APPROXIMATIONS TO RUIN PROBABILITY IN THE PRESENCE OF AN ABSORBING UPPER BARRIER

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ABSTRACT

Segerdahl (1970) studied a risk process with two absorbing barriers and presented solutions in [13] when single claim amounts follow an exponential distribution. The methods of that paper only lead to explicit results for a few particular single claim amount distributions. The main purpose of this thesis is to develop approximate methods of calculating the probability of ruin in the presence of an absorbing upper barrier which can be applied to any single claim amount distribution.

In Chapter 2, an explicit solution for the probability of ruin, when there is an absorbing upper barrier, is developed in terms of the probability of ruin $\psi(\cdot)$ in the unrestricted process. Explicit solutions are also derived, by extending the methods of Segerdahl, when single claim amounts follow particular types of gamma distribution.

The main approximation method is presented in Chapters 3 and 4. The method uses a limiting property of Markov chains to produce approximations to ruin probabilities. Comparisons are made between approximations by this method, when single claims follow the gamma distributions of Chapter 2, and exact values obtained through the results of that chapter to provide a measure of the accuracy of the method.

Approximations to the probability of ruin in the presence of an absorbing upper barrier are obtained for any single claim amount distribution in Chapter 5. This is done in two ways: by approximating the function $\psi(\cdot)$ and using the relationship between the probability

of ruin in the restricted process and $\psi(\cdot)$, and by using the Markov chain method. A possible approach to calculating finite time ruin probabilities in the presence of an absorbing upper barrier is presented in the final chapter.

1.0 INTRODUCTION

The risk reserve of a non-life insurance company is subject to two different types of change with time. Firstly, the reserves are increased by premium income and, secondly, they are reduced on the occurrence and immediate payment of claims. Formally, we define Z_t, the reserve level at time t, as

$$Z_t = Z_o + C_t - X_t$$

where Z_o is the initial reserve level, i.e. the reserves at time 0, C_t is the net premium income by time t and X_t is the total amount of claims paid by time t. The model is usually simplified by assuming that the premium income in received continuously at a constant rate c, so that

$$Z_t = Z_o + ct - X_t$$

The behaviour of Z_t is modelled statistically by considering two independent distributions:

i that of the number of claims, N(t), which occur by time t; and

ii that of the single claim amounts, which we denote by V(x).

The distribution of N(t) is obtained by making the following assumptions about the occurrence of claims:

a. claims are independent of each other;

b. the probability of a single claim in a time interval $(\tau, \tau + d\tau)$, where $d\tau$ is small, is $\rho d\tau + o(d\tau)$, where ρ is a constant; and

c. the probability of more than one claim in $(\tau, \tau + d\tau)$ is $o(d\tau)$.

For convenience, we introduce operational time as defined by Cramer [4].

Natural time τ is transformed to an operational time t by the relation

$$t = t(\tau) = \int_0^{\tau} \rho_u du$$

where, in our case, $\rho_u = \rho$. Then $dt = \rho d\tau$ and the interval $(\tau, \tau + d\tau)$ on the natural time scale becomes $(t, t + dt)$ on the operational time scale.

The above assumptions, together with this transformed time scale, lead to a Poisson distribution for N(t) with probability function

$$Pr[N(t) = n] = e^{-t} \frac{t^n}{n!} \qquad \text{for } n = 0, 1, 2, \ldots\ldots$$

Thus the expected number of claims in any operational time interval t is simply equal to t. Throughout this thesis, time will be measured in operational units.

Individual claim amounts are assumed to be distributed independently of each other and of the time at which they occur. They have common distribution function V(x). We define F(x,t) to be the distribution function of the total amount of claims by time t, i.e. $F(x,t) = Pr(X_t < x)$. This distribution function is obtained by considering the events satisfying $X_t \leqslant x$. These are:

i no claims occur in (0,t);

ii a single claim occurs in (0,t) which is of an amount less than or equal to x; and

iii n claims occur in (0,t) where $n \geqslant 2$, and the total amount of these n claims is less than or equal to x.

Then

$$F(x,t) = \sum_{n\geq 0} e^{-t} \frac{t^n}{n!} V^{n*}(x)$$

where $V^{n*}(x)$ is the n-fold convolution of $V(x)$ with itself, satisfying

$$V^{n*}(x) = \int_0^x V^{(n-1)*}(x - z)dV(z) \qquad [1.1]$$

with $V^{0*}(x) = 1$ and $V^{1*}(x) = V(x)$.

We have defined the range of integration in [1.1] to be (0,x) as we shall only consider non-negative claims. Thus $V(0) = 0$ throughout. $F(x,t)$ is a generalised Poisson distribution, and has characteristic function (see Beard et al [1]),

$$\phi_F(\alpha) = \exp\{(\phi(\alpha) - 1)t\}$$

where

$$\phi(\alpha) = \int_0^\infty e^{i\alpha x} dV(x)$$

Hence, it can be shown that

$$E(X_t) = \nu_1 t$$

and

$$Var(X_t) = \nu_2 t$$

where

$$\nu_K = \int_0^\infty x^K dV(x)$$

It is customary to scale $V(x)$ so that the mean single claim amount is

unity. Thus, the expected total claim amount by time t is equal to t. This is the framework which we shall use to study the behaviour of Z_t.

Risk theory's classical problem concerning the behaviour of Z_t is to identify the probability that, at some time in the future, the reserve level ever becomes negative. If this happens, we say that ruin has occurred. Let $\psi(u)$ denote the probability of ruin from an initial reserve level u. Then

$$\psi(u) = \Pr\{Z_\tau < 0 \text{ for some } \tau > 0 \,|\, Z_0 = u\}$$

It can be shown that, for any value of u, $\psi(u) = 1$ if the premium income per unit time does not exceed the expected total claim amount over that period, i.e. if $c \leqslant 1$. Thus we let $c = 1 + \lambda$, where $\lambda > 0$ is called the premium loading.

Generally, it is not possible to find explicit solutions for $\psi(u)$. A first approach to the problem is Lundberg's inequality, which states that

$$\psi(u) \leqslant e^{-Ru}$$

where R, the insurer's insolvency constant, is the unique positive root of

$$1 + (1 + \lambda)R = \int_0^\infty e^{Rx} dV(x)$$

Lundberg's inequality only provides an upper bound for the ruin probability. However, it can be shown that, for large values of u, $\psi(u)$ is related to this upper bound, as shown in Gerber [6], by the

asymptotic result

$$\psi(u) \sim \eta e^{-Ru}$$

where

$$\eta = \frac{c - \nu_1}{R\int_0^\infty ye^{Ry}[1 - V(y)]dy}$$

Exact solutions for $\psi(u)$ can be obtained for a few particular distributions $V(x)$ in the following way. Suppose that the first claim occurs at time τ and is of an amount x. If we distinguish between ruin occurring at this or at a subsequent claim, we have

$$\psi(u) = \int_0^\infty e^{-\tau}\int_0^{u+c\tau} \psi(u+c\tau-x)v(x)dxd\tau + \int_0^\infty e^{-\tau}\int_{u+c\tau}^\infty v(x)dxd\tau \qquad [1.2]$$

where we have written $v(x)dx$ for $dV(x)$. Hereafter, we shall assume, out of convenience rather than necessity, that $V(x)$ is differentiable and has density function $v(x)$. By making the substitution $s = u + c\tau$ in the above equation and differentiating with respect to u, we obtain

$$-c\psi'(u) + \psi(u) = \int_0^u \psi(u - x)v(x)dx + 1 - V(u)$$

This readily yields a solution for $\psi(u)$ when $V(x)$ takes an exponential or mixed exponential distribution (see Gerber [6]). In particular, if $V(x) = 1 - e^{-x}$, we have that

$$\psi(u) = \frac{1}{1 + \lambda}\exp(-\frac{\lambda u}{1 + \lambda}) \qquad [1.3]$$

Explicit solutions can also be obtained by this method when $V(x)$ takes a gamma distribution, given by

$$dV(x) = \frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^{\alpha}}\,dx$$

provided the parameter α is an integer. We denote this distribution by $\gamma(\alpha)$; β is simply a scale parameter.

Seal describes numerical methods of calculating $\delta(u,t)$, the probability of non-ruin over a finite time interval $(0,t)$ when the initial reserve level is u, in his book [12]. It is possibly more realistic to study $\psi(u,t)$ (the complement of $\delta(u,t)$) than $\psi(u)$. Referring to early Swedish contributions to ruin theory, Seal points out that "even Swedish insurance companies cannot plan to last forever". Seal's formula is

$$\delta(u,t) = F(u + ct,t) - \int_0^t \delta(0,t - s)dF(u + cs,s) \qquad [1.4]$$

A simple derivation and interpretation of this formula is given in [9], p106-107.

Just an an indefinite future for an insurance company could be criticised as unrealistic, so too could the resulting infinite risk reserve. For, if the premium loading is positive, then the risk reserve is certain to drift ultimately to infinity. The continued growth of reserves can be prevented by introducing an upper limit to the amount of the reserve level. Segerdahl [13] studied a risk process with an upper barrier, which we denote by K, allowing this barrier to be either reflecting or absorbing in nature. In the case of a reflecting upper barrier,when the reserve level reaches K, all further income is removed from the process and used for other purposes such as the payment of dividends. The reserve level remains at K until a claim occurs. If the upper barrier is absorbing, the process terminates when the reserve level reaches K, with the possibility of premium reassessment. Thus, the

insurer may use an absorbing upper barrier to control his profits.

In this thesis, we consider just one of the problems tackled by Segerdahl: the probability of ruin in the presence of an upper barrier. Since it can be shown that ultimate ruin is certain if the upper barrier is reflecting, (see [9], p122-123), we restrict our attention to an absorbing upper barrier. In Segerdahl's notation, let $\xi(u,K)$ denote the probability of ruin in the presence of an absorbing upper barrier. We shall only concentrate on finding solutions for $\xi(u,K)$. Problems such as suitable values of K for a given initial reserve level or strategies depending on the value of K will not be considered.

Although Segerdahl produced general equations for $\xi(u,K)$, he only obtained explicit solutions when single claim amounts took an exponential distribution. However, he indicated that it is possible to extend his methods to related distributions to find further explicit solutions for $\xi(u,K)$. The main purpose of this thesis is to find approximate solutions for $\xi(u,K)$ which can be applied to different forms of $V(x)$. If we can identify further explicit solutions for $\xi(u,K)$, then we can compare exact and approximate ruin probabilities. Hence, we can deduce to which forms of $F(x,t)$ these approximation methods are applicable and find how accurate our approximate ruin probabilities are likely to be when exact solutions for $\xi(u,K)$ are unknown.

We begin our study of the problem in the next chapter. By taking an alternative approach to that of Segerdahl we find that there exists a general solution for $\xi(u,K)$ in terms of the function $\psi(\cdot)$. However, as we noted earlier, there are few explicit solutions for $\psi(u)$. Hence we find that this general solution does not lead to new solutions for $\xi(u,K)$, as exact solutions for $\xi(u,K)$ also exist for forms of $V(x)$ which

yield exact solutions for $\psi(u)$ through [1.2]. The solution is, however, useful in approximating $\xi(u,K)$ through $\psi(u)$ as there exist various approximations to $\psi(u)$. By extending Segerdahl's methods, we also show how explicit solutions for $\xi(u,K)$ can be achieved if $V(x)$ takes a $\gamma(\alpha)$ distribution, provided the parameter α is an integer. The procedure in this case is similar to, but much more complicated than, the relatively simple case of $V(x) = 1 - e^{-x}$, solved by Segerdahl.

The following three chapters are devoted to approximate methods of calculating $\xi(u,K)$. In Chapters 3 and 4 we develop a method which uses properties of Markov chains. We can create a partition of all possible values which the reserve level can take by splitting the interval $[0,K)$ into a fixed number of equal intervals, with the intervals $(-\infty,0)$ and $[K,\infty)$ completing the partition. By letting these intervals be states of a Markov chain, with the latter two absorbing, we can construct a transition matrix by considering the behaviour of the reserves over an appropriately chosen finite time interval. We approximate $\xi(u,K)$, where u is in one of the non-absorbing states, by calculating the probability of eventual absorption into $(-\infty,0)$ from that non-absorbing state. In Chapter 3, we produce approximations by using only the distribution function $F(x,t)$. The refinements to the method introduced in Chapter 4 require that we can also calculate accurately numerical values for $f(x,t)$, the density function of $F(x,t)$. Thus we only need a knowledge of the functions $F(x,t)$ and $f(x,t)$ to be able to employ this method.

In Chapter 5, we consider approximations to $\xi(u,K)$ obtained by approximating the function $\psi(u)$. This approach can lead to approximations for a wide variety of single claim amount distributions $V(x)$. We also show that the Markov chain method can be applied to distributions $V(x)$

whose convolutions do not exist in an exact form. Finally, in Chapter 6, we briefly tackle the problem of finite time ruin probabilities in the presence of an absorbing upper barrier. We show that accurate approximations to these ruin probabilities can be obtained by adapting the Markov chain method of approximating $\xi(u,K)$. If accurate numerical values of $F(x,t)$ and $f(x,t)$ can be produced then our approach leads to more general solutions than those presented in Segerdahl's paper, and to more accurate approximations than can be produced by computer simulation of the risk process.

2.0 EXACT SOLUTIONS FOR $\xi(u,K)$

2.1 Introduction

In this chapter we find exact results for $\xi(u,K)$. Firstly, we show in Section 2.2 that, under certain conditions, there exists a simple relationship between $\xi(u,K)$ and $\psi(u)$. Further, if $\chi(u,K)$ is the probability of absorption by the upper barrier from initial reserve level u, then it is shown that $\xi(u,K)$ and $\chi(u,K)$ are complementary functions. Using the relationship between $\xi(u,K)$ and $\psi(u)$, we deduce an upper bound for $\xi(u,K)$ based on Lundberg's inequality, when X_t takes a generalised Poisson distribution.

In Section 2.3, the exact result for $\xi(u,K)$ is derived when $V(x) = 1-e^{-x}$. This is not a new result. Segerdahl derived $\chi(u,K)$ for this distribution in [13]. However, the method used to obtain this solution indicates the approach to find $\xi(u,K)$ when $V(x)$ takes any $\gamma(\alpha)$ distribution, where α is an integer. As in the case of exponential, i.e. $\gamma(1)$, single claim amounts, the solution for $\xi(u,K)$ is found by solving a linear differential equation of order $\alpha + 1$.

Depending on the values of the gamma parameter α and the premium loading λ, the roots of the characteristic equation of this differential equation may be real or complex, distinct or multiple. In Section 2.4 we present the solution for $\xi(u,K)$ in two cases, assuming in each case that the roots are all distinct. In Section 2.4.1 we consider the case of real roots only, and in the final section we consider the case of both real and complex roots.

2.2 The Relationship Between $\xi(u,K)$ and $\psi(u)$

Let $\delta(u) = 1 - \psi(u)$. Then $\delta(u)$ is the probability of non-ruin for the unrestricted process. If the premium loading λ is positive, then the unrestricted process is certain to drift to $+\infty$ as $t \to \infty$. Thus, if the process starts at u, it is certain to have eventually passed through the fixed value K, where $u < K < \infty$.

We note that $\xi(u,K)$ is equivalent to the probability of ruin occurring in the unrestricted process without the reserve level ever having reached K, and that $\chi(u,K)$ is equivalent to the probability that the reserve level of the unrestricted process reaches K without previously having been negative. Thus we may express $\delta(u)$ as

$$\delta(u) = \chi(u,K)\,\delta(K)$$

which gives

$$\chi(u,K) = \frac{\delta(u)}{\delta(K)}$$

or

$$\chi(u,K) = \frac{1 - \psi(u)}{1 - \psi(K)}$$

Similarly, we may express $\psi(u)$ as

$$\psi(u) = \xi(u,K) + \chi(u,K)\psi(K) \qquad [2.1]$$

Inserting for $\chi(u,K)$ in terms of $\psi(\cdot)$ in [2.1] and rearranging yields

$$\xi(u,K) = \frac{\psi(u) - \psi(K)}{1 - \psi(K)} \qquad [2.2]$$

Clearly $\xi(u,K) + \chi(u,K) = 1$, which means that the process is certain to be absorbed eventually by one of the barriers.

If X_t has a generalised Poisson distribution, then

$$\xi(u,K) \leqslant \frac{e^{-Ru} - e^{-RK}}{1 - e^{-RK}} \qquad [2.3]$$

where R is as defined in Chapter 1.

This can be proved by induction, as follows. Define $\xi_n(u,K)$ to be the probability of ruin in the presence of the upper barrier K, at or before the n^{th} claim. Since $\xi(u,K) = \lim_{n\to\infty} \xi_n(u,K)$ it is sufficient to show that $\xi_n(u,K)$ satisfies [2.3] for all $n \geqslant 0$.

Clearly $\xi_o(u,K) = 0$. Assume $\xi_{n-1}(u,K)$ satisfies [2.3]. Distinguishing between ruin occurring at the first claim, or in the subsequent n-1 claims, we may write

$$\xi_n(u,K) = \int_0^{t_o} e^{-\tau} \int_0^{u+c\tau} \xi_{n-1}(u+c\tau-x,K)v(x)\,dx\,d\tau + \int_0^{t_o} e^{-\tau} \int_{u+c\tau}^{\infty} v(x)\,dx\,d\tau$$

where $t_o = (K-u)/c$. There must be a claim by time t_o if ruin is to occur. Otherwise $Z_{t_o} = K$. Then

$$\xi_n(u,K) \leqslant \int_0^{t_o} e^{-\tau} \int_0^{u+c\tau} \frac{e^{-R(u+c\tau-x)} - e^{-RK}}{1 - e^{-RK}} v(x)\,dx\,d\tau$$
$$+ \int_0^{t_o} e^{-\tau} \int_{u+c\tau}^{\infty} \frac{e^{-R(u+c\tau-x)} - e^{-RK}}{1 - e^{-RK}} v(x)\,dx\,d\tau$$

Since $\xi_{n-1}(u,K)$ satisfies [2.3] in the first term, while $u+c\tau-x \leqslant 0$ in the second term implies $e^{-R(u+c\tau-x)} \geqslant 1$.

So

$$\xi_n(u,K) \leqslant \int_0^{t_o} e^{-\tau} \int_0^{\infty} \frac{e^{-R(u+c\tau-x)} - e^{-RK}}{1 - e^{-RK}} v(x)\,dx\,d\tau$$

$$= \frac{(1 + cR)e^{-Ru}}{1 - e^{-RK}} \int_0^{t_o} e^{-(1+cR)\tau} d\tau - \frac{e^{-RK}}{1 - e^{-RK}} (1 - e^{-t_o})$$

$$= \frac{e^{-Ru} - e^{-RK}}{1 - e^{-RK}} + \frac{e^{-t_o}}{1 - e^{-RK}} [e^{-RK} - e^{-R(u+ct_o)}]$$

$$= \frac{e^{-Ru} - e^{-RK}}{1 - e^{-RK}} \quad \text{by definition of } t_o.$$

Thus

$$\xi_n(u,K) \leqslant \frac{e^{-Ru} - e^{-RK}}{1 - e^{-RK}}$$

We note that as $K \to \infty$, this result tends to Lundberg's upper bound for $\psi(u)$, namely e^{-Ru}.

Gerber shows in [5] that, if the total claim amount follows a compound Poisson process, $\psi(u)$ and Lundberg's upper bound are related by the equality

$$\psi(u) = \frac{e^{-Ru}}{E[e^{-R.Z_T} \mid T < \infty \text{ and } Z_o = u]}$$

where T is the time at which ruin occurs and Z_T is the negative reserve level at time T. Inserting this expression for $\psi(\cdot)$ in [2.2] would yield a similar sort of relationship between $\xi(u,K)$ and the upper bound which we have just derived. However, this new identity would not lead to any new solutions for $\xi(u,K)$ as the distribution of the negative reserve level can only be found when V(x) takes an exponential distribution.

2.3 The Solution for Exponential Single Claim Amounts

We can construct an equation for $\xi(u,K)$, analogous to [1.2] of Chapter 1 for $\psi(u)$, by considering whether absorption by the lower barrier occurs at the first or at a subsequent claim. As explained in the previous section, the first claim must occur by time t_o.

