AN APPLICATION OF STOCHASTIC INTEREST RATES MODELS IN LIFE ASSURANCE

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Abstract

Although assurance companies are pooling many risks, the law of large numbers does not fully apply. This leaves the companies with a possibility of insolvency and a corresponding need for contingency reserves which are matters of serious concern. In this thesis we derive some fundamental results that are useful when the time comes to set contingency reserves or to assess solvency. We use a model where both the mortality and the interest rates are random variables. We choose to model the force of interest by the Ornstein-Uhlenbeck process. For temporary assurances and endowment assurances we derive an efficient recursive method to find the first three moments of the present value of a portfolio of identical policies. We then use these moments to approximate accurately the distribution of the present value of such a portfolio, firstly when the number of policies in the portfolio tends to infinity, and secondly, for a portfolio of finite size.

Key Notation

Notation	Page*	Description
A	47	Net single premium for a life assurance policy (a general symbol). Examples of particular cases are: $A_{x:\overline{n}\vert}$, $A^1_{x:\overline{n}\vert}$.
^{m}A	47	Same as A but discounted with forces of interest of $m\cdot\delta_t$.
ACPP(*,n,c)	180	Average cost (or present value) per policy for a portfolio. Where * identifies the type of assurance contracts (T for temporary and E for endowment), n is the term of each policy, and c is the number of policies in the portfolio.
b_k	46	Benefit payable at time k under a life assurance contract.
c	79	Number of policies in a portfolio.
c_i	114	Random variable denoting the number of policies with a benefit payable at time i+1 for a portfolio of c policies.
c_i	179	A realization of c_i.
$\mathfrak{C}_i$	180	Random variable denoting the number of policies with a benefit payable between time 0 and time i for a portfolio of c policies.
c_i	181	A realization of $\mathfrak{C}_i$.
cdf	49	Cumulative distribution function.
cov(·,·)	12	Covariance function.
cv[·]	41	Coefficient of variation (expected value divided by standard deviation).

$\exp\{\cdot\}$	12	Exponential function. (same as $e^{\cdot}$)
$E[\cdot]$	11	Expected value.
$\hat{E}[\cdot]$	139	Approximated expected value.
$E_n[\cdot]$	96	Expected value of a function of random present values of benefits for assurance contracts of term n (used in recursive equations).
g_n	117	A function based on the distribution of ζ_n and the density of y(n).
$\mathcal{g}_n$	181	A function based on the distribution of ACPP(T,n,c), the density of $\mathfrak{C}_n$ and the density of y(n).
h_n	150	A function based on the distribution of $\mathfrak{Z}_n$ and the density of y(n).
$\hbar_n$	202	A function based on the distribution of ACPP(E,n,c), the density of $\mathfrak{C}_n$ and the density of y(n).
K	46	Random variable denoting the curtate-future-lifetime of someone aged x.
k	47	A realization of K (or a summation indice).
pdf	49	Probability density function.
${}_np_x$	56	Probability that someone aged x survives to age x+n.
${}_{k\|}q_x$	47	Probability that someone aged x dies between age x+k and x+k+1.
$sd[\cdot]$	35	Standard deviation of a random variable.
$sk[\cdot]$	35	Skewness of a random variable.
t(k)	46	Time of payment function.
$V[\cdot]$	12	Variance.
W_t	11	Wiener process.
x	46	Age of a life assured at time of issue.
y(t)	23	Random variable of the integral of δ_s between 0 and t.
y_t	116	A realization of y(t).

$y_t[i]$	128	i^{th} element of a vector of realizations of y(t).
$\tilde{y}(t)$	26	Random variable of the integral of $\delta_s-\delta$ between 0 and t.
$\mathcal{Z}$	46	Random variable denoting the present value of the benefit payable under a life assurance contract.
$\mathfrak{z}$	61	A realization of $\mathcal{Z}$.
$\mathcal{Z}_n$	55	Random variable denoting the present value of a n-year life assurance contract (used only in Chapter 4).
$\mathcal{Z}_i$	79	Random variable denoting the present value of the benefit payable to life assured i of a portfolio.
${}_c\mathcal{Z}$	79	Random variable denoting the present value of all the benefits for a portfolio of assurance contracts.
ζ_n	115	Random variable of the average cost per policy for a limiting portfolio of temporary assurance contracts.
z_n	116	A realization of ζ_n.
$z_n[i]$	139	i^{th} element of a vector of realization of ζ_n.
$\mathfrak{Z}_n$	148	Random variable of the average cost per policy for a limiting portfolio of endowment assurance contracts.
$\mathfrak{z}_n$	150	A realization of $\mathfrak{Z}_n$.
$\mathfrak{z}_n[i]$	169	i^{th} element of a vector of realization of $\mathfrak{Z}_n$.
α	12	Friction parameter of the Ornstein-Uhlenbeck process.
α_0, α_1	13	Parameters of the second order SDE.
δ	12	Ultimate force of interest.
δ_0	11	Current force of interest (at time 0).
δ_t	11	Force of interest per unit time at time t.
δ_0'	14	Derivative of the force of interest evaluated at time 0.
δ_t'	14	Derivative of the force of interest.
$\phi(\cdot)$	118	Probability density of the standardized normal.
$\Phi(\cdot)$	131	Cumulative distribution of the standardized normal.
$\rho(\cdot,\cdot)$	85	Correlation coefficient.

ρ_1	189	Multiple correlation coefficient between ACPP(T,n-1,c) and $(e^{-y(n)}, e^{-y(n-1)}, \mathfrak{C}_{n-1})$.
ρ_2	189	Multiple correlation coefficient between ACPP(T,n-1,c) and $(e^{-y(n-1)}, \mathfrak{C}_{n-1})$.
$\rho_{Y \cdot X}$	190	Multiple correlation coefficient between Y and X.
σ	11	Diffusion coefficient.
ω	46	Limiting age. Least integer where there is no survivor.

* Page where the notation is used for the first time.

CHAPTER 1

INTRODUCTION

1.1 Contingency Reserve and Solvency.

Very briefly, assurance may be defined as a transfer of financial risk which involves a sharing of losses, see, for example, Black and Skipper (1987, p.13). The policyholder pays a premium to the insurer in return for which the insurer agrees to pay a defined amount in case of loss. So, for a premium, an assurance company is assuming the policyholder's risk. And although the company is pooling many risks, the law of large numbers does not fully apply. There is always a possibility of insolvency or ruin of the company and even if this possibility is thought to be negligible, there remains a need for a contingency reserve to cover possible adverse fluctuations.

The necessary contingency reserve and the related risk of insolvency are matters of concern to assurance companies as well as to regulating authorities. The need exists for more study in these areas.

One approach to study the ruin probabilities of assurance companies is the subject of ruin theory or risk theory. Ruin theory is mainly concerned with a portfolio of policies where each policy faces the possibility that more than one loss occur in the given (usually short) period of coverage. This approach is certainly appropriate for casualty assurance and short term sickness and disability assurance. Many authors have used this approach, see for example, Beekman (1985), Panjer (1986), DePril (1986), De Vylder and Goovaerts (1988), Shiu (1988), Dufresne and Gerber (1988, 1989).

Since those policies are usually of short term nature, they seldom involve interest rates. Some exceptions to this are Schnieper (1983), Waters (1983) and Dufresne F. (1989) who have studied ruin theory with different approaches regarding interest rates or inflation. Waters (1990) calculated the moments of the present value of the profit on a sickness insurance policy.

For life assurance, the situation is usually the reverse in the sense that the loss will only occur once (for example at time of death) and the term of the contract may be very long, up to a lifetime, making the interest rates an essential factor. These characteristics make the ruin theory approach inappropriate.

The mortality risk is fairly easy to model. A generally accepted model exists where future lifetime is simply regarded as a random variable with some analytical distribution (for example, Gompertz, Makeham) or a non-parametric distribution given by a life table. The mortality risk is also a risk that is possible to spread. The larger the number of policies sold, the smaller the mortality risk.

One of the problems facing actuaries working with life assurance products is the random nature of future interest rates. This is something they have to deal with every time they have to price or value a product. That random nature of interest is a vital consideration by valuation actuaries and government actuaries when the time comes to determine a contingency reserve or to assess the solvency of a company.

The biggest problem with the interest risk lies in the fact that, as opposed to the mortality risk, it is not possible to spread it by selling a large number of policies. This is so because, even for very large companies, every policy is generally subjected to the same interest rates, or at least to highly correlated interest rates. This makes the interest risk a much more important one than the mortality risk.

Only a few years back, Bowers et al. (1986, p.xviii), in their book entitled *Actuarial Mathematics*, wrote:

> *"... there are some technical problems in building models to combine random interest and random time at claim that are only in the process of being solved at the time of this publication."*

One of the technical problems may be that there is no generally accepted model for interest rates. Another technical problem may be that, even for simple interest rates models, studying actuarial functions is often a very complicated task.

With regard to this problem, one broad objective of this thesis is to propose a way of combining random interest and random time at claim for life assurance.

The following section briefly outlines some of the papers published so far about interest rate modelling and stochastic life assurance. It is not intended to be an exhaustive literature review but merely a presentation of some of the approaches that have been suggested to model interest rates and their applications in life assurance.

1.2 Some Proposed Stochastic Models.

Many early papers (see, for example, Pollard and Pollard (1969), Fibiger and Kellison (1971), Tenenbein (1978)) and some textbooks like

Actuarial Mathematics by Bowers et al. (1986) and *Life Insurance Mathematics* by Gerber (1990) use a stochastic approach for the mortality risk. There is a generally accepted approach to modelling this risk which is already being taught in university courses and examined by professional actuarial associations.

More recently, numerous papers have been treating interest rates as random variables. Some present sophisticated models for interest rates with few actuarial applications while others use simpler models for interest rates and put more emphasis on actuarial applications. The following are some of the papers which study actuarial functions when the interest rates are treated as random variables:

Boyle (1976) uses an autoregressive model of order one for interest rates and he finds the moments of the present value of the benefit payable under certain types of assurance policies.

Panjer and Bellhouse (1980, 1981) do something similar but with a more general approach. They use first and second order autoregressive models for interest rates and look at moments of assurance and annuity functions. In the first paper they use stationary processes while in the second they use conditional processes.

Waters (1978) presents a way of finding moments of assurance and annuity functions when interest rates are gaussian, independent and identically distributed. He also presents moments of portfolios of policies and percentage points for limiting distributions computed by fitting Pearson curves to the known moments of the distributions.

Parker (1985) finds the first two moments of some elementary actuarial functions when the force of interest is assumed to be a Wiener process. He also presents results for the variance of the present value of the benefits for a portfolio of identical temporary assurance contracts.

Devolder (1986) presents a new principle of premium calculation for capitalization operations when the capitalization process is being modelled by a Wiener process.

Stationary as well as non-stationary ARIMA(p,0,q) and ARIMA(p,1,q) processes have been used by Giacotto (1986) to model interest rates when analyzing actuarial functions. Dhaene (1989) suggests a more efficient method to compute the moments of the present value function when the force of interest evolves according to an ARIMA(p,d,q) process. He uses an approach similar to the one used by Panjer and Bellhouse (1980, 1981).

Dufresne D. (1988) studies the variability of contribution rates and fund levels when rates of return $(i(t), t\geq 1)$ are independent and identically distributed.

Using the time reversibility property, Dufresne D. (1990) derives a method to obtain the distribution of a perpetuity when the rates of return are independent and identically distributed.

Wilkie (1976) reconsiders some basic results of compound interest when the force of interest is assumed to follow a gaussian random walk. He considers the distributions of the accumulated value of 1 after n periods and the accumulated value of 1 per period after n periods. And, by means of simulations, he presents some results for equity-linked endowment assurances.

Some more sophisticated stochastic models for interest rates (and other financial series) and different applications to actuarial functions are presented in Wilkie (1986a, 1986b and 1987).

Beekman and Fuelling (1990) model the integral of the force of interest by an Ornstein-Uhlenbeck process. They find the first two moments of life annuity contracts and also derive certain boundary crossing probabilities for the Ornstein-Uhlenbeck process.

By consideration of the results of the above and other papers, some more specific objectives of this thesis were determined. These objectives are described in the next section which also provides an outline of the content of the entire thesis.

1.3 Objectives.

As stated earlier, a broad objective is to combine random interest rates and mortality into a general stochastic model applicable to life assurance. Such a model could then be used to determine a contingency reserve and the probability of insolvency of a given life assurance company.

The more specific objectives are, firstly, to suggest a rather general way of finding the moments and the distribution of some simple actuarial functions such as the present value function and the net single premium when considering both interest rates and mortality as random variables.

Secondly, to generalize those results to portfolios of identical assurance policies.

And thirdly, to illustrate the results numerically and graphically as well as trying to understand them by general reasoning whenever possible.

In chapter 2 we consider different stochastic models for interest rates and choose one for our illustration purposes. It would be utopian to think that the chosen model would be the "*best*". It is not our intention to look for the best model or even to recommend any model more than others. The author however believes that, under some criteria presented in this chapter 2, the Ornstein-Uhlenbeck process is an acceptable one and will use it in the following chapters to illustrate the results obtained. The same could be said about the parameters used in the illustrations. Most of the illustrations will involve different parameter values and it is believed that the fitted parameters of most financial series that an actuary might be willing to use will fall in the range of values presented.

We then look, in chapter 3, at the present value function and derive some basic results that will be used extensively in all the subsequent chapters. We also study the distribution of a general linear combination of discounting factor and its exponential. This last result is useful in later chapters.

Studying the first three moments and the cumulative distribution of the net single premium for one assurance contract (temporary, endowment or whole life) is done in chapter 4.

Chapter 5 generalizes the results of chapter 4 to portfolios. We first define a portfolio of identical assurance policies and then derive a recursive method to find the first three moments of the present value of the benefits payable under such portfolios. We also look at the effect of the size of the portfolio on these moments and at the limiting moments when the size of the portfolio tends to infinity.

Chapter 6 presents, for temporary assurance contracts, a way of approximating the distribution of the present value of benefits payable under a limiting portfolio (i.e. when the number of policies in the portfolio tends to infinity).

Chapter 7 is similar to chapter 6 in all points except that the portfolio now consists of endowment assurance contracts.

We then generalize the results of chapters 6 and 7 to portfolios of finite size. In the first part of chapter 8, we illustrate a method to obtain the distribution of the present value of finite portfolios of temporary assurance contracts. In its second part, this method is adapted to finite portfolios of endowment assurance contracts.

Finally, the concluding chapter, chapter 9, summarizes the main results of this thesis. It also presents some possible extensions for these results and suggests some applications. Some ideas for further areas of research in connection with this thesis are also presented.

CHAPTER 2

FORCE OF INTEREST

2.1 Considerations.

Although most of the results that will be presented in the remainder of this thesis do not depend on the use of any particular stochastic process for the force of interest, it will be necessary to choose a specific model in order to present some illustrations of the results.

Bearing in mind the purpose of the particular stochastic process chosen here for the force of interest, it seems appropriate to restrict the possibilities considerably. Firstly, as to whether the process should be discrete or continuous is not really relevant, since most continuous processes will have equivalent processes in the discrete case and vice versa. Somewhat arbitrarily we have decided to look at continuous processes, but this does not affect in any way the conclusions that will be reached.

Secondly, the process should not be unnecessarily complex. Since we are interested in only one variable, the force of interest that will apply on future investments of an insurance company, it is not necessary to look at multivariate models. A multivariate model would be desirable (and perhaps essential) if we were to need consistency between different economic and/or financial variables, but this is not the case here. We will, therefore, restrict the possible processes even more by considering only univariate models.

For simplicity and for easier understanding of the process we will try to keep the number of parameters at a reasonable level.

Thirdly, a very subjective consideration is the realism of the process. This is considered to be subjective, because there is no uniformly accepted realistic model for the force of interest. In light of the considerations discussed so far, the realism aspect of the model will be taken into account only in its most elementary form. More specifically, we will be interested here in determining, for example, how the process is expected to behave in the light of its recent past.

Fourthly, we will require that the chosen process leads to both theoretical and numerical results. By this it is meant that at least for the simplest actuarial functions (such as the present value and the net single premium) we will be able to find theoretical formulae involving only well-known functions and that we will be able to evaluate the resulting expression with the help of a computer.

Under all these considerations, it was decided to look more closely at three stochastic processes: i) the Wiener process, ii) the Ornstein-Uhlenbeck process and iii) a second order stochastic differential equation. These all appear to be acceptable processes that could be used for our illustrative purposes, although they are certainly not the only possible choices.

Note that each of the three processes considered is gaussian, which implies that the force of interest is normally distributed.

In the remainder of this chapter these three processes will be described in more detail using very well-known results. Finally we will choose one particular process to be used to illustrate the results of the later chapters.

2.2 Definitions.

2.2.1 Wiener process.

Let δ_t be the force of interest per unit time at time t. Then, as a Wiener process with parameter σ, δ_t is defined as:

$$d\delta_t = \sigma \cdot dW_t \qquad 2.1$$

where W_t is the standard Wiener process with parameter $\sigma=1$. The stochastic process W_t therefore satisfies the following properties: (Hoel, Port and Stone (1971, p.123))

i) $W_0 = 0$

ii) $W_t - W_s \sim N(0, t-s)$

iii) $W_{t_2}-W_{t_1}$, $W_{t_3}-W_{t_2}, \ldots, W_{t_n}-W_{t_{n-1}}$ are independent for $t_1 \le t_2 \le \ldots \le t_n$.

The Wiener process has been used to model the so-called Brownian motion with no friction, such as the motion of very small particles in a homogeneous medium (Hoel, Port and Stone (1971, p.123), Arnold (1974, p.39)). The parameter σ is called the diffusion coefficient.

The solution of this process is

$$\delta_t = \delta_0 + \int_0^t \sigma \cdot dW_s$$

or $$\delta_t = \delta_0 + \sigma \cdot W_t \qquad 2.2$$

The expected value of δ_t is constant and equal to the initial value of the process, thus, for all values of $t>0$,

$$E[\delta_t] = \delta_0 \qquad 2.3$$

The autocovariance function is:

$$\text{cov}(\delta_s,\delta_t) = \sigma^2\cdot\min(s,t). \qquad 2.4$$

The variance of the force of interest is linear in t and consequently unbounded as t tends to infinity. It is given by:

$$V[\delta_t] = \sigma^2\cdot t. \qquad 2.5$$

2.2.2 Ornstein-Uhlenbeck process.

If we define δ_t such that

$$d\delta_t = -\alpha\cdot(\delta_t-\delta)\cdot dt + \sigma\cdot dW_t \qquad 2.6$$

where α, σ and δ are constants with $\alpha\geq0$ and $\sigma\geq0$, then the force of interest is a Ornstein-Uhlenbeck process. (Arnold (1974, section 8.3))

This has been used to model the Brownian motion of a particle under the influence of friction. (Arnold (1974, p.134))

The parameters of 2.6 may be given some meanings with respect to the Brownian motion with friction. Thus α is a friction force bringing the process back towards δ, which is the long term mean of the process. The diffusion coefficient is σ.

The solution for δ_t is:

$$\delta_t = \delta + e^{-\alpha t}\cdot(\delta_0-\delta) + \int_0^t e^{-\alpha(t-s)}\ \sigma\cdot dW_s \qquad 2.7$$

The expected force of interest is its long term mean plus an exponentially decreasing term that is proportional to the difference between the current value at time 0, δ_0, and its long term mean.

Algebraically, we have:

$$E[\delta_t] = \delta + e^{-\alpha t}\cdot(\delta_0-\delta) \qquad 2.8$$

so this expectation starts at δ_0 and tends monotonically to δ as $t\to\infty$.

The autocovariance function is:

$$\text{cov}(\delta_s,\delta_t) = e^{-\alpha(t+s)}\cdot \frac{\sigma^2}{2\alpha}\cdot(e^{2\alpha s} - 1) \qquad s\le t. \qquad 2.9$$

So its variance is:

$$V[\delta_t] = \text{cov}(\delta_t,\delta_t) = \frac{\sigma^2}{2\alpha}\cdot(1 - e^{-2\alpha t}) \qquad 2.10$$

which is approximately $\sigma^2\cdot t$ if t is small and tends to $\frac{\sigma^2}{2\alpha}$ as t tends to infinity, so it is bounded.

2.2.3 Second order stochastic differential equation.

If we let the force of interest be defined by the equation:

$$d(\frac{d}{dt}\delta_t) = \alpha_1\cdot d(\delta_t-\delta) + \alpha_0\cdot(\delta_t-\delta)\cdot dt + \sigma\cdot dW_t \qquad 2.11$$

then δ_t satisfies a second order stochastic differential equation (that we will sometimes abbreviate to SDE). (Arnold (1974, section 8.2))

Equation 2.11 may be written in a matrix form as:

$$d\begin{pmatrix}\frac{d}{dt}\delta_t \\ \delta_t - \delta\end{pmatrix} = \begin{pmatrix}\alpha_1 & \alpha_0 \\ 1 & 0\end{pmatrix}\cdot\begin{pmatrix}\frac{d}{dt}(\delta_t-\delta) \\ (\delta_t-\delta)\end{pmatrix}\cdot dt + \begin{pmatrix}\sigma \\ 0\end{pmatrix}\cdot dW_t \qquad 2.12$$

With this process, the second derivative of the force of interest depends on its first derivative, on how far it is from its long term mean, δ, and on a diffusion term.

The solution of this is given by: (Arnold (1974, p.129))

$$\begin{pmatrix} \delta_t' \\ \delta_t - \delta \end{pmatrix} = e^{M\cdot t}\cdot\begin{pmatrix} \delta_0' \\ (\delta_0 - \delta) \end{pmatrix} + \int_0^t e^{M(t-s)}\,\sigma\cdot dW_s \qquad 2.13$$

$$\text{where } M = \begin{pmatrix} \alpha_1 & \alpha_0 \\ 1 & 0 \end{pmatrix} \qquad 2.14$$

$$\delta_t' = \frac{d}{dt}\,\delta_t \qquad \text{and} \qquad \delta_0' = \frac{d}{dt}\,\delta_t\Big|_{t=0} \qquad 2.15$$

The expectation is given by: (Arnold (1974, p.131))

$$E\begin{bmatrix} \delta_t' \\ \delta_t - \delta \end{bmatrix} = e^{Mt}\cdot\begin{pmatrix} \delta_0' \\ (\delta_0 - \delta) \end{pmatrix} \qquad 2.16$$

Alternatively, we can obtain this expectation by solving the following set of ordinary differential equations:

$$dE\begin{bmatrix} \delta_t' \\ \delta_t - \delta \end{bmatrix} = M\cdot E\begin{bmatrix} \delta_t' \\ (\delta_t - \delta) \end{bmatrix}\cdot dt \qquad 2.17$$

The autocovariance function is: (Arnold (1974, p.131))

$$\operatorname{cov}\left[\begin{pmatrix} \delta_s' \\ (\delta_s - \delta) \end{pmatrix}, \begin{pmatrix} \delta_t' \\ (\delta_t - \delta) \end{pmatrix}\right] = \begin{bmatrix} \operatorname{cov}(\delta_s', \delta_t') & \operatorname{cov}(\delta_s', \delta_t) \\ \operatorname{cov}(\delta_s, \delta_t') & \operatorname{cov}(\delta_s, \delta_t) \end{bmatrix} \qquad 2.18$$

$$= e^{Ms}\cdot\left[\int_0^s \left(e^{Mu}\right)^{-1}\cdot\begin{pmatrix} \sigma^2 & 0 \\ 0 & 0 \end{pmatrix}\cdot\left(\left(e^{Mu}\right)^{-1}\right)^T\cdot du\right]\cdot\left(e^{Mt}\right)^T \qquad s\le t \qquad 2.19$$

The variance of δ_t is given by the element (2,2) of the autocovariance function (2.19) when s=t.

To get explicit results when M is defined as in 2.14 we start by solving 2.17 with initial values δ_0 and δ_0'.

The eigenvalues of M are

$$\lambda_1 = \frac{\alpha_1 - \sqrt{\alpha_1^2 + 4\alpha_0}}{2} \qquad 2.20(a)$$

$$\lambda_2 = \frac{\alpha_1 + \sqrt{\alpha_1^2 + 4\alpha_0}}{2} \qquad 2.20(b)$$

and the eigenvectors are: $\begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}$

so that

$$E\begin{bmatrix} \delta_t' \\ \delta_t - \delta \end{bmatrix} = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} c_1 \cdot \exp\{\lambda_1 t\} \\ c_2 \cdot \exp\{\lambda_2 t\} \end{pmatrix} \qquad 2.21$$

where c_1 and c_2 are determined such that

$$E\begin{bmatrix} \delta_t' \\ \delta_t - \delta \end{bmatrix}\Bigg|_{t=0} = \begin{pmatrix} \delta_0' \\ (\delta_0 - \delta) \end{pmatrix} \qquad 2.22$$

which gives

$$c_1 = \frac{-\delta_0' + \lambda_2 \cdot (\delta_0 - \delta)}{\lambda_2 - \lambda_1} \qquad 2.23$$

and

$$c_2 = \frac{\delta_0' - \lambda_1 \cdot (\delta_0 - \delta)}{\lambda_2 - \lambda_1} \qquad 2.24$$

Now by 2.16 and 2.21 it follows that

$$e^{Mt} = \begin{pmatrix} \dfrac{\lambda_2\exp\{\lambda_2 t\}-\lambda_1\exp\{\lambda_1 t\}}{\lambda_2-\lambda_1} & \dfrac{\lambda_1\lambda_2(\exp\{\lambda_1 t\}-\exp\{\lambda_2 t\})}{\lambda_2-\lambda_1} \\ \dfrac{\exp\{\lambda_2 t\}-\exp\{\lambda_1 t\}}{\lambda_2-\lambda_1} & \dfrac{\lambda_2\exp\{\lambda_1 t\}-\lambda_1\exp\{\lambda_2 t\}}{\lambda_2-\lambda_1} \end{pmatrix} \qquad 2.25$$

and that its inverse is:

$$(e^{Mt})^{-1} = \begin{pmatrix} \dfrac{\lambda_2\exp\{-\lambda_2 t\}-\lambda_1\exp\{-\lambda_1 t\}}{\lambda_2-\lambda_1} & \dfrac{\lambda_1\lambda_2(\exp\{-\lambda_1 t\}-\exp\{-\lambda_2 t\})}{\lambda_2-\lambda_1} \\ \dfrac{\exp\{-\lambda_2 t\}-\exp\{-\lambda_1 t\}}{\lambda_2-\lambda_1} & \dfrac{\lambda_2\exp\{-\lambda_1 t\}-\lambda_1\exp\{-\lambda_2 t\}}{\lambda_2-\lambda_1} \end{pmatrix} \qquad 2.26$$

We then have that the integral in 2.19, which we will denote by I(s), becomes

$$I(s) = \begin{pmatrix} I_{1,1}(s) & I_{1,2}(s) \\ I_{2,1}(s) & I_{2,2}(s) \end{pmatrix} \qquad 2.27$$

$$\text{with } I_{1,1}(s) = \frac{\sigma^2}{(\lambda_2-\lambda_1)^2}\cdot\left\{\frac{-\lambda_2}{2}\cdot\left(\exp\{-2\lambda_2 s\}-1\right) + \frac{2\lambda_1\lambda_2}{\lambda_1+\lambda_2}\cdot\left(\exp\{-(\lambda_1+\lambda_2)s\}-1\right)\right.$$
$$\left. + \frac{-\lambda_1}{2}\cdot\left(\exp\{-2\lambda_1 s\}-1\right)\right\} \qquad 2.28$$

$$I_{1,2}(s) = I_{2,1}(s) = \frac{\sigma^2}{(\lambda_2-\lambda_1)^2}\cdot\left\{\frac{-1}{2}\cdot\left(\exp\{-2\lambda_2 s\}-1\right) + \left(\exp\{-(\lambda_1+\lambda_2)s\}-1\right)\right.$$
$$\left. + \frac{-1}{2}\cdot\left(\exp\{-2\lambda_1 s\}-1\right)\right\} \qquad 2.29$$

and $I_{2,2}(s) = \dfrac{\sigma^2}{(\lambda_2-\lambda_1)^2}\cdot\left\{\dfrac{-1}{2\lambda_2}\cdot\left(\exp\{-2\lambda_2 s\}-1\right) + \dfrac{2}{\lambda_1+\lambda_2}\cdot\left(\exp\{-(\lambda_1+\lambda_2)s\}-1\right)\right.$

$$\left. + \frac{-1}{2\lambda_1}\cdot\left(\exp\{-2\lambda_1 s\}-1\right)\right\} \qquad 2.30$$

and from 2.19, we finally get the autocovariance function as follows:

$$\mathrm{cov}\left[\begin{pmatrix}\delta_s' \\ (\delta_s-\delta)\end{pmatrix}, \begin{pmatrix}\delta_t' \\ (\delta_t-\delta)\end{pmatrix}\right] = e^{Ms}\cdot I(s)\cdot\left(e^{Mt}\right)^T \qquad 2.31$$

which is simply the product of three two-by-two matrices.

