = (\mathbf{r}(a)\mathbf{d}(b)|b)$. Then

$$(a * b)\mu' = (\tilde{a}\tilde{b})\mu = (\tilde{a}\mu)(\tilde{b}\mu) = (a\mu)(b\mu) = (a\mu')(b\mu')$$

since μ is levelling.

It follows that μ' induces a functor $\mu_{\updownarrow} : G_{\updownarrow} \to H$ such that $a\lambda = a\mu$ which is therefore uniquely defined on $G_{\updownarrow}$. □

An ordered groupoid G is defined to be *incompressible* if $\lambda : G \to G_{\updownarrow}$ is star injective.

Let S be an inverse semigroup with zero, and let $\mathbb{G}(S)$ be the associated groupoid. Delete the zero from $\mathbb{G}(S)$ to get $\mathbb{G}(S)^*$.

Lemma 4.3.3. *[5, Lemma 2.5].*

1. *The inverse semigroup S is E-unitary if and only if $\mathbb{G}(S)$ is incompressible.*

2. *The inverse semigroup with zero S is strongly E^*-unitary if and only if $\mathbb{G}(S)^*$ is incompressible.*

Proof. (1) For inverse semigroup S the maximum group image is $\hat{S} = \mathbb{G}(S)_{\updownarrow}$. Now S is E-unitary if and only if $\sigma : S \to \hat{S}$ is idempotent pure, by theorem 1.1.22. By lemma 1.1.29, $\sigma : S \to \hat{S}$ is idempotent pure if and only if $\sigma : S \to \hat{S}$ is $\mathcal{R}$-injective. Converting via theorem 1.5.3, $\mathcal{R}$-injective map $\sigma : S \to \hat{S}$ becomes $\sigma : \mathcal{G}(S) \to \mathbb{G}(S)_{\updownarrow}$. Thus S is E-unitary if and only if $\mathbb{G}(S)$ is incompressible.

(2) Let S be strongly E^*-unitary. Then there exists a group G and a function $\theta : S \to G^0$, where G^0 is a group with zero adjoined, such that

(a) $s\theta = 0 \Leftrightarrow s = 0$,

(b) $s\theta = 1 \Leftrightarrow 0 \neq s \in E(S)$,

(c) $(st)\theta = (s\theta)(t\theta)$ if $st \neq 0$.

Regard θ as a functor $\mathbb{G}(S)^* \to G$. If $g, h \in \mathbb{G}(S)^*$ and $g \leqslant h$ then there exists an $e \in E(\mathbb{G}(S))$ with $g = eh$ so

$$g\theta = (eh)\theta = (e\theta)(h\theta) = 1(h\theta) = h\theta.$$

Thus θ is levelling. Suppose $s, t \in star_{\mathbb{G}(S)^*}(e)$ such that $s\theta = t\theta$. Then

$$(s^{-1}t)\theta = (s^{-1}\theta)(t\theta) = (s\theta)^{-1}(t\theta) = (t\theta)^{-1}(t\theta) = (t^{-1}t)\theta = 1.$$

Since $s^{-1}t \neq 0$ in $\mathbb{G}(S)^*$ and $(s^{-1}t)\theta = 1$, we have $s^{-1}t \in E(\mathbb{G}(S)^*)$. Thus $s = t$ and so θ is star injective. Hence $\mathbb{G}(S)^*$ is incompressible.

Let $\mathbb{G}(S)^*$ be incompressible. Construct the universal group, denoted $\underline{G}(S)$, of the level groupoid $(\mathbb{G}(S)^*)_\updownarrow$ by mapping all the identities to 1. Let $\pi : (\mathbb{G}(S)^*)_\updownarrow \to \underline{G}(S)$ be the canonical map. If $gh = k$ in $(\mathbb{G}(S)^*)_\updownarrow$ then $(g\pi)(h\pi) = (k\pi)$ in $\underline{G}(S)$. Let $\theta = \lambda\pi$. If $s, t \in S$ with $st \neq 0$ then $s^{-1}s \updownarrow tt^{-1}$ in $\mathbb{G}(S)^*$ and in $(\mathbb{G}(S)^*)_\updownarrow$ we have $(st)\lambda = (s\lambda)(t\lambda)$. Hence $(st)\theta = (s\theta)(t\theta)$. Suppose that $s \neq 0$ and that $s\lambda\pi = s\theta = 1$. This implies, by Higgins's solution to the word problem in $\underline{G}(S)$, [8, Cor 1 to Thrm 4, pg76], that $s\lambda$ must be an identity in $(\mathbb{G}(S)^*)_\updownarrow$. Since $\mathbb{G}(S)$ is incompressible, s must be an identity of $\mathbb{G}(S)^*$. It follows that θ, considered as a map $S \to \underline{G}(S)$, satisfies the conditions (1), (2) and (3) and so S is strongly E^*-unitary. $\square$

4.3.3 Incompressibility and the Margolis-Meakin Expansion

Margolis and Meakin [14] show that the expansion $(G, \Delta)^{MM}$ of a Δ–generated group G is again Δ-generated (as an inverse semigroup), and is an E-unitary inverse semigroup with maximum group image G. As we shall see below in Theorem 4.3.7, the E–unitary property arises because they begin with trivially E-unitary ingredients, namely groups. We show that a Margolis-Meakin expansion of an ordered groupoid G is incompressible if and only if G is incompressible, and that G and its expansion always have the same level groupoid (our analogue of the maximum group image). Firstly, we consider the property of Δ–generation.

Suppose that G is an ordered groupoid generated by (Δ, γ). Denote by $\Gamma(G, \Delta)$ the Cayley graph of the ordered groupoid G.

Lemma 4.3.4. *Let $(P, h) \in (G, \Delta)^{MM}$ and let $x \in \Delta$ be such that $\mathbf{r}(h) \leqslant \mathbf{d}(x\gamma)$. Denote by K the pattern of $\Gamma(G, \Delta)$ consisting of the single edge $(\mathbf{d}(x\gamma), x, x\gamma)$. Then*

$$(P, h) * (K, x\gamma) = (P', h * x\gamma)$$

where P' is obtained by adjoining to P the edge labelled x starting at h.

Proof. The pseudoproduct $(P, h) * (K, x\gamma)$ exists if $\mathbf{r}(P, h)$ and $\mathbf{d}(K, x\gamma)$ have a greatest lower bound. Now $\mathbf{r}(P, h) = (h^{-1} \cdot P, \mathbf{r}(h))$ and $\mathbf{d}(K, x\gamma) = (K, \mathbf{d}(x\gamma))$. As $\mathbf{r}(h) \leqslant \mathbf{d}(x\gamma)$ we can construct $(\mathbf{r}(h)|K)$ which is the pattern consisting of the single edge $(\mathbf{r}(h), x, \mathbf{r}(h)*x\gamma)$. We can then form the pattern $R = h^{-1} \cdot P \cup (\mathbf{r}(h)|K)$ and mark it at $\mathbf{r}(h)$. See Fig. 4.3.1.

Since $\mathbf{r}(h) = \mathbf{r}(h)$ and $(\mathbf{r}(h)|h^{-1} \cdot P) = h^{-1} \cdot P \subseteq R$ then $(R, \mathbf{r}(h)) \leqslant (h^{-1} \cdot P, \mathbf{r}(h))$. Also since $\mathbf{r}(h) \leqslant \mathbf{d}(x\gamma)$ and $(\mathbf{r}(h)|K) \subseteq R$ then $(R, \mathbf{r}(h)) \leqslant$

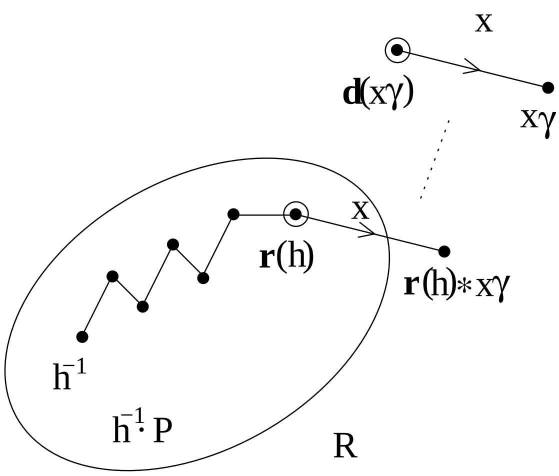

Figure 4.3.1: pattern R.

$(K, \mathbf{d}(x\gamma))$. So $(R, \mathbf{r}(h))$ is a lower bound of $\mathbf{r}(P, h)$ and $\mathbf{d}(K, x\gamma)$. It is clear by the construction of R that there is no smaller pattern that contains both $h^{-1} \cdot P$ and a copy of K. Thus $(R, \mathbf{r}(h)) = \mathbf{r}(P, h) \wedge \mathbf{d}(K, x\gamma)$. Now we can construct the pseudoproduct:

$$\begin{aligned}
(P, h) * (K, x\gamma) &= ((P, h)|(R, \mathbf{r}(h)))((R, \mathbf{r}(h))|(K, x\gamma)) \\
&= ((h|\mathbf{r}(h)) \cdot R, (h|\mathbf{r}(h)))(R, (\mathbf{r}(h)|x\gamma)) \\
&= (h \cdot R, h)(R, (\mathbf{r}(h)|x\gamma)).
\end{aligned}$$

Composition occurs because $h^{-1}h \cdot R = R$ and $\mathbf{r}(h) = \mathbf{d}(\mathbf{r}(h)|x\gamma)$. So

$$\begin{aligned}
(P, h) * (K, x\gamma) &= (h \cdot R, h(\mathbf{r}(h)|x\gamma)) \\
&= (h \cdot R, h * x\gamma).
\end{aligned}$$

The pattern $h \cdot R = h \cdot (h^{-1} \cdot P \cup (\mathbf{r}(h)|K))$ is then the pattern obtained by adjoining the edge $(h, x, h * x\gamma)$ to P. See Fig. 4.3.2.

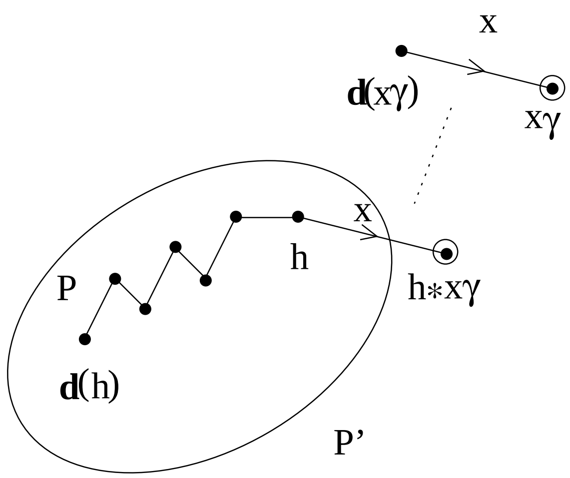

Figure 4.3.2: marked pattern $(P', h * x\gamma)$.

Therefore $(P, h) * (K, x\gamma) = (P', h * x\gamma)$. $\square$

Lemma 4.3.5. *Let K be the pattern of $\Gamma(G, \Delta)$ consisting of the single edge $(\mathbf{d}(x\gamma), x, x\gamma)$. Let K' be the pattern of $\Gamma(G, \Delta)$ consisting of the single edge $(\mathbf{d}(y\gamma), y, y\gamma)$. If $x\gamma * y\gamma$ exists and $\mathbf{r}(x\gamma) \wedge \mathbf{d}(y\gamma) = e$ then $(K, x\gamma), (K', y\gamma) \in (G, \Delta)^{MM}$ and*

$$(K, x\gamma) * (K', y\gamma) = (P', x\gamma * y\gamma)$$

where P' consists of the two edges $(\mathbf{d}(x\gamma|e), x, (x\gamma|e))$ and $((x\gamma|e), y, x\gamma * y\gamma)$.

Proof. The pseudoproduct $(K, x\gamma) * (K', y\gamma)$ exists if $\mathbf{r}(K, x\gamma) \wedge \mathbf{d}(K', y\gamma)$ exists. Now $\mathbf{r}(K, x\gamma) = ((x\gamma)^{-1} \cdot K, \mathbf{r}(x\gamma))$ and $\mathbf{d}(K', y\gamma) = (K', \mathbf{d}(y\gamma))$. As $\mathbf{r}(x\gamma) \geqslant e$ we can construct the pattern $((x\gamma)^{-1} \cdot K|e)$ which consists of the edge $(((x\gamma)^{-1}|e), x, e)$. Also, as $\mathbf{d}(y\gamma) \geqslant e$, we can construct the pattern $(e|K')$ which is the edge $(e, y, (e|(y\gamma)))$. We can combine these two edges at vertex e, denote the resulting pattern by R. See Fig. 4.3.3.

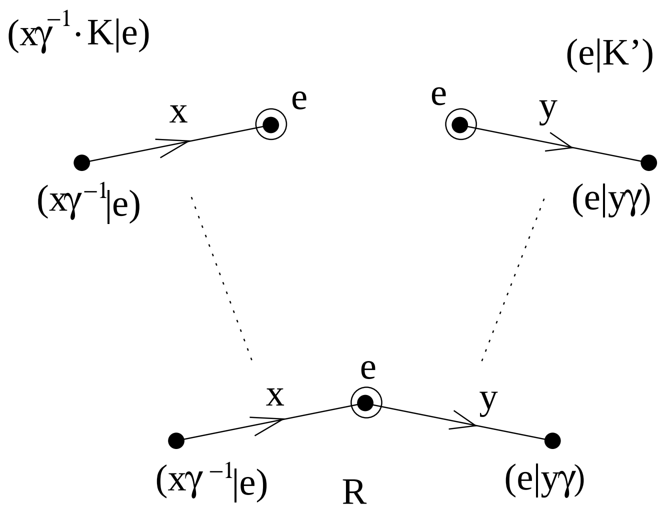

Figure 4.3.3: pattern R.

It is clear by the construction of R that $(R, e) = \mathbf{r}(K, x\gamma) \wedge \mathbf{d}(K', y\gamma)$. Thus

$$\begin{aligned}
&(K, x\gamma) * (K', y\gamma) \\
&= ((K, x\gamma)|(R, e))((R, e)|(K', y\gamma)) \\
&= ((x\gamma|e) \cdot R, (x\gamma|e))(R, (e|y\gamma)) \\
&= ((x\gamma|e) \cdot R, (x\gamma|e)(e|y\gamma)) \\
&= ((x\gamma|e) \cdot R, x\gamma * y\gamma) \\
&= (P', x\gamma * y\gamma)
\end{aligned}$$

where P' consists of the two edges $(\mathbf{d}(x\gamma|e), x, (x\gamma|e))$ and $((x\gamma|e), y, x\gamma{*}y\gamma)$. See Fig. 4.3.4.

□

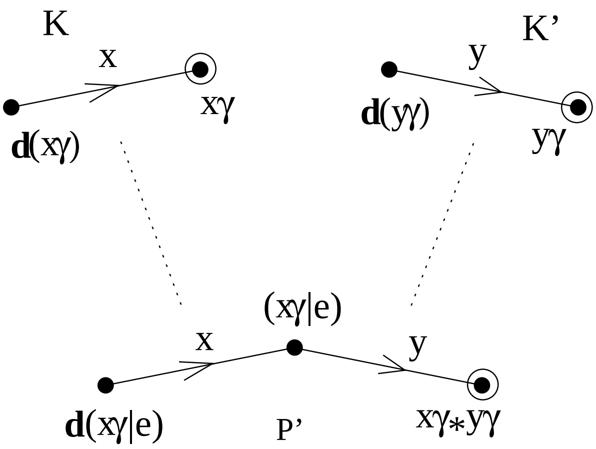

Figure 4.3.4: marked pattern $(P', x\gamma * y\gamma)$.

Theorem 4.3.6. *Let G be an ordered groupoid generated by Δ. Then the Margolis-Meakin expansion $(G, \Delta)^{MM}$ is an ordered groupoid generated by Δ.*

Proof. We define a function $\bar{\gamma} : \Delta \to (G, \Delta)^{MM}$ as follows: $x\bar{\gamma}$ is the marked pattern consisting of the single edge $(\mathbf{d}(x\gamma), x, x\gamma)$, marked at $x\gamma$.

Let (P, g) be a marked pattern in $(G, \Delta)^{MM}$. Then for some representation of g as a left-normed pseudoproduct of generating elements $g = (x_1\gamma)^{\varepsilon_1} * (x_2\gamma)^{\varepsilon_2} * \cdots * (x_m\gamma)^{\varepsilon_m}$ there exists a path $\mathbf{x}$ corresponding to the pseudoproduct. Further $(\mathbf{x}, g)$ is a marked pattern, and in $(G, \Delta)^{MM}$ we have, by the repeated use of lemma 4.3.5, $(\mathbf{x}, g) = (x_1\bar{\gamma})^{\varepsilon_1} * (x_2\bar{\gamma})^{\varepsilon_2} * \cdots * (x_m\bar{\gamma})^{\varepsilon_m}$, a left-normed pseudoproduct of elements of $(\Delta\bar{\gamma}) \cup (\Delta\bar{\gamma})^{-1}$.

Let (P, g) be a marked pattern that can be expressed as a left-normed pseudoproduct of elements of $(\Delta\bar{\gamma}) \cup (\Delta\bar{\gamma})^{-1}$:

$$(P, g) = a_1\bar{\gamma} * a_2\bar{\gamma} * \cdots * a_k\bar{\gamma}.$$

By lemma 3.7.2 we then have that $\mathbf{d}(P, g)$ equals the left-normed pseudoproduct:

$$\begin{aligned}\mathbf{d}(P,g) &= (P,\mathbf{d}(g))\\ &= a_1\bar{\gamma} * a_2\bar{\gamma} * \cdots * a_k\bar{\gamma} * (a_k\bar{\gamma})^{-1} * \cdots * (a_2\bar{\gamma})^{-1} * (a_1\bar{\gamma})^{-1}\end{aligned}$$

Now we consider adding edges onto patterns. Let (P,g) be a marked pattern, let h be a vertex of P and suppose $\mathbf{r}(h) \leqslant \mathbf{d}(x)$ for some $x \in \Delta$. There exists a path in P from $\mathbf{d}(g)$ to h, and so, by lemma 3.7.3, we get a left-normed pseudoproduct $\mathbf{d}(g) * (z_1\gamma)^{\varepsilon_1} * \cdots * (z_m\gamma)^{\varepsilon_m}$ equal to h. By lemma 4.3.4 we have $(P,\mathbf{d}(g)) * (z_1\bar{\gamma})^{\varepsilon_1} = (P', \mathbf{d}(g) * z_1\gamma^{\varepsilon_1})$ but the edge we added to P to get P' already exists in P, so $P' = P$. Hence $(P,\mathbf{d}(g)) * (z_1\bar{\gamma})^{\varepsilon_1} = (P, \mathbf{d}(g) * z_1\gamma^{\varepsilon_1})$. By repeated use of lemma 4.3.4 it follows that the left-normed pseudoproduct $(P,\mathbf{d}(g)) * (z_1\bar{\gamma})^{\varepsilon_1} * \cdots * (z_m\bar{\gamma})^{\varepsilon_m}$ equals $(P, \mathbf{d}(g) * z_1\gamma^{\varepsilon_1} * \cdots * z_m\gamma^{\varepsilon_m}) = (P,h)$. Thus (P,h) can be written as a left-normed pseudoproduct of elements of $(\Delta\bar{\gamma}) \cup (\Delta\bar{\gamma})^{-1}$. Since $\mathbf{r}(h) \leqslant \mathbf{d}(x\gamma)$ we have, by lemma 4.3.4, that

$$(P,h) * (x\bar{\gamma}) = (P', h * x\gamma)$$

where P' is the pattern P with edge $(h, x, h*x\gamma)$ adjoined to P. As (P,h) is a left-normed pseudoproduct then $(P', h * x\gamma)$ is a left-normed pseudoproduct of elements of $(\Delta\bar{\gamma}) \cup (\Delta\bar{\gamma})^{-1}$. It follows, by lemma 3.7.2 and because vertices from the same component of the Cayley graph come from the same star, that $\mathbf{d}(P', h * x\gamma) = (P', \mathbf{d}(h * x\gamma)) = (P', \mathbf{d}(g))$ is a left-normed pseudoproduct. Now because P' contains P which contains the path $\mathbf{x}$ from $\mathbf{d}(g)$ to g,

$$(P', \mathbf{d}(g)) * (\mathbf{x}, g) = (P', g).$$

So (P', g) is left-normed pseudoproduct of elements of $(\Delta\bar{\gamma}) \cup (\Delta\bar{\gamma})^{-1}$. We can similarly add an inverse edge to a marked pattern P.

So we can add an edge to a pattern (P,g) that is a left-normed pseudoproduct of elements of $(\Delta\bar{\gamma}) \cup (\Delta\bar{\gamma})^{-1}$ and get another pattern (P', g) which is also a left-normed pseudoproduct of elements of $(\Delta\bar{\gamma}) \cup (\Delta\bar{\gamma})^{-1}$. We can build a pattern P marked at g from the path $\mathbf{x}$ by adding finitely many edges $x\bar{\gamma}$ to it. Thus (P, g) can be written as a left-normed pseudoproduct of elements of $(\Delta\bar{\gamma}) \cup (\Delta\bar{\gamma})^{-1}$. Therefore $(G, \Delta)^{MM}$ is generated by $(\Delta, \bar{\gamma})$. □

Now we come to our main theorem of this chapter.

Theorem 4.3.7. *Let G be an ordered groupoid generated by Δ. Then the level groupoids $G_{\updownarrow}$ and $(G, \Delta)^{MM}_{\updownarrow}$ are isomorphic. Further, $(G, \Delta)^{MM}$ is incompressible if and only if G is incompressible.*

Proof. Let $\lambda : G \to G_{\updownarrow}$ and $\lambda_\star : (G, \Delta)^{MM} \to (G, \Delta)^{MM}_{\updownarrow}$ be the levelling functors.

First let us consider the element $g \in G$. We can write g in terms of the pseudoproduct of generating elements, $g = a_1 * a_2 * \cdots * a_m$ where $a_j \in (\Delta\gamma \cup \Delta^{-1}\gamma)$. Then in the Cayley graph $\Gamma(G, \Delta)$ there exists a path $\mathbf{a}$ starting at vertex $\mathbf{d}(g)$, ending at the vertex g and labelled by $(a_1, a_2, \ldots, a_m)$. Hence we have a marked pattern $(\mathbf{a}, g)$. If we also have $g = b_1 * b_2 * \cdots * b_n$ with $b_i \in \Delta\gamma \cup \Delta^{-1}\gamma$ giving rise to the marked pattern $(\mathbf{b}, g)$, then $(\mathbf{a} \cup \mathbf{b}, g)$ is also a marked pattern. Further, $(\mathbf{a} \cup \mathbf{b}, g) \leqslant (\mathbf{a}, g)$ and $(\mathbf{a} \cup \mathbf{b}, g) \leqslant (\mathbf{b}, g)$, so $(\mathbf{a}, g) \updownarrow (\mathbf{b}, g)$. Moreover, for any marked pattern (P, g), the subgraph P must contain some path $\mathbf{c}$ from $\mathbf{d}(g)$ to g so that $(P, g) \leqslant (\mathbf{c}, g)$. It follows that for two marked patterns with the same marked element $g \in G$, $(P, g), (Q, g) \in (G, \Delta)^{MM}$, we have that $(P, g) \updownarrow (Q, g)$ and so $(P, g)\lambda_\star = (Q, g)\lambda_\star$

Define $\alpha : G \to (G, \Delta)^{MM}_{\updownarrow}$ by $g \mapsto (P, g)\lambda_\star$ where P is any pattern with marked vertex g. By the argument above α is well-defined.

Now suppose that $g, h \in G$ with $g \leqslant h$. Consider a marked pattern (Q, h). Then $((\mathbf{d}(g)|Q), g) \leqslant (Q, h)$ in $(G, \Delta)^{MM}$ and so $((\mathbf{d}(g)|Q), g)\lambda_\star = (Q, h)\lambda_\star$. It follows that $g\alpha = h\alpha$, and so α is levelling. Then α induces a functor β :

$G_\updownarrow \to (G,\Delta)^{MM}_\updownarrow$ carrying $g\lambda \mapsto (P,g)\lambda_\star$. We show that β is indeed a functor. Let $(P,g),(Q,h) \in (G,\Delta)^{MM}_\updownarrow$. By the construction of the levelling groupoid $(g\lambda)(h\lambda) = (gh)\lambda$, similarly for $\lambda_\star$. Assume $\mathbf{r}(g) = \mathbf{d}(h)$ and $g^{-1} \cdot P = Q$. Then,

$$\begin{aligned}
(g\lambda)\beta(h\lambda)\beta &= (P,g)\lambda_\star(Q,h)\lambda_\star \\
&= ((P,g)(Q,h))\lambda_\star \\
&= (P,gh)\lambda_\star \\
&= ((gh)\lambda)\beta \\
&= ((g\lambda)(h\lambda))\beta.
\end{aligned}$$

Take the map $\theta : (G,\Delta)^{MM} \to G$ to be defined as $(P,g) \mapsto g$.

We show that the map θ is star injective. Let $(P,g),(Q,h) \in \mathrm{star}_{(G,\Delta)^{MM}}(R,e)$ and suppose $(P,g)\theta = (Q,h)\theta$. Then $\mathbf{d}(P,g) = (P,\mathbf{d}(g)) = (R,e)$ and $\mathbf{d}(Q,h) = (Q,\mathbf{d}(h)) = (R,e)$, so $R = P = Q$. Further $\mathbf{d}(g) = \mathbf{d}(h) = e$ so g and h belong to the star of e in G. Also, $(P,g)\theta = g = h = (Q,h)\theta$ so $(P,g) = (Q,h)$ and θ is star injective.

The composition $\theta\lambda$ is levelling, for if $(P,g) \leqslant (Q,h)$ then $g \leqslant h$ so $g\lambda = h\lambda$ thus $(P,g)\theta\lambda = (Q,h)\theta\lambda$. Then $\theta\lambda$ induces a functor $\gamma : (G,\Delta)^{MM}_\updownarrow \to G_\updownarrow$ that carries $(P,g)\lambda_\star \mapsto g\lambda$.

Clearly β and γ are inverse functors, and $G_\updownarrow$ is isomorphic to $(G,\Delta)^{MM}_\updownarrow$.

We have the following commutative diagram:

$$\begin{tikzcd}
(G,\Delta)^{MM} \arrow[r, "\lambda_\star"] \arrow[d, "\theta"'] & (G,\Delta)^{MM}_\updownarrow \arrow[d, "\gamma"] \\
G \arrow[r, "\lambda"'] & G_\updownarrow
\end{tikzcd}$$

with $(P,g)\lambda_\star\gamma = (P,g)\theta\lambda = g\lambda$.

Assume G is incompressible, then λ is star injective. As θ is star injective and γ is an isomorphism, we have that $\lambda_\star$ is star injective so $(G, \Delta)^{MM}$ is incompressible.

Similarly if we suppose $(G, \Delta)^{MM}$ is incompressible this implies that G is incompressible. $\square$

Chapter 5

The $\boldsymbol{P}$-theorem and $\mathbb{L}$-systems

McAlister's P-theorem, [16], classifies all E-unitary inverse semigroups. This classification relies upon an action of the maximal group image of the semigroup on some poset. A more general P-theorem is that of Gilbert [5] for ordered groupoids which classifies all incompressible ordered groupoids. This classification employs the use of an action of a groupoid on some poset. There are many proofs of McAlister's P-theorem for inverse semigroups, some of which are reviewed in [12]. We give an alternative proof for the P-theorem for ordered groupoids based on Steinberg's succinct proof involving Schützenberger graphs [26].

E-unitary inverse semigroups are semigroups that are idempotent pure extensions by groups. O'Carroll [21] proved a structure theorem for inverse semigroups that are idempotent pure extensions by inverse semigroups. O'Carroll's theorem generalises McAlister's P-theorem and involves an action of a semigroup on a poset. Idempotent pure homomorphisms of inverse semigroups correspond to immersions of inductive groupoids. Given an immersion of ordered groupoids $\nu : G \to T$ we wish to formulate a structure theorem for G similar to the P-theorem. The result uses the groupoid analogue of

O'Carroll's $\mathbb{L}$-semigroups, $\mathbb{L}$-systems, along with a version of Ehresmann's Maximum Enlargement Theorem. Ehresmann's Maximum Enlargement Theorem is a structure theorem for immersions of ordered groupoids and generalises the role of the *action groupoid* for coverings. We discuss the relationship between the action groupoid and coverings of ordered groupoids before giving our version of the Maximum Enlargement Theorem. Then we state our $\mathbb{L}$-system structure theorem for ordered groupoids and we show how this theorem generalises the P-theorem for ordered groupoids.

5.1 McAlister's P-theorem for Inverse Semigroups

McAlister's P-theorem classifies all E-unitary inverse semigroups. The P-theorem describes the structure of an E-unitary inverse semigroup in terms of an action of a group on a poset. The details are as follows.

We suppose a group G acts on a poset X. Suppose that X has an order ideal Y which is a semilattice. Then if $G \cdot Y = X$ and $g \cdot Y \cap Y \neq \emptyset$ for every $g \in G$ we say that (X, Y, G) is a *McAlister triple*. Given a McAlister triple we define a P-semigroup, [16],

$$P(X, Y, G) = \{(y, g) \in Y \times G : g^{-1} \cdot y \in Y\}$$

with composition

$$(y, g)(x, h) = (g \cdot (g^{-1} \cdot y \wedge x), gh).$$

Proposition 5.1.1. *$P(X, Y, G)$ is an E-unitary inverse semigroup with natural partial order given as follows,*

$$(y, g) \leqslant (x, h) \Leftrightarrow y = x \wedge h \cdot p \text{ for some } p \in Y, \text{ and } g = h.$$

Proof. Let $(y, g), (x, h) \in P(X, Y, G)$. Then $g^{-1} \cdot y \in Y$ by definition of $P(X, Y, G)$ and, as Y is a semilattice, $g^{-1} \cdot y \wedge x \in Y$. Now $g^{-1} \cdot y \wedge x \leqslant g^{-1} \cdot y$ so $g \cdot (g^{-1} \cdot y \wedge x) \leqslant g \cdot g^{-1} \cdot y = y \in Y$. Y is an order ideal so $g \cdot (g^{-1} \cdot y \wedge x) \in Y$. Also

$$\begin{aligned}
&(gh)^{-1} \cdot g \cdot (g^{-1} \cdot y \wedge x) \\
&= h^{-1} \cdot (g^{-1}g) \cdot (g^{-1} \cdot y \wedge x) \\
&= h^{-1} \cdot 1 \cdot (g^{-1}y \wedge x) \\
&= h^{-1} \cdot (g^{-1} \cdot y \wedge x) \\
&\leqslant h^{-1} \cdot x \in Y.
\end{aligned}$$

Y is an order ideal so $(gh)^{-1} \cdot g \cdot (g^{-1} \cdot y \wedge x) \in Y$ and so composition is well-defined. It is also associative. Now,

$$\begin{aligned}
&(y, g)(g^{-1}y, g^{-1})(y, g) \\
&= (g \cdot (g^{-1} \cdot y \wedge g^{-1} \cdot y), gg^{-1})(y, g) \\
&= (g \cdot (g^{-1} \cdot y), 1)(y, g) \\
&= (y, 1)(y, g) \\
&= (1 \cdot (1 \cdot y \wedge y), 1) \\
&= (y, g)
\end{aligned}$$

so P is regular. The idempotents are of the form $(y, 1)$ where 1 is the identity element of G. Given $(y, 1), (x, 1) \in P(X, Y, G)$ we see that

$$(y, 1)(x, 1) = (1 \cdot (1^{-1} \cdot y \wedge x), 1) = (y \wedge x, 1) = (x \wedge y, 1) = (x, 1)(y, 1)$$

so idempotents commute. Therefore $P(X, Y, G)$ is an inverse semigroup.

Let $(y, 1) \leqslant (x, h)$. Then $1 = h$ and $(x, h) = (x, 1) \in E(P(X, Y, G))$ and $P(X, Y, G)$ is E-unitary. $\square$

McAlister's P-theorem [16] states that any E-unitary inverse semigroup is isomorphic to $P(X, Y, G)$ for some McAlister triple (X, Y, G). Thus, up to isomorphism and equivalence of group actions, McAlister has classified all E-unitary inverse semigroups.

5.2 P-theorem for Ordered Groupoids

As seen in section 4.3.2, the incompressibility concept for ordered groupoids is analogous to the E-unitary property. Gilbert gives an analogue to McAlister's P-theorem in [5]. We describe Gilbert's analogue to the P-theorem and give a detailed alternative proof. This proof is inspired by Steinberg's succinct proof of McAlister's P-theorem using Schützenberger graphs [26].

5.2.1 The Ordered Groupoid $P(X, Y, G)$

Gilbert gives a P-theorem for ordered groupoids in [5]. We now give the details of Gilbert's construction of an incompressible ordered groupoid $P(X, Y, G)$ and then state the P-theorem for ordered groupoids.

Consider a contravariant functor from a groupoid G to the category of posets. For each $e \in E(G)$ we have a poset X_e. Each arrow $g \in G$ determines an isomorphism $g : X_{\mathbf{r}(g)} \to X_{\mathbf{d}(g)}$. We say G acts on the disjoint union

$$X = \bigsqcup_{e \in E(G)} X_e.$$

If Y is an order ideal in X we set $Y_e = Y \cap X_e$. In the construction of the ordered groupoid $P(X, Y, G)$ only elements of the G-invariant poset $G \cdot Y$ need be considered, and so without loss of generality, we may assume that $X = G \cdot Y$.

As in [5], we define an ordered groupoid $P(X, Y, G)$ as follows:

$$P(X, Y, G) = \{(y, a) \in Y \times G : y \in Y_{\mathbf{d}(a)}, a^{-1} \cdot y \in Y\}$$

with composition between elements (x, a) and (z, b) occuring if $\mathbf{r}(a) = \mathbf{d}(b)$ and $a^{-1} \cdot x = z$, in which case $(x, a)(z, b) = (x, ab)$. The element $(y, a) \in P(X, Y, G)$ has domain $\mathbf{d}(y, a) = (y, \mathbf{d}(a))$ and range $\mathbf{r}(y, a) = (a^{-1} \cdot y, \mathbf{r}(a))$. The inverse of (y, a) is $(y, a)^{-1} = (a^{-1} \cdot y, a^{-1})$. The set of identites is

$$E(P(X, Y, G)) = \{(y, e) : y \in Y_e, e \in E(G)\}.$$

The ordering is given by

$$(x, a) \leqslant (z, b) \Leftrightarrow x \leqslant z \text{ and } a = b.$$

Lemma 5.2.1. *[5, Lemma 3.1]. $P = P(X, Y, G)$ is an incompressible ordered groupoid.*

Proof. First we check the axioms for an ordered groupoid.

(OG1): Suppose $(x, a) \leqslant (z, b)$. Then $x \leqslant z$ and $a = b$ so $a^{-1} = b^{-1}$ and $a^{-1} \cdot x \leqslant a^{-1} \cdot z = b^{-1} \cdot z$. Therefore $(x, a)^{-1} = (a^{-1} \cdot x, a^{-1}) \leqslant (b^{-1} \cdot z, b^{-1}) = (z, b)^{-1}$.

(OG2): Suppose (x, a) and (z, b) are composable, so $\mathbf{r}(a) = \mathbf{d}(b)$, $a^{-1} \cdot x = z$ and $(x, a)(z, b) = (x, ab)$, and suppose (v, c) and (y, d) are composable, so $\mathbf{r}(c) = \mathbf{d}(d)$, $c^{-1} \cdot v = y$ and $(v, c)(y, d) = (v, cd)$. Suppose also that $(x, a) \leqslant (v, c)$ and $(z, b) \leqslant (y, d)$. Then $x \leqslant v$, $a = c$ and $z \leqslant y$, $b = d$. Thus $ab = cd$ and $x \leqslant v$. Hence $(x, a)(z, b) = (x, ab) \leqslant (v, cd) = (v, c)(y, d)$.

(OG3): Suppose $(x, a) \in P(X, Y, G)$ and $(y, e) \in E(P)$ such that $(y, e) \leqslant \mathbf{d}(x, a) = (x, \mathbf{d}(a))$. This is true if and only if $y \leqslant x$ and $e = \mathbf{d}(a)$. We require $(z, c) \in P(X, Y, G)$ such that $\mathbf{d}(z, c) = (y, e) = (y, \mathbf{d}(a))$ and $(z, c) \leqslant (x, a)$. Now $\mathbf{d}(z, c) = (z, \mathbf{d}(c)) = (y, \mathbf{d}(a))$ and so z must equal y. Also $(z, c) \leqslant (x, a)$

if and only if $z \leqslant x$ and $c = a$, so c must equal a. Thus the restriction of (x, a) to (y, e) is $(z, c) = (y, a)$ and by construction is unique.

Next consider $\mu : P(X, Y, G) \to G$ carrying $(y, g) \mapsto g$. We show that μ is a levelling functor. Let $(y, g), (x, h) \in P(X, Y, G)$. Then $(y, g)\mu(x, h)\mu = gh$ provided $\mathbf{r}(g) = \mathbf{d}(h)$. Also $[(y, g)(x, h)]\mu = (y, gh)\mu = gh$ provided $\mathbf{r}(g) = \mathbf{d}(h)$ and $g^{-1} \cdot y = x$. If $(y, g) \leqslant (x, h)$ then $y \leqslant x$ and $g = h$. Hence $(y, g)\mu = g = h = (x, h)\mu$. So μ is a levelling functor.