We have

$$\xi(u,K) = \int_0^{t_o} e^{-\tau} \int_0^{u+c\tau} \xi(u+c\tau-x,K)\,v(x)\,dx\,d\tau + \int_0^{t_o} e^{-\tau}[1-V(u+c\tau)]\,d\tau \qquad [3.1]$$

Writing $s = u+c\tau$, [3.1] becomes

$$c.\xi = \int_u^K e^{-\frac{1}{c}(s-u)} \int_0^s \xi(s-x,K)\,v(x)\,dx\,ds + \int_u^K e^{-\frac{1}{c}(s-u)}[1-V(s)]\,ds$$

Differentiation with respect to u yields

$$c.\xi' = \xi - \int_0^u \xi(u-x,K)\,v(x)\,dx - 1 + V(u)$$

or

$$-c.\xi' + \xi = \int_0^u \xi(x,K)\,v(u-x)\,dx + 1 - V(u) \qquad [3.2]$$

Equation [3.2] holds for any single claim amount distribution $V(x)$, and has boundary condition

$$\xi(K,K) = 0 \qquad [3.3]$$

Using Segerdahl's technique, [3.2] is readily solved when $V(x)$ is exponentially distributed. Inserting $V(x) = 1 - e^{-x}$, so that $\nu_1 = 1$, and writing $c = 1 + \lambda$ in [3.2] gives

$$-(1 + \lambda)\xi' + \xi = \int_0^u \xi(x,K)e^{-(u-x)}dx + e^{-u} \qquad [3.4]$$

Multiplication by e^u, followed by differentiation with respect to u yields

$$\xi'' + \frac{\lambda}{1 + \lambda}\xi' = 0 \qquad [3.5]$$

The solution of this second order linear differential equation is

$$\xi(u,K) = c_1(\lambda,K)e^{\frac{-\lambda u}{1+\lambda}} + c_2(\lambda,K)$$

The boundary condition [3.3] implies that

$$c_2(\lambda,K) = -c_1(\lambda,K)e^{\frac{-\lambda K}{1+\lambda}}$$

Inserting this solution for $\xi(u,K)$ in [3.4] we have

$$-(1 + \lambda)\xi' + \xi = c_1(\lambda,K)[(1 + \lambda)e^{\frac{-\lambda u}{1+\lambda}} - e^{\frac{-\lambda K}{1+\lambda}}] \qquad [3.6]$$

and

$$\int_0^u \xi(x,K)e^{-(u-x)}dx + e^{-u} =$$

$$c_1(\lambda,K)[(1+\lambda)e^{\frac{-\lambda u}{1+\lambda}} - e^{\frac{-\lambda K}{1+\lambda}}] + e^{-u}[1+c_1(\lambda,K)e^{\frac{-\lambda K}{1+\lambda}} - (1+\lambda)c_1(\lambda,K)] \qquad [3.7]$$

Equating the right hand sides of [3.6] and [3.7] gives

$$c_1(\lambda,K) = (1 + \lambda - e^{\frac{-\lambda K}{1+\lambda}})^{-1}$$

Hence

$$\xi(u,K) = \frac{e^{\frac{-\lambda u}{1+\lambda}} - e^{\frac{-\lambda K}{1+\lambda}}}{1 + \lambda - e^{\frac{-\lambda K}{1+\lambda}}} \qquad [3.8]$$

As expected, this is the complement of Segerdahl's result (19.12) in [13] for $\chi(u,K)$. Two simple corollaries emerge from [3.8]. Firstly,

$$\lim_{K\to\infty} \xi(u,K) = \frac{1}{1 + \lambda} e^{\frac{-\lambda u}{1+\lambda}}$$

which is simply the result obtained in Chapter 1 for $\psi(u)$ in this situation. Secondly,

$$\lim_{\lambda\to 0} \xi(u,K) = \frac{K - u}{1 + K}$$

i.e. as the premium loading tends to zero, the ruin probability depends on the difference between u and K. Clearly, as the value of the upper barrier increases to $+\infty$, the ruin probability tends to 1 as in the unrestricted process.

2.4 The Solution for Gamma Single Claim Amounts

In this section we apply the same procedure as in the previous section to find the solution for $\xi(u,K)$ when $V(x)$ takes a $\gamma(\alpha)$ distribution. The distribution function is given by

$$V(x) = \int_0^x \frac{y^{\alpha-1} e^{-y/\beta}}{\Gamma(\alpha)\beta^{\alpha}}\, dy$$

where β is the scale parameter of the distribution. For this distribution, $\nu_1 = \alpha\beta$ so that $\beta = 1/\alpha$ when the distribution has unit mean. For clarity the solution is presented for the general case in terms of both α and β. The premium income is now equal to $(1 + \lambda)\alpha\beta$.

Further, as we only consider integral values of α, V(x) can be expressed as

$$V(x) = 1 - \sum_{j=0}^{\alpha-1} e^{-x/\beta} \frac{x^j}{\beta^j . j!}$$

which is easily proved by induction. Proceeding from equation [3.2] we have

$$-(1+\lambda)\alpha\beta\xi' + \xi = \int_0^u \xi(x,K) \frac{(u-x)^{\alpha-1} e^{-(u-x)/\beta}}{\Gamma(\alpha)\beta^\alpha} dx + \sum_{j=0}^{\alpha-1} e^{-u/\beta} \frac{u^j}{\beta^j . j!}$$

which, on multiplication by $e^{u/\beta}$, gives

$$(-(1+\lambda)\alpha\beta\xi' + \xi)e^{\mu/\beta} = \int_0^u \xi(x,K) \frac{(u-x)^{\alpha-1} e^{x/\beta}}{\Gamma(\alpha)\beta^\alpha} dx + \sum_{j=0}^{\alpha-1} \frac{u^j}{\beta^j . j!} \qquad [4.1]$$

As before, we must eliminate the integral from [4.1]. To do so, we differentiate the equation α times with respect to u. Let

$$I_\alpha = \int_0^u \xi(x,K) \frac{(u-x)^{\alpha-1} e^{x/\beta}}{\Gamma(\alpha)} dx$$

Then

$$\frac{d}{du} I_\alpha = \int_0^u \xi(x,K) \frac{(u-x)^{\alpha-2} e^{x/\beta}}{\Gamma(\alpha-1)} dx = I_{\alpha-1}$$

Hence

$$\frac{d^\alpha}{du^\alpha} \int_0^u \xi(x,K) \frac{(u-x)^{\alpha-1} e^{x/\beta}}{\Gamma(\alpha)\beta^\alpha} dx = \frac{\xi(u,K) e^{u/\beta}}{\beta^\alpha} \qquad [4.2]$$

The terms in u on the right hand side of [4.1] all disappear after differentiation α times with respect to u. We can also show by an inductive argument that

$$\frac{d^{\alpha}}{du^{\alpha}} \xi(u,K)e^{u/\beta} = e^{u/\beta} \sum_{j=0}^{\alpha} \frac{1}{\beta^{\alpha-j}} \binom{\alpha}{j} \xi^{(j)}$$

where

$$\xi^{(j)} = \frac{d^{j}}{du^{j}} \xi(u,K) \quad \text{and} \quad \xi^{(0)} = \xi(u,K).$$

Hence we have

$$\frac{d^{\alpha}}{du^{\alpha}} [-(1+\lambda)\alpha\beta e^{u/\beta}\xi' + e^{u/\beta}\xi] = e^{u/\beta} \sum_{j=0}^{\alpha} \frac{1}{\beta^{\alpha-j}} \binom{\alpha}{j} [-(1+\lambda)\alpha\beta\xi^{(j+1)} + \xi^{(j)}]$$

$$= -(1+\lambda)\alpha\beta e^{u/\beta}\xi^{(\alpha+1)} + e^{u/\beta} \sum_{j=1}^{\alpha} \frac{1}{\beta^{\alpha-j}} [-(1+\lambda)\alpha\binom{\alpha}{j-1} + \binom{\alpha}{j}]\xi^{(j)} + \frac{e^{u/\beta}\xi}{\beta^{\alpha}} \qquad [4.3]$$

Equating [4.2] and [4.3] shows that

$$(1 + \lambda)\alpha\beta\xi^{(\alpha+1)} + \sum_{j=1}^{\alpha} \frac{1}{\beta^{\alpha-j}} [(1+\lambda)\alpha\binom{\alpha}{j-1} - \binom{\alpha}{j}]\xi^{(j)} = 0 \qquad [4.4]$$

Again $\xi(u,K)$ is found as the solution of a linear differential equation, this time of order $\alpha + 1$. If both α and β equal 1, i.e. $V(x) = 1 - e^{-x}$, then [4.4] simply reduces to [3.5].

The characteristic equation of [4.4] is

$$(1 + \lambda)\alpha\beta\ell^{\alpha+1} + \sum_{j=1}^{\alpha} \frac{1}{\beta^{\alpha-j}} [(1+\lambda)\alpha\binom{\alpha}{j-1} - \binom{\alpha}{j}]\ell^{j} = 0 \qquad [4.5]$$

which implies that

$$\ell(\ell + R_1)(\ell + R_2) \ldots\ldots (\ell + R_{\alpha}) = 0$$

where $\{-R_i\}$ are the roots of [4.5]. If all the roots are distinct, then the solution for $\xi(u,K)$ is of the form

$$\xi(u,K) = b_o(\lambda,K) + \sum_{i=1}^{\alpha} b_i(\lambda,K)e^{-R_i u} \qquad [4.6]$$

This solution is general, whether the roots are real or complex. To complete the solution for $\xi(u,K)$, the coefficients $b_i(\lambda,K)$, hereafter denoted by b_i for brevity, must be determined. In the following two sections, on the assumption that all the roots are distinct, we determine the b_i when (a) the roots are all real, and (b) the roots are both real and complex.

2.4.1 The Characteristic Equation With Real Roots Only

In equation [4.5], all coefficients of powers in ℓ are strictly positive since $\alpha(1+\lambda)\binom{\alpha}{j-1} > \binom{\alpha}{j}$ for any combination of α and j. Therefore all real roots of the characteristic equation are strictly negative.

Throughout this section, $-R_i$ denotes a root of the characteristic equation where $R_i > 0$.

We now solve

$$-(1+\lambda)\alpha\beta\xi'+\xi = \int_0^u \xi(u-x,K)v(x)dx + 1 - V(u)$$

by inserting the gamma distribution for $V(x)$ and [4.6] for $\xi(u,K)$

$$\xi(u,K) = b_o + \sum_{i=1}^{\alpha} b_i e^{-R_i u}$$

$$\Rightarrow -(1+\lambda)\alpha\beta\xi'+\xi = b_o + \sum_{i=1}^{\alpha} [1 + (1+\lambda)\alpha\beta R_i]b_i e^{-R_i u}$$

Now

$$\int_0^u \xi(u-x,K)v(x)dx =$$

$$= b_o V(u) + \sum_{i=1}^{\alpha} \frac{b_i e^{-R_i u}}{(1-\beta R_i)^{\alpha}} \int_0^u \frac{x^{\alpha-1}}{\Gamma(\alpha)} \left(\frac{1-\beta R_i}{\beta}\right)^{\alpha} e^{-\left(\frac{1}{\beta} - R_i\right)x} dx$$

Consider

$$\int_0^u \frac{x^{\alpha-1}}{\Gamma(\alpha)} \left(\frac{1 - \beta R_i}{\beta}\right)^{\alpha} e^{-\left(\frac{1}{\beta} - R_i\right)x} dx$$

If $1 - \beta R_i > 0$, it can be expressed as

$$1 - \sum_{j=0}^{\alpha-1} e^{-\left(\frac{1}{\beta} - R_i\right)u} \frac{[(1 - \beta R_i)u]^j}{\beta^j . j!} \qquad [4.7]$$

If $1 - \beta R_i < 0$, the expression becomes

$$1 - \sum_{j=0}^{\alpha-1} e^{(R_i - 1/\beta)u} \frac{[(\beta R_i - 1)u]^j}{\beta^j . j!} (-1)^j$$

which is equivalent to [4.7].

We note that $1 - \beta R_i = 0$ does not satisfy [4.5]. Thus

$$b_o + \sum_{i=1}^{\alpha} [1 + (1+\lambda)\alpha\beta R_i] b_i e^{-R_i u}$$

$$= b_o + (1-b_o) \sum_{j=0}^{\alpha-1} e^{-u/\beta} \frac{u^j}{\beta^j . j!} + \sum_{i=1}^{\alpha} \frac{b_i e^{-R_i u}}{(1-\beta R_i)^{\alpha}}$$

$$- \sum_{i=1}^{\alpha} \frac{b_i}{(1-\beta R_i)^{\alpha}} \sum_{j=0}^{\alpha-1} e^{-u/\beta} \frac{[(1 - \beta R_i)u]^j}{\beta^j . j!}$$

Equating the coefficients of $e^{-R_i u}$, for $i = 1, \ldots, \alpha$, gives

$$1 + (1+\lambda)\alpha\beta R_i = (1 - \beta R_i)^{-\alpha} \qquad [4.8]$$

leaving

$$(1-b_o)\sum_{j=0}^{\alpha-1} e^{-u/\beta}\frac{u^j}{\beta^j.j!} - \sum_{i=1}^{\alpha}\frac{b_i}{(1-\beta R_i)^{\alpha}}\sum_{j=0}^{\alpha-1} e^{-u/\beta}\frac{[(1-\beta R_i)u]^j}{\beta^j.j!} = 0$$

Equating coefficients of the powers in u, we have

$$b_o + \sum_{i=1}^{\alpha}\frac{b_i}{(1-\beta R_i)^j} = 1 \quad \text{for } j = 1, \ldots, \alpha \qquad [4.9]$$

The boundary condition [3.3] implies that

$$b_o = -\sum_{i=1}^{\alpha} b_i e^{-R_i K} \qquad [4.10]$$

The equations [4.9], together with [4.10], are sufficient to yield a solution for the b_i, and hence a complete solution for $\xi(u,K)$.

We remark that equations [4.8] do not imply α solutions for the insurers insolvency constant, R, which in this case is the unique positive root of

$$1 + \alpha\beta(1+\lambda)R = (1 - \beta R)^{-\alpha}$$

This is because the moment generating function for the gamma distribution is only defined if $1 - \beta R > 0$. Thus, the insolvency constant will be the R_i satisfying $1 - \beta R_i > 0$.

Further, we note that if R_i is replaced by $-R_i$ in [4.8], then the equations characterise [4.4] as follows:

$$1 - (1+\lambda)\alpha\beta R_i = (1 + \beta R_i)^{-\alpha} \quad \text{for } i = 1, \ldots, \alpha$$

$$\Rightarrow [1 - (1+\lambda)\alpha\beta R_i]\sum_{j=0}^{\alpha}\binom{\alpha}{j}(\beta R_i)^j = 1$$

$$\Rightarrow\ (1+\lambda)\alpha\beta R_i \sum_{j=0}^{\alpha} \binom{\alpha}{j} (\beta R_i)^j - \sum_{j=1}^{\alpha} \binom{\alpha}{j} (\beta R_i)^j = 0$$

$$\Rightarrow\ (1+\lambda)\alpha(\beta R_i)^{\alpha+1} + (1+\lambda)\alpha \sum_{j=1}^{\alpha} \binom{\alpha}{j-1} (\beta R_i)^j - \sum_{j=1}^{\alpha} \binom{\alpha}{j} (\beta R_i)^j = 0$$

$$\text{i.e. } (1+\lambda)\alpha\beta R_i^{\alpha+1} + \sum_{j=1}^{\alpha} \frac{1}{\beta^{\alpha-j}} \left[\alpha(1+\lambda)\binom{\alpha}{j-1} - \binom{\alpha}{j}\right] R_i^j = 0$$

This is [4.5], with ℓ replaced by R_i.

2.4.2 The Characteristic Equation With Real and Complex Roots

When there are complex roots of the characteristic equation, they exist in conjugate pairs. Therefore, to illustrate the solution for $\xi(u,K)$ when the roots of the characteristic equation are both real and complex, we assume that there are r real roots and 2c complex roots, such that $r + 2c = \alpha$.

In this case we may write

$$\xi(u,K) = b_o + \sum_{i=1}^{r} b_i e^{-R_i u} + \sum_{j=1}^{c} (d_{2j-1} e^{-iq_j u} + d_{2j} e^{iq_j u}) e^{-\rho_j u} \qquad [4.11]$$

$$= b_o + \sum_{i=1}^{r} b_i e^{-R_i u} + \sum_{j=1}^{c} [(d_{2j-1}+d_{2j})\cos q_j u - (d_{2j-1}-d_{2j}) i \sin q_j u] e^{-\rho_j u} \qquad [4.12]$$

where the complex roots of [4.5] are $-\rho_j \pm iq_j$, for $j = 1, \ldots, c$. Both sets of coefficients b_i and d_j depend on both λ and K. The b_i will all be real, the d_j may be real or complex.

Using [4.11] we have

$$-(1+\lambda)\alpha\beta\xi'+\xi = b_o + \sum_{i=1}^{r} (1 + (1+\lambda)\alpha\beta R_i)b_i e^{-R_i u}$$

$$+ \sum_{j=1}^{c} [(1 + (1+\lambda)\alpha\beta(\rho_j + iq_j)d_{2j-1}e^{-(\rho_j+iq_j)u}$$

$$+ (1 + (1+\lambda)\alpha\beta(\rho_j - iq_j)d_{2j}e^{-(\rho_j-iq_j)u}] \qquad [4.13]$$

Let $P(x,\alpha)$ denote the incomplete gamma function ratio defined by

$$P(x,\alpha) = \int_0^x \frac{z^{\alpha-1}e^{-z}}{\Gamma(\alpha)}\,dz$$

Then, if we insert [4.11] for $\xi(u,K)$ in the right hand side of [3.2] we find, after rearranging the terms, that

$$\int_0^u \xi(u-x,K)v(x)dx = b_o P(u/\beta,\alpha) + \sum_{i=1}^{r} \frac{b_i e^{-R_i u}}{(1-\beta R_i)^{\alpha}} P(\frac{1-\beta R_i}{\beta} u,\alpha)$$

$$+ \sum_{j=1}^{c} \frac{d_{2j-1}e^{-(\rho_j+iq_j)u}}{[1-\beta(\rho_j+iq_j)]^{\alpha}} P(\frac{1-\beta(\rho_j+iq_j)}{\beta} u,\alpha)$$

$$+ \sum_{j=1}^{c} \frac{d_{2j}e^{-(\rho_j-iq_j)u}}{[1-\beta(\rho_j-iq_j)]^{\alpha}} P(\frac{1-\beta(\rho_j-iq_j)}{\beta} u,\alpha) \qquad [4.14]$$

If we add $1-V(u)$ to [4.14] we can equate this with [4.13]. As before, we equate coefficients of exponential terms and then powers of u as

$$P(x,\alpha) = 1 - \sum_{j=0}^{\alpha-1} e^{-x}\frac{x^j}{j!}$$

since α is an integer.

Firstly, equating exponential terms yields

$$1 + (1+\lambda)\alpha\beta R_i = (1 - \beta R_i)^{-\alpha} \qquad \text{for } i = 1, \ldots\ldots, r$$

and

$$1 + (1+\lambda)\alpha\beta(\rho_j \pm iq_j) = (1 - \beta(\rho_j \pm iq_j))^{-\alpha} \quad \text{for } j = 1, \ldots, c$$

These equations have the same interpretation as [4.8]. The result of equating coefficients of powers in u is

$$1-b_o = \sum_{i=1}^{r} \frac{b_i}{(1-\beta R_i)^n} + \sum_{j=1}^{c} \frac{d_{2j-1}}{[(1-\beta\rho_j)-i\beta q_j]^n} + \sum_{j=1}^{c} \frac{d_{2j}}{[(1-\beta\rho_j)+i\beta q_j]^n} \qquad [4.15]$$

for $n = 1, \ldots, \alpha$.