2.3 Relationships.

2.3.1 Ornstein-Uhlenbeck process and second order SDE.

If we let $\alpha_0=0$ in the second order SDE (2.11), we get:

$$d(\frac{d}{dt}\delta_t) = \alpha_1\cdot d(\delta_t-\delta) + \sigma\cdot dW_t \qquad 2.32$$

which means that $\frac{d}{dt}\delta_t$ becomes a Ornstein-Uhlenbeck process with its parameter α being equal to $-\alpha_1$.

2.3.2 Wiener process and second order SDE.

If the parameters α_0 and α_1 of the second order SDE are such that the eigenvalues of M given by 2.20 tend to 0 and $-\infty$, then the limiting second order SDE is a Wiener process. (Pandit and Wu (1983, section 7.6.4))

2.3.3 Wiener process and Ornstein-Uhlenbeck process.

If we set $\alpha=0$ in 2.6 we get 2.1 which is a Wiener process. We can also check that the results given in section 2.2.2, when α tends to 0, are those of a Wiener process.

For example, the expected force of interest becomes:

$$E[\delta_t] = \delta + (\delta_0-\delta) = \delta_0.$$

the autocovariance function 2.9 may be written as:

$$\text{cov}(\delta_s,\delta_t) = \frac{\sigma^2}{2\alpha}\cdot(e^{-\alpha\cdot(t-s)}-e^{-\alpha\cdot(t+s)}) \qquad s\le t \qquad 2.33$$

which gives:

$$\lim_{\alpha\to 0} \text{cov}(\delta_s,\delta_t) = \sigma^2\cdot s \qquad s\le t \qquad 2.34$$

2.4 Illustrations.

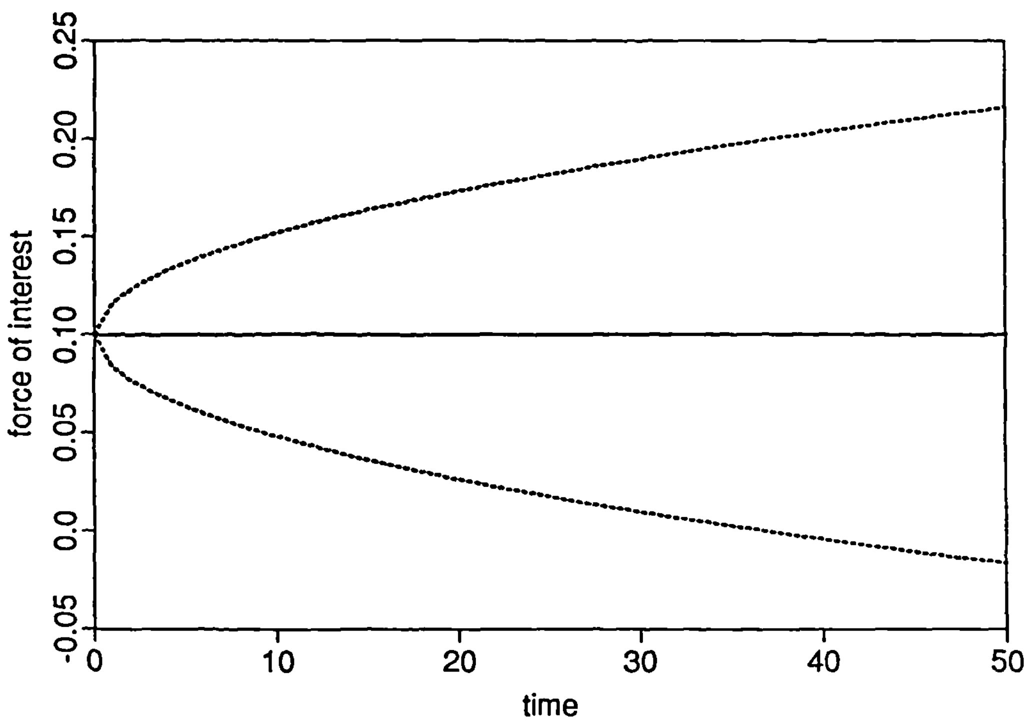
In this section we will look at the expectation and a 95% confidence interval of the force of interest when it is modelled by each of the three stochastic processes that we are considering. The parameters of the processes are chosen somewhat arbitrarily (i.e. they are not meant to represent any particular market) but are thought to be reasonable in terms of what an insurance company might find by estimating the parameters from its past investment experience. For example, the chosen values are comparable to the equivalent discrete estimations obtained by Panjer and Bellhouse (1980, table 1). It is possible that some companies and/or circumstances might call for fairly different values of the parameters then the ones to be used in this section. However since one can easily obtain the results for different values of the parameters from section 2.2 and since the conclusions would most certainly not be affected, it does not appear necessary to present results for a whole range of values of the parameters.

2.4.1 Wiener process.

Figure 2.1 presents the expected future force of interest and a 95% confidence interval assuming that the current force of interest is 10% and that the diffusion coefficient is 1% (i.e. δ_0=.1, σ=.01 in equation 2.6).

Figure 2.1 Expectation and 95% confidence interval of δ_t

Wiener process with δ_0=.1 and σ=.01



The expectation is given by 2.3, it is simply 10% at all times. The 95% confidence interval is given by:

$$E[\delta_t] \mp 1.645\cdot(V[\delta_t])^{.5} \qquad 2.35$$

since δ_t, the force of interest, is normally distributed. The variance of the force of interest is given by 2.5.

In this case the width of the confidence interval is proportional to the square root of t and hence tends to infinity as t becomes very large. This means that we can always find a finite time t where the width of the confidence interval becomes greater than a given value, however unrealistic this value may be. For example, there exist a time where the confidence interval will be say 10% ∓ 1000%. This is highly unrealistic for the force of interest but as long as say up to 100 years it is not too wide, the model may still be used.

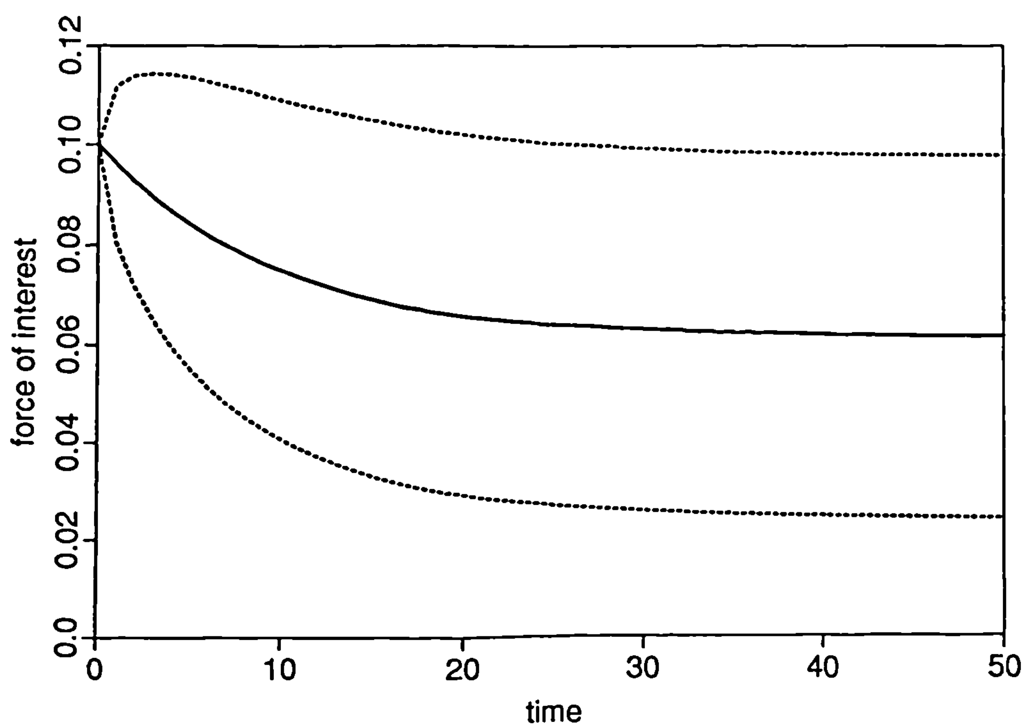
2.4.2 Ornstein-Uhlenbeck process.

The expectation of the force of interest at future times, given by 2.8, and the corresponding 95% confidence interval, given by 2.35, are illustrated in figure 2.2. The variance is given by 2.10. The illustrative values used for the parameters are a long term mean for the force of interest of .06, a current value of .1, a friction parameter of .1 and a diffusion coefficient of .01.

In this example the force of interest is expected to decrease exponentially from 10% to an asymptotic value of 6%. The greater the friction force, the more rapidly it will approach this 6% value. The width of the confidence interval increases asymptotically with time to a limiting value of .0735663.

Figure 2.2 Expectation and 95% confidence interval of δ_t

Ornstein-Uhlenbeck process with δ=.06 δ_0=.1 α=.1 and σ=.01



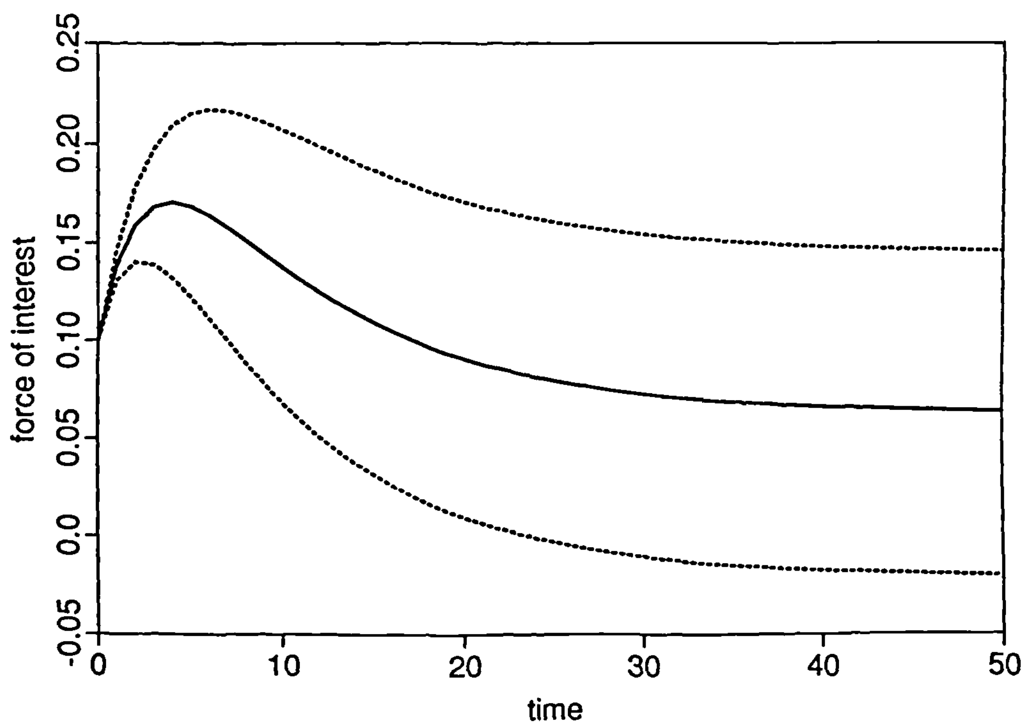
2.4.3 Second order stochastic differential equation.

In figure 2.3 the expected value of the future force of interest, which is the element in the second row of 2.21 and the 95% confidence interval, given by 2.35 are illustrated. Note that the variance is given by the element in the second row second column of 2.31 using the explicit results of 2.25 and 2.27 to 2.30. The illustrative parameters of 2.11 were chosen to be: 5% per unit time for the derivative of the force of interest at time 0, a current value of 10%, a long term mean of 6%, α_1 and α_2 being -.5 and -.04 respectively, and a diffusion coefficient of 1%.

Figure 2.3 Expectation and 95% confidence interval of δ_t

Second order Stochastic Differential Equation

$\delta'_0=.05$ $\delta_0=.1$ $\delta=.06$ $\alpha_1=-.5$ $\alpha_2=-.04$ $\sigma=.01$



Note that in this example, the force of interest is expected to increase for a while before tending asymptotically to its long term mean which is smaller than its current value. This is due to the fact that initially, the influence of the parameter α_1 (-.5) with a relatively high first derivative at origin (.05) added to the influence of the parameter α_2 (-.04) with the process being initially 4% (10%-6%) away from its long term mean, are not enough to bring the first derivative of the process negative although the second derivative is negative.

2.5 Present Value.

2.5.1 Definitions.

The present value (i.e. the value at time 0) of 1 payable at time t is

$$\text{present value} = e^{-y(t)} \qquad 2.36$$

where

$$y(t) = \int_0^t \delta_s \cdot ds \qquad 2.37$$

(see, for example, McCutcheon and Scott (1986, p.18)).

Looking first at the function y(t), we can find its expected value in terms of the expected future forces of interest. It is given by:

$$E[y(t)] = E\left[\int_0^t \delta_s \cdot ds\right] = \int_0^t E[\delta_s] \cdot ds \qquad 2.38$$

The autocovariance function of y(t) can be expressed in terms of the autocovariance of the force of interest as:

$$\mathrm{cov}(y(s),y(t)) = \mathrm{cov}\left(\int_0^s \delta_u \cdot du,\ \int_0^t \delta_v \cdot dv\right)$$

$$= \int_0^s \int_0^t \mathrm{cov}\left(\delta_u,\ \delta_v\right) \cdot dvdu \qquad 2.39$$

Its variance is then obtained from:

$$V[y(t)]=\mathrm{cov}(y(t),y(t)) \qquad 2.40$$

Now, if the force of interest is normally distributed (as is the case with the three stochastic processes considered here), then y(t) is also normally distributed. Also if y(t) is gaussian, then $e^{-y(t)}$ is log-normally distributed with mean: (Aitchison and Brown (1963, p.8))

$$E[e^{-y(t)}] = \exp\{-E[y(t)] + .5V[y(t)]\} \qquad 2.41$$

and variance:

$$V[e^{-y(t)}] = \exp\{-2E[y(t)] + V[y(t)]\}\cdot(\exp\{V[y(t)]\}-1) \qquad 2.42$$

2.5.2 Wiener process.

Under the assumption that the force of interest follows a Wiener process, the expectation of y(t) is obtained by using 2.38 and 2.3 and is:

$$E[y(t)] = \int_0^t E[\delta_s]\cdot ds = \int_0^t \delta_0\cdot ds = \delta_0\cdot t \qquad 2.43$$

Its autocovariance function will be, by using 2.39 and 2.4:

$$\mathrm{cov}(y(s),y(t)) = \int_0^s \int_0^t \mathrm{cov}\left(\delta_u, \delta_v\right)\cdot dvdu$$

$$= \int_0^s \int_0^u \sigma^2\cdot v\cdot dvdu + \int_0^s \int_u^t \sigma^2\cdot u\cdot dvdu \qquad \text{if } s\le t \qquad 2.44$$

$$= \sigma^2\cdot\left(\frac{s^2\cdot t}{2} - \frac{s^3}{6}\right) \qquad s\le t \qquad 2.45$$

Its variance will then be:

$$V[y(t)] = \mathrm{cov}(y(t),y(t)) = \frac{1}{3}\sigma^2\cdot t^3 \qquad 2.46$$

2.5.3 Ornstein-Uhlenbeck process.

If we assume that the force of interest is modelled by a Ornstein-Uhlenbeck process, the function y(t) has an expectation of:

$$E[y(t)] = \int_0^t E[\delta_s]\cdot ds \quad = \int_0^t (\delta + (\delta_0-\delta)\cdot e^{-\alpha s})\ ds$$

$$= \delta\cdot t + (\delta_0-\delta)\cdot\left(\frac{1 - e^{-\alpha t}}{\alpha}\right) \qquad 2.47$$

from 2.38 and 2.8.

Its autocovariance is obtained by combining the results of 2.39 and 2.9 and is:

$$\mathrm{cov}(y(s),y(t))=\frac{\sigma^2}{\alpha^2}\min(s,t)+\frac{\sigma^2}{2\alpha^3}\left[-2+2e^{-\alpha s}+2e^{-\alpha t}-e^{-\alpha|t-s|}-e^{-\alpha(t+s)}\right] \qquad 2.48$$

To derive this last result we start with 2.39:

$$\mathrm{cov}(y(s),y(t)) = \int_0^s \int_0^t \mathrm{cov}(\delta_u,\ \delta_v)\cdot dvdu$$

and from 2.33 assuming $s\le t$, we get:

$$\mathrm{cov}(y(s),y(t)) = \frac{\sigma^2}{2\alpha}\cdot\left\{\int_0^s \int_0^u (e^{-\alpha\cdot(u-v)}-e^{-\alpha\cdot(u+v)})\cdot dvdu + \int_0^s\int_u^t (e^{-\alpha\cdot(v-u)}-e^{-\alpha\cdot(u+v)})\cdot dvdu\right\} \qquad 2.49$$

This is straightforward to integrate and after some simplification gives the desired result for $s\le t$. The same argument can be made for $t\le s$ and hence we get 2.48. An alternative proof of this last result can be found in appendix A.

The function $y(t)$ has a variance of:

$$V[y(t)]= \mathrm{cov}(y(t),y(t)) = \frac{\sigma^2}{\alpha^2}\cdot t + \frac{\sigma^2}{2\alpha^3}\cdot\left[-3+4e^{-\alpha t}-e^{-2\alpha t}\right] \qquad 2.50$$

2.5.4 Second order stochastic differential equation.

Here, to study the function y(t), we consider the system of SDE's:

$$d\begin{pmatrix}\frac{d}{dt}\delta_t \\ \delta_t-\delta \\ \tilde{y}(t)\end{pmatrix} = \begin{pmatrix}\alpha_1 & \alpha_0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0\end{pmatrix}\cdot\begin{pmatrix}\frac{d}{dt}\delta_t \\ \delta_t-\delta \\ \tilde{y}(t)\end{pmatrix}\cdot dt + \begin{pmatrix}\sigma \\ 0 \\ 0\end{pmatrix}\cdot dW_t \tag{2.51}$$

where

$$\tilde{y}(t) = \int_0^t (\delta_s-\delta)\cdot ds \tag{2.52}$$

$$= y(t) - \delta\cdot t \tag{2.53}$$

Note that

$$d\tilde{y}(t) = dy(t) - \delta\cdot dt$$

or by using A.2 for dy(t),

$$d\tilde{y}(t) = (\delta_t-\delta)\cdot dt \tag{2.54}$$

We first study the system of SDE's involving $\tilde{y}(t)$ and then adapt the results to get corresponding results for y(t), since the expectations are related by the following equation:

$$E[y(t)] = E[\tilde{y}(t)] + \delta\cdot t \tag{2.55}$$

and the autocovariance is not affected by the substitution, i.e.

$$\text{cov}\left[\begin{pmatrix}\frac{d}{ds}\delta_s \\ \delta_s-\delta \\ y(s)\end{pmatrix}, \begin{pmatrix}\frac{d}{dt}\delta_t \\ \delta_t-\delta \\ y(t)\end{pmatrix}\right] = \text{cov}\left[\begin{pmatrix}\frac{d}{ds}\delta_s \\ \delta_s-\delta \\ \tilde{y}(s)\end{pmatrix}, \begin{pmatrix}\frac{d}{dt}\delta_t \\ \delta_t-\delta \\ \tilde{y}(t)\end{pmatrix}\right] \tag{2.56}$$

Now, using the results in Arnold (1974, section 8.2) we can find the expectation, the autocovariance function and the variance of this system defined in 2.51.

If we let

$$X_t = \begin{pmatrix} \frac{d}{dt}\delta_t \\ \delta_t - \delta \\ \tilde{y}(t) \end{pmatrix} \tag{2.57}$$

and

$$X_0 = \begin{pmatrix} \delta_0' \\ \delta_0 - \delta \\ 0 \end{pmatrix} \tag{2.58}$$

then we have that its expectation is given by:

$$E[X_t] = e^{Mt} \cdot X_0 \tag{2.59}$$

where

$$M = \begin{pmatrix} \alpha_1 & \alpha_0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.60}$$

and its autocovariance function is given by:

$$\mathrm{cov}(X_s, X_t) = e^{Ms} \cdot I(s) \cdot \left(e^{Mt}\right)^T \quad \text{for } s \le t \tag{2.61}$$

where

$$I(s) = \int_0^s \left(e^{Mu}\right)^{-1} \cdot \begin{pmatrix} \sigma^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \left(\left(e^{Mu}\right)^{-1}\right)^T \cdot du \tag{2.62}$$

Proceeding as in section 2.2.3, we can find I(s) indirectly by first solving the ordinary differential system:

$$dE[X_t] = M \cdot E[X_t] \cdot dt \qquad 2.63$$

subject to the initial conditions for X_0.

Then we use this result to determine e^{Mt} knowing that $E[X_t]$ is also given by 2.59.

This gives:

$$e^{Mt} = \begin{pmatrix} \dfrac{\lambda_2 \exp\{ \lambda_2 t\} - \lambda_1 \exp\{ \lambda_1 t\}}{\lambda_2 - \lambda_1} & \dfrac{\lambda_1 \lambda_2 (\exp\{ \lambda_1 t\} - \exp\{ \lambda_2 t\})}{\lambda_2 - \lambda_1} & 0 \\ \dfrac{\exp\{ \lambda_2 t\} - \exp\{ \lambda_1 t\}}{\lambda_2 - \lambda_1} & \dfrac{\lambda_2 \exp\{ \lambda_1 t\} - \lambda_1 \exp\{ \lambda_2 t\}}{\lambda_2 - \lambda_1} & 0 \\ \dfrac{1-\exp\{\lambda_1 t\}}{\lambda_1(\lambda_2-\lambda_1)} - \dfrac{1-\exp\{\lambda_2 t\}}{\lambda_2(\lambda_2-\lambda_1)} & \dfrac{\lambda_1(1-\exp\{\lambda_2 t\})}{\lambda_2(\lambda_2-\lambda_1)} - \dfrac{\lambda_2(1-\exp\{\lambda_1 t\})}{\lambda_1(\lambda_2-\lambda_1)} & 1 \end{pmatrix} \qquad 2.64$$

and that its inverse is:

$$(e^{Mt})^{-1} =$$

$$\begin{pmatrix} \dfrac{\lambda_2 \exp\{-\lambda_2 t\} - \lambda_1 \exp\{-\lambda_1 t\}}{\lambda_2 - \lambda_1} & \dfrac{\lambda_1 \lambda_2 (\exp\{-\lambda_1 t\} - \exp\{-\lambda_2 t\})}{\lambda_2 - \lambda_1} & 0 \\ \dfrac{\exp\{-\lambda_2 t\} - \exp\{-\lambda_1 t\}}{\lambda_2 - \lambda_1} & \dfrac{\lambda_2 \exp\{-\lambda_1 t\} - \lambda_1 \exp\{-\lambda_2 t\}}{\lambda_2 - \lambda_1} & 0 \\ \dfrac{1-\exp\{-\lambda_1 t\}}{\lambda_1(\lambda_2-\lambda_1)} - \dfrac{1-\exp\{-\lambda_2 t\}}{\lambda_2(\lambda_2-\lambda_1)} & \dfrac{\lambda_1(1-\exp\{-\lambda_2 t\})}{\lambda_2(\lambda_2-\lambda_1)} - \dfrac{\lambda_2(1-\exp\{-\lambda_1 t\})}{\lambda_1(\lambda_2-\lambda_1)} & 1 \end{pmatrix} \qquad 2.65$$

Finally, after lengthy algebra, we find that $I_{1,1}(s)$ is given by 2.28, $I_{2,1}(s)$ and $I_{1,2}(s)$ are given by 2.29, $I_{2,2}(s)$ is given by 2.30 and the other elements of I(s) are:

$$I_{1,3}(s) = I_{3,1}(s) = \frac{\sigma^2 \cdot \exp\{-(\lambda_1+\lambda_2)s\}}{(\lambda_2-\lambda_1)^2} \cdot \left\{ \frac{-(\lambda_2-\lambda_1)}{\lambda_1\lambda_2} \cdot \exp\{\lambda_1 s\} - \right.$$

$$\frac{\exp\{(\lambda_1-\lambda_2)s\}-\exp\{(\lambda_1+\lambda_2)s\}}{2\lambda_2} + \frac{(\lambda_2-\lambda_1)}{\lambda_1\lambda_2} \cdot \exp\{\lambda_2 s\} -$$

$$\left. \frac{\exp\{(\lambda_2-\lambda_1)s\}-\exp\{(\lambda_1+\lambda_2)s\}}{2\lambda_1} + \frac{\lambda_1^2+\lambda_2^2}{\lambda_1\lambda_2(\lambda_1+\lambda_2)} \cdot (1-\exp\{(\lambda_1+\lambda_2)s\}) \right\} \qquad 2.66$$

$$I_{2,3}(s) = I_{3,2}(s) = \frac{\sigma^2 \cdot \exp\{-(\lambda_1+\lambda_2)s\}}{(\lambda_2-\lambda_1)^2} \cdot \left\{ \frac{-(\lambda_2-\lambda_1)}{\lambda_1\lambda_2} \cdot \frac{\exp\{\lambda_1 s\}-\exp\{(\lambda_1+\lambda_2)s\}}{\lambda_2} - \right.$$

$$\frac{\exp\{(\lambda_1-\lambda_2)s\}-\exp\{(\lambda_1+\lambda_2)s\}}{2\lambda_2^2} + \frac{(\lambda_2-\lambda_1)}{\lambda_1\lambda_2} \cdot \frac{\exp\{\lambda_2 s\}-\exp\{(\lambda_1+\lambda_2)s\}}{\lambda_1} -$$

$$\left. \frac{\exp\{(\lambda_2-\lambda_1)s\}-\exp\{(\lambda_1+\lambda_2)s\}}{2\lambda_1^2} + \frac{1-\exp\{(\lambda_1+\lambda_2)s\}}{\lambda_1\lambda_2} \right\} \qquad 2.67$$

$$I_{3,3}(s) = \frac{\sigma^2\cdot\exp\{-(\lambda_1+\lambda_2)s\}}{(\lambda_2-\lambda_1)^2}\cdot$$

$$\left\{\frac{(2s\lambda_2-3)\cdot\exp\{(\lambda_1+\lambda_2)s\}+4\exp\{\lambda_1 s\}-\exp\{(\lambda_1-\lambda_2)s\}}{2\lambda_2^3}\right.$$

$$+\frac{(2s\lambda_1-3)\cdot\exp\{(\lambda_1+\lambda_2)s\}+4\exp\{\lambda_2 s\}-\exp\{(\lambda_2-\lambda_1)s\}}{2\lambda_1^3}$$

$$-\frac{-2}{\lambda_1\lambda_2}\cdot\left(s\cdot\exp\{(\lambda_1+\lambda_2)s\}+\frac{\exp\{\lambda_1 s\}-\exp\{(\lambda_1+\lambda_2)s\}}{\lambda_2}+\right.$$

$$\left.\left.\frac{\exp\{\lambda_2 s\}-\exp\{(\lambda_1+\lambda_2)s\}}{\lambda_1}-\frac{1-\exp\{(\lambda_1+\lambda_2)s\}}{\lambda_1+\lambda_2}\right)\right\} \qquad 2.68$$

Having now found explicit expression for $e^{M\cdot s}$ and I(s), we can find the autocovariance function of y(t), that is element (3,3) of 2.61 which is a product of three three-by-three matrices. This result is the autocovariance of X_t as defined in 2.57; it gives the covariance between all combinations of two random variables chosen from the six that are, for time s and time t, the derivative of the force of interest ($\frac{d}{ds}\delta_s$ and $\frac{d}{dt}\delta_t$), its position (δ_s and δ_t), and its integral ($\tilde{y}(s)$ and $\tilde{y}(t)$ or equivalently, y(s) and y(t)).