Now μ is star injective, for suppose that $(y, g), (x, h) \in \mathrm{star}_P(z, e)$ with $(y, g)\mu = (x, h)\mu$. Then $\mathbf{d}(y, g) = \mathbf{d}(x, h) = (z, e)$ which implies $(y, \mathbf{d}(g)) = (x, \mathbf{d}(h)) = (z, e)$ and so $y = x = z$. Also $\mathbf{d}(g) = \mathbf{d}(h) = e$ so $g, h \in \mathrm{star}_G(e)$. Further $(y, g)\mu = (x, h)\mu$ implies $g = h$. Thus $(y, g) = (x, h)$.

Let $(y, a), (x, b) \in \mathrm{star}_P(z, e)$. Take $\lambda : P(X, Y, G) \to P(X, Y, G)_{\updownarrow}$ to be the levelling functor and suppose $(y, a)\lambda = (x, b)\lambda$. Then by lemma 4.3.2, there exists a functor $\mu_{\updownarrow} : P(X, Y, G)_{\updownarrow} \to G$ such that $\mu = \lambda\mu_{\updownarrow}$. Since μ is star injective then so is λ, and therefore $P(X, Y, G)$ is an incompressible ordered groupoid. $\square$

In [5] Gilbert proves that any incompressible ordered groupoid is isomorphic to $P(X, Y, G)$ for some poset X, some order ideal Y of X and some groupoid G. Hence all incompressible ordered groupoids are classified up to isomorphism. We shall give an alternative proof in the next section.

5.2.2 The P-theorem for Ordered Groupoids

Now that we have constructed the ordered groupoid $P(X, Y, G)$ we shall give a proof of the P-theorem for ordered groupoids using components of Cayley graphs, which we described earlier in section 3.7. Steinberg proves McAlister's P-theorem for inverse semigroups in [26] using Schützenberger graphs.

Gilbert's approach to classifying all incompressible ordered groupoids, [5], follows McAlister's description [17] of Munn's proof [20]. In this section we reprove Gilbert's result but following Steinberg's method using Schützenberger graphs and so give an alternative proof of the P-theorem for ordered groupoids.

Let G be an incompressible ordered groupoid generated by (Δ, γ), so every element $g \in G$ is a left-normed pseudoproduct of arrows of $\Delta\gamma \cup \Delta^{-1}\gamma$, say $g = a_1 * \cdots * a_m$. Then the composition $(a_1\lambda) \ldots (a_m\lambda)$ is defined in $G_{\updownarrow}$ and so the level groupoid is generated by $(\Delta, \gamma\lambda)$. From now on we suppress mention of γ and $\gamma\lambda$ when dealing with the generating set Δ. If G is incompressible then $\lambda : G \to G_{\updownarrow}$, induces an embedding of each component Γ_e of the Cayley graph of G into the Cayley graph of $G_{\updownarrow}$ (as $\lambda : G \to G_{\updownarrow}$ is star injective). Let Y be the set of all such embedded components of the Cayley graph of G. Now $G_{\updownarrow}$ acts on Y: if $g \in G_{\updownarrow}$ has $\mathbf{d}(g) = d$ and $\mathbf{r}(g) = e\lambda$ then g acts on the embedded copy Γ_e by left multiplication on the vertex sets, and $g \cdot \Gamma_e$ is some connected subgraph of the component $\bar{\Gamma}_d$ of $\Gamma(G_{\updownarrow}, \Delta)$. We let $X = G_{\updownarrow} \cdot Y$. Then X is a poset ordered by reverse inclusion.

Theorem 5.2.2. *An incompressible ordered groupoid G is isomorphic to the ordered groupoid $P(X, Y, G_{\updownarrow})$.*

Proof. We begin by showing that Y is an order ideal of X. For this we need to show that if $g \cdot \Gamma_e$ contains an embedded copy of some Γ_f, then it is itself an embedded copy of some Γ_x.

By assumption we have, for $g \in G_{\updownarrow}(d, e\lambda)$,

$$\bar{\Gamma}_d \supseteq g \cdot \Gamma_e \supseteq \Gamma_f \in Y$$

where $\bar{\Gamma}_d$ is a component of $\Gamma(G_{\updownarrow}, \Delta)$. Hence $f\lambda = d$ and so $d \in g \cdot \Gamma_e$.

There exists a path in $g \cdot \Gamma_e$ from vertex g to vertex d. Suppose $(g_i, a_i^{\epsilon_i}, g_i a_i^{\epsilon_i})$ for $1 \leqslant i \leqslant r$ are the successive edges of this path. Then $(a_1^{\epsilon_1}, \ldots, a_r^{\epsilon_r})$ labels a path u in $\Gamma(G,\Delta)$ and so the left-normed pseudoproduct $u = e * a_1^{\epsilon_1} * \cdots * a_r^{\epsilon_r}$ exists in G and $u\lambda = g^{-1}$. We take x to be $u^{-1}eu$ and show that, as subgraphs of the Schützenberger graph of $G_{\updownarrow}$,

$$g \cdot \Gamma_e = \Gamma_{u^{-1}eu}.$$

It is enough to check that their vertex sets are the same. A typical vertex of $g{\cdot}\Gamma_e$ is $g(h\lambda)$ where $h \in G$ and $\mathbf{d}(h) = e$. Now $g(h\lambda) = (u\lambda)^{-1}(h\lambda) = (u^{-1}h)\lambda$ and $\mathbf{d}(u^{-1}h) = (u^{-1}h)(u^{-1}h)^{-1} = u^{-1}(hh^{-1})u = u^{-1}eu$. Hence $g(h\lambda)$ is also a vertex of $\Gamma_{u^{-1}eu}$. Conversely, a typical vertex of $\Gamma_{u^{-1}eu}$ has the form $k\lambda$ where $\mathrm{d}(k) = u^{-1}eu$. Therefore $\mathrm{d}(k\lambda) = (u^{-1}eu)\lambda = (u\lambda)^{-1}(e\lambda)(u\lambda) = g(e\lambda)g^{-1} = gg^{-1} = d$, and so $g^{-1}(k\lambda)$ is defined in $G_{\updownarrow}$. So we may write $k\lambda = g(g^{-1}(k\lambda))$ with $g^{-1}(k\lambda)$ a vertex of Γ_e. Thus Y is an order ideal of X.

Now we have constructed the ingredients for an ordered groupoid $P(X,Y,G_{\updownarrow})$. Indeed,

$$P(X,Y,G_{\updownarrow}) = \{(\Gamma_{\mathbf{d}(g)}, g\lambda) \in Y \times G_{\updownarrow} : (g^{-1})\lambda \cdot \Gamma_{\mathbf{d}(g)} \in Y\}.$$

Let $\phi : G \to P(X,Y,G_{\updownarrow})$ map $g \mapsto (\Gamma_{\mathbf{d}(g)}, g\lambda)$. We show ϕ is an isomorphism. Let $g,h \in G$. Provided $\mathbf{r}(g) = \mathbf{d}(h)$ and $g^{-1}\lambda \cdot \Gamma_{\mathbf{d}(g)} = \Gamma_{\mathbf{d}(h)}$, then

$$\begin{aligned}(g\phi)(h\phi) &= (\Gamma_{\mathbf{d}(g)}, g\lambda)(\Gamma_{\mathbf{d}(h)}, h\lambda)\\ &= (\Gamma_{\mathbf{d}(g)}, g\lambda h\lambda)\\ &= (\Gamma_{\mathbf{d}(g)}, (gh)\lambda)\\ &= (gh)\phi\end{aligned}$$

and so ϕ is a functor. If $g \leqslant h$ in G then $(\mathbf{d}(g)|\Gamma_{\mathbf{d}(h)}) \subseteq \Gamma_{\mathbf{d}(h)}$, so $(\Gamma_{\mathbf{d}(g)}, g\lambda) \leqslant (\Gamma_{\mathbf{d}(h)}, h\lambda)$ and so ϕ is an ordered functor.

We show next that ϕ is bijective. Given $(\Gamma_{\mathbf{d}(a)}, a\lambda) \in P(X,Y,G_{\updownarrow})$ then clearly we have $a \in G$ that maps via λ to $a\lambda$. Hence a maps to $(\Gamma_{\mathbf{d}(a)}, a\lambda)$ and ϕ

is surjective. We wish to show injectivity. Consider $\Gamma_{\mathbf{d}(g)}$ with distinguished vertex $g\lambda$. Now $\Gamma_{\mathbf{d}(g)}$ contains only one vertex labelled by an identity of $G_{\updownarrow}$, namely $d = \mathbf{d}(g)\lambda$. As g is equal to some left-normed pseudoproduct $a_1^{\epsilon_1} * \cdots * a_m^{\epsilon_m}$ and by lemma 3.7.3 there is a corresponding path in $\Gamma(G, \Delta)$ from $\mathbf{d}(g)$ to g, and so in $\bar{\Gamma}_{\mathbf{d}(g)}$ there is a path from d to $g\lambda$ whose label is $(a_1^{\epsilon_1}, \ldots, a_m^{\epsilon_m})$. Consider only those paths in $\Gamma_{\mathbf{d}(g)}$ from d to $g\lambda$ whose labels possess a left-normed pseudoproduct in G. If $h = b_1 * \cdots * b_n$ is the left-normed pseudoproduct of the label of such a path there will exist a path in $\Gamma(G, \Delta)$ from $\mathbf{d}(g)$ to g with the same label. Therefore $g = \mathbf{d}(g) * b_1 * \cdots * b_n$, and so $g \leqslant b_1 * \cdots * b_n = h$, and g is the minimum element of G obtained as the evaluation of a left-normed pseudoproduct of the label of a path from d to $g\lambda$ in $\Gamma_{\mathbf{d}(g)}$. Hence the pair $(\Gamma_{\mathbf{d}(g)}, g\lambda)$ determines $g \in G$, and therefore ϕ is injective.

Therefore ϕ is an isomorphism. $\square$

Example 5.2.3. Let G be the following incompressible ordered groupoid consisting of two copies of $\mathbb{I}^1$ as shown in Fig. 5.2.1, with $\Delta = \{\alpha, \alpha^{-1}, \beta, \beta^{-1}\}$ the generating set for G. The ordering is $e_0 \geqslant f_0$, $\alpha \geqslant \beta$, $\alpha^{-1} \geqslant \beta^{-1}$, $e_1 \geqslant f_1$. The level groupoid $G_{\updownarrow}$ is then a single copy of $\mathbb{I}^1$ as shown in Fig. 5.2.2. There are four components of the Cayley graph corresponding to the four identites of G, see Fig. 5.2.3. $\Gamma(G_{\updownarrow}, \Delta)$ is shown in Fig. 5.2.4. We consider each component of (G, Δ) as embedded in $\Gamma(G_{\updownarrow}, \Delta)$. Recall Y is the set of all such Schützenberger graphs and $X = G_{\updownarrow} \cdot Y$. For each $g \in \Gamma(G_{\updownarrow}, \Delta)$ we have isomorphism of posets $X_{\mathbf{r}(g)} \to X_{\mathbf{d}(g)}$. In this example $g \cdot \Gamma_{\mathbf{r}(g)} = \Gamma_{\mathbf{d}(g)}$ for all $g \in G_{\updownarrow}$, so we can clearly see that Y is an order ideal in X.

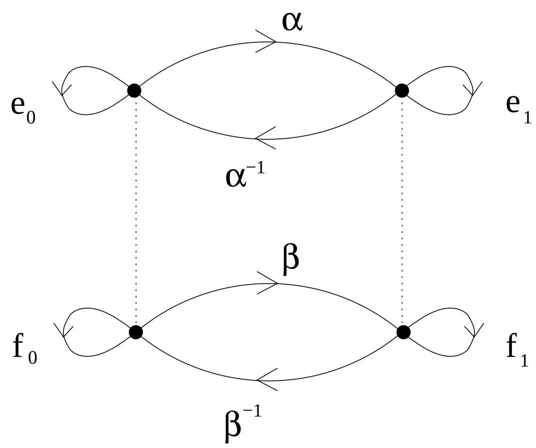

Figure 5.2.1: groupoid G.

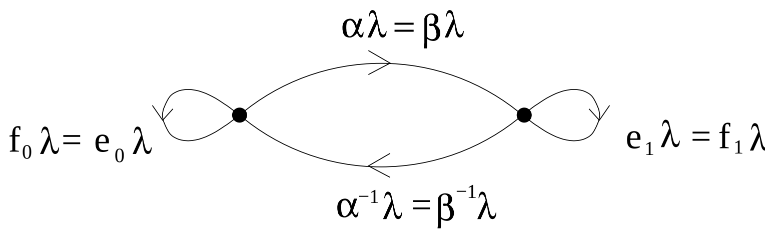

Figure 5.2.2: level groupoid $G_{\updownarrow}$

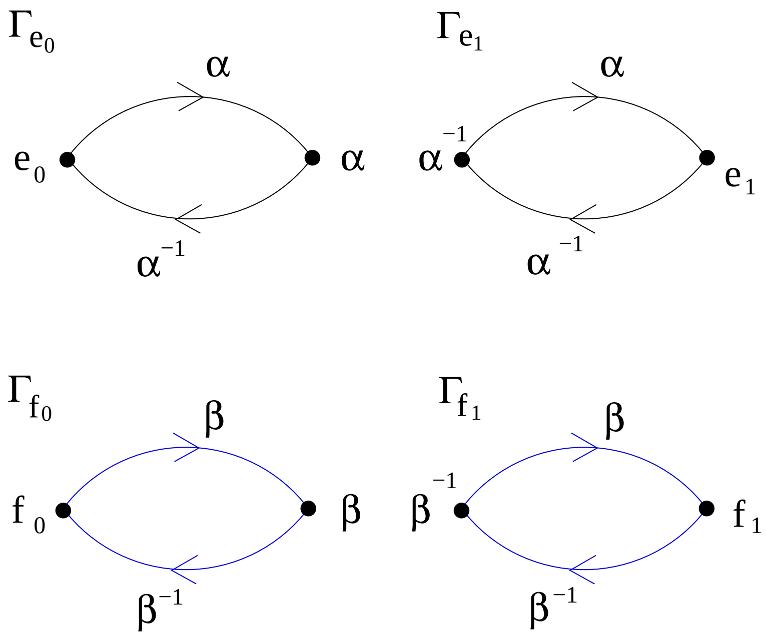

Figure 5.2.3: components.

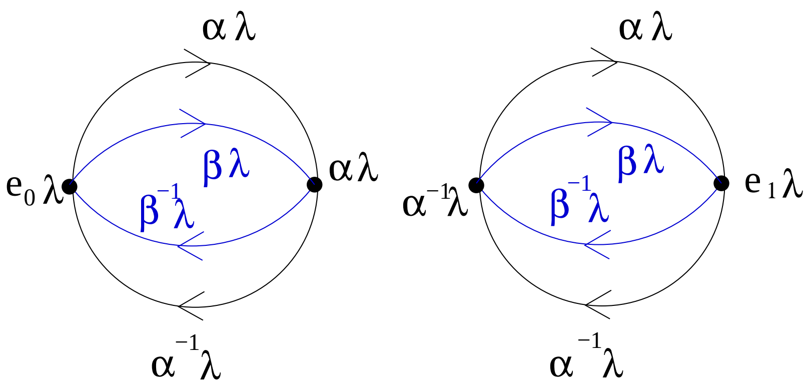

Figure 5.2.4: $\Gamma(G_{\updownarrow}, \Delta)$.

$$\begin{aligned}P(X,Y,G_{\updownarrow}) &= \{(y,g)\in Y\times G_{\updownarrow} : y\in Y_{\mathbf{d}(g)}, g^{-1}\cdot y\in Y\}\\ &= \{(\Gamma_{e_0}, e_0\lambda), (\Gamma_{e_0}, \alpha\lambda), (\Gamma_{e_1}, \alpha^{-1}\lambda), (\Gamma_{e_1}, e_1\lambda),\\ &\quad (\Gamma_{f_0}, f_0\lambda), (\Gamma_{f_0}, \beta\lambda), (\Gamma_{f_1}, \beta^{-1}\lambda), (\Gamma_{f_1}, f_1\lambda)\}\end{aligned}$$

We see here that $P(X,Y,G_{\updownarrow})$ and G contain the same number of elements, an excellent start if the two are isomorphic. $P(X,Y,G_{\updownarrow})$ is indeed isomorphic to G, $\phi : G \to P(X,Y,G_{\updownarrow})$ mapping $g \mapsto (\Gamma_{\mathbf{d}(g)}, g\lambda)$ is an isomorphism.

5.3 Fibred Actions and Ordered Coverings

Lawson [10] proves McAlister's P-theorem using Ehresmann's Maximum Enlargement Theorem, a structure theorem for immersions of ordered groupoids. When given an immersion from an ordered groupoid G to some "simpler" ordered groupoid T we want a structure theorem for G somewhat like the P-theorem. We give a version of the Maximum Enlargement Theorem following Lawson, [10, pg 256]. Then we introduce the analogue of O'Carroll's $\mathbb{L}$-semigroups for groupoids and give the structure theorem in section 5.5.

Before we discuss the Maximum Enlargement Theorem or our generalisation of the P-theorem we first introduce fibred actions and ordered coverings. In this section we plan to show an equivalence between two categories, one of fibred actions of ordered groupoids on posets and the other of ordered coverings of ordered groupoids. We begin with a description of fibred actions and action groupoids. Following this we look at coverings of ordered groupoids.

5.3.1 Fibred Actions

In this subsection we define a left fibred action of an ordered groupoid on a poset and after this we will construct an action groupoid from this fibred action.

We begin with the introduction of some necessary notation. Given a poset $(X, \leqslant)$ we denote the set of all isomorphisms between principal order ideals of X by $\Sigma(X)$. For any $e \in X$ let $X(e) = \{x : x \leqslant e\} \subseteq X$. Let $\alpha : X(e) \to X(f), \beta : X(m) \to X(n) \in \Sigma(X)$. Then composition between α and β occurs if $f = m$, in which case,

$$\alpha\beta : X(e) \to X(n).$$

The ordering is given by

$$\beta \leqslant \alpha \Leftrightarrow \beta = \alpha|_{X(m)}.$$

So if $\beta \leqslant \alpha$ then $X(m) \subseteq X(e)$ and $X(n) \subseteq X(f)$, equivalently $m \leqslant e$ and $n \leqslant f$.

Lemma 5.3.1. *$\Sigma(X)$ is an ordered groupoid. If X is a meet semilattice then $\Sigma(X)$ is an inductive groupoid.*

Proof. First we note that the identities are the identity maps $\iota_k : X(k) \to X(k)$. Arrow $\alpha : X(e) \to X(f) \in \Sigma(X)$ has domain ι_e and range ι_f. The restriction of the order of $\Sigma(X)$ to the set of identites gives

$$\iota_e \leqslant \iota_f \Leftrightarrow X(e) \subseteq X(f) \Leftrightarrow e \leqslant f.$$

We check the axioms for an ordered groupoid.

(OG1) Let $\alpha : X(e) \to X(f)$ and $\beta : X(m) \to X(n)$ and suppose $\beta \leqslant \alpha$. Then $\beta = \alpha|_{X(m)}$ so $\beta^{-1} = (\alpha|_{X(m)})^{-1} = \alpha^{-1}|_{X(n)}$. Thus $\beta^{-1} \leqslant \alpha^{-1}$.

(OG2) Let $\alpha : X(e) \to X(f)$, $\gamma : X(f) \to X(g)$, $\beta : X(m) \to X(n)$ and $\delta : X(n) \to X(p)$ be such that $\beta \leqslant \alpha$ and $\delta \leqslant \gamma$. Then $\beta = \alpha|_{X(m)}$ and $\delta = \gamma|_{X(n)}$. For any $x \in X(m)$,

$$(x)(\beta\delta) = (x)(\alpha|_{X(m)}\gamma|_{X(n)}) = (x\alpha)\gamma|_{X(n)} = (x)(\alpha\gamma) = (x)(\alpha\gamma)|_{X(m)}.$$

Whence $\beta\delta \leqslant \alpha\gamma$.

(OG3) Let $\alpha : X(e) \to X(f) \in \Sigma(X)$. Let $\iota_k \in E(\Sigma(X))$. Suppose $\mathbf{d}(\alpha) = \iota_e \geqslant \iota_k$. Then the restriction of α to ι_k is given by $\alpha|_{X(k)}$. We see $\alpha|_{X(k)} \leqslant \alpha$ and $\mathbf{d}(\alpha|_{X(k)}) = \iota_k$. As the restiction of isomorphism $\alpha|_{X(k)}$ is unique, then so $(\iota_{X_k}|\alpha)$ is unique.

Therefore $\Sigma(X)$ is an ordered groupoid.

If X is a meet semilattice we show that the identities form a meet semilattice. Let $\iota_e, \iota_f \in E(\Sigma(X))$. It is clear that $X(ef) = X(e) \cap X(f)$. Now $\iota_{e\wedge f} : X(e\wedge f) \to X(e \wedge f) = \iota_{ef} : X(ef) \to X(ef) = \iota_{ef} : X(e) \cap X(f) \to X(e) \cap X(f)$ is a lower bound for ι_e and for ι_f. Further, if $\iota_r \leqslant \iota_e$ and ι_f then $r \leqslant e, f$ so $r \leqslant e \wedge f$. Hence $\iota_r \leqslant \iota_{e\wedge f}$ and $\iota_{e\wedge f}$ is the greatest lower bound for ι_e and ι_f. Therefore $E(\Sigma(X))$ is a meet semilattice. $\square$

A *fibred action* of an ordered groupoid T on a poset X, [10, pg 262], is determined by the following data:

- an order preserving function $\mu : X \to E(T)$;
- an order ideal $X_e = \{x : x\mu \leqslant e\}$ for each $e \in E(T)$;
- an ordered functor $\eta : T \to \Sigma(X)$ that maps t to $\eta(t) : X_{\mathbf{r}(t)} \to X_{\mathbf{d}(t)}$;
- an action of T on X: $t \cdot x = \eta(t)(x)$;
- $\mu : X \to E(T)$ is such that:

1. t acts on x if and only if $x\mu \leqslant \mathbf{r}(t)$;
2. if $x\mu = \mathbf{r}(t)$ then $(t \cdot x)\mu = \mathbf{d}(t)$.

If such an order preserving function $\mu : X \to E(T)$ exists we say X *is fibred over* $E(T)$ or alternatively we say *the fibred action of* T *on* X *is fibred by* μ. We give diagrams Fig. 5.3.1 and Fig. 5.3.2 depicting axioms (1) and (2).

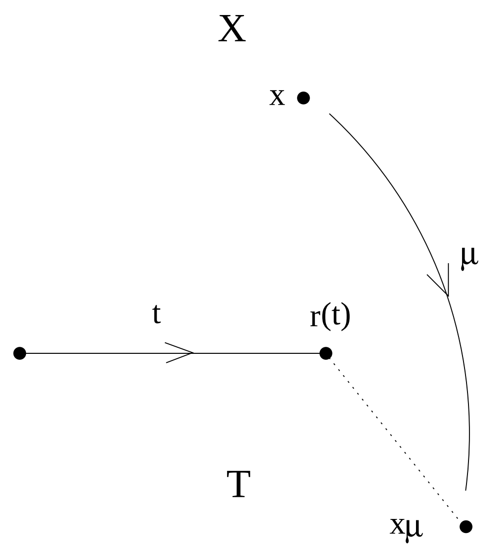

Figure 5.3.1: axiom (1).

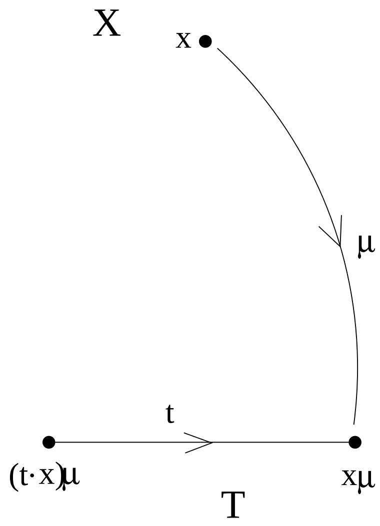

Figure 5.3.2: axiom (2).

Given the fibred action above we can construct the *action groupoid* as follows:

$$T \ltimes X = \{(t,x) \in T \times X : x\mu = \mathbf{d}(t)\}$$

with composition

$$(t,x)(s,y) = (ts,x) \text{ if } t^{-1} \cdot x = y \text{ and } \mathbf{r}(t) = \mathbf{d}(s).$$

The arrow (t,x) has domain $\mathbf{d}(t,x) = (\mathbf{d}(t),x)$ and range $\mathbf{r}(t,x) = (\mathbf{r}(t), t^{-1} \cdot x)$ and inverse $(t,x)^{-1} = (t^{-1}, t^{-1} \cdot x)$. The set of identities of $T \ltimes X$ is

$$E(T \ltimes X) = \{(e,x) \in E(T) \times X : x\mu = e\}.$$

The partial ordering is given by

$$(t,x) \leqslant (s,y) \text{ if and only if } t \leqslant s \text{ and } x \leqslant y.$$

Lemma 5.3.2. *$T \ltimes X$ is an ordered groupoid.*

Proof. We verify the axioms for an ordered groupoid.

(OG1) Let $(t,x) \leqslant (s,y)$. Now $(t,x)^{-1} = (t^{-1}, t^{-1} \cdot x) = (t^{-1}, \eta(t^{-1})(x))$ and $(s,y)^{-1} = (s^{-1}, s^{-1} \cdot y) = (s^{-1}, \eta(s^{-1})(y))$. As $t \leqslant s$, then $t^{-1} \leqslant s^{-1}$. Also, as η is order preserving $\eta(t) \leqslant \eta(s)$. So $\eta(t^{-1}) = \eta|_{X_{\mathbf{d(t)}}}(s^{-1})$. Thus $t^{-1} \cdot x = \eta(t^{-1})(x) = \eta(s^{-1})(x) = s^{-1} \cdot x$. Now $x \leqslant y$ so $x\mu \leqslant y\mu$. Thus $X_{x\mu} \subseteq X_{y\mu}$ and so $x, y \in X_{y\mu}$. Now T acts on X by order isomorphisms so $s^{-1} \cdot x \leqslant s^{-1} \cdot y$. Therefore $t^{-1} \cdot x \leqslant s^{-1} \cdot y$.

(OG2) Let $(t,x) \leqslant (s,y)$ and $(u,w) \leqslant (v,z)$. Then $t \leqslant s$, $x \leqslant y$ and $u \leqslant v$, $w \leqslant z$. Assume $t^{-1} \cdot x = w$, $\mathbf{r}(t) = \mathbf{d}(u)$, $s^{-1} \cdot y = z$ and $\mathbf{r}(s) = \mathbf{d}(v)$. So $(t,x)(u,w) = (tu, x)$ and $(s,y)(v,z) = (sv, y)$. We require $x \leqslant y$ and $tu \leqslant sv$. Now $t \leqslant s$ and $u \leqslant v$ imply that $tu \leqslant sv$. Therefore $(tu, x) \leqslant (sv, y)$.

(OG3) Let $(t,x) \in T \ltimes X$ and $(e,y) \in E(T \ltimes X)$ and suppose $(e,y) \leqslant \mathbf{d}(t,x) = (\mathbf{d}(t), x)$. As $y\mu = e \leqslant \mathbf{d}(t)$ we can restrict t to e. Then $y\mu = \mathbf{d}(e|t)$ so $((e|t), y) \in T \ltimes X$. Also $\mathbf{d}((e|t), y) = (\mathbf{d}(e|t), y) = (e, y)$ and $((e|t), y) \leqslant (t,x)$. We show now that $((e|t), y)$ is the unique restriction of (t,x) to (e,y). Assume (v,c) is also a restriction of (t,x) to (e,y). Then $\mathbf{d}(v,c) = (\mathbf{d}(v), c) = (e,y)$ and so $\mathbf{d}(v) = e$ and $c = y$. Also $(v,c) \leqslant (t,x)$ so $v \leqslant t$, but $\mathbf{d}(v) = e$ and restriction from t to e is unique thus $v = (e|t)$. Therefore $(v,c) = ((e|t), y)$. $\square$

5.3.2 From Action to Covering

Given a fibred action of an ordered groupoid on a poset we show that one can construct an ordered covering of ordered groupoids from such a fibred action. Further we will show that given an arbitrary ordered covering we can construct a fibred action. So equipped with a fibred action we get an ordered covering, and then we can construct another fibred action from this covering.

It transpires that the fibred action constructed is actually isomorphic to the original.

Recall from section 1.4.2 that an ordered covering is a star bijective functor between two ordered groupoids that is order preserving.

Take a fibred action of an ordered groupoid T on a poset X that is fibred by μ. Given the fibred action we can convert this into an ordered covering using the action groupoid $T \ltimes X$.

Lemma 5.3.3. *Given a fibred action of T on X, then $\pi_2 : T \ltimes X \to T$ taking $(t, x) \mapsto t$ is an ordered covering.*

Proof. Let $(t, x), (s, y) \in T \ltimes X$ with $t^{-1} \cdot x = y$ and $\mathbf{r}(t) = \mathbf{d}(s)$, then $((t, x)(s, y))\pi_2 = (ts, x)\pi_2 = ts = (t, x)\pi_2(s, y)\pi_2$. So π_2 is a functor.

Restrict π_2 to $\mathrm{star}_{T\ltimes X}(j, z) \to \mathrm{star}_T(j)$ and let $(t, x), (s, y) \in \mathrm{star}_{T\ltimes X}(j, z)$. Then $x = y = z$ and $\mathbf{d}(t) = \mathbf{d}(s) = j$ so $t, s \in \mathrm{star}_T(j)$. Assume $(t, x)\pi_2 = (s, y)\pi_2$. Then $t = s$ and so $(t, x) = (s, y)$ and π_2 is star injective.

Next we show π_2 is star surjective. Restrict π_2 to $\mathrm{star}_{T\ltimes X}(j, z) \to \mathrm{star}_T(j)$ and let $t \in \mathrm{star}_T(j)$. Then $\mathbf{d}(t) = j$. Now

$$\begin{aligned} \mathrm{star}_{T\ltimes X}(j, z) &= \{(s, x) : x\mu = \mathbf{d}(s), (\mathbf{d}(s), x) = (j, z)\} \\ &= \{(s, z) : z\mu = \mathbf{d}(s) = j\}. \end{aligned}$$

Hence $(t, z) \in \mathrm{star}_{T\ltimes X}(j, z)$ and $(t, z)\pi_2 = t$.

To show π_2 is order preserving assume that $(t, x) \leqslant (s, y)$. Then $t \leqslant s$ and $x \leqslant y$ so $(t, x)\pi_2 = t \leqslant s = (s, y)\pi_2$. $\square$

Lemma 5.3.4. *Let $\gamma : C \to T$ be an ordered covering of ordered groupoids. Let $e \in E(C)$ and $t \in T$. We define a fibred action of T on $E(C)$ as follows:*

if $e\gamma \leqslant \mathbf{r}(t)$ *then* $t \cdot e = \mathbf{d}(c)$ *where* $c\gamma = (t|e\gamma)$. *The action is fibred by* γ *restricted to* $E(C) \to E(T)$.

Proof. Let $e \in E(C)$ and $t \in T$. If $e\gamma \leqslant \mathbf{r}(t)$ then we can corestrict t to e. Now γ is a covering so for $(t|e\gamma) \in T$ there exists a unique element $c \in \mathrm{costar}_C(e)$ such that $c\gamma = (t|e\gamma)$. Element c has range e. The action $t \cdot e$ is then defined as $t \cdot e = \mathbf{d}(c)$, see Fig. 5.3.3. If $e\gamma = \mathbf{r}(t)$ then $(t|e\gamma) = t$ and

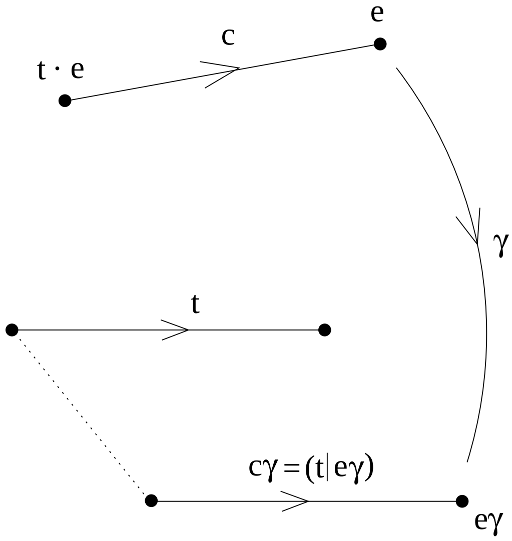

Figure 5.3.3: fibred action.

then the unique element c that maps to $(t|e\gamma)$ maps to t. As $t \cdot e = \mathbf{d}(c)$ and $c\gamma = t$ we have that $\mathbf{d}(c)\gamma = (t \cdot e)\gamma = \mathbf{d}(t)$. Thus this action of T on $E(C)$ is fibred by γ restricted to $E(C) \to E(T)$. $\square$

Given the fibred action of T on X we constructed the ordered covering π_2. By lemma 5.3.4 we then have a fibred action of T on $E(T \ltimes X)$. If $t \in T$, $(x\mu, x) \in E(T \ltimes X)$ and $(x\mu, x)\pi_2 = x\mu \leqslant \mathbf{r}(t)$ then we have a unique $(v, c) \in \mathrm{costar}_{T \ltimes X}(x\mu, x)$ such that $(v, c)\pi_2 = (t|x\mu)$. Then $v = (t|x\mu)$ and further

$\mathbf{r}(v,c) = (\mathbf{r}(v), v^{-1} \cdot c) = (x\mu, x)$ so $c = v \cdot x$. Thus $(v,c) = ((t|x\mu), (t|x\mu) \cdot x)$. The action then becomes

$$t \cdot (x\mu, x) = \mathbf{d}((t|x\mu), (t|x\mu) \cdot x) = (\mathbf{d}(t|x\mu), (t|x\mu) \cdot x)$$

provided $(x\mu, x)\pi_2 \leqslant \mathbf{r}(t)$. The action is fibred by π_2 restricted to $E(T \ltimes X) \to E(T)$.

If T is an ordered groupoid and if X is a poset with a given fibred action of T on X fibred by μ, then we call X a *T-poset*. Take X and Y to be T-posets with Y fibred by $\bar{\mu}$. The map $\delta : X \to Y$ is a *map of T-posets* if $\delta\bar{\mu} = \mu$ and, for $x \in X$ and $t \in T$, δ is such that:

1. if $t \cdot x$ is defined in X then $t \cdot x\delta$ is defined in Y;
2. $(t \cdot x)\delta = t \cdot (x\delta)$.

Given two T-posets X and Y the two fibred actions involved are *isomorphic* if the map of T-posets is bijective.

Lemma 5.3.5. *The fibred action of T on $E(T \ltimes X)$ fibred by π_2 is isomorphic to the original fibred action of T on X fibred by μ.*

Proof. Let $\delta : E(T \ltimes X) \to X$ be defined as $(e,x) \mapsto x$. Let $(e,x) \in E(t \ltimes X)$, then $x\mu = e$. So $(e,x)\delta\mu = x\mu = e = (e,x)\pi_2$, thus $\delta\mu = \pi_2$. We show the axioms of a map of T-posets hold.

(1) Suppose $t \in T$ acts on $(s,y) \in E(T \ltimes X)$. Then $(s,y)\pi_2 \leqslant \mathbf{r}(t)$ which implies that $s \leqslant \mathbf{r}(t)$. Now $(s,y) \in E(T \ltimes X)$ implies that $y\mu = \mathbf{d}(s) = s$ and so $y\mu \leqslant \mathbf{r}(t)$. Thus t acts on $y = (s,y)\delta$.

(2) Now that the actions are defined we show that $(t \cdot (s,y))\delta = t \cdot (s,y)\delta$. We have seen that $s = y\mu$ so let us consider $t \cdot (y\mu, y) = (\mathbf{d}(t|y\mu), (t|y\mu) \cdot y)$. Now $(t|y\mu) \cdot y = \eta(t|y\mu)(y)$. We have $\eta(t), \eta(t|y\mu) \in \Sigma(X)$ and $\eta(t|y\mu) \leqslant \eta(t)$ so

$\eta(t|y\mu) = \eta|_{X_{\mathbf{r}(t|y\mu)}}(t) = \eta|_{X_{y\mu}}(t)$. Thus for all $z \in X_{y\mu}$, $\eta(t|y\mu) = \eta(t)$. As $y \in X_{y\mu}$ we have $\eta(t|y\mu)(y) = \eta(t)(y)$. So $(t|y\mu) \cdot y = t \cdot y$ and

$$(t \cdot (y\mu, y))\delta = (\mathbf{d}(t|y\mu), (t|y\mu) \cdot y)\delta = (t|y\mu) \cdot y = t \cdot y = t \cdot (y\mu, y)\delta.$$

We now show that δ is bijective. Let $(e, x), (f, y) \in E(T \ltimes X)$ and suppose $(e, x)\delta = (f, y)\delta$, i.e. $x = y$. Now $x\mu = e$ and $y\mu = f$ and since $x = y$ implies $x\mu = y\mu$, we have $e = f$. So $(e, x) = (f, y)$ and δ is injective. Now suppose $x \in X$. Then $(x\mu, x) \in E(T \ltimes X)$ and $(x\mu, x)\delta = x$. Therefore δ is surjective.