We proceed by separating [4.15] into real and complex terms. Multiplication of the numerators and denominators of the final two terms by the complex conjugates of their respective denominators yields

$$1-b_o = \sum_{i=1}^{r} \frac{b_i}{(1-\beta R_i)^n} + \sum_{j=1}^{c} \frac{d_{2j-1}[(1-\beta\rho_j)+i\beta q_j]^n}{[(1-\beta\rho_j)^2+(\beta q_j)^2]^n}$$

$$+ \sum_{j=1}^{c} \frac{d_{2j}[(1-\beta\rho_j)-i\beta q_j]^n}{[(1-\beta\rho_j)^2+(\beta q_j)^2]^n} \qquad [4.16]$$

We now apply the following identities to [4.16]:

$$(x\pm iy)^n = \sum_{j=0}^{[n/2]} (-1)^j \binom{n}{2j} x^{n-2j} y^{2j} \pm i \sum_{j=1}^{\{n/2\}} (-1)^{j+1} \binom{n}{2j-1} x^{n-2j+1} y^{2j-1}$$

where $[n/2]$ is the largest integer not greater than $n/2$, and $\{n/2\}$ is the least integer not less than $n/2$. Hence

$$1-b_o = \sum_{i=1}^{r} \frac{b_i}{(1-\beta R_i)^n} + \sum_{j=1}^{c} \frac{d_{2j-1}+d_{2j}}{g^n} \sum_{t=0}^{[n/2]} \binom{n}{2t} (-1)^t (1-\beta\rho_j)^{n-2t} (\beta q_j)^{2t}$$

$$+ i \sum_{j=1}^{c} \frac{d_{2j-1}-d_{2j}}{g^n} \sum_{t=1}^{\{n/2\}} \binom{n}{2t-1} (-1)^{t+1} (1-\beta\rho_j)^{n-2t+1} (\beta q_j)^{2t-1} \qquad [4.17]$$

where $g = (1 - \beta\rho_j)^2 + (\beta q_j)^2$.

The solution for $\xi(u,K)$ must be real. Therefore, we require, by [4.12], that $d_{2j-1} + d_{2j}$ is real and that $d_{2j-1} - d_{2j}$ is imaginary or zero, for each j. Let

$$d_{2j-1} = \frac{1}{2}(\gamma_j + \delta_j i) \quad \text{and} \quad d_{2j} = \frac{1}{2}(\gamma_j - \delta_j i)$$

for $j = 1, \ldots\ldots, c$. Then

$$d_{2j-1} + d_{2j} = \gamma_j \quad \text{and} \quad d_{2j-1} - d_{2j} = \delta_j i$$

Thus

$$1-b_o = \sum_{i=1}^{r} \frac{b_i}{(1-\beta R_i)^n} + \sum_{j=1}^{c} \frac{\gamma_j}{g^n} \sum_{t=0}^{[n/2]} \binom{n}{2t} (-1)^t (1-\beta\rho_j)^{n-2t} (\beta q_j)^{2t}$$

$$- \sum_{j=1}^{c} \frac{\delta_j}{g^n} \sum_{t=1}^{\{n/2\}} \binom{n}{2t-1} (-1)^{t+1} (1-\beta\rho_j)^{n-2t+1} (\beta q_j)^{2t-1} \qquad [4.18]$$

Since this equation holds for $n = 1, \ldots\ldots, \alpha$, along with the boundary condition [3.3], we have sufficient equations to yield a solution of the form

$$\xi(u,K) = b_o + \sum_{i=1}^{r} b_i e^{-R_i u} + \sum_{j=1}^{c} e^{-\rho_j u}(\gamma_j \cos q_j u + \delta_j \sin q_j u) \qquad [4.19]$$

Although equation [4.18] is rather complicated, it is reasonably simple to obtain the solution for small values of α. Suppose $\alpha = 3$, with $r = 1$. Then

$$\xi(u,K) = b_o + b_1 e^{-Ru} + e^{-\rho u}(\gamma \cos qu + \delta \sin qu)$$

where

$$b_o = -b_1 e^{-RK} - e^{-\rho K}(\gamma \cos qK + \delta \sin qK)$$

and b_1, γ and δ are found from the following equations:

$$1-b_o = \frac{b_1}{(1-\beta R)^3} + \frac{\gamma}{g^3}\left[(1-\beta\rho)^3 - 3(1-\beta\rho)(\beta q)^2\right] - \frac{\delta}{g^3}\left[3(1-\beta\rho)^2\beta q - (\beta q)^3\right]$$

$$1-b_o = \frac{b_1}{(1-\beta R)^2} + \frac{\gamma}{g^2}\left[(1-\beta\rho)^2 - (\beta q)^2\right] - \frac{\delta}{g^2} 2(1-\beta\rho)\beta q$$

$$1-b_o = \frac{b_1}{1-\beta R} + \frac{\gamma}{g}(1 - \beta\rho) - \frac{\delta}{g}\beta q$$

with

$$g = (1 - \beta\rho)^2 + (\beta q)^2 \qquad [4.20]$$

The same approach to the problem is still valid if the roots of the characteristic equation are of a different type, e.g. double roots, but we shall not consider the solution here.

The above methods could also be used to find $\psi(u)$ as the solution of

$$-c\psi' + \psi = \int_0^u \psi(u-x)v(x)dx + 1 - V(u)$$

when the single claim amount is $\gamma(\alpha)$. In the case of exponential single claim amounts, the solution for $\xi(u,K)$ could have been obtained as a trivial consequence of [1.3] of Chapter 1 and [2.2]. This is not so for $\gamma(\alpha)$ claims. Although it is possible to obtain an explicit solution for $\psi(u)$ through its Laplace transform when $V(x)$ takes a $\gamma(\alpha)$ distribution, with integral valued α (see, for example, Seal [12]), the general

expression for the Laplace transform is

$$\frac{1}{S} - \frac{\lambda\alpha\beta(1+\beta s)^{\alpha}}{(1+\lambda)\alpha\beta s(1+\beta s)^{\alpha} - (1+\beta s)^{\alpha} + 1}$$

The denominator of this expression is [4.5], with s replacing ℓ, and leads to a solution by partial fractions for $\psi(u)$. However, solutions by this method seem best achieved for specific, rather than general, values of α.

Finally we note that the solution for $\xi(u,K)$ as $\lambda \to 0$ cannot be found from the results derived. The solution is obtained by letting $\lambda = 0$ in [4.4] and proceeding as before. Then, for $\alpha \geqslant 2$, we have

$$\alpha\beta\xi^{(\alpha+1)} + \sum_{j=2}^{\alpha} \frac{1}{\beta^{\alpha-j}} [\alpha\binom{\alpha}{j-1} - \binom{\alpha}{j}]\xi^{(j)} = 0 \qquad [4.21]$$

The general solution for $\xi(u,K)$ is now of the form

$$\xi(u,K) = b_o + b_1 u + \sum_{i=1}^{\alpha-1} b_i e^{-R_i u}$$

where $\{-R_i\}$ are the roots, assumed here to be distinct, of the characteristic equation of [4.21].

3.0 THE MARKOV CHAIN METHOD

3.1 Introduction

The methods of solution for $\xi(u,K)$ presented in Chapter 2 are only applicable to specific forms of $V(x)$. Therefore, we now turn our attention to finding numerical approximations to $\xi(u,K)$ which may be applied to a wider variety of single claim amount distributions.

In this and the following chapter, we present a method of approximating $\xi(u,K)$ which uses a limiting property of Markov chains. We form a partition of all possible values of the reserve level by dividing the interval $[0,K)$ into a fixed number of equal intervals and complete the partition with the intervals $(-\infty,0)$ and $[K,\infty)$. These intervals form the states of our Markov chain. If the reserve level is in the state $(-\infty,0)$, then ruin has occurred; if it is in the state $[K,\infty)$, then the upper barrier has been crossed. Hence we let these states be absorbing. By considering the possible changes in the reserve level over a fixed time interval, we can construct a matrix P of transition probabilities. We find approximations to the probabilities of absorption by the respective barriers by calculating $\lim_{n\to\infty} P^n$.

In Section 3.2 we indicate a method of construction of the transition matrix. The following section deals with the numerical procedure used to calculate the limiting matrix. The success of the approximation method is particularly dependent on two parameters: the number of states in the Markov chain and the length of time interval considered. In Section 3.4, it is shown how these parameters should be chosen in order to justify assumptions concerning the transition probabilities. We present a numerical illustration of the method in the final section, where exact

and approximate solutions are calculated for a $\gamma(\alpha)$ single claim amount distribution, with α taking the values 1, 2, 3 and 4.

3.2 The Transition Matrix

The states of the Markov chain are defined by forming a partition of all possible values that the risk reserve can take.

Define

$$E_j = [(j-2)w,(j-1)w), \quad j = 2, \ldots, N-1$$

$$E_1 = (-\infty,0) \quad \text{and} \quad E_N = [K,\infty)$$

where $w = K/(N-2)$.

Clearly Z_t can only be in one of these N intervals at any fixed time t. These intervals $\{E_j\}$ are the states of the Markov chain. States E_1 and E_N are absorbing. Entry into these states represents ruin and absorption by the upper barrier respectively. Every other state is transient. For an initial reserve level u, in one of the transient states, the approximation to $\xi(u,K)$ is found by calculating the probability of eventual absorption in E_1.

Define $\rho_{i,j}(\tau)$ to be the probability of a transition from E_i to E_j in a time interval of length τ. In order to calculate transition probabilities, we make the following two assumptions:

i if the initial reserve level u is in the transient state E_j, then $u = M_j$, where $M_j = (j-1.5)w$ is the midpoint of E_j; and

ii if a transition from E_i to E_j takes place, then the reserve level in E_j takes the value of the midpoint M_j.

The first assumption simply states that our approximations will be for initial reserve levels which are midpoints. It will be shown later that these can easily be converted into ruin probabilities for any given initial reserve level u. It is our second assumption which seems likely to lead to inaccuracies. We defer discussion of these inaccuracies until later sections when we have numerical results. It will then be easier to assess the effect of this assumption.

By consideration of the reserve level at the end of the time interval τ, we can clearly define transition probabilities in the following way.

For $i,j = 2, \ldots, N-1$

$$\rho_{i,j}(\tau) = \Pr\{c\tau + (i-j-.5)w < X_\tau \leqslant c\tau + (i-j+.5)w\}$$

$$= F(c\tau + (i-j+.5)w,\tau) - F(c\tau + (i-j-.5)w,\tau) \qquad [2.1]$$

For $i = 2, \ldots, N-1$

$$\rho_{i,1}(\tau) = \Pr\{c\tau + (i-1.5)w < X_\tau\}$$

$$= 1 - F(c\tau + (i-1.5)w,\tau)$$

$$\rho_{i,N}(\tau) = \Pr\{X_\tau \leqslant c\tau + (i-N+.5)w\}$$

$$= F(c\tau + (i-N+.5)w,\tau)$$

Since E_1 and E_N are absorbing, we have

$$\rho_{1,1}(\tau) = \rho_{N,N}(\tau) = 1$$

$$\rho_{1,i}(\tau) = 0 \quad , \quad i \neq 1$$

$$\rho_{N,i}(\tau) = 0 \quad , \quad i \neq N$$

Thus the transition matrix is clearly obtainable whenever exact numerical values of F(x,t) can be calculated.

However, we note that this matrix does not give an exact account of what happens in the continuous model, as we have based transition probabilities on the reserve level after a time τ. During this time interval, the process may have temporarily crossed either, or both, of the absorbing barriers prior to finishing in a transient state at τ. Thus, sample paths of this type, which would be absorbed by the barriers in the continuous model, are not being absorbed in the Markov chain model. Hence certain transition probabilities will be inexact, particularly those to the absorbing states. Nevertheless, it is worth starting our investigation of this approximation method by using this inexact transition matrix since the transition probabilities are so clearly defined in terms of F(x,t). Where possible, we shall ensure that the number of transition probabilities which do not exactly mirror the continuous model is kept to a minimum.

3.3 Calculation of π

A class of states C of a Markov chain is said to be closed if states outside C cannot be reached from any state inside C. In particular, a single state forming a closed class is said to be absorbing. If the probability that a system which starts in a class C will ever pass through C in the future is one, then the class is said to be recurrent.

For a transition matrix P, let $\pi = \lim_{n\to\infty} P^n$. We state here the general method for the calculation of π; a proof is included in Appendix 1.

Theorem

For a finite aperiodic Markov chain

$$\pi = \sum_{c_j} \underline{y}^{c_j} \underline{x}^{c_j}$$

where the sum is taken over all recurrent classes $\{c_j\}$, and

i $\underline{x}^{c_j}$ is the unique stationary distribution over class c_j, i.e.

$$\underline{x}^{c_j} P = \underline{x}^{c_j}$$

where

$$x_i^{c_j} \geqslant 0 \quad \text{if} \quad i \varepsilon c_j$$

$$= 0 \quad \text{if} \quad i \not\varepsilon c_j$$

such that

$$\sum_{i \varepsilon c_j} x_i^{c_j} = 1.$$

ii $\underline{y}^{c_j}$ is the solution of $P\underline{y}^{c_j} = \underline{y}^{c_j}$, with

$$y_i^{c_j} = 1 \quad \text{if} \quad i \varepsilon c_j$$

$$= 0 \quad \text{if} \quad i \varepsilon c_K$$

where c_K is any other recurrent class.

In our situation, there are two recurrent classes. These are the two closed classes comprising the single states E_1 and E_N. Let us denote these by $c_1 = \{E_1\}$ and $c_2 = \{E_N\}$. Then, by the above theorem

$$\pi = \underline{y}^{c_1}\underline{x}^{c_1} + \underline{y}^{c_2}\underline{x}^{c_2}$$

Clearly

$$\underline{x}^{c_1} = (1, 0, 0, \ldots, 0)$$

and

$$\underline{x}^{c_2} = (0, 0, 0, \ldots, 1)$$

By the nature of $\underline{x}^{c_1}$ and $\underline{x}^{c_2}$, we see that the matrix π will only have two non-zero columns, the first and the N^{th}. Since the first column of π gives the ruin probabilities from the states $\{E_j\}$, it is sufficient to solve $P\underline{y}^{c_1} = \underline{y}^{c_1}$ to obtain these probabilities.

We note that $y_1^{c_1} = 1$ and $y_N^{c_1} = 0$, so that we must solve

$$\begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ \rho_{2,1} & \rho_{2,2} & \rho_{2,3} & \ldots & \rho_{2,N} \\ \rho_{3,1} & \rho_{3,2} & \rho_{3,3} & \ldots & \rho_{3,N} \\ \vdots & \vdots & \vdots & & \vdots \\ \rho_{N-1,1} & \rho_{N-1,2} & \rho_{N-1,3} & \ldots & \rho_{N-1,N} \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix} \begin{bmatrix} 1 \\ y_1 \\ y_2 \\ \vdots \\ y_{N-2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ y_1 \\ y_2 \\ \vdots \\ y_{N-2} \\ 0 \end{bmatrix}$$

where $\rho_{i,j} = \rho_{i,j}(\tau)$.

We can rearrange this equation as

$$\begin{bmatrix} (\rho_{2,2}{}^{-1}) & \rho_{2,3} \; . \; . & \rho_{2,N-1} \\ \rho_{3,2} & (\rho_{3,3}{}^{-1}) \; . & \rho_{3,N-1} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \rho_{N-1,2} & \rho_{N-1,3} \; . \; . & (\rho_{N-1,N-1}{}^{-1}) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_{N-2} \end{bmatrix} = \begin{bmatrix} -\rho_{2,1} \\ -\rho_{3,1} \\ \cdot \\ \cdot \\ \cdot \\ -\rho_{N-1,1} \end{bmatrix}$$

If we define this second equation as $\underline{Q}\underline{y} = \underline{s}$ where

$$q_{i,j} = \rho_{i+1,j+1}, \quad i \neq j \qquad i,j = 1, \ldots, N-2$$

$$\left.\begin{aligned} q_{i,i} &= \rho_{i+1,i+1}{}^{-1} \\ y_i &= y_{i+1}{}^{c_1} \\ s_i &= -\rho_{i+1,1} \end{aligned}\right\} \quad i = 1, \ldots, N-2$$

then the equations are easily solved by computer using the Numerical Algorithms Group (NAG) routine FØ4ATF.

3.4 Choice of Parameters

In choosing the parameters N and τ for the Markov chain model, there are two factors which should be considered. Firstly, as indicated in Section 3.2, we should try to keep the number of inexact transition probabilities to a minimum if possible. Secondly, we must consider what effect these parameters have on our assumption that the reserve level in any transient state E_i becomes M_i if a transition to that state occurs.

It is obvious that the premium income over the time interval τ must exceed one half of the width of a state. Otherwise, transitions of the form

$E_i \rightarrow E_j$, where $j > i$, would be impossible and, hence, absorption by the lower barrier would be certain. Suppose $Z_o = M_i$ and no claims occur by time τ. Then $Z_\tau = M_i + c\tau$. If $c\tau$ is a multiple of the state width, then Z_τ will be at the midpoint of a state higher than E_i. Thus, if we set $c\tau$ as a multiple of the state width, then with probability $F(0,\tau)$ Z_τ will be at a midpoint. As the value of $F(0,\tau)$ increases, there will be less error due to our assumption. For the generalised Poisson distribution, $F(0,\tau) = e^{-\tau}$. Clearly this value increases as τ decreases. If $c\tau = rw$, where r is an integer, then the smallest value of τ, in terms of the other parameters, is achieved when $r = 1$. Thus we let the premium income in $(0,\tau)$ be equal to w, i.e.

$$c\tau = w \qquad [4.1]$$

If we let the probability of hitting M_{i+1} from M_i take a large value then there should only be a little inaccuracy caused by letting Z_τ take the value of the midpoints of the other transient states.

Therefore, to determine values of N and τ, we set a minimum level ε, such that

$$\rho_{i,i+1}(\tau) \geqslant F(0,\tau) \geqslant \varepsilon$$

This ensures that there is a probability of at least ε that the reserve level will hit a midpoint at τ. Naturally the expected reserve level in E_{i+1} will be less than the midpoint, but will tend to M_{i+1} as $\rho_{i,i+1}(\tau) \rightarrow F(0,\tau)$. We have

$$F(0,\tau) = e^{-\tau}$$

Setting $e^{-\tau} \geqslant \varepsilon$ gives $\tau \leqslant \ln 1/\varepsilon$. Inserting this in [4.1] yields

$$N - 2 \geqslant \frac{K}{c \ln 1/\varepsilon} \qquad [4.2]$$

Clearly N must take an integral value. Thus the minimum value for N should be the least integer not less than $\frac{K}{c \ln 1/\varepsilon} + 2$. In our numerical examples, we shall always give N an odd value in order to give the model a sense of symmetry. One advantage of using an odd number of states is that we can always approximate $\xi(.5K,K)$ since .5K is always a midpoint when N is odd. This is useful when comparing results using different parameters. Once a value of N is chosen, τ can be established by [4.1].

N may be chosen to take a much larger value than the minimum required by [4.2]. This would lead to smaller values of τ and hence larger values of $F(0,\tau)$. While this would seem desirable theoretically, there are practical restrictions on the size of N. In particular, there are limits to the size of matrix a computer can hold. Since our method of calculating π requires a computer to hold two matrices, this restricts further the value of N.

By letting $c\tau = w$ we have established a method of evaluating N and τ. Returning to our first consideration, we find that, by making this choice, we have also minimised the number of inexact elements of the transition matrix. Since $c\tau = w$, the only possible transitions to higher states are of the form $E_i \rightarrow E_{i+1}$, for each transient state. Thus, E_N may be reached from only one state, namely E_{N-1}. All transition probabilities from E_{N-1} are inexact. In particular, the probability of absorption by the upper barrier, i.e. $\rho_{N-1,N}(\tau)$, will be too small. Nevertheless, this

is clearly the smallest number of inexact transition probabilities with regard to the upper barrier.

A similar situation exists at the lower barrier. Let the initial reserve be M_i. Suppose Z_τ is positive, but for some value of s, where $0 < s < \tau$, Z_s is negative. Any sample path of this type which crosses the lower barrier prior to τ, could only be in E_2 at τ, as the premium income in $(0,\tau)$ is now insufficient for Z_τ to exceed w. Sample paths which cross the lower barrier in $(0,\tau)$ do not affect transition probabilities to E_3 or higher states. Thus it is only our transition probabilities $\rho_{i,1}(\tau)$ and $\rho_{i,2}(\tau)$ which are inexact. Clearly this is the minimum number of inexact transition probabilities at the lower barrier.

We have reduced the number of inexact elements of our NxN transition matrix to 3N-6. This is the minimum possible number of inexact transition probabilities. The results obtained using the above combination of N and τ can, in a sense, be regarded as the best attainable for this form of P.