2.6 Analysis.

2.6.1 Wiener process.

Assuming the Wiener process for the force of interest gives us a very simple model from which we can easily obtain theoretical and numerical results.

The expectation of the force of interest being constant for all future times has among its disadvantages the fact that its current value is also its long term expectation. For example, if we let the parameter δ_0 be the current force of interest, it might be an unrealistic value for the expected future rates. Also, if we choose anything else than the current value of the force of interest for the parameter δ_0, we may have more realistic expected rates for the future but the model will fail to reflect appropriately the process in its early stage. The consequences of this could be important.

The variance of the force of interest is unbounded, it is linearly proportional to time. This causes the process to take negative values (when t is large enough) with a relatively high probability.

It also causes the variance of the function y(t) to be unbounded. This in turn will imply that the expected present value for very large t will be increasing while it is usually considered to be a monotonic decreasing function of time. Note that depending on the choice of the parameter σ, this might not happen until over 100 years which means that it would not cause any serious problems when studying any typical assurance contract.

However, there are situations where the unbounded variance of the Wiener process may give inconsistent results for the expected present value.

2.6.2 Ornstein-Uhlenbeck process.

For the Ornstein-Uhlenbeck process, the expected force of interest starts at its current value, δ_0, and tends asymptotically and exponentially towards its long term mean δ. This is a more realistic model than the Wiener process as it allows the use of a realistic expected future force of interest while reflecting the actual state of the process.

The friction parameter of this model will imply that the variance of the force of interest will be bounded, it will tend to $\sigma^2/2\alpha$. However, if α is close to 0 and/or σ is very large, we can still get an expected present value that increases with time for some large values of t. This is a consequence of the fact that if α is 0 we obtain the Wiener process.

The model resulting from the use of the Ornstein-Uhlenbeck process for the force of interest is still reasonably simple and we can obtain both theoretical and numerical results from it.

2.6.3 Second order stochastic differential equation.

Now here we will assume the the force of interest is modelled by a second order stochastic differential equation with the two parameters α_1 and α_0 being negative. This is so that a large increase in the force of interest is unlikely to be followed by an even larger increase and so that a force of interest different from its long term mean has a tendency to move towards it rather than to move away from it.

The expected force of interest starts at its current value, δ_0, and is eventually going towards its long term mean, δ. But in the short term it has a tendency to continue its recent trend to some extent before moving smoothly towards its long term mean. So if the process is above the long term mean at time 0 and if it was recently increasing, it might be expected to continue its increasing trend in the immediate future before starting to decrease. This model, by offering more flexibility than the other two, might be considered more realistic.

The variance of the force of interest is bounded, it tends asymptotically to $\sigma^2/2\alpha_0\alpha_1$ when the time tends to infinity.

The probability of having high negative values for the force of interest is reduced even further with this model. The expected present value is now a decreasing function of time for almost all situations for at least 100 years.

This model is more complex than the other two. The computing time required to obtain numerical results is much longer than the other two. We may even face practical problems of overflow, though these may be overcome, when trying to evaluate numerically the variance of the present value as early as duration 25 (depending on the choice of parameters and the computer used).

Also, although it is possible to write down explicit expression for the results that we are interested in, the variances were left in this chapter as a product of three matrices, each of them having fairly long algebraic expressions for its elements.

2.7 Conclusion.

Taking account of all the considerations presented in section 2.1, it seems that the Ornstein-Uhlenbeck process is a reasonable model. It provides a good trade-off between realism and simplicity. It leads to both theoretical and numerical results.

The improvement, in terms of the expected behavior of the present value, that we obtain by moving from a model with unbounded variance to one with bounded variance is considerable. The Ornstein-Uhlenbeck appears to be only slightly more complicated than the Wiener process but a lot more realistic.

The difference between the expected present value when the force of interest is modelled by the Ornstein-Uhlenbeck process and when it is modelled by the second order stochastic differential equation is not significant. The latter is far more complicated and one could face the difficulties in evaluating some results numerically.

Accordingly, we conclude that, among the models considered, the Ornstein-Uhlenbeck process is the most suitable to use for the force of interest, at least for illustrative purposes in the coming chapters.

CHAPTER 3

PRESENT VALUE

3.1 Moments about the origin of the present value.

The present value, i.e. the value at time 0, of 1 payable at t was defined in chapter 2 by the function $e^{-y(t)}$ (see 2.36). We know also from chapter 2 that, if the force of interest is gaussian, this function, $e^{-y(t)}$, is log-normally distributed with parameters -E[y(t)] and V[y(t)]. Consequently, its m^{th} moment about the origin is given by: (see, for example, Aitchison and Brown (1963), p.8))

$$E[(e^{-y(t)})^{m}] = E[e^{-m\cdot y(t)}] = \exp\{-m\cdot E[y(t)] + .5\cdot m^{2}\cdot V[y(t)]\}. \qquad 3.1$$

This last result may be used to get different statistics of the random present value function. For example, its standard deviation is:

$$sd[e^{-y(t)}] = \left[E[e^{-2\cdot y(t)}] - (E[e^{-y(t)}])^{2} \right]^{.5}, \qquad 3.2$$

its skewness is:

$$sk[e^{-y(t)}] = E\left[(e^{-y(t)} - E[e^{-y(t)}])^{3}\right]\cdot (sd(e^{-y(t)}))^{-3}$$

$$= \frac{E[e^{-3\cdot y(t)}] - 3\cdot E[e^{-2\cdot y(t)}]\cdot E[e^{-y(t)}] + 2\cdot E[e^{-y(t)}]^{3}}{(s.d.(e^{-y(t)}))^{3}} \qquad 3.3$$

Note that we will use the notations sd[X] and sk[X] for the standard deviation and skewness respectively of the random variable X.

3.2 Illustrations.

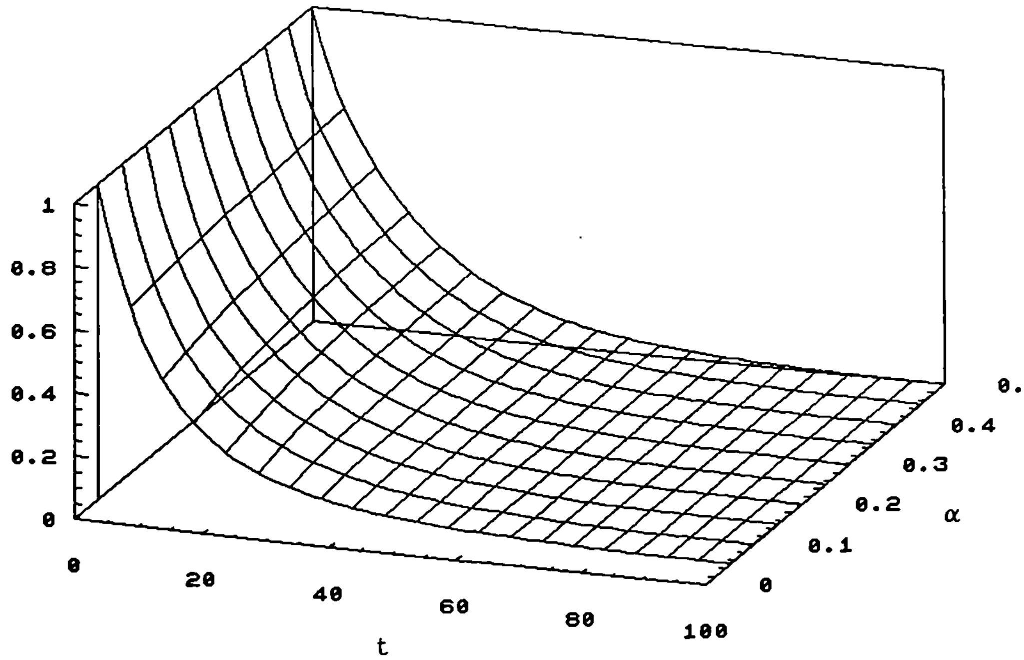
As was mentioned in chapter 2, we will use the Ornstein-Uhlenbeck process to illustrate some of the results obtained in this and later chapters. Figure 3.1 shows the expected value of $e^{-y(t)}$ for different sets of parameters. In part (a), σ is equal to one percent and the parameter α is varied from .005 to .5. In trying to determine the sign of the derivative of the expected value of $e^{-y(t)}$ with respect to α one has to be careful. From equation 2.50, increasing (reducing) the parameter α, without changing the other parameters, reduces (increases) the variance of y(t). And from equation 2.47, if δ_0 is greater than δ, increasing (reducing) the parameter α reduces (increases) the expectation of y(t). This means that the combined effect on the expected present value, given by (3.1) with m=1, is not that easy to predict. For the situation presented in figure 3.1 (a), it happens to be that a greater α produces a greater expected present value.

If δ_0 were equal to δ, then an increase in α would reduce the variance of y(t) without affecting its expectation. In this case, the expected present value would decrease as α increases. If δ_0 were smaller than δ, an increase in α would still reduce the variance of y(t) and would increase its expectation. So, in this latter case also, the expected present value would decrease as α increases.

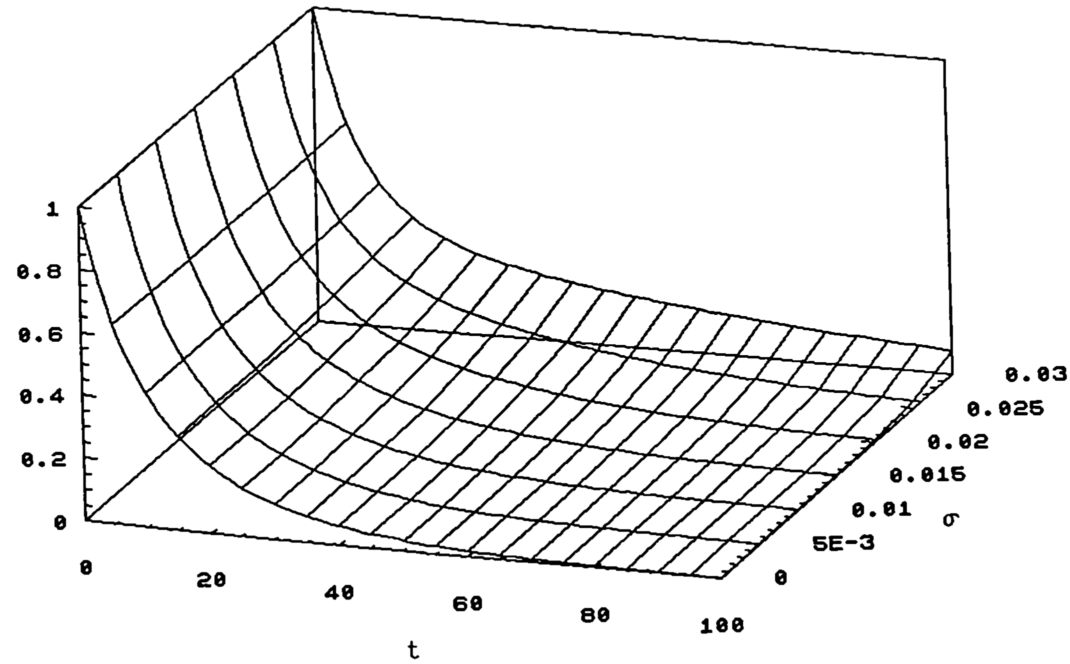
In part (b), α is equal to .1 and the parameter σ is varied from 0 to 3 percent. Here, increasing (reducing) σ without changing the other parameters, increases (reduces) the variance of y(t) (see 2.50) but it

Figure 3.1 Expected Present Value : $E[e^{-y(t)}]$

(a) varying α $\delta=.06$ $\delta_0=.1$ $\sigma=.01$



(b) varying σ $\delta=.06$ $\delta_0=.1$ $\alpha=.1$



has no effect on its expectation (see 2.47). Consequently, for any δ_0 and δ, increasing σ will produce a greater expected present value.

The standard deviation of the present value for the corresponding sets of parameters is illustrated in figure 3.2. In part (a), σ is equal to one percent while α varies from .005 to .5. Here increasing α reduces the standard deviation of the present value. Note that for the same reasons that were explained for the expected present value, when δ_0 is greater than δ, it is not a straightforward exercise to try to predict the direction of change in the standard deviation of the present value as α varies.

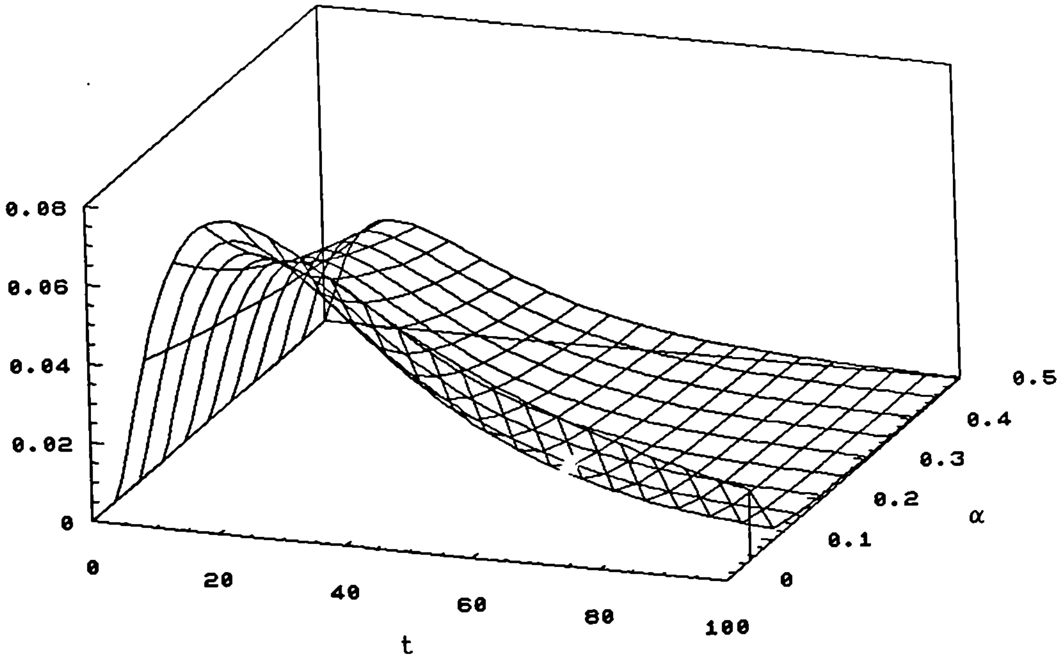
In part (b), α is equal to .1 and σ is varied from 0 to 3 percent. Increasing the parameter σ increases the variability in the present value. This is exactly what one would expect. Since increasing σ means that there is a greater uncertainty in the future force of interest without affecting its expected value, the uncertainty in the present value should be greater.

For most values of α and σ, the graph of the standard deviation of the present value function is increasing from t=0 to a certain value of t that maximize its value and is then decreasing.

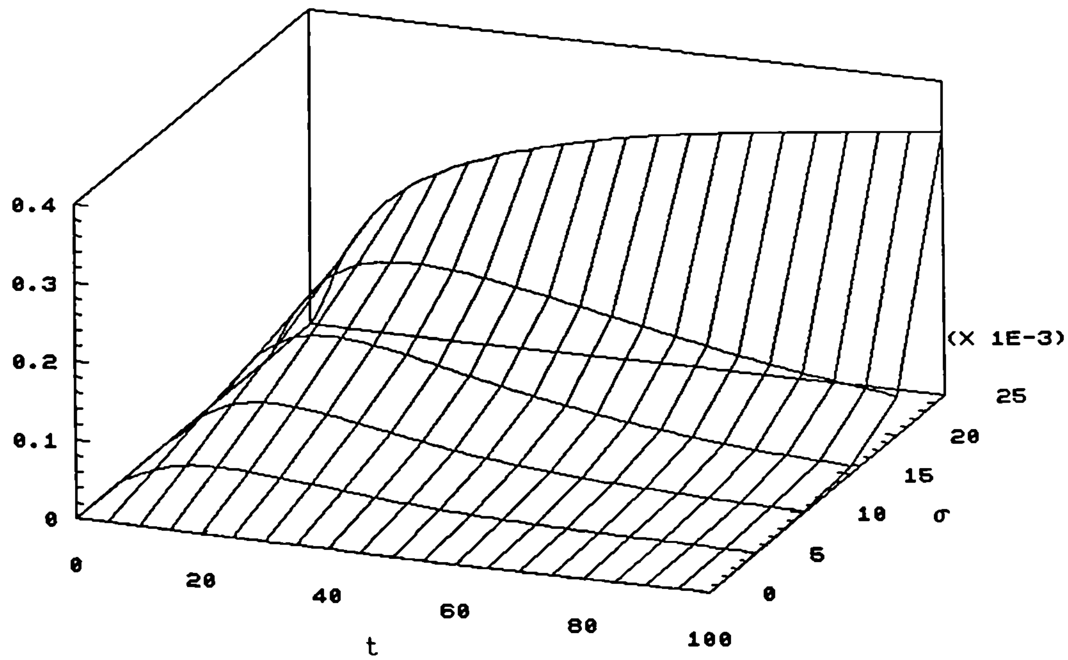
This is understandable if δ_t is positive for all t since, for t=0, the present value is 1 with a standard deviation of 0, and for very large values of t, y(t) tends to infinity and the present value function tends to 0 (i.e. $\lim_{t\to\infty} e^{-y(t)} = 0$) regardless of the path of positive forces of interest. This means that $sd[e^{-y(t)}]$ tends to 0. Now, since $sd[e^{-y(t)}]$ is positive and continuous, starting at 0 for t=0 and going back to 0 for large values of t, there has to be a value of t that maximizes this standard deviation.

Figure 3.2 Standard Deviation of the Present Value : $sd[e^{-y(t)}]$

(a) varying α $\delta=.06$ $\delta_0=.1$ $\sigma=.01$



(b) varying σ $\delta=.06$ $\delta_0=.1$ $\alpha=.1$



If the force of interest is not strictly positive, the limiting standard deviation of the present value function is not equal to 0. Whenever the force of interest is negative, the present value function is an increasing function. So the present value function is not necessarily going to 0 as t tends to infinity and this implies that its standard deviation is strictly positive. However, the same reasoning still holds as long as the probability of experiencing negative forces of interest for large values of t is not too high.

More precisely, $\delta > \frac{\sigma^2}{\alpha^2}$ is a sufficient condition for the standard deviation of $e^{-y(t)}$ to have a maximum value since it implies that this standard deviation tends to 0. Otherwise the standard deviation does not tend to 0 as t tends to infinity and we cannot conclude from this analysis whether it has a maximum value or not.

This may be seen by studying the limiting variance of $e^{-y(t)}$. Combining the results of 2.42, 2.47 and 2.50 we can express this variance as:

$$V[e^{-y(t)}] = \exp\left\{ 2\cdot\left(-\delta t + \frac{\sigma^2}{\alpha^2}\cdot t\right)\right\}\cdot$$

$$\exp\left\{-2(\delta_0-\delta)\cdot\left(\frac{1-e^{-\alpha t}}{\alpha}\right) + \frac{\sigma^2}{\alpha^3}\cdot\left[-3+4e^{-\alpha t}-e^{-2\alpha t}\right]\right\}$$

$$-\exp\left\{\left(-2\cdot\delta t + \frac{\sigma^2}{\alpha^2}\cdot t\right)\right\}\cdot$$

$$\exp\left\{-2(\delta_0-\delta)\cdot\left(\frac{1-e^{-\alpha t}}{\alpha}\right) + \frac{\sigma^2}{2\alpha^3}\cdot\left[-3+4e^{-\alpha t}-e^{-2\alpha t}\right]\right\} \qquad 3.4$$

And taking the limit as t tends to infinity we obtain:

$$\lim_{t\to\infty} V[e^{-y(t)}] = \begin{cases} 0 & \delta > \dfrac{\sigma^2}{\alpha^2} \\ \exp\left\{\dfrac{-2(\delta_0-\delta)}{\alpha} - \dfrac{3\sigma^2}{\alpha^3}\right\} & \delta = \dfrac{\sigma^2}{\alpha^2} \\ \infty & \delta < \dfrac{\sigma^2}{\alpha^2} \end{cases} \qquad 3.5$$

The coefficient of variation (standard deviation divided by expected value) of the present value function, however, is a strictly increasing function of t for any set of parameters. The coefficient of variation of $e^{-y(t)}$, denoted by $cv[e^{-y(t)}]$, will have a positive derivative if:

$$cv[e^{-y(t)}] > 0 \qquad 3.6$$

and

$$\frac{d}{dt}\left((cv[e^{-y(t)}])^2\right) > 0 \qquad 3.7$$

since

$$\frac{d}{dt}\left((cv[e^{-y(t)}])^2\right) = 2\cdot cv[e^{-y(t)}]\cdot \frac{d}{dt}\left(cv[e^{-y(t)}]\right) \qquad 3.8$$

Now from 2.41 and 2.42, we know that

$$(cv[e^{-y(t)}])^2 = e^{V[y(t)]} - 1 \qquad 3.9$$

which has a derivative with respect to t equal to:

$$\frac{d}{dt}\left((cv[e^{-y(t)}])^2\right) = e^{V[y(t)]} \cdot \frac{d}{dt}\left(V[y(t)]\right) \qquad 3.10$$

and this is the product of two positive terms.

Note that the derivative of V[y(t)] is positive since:

$$\frac{d}{dt}\left(V[y(t)]\right) = \frac{d}{dt}\left(\int_0^t\int_0^t \operatorname{cov}\left(\delta_u, \delta_v\right) \cdot du\ dv\right) \qquad 3.11$$

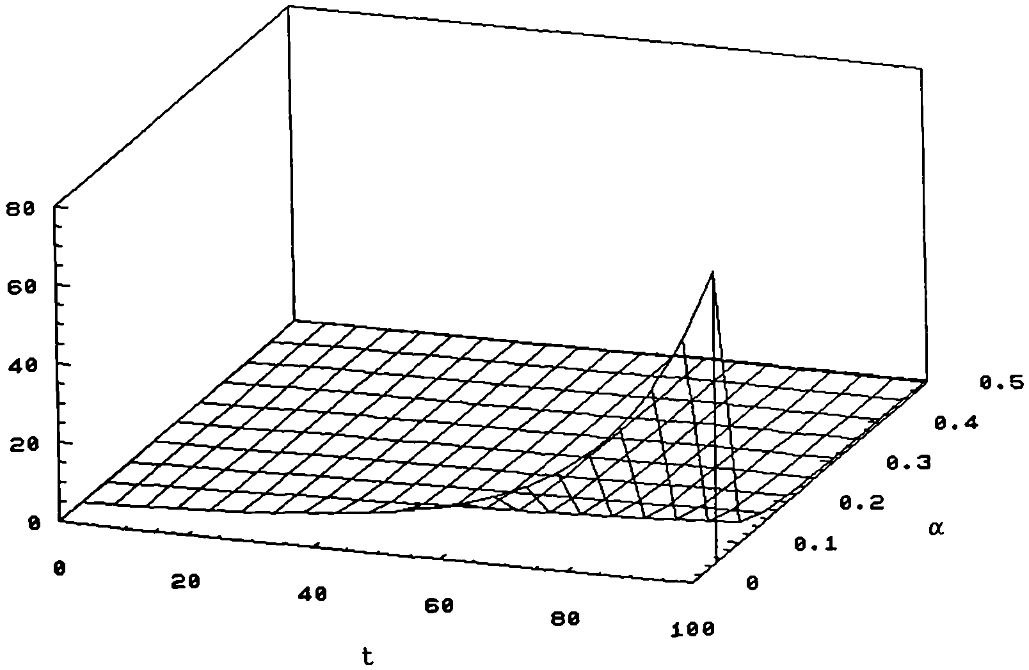
$$= 2\int_0^t \operatorname{cov}\left(\delta_u, \delta_t\right) \cdot du \qquad 3.12$$

where $\operatorname{cov}\left(\delta_u, \delta_t\right)$ is positive for all values of u.

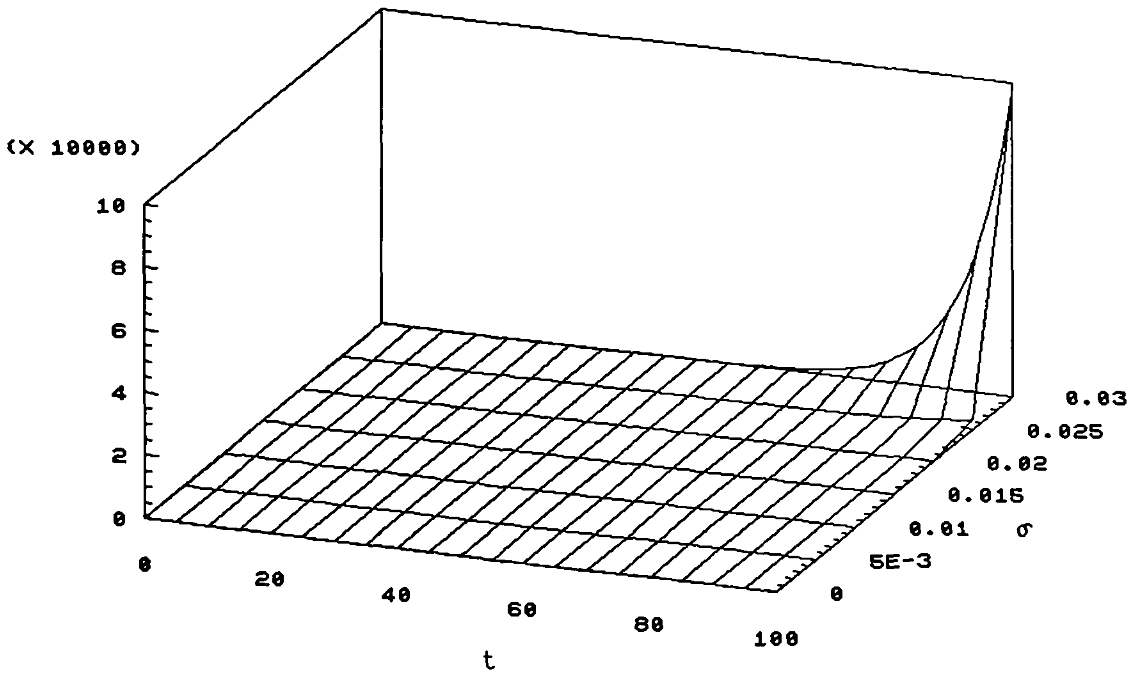
Finally, in figure 3.3, one can see graphically the skewness of the present value. As before, in part (a) it is the parameter α that is varied (from .005 to .5) while in part (b) we vary the parameter σ (from 0 to 3 percent). An increase in the variability of y(t), when the mean of y(t) remains the same, produces a greater skewness of the present value. It is interesting to note how quickly the skewness of the present value varies when one changes one of the parameters in a situation of high variability. If figure 3.3 (a) had been plotted for different ranges of values for t and α (say, $0 \le t \le 40$, $.2 \le \alpha \le .5$), the surface would still look the same i.e. flat except for a peak for the highest value of t and lowest value of α. A similar point could be made about figure 3.3 (b).