Therefore δ is a bijective map of T-posets. $\square$

5.3.3 From Covering to Action

We have shown that we can construct a fibred action from an ordered covering. Further, given an ordered covering we can construct a fibred action. In this subsection we show that, equipped with an ordered covering we get a fibred action from which we can create another ordered covering that is isomorphic to the original map.

Let $\gamma : C \to T$ be an ordered covering of ordered groupoids. By lemma 5.3.4, we get a fibred action of T on $E(C)$: if $e\gamma \leqslant \mathbf{r}(t)$ then $t \cdot e = \mathbf{d}(c)$ where $c\gamma = (t|e\gamma)$. With this fibred action we construct the action groupoid

$$T \ltimes E(C) = \{(t, c) \in T \times E(C) : c\gamma = \mathbf{d}(t)\}.$$

Then $\tilde{\pi_2} : T \ltimes E(C) \to T$ defined by $(t, c) \mapsto t$ is, by lemma 5.3.3, an ordered covering.

Lemma 5.3.6. *Given ordered covering $\gamma : C \to T$, we have that C is isomorphic to $T \ltimes E(C)$. Furthermore, γ is isomorphic to $\tilde{\pi_2} : T \ltimes E(C) \to T$.*

Proof. Let $\phi : C \to T \ltimes E(C)$ be given by $c \mapsto (c\gamma, \mathbf{d}(c))$. Let $a, c \in C$ with $a\gamma^{-1} \cdot \mathbf{d}(a) = \mathbf{d}(c)$ and $\mathbf{r}(a\gamma) = \mathbf{d}(c\gamma)$ then

$$\begin{aligned}(a\phi)(c\phi) &= (a\gamma, \mathbf{d}(a))(c\gamma, \mathbf{d}(c)) \\ &= (a\gamma c\gamma, \mathbf{d}(a)) \\ &= ((ac)\gamma, \mathbf{d}(ac)) \\ &= (ac)\phi\end{aligned}$$

Assume now that $a, c \in C$ such that $a\phi = c\phi$, then $(a\gamma, \mathbf{d}(a)) = (c\gamma, \mathbf{d}(c))$. So $a\gamma = c\gamma$ and $\mathbf{d}(a) = \mathbf{d}(c)$ so a and c belong to the same star in C. As γ is an ordered covering, $a = c$.

Given $(t, e) \in T \ltimes E(C)$ then $e\gamma = \mathbf{d}(t)$ implies $(t^{-1} \cdot e)\gamma = \mathbf{r}(t)$. As γ is a covering there exists a $c \in \mathrm{star}_C(e)$ such that $c\gamma = t$. Hence $c\phi = (c\gamma, \mathbf{d}(c)) = (t, e)$ and ϕ is an isomorphism.

Together with the identity on T, ϕ furnishes an isomorphism of coverings. $\square$

5.3.4 Equivalence of Categories of Actions and Coverings

Fix an ordered groupoid T, and let $\mathrm{Act}(T)$ be the category whose objects are T-posets, and whose morphisms are maps of T-posets. Let $\gamma : G \to T$ and $\beta : H \to T$ be ordered coverings to T. If there exists a map of ordered groupoids $\theta : G \to H$ such that $\gamma = \theta\beta$ then we call such a map a *map of coverings*. Define $\mathrm{Cov}(T)$ to be the category whose objects are ordered groupoids with an ordered covering to T. The morphisms of this category are maps of coverings. We will establish an equivalence between the categories $\mathrm{Act}(T)$ and $\mathrm{Cov}(T)$.

In order to show that the categories $\mathrm{Act}(T)$ and $\mathrm{Cov}(T)$ are equivalent we require functors $F_A : \mathrm{Act}(T) \to \mathrm{Cov}(T)$ and $F_C : \mathrm{Cov}(T) \to \mathrm{Act}(T)$, as discussed in section 1.3.2.

Define $F_A : \mathrm{Act}(T) \to \mathrm{Cov}(T)$ on the objects of $\mathrm{Act}(T)$ to be

$$F_A : X \mapsto T \ltimes X$$

where X is fibred by $\mu : X \to E(T)$ and, by lemma 5.3.3, $T \ltimes X$ has the associated covering $\pi_2 : T \ltimes X \to T$. On the morphisms of Act(T) define

$$\begin{aligned} F_A : [\alpha : X \to Y] \mapsto & [\theta : T \ltimes X \to T \ltimes Y] \\ & [\theta : (t, x) \mapsto (t, x\alpha)]. \end{aligned}$$

If $(t, x) \in T \ltimes X$ then $x\mu = \mathbf{d}(t)$ and so t^{-1} acts on x. By definition of α, if t^{-1} acts on x then t^{-1} acts on $x\alpha$. So, if Y is fibred by $\bar{\mu} : Y \to E(T)$, $(x\alpha)\bar{\mu} = x\mu = \mathbf{d}(t)$ and so $(t, x\alpha) \in T \ltimes Y$ and θ is well defined.

Define $F_C : \mathrm{Cov}(T) \to \mathrm{Act}(T)$ on the objects of $\mathrm{Cov}(T)$ to be the fibred action we have previously constructed from a covering in lemma 5.3.4:

$$F_C : G \mapsto E(G)$$

where G has the associated covering $\gamma : G \to T$ and $E(G)$ is fibred by the restriction of γ to $E(G) \to E(T)$. On the morphisms of $\mathrm{Cov}(T)$ define

$$F_C : [\theta : G \to H] \mapsto [\theta|_{E(G)} : E(G) \to E(H)].$$

We check that F_C is well-defined. Let $\theta : G \to H$ be a morphism in $\mathrm{Cov}(T)$. Then there are two ordered coverings $\gamma : G \to T$ and $\beta : H \to T$ such that $\gamma = \theta\beta$. Take $\theta|_{E(G)} : E(G) \to E(H)$. Let $e, f \in E(G)$ and $t \in T$ be such that $t \cdot e = f$. Let $g \in G$ be such that $g\gamma = (t|e\gamma)$. We require $(t \cdot e)\theta = t \cdot (e\theta)$. We have that $(t \cdot e)\theta = f\theta$. Now $e\theta$ gets mapped by β to $e\theta\beta = e\gamma$ and $g\theta$ gets mapped by β to $g\theta\beta = g\gamma = (t|e\gamma)$, so $t \cdot (e\theta) = \mathbf{d}(g\theta) = f\theta$. Hence $(t \cdot e)\theta = t \cdot (e\theta)$ and F_C is well-defined.

Given morphism $\alpha : X \to Y$ in $\mathrm{Act}(T)$ we need a natural equivalence τ such that the diagram in Fig. 5.3.4 commutes.

$$\begin{tikzcd}
(X)F_A F_C \arrow[rr, "(\alpha)F_A F_C"] \arrow[d, "\tau_X"'] & & (Y)F_A F_C \arrow[d, "\tau_Y"] \\
X1_A = X \arrow[rr, "\alpha 1_A = \alpha"'] & & Y1_A = Y
\end{tikzcd}$$

Figure 5.3.4: equivalence of categories.

Now $(X)F_A F_C = (T \ltimes X)F_C = E(T \ltimes X)$ and similarly $(Y)F_A F_C = E(T \ltimes Y)$. Also $(\alpha)F_A F_C = (\theta : T \ltimes X \to T \ltimes Y)F_C = \theta|_{E(T \ltimes X)}$, so Fig. 5.3.4 becomes Fig. 5.3.5.

Lemma 5.3.5 tells us that $E(T \ltimes X)$ and its fibred action is isomorphic to X with its fibred action. Similarly with $E(T \ltimes Y)$ and Y. So taking $\tau_X = \delta_X$ to be defined as in the proof of lemma 5.3.5, $\delta_X : E(T \ltimes X) \to X$, $(t, x) \mapsto x$, and similarly $\tau_Y = \delta_Y$, we have that if $(t, x) \in E(T \ltimes X)$, $(t, x)\delta_X \alpha = x\alpha$ and $(t, x)\theta\delta_Y = (t, x\alpha)\delta_Y = x\alpha$. Thus the diagram commutes.

Now given a morphism $\theta : G \to H$ in $\mathrm{Cov}(T)$ we require a natural equivalence $\tilde{\tau}$ such that the diagram Fig. 5.3.6 commutes.

$$\begin{tikzcd}
E(T \ltimes X) \arrow[r, "\theta : (t,x) \mapsto (t,x\alpha)"] \arrow[d, "\tau_X"'] & E(T \ltimes Y) \arrow[d, "\tau_Y"] \\
X \arrow[r, "\alpha"'] & Y
\end{tikzcd}$$

Figure 5.3.5: equivalence of categories.

Now $(G)F_CF_A = (E(G))F_A = T \ltimes E(G)$ and $(H)F_CF_A = T \ltimes E(H)$. Also, $(\theta)F_CF_A = (\theta|_{E(G)})F_A = \theta' : T \ltimes E(G) \to T \ltimes E(H)$, $\theta' : (t,e) \mapsto (t, e\theta)$. So Fig. 5.3.6 becomes Fig. 5.3.7.

By lemma 5.3.6, $T \ltimes E(G)$ and its associated covering $\tilde{\pi_2}$ is isomorphic to G and its associated covering γ. Similarly with $T \ltimes E(H)$ and H. By the proof of lemma 5.3.6, if $(t,e) \in T \ltimes E(G)$ then there exists a $g \in \text{star}_G(e)$ such that $g\gamma = t$ and $\mathbf{d}(g) = e$ and so we can take $\tilde{\tau_G}$ to be $\phi_G : (t,e) = (g\gamma, \mathbf{d}(g)) \mapsto g$. Similarly take $\tilde{\tau_H}$ to be ϕ_H. Let $(t,e) \in T \ltimes E(G)$ with $g\gamma = t$ and $\mathbf{d}(g) = e$. Then $(t,e)\phi_G\theta = (g\gamma, \mathbf{d}(g))\phi_G\theta = g\theta$. If β is the ordered covering from H to T then $\gamma = \theta\beta$ and so $(t,e)\theta'\phi_H = (g\gamma, \mathbf{d}(g))\theta'\phi_H = (g\gamma, \mathbf{d}(g)\theta)\phi_H = (g\theta\beta, \mathbf{d}(g)\theta)\phi_H = g\theta$.

$$\begin{tikzcd}
(G)F_CF_A \arrow[r, "(\theta)F_CF_A"] \arrow[d, "\tilde{\tau}_G"'] & (H)F_CF_A \arrow[d, "\tilde{\tau}_H"] \\
(G)1_C = G \arrow[r, "(\theta)1_C = \theta"'] & (H)1_C = H
\end{tikzcd}$$

Figure 5.3.6: equivalence of categories.

$$
\begin{tikzcd}
T \ltimes E(G) \arrow[r, "\theta'"] \arrow[d, "\tilde{\tau}_G"'] & T \ltimes E(H) \arrow[d, "\tilde{\tau}_H"] \\
G \arrow[r, "\theta"'] & H
\end{tikzcd}
$$

Figure 5.3.7: equivalence of categories.

Therefore the categories $\mathrm{Act}(T)$ and $\mathrm{Cov}(T)$ are equivalent.

5.4 Enlargements

Now that we have established our definitions for fibred actions and the relationship between fibred actions and ordered coverings, we may continue our quest for a generalised version of the P-theorem for an ordered groupoid G given an immersion $\nu : G \to T$ by discussing "enlargements". The Maximum Enlargement Theorem is a structure theorem for immersions of ordered groupoids. We first construct from $E(G)$ and T a poset on which T acts, and then construct the action groupoid. The immersion ν then induces an embedding of G into this action groupoid, giving an embedding of G into a larger structure. This larger structure has nice properties, and is said to be an *enlargement* of G. Moreover, G can be reconstructed from the enlargement.

We begin with the definition of an enlargement, as in [10], and then we construct our action groupoid and give our version of Ehresmann's Maximum Enlargement Theorem.

Let G be an ordered subgroupoid of the ordered groupoid H. Then H is an *enlargement* of G if the following three axioms hold.

(E1) $E(G)$ is an order ideal of $E(H)$.

(E2) If $h \in H$ and $\mathbf{d}(h), \mathbf{r}(h) \in G$ then $h \in G$.

(E3) If $e \in E(H)$ then there exists an $h \in H$ with $\mathbf{r}(h) = e$ and $\mathbf{d}(h) \in G$.

Let $\nu : G \to T$ be an immersion of ordered groupoids. We construct a fibred action on the poset $E(G)$ and an action on the groupoid T. Suppose that $g \in G(e, f)$ and $t \in T$ is such that $\mathbf{r}(t) = e\nu$. Then G acts on the left of $E(G)$: $g \cdot f = e$. Also G acts on the right of T: $t \cdot g = t(g\nu)$.

The pullback $T \between E(G)$ is defined as

$$T \between E(G) = \{(s, e) \in T \times E(G) : \mathbf{r}(s) = e\nu\}.$$

We define a relation on the pullback as follows: $(s, e) \simeq_G (t, f)$ if there exists $g \in G(e, f)$ such that $t = s(g\nu)$. It is easy to see that this is an equivalence relation. The equivalence class of (s, e) is denoted by $s \otimes e$ and the quotient set is denoted $T \otimes E(G)$.

Define an ordering on $T \otimes E(G)$ as follows: $t \otimes f \leqslant s \otimes e$ if and only if there exists $k \leqslant e$ and there exists an $h \in G(f, k)$ such that $t(h\nu) \leqslant s$. See Fig. 5.4.1.

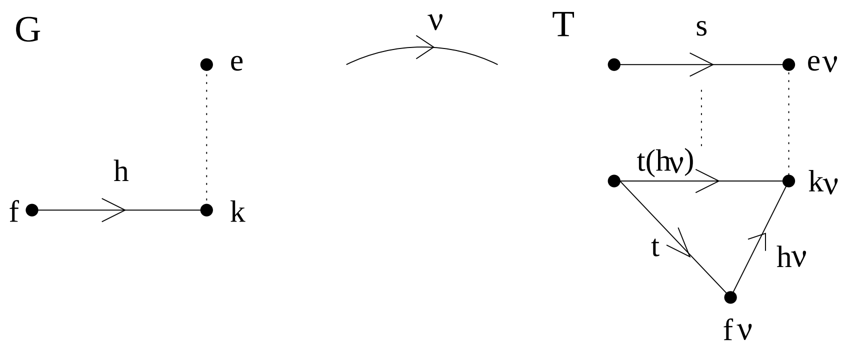

Figure 5.4.1: ordering.

Equivalently, $t \otimes f \leqslant s \otimes e$ if and only if there exists $w \otimes k \in T \otimes E(G)$ such that $w \otimes k = t \otimes f$ with $k \leqslant e$ and $w \leqslant s$. See Fig. 5.4.2.

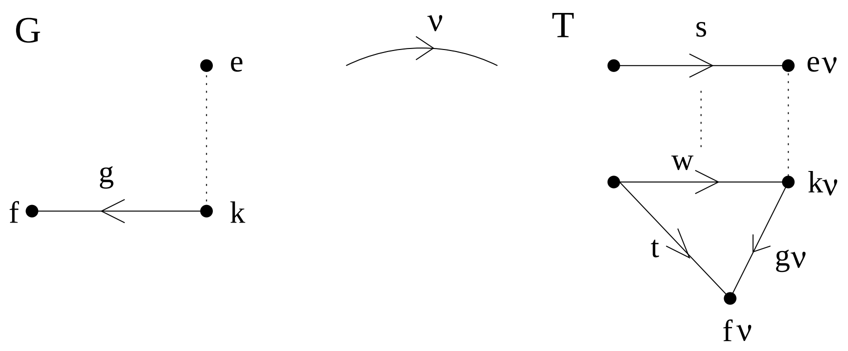

Figure 5.4.2: equivalent ordering.

We show that the ordering is well defined. Suppose that $t \otimes f \leqslant s \otimes e$. Owing to the equivalent definition of the ordering on $T \otimes E(G)$ it is clear that any $t' \otimes f'$ that is equal to $t \otimes f$ will also be equal to $w \otimes k$ with $w \leqslant s$ and $k \leqslant k$, so $t' \otimes f' \leqslant s \otimes e$. It is then enough to show that if $s \otimes e = s' \otimes e'$ and $t \otimes f \leqslant s \otimes e$ then $t \otimes f \leqslant s' \otimes e'$. As $s \otimes e = s' \otimes e'$, there exists $g \in G(e, e')$ such that $s' = s(g\nu)$. Then as $t \otimes f \leqslant s \otimes e$, we have $h \in G(f, k)$ with $k \leqslant e$ such that $t(h\nu) \leqslant s$. Take $h' = h(k|g)$ with $\mathbf{r}(k|g) = k'$. Then $h' \in G(f, k')$ with $k' \leqslant e'$. Also, $t(h'\nu) = t(h\nu)(k|g)\nu \leqslant s(g\nu) = s'$. Thus $t \otimes f \leqslant s' \otimes e'$. Therefore the ordering is well-defined.

Lemma 5.4.1. *$T \otimes E(G)$ is a poset.*

Proof. If $t \otimes f \in T \otimes E(G)$ then clearly $t \otimes f = t \otimes f$ so $\leqslant$ is reflexive.

To prove transitivity suppose that $t \otimes f \leqslant s \otimes e$ and $s \otimes e \leqslant u \otimes j$. So there exists $k \leqslant e$ and $h \in G(f, k)$ such that $t(h\nu) \leqslant s$. Also there exists $k' \leqslant j$ and $h' \in G(e, k')$ such that $s(h'\nu) \leqslant u$. As $k \leqslant e$ we can restrict h' to k and we let i be the range of $(h'|k)$. Letting $p = t(h\nu)$ and $q = s(h'\nu)$ we have the diagram shown in Fig. 5.4.3.

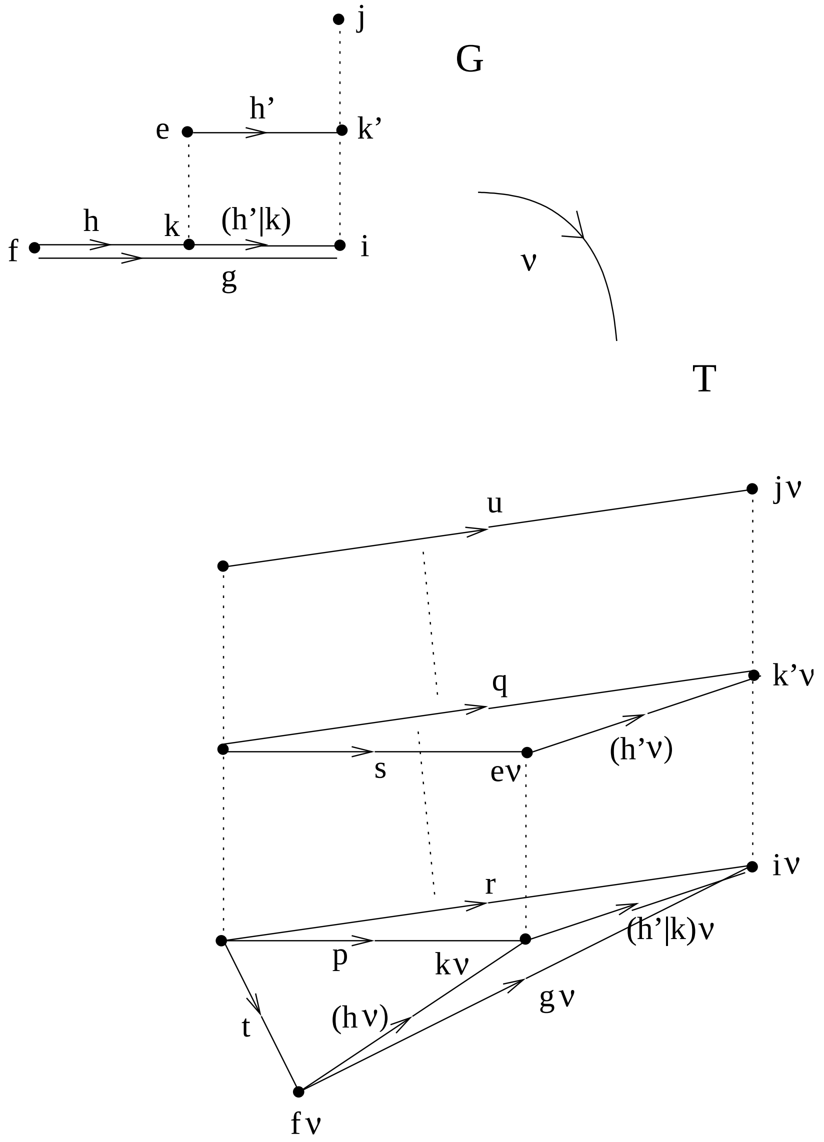

Figure 5.4.3: transitivity.

Let $g = h(h'|k)$ and $r = t(g\nu)$. Then $i \leqslant k' \leqslant j$ and $g \in G(f,i)$. Further $r = t(g\nu) = t(h(h'|k))\nu = t(h\nu)(h'|k)\nu = p(h'|k)\nu$. Since $(h'|k) \leqslant h'$ and $p \leqslant s$ we have $r \leqslant s(h'\nu) = q \leqslant u$. Therefore $t(g\nu) \leqslant u$ and $t \otimes f \leqslant u \otimes j$. The relation $\leqslant$ is transitive.

We now show that $\leqslant$ is antisymmetric. Suppose that $t \otimes f \leqslant t' \otimes f'$ and $t' \otimes f' \leqslant t \otimes f$. Now $t \otimes f \leqslant t' \otimes f'$ if and only if there exists $e \leqslant f'$ and $h \in G(f,e)$ such that $t(h\nu) \leqslant t'$, so $t \otimes f = t(h\nu) \otimes e$. We can therefore assume $f \leqslant f'$ and $t \leqslant t'$. Now $t' \otimes f' \leqslant t \otimes f$ if and only if there exists $k \leqslant f$ and $h' \in G(f',k)$ such that $t'(h'\nu) \leqslant t$. Then $k \leqslant f \leqslant f'$ and $t'(h'\nu) \leqslant t \leqslant t'$. Obviously $\mathbf{d}(t'(h'\nu)) = \mathbf{d}(t')$ so $t'(h'\nu) = (\mathbf{d}(t'(h'\nu))|t) = (\mathbf{d}(t'(h'\nu))|t') = (\mathbf{d}(t')|t') = t'$. Then, as $\mathbf{r}(t') = f'\nu$, we have $h'\nu = f'\nu$. Since ν is an immersion $h' = f'$ and since $f' \in E(G)$ and $h' \in G(f',k)$ then $k = f'$. So $k \leqslant f \leqslant f'$ implies that $f' \leqslant f \leqslant f'$ so $f = f'$. Also $t'(h'\nu) \leqslant t \leqslant t'$ implies that $t' \leqslant t \leqslant t'$ so $t' = t$. Therefore $t \otimes f = t' \otimes f'$. $\square$

Now $t \otimes f = s \otimes e$ if $(s,e) \simeq_G (t,f)$, which happens if there exists a $g \in G(e,f)$ such that $t = s(g\nu)$, so then $\mathbf{d}(s) = \mathbf{d}(t)$. So $\mathbf{d}(t)$ is an invariant of the class $t \otimes f$.

Take $\mu : T \otimes E(G) \to E(T)$ to be $\mu : s \otimes e \mapsto \mathbf{d}(s)$. Let $t \in T$ and $s \otimes e \in T \otimes E(G)$. If $\mathbf{d}(s) \leqslant \mathbf{r}(t)$ then t acts on $s \otimes e$ as follows

$$t \cdot (s \otimes e) = (t|\mathbf{d}(s))s \otimes e.$$

We show that μ is order preserving. Take $u \otimes f, s \otimes e \in T \otimes E(G)$ such that $u \otimes f \leqslant s \otimes e$. Then there exists some $k \leqslant e$ and an $h \in G(f,k)$ such that $u(h\nu) \leqslant s$. Equivalently $u \otimes f = w \otimes k$ with $k \leqslant e$ and $w \leqslant s$. So we can assume $f \leqslant e$ and $u \leqslant s$. Then as $u \leqslant s$ we have $(u \otimes f)\mu = \mathbf{d}(u) \leqslant \mathbf{d}(s) = (s \otimes e)\mu$ and μ is order preserving. The action of T on the poset $E(G) \otimes T$ is then a fibred action, fibred by μ. We check that the axioms, as given in section 5.3.1, for a fibred action hold. First, $t \in T$ acts on $s \otimes e \in T \otimes E(G)$ if

$\mathbf{d}(s) = (s \otimes e)\mu \leqslant \mathbf{r}(t)$ so axiom (1) holds. Next if $(s \otimes e)\mu = \mathbf{d}(s) = \mathbf{r}(t)$ then $t \cdot (s \otimes e) = (t|\mathbf{d}(s))s \otimes e = ts \otimes e$ so $(t \cdot (s \otimes e))\mu = (ts \otimes e)\mu = \mathbf{d}(ts) = \mathbf{d}(t)$ and axiom (2) holds.

From this fibred action we can construct the action groupoid as follows:

$$\begin{aligned} T \ltimes (T \otimes E(G)) &= \{(t, s \otimes e) \in T \times (T \otimes E(G)) : (s \otimes e)\mu = \mathbf{d}(t)\} \\ &= \{(t, s \otimes e) \in T \times (T \otimes E(G)) : \mathbf{d}(s) = \mathbf{d}(t)\} \end{aligned}$$

We denote this action groupoid by $\tilde{T}_\nu$. Let $(u, t \otimes f), (v, s \otimes e) \in \tilde{T}_\nu$. Then $\mathbf{d}(t) = \mathbf{d}(u)$ and $\mathbf{d}(s) = \mathbf{d}(v)$. Then the composition is given by

$$(u, t \otimes f)(v, s \otimes e) = (uv, t \otimes f)$$

provided $\mathbf{r}(u) = \mathbf{d}(v)$ and $u^{-1} \cdot (t \otimes f) = (u^{-1}|\mathbf{d}(t))t \otimes f = u^{-1}t \otimes f = s \otimes e$. Equivalently, composition occurs if there exists a $g \in G(f, e)$ such that $s = u^{-1}t(g\nu)$. An element $(u, t \otimes f)$ of $\tilde{T}_\nu$ has domain $(\mathbf{d}(u), t \otimes f) = (\mathbf{d}(t), t \otimes f)$ and range $(\mathbf{r}(u), u^{-1}t \otimes f)$. The inverse of $(u, t \otimes f)$ is $(u, t \otimes f)^{-1} = (u^{-1}, u^{-1}t \otimes f)$. The set of identities is

$$E(\tilde{T}_\nu) = \{(\mathbf{d}(t), t \otimes f) \in T \ltimes (T \otimes E(G))\}.$$

Ordering is simply $(u, t \otimes f) \leqslant (w, s \otimes e)$ if and only if $u \leqslant w$ and $t \otimes f \leqslant s \otimes e$. These properties have all been derived from the general properties of an action groupoid, as given in section 5.3.1.

We can now give our version of Ehresmann's Maximum Enlargement Theorem (following [10, pg256]).

Theorem 5.4.2. *Let $\nu : G \to T$ be an immersion.*
1. $\tilde{T}_\nu$ is an ordered groupoid and $\pi_2 : T \ltimes (T \otimes E(G)) \to T$ taking $(t, s \otimes e) \mapsto t$

is an ordered covering of T.

2. There is an ordered embedding of G into $\tilde{T}_\nu$ given by $\iota : g \mapsto (g\nu, \mathbf{d}(g)\nu \otimes \mathbf{d}(g))$ such that $\iota\pi_2 = \nu$, and $\tilde{T}_\nu$ is an enlargement of $G\iota$.

3. Suppose that $j : G \to C$ is an ordered embedding and that $\pi : C \to T$ is an ordered covering such that $\nu = j\pi$. Then there exists a unique ordered functor $\theta : \tilde{T}_\nu \to C$ such that $j = \iota\theta$ and $\pi_2 = \theta\pi$.

Proof. 1. By lemma 5.3.2, $\tilde{T}_\nu$ is an ordered groupoid and by lemma 5.3.3, π_2 is an ordered covering.

2. We note that ι maps $E(G) \to E(\tilde{T}_\nu)$, for if $f \in E(G)$ then $\iota : f \mapsto (f\nu, \mathbf{d}(f)\nu \otimes \mathbf{d}(f)) = (f\nu, f\nu \otimes f) = (\mathbf{d}(f\nu), f\nu \otimes f) \in E(\tilde{T}_\nu)$. Let $g, h \in G$ and suppose gh is defined. Then $(g\iota)(h\iota)$ is defined if $\mathbf{r}(g) = \mathbf{d}(h)$ and if there exists a $k \in G(\mathbf{d}(g), \mathbf{d}(h))$ such that $\mathbf{d}(h)\nu = (g\nu)^{-1}\mathbf{d}(g)\nu(k\nu)$. As gh is defined then $\mathbf{r}(g) = \mathbf{d}(h)$. Take $k = g$, then clearly $g \in G(\mathbf{d}(g), \mathbf{r}(g))$ and $(g\nu)^{-1}\mathbf{d}(g)\nu(k\nu) = (g\nu)^{-1}g\nu = \mathbf{r}(g)\nu = \mathbf{d}(h)\nu$. Then

$$\begin{aligned}
(g\iota)(h\iota) &= (g\nu, \mathbf{d}(g)\nu \otimes \mathbf{d}(g))(h\nu, \mathbf{d}(h)\nu \otimes \mathbf{d}(h)) \\
&= (g\nu h\nu, \mathbf{d}(g)\nu \otimes \mathbf{d}(g)) \\
&= ((gh)\nu, \mathbf{d}(gh)\nu \otimes \mathbf{d}(gh)) \\
&= (gh)\iota
\end{aligned}$$

and ι is a functor.

If $g \leqslant h$ in G then $g\nu \leqslant h\nu$, $\mathbf{d}(g) \leqslant \mathbf{d}(h)$ and $\mathbf{d}(g) \in G(\mathbf{d}(g), \mathbf{d}(g))$ with $\mathbf{d}(g)\nu\mathbf{d}(g)\nu = \mathbf{d}(g)\nu \leqslant \mathbf{d}(h)\nu$ so $(g\nu, \mathbf{d}(g)\nu \otimes \mathbf{d}(g)) \leqslant (h\nu, \mathbf{d}(h)\nu \otimes \mathbf{d}(h))$, i.e. $g\iota \leqslant h\iota$.

Assume now that $(g\nu, \mathbf{d}(g)\nu \otimes \mathbf{d}(g)) \leqslant (h\nu, \mathbf{d}(h)\nu \otimes \mathbf{d}(h))$. Then $g\nu \leqslant h\nu$ and $\mathbf{d}(g)\nu \otimes \mathbf{d}(g) \leqslant \mathbf{d}(h)\nu \otimes \mathbf{d}(h)$. So there exists $e \leqslant \mathbf{d}(h)$ and $a \in G(\mathbf{d}(g), e)$

such that $\mathbf{d}(g)\nu(a\nu) \leqslant \mathbf{d}(h)\nu$. It follows that $a\nu \leqslant \mathbf{d}(h)\nu$ so $a\nu \in E(T)$. Thus $a \in E(G)$ and so $a = e = \mathbf{d}(g)$. Therefore $\mathbf{d}(g) \leqslant \mathbf{d}(h)$. Now $g\nu \leqslant h\nu$ and $(\mathbf{d}(g)|h)\nu \leqslant h\nu$. Also, $\mathbf{d}(\mathbf{d}(g)|h) = \mathbf{d}(g)$ so $\mathbf{d}(\mathbf{d}(g)|h)\nu = \mathbf{d}(g)\nu$ and by the uniqueness of restriction $\mathbf{d}(g)|h)\nu = g\nu$. Since ν is an immersion, $(\mathbf{d}(g)|h) = g$. As $(\mathbf{d}(g)|h) \leqslant h$, then $g \leqslant h$.

Therefore $g \leqslant h$ if and only if $g\iota \leqslant h\iota$. Hence ι preserves order and is an embedding.

Next we check that $\tilde{T}_\nu$ is an enlargement of $G\iota$. The above considerations show that $G\iota$ is an ordered subgroupoid of $\tilde{T}_\nu$.

(E1) Suppose $(\mathbf{d}(t), t \otimes f) \leqslant (k\nu, k\nu \otimes k) \in E(G\iota)$. Then $\mathbf{d}(t) \leqslant k\nu$ and there exists $e \leqslant k$ and $g \in G(f, e)$ such that $t(g\nu) \leqslant k\nu$. As $k\nu \in E(T)$, $t(g\nu) \in E(T)$ which implies that $g\nu = t^{-1}$. So $\mathbf{d}(t) = e\nu$. Then $g \in G(f, e)$ is such that $t(g\nu) = tt^{-1} = \mathbf{d}(t) = e\nu$ which implies that $t \otimes f = e\nu \otimes e$. Therefore $(\mathbf{d}(t), t \otimes f) = (e\nu, e\nu \otimes e) \in E(G\iota)$ so $E(G\iota)$ is an order ideal in $E(\tilde{T}_\nu)$.

(E2) Take $(u, t \otimes f) \in \tilde{T}_\nu$ with $\mathbf{d}(u, t \otimes f) = (\mathbf{d}(u), t \otimes f) \in E(G\iota)$ and $\mathbf{r}(u, t \otimes f) = (\mathbf{r}(u), u^{-1}t \otimes f) \in E(G\iota)$. So there must exist $x, y \in E(G)$ such that $(x\nu, x\nu \otimes x) = (\mathbf{d}(u), t \otimes f)$ and $(y\nu, y\nu \otimes y) = (\mathbf{r}(u), u^{-1}t \otimes f)$. This implies $\mathbf{d}(t) = \mathbf{d}(u) = x\nu$ and there exists $g \in G(x, f)$ such that $t = (x\nu)(g\nu) = (g\nu)$. Also, $\mathbf{r}(u) = y\nu$ and there exists $h \in G(y, f)$ such that $u^{-1}t = (y\nu)(h\nu) = (h\nu)$. It follows that $u = (g\nu)(h\nu)^{-1} = (gh^{-1})\nu$. Since $g \in G(x, f)$, $\mathbf{d}(gh^{-1}) = x$. So $(u, t \otimes f) = ((gh^{-1})\nu, x\nu \otimes x) = ((gh^{-1})\nu, \mathbf{d}(gh^{-1})\nu \otimes \mathbf{d}(gh^{-1})) \in G\iota$.

(E3) Let $(\mathbf{d}(t), t \otimes f) \in E(\tilde{T}_\nu)$. Then $f\nu = \mathbf{r}(t)$ so $(t^{-1}, f\nu \otimes f) \in \tilde{T}_\nu$. Now $\mathbf{r}(t^{-1}, f\nu \otimes f) = (\mathbf{r}(t^{-1}), tf\nu \otimes f) = (\mathbf{d}(t), t \otimes f)$. The domain of $(t^{-1}, f\nu \otimes f) \in \tilde{T}_\nu$ is $\mathbf{d}(t^{-1}, f\nu \otimes f) = (\mathbf{d}(t^{-1}), f\nu \otimes f) = (\mathbf{r}(t), f\nu \otimes f) = (f\nu, f\nu \otimes f) \in E(G\iota)$.

Therefore $\tilde{T}_\nu$ is the enlagement of $G\iota$.

3. Let $(u, t \otimes f) \in \tilde{T}_\nu$. Then $f\nu = \mathbf{r}(t) = fj\pi$ and so there exists a unique element $c_{(t,f)} \in C$ such that $\mathbf{r}(c_{(t,f)}) = fj$ and $c_{(t,f)}\pi = t$. Now $\mathbf{d}(c_{(t,f)})\pi = \mathbf{d}(c_{(t,f)}\pi) = \mathbf{d}(t) = \mathbf{d}(u)$. Define $(u, t \otimes f)\theta$ to be the unique element $a \in C$ such that $\mathbf{d}(a) = \mathbf{d}(c_{(t,f)})$ and $a\pi = u$.

We show that θ is well-defined. If $t \otimes f = s \otimes e$ then there exists $g \in G(f, e)$ with $s = t(g\nu)$. Then $c_{(t,f)}(gj)$ has range ej and $(c_{(t,f)}(gj))\pi = ((c_{(t,f)}\pi)(gj\pi) = t(g\nu) = s$. Therefore $c_{(t,f)}(gj) = c_{(s,e)}$ because $\mathbf{r}(c_{(s,e)}) = ej$ and $c_{(s,e)}\pi = s$. Hence $\mathbf{d}(c_{(s,e)}) = \mathbf{d}(c_{(t,f)}) = \mathbf{d}(a)$ where $a\pi = u$. So $(u, t \otimes f)\theta = a = (u, s \otimes e)\theta$.

Two elements $(u, t \otimes f)$ and $(v, t' \otimes f')$ are composable if $\mathbf{r}(u) = \mathbf{d}(v)$ and there exists $g \in G(f, f')$ such that $t' = u^{-1}t(g\nu)$. Thus for the elements to be composable $t' \otimes f'$ must equal $u^{-1}t \otimes f$. So consider a pair of composable elements $(u, t \otimes f)$ and $(v, u^{-1}t \otimes f)$ in $\tilde{T}_\nu$. Now $\mathbf{d}(u, t \otimes f)\theta = (\mathbf{d}(u), t \otimes f)\theta = (\mathbf{d}(t), t \otimes f)\theta = \mathbf{d}(c_{(t,f)})$. Then $(u, t \otimes f)\theta^{-1}c_{(t,f)}$ has range fj and is mapped to $u^{-1}t$ by π, so $(u, t \otimes f)\theta^{-1}c_{(t,f)} = c_{(u^{-1}t,f)}$. Now $(v, u^{-1}t \otimes f)\theta$ is the unique element $b \in C$ such that $(u, t \otimes f)\theta b$ exists and $b\pi = v$. So $\mathbf{d}((u, t \otimes f)\theta(v, u^{-1}t \otimes f)\theta) = \mathbf{d}(c_{(t,f)})$. Therefore $((u, t \otimes f)\theta(v, u^{-1}t \otimes f)\theta)\pi = uv$ and so

$$[(u, t \otimes f)\theta][(v, u^{-1}t \otimes f)\theta] = (uv, t \otimes f)\theta = [(u, t \otimes f)(v, u^{-1}t \otimes f)]\theta.$$

Therefore θ is a functor.