3.5 A Numerical Illustration

In this section we compare approximations by the Markov chain method with exact results for $\xi(u,K)$ when the single claim amount distribution is $\gamma(\alpha)$. In this case it is easy to calculate F(x,t) exactly as the n^{th} convolution of a $\gamma(\alpha)$ distribution with itself is a $\gamma(n\alpha)$ distribution. Since we are only dealing with values of α which are integers, we can easily calculate $V^{n*}(x)$ to any desired accuracy as

$$V^{n*}(x) = 1 - \sum_{j=0}^{n\alpha-1} e^{-x/\beta} \frac{x^j}{\beta^j . j!} \qquad \text{for } n \geqslant 1$$

where β is the scale parameter of V(x). In the following example F(x,t) is calculated accurately to ten decimal places.

The selected values of α are 1, 2, 3 and 4, with $\beta = 1/\alpha$, so that in each case the mean claim amount is 1. In calculating the exact results for $\xi(u,K)$ for $\alpha \geqslant 2$, the roots of the characteristic equation, [4.5] of Chapter 2, are found using the NAG routine CØ2AEF. The simultaneous equations to find the coefficients b_i, γ_j and δ_j are solved by the NAG routine FØ4ATF. In this illustration, we choose a loading of $\lambda = .1$, giving a premium income per unit time of 1.1, and we set the upper barrier at K = 5.

For the Markov chain model, the value of ε is .95. This means that with a probability of at least .95 the process is at a midpoint following a transition from any transient state Ej, for j = 2, ... , N-2. Now

$$\varepsilon = .95 \Rightarrow \ln 1/\varepsilon = .051293$$

By [4.2], we find $N \geqslant 90.617$. Taking N to be the minimum value 91 yields

$$\tau = .051073 \quad \text{with} \quad F(0,\tau) = .950210$$

So $F(0,\tau)$ is very close to ε in this case.

The results are presented in Table 3.1 for selected starting states E_j.

The results follow the same pattern for each value of α. If the initial reserve level is low, the method is producing approximations which are too small; if the initial reserve level is close to K, the approximations are too large. It is only when the initial reserve level

α	1		2		3		4	
State	Exact	Approx	Exact	Approx	Exact	Approx	Exact	Approx
4	.75780	.75400	.79445	.79072	.80796	.80432	.81496	.81140
11	.68327	.68018	.71890	.71524	.73247	.72849	.73976	.73559
16	.63165	.62905	.66319	.65993	.67445	.67080	.68017	.67626
22	.57142	.56938	.59727	.59460	.60538	.60235	.60900	.60575
28	.51300	.51152	.53334	.53130	.53870	.53636	.54070	.53818
34	.45635	.45541	.47175	.47033	.47491	.47324	.47572	.47391
40	.40140	.40099	.41256	.41174	.41401	.41298	.41393	.41278
46	.34812	.34821	.35573	.35549	.35589	.35547	.35516	.35465
52	.29644	.29703	.30118	.30150	.30042	.30058	.29926	.29934
58	.24632	.24739	.24884	.24970	.24748	.24820	.24609	.24673
64	.19772	.19925	.19862	.19998	.19696	.19821	.19551	.19669
70	.15058	.15257	.15043	.15228	.14874	.15049	.14739	.14908
76	.10486	.10729	.10418	.10650	.10273	.10496	.10162	.10379
81	.06782	.07061	.06708	.06977	.06599	.06860	.06518	.06775
88	.01753	.02081	.01723	.02043	.01689	.02002	.01666	.01973

TABLE 3.1

is around .5K that there is agreement to about three decimal places between the exact and approximate results.

We can identify the inexact transition probabilities as the main reason for the difference between the two sets of results. Transition probabilities to the two absorbing states are less than the exact absorption probabilities because we have based our transition matrix only on the reserve level at τ. Hence we have not included in these absorption probabilities all realisations of the process which have crossed a barrier prior to τ, but take a value in $[0,K)$ at τ. Hence, at each transition, we are reducing the true probability of absorption by either barrier. We can easily see how these inexact transition probabilities affect our approximations.

For lower values of u, the approximations are too small because the transition probabilities to E_1 are less than the exact ruin probabilities over $(0,\tau)$. Thus we are increasing the true probability of non-ruin at each transition, producing approximations to $\xi(u,K)$ which are less than the exact ruin probabilities. Clearly, as the value of u decreases this error is more significant as the true probability of non-ruin over $(0,\tau)$ decreases with u. Similarly, when u takes larger values, approximations to $\xi(u,K)$ are too big, corresponding to approximations to $\chi(u,K)$ which are too small. This is because our probability of absorption in E_N from E_{N-1} is smaller than the true absorption probability. Again this has more effect the closer u becomes to K, as the probability of reaching E_{N-1} is increasing for these values. When the initial reserve level moves further from the barriers, these two errors have less effect. Indeed, when u is around halfway between the two barriers the errors appear to compensate for each other.

On the whole, these approximations are reasonably good. The percentage error of the approximations is less than 1% in most cases. It is only for larger values of u that this error increases to around 5%. At this stage, it is impossible to assess the effect of centring the process on the midpoints of the transient states after a transition. This problem will be discussed in the following chapter, where we consider a modified transition matrix which reflects exactly the continuous process over $(0,\tau)$.

4.0 AN EXACT TRANSITION MATRIX

4.1 Introduction

It is clear that the approximations of the previous chapter could be improved by using a transition matrix which reflects exactly the continuous process over $(0,\tau)$. In Section 4.2 we consider how this can be achieved and indicate how transition probabilities should be modified. Using the modified transition matrix, we repeat the numerical example of the previous chapter in Section 4.3. The results help indicate the effect of centring the process after each transition and suggest to which forms of $F(x,t)$ the method is applicable. It is shown in Section 4.4 that the method can easily be applied to any initial reserve level u, and not just values which are midpoints of the states. Finally, we consider the effects of varying two important parameters in Section 4.5.

4.2 Exact Transition Probabilities

We start by considering transition probabilities to E_1. Let the reserve level at the beginning of the time interval be M_j, where $2 \leqslant j \leqslant N-2$. Then the process cannot be absorbed by the upper barrier in $(0,\tau)$ because we are still considering the situation when $c\tau = w$. Hence the probability of absorption in E_1 over $(0,\tau)$ is simply $\psi(M_j,\tau)$, the probability of ruin by time τ from initial reserve level M_j. In the numerical examples, we calculate this as the complement of Seal's formula for $\delta(M_j,\tau)$, i.e.

$$\rho_{j,1}(\tau) = 1 - F(M_j+c\tau,\tau) + c\int_0^\tau \delta(0,t-s)f(M_j+cs,s)ds \qquad [2.1]$$

In the above equation, we have rewritten [1.4] of Chapter 1 in terms of $f(x,t)$, the density function of $F(x,t)$. As before, $\delta(0,\cdot)$ is the probability of non-ruin over a finite time interval from an initial reserve of zero.

The only other transition probabilities from E_j to be altered are those to E_2. Since the transition probabilities to states E_3 and higher are unaffected if the process crosses the lower barrier during $(0,\tau)$, we still have that

$$\sum_{\ell=3}^{j+1} \rho_{j,\ell}(\tau) = F(M_j,\tau) \quad \text{for} \quad j = 2, \ldots, N-2$$

Then $\rho_{j,2}(\tau)$ is clearly defined as

$$\rho_{j,2}(\tau) = 1 - \psi(M_j,\tau) - F(M_j,\tau) \quad \text{for} \quad j = 2, \ldots, N-2$$

since the sum of the elements of each row j must add to one. Thus, as before, we have

$$\rho_{j,1}(\tau) + \rho_{j,2}(\tau) = 1 - F(M_j,\tau) = \Pr[X_\tau > M_j]$$

i.e. the sum of the probabilities of transition to E_1 and E_2 is simply the probability that the total claim amount in $(0,\tau)$ exceeds M_j.

To complete the matrix of exact transition probabilities we must now modify transition probabilities from E_{N-1}. This can be done by considering the following more general situation.

Let u be the initial reserve level and K be the absorbing upper barrier. We let the process continue if the reserve level becomes negative, i.e. we are removing the lower barrier. As before, t_o is the first time at

which the reserve level can reach K. Using a result due to Keilson [8], it can be shown that the density function of the first passage time to K from u is given by

$$g(u,t) = \frac{K-u}{ct}\left[\delta(t-t_o)F(0,t_o) + cf(u+ct-K,t)\right] \quad , \; t \geqslant t_o$$

$$0 \text{ otherwise} \qquad [2.2]$$

where $\delta(\cdot)$ is the Dirac Delta function.

Let us now reintroduce the absorbing lower barrier. Then the density function of the first passage time to K is still given by g(u,t) provided ct < K, i.e. provided it is not possible for the reserve level to be negative prior to reaching K. Hence, the probability of absorption by the upper barrier in (0,t), where t < K/c is given by

$$\int_{t_o}^{t} g(u,s)ds = F(0,t_o) + \int_{t_o}^{t} \frac{K-u}{s} f(u+cs-K,s)ds \qquad [2.3]$$

It does not appear possible to extend this to the general situation when ct > K, since it is then possible for the process to hit the lower barrier first. Nevertheless, we can use [2.2] and [2.3] as a basis for calculating transition probabilities from E_{N-1}.

As $\rho_{N-1,N}(\tau)$ is simply the probability of absorption from M_{N-1} in $(0,\tau)$ we can immediately write

$$\rho_{N-1,N}(\tau) = e^{-t_o} + \int_{t_o}^{\tau} \frac{K-M_{N-1}}{s} f(M_{N-1}+cs-K,s)ds \qquad [2.4]$$

For the Markov chain model it is easily seen that $t_o = .5\tau$ since

$$K - M_{N-1} = .5w \qquad \text{and} \qquad M_{N-1} + ct_o = K$$

$\Rightarrow \quad ct_o = .5w$

Clearly $t_o = .5\tau$ as $\tau = w/c$.

To obtain $\rho_{N-1,1}(\tau)$ we consider the process over the interval $(0,\tau)$, allowing it to continue if it crosses either barrier. If the process has crossed the lower barrier by τ, this can only have arisen in one of the following two ways. Either

i the reserve level has never reached K in $(0,\tau)$ and for some value of s, where $0 < s \leqslant \tau$, Z_s is negative; or

ii the reserve level reaches K for the first time at s, where $t_o \leqslant s < \tau$ and then crosses the lower barrier in the subsequent time interval $\tau-s$.

The probability of (i) is the required probability $\rho_{N-1,1}(\tau)$. The probability of (ii) is obtained by multiplying together the probabilities of the following events, and integrating over the range t_o to τ:

a. the reserve level reaches K for the first time at s, where $t_o \leqslant s < \tau$; and

b. ruin occurs from the reserve level K in the remaining time interval $\tau-s$.

The probability of (a) is defined by [2.2], the probability of (b) is simply $\psi(K,\tau-s)$. Since the sum of the probabilities i and ii is $\psi(M_{N-1},\tau)$, we have that

$$\rho_{N-1,1}(\tau) = \psi(M_{N-1},\tau) - e^{-t_o}\psi(K,t_o) - \int_{t_o}^{\tau} \frac{K-M_{N-1}}{s} f(M_{N-1}+cs-K,s)\psi(K,t-s)ds \qquad [2.5]$$

Finally, we must evaluate the probabilities of transition to the transient states from E_{N-1}. Consider transitions from E_{N-1} to E_j, for $j = 3, \ldots, N-1$.

The only barrier which the process could cross in $(0,\tau)$ prior to finishing in E_j at τ, is the upper one. In the previous chapter, we defined $\rho_{N-1,j}(\tau)$ as

$$\rho_{N-1,j}(\tau) = F((N-j+1/2)w,\tau) - F((N-j-1/2)w,\tau) \qquad [2.6]$$

To obtain exact transition probabilities we must exclude from this the probability that the process hits K for the first time at s, where $t_o \leqslant s < \tau$, and in the remaining time interval τ-s, the total claim amount lies between $K + c(\tau-s) - (j-2)w$ and $K + c(\tau-s) - (j-1)w$. Thus we must substract the following amount from [2.6] to find the exact transition probabilities:

$$e^{-t_o}[F(K + ct_o - (j-2)w,t_o) - F(K + ct_o - (j-1)w,t_o)]$$

$$+ \int_{t_o}^{t} \frac{K-M_{N-1}}{s} f(M_{N-1}+cs-K,s)[F(K + c(\tau-s) - (j-2)w,\tau-s)$$

$$- F(K + c(\tau-s) - (j-1)w,\tau-s)]ds \qquad [2.7]$$

We calculate $\rho_{N-1,2}(\tau)$ as $1 - \sum_{j\neq 2} \rho_{N-1,j}(\tau)$ as direct calculation is not possible. Again the choice of $c\tau = w$ appears to be sensible. If $c\tau$ had been chosen as a multiple of w, say $c\tau = rw$ where r is a positive integer, then it would not have been possible to exactly identify every transition probability. For any initial reserve level M_j, when $c\tau = w$, the reserve level at τ is in either E_1 or E_2 if the reserves have been negative prior to τ. But, if $c\tau = rw$, then the reserve level could be in any one of the transient states $E_2, E_3, \ldots, E_{r+1}$ at τ if the reserves have been negative prior to τ. Just as we have been unable to directly calculate $\rho_{j,2}(\tau)$ for all j, we would now be unable to calculate $\rho_{j,i}(\tau)$

for $i = 2, \ldots, r+1$ if $c\tau = rw$.

4.3 A Numerical Illustration

To evaluate the exact transition probabilities, we now require numerical values for the functions $f(x,t)$ and $\delta(0,t)$ which we introduced in the previous section. We shall again use the gamma distribution for $V(x)$ so that

$$F(x,t) = e^{-t} + \sum_{n\geqslant 1} e^{-t}\frac{t^n}{n!}\int_0^x \frac{y^{n\alpha-1}e^{-y/\beta}}{\Gamma(n\alpha)\beta^{n\alpha}}\,dy \qquad [3.1]$$

which in the case under consideration, i.e. integral values of α, can be expressed as

$$F(x,t) = e^{-t} + \sum_{n\geqslant 1} e^{-t}\frac{t^n}{n!}\left[1 - \sum_{j=0}^{n\alpha-1} e^{-x/\beta}\frac{x^j}{\beta^j.j!}\right] \qquad [3.2]$$

Now $f(x,t)$ is obtained as the derivative with respect to x of $F(x,t)$ as

$$f(x,t) = \sum_{n\geqslant 1} e^{-t}\frac{t^n}{n!}\frac{x^{n\alpha-1}e^{-x/\beta}}{\Gamma(n\alpha)\beta^{n\alpha}} \quad \text{for} \quad x > 0 \qquad [3.3]$$

Clearly there is no difficulty in evaluating $f(x,t)$ to any desired accuracy for this form of $V(x)$. In this case it is also possible to represent $\delta(0,t)$ as an infinite series. As defined by Seal in [12], we have

$$\begin{aligned}
\delta(0,t) &= \frac{1}{ct}\int_0^{ct} F(x,t)\,dx \\
&= \frac{1}{ct}\int_0^{ct}\left(e^{-t} + \sum_{n\geqslant 1} e^{-t}\frac{t^n}{n!}\left[1 - \sum_{j=0}^{n\alpha-1} e^{-x/\beta}\frac{x^j}{\beta^j.j!}\right]\right)dx \\
&= \frac{1}{ct}\int_0^{ct}\sum_{n\geqslant 0} e^{-t}\frac{t^n}{n!}\,dx - \frac{e^{-t}}{ct}\sum_{n\geqslant 1}\frac{t^n}{n!}\sum_{j=0}^{n\alpha-1}\int_0^{ct}\frac{x^j e^{-x/\beta}}{\beta^j.j!}\,dx
\end{aligned}$$

$$= 1 - \frac{\beta e^{-t}}{ct} \sum_{n \geqslant 1} \frac{t^n}{n!} \sum_{j=1}^{n\alpha} P(ct/\beta, j) \qquad [3.4]$$

Thus there is no problem in accurately evaluating $\delta(0,t)$.

In calculating the matrix of exact transition probabilities, the following procedure was adopted. As in the last example, $F(x,t)$ was evaluated by [3.2] to 10 decimal places. Using [2.1] to find the transition probabilities to E_1, the integral was found numerically using the trapezoidal method with 128 panels. Both terms of the integrand were calculated to 10 decimal places by [3.3] and [3.4] respectively. The transition probabilities from E_{N-1} were calculated in a similar fashion. Using [2.4], $\rho_{N-1,N}(\tau)$ was obtained by the above numerical integration method. Evaluating $\rho_{N-1,1}(\tau)$ by [2.5] required that $\psi(K,\cdot)$ be found by the same procedure as used to evaluate $\psi(M_j,\tau)$ by [2.1] and these values were used to obtain the integral by the trapezoidal method, again with 128 panels. The adjustments to the transition probabilities from E_{N-1} to the transient states, defined by [2.7], were found by the same numerical integration method, with $F(x,t)$ and $f(x,t)$ being found as before.

We can now repeat the approximation of the previous chapter, this time using the matrix of exact transition probabilities. Keeping the parameters as before, $K = 5$, $\lambda = .1$ and $\varepsilon = .95$, with $\gamma(\alpha)$ single claim amounts, the new approximations are shown in Table 4.1 for $\alpha = 1, 2, 3$ and 4.

It is immediately obvious that the new approximations are vastly superior to those of Table 3.1. However, while the results of that table followed a similar pattern for each value of α, it is interesting to note that this is not the case with the new approximations. The new results fall into

α	1		2		3		4	
State	Exact	Approx	Exact	Approx	Exact	Approx	Exact	Approx
4	.75780	.75774	.79445	.79445	.80796	.80796	.81496	.81495
11	.68327	.68321	.71890	.71890	.73247	.73247	.73976	.73976
16	.63165	.63158	.66319	.66320	.67445	.67445	.68017	.68017
22	.57142	.57134	.59727	.59727	.60538	.60538	.60900	.60900
28	.51300	.51293	.53334	.53334	.53870	.53871	.54070	.54070
34	.45635	.45627	.47175	.47175	.47491	.47492	.47572	.47572
40	.40140	.40133	.41256	.41256	.41401	.41402	.41393	.41394
46	.34812	.34805	.35573	.35573	.35589	.35590	.35516	.35517
52	.29644	.29638	.30118	.30119	.30042	.30043	.29926	.29927
58	.24632	.24627	.24884	.24885	.24748	.24749	.24609	.24610
64	.19772	.19767	.19862	.19863	.19696	.19697	.19551	.19551
70	.15058	.15054	.15043	.15043	.14874	.14875	.14739	.14739
76	.10486	.10484	.10418	.10419	.10273	.10273	.10162	.10162
81	.06782	.06781	.06708	.06708	.06599	.06599	.06518	.06518
88	.01753	.01753	.01723	.01723	.01689	.01690	.01666	.01666

TABLE 4.1

two different categories when (a) $\alpha = 1$, and (b) $\alpha > 1$.

In the former case, we note that all approximations are less than the exact values. However, in the latter case, all bar one of the tabulated approximations exceed the exact values. Indeed, around half of the tabulated approximations agree with the exact values to 5 decimal places when $\alpha > 1$.