Figure 3.3 Skewness of the Present Value : $sk[e^{-y(t)}]$

(a) varying α $\delta=.06$ $\delta_0=.1$ $\sigma=.01$



(b) varying σ $\delta=.06$ $\delta_0=.1$ $\alpha=.1$



3.3 A linear combination of y(t) for different values of t.

One particular function that will be needed in order to obtain the higher moments of actuarial functions is the exponential of certain linear combinations of the function y(t), defined in 2.37, for different values of t. It is presented here for completeness.

Accordingly suppose that $0 \le t_1 \le t_2 \le \ldots \le t_m$ and consider the following random variable:

$$\exp\left\{-\sum_{i=1}^{m} \beta_i \cdot y(t_i)\right\} \tag{3.13}$$

where $\beta_1, \beta_2, \ldots, \beta_m$ are given.

Since any linear combination of y(t)'s normally distributed is also normally distributed, the exponential of any linear combination of y(t)'s is log-normally distributed. Thus the random variable defined in 3.13 is log-normally distributed.

Therefore, its expectation is:

$$E\left[\exp\left\{-\sum_{i=1}^{m} \beta_i \cdot y(t_i)\right\}\right] = \exp\left\{E\left[-\sum_{i=1}^{m} \beta_i \cdot y(t_i)\right] + .5 \cdot V\left[\sum_{i=1}^{m} \beta_i \cdot y(t_i)\right]\right\} \tag{3.14}$$

where

$$E\left[-\sum_{i=1}^{m} \beta_i \cdot y(t_i)\right] = -\sum_{i=1}^{m} \beta_i \cdot E[y(t_i)] \tag{3.15}$$

and

$$V\left[\sum_{i=1}^{m} \beta_i \cdot y(t_i)\right] = \sum_{i=1}^{m} \sum_{j=1}^{m} \beta_i \beta_j \cdot \mathrm{cov}\left(y(t_i), y(t_j)\right) \qquad 3.16$$

Now, if the force of interest follows a Ornstein-Uhlenbeck process, the expectation of y(t) is given by 2.47 and its autocovariance function is given by 2.48.

From now on, whenever we use an expression involving the expectation of a particular case of the random variable defined in 3.13, we shall assume that the expected value is known and given by the results of this section (without further reference to them).

CHAPTER 4

NET SINGLE PREMIUM ASSURANCE CONTRACTS

4.1 Life Assurance.

4.1.1 Definitions.

Let Z be the present value of the benefit payable under a given life assurance contract, i.e. the net single premium for this contract. Note that this definition of Z is a rather general one. The precise definition of Z will depend on the specific assurance contract under consideration. This approach is very similar to the one taken by Bowers et al. (1986).

We denote by K the integer-valued discrete random variable representing the number of completed years to be lived by a life assured now aged exactly x (an integer). It is called the curtate-future-lifetime of the life assured aged x (Bowers et al. (1986, p.48)) and may take the values $0,1,2,\ldots,\omega-x-1$ where ω is the least integer y with $l_y = 0$.

The benefit payable may be a function of K and it will be denoted b_K.

We assume that the time of payment of the benefit may also depend on K, in which case it will be denoted by t(K). The class of function that is valid for t(K) is somewhat restricted. For example, it would be impossible to have a time of payment of 5 for a death in the 11^{th} policy year (i.e. t(10)=5) and to have a time of payment of 10 for a death in the 21^{th} policy year (i.e. t(20)=10). No attempt will be made here to define this class of function in complete generality; instead we will look at some specific contracts and define t(K) accordingly.

The net single premium of a life assurance that we will refer to as A, is given by:

$$A = E[\mathcal{Z}] = E[E[\mathcal{Z}|K]] = \sum_{k=0}^{\omega-x-1} b_k \cdot E\left[e^{-y(t(k))}\right] \cdot {}_{k|}q_x \qquad 4.1$$

4.1.2 The moments about the origin of $\mathcal{Z}$.

As $\mathcal{Z}$ is a random variable, one might be interested not only in its expected value but also in its higher moments. If this is so, the following theorem will prove to be very useful, as it gives a straightforward way of evaluating these higher moments of $\mathcal{Z}$.

THEOREM 4.1 : Let $\mathcal{Z}$ be the present value of a given life assurance contract for which:

i- the death benefit (which may depend on the time of death) is either 0 or 1,

ii- the time of payment of the benefit is a function of the curtate-future-lifetime of the life assured,

iii- δ_t, the force of interest per unit time at time t, follows any stochastic process.

For each positive integer m let ${}^{m}A$ denote the net single premium for the given assurance when the force of interest per unit time at time t is $m \cdot \delta_t$.

Then,

$$E[\mathcal{Z}^m] = {}^{m}A. \qquad 4.2$$

Proof: By definition we have $A = E[Z]$ so,

$$A = E[E[Z|K]] = \sum_{k=0}^{\omega-x-1} b_k \cdot E\left[e^{-y(t(k))}\right] \cdot {}_{k|}q_x. \qquad 4.1$$

Now, ${}^{m}A$ represents the net single premium at a force of interest per unit time of $m \cdot \delta_t$ at time t. Hence

$${}^{m}A = \sum_{k=0}^{\omega-x-1} b_k \cdot E\left[e^{-\mathring{y}(t(k))}\right] \cdot {}_{k|}q_x \qquad 4.3$$

where

$$\mathring{y}(t(k)) = \int_0^{t(k)} m \cdot \delta_s \cdot ds = m \cdot y(t(k)), \qquad 4.4$$

then,

$${}^{m}A = \sum_{k=0}^{\omega-x-1} b_k \cdot E\left[e^{-m \cdot y(t(k))}\right] \cdot {}_{k|}q_x \qquad 4.5$$

and since, by hypothesis, $b_k = 0$ or 1,

$${}^{m}A = \sum_{k=0}^{\omega-x-1} (b_k)^m \cdot E\left[e^{-m \cdot y(t(k))}\right] \cdot {}_{k|}q_x \qquad 4.6$$

This last equation can be rewritten as:

$${}^{m}A = \sum_{k=0}^{\omega-x-1} E\left[\left(b_k \cdot e^{-y(t(k))}\right)^m\right] \cdot {}_{k|}q_x \qquad 4.7$$

$${}^{m}A = \sum_{k=0}^{\omega-x-1} E[Z^m|K=k] \cdot {}_{k|}q_x \qquad 4.8$$

or

$${}^{m}A = E[Z^m]. \;\square \qquad 4.2$$

Although in theory theorem 4.1 is valid for any stochastic process for the force of interest, in practice it may be extremely difficult, if not impossible, to evaluate $E[e^{-m\cdot y(t(k))}]$ in 4.5. Even with a gaussian process for the force of interest δ_t, which lead to a lognormal distribution of $e^{-m\cdot y(t(k))}$, it may be difficult to evaluate numerically, due for example to overflow problems. This may also be the case for the second order stochastic differential equation looked at in chapter 2.

4.1.3 The pdf and cdf of $\mathcal{Z}$.

By conditioning on K, we can find the probability density function (pdf) and the cumulative distribution function (cdf) of $\mathcal{Z}$.

The pdf of $\mathcal{Z}$ is:

$$f_{\mathcal{Z}}(z) = \sum_{k=0}^{\omega-x-1} P(K=k)\cdot f_{\mathcal{Z}}(z|K=k) = \sum_{k=0}^{\omega-x-1} {}_{k|}q_x\cdot f_{\mathcal{Z}}(z|K=k). \qquad 4.9$$

The cdf of $\mathcal{Z}$ is:

$$F_{\mathcal{Z}}(z) = \sum_{k=0}^{\omega-x-1} P(K=k)\cdot F_{\mathcal{Z}}(z|K=k) = \sum_{k=0}^{\omega-x-1} {}_{k|}q_x\cdot F_{\mathcal{Z}}(z|K=k). \qquad 4.10$$

In the next three sections, we consider specific contracts, namely the n-year temporary assurance contract, the n-year endowment assurance contract and the whole-life assurance contract, with sum assured 1 in each case.

4.2 Temporary Assurance.

4.2.1 Moments of $\mathcal{Z}$.

Under the n-year temporary assurance, the benefit of 1 is payable at the end of the year of death of a life assured, if death occurs within n years from the issue date. Thus t(k)=k+1 for k=0,1,...,n-1.

Thus we have

$$\mathcal{Z} = \begin{cases} e^{-y(K+1)} & K=0,1,\dots,n-1 \\ 0 & K=n,n+1,\dots \end{cases} \qquad 4.11$$

Using theorem 4.1, the m^{th} moment about the origin of $\mathcal{Z}$ is given by:

$$E[\mathcal{Z}^m] = {}^{m}A^{1}_{x:\overline{n}|} \qquad 4.12$$

$$E[\mathcal{Z}^m] = \sum_{k=0}^{n-1} E\left[e^{-m\cdot y(k+1)}\right]\cdot {}_{k|}q_x \qquad 4.13$$

where, by definition, ${}^{m}A^{1}_{x:\overline{n}|}$ denotes the net single premium for a n-year temporary assurance, valued at a force of interest of $m\cdot\delta_t$. (Note that ${}^{1}A^{1}_{x:\overline{n}|} = A^{1}_{x:\overline{n}|}$)

Note that with a gaussian process for the force of interest,

$$E[\mathcal{Z}^m] = \sum_{k=0}^{n-1} \exp\left\{ -m\cdot E[y(k+1)] + .5\cdot m^2\cdot V[y(k+1)] \right\}\cdot {}_{k|}q_x \qquad 4.14$$

4.2.2 Some statistics of $\mathcal{Z}$.

The expected value of $\mathcal{Z}$, denoted by $A^{1}_{x:\overline{n|}}$, for a force of interest that is a gaussian process is:

$$E[\mathcal{Z}] = A^{1}_{x:\overline{n|}} = \sum_{k=0}^{n-1} \exp\left\{ -E[y(k+1)] + .5 \cdot V[y(k+1)] \right\} \cdot {}_{k|}q_x . \qquad 4.15$$

The standard deviation is:

$$sd[\mathcal{Z}] = \left\{ E[\mathcal{Z}^2] - E[\mathcal{Z}]^2 \right\}^{.5} = \left\{ {}^{2}A^{1}_{x:\overline{n|}} - A^{1\;2}_{x:\overline{n|}} \right\}^{.5}. \qquad 4.16$$

The skewness is:

$$sk[\mathcal{Z}] = \frac{E[(\mathcal{Z} - E[\mathcal{Z}])^3]}{(sd[\mathcal{Z}])^3} \qquad 4.17$$

$$= \frac{E[\mathcal{Z}^3] - 3 \cdot E[\mathcal{Z}^2] \cdot E[\mathcal{Z}] + 2 \cdot E[\mathcal{Z}]^3}{(sd[\mathcal{Z}])^3} \qquad 4.18$$

$$= \frac{{}^{3}A^{1}_{x:\overline{n|}} - 3 \cdot {}^{2}A^{1}_{x:\overline{n|}} \cdot A^{1}_{x:\overline{n|}} + 2 \cdot (A^{1}_{x:\overline{n|}})^3}{\left[{}^{2}A^{1}_{x:\overline{n|}} - A^{1\;2}_{x:\overline{n|}} \right]^{3/2}} . \qquad 4.19$$

In order to present some illustrations of the expected value, the standard deviation and the skewness of $\mathcal{Z}$ we use the CA80-82 male mortality table produced by the Canadian Institute of Actuaries (Coward (1988, pp.227-231)). The mortality rates can be found in appendix B. The stochastic process used for the force of interest is the

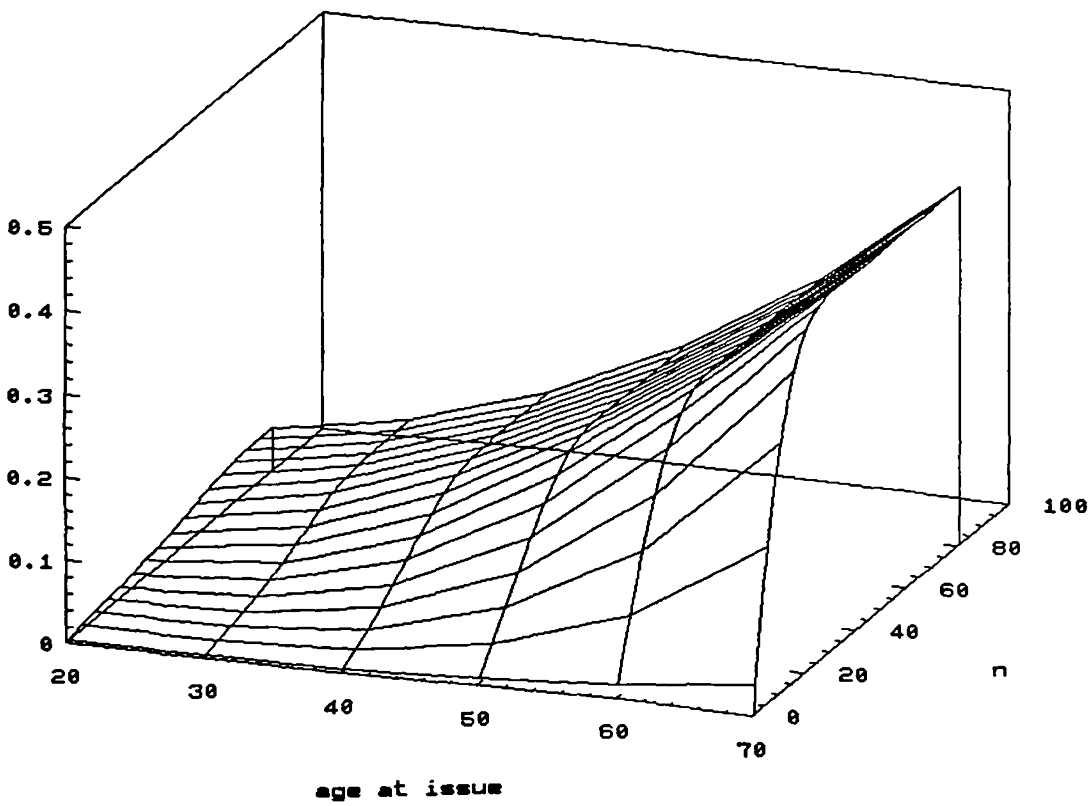
Ornstein-Uhlenbeck process with a current value of ten percent, a long term mean of six percent, a friction coefficient of .1 and a diffusion coefficient of one percent.

Figure 4.1 illustrates the expectation of $\mathcal{Z}$ (i.e. the net single premium) for different ages at issue and different terms of a temporary assurance contract.

Figure 4.1 Expected value of $\mathcal{Z}$

n-year temporary assurance

Ornstein-Uhlenbeck $\delta=.06$ $\delta_0=.1$ $\alpha=.1$ $\sigma=.01$



The net single premium, as expected, is an increasing function of the term n of the contract for every age at issue. From 4.13, with m equal 1, the increase in the net single premium when the term increases from n to n+1 is given by:

$$A1_{x:\overline{n+1}|} - A1_{x:\overline{n}|} = E[e^{-y(n+1)}] \cdot {}_{n|}q_x \qquad 4.20$$

This last expression is positive since $E[e^{-y(n+1)}]$ is positive and, obviously, ${}_{n|}q_x$ is positive.

The expected present value, $E[e^{-y(n+1)}]$, is usually a decreasing function of n for a reasonnable choice of parameters. So, assuming that ${}_{n|}q_x$ does not change much with n, one should observe smaller increases in the net single premium as n gets larger. Referring to figure 4.1, this appears to be the case. However, since ${}_{n|}q_x$ is generally increasing with n for small values of n, one cannot conclude that smaller increases in the net single premium will always result when one increases the term, n.

Now one may compare the increases in the net single premium with n for different ages at issue. For younger ages, the probability of dying in the near future is smaller than for older ages. For example, ${}_{n|}q_{20} < {}_{n|}q_{70}$ for small values of n. So, for an age at issue of 20, the net single premium increases more slowly with n, when n is small, than for an age at issue of 70. For the larger values of n, the expected present value, $E[e^{-y(n+1)}]$, is relatively small hence the influence of ${}_{n|}q_x$ in 4.20 is less important. Consequently, the increases in the net single premium with n, when n is large, are relatively small.

As a function of the age at issue, because of the mortality table used, the net single premium is somewhat unusual. For contracts of longer term the net single premium increases with age at issue. For the single premium decreases with age at issue from about age 20 to age 30 due to the decreasing mortality rates over that range.

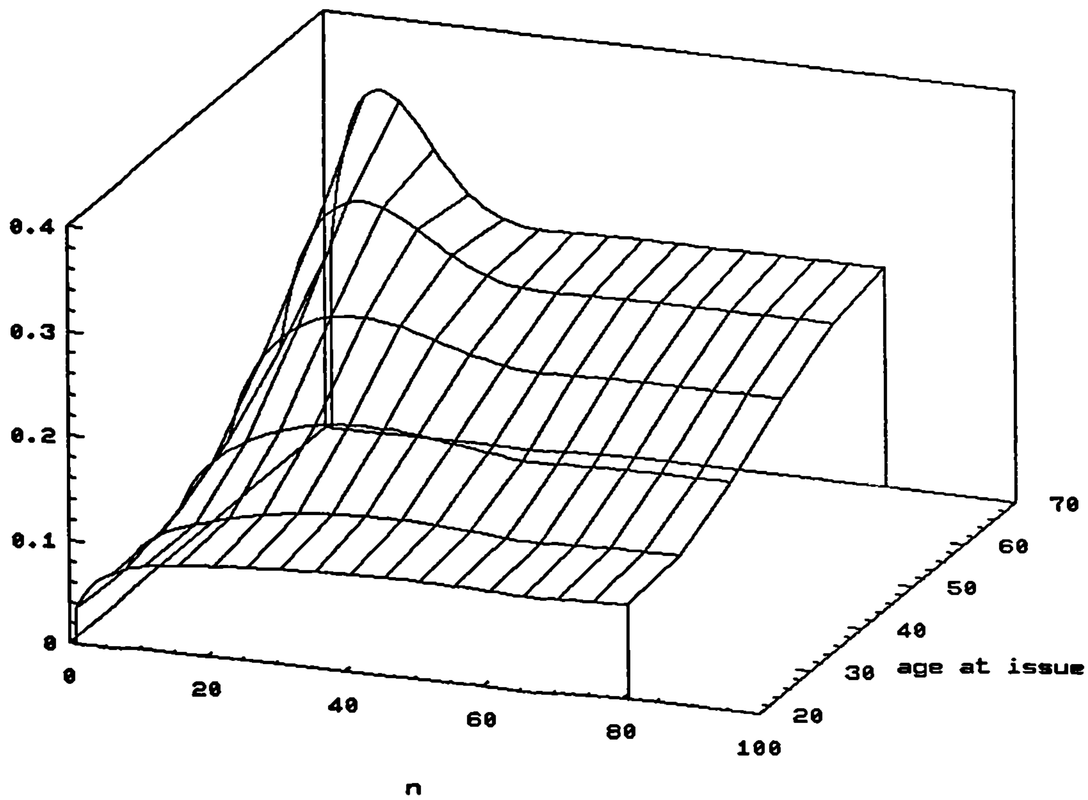
The plateau that appears for the older ages at issue is simply the consequence of the limiting age, ω=103, of the mortality table used. In fact, for every $n \geq \omega - x$, the temporary assurance contract becomes a whole-life one.

Figure 4.2 illustrates the standard deviation of Z for different ages at issue and different terms of a temporary assurance contract. Note that for this and some other graphs, the axes are reversed.

Figure 4.2 Standard deviation of Z

n-year temporary assurance

Ornstein-Uhlenbeck δ=.06 δ_0=.1 α=.1 σ=.01



As $\mathcal{Z}$ is a function of 2 random variables, namely, the force of interest (δ) and the curtate-future-lifetime (K) of the life assured, its standard deviation will depend on the distribution of these 2 random variables.

In order to gain insight about the variance of $\mathcal{Z}$, one may condition on K and use the following well-known result:

$$V[\mathcal{Z}] = E[V[\mathcal{Z}|K]] + V[E[\mathcal{Z}|K]] \qquad 4.21$$

In this last expression, 4.21, the variance of $\mathcal{Z}$ may be interpreted as the sum of two components. First, $E[V[\mathcal{Z}|K]]$ may be considered as a measure of the uncertainty due to the unknown force of interest. It is a weighted average of the variance of the present value function, the weights being the probability of death in each of year 1 to n. Second, $V[E[\mathcal{Z}|K]]$ may be considered as a measure of the uncertainty due to the unknown year of death of the life assured. It is the variance of the present value of the benefit discounted at a deterministic force of interest.

The first term, $E[V[\mathcal{Z}|K]]$, may be obtained as:

$$E[V[\mathcal{Z}|K]] = \sum_{k=0}^{n-1} {}_{k|}q_x \cdot V[e^{-y(k+1)}] \qquad 4.22$$

therefore, the change in this term when n is increased is given by:

$$E[V[\mathcal{Z}_{n+1}|K]] - E[V[\mathcal{Z}_n|K]] = {}_{n|}q_x \cdot V[e^{-y(n+1)}] \qquad 4.23$$

where Z_n is the present value of the benefit of a n-year temporary assurance contract and Z_{n+1} is the present value of the benefit of a (n+1)-year temporary assurance contract.

From 4.23, the increase in the uncertainty due to the force of interest is positive since each of ${}_{n|}q_x$ and $V[e^{-y(n+1)}]$ is positive. Note that since $V[e^{-y(n+1)}]$ has a maximum value around n equal 15 for the chosen parameters (see figure 3.2, for δ=.06, δ_0=.1, α=.1 and σ=.01), the largest increase in the uncertainty due to the force of interest will tend to occur near n equal 15.

The second term, $V[E[Z|K]]$, is increasing from n=1 to a given value which is near the value of n such that ${}_np_x$ is about .5, and is decreasing from that given value to ω-x.

This is so, because for a temporary assurance contract, the largest uncertainty due to mortality should logically occur near the term of the contract, n, which is such that there is a probability of .5 of paying a benefit and a probability of .5 of paying nothing. Away from this particular value of n, the uncertainty is smaller since the outcome is more predictable in the sense that the odds are necessarily for one of the outcomes (paying a benefit or not) against the other. The maximum is not exactly at n such that ${}_np_x$=.5 because Z is not exactly a Bernoulli random variable.

Finally, for the particular mortality table and Ornstein-Uhlenbeck process used, the uncertainty due to mortality (i.e. the last term of 4.21) is much larger than the uncertainty due to the force of interest, i.e. the first term of 4.21. For this reason the shape of $V[Z]$ when n is varied will tend to be given by the uncertainty due to mortality. the maximum being made more or less apparent by the uncertainty due to the force of interest. For example, for an age at issue of 70, ${}_{15}p_{70}$ is about .5 so the uncertainty due to mortality is at its highest and the uncertainty due to the force of interest, although strictly increasing with n, will tend to increase more rapidly around n equal 15. Hence, for this age at issue, the combined effect should give a clear maximum around n equal 15, and this is actually the case in figure 4.2.

Another approach to study the behavior of $V[Z]$, would be to start with a situation where there is no discounting factor, i.e. a force of interest of 0 for all t ($\delta=\delta_0=\sigma=0$). In this case, Z would simply be a Bernoulli random variable (see 4.11 with $y(k+1)=0$, $\forall k$). It then follows that the n which maximizes the variance if Z would be such that ${}_np_x$ is closest to .5. This happens at n equal 56 for an age at issue of 20 and at n equal 11 for an age at issue of 70. The variance of Z is zero for n equal 0 and for n equal ω-x and its maximum value is about .25.

If one now assumes a deterministic non-negative force of interest ($\delta_0>0$, $\sigma=0$), then the present value of the benefit in case of death would be smaller than 1 while the zeros in case of survival would remain unchanged. The variance of Z would therefore be smaller than in the previous case except for very large values of n. It would still start at 0 for n equal 0 and, as the effect of the positive force of interest is less important for small values of n, one would expect the shape of the graph of $V[Z]$ to remain about the same for small n. Obviously, the maximum would no longer be at n such that ${}_np_x$ is closest to .5. It would tend to occur at a smaller value of n. Note that the maximum value of $V[Z]$ is more affected when it occurs at larger values of n.

If one finally assumes a random force of interest, the present value of the benefit in case of death would become unknown with a maximum variance (for the parameters used) occurring near 15 years (see figure 3.2).

Since here this added uncertainty is relatively smaller than the one due to mortality, the maximum $V[Z]$ will occur at about the same value of n as the last case. The maximum will also be more apparent if it occurs near 15 years since both uncertainties due to mortality and to the force of interest would be at their maximum.

The same arguments about the uncertainty due to mortality and the one due to the force of interest could be used to explain the shape of $V[Z]$ for varying ages at issue, x, when the term of the contract, n, is kept constant.

As for the coefficient of variation of Z (i.e. the standard deviation of Z divided by the expected value of Z), in our illustrations it decreases rapidly with n when n is small and more slowly when n is large. It also decreases, for a fixed n, as the probability of paying a death benefit increases.

For the parameters used to produce figures 4.1 and 4.2, the coefficient of variation of Z varies from 25.55 for a 1-year temporary contract to 1.77 for a 80-year temporary contract issued at age 20. At issue age 70, the coefficient of variation of Z is 4.96 and .50 for a 1-year and 30-year temporary contract respectively.

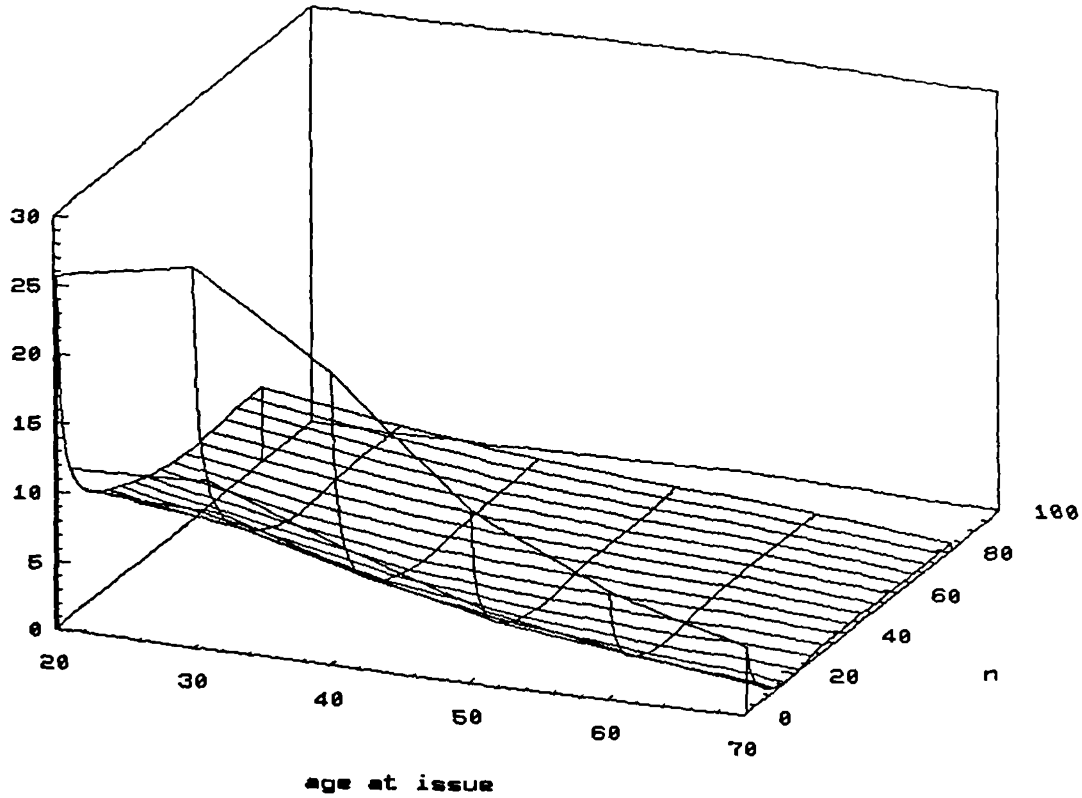
Figure 4.3 illustrates the skewness of Z for different ages at issue and different terms of a temporary assurance contract.