If $(u, t \otimes f) \leqslant (v, s \otimes e)$ in $\tilde{T}_\nu$ then $u \leqslant v$ and we may assume that $f \leqslant e$ and $t \leqslant s$. Then $c_{(t,f)} = (c_{(s,e)}|fj)$. If we let $(u, t \otimes f)\theta = a$ and $(v, s \otimes e)\theta = b$ then $(\mathbf{d}(c_{(t,f)})|b)\pi \leqslant b\pi = v$ and $\mathbf{d}((\mathbf{d}(c_{(t,f)})|b)\pi) = \mathbf{d}(c_{(t,f)})\pi = \mathbf{d}(t) = \mathbf{d}(u)$. Hence by uniqueness of restriction $(\mathbf{d}(c_{(t,f)})|b)\pi = u$ and so $(\mathbf{d}(c_{(t,f)})|b) = (u, t \otimes f)\theta = a$. Therefore $a \leqslant b$ and θ is an ordered functor.

To show that θ is unique, suppose $\phi : \tilde{T}_\nu \to C$ is such that $j = \iota\phi$ and

$\pi_2 = \phi\pi$. If $z \in \tilde{T}_\nu$ and $(\mathbf{r}(z))\theta = (\mathbf{r}(z))\phi$ then, since $(z\theta)\pi = z\pi_2 = (z\phi)\pi$ and π is a covering, we have that $z\theta = z\phi$. Now since $\iota\theta = j = \iota\phi$ then $\phi|_{G\iota} = \theta|_{G\iota}$. Given any identity of $\tilde{T}_\nu$, $(\mathbf{d}(t), t\otimes f) \notin G\iota$, let $z = (u, t\otimes f) \in \tilde{T}_\nu$ be any arrow with $\mathbf{r}(z) \in G\iota$. At least one such arrow exists, as $\tilde{T}_\nu$ is an enlargement of $G\iota$. Hence $(\mathbf{r}(z))\theta = (\mathbf{r}(z))\phi$ so $z\theta = z\phi$. In particular, $(\mathbf{d}(z))\theta = (\mathbf{d}(z))\phi$. Therefore θ and ϕ are equal on $E(\tilde{T}_\nu)$ and it follows that $\theta = \phi$ on $\tilde{T}_\nu$. $\square$

We give a commutative diagram in Fig. 5.4.4 to depict the information in theorem 5.4.2 clearly.

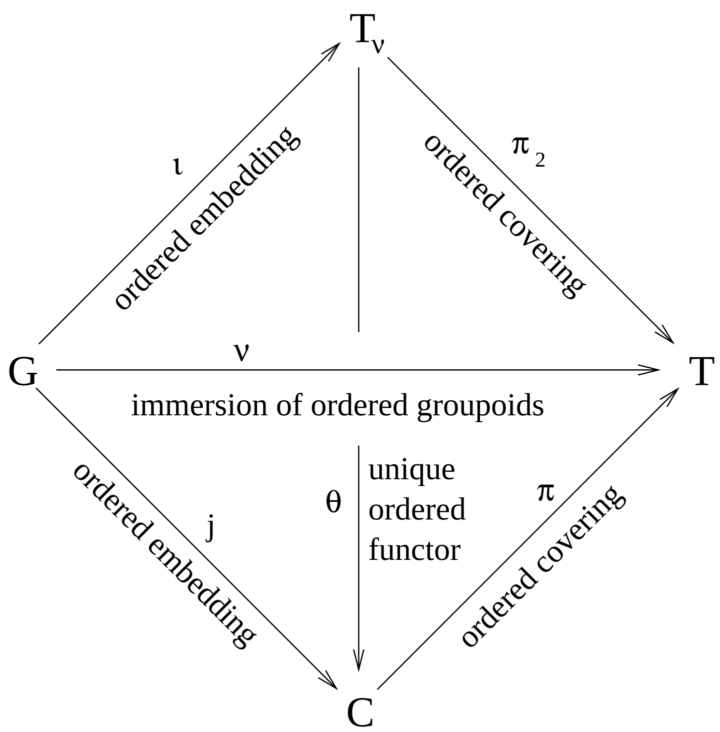

Figure 5.4.4: Theorem 5.4.2.

The Maximum Enlargement Theorem generalises the role of the action groupoid for coverings. If $\nu : G \to T$ is a covering then $T \otimes E(G)$ is isomorphic to

$E(G)$ and $\tilde{T}_\nu$ is isomorphic to G. We prove these isomorphisms below. First we show that $\phi : E(G) \to T \otimes E(G)$, $e \mapsto e\nu \otimes e$ is bijective. Let $e, f \in E(G)$ and suppose $e\phi = f\phi$. Then $e\nu \otimes e = f\nu \otimes f$. So there exists $k \in G(e, f)$ such that $e\nu = (f\nu)(k\nu)^{-1}$. Now $e\nu, f\nu \in E(T)$ and $\mathbf{r}(k) = f$ so $e\nu = (k\nu)^{-1}$. As ν is a covering, $e = k^{-1}$ and so $k \in E(G)$. It follows that $e = f$. Thus ϕ is injective.

Now we let $s \otimes e \in T \otimes E(G)$. Then $e\nu = \mathbf{r}(s)$. Now ν is a covering so there exists $g \in G$ such that $g\nu = s$. So $e\nu = \mathbf{r}(g\nu) = \mathbf{r}(g)\nu$. As ν is a covering, $e = \mathbf{r}(g)$. Then $s \otimes e = g\nu \otimes \mathbf{r}(g) = \mathbf{d}(g)\nu \otimes \mathbf{d}(g)$ because $g \in G(\mathbf{d}(g), \mathbf{r}(g))$ and $g\nu = \mathbf{d}(g)\nu(g\nu)$. So we have $\mathbf{d}(g) \in E(G)$ such that $\mathbf{d}(g) \mapsto s \otimes e$. Thus ϕ is surjective.

Therefore $E(G)$ is isomorphic to $T \otimes E(G)$.

As ν is an ordered covering, by lemma 5.3.6 we have that G is isomorphic to $T \ltimes E(G)$. It follows that G is isomorphic to $T \ltimes (T \otimes E(G))$. $\square$

5.5 $\mathbb{L}$-Systems

As E-unitary inverse semigroups are those semigroups that are idempotent pure extensions by groups, lemma 1.1.33, O'Carroll's structure theorem for inverse semigroups that are idempotent pure extensions by inverse semigroups [21] generalises McAlister's P-theorem. In the construction of McAlister's P-theorem we create a triple from a poset, an order ideal and an action, and from this a semigroup is constructed, as we have already seen in section 5.1. O'Carroll's structure theorem develops in a similar manner. His semigroup, an "$\mathbb{L}$-semigroup" is also constructed from a triple similar to McAlister's triple.

Our aim is, when given an immersion from an ordered groupoid G to some "simpler" ordered groupoid T, to construct a structure theorem for G somewhat like the P-theorem. The result uses Ehresmann's Maximum Enlargement Theorem and the corresponding groupoid definition of O'Carroll's $\mathbb{L}$-semigroups. We will demonstrate, by means of an example, that our result generalises the P-theorem for ordered groupoids.

To start then we need an analogue of O'Carroll's notion of an $\mathbb{L}$-semigroup. For this we require a poset X and an order ideal Y of X. We also need a fibred action of T on X, fibred by $\mu : X \to E(T)$ say, such that $X = T \cdot Y$. If we have such a poset, order ideal and fibred action then we call (X, Y, T) an $\mathbb{L}$-*system*.

An $\mathbb{L}$-system determines an ordered groupoid

$$\mathbb{L}(X, Y, T) = \{(t, y) \in T \times Y : y\mu = \mathbf{d}(t), t^{-1} \cdot y \in Y\}$$

with composition

$$(t, x)(s, y) = (ts, x) \text{ if } t^{-1} \cdot x = y \text{ and } \mathbf{r}(t) = \mathbf{d}(s)$$

and ordering

$$(t, x) \leqslant (s, y) \text{ if and only if } t \leqslant s \text{ and } x \leqslant y.$$

In section 5.2.1 we define the ordered groupoid $P(X, Y, G)$ given a groupoid G and action on X. The $\mathbb{L}$-system determines an ordered groupoid $\mathbb{L}(X, Y, T)$ given an *ordered* groupoid T and an action on X. Although composition in both $P(X, Y, G)$ and $\mathbb{L}(X, Y, T)$ is the same, ordering differs. Both $P(X, Y, G)$ and $\mathbb{L}(X, Y, T)$ use the ordering of the poset, $\mathbb{L}(X, Y, T)$ also uses the ordering of the groupoid whereas $P(X, Y, G)$ has the trivial ordering on the groupoid element because T is an ordered groupoid and G is not.

Lemma 5.5.1. $\mathbb{L}(X, Y, T)$ *is an ordered subgroupoid of the action groupoid* $T \ltimes X$.

Proof. Firstly, $\mathbb{L}(X, Y, T) = \{(t, y) : y\mu = \mathbf{d}(t), t^{-1} \cdot y \in Y\} \subseteq \{(t, x) : x\mu = \mathbf{d}(t)\} = T \ltimes X$. We now check closure under composition. Let $(t, x), (s, y) \in \mathbb{L}(X, Y, T)$ and assume $t^{-1} \cdot x = y$ and $\mathbf{r}(t) = \mathbf{d}(s)$. Then $x\mu = \mathbf{d}(t) = \mathbf{d}(ts)$ and $(ts)^{-1} \cdot x = s^{-1} \cdot (t^{-1} \cdot x) = s^{-1} \cdot y \in Y$. Thus $(ts, x) \in \mathbb{L}(X, Y, T)$.

We show next that $\mathbb{L}(X, Y, T)$ is closed under inverses. Let $(t, y) \in \mathbb{L}(X, Y, T)$. Now $(t, y)^{-1} = (t^{-1}, t^{-1} \cdot y)$. Then $(t^{-1} \cdot y)\mu = \mathbf{d}(t^{-1})$ and $t \cdot (t^{-1} \cdot y) = (tt^{-1}) \cdot y = y \in Y$ so $(t, y)^{-1} \in \mathbb{L}(X, Y, T)$.

Next we establish that $\mathbb{L}(X, Y, T)$ is closed under the restriction operation. Let $(t, y) \in \mathbb{L}(X, Y, T)$ and let $(e, z) \in E(\mathbb{L}(X, Y, T))$ be such that $(e, z) \leqslant \mathbf{d}(t, y) = (\mathbf{d}(t), y)$. Then $((e, z)|(t, y)) = ((e|t), z)$. Now $(e|t) \leqslant t$ and $z \leqslant y$ so $(e|t)^{-1} \cdot z \leqslant t^{-1} \cdot y \in Y$ but Y is an order ideal so $(e|t)^{-1} \cdot z \in Y$. Also, $z\mu = e = \mathbf{d}(e|t)$. Hence $((e|t), z) \in \mathbb{L}(X, Y, T)$.

Therefore $\mathbb{L}(X, Y, T)$ is a subgroupoid of $T \ltimes X$. $\square$

Proposition 5.5.2. *(a) Given an* $\mathbb{L}$*-system* (X, Y, T)*, the action groupoid* $T \ltimes X$ *is an enlargement of* $\mathbb{L}(X, Y, T)$.
(b) If the action groupoid $T \ltimes X$ *is an enlargement of the subgroupoid* H *then there exists an* $\mathbb{L}$*-system* $(\bar{X}, Y_H, H\pi_2)$*, where* $\bar{X} = H\pi_2 \cdot Y_H$*, such that*

$$H = \mathbb{L}(\bar{X}, Y_H, H\pi_2).$$

Proof. (a) By lemma 5.5.1, $\mathbb{L}(X, Y, T)$ is a subgroupoid of $T \ltimes X$.

We must now check the enlargement axioms are satisfied.

(E1) Suppose $(e, x) \leqslant (f, y) \in E(\mathbb{L}(X, Y, T))$, then $e \leqslant f$ and $x \leqslant y$. Now Y is an order ideal, so $x \leqslant y \in Y$ implies that $x \in Y$. Now $e^{-1} \cdot x = x \in Y$

and $x\mu = e$ so $(e, x) \in E(\mathbb{L}(X, Y, T))$ and $E(\mathbb{L}(X, Y, T))$ is an order ideal of $E(T \ltimes X)$.

(E2) Let $(t, x) \in T \ltimes X$, $\mathbf{d}(t, x) = (\mathbf{d}(t), x) \in \mathbb{L}(X, Y, T)$ and $\mathbf{r}(t, x) = (\mathbf{r}(t), t^{-1} \cdot x) \in \mathbb{L}(X, Y, T)$. Now $(t, x) \in T \ltimes X$ implies that $x\mu = \mathbf{d}(t)$ and $(\mathbf{r}(t), t^{-1} \cdot x) \in \mathbb{L}(X, Y, T)$ implies that $\mathbf{r}(t)^{-1} \cdot (t^{-1} \cdot x) = t^{-1} \cdot x \in Y$. Thus $(t, x) \in \mathbb{L}(X, Y, T)$.

(E3) Let $(e, v) \in E(T \ltimes X)$. We require an $(t, x) \in T \ltimes X$ such that $\mathbf{r}(t, x) = (e, v)$ and $\mathbf{d}(t, x) \in \mathbb{L}(X, Y, T)$. Now $X = T \cdot Y$ so $v = s \cdot w$, for some $w \in Y$, $s \in T$, such that $w\mu = \mathbf{r}(s)$. Since $(e, v) = (e, s \cdot v) \in E(T \ltimes X)$, $(s \cdot w)\mu = \mathbf{d}(s) = e$. Consider (s^{-1}, w). Now $w\mu = \mathbf{r}(s) = \mathbf{d}(s^{-1})$ so $(s^{-1}, w) \in T \ltimes X$. Also, $\mathbf{r}(s^{-1}, w) = (\mathbf{r}(s^{-1}), s \cdot w) = (\mathbf{d}(s), s \cdot w) = (e, v)$ and $\mathbf{d}(s^{-1}, w) = (\mathbf{d}(s^{-1}), w)$. Now $w\mu = \mathbf{d}(s^{-1})$ and $\mathbf{d}(s^{-1}) \cdot w = w \in Y$ so $\mathbf{d}(s^{-1}, w) \in \mathbb{L}(X, Y, T)$. Thus (s^{-1}, w) is the required $(t, x) \in T \ltimes X$.

Therefore $T \ltimes X$ is an enlargement of $\mathbb{L}(X, Y, T)$.

(b) Take $Y_H = \{x \in X : (t, x) \in H \,\text{for some}\, t \in T\}$ and $\bar{X} = H\pi_2 \cdot Y_H \subseteq X$. We show that Y_H is an order ideal of $\bar{X}$. For $x \in \bar{X}$ let $x \leqslant y \in Y_H$. As $y \in Y_H$ there exists $t \in T$ such that $(t, y) \in H \subseteq T \ltimes X$, so $y\mu = \mathbf{d}(t)$. Also $(\mathbf{d}(t), y) \in E(H)$. Now $x \leqslant y$ implies $x\mu \leqslant y\mu = \mathbf{d}(t)$ as μ is order preserving, so $(x\mu, x) \leqslant (\mathbf{d}(t), y) \in E(H)$. Now $E(H)$ is an order ideal of $E(T \ltimes X)$ so $(x\mu, x) \in E(H)$. This gives us $x\mu \in T$ such that $(x\mu, x) \in H$, therefore $x \in Y_H$ and so Y_H is an order ideal of $\bar{X}$.

We show next that $\mathbb{L}(\bar{X}, Y_H, H\pi_2) \subseteq H$. Let $(t, y) \in \mathbb{L}(\bar{X}, Y_H, H\pi_2)$. Now $y\mu = \mathbf{d}(t)$, and since $y \in Y_H$ there exists an $s \in T$ such that $(s, y) \in H$, so $y\mu = \mathbf{d}(s)$. Also $t^{-1} \cdot y \in Y_H$ so there exists an $s' \in T$ such that $(s', t^{-1} \cdot y) \in H$. Then $(t^{-1} \cdot y)\mu = \mathbf{d}(s')$ but, as $y\mu = \mathbf{d}(t)$, $(t^{-1} \cdot y)\mu = \mathbf{d}(t^{-1}) = \mathbf{r}(t)$ so $\mathbf{d}(s') = \mathbf{r}(t)$. Also, $\mathbf{d}(t) = y\mu = \mathbf{d}(s)$. Now $(s, y) \in H$ implies that $\mathbf{d}(s, y) \in H$ because H is a subgroupoid of $T \ltimes X$. Then $\mathbf{d}(s, y) = (\mathbf{d}(s), y) =$

$(\mathbf{d}(t), y) \in H$. Similarly, $(s', t^{-1} \cdot y) \in H$ implies $\mathbf{d}(s', t^{-1} \cdot y) \in H$. Then $\mathbf{d}(s', t^{-1} \cdot y) = (\mathbf{d}(s'), t^{-1} \cdot y) = (\mathbf{r}(t), t^{-1} \cdot y) \in H$. Now $(\mathbf{d}(t), y) = \mathbf{d}(t, y) \in H$ and $(\mathbf{r}(t), t^{-1} \cdot y) = \mathbf{r}(t, y) \in H$ and $T \ltimes X$ is an enlargement of H so by property (E3), $(t, y) \in H$.

Finally we show that $H \subseteq \mathbb{L}(\bar{X}, Y_H, H\pi_2)$. Let $(s, x) \in H \subseteq T \ltimes X$. Now

$$\begin{aligned}\mathbb{L}(\bar{X}, Y_H, H\pi_2) &= \{(t, y) \in H\pi_2 \times Y_H : y\mu = \mathbf{d}(t), t^{-1} \cdot y \in Y_H\} \\ &= \{(t, y) : (s, y) \in H \text{ for some } s \in T, \\ &\quad (t, z) \in H \text{ for some } z \in X, y\mu = \mathbf{d}(t), \\ &\quad (s', t^{-1} \cdot y) \in H \text{ for some } s' \in T\}.\end{aligned}$$

As $(s, x) \in H$, $(s, x) \in H\pi_2 \times Y_H$. Also, $(s, x) \in T \ltimes X$ implies that $x\mu = \mathbf{d}(s)$. Finally, as H is a subgroupoid of $T \ltimes X$, $\mathbf{r}(s, x) = (\mathbf{r}(s), s^{-1} \cdot x) \in H$. Thus $(s, x) \in \mathbb{L}(\bar{X}, Y_H, H\pi_2)$ and $H \subseteq \mathbb{L}(\bar{X}, Y_H, H\pi_2)$.

Therefore $H = \mathbb{L}(\bar{X}, Y_H, H\pi_2)$. $\square$

To summarize, we have:

- immersion $\nu : G \to T$
- poset $T \otimes E(G)$
- action groupoid $\tilde{T}_\nu = T \ltimes (T \otimes E(G))$
- covering $\pi_2 : \tilde{T}_\nu \to T$, $(u, t \otimes f) \mapsto u$
- embedding $\iota : G \to \tilde{T}_\nu$, $g \mapsto (g\nu, \mathbf{d}(g)\nu \otimes \mathbf{d}(g))$
- $\tilde{T}_\nu$ is an enlargement of $G\iota$

- $\iota\pi_2 = \nu$ so $G\iota\pi_2 = G\nu$.

Now that we have all the jigsaw pieces we can put them together. Following proposition 5.5.2 we have :

Theorem 5.5.3. *1. Given an $\mathbb{L}$-system $(T \otimes E(G), Y, T)$, then $\tilde{T}_\nu$ is an enlargement of* $\mathbb{L}(T \otimes E(G), Y, T)$
2. $\tilde{T}_\nu$ is an enlargement of $G\iota$ so there exists an order ideal

$$Y_{G\iota} = \{(t \otimes e) \in T \otimes E(G) : (u, t \otimes e) \in G\iota \text{ for some } u \in T\}$$

of $T \otimes E(G)$ and an $\mathbb{L}$-system $(T \otimes E(G), Y_{G\iota}, G\iota\pi_2)$ such that

$$G\iota = \mathbb{L}(T \otimes E(G), Y_{G\iota}, G\iota\pi_2).$$

We give an example to illustrate some of the details involved in the construction of the $\mathbb{L}$-system.

Example 5.5.4. Let G be the groupoid consisting of two copies of the interval groupoid $\mathbb{I}^1$ as shown in Fig. 5.5.1. The ordering is $e_0 \geqslant f_0$, $\alpha \geqslant \beta$, $\alpha^{-1} \geqslant \beta^{-1}$, and $e_1 \geqslant f_1$. We take $\nu : G \to T$ to be an immersion of ordered groupoids. Then $\iota : G \to \tilde{T}_\nu$ defined by $g \mapsto (g\nu, \mathbf{d}(g)\nu \otimes \mathbf{d}(g))$ gives us

$$\begin{aligned} G\iota = \{&(e_0\nu, e_0\nu \otimes e_0), (\alpha\nu, e_0\nu \otimes e_0), (\alpha^{-1}\nu, e_1\nu \otimes e_1\nu), (e_1\nu, e_1\nu \otimes e_1), \\ &(f_0\nu, f_0\nu \otimes f_0), (\beta\nu, f_0\nu \otimes f_0), (\beta^{-1}\nu, f_1\nu \otimes f_1\nu), (f_1\nu, f_1\nu \otimes f_1)\} \end{aligned}$$

Then $Y_{G\iota} = \{e_0\nu \otimes e_0, e_1\nu \otimes e_1, f_0\nu \otimes f_0, f_1\nu \otimes f_1\}$.

Now $u^{-1} \in T$ acts on $t \otimes f$ only if $(t \otimes f)\mu = \mathbf{d}(t) \leqslant \mathbf{d}(u)$, in which case $u^{-1} \cdot (t \otimes f) = (u^{-1}|\mathbf{d}(t))t \otimes f$. So if $\mathbf{d}(t) = \mathbf{d}(u)$ then $u^{-1} \cdot (t \otimes f) = u^{-1}t \otimes f$. Given $\mathbf{d}(g)\nu \otimes \mathbf{d}(g)$ we can act on this by $g\nu$ giving $(g\nu)^{-1} \cdot (\mathbf{d}(g)\nu \otimes \mathbf{d}(g)) =$

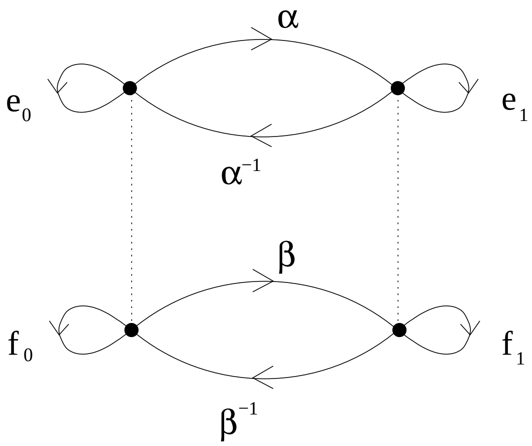

Figure 5.5.1: groupoid G.

$(g\nu)^{-1}\mathbf{d}(g)\nu \otimes \mathbf{d}(g) = (g\nu)^{-1} \otimes \mathbf{d}(g)$. Now we note that $g \in G(\mathbf{d}(g), \mathbf{r}(g))$ and $(g\nu)^{-1} = \mathbf{r}(g)\nu(g\nu)^{-1}$ so $(g\nu)^{-1} \otimes \mathbf{d}(g) = \mathbf{r}(g)\nu \otimes \mathbf{r}(g)$.

Consider $e_0\nu \otimes e_0 \in Y_{G\iota}$. Then $\alpha\nu^{-1} \cdot (e_0\nu \otimes e_0) = \alpha\nu^{-1} \otimes e_0 = e_1\nu \otimes e_1 \in Y_{G\iota}$ and $(e_0\nu \otimes e_0)\mu = \mathbf{d}(e_0\nu) = e_0\nu = \mathbf{d}(\alpha\nu)$, which implies $(\alpha\nu, e_0\nu \otimes e_0) \in \mathbb{L}(T \otimes E(G), Y_{G\iota}, G\iota\pi_2)$. Similarly, $(e_0\nu, e_0\nu \otimes e_0) \in \mathbb{L}(T \otimes E(G), Y_{G\iota}, G\iota\pi)$. Assume that $t^{-1} \in T$ such that t^{-1} acts on $e_0\nu \otimes e_0$ but $t^{-1} \neq \alpha^{-1}\nu$ and $t^{-1} \neq e_0\nu$. Then $t^{-1} \cdot (e_0\nu \otimes e_0) = (t^{-1}|e_0\nu)e_0\nu \otimes e_0 \neq Y_{G\iota}$ so $(t, e_0\nu \otimes e_0) \notin \mathbb{L}(T \otimes E(G), Y_{G\iota}, G\iota\pi)$. We can repeat this process for all the elements of $Y_{G\iota}$ to show

$$\mathbb{L}(T \otimes E(G), Y_{G\iota}, G\iota\pi_2) = G\iota.$$

We now give another example. We show that when G is an incompressible ordered groupoid, and the immersion in question is the levelling functor $\lambda : G \to G_{\updownarrow}$, we recover the P-theorem for ordered groupoids from theorem 5.5.3 with the $\mathbb{L}$-system $(G_{\updownarrow} \otimes E(G), Y_{G\iota}, G\iota\pi_2)$. Thus theorem 5.5.3 generalises the P-theorem for ordered groupoids.

Example 5.5.5. Let G be an incompressible ordered groupoid and let $\lambda :$

$G \to G_{\updownarrow}$ be the levelling immersion. We wish to show that $\mathbb{L}(G_{\updownarrow} \otimes E(G), Y_{G\iota}, G\iota\pi_2) = P(G_{\updownarrow} \otimes E(G), Y_{G\iota}, G_{\updownarrow})$. We begin by describing the ordered groupoid $P(G_{\updownarrow} \otimes E(G), Y_{G\iota}, G_{\updownarrow})$, then we describe the ordered groupoid $\mathbb{L}(G_{\updownarrow} \otimes E(G), Y_{G\iota}, G\iota\pi_2)$ and finally we compare the two.

Recall that

$$P(X, Y, G_{\updownarrow}) = \{(y, a) \in Y \times G_{\updownarrow} : y \in Y_{\mathbf{d}(a)}, a^{-1} \cdot y \in Y\}.$$

For each element $e\lambda \in E(G_{\updownarrow})$ we have poset $X_{e\lambda} = \{s \otimes f \in G_{\updownarrow} \otimes E(G) : \mathbf{d}(s) = e\lambda\}$. Each arrow $g\lambda \in G_{\updownarrow}$ then determines an isomorphism $g\lambda : X_{\mathbf{r}(g)\lambda} \to X_{\mathbf{d}(g)\lambda}$. Then take the poset X to be

$$X = \bigsqcup_{e \in E(G_{\updownarrow})} X_{e\lambda} = G_{\updownarrow} \otimes E(G).$$

$X = G_{\updownarrow} \otimes E(G)$ is a disjoint union because $\mathbf{d}(t)$ is an invariant of $t \otimes k$. Now

$$\begin{aligned} Y_{G\iota} &= \{t \otimes e \in G_{\updownarrow} \otimes E(G) : (u, t \otimes e) \in G\iota \text{ for some } u \in G_{\updownarrow}\} \\ &= \{\mathbf{d}(g)\lambda \otimes \mathbf{d}(g) \in G_{\updownarrow} \otimes E(G) : g \in G\}. \end{aligned}$$

Further, $Y_{e\lambda} = Y_{G\iota} \cap X_{e\lambda}$ so

$$\begin{aligned} Y_{e\lambda} &= \{\mathbf{d}(g)\lambda \otimes \mathbf{d}(g) \in G_{\updownarrow} \otimes E(G) : g \in G, \mathbf{d}(\mathbf{d}(g)\lambda) = e\lambda\} \\ &= \{\mathbf{d}(g)\lambda \otimes \mathbf{d}(g) \in G_{\updownarrow} \otimes E(G) : g \in G, \mathbf{d}(g)\lambda = e\lambda\} \\ &= \{e\lambda \otimes \mathbf{d}(g) \in G_{\updownarrow} \otimes E(G)\}. \end{aligned}$$

Then

$$\begin{aligned} &P(G_{\updownarrow} \otimes E(G), Y_{G\iota}, G_{\updownarrow}) \\ &\quad = \{(\mathbf{d}(g)\lambda \otimes \mathbf{d}(g), h\lambda) : \mathbf{d}(g)\lambda \otimes \mathbf{d}(g) \in Y_{\mathbf{d}(h\lambda)}, h\lambda^{-1} \cdot (\mathbf{d}(g)\lambda \otimes \mathbf{d}(g)) \in Y_{G\iota}\}. \end{aligned}$$

As $\mathbf{d}(g)\lambda \otimes \mathbf{d}(g) \in Y_{\mathbf{d}(h\lambda)}$ we have $\mathbf{d}(g)\lambda \otimes \mathbf{d}(g) = \mathbf{d}(h)\lambda \otimes \mathbf{d}(g)$. Further,

$$\begin{aligned} & h\lambda^{-1} \cdot (\mathbf{d}(g)\lambda \otimes \mathbf{d}(g)) \\ & \quad = h^{-1}\lambda \cdot (\mathbf{d}(h)\lambda \otimes \mathbf{d}(g)) \\ & \quad = h^{-1}\lambda \mathbf{d}(h)\lambda \otimes \mathbf{d}(g) \\ & \quad = h^{-1}\lambda \otimes \mathbf{d}(g) \end{aligned}$$

and so $h^{-1}\lambda \otimes \mathbf{d}(g) \in Y_{G\iota}$. So, $h^{-1}\lambda \otimes \mathbf{d}(g) = \mathbf{d}(m)\lambda \otimes \mathbf{d}(m)$ for some $m \in G$. Thus

$$\begin{aligned} & P(G_{\updownarrow} \otimes E(G), Y_{G\iota}, G_{\updownarrow}) \\ & \quad = \{(\mathbf{d}(h)\lambda \otimes \mathbf{d}(g), h\lambda) : h^{-1}\lambda \otimes \mathbf{d}(g) = \mathbf{d}(m)\lambda \otimes \mathbf{d}(m) \text{ for some } m \in G\}. \end{aligned}$$

Now we describe $\mathbb{L}(G_{\updownarrow} \otimes E(G), Y_{G\iota}, G\iota\pi_2)$. As $\lambda : G \to G_{\updownarrow}$ is our given immersion then $\iota\pi_2 = \lambda$. So we have

$$\begin{aligned} & \mathbb{L}(G_{\updownarrow} \otimes E(G), Y_{G\iota}, G\iota\pi_2) \\ & \quad = \mathbb{L}(G_{\updownarrow} \otimes E(G), Y_{G\iota}, G\lambda) \\ & \quad = \mathbb{L}(G_{\updownarrow} \otimes E(G), Y_{G\iota}, G_{\updownarrow}) \\ & \quad = \{(u, t \otimes e) \in G_{\updownarrow} \times Y_{G\iota} : \mathbf{d}(t) = \mathbf{d}(u), u^{-1} \cdot (t \otimes e) \in Y_{G\iota}\} \\ & \quad = \{(h\lambda, \mathbf{d}(g)\lambda \otimes \mathbf{d}(g)) : \mathbf{d}(g)\lambda = \mathbf{d}(h)\lambda, h\lambda^{-1} \cdot (\mathbf{d}(g)\lambda \otimes \mathbf{d}(g)) \in Y_{G\iota}\}. \end{aligned}$$

Then as $\mathbf{d}(g)\lambda = \mathbf{d}(h)\lambda$ we have that $h\lambda^{-1} \cdot (\mathbf{d}(g)\lambda \otimes \mathbf{d}(g) = h^{-1}\lambda \otimes \mathbf{d}(g)$. So $h^{-1}\lambda \otimes \mathbf{d}(g) \in Y_{G\iota}$ and so $h^{-1}\lambda \otimes \mathbf{d}(g) = \mathbf{d}(m)\lambda \otimes \mathbf{d}(m)$ for some $m \in G$.

Thus

$$\mathbb{L}(G_{\updownarrow} \otimes E(G), Y_{G\iota}, G_{\updownarrow})$$
$$= \{(h\lambda, \mathbf{d}(h)\lambda \otimes \mathbf{d}(g)) : h^{-1}\lambda \otimes \mathbf{d}(g)) = \mathbf{d}(m)\lambda \otimes \mathbf{d}(m) \text{ for some } m \in G\}.$$

Hence $\mathbb{L}(G_{\updownarrow} \otimes E(G), Y_{G\iota}, G\iota\pi_2) = P(G_{\updownarrow} \otimes E(G), Y_{G\iota}, G_{\updownarrow})$. Therefore by theorem 5.5.3 we have that $P(G_{\updownarrow} \otimes E(G), Y_{G\iota}, G_{\updownarrow})$ is isomorphic to $G\iota$. Thus the P-theorem is recovered.

5.6 Gilbert's Proof of the P-theorem

In [5] Gilbert proves that any incompressible ordered groupoid is isomorphic to the groupoid $P(X, Y, H)$ for some poset X, some order ideal Y of X and some groupoid H. Hence all incompressible ordered groupoids are classified up to isomorphism. Gilbert models his proof of this main theorem of [5] on that of Munn [20]. To construct the required poset Gilbert introduces a left cancellative category. The actions of this category are used to define a quasiorder on the pullback and so an equivalence relation. The set of equivalence classes is the required poset. In the previous section we gave an $\mathbb{L}$-system description of an ordered groupoid G given an immersion, and we showed that it generalised the P-theorem. In the construction of the poset we used in the $\mathbb{L}$-system we created a set of equivalence classes from a pullback. This pullback was constructed from the actions of the groupoid G, unlike Gilbert's which uses the actions of a left cancellative category. We show that the introduction of the left cancellative category is an unnecessary complication. After we reconstruct Gilbert's poset we will compare it to the $\mathbb{L}$-system poset for immersion and levelling functor $\lambda : G \to G_{\updownarrow}$ as given in example 5.5.5.

Recall from section 5.2.1 that for an ordered groupid G, poset X and order ideal Y of X we define the ordered groupoid

$$P(X, Y, G) = \{(y, a) \in Y \times G : y \in Y_{\mathbf{d}(a)}, a^{-1} \cdot y \in Y\}.$$

As only elements of the poset $G \cdot Y$ need be considered we may assume $X = G{\cdot}Y$. Compostion between elements (x, a) and (z, b) occur if $\mathbf{r}(a) = \mathbf{d}(b)$ and $a^{-1} \cdot x = z$, in which case

$$(x, a)(z, b) = (x, ab).$$

The ordering is

$$(x, a) \leqslant (z, b) \Leftrightarrow x \leqslant z \text{ and } a = b.$$

To prove his P-theorem for ordered groupoids Gilbert constructs a left cancellative category and from this constructs the required poset. This is done as follows.