To help assess why there is a discrepancy between the two cases, let us study Tables 4.2 to 4.5. Each table shows what is happening in the model for a single transition from the starting state E_{46}, for each of the four values of α. In each table, the columns are as follows:

i the states E_j, for $j = 2, \ldots, 47$;

ii the transition probabilities $\rho_{46,j}(\tau)$;

iii the expected reserve level in E_j if a transition from E_{46} to E_j occurs. This is calculated as

$$\frac{1}{F(K_1,\tau)-F(K_2,\tau)} \int_{K_2}^{K_1} xf(x,\tau)dx$$

where $K_1 = M_{46} + c\tau - (j-2)w = M_{46} - (j-3)w$

and $K_2 = M_{46} + c\tau - (j-1)w = M_{46} - (j-2)w$

In this particular case, we can show that the above equals

$$\frac{\alpha\beta}{F(K_1,\tau)-F(K_2,\tau)} \sum_{n\geq 1} e^{-\tau} \frac{\tau^n}{(n-1)!} [P(K_1/\beta,n\alpha+1) - P(K_2/\beta,n\alpha+1)]$$

we can only calculate this for $j = 3, \ldots, 47$. It is not possible to find the expected reserve level in E_2. We denote this conditional expected value by $E(Z_\tau)$ in Tables 4.2 to 4.5;

iv the midpoints M_j - for comparison with (iii). This comparison should

E_j	$\rho_{46,j}(\tau)$	$E(Z_\tau)$	M_j
2	.00012	-	.02809
3	.00025	.08450	.08427
4	.00026	.14072	.14045
5	.00027	.19686	.19663
6	.00029	.25305	.25281
7	.00031	.30925	.30899
8	.00032	.36541	.36517
9	.00034	.42162	.42135
10	.00036	.47778	.47753
11	.00038	.53395	.53371
12	.00040	.59014	.58989
13	.00042	.64631	.64607
14	.00045	.70249	.70225
15	.00047	.75870	.75843
16	.00050	.81487	.81461
17	.00053	.87104	.87079
18	.00056	.92723	.92697
19	.00059	.98341	.98315
20	.00062	1.03958	1.03933
21	.00066	1.09576	1.09551
22	.00069	1.15194	1.15169
23	.00073	1.20812	1.20787
24	.00077	1.26430	1.26404
25	.00082	1.32048	1.32022
26	.00086	1.37666	1.37640
27	.00091	1.43284	1.43258
28	.00096	1.48902	1.48876
29	.00102	1.54520	1.54494
30	.00108	1.60138	1.60112
31	.00114	1.65756	1.65730
32	.00120	1.71374	1.71348
33	.00127	1.76992	1.76966
34	.00134	1.82610	1.82584
35	.00141	1.88228	1.88202
36	.00149	1.93846	1.93820
37	.00158	1.99464	1.99438
38	.00167	2.05082	2.05056
39	.00176	2.10700	2.10674
40	.00186	2.16318	2.16292
41	.00196	2.21936	2.21910
42	.00207	2.27554	2.27528
43	.00219	2.33172	2.33146
44	.00231	2.38790	2.38764
45	.00244	2.44408	2.44382
46	.00258	2.50026	2.50000
47	.95155	2.55616	2.55618

TABLE 4.2

$\alpha = 1$

E_j	$\rho_{46,j}(\tau)$	$E(Z_\tau)$	M_j
2	.00010	-	.02809
3	.00021	.08467	.08427
4	.00023	.14083	.14045
5	.00025	.19701	.19663
6	.00027	.25320	.25281
7	.00030	.30941	.30899
8	.00032	.36555	.36517
9	.00035	.42174	.42135
10	.00038	.47791	.47753
11	.00041	.53409	.53371
12	.00045	.59027	.58989
13	.00049	.64643	.64607
14	.00053	.70261	.70225
15	.00057	.75879	.75843
16	.00061	.81496	.81461
17	.00066	.87115	.87079
18	.00071	.92731	.92697
19	.00077	.98350	.98315
20	.00083	1.03967	1.03933
21	.00089	1.09584	1.09551
22	.00095	1.15202	1.15169
23	.00102	1.20819	1.20787
24	.00109	1.26436	1.26404
25	.00117	1.32052	1.32022
26	.00124	1.37670	1.37640
27	.00132	1.43287	1.43258
28	.00140	1.48903	1.48876
29	.00148	1.54520	1.54494
30	.00157	1.60137	1.60112
31	.00165	1.65753	1.65730
32	.00172	1.71369	1.71348
33	.00182	1.76985	1.76966
34	.00186	1.82600	1.82584
35	.00192	1.88215	1.88202
36	.00197	1.93830	1.93820
37	.00200	1.99443	1.99438
38	.00201	2.05056	2.05056
39	.00200	2.10668	2.10674
40	.00196	2.16277	2.16292
41	.00187	2.21884	2.21910
42	.00175	2.27487	2.27528
43	.00156	2.33081	2.33146
44	.00131	2.38660	2.38746
45	.00097	2.44199	2.44382
46	.00054	2.49580	2.50000
47	.95028	2.55618	2.55618

TABLE 4.3

$\alpha = 2$

E_j	$\rho_{46,j}(\tau)$	$E(Z_\tau)$	M_j
2	.00008	-	.02809
3	.00016	.08484	.08427
4	.00018	.14097	.14045
5	.00020	.19718	.19663
6	.00022	.25334	.25281
7	.00025	.30948	.30899
8	.00028	.36567	.36517
9	.00031	.42186	.42135
10	.00034	.47802	.47753
11	.00038	.53419	.53371
12	.00042	.59037	.58989
13	.00047	.64654	.64607
14	.00052	.70274	.70225
15	.00058	.75891	.75843
16	.00064	.81507	.81461
17	.00070	.87124	.87079
18	.00077	.92741	.92697
19	.00085	.98358	.98315
20	.00093	1.03975	1.03933
21	.00102	1.09592	1.09551
22	.00111	1.15209	1.15169
23	.00120	1.20825	1.20787
24	.00131	1.26441	1.26404
25	.00141	1.32057	1.32022
26	.00152	1.37673	1.37640
27	.00162	1.43289	1.43258
28	.00173	1.48905	1.48876
29	.00183	1.54520	1.54494
30	.00193	1.60135	1.60112
31	.00202	1.65750	1.65730
32	.00210	1.71364	1.71348
33	.00216	1.76978	1.76966
34	.00220	1.82591	1.82584
35	.00222	1.88203	1.88202
36	.00221	1.93814	1.93820
37	.00216	1.99423	1.99438
38	.00207	2.05031	2.05056
39	.00193	2.10636	2.10674
40	.00175	2.16237	2.16292
41	.00152	2.21833	2.21910
42	.00125	2.27420	2.27528
43	.00095	2.32991	2.33146
44	.00063	2.38532	2.38746
45	.00033	2.43998	2.44382
46	.00010	2.49198	2.50000
47	.95021	2.55618	2.55618

TABLE 4.4

$\alpha = 3$

E_j	$\rho_{46,j}(\tau)$	$E(Z_\tau)$	M_j
2	.00006	-	.02809
3	.00012	.08484	.08427
4	.00013	.14111	.14045
5	.00015	.19732	.19663
6	.00017	.25346	.25281
7	.00020	.30963	.30899
8	.00023	.36581	.36517
9	.00026	.42194	.42135
10	.00029	.47812	.47753
11	.00033	.53430	.53371
12	.00038	.59050	.58989
13	.00043	.64664	.64607
14	.00049	.70284	.70225
15	.00055	.75899	.75843
16	.00062	.81517	.81461
17	.00070	.87134	.87079
18	.00078	.92750	.92697
19	.00088	.98367	.98315
20	.00098	1.03983	1.03933
21	.00109	1.09599	1.09551
22	.00121	1.15215	1.15169
23	.00133	1.20831	1.20787
24	.00146	1.26447	1.26404
25	.00159	1.32063	1.32022
26	.00173	1.37677	1.37640
27	.00187	1.43292	1.43258
28	.00200	1.48907	1.48876
29	.00212	1.54521	1.54494
30	.00224	1.60134	1.60112
31	.00233	1.65747	1.65730
32	.00240	1.71359	1.71348
33	.00244	1.76971	1.76966
34	.00244	1.82581	1.82584
35	.00241	1.88190	1.88202
36	.00232	1.93797	1.93820
37	.00218	1.99403	1.99438
38	.00199	2.05005	2.05056
39	.00175	2.10604	2.10674
40	.00147	2.16197	2.16292
41	.00116	2.21781	2.21910
42	.00084	2.27353	2.27528
43	.00054	2.32902	2.33146
44	.00029	2.38406	2.38746
45	.00011	2.43807	2.44382
46	.00002	2.48887	2.50000
47	.95021	2.55618	2.55618

TABLE 4.5

$\alpha = 4$

give some indication as to the validity of our assumption that the reserve level takes the value of the midpoint after a transition.

Firstly, we note that the density function takes a different shape in each case. When $\alpha = 1$, the density is a monotone increasing function. As $x \to 0^+$, $f(x,\tau)$ tends to $\tau e^{-\tau}$ and as $x \to \infty$, it tends to zero. However, in the second case, the distribution is unimodal, with the mode occurring at a value of x greater than zero. In this situation, $f(x,\tau) \to 0$ as $x \to 0^+$ and again goes to zero as $x \to \infty$. This explains why the transition probabilities have a different structure in each case. Table 4.2 shows that when $\alpha = 1$, the transition probabilities $\rho_{46,j}(\tau)$ decrease as j decreases from 47 to 2. In Tables 4.3 to 4.5, we find that this is not so. The transition probabilities $\rho_{46,j}(\tau)$ increase as j decreases from 46 to M, where E_M is the state such that the modal total claim amount in $(0,\tau)$ lies between u-(M-2)w and u-(M-3)w. Thereafter, the probabilities $\rho_{46,j}(\tau)$ decrease with j, as $f(x,\tau)$ is a decreasing function over $(u-(M-3)w,\infty)$.

For example, in Table 4.3, M = 38. Clearly the transition probabilities $\rho_{46,j}(\tau)$ increase as j decreases from 46 to 38, and then decrease with j. The mode of the distribution $F(x,\tau)$ in this example is found to be .5043 which lies between .4775 and .5337.

Thus, in case (a) the conditional expected reserve level in each state, excluding E_{47}, is greater than the midpoint. In case (b), the relation between the two depends on whether the transition probabilities are increasing or decreasing as j decreases. In both cases, the conditional expected reserve level in E_{47} is less than M_{47}, since the reserve level

after time τ cannot exceed M_{47}.

Of most interest are the figures for the transition from E_{46} to E_{47}, as we have placed so much emphasis on transitions to states one above the starting state in an attempt to justify our assumption of centring the process on the midpoints after a transition. Table 4.6 summarises the results of this transition for all four values of α.

α	$\rho_{46,47}(\tau)$	$F(0,\tau)$	%	(5)	M_{47}	(7)
1	.951554	.950210	99.86	2.556160	2.556180	-.000019
2	.950284		99.99	2.556178		-.0000019
3	.950214		100.00	2.556180		0
4	.950210		100.00	2.556180		0

TABLE 4.6

The column headed % shows $F(0,\tau)$ as a percentage of $\rho_{46,47}(\tau)$; the column headed (5) gives the respective figures from the third column of Tables 4.2 to 4.5 and the column headed (7) shows the quantity

$$\rho_{46,47}(\tau)[(5) - M_{47}]$$

This is a simple measure of the deviation of the expected reserve level in E_{47} from the midpoint M_{47} if the transition $E_{46} \rightarrow E_{47}$ occurs. Obviously we would like this quantity to be as small as possible, so that the expected reserve level in E_{47} is taking almost the same value as M_{47}. The largest deviation occurs when $\alpha = 1$. This is not surprising. In Section 3.4 we stated that the expected reserve level in E_{47} would tend to M_{47} as $F(0,\tau)$ increased as a percentage of $\rho_{46,47}(\tau)$. When $\alpha = 1$,

$F(0,\tau)$ is 99.86% of $\rho_{46,47}(\tau)$. For the other three values of α, it is almost 100%. The reason for this can be seen by writing

$$\rho_{46,47}(\tau) = e^{-\tau} + \int_0^{.5w} f(x,\tau)dx \ .$$

When $\alpha > 1$, the values of $f(x,\tau)$ are very small over the range $(0,.5w)$. Thus the integral is making little contribution to $\rho_{46,47}(\tau)$. But when $\alpha = 1$, the values of $f(x,\tau)$ are much larger and the integral is now making a considerable contribution. However, it seems that our assumption is reasonably justified for all four values of α. Similarly a comparison of the columns headed $E(Z_\tau)$ and M_j in Tables 4.2 to 4.5 shows good agreement between the expected reserve levels in the states E_j and the midpoints M_j. Obviously we have exactly the same agreement when the process starts in any other transient state E_j, $j = 2, \ldots, N-2$.

Thus we conclude that there is little error caused by our assumption about centring the process on the midpoints after a transition. However, this is the reason for the approximations being less than the exact results in case (a) and, mostly greater than the exact results in case (b).

In case (a), it appears that the effect of centring the process after a transition is to increase the true reserve level. This is because the reserve level in E_{j+1} is increased, following a transition of the form $E_j \rightarrow E_{j+1}$. Since the probability of this transition is very much larger than any of the other transition probabilities from E_j, it outweighs the effect of decreasing the expected reserve level if a transition to any of the other transient states occurs. In summary, we are creating a slight upward drift in the true reserve level at each transition. Compounding this error by successively multiplying the transition matrix

with itself is therefore producing ruin probabilities which are less than the exact ones.

The situation is not so clearly defined in case (b). If the transition $E_j \rightarrow E_{j+1}$ occurs, the error of the assumption is negligible as the process is effectively hitting M_{j+1}. We note that for transitions from higher starting states, the expected reserve level exceeds the midpoints in more cases than it takes a lower value. When the process starts in the lower states, the opposite is true. Thus we are creating a slight downward drift in this case, although much less pronounced than the upward drift in case (a). Overall the method produces approximations which are slightly larger than the exact values. However, we note that in Table 4.1 the approximate ruin probability from initial reserve level M_4, when $V(x)$ takes the $\gamma(4)$ distribution, is less than the exact value. Here, if ruin occurs from M_4, then only a few upward transitions are likely to have occurred. Thus, following each transition, we would expect to increase the true reserve level on almost all occasions, since transitions will be to states no more than, say, ten below. This makes the process "safer" and hence results in an under approximation.

The results of Table 4.1 suggest that the method can successfully be applied to distributions $F(x,\tau)$ similar to case (b) where the density function increases from 0 at $x = 0^+$, with a mode occurring at some strictly positive value of x, and tails off to 0 as $x \rightarrow \infty$. In such cases, $F(0,\tau)$ should represent a sufficiently large percentage of $\rho_{j,j+1}(\tau)$ to justify our assumption. In our considerations in the remainder of this thesis, we shall only use distribution functions of this type.

It is perhaps surprising that adjusting our transition probabilities to the absorbing states has made such a great improvement to our approximations. In Chapter 3, we calculated $\rho_{j,1}(\tau)$ as

$$\rho_{j,1}(\tau) = 1 - F(c\tau + (j-1.5)w,\tau)$$

$$= 1 - F(M_j+c\tau,\tau) \quad \text{for} \quad j = 2, \ldots, N-1$$

In this chapter we have merely increased this by the amount

$$c\int_0^\tau \delta(0,\tau-s)f(M_j+cs,s)ds \quad \text{for} \quad j = 2, \ldots, N-2$$

Similarly, we previously defined $\rho_{N-1,N}(\tau)$ as

$$\rho_{N-1,N}(\tau) = F(.5w,\tau)$$

This was redefined by [2.4]. However, by considering the process retrospectively, we could have made the following definition

$$\rho_{N-1,N}(\tau) = F(.5w,\tau) + e^{-t_o}[1 - F(ct_o,t_o)]$$

$$+ \int_{t_o}^{\tau} \frac{K-M_{N-1}}{s} f(M_{N-1}+cs-K,s)[1 - F(c(t-s),t-s)]ds$$

Thus, all exact transition probabilities to the absorbing states (including $\rho_{N-1,1}(\tau)$) can be partly defined in terms of the inexact ones of the last chapter. Since we are using small values of τ, the values of the exact and inexact transition probabilities are fairly close. While the inexact transition probabilities may thus be acceptable over $(0,\tau)$, it is clear that compounding this inaccuracy is unacceptable.

For the remainder of this thesis we shall use this modified transition

matrix to produce approximations by the Markov chain method.

4.4 Approximation for General Values of u

So far we have produced approximations for initial reserve levels which are midpoints of the transient states. In this section we show that it is still possible to apply the Markov chain method to approximate $\xi(u,K)$ for values of u which are not midpoints.

Suppose we have approximated $\xi(M_j,K)$ by the Markov chain method for each value of j under some fixed combination of N and τ. Let M_i be the least midpoint greater than u, and let s be such that $u+cs = M_i$. Then $Z_s = M_i$ if no claims occur in (0,s). We can calculate transition probabilities $\rho_j(s)$ that Z_s is in state E_j of the Markov chain at time s. As before we assume that the reserve level at time s takes the value of the midpoint of the state which the reserve level has entered. The reserve level at s will be at the midpoint M_i with probability $F(0,s)$. Since s is less than τ by definition, this assumption does not lead to any great error. Our approximation to $\xi(u,K)$ is obtained by summing the product of the probabilities of entering E_j from u together with the respective approximate ruin probabilities from M_j. Thus, our approximation is given by

$$\xi(u,K) = \rho_1(s) + \sum_{j=2}^{i} \rho_j(s)\,\xi(M_j,K)$$

Using the same parameters as in Table 4.1, the results of Table 4.7 show the same agreement as previously achieved. In this illustration the single claim amount distribution is $\gamma(2)$.

u	Exact	Approx
.5	.72556	.72557
1.5	.53125	.53125
2.5	.35573	.35574
3.5	.20026	.20027
4.5	.06271	.06271

TABLE 4.7

4.5 Variation of Parameters

In choosing our parameters N and τ, we have used the inequality

$$N-2 \geqslant \frac{K}{(1+\lambda)\ln 1/\varepsilon} \qquad [5.1]$$

to determine N, and have then found τ by the identity $c\tau = w$. So far the parameters ε and λ have been fixed at .95 and .1 respectively. In this section we examine the effect on the approximation method of varying these two parameters. Let us start by considering smaller values of ε.

As a first illustration, we reduce ε to .9. Since we are reducing the value of ε we would expect our approximations to be less accurate than before as we are introducing a larger error into the model through our assumption of centring the process after a transition. Keeping the upper barrier at 5 and the premium income per unit time at 1.1, we find that the minimum value for N when $\varepsilon = .9$ is 47. Using this minimum value for N gives $\tau = .10101$. Hence $F(0,\tau) = .90392$, a drop of around .05 from the previous example. However, we are now using just over half the number of states of the previous example so that we require much less computation for this approximation.

α	1		2		3		4	
State	Exact	Approx	Exact	Approx	Exact	Approx	Exact	Approx
3	.7528	.7525	.78967	.78967	.80337	.80337	.81051	.81050
6	.6896	.6893	.72556	.72557	.73933	.73933	.73675	.74674
9	.6283	.6280	.65949	.65950	.67058	.67058	.67617	.67617
12	.5688	.5685	.59438	.59440	.60236	.60237	.60589	.60590
15	.5111	.5108	.53124	.53127	.53653	.53655	.53848	.53849
18	.4551	.4548	.47040	.47043	.47353	.47356	.47431	.47433
21	.4008	.4005	.41191	.41194	.41335	.41338	.41326	.41329
24	.3481	.3479	.35573	.35576	.35589	.35592	.35516	.35520
27	.2970	.2968	.30178	.30181	.30102	.30106	.29987	.29990
30	.2474	.2472	.24998	.25001	.24863	.24867	.24724	.24728
33	.1993	.1991	.20026	.20029	.19861	.19864	.19715	.19718
36	.1527	.1525	.15253	.15255	.15084	.15086	.14948	.14950
39	.1074	.1073	.10670	.10672	.10523	.10525	.10410	.10412
42	.0634	.0634	.06271	.06272	.06167	.06169	.06091	.06092
45	.0208	.0208	.02048	.02048	.02008	.02009	.01980	.01981

TABLE 4.8

Our approximations to $\xi(u,K)$ under the above combination of N and τ are shown in Table 4.8 for $\gamma(\alpha)$ single claim amounts. The results in this table follow exactly the same pattern as those of Table 4.1, for the same reasons. These approximations are, as expected, slightly less accurate than those obtained when ε was set at .95. Nevertheless, when $\alpha \geqslant 2$ we are still producing approximations which are accurate to four decimal places.

Suppose we further reduce the value of ε to .75, with the upper barrier and premium as above. In this situation, we only require a minimum of 19 states for our Markov chain. Setting N = 19 gives τ = .26738 so that $F(0,\tau)$ = .76538. There is now a probability of just over .76 that the reserve level is at a midpoint following a single transition. This certainly reduces the accuracy of our assumption that the process is at a midpoint following each transition. Further, as the intervals w become wider, if a transition of the form $E_j \rightarrow E_\ell$ occurs, where $\ell \leqslant j$, the conditional expected reserve level in E_ℓ will generally be proportionally further removed from M_ℓ as the value of w increases and the value of ℓ decreases. We illustrate this in the case of $\gamma(4)$ single claim amounts in Table 4.9. This table shows the state of the process following a single transition from state E_{10} and can be compared with Table 4.5. As in that table, the column in Table 4.9 headed $E(Z_\tau)$ gives the conditional expected reserve level in E_j if a transition from E_{10} to E_j occurs.

Here, w = .294, compared to w = .056 when ε = .95.