The random variable Z for a short-term temporary assurance contract is highly skewed, due to the important mass probability at Z equal 0 corresponding to survival to the end of the term of the contract. In case of death, which usually happens with a small probability for a short-term temporary assurance contract, Z takes a value near 1 with a

Figure 4.3 Skewness of Z

n-year temporary assurance

Ornstein-Uhlenbeck $\delta=.06$ $\delta_0=.1$ $\alpha=.1$ $\sigma=.01$



probability density function that is a mixture of lognormal distributions.

The higher the probability of survival from age x to age x+n, i.e. Z taking the value 0, the higher the skewness. For example, sk[Z] is greater for an age at issue of 20 than for an age at issue of 70. The increase in the skewness between ages at issue 20 and 30 is due to the decreasing mortality rates that we find in that range.

For small values of n, when one increases the term of the temporary contract from n to n+1, the skewness (which is positive) decreases because the large mass probability at $\mathcal{Z}$ equal 0 is reduced and the reduction is replaced by a weighted lognormal distribution with mean $E[e^{-y(n+1)}]$ (which is greater than $E[\mathcal{Z}]$) and variance $V[e^{-y(n+1)}]$, the weight being ${}_{n|}q_x$. Recalling that $\mathcal{Z}$ has a lower bound at 0 and a practical upper bound at 1, it makes sense that reducing the mass probability at the lower bound 0 and replacing it by a lognormal distribution situated somewhere to the right of 0 but to the left of the remaining part of the probability density function, would increase the symmetry of the distribution.

For large values of n, however, the positive skewness will increase with n because part of the small mass probability at $\mathcal{Z}$ equal 0 is replaced by a weighted lognormal distribution to the right of 0.

Essentially, one could view the probability density function of $\mathcal{Z}$ as a left part, the mass probability at 0, and a right part, a mixture of weighted lognormal distribution. When the left part is more important, transferring part of it to the right would logically make the distribution more symmetrical. But when the left part is less important, transferring part of it to the right would reduce the symmetry of the distribution. When both parts are about even, the resulting effect is not so clear because increasing n replaces part of the constant left side (at 0) by a spread part to the right.

As a rule of thumb, the skewness of $\mathcal{Z}$ for a temporary contract decreases with the term of the contract from 1 to n such that ${}_np_x$ is about .5 and increases thereafter.

Consideration of the probability density function of $\mathcal{Z}$ (see figure 4.4) might help our understanding of all this.

4.2.3 The pdf and cdf of $\mathcal{Z}$.

Using 4.9 with $\mathcal{Z}$ defined as in 4.11, we get for the pdf of $\mathcal{Z}$:

$$f_{\mathcal{Z}}(z) = \sum_{k=0}^{n-1} {}_{k|}q_x \cdot f_{e^{-y(k+1)}}(z) \tag{4.24}$$

with a mass probability of ${}_np_x$ at z=0.

Using 4.10 with $\mathcal{Z}$ defined as in 4.11, we have that the cdf of $\mathcal{Z}$ is:

$$F_{\mathcal{Z}}(z) = P(\mathcal{Z} \le z) = {}_np_x + \sum_{k=0}^{n-1} {}_{k|}q_x \cdot P\left(e^{-y(k+1)} \le z\right). \tag{4.25}$$

The pdf of $\mathcal{Z}$ for 5-year and for 25-year temporary assurances issued at age 30 are presented in figure 4.4.

The pdf of $\mathcal{Z}$ is a weighted sum of lognormal distributions with a mass probability at 0. The lognormal distributions are those of $e^{-y(k+1)}$ for k=0,1,...,n-1 and the weights are ${}_{k|}q_x$.

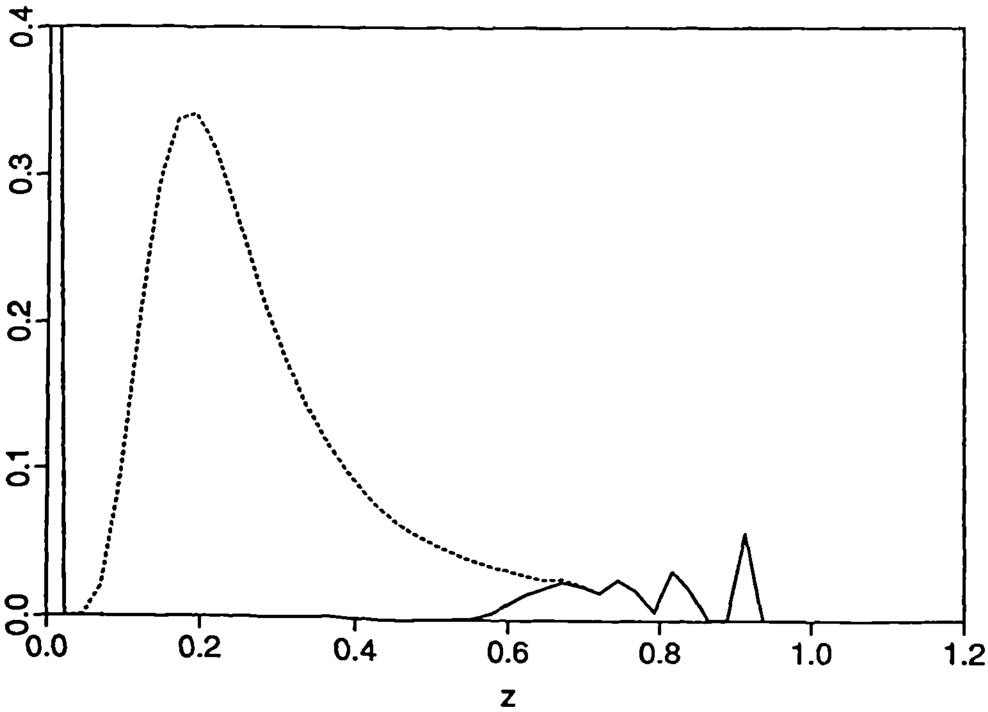
For the 5-year temporary contract, the first peak near $\mathcal{Z}$ equal 1 is the lognormal distribution of $e^{-y(1)}$ times q_{30}. The second peak around $\mathcal{Z}$ equal .82 is mainly the pdf of $e^{-y(2)}$ times ${}_{1|}q_{30}$. And for the first two or three years, $e^{-y(k+1)}$ for k=0,1,2 have distributions that do not overlap much. As a consequence, we can observe specific weighted lognormal distributions around different values of $\mathcal{Z}$ near 1.

But as n increases, $e^{-y(n)}$ and $e^{-y(n+1)}$ have closer expected values and larger variances. The overlapping of the distributions then becomes more important and this tend to smooth the pdf of $\mathcal{Z}$. For example, most values that the random variables $e^{-y(4)}$ and $e^{-y(5)}$ may take are in the same range. In other words, $\mathcal{Z}$ may take a given value because either K=3 and δ_t for $t\in[0,4]$ is relatively high or because K=4 and δ_t for $t\in[0,5]$ is relatively low.

Figure 4.4 Pdf of Z

5 and 25 years temporary assurance issued at age 30

Ornstein-Uhlenbeck $\delta=.06$ $\delta_0=.1$ $\alpha=.1$ $\sigma=.01$



... term 25 years [P(Z=0) = .9125]

—— term 5 years [P(Z=0) = .9932]

Note that for a 5-year temporary contract, Z will either be 0 or be between .5 and 1 with a probability of almost 1. As the term increases from 5 to 25, Z may take the value 0 or any value between a value close to 0 and 1.

4.3 Endowment Assurance.

4.3.1 Moments of Z.

Under the endowment assurance contract, the benefit is payable at the end of the year of death if death occurs within n years of the issue date or, if the insured life survives n years, the benefit is payable at time n. Thus $t(k) = k+1$ for $k=0,1,\ldots,n-1$ and $t(k)=n$ for $k=n,n+1,\ldots$ Accordingly, the variable Z for the present value of an endowment assurance is defined as:

$$Z = \begin{cases} e^{-y(K+1)} & K=0,1,\ldots,n-1 \\ e^{-y(n)} & K=n,n+1,\ldots \end{cases} \qquad 4.26$$

Note that $P(K \geq n) = {}_np_x$.

From the result of theorem 4.1, the m^{th} moment of Z is given by:

$$E[Z^m] = {}^mA_{x:\overline{n}|}$$

$$= \sum_{k=0}^{n-1} E\left[e^{-m\cdot y(k+1)}\right]\cdot {}_{k|}q_x + E\left[e^{-m\cdot y(n)}\right]\cdot {}_np_x. \qquad 4.27$$

where ${}^mA_{x:\overline{n}|}$, is the net single premium for an n-year endowment assurance valued at a force of interest of $m\cdot\delta_t$.

4.3.2 Some statistics of Z.

The expected value of Z when the force of interest is gaussian is:

$$E[Z] = A_{x:\overline{n}|} = \sum_{k=0}^{n-1} \exp\left\{ -E[y(k+1)] + .5\cdot V[y(k+1)] \right\}\cdot {}_{k|}q_x$$

$$+ \exp\left\{ -E[y(n)] + .5\cdot V[y(n)] \right\}\cdot {}_np_x. \qquad 4.28$$

The standard deviation is:

$$sd[Z] = \left\{E[Z^2] - E[Z]^2\right\}^{.5} = \left\{{}^2A_{x:\overline{n}|} - A_{x:\overline{n}|}{}^2\right\}^{.5}. \qquad 4.29$$

Using 4.17 with the appropriate definition of Z (4.26), the skewness is:

$$sk[Z] = \frac{{}^{3}A_{x:\overline{n}|} - 3\cdot{}^{2}A_{x:\overline{n}|}\cdot A_{x:\overline{n}|} + 2\cdot(A_{x:\overline{n}|})^{3}}{\left[{}^{2}A_{x:\overline{n}|} - A_{x:\overline{n}|}^{2}\right]^{3/2}} . \qquad 4.30$$

Some illustrations of the expected value, the standard deviation and the skewness of Z are presented in figures 4.5, 4.6 and 4.7 respectively.

Among the salient features of figure 4.5, we note that for n equal 1, the expected value of Z is the expected value of $e^{-y(1)}$ for every age at issue. This is so, because a 1-year endowment contract is always payable at time 1. We also note that, for a given age at issue, the expected value of Z decreases with n. Increasing n by 1 means that the payment of the pure endowment benefit (if applicable) is postponed one more year and this later benefit has a smaller expected present value.

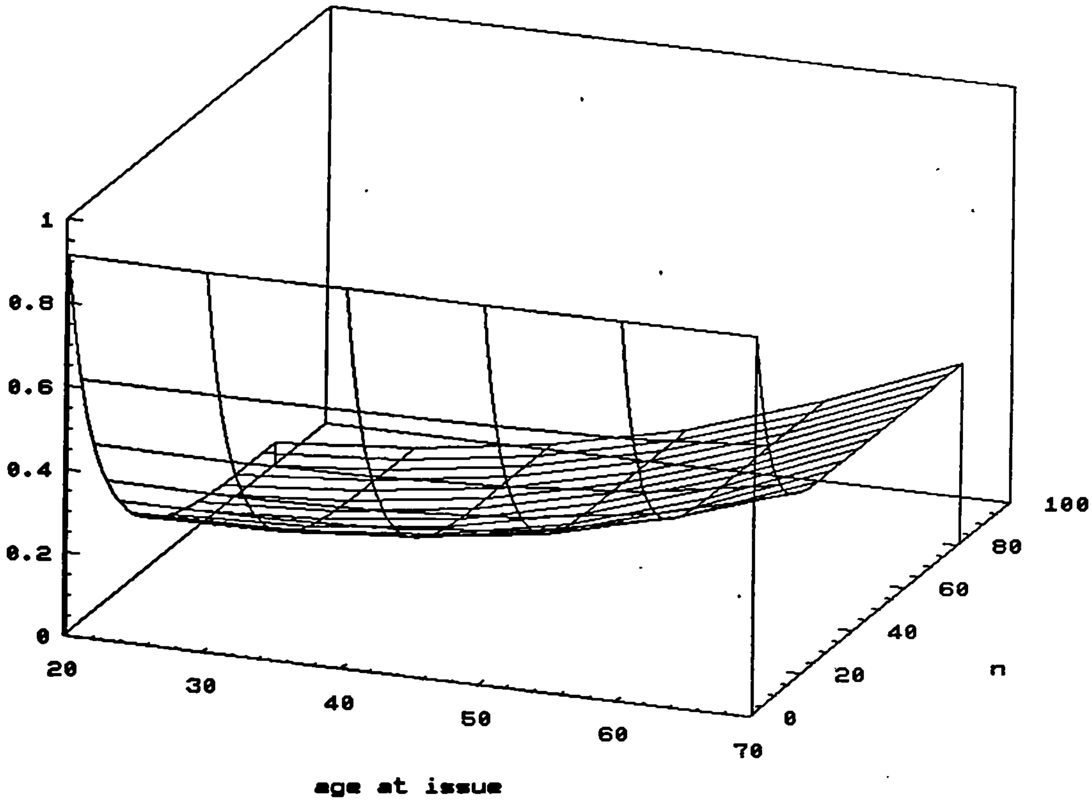
Here again, as was the case in figure 4.1 for the n-year temporary assurance contracts, we have a plateau for very large terms. Moreover, the plateau is exactly the same as the one in figure 4.1. This is so because for every $n \geq \omega-x$, the n-year endowment assurance contract becomes a whole-life one.

Comparing figures 4.1 and 4.5, the expected value of Z for a n-year endowment is always greater than that of a n-year temporary (except for $n \geq \omega-x$, where they are equal). This is so, because the difference between the two contracts is their provisions in case of survival to the end of the contract. The endowment assurance contract allows for the payment of the sum assured in case of survival to age x+n, whereas the

Figure 4.5 Expected value of Z

n-year endowment assurance

Ornstein-Uhlenbeck $\delta=.06$ $\delta_0=.1$ $\alpha=.1$ $\sigma=.01$



temporary assurance allows no benefit in such a case. Finally, since the expected present value of the survival benefit decreases with n, the difference between the expected present values for an endowment assurance and a temporary assurance will be decreasing with n.

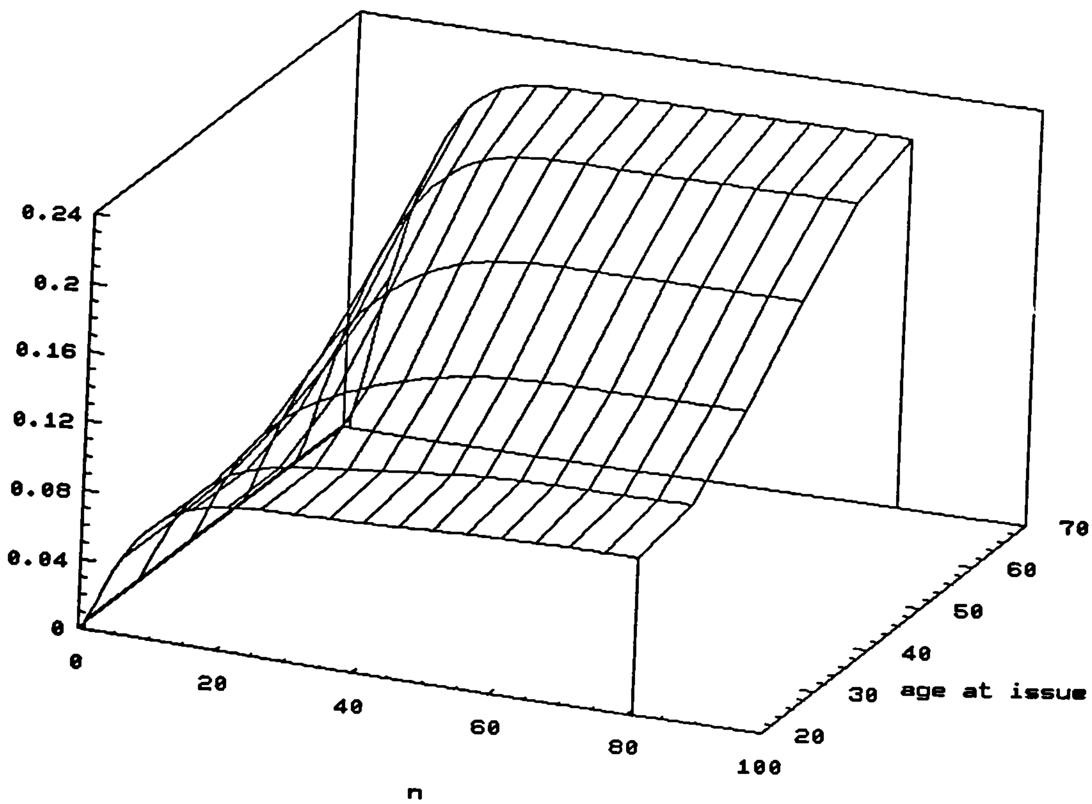
Since the probability of paying a death benefit usually increases with the age at issue and since the expected present value of an eventual death benefit is greater than that of the pure endowment benefit at the end of the term of the contract, the expected value of Z for an endowment contract increases with age at issue. Note, however, that for reasons mentioned earlier, E[$\mathcal{Z}$] is smaller for an age at issue of 30 than for an age at issue of 20.

From figure 4.6, one can see that the standard deviation of $\mathcal{Z}$ for a n-year endowment assurance contract increases with n.

Figure 4.6 Standard deviation of $\mathcal{Z}$

n-year endowment assurance

Ornstein-Uhlenbeck $\delta=.06$ $\delta_0=.1$ $\alpha=.1$ $\sigma=.01$



As opposed to the behavior of the standard deviation of Z for a temporary contract, this behavior for an endowment is fairly easy to predict. In fact, when n increases, the change in the pdf of Z for an endowment contract is only to move part of it slightly to the left without affecting much its right part. As a result, the dispersion of the possible values for Z is greater. This may be more evident when one looks at the pdf of Z (see figure 4.8).

If one wants to break down the variance of Z into two components, one broadly corresponding to the unknown force of interest, the other to the unknown year of death of the life assured, formula 4.21 can be used with Z now defined as in 4.26.

The first term, $E[V[Z|K]]$, is given by:

$$E[V[Z|K]] = \sum_{k=0}^{n-1} {}_{k|}q_x \cdot V[e^{-y(k+1)}] + {}_n p_x \cdot V[e^{-y(n)}] \qquad 4.31$$

and the change in this term when n is increased by one, is given by:

$$E[V[Z_{n+1}|K]] - E[V[Z_n|K]] = {}_{n|}q_x \cdot V\left[e^{-y(n+1)}\right] + {}_{n+1}p_x \cdot V\left[e^{-y(n+1)}\right] - {}_n p_x \cdot V\left[e^{-y(n)}\right]$$

$$= {}_n p_x \cdot \left\{ V\left[e^{-y(n+1)}\right] - V\left[e^{-y(n)}\right] \right\} \qquad 4.32$$

where Z_n is the present value of the benefit of a n-year endowment assurance contract and Z_{n+1} is the present value of the benefit of a (n+1)-year endowment assurance contract.

Now since ${}_{n}p_{x}$ is positive (it is a probability) and since $V[e^{-y(n)}]$ increases until about duration 15 and decreases thereafter, $E[V[Z|K]]$ will be increasing with the term until about 15 years and will then be decreasing.

The second term, $V[E[Z|K]]$, representing the uncertainty due to mortality is increasing with n. This term is simply the variance of the present value of the benefit when discounting is done at some deterministic force of interest. Since increasing n only shifts some of the left part of the distribution of Z further to the left, this will increase its standard deviation. Note that whether the force of interest is deterministic or stochastic does not affect the conclusion here.

Combining the effects, when increasing n, of the two terms would require more investigation, but, as we already know that the total variance of Z should be increasing with n, this further investigation is not necessary.

Part of the insight that one gains from breaking down the variance of Z in the two terms of 4.21 is that since the uncertainty due to the force of interest is increasing for terms of up to about 15 years and then decreasing, one could expect the total increase in the variance of Z to be more important when n is small. This is actually the case in figure 4.6.

Comparing figures 4.2 and 4.6, one observes that the standard deviation of a n-year temporary assurance is always greater than that of a n-year endowment assurance. Although, in case of survival to the end of the contract, the endowment will pay a survival benefit with a random present value (therefore having a positive standard deviation) as opposed to no such benefit for the temporary contract (with a standard deviation of 0), this is more than counterbalanced by the reduction in the uncertainty caused by replacing the mass probability at 0 by a weighted lognormal distribution near the remaining part of the pdf of Z (even though the weighted lognormal distribution is placed to the left of the pdf).

Here, as a function of the age at issue, the standard deviation of Z is increasing except, of course, for some ages between 20 and 30.

In our illustrations, the coefficient of variation of Z for a n-year endowment assurance contract increases with n. The increases are larger when n is small than when n is large.

For the parameters used to produce figures 4.5 and 4.6, the coefficient of variation of Z for a 1-year endowment contract is .0056 (recall that it was 25.55 for a 1-year temporary contract issued at age 20) for every age at issue. When the age at issue is 20, it is .058, .640 and 1.77 for 5-year, 30-year and 80-year endowment contracts respectively. For an age at issue of 70, it is .112 and .494 for 5-year and 30-year endowment.

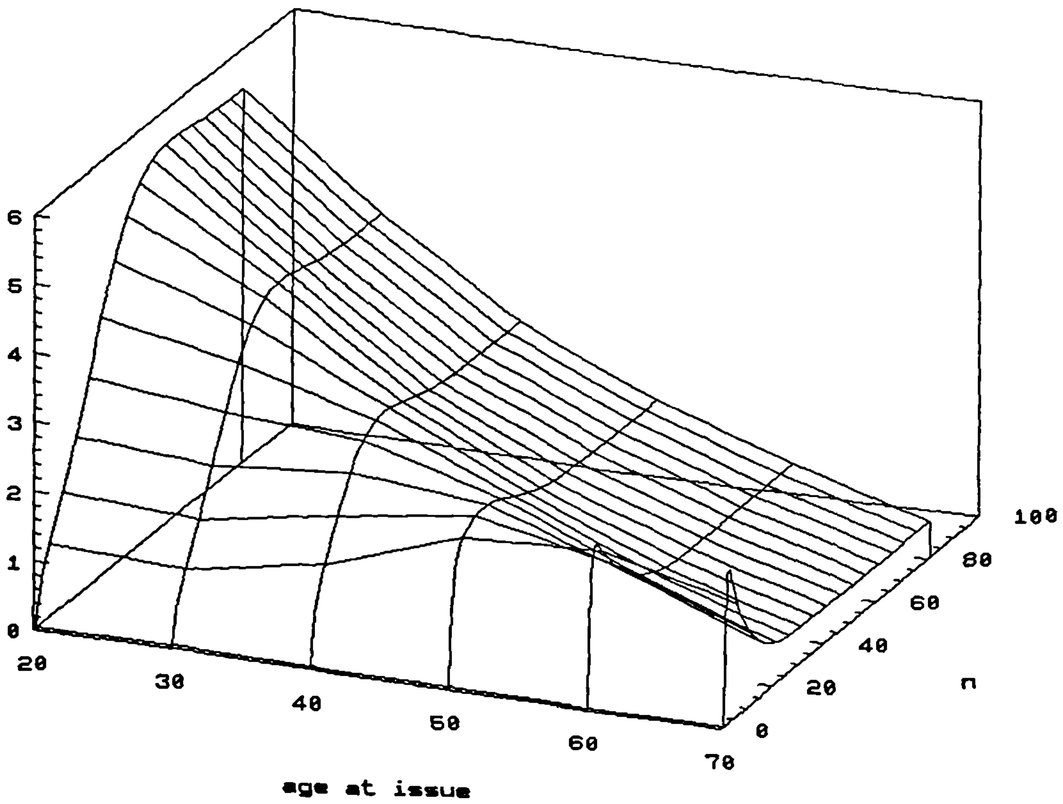
As it was the case for the expected value and standard deviation of Z, for a 1-year endowment contract, the skewness is independent of the age at issue (see figure 4.7). In this case the skewness of Z is that of $e^{-y(1)}$ and is .0167.

For n equal 2, the pdf of Z is composed of two weighted lognormal distributions, those of $e^{-y(1)}$ and $e^{-y(2)}$. If q_x is very small, Z is almost equivalent to $e^{-y(2)}$ in the sense that the pdf of Z would almost be that of $e^{-y(2)}$ (the difference being that a small part of the latter is transferred to the right of it). Since $e^{-y(2)}$ is positively skewed, so is Z. Now if q_x is increased, this would increase the importance of the right part of the pdf of Z and this in turn should logically increase its skewness. However, there is a point where increasing q_x,

Figure 4.7 Skewness of Z

n-year endowment assurance

Ornstein-Uhlenbeck $\delta=.06$ $\delta_0=.1$ $\alpha=.1$ $\sigma=.01$



and therefore increasing the importance of the right part of the pdf of Z, would make the pdf more symmetric, thus reducing the skewness. Note that if q_x were about .5, then it makes sense that the skewness of Z should be near 0, both the right and left part of its pdf having about the same weight. Also note that if the mortality rate, q_x, were to be greater than .5, the skewness would be negative. This is usually not the case, except at extremely old ages at issue for which no policy would be sold anyway.

As n is increased, the probability of paying a death benefit, ${}_nq_x$, increases and following the reasoning of the last paragraph, the age at issue at which the right part of the pdf of $\mathcal{Z}$ becomes important enough to make it more symmetric should be reduced. In other words, as n is increased, the maximum skewness of $\mathcal{Z}$ should be attained at younger ages at issue.

For example, in figure 4.7, the maximum skewness for n equal 2, 3 and 20 are 2.1988, 2.3187 and 2.8825 and are attained at ages at issue 70, 64 and 21 respectively.

When n is large, the skewness of $\mathcal{Z}$ is mainly determined by the distribution of K, the more symmetric the distribution of K is, the smaller should be the skewness of $\mathcal{Z}$. This explains why, for large values of n, the skewness of $\mathcal{Z}$ at age at issue 70 is smaller than that of age at issue 20.

Finally, it is difficult to compare the skewness of $\mathcal{Z}$ for a n-year endowment assurance contract with that for a n-year temporary assurance contract. Replacing the mass probability at 0 for the temporary contract by a weighted lognormal distribution located at the right of 0 to get the endowment contract, clearly increases the expected value and reduces the standard deviation of $\mathcal{Z}$ but this may affect the skewness of its pdf in both directions.