Following Lawson [11], Gilbert constructs a left cancellative category from an ordered groupoid G,

$$\mathbb{C}(G) = \{(f, a) \in E(G) \times G : \mathbf{d}(a) \leqslant f\}.$$

An arrow $(f, a) \in \mathbb{C}(G)$ has domain (f, f) and range $(\mathbf{r}(a), \mathbf{r}(a))$. Identities have the form (e, e) where $e \in E(G)$. Composition between arrows (f, a) and (e, b) occurs if $e = \mathbf{r}(a)$, in which case

$$(f, a)(e, b) = (f, (a|\mathbf{d}(b))b) = (f, a * b).$$

Lemma 5.6.1. *$\mathbb{C}(G)$ is a left cancellative category.*

Proof. $\mathbb{C}(G)$ is clearly a category. We show it is left cancellative. Let $(f, a), (e, b), (i, c) \in \mathbb{C}(G)$. Now let $f = \mathbf{r}(c)$ so that $(i, c)(f, a)$ is defined and

$$(i, c)(f, a) = (i, (c|\mathbf{d}(a))a) = (i, c * a)$$

and let $e = \mathbf{r}(c)$ so that $(i, c)(e, b)$ is defined and

$$(i, c)(e, b) = (i, (c|\mathbf{d}(b))b) = (i, c * b).$$

Suppose $(i, c)(f, a) = (i, c)(e, b)$. Then $(i, c * a) = (i, c * b)$ so $c * a = c * b$. Let $c' = (c|\mathbf{d}(a))$ and $c'' = (c|\mathbf{d}(b))$. Then $c'a = c''b$ so $\mathbf{d}(c') = \mathbf{d}(c'')$. As restriction of c to $\mathbf{d}(c')$ is unique, $c' = c''$. Then, as $c'a = c''b$ and $c' = c''$, we have that $a = b$. Therefore $(f, a) = (e, b)$ $\square$

$\mathbb{C}(G)$ acts on the left of $E(G)$: if $e = \mathbf{r}(a)$ then $(f, a) \cdot e = f$. Also if $\lambda : G \to G_{\updownarrow}$ is the levelling functor then $\mathbb{C}(G)$ acts on the right of $G_{\updownarrow}$: $w \cdot (f, a) = w(a\lambda)$ whenever this is defined. Combining these actions gives us a quasiorder $\preccurlyeq$ on the pullback of $G_{\updownarrow}$ and $E(G)$,

$$G_{\updownarrow} \between E(G) = \{(w, e) \in G_{\updownarrow} \times E(G) : e\lambda = \mathbf{r}(w)\},$$

defined by

$$(w \cdot (f, a), e) \preccurlyeq (w, (f, a) \cdot e)$$

if $(f, a) \in \mathbb{C}(G)$ and $e = \mathbf{r}(a)$. So $(u, e) \preccurlyeq (w, k)$ if there exists $a \in G$ such that $\mathbf{r}(a) = e$, $\mathbf{d}(a) \leqslant k$ and $u = w(a\lambda)$. Denote the equivalence class of (u, e) induced by the quasiorder by $u \otimes_C e$ and let $G_{\updownarrow} \otimes_C E(G)$ denote the set of equivalence classes. So

$$u \otimes_C e = v \otimes_C f \Leftrightarrow (u, e) \preccurlyeq (v, f) \text{ and } (v, f) \preccurlyeq (u, e).$$

Lemma 5.6.2. *[5, Lemma 4.1]. If G is incompressible then $u \otimes_C e = v \otimes_C f$ if and only if there exists $c \in G(f, e)$ such that $u = v(c\lambda)$.*

Proof. Assume $u \otimes_C e = v \otimes_C e$, then $(u, e) \preccurlyeq (v, f)$ and $(v, f) \preccurlyeq (u, e)$. So there exists an $a, b \in G$ with $\mathbf{r}(a) = e \geqslant \mathbf{d}(b)$ and $\mathbf{r}(b) = f \geqslant \mathbf{d}(a)$ such that $u = v(a\lambda)$ and $v = u(b\lambda)$. Now $(a * b)\lambda = ((a|\mathbf{d}(b))b)\lambda = (a|\mathbf{d}(b))\lambda(b\lambda) \leqslant a\lambda b\lambda = f\lambda \in E(G)$. This implies $(a|\mathbf{d}(b))\lambda(b\lambda) \in E(G_{\updownarrow})$. Thus, as λ is an immersion, $a * b \in E(G)$. Therefore $a * b = \mathbf{r}(a * b) = \mathbf{r}(b) = f$. Then $f = \mathbf{d}(a * b) \leqslant \mathbf{d}(a) \leqslant f$ so $\mathbf{d}(a) = f$. Then $ba \in G$ and $(ba)\lambda = e\lambda$ so $ba = f \in E(G)$ and $ba = e$. Thus $\mathbf{d}(b) = e$. Hence $\mathbf{r}(a) = e = \mathbf{d}(b)$ and $\mathbf{r}(b) = f = \mathbf{d}(a)$ and so $a = b^{-1}$. Take $c = a$. $\square$

Lemma 5.6.3. *[5, Lemma 4.3]. Let $\lambda_* : E(G) \to G_{\updownarrow} \otimes_C E(G)$, $e \mapsto e\lambda \otimes_C e$. If $g \in G$ then*

$$(\mathbf{d}(g))\lambda_* = \mathbf{d}(g)\lambda \otimes_C \mathbf{d}(g) = g\lambda \otimes_C \mathbf{r}(g).$$

Lemma 5.6.4. *$G_{\updownarrow} \otimes_C E(G)$, with ordering*

$$u \otimes_C e \leqslant v \otimes_C f \Leftrightarrow (u, e) \preccurlyeq (v, f),$$

is a partially ordered set.

The above poset $X = G_{\updownarrow} \otimes_C E(G)$ is the one used in the proof of Gilbert's P-theorem for incompressible ordered groupoids. The right action of $G_{\updownarrow}$ on X is given by $w \cdot (u \otimes_C e) = wu \otimes_C e$. The map λ_* gives us $Y = E(G)\lambda_*$, the order ideal of X, [5, Lemma 4.4].

Theorem 5.6.5. *[5, Theorem 4.5]. Let G be an incompressible ordered groupoid. Then G is isomorphic to the ordered groupoid $P =$ $P(G_{\updownarrow} \otimes_C E(G), E(G)\lambda_*, G_{\updownarrow})$.*

We recall the construction of the poset used in the $\mathbb{L}$-system structure in section 5.5. We take our immersion from incompressible ordered groupoid G to be the levelling functor $\lambda : G \to G_{\updownarrow}$ as we did in example 5.5.5. For $g \in G(e, f)$ and $t \in G_{\updownarrow}$ with $\mathbf{r}(t) = e\lambda$, G acts on the left of $E(G)$: $g \cdot f = e$. Also, G acts on the right of $G_{\updownarrow}$: $t \cdot g = t(g\lambda)$. The pullback is

$$G_{\updownarrow} \between E(G) = \{(s, e) \in G_{\updownarrow} \times E(G) : e\lambda = \mathbf{r}(s)\}.$$

The equivalence relation on $G_{\updownarrow} \between E(G)$ is defined as follows: $(u, e) \simeq_G (t, f)$ of there exists $g \in G(e, f)$ such that $t = u(g\lambda)$. The equivalence classes are denoted by $G_{\updownarrow} \otimes E(G)$. By theorem 5.4.1, $G_{\updownarrow} \otimes E(G)$ is a poset with the ordering $t \otimes f \leqslant s \otimes e$ if and only if there exists $k \leqslant e$ and $h \in G(f, k)$ such that $t(h\lambda) \leqslant s$. Since $k \leqslant e$ then $k\lambda = e\lambda$ in $G_{\updownarrow}$. Further, we have that $t(h\lambda) = s$ in $G_{\updownarrow}$. Thus the ordering on $G_{\updownarrow} \otimes E(G)$ is the same as the ordering on $G_{\updownarrow} \otimes_C E(G)$.

We want to know if $G_{\updownarrow} \otimes E(G) = G_{\updownarrow} \otimes_C E(G)$. It is clear that $G_{\updownarrow} \between E(G) = G_{\updownarrow} \between_C E(G)$. Our question becomes, is it true that

$$u \otimes e = t \otimes f \Leftrightarrow u \otimes_C e = t \otimes_C f \ ?$$

Our answer follows.

Theorem 5.6.6. *Let G be an incompressible ordered groupoid with levelling functor and immersion $\lambda : G \to G_{\updownarrow}$. If $(u, e), (t, f) \in G_{\updownarrow} \between E(G)$ then*

$$u \otimes e = t \otimes f \Leftrightarrow u \otimes_C e = t \otimes_C f.$$

Proof. By definition $u \otimes e = t \otimes f$ if and only if there exists $g \in G(e, f)$ such that $t = u(g\lambda)$. As G is incompressible the defintion of $u \otimes_C e = t \otimes_C f$ becomes, by lemma 5.6.2, $u \otimes_C e = t \otimes_C f$ if and only if there exists $c \in G(f, e)$ such that $u = t(c\lambda)$. Take c to be g^{-1} and the equivalence becomes clear. $\square$

Hence in the proof of the P-theorem for incompressible ordered groupoids the introduction of $\mathbb{C}(G)$ is an unnecessary complication.

Chapter 6

Quivers and $\mathbb{L}$-systems

McAlister's P-theorem classifies all inverse semigroups that are idempotent pure extensions by a group. O'Carroll classifies all inverse semigroups that are idempotent pure extensions by an inverse semigroup, generalising McAlister's P-theorem. Gomes and Szendrei [7] describe, by means of *quivers*, the structure of regular semigroups that are idempotent pure regular extensions by inverse semigroups.

Gomes and Szendrei use quivers, category-like structures, to formulate their structure theorem. Given a regular semigroup S that is an idempotent pure regular extension by an inverse semigroup, Gomes and Szendrei construct a derived quiver and an action on this quiver. From their derived quiver they construct a regular semigroup that is an idempotent pure regular extension by an inverse semigroup. Further, they show this new regular semigroup is isomorphic to the original S. Using the properties of quivers, Gomes and Szendrei then construct an $\mathbb{L}$-semigroup and so reprove O'Carroll's structure theorem.

Given an immersion of ordered groupoids $\rho : G \to T$ we described, in section

5.5, the ordered groupoid G in terms of an $\mathbb{L}$-system. If the ordered groupoid G is inductive we can convert G and the $\mathbb{L}$-system into an inverse semigroup and an $\mathbb{L}$-semigroup respectively. We will show that this $\mathbb{L}$-semigroup is isomorphic to that constructed by Gomes and Szendrei.

6.1 Quivers

All the lemmas and theorems in this section come from Gomes and Szendrei's paper [7] and all proofs can be found in this paper.

6.1.1 Quiver Types and Properties

In this section we will give Gomes' and Szendrei's definition of a quiver and a T-quiver with some properties. This is followed by a section on the derived quiver $\mathcal{D}$ of an epimorphism ρ, after which the semigroup $S(\mathcal{D})$ is defined. From this semigroup Gomes and Szendei construct their structure theorem for regular semigroups that are extensions of inverse semigroups.

A *quiver* $\mathcal{C}$ consists of a set of objects $\mathrm{Obj}\mathcal{C}$ and, for any $u, v \in \mathrm{Obj}\mathcal{C}$, a set of arrows from object u to object v denoted $\mathcal{C}(u, v)$. The set of all arrows is denoted $\mathrm{Arr}\mathcal{C}$ and has a partial binary operation $+$ such that

(Q1) If $u, v, w \in \mathrm{Obj}\mathcal{C}$, $p \in \mathcal{C}(u, v)$ and $q \in \mathcal{C}(v, w)$ then $p + q$ is defined and $p + q \in \mathcal{C}(u, w)$.

(Q2) If $p, q, r \in \mathrm{Arr}\mathcal{C}$ with $p + q$ and $q + r$ defined then $(p + q) + r$ and $p + (q + r)$ are defined and $(p + q) + r = p + (q + r)$.

We note that an addition of arrows $p + q$ may be defined even if p and q do not match up: that is, where $p \in \mathcal{C}(u, v)$ and $q \in \mathcal{C}(z, w)$ with $v \neq z$.

An inverse semigroup T acts partially on a quiver $\mathcal{C}$ if there is a partial mapping

$$T \times \mathrm{Obj}\mathcal{C} \to \mathrm{Obj}\mathcal{C}$$

$$(a, u) \mapsto a \cdot u$$

such that, for all $a, b \in T$ and $u \in \mathrm{Obj}\mathcal{C}$,

$$a \cdot (b \cdot u) = (ab) \cdot u$$

and a partial mapping

$$T \times \mathrm{Arr}\mathcal{C} \to \mathrm{Arr}\mathcal{C}$$

$$(a, p) \mapsto a \cdot p$$

such that, for all $a, b \in T$ and $p, q \in \mathrm{Arr}\mathcal{C}$,

$$a \cdot (b \cdot p) = (ab) \cdot p \quad \text{and} \quad a \cdot (p + q) = a \cdot p + a \cdot q.$$

Note that if $p \in \mathcal{C}(u, v)$ then $a \cdot p \in \mathcal{C}(au, av)$.

Let $\mathcal{C}$ be a quiver and suppose that its object set $\mathrm{Obj}\mathcal{C}$ is an inverse monoid T that acts partially on $\mathcal{C}$. Then $\mathcal{C}$ is a T-*quiver* if, for all $a, b, c, d \in T$,

(T1) If $p \in \mathcal{C}(a, b)$, $q \in \mathcal{C}(c, d)$ then $p + q$ is defined provided $bc^{-1} \in E(T)$, in which case $p + q \in \mathcal{C}(bc^{-1}a, bc^{-1}d)$.

(T2) If $p \in \mathcal{C}(a, b)$ then $c \cdot p$ is defined if and only if $c^{-1}ca = a$.

(T3) The set of arrows of $\mathcal{C}$ starting at a is a subset of $a\mathrm{Arr}\mathcal{C}$ and the set of arrows of $\mathcal{C}$ ending at b is contained in $b\mathrm{Arr}\mathcal{C}$.

Note that axiom (T1) states for any $p \in \mathcal{C}(a_1, b_1)$ and $q \in \mathcal{C}(a_2, b_2)$, $p + q \in \mathcal{C}(b_1 a_2^{-1} a_1, b_1 a_2^{-1} b_2)$ is defined if $b_1 a_2^{-1} \in E(T)$. Then, as $b_1 a_2^{-1} = (b_1 a_2^{-1})^{-1} = a_2 b_1^{-1}$, we have

$$b_1 a_2^{-1} b_1 = b_1 a_2^{-1} b_1 a_2^{-1} b_1 = b_1 a_2^{-1} a_2 b_1^{-1} b_1 = b_1 b_1^{-1} b_1 a_2^{-1} a_2 = b_1 a_2^{-1} a_2.$$

Therefore

$$b_1 a_2^{-1} b_1 = b_1 a_2^{-1} a_2. \tag{6.1.1}$$

The following lemma generalises the above note for n arrows in $\mathcal{C}$.

Lemma 6.1.1. *[7, Lemma 2.1]. Let T be an inverse monoid and $\mathcal{C}$ a T-quiver. Let $p_i \in \mathcal{C}(a_i, b_i)$, for $i = 1, 2, \ldots, n$ and $n \geqslant 2$, be such that, for any $i = 1, 2, \ldots, n-1$, the sum $p_i + p_{i+1}$ is defined. Then each sum of $p_1, \ldots, p_n$ obtained by inserting parentheses into $p_1 + \cdots + p_n$ in a meaningful way exists, and it is independent of the place of the parentheses. Moreover, $f = b_1 a_2^{-1} b_2 a_3^{-1} \ldots b_{n-1} a_n^{-1}$ is an idempotent of T and $p_1 + \cdots + p_n \in \mathcal{C}(f a_1, f b_n)$. Also, for any $i = 1, \ldots, n-1$, we have $f b_i = f a_{i+1}$.*

Let p be an arrow of a quiver. An arrow is *regular* if $p = p + p' + p$ and $p' = p' + p + p'$ for some arrow p' of the quiver, in which case p' is called an *inverse* of p. The set of inverses of p is denoted $V(p)$. A quiver in which every arrow is regular is called a *regular quiver.* If each arrow is regular and has a unique inverse element then the quiver is *inverse.* The set of idempotents of a quiver $\mathcal{C}$ is denoted by $E(\mathrm{Arr}\mathcal{C})$. For every $u \in \mathrm{Obj}\mathcal{C}$, $\mathcal{C}(u, u) \subseteq \mathrm{Arr}\mathcal{C}$ is a semigroup. A *local subsemigroup* of a quiver $\mathcal{C}$ is a subset $\mathcal{C}(u, u)$. If every local subsemigroup $\mathcal{C}(u, u)$ of a quiver is idempotent, i.e. if $\mathcal{C}(u, u) = E(\mathcal{C}(u, u))$, then the quiver is said to be *idempotent.* If every local subsemigroup of a quiver is commutative then the quiver is called *commutative.* A T-quiver $\mathcal{C}$ is *weakly connected* if $\mathcal{C}(u, uu^{-1}) \neq \emptyset$ for any $u \in T \setminus \{1\}$. (Note that we may have $\mathcal{C}(1, 1) = \emptyset$.)

Take $\mathcal{C}$ to be a T-quiver. The Green's relations on $\mathcal{C}$ are as follows.

$$(p,q) \in \mathcal{R} \Leftrightarrow p + \mathcal{C} = q + \mathcal{C}$$

$$(p,q) \in \mathcal{L} \Leftrightarrow \mathcal{C} + p = \mathcal{C} + q$$

$$(p,q) \in \mathcal{J} \Leftrightarrow \mathcal{C} + p + \mathcal{C} = \mathcal{C} + q + \mathcal{C}$$

Equivalently, $(p,q) \in \mathcal{J}$ if and only if there exists $a,b,c,d \in \mathrm{Arr}\mathcal{C}$ with $a+p$, $p+b$, $c+q$ and $q+d$ defined such that $p = c+q+d$ and $q = a+p+b$.

Lemma 6.1.2. *[7, Lemma 2.2]. Let T be an inverse monoid and $\mathcal{C}$ a T-quiver.*
(a) If $\mathcal{C}(a,b) \neq \emptyset$ then $(a,b) \in \mathcal{R}$.
(b) If $p \in \mathcal{C}(a,b)$ and $p' \in V(p)$ then $p' \in \mathcal{C}(b,a)$.
(c) If $p \in E(\mathrm{Arr}\mathcal{C})$ then $p \in \mathcal{C}(a,a)$ for some $a \in T$.

Lemma 6.1.3. *[7, Cor 2.5]. Let T be an inverse monoid and $\mathcal{C}$ a regular T-quiver. If $\mathcal{C}$ is commutative then $\mathcal{C}$ is inverse.*

Proposition 6.1.4. *[7, Prop 2.9]. Let $\mathcal{C}$ be an idempotent commutative regular T-quiver. Let $p,q \in \mathcal{C}(a,b)$ for some $a,b \in \mathrm{Obj}\mathcal{C}$. Then*

$$(p,q) \in \mathcal{J} \Leftrightarrow p = q.$$

6.1.2 The Derived Quiver

Let S be a regular semigroup and T an inverse semigroup with epimorphism $\rho : S \twoheadrightarrow T$. The *derived quiver* $\mathcal{D}$ *of* ρ has $\mathrm{Obj}\mathcal{D} = T$ and arrows are pairs $(u,m) \in T \times S$ from u to $u(m\rho)$ such that $\mathbf{r}(u) = \mathbf{d}(m\rho)$. The partial operation is defined as follows: $(u,m) + (v,n)$ is defined if $u(m\rho)v^{-1} \in E(T)$ in which case

$$(u,m) + (v,n) = (u(m\rho)v^{-1}u, mn).$$

We note that $(m\rho)(m\rho)^{-1} = (mm')\rho$ where $m' \in V(m)$.

Gomes and Szendrei define a partial action of a monoid T^1 on the derived quiver $\mathcal{D}$. T acts on Obj$\mathcal{D}$ by multiplication in T and T acts on the arrows of $\mathcal{D}$ as follows. Let $(u, m) \in \text{Arr}\mathcal{D}$ and $t \in T$ then $t \cdot (u, m)$ exists if and only if $t^{-1}tu = u$, if so $t \cdot (u, m) = (tu, m)$.

Lemma 6.1.5. *[7, Lemmas 4.1 and 4.2]. $\mathcal{D}$ is a regular T^1-quiver.*

Lemma 6.1.6. *[7, Cor 4.5]. Let S be a regular semigroup which is an idempotent pure extension by an inverse semigroup T. Then $\mathcal{D}$ is an idempotent, regular T-quiver.*

If S is also inverse then $\mathcal{D}$ is also commutative.

Gomes and Szendrei define, for an inverse monoid T and derived quiver $\mathcal{D}$,

$$S(\mathcal{D}) = \{((u, m), a) : a \in T, (u, m) \in \mathcal{D}(aa^{-1}, a)\}$$

with multiplication

$$((u, m), a)((v, n), b) = (a \cdot (a^{-1} \cdot (u, m) + (v, n)), ab).$$

Lemma 6.1.7. *[7, Lemmas 3.1 and 3.3]. Let T be an inverse monoid and $\mathcal{D}$ be the derived T-quiver of $\rho : S \twoheadrightarrow T$, where S is a regular semigroup. Then $S(\mathcal{D})$ is a regular semigroup.*

If in addition S is an inverse semigroup then $S(\mathcal{D})$ is also an inverse semigroup and

$$E(S(\mathcal{D})) = \{((u, m), e) : e \in E(T), (u, m) \in E(\mathcal{D}(e, e))\}.$$

Proposition 6.1.8. *[7, Prop 3.4] Let S be a regular (inverse) semigroup and let T be an inverse monoid and let $\mathcal{D}$ be the derived regular (inverse) T-quiver of epimorphism $\rho : S \twoheadrightarrow T$. Then $S(\mathcal{D})$ is a regular (inverse) semigroup which is an exension by an inverse semigroup.*

Proof, [7]. We define the congruence δ on $S(\mathcal{D})$ as follows:

$$((u,m),a)\delta((v,n),b) \Leftrightarrow a = b.$$

Denote the congruence class of $((u,m),a)$ by $((u,m),a)\delta$. Take $\delta^\natural : S(\mathcal{D}) \to S(\mathcal{D})/\delta$ to be $((u,m),a) \mapsto ((u,m),a)\delta$. Then $\delta^\natural$ is a surjection. $S(\mathcal{D})/\delta$ can be embedded into T via $\theta : ((u,m),a)\delta \mapsto a$ and so $(S(\mathcal{D})/\delta)\theta$ is a subsemigroup of T. Therefore $S(\mathcal{D})$ is an extension by an inverse subsemigroup $(S(\mathcal{D})/\delta)\theta$ of T. $\square$

Proposition 6.1.9. *[7, Prop 4.6]. Let S be an regular semigroup. If S is an extension by an inverse semigroup then S is isomorphic to $S(\mathcal{D})$.*

Proof, [7]. Let S be the regular extension by inverse semigroup T and let $\rho : S \twoheadrightarrow T$ be the epimorphism. Let $\mathcal{D}$ be the derived quiver of ρ. Then by lemma 6.1.7, $S(\mathcal{D})$ is a regular (inverse) semigroup. Then the isomorphism $\theta : S \to S(\mathcal{D})$ is defined as $m \mapsto ((m\rho m\rho^{-1}, m), m\rho)$. $\square$

We can now give Gomes' and Szendrei's structure theorem.

Theorem 6.1.10. *[7, Theorem 4.7] A regular semigroup S is an (idempotent pure) extension by an inverse semigroup if and only if S is isomorphic to $S(\mathcal{D})$ where $\mathcal{D}$ is an (idempotent) regular T-quiver and T is an inverse monoid.*

Proof, [7]. The proof follows directly from propositions 6.1.8 and 6.1.9 $\square$

Lemma 6.1.11. *[7]. Let inverse semigroup S be an extension by an inverse semigroup T. Let $\rho : S \twoheadrightarrow T$ be the epimorphism and $\mathcal{D}$ be the derived quiver ρ. Then the T-quiver $\mathcal{D}$ is weakly connected.*

6.1.3 T-quiver Properties

In order to show that their structure theorem 6.1.10 generalises O'Carroll's for inverse semigroups, Gomes and Szendrei show that $S(\mathcal{D})$ is isomorphic to an $\mathbb{L}$-semigroup. The triple that is used for the $\mathbb{L}$-semigroup construction is derived from certain T-quiver properties. We discuss these properties in this section and then, in the next section, we construct the required triple and establish the connection with O'Carroll's structure theorem for idempotent pure extensions of inverse semigroups.

Let $\mathcal{C}$ be a T-quiver. Let I be a subset of Arr$\mathcal{C}$. If, for all $p \in I$ and $q \in \text{Arr}\mathcal{C}$ with $p+q$ defined, $p+q \in I$ then I is a *right ideal*. Similarly we can define a *left ideal*. If I is both a left and right ideal we call I an *ideal*. The intersection of all ideals of $\mathcal{C}$ containing arrow p is called the *principal ideal*. Gomes and Szendei denote the principal ideal of $p \in \text{Arr}\mathcal{C}$ by $I(p)$.

Lemma 6.1.12. *[7, Lemma 5.3]. For $p \in \text{Arr}\mathcal{C}$,*

$$I(p) = \{r + p + s : r, s \in \text{Arr}\mathcal{C}, r + p, p + s \,\text{are defined}\,\}.$$

Proof. Denote by M the set

$$\{r + p + s : r, s \in \text{Arr}\mathcal{C}, r + p, p + s \,\text{are defined}\,\}.$$

Let $p \in M$. Then for some $r, s \in \text{Arr}\mathcal{C}$, $r + p$ and $p + s$ are defined. Then by lemma 6.1.1, $r + p + s$ is defined. So $r + p + s \in I(p)$. Thus $M \subseteq I(p)$.

We next show that M is a principal ideal of $p \in \text{Arr}\mathcal{C}$. We first show that M contains p then prove M is an ideal. Now, $p = p + p^{-1} + p$ for any $p^{-1} \in V(p)$. Thus $p \in M$. Next let $r + p + s \in M$ with $r + p$ and $p + s$ defined. Let $z \in \text{Arr}\mathcal{C}$ be such that $(r + p + s) + z$ is defined. We require $(r + p + s) + z$ belong to M. Let $q \in V(r + p)$. By lemma 6.1.1 we have

$(r+p+s)+z = (r+p+q+r+p+s)+z = r+p+(q+(r+p+s)+z) \in M$. Thus M is a right ideal. Similarly M is a left ideal. Therefore $M = I(p)$. $\square$

Gomes and Szendrei define an action of T on any ideal I of Arr$\mathcal{C}$ as follows:

$$t \cdot I = \{t \cdot p : p \in I, t \cdot p \,\text{is defined}\,\}.$$

Lemma 6.1.13. *[7, Lemma 5.5]. Let T be an inverse semigroup and $\mathcal{C}$ a regular T-quiver. For $t \in T$, $p \in$ Arr$\mathcal{C}$ where $t \cdot p$ is defined then $t \cdot I(p) = I(t \cdot p)$.*

The Green's relation $\mathcal{J}$ on $\mathcal{C}$ can then be defined as follows:

$$(p, q) \in \mathcal{J} \text{ if and only if } I(p) = I(q).$$

6.1.4 $\mathbb{L}$-semigroups

O'Carroll, [21], describes inverse semigroups that are idempotent pure extensions by inverse semigroups as $\mathbb{L}_M$-semigroups. This description is a generalisation of McAlister's P-theorem. We have already given the analogous $\mathbb{L}$-system groupoid description of the $\mathbb{L}_M$-semigroup in section 5.5. We now give O'Carroll's original description of an $\mathbb{L}$-semigroup.

O'Carroll's $\mathbb{L}$-triple consists of a poset X and an order ideal Y of X that is also a subsemilattice of X. The final component of the triple is an inverse semigroup T that acts partially on X. As with the McAlister triple, (X, Y, T) must satisfy the two properties $X = T \cdot Y$ and $t \cdot X \neq \emptyset$ for every $t \in T$.

If in addition T satisfies the extra condition that for all $a \in Y$ there exists an $e_a \in E(T)$ such that $e_a \cdot a$ is defined and, for all $k \in E(T)$ such that $k \cdot a$ is defined, we have $e_a \leqslant k$ and for $a, b \in Y$, $e_{a \wedge b} = e_a \wedge e_b$ then the $\mathbb{L}$-triple (X, Y, T) is said to be *strict*.

Given an $\mathbb{L}$-triple (X, Y, T) the $\mathbb{L}$-semigroup is defined as

$$\mathbb{L}(X, Y, T) = \{(y, t) \in Y \times T : t^{-1} \cdot y \in Y\}$$

with composition

$$(y, t)(x, u) = (t \cdot (t^{-1} \cdot y \wedge x), tu).$$

This differs from the $\mathbb{L}$-system definition given in section 5.5 because, unlike in a groupoid, in an inverse semigroup composition and actions are defined everywhere.

Given a *strict* $\mathbb{L}$-triple (X, Y, T) the $\mathbb{L}_M$-semigroup is defined as

$$\begin{aligned} \mathbb{L}_M(X, Y, T) &= \{(y, t) \in \mathbb{L}(X, Y, T) : tt^{-1} = e_y\} \\ &= \{(y, t) \in Y \times T : t^{-1} \cdot y \in Y, tt^{-1} = e_y\} \end{aligned}$$

Gomes and Szendrei show that for any inverse semigroup S that is an extension by an inverse semigroup T via the surjection $\rho : S \twoheadrightarrow T$, the inverse semigroup $S(\mathcal{D})$ of the derived quiver (which is isomorphic to S) is isomorphic to an $\mathbb{L}_M$-semigroup. We construct the required $\mathbb{L}$-triple as follows.

Let X_J be the set of all $\mathcal{J}$-classes in Arr$\mathcal{D}$. Denote by J_p the $\mathcal{J}$-class of $p \in$ Arr$\mathcal{D}$. The partial order is $J_p \leqslant J_q$ if and only if $I(p) \subseteq I(q)$. Then X_J is a poset.

Let

$$Y_J = \{J_p \in X_J : p \in \mathcal{D}(\bar{e}, \bar{e}) \text{ for some } \bar{e} \in E(T)\}.$$

If $p \in Y_J$ then $p = (u, m) \in \mathcal{D}(\bar{e}, \bar{e})$ such that $u^{-1}u = m\rho m\rho^{-1}$ with $u = \bar{e}$ and $u(m\rho) = \bar{e}$. Now ρ is sujective and idempotent pure so for $\bar{e} \in E(T)$

there exists an $e \in E(S)$ such that $e\rho = \bar{e}$. Then $u = e\rho$ and $e\rho m\rho = e\rho$ but $e\rho^{-1}e\rho = e\rho = m\rho m\rho^{-1}$ so $e\rho m\rho = m\rho m\rho^{-1}m\rho = m\rho = e\rho$. Whence Y_J becomes

$$Y_J = \{J_{(e\rho,e)} \in X_J : e \in E(S)\}.$$

Lemma 6.1.14. *[7, Lemma 6.1]. Y_J is an order ideal and a subsemilattice of X_J. In particular, if $p \in \mathcal{D}(e,e)$ and $q \in \mathcal{D}(f,f)$ for $e, f \in E(T)$ then $J_p \wedge J_q$ exists and equals $J_{p+q} \in Y$.*

Gomes and Szendrei define a partial action of T on the poset X_J as follows

$$t \cdot J_p = J_{t \cdot p} \text{ if and only if } t \cdot p \text{ is defined.}$$

Lemma 6.1.15. *[7, Lemma 6.3]. (X_J, Y_J, T) is a strict $\mathbb{L}$-triple.*

Lemma 6.1.16. *[7, Lemma 6.4]. Y_J is isomorphic to $E(S(\mathcal{D}))$ as a semilattice.*

Proof. The isomorphism is given by $\theta : Y_J \to E(S(\mathcal{D}))$, $\theta : J_{(e\rho,e)} \mapsto ((e\rho, e), e\rho)$. $\square$

Gomes and Szendrei remark that if $J \in Y_J$ then there exists a unique $e \in E(T)$ and a unique $p \in \mathcal{D}(e,e)$ such that $J = J_p$. Further, e is the least idempotent in $E(T)$ such that $e \cdot J$ is defined. For if $J_p \in Y$ with $p \in \mathcal{D}(e,e)$ for some $e \in E(T)$ then $e \cdot p$ exists and $e \cdot J_p$ exists. If $f \cdot J_p$ then exists for some $f \in E(T)$ then $f \cdot p$ exists so $f^{-1}fe = fe = e$ and $e \leqslant f$. Also we have seen that for $p \in \mathcal{D}(e,e)$, $p = (k\rho, k)$ for $k \in E(S)$ such that $k\rho = e$.

We now give the second main theorem of Gomes's and Szendrei's paper [7]. This theorem reproves O'Carroll's [21].

Theorem 6.1.17. *[7, Theorem 6.5]. Let S and T be inverse semigroups. Let $\rho : S \twoheadrightarrow T$ be an epimorphism and $\mathcal{D}$ be the weakly connected, idempotent,* *commutative, regular derived T^1-quiver of ρ. Then $S(\mathcal{D})$ is isomorphic to $\mathbb{L}(X_J, Y_J, T)$.*

Proof. The isomorphism is given by $\phi : S(\mathcal{D}) \to \mathbb{L}_Q$, $(p, t) \mapsto (J_p, t)$. $\square$

6.2 Comparing Quivers and $\mathbb{L}$-systems

Gomes and Szendrei showed that for inverse semigroups S and T and epimorphism $\rho : S \to T$ then S is isomorphic to $S(\mathcal{D})$ where $\mathcal{D}$ is the derived quiver of ρ. Further $S(\mathcal{D})$, and so S, can be described as an $\mathbb{L}$-semigroup. For an ordered groupoid G and an immersion from G to an ordered groupoid T we have already shown that we can describe $G\iota$, and so G, as an $\mathbb{L}$-system. Taking G to be an inductive groupoid we can convert G into an inverse semigroup. Likewise we can convert the poset $T \otimes E(G)$, action groupoid $T \ltimes (T \otimes E(G))$ and the $\mathbb{L}$-system into semigroup terms giving us an $\mathbb{L}$-semigroup description for G. A natural question is whether or not this $\mathbb{L}$-semigroup is isomorphic to Gomes' and Szendrei's $\mathbb{L}$-semigroup. We will give an isomorphism that answers this question. Before we do this we give some alternative notation for Gomes' and Szendrei's $\mathbb{L}$-semigroup and remind ourselves of the $\mathbb{L}$-system description along with a conversion from groupoid to semigroup terms.

6.2.1 Quiver Notation

In Gomes' and Szendrei's quiver construction the poset X_J comprises the set of all $\mathcal{J}$-classes of Arr$\mathcal{D}$. As

$$J_{(u,m)} = J_{(v,n)} \text{ if and only if } I(u, m) = I(v, n)$$

it is clear that X_J is isomorphic as a poset to the set of all principal ideals with partial order

$$I(u,m) \leqslant I(v,n) \text{ if and only if } I(u,m) \subseteq I(v,n).$$

We also denote this poset by X_J. We will use this alternative description of X_J in order to show the $\mathbb{L}$-semigroup description of inverse semigroup S is isomorphic to the $\mathbb{L}$-system description.

Given an inverse semigroup S and another inverse semigroup T with epimorphism $\rho : S \twoheadrightarrow T$ and derived quiver $\mathcal{D}$, for $u \in T$ and $m \in S$ we define

$$\begin{aligned} I(u,m) = \{(t,r) + (u,m) + (v,s) :& r,s \in S, t,v \in T, \\ & (t,r),(u,m),(v,s) \in \mathcal{D} \\ & (t,r) + (u,m) \,\text{is defined}, \\ & (u,m) + (v,s) \,\text{is defined}\} \end{aligned}$$

Alternatively,

$$\begin{aligned} I(u,m) = \{(t,r) + (u,m) + (v,s) :& r,s \in S, t,v \in T, \mathbf{r}(u) = \mathbf{d}(m\rho), \\ & \mathbf{r}(t) = \mathbf{d}(r\rho), \mathbf{r}(v) = \mathbf{d}(s\rho), \\ & t(r\rho)u^{-1}, u(m\rho)v^{-1} \in E(T)\} \end{aligned}$$

Now

$$\begin{aligned} (t,r) + (u,m) + (v,s) &= (t(r\rho)u^{-1}t, rm) + (v,s) \\ &= (t(r\rho)u^{-1}t(rm)\rho v^{-1}t(r\rho)u^{-1}t, rms). \end{aligned}$$

This is defined as $t(r\rho)u^{-1}, u(m\rho)v^{-1} \in E(T)$ and

$$\begin{aligned} t(r\rho)u^{-1}t(rm)\rho v^{-1} &= t(r\rho)u^{-1}t(r\rho)(m\rho)v^{-1} \\ &= t(r\rho)u^{-1}t(r\rho)(m\rho)(m\rho)^{-1}(m\rho)v^{-1} \\ &= t(r\rho)u^{-1}t(r\rho)u^{-1}u(m\rho)v^{-1} \\ &= (t(r\rho)u^{-1})(t(r\rho)u^{-1})u(m\rho)v^{-1} \\ &= t(r\rho)u^{-1}u(m\rho)v^{-1} \\ &\in E(T). \end{aligned}$$

Then

$$\begin{aligned} &(t(r\rho)u^{-1}t(rm)\rho v^{-1})t(r\rho)u^{-1}t \\ &\qquad = (t(r\rho)u^{-1})(u(m\rho)v^{-1})(t(r\rho)u^{-1})t \\ &\qquad = (t(r\rho)u^{-1})(u(m\rho)v^{-1})t \\ &\qquad = t(r\rho)(u^{-1}u)(m\rho)v^{-1}t \\ &\qquad = t(r\rho)(m\rho)(m\rho)^{-1}(m\rho)v^{-1}t \\ &\qquad = t(r\rho)(m\rho)v^{-1}t \end{aligned}$$

Hence

$$(t,r) + (u,m) + (v,s) = (t(r\rho)(m\rho)v^{-1}t, rms)$$

and

$$\begin{aligned} I(u,m) = \{(t(r\rho)(m\rho)v^{-1}t, rms) :& r, s \in S, t, v \in T, \mathbf{r}(u) = \mathbf{d}(m\rho), \\ & \mathbf{r}(t) = \mathbf{d}(r\rho), \mathbf{r}(v) = \mathbf{d}(s\rho), \\ & t(r\rho)u^{-1}, u(m\rho)v^{-1} \in E(T)\}. \end{aligned}$$

Lemma 6.2.1. *$I(u,m) = I(u,mm^{-1})$.*

Proof. We prove first the inclusion $I(u,m) \subseteq I(u,mm^{-1})$. Let $(t(rm)\rho v^{-1}t, rms) \in I(u,m)$. Let $u' = u$, $m' = mm^{-1}$, $s' = ms$ and $r' = r$. Then $r'm's' = rmm^{-1}ms = rms$. Now consider the first component.

$$\mathbf{d}(s'\rho) = \mathbf{d}((ms)\rho) = (ms)\rho(ms)\rho^{-1} = m\rho s\rho s\rho^{-1}m\rho^{-1} = m\rho v^{-1}vm\rho^{-1}.$$

Since $\mathbf{r}(v') = \mathbf{d}(s'\rho)$, we set $v' = vm\rho^{-1}$. Also, set $t' = t$. Then

$$\begin{aligned} t'(r'm')\rho(v')^{-1}t' &= t(rmm^{-1})\rho(v(m\rho)^{-1})^{-1}t \\ &= t(r\rho)(m\rho)(m^{-1}\rho)(m\rho)v^{-1}t \\ &= t(r\rho)(m\rho)v^{-1}t \\ &= t(rm)\rho v^{-1}t. \end{aligned}$$

We check $(t'(r'm')\rho(v')^{-1}t', r'm's') \in I(u',m')$. So, $t'(r'\rho)(u')^{-1} = t(r\rho)u^{-1} \in E(T)$. Also,

$$\begin{aligned} u'(m'\rho)(v')^{-1} &= u(mm^{-1})\rho(v(m\rho)^{-1})^{-1} \\ &= u(m\rho)(m^{-1}\rho)(m\rho)v^{-1} \\ &= u(m\rho)v^{-1} \\ &\in E(T). \end{aligned}$$

Hence $(t(rm)\rho v^{-1}t, rms) = (t'(r'm')\rho(v')^{-1}t', r'm's') \in I(u',m') = I(u,mm^{-1})$. Thus $I(u,m) \subseteq I(u,mm^{-1})$.