E_j	$\rho_{10,j}(\tau)$	$E(Z_\tau)$	M_j
2	.00265	-	.14706
3	.00652	.45312	.44118
4	.01062	.74717	.73529
5	.01721	1.04108	1.02941
6	.02733	1.33435	1.32353
7	.04075	1.62618	1.61765
8	.05239	1.91512	1.91176
9	.04868	2.19747	2.20588
10	.02047	2.45851	2.50000
11	.76602	2.79402	2.79412

TABLE 4.9

We note that, in this example, $F(0,\tau)$ represents 99.92% of $\rho_{j,j+1}(\tau)$ and that the conditional expected reserve level in E_{j+1} (following the transition $E_j \rightarrow E_{j+1}$) is not as close to M_{j+1} as in previous examples. As before, we are producing an upward drift in the true reserve level at each transition. The upward drift is more pronounced than in the two previous cases of $\varepsilon = .9$ and $\varepsilon = .95$. This is reflected in the results of Table 4.10 which shows approximations to $\xi(u,K)$ (when $V(x)$ takes a $\gamma(4)$ distribution) which exceed the exact values by as much as 3 in the fourth decimal place. Nevertheless, the model still seems to be working reasonably well and we do not appear to have lost as much accuracy as might have been expected considering the large reduction in the number of states.

State	Exact	Approx
3	.7588	.7588
5	.6346	.6347
7	.5155	.5157
9	.4063	.4066
11	.3062	.3065
13	.2145	.2148
15	.1305	.1307
17	.0535	.0536

TABLE 4.10

The results of Tables 4.8 and 4.10 suggest that if there were practical restrictions on, say, the size of N, then it is acceptable to use smaller values of ε. In particular, if we increase the value of K for a fixed value of ε, then the increase in the minimum number of states required is almost directly proportional to the increase in K. For example, under the combination of $K = 5$, $c = 1.1$ and $\varepsilon = .95$ we require a minimum of 91 states. If K takes the values 10, 15 and 20, then the minimum numbers of states required are 181, 269 and 359 respectively. In Table 4.11 we show that good approximation can be achieved for larger values of K using values of ε less than .95. The first set of results shows approximations when $K = 10$ and $\varepsilon = .9$, requiring a minimum of 89 states; the second when $K = 15$ and $\varepsilon = .85$, requiring a minimum of 87 states. In each case the single claim amount distribution is $\gamma(2)$ and the loading is $\lambda = .1$.

K=10			K=15		
State	Exact	Approx	State	Exact	Approx
10	.7465	.7465	11	.7053	.7055
21	.5869	.5870	23	.5051	.5053
33	.4382	.4383	34	.3619	.3622
45	.3126	.3127	44	.2582	.2584
57	.2065	.2065	54	.1747	.1749
69	.1169	.1169	65	.1014	.1016
80	.0470	.0470	77	.0391	.0391

TABLE 4.11

Let us now consider the effect of increasing the premium loading λ. For fixed values of ε and K, it is obvious by [5.1] that the minimum number of states required decreases as λ increases. This should not have any great effect on the transition probabilities $\rho_{j,j+1}(\tau)$. However, we would anticipate that approximations to $\xi(u,K)$ for increased λ will be slightly poorer simply because we require to use a smaller number of states. The states of the Markov chain are now wider and hence more inaccuracy is introduced into the model by centring the process on the midpoints. In the following illustration, ε and K are fixed at .95 and 5 respectively and the single claim amounts are distributed as $\gamma(3)$. Loadings of λ = .1, .2 and .3 have been used, requiring minimum values of N of 91, 85 and 77 respectively. The results in Table 4.12 show that as λ increases, the approximations become less accurate in the fifth decimal place. However, as the percentage error of these approximations is negligible, it seems that [5.1] is satisfactory for all values of λ. Increased accuracy of our approximations for larger values of λ can always be obtained by using a larger number of states than the minimum required by [5.1].

	$\lambda = .1$			$\lambda = .2$			$\lambda = .3$	
State	Exact	Approx	State	Exact	Approx	State	Exact	Approx
4	.80796	.80796	4	.75232	.75233	4	.69777	.69778
11	.73247	.73247	10	.67234	.67235	9	.61658	.61660
16	.67445	.67445	15	.60272	.60273	14	.53413	.53415
22	.60538	.60538	21	.52263	.52265	19	.45742	.45744
28	.53870	.53871	27	.44868	.44869	24	.38841	.38844
34	.47491	.47492	33	.38111	.38113	29	.32701	.32704
40	.41401	.41402	38	.32940	.32942	34	.27255	.27258
46	.35589	.35590	43	.28155	.28157	39	.22427	.22430
52	.30042	.30043	48	.23727	.23729	44	.18147	.18150
58	.24728	.24729	53	.19631	.19632	49	.14353	.14356
64	.19696	.19697	59	.15116	.15118	54	.10990	.10993
70	.14874	.14875	65	.11004	.11005	59	.08009	.08011
76	.10273	.10273	71	.07257	.07258	64	.05367	.05368
81	.06599	.06599	76	.04390	.04391	69	.02324	.02325
88	.01689	.01690	82	.01232	.01232	74	.00948	.00948

TABLE 4.12

5.0 FURTHER APPROXIMATION METHODS

5.1 Introduction

The approximations produced so far by the Markov chain method, using the exact transition matrix, have all been good. However, it has required a large amount of numerical calculation to produce these approximations. In Chapter 2 we showed that when the premium loading is positive, there exists a simple expression for $\xi(u,K)$ in terms of $\psi(u)$ and $\psi(K)$. Thus, it should be possible to approximate $\xi(u,K)$ by approximating the function $\psi(\cdot)$. In Section 5.2 we use the readily calculated approximation to $\psi(u)$ proposed by Beekman and Bowers in [2] to produce alternative approximations to $\xi(u,K)$. The results obtained suggest that the longer calculations required for the Markov chain method are justified, particularly when a large value of ε is used.

This alternative approximation method is, however, valid for more general forms of $V(x)$. In Section 5.3, we show that by approximating $F(x,\tau)$ the Markov chain method is also applicable to general forms of $V(x)$, and not just distributions whose n-fold convolutions $V^{n*}(x)$ exist in an exact form.

5.2 An Alternative Approach

The Beekman-Bowers approximation to $\psi(u)$ is given by

$$\psi(u) \doteq \frac{\nu_1}{\nu_1 + \lambda}\,[1 - P(u/\beta,\alpha)]$$

where

$$P(u/\beta,\alpha) = \int_0^{u/\beta} \frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)}\,dx$$

$$\beta = \frac{2\nu_3}{3\nu_2} + \frac{\nu_2}{2\lambda}\left[1 - \frac{\lambda}{\nu_1}\right]$$

and

$$\alpha = \frac{\nu_2(\lambda + \nu_1)}{2\lambda\beta\nu_1}$$

This approximation can be calculated for any single claim amount distribution $V(x)$ whose first three moments about zero exist. By substituting the above expression for $\psi(u)$ in equation [2.2] of Chapter 2, we obtain the approximation

$$\xi(u,K) \doteq \frac{P(K/\beta,\alpha) - P(u/\beta,\alpha)}{\lambda/\nu_1 + P(K/\beta,\alpha)} \qquad [2.1]$$

As a numerical illustration, we compare approximations obtained by [2.1] and by the Markov chain method. We shall use the following three distributions for $V(x)$:

i the gamma distribution, with parameters $\alpha = 2$ and $\beta = .5$;

ii the inverse Gaussian distribution, given by

$$V(x) = \Phi\left(\left(\frac{x}{\mu} - 1\right)\sqrt{\frac{\theta}{x}}\right) + e^{2\theta/u}\left[1 - \Phi\left(\left(\frac{x}{\mu} + 1\right)\sqrt{\frac{\theta}{x}}\right)\right], \quad x > 0$$

where $\Phi(\cdot)$ is the standard normal distribution function. For the inverse Gaussian distribution, an exact form of $F(x,t)$ exists and is given in e.g. Seal [11]. The mean and variance of the distribution $V(x)$ are μ and μ^3/θ respectively. We have chosen $\mu = 1$ and $\theta = 3$, giving $\nu_1 = 1$ with $\nu_2 = 4/3$;

iii the two parameter exponential distribution

$$V(x) = 1 - e^{-\alpha(x-b)}, \quad x \geqslant b$$

$$0 \text{ otherwise.}$$

For the two parameter exponential distribution, we can calculate $F(x,t)$ exactly as

$$F(x,t) = e^{-t}, \quad x \leqslant b$$

$$= e^{-t} + e^{-t} \sum_{j=1}^{r} \frac{t^j}{j!} W^{j*}(x-jb), \quad rb < x \leqslant (r+1)b \text{ for } r = 1,2,..$$

$W(y)$ is the distribution function representing the amount by which a single claim exceeds the guaranteed claim amount b. Thus if $Y = X-b$, where X is a random variable denoting the single claim amount, then Y is distributed as W, with $W(y) = V(y+b)$ for $y > 0$. Hence $V^{j*}(x) = W^{j*}(x-jb)$ for $j \geqslant 1$ and $x > jb$. In this instance, W is simply the exponential distribution with parameter α. Here $V(x)$ has mean $b + 1/\alpha$ and variance $1/\alpha^2$. We set $\alpha = (1-b)^{-1}$ so that the distribution has unit mean.

Although we find that the least accurate approximations by the Markov chain method occurred when $V(x)$ took an exponential distribution, it is worth applying the method to the two parameter exponential distribution. Suppose $b > c\tau$. Then a transition of the form $E_j \to E_{j+1}$ can only occur if there are no claims in $(0,\tau)$. Hence $\rho_{j,j+1}(\tau)$ and $F(0,\tau)$ are identical, making the assumption on centring more accurate than in any of the previous examples. A similar situation as before exists for transitions of the form $E_j \to E_\ell$, where $\ell \leqslant j$.

As a brief aside, let us consider how we must alter transition probabilities for single claim amount distributions $V(x)$ such that $V(b) = 0$. As above, we assume that $b > c\tau$. In particular, we let $b = (r + 1/2)w$, where r is a positive integer, taking a value much less than N. We let b take this value for the following reason. Suppose the initial reserve level is M_j

where $j \geqslant r + 2$. Then if a claim occurs in $(0,\tau)$, the reserve level may take any value in the transient states $E_2, E_3, \ldots, E_{j-r}$ at τ. In our numerical illustrations we shall fix b according to w. In practice, the parameter b will be given and we shall have to fix w and r according to b. In such instances it may not always be possible to choose w and r to satisfy $b = (r+1/2)w$. In this case we should choose values of w and r such that $(r + 1/2)w$ is slightly larger than b. If we let $(r + 1/2)w$ take a smaller value than b, then a transition from E_j to E_{j-r+1} would be possible, but the reserve level in E_{j-r+1} would always be much less than M_{j-r+1}. In this instance, our assumption that the reserve level takes the value of the midpoint would be totally unrealistic.

The following changes should be made to the transition matrix described in Chapter 4:

a. If the initial reserve level is M_j, $j = 2, \ldots, r+1$ then

   (i) ruin occurs if there is a claim in $(0,\tau)$ since $b > M_j + c\tau$;

   (ii) survival over $(0,\tau)$ occurs if there are no claims in $(0,\tau)$.

   Hence

$$\rho_{j,j+1}(\tau) = e^{-\tau}$$

$$\rho_{j,i}(\tau) = 0, \quad i = 2, \ldots, j$$

$$\rho_{j,1}(\tau) = 1 - e^{-\tau}$$

b. If the initial reserve level is M_{N-1}, then the process can only reach the upper barrier in $(0,\tau)$ if there are no claims by t_o. Thus

$$\rho_{N-1,N}(\tau) = e^{-t_o}$$

Similarly,

$$\rho_{N-1,1}(\tau) = \psi(M_{N-1},\tau) - e^{-t_o}\psi(K,t_o)$$

The transition probabilities from E_{N-1} to the transient states are adjusted in a similar way. Finally we note that in calculating transition probabilities to E_1 from the other transient states, $\delta(0,t)$ is simply given by e^{-t}.

In Table 5.1 we present a comparison of the two methods for each of the above distributions. The upper barrier is K = 5 and the number of states is 91. For the two parameter exponential distribution, r = 8, giving α = 1.914 and b = .478. The premium loading in each case is λ = .1. MC and B-B denote the Markov chain and Beekman-Bowers approximations respectively.

We do not have exact results for $\xi(u,K)$ when V(x) takes the inverse Gaussian or two parameter exponential distribution. However, we would expect that the Markov chain approximations in these two cases are as close to the exact, but unknown, ruin probabilities as the Markov chain approximations are to the exact results when V(x) takes the $\gamma(2)$ distribution. For each of these three distributions, it is interesting to note that a similar pattern exists between the two sets of approximations. When the initial reserve level is near to the upper barrier, there is fairly close agreement between the two methods, but as u decreases, the Markov chain approximations are considerably larger than those obtained by the Beekman-Bowers approach. If the Markov chain approximations are "exact" to four decimal places, then these results agree with numerical tests conducted by Seal in [12], which indicate that the Beekman-Bowers method gives poor approximations for small values of u. Therefore let us increase our upper barrier and compare approximations. In Table 5.2 we

Distribution:	Gamma		Inverse Gaussian		Two Parameter Exponential	
State	MC	B-B	MC	B-B	MC	B-B
4	.7945	.7901	.8082	.8012	.8135	.8052
11	.7189	.7100	.7299	.7177	.7338	.7204
16	.6632	.6543	.6702	.6598	.6722	.6617
22	.5973	.5894	.6007	.5927	.6019	.5937
28	.5333	.5268	.5344	.5283	.5349	.5285
34	.4718	.4664	.4711	.4664	.4710	.4661
40	.4126	.4083	.4107	.4072	.4102	.4064
46	.3557	.3524	.3531	.3505	.3523	.3494
52	.3012	.2987	.2981	.2962	.2971	.2949
58	.2489	.2470	.2456	.2442	.2445	.2429
64	.1986	.1973	.1955	.1945	.1944	.1933
70	.1504	.1495	.1476	.1470	.1466	.1459
76	.1042	.1036	.1020	.1016	.1012	.1007
81	.0671	.0667	.0655	.0653	.0649	.0647
88	.0172	.0172	.0168	.0167	.0166	.0166

TABLE 5.1

present the results only for the case of $\gamma(2)$ single claim amounts. The upper barrier is now at 10, with the premium per unit time at 1.1 and we have used 89 states for the Markov chain approximations. Although the Markov chain approximations are still superior, there is a definite improvement in the Beekman-Bowers approximations, particularly for larger values of u.

State	Exact	MC	B-B
5	.8246	.8246	.8189
15	.6712	.6712	.6669
27	.5094	.5095	.5079
39	.3727	.3728	.3725
51	.2573	.2574	.2575
63	.1598	.1598	.1601
75	.0774	.0775	.0777
85	.0187	.0187	.0188

TABLE 5.2

So far we have only considered an upper barrier which is up to ten times the mean claim amount. Although we could still apply the Markov chain method to approximate $\xi(u,K)$ when K takes larger values, we would clearly require a much larger transition matrix if the value of ε is to remain at least .9. This would obviously involve considerably more calculation. In such cases, it would therefore seem acceptable to approximate $\xi(u,K)$ through the function $\psi(\cdot)$. In addition to the Beekman-Bowers approximation method, we could also apply approximation methods discussed in the paper by Grandell and Segerdahl [7] when both u and K are much larger.

5.3 The Markov Chain Method for General Forms of V(x)

In our previous applications of the Markov chain method, we have only considered single claim amount distributions, V(x), whose convolutions are obtainable in an exact form, yielding explicit forms of F(x,t). In this section, we show that it is possible to produce good approximations to $\xi(u,K)$ by the Markov chain method when F(x,t) is approximated.

The value of τ has been small in most of our examples on the Markov chain method. Since

$$F(x,\tau) = e^{-\tau} + \sum_{n\geqslant 1} e^{-\tau} \frac{\tau^n}{n!} V^{n*}(x)$$

we note that, for small values of τ, terms in the summation contribute little to $F(x,\tau)$ as n increases. Consider our very first numerical illustration, where τ took the value .05107. In this case we find that the third term in the above summation is $.000021\ V^{3*}(x)$, while the fourth term in the summation makes no contribution to $F(x,\tau)$ until the seventh decimal place. Thus we could approximate $F(x,\tau)$ by

$$\hat{F}(x,\tau) = e^{-\tau}\ [1 + \tau V(x) + .5\tau^2 V^{2*}(x)]$$

when τ is small since it is possible to produce numerical values of $V^{2*}(x)$ for most distributions. For the above value of τ, $\hat{F}(x,\tau)$ would always be less than $F(x,\tau)$, the largest discrepancy being two in the fifth decimal place, which would only occur for large values of x.

Unfortunately $\hat{F}(x,\tau)$ is unsuitable for use in the Markov chain method, as it is an improper distribution function. If we were to use $\hat{F}(x,\tau)$ to obtain a transition matrix, then the matrix would be sub-stochastic. This problem can be overcome by defining the proper distribution function

$G(x,\tau)$ by

$$G(x,\tau) = e^{-\tau} + \phi e^{-\tau}[\tau V(x) + .5\tau^2 V^{2*}(x)]$$

where

$$\phi = (e^{\tau} - 1)/(\tau + .5\tau^2)$$

Note that ϕ has only been applied to the last two terms of $\hat{F}(x,\tau)$ to make $G(x,\tau)$ a proper distribution function so that equality exists between $F(0,\tau)$ and $G(0,\tau)$. Clearly this is desirable if we are using $G(x,\tau)$ to obtain the transition matrix, as it maintains the accuracy of the transitions $E_j \to E_{j+1}$. While $\hat{F}(x,\tau)$ is less than $F(x,\tau)$ for all values of x, $G(x,\tau)$ will mostly be larger than $F(x,\tau)$ but the difference between the two should decrease for larger values of x.

Using the above value of $\tau = .05107$, we have tabulated values of $F(x,\tau)$ and $G(x,\tau)$ in Table 5.3, when $V(x)$ is distributed as

i gamma, with $\alpha = 2$ and $\beta = .5$; and

ii inverse Gaussian, with $\mu = 1$ and $\theta = 3$.

The value of ϕ in this case is 1.00043.

The agreement between $F(x,\tau)$ and $G(x,\tau)$ is sufficiently good to suggest that a transition matrix based on $G(x,\tau)$ could produce good approximations to $\xi(u,K)$. Let us apply this to the above distributions when $K = 5$, $\lambda = .1$ and the number of states of the Markov chain is 91. In addition we shall use the two parameter exponential distribution for $V(x)$ with parameters $\alpha = 1.914$ and $b = .478$ (i.e. the distribution has unit mean with $b = 8.5w$). Here, we calculate $G(x,\tau)$ as

	(i)		(ii)	
x	$F(x,\tau)$	$G(x,\tau)$	$F(x,\tau)$	$G(x,\tau)$
0	.95021	.95021	.95021	.95021
.4	.95950	.95950	.95388	.95388
.8	.97336	.97337	.97198	.97199
1.2	.98405	.98406	.98560	.98562
1.6	.99093	.99094	.99283	.99285
2.0	.99500	.99502	.99644	.99646
2.4	.99730	.99732	.99822	.99824
2.8	.99857	.99858	.99911	.99912
3.2	.99925	.99926	.99955	.99956
3.6	.99961	.99961	.99977	.99978
4.0	.99980	.99980	.99988	.99989
4.4	.99990	.99990	.99994	.99994
4.8	.99995	.99995	.99997	.99997

TABLE 5.3

$$G(x,\tau) = F(x,\tau) \ , \quad 0 \leqslant x \leqslant 3b$$

$$= F(3b,\tau) + \phi[\hat{F}(x,\tau) - F(3b,\tau)] \ , \quad x > 3b$$

where

$$\phi = \frac{1 - F(3b,\tau)}{\hat{F}(\infty,\tau) - F(3b,\tau)}$$

We make this definition as $F(x,\tau)$ may be calculated exactly with only a knowledge of $V(x)$ and $V^{2*}(x)$ if $x \leqslant 3b$. Agreement between $F(x,\tau)$ and $G(x,\tau)$ when $V(x)$ takes this distribution is similar to that of Table 5.3 when $x > 3b$.

The following approximations to $\xi(u,K)$ by the Markov chain method were obtained where

1. the transition matrix is based on $F(x,\tau)$; and
2. the transition matrix is based on $G(x,\tau)$

with the single claim amount distributions being (a) gamma, (b) inverse Gaussian, and (c) two parameter exponential, their respective parameters being as above.