4.3.3 The pdf and cdf of $\mathcal{Z}$.

Using 4.9 with $\mathcal{Z}$ defined as in 4.26, we get for the pdf of $\mathcal{Z}$:

$$f_{\mathcal{Z}}(z) = \sum_{k=0}^{n-1} {}_{k|}q_x \cdot f_{e^{-y(k+1)}}(z) + {}_np_x \cdot f_{e^{-y(n)}}(z). \qquad 4.33$$

Using 4.10 with $\mathcal{Z}$ defined as in 4.26, we have that the cdf of $\mathcal{Z}$ is:

$$F_{\mathcal{Z}}(z) = P(\mathcal{Z} \le z)$$

$$= \sum_{k=0}^{n-1} {}_{k|}q_x \cdot P\left(e^{-y(k+1)} \le z\right) + {}_np_x \cdot P\left(e^{-y(n)} \le z\right). \qquad 4.34$$

The pdf of $\mathcal{Z}$ for 5-year and for 25-year endowment assurances are presented in figure 4.8.

The pdf of $\mathcal{Z}$ for a n-year endowment assurance contract is a weighted sum of lognormal distributions. The lognormal distributions are those of $e^{-y(k+1)}$ for k=0,1,...,n-1 and $e^{-y(n)}$ with weights ${}_{k|}q_x$ for k=0,1,...,n-1 and ${}_np_x$ respectively.

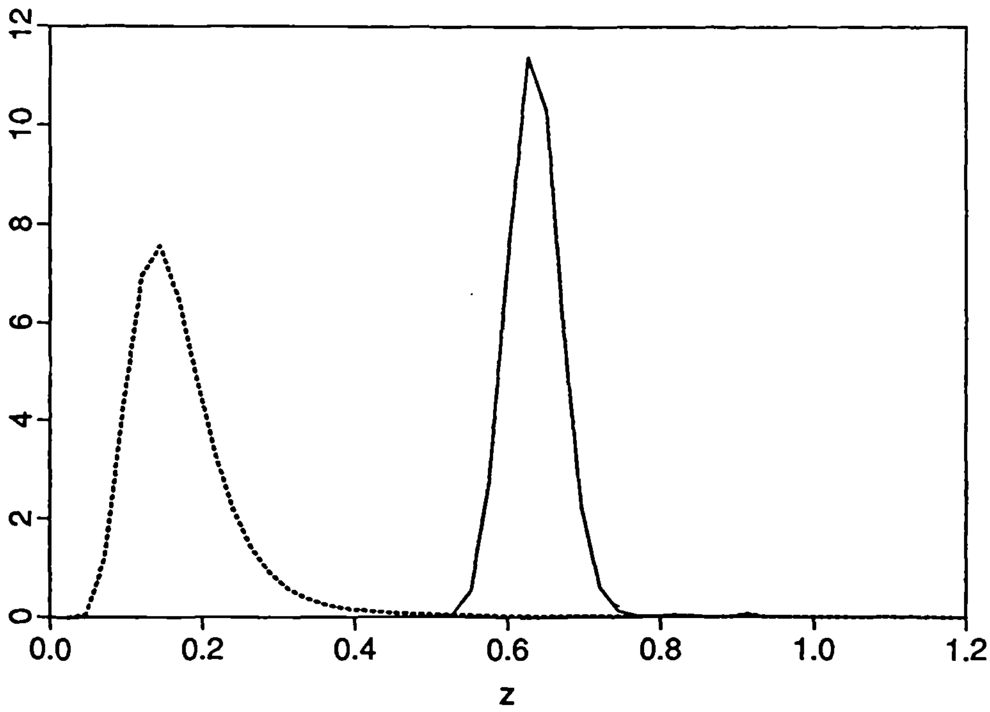
The only difference between figure 4.4, the pdf of $\mathcal{Z}$ for 5-year and 25-year temporary contracts, and figure 4.8 is the replacement of the mass probability at 0 in the former by a weighted lognormal distribution with mean $E[e^{-y(n)}]$ in the latter. Accordingly, the pdf of $\mathcal{Z}$ will have the same peaks and overlapping phenomena as those described in section 4.2.3 (except for the mass probability at 0 obviously). The peaks for $\mathcal{Z}$ near 1 are not as apparent in figure 4.8 as they were in figure 4.4 simply because of the different scale used.

For small values of n, the possible values that $\mathcal{Z}$ may take are all relatively near 1. As n increases the range of possible values extend to include smaller values. For example, $\mathcal{Z}$ for a 5-year endowment assurance contract issued at 30 will take a value between .5 and 1 with probability almost 1. As the term increases from 5 to 25, $\mathcal{Z}$ may take the value 0 or any value between a value close to 0 and 1.

Figure 4.8 Pdf of Z

5 and 25 years endowment assurance issued at age 30

Ornstein-Uhlenbeck δ=.06 δ_0=.1 α=.1 σ=.01



... term 25 years

—— term 5 years

Finally, one should be aware that for an age at issue of 30, the most probable events are a survival to the end of the term of the contract or a death late in the contract. This explains the important mode of the distribution at its far left. For very advanced ages at issue, where death in early years of the contract is most likely, the pdf of Z would have a mode in its right part.

4.4 Whole-Life Assurance.

4.4.1 Moments of $\mathcal{Z}$.

For this type of contract, the benefit is payable at the end of the year of death (whenever the death occurs). Thus, for all values of k, t(k)=k+1. We then have $\mathcal{Z}$, the random variable representing the present value of such a benefit, defined as:

$$\mathcal{Z} = e^{-y(K+1)} \qquad K=0,1,\ldots,\omega-x-1. \tag{4.35}$$

where ω, for the CA80-82 male, is taken as 103.

Using the result of theorem 4.1, the m^{th} moment of $\mathcal{Z}$ is given by:

$$E[\mathcal{Z}^m] = {}^{m}A_x$$

$$= \sum_{k=0}^{\omega-x-1} E\left[e^{-m\cdot y(k+1)}\right]\cdot {}_{k|}q_x \tag{4.36}$$

$$= \sum_{k=0}^{\omega-x-1} \exp\left\{ -m\cdot E[y(k+1)] + .5\cdot m^2\cdot V[y(k+1)] \right\}\cdot {}_{k|}q_x. \tag{4.37}$$

${}^{m}A_x$ being by definition the net single premium on the basis of a force of interest $m\cdot\delta_t$.

4.4.2 Some statistics of $\mathcal{Z}$.

Since the force of interest is assumed to be gaussian, the expected value of $\mathcal{Z}$ is:

$$E[\mathcal{Z}] = A_x = \sum_{k=0}^{\omega-x-1} \exp\left\{ -E[y(k+1)] + .5\cdot V[y(k+1)] \right\}\cdot {}_{k|}q_x. \tag{4.38}$$

The standard deviation is:

$$sd[Z] = \left\{E[Z^2] - E[Z]^2\right\}^{.5} = \left\{{}^2A_x - A_x{}^2\right\}^{.5}. \qquad 4.39$$

Using 4.18 with the appropriate definition of Z (i.e. 4.35), the skewness is:

$$sk[Z] = \frac{{}^3A_x - 3\cdot{}^2A_x\cdot A_x + 2\cdot(A_x)^3}{\left[{}^2A_x - A_x{}^2\right]^{3/2}}. \qquad 4.40$$

Some numerical values of the expected value, the standard deviation and the skewness of Z are presented in table 4.1.

Table 4.1 Mean, standard deviation, skewness and coefficient of variation of the net single premium of a whole-life assurance

Ornstein-Uhlenbeck $\delta=.06$ $\delta_0=.1$ $\alpha=.1$ $\sigma=.01$

age at issue	mean	standard deviation	skewness	coefficient of variation
20	.051187	.090805	5.41185	1.77398
30	.076342	.097460	3.91518	1.27662
40	.123992	.127706	2.63290	1.02995
50	.199394	.167886	1.78311	.84198
60	.303412	.200298	1.10098	.66015
70	.432234	.213380	0.52339	.49367
80	.573185	.200033	-0.00956	.34899
90	.698856	.161555	-0.38825	.23117
100	.883526	.041425	-1.50227	.00469

Since, for $n \geq \omega-x$, n-year temporary and n-year endowment assurance contracts are in fact equivalent to a whole-life contract, the values in table 4.1 are those of the different plateaus of figures 4.1 to 4.3 or figures 4.5 to 4.7.

For a whole-life assurance, increasing the age at issue usually increases the probability of early death during the contract (with some exceptions between ages 20 and 30) but also shortens the maximum duration of the contract (since it is $\omega-x$). These two effects work together to increase the expected value of $\mathcal{Z}$ with age at issue.

They somehow work in opposite directions for the standard deviation of $\mathcal{Z}$. First, the standard deviation of $\mathcal{Z}$ increases with age at issue for younger ages, then it is approximately constant for ages between 60 and 80, and finally it decreases to 0 for age at issue $\omega-1$.

The coefficient of variation decreases from 1.774 for age at issue 20 to .4937 for age at issue 70, to .0469 for age at issue 100.

The skewness which is highly positive at younger ages at issue (5.41 at age 20) decreases when the contract is sold to an older life assured and ends up being negative at very old ages at issue (-1.50 at age 100).

4.4.3 The pdf and cdf of $\mathcal{Z}$.

Using 4.9 with $\mathcal{Z}$ defined as in 4.35, we get for the pdf of $\mathcal{Z}$:

$$f_{\mathcal{Z}}(z) = \sum_{k=0}^{\omega-x-1} {}_{k|}q_x \cdot f_{e^{-y(k+1)}}(z). \qquad 4.41$$

Using 4.10 with $\mathcal{Z}$ defined as in 4.35, we have that the cdf of $\mathcal{Z}$ is:

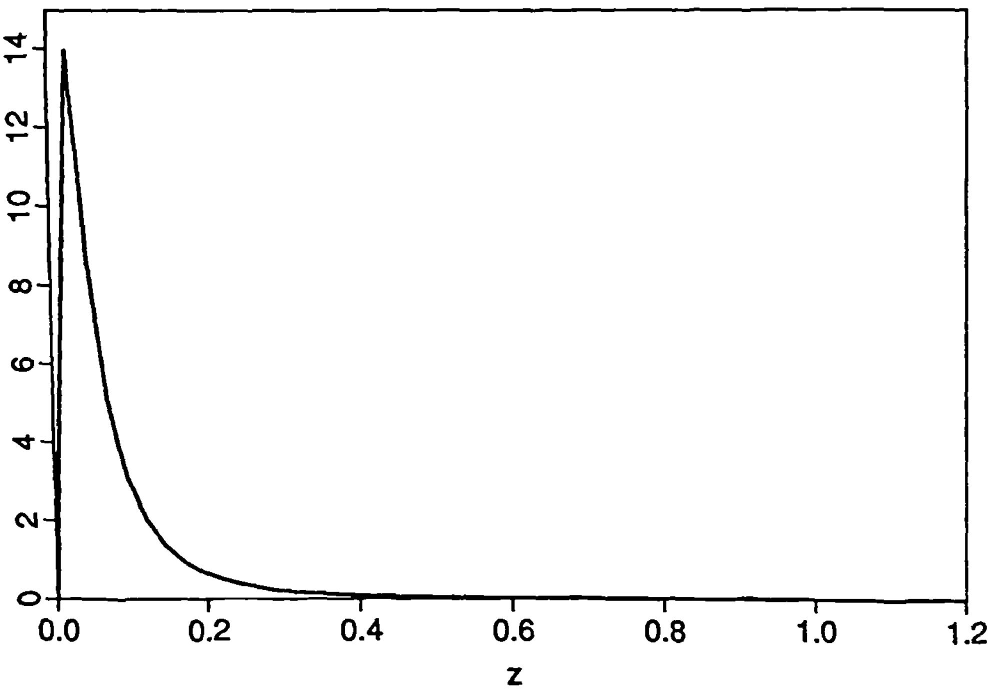
$$F_{\mathcal{Z}}(z) = P(\mathcal{Z} \leq z) = \sum_{k=0}^{\omega-x-1} {}_{k|}q_x \cdot P\left(e^{-y(k+1)} \leq z\right). \qquad 4.42$$

The pdf of $\mathcal{Z}$ for a whole-life assurance issued to a life assured aged 30 is presented in figure 4.9.

Figure 4.9 Pdf of $\mathcal{Z}$

whole-life assurance issued at age 30

Ornstein-Uhlenbeck $\delta=.06$ $\delta_0=.1$ $\alpha=.1$ $\sigma=.01$



The pdf of Z for a whole life assurance contract issued at age x is a weighted sum of lognormal distributions. As shown by equation 4.41, the lognormal distributions are those of $e^{-y(k+1)}$ and the weights are ${}_{k|}q_x$ for $k=0,1,\ldots,\omega-x-1$.

Note that although not apparent in figure 4.9 because of the scale used, the peaks that we can see in figure 4.4 near Z equal .9 (and lower) corresponding to an early death, i.e. k=0,1 or 2, are still present.

The mode is located at the far left of the graph since a life assured aged 30 is more likely to live up to 70 or 80 than to die before those ages. As a consequence, Z is more likely to take a value near $E[e^{-y(70)}]$ or even smaller, and this value is already fairly close to 0.

CHAPTER 5

A PORTFOLIO OF POLICIES

5.1 General Results.

5.1.1 Definitions.

By a portfolio of assurance policies, we mean a group of identical life assurance contracts issued at the same time to independent lives assured of the same age.

In general terms, a life assurance contract is modelled by the benefit function, b_k, and the time of payment function, $t(k)$. These two functions were defined in section 4.1.1. Note that appropriate definitions of b_k and $t(k)$ will represent the specific contracts that we will study, namely, the n-year temporary assurance contract, the n-year endowment assurance contract and the whole-life assurance contract.

Consider a portfolio of c identical policies, one policy being issued to each of c independent lives of the same age. Let K_i be the curtate-future-lifetime of the i^{th} life assured, let $\mathcal{Z}_i$ be the random variable denoting the present value of the benefit that is payable with respect to the i^{th} life assured of the portfolio, and let ${}_c\mathcal{Z}$ be the random variable representing the total present value of all the benefits to be paid within the portfolio.

Then

$$ {}_c\mathcal{Z} = \sum_{i=1}^{c} \mathcal{Z}_i . \qquad 5.1 $$

When studying the random variable ${}_c Z$, we will make the following three assumptions :

$$\left.\begin{array}{ll} \text{(i)-} \left\{K_i\right\}_{i=1}^{c} & \text{are i.i.d} \\ \text{(ii)-} \left\{Z_i\right\}_{i=1}^{c} & \text{are identically distributed} \\ \text{(iii)-} \left\{Z_i \mid \{y(t)\}\right\}_{i=1}^{c} & \text{are i.i.d.} \end{array}\right\} \qquad 5.2$$

Assumption 5.2(i) is a realistic and usual assump⁺ion (see, for example, Bowers et al. (1986, pp.52, 87)) unless we are dealing with joint-life assurance or annuity. Even for lives covered by a joint-life contract (who presumably have some association, implying that their curtate-future-lifetimes are not independent) it is usual to assume independence (see, for example, Bowers et al. (1986, p.232)), because the dependence is difficult to quantify.

Assumptions 5.2(ii) and (iii) simply mean that the force of interest used to discount the benefit payable is the same for each policy in the portfolio and that we look at a portfolio as a whole. This does not mean that the funds of the portfolio cannot be split and invested in different investment vehicles. It means that we will consider and model only the global rate of return of the entire portfolio.

We refer to this set of three assumptions as "assumptions 5.2".

It is important to note that under assumptions 5.2, the random variables Z_i are not independent. Although independent with respect to mortality of the lives assured, they are all discounted using the same future forces of interest.

Like Waters (1978, section 4), we shall be interested in the average cost (or present value) per policy for the portfolio, (i.e. the random variable ${}_c\mathcal{Z}/c$).

5.1.2 Expected value of ${}_c\mathcal{Z}$.

The expected value of ${}_c\mathcal{Z}$ is simply c times the expected value of $\mathcal{Z}$. This result follows since:

$$E[{}_c\mathcal{Z}] = E\left[\sum_{i=1}^{c} \mathcal{Z}_i\right] = \sum_{i=1}^{c} E[\mathcal{Z}_i] = c\cdot E[\mathcal{Z}_i] = c\cdot A. \qquad 5.3$$

where $A = E[\mathcal{Z}_i]$ is the net single premium for one contract.

So the expected average cost per policy is independent of the number of policies in the portfolio, that is:

$$E\left[{}_c\mathcal{Z}/c\right] = A \qquad 5.4$$

Note that assumptions 5.2 were not required to obtain equations 5.3 and 5.4.

5.1.3 Second moment of ${}_c\mathcal{Z}$.

<u>THEOREM</u> <u>5.1:</u> The second moment of ${}_c\mathcal{Z}$ under assumptions 5.2 is given by: (a similar result may be found in Waters (1978, p.69))

$$E[{}_c\mathcal{Z}^2] = c\cdot(c-1)\cdot E\left[\mathcal{Z}_1\cdot\mathcal{Z}_2\right] + c\cdot{}^2A \qquad 5.5$$

where, as before, A is the net single premium for one contract, and

$$E\left[Z_1 \cdot Z_2\right] = \sum_{k_1=0}^{\omega-x-1} \sum_{k_2=0}^{\omega-x-1} b_{k_1} \cdot b_{k_2} \cdot E\left[e^{-y(t(k_1))-y(t(k_2))}\right] \cdot {}_{k_1|}q_x \cdot {}_{k_2|}q_x \qquad 5.6$$

Proof: To prove 5.5, we start by expanding ${}_cZ^2$ into a double summation, we then have:

$$E[{}_cZ^2] = E\left[\left(\sum_{i=1}^{c} Z_i\right)^2\right] = E\left[\sum_{i=1}^{c} \sum_{j=1}^{c} Z_i \cdot Z_j\right] = \sum_{i=1}^{c} \sum_{j=1}^{c} E\left[Z_i \cdot Z_j\right]. \qquad 5.7$$

Since $E\left[Z_i \cdot Z_j\right]$ depends on whether i and j represent the same life assured or not, we have to consider the two possibilities in the double summation. This implies that we have to write it as:

$$E[{}_cZ^2] = \sum_{i=1}^{c} \sum_{\substack{j=1 \\ i \neq j}}^{c} E\left[Z_i \cdot Z_j\right] + \sum_{i=1}^{c} E\left[Z_i^{\,2}\right]. \qquad 5.8$$

Now from assumptions 5.2, we have that:

$$E\left[Z_i \cdot Z_j\right] = E\left[Z_1 \cdot Z_2\right] \qquad \forall (i,j)\ i \neq j \qquad 5.9$$

Using the result of theorem 4.1 for the second term of 5.8, we immediately obtain equation 5.5.

Note that

$$E\left[\mathcal{Z}_1 \cdot \mathcal{Z}_2\right] = E\left[b_{K_1} \cdot b_{K_2} \cdot e^{-y(t(K_1))-y(t(K_2))}\right]$$

$$= E\left[E\left[b_{K_1} \cdot b_{K_2} \cdot e^{-y(t(K_1))-y(t(K_2))} \mid K_1, K_2\right]\right] \qquad 5.10$$

As K_1 and K_2 are independent, their joint probability function is the product of their probability functions. Equation 5.6 therefore follows immediately from equation 5.10.□

In relation to equation 5.6, note that $E\left[e^{-y(t(k_1))-y(t(k_2))}\right]$ for a gaussian process has been studied in chapter 3 (section 3.3).

In relation to this second moment of $_c\mathcal{Z}$, it is of interest to consider the variance of the average cost per policy, when the number of policies, c, becomes very large. It is a known fact that for a deterministic force of interest and for c mutually independent lives assured, this variance tends to 0 when c tends to infinity. The answer to this question for when the force of interest is stochastic is found in the next theorem.

THEOREM 5.2: Under assumptions 5.2, the limiting variance of the average cost per policy as c tends to infinity is:

$$\lim_{c\to\infty} V\left[{}_c\mathcal{Z}/_c\right] = E\left[\mathcal{Z}_1 \cdot \mathcal{Z}_2\right] - A^2 \qquad 5.11$$

where $E\left[\mathcal{Z}_1 \cdot \mathcal{Z}_2\right]$ is given by 5.6.

Proof:

$$V\left[{}_c\mathcal{Z}/_c\right] = \left\{E[{}_c\mathcal{Z}^2] - E[{}_c\mathcal{Z}]^2\right\} / c^2. \qquad 5.12$$

From theorem 5.1 for $E[{}_c\mathcal{Z}^2]$,

$$V\left[{}_c\mathcal{Z}/_c\right] = \frac{c\cdot(c-1)\cdot E\left[\mathcal{Z}_1\cdot\mathcal{Z}_2\right] + c\cdot{}^2A - c^2\cdot A^2}{c^2} \quad . \qquad 5.13$$

$$V\left[{}_c\mathcal{Z}/_c\right] = (1-1/c)\cdot E\left[\mathcal{Z}_1\cdot\mathcal{Z}_2\right] + {}^2A_{/c} - A^2. \qquad 5.14$$

and taking the limit, we get equation 5.11:

$$\lim_{c\to\infty} V\left[{}_c\mathcal{Z}/_c\right] = E\left[\mathcal{Z}_1\cdot\mathcal{Z}_2\right] - A^2. \square$$

Note that, if the force of interest is deterministic, the expression 5.11 is 0, as in this case the expectation of $\mathcal{Z}_1$ times $\mathcal{Z}_2$ would simplify to A^2.

Another interesting problem is to consider the rate of decrease of the variance of the average cost per policy, as additional lives assured are included in the portfolio. The next theorem gives an expression for the derivative of this variance.

THEOREM 5.3: Assuming $E[V[\mathcal{Z}|\{y(t)\}]] > 0$ and under assumptions 5.2, when c is increased the variance of the average cost per policy decreases at a rate of:

$$\left(1/c^2\right)\cdot\ ({}^2A - E[\mathcal{Z}_1\cdot\mathcal{Z}_2]),$$

Proof: The result is immediate in treating 5.14 as a continuous function of c and by taking the derivative of 5.14 with respect to c.

The derivative of 5.14 is:

$$\frac{d}{dc}\left\{V\left[{}_{c}\mathcal{Z}/_{c}\right]\right\} = \frac{1}{c^2}\cdot\left(E\left[\mathcal{Z}_1\cdot\mathcal{Z}_2\right] - E[\mathcal{Z}_1]\cdot E[\mathcal{Z}_2] - {}^{2}A + E[\mathcal{Z}_1]\cdot E[\mathcal{Z}_2]\right)$$

and since $E[\mathcal{Z}_1]=E[\mathcal{Z}_2]$,

$$\frac{d}{dc}\left\{V\left[{}_{c}\mathcal{Z}/_{c}\right]\right\} = \frac{1}{c^2}\cdot\left(E\left[\mathcal{Z}_1\cdot\mathcal{Z}_2\right] - {}^{2}A\right). \qquad 5.15$$

where

$$E\left[\mathcal{Z}_1\cdot\mathcal{Z}_2\right] - {}^{2}A = E\left[E\left[\mathcal{Z}_1\cdot\mathcal{Z}_2|y(t)\right]\right] - E\left[E\left[\mathcal{Z}_1^2|y(t)\right]\right] \qquad 5.16$$

by assumption 5.2 (iii), we have:

$$= E\left[E\left[\mathcal{Z}_1|y(t)\right]\cdot E\left[\mathcal{Z}_2|y(t)\right]\right] - E\left[E\left[\mathcal{Z}_1^2|y(t)\right]\right]$$

$$= E\left[E^2\left[\mathcal{Z}_1|y(t)\right]\right] - E\left[E\left[\mathcal{Z}_1^2|y(t)\right]\right]$$

$$= -\ E\left[V\left[\mathcal{Z}_1|y(t)\right]\right] < 0 \text{ by assumption.}\ \square$$

5.1.4 Third moment of ${}_c\mathcal{Z}$.

<u>THEOREM</u> <u>5.4:</u> The third moment of ${}_c\mathcal{Z}$ under assumptions 5.2 is given by: (a similar result may be found in Waters (1978, p.69))

$$E[{}_c\mathcal{Z}^3] = c(c-1)(c-2)\cdot E\left[\mathcal{Z}_1\cdot\mathcal{Z}_2\cdot\mathcal{Z}_3\right] + 3c(c-1)\cdot E\left[\mathcal{Z}_1^2\cdot\mathcal{Z}_2\right] + c\cdot{}^3A \qquad 5.17$$

where

$$E\left[\mathcal{Z}_1\cdot\mathcal{Z}_2\cdot\mathcal{Z}_3\right] = \sum_{k_1=0}^{\omega-x-1}\sum_{k_2=0}^{\omega-x-1}\sum_{k_3=0}^{\omega-x-1} b_{k_1}\cdot b_{k_2}\cdot b_{k_3}\cdot$$

$$E\left[e^{-y(t(k_1))-y(t(k_2))-y(t(k_3))}\right]\cdot{}_{k_1|}q_x\cdot{}_{k_2|}q_x\cdot{}_{k_3|}q_x \qquad 5.18$$

and

$$E\left[\mathcal{Z}_1{}^2\cdot\mathcal{Z}_2\right] = \sum_{k_1=0}^{\omega-x-1}\sum_{k_2=0}^{\omega-x-1} b_{k_1}^2\cdot b_{k_2}\cdot$$

$$E\left[e^{-2y(t(k_1))-y(t(k_2))}\right]\cdot{}_{k_1|}q_x\cdot{}_{k_2|}q_x \qquad 5.19$$

<u>Proof:</u>

$$E[{}_c\mathcal{Z}^3] = E\left[\left(\sum_{i=1}^{c}\mathcal{Z}_i\right)^3\right] = E\left[\sum_{i=1}^{c}\sum_{j=1}^{c}\sum_{l=1}^{c}\mathcal{Z}_i\cdot\mathcal{Z}_j\cdot\mathcal{Z}_l\right]$$

$$= \sum_{i=1}^{c}\sum_{j=1}^{c}\sum_{l=1}^{c}E\left[\mathcal{Z}_i\cdot\mathcal{Z}_j\cdot\mathcal{Z}_l\right] \qquad 5.20$$

Since $E\left[\mathcal{Z}_i\cdot\mathcal{Z}_j\cdot\mathcal{Z}_l\right]$ depends on whether i, j and l represent three different lives assured, two different lives assured or just one life assured, we have to consider the three possibilities in the triple summation of 5.20. This implies that

$$E[{}_cZ^3] = \sum_{i=1}^{c} \sum_{\substack{j=1 \\ i\neq j}}^{c} \sum_{\substack{l=1 \\ i\neq l \\ j\neq l}}^{c} E\left[Z_i \cdot Z_j \cdot Z_l\right]$$

$$+ 3\cdot\sum_{i=1}^{c} \sum_{\substack{j=1 \\ i\neq j}}^{c} E\left[Z_i^{\,2} \cdot Z_j\right] + \sum_{i=1}^{c} E\left[Z_i^{\,3}\right] \qquad 5.21$$

Now from assumptions 5.2, we have that:

$$E\left[Z_i \cdot Z_j \cdot Z_l\right] = E\left[Z_1 \cdot Z_2 \cdot Z_3\right] \text{ if } i, j \text{ and } l \text{ are all different} \qquad 5.22$$

and

$$E\left[Z_i^2 \cdot Z_j\right] = E\left[Z_1^2 \cdot Z_2\right] \qquad \text{if } i\neq j \qquad 5.23$$

Using theorem 4.1 for the last term of 5.21, equation 5.17 follows:

$$E[{}_cZ^3] = c(c-1)(c-2)\cdot E\left[Z_1 \cdot Z_2 \cdot Z_3\right] + 3c(c-1)\cdot E\left[Z_1^2 \cdot Z_2\right] + c\cdot {}^3A\ .$$

where

$$E\left[Z_1 \cdot Z_2 \cdot Z_3\right] = E\left[E\left[Z_1 \cdot Z_2 \cdot Z_3 | K_1, K_2, K_3\right]\right] \qquad 5.24$$

and assuming independence of K_1, K_2 and K_3, 5.18 is obtained from equation 5.24:

$$E\left[\mathcal{Z}_1 \cdot \mathcal{Z}_2 \cdot \mathcal{Z}_3\right] = \sum_{k_1=0}^{\omega-x-1} \sum_{k_2=0}^{\omega-x-1} \sum_{k_3=0}^{\omega-x-1} b_{k_1} \cdot b_{k_2} \cdot b_{k_3} \cdot$$

$$E\left[e^{-y(t(k_1))-y(t(k_2))-y(t(k_3))}\right] \cdot {}_{k_1|}q_x \cdot {}_{k_2|}q_x \cdot {}_{k_3|}q_x$$

and where

$$E\left[\mathcal{Z}_1^{\,2} \cdot \mathcal{Z}_2\right] = E\left[E\left[\mathcal{Z}_1^{\,2} \cdot \mathcal{Z}_2 | K_1, K_2\right]\right] \qquad 5.25$$

and again assuming independence of K_1 and K_2, 5.19 is obtained from equation 5.25:

$$E\left[\mathcal{Z}_1^{\,2} \cdot \mathcal{Z}_2\right] = \sum_{k_1=0}^{\omega-x-1} \sum_{k_2=0}^{\omega-x-1} b_{k_1}^{\,2} \cdot b_{k_2} \cdot E\left[e^{-2y(t(k_1))-y(t(k_2))}\right] \cdot {}_{k_1|}q_x \cdot {}_{k_2|}q_x \cdot \square$$

The skewness of the limiting average cost per policy is given by the following theorem:

<u>THEOREM</u> <u>5.5:</u> Under assumptions 5.2, the limiting skewness of the average cost per policy as c tends to infinity is:

$$\lim_{c\to\infty} sk\left[{}^{c}\mathcal{Z}/_{c}\right] = \frac{E\left[\mathcal{Z}_1 \cdot \mathcal{Z}_2 \cdot \mathcal{Z}_3\right] - 3 \cdot E\left[\mathcal{Z}_1 \cdot \mathcal{Z}_2\right] \cdot A + 2 \cdot A^3}{\left(E\left[\mathcal{Z}_1 \cdot \mathcal{Z}_2\right] - A^2\right)^{1.5}} \qquad 5.26$$

where $E\left[\mathcal{Z}_1 \cdot \mathcal{Z}_2 \cdot \mathcal{Z}_3\right]$ is given by 5.18 and $E\left[\mathcal{Z}_1 \cdot \mathcal{Z}_2\right]$ is given by 5.6.