Now the converse inclusion, let $(t(rmm^{-1})\rho v^{-1}t, rmm^{-1}s) \in I(u,mm^{-1})$. Let $\tilde{u} = u$, $\tilde{m} = m$, $\tilde{s} = m^{-1}s$, $\tilde{r} = r$, $\tilde{t} = t$ and $\tilde{v} = v(m\rho)$. Then the second component, $\tilde{r}\tilde{m}\tilde{s} = rmm^{-1}s$. The first component,

$$\begin{aligned} t(rmm^{-1})\rho v^{-1}t &= t(r\rho)(m\rho)(m\rho)^{-1}v^{-1}t \\ &= t(r\rho)(m\rho)(v(m\rho))^{-1}t \\ &= \tilde{t}(\tilde{r}\rho)(\tilde{m}\rho)\tilde{v}^{-1}\tilde{t}. \end{aligned}$$

We check $(\tilde{t}(\tilde{r}\tilde{m})\rho\tilde{v}^{-1}\tilde{t}, \tilde{r}\tilde{m}\tilde{s}) \in I(\tilde{u}, \tilde{m})$. Now,

$$\begin{aligned} \mathbf{r}(\tilde{v}) &= \tilde{v}^{-1}\tilde{v} \\ &= (v(m\rho))^{-1}v(m\rho) \\ &= (m\rho)^{-1}v^{-1}v(m\rho) \\ &= (m\rho)^{-1}(s\rho)(s\rho)^{-1}(m\rho) \\ &= \mathbf{d}(\tilde{s}\rho) \end{aligned}$$

and

$$\tilde{t}(\tilde{r}\rho)\tilde{u}^{-1} = t(r\rho)u^{-1} \in E(T)$$

further

$$\tilde{u}(\tilde{m}\rho)\tilde{v}^{-1} = u(m\rho)(m\rho)^{-1}v^{-1} \in E(T).$$

Hence $(t(rmm^{-1})\rho v^{-1}t, rmm^{-1}s) = (\tilde{t}(\tilde{r}\tilde{m})\rho\tilde{v}^{-1}\tilde{t}, \tilde{r}\tilde{m}\tilde{s}) \in I(\tilde{u}, \tilde{m}) = I(u, m)$. Thus $I(u, mm^{-1}) \subseteq I(u, m)$.

Therefore $I(u, m) = I(u, mm^{-1})$. $\square$

As a result of the previous lemma we can assume that $m \in E(S)$. So

$$\begin{aligned} I(u, m) = \{(t(rm)\rho v^{-1}t, rms) :& r, s \in S, t, v \in T, \mathbf{r}(u) = \mathbf{d}(m\rho) = (m\rho), \\ & \mathbf{r}(t) = \mathbf{d}(r\rho), \mathbf{r}(v) = \mathbf{d}(s\rho), \\ & t(r\rho)u^{-1} \in E(T), u(m\rho)v^{-1} = uv^{-1} \in E(T)\} \end{aligned}$$

As $m \in E(S)$ we now have that

$$\begin{aligned}
t(rm)\rho v^{-1}t &= t(r\rho)(m\rho)v^{-1}t \\
&= t(r\rho)(u^{-1}u)v^{-1}t \\
&= (t(r\rho)u^{-1})(uv^{-1})t \\
&= (uv^{-1})(t(r\rho)u^{-1})t \\
&= (vu^{-1})(u(r\rho)^{-1}t^{-1})t \\
&= v(u^{-1}u)(r\rho)^{-1}(t^{-1}t) \\
&= v(m\rho)(r\rho)^{-1}(r\rho)(r\rho)^{-1} \\
&= v(m\rho)(r\rho)^{-1}
\end{aligned}$$

Further, as u and t are no longer in the first component, for $(u, m) \in \mathcal{D}$ and $m \in E(S)$,

$$\begin{aligned}
I(u, m) = \{(v(mr^{-1})\rho, rms) :& r, s \in S, v \in T, \\
& \mathbf{r}(v) = \mathbf{d}(s\rho), uv^{-1} \in E(T), \\
& \text{and } t(r\rho)u^{-1} \in E(T) \\
& \text{for some } t \in T \text{ with } \mathbf{r}(t) = \mathbf{d}(r\rho)\}.
\end{aligned}$$

Take, for $(u, m) \in \mathcal{D}$ and $m \in E(S)$.

$$\begin{aligned}
J(u, m) = \{(v(ma^{-1})\rho, ams) :& a \in Sm, s \in S, v \in T, \\
& \mathbf{r}(v) = \mathbf{d}(s\rho), uv^{-1} \in E(T), \\
& \text{and } t(a\rho)u^{-1} \in E(T) \\
& \text{for some } t \in T \text{ with } \mathbf{r}(t) = \mathbf{d}(a\rho)\}.
\end{aligned}$$

$J(u,m)$ differs from $I(u,m)$ as the element a belongs to the left ideal Sm whereas element $r \in S$. By definition $J(u,m) \subseteq I(u,m)$. We will show that $I(u,m) \subseteq J(u,m)$ but we show first that the t element always exist in $J(u,m)$.

Notice first that if $a \in Sm$ then $a = \bar{s}m$ for some $\bar{s} \in S$ and, because $m \in E(S)$,

$$am = \bar{s}mm = \bar{s}m = a.$$

Therefore,

$$\begin{aligned} J(u,m) = \{(v(ma^{-1})\rho, ams) : & a \in Sm, s \in S, v^{-1}T \\ & \mathbf{r}(v) = \mathbf{d}(s\rho), \\ & uv^{-1} \in E(T)\} \end{aligned}$$

Consider now any element $(v(mr^{-1})\rho, rms)$ of $I(u,m)$. Here $r \in S$ is arbitrary but $m \in E(S)$. So

$$\begin{aligned} (v(mr^{-1})\rho, rms) &= (v(mmr^{-1})\rho, rmms) \\ &= (v(mm^{-1}r^{-1})\rho, rmms) \\ &= (v(m\rho)(rm)^{-1}\rho, (rm)ms). \end{aligned}$$

Thus $a = rm \in Sm$ and $(v(mr^{-1})\rho, rms) \in J(u,m)$. So $I(u,m) \subseteq J(u,m)$.

Therefore $I(u,m) = J(u,m)$ giving us the definition

$$\begin{aligned} I(u,m) = \{(v(mr^{-1})\rho, rms) : & r, s \in S, v \in T, \\ & \mathbf{r}(v) = \mathbf{d}(s\rho), \\ & uv^{-1} \in E(T)\}. \end{aligned}$$

Recall that elements a and b in an inverse semigroup are right compatible if $ab^{-1} \in E(T)$. In which case, by [10, lemma 12, page 25],

$$ab^{-1}b = bab^{-1} = ba^{-1}a = ab^{-1}a.$$

In $I(u,m)$ elements u and v of T are right compatible and so

$$\begin{aligned}
v(mr^{-1})\rho &= v(m\rho)(r\rho)^{-1} \\
&= v(u^{-1}u)(r\rho)^{-1} \\
&= (vu^{-1}u)(r\rho)^{-1} \\
&= (uv^{-1}v)(r\rho)^{-1} \\
&= u(v^{-1}v)(r\rho)^{-1} \\
&= u(s\rho)(s\rho)^{-1}(r\rho)^{-1} \\
&= u(ss^{-1}r^{-1})\rho.
\end{aligned}$$

Now $u^{-1}u = m\rho$ so we have $uu^{-1}u = u = u(m\rho)$ giving us

$$\begin{aligned}
u(ss^{-1}r^{-1})\rho &= u(m\rho)(ss^{-1}r^{-1})\rho \\
&= u(mss^{-1}r^{-1})\rho \\
&= u(ss^{-1}mr^{-1})\rho.
\end{aligned}$$

Thus

$$\begin{aligned}
I(u,m) = \{(u(ss^{-1}mr^{-1})\rho, rms) &: r, s \in S, \\
& uv^{-1} \in E(S) \,\text{for some}\, v \in T \,\text{with}\, \mathbf{r}(v) = \mathbf{d}(s\rho)\}.
\end{aligned}$$

We have eliminated v from the first component as we did with t and, as before, we can show the element v always exists. Take, for $(u,m) \in \mathcal{D}$ and

$m \in E(S)$,

$$\bar{J}(u,m) = \{(u(zz^{-1}m^{-1})\rho, rmz) : z \in mS, r \in S, v \in T,$$
$$uv^{-1} \in E(S) \,\text{for some}\, v \in T \,\text{with}\, \mathbf{r}(v) = \mathbf{d}(z\rho)\}.$$

By definition, $\bar{J}(u,m) \subseteq I(u,m)$. Note that $z \in mS$ so $mz = z$. Take $v = u(zz^{-1})\rho$. Then $uv^{-1} = u(u(zz^{-1})\rho)^{-1} = u(zz^{-1})\rho^{-1}u^{-1} = u(zz^{-1})\rho u^{-1}$ so $uv^{-1} \in E(T)$ because if $e \in E(S)$ then $xex^{-1} \in E(T)$ as $xex^{-1}xex^{-1} = xx^{-1}xe^2x^{-1} = xex^{-1}$. Also,

$$\begin{aligned}
\mathbf{r}(v) &= v^{-1}v \\
&= (zz^{-1})^{-1}\rho u^{-1}u(zz^{-1})\rho \\
&= (zz^{-1})\rho(m\rho)(zz^{-1})\rho \\
&= (zz^{-1}mzz^{-1})\rho \\
&= (mzz^{-1})\rho \\
&= ((mz)z^{-1})\rho \\
&= (zz^{-1})\rho \\
&= \mathbf{d}(z\rho)
\end{aligned}$$

Therefore,

$$\bar{J}(u,m) = \{u(zz^{-1}mr^{-1})\rho, rmz) : z \in mS, r \in S\}.$$

Consider now any element $(u(ss^{-1}mr^{-1})\rho, rms)$ of $I(u,m)$. Here $s \in S$ is arbitrary but $m \in E(S)$. So

$$\begin{aligned}
(u(ss^{-1}mr^{-1})\rho, rms) &= (u(ss^{-1}mmr^{-1})\rho, rmms) \\
&= (u(ss^{-1}mmmr^{-1})\rho, rmms) \\
&= (u(mss^{-1}mmr^{-1})\rho, rm(ms)) \\
&= (u(mss^{-1}m^{-1}mr^{-1})\rho, rm(ms)) \\
&= (u((ms)(ms)^{-1}mr^{-1})\rho, rm(ms)).
\end{aligned}$$

Thus $z = ms$ and $(u(ss^{-1}mr^{-1})\rho, rms) \in \bar{J}(u,m)$. So $I(u,m) \subseteq \bar{J}(u,m)$.

Therefore $I(u,m) = \bar{J}(u,m)$ giving us the definition

$$I(u,m) = \{(u(ss^{-1}mr^{-1})\rho, rms) : r, s \in S\}.$$

Lemma 6.2.2. *If $(w,k) \in I(u,m)$ then $ww^{-1} \leqslant uu^{-1}$.*

Proof. $(w,k) \in I(u,m)$ so $w = u(ss^{-1}mr^{-1})\rho$ for some $s, r \in S$. Recall that $m \in E(S)$ and, because $(u,m) \in \mathcal{D}$, $\mathbf{r}(u) = m\rho$. Then, as idempotents in an inverse semigroup commute,

$$\begin{aligned}
w &= u(ss^{-1}mr^{-1})\rho \\
&= u(ss^{-1})\rho(m\rho)(r^{-1})\rho \\
&= u(m\rho)(ss^{-1})\rho(r^{-1})\rho \\
&= u(ss^{-1}r^{-1})\rho.
\end{aligned}$$

Now $w = ux$ where $x = (ss^{-1}r^{-1})\rho$ so

$$ww^{-1} = uxx^{-1}u^{-1} = uu^{-1}uxx^{-1}u^{-1} = uu^{-1}(uxx^{-1}u^{-1})$$

and $uxx^{-1}u^{-1} \in E(T)$. Thus $ww^{-1} \leqslant uu^{-1}$. $\square$

Suppose $I(w, f) = I(u, m)$. Since $(w, f) \in I(u, m)$, by lemma 6.2.2, $ww^{-1} \leqslant uu^{-1}$. Similarly $uu^{-1} \leqslant ww^{-1}$. So $uu^{-1} = ww^{-1}$. If $w = u(ss^{-1}r^{-1})\rho$ then $uu^{-1} = ww^{-1}$ implies

$$\begin{aligned} uu^{-1}u &= ww^{-1}u \\ &= u((ss^{-1}r^{-1})\rho)((ss^{-1}r^{-1})\rho)^{-1}u^{-1}u \\ &= u(ss^{-1}r^{-1}rss^{-1})\rho u^{-1}u \\ &= u(ss^{-1}r^{-1}r)\rho(u^{-1}u) \\ &= u(u^{-1}u)(ss^{-1}r^{-1}r)\rho \\ &= u(ss^{-1}r^{-1}r)\rho \end{aligned}$$

and so

$$u = u(ss^{-1}r^{-1}r)\rho. \tag{6.2.1}$$

Thus

$$w = u(ss^{-1}r^{-1})\rho = u(ss^{-1}r^{-1}rr^{-1})\rho = u(ss^{-1}r^{-1}r)\rho(r^{-1})\rho = u(r^{-1}\rho). \tag{6.2.2}$$

Now Y_J becomes

$$\begin{aligned} Y_J &= \{I(u, m) \in X_J : (u, m) \in \mathcal{D}(e, e) \,\text{for some}\, e \in E(T)\} \\ &= \{I(u, m) \in X_J : (u, m) \in \mathcal{D}(u, u), u \in E(T)\} \end{aligned}$$

Gomes and Szendrei remark that if $I(u, m) \in Y_J$ then $u \in E(T)$ is unique. We show this is true. Suppose then that $u \in E(T)$. Then $uu^{-1} = m\rho$ becomes $u^2 = u = m\rho$ and $u(ss^{-1}mr^{-1})\rho = m\rho(ss^{-1}mr^{-1})\rho = (mss^{-1}mr^{-1})\rho =$

$(ss^{-1}mr^{-1})\rho$. In this case,

$$I(u,m) = I(m\rho, m) = \{((ss^{-1}mr^{-1})\rho, rms) : r, s \in S\}.$$

Now suppose that $I(m\rho, m) = I(f\rho, f)$ for some other $f \in E(T)$. Then, since $(f\rho, f) \in I(f\rho, f)$ we have $f = rms$ for some $r, s \in S$ and by (6.2.2), $f\rho = m\rho(r^{-1})\rho$. Since S is an inverse semigroup that is an idempotent pure extension by inverse semigroup T, ρ is idempotent pure. Thus $f\rho = (m\rho)(r^{-1}\rho) = (mr^{-1})\rho$ implies $mr^{-1} \in E(S)$, and so $mr^{-1} = (mr^{-1})^{-1} = rm^{-1} = rm$. Hence $f = rms = mr^{-1}s$ implies $mf = mmr^{-1}s = mr^{-1}s = f$ so $f \leqslant m$. By symmetry $m \leqslant f$ and so $f = m$. Therefore, as Gomes and Szendrei remark, if $I(m\rho, m) = I(f\rho, f)$ then $f = m$.

The order ideal Y_J of X_J then has form

$$Y_J = \{I(m\rho, m) \in X_J : m \in E(S)\}.$$

By lemma 6.1.13, the action of T on X_J is given by

$$t \cdot I(u,m) = I(tu, m)$$

provided $t^{-1}tu = u$. Then

$$\mathbb{L}_Q = \mathbb{L}_M(X_J, Y_J, T) = \{(I(u,m), t) \in Y_J \times T : t^{-1} \cdot I(u,m) \in Y_J\}$$

with composition

$$(I(u,m), t)(I(v,n), k) = (t \cdot (t^{-1} \cdot I(u,m) \wedge I(v,n)), tk).$$

Alternatively, as Gomes and Szendrei describe in [7],

$$\begin{aligned}\mathbb{L}_Q &= \mathbb{L}_M(X_J, Y_J, T) \\ &= \{(I(u,m), t) \in Y_J \times T : t^{-1} \cdot I(u,m) \in Y_J, (u,m) \in \mathcal{D}(tt^{-1}, tt^{-1})\}.\end{aligned}$$

6.2.2 $\mathbb{L}$-system Notation

We recall the $\mathbb{L}$-system construction of section 5.5. Given an immersion of ordered groupoids $\rho : G \to T$ with $g \in G(e,f)$ and $t \in T$ with $\mathbf{r}(t) = e\rho$ then G acts on the left of $E(G)$: $g \cdot f = e$. Also G acts on the right of T: $t \cdot g = t(g\rho)$. This gives us the pullback

$$T \between E(G) = \{(s,e) \in T \times E(G) : e\rho = \mathbf{r}(s)\}.$$

Equivalence relation $(s,e) \simeq_G (t,f)$ if there exists $g \in G(e,f)$ such that $t = s(g\rho)$ gives us the quotient set and poset $T \otimes E(G)$.

The fibred action of T on $T \otimes E(G)$ is given by

$$t \cdot (s \otimes e) = (t|\mathbf{d}(s))s \otimes e$$

if $\mathbf{d}(s) \leqslant \mathbf{r}(t)$. The action is fibred by $\mu : s \otimes e \mapsto \mathbf{d}(s)$. Note that

$$g\rho \cdot (\mathbf{r}(g)\rho \otimes \mathbf{r}(g)) = g\rho \otimes \mathbf{r}(g) = \mathbf{d}(g)\rho \otimes \mathbf{d}(g)$$

because $g^{-1} \in G(\mathbf{r}(g), \mathbf{d}(g))$ and $\mathbf{d}(g)\rho = g\rho(g^{-1}\rho)$. Then we have the action groupoid $\tilde{T}_\rho = T \ltimes (T \otimes E(G))$, a covering $\pi_2 : \tilde{T}_\rho \to T$, $(u, t \otimes f) \mapsto u$ and an embedding $\iota : G \to \tilde{T}_\rho$, $g \mapsto (g\rho, \mathbf{d}(g)\rho \otimes \mathbf{d}(g))$. Further $G\iota\pi_2 = G\rho$.

Now,

$$Y_{G\iota} = \{t \otimes e \in T \otimes E(G) : (u, t \otimes e) \in G\iota \,\text{for some}\, u \in T\}.$$

So,

$$\begin{aligned} \mathbb{L}_G &= \mathbb{L}(T \otimes E(G), Y_{G\iota}, G\iota\pi_2) \\ &= \{(u, t \otimes e) \in G\iota\pi_2 \times Y_{G\iota} : \mathbf{d}(t) = \mathbf{d}(u), u^{-1} \cdot (t \otimes e) \in Y_{G\iota}\} \end{aligned}$$

with composition

$$(u, t \otimes e)(w, s \otimes f) = (uw, t \otimes e)$$

provided $u^{-1} \cdot (t \otimes e) = s \otimes f$ and $\mathbf{r}(u) = \mathbf{d}(w)$. Recall that theorem 5.5.3 tells us $G\iota = \mathbb{L}_G$.

Taking G to be an inductive groupoid we can then convert G into an inverse semigroup S. We can convert the ideas above into semigroups terms. So the immersion ρ becomes an $\mathcal{R}$-injective homomorphism and then by proposition 1.1.29 this is equivalent to an idempotent pure homomorphism of inverse semigroups $\rho : S \to T$.

The pullback $T \between E(S)$ has equivalence $(u, e) \simeq_G (t, f)$ if there exists $s \in S$ with $\mathbf{d}(s) = e$ and $\mathbf{r}(s) = f$ such that $t = u(s\rho)$. The quotient set is $T \otimes E(S)$. The fibred action of inverse semigroup T on the poset $T \otimes E(S)$ is then given by

$$t \cdot (s \otimes e) = ts \otimes e$$

if $\mathbf{d}(s) \leqslant \mathbf{r}(t)$. Then $\tilde{T}_\rho = T \ltimes (T \otimes E(S))$ and $S\iota\pi_2 = S\rho$. The order ideal of $T \otimes E(S)$ is

$$Y_{S\iota} = \{t \otimes e \in T \otimes E(S) : (u, t \otimes e) \in S\iota \,\text{for some}\, u \in T\}$$

and

$$\begin{aligned} \mathbb{L}_S &= \mathbb{L}(T \otimes E(S), Y_{S\iota}, S\iota\pi_2) \\ &= \{(u, t \otimes e) \in S\iota\pi_2 \times Y_{S\iota} : \mathbf{d}(t) = \mathbf{d}(u), u^{-1} \cdot (t \otimes e) \in Y_{S\iota}\}. \end{aligned}$$

Again theorem 5.5.3 says, $S\iota = \mathbb{L}_S$.

6.2.3 Isomorphism of Quivers and $\mathbb{L}$-systems

We plan to show that Gomes' and Szendrei's $\mathbb{L}$-semigroup, $\mathbb{L}_Q$, is isomorphic to our $\mathbb{L}$-semigroup, $\mathbb{L}_S$. The isomorphism relies on another map between the posets $T \otimes E(S)$ and X_J. We describe, in detail, the properties of this map and then go on to give the required isomorphism. To illustrate the isomorphism of the two $\mathbb{L}$-semigroups a few examples are given at the end of this section.

We first construct a map between the two posets $\bar{\theta} : T \otimes E(S) \to X_J$, $u \otimes m \mapsto I(u, m)$. We show this map to be well-defined.

Let $u \otimes m \in T \otimes E(S)$ then $(u, m) \in T \between E(S)$ so $m\rho = \mathbf{r}(u)$. As $m \in E(S)$ we have $m\rho \in E(T)$ and $m\rho = m\rho(m\rho)^{-1} = \mathbf{d}(m\rho) = \mathbf{r}(u)$. As a result $(u, m) \in \mathcal{D}$ and so $I(u, m) \in X_J$.

To show $\bar{\theta}$ is well-defined we also require $I(u, m)$ to equal $I(w, n)$ whenever $u \otimes m = w \otimes n$. Assume then that $u \otimes m = w \otimes n$. So there exists an $a \in S$ with $\mathbf{d}(a) = m$ and $\mathbf{r}(a) = n$ such that $w = u(a\rho)$.

To show the first inclusion let $(u(ss^{-1}mr^{-1})\rho, rms) \in I(u, m)$. Then we require $r', s' \in S$ such that $(u(ss^{-1}mr^{-1})\rho, rms) = (w(s'(s')^{-1}n(r')^{-1})\rho, r'ns') \in I(w, n)$. Now let $r' = ra$ and $s' = a^{-1}s$. Then

$$\begin{aligned} w(s'(s')^{-1}n(r')^{-1})\rho &= u(a\rho)(a^{-1}ss^{-1}aa^{-1}aa^{-1}r^{-1})\rho \\ &= u(aa^{-1}ss^{-1}aa^{-1}r^{-1})\rho \\ &= u(ss^{-1}aa^{-1}r^{-1})\rho \\ &= u(ss^{-1}mr^{-1})\rho \end{aligned}$$

and

$$r'ns' = raa^{-1}aa^{-1}s = raa^{-1}s = rms.$$

Thus $(u(ss^{-1}mr^{-1})\rho, rms) = (w(s'(s')^{-1}n(r')^{-1})\rho, r'ns') \in I(w, n)$ and so $I(u, m) \subseteq I(w, n)$.

For the converse inclusion let $(w(s'(s')^{-1}n(r')^{-1})\rho, r'ns') \in I(w, n)$. Then we need $r, s \in S$ such that $(w(s'(s')^{-1}n(r')^{-1})\rho, r'ns') = (u(ss^{-1}mr^{-1})\rho, rms) \in$

$I(u, m)$. Let $r = r'a^{-1}$ and $s = as'$. Then

$$\begin{aligned} u(ss^{-1}mr^{-1})\rho &= u(as'(s')^{-1}a^{-1}aa^{-1}a(r')^{-1})\rho \\ &= u(a\rho)(s'(s')^{-1}a^{-1}a(r')^{-1})\rho \\ &= w(s'(s')^{-1}n(r')^{-1})\rho \end{aligned}$$

and

$$rms = r'a^{-1}aa^{-1}as' = r'a^{-1}as' = r'ns'.$$

Thus $(w(s'(s')^{-1}n(r')^{-1})\rho, r'ns') = (u(ss^{-1}mr^{-1})\rho, rms) \in I(u, m)$ and so $I(w, n) \subseteq I(u, m)$.

Therefore $I(u, m) = I(w, n)$ and $\bar{\theta}$ is well defined.

For sets A and B on which T acts, a map $\alpha : A \to B$ is said to be T-*equivariant* if, for $a \in A$, $t \in T$,

$$[t \cdot a]\alpha = t \cdot [a\alpha]$$

where these actions are defined.

Lemma 6.2.3. *$\bar{\theta}$ is T-equivariant.*

Proof. The action of T on $T \otimes E(S)$ is fibred by $\mu : T \otimes E(S) \to T$, $s \otimes e \mapsto \mathbf{d}(s)$. Let $t \in T$, $s \otimes e \in T \otimes E(S)$. Then t acts on $s \otimes e$ if $(s \otimes e)\mu = \mathbf{d}(s) \leqslant \mathbf{r}(t)$. For semigroups, $t \cdot (s \otimes e) = ts \otimes e$ and so

$$[t \cdot (s \otimes e)]\bar{\theta} = [ts \otimes e]\bar{\theta} = I(ts, e).$$

On the other hand $(s \otimes e)\bar{\theta} = I(s, e)$ and $t \in T$ acts on $I(s, e)$ if t acts on $(s, e) \in \mathcal{D}$. This occurs if $t^{-1}ts = s$. Now,

$$t^{-1}ts = s \Rightarrow t^{-1}tss^{-1} = ss^{-1} \Rightarrow t^{-1}t \geqslant ss^{-1}$$

so $t \cdot (s, e) = (ts, e)$ and $t \cdot I(s, e) = I(ts, e)$.

Therefore,

$$[t \cdot (s \otimes e)]\bar{\theta} = I(ts, e) = t \cdot [(s \otimes e)\bar{\theta}].$$

□

Restrict $\bar{\theta}$ to $Y_{S\iota}$. Recall $\iota : S \to T \ltimes (T \otimes E(S))$, $s \mapsto (s\rho, \mathbf{d}(s)\rho \otimes \mathbf{d}(s))$. Then

$$\begin{aligned} Y_{S\iota} &= \{t \otimes e \in T \otimes E(S) : (u, t \otimes e) \in S\iota \,\text{for some}\, u \in T\} \\ &= \{\mathbf{d}(s)\rho \otimes \mathbf{d}(s) \in T \otimes E(S) : s \in S\}. \end{aligned}$$

Then

$$\bar{\theta}|_{Y_{S\iota}} : \mathbf{d}(s)\rho \otimes \mathbf{d}(s) \mapsto I(\mathbf{d}(s)\rho, \mathbf{d}(s)).$$

Recall $Y_J = \{I(m\rho, m) \in X_J : m \in E(S)\}$ then $Y_{S\iota}\bar{\theta} \subseteq Y_J$ so $\bar{\theta} : Y_{S\iota} \to Y_J$.

Lemma 6.2.4. *$\bar{\theta}|_{Y_{S\iota}} : Y_{S\iota} \to Y_J$, $\bar{\theta}|_{Y_{S\iota}} : \mathbf{d}(s)\rho \otimes \mathbf{d}(s) \mapsto I(\mathbf{d}(s)\rho, \mathbf{d}(s))$ is bijective.*

Proof. Let $s, z \in S$ be such that $(\mathbf{d}(s)\rho \otimes \mathbf{d}(s))\bar{\theta}|_{Y_{S\iota}} = (\mathbf{d}(z)\rho \otimes \mathbf{d}(z))\bar{\theta}|_{Y_{S\iota}}$. Then $I(\mathbf{d}(s)\rho, \mathbf{d}(s)) = I(\mathbf{d}(z)\rho, \mathbf{d}(z))$. As Gomes and Szendrei remarked and because $\mathbf{d}(s)\rho, \mathbf{d}(z)\rho \in E(T)$, we have that $\mathbf{d}(s) = \mathbf{d}(z)$. Therefore $\bar{\theta}|_{Y_{S\iota}}$ is injective.

Assume now that $I(m\rho, m) \in Y_J$. Then $m \in E(S)$ and so $m = \mathbf{d}(m)$. Thus $\mathbf{d}(m)\rho \otimes \mathbf{d}(m) = m\rho \otimes m$ and $(m\rho \otimes m)\bar{\theta}|_{Y_{S\iota}} = I(m\rho, m)$. So $\bar{\theta}|_{Y_{S\iota}}$ is surjective.

Therefore $\bar{\theta}|_{Y_{S\iota}}$ is a bijection. □

Lemma 6.2.5. *As a semilattice, Y_J is isomorphic to $E(S)$.*

Proof. Let $\phi : E(S) \to Y_J$ be defined by $e \mapsto I(e\rho, e)$. By defintion of Y_J, ϕ is surjective. Following Gomes's and Szendrei's remark that there exists a unique $e \in E(T)$ and a unique $p \in \mathcal{D}(e, e)$ such that $J = J_p$, ϕ is also injective.

Lemma 6.1.14 states Y_J is a subsemilattice of X_J and defines for $e, f \in E(S)$, $I(e\rho, e) \wedge I(f\rho, f)$ to be $I((e\rho, e) + (f\rho, f))$. Now $(e\rho)(e\rho)(f\rho)^{-1} = (e\rho)(f\rho) \in E(T)$ so $(e\rho, e) + (f\rho, f) = ((e\rho)(f\rho)(e\rho), ef) = ((ef)\rho, ef)$.

We then show that ϕ is a semilattice isomorphism. Let $e, f \in E(S)$. In an inverse semigroup S, $e \wedge f = ef = fe = f \wedge e$. Then

$$\begin{aligned} e\phi \wedge f\phi &= I(e\rho, e) \wedge I(f\rho, f) \\ &= I((e\rho, e) + (f\rho, f)) \\ &= I((ef)\rho, ef) \\ &= (ef)\phi \\ &= (e \wedge f)\phi \end{aligned}$$

Thus ϕ is a semilattice isomorphism. $\square$

Next we restrict the order on $T \otimes E(S)$ to $Y_{S\iota}$. Recall

$$Y_{S\iota} = \{\mathbf{d}(s)\rho \otimes \mathbf{d}(s) \in T \otimes E(S) : s \in S\}.$$

Let $w, e \in E(S)$ and suppose $w\rho \otimes w \leqslant e\rho \otimes e$. Then there exists an $h \in S$ with $\mathbf{d}(h) = w$ and $\mathbf{r}(h) = k \leqslant e$ such that $w\rho(h\rho) \leqslant e\rho$. Now $w\rho(h\rho) = (wh)\rho \leqslant e\rho$ implies $(wh)\rho \in E(T)$. Then $wh \in E(S)$ so

$$whh^{-1}w^{-1} = wh(wh)^{-1} = (wh)^{-1}wh = h^{-1}w^{-1}wh$$

but

$$whh^{-1}w^{-1} = www^{-1} = www = w$$

and

$$h^{-1}w^{-1}wh = h^{-1}hh^{-1}hh^{-1}h = hh^{-1} = k$$

so $w = k$. Thus $w\rho \otimes w \leqslant e\rho \otimes e$ if and only if there exists an h with $\mathbf{d}(h) = w = \mathbf{r}(h)$ such that $w \leqslant e$ and $w\rho(h\rho) \leqslant e\rho$. Since $w \leqslant e$ then $w\rho \leqslant e\rho$ and $h \in E(S)$ can be given by $h = w$ because $w\rho(w\rho) = w\rho \leqslant e\rho$. So $w\rho \otimes w \leqslant e\rho \otimes e$ if and only if $w \leqslant e$.

Lemma 6.2.6. *For $e, f \in E(S)$,*

$$(e\rho \otimes e) \wedge (f\rho \otimes f) = (e \wedge f)\rho \otimes (e \wedge f).$$

Proof. Now $e \wedge f \leqslant e, f$ so $(e \wedge f)\rho \otimes (e \wedge f) \leqslant e\rho \otimes e, f\rho \otimes f$.

Suppose $w\rho \otimes w \leqslant e\rho \otimes e, f\rho \otimes f$. Then $w \leqslant e, f$. Hence $w \leqslant e \wedge f$ and $w\rho \otimes w \leqslant (e \wedge f)\rho \otimes (e \wedge f)$. Thus $(e\rho \otimes e) \wedge (f\rho \otimes f) = (e \wedge f)\rho \otimes (e \wedge f)$. □

Lemma 6.2.7. *As a semilattice $Y_{S\iota}$ is isomorphic to $E(S)$.*

Proof. Let $\phi : E(S) \to Y_{S\iota}$ be defined by $e \mapsto e\rho \otimes e$. By definition of $Y_{S\iota}$, ϕ is surjective.

Let $e, f \in E(S)$ and suppose $e\phi = f\phi$. Then $e\rho \otimes e = f\rho \otimes f$. So there exists $g \in S$ with $\mathbf{d}(g) = e$ and $\mathbf{r}(g) = f$ such that $f\rho = e\rho(g\rho)$. Now $f\rho = e\rho(g\rho) = (gg^{-1})\rho(g\rho) = (gg^{-1}g)\rho = g\rho$. This implies $f = g$ and so $f\rho = (ff)\rho = (ff^{-1})\rho = (gg^{-1})\rho = e\rho$, Thus $e = f$. So ϕ is injective.

Let $e, f \in E(T)$. Then, by lemma 6.2.6,

$$\begin{aligned} e\phi \wedge f\phi &= (e\rho \otimes e) \wedge (f\rho \otimes f) \\ &= (e \wedge f)\rho \otimes (e \wedge f) \\ &= (e \wedge f)\phi. \end{aligned}$$

Therefore ϕ is a semilattice isomorphism. $\square$

By lemmas 6.2.7 and 6.2.5 we see that $Y_{s\iota}$ is isomorphic $E(S)$ which in turn is isomorphic to Y_J.