In this example, when $f(x,\tau)$ was required for the calculation of transition probabilities, we simply used the derivative with respect to x of $G(x,\tau)$.

We see that there is agreement to three decimal places between the two sets of approximations in most instances. The agreement is best when $V(x)$ takes the two parameter exponential distribution. This is not surprising since both matrices have common transition probabilities due to our definition of $G(x,\tau)$ for this distribution. The approximations

	(a)		(b)		(c)	
State	(1)	(2)	(1)	(2)	(1)	(2)
4	.7945	.7941	.8082	.8079	.8135	.8134
11	.7189	.7185	.7299	.7294	.7338	.7337
16	.6632	.6627	.6702	.6696	.6722	.6720
22	.5973	.5967	.6007	.6001	.6019	.6016
28	.5333	.5327	.5344	.5337	.5349	.5346
34	.4718	.4711	.4711	.4703	.4710	.4707
40	.4126	.4119	.4107	.4099	.4102	.4098
46	.3557	.3551	.3531	.3523	.3523	.3519
52	.3012	.3006	.2981	.2974	.2971	.2967
58	.2489	.2483	.2456	.2449	.2445	.2441
64	.1986	.1981	.1955	.1949	.1944	.1941
70	.1504	.1500	.1476	.1472	.1466	.1464
76	.1042	.1039	.1020	.1016	.1012	.1010
81	.0671	.0669	.0655	.0653	.0649	.0648
88	.0172	.0172	.0168	.0167	.0166	.0166

TABLE 5.4

using the matrix based on $G(x,\tau)$ are all less than those using the matrix based on $F(x,\tau)$. A comparison of transition probabilities reveals that the probability of a transition from E_j to E_ℓ, where ℓ is not much less than j, is larger when using $G(x,\tau)$. Otherwise there is not much difference between the two matrices. However, this appears to make the process slightly "safer" when using $G(x,\tau)$, resulting in smaller ruin probabilities.

Of course, the accuracy of $G(x,\tau)$ could be improved by decreasing the value of τ. As $\tau \to 0$, we find that $\phi \to 1$, so that $F(x,\tau)$ and $G(x,\tau)$ should coincide to a greater number of decimal places as τ decreases. In the above example we used 91 states, which is the minimum required under the combination of $\varepsilon = .95$, $K = 5$ and $\lambda = .1$. By using more than the minimum number of states, we can decrease τ. In the following example, using the same upper barrier and premium loading, we have used 183 states giving $\tau = .025113$ and $\phi = 1.00011$. This gives five figure agreement between $F(x,\tau)$ and $G(x,\tau)$. Using the $\gamma(2)$ distribution for $V(x)$ as an illustration, we see in Table 5.5 that there is a significant improvement in the approximations to $\xi(u,K)$ when the transition matrix based on $G(x,\tau)$ is used. (1) and (2) are as in Table 5.4.

Obviously, if there was greater agreement between $F(x,\tau)$ and $G(x,\tau)$ we could obtain better approximations to $\xi(u,K)$ using a matrix based on $G(x,\tau)$. However, this would generally require the use of a very large transition matrix. The five figure agreement illustrated here between $F(x,\tau)$ and $G(x,\tau)$ would seem to be sufficient, as our approximations based on $G(x,\tau)$ here are in error by no more than two in the fourth decimal place.

State	Exact	(1)	(2)
8	.7873	.7873	.7872
20	.7234	.7234	.7233
32	.6577	.6577	.6575
44	.5930	.5930	.5928
56	.5302	.5302	.5301
68	.4697	.4697	.4696
80	.4116	.4116	.4114
92	.3557	.3557	.3556
104	.3021	.3021	.3019
116	.2505	.2505	.2504
128	.2011	.2011	.2009
140	.1536	.1536	.1535
152	.1079	.1079	.1079
164	.0641	.0641	.0641
176	.0221	.0221	.0221

TABLE 5.5

6.0 FINITE TIME RUIN PROBABILITIES

6.1 Introduction

In this chapter we consider the calculation of $\xi(u,K,t)$, the probability of ruin from initial reserve level u by time t when there is an absorbing upper barrier K. By following Segerdahl's approach to the problem, we find an equation for $\xi(u,K,t)$ in Section 6.2, and show that both explicit and implicit solutions for the Laplace transform of $\xi(u,K,t)$ can be achieved when single claim amounts follow the gamma distributions of previous chapters. These solutions may lead to numerical values of $\xi(u,K,t)$. By generalising equation [2.5] of Chapter 4, we find an expression for $\xi(u,K,t)$ which is valid when the premium income in (0,t) is less than K.

In Section 6.3 we consider computer simulation as a method of calculating $\xi(u,K,t)$, using a similar approach to that of Seal in [10]. Finally, we show in Section 6.4 that it is possible to adapt the Markov chain method of approximating $\xi(u,K)$ to approximate $\xi(u,K,t)$ for any value of t. This approach can be applied to a wider range of situations than those described in Section 6.2.

6.2 Exact Equations for $\xi(u,K,t)$

We may construct a general equation for $\xi(u,K,t)$ in the following way. Consider the probability $\xi(u,K,t+dt)$, where dt is small and the possible changes in the process during the interval (0,dt). The following events represent the different possibilities:

a. No claims occur. This has probability 1-dt. The reserve level at time dt is u + cdt and the probability of ruin from this new reserve level by t + dt is $\xi(u+cdt,K,t)$.

b. A claim occurs, with probability dt. In this event, either

(i) the claim size exceeds u + cdt, resulting in immediate ruin, which has probability 1 - V(u + cdt); or

(ii) the claim is of an amount z, where 0 < z < u + cdt. The reserve level at time dt becomes u + cdt - z. The probability of a claim of amount z is $v(z)dz$ and the probability of ruin by time t + dt from the new reserve level is $\xi(u+cdt-z,K,t)$. Integration over the range of z gives the total probability of all such realisations of the process.

c. More than one claim occurs. This has probability o(dt).

Combining these probabilities we have that

$$\xi(u,K,t+dt) = (1 - dt)\xi(u+cdt,K,t) + dt[1 - V(u+cdt)] + dt\int_0^{u+cdt} \xi(u+cdt-z,K,t)v(z)dz + o(dt) \qquad [2.1]$$

We now expand $\xi(u+cdt,K,t)$ as a Taylor series about u. Dividing the resulting equation throughout by dt, then letting $dt \to 0$ we find that

$$\xi'_t - c\xi'_u + \xi = \int_0^u \xi(z,K,t)v(u-z)dz + 1 - V(u) \qquad [2.2]$$

where, in an obvious notation ξ'_t and ξ'_u denote the derivatives with respect to t and u of $\xi(u,K,t)$. Segerdahl's approach to finding $\xi(u,K,t)$ from equation [2.2] starts by taking the Laplace transform of $\xi(u,K,t)$.

Define

$$\rho(u,K,s) = \int_0^\infty e^{-st}\xi(u,K,t)dt \qquad \text{where } s = a + ib$$

$$= \frac{1}{s}\int_0^\infty e^{-st}\xi'_t dt$$

Then, taking the Laplace transform of [2.2], we have

$$s\rho - c\rho'_u + \rho = \int_0^u \rho(z,K,s)v(u-z)dz + \frac{1}{s}[1 - V(u)] \qquad [2.3]$$

In the simple case $V(x) = 1 - e^{-x}$, [2.3] reduces to

$$-(1+\lambda)\rho'_u + (1+s)\rho = \int_0^u \rho(z,K,s)e^{-(u-z)}dz + \frac{1}{s}e^{-u}$$

where $c = 1+\lambda$.

Multiplication by e^u, followed by differentiation with respect to u yields

$$\rho''_u + \frac{\lambda-s}{1+\lambda}\rho'_u - \frac{s}{1+\lambda}\rho = 0 \qquad [2.4]$$

Hence

$$\rho(u,K,s) = b_1(\lambda,s)e^{-R_1u} + b_2(\lambda,s)e^{-R_2u} \qquad [2.5]$$

where

$$R_1,R_2 = \frac{1}{2(1+\lambda)}[(\lambda-s) \pm \sqrt{(\lambda+s)^2 + 4s}]$$

Further manipulation (see [13]) yields a complete solution for $\rho(u,K,s)$ but this solution can only be inverted numerically to gives values of $\xi(u,K,t)$. Suppose we had inserted a $\gamma(\alpha)$ distribution for $V(x)$ in [2.3], where α is an integer. Then [2.4] would become a linear differential equation of order $\alpha + 1$. To produce a solution of the form [2.5] for $\rho(u,K,s)$ would require finding the roots of the characteristic equation (of the new [2.4]) in terms of λ and s. It does not seem possible to produce a general solution in this case, but solutions for specific values

of α may be obtainable. We shall not pursue this method to obtain exact numerical values for $\xi(u,K,t)$ since, as before, the approach is only applicable to these specific distributions.

A more general equation for $\xi(u,K,t)$ is found by adapting equation [2.5] of Chapter 4. In that equation, $\rho_{N-1,1}(\tau)$ is the probability $\xi(M_{N-1},K,\tau)$. We could develop that equation because $c\tau < K$ and hence the density function of the first passage time from M_{N-1} to K was defined. Replacing M_{N-1} by u, and τ by t in the equation, we have

$$\xi(u,K,t) = \psi(u,t) - e^{-t_o}\psi(K,t-t_o) - \int_{t_o}^{t} \frac{K-u}{s} f(u+cs-K,s)\psi(K,t-s)ds \qquad [2.6]$$

where $t_o = (K-u)/c$. Equation [2.6] holds for all values of t satisfying $ct < K$.

6.3 Computer Simulation of $\xi(u,K,t)$

Our model for the risk process is easily simulated by computer. Seal [10] gives a full description of the calculation of finite time ruin probabilities for an unrestricted process by the use of computer simulation. To adopt this procedure we require to simulate variables from two distributions: that of the time interval between successive claims and that of the single claim amounts. Since the claim number distribution is Poisson, it is easily shown that the former distribution is exponential, with the same parameter as the Poisson distribution. We may adopt Seal's procedure to incorporate an absorbing upper barrier, and calculate $\xi(u,K,t)$ as follows:

i Compute a pair of pseudorandom variables (t_1,x_1) from the exponential and single claim amount distributions respectively.

ii Calculate $\hat{Z}_{t_1} = u + ct_1$. If $\hat{Z}_{t_1} \geqslant K$, then the process has been absorbed by the upper barrier before the occurrence of the first claim. In this event, we start another realisation of the process. Otherwise, calculate $Z_{t_1} = u + ct_1 - x_1$. If Z_{t_1} is negative, ruin has occurred, so we start anew. If $t_1 > t$, we also start anew.

iii If $0 \leqslant Z_{t_1} < K$, we compute two new pseudorandom variables (t_2, x_2), calculate $\hat{Z}_{t_1+t_2} = Z_{t_1+ct_2}$ and proceed as in (ii).

iv We continue to compute pairs of variables (t_j, x_j) for $j = 3, 4, \ldots$, and the corresponding values of $\hat{Z}_\tau$ and Z_τ where $\tau = \sum_{i=1}^{j} t_i$, until absorption by either barrier occurs or τ exceeds t.

If we simulate n realisations of the process over the interval (0,t), and ℓ of these realisations result in ruin, then our estimate $\hat{\xi}$ of $\xi(u,K,t)$ is ℓ/n. On the assumption that the number of realisations which result in ruin is a binomial random variable, the estimated standard error of ξ is $[(\ell/n)(1-\ell/n)/n]^{1/2}$.

As an illustration of this procedure, we have calculated $\xi(4,5,t)$ in Table 6.1 for t = 1, 2, 3 and 4. For these values of t, we can also produce $\xi(4,5,t)$ by equation [2.6], using numerical integration. The single claim amount distribution is $\gamma(2)$ (simulated by the NAG routine GØ5DGF) and the premium loading is $\lambda = .1$. Variables from the exponential distribution were simulated by the NAG routine GØ5DBF. In this example, n = 25,000.

We have only achieved three figure accuracy by this method. Even by increasing n, we are not certain to improve the accuracy of the approximation. However, a larger value of n should reduce the standard error of $\hat{\xi}$, resulting in a shorter confidence interval for the true ruin

probability $\xi(u,K,t)$. We shall not produce any further results for $\xi(u,K,t)$ by this method. The results of Table 1 of [10], where $n = 60,000$ and $t = 20$, suggest that we are unlikely to achieve further places of decimal accuracy by increasing either n or t.

t	$\xi(4,5,t)$	$\hat{\xi}$	s.e.(ξ)
1	.0143	.0137	.0007
2	.0372	.0366	.0012
3	.0579	.0581	.0014
4	.0747	.0752	.0016

TABLE 6.1

6.4 Markov Chain Approximations

We have approximated $\xi(M_j,K)$, the probability of eventual ruin from initial reserve level M_j when there is an absorbing upper barrier K, by calculating $\lim_{n\to\infty} P^n$ and finding the probability of absorption in E_1 from E_j. If we calculate $Q = P^n$ for some finite value of n, then $q_{j,1}$ gives the probability that absorption in E_1 from E_j has occurred by time $n\tau$, where τ is the length of time interval used in the Markov chain model. Thus, by calculating certain powers of the matrix P, we should be able to approximate finite time ruin probabilities.

Let τ be as above. If the process starts in E_j, then the probability that the process is in E_i after n transitions is $\rho_{j,i}(n\tau)$. Thus, if $n\tau < t < (n+1)\tau$, we may approximate $\xi(M_j,K,t)$ by multiplying the probability of being in E_i at $n\tau$ with the probability of ruin from M_i in time $t-n\tau$, and summing over all values of i. Thus, our approximation is

$$\xi(M_j,K,t) = \sum_{i=1}^{N-2} \rho_{j,i}(n\tau)\psi(M_i,t-n\tau) + \rho_{j,N-1}(n\tau)\xi(M_{N-1},K,t-n\tau)$$

where $\psi(M_1,\cdot) = 1$. Obviously $\xi(M_{N-1},K,t-n\tau)$ can be calculated by [2.6]. If $t-n\tau < .5\tau$, then $\xi(M_{N-1},K,t-n\tau)$ is replaced by $\psi(M_{N-1},t-n\tau)$ as the upper barrier cannot be reached from M_{N-1} in this case. This approximation may be calculated for any value of t. However, to produce this approximation we must now find powers of the matrix P, so that computation of approximate finite time ruin probabilities will actually take slightly longer than approximating $\xi(u,K)$ by the Markov chain method once the transition matrix has been calculated.

As an illustration of this approach, we shall approximate finite time ruin probabilities from the fixed initial reserve level of u = 10, with the upper barrier K = 12. We shall consider $\xi(u,K,t)$ when the single claim amount distribution is $\gamma(2)$ and the premium income per unit time is 1.1. These values of K and c have been selected so that we may calculate $\xi(10,12,t)$ by [2.6] for t = 2, ... , 10. Thus we may compare ruin probabilities obtained by both approaches over a reasonable range of t. Setting $\varepsilon = .9$, we require a minimum of 107 states. Using this number of states, we find that 10 is a midpoint, although we could easily have adapted our approach in a similar manner to that of Section 4.4 if it had not been a midpoint. The results are given in Table 6.2.

The figures by [2.6] should be exact to five decimal places, $\xi(10,12)$ is an exact figure. The agreement between the two sets of figures is good when t is less than 10 and when $t = \infty$. Although the approximations will become less accurate as t increases, we would expect that the approximations for the values of t between 15 and 50 will be no less accurate than the approximation to $\xi(10,12)$. In this example we have set

$\varepsilon = .9$. By using a larger value of ε and hence a larger number of states, further places of decimal accuracy could be achieved.

t	Exact	Approx
2	.00017	.00017
3	.00059	.00059
4	.00138	.00139
5	.00256	.00256
6	.00409	.00409
7	.00593	.00594
8	.00801	.00803
9	.01027	.01029
10	.01266	.01268
15	-	.02507
20	-	.03620
25	-	.04521
30	-	.05224
35	-	.05761
40	-	.06170
45	-	.06480
50	-	.06715
∞	.07440	.07443

TABLE 6.2

CONCLUSION

We have shown that the Markov chain method is effective in producing accurate approximations to $\xi(u,K)$, particularly when large values of ε are used. To produce these approximations we have not required any sophisticated mathematical techniques, simply that we can accurately calculate $F(x,t)$ and its density function.

If we employ the approximation to $F(x,t)$ introduced in Chapter 5, it should be possible to expand the ideas of Chapter 6 to produce approximations to finite time ruin probabilities in the presence of an absorbing upper barrier for any form of $V(x)$ by the Markov chain method. In addition, under a suitable choice of K, the method could be used to approximate finite time ruin probabilities for the unrestricted process over a limited range of t, as $\xi(u,K,t) = \psi(u,t)$ for values of t such that $u + ct < K$.

An upper limit to the amount of the risk reserve was introduced to prevent the continued growth of the reserves. It is possible to further curtail the growth of the reserves by letting the premium loading be a function of the reserve level at time t so that $\lambda = \lambda(Z_t)$, where $\lambda(Z_t)$ is a decreasing function of Z_t. This could be built into the Markov chain model provided that $\lambda(Z_t)$ is not given by a continuous function. A step function for $\lambda(Z_t)$ would be most easily included in the model. To incorporate such a change would mean that the states of the Markov chain would become narrower as the reserve level increases, assuming $(1 + \lambda(Z_\tau))\tau = w$ as before, but the basics of the model would remain the same.

We have assumed throughout that the claim number distribution is Poisson,

so that the distribution of the time intervals between successive claims is exponential. Thus, given the reserve level at any time t, the future behaviour of the reserves is independent of the reserve level at time s, where s < t, due to the "memoryless" property of the exponential distribution. Clearly the Markov property is satisfied in this case. This would not be true of a different claim number distribution. Although we are thus restricted to applying the Markov chain model to this particular claim number distribution, we have shown that it is otherwise adaptable to situations where exact analysis does not lead to numerical values for ruin probabilities.

APPENDIX 1 : PROOF OF THEOREM OF SECTION 3.3

We start by making the following definitions:

$\rho_{ij}^{(n)}$ is the probability that a system, starting in state i, is in state j after n transitions.

$f_{ij}^{(n)}$ is the probability that a system, starting in state i, first visits state j after n transitions. If j = i, then $f_{ii}^{(n)}$ is the probability of a first return to state i after n transitions.

$F_{ij}(s) = \sum_{n\geq 1} f_{ij}^{(n)} s^n$, (similarly $F_{ii}(s)$).

$\mu_j = \sum_{n\geq 1} n f_{ij}^{(n)}$ is the mean first passage time from state i to state j. If j = i, we call μ_i the mean recurrence time for returns to state i.

We have that $\lim_{n\to\infty} \rho_{ij}^{(n)} = \pi_{ij} = \frac{F_{ij}(1)}{\mu_j}$, (see [3], p96). We shall show that for a recurrent class c,

i if $x_i^c = 1/\mu_i$, $i \varepsilon c$

$= 0$, $i \not\varepsilon c$

then $\sum_c x_i^c = 1$ and $\underline{x}^c P = \underline{x}^c$

ii if, for $j \varepsilon c$, $y_i^c = F_{ij}(1)$ independently of our choice of j, then $\underline{y}^c$ satisfies $P\underline{y}^c = \underline{y}^c$, and

iii $\pi_{ij} = \sum_c y_i^c x_j^c$, and the above solutions are unique.

Proof

i Let $x_i^c = 1/\mu_i$, $i \varepsilon c$

$= 0$, $i \not\varepsilon c$

For $j \varepsilon c$, $F_{ij}(1) = 1$, so that $\pi_{ij} = 1/\mu_j$.

Now, $\sum_{j \varepsilon c} \pi_{ij} = \lim_{n \to \infty} \sum_{j \varepsilon c} \rho_{ij}^{(n)}$

and the recurrent class c forms a Markov chain in its own right, so that

$$\sum_{j \varepsilon c} \rho_{ij}^{(n)} = 1 \quad \text{for all } n.$$

Hence

$$\sum_{i \varepsilon c} x_i^c = \sum_{j \varepsilon c} \pi_{ij} = 1$$

Now $\rho_{ij}^{(n+1)} = \sum_K \rho_{iK}^{(n)} \rho_{Kj}$ for all n

So as $n \to \infty$, we have $\pi_{ij} = \sum_K \pi_{iK} \rho_{Kj}$

But $x_j^c = \pi_{ij}$, so that $x_j^c = \sum_K x_K^c \rho_{Kj}$

i.e. $\underline{x}^c = \underline{x}^c P$.

ii Let j be any other state in c, and let $Y_i^c = F_{ij}(1)$, for all i. This is independent of our choice of j as it is easily shown that if ℓ is another state in c, then $F_{ij}(1) = F_{i\ell}(1)$.