Proof:

$$sk\left[{}_c\mathcal{Z}/c\right] = sk({}_c\mathcal{Z}) \qquad 5.27$$

$$= \frac{E[{}_c\mathcal{Z}^3] - 3\cdot E[{}_c\mathcal{Z}^2]\cdot E[{}_c\mathcal{Z}] + 2\cdot E[{}_c\mathcal{Z}]^3}{(E[{}_c\mathcal{Z}^2] - E[{}_c\mathcal{Z}]^2)^{1.5}} \qquad 5.28$$

$$= \frac{\left\{c(c-1)(c-2)\cdot E\left[\mathcal{Z}_1\cdot\mathcal{Z}_2\cdot\mathcal{Z}_3\right] + 3c(c-1)\cdot E\left[\mathcal{Z}_1^2\cdot\mathcal{Z}_2\right] + c\cdot{}^3A - 3\left(c(c-1)\cdot E\left[\mathcal{Z}_1\cdot\mathcal{Z}_2\right] + c\cdot{}^2A\right)\cdot c\cdot A + 2c^3\cdot A^3\right\}}{\left\{c(c-1)\cdot E\left[\mathcal{Z}_1\cdot\mathcal{Z}_2\right] + c\cdot{}^2A - c^2\cdot A^2\right\}^{1.5}} \qquad 5.29$$

dividing both numerator and denominator by c^3, we get:

$$= \frac{\left\{(1-\frac{1}{c})(1-\frac{2}{c})\cdot E\left[\mathcal{Z}_1\cdot\mathcal{Z}_2\cdot\mathcal{Z}_3\right] + \frac{3}{c}(1-\frac{1}{c})\cdot E\left[\mathcal{Z}_1^2\cdot\mathcal{Z}_2\right] + \frac{1}{c}2\cdot{}^3A - 3\left((1-\frac{1}{c})\cdot E\left[\mathcal{Z}_1\cdot\mathcal{Z}_2\right] + \frac{1}{c}\cdot{}^2A\right)\cdot A + 2\,A^3\right\}}{\left\{(1-\frac{1}{c})\cdot E\left[\mathcal{Z}_1\cdot\mathcal{Z}_2\right] + \frac{1}{c}\cdot{}^2A - A^2\right\}^{1.5}} \qquad 5.30$$

Finally, if $E\left[\mathcal{Z}_1\cdot\mathcal{Z}_2\right] - A^2$ is different than 0, by taking the limit as c tends to infinity, we obtain equation 5.26.□

Note that if the force of interest is deterministic, the limiting variance is 0 and 5.26 becomes 0 (the skewness of a constant is 0).

The above results relate to general contracts. In the next section we study the second and third moments of ${}_{c}\mathcal{Z}$ (and of the corresponding average cost per policy) for a portfolio of n-year temporary assurance contracts, each with sum assured 1.

5.2 Temporary Assurance.

5.2.1 Second moment of ${}_{c}\mathcal{Z}$.

For the particular portfolio of n-year temporary assurance contracts, all with sum assured 1, issued to c independent lives assured aged x, the total present value of all the benefits to be paid, ${}_{c}\mathcal{Z}$, is defined as in 5.1 with $\mathcal{Z}_i$ given by 4.11.

From theorem 5.1, we can find the second moment of ${}_{c}\mathcal{Z}$ by using 5.5 with the appropriate version of 5.6. Note that for n-year temporary assurance contracts equation 5.6 becomes:

$$E\left[\mathcal{Z}_1 \cdot \mathcal{Z}_2\right] = \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-1} E\left[e^{-y(k_1+1)-y(k_2+1)}\right] \cdot {}_{k_1|}q_x \cdot {}_{k_2|}q_x \qquad 5.31$$

As evaluating the double summation in 5.31 is relatively time-consuming, it is more economical to use a recursive approach (based on the term of assurance) to evaluate $E\left[\mathcal{Z}_1 \cdot \mathcal{Z}_2\right]$. It is possible to use the expected value of $\mathcal{Z}_1 \cdot \mathcal{Z}_2$ for (n-1)-year temporary assurance contracts to obtain that of $\mathcal{Z}_1 \cdot \mathcal{Z}_2$ for n-year temporary assurance contracts, thereby avoiding the double summation.

A new notation must be introduced to avoid confusion. Let $E_n\left[Z_1 \cdot Z_2\right]$ be the expected value of $Z_1 \cdot Z_2$ where Z_1 and Z_2 are the random variables of the present value of the benefits payable under n-year temporary assurance contracts issued to two lives assured aged x. So $E_n\left[Z_1 \cdot Z_2\right]$ is given by 5.31.

It is fairly straightforward to verify that the following recursive equation holds:

$$E_n\left[Z_1 \cdot Z_2\right] = E_{n-1}\left[Z_1 \cdot Z_2\right] + 2\left(\sum_{k=0}^{n-2} E\left[e^{-y(n)-y(k+1)}\right] \cdot {}_{k|}q_x\right) \cdot {}_{n-1|}q_x + E\left[e^{-2y(n)}\right] \cdot \left({}_{n-1|}q_x\right)^2 \qquad 5.32$$

Note that the starting value for 5.32 is:

$$E_1\left[Z_1 \cdot Z_2\right] = E\left[e^{-2y(1)}\right] \cdot q_x^{\,2} \qquad 5.33$$

The specific result in 5.5 gives:

$$E[{}_cZ^2] = c \cdot (c-1) \cdot E_n\left[Z_1 \cdot Z_2\right] + c \cdot {}^2A^1_{x:\overline{n}|} \qquad 5.34$$

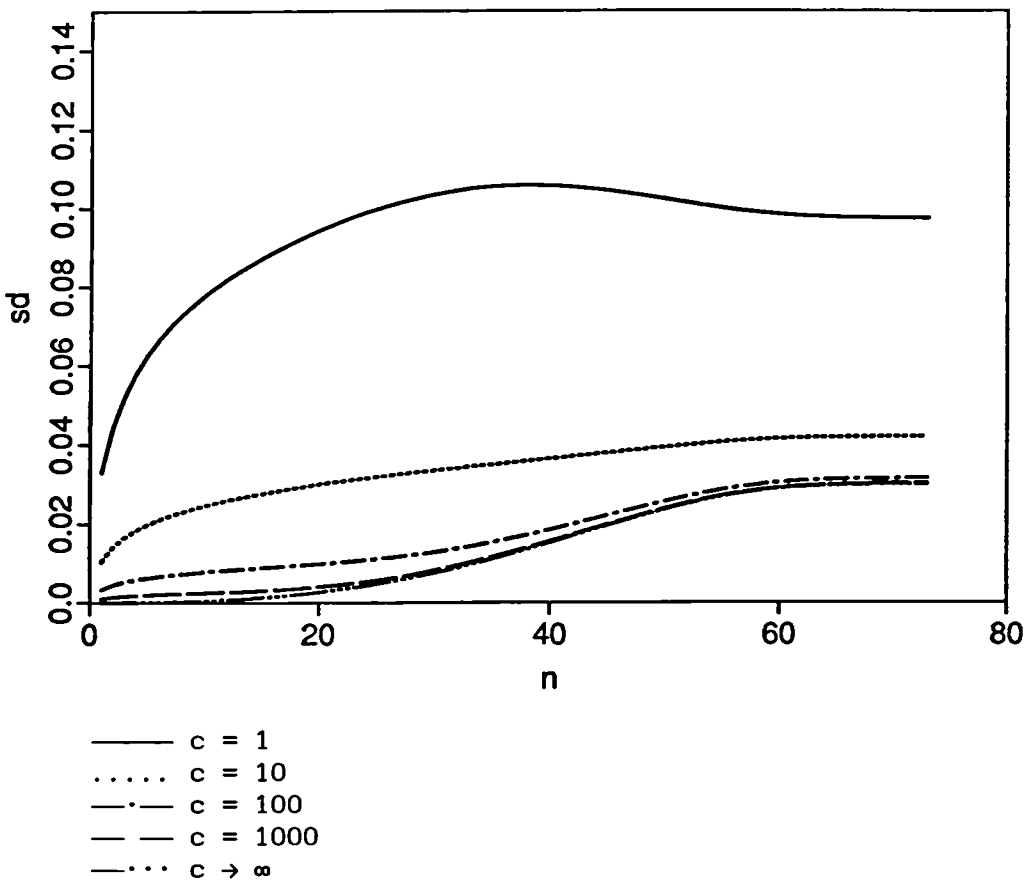
Finally, the variance of the average cost per policy is given by 5.12 with $E[{}_cZ^2]$ given by 5.34 and $E[{}_cZ]$ being $A^1_{x:\overline{n}|}$.

Figure 5.1 illustrates the standard deviation of the average cost per policy for portfolios of n-year temporary assurance contracts issued to c lives assured aged 30.

Figure 5.1 Standard deviation of the average cost per policy

n-year temporary assurance contracts issued at age 30

Ornstein-Uhlenbeck $\delta=.06$ $\delta_0=.1$ $\alpha=.1$ $\sigma=.01$



For the portfolio of size one, i.e. c=1, the curve in figure 5.1 is exactly the curve for age at issue 30 in figure 4.2. The shape of this curve has therefore been explained in section 4.2.2.

For the limiting portfolio (i.e. as c tends to infinity) the variance of the average cost per policy, ${}_c\mathcal{Z}/c$, is given, from theorem 5.2, by equation 5.11 and using 5.31 for $E\left[\mathcal{Z}_1 \cdot \mathcal{Z}_2\right]$. Thus

$$\lim_{c\to\infty} V\left[{}_c\mathcal{Z}/c\right] = \sum_{k_1=0}^{n-1}\sum_{k_2=0}^{n-1} E\left[e^{-y(k_1+1)-y(k_2+1)}\right]\cdot {}_{k_1|}q_x \cdot {}_{k_2|}q_x$$

$$- \left(\sum_{k_1=0}^{n-1} E\left[e^{-y(k_1+1)}\right]\cdot {}_{k_1|}q_x\right)^2 \qquad 5.35$$

$$= \sum_{k_1=0}^{n-1}\sum_{k_2=0}^{n-1} {}_{k_1|}q_x \cdot {}_{k_2|}q_x \cdot \left(E\left[e^{-y(k_1+1)-y(k_2+1)}\right]\right.$$

$$\left. - E\left[e^{-y(k_1+1)}\right]\cdot E\left[e^{-y(k_2+1)}\right]\right) \qquad 5.36$$

$$\lim_{c\to\infty} V\left[{}_c\mathcal{Z}/c\right] = \sum_{k_1=0}^{n-1}\sum_{k_2=0}^{n-1} {}_{k_1|}q_x \cdot {}_{k_2|}q_x \cdot \operatorname{cov}\left(e^{-y(k_1+1)},\ e^{-y(k_2+1)}\right) \qquad 5.37$$

From 5.37, the limiting variance of the average cost per policy is increasing with n since $\operatorname{cov}\left(e^{-y(k_1+1)},\ e^{-y(k_2+1)}\right)$ is positive. That this last covariance is positive may be seen from the following steps:

First, note that

$$\operatorname{cov}\left(e^{-y(k_1+1)},\ e^{-y(k_2+1)}\right) = E\left[e^{-y(k_1+1)-y(k_2+1)}\right]$$

$$- E\left[e^{-y(k_1+1)}\right]\cdot E\left[e^{-y(k_2+1)}\right] \qquad 5.38$$

from 3.14, this may be written as:

$$= \exp\left\{-E[y(k_1+1)+y(k_2+1)] + .5\ V[y(k_1+1)+y(k_2+1)]\right\}$$

$$- \exp\left\{-E[y(k_1+1)]+.5\ V[y(k_1+1)]\right\}\cdot\exp\left\{-E[y(k_2+1)]+.5\ V[y(k_2+1)]\right\} \quad 5.39$$

$$= \exp\left\{-E[y(k_1+1)+y(k_2+1)] + .5\left(V[y(k_1+1)]+V[y(k_2+1)]\right)\right\}\cdot$$

$$\left(\exp\left\{cov(y(k_1+1),y(k_2+1))\right\} - 1\right) \quad 5.40$$

Second, note that 5.40 is positive if $cov(y(k_1+1),y(k_2+1))$ is positive. And referring to 2.39 and 2.9, it is clear that this covariance is positive, so 5.37 is increasing with n.

In other words, for the limiting portfolio, when n is small, the average cost per policy is made up of a large proportion of the portfolio for which no benefit is payable (with a variance of 0) and a small proportion of the portfolio for which a death benefit having a random present value is payable. Globally, the average cost per policy will then have a small variance. When n increases, there is a reduction in the proportion of the portfolio receiving no benefit but an increase in the proportion receiving a death benefit (with a random present value). There is, therefore, an increase in the variance of the average cost per policy.

The reason why the limiting curve (corresponding to c tending to infinity) increases slowly with n for n small (1 to 20), then more rapidly (20 to 60) and slowly again (60 to 73) is the existence of a high mode in the probability function of K(30), the curtate-future-lifetime of a life assured aged 30, around 50.

It is of interest to note from figure 5.1 that the average cost per policy for a portfolio of only 100 policies has about the same variance as that of a very large portfolio (c tending to infinity). This variance is almost identical for any portfolio of 1000 policies or more.

It is also interesting to note that although the standard deviation of the average cost per policy is not sensitive to the size of the portfolio when the portfolio is large, it is very sensitive to the number of policies for small portfolios.

5.2.2 Third moment of ${}_c\mathcal{Z}$.

From theorem 5.4, we can find the third moment of ${}_c\mathcal{Z}$ by using 5.17 with the appropriate versions of 5.18 and 5.19. For n-year temporary assurance contracts, the appropriate version of 5.18 is given by:

$$E\left[\mathcal{Z}_1\cdot\mathcal{Z}_2\cdot\mathcal{Z}_3\right] =$$

$$\sum_{k_1=0}^{n-1}\sum_{k_2=0}^{n-1}\sum_{k_3=0}^{n-1} E\left[e^{-y(k_1+1)-y(k_2+1)-y(k_3+1)}\right]\cdot {}_{k_1|}q_x\cdot {}_{k_2|}q_x\cdot {}_{k_3|}q_x \qquad 5.41$$

and the appropriate version of 5.19 is given by:

$$E\left[\mathcal{Z}_1^2\cdot\mathcal{Z}_2\right] = \sum_{k_1=0}^{n-1}\sum_{k_2=0}^{n-1} E\left[e^{-2y(k_1+1)-y(k_2+1)}\right]\cdot {}_{k_1|}q_x\cdot {}_{k_2|}q_x \qquad 5.42$$

Looking at 5.41 we notice that there is a triple summation which must be very time consuming to evaluate. This suggests that a recursive approach to evaluate $E\left[Z_1 \cdot Z_2 \cdot Z_3\right]$ could be more effective. In fact, it is possible to use the expected value of $Z_1 \cdot Z_2 \cdot Z_3$ for (n-1)-year temporary assurance contracts to obtain that of $Z_1 \cdot Z_2 \cdot Z_3$ for n-year temporary assurance contracts, therefore avoiding the triple summation. The same applies to 5.42 with its double summation.

As we did in section 5.2.1, we must introduce a new notation to avoid confusion. Let $E_n\left[Z_1 \cdot Z_2 \cdot Z_3\right]$ be the expected value of $Z_1 \cdot Z_2 \cdot Z_3$ where Z_1, Z_2 and Z_3 are the random variables of the present value of the benefits payable under n-year temporary assurance contracts issued to three lives assured aged x. So $E_n\left[Z_1 \cdot Z_2 \cdot Z_3\right]$ is given by 5.41.

Also let $E_n\left[Z_1^2 \cdot Z_2\right]$ be the expected value of $Z_1^2 \cdot Z_2$ for two n-year temporary assurance contracts issued to two independent lives assured aged x. Then $E_n\left[Z_1^2 \cdot Z_2\right]$ is given by 5.42.

It is fairly straightforward to verify that the following recursive equations hold:

$$E_n\left[Z_1 \cdot Z_2 \cdot Z_3\right] = E_{n-1}\left[Z_1 \cdot Z_2 \cdot Z_3\right]$$

$$+3\sum_{k_1=0}^{n-2}\sum_{k_2=0}^{n-2} E\left[e^{-y(k_1+1)-y(k_2+1)-y(n)}\right] \cdot {}_{k_1|}q_x \cdot {}_{k_2|}q_x \cdot {}_{n-1|}q_x$$

$$+3\sum_{k=0}^{n-2} E\left[e^{-y(k+1)-2y(n)}\right] \cdot {}_{k|}q_x \left({}_{n-1|}q_x\right)^2 + E\left[e^{-3y(n)}\right] \cdot \left({}_{n-1|}q_x\right)^3 \qquad 5.43$$

and

$$E_n\left[Z_1^2 \cdot Z_2\right] = E_{n-1}\left[Z_1^2 \cdot Z_2\right] + \sum_{k=0}^{n-2} E\left[e^{-2y(k+1)-y(n)}\right] \cdot {}_{k|}q_x \cdot {}_{n-1|}q_x$$

$$+ \sum_{k=0}^{n-2} E\left[e^{-y(k+1)-2y(n)}\right] \cdot {}_{k|}q_x \cdot {}_{n-1|}q_x + E\left[e^{-3y(n)}\right] \cdot \left({}_{n-1|}q_x\right)^2 \qquad 5.44$$

Note that the starting value for 5.43 is:

$$E_1\left[Z_1 \cdot Z_2 \cdot Z_3\right] = E\left[e^{-3y(1)}\right] \cdot \ (q_x)^3 \qquad 5.45$$

and the starting value for 5.44 is:

$$E_1\left[Z_1^2 \cdot Z_2\right] = E\left[e^{-3y(1)}\right] \cdot \ (q_x)^2 \qquad 5.46$$

Those specific results in 5.17 gives:

$$E[{}_cZ^3] = c(c-1)(c-2) \cdot E_n\left[Z_1 \cdot Z_2 \cdot Z_3\right] + 3c(c-1) \cdot E_n\left[Z_1^2 \cdot Z_2\right] + c \cdot {}^3A^1_{x:\overline{n}|} \qquad 5.47$$

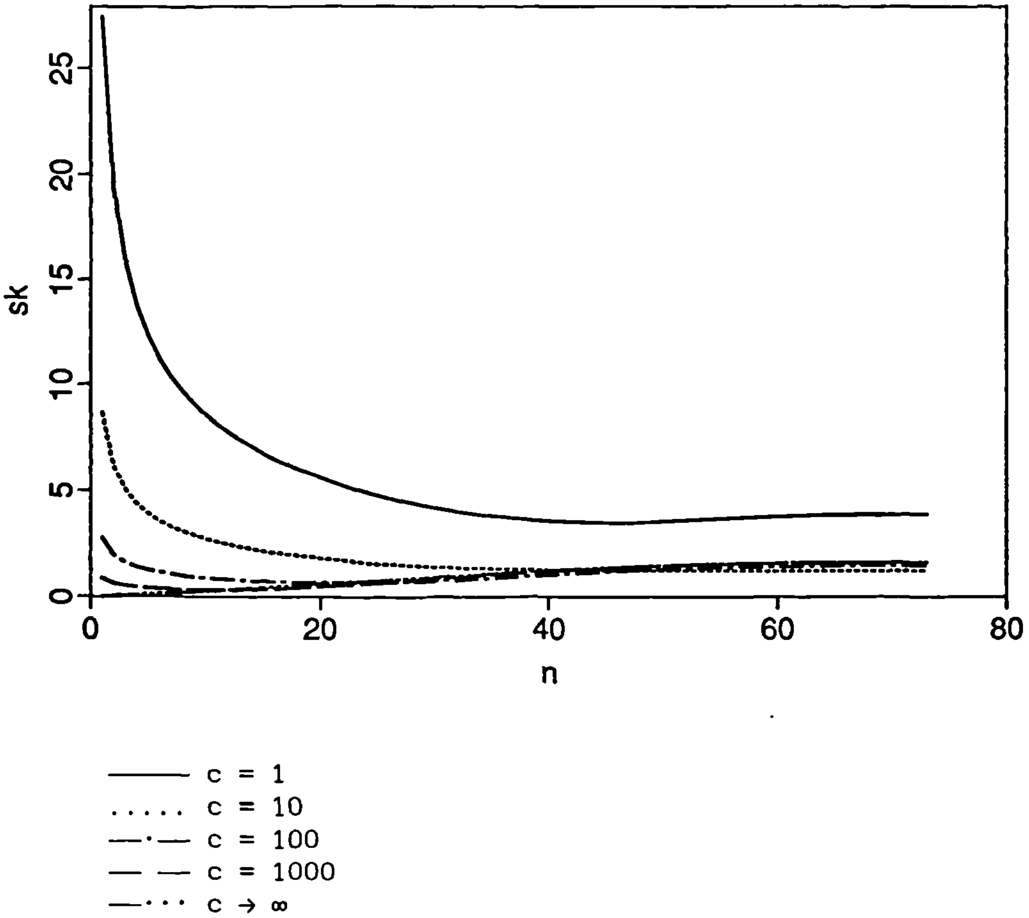
Finally, the skewness of the average cost per policy is given by 5.28 with $E[{}_cZ^3]$ given by 5.47, $E[{}_cZ^2]$ given by 5.34 and $E[{}_cZ]$ by $A^1_{x:\overline{n}|}$.

Figure 5.2 illustrates the skewness of the average cost per policy for portfolios of n-year temporary assurance contracts issued to c lives assured aged 30.

Figure 5.2 Skewness of the average cost per policy

n-year temporary assurance contracts issued at age 30

Ornstein-Uhlenbeck $\delta=.06$ $\delta_0=.1$ $\alpha=.1$ $\sigma=.01$



The curve for c=1 in figure 5.2 corresponding to only one n-year temporary assurance contract is the same as the one for age at issue 30 in figure 4.3. The skewness of the present value of the benefit for one temporary contract has been discussed in section 4.2.2.

For the limiting portfolio, i.e. c tending to infinity, the skewness of the average cost per policy, ${}_cZ/c$, is given, from theorem 5.5, by equation 5.26 and using 5.31 for $E\left[Z_1 \cdot Z_2\right]$ and 5.41 for $E\left[Z_1 \cdot Z_2 \cdot Z_3\right]$.

For short term temporary assurance contracts, the skewness of ${}_c\mathcal{Z}/c$ for c equal 1 is quite different and larger than that for c tending to infinity. For longer term temporary contracts, the variations in this skewness when the number of policies in the portfolio increases are not as important as for short term contracts. Note that the skewness of ${}_c\mathcal{Z}/c$ has a minimum.

From figure 5.2, we see that, at least for terms in excess of 20 years, a portfolio with as few as ten temporary contracts issued at age 30 has about the same asymmetry as a very large portfolio (c tending to infinity). And for any portfolio of 100 policies or more, the asymmetries are about identical.

As a final remark concerning a portfolio of n-year temporary assurance contracts issued at age 30, figures 5.1 and 5.2 suggest that all portfolios of at least 1000 policies will have about the same distribution for their average cost per policy. This suggestion comes from the fact that the average cost per policy for such portfolios would have exactly the same expected value and approximately the same value for its standard deviation and skewness.

Our next section considers the results corresponding to those above for a portfolio of endowment assurances, each with sum assured 1.

5.3 Endowment Assurance.

5.3.1 Second moment of ${}_c\mathcal{Z}$.

For the particular portfolio of n-year endowment assurance contracts, all with sum assured 1, issued to c independent lives assured aged x, the total present value of all the benefits to be paid, ${}_c\mathcal{Z}$, is defined as in 5.1 with $\mathcal{Z}_i$ given by 4.26.

From theorem 5.1, we can find the second moment of $_c\mathcal{Z}$ by using 5.5 with a modified version of 5.6, which for n-year endowment assurance contracts becomes:

$$E\left[\mathcal{Z}_1\cdot\mathcal{Z}_2\right] = \sum_{k_1=0}^{n-1}\ \sum_{k_2=0}^{n-1} E\left[e^{-y(k_1+1)-y(k_2+1)}\right]\cdot {}_{k_1|}q_x\cdot {}_{k_2|}q_x$$

$$+\ 2\sum_{k=0}^{n-1} E\left[e^{-y(n)-y(k+1)}\right]\cdot {}_{k|}q_x\cdot {}_np_x + E\left[e^{-2y(n)}\right]\cdot\left({}_np_x\right)^2 \qquad 5.48$$

Noting that the double summation in 5.48, i.e.:

$$\sum_{k_1=0}^{n-1}\ \sum_{k_2=0}^{n-1} E\left[e^{-y(k_1+1)-y(k_2+1)}\right]\cdot {}_{k_1|}q_x\cdot {}_{k_2|}q_x\ ,$$

is exactly the expected value of $\mathcal{Z}_1\cdot\mathcal{Z}_2$ for two n-year temporary assurance contracts (see 5.31), we have that the expected value of $\mathcal{Z}_1\cdot\mathcal{Z}_2$ for two endowment contracts is that of $\mathcal{Z}_1\cdot\mathcal{Z}_2$ for two temporary contracts plus extra terms due to the survival benefit.

Since $E\left[\mathcal{Z}_1\cdot\mathcal{Z}_2\right]$ for two temporary assurance contracts has already been studied in section 5.2.1, we can use the results of this last section, namely the recursive equation 5.32 with the starting value given by 5.33, to evaluate the double summation in 5.48. And adding the other terms of 5.48, i.e.:

$$2\sum_{k=0}^{n-1} E\left[e^{-y(n)-y(k+1)}\right]\cdot {}_{k|}q_x\cdot {}_np_x + E\left[e^{-2y(n)}\right]\cdot\left({}_np_x\right)^2\ ,$$

we obtain $E\left[\mathcal{Z}_1\cdot\mathcal{Z}_2\right]$ for two n-year endowment assurance contracts.