Lemma 6.2.8. *Let S be an inverse semigroup. If $x, y \in S$ are such that $xy = x$ and $x^{-1}x = yy^{-1}$ then $y = x^{-1}x$ (and so $y \in E(S)$).*

Proof. $xy = x \Rightarrow x^{-1}xy = x^{-1}x \Rightarrow yy^{-1}y = x^{-1}x \Rightarrow y = x^{-1}x$. $\square$

Lemma 6.2.9. *$\bar{\theta} : T \otimes E(S) \to X_J$, $u \otimes m \mapsto I(u, m)$ is bijective.*

Proof. We first establish that $\bar{\theta}$ is injective. Let $(u \otimes m)\bar{\theta} = (w \otimes f)\bar{\theta}$. Then $I(u, m) = I(w, f)$ where $m, f \in E(S)$. As $(w, f) \in I(u, m)$ we let $(w, f) = (u(ss^{-1}r^{-1})\rho, rms)$ for some $s, r \in S$. Then using 6.2.1

$$\begin{aligned} m\rho &= u^{-1}u \\ &= u^{-1}u(ss^{-1}r^{-1}r)\rho \\ &= m\rho(ss^{-1}r^{-1}r)\rho. \end{aligned}$$

Hence, $m\rho \leqslant (ss^{-1}r^{-1}r)\rho$. As $m\rho, (ss^{-1})\rho, (r^{-1}r)\rho \in E(T)$ we have

$$m\rho \leqslant (ss^{-1})\rho \Leftrightarrow m\rho = m\rho(ss^{-1})\rho = (ss^{-1})\rho m\rho \tag{6.2.3}$$

and

$$m\rho \leqslant (r^{-1}r)\rho \Leftrightarrow m\rho = m\rho(r^{-1}r)\rho = (r^{-1}r)\rho m\rho. \tag{6.2.4}$$

Similarly, as $(u, m) \in I(w, f)$ and letting $(u, m) = (w(\bar{s}\bar{s}^{-1}\bar{r} \in)\rho, \bar{r}f\bar{s})$ for $\bar{s}, \bar{r} \in S$, we have

$$f\rho \leqslant (\bar{s}\bar{s}^{-1})\rho \Leftrightarrow f\rho = f\rho(\bar{s}\bar{s}^{-1})\rho = (\bar{s}\bar{s}^{-1})\rho f\rho \tag{6.2.5}$$

and

$$f\rho \leqslant (\bar{r}\bar{r}^{-1})\rho \Leftrightarrow f\rho = f\rho(\bar{r}\bar{r}^{-1})\rho = (\bar{r}\bar{r}^{-1})\rho f\rho.$$

Consider the element $a = rmss^{-1} \in S$. Then we have $a = rmss^{-1} = fs^{-1}$. Now,

$$aa^{-1} = rmss^{-1}ss^{-1}m^{-1}r^{-1} = rmss^{-1}m^{-1}r^{-1} = (rms)(rms)^{-1} = ff^{-1} = f$$

and

$$a^{-1}a = ss^{-1}m^{-1}r^{-1}rmss^{-1} = r^{-1}rm^2ss^{-1}ss^{-1} = r^{-1}rmss^{-1}.$$

Also, we have

$$a\rho = (r\rho)(m\rho)(ss^{-1})\rho = (r\rho)(m\rho).$$

Similarly consider the element $b = \bar{r}f\bar{s}\bar{s}^{-1} = m\bar{s}^{-1}$. Then

$$bb^{-1} = m$$

$$b^{-1}b = \bar{r}^{-1}\bar{r}f\bar{s}\bar{s}^{-1}$$

and

$$b\rho = (\bar{r}\rho)(f\rho).$$

Now consider $y = ab$. Then

$$\begin{aligned} yy^{-1} &= abb^{-1}a^{-1} \\ &= ama^{-1} \\ &= a(a^{-1}am)a^{-1} \\ &= a(r^{-1}rmss^{-1}m)a^{-1} \\ &= a(r^{-1}rmss^{-1})a^{-1} \\ &= a(a^{-1}a)a^{-1} \\ &= aa^{-1} \end{aligned}$$

So

$$
\begin{aligned}
(y\rho)(y\rho)^{-1} &= (yy^{-1})\rho \\
&= (aa^{-1})\rho \\
&= f\rho \\
&= w^{-1}w.
\end{aligned}
$$

Now,

$$
\begin{aligned}
w(y\rho) &= w(ab)\rho \\
&= w(r\rho)(m\rho)(\bar{r}\rho)(f\rho) \\
&= u(r^{-1})\rho(r\rho)(m\rho)(\bar{r}\rho)(f\rho) \quad \textit{by } (6.2.2) \\
&= u(r^{-1}r)\rho(m\rho)(\bar{r}\rho)(f\rho) \\
&= u(m\rho)(\bar{r}\rho)(f\rho) \qquad \textit{by } (6.2.4)
\end{aligned}
$$

Recall $u^{-1}u = m\rho$ which implies $uu^{-1}u = u = u(m\rho)$. So

$$
\begin{aligned}
w(y\rho) &= u(m\rho)(\bar{r}\rho)(f\rho) \\
&= u(\bar{r}\rho)(f\rho) \\
&= w(\bar{r}^{-1}\rho)(\bar{r}\rho)(f\rho) \quad \textit{by } (6.2.2) \\
&= w(\bar{r}^{-1}\bar{r})\rho(f\rho) \\
&= w(f\rho) \qquad \textit{by } (6.2.5) \\
&= w
\end{aligned}
$$

since $w^{-1}w = f\rho$ implies $ww^{-1}w = w = w(f\rho)$. Note that

$$
u(\bar{r}\rho)(f\rho) = w. \tag{6.2.6}
$$

By lemma 6.2.8, $y\rho = w^{-1}w = f\rho$. Since ρ is idempotent pure $y\rho = f\rho$ implies $y = ab \in E(S)$. Hence $y = yy^{-1} = f$.

Now $b^{-1}b = \bar{r}^{-1}\bar{r}f\bar{s}\bar{s}^{-1} \leqslant f$ and $f = y^{-1}f = b^{-1}a^{-1}ab = b^{-1}a^{-1}abb^{-1}b \leqslant b^{-1}b$ because $b^{-1}a^{-1}ab \in E(S)$. Hence $b^{-1}b = f$.

Thus $bb^{-1} = m$ and $b^{-1}b = f$ and by (6.2.6) $u(b\rho) = u(\bar{r})(f\rho) = w$. In other words, if $I(u,m) = I(w,f)$ then there exists $b \in S$ with $\mathbf{d}(b) = m$ and $\mathbf{r}(b) = f$ such that $w = u(b\rho)$. Thus $u \otimes m = w \otimes f$. Therefore $\bar{\theta}$ is injective.

We now show $\bar{\theta}$ is surjective. Let $I(u,m) \in X_J$. Then by lemma 6.2.1, $I(u,m) = I(u, mm^{-1})$. Now $(u,m) \in \mathcal{D}$ so $u^{-1}u = m\rho m\rho^{-1}$ then $\mathbf{r}(u) = (mm^{-1})\rho$. So $(u, mm^{-1}) \in T \between E(S)$ and $u \otimes mm^{-1} \in T \otimes E(S)$. Then $(u \otimes mm^{-1})\bar{\theta} = I(u, mm^{-1}) = I(u,m)$. Therefore $\bar{\theta}$ is surjective.

Therefore $\bar{\theta}$ is bijective. $\square$

Lemma 6.2.10. $\bar{\theta} : T \otimes E(S) \to X_J$, $u \otimes m \mapsto I(u,m)$ *is order preserving.*

Proof. Let $u \otimes m \leqslant w \otimes n$. Then there exists an $h \in S$ with $\mathbf{d}(h) = m$ and $\mathbf{r}(h) = e$ such that $e \leqslant n$ and $u(h\rho) \leqslant w$. If we consider S and T to be inductive groupoids the information above would be depicted in the diagram shown in Fig. 6.2.1. Hence $(w|e\rho) = u(h\rho)$ because restriction in an inductive groupoid is unique. Converting $(w|e\rho) = u(h\rho)$ in inductive groupoid T to the associated inverse semigroup T we get

$$w(e\rho) = u(h\rho).$$

Now $e\rho \in E(T)$ so $u(h\rho) \leqslant w$ in inverse semigroup T. In inductive groupoid S, $(n|e)h = eh = h$ so in the associated inverse semigroup S we have

$$nh = h.$$

We want $I(u,m) \leqslant I(w,n)$, equivalently $I(u,m) \subseteq I(w,n)$. So let $(u(ss^{-1}mr^{-1})\rho, rms) \in I(u,m)$ then we require an $r', s' \in S$ such that

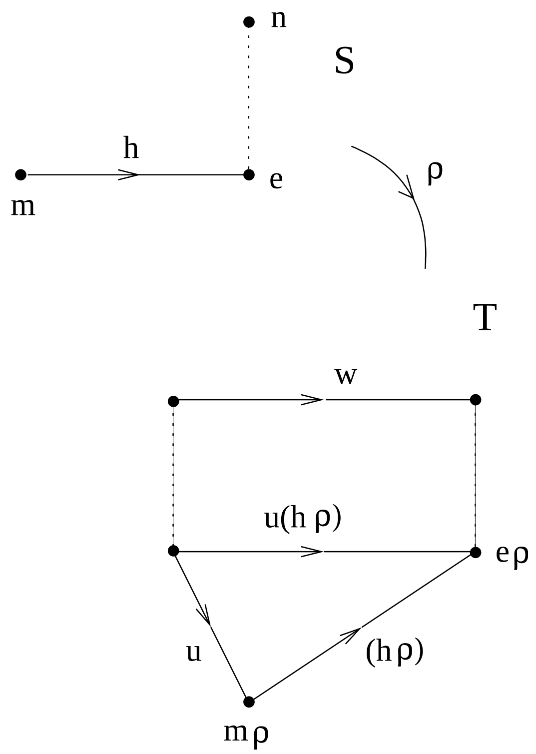

Figure 6.2.1: inductive groupoids S and T.

$(u(ss^{-1}mr^{-1})\rho, rms) = (w(s'(s')^{-1}n(r')^{-1}), r'ns') \in I(w,n)$. Now $hh^{-1} = m$ and $h^{-1}h = e$. Also $e \leqslant n$ so $e = ne = en$. Let $r' = rh$ and $s' = h^{-1}s$. Then

$$\begin{aligned}
r'ns' &= rhnh^{-1}s \\
&= rhnh^{-1}hh^{-1}s \\
&= rhneh^{-1}s \\
&= rheh^{-1}s \\
&= rhh^{-1}hh^{-1}s \\
&= rhh^{-1}s \\
&= rms
\end{aligned}$$

and

$$\begin{aligned}
w(s'(s')^{-1}n(r')^{-1})\rho &= w(h^{-1}ss^{-1}hnh^{-1}r^{-1})\rho \\
&= w(h^{-1}hh^{-1}ss^{-1}hnh^{-1}hh^{-1}r^{-1})\rho \\
&= w(eh^{-1}ss^{-1}hneh^{-1}r^{-1})\rho \\
&= w(e\rho)(h^{-1}ss^{-1}heh^{-1}r^{-1})\rho \\
&= u(h\rho)(h^{-1}ss^{-1}hh^{-1}hh^{-1}r^{-1})\rho \\
&= u(hh^{-1}ss^{-1}hh^{-1}r^{-1})\rho \\
&= u(ss^{-1}hh^{-1}r^{-1})\rho \\
&= u(ss^{-1}mr^{-1})\rho
\end{aligned}$$

Thus $(u(ss^{-1}mr^{-1})\rho, rms) = (w(s'(s')^{-1}n(r')^{-1})\rho, r'ns') \in I(w,n)$ and $I(u,m) \subseteq I(w,n)$. Therefore $I(u,m) \leqslant I(w,n)$ and $\bar{\theta}$ is order preserving. $\square$

Theorem 6.2.11. *For inverse semigroups S and T and surjective idempotent pure homomorphism $\rho : S \to T$, $\mathbb{L}_S = \mathbb{L}(T \otimes E(S), Y_{S\iota}, T)$ is isomorphic to $\mathbb{L}_Q = \mathbb{L}(X_J, Y_J, T)$.*

Proof. Let $\theta : \mathbb{L}_S \to \mathbb{L}_Q$ be defined by

$$(t, u \otimes m) \mapsto ((u \otimes m)\bar{\theta}, t) = (I(u, m), t).$$

We show θ is well-defined. Let $(t, u \otimes m) \in \mathbb{L}_S$ then $u \otimes m \in Y_{S\iota}$ so $(u \otimes m)\bar{\theta} = I(u, m) \in Y_J$ by lemma 6.2.4. Now $t^{-1} \cdot (u \otimes m) \in Y_{S\iota}$ so $[t^{-1} \cdot (u \otimes m)]\bar{\theta} \in Y_J$. $\bar{\theta}$ is T-equivariant, hence

$$[t^{-1} \cdot (u \otimes m)]\bar{\theta} = t^{-1} \cdot [(u \otimes m)\bar{\theta}] = t^{-1} \cdot I(u, m) \in Y_J.$$

So $(I(u, m), t) \in \mathbb{L}_Q$ and θ is well-defined.

Next we show that θ is injective. Let $(t, u \otimes m), (s, w \otimes n) \in \mathbb{L}_S$. Suppose $(t, u \otimes m)\theta = (s, w \otimes n)\theta$ then $(I(u, m), t) = (I(w, n), s)$. So $t = s$ in T and $(u \otimes m)\bar{\theta} = (w \otimes n)\bar{\theta}$ in Y_J. $\bar{\theta}|_{Y_{S\iota}}$ is bijective by lemma 6.2.4, so $u \otimes m = w \otimes n$ in $Y_{S\iota}$. Hence $(t, u \otimes m) = (s, w \otimes n)$ and θ is injective.

We now wish to determine that θ is surjective. Let $(I(u, m), t) \in \mathbb{L}_Q$ then $t \in T$ and $I(u, m) \in Y_J$. We have already shown $\bar{\theta}|_{Y_{S\iota}}$ is bijective and $(u \otimes m)\bar{\theta}|_{Y_{S\iota}} = I(u, m)$ with $u \otimes m \in T \otimes E(S)$. Now $t^{-1} \cdot I(u, m) \in Y_J$, so

$$t^{-1} \cdot I(u, m) = t^{-1} \cdot [(u, m)\bar{\theta}] = [t^{-1} \cdot (u, m)]\bar{\theta} \in Y_J.$$

As $\bar{\theta}|_{Y_{S\iota}} : Y_{S\iota} \to Y_J$ we have $t^{-1}(u \otimes m) \in Y_{S\iota}$. Then $(t, u \otimes m)\theta = ((u \otimes m)\bar{\theta}|_{Y_{S\iota}}, t) = (I(u, m), t)$ for $(t, u \otimes m) \in \mathbb{L}_S$. Thus θ is surjective.

Therefore θ is bijective.

Finally we require θ to be a homomorphism. Let $(t, u \otimes m), (s, w \otimes n) \in \mathbb{L}_S$. Then $m \in E(S)$, $u = m\rho \in E(T)$ and $\mathbf{d}(t) = tt^{-1} = uu^{-1} = \mathbf{d}(u)$ so $\mathbf{d}(t) = tt^{-1} = u = m\rho$. Also $n \in E(S)$, $w = n\rho \in E(T)$ and $\mathbf{d}(s) = ss^{-1} = ww^{-1} = \mathbf{d}(w)$ so $\mathbf{d}(s) = ss^{-1} = w = n\rho$.

Assume composition can occur in $\mathbb{L}_S$ so $\mathbf{r}(t) = t^{-1}t = ss^{-1} = \mathbf{d}(s)$ and

$t^{-1} \cdot (u \otimes m) = w \otimes n$. Then

$$\begin{aligned}[(t, u \otimes m)(s, w \otimes n)]\theta &= [(ts, u \otimes m)]\theta \\ &= ((u \otimes m)\bar{\theta}, ts) \\ &= (I(u, m), ts).\end{aligned}$$

Now

$$\begin{aligned}(t, u \otimes m)\theta(s, w \otimes n)\theta &= ((u \otimes m)\bar{\theta}, t)((w \otimes n)\bar{\theta}, s) \\ &= (I(u, m), t)(I(w, n), s).\end{aligned}$$

$\bar{\theta}$ is T-equivariant and bijective so

$$\begin{aligned}&t^{-1} \cdot (u \otimes m) = w \otimes n \\ \Rightarrow &[t^{-1} \cdot (u \otimes m)]\bar{\theta} = [w \otimes n]\bar{\theta} \\ \Rightarrow &t^{-1} \cdot [(u \otimes m)\bar{\theta}] = [w \otimes n]\bar{\theta} \\ \Rightarrow &t^{-1} \cdot I(u, m) = I(w, n)\end{aligned}$$

As $t^{-1} \cdot I(u, m) = I(w, n)$, then

$$t \cdot (t^{-1} \cdot I(u, m)) = (tt^{-1}) \cdot I(u, m) = I(u, m) = t \cdot I(w, n).$$

Whence,

$$\begin{aligned}&(I(u, m), t)(I(w, n), s) \\ &\qquad = (t \cdot (t^{-1} \cdot I(u, m) \wedge I(w, n)), ts) \\ &\qquad = (t \cdot (I(w, n) \wedge I(w, n)), ts) \\ &\qquad = (t \cdot I(w, n), ts) \\ &\qquad = (I(u, m), ts).\end{aligned}$$

Therefore θ is an isomorphism. $\square$

We now give two examples of the isomorphism given in the previous theorem. The first is the most simple case, where ρ is an isomorphism of groups, and so the first considered when researching this isomorphism.

Example 6.2.12. Let S and T be groups with respective identities $1_S, 1_T$. Take $\rho : S \to T$ to be idempotent pure and bijective. Then

$$\begin{aligned} \iota :& S \to T \ltimes (T \otimes E(S)) \\ & s \mapsto (s\rho, \mathbf{d}(s)\rho \otimes \mathbf{d}(s)) = (s\rho, 1_S\rho \otimes 1_s) = (s\rho, 1_T \otimes 1_S) \end{aligned}$$

and

$$S\iota = \{(s\rho, 1_T \otimes 1_S) : s \in S\}.$$

Also

$$T \otimes E(S) = T \otimes \{1_S\}.$$

Further

$$\begin{aligned} \pi_2 :& T \ltimes (T \otimes \{1_S\}) \to T \\ & (u, t \otimes 1_S) \mapsto u \end{aligned}$$

and so

$$S\iota\pi_2 = \{s\rho : s \in S\} = S\rho.$$

Then

$$\begin{aligned} Y_{S\iota} &= \{t \otimes 1_S \in T \otimes \{1_S\} : (u, t \otimes 1_S) \in S\iota \, \text{for some} \, u \in T\} \\ &= \{1_T \otimes 1_S\} \end{aligned}$$

and so

$$\begin{aligned}
&\mathbb{L}_S \\
&= \mathbb{L}(T \otimes \{1_S\}, Y_{S\iota}, S\iota\pi_2) \\
&= \{(u, 1_T \otimes 1_S) \in S\iota\pi_2 \times Y_{S\iota} : (1_T \otimes 1_S)\mu = 1_T = \mathbf{d}(u), u^{-1} \cdot (1_T \otimes 1_S) \in Y_{S\iota}\} \\
&= \{(s\rho, 1_T \otimes 1_S) : s\rho^{-1} \cdot (1_T \otimes 1_S) \in Y_{S\iota}\} \\
&= \{(s\rho, 1_T \otimes 1_S) : s\rho^{-1} \otimes 1_S \in Y_{S\iota}\} \\
&= \{(s\rho, 1_T \otimes 1_S) : 1_T \otimes 1_S \in Y_{S\iota}\} \\
&= \{(s\rho, 1_T \otimes 1_S) : s \in S\} \\
&= S\iota
\end{aligned}$$

Next we consider $\mathbb{L}_Q$,

$$\begin{aligned}
\mathcal{D}(S) &= \{(u, m) : u^{-1}T, m \in S, \mathbf{r}(u) = \mathbf{d}(m\rho)\} \\
&= \{(u, m) : u \in T, m \in S, 1_T = \mathbf{d}(m)\rho = 1_{s\rho} = 1_T\} \\
&= \{(u, m) : u \in T, m \in S\}
\end{aligned}$$

and composition $(u, m) + (x, n) = (u(m\rho)x^{-1}u, mn)$ exists if $u(m\rho)x^{-1} = 1_T$. Now

$$S(\mathcal{D}) = \{((u, m), a) : a \in T, (u, m) \in \mathcal{D}(aa^{-1}, a)\}.$$

As $(u, m) \in \mathcal{D}(u, u(m\rho))$ then $u = aa^{-1} = 1_T$ so $u(m\rho) = a$ implies $aa^{-1}(m\rho) = a$ which in turn implies $m\rho = a$. So

$$S(\mathcal{D}) = \{((1_T, m), m\rho) : m \in S\}.$$

Now

$$\begin{aligned} I(u,m) &= I(u,mm^{-1}) = I(u,1_S) \\ &= \{(u(ss^{-1}1_Sr^{-1})\rho, r1_Ss) : r,s \in S\} \\ &= \{(u(1_S1_Sr^{-1})\rho, rs) : r,s \in S\} \\ &= \{(u(r\rho)^{-1}, rs) : r,s \in S\} \end{aligned}$$

and

$$Y_J = \{I(u,m) \in X_J : (u,m) \in \mathcal{D}(1_T,1_T)\}.$$

As $(u,m) \in \mathcal{D}(u,u(m\rho))$ then $u = 1_T$ so $u(m\rho) = 1_T$ implies $m\rho = 1_T$ which in turn implies $m = 1_S$. So

$$Y_J = \{I(1_T,1_S)\}.$$

Then

$$\begin{aligned} \mathbb{L}_Q &= \mathbb{L}(X_J,Y_J,T) \\ &= \{(I(u,m),t) \in Y_J \times T : t^{-1} \cdot I(u,m) \in Y_J\} \\ &= \{(I(1_T,1_S),t) \in Y_J \times T : t^{-1} \cdot I(1_T,1_S) \in Y_J\}. \end{aligned}$$

We want to know for which $t \in T$ does $t^{-1} \cdot I(1_T,1_S) \in Y_J$. Now $t^{-1} \cdot I(1_T,1_S) \in Y_J$ if $t^{-1} \cdot I(1_T,1_S) = I(1_{T},_S)$. As $tt^{-1}1_T = 1_T1_T = 1_T$ then $t^{-1} \cdot (1_T,1_S)$ exists and so $t^{-1} \cdot I(1_T,1_S)$ exists and equals $I(t^{-1},1_S)$. Let

$$I_1 = I(1_T,1_S) = \{(1_T(r\rho)^{-1}, rs) : r,s \in S\} = \{(r\rho^{-1}, rs) : r,s \in S\}$$

and let

$$I_2 = I(t^{-1},1_S) = \{(t^{-1}(r\rho)^{-1}, rs) : r,s \in S\}.$$

Now $I_1 \subseteq I_2$ if $(r\rho^{-1}, rs) = (t^{-1}(\tilde{r}\rho)^{-1}, \tilde{r}\tilde{s})$ for some $\tilde{r}, \tilde{s} \in S$. So we must have $r\rho^{-1} = t^{-1}(\tilde{r}\rho)^{-1} = (\tilde{r}\rho t)^{-1}$ which implies $r\rho = (\tilde{r}\rho)t$ and $(r\rho)t^{-1} = (\tilde{r}\rho)$. As ρ is surjective for $t^{-1} \in T$ there is an $c \in S$ such that $c\rho = t^{-1}$. Then we can take $\tilde{r} = rc$. So $rs = \tilde{r}\tilde{s} = rq\tilde{s}$ and so we can take $\tilde{s} = c^{-1}s$. Hence $I(t^{-1}, 1_S) = I(1_T, 1_S)$ if there exists $a \in S$ such that $a\rho = t$. Thus

$$\mathbb{L}_Q = \{(I(1_T, 1_S), a\rho) : a \in S\}.$$

For completeness we show $\mathbb{L}_Q$ is isomorphic to $S(\mathcal{D})$ before we show $\mathbb{L}_Q$ is isomorphic to $\mathbb{L}_S$.

Let $\phi : S(\mathcal{D}) \to \mathbb{L}_Q$, $((1_T, m), m\rho) \mapsto (I(1_T, 1_S), m\rho)$. Now $m\rho^{-1}m\rho 1_T = 1_T \in E(T)$, hence

$$\begin{aligned}
&[((1_T, m), m\rho)((1_T, s), s\rho)]\phi \\
=&[m\rho(m\rho^{-1}(1_T, m) + (1_T, s)), m\rho s\rho]\phi \\
=&[m\rho((m\rho^{-1}, m) + (1_T, s)), m\rho s\rho]\phi \\
=&[m\rho(m\rho^{-1}m\rho 1_T m\rho^{-1}, ms), (ms)\rho]\phi \\
=&[m\rho(m\rho^{-1}, ms), (ms)\rho]\phi \\
=&[(1_T, ms), (ms)\rho]\phi \\
=&(I(1_T, 1_S), (ms)\rho)
\end{aligned}$$

and

$$
\begin{aligned}
&[((1_T, m), m\rho)\phi][((1_T, s), s\rho)\phi] \\
=&(I(1_T, 1_S), m\rho)(I(1_T, 1_S), s\rho) \\
=&(m\rho(m\rho^{-1}I(1_T, 1_S) \wedge I(1_T, 1_S)), m\rho s\rho) \\
=&(m\rho(I(m\rho^{-1}, 1_S) \wedge I(1_T, 1_S)), (ms)\rho) \\
=&(m\rho(I(m^{-1}\rho, 1_S) \wedge I(1_T, 1_S)), (ms)\rho) \\
=&(m\rho(I(1_T, 1_S) \wedge I(1_T, 1_S)), (ms)\rho) \\
=&(m\rho I(1_T, 1_S), (ms)\rho) \\
=&(I(m\rho, 1_S), (ms)\rho) \\
=&(I(1_T, 1_S), (ms)\rho)
\end{aligned}
$$

so ϕ is a homomorphism. As ρ is bijective then so is ϕ. So ϕ is an isomorphism between $S(\mathcal{D})$ and $\mathbb{L}_Q$.

We define $\theta : \mathbb{L}_S \to \mathbb{L}_Q$ as $(t, u \otimes m) \mapsto (I(u, m), t)$. In this example θ becomes $(s\rho, 1_T \otimes 1_S) \mapsto (I(1_T, 1_S), s\rho)$. As ρ is bijective then so is θ. Also,

$$
\begin{aligned}
&[(s\rho, 1_T \otimes 1_S)\theta][(m\rho, 1_T \otimes 1_S)\theta] \\
=&(I(1_T, 1_S), s\rho)(I(1_T, 1_S), m\rho) \\
=&(I(1_T, 1_S), (sm)\rho)
\end{aligned}
$$

and, as $s\rho^{-1} \cdot (1_T \otimes 1_S) = s\rho^{-1}1_T \otimes 1_S = s\rho^{-1} \otimes 1_S = \mathbf{d}(s^{-1}\rho) \otimes \mathbf{d}(s) = 1_T \otimes 1_S$, we also have

$$\begin{aligned}
&[(s\rho, 1_T \otimes 1_S)(m\rho, 1_T \otimes 1_S)]\theta \\
=&(s\rho m\rho, 1_T \otimes 1_S)\theta \\
=&((sm)\rho, 1_T \otimes 1_S)\theta \\
=&(I(1_T, 1_S), (sm)\rho).
\end{aligned}$$

Therefore $\mathbb{L}_S$ is isomorphic to $\mathbb{L}_Q$.

Example 6.2.13. For a set I and a group G the Brandt semigroup is defined to be

$$B(G, I) = \{(i, g, j) : i, j \in I, g \in G\} \cup \{0\}$$

with composition $(i, g, j)(k, h, l) = (i, gh, l)$ if $j = k$ and $(i, g, j)(k, h, l) = 0$ if $j \neq k$. Idempotents are of the form $(i, 1, i)$. Also $(i, g, j)^{-1} = (j, g^{-1}, i)$ so $\mathbf{d}(i, g, j) = (i, 1, i)$ and $\mathbf{r}(i, g, j) = (j, 1, j)$.

Let H be a group and let $\alpha : H \to G$ be a homomorphism. Then define $\rho : B(H, I) \to B(G, I)$ to be $(i, h, j) \mapsto (i, h\alpha, j)$. We take ρ to be idempotent pure. For ρ to be idempotent pure the set $\{(i, h, i) : h\alpha \in \ker\alpha\}$ must equal $E(B(H.I))$. In other words ρ is idempotent pure if and only if $h \in \ker\alpha$ implies $h = 1_H$. This occurs if and only if α is injective.

First we construct $\mathbb{L}_S$, where $S = B(H, I)$, and show it is equal to $S\iota$.

Now $\iota : B(H, I) \to B(G, I) \ltimes (B(G, I) \otimes E(B(H, I)))$ is given by

$$\begin{aligned}
(i, h, j) \mapsto &((i, h, i)\rho, \mathbf{d}(i, h, j)\rho \otimes (i, h, j)) \\
&= ((i, h\alpha, j)\rho, (i, 1_G, i)\rho \otimes (i, 1_H, i)).
\end{aligned}$$

So

$$B(H,I)\iota = \{((i,h\alpha,j)\rho,(i,1_G,i)\rho \otimes (i,1_H,i)) : i,j \in I, h \in H\}$$

Also

$$B(G,I) \otimes E(B(G,I)) = \{(j,g,k) \otimes (i,1_H,i) : i,j,k \in I, g \in G\}.$$

Further

$$\begin{aligned} \pi_2 :& B(G,I) \ltimes (B(G,I) \otimes E(B(H,I))) \to B(G,I) \\ & ((m,a,n),(j,g,k) \otimes (i,1_H,i)) \mapsto (m,a,n) \end{aligned}$$

so

$$\begin{aligned} B(H,I)\iota\pi_2 &= \{(i,h\alpha,j) : i,j \in I, h \in H\} \\ &= \{(i,h,j)\rho : i,j \in I, h \in H\} \\ &= B(H,I)\rho. \end{aligned}$$

We have that

$$\begin{aligned} & Y_{B(H,I)\iota} \\ & = \{(j,g,k) \otimes (i,1_H,i) \in B(G,I) \otimes E(B(H,I)) : \\ & \quad ((m,a,n),(j,g,k) \otimes (i,1_H,i) \in B(H,I)\iota \text{ for some } (m,a,n) \in B(G,I)\} \\ & = \{(i,1_G,i) \otimes (i,1_H,i) : i \in I\} \end{aligned}$$

and so

$$\begin{aligned}
\mathbb{L}_S &= \mathbb{L}(B(G,I) \otimes E(B(H,I)), Y_{B(H,I)\iota}, B(H,I)\iota\pi_2) \\
&= \{((j,h,k)\rho, (i,1_G,i) \otimes (i,1_H,i)) \in B(H,I)\iota\pi_2 \times Y_{B(H,I)\iota} : \\
&\quad ((i,1_G,i) \otimes (i,1_H,i))\mu = \mathbf{d}(i,1_G,i) = \mathbf{d}(j,h,k)\rho, \\
&\quad (j,h,k)\rho^{-1} \cdot ((i,1_G,i) \otimes (i,1_H,i)) \in Y_{B(H,I)\iota}\}.
\end{aligned}$$

Now $\mathbf{d}(i,1_G,i) = \mathbf{d}(j,h,k)\rho$ implies that $(i,1_G,i) = (j,1_G,j)$ which in turn implies that $i = j$. Then $(j,h,k) = (i,h,k)$ and

$$\begin{aligned}
&(j,h,k)\rho^{-1} \cdot ((i,1_G,i) \otimes (i,1_H,i)) \\
&= (i,h,k)\rho^{-1} \cdot ((i,1_G,i)\rho \otimes (i,1_H,i)) \\
&= (k,h^{-1},i)\rho \cdot ((i,1_G,i) \otimes (i,1_H,i)) \\
&= (k,h^{-1}\alpha,i) \cdot ((i,1_G,i) \otimes (i,1_H,i)) \\
&= (k,h^{-1}\alpha,i) \otimes (i,1_H,i)) \\
&= (k,h^{-1}\alpha,i) \otimes \mathbf{r}(k,h^{-1},i) \\
&= \mathbf{d}(k,h^{-1}\alpha,i) \otimes \mathbf{d}(k,h^{-1},i) \\
&= (k,1_G,k) \otimes (k,1_H,k) \\
&\in Y_{B(H,I)\iota}.
\end{aligned}$$

So

$$\begin{aligned}
\mathbb{L}_S &= \{((i,h,k)\rho, (i,1_G,i)\rho \otimes (i,1_H,i)) : i,k \in I, h \in H\} \\
&= \{((i,h\alpha,k), (i,1_G,i) \otimes (i,1_H,i) : i,k \in I, h \in H\} \\
&= B(H,I)\iota.
\end{aligned}$$

The $\mathbb{L}_S$ construction for an idempotent pure map $\rho : B(H, I) \to B(G, I)$, involving an injective group homomorphism $\alpha : H \to G$, is then identical to the construction for $\rho : B(H, I) \to B(H\alpha, I)$ which is surjective. We show that the $\mathbb{L}_S$ construction is isomorphic to the $\mathbb{L}_Q$ construction for the surjective idempotent pure map (an isomorphism in this case) $\rho : B(H, I) \to B(H\alpha, I)$.

We begin with

$$\begin{aligned}\mathcal{D} = \{((i, g, j), (k, h, l)) :& (i, g, j) \in B(H\alpha, I), (k, h, l) \in B(H, I), \\ & \mathbf{r}(i, g, j) = \mathbf{d}(k, h, l)\rho\}\end{aligned}$$

but $\mathbf{r}(i, g, j) = (j, 1_G, j) = \mathbf{d}(k, h, l)\rho = (k, 1_G, k)$ implies $j = k$ so

$$\mathcal{D} = \{((i, g, j), (j, h, l)) : i, j, l \in I, g \in H\alpha, h \in H\}.$$

Then

$$\begin{aligned}S(\mathcal{D}) = \{(((i, g, j), (j, h, l)), (m, a, n)) : (m, a, n) \in B(H\alpha, I), \\ ((i, g, j), (j, h, l)) \in \mathcal{D}((m, a, n)(m, a, n)^{-1}, (m, a, n))\}\end{aligned}$$

and as $((i, g, j), (j, h, l)) \in \mathcal{D}((i, g, j), (i, g, j)(j, h, l)\rho)$ we have $(i, g, j) = (m, 1_G, m)$ which implies $i = m = j$ and $g = 1_G$. Also $(i, gh\alpha, l) = (m, a, n)$ implies $i = m$, $gh\alpha = a$ and $l = n$. Now $gh\alpha = a$ becomes $h\alpha = a$ because $g = 1_G$. Hence

$$S(\mathcal{D}) = \{(((m, 1_G, m), (m, h, n)), (m, h\alpha, n)) : m, n \in I, h \in H\}.$$

Now

$$\begin{aligned}
&I((i,g,j),(j,h,l))\\
&=I((i,g,j),(j,h,l)(j,h,l)^{-1})\\
&=I((i,g,j),(j,1_H,j))\\
&=\{((i,g,j)[(y,s,z)(y,s,z)^{-1}(j,1_H,j)(w,r,x)^{-1}]\rho,(w,r,x)(j,1_H,j)(y,s,z)):\\
&\quad y,z,w,x\in I,r,s\in H\}
\end{aligned}$$

For $(w,r,x)(j,1_H,j)(y,s,z)$ to exist we must have $x=j=y$ and then $(w,r,x)(j,1_H,j)(y,s,z)=(w,rs,z)$. Also

$$\begin{aligned}
[(y,s,z)(z,s^{-1},y)(j,1_H,j)(x,r^{-1},w)]\rho &= (y,ss^{-1}1_Hr^{-1},w)\rho\\
&=(y,r^{-1},w)\rho\\
&=(y,r^{-1}\alpha,w).
\end{aligned}$$

So $(i,g,j)(y,r^{-1}\alpha,w)=(i,g(r^{-1}\alpha),w)$ and

$$I((i,g,j),(j,1_H,j))=\{((i,g(r^{-1}\alpha),w),(w,rs,z)):w,z\in I,r,s\in H\}.$$

Now

$$\begin{aligned}
Y_J=\{I((i,g,j),(j,1_H,j))\in X_J:&((i,g,j),(j,1_H,j))\in\mathcal{D}((e,1_G,e),(e,1_H,e))\\
&\text{for some }(e,1_G,e)\in E(B(H\alpha,I))\}
\end{aligned}$$

and then $((i,g,j),(j,1_H,j))\in\mathcal{D}((i,g,j),(i,g,j)(j,1_G,j))=\mathcal{D}((i,g,j),(i,g,j))$ so $(i,g,j)=(e,1_G,e)$ which implies $i=e=j$ and $g=1_G$. Then we have

$$Y_J=\{I((e,1_G,e),(e,1_H,e)):e\in I\}.$$

So

$$\begin{aligned}\mathbb{L}_Q &= \mathbb{L}(X_J, Y_J, B(H\alpha, I)) \\ &= \{(I((e, 1_G, e), (e, 1_H, e)), (m, a, n)) \in Y_J \times B(H\alpha, I) : \\ &\quad (m, a, n)^{-1} \cdot I((e, 1_G, e), (e, 1_H, e)) \in Y_J\}.\end{aligned}$$

We have that

$$\begin{aligned}&(m, a, n)^{-1} \cdot I((e, 1_G, e), (e, 1_H, e)) \\ &= (n, a^{-1}m) \cdot I((e, 1_G, e), (e, 1_H, e)) \\ &= I((n, a^{-1}, m)(e, 1_G, e), (e, 1H, e))\end{aligned}$$

provided $(n, a^{-1}, m)^{-1}(n, a^{-1}, m)(e, 1_G, e) = (e, 1_G, e)$. Now $(n, a^{-1}, m)^{-1}(n, a^{-1}, m)(e, 1_G, e) = (m, 1_G, m) = (e, 1_G, e)$ if $m = e$. Taking $m = e$ we have

$$\begin{aligned}&(m, a, n)^{-1} \cdot I((m, 1_G, e), (m, 1_H, m)) \\ &= I((n, a^{-1}, m)(m, 1_G, m), (m, 1_H, m)) \\ &= I((n, a^{-1}m), (m, 1_H, m)).\end{aligned}$$

Now $I((n, a^{-1}, m), (m, 1_H, m)) \in Y_J$ if there exists an $\bar{e} \in I$ such that $I((n, a^{-1}, m), (m, 1_H, m)) = I((\bar{e}, 1_G, \bar{e}), (\bar{e}, 1_H, \bar{e}))$.

Let

$$\begin{aligned}I_1 &= I((n, a^{-1}, m), (m, 1_H, m)) \\ &= \{((n, a^{-1}(r^{-1}\alpha), w), (w, rs, z)) : w, z \in I, r, s \in H\}\end{aligned}$$

and let

$$\begin{aligned} I_2 &= I((\bar{e}, 1_G, \bar{e}), (\bar{e}, 1_H, \bar{e})) \\ &= \{((\bar{e}, (\tilde{r}^{-1}\alpha), \tilde{w}), (\tilde{w}, \tilde{r}\tilde{s}, \tilde{z})) : \tilde{w}, \tilde{z} \in I, \tilde{r}, \tilde{s} \in H\}. \end{aligned}$$

Then $I_1 \subseteq I_2$ if $(n, a^{-1}(r^{-1}\alpha), w) = (\bar{e}, (\tilde{r}^{-1}\alpha), \tilde{w})$ and $(w, rs, z) = (\tilde{w}, \tilde{r}\tilde{s}, \tilde{z})$ for some $\tilde{w}, \tilde{z} \in I$ and $\tilde{r}, \tilde{s} \in H$. For this to hold we require $\tilde{w} = w$, $\tilde{z} = z$ and $\bar{e} = n$. As $a \in H\alpha$ then there exists $h \in H$ such that $h\alpha = a$. Take $\tilde{r} = rh$ and $\tilde{s} = h^{-1}s$. Then $\tilde{r}^{-1}\alpha = (rh)^{-1}\alpha = (h\alpha)^{-1}(r^{-1}\alpha) = a^{-1}(r^{-1}\alpha)$ and $\tilde{r}\tilde{s} = rhh^{-1}s = r1_Hs = rs$. Thus $I_1 \subseteq I_2$.