If i and j are in the same recurrent class then $F_{ij}(1) = 1$ and if they are in different recurrent classes then $F_{ij}(1) = 0$.

Thus $y_i^c = 1$ if $i \varepsilon c$, (j is in c)

$= 0$ if $i \varepsilon c'$, where c' is a different recurrent class.

By considering the first time step we have

$$y_i^c = F_{ij}(1) = \sum_{K \neq j} \rho_{iK} F_{Kj}(1) + \rho_{ij}.1$$

$$= \sum_K \rho_{iK} F_{Kj}(1) \quad \text{since j is recurrent}$$

$$= \sum_K \rho_{iK}\, y_K^c$$

i.e. $\underline{y}^c = P\underline{y}^c$.

iii By (i) and (ii) we have

$$\pi_{ij} = y_i^c\, x_j^c \quad \text{for} \quad j \varepsilon c$$

and that $\underline{y}^c$ and $\underline{x}^c$ satisfy the conditions of the theorem. Further, $\pi_{ij} = 0$ if j is a transient state, since $\mu_j = \infty$. Now $y_i^c x_j^c$ has non-zero terms only for $j \varepsilon c$, so that

$$\pi_{ij} = \sum_{c_K} y_i^{c_K} x_j^{c_K} \Rightarrow \pi = \sum_{c_K} \underline{y}^{c_K} \underline{x}^{c_K}$$

where the summation is over all recurrent classes $\{c_K\}$. This is true because a non-zero value of $y_i^{c_K} x_j^{c_K}$ implies a zero value of $y_i^{c_\ell} x_j^{c_\ell}$ where c_K and c_ℓ are different recurrent classes.

Finally, we show that the above solutions are unique.

Let $\underline{Z}$ be any solution of $\underline{Z} = \underline{Z}P$.

Then $\underline{Z} = \underline{ZP} = (\underline{ZP})P = \underline{ZP}^2 = \underline{ZP}^3 = \ldots\ldots = \underline{ZP}^n = \ldots.$

Since $P^n \to \pi$ as $n \to \infty$, we have $\underline{Z} = \underline{Z\pi}$

Thus $\underline{Z} = \underline{Z}\sum_{c_K} \underline{y}^{c_K}\underline{x}^{c_K} = \sum_{c_K} \alpha_{c_K}\underline{x}^{c_K}$ where $\alpha_{c_K} = \underline{Zy}^{c_K}$

So Z is simply a linear combination of the x^{c_K}.

Hence the solution is unique. A similar proof holds for the $\underline{y}^{c_K}$.

APPENDIX 2

COMPUTER PROGRAMS

1. GAMMA3

The following program illustrates the calculation of exact values of $\xi(u,K)$ when single claim amounts follow a $\gamma(3)$ distribution. In this example, one root of the characteristic equation [4.5] is real and two are complex. These roots are found by the NAG routine CØ2AEF. The simultaneous equations defined by [4.20] (i.e. [4.18] with r = 1 and c = 1), are solved by the NAG routine FØ4ATF. Once we have obtained b_1, γ and δ, as defined by [4.20], we find $\xi(u,K)$ by [4.19].

Note: The references are all to equations in Chapter 2.

```
*       CALCULATES EXACT VALUES OF XI WHEN SINGLE CLAIM
*       AMOUNTS ARE DISTRIBUTED AS GAMMA(3) WITH SCALE
*       PARAMETER BETA, WHEN ONE ROOT OF THE CHARACTERISTIC
*       EQUATION IS REAL AND TWO ARE COMPLEX.
*
        DOUBLE PRECISION B(3),C(3),D(3),IMZ(3),REZ(3)
        DOUBLE PRECISION WKS1(3),WKS2(3),A(3,3),AA(3,3)
        DOUBLE PRECISION G,K,P,Q,R,U,B0,BP,BQ,BR,XI
*
*       INPUT THE COEFFICIENTS D(I) OF THE POWERS IN L
*       IN THE CHARACTERISTIC EQUATION (4.5).
*
        WRITE(6,2)
        READ(5,1)(D(I),I=1,4)
        TOL=X02AAF(X)
        CALL C02AEF(D,4,REZ,IMZ,TOL,0)
*
*       THE ROUTINE C02AEF CALCULATES THE ROOTS OF THE
*       CHARACTERISTIC EQUATION,PLACING THE REAL PARTS
*       IN THE VECTOR REZ AND THE IMAGINARY PARTS IN IMZ.
*
        WRITE(6,3)
        WRITE(6,4)(L,REZ(L),IMZ(L),L=1,3)
        P=-REZ(1)
        Q=IMZ(2)
        R=-REZ(3)
        WRITE(6,5)R,P,Q
*
*       P,Q AND R ARE AS DEFINED IN EQUATIONS (4.20).
*       INPUT THE SCALE PARAMETER BETA, THE UPPER BARRIER K
*       AND THE NUMBER OF STATES NS.
*
        WRITE(6,6)
        READ(5,1)BETA,K,NS
    1 FORMAT(V)
    2 FORMAT(18H INPUT THE COEFFTS)
    3 FORMAT(/4X,1HK,10X,6HREZ(K),12X,6HIMZ(K))
    4 FORMAT(I5,4X,D15.8,4X,D15.8)
    5 FORMAT(/3H R=,D15.8,2X,2HP=,D15.8,2X,2HQ=,D15.8/)
    6 FORMAT(27H INPUT BETA,K,&NO.OF STATES)
*
        BP=1.-(BETA*P)
        BR=1.-(BETA*R)
        BQ=BETA*Q
        G=(BP*BP)+(BQ*BQ)
        Z=DEXP(-R*K)
        Z1=DEXP(-P*K)*DCOS(Q*K)
        Z2=DEXP(-P*K)*DSIN(Q*K)
```

```
*     CREATE THE MATRIX A, WHERE COLUMNS 1,2 AND 3 OF A
*     GIVE THE COEFFICIENTS OF B,GAMMA AND DELTA AS
*     DEFINED IN EQUATIONS (4.20).
*
      DO 10 L=1,3
      B(L)=1.
      A(L,1)=(1./(BR^L))-Z
   10 CONTINUE
      A(1,2)=(BP/G)-Z1
      A(2,2)=(((BP^2)-(BQ^2))/(G^2))-Z1
      A(3,2)=(((BP^3)-3.*BP*(BQ^2))/(G^3))-Z1
      A(1,3)=(-BQ/G)-Z2
      A(2,3)=-(2*BP*BQ/(G^2))-Z2
      A(3,3)=-((3*(BP^2)*BQ-(BQ^3))/(G^3))-Z2
      WRITE(6,7)
      WRITE(6,8)((A(I,J),J=1,3),I=1,3)
      CALL F04ATF(A,3,B,3,C,AA,3,WKS1,WKS2,0)
*
*     THE ROUTINE F04ATF SOLVES THE EQUATIONS (4.20) AND
*     PLACES B,GAMMA AND DELTA IN THE VECTOR C.
*
      WRITE(6,9)
      WRITE(6,11)(C(I),I=1,3)
      WRITE(6,12)
    7 FORMAT(/5X,8HMATRIX A)
    8 FORMAT(3(2X,D15.8))
    9 FORMAT(/5X,7HCOEFFTS)
   11 FORMAT(2X,D15.8)
   12 FORMAT(/3X,1HK,9X,1HU,18X,2HXI)
*
*     WE PRODUCE VALUES OF XI BY (4.19) FOR VALUES OF U
*     WHICH ARE MIDPOINTS OF EACH STATE.
*
      B0=-(C(1)*DEXP(-R*K))-(C(2)*Z1)-(C(3)*Z2)
      DO 20 M=2,NS-1
      U=K*(FLOAT(M)-1.5)/FLOAT(NS-2)
      XI=B0+(C(1)*DEXP(-R*U))+(DEXP(-P*U)*(C(2)*DCOS(Q*U)
     &+C(3)*DSIN(Q*U)))
      WRITE(6,13)M,U,XI
   13 FORMAT(I4,2X,D15.8,2X,D15.8)
   20 CONTINUE
*
      STOP
      END
```

2. MATRIX

In the following pages we present the computer program which was used to calculate the transition matrix in Section 4.3. The parameters N, K and λ are fixed; α can be varied. The first two pages of the program give the structure of the transition matrix, and are easily adapted for different forms of F(x,t). Likewise, the subprograms in the remaining pages can be altered when the subprograms FX and PDF give a different total claim amount and density. In this program we calculate $\delta(0,t)$ by the exact formula [3.4] of Chapter 4 in the subprogram UOT. For the other distributions used in this thesis, we have to calculate $\delta(0,t)$ by numerical integration, using the same procedure as in the subprograms UWT, CHI, RUIN and ADJ.

```
*       CALCULATES THE EXACT TRANSITION MATRIX FOR THE
*       POISSON/GAMMA MODEL.
*
        DOUBLE PRECISION XM(91),YM(91),P(91,91)
        IMPLICIT DOUBLE PRECISION (A-H,O-Z)
        DOUBLE PRECISION LLIM,LAMBDA
*
*
*
*       INPUT: THE TOTAL NUMBER OF STATES,N, THE ABSORBING
*       BARRIER,ABSBAR, THE GAMMA PARAMETER,AL AND THE PREMIUM
*       LOADING,LAMBDA.
*
          READ(12,1)AL
        1 FORMAT(V)
          BE=1./AL
          N=91
          ABSBAR=5.
          LAMBDA=.1
          W=ABSBAR/FLOAT(N-2)
          PREM=1.+LAMBDA
          T=W/PREM
          CT=W
          U=ABSBAR-.5*W
*
*       U IS THE MIDPOINT OF STATE E(N-1).
*       W IS THE WIDTH OF THE NON-ABSORBING STATES.
*       PREM IS THE PREMIUM INCOME PER UNIT TIME.
*       CT IS THE INCOME IN (0,T), WHERE T IS SUCH THAT CT=W.
*
*
*       SET TRANSITION PROBABILITIES FOR STATES E(1) AND E(N).
*
          P(1,1)=1.0
          P(N,N)=1.0
          DO 10 I=2,N
          P(1,I)=0.
          P(N,I-1)=0.
       10 CONTINUE
*
*
*
*       CALCULATE TRANSITION PROBABILITIES TO STATE E(1)
*       FROM STATES E(2),...,E(N-2).
*
          DO 20 J=2,N-2
          XM(J)=W*(FLOAT(J)-1.5)
          P(J,1)=1.-UWT(XM(J),T,AL,BE,PREM,128)
       20 CONTINUE
```

```
*       CALCULATE TRANSITION PROBABILITIES  P(K,L)  FOR
*       K=2,...,(N-1), L=3,...,(N-1).
*
        DO 30 K=2,N-1
        DO 40 L=3,N-1
        M=K-L
        IF (M.LT.-1) GO TO 50
        A=CT+(W*(FLOAT(M)+.5))
        B=CT+(W*(FLOAT(M)-.5))
        P(K,L)=FX(AL,BE,T,A)-FX(AL,BE,T,B)
        GO TO 40
   50 P(K,L)=0.
   40 CONTINUE
   30 CONTINUE
*
        DO 60 L1=2,N-2
        YM(L1)=W*(FLOAT(L1)-1.5)
        P(L1,2)=1.-FX(AL,BE,T,YM(L1))-P(L1,1)
   60 CONTINUE
*
*
*
*       CALCULATE TRANSITION PROBABILITIES TO THE ABSORBING
*       STATES FROM STATE E(N-1), AND ADJUST TRANSITION
*       PROBABILITIES FROM E(N-1) TO THE TRANSIENT STATES.
*
        P(N-1,N)=CHI(U,ABSBAR,T,AL,BE,PREM,128)
        P(N-1,1)=RUIN(U,ABSBAR,T,AL,BE,PREM,128)
        P(N-1,2)=1.-P(N-1,N)-P(N-1,1)
*
        DO 70 J7=3,N-1
        LLIM=(J7-2)*W
        ULIM=(J7-1)*W
        P(N-1,J7)=P(N-1,J7)-ADJ(U,ABSBAR,T,AL,BE,PREM,128,LLIM,ULIM)
        P(N-1,2)=P(N-1,2)-P(N-1,J7)
   70 CONTINUE
*
        WRITE(13)((P(I,J),J=1,91),I=1,91)
        STOP
        END
```

```
*       FUNCTION SUBPROGRAMME TO CALCULATE F(x,t).
*
        DOUBLE PRECISION FUNCTION FX(A,B,T,X)
        IMPLICIT DOUBLE PRECISION (A-H,O-Z)
        IF (X.LT.0.) GO TO 10
        IF (X.LT.1.0D-10) GO TO 20
        Y=X/B
        FX=1.
        T1=T
        K=1
   30   KA=K*A
        TERM=T1*GAM(Y,KA)
        IF (TERM.LT.1.0D-11) GO TO 40
        FX=FX+TERM
        K=K+1
        T1=T*T1/FLOAT(K)
        GO TO 30
   40   FX=FX*DEXP(-T)
        GO TO 50
   10   FX=0.
        GO TO 50
   20   FX=DEXP(-T)
   50   RETURN
        END
*
*
*
*       FUNCTION SUBPROGRAMME TO CALCULATE f(x,t).
*
        DOUBLE PRECISION FUNCTION PDF(A,B,T,X)
        IMPLICIT DOUBLE PRECISION (A-H,O-Z)
        T1=T
        Z=(DEXP(-T-X/B))/B
        PDF=0.
        K=1
   20   TERM=T1*TKA(X,A,B,K)
        IF (TERM.LT.1D-11) GO TO 10
        PDF=PDF+TERM
        K=K+1
        T1=T1*T/FLOAT(K)
        GO TO 20
   10   PDF=PDF*Z
        RETURN
        END
```

```
*       FUNCTION SUBPROGRAMME TO CALCULATE THE PROBABILITY
*       OF SURVIVAL TO TIME T, FROM INITIAL RESERVE LEVEL U,
*       WHEN THE UPPER BARRIER CANNOT BE REACHED BY TIME T.
*
      DOUBLE PRECISION FUNCTION UWT(U,T,A,B,PREM,N)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      INTEGER N
      H=T/FLOAT(N)
      W=U+PREM*T
      ZERO=0.
      DO 10 J=1,N-1
      S=H*FLOAT(J)
      TS=T-S
      X=U+PREM*S
      TERM=UOT(A,B,PREM,TS)*PDF(A,B,S,X)
      ZERO=ZERO+TERM
   10 CONTINUE
      ZERO=PREM*H*(ZERO+.5*PDF(A,B,T,W))
      UWT=FX(A,B,T,W)-ZERO
      RETURN
      END
*
*
*
*       FUNCTION SUBPROGRAMME TO CALCULATE THE PROBABILITY
*       OF ABSORPTION BY THE UPPER BARRIER K, FROM INITIAL
*       RESERVE LEVEL U, BY TIME T.
*
      DOUBLE PRECISION FUNCTION CHI(U,K,T,A,B,PREM,N)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION K
      INTEGER N
      T0=(K-U)/PREM
      H=(T-T0)/FLOAT(N)
      Z=DEXP(-T0)
      CHI=0.
      DO 10 J=1,N-1
      S=T0+(H*FLOAT(J))
      X=U-K+(PREM*S)
      TERM=(K-U)*PDF(A,B,S,X)/S
      CHI=CHI+TERM
   10 CONTINUE
      CT=U-K+(PREM*T)
      CHI=Z+H*(CHI+.5*(K-U)*PDF(A,B,T,CT)/T)
      RETURN
      END
```

```
*       FUNCTION SUBPROGRAMME TO CALCULATE THE PROBABILITY
*       OF ABSORPTION BY THE LOWER BARRIER FROM STATE E(N-1)
*       BY TIME T.
*
      DOUBLE PRECISION FUNCTION RUIN(U,K,T,A,B,PREM,N)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION K
      INTEGER N
      T0=(K-U)/PREM
      H=T0/FLOAT(N)
      Z=DEXP(-T0)*(1.-UWT(K,T0,A,B,PREM,N))
      RUIN=0.
      DO 10 J=1,N-1
         S=T0+(H*FLOAT(J))
         TS=T-S
         X=U-K+PREM*S
         TERM=(K-U)*PDF(A,B,S,X)*(1.-UWT(K,TS,A,B,PREM,N))/S
         RUIN=RUIN+TERM
   10 CONTINUE
      RUIN=1.-UWT(U,T,A,B,PREM,N)-Z-(H*RUIN)
      RETURN
      END
*
*
*
*       FUNCTION SUBPROGRAMME TO CALCULATE THE ADJUSTMENTS
*       TO TRANSITION PROBABILITIES FROM STATE E(N-1) TO
*       INCLUDE THE POSSIBILITY OF HITTING THE UPPER BARRIER
*       PRIOR TO T.
*
      DOUBLE PRECISION FUNCTION ADJ(U,K,T,A,B,PREM,N,L1,L2)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION K,L1,L2
      INTEGER N
      T0=(K-U)/PREM
      R1=K+(PREM*T0)-L1
      R2=K+(PREM*T0)-L2
      Z=DEXP(-T0)*(FX(A,B,T0,R1)-FX(A,B,T0,R2))
      H=T0/FLOAT(N)
      ADJ=0.
      DO 10 J=1,N-1
      S=T0+(H*FLOAT(J))
      X=U-K+(PREM*S)
      TS=T-S
      Y1=K+(PREM*TS)-L1
      Y2=K+(PREM*TS)-L2
      TERM=(K-U)*PDF(A,B,S,X)*(FX(A,B,TS,Y1)-FX(A,B,TS,Y2))/S
      ADJ=ADJ+TERM
   10 CONTINUE
      ADJ=Z+(H*ADJ)
      RETURN
      END
```

```
*       FUNCTION SUBPROGRAMME TO CALCULATE THE PROBABILITY
*       OF SURVIVAL TO TIME T FROM AN INITIAL RESERVE LEVEL
*       OF ZERO.
*
      DOUBLE PRECISION FUNCTION UOT(A,B,PREM,T)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      S=0.
      K=1
      T1=T
      BCT=PREM*T/B
   20 KA=K*A
      TERM=T1*SUM(BCT,KA)
      IF (TERM.LT.1D-10) GO TO 10
      S=S+TERM
      K=K+1
      T1=T*T1/FLOAT(K)
      GO TO 20
   10 UOT=1.-S*DEXP(-T)/BCT
      RETURN
      END
*
*
*
*       FUNCTION SUBPROGRAMME TO CALCULATE TERMS IN
*       THE SUBPROGRAMME UOT.
*
      DOUBLE PRECISION FUNCTION SUM(Y,K)
      DOUBLE PRECISION Y,GAM,SUM
      SUM=0.
      J=1
   20 SUM=SUM+GAM(Y,J)
      J=J+1
      IF (J.GT.K) GO TO 10
      GO TO 20
   10 RETURN
      END
*
*
*
*       FUNCTION SUBPROGRAMME TO CALCULATE THE INCOMPLETE
*       GAMMA FUNCTION RATIO P(x,a), WHEN a IS AN INTEGER.
*
      DOUBLE PRECISION FUNCTION GAM(X,I)
      DOUBLE PRECISION T,X,GAM
      GAM=1.
      M=0
      T=DEXP(-X)
   20 GAM=GAM-T
      M=M+1
      IF (M.EQ.I) GO TO 10
      T=T*(X/FLOAT(M))
      GO TO 20
   10 RETURN
      END
```

```
*       FUNCTION SUBPROGRAMME TO CALCULATE TERMS IN
*       THE SUBPROGRAMME PDF.
*
        DOUBLE PRECISION FUNCTION TKA(X,A,B,K)
        DOUBLE PRECISION A,B,X,Y,TKA,TERM
        Y=X/B
        KA=K*A-1
        IF (KA.EQ.0) GO TO 10
        L=1
        TKA=Y/FLOAT(L)
   20   L=L+1
        IF (L.GT.KA) GO TO 30
        TKA=TKA*Y/FLOAT(L)
        IF (TKA.LT.1D-10) GO TO 40
        GO TO 20
   10   TKA=1.
        GO TO 30
   40   TKA=0.
   30   RETURN
        END
*
        STOP
        END
```

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