Similarly we can show $I_2 \subseteq I_1$ if $\bar{e} = n$.

So for $h \in H$ with $h\alpha = a$ and $\bar{e} = n$ we have

$$\begin{aligned} &(m, a, n)^{-1} \cdot I((e, 1_G, e), (e, 1_H, e)) \\ &= (n, a^{-1}, m) \cdot I((m, 1_G, m), (m, 1_H, m)) \\ &= I((n, a^{-1}, m), (m, 1_H, m)) \\ &= I((n, 1_G, n), (n, 1_H, n)). \end{aligned}$$

Thus

$$\mathbb{L}_Q = \{(I((m, 1_G, m), (m, 1_H, m)), (m, h\alpha, n)) : m, n \in I, h \in H\}.$$

Define the isomorphism $\phi : S(\mathcal{D}) \to \mathbb{L}_Q$ by $(((m, 1_G, m), (m, h, n)), (m, h\alpha, n))$ $\mapsto (I((m, 1_G, m), (m, h, n)), (m, h\alpha, n))$.

Now

$$\begin{aligned}
&I((m,1_G,m),(m,h,n))\\
&=I((m,1_G,m),(m,h,n)(m,h,n)^{-1})\\
&=I((m,1_G,m),(m,h,n)(n,h^{-1},m))\\
&=I((m,1_G,m),(m,1_H,m))
\end{aligned}$$

so ϕ is well-defined.

As α is bijective then so is ϕ.

We show ϕ is a homomorphism. Now

$$\begin{aligned}
&(((m,1_G,m),(m,h,n)),(m,h\alpha,n))\phi(((p,1_G,p),(p,b,q)),(p,b\alpha,q))\phi\\
&\quad=(I((m,1_G,m),(m,h,n)),(m,h\alpha,n))(I((p,1_G,p),(p,b,q)),(p,b\alpha,q)).
\end{aligned}$$

Now $(m,h\alpha,n)(p,b\alpha,q)$ is defined if $n=p$ in which case $(m,h\alpha,n)(p,b\alpha,q)=(m,h\alpha b\alpha,q)$. Recall that $I((m,1_G,m),(m,h,n))=I((m,1_G,m),(m,1_H,m))$ and $I((p,1_G,p),(p,b,q))=I((p,1_G,p),(p,1_H,p))$. Then $(m,h\alpha,n)^{-1}\cdot I((m,1_G,m),(m,1_H,m))$ is defined because $(m,h\alpha,n)(m,h\alpha,n)^{-1}(m,1_G,m)=(m,1_G,m)$ and so $(m,h\alpha,n)^{-1}\cdot I((m,1_G,m),(m,1_H,m))=I((n,h\alpha^{-1},m),(m,1_H,m))$. Further $I((n,h\alpha^{-1},m),(m,1_H,m))=I((n,1_G,n),(n,1_H,n))$. Since $n=p$ we have $I((p,1_G,p),(p,1_H,p))=I((n,1_G,n),(n,1_H,n))$. Hence

$$\begin{aligned}
&I((n,h\alpha^{-1},m),(m,1_H,m))\wedge I((p,1_G,p),(p,1_H,p))\\
&=I((n,1_G,n),(n,1_H,n))\wedge I((n,1_G,n),(n,1_H,n))\\
&=I((n,1_G,n),(n,1_H,n)).
\end{aligned}$$

As $(m, h\alpha, n)^{-1}(m, h\alpha, n)(n, 1_G, n) = (n, h\alpha^{-1}h\alpha 1_G, n) = (n, 1_G, n)$ we have that

$$\begin{aligned}&(m, h\alpha, n) \cdot I((n, 1_G, n), (n, 1_H, n))\\ &= I((m, h\alpha, n), (n, 1_H, n))\\ &= I((m, 1_G, m), (m, 1_H, m)).\end{aligned}$$

So

$$\begin{aligned}&(((m,1_G, m), (m, h, n)), (m, h\alpha, n))\phi(((p, 1_G, p), (p, b, q)), (p, b\alpha, q))\phi\\ &\quad= (I((m, 1_G, m), (m, h, n)), (m, h\alpha, n))(I((p, 1_G, p), (p, b, q)), (p, b\alpha, q))\\ &\quad= (I((m, 1_G, m), (m, 1_H, m)), (m, (h\alpha)(b\alpha), q)).\end{aligned}$$

Next consider

$$[(((m, 1_G, m), (m, h, n)), (m, h\alpha, n))(((p, 1_G, p), (p, b, q)), (p, b\alpha, q))]\phi.$$

Again $(m, h\alpha, n)(p, b\alpha, q)$ is defined if $n = p$ in which case $(m, h\alpha, n)(p, b\alpha, q) = (m, (h\alpha)(b\alpha), q)$. Also as $(m, h\alpha, n)(m, h\alpha, n)^{-1}(m, 1_G, m) =$ $(m, (h\alpha)(h\alpha)^{-1}1_G, m) = (m, 1_G, m)$ we have

$$\begin{aligned}&(m, h\alpha, n)^{-1} \cdot ((m, 1_G, m), (m, h, n))\\ &= (n, h\alpha^{-1}, m) \cdot ((m, 1_G, m), (m, h, n))\\ &= ((n, h\alpha^{-1}, m), (m, h, n)).\end{aligned}$$

As

$$\begin{aligned}
&(n, h\alpha^{-1}, m)(m, h, n)\rho(p, 1_G, p)^{-1} \\
&= (n, h\alpha^{-1}, m)(m, h\alpha, n)(n, 1_G, n) \\
&= (n, (h\alpha)^{-1}(h\alpha)1_G, n) \\
&= (n, 1_G, n) \in E(B(H\alpha, I))
\end{aligned}$$

we have

$$\begin{aligned}
&((n, h\alpha^{-1}, m), (m, h, n)) + ((p, 1_G, p), (p, b, q)) \\
&= ((n, 1_G, n)(n, h\alpha^{-1}, m), (m, h, n)(p, b, q)) \\
&= ((n, h\alpha^{-1}, m), (m, hb, q)).
\end{aligned}$$

Then as $(m, h\alpha, n)^{-1}(m, h\alpha, n)(n, h\alpha^{-1}m)$
$= (n, (h\alpha)^{-1}(h\alpha)(h\alpha)^{-1}, m) = (n, (h\alpha)^{-1}m)$ we get

$$(m, h\alpha, n) \cdot ((n, h\alpha^{-1}, m), (m, hb, q)) = ((m, 1_G, m), (m, hb, q)).$$

So

$$\begin{aligned}
&[(((m, 1_G, m), (m, h, n)), (m, h\alpha, n))(((p, 1_G, p), (p, b, q)), (p, b\alpha, q))]\phi \\
&= [(((m, 1_G, m), (m, hb, q)), (m, (h\alpha)(b\alpha), q))]\phi \\
&= (I((m, 1_G, m), (m, hb, q)), (m, (h\alpha)(b\alpha), q)) \\
&= (I((m, 1_G, m), (m, 1_H, m)), (m, (h\alpha)(b\alpha), q)) \\
&= (((m, 1_G, m), (m, h, n)), (m, h\alpha, n))\phi(((p, 1_G, p), (p, b, q)), (p, b\alpha, q))\phi.
\end{aligned}$$

Therefore $S(\mathcal{D})$ is isomorphic to $\mathbb{L}_Q$.

Let $\theta : \mathbb{L}_S \to \mathbb{L}_Q$ be defined by $\theta : ((i, h\alpha, k), (i, 1_G, i) \otimes (i, 1_H, i))$
$\mapsto (I((i, 1_G, i), (i, 1_H, i)), (i, h\alpha, k))$. As α is bijective then so is θ.

We show that θ is a homomorphism. Now

$$\begin{aligned} &((i, h\alpha, k), (i, 1_G, i) \otimes (i, 1_H, i))\theta((j, b\alpha, y), (j, 1_G, j) \otimes (j, 1_H, j))\theta \\ &= (I((i, 1_G, i), (i, 1_H, i)), (i, h\alpha, k))(I((j, 1_G, j), (j, 1_H, j), (j, b\alpha, y))) \end{aligned}$$

Let $k = j$ so $(i, h\alpha, k)(j, b\alpha, y) = (i, (h\alpha)(b\alpha), y)$. As
$(i, h\alpha, k)(i, h\alpha, k)^{-1}(i, 1_G, i) = (i, (h\alpha)(h\alpha)^{-1}1_G, i) = (i, 1_G, i)$ we have

$$\begin{aligned} &(i, h\alpha, k)^{-1} \cdot I((i, 1_G, i), (i, 1_H, i)) \\ &= (k, h\alpha^{-1}, i) \cdot I((i, 1_G, i), (i, 1_H, i)) \\ &= I((k, h\alpha^{-1}i), (i, 1_H, i)) \\ &= I((k, 1_G, k), (k, 1_H, k)). \end{aligned}$$

So

$$\begin{aligned} &I((k, h\alpha^{-1}, i), (i, 1_H, i)) \wedge I((j, 1_G, j), (j, 1_H, j)) \\ &= I((k, 1_G, k), (k, 1_H, k)) \wedge I((k, 1_G, k), (k, 1_H, k)) \\ &= I((k, 1_G, k), (k, 1_H, k)). \end{aligned}$$

Since $(i, h\alpha, k)^{-1}(i, h\alpha, k)(k, 1_G, k) = (k, (h\alpha)^{-1}(h\alpha)1_G, k) = (k, 1_G, k)$ we have

$$\begin{aligned} &(i, h\alpha, k) \cdot I((k, 1_G, k), (k, 1_H, k)) \\ &= I((i, h\alpha, k), (k, 1_H, k)) \\ &= I((i, 1_G, i), (i, 1_H, i)). \end{aligned}$$

So

$$\begin{aligned}
&((i,h\alpha,k),(i,1_G,i)\otimes(i,1_H,i))\theta((j,b\alpha,y),(j,1_G,j)\otimes(j,1_H,j))\theta \\
&\quad=(I((i,1_G,i),(i,1_H,i)),(i,h\alpha,k))(I((j,1_G,j),(j,1_H,j),(j,b\alpha,y))) \\
&\quad=(I((i,1_G,i),(i,1_H,i)),(i,(h\alpha)(b\alpha),y)).
\end{aligned}$$

Next consider

$$[((i,h\alpha,k),(i,1_G,i)\otimes(i,1_H,i))((j,b\alpha,y),(j,1_G,j)\otimes(j,1_H,j))]\theta.$$

Let $\mathbf{r}(i,h\alpha,k)=\mathbf{d}(j,b\alpha,y)$, so $(k,1_G,k)=(j,1_G,j)$ and $k=j$. Then

$$\begin{aligned}
&(i,h\alpha,k)^{-1}\cdot[(i,1_G,i)\otimes(i,1_H,i)] \\
&\quad=(k,h\alpha^{-1},i)\cdot[(i,1_G,i)\otimes(i,1_H,i)] \\
&\quad=(k,h^{-1},i)\rho\cdot[\mathbf{r}(k,h^{-1},i)\rho\otimes\mathbf{r}(k,h^{-1},i)] \\
&\quad=(k,h^{-1},i)\rho\otimes\mathbf{r}(k,h^{-1},i)] \\
&\quad=\mathbf{d}(k,h^{-1},i)\rho\otimes\mathbf{d}(k,h^{-1},i) \\
&\quad=(k,1_G,k)\otimes(k,1_H,k) \\
&\quad=(j,1_G,j)\otimes(j,1_H,j).
\end{aligned}$$

So

$$\begin{aligned}
&[((i,h\alpha,k),(i,1_G,i)\otimes(i,1_H,i))((j,b\alpha,y),(j,1_G,j)\otimes(j,1_H,j))]\theta \\
&\quad=((i,h\alpha,k)(k,b\alpha,y),(i,1_G,i)\otimes(i,1_H,i))\theta \\
&\quad=((i,(h\alpha)(b\alpha),y),(i,1_G,i)\otimes(i,1_H,i))\theta \\
&\quad=(I((i,1_G,i),(i,1_H,i)),(i,(h\alpha)(b\alpha),y)) \\
&\quad=((i,h\alpha,k),(i,1_G,i)\otimes(i,1_H,i))\theta((j,b\alpha,y),(j,1_G,j)\otimes(j,1_H,j))\theta.
\end{aligned}$$

Hence θ is a homomorphism and $\mathbb{L}_S$ is isomorphic to $\mathbb{L}_Q$.

Appendix to chapter 2

Chapter 2, Proof of Theorem 2.2.6

We note that if $r = m$ we assume that $(r|(m, g, n)) = (m, g, n)$ whether we restrict in $\mathcal{G}(\theta, \mathbf{a})$ or in $\mathcal{G}(\theta)$. Hence applying α then restricting gives

$$(m|[(m, g, n)\alpha]) = (m|(m, g', n)) = (m, g', n)$$

for some $g' \in G$. Whereas restricting first then applying α gives

$$[(m|(m, g, n))]\alpha = (m, g, n)\alpha = (m, g', n).$$

Therefore $(m|[(m, g, n)\alpha]) = [(m|(m, g, n))]\alpha$.

We now have three cases to consider:

(i) $m \geqslant 2$, $n < 2$;

(ii) $m < 2$, $n \geqslant 2$;

(iii) $m < 2$, $n < 2$.

Within these cases we have subcases which we show in the following table.

m,n \ r	m,n $\geqslant$ 2	m $\geqslant$ 2		n $\geqslant$ 2		m,n < 2			
		n=0	n=1	m=0	m=1	m=0 n=0	m=1 n=0	m−0 n=1	m=1 n=1
r $\geqslant$ 2	proved as part of main theorem	Case 1 (i)	Case 1 (ii)	Case 2 (i)	Case 2 (ii)	Case 3 (i)	Case 3 (iii)	Case 3 (iv)	Case 3 (vi)
r=1				Case 2 (iii)	r=m	Case 3 (ii)	r=m	Case 3 (v)	r=m
r=0				r=m		r=m		r=m	

Case 1(i): $\boldsymbol{m \geqslant 2, n = 0}$ and $\boldsymbol{r \geqslant 2}$

First, $\boldsymbol{r - m \geqslant 2}$.

Apply α,

$$\begin{aligned}
&(m, g, 0)\alpha \\
&= (m, v_m^{-1} g v_0, 0) \\
&= (m, w_{1,m-2} g, 0) \qquad \text{by equation (2.2.11) and as } v_0 = 1
\end{aligned}$$

then restrict in $\mathcal{G}(\theta)$,

$$(r|(m, w_{1,m-2} g, 0)) = (r, w_{1,m-2}\theta^{r-m}(g\theta^{r-m}), r - m).$$

Restrict in $\mathcal{G}(\theta, \mathbf{a})$,

$$(r|(m, g, 0)) = (r, w_{m,r-m-1}^{-1}(g\theta^{r-m}) w_{0,r-m-1}, r - m)$$

then apply α using (2.2.12),

$$\begin{aligned}&(r,w_{m,r-m-1}^{-1}(g\theta^{r-m})w_{0,r-m-1}, r-m)\alpha\\&\quad = (r, w_{1,r-2}w_{m,r-m-1}^{-1}(g\theta^{r-m})w_{0,r-m-1}w_{1,r-m-2}^{-1}, r-m).\end{aligned}$$

Now,

$$\begin{aligned}&w_{1,r-2}w_{m,r-m-1}^{-1}\\&= [u_1\theta^{r-2}u_2\theta^{r-3}\dots u_{r-1}][u_m\theta^{r-m-1}u_m\theta^{r-m-2}\dots u_{r-1}]^{-1} && \text{using (2.2.7)}\\&= [u_1\theta^{r-2}u_2\theta^{r-3}\dots u_{m-1}\theta^{r-m}]\\&= [u_1\theta^{m-2}u_2\theta^{m-3}\dots u_0]\theta^{r-m}\\&= w_{1,m-2}\theta^{r-m} && \text{using (2.2.7)}\end{aligned}$$

and

$$\begin{aligned}&w_{0,r-m-1}w_{1,r-m-2}^{-1}\\&= [u_0\theta^{r-m-1}u_1\theta^{r-m-2}\dots u_{r-m-1}][u_1\theta^{r-m-2}u_2\theta^{r-m-3}\dots\\&\quad \dots u_{r-m-1}]^{-1} && \text{using (2.2.7)}\\&= u_0\theta^{r-m-1}\\&= 1\theta^{r-m-1} && \text{as } u_0 = 1\\&= 1.\end{aligned}$$

Therefore,

$$\begin{aligned}&(r,w_{1,r-2}w_{m,r-m-1}^{-1}(g\theta^{r-m})w_{0,r-m-1}w_{1,r-m-2}^{-1}, r-m)\\&\quad = (r, w_{1,m-2}\theta^{r-m}(g\theta^{r-m}), r-m)\end{aligned}$$

Secondly, $\boldsymbol{r - m = 1}$.

Apply α,

$$(m, g, 0)\alpha = (m, w_{1,m-2}g, 0)$$

then restrict in $\mathcal{G}(\theta)$,

$$(r|(m, w_{1,m-2}g, 0)) = (r, w_{1,m-2}\theta(g\theta), 1).$$

Restrict in $\mathcal{G}(\theta, \mathbf{a})$,

$$(r|(m, g, 0)) = (r, w_{m,0}^{-1}(g\theta)w_{0,0}, 1)$$

then apply α,

$$\begin{aligned}
&(r, w_{m,0}^{-1}(g\theta)w_{0,0}, 1)\alpha \\
&= (r, v_r^{-1}w_{m,0}^{-1}(g\theta)w_{0,0}v_1, 1) \\
&= (r, w_{1,r-2}w_{m,0}^{-1}(g\theta)u_0v_1, 1) && \text{by equation (2.2.11)} \\
&= (r, w_{1,r-2}w_{m,0}^{-1}(g\theta)11, 1) && \text{as } u_0 = 1 \text{ and } v_0 = 1 \\
&= (r, w_{1,r-2}w_{m,0}^{-1}(g\theta), 1)
\end{aligned}$$

Now, $r - m = 1 \Rightarrow r = m + 1$ so

$$\begin{aligned}
&w_{1,r-2}w_{m,0}^{-1} \\
&= w_{1,m-1}w_{m,0}^{-1} \\
&= [u_1\theta^{m-1}u_2\theta^{m-2}\ldots u_m][u_m]^{-1} && \text{by equation (2.2.7) and } w_{m,0} = u_m \\
&= u_1\theta^{m-1}u_2\theta^{m-2}\ldots u_{m-1}\theta^1 \\
&= [u_1\theta^{m-2}u_2\theta^{m-3}\ldots u_{m-1}]\theta \\
&= w_{1,m-2}\theta && \text{by equation (2.2.7).}
\end{aligned}$$

Therefore,

$$(r, w_{1,r-2}w_{m,0}^{-1}(g\theta), 1) = (r, w_{1,m-2}\theta(g\theta), 1).$$

Case 1(ii): $\boldsymbol{m \geqslant 2, n = 1}$ and $\boldsymbol{r \geqslant 2}$.

Apply α,

$$\begin{aligned}
&(m, g, 1)\alpha \\
&= (m, v_m^{-1} g v_1, 1) \\
&= (m, w_{1,m-2}g1, 1) && \text{by equation (2.2.11) and as } v_1 = 1 \\
&= m, w_{1,m-2}g, 1)
\end{aligned}$$

then restrict in $\mathcal{G}(\theta)$,

$$(r|(m, w_{1,m-2}g, 1)) = (r, w_{1,m-2}\theta^{r-m}(g\theta^{r-m}), r - m + 1).$$

Restrict in $\mathcal{G}(\theta, \mathbf{a})$,

$$(r|(m, g, 1)) = (r, w_{m,r-m-1}^{-1}(g\theta^{r-m})w_{1,r-m-1}, r - m + 1)$$

then apply α using (2.2.12),

$$\begin{aligned}
&(r, w_{m,r-m-1}^{-1}(g\theta^{r-m})w_{1,r-m-1}, r - m + 1)\alpha \\
&\quad = (r, w_{1,r-2}w_{m,r-m-1}^{-1}(g\theta^{r-m})w_{1,r-m-1}w_{1,r-m-1}^{-1}, r - m + 1)
\end{aligned}$$

Now,

$$w_{1,r-2}w_{m,r-m-1}^{-1} = w_{1,m-2}\theta^{r-m}$$

and

$$w_{1,r-m-1}w^{-1}_{1,r-m-1} = 1.$$

Therefore,

$$\begin{aligned}(r,&w_{1,r-2}w^{-1}_{m,r-m-1}(g\theta^{r-m})w_{1,r-m-1}w^{-1}_{1,r-m-1}, r-m+1)\\ &(r, w_{1,m-2}\theta^{r-m}(g\theta^{r-m}), r-m+1)\end{aligned}$$

Case 2(i): $n \geqslant 2, m = 0$ and $r \geqslant 2$.

Apply α,

$$\begin{aligned}&(0, g, n)\alpha\\ &= (0, v_0^{-1}gv_n, n)\\ &= (0, gw^{-1}_{1,n-2}, n) \qquad \text{by equation (2.2.11) and as } v_0 = 1\end{aligned}$$

then restrict in $\mathcal{G}(\theta)$,

$$(r|(0, gw^{-1}_{1,n-2}, n)) = (r, (g\theta^r)(w^{-1}_{1,n-2}\theta^r), r+n).$$

Restrict in $\mathcal{G}(\theta, \mathbf{a})$,

$$(r|(0, g, n)) = (r, w^{-1}_{0,r-1}(g\theta^r)w_{n,r-1}, r+n)$$

then apply α using (2.2.12),

$$\begin{aligned}(r,&w^{-1}_{0,r-1}(g\theta^r)w_{n,r-1}, r+n)\alpha\\ &= (r, w_{1,r-2}w^{-1}_{0,r-1}(g\theta^r)w_{n,r-1}w^{-1}_{1,r+n-2}, r+n).\end{aligned}$$

Now,

$$
\begin{aligned}
& w_{1,r-2}w_{0,r-1}^{-1} \\
& = [u_1\theta^{r-2}u_2\theta^{r-3}\ldots u_{r-1}][u_0\theta r-1u_1\theta^{r-2}\ldots u_{r-1}]^{-1} && \text{using } (2.2.7) \\
& = u_1\theta^{r-2}u_2\theta^{r-3}\ldots u_{r-1}u_{r-1}^{-1}\ldots u_1^{-1}\theta^{r-2}u_0^{-1}\theta r-1 \\
& = u_0^{-1}\theta^{r-1} \\
& = 1\theta^{r-1} && \text{as } u_0 = 1 \\
& = 1
\end{aligned}
$$

and

$$
\begin{aligned}
& w_{n,r-1}w_{1,r+n-2}^{-1} \\
& = [u_n\theta^{r-1}u_{n+1}\theta^{r-2}\ldots u_{n+r-1}][u_1\theta^{r+n-2}u_2\theta^{r+n-3}\ldots \\
& \quad \ldots u_{r+n-1}]^{-1} && \text{by equation } (2.2.7) \\
& = [u_1\theta^{r+n-2}u_2\theta^{r+n-3}\ldots u_{n-1}\theta^{r}]^{-1} \\
& = [u_1\theta^{n-2}u_2\theta^{n-3}\ldots u_{n-1}]^{-1}\theta^{r} \\
& = w_{1,n-2}^{-1}\theta^{r} && \text{by equation } (2.2.7).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& (r, w_{1,r-2}w_{0,r-1}^{-1}(g\theta^{r})w_{n,r-1}w_{1,r+n-2}^{-1}, r+n) \\
& \quad = (r, (g\theta^{r})(w_{1,n-2}^{-1}\theta^{r}), r+n).
\end{aligned}
$$

Case 2(ii): $\boldsymbol{n \geqslant 2, m = 1}$ and $\boldsymbol{r \geqslant 2}$.

Apply α,

$$\begin{aligned}&(1, g, n)\alpha\\ &= (1, v_1^{-1} g v_n, n)\\ &= (1, g w_{1,n-2}^{-1}, n) \qquad \text{by equation } (2.2.11) \text{ and as } v_1 = 1\end{aligned}$$

then restrict in $\mathcal{G}(\theta)$,

$$(r|(1, g w_{1,n-2}^{-1}, n)) = (r, (g\theta^{r-1})(w_{1,n-2}^{-1}\theta^{r-1}), r+n-1).$$

Restrict in $\mathcal{G}(\theta, \mathbf{a})$,

$$(r|(1, g, n)) = (r, w_{1,r-2}^{-1}(g\theta^{r-1})w_{n,r-2}, r+n-1)$$

then apply α using (2.2.12),

$$\begin{aligned}&(r, w_{1,r-2}^{-1}(g\theta^{r-1})w_{n,r-2}, r+n-1)\alpha\\ &\quad = (r, w_{1,r-2}w_{1,r-2}^{-1}(g\theta^{r-1})w_{n,r-2}w_{1,r+n-3}^{-1}, r+n-1)\end{aligned}$$

Now,

$$w_{1,r-2}w_{1,r-2}^{-1} = 1$$

and

$$
\begin{aligned}
& w_{n,r-2}w^{-1}_{1,r+n-3} \\
& = [u_n\theta^{r-2}u_{n+1}\theta^{r-3}\dots u_{r+n-2}][u_1\theta^{r+n-3}u_2\theta^{r+n-4}\dots \\
& \quad \dots u_{r+n-2}]^{-1} && \text{using (2.2.7)} \\
& = [u_1\theta^{r+n-3}u_2\theta^{r+n-4}\dots u_{n-1}\theta^{r-1}]^{-1} \\
& = [u_1\theta^{n-2}u_2\theta^{n-3}\dots u_{n-1}]^{-1}\theta^{r-1} \\
& = w^{-1}_{1,n-2}\theta^{r-1} && \text{using (2.2.7).}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(r, & w_{1,r-2}w^{-1}_{1,r-2}(g\theta^{r-1})w_{n,r-2}w^{-1}_{1,r+n-3}, r+n-1) \\
& = (r, (g\theta^{r-1})(w^{-1}_{1,n-2}\theta^{r-1}), r+n-1).
\end{aligned}
$$

Case 2(iii): $n \geqslant 2, m = 0$ and $r = 1$.

Apply α,

$$
\begin{aligned}
& (0, g, n)\alpha \\
& = (0, v_0^{-1}gv_n, n) \\
& = (0, gw^{-1}_{1,n-2}, n) && \text{by equation (2.2.11) and as } v_0 = 1
\end{aligned}
$$

then restrict in $\mathcal{G}(\theta)$,

$$(1|(0, gw^{-1}_{1,n-2}, n)) = (1, (g\theta)(w^{-1}_{1,n-2}\theta), n+1).$$

Restrict in $\mathcal{G}(\theta, \mathbf{a})$,

$$(1|(0, g, n)) = (1, w^{-1}_{0,0}(g\theta)w_{n,0}, n+1)$$

then apply α,

$$\begin{aligned}
&(1, w_{0,0}^{-1}(g\theta)w_{n,0}, n+1)\alpha \\
&= (1, v_1^{-1}w_{0,0}^{-1}(g\theta)w_{n,0}v_{n+1}, n+1) \\
&= (1, u_0(g\theta)w_{n,0}v_{n+1}, n+1) && \text{by (2.2.11) and as } v_1 = 1 \text{and } w_{m,0} = u_m \\
&= (1, (g\theta)w_{n,0}v_{n+1}, n+1) && \text{as } u_0 = 1
\end{aligned}$$

Now,

$$\begin{aligned}
&w_{n,0}w_{1,n-1}^{-1} \\
&= [u_n][u_1\theta^{n-1}u_2\theta^{n-2}\ldots u_n]^{-1} && \text{by equation (2.2.7)} \\
&= [u_1\theta^{n-1}u_2\theta^{n-2}\ldots u_{n-1}\theta]^{-1} \\
&= [u_1\theta^{n-2}u_2\theta^{n-3}\ldots u_{n-1}]^{-1}\theta \\
&= w_{1,n-2}^{-1}\theta && \text{by equation (2.2.7).}
\end{aligned}$$

Therefore,

$$(1, (g\theta)w_{n,0}w_{1,n-1}^{-1}, n+1) = (1, (g\theta)(w_{1,n-2}^{-1}\theta), n+1).$$

Case 3(i): $m = 0, n = 0$ and $r \geqslant 2$.

Apply α,

$$\begin{aligned}
&(0, g, 0)\alpha \\
&= (0, v_0^{-1}gv_0, 0) \\
&= (0, 1^{-1}g1, 0) && \text{as } v_0 = 1 \\
&= (0, g, 0)
\end{aligned}$$

then restrict in $\mathcal{G}(\theta)$,

$$(r|(0,g,0)) = (r, g\theta^r, r).$$

Restrict in $\mathcal{G}(\theta, \mathbf{a})$,

$$(r|(0,g,0)) = (r, w_{0,r-1}^{-1}(g\theta^r)w_{0,r-1}, r)$$

then apply α,

$$\begin{aligned}
&(r, w_{0,r-1}^{-1}(g\theta^r)w_{0,r-1}, r)\alpha \\
&= (r, v_r^{-1}w_{0,r-1}^{-1}(g\theta^r)w_{0,r-1}v_r, r) \\
&= (r, w_{1,r-2}w_{0,r-1}^{-1}(g\theta^r)w_{0,r-1}w_{1,r-2}^{-1}, r) \qquad \text{by equation (2.2.11).}
\end{aligned}$$

Now,

$$w_{1,r-2}w_{0,r-1}^{-1} = 1$$

and

$$w_{0,r-1}w_{1,r-2}^{-1} = [w_{1,r-2}w_{0,r-1}^{-1}]^{-1} = 1^{-1} = 1.$$

Therefore,

$$(r, w_{1,r-2}w_{0,r-1}^{-1}(g\theta^r)w_{0,r-1}w_{1,r-2}^{-1}, r) = (r, g\theta^r, r).$$

Case 3(ii): $m = 0, n = 0$ and $r = 1$.

Apply α,

$$\begin{aligned}
&(0, g, 0)\alpha \\
&= (0, v_0^{-1}gv_0, 0) \\
&= (0, g, 0) \qquad \text{as } v_0 = 1
\end{aligned}$$

then restrict in $\mathcal{G}(\theta)$,

$$(1|(0,g,0)) = (1, g\theta, 1).$$

Restrict in $\mathcal{G}(\theta, \mathbf{a})$,

$$\begin{aligned}
&(1|(0,g,0)) \\
&= (1, w_{0,0}^{-1}(g\theta)w_{0,0}, 1) \\
&= (1, g\theta, 1) && \text{as } w_{0,0} = u_0 = 1
\end{aligned}$$

then apply α,

$$\begin{aligned}
&(1, g\theta, 1)\alpha \\
&= ((1, v_1^{-1}(g\theta)v_1, 1) \\
&= (1, g\theta, 1) && \text{as } v_1.
\end{aligned}$$

Case 3(iii): $\boldsymbol{m = 1, n = 0}$ and $\boldsymbol{r \geqslant 2}$.

Firstly $\boldsymbol{r - 1 \geqslant 2}$ ($\boldsymbol{\Rightarrow r \geqslant 3}$).

Apply α,

$$\begin{aligned}
&(1, g, 0)\alpha \\
&= (1, v_1^{-1}gv_0, 0) \\
&= (1, g, 0) && \text{as } v_1 = v_0 = 1
\end{aligned}$$

then restrict in $\mathcal{G}(\theta)$,

$$(r|(1,g,0)) = (r, g\theta^{r-1}, r-1).$$

Restrict in $\mathcal{G}(\theta, \mathbf{a})$,

$$(r|(1,g,0)) = (r, w_{1,r-2}^{-1}(g\theta^{r-1})w_{0,r-2}, r-1)$$

then apply α using (2.2.12),

$$\begin{aligned}(r,&w_{1,r-2}^{-1}(g\theta^{r-1})w_{0,r-2}, r-1)\alpha \\ &= (r, w_{1,r-2}w_{1,r-2}^{-1}(g\theta^{r-1})w_{0,r-2}w_{1,r-3}^{-1}, r-1) \\ &= (r, (g\theta^{r-1})w_{0,r-2}w_{1,r-3}^{-1}, r-1)\end{aligned}$$

Now,

$$\begin{aligned}&w_{0,r-2}w_{1,r-3}^{-1} \\ &= [u_0\theta^{r-2}u_1\theta^{r-3}\dots u_{r-2}][u_1\theta^{r-3}u_2\theta^{r-4}\dots \\ &\quad \dots u_{r-2}]^{-1} && \text{by equation (2.2.7)} \\ &= u_0\theta^{r-2} \\ &= 1\theta^{r-2} && \text{as } u_0 = 1 \\ &= 1.\end{aligned}$$

Therefore,

$$(r, (g\theta^{r-1})w_{0,r-2}w_{1,r-3}^{-1}, r-1) = (r, g\theta^{r-1}, r-1).$$

Secondly $r-1 = 1$ ($\Rightarrow r = 2$).

Apply α,

$$(1,g,0)\alpha = (1,g,0)$$

then restrict in $\mathcal{G}(\theta)$,

$$(2|(1,g,0)) = (2, g\theta, 1).$$

Restrict in $\mathcal{G}(\theta,\mathbf{a})$,

$$(2|(1,g,0)) = (2, w_{1,0}^{-1}(g\theta)w_{0,0},1)$$

then apply α,

$$\begin{aligned}
&(2, w_{1,0}^{-1}(g\theta)w_{0,0},1)\alpha \\
&= (2, v_2^{-1}w_{1,0}^{-1}(g\theta)w_{0,0}v_1,1) \\
&= (2, w_{1,0}w_{1,0}^{-1}(g\theta)u_0v_1,1) && \text{by equation (2.2.11) as } w_{m,0}=u_m \\
&= (2,(g\theta),1) && \text{as } u_0=v_0=1
\end{aligned}$$

Case 3(iv): $m=0, n=1$ and $r \geqslant 2$.

Apply α,

$$\begin{aligned}
&(0,g,1)\alpha \\
&= (0, v_0^{-1}gv_1,1) \\
&= (0,g,1) && \text{as } v_0=v_1=1
\end{aligned}$$

then restrict in $\mathcal{G}(\theta)$,

$$(r|(0,g,1)) = (r, g\theta^r, r+1).$$

Restrict in $\mathcal{G}(\theta,\mathbf{a})$,

$$(r|(0,g,1)) = (r, w_{0,r-1}^{-1}(g\theta^r)w_{1,r-1}, r+1)$$

then apply α using (2.2.12),

$$\begin{aligned}
&(r,w_{0,r-1}^{-1}(g\theta^r)w_{1,r-1}, r+1)\alpha \\
&\quad = (r, w_{1,r-2}w_{0,r-1}^{-1}(g\theta^r)w_{1,r-1}w_{1,r-1}^{-1}, r+1).
\end{aligned}$$

Now,

$$w_{1,r-2}w_{0,r-1}^{-1} = 1$$

and

$$w_{1,r-1}w_{1,r-1}^{-1} = 1.$$

Therefore,

$$(r, w_{1,r-2}w_{0,r-1}^{-1}(g\theta^r)w_{1,r-1}w_{1,r-1}^{-1}, r+1) = (r, g\theta^r, r+1).$$

Case 3(v): $m = 0, n = 1$ and $r = 1$.

Apply α,

$$(0, g, 1)\alpha = (0, g, 1)$$

then restrict in $\mathcal{G}(\theta)$,

$$(1|(0, g, 1)) = (1, g\theta, 2).$$

Restrict in $\mathcal{G}(\theta, \mathbf{a})$,

$$\begin{aligned}
&(1|(0, g, 1)) \\
&= (1, w_{0,0}^{-1}(g\theta)w_{1,0}, 2) \\
&= (1, (g\theta)w_{1,0}, 2) && \text{as } w_{0,0} = u_0 = 1
\end{aligned}$$

then apply α,

$$\begin{aligned}
&(1, (g\theta)w_{1,0}, 2)\alpha \\
&= (1, v_1^{-1}(g\theta)w_{1,0}v_2, 2) \\
&= (1, 1g\theta w_{1,0}w_{1,0}^{-1}, 2) && \text{by equation (2.2.11) and as } v_1 = 1 \\
&= (1, (g\theta), 2)
\end{aligned}$$

Case 3(vi): $\boldsymbol{m = 1, n = 1}$ and $\boldsymbol{r \geqslant 2}$.

Apply α,

$$\begin{aligned}
&(1, g, 1)\alpha \\
&= (1, v_1^{-1} g v_1, 1) \\
&= (1, 1g1, 1) && \text{as } v_1 = 1 \\
&= (1, g, 1)
\end{aligned}$$

then restrict in $\mathcal{G}(\theta)$,

$$(r|(1, g, 1)) = (r, g\theta^{r-1}, r).$$

Restrict in $\mathcal{G}(\theta, \mathbf{a})$,

$$(r|(1, g, 1)) = (r, w_{1,r-2}^{-1}(g\theta^{r-1})w_{1,r-2}, r)$$

then apply α using (2.2.12),

$$\begin{aligned}
(r, &w_{1,r-2}^{-1}(g\theta^{r-1})w_{1,r-2}, r)\alpha \\
&= (r, w_{1,r-2} w_{1,r-2}^{-1}(g\theta^{r-1})w_{1,r-2} w_{1,r-2}^{-1}, r) \\
&= (r, g\theta^{r-1}, r).
\end{aligned}$$

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