The second moment about the origin of ${}_c\mathcal{Z}$ for a portfolio of n-year endowment contracts is then, from 5.5:

$$E[{}_c\mathcal{Z}^2] = c\cdot(c-1)\cdot E\left[\mathcal{Z}_1\cdot\mathcal{Z}_2\right] + c\cdot {}^2A_{x:\overline{n}|} \qquad 5.49$$

Finally, the variance of the average cost per policy is given by 5.12 with $E[{}_c\mathcal{Z}^2]$ given by 5.49 and $E[{}_c\mathcal{Z}]$ being $A_{x:\overline{n}|}$.

Figure 5.3 illustrates the standard deviation of the average cost per policy for portfolios of n-year endowment assurance contracts issued to c lives assured aged 30.

For the portfolio of size one, i.e. c=1, the curve in figure 5.3 is exactly the curve for age at issue 30 in figure 4.6. Some discussion relating to this curve has been given in section 4.3.2.

For the limiting portfolio, i.e. c tending to infinity, the variance of the average cost per policy, ${}_c\mathcal{Z}/_c$, is given, from theorem 5.2, by equation 5.11 and using 5.48 for $E\left[\mathcal{Z}_1\cdot\mathcal{Z}_2\right]$ as follows:

$$\lim_{c\to\infty} V\left[{}_c\mathcal{Z}/_c\right] = \sum_{k_1=0}^{n-1}\sum_{k_2=0}^{n-1} E\left[e^{-y(k_1+1)-y(k_2+1)}\right]\cdot {}_{k_1|}q_x\cdot {}_{k_2|}q_x$$

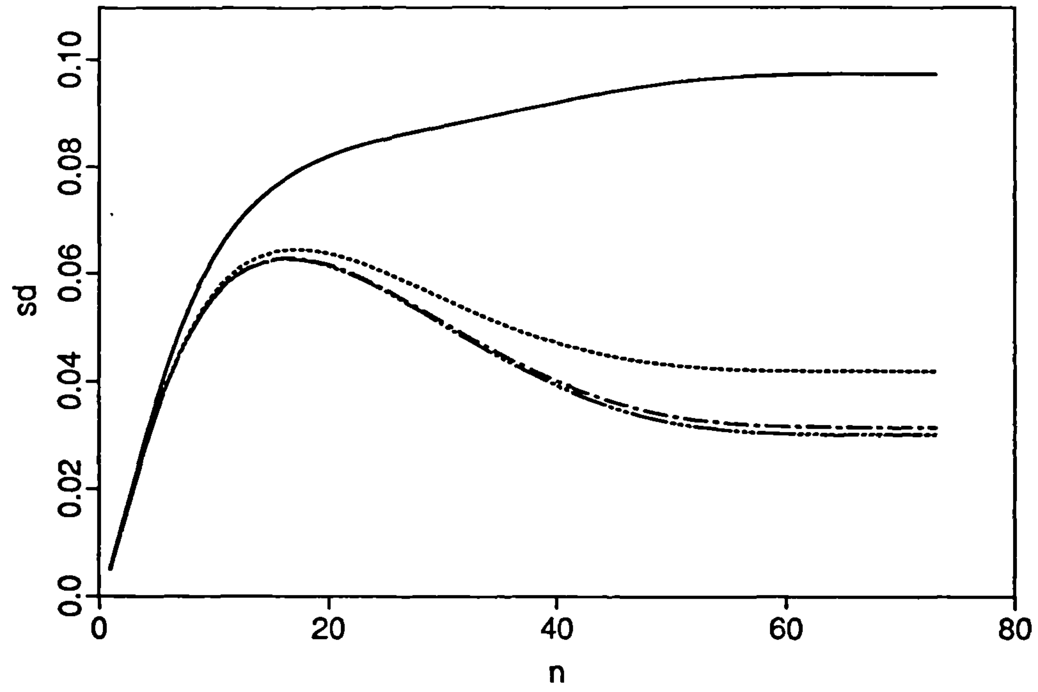
$$+ 2\sum_{k=0}^{n-1} E\left[e^{-y(n)-y(k+1)}\right]\cdot {}_{k|}q_x\cdot {}_np_x + E\left[e^{-2y(n)}\right]\cdot\left({}_np_x\right)^2$$

$$- \left(\sum_{k=0}^{n-1} E\left[e^{-y(k+1)}\right]\cdot {}_{k|}q_x + E\left[e^{-y(n)}\right]\cdot {}_np_x\right)^2 \qquad 5.50$$

Figure 5.3 Standard deviation of the average cost per policy

n-year endowment assurance contracts issued at age 30

Ornstein-Uhlenbeck $\delta=.06$ $\delta_0=.1$ $\alpha=.1$ $\sigma=.01$



——— c = 1
. . . . . c = 10
—·— c = 100
—· · · c → ∞

The last term of 5.50 may be written, on expansion, as:

$$\sum_{k_1=0}^{n-1}\sum_{k_2=0}^{n-1} E\left[e^{-y(k_1+1)}\right]\cdot E\left[^{-y(k_2+1)}\right]\cdot {}_{k_1|}q_x\cdot {}_{k_2|}q_x$$

$$+\ 2\sum_{k=0}^{n-1} E\left[e^{-y(n)}\right]\cdot E\left[^{-y(k+1)}\right]\cdot {}_{k|}q_x\cdot {}_{n}p_x\ +\ E\left[e^{-y(n)}\right]^2\cdot\left({}_{n}p_x\right)^2 \qquad 5.51$$

Then, appropriate regrouping of the terms in 5.50 and using 5.51, we easily obtain the following:

$$\lim_{c\to\infty} V\left[{}_c\mathcal{Z}/c\right] = \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-1} {}_{k_1|}q_x \cdot {}_{k_2|}q_x \cdot \mathrm{cov}\left(e^{-y(k_1+1)},\ e^{-y(k_2+1)}\right)$$

$$+2\sum_{k=0}^{n-1} {}_{k|}q_x \cdot {}_n p_x \cdot \mathrm{cov}\left(e^{-y(n)}, e^{-y(k+1)}\right) + \left({}_n p_x\right)^2 \cdot V\left[e^{-y(n)}\right] \qquad 5.52$$

For n small, ${}_{k|}q_x$ for k=0,1,...,n will be relatively small in comparison with ${}_n p_x$. Accordingly, in this case, $\left({}_n p_x\right)^2 \cdot V\left[e^{-y(n)}\right]$ will be the dominant term in 5.52. Note that $V\left[e^{-y(n)}\right]$ has a maximum value for n about 15 (see figure 3.3).

As n increases, the other two components of 5.52 become more important and $\left({}_n p_x\right)^2 \cdot V\left[e^{-y(n)}\right]$ will become relatively small, since ${}_n p_x$ decreases with n and $V\left[e^{-y(n)}\right]$ also decreases after duration 15.

Summarizing, we may say that the limiting curve in figure 5.3 has such a shape because, for n small, the pure endowment benefit is most important. This implies that the standard deviation of the average cost per policy will be approximately the standard deviation of the present value of this pure endowment benefit multiplied by the square of the probability of survival to the end of the term of the contract. For n large, the death benefits are more important and the standard deviation of the average cost per policy is a weighted sum of the standard deviations of their present values.

Other interesting observations that we can make from figure 5.3 are that, firstly, for n equal one, the standard deviation of the average cost per policy is the same for portfolios of any size. It is the standard deviation of $e^{-y(1)}$.

Secondly, the average cost per policy for a portfolio of only 100 policies has about the same variance as that of a very large portfolio (c tending to infinity). Moreover, a portfolio of even as few as 10 policies has about the same variance of the average cost per policy as that of a limiting portfolio as long as the term of the endowment contracts is not more than 20 years.

Thirdly, although the standard deviation of the average cost per policy is not sensitive to the size of the portfolio when the portfolio counts at least 10 policies, it is very sensitive to the number of policies for smaller portfolios.

5.3.2 Third moment of ${}_{c}Z$.

From theorem 5.4, we can find the third moment of ${}_{c}Z$ by using 5.17 with the appropriate versions of 5.18 and 5.19. Equation 5.18 for n-year endowment assurance contracts becomes:

$$E\left[Z_1 \cdot Z_2 \cdot Z_3\right] =$$

$$\sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-1} \sum_{k_3=0}^{n-1} E\left[e^{-y(k_1+1)-y(k_2+1)-y(k_3+1)}\right] \cdot {}_{k_1|}q_x \cdot {}_{k_2|}q_x \cdot {}_{k_3|}q_x$$

$$+ 3 \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-1} E\left[e^{-y(k_1+1)-y(k_2+1)-y(n)}\right] \cdot {}_{k_1|}q_x \cdot {}_{k_2|}q_x \cdot {}_n p_x$$

$$+ 3 \sum_{k=0}^{n-1} E\left[e^{-y(k+1)-2y(n)}\right] \cdot {}_{k|}q_x \cdot \left({}_n p_x\right)^2 + E\left[e^{-3y(n)}\right] \cdot \left({}_n p_x\right)^3 \qquad 5.53$$

and equation 5.19 becomes:

$$E\left[Z_1^2 \cdot Z_2\right] = \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-1} E\left[e^{-2y(k_1+1)-y(k_2+1)}\right] \cdot {}_{k_1|}q_x \cdot {}_{k_2|}q_x$$

$$+ \sum_{k=0}^{n-1} E\left[e^{-2y(k+1)-y(n)}\right] \cdot {}_{k|}q_x \cdot {}_n p_x$$

$$+ \sum_{k=0}^{n-1} E\left[e^{-y(k+1)-2y(n)}\right] \cdot {}_{k|}q_x \cdot {}_n p_x + E\left[e^{-3y(n)}\right] \cdot \left({}_n p_x\right)^2 \qquad 5.54$$

Looking at 5.53, we notice that the triple summation, i.e.:

$$\sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-1} \sum_{k_3=0}^{n-1} E\left[e^{-y(k_1+1)-y(k_2+1)-y(k_3+1)}\right] \cdot {}_{k_1|}q_x \cdot {}_{k_2|}q_x \cdot {}_{k_3|}q_x ,$$

is exactly the expected value of $Z_1 \cdot Z_2 \cdot Z_3$ for three n-year temporary assurance contracts (see 5.41). We then have that the expected value of $Z_1 \cdot Z_2 \cdot Z_3$ for three endowment contracts is that of $Z_1 \cdot Z_2 \cdot Z_3$ for three temporary contracts plus three extra terms due to the survival benefit.

Since $E\left[Z_1 \cdot Z_2 \cdot Z_3\right]$ for three temporary assurance contracts has already been studied in section 5.2.2, we can use the results of this last section, namely the recursive equation 5.43 with the starting value given by 5.45, to evaluate the triple summation in 5.53. By adding the other terms of 5.53, i.e.:

$$3 \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-1} E\left[e^{-y(k_1+1)-y(k_2+1)-y(n)}\right] \cdot {}_{k_1|}q_x \cdot {}_{k_2|}q_x \cdot {}_np_x$$

$$+ 3 \sum_{k=0}^{n-1} E\left[e^{-y(k+1)-2y(n)}\right] \cdot {}_{k|}q_x \cdot \left({}_np_x\right)^2 + E\left[e^{-3y(n)}\right] \cdot \left({}_np_x\right)^3 ,$$

we obtain $E\left[Z_1 \cdot Z_2 \cdot Z_3\right]$ for three n-year endowment assurance contracts.

Now looking at 5.54, we notice that the double summation, i.e.:

$$\sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-1} E\left[e^{-2y(k_1+1)-y(k_2+1)}\right] \cdot {}_{k_1|}q_x \cdot {}_{k_2|}q_x ,$$

is exactly the expected value of $Z_1^2 \cdot Z_2$ for two n-year temporary assurance contracts (see 5.42).Then the expected value of $Z_1^2 \cdot Z_2$ for two endowment contracts is that of $Z_1^2 \cdot Z_2$ for two temporary contracts plus extra terms due to the survival benefit.

Since $E\left[\mathcal{Z}_1^2 \cdot \mathcal{Z}_2\right]$ for two temporary assurance contracts has already been studied in section 5.2.2, we can use the results of this last section, namely the recursive equation 5.44 with the starting value given by 5.46, to evaluate the double summation in 5.54. And adding the other terms of 5.54, i.e.:

$$\sum_{k=0}^{n-1} E\left[e^{-2y(k+1)-y(n)}\right] \cdot {}_{k|}q_x \cdot {}_n p_x$$

$$+ \sum_{k=0}^{n-1} E\left[e^{-y(k+1)-2y(n)}\right] \cdot {}_{k|}q_x \cdot {}_n p_x + E\left[e^{-3y(n)}\right] \cdot \left({}_n p_x\right)^2 ,$$

we obtain $E\left[\mathcal{Z}_1^2 \cdot \mathcal{Z}_2\right]$ for two n-year endowment assurance contracts.

The third moment about the origin of ${}_c\mathcal{Z}$ for a portfolio of n-year endowment contracts is then, from 5.17:

$$E[{}_c\mathcal{Z}^3] = c(c-1)(c-2) \cdot E\left[\mathcal{Z}_1 \cdot \mathcal{Z}_2 \cdot \mathcal{Z}_3\right] + 3c(c-1) \cdot E\left[\mathcal{Z}_1^2 \cdot \mathcal{Z}_2\right] + c \cdot {}^3A_{x:\overline{n|}} \qquad 5.55$$

Finally, the skewness of the average cost per policy is given by 5.28 with $E[{}_c\mathcal{Z}^3]$ given by 5.55, $E[{}_c\mathcal{Z}^2]$ given by 5.49 and $E[{}_c\mathcal{Z}]$ by $A_{x:\overline{n|}}$.

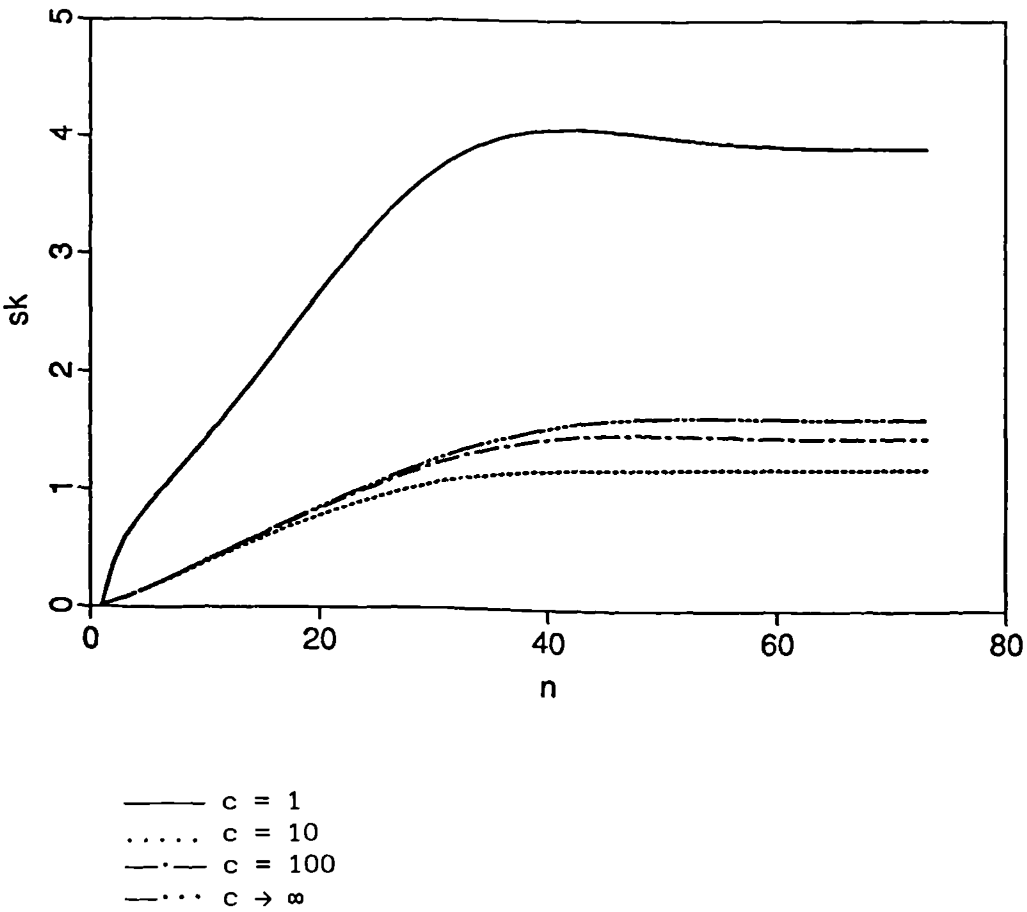
Figure 5.4 illustrates the skewness of the average cost per policy for portfolios of n-year endowment assurance contracts issued to c lives assured aged 30.

The skewness of only one n-year endowment assurance contract, illustrated by the curve with c=1 in figure 5.4, has been discussed in more general terms in section 4.3.2, as it is a particular case (age at issue of 30) of figure 4.7.

Figure 5.4 Skewness of the average cost per policy

n-year endowment assurance contracts issued at age 30

Ornstein-Uhlenbeck $\delta=.06$ $\delta_0=.1$ $\alpha=.1$ $\sigma=.01$



For the limiting portfolio, i.e. c tending to infinity, the skewness of the average cost per policy, ${}_c\mathcal{Z}/_c$, is given, from theorem 5.5, by equation 5.26 and using 5.48 for $E\left[\mathcal{Z}_1 \cdot \mathcal{Z}_2\right]$ and 5.53 for $E\left[\mathcal{Z}_1 \cdot \mathcal{Z}_2 \cdot \mathcal{Z}_3\right]$.

Note that for n small, ${}_np_x$ will be relatively large and the dominant term of 5.53 will therefore be $E\left[e^{-3y(n)}\right] \cdot \left({}_np_x\right)^3$ where $E\left[e^{-3y(n)}\right]$ is the third moment about the origin of the present value of the pure endowment benefit.

For short term endowment contracts, where the pure endowment benefit is most important, the size of the portfolio does not influence the asymmetry of the average cost per policy as much as it does for long term endowment contract. Note that for n-year temporary assurance contract we had the opposite.

We can see from figure 5.4 that for n equal one, the skewness of the average cost per policy is the same for portfolios of any size. It is the skewness of $e^{-y(1)}$.

For the parameters used to obtain figure 5.4, the number of policies which minimizes the skewness of the average cost per policy is between 8 and 14 depending on the value of n.

Finally from figure 5.4, we note that a portfolio with as few as 100 endowment contracts issued at age 30 has approximately the same asymmetry as a very large portfolio (c tending to infinity). Note that for endowment assurance contracts with terms of 40 years or less, the first three moments of ${}_c\mathcal{Z}/c$ for portfolios of at least 100 policies are almost identical. This suggests that for terms of 40 years or less the distribution of the average cost per policy may not vary greatly with the number of policies, provided that this is at least 100.

Our final section considers a portfolio of whole-life assurances.

5.4 Whole-Life Assurance.

5.4.1 Second moment of ${}_c\mathcal{Z}$.

For the particular portfolio of whole-life assurance contracts, each with sum assured 1, issued to c independent lives assured aged x, the total present value of all the benefits to be paid, ${}_c\mathcal{Z}$, is defined as in 5.1 with $\mathcal{Z}_i$ given by 4.35.

We can find the second moment of ${}_c\mathcal{Z}$ for a portfolio of whole-life assurance contracts from theorem 5.1. We then use equation 5.5 where $E\left[\mathcal{Z}_1 \cdot \mathcal{Z}_2\right]$ is now given by 5.31 with n = ω-x. Algebraically, we have:

$$E[{}_c\mathcal{Z}^2] = c \cdot (c-1) \cdot E\left[\mathcal{Z}_1 \cdot \mathcal{Z}_2\right] + c \cdot {}^2A_x \qquad 5.56$$

where

$$E\left[\mathcal{Z}_1 \cdot \mathcal{Z}_2\right] = \sum_{k_1=0}^{\omega-x-1} \sum_{k_2=0}^{\omega-x-1} E\left[e^{-y(k_1+1)-y(k_2+1)}\right] \cdot {}_{k_1|}q_x \cdot {}_{k_2|}q_x \qquad 5.57$$

Note that 5.57 may also be evaluated by using the recursive equation 5.32 with the starting value given by 5.33.

Finally, the variance of the average cost per policy is given by 5.12 with $E[{}_c\mathcal{Z}^2]$ given by 5.56 and $E[{}_c\mathcal{Z}]$ being A_x.

Table 5.1 presents the standard deviation of the average cost per policy for different sizes of portfolio of whole-life assurance contracts issued to independent lives assured aged 30.

For n = ω-x, a temporary assurance or an endowment assurance with term n years is in fact a whole-life contract. Accordingly, with an issue age of 30, for n = 73, the values for each curve in figures 5.1 and 5.3 are given in table 5.1.

Table 5.1 Standard Deviation of ${}_{c}\mathcal{Z}/c$

Whole-Life issued at age 30.

Ornstein-Uhlenbeck $\delta_0=.06$ $\delta=.1$ $\alpha=.1$ $\sigma=.01$

c	$SD\left[{}_{c}\mathcal{Z}/c\right]$
1	.0974602
10	.0419695
100	.0314283
1000	.0301723
10000	.0300438
∞	.0300295

As one would expect, from the earlier comments relating to figures 5.1 and 5.3, the standard deviation of the average cost per policy is fairly constant for portfolios of 100 policies or more. When c is small, however, the standard deviation decreases rapidly as the size of the portfolio is increased.

5.4.2 Third moment of ${}_{c}\mathcal{Z}$.

From theorem 5.4, we can find the third moment of ${}_{c}\mathcal{Z}$ for a portfolio of whole-life assurance contracts by using equation 5.17 with the appropriate versions of 5.18 and 5.19. These last two equations being given by 5.41 and 5.42 respectively with n replaced by ω-x in each case. We then have:

$$E\left[\mathcal{Z}_1 \cdot \mathcal{Z}_2 \cdot \mathcal{Z}_3\right] =$$

$$\sum_{k_1=0}^{\omega-x-1} \sum_{k_2=0}^{\omega-x-1} \sum_{k_3=0}^{\omega-x-1} E\left[e^{-y(k_1+1)-y(k_2+1)-y(k_3+1)}\right] \cdot {}_{k_1|}q_x \cdot {}_{k_2|}q_x \cdot {}_{k_3|}q_x \qquad 5.58$$

and

$$E\left[\mathcal{Z}_1^2 \cdot \mathcal{Z}_2\right] = \sum_{k_1=0}^{\omega-x-1} \sum_{k_2=0}^{\omega-x-1} E\left[e^{-2y(k_1+1)-y(k_2+1)}\right] \cdot {}_{k_1|}q_x \cdot {}_{k_2|}q_x \qquad 5.59$$

Note that 5.58 may also be evaluated by using the recursive equation 5.43 with the starting value given by 5.45. The same applies to 5.59 with equations 5.44 and 5.46.

From 5.17, the third moment about the origin of ${}_c\mathcal{Z}$ is then:

$$E[{}_c\mathcal{Z}^3] = c(c-1)(c-2) \cdot E\left[\mathcal{Z}_1 \cdot \mathcal{Z}_2 \cdot \mathcal{Z}_3\right] + 3c(c-1) \cdot E\left[\mathcal{Z}_1^2 \cdot \mathcal{Z}_2\right] + c \cdot {}^3A_x \qquad 5.60$$

Finally, the skewness of the average cost per policy is given by 5.28 with $E[{}_c\mathcal{Z}^3]$ given by 5.60, $E[{}_c\mathcal{Z}^2]$ given by 5.56 and $E[{}_c\mathcal{Z}]$ by A_x.

Table 5.2 presents the skewness of the average cost per policy for different sizes of portfolio of whole-life assurance contracts issued to independent lives assured aged 30.

Table 5.2 Skewness of ${}_cZ/c$

Whole-Life issued at age 30.

Ornstein-Uhlenbeck $\delta_0=.06$ $\delta=.1$ $\alpha=.1$ $\sigma=.01$

c	$sk\left[{}_cZ/c\right]$
1	3.9152
10	1.2046
14*	1.1718
100	1.4695
1000	1.6155
10000	1.6328
∞	1.6348

* c=14 gives the minimum skewness.

Again, for obvious reasons, the values for n = 73 on the curves in figures 5.2 and 5.4 appear in table 5.2.

As one would expect, from the comments made about figures 5.2 and 5.4, the skewness of the average cost per policy is fairly constant for portfolios of 1000 policies or more. But when c is small, the skewness decreases rapidly with the size of the portfolio.

Since the first three moments of portfolios of at least 1000 whole-life policies are almost identical, they should all have fairly similar distributions for their average cost per policy.

CHAPTER 6

LIMITING DISTRIBUTION FOR A n-YEAR TEMPORARY ASSURANCE

6.1 Limiting Distribution.

In this chapter, we will study the limiting distribution of the average present value of the benefits for a portfolio of temporary assurances, each with sum assured 1, as the number of policies tends to infinity. The preceding chapter used a definition of ${}_c\mathcal{Z}$ involving a summation over the c contracts of the portfolio. This definition was convenient for calculating the moments of ${}_c\mathcal{Z}$ because it was possible to simplify the expressions for these moments due to the mutual independence of the time at death of the c policyholders. However, another definition which is equivalent appears to be more appropriate for studying the limiting distribution of the random variable, ${}_c\mathcal{Z}$.

Instead of summing over the c policies, one could consider summing the present value of the benefits in a given year over the n policy-years of the contract. Algebraically, an expression equivalent to 5.1 is the following:

$$ {}_c\mathcal{Z} = \sum_{i=0}^{n-1} c_i \cdot e^{-y(i+1)} \qquad 6.1 $$

where c_i, i=0,1,...,n-1 is the random variable denoting the number of policies where the death benefit is actually paid at time i+1. We let c_n be the number of lives assured surviving to the end of the term, n.

Note that the sum of the c_i's from i equal 0 to n is c, the total number of policies in the portfolio. Thus,

$$\sum_{i=0}^{n} c_i = c \tag{6.2}$$

Using 6.1, one could intuitively derive that the average cost per policy as the number of such policies tends to infinity would simply be a weighted average of the present value functions from year 1 to year n. The weights being the expected proportion of contracts payable in each year, i.e. $_{i|}q_x$. The probabilistic version of this intuition is presented in theorem 6.1.

<u>THEOREM</u> <u>6.1:</u> As c tends to infinity, the average cost per policy for a portfolio of n-year temporary assurance contracts tends in distribution to: (see also Frees (1991, proposition 6))

$$\zeta_n = \sum_{i=0}^{n-1} {}_{i|}q_x \cdot e^{-y(i+1)} \tag{6.3}$$

<u>Proof:</u> This result is true if

$$ {}_c Z/_c - \zeta_n = \sum_{i=0}^{n-1} \left({}^{c_i}/_{c} - {}_{i|}q_x \right) \cdot e^{-y(i+1)} \tag{6.4}$$

tends in probability to 0.

We use the well-known result that if X tends in probability to 0 and Y has finite mean and variance, then X·Y tends in probability to 0. (see, for example, Chung (1974, p.92))

Here, c_i is binomial$(c, {}_{i|}q_x)$ so, $\left(c_{i}/c - {}_{i|}q_x\right)$ tends in probability to 0 for each i. And as $e^{-y(i+1)}$ is log-normally distributed with finite mean and variance, it follows that:

$$\sum_{i=0}^{n-1} \left(c_{i}/c - {}_{i|}q_x\right) \cdot e^{-y(i+1)}$$

tends in probability to 0. □

Therefore, one could think of ζ_n as the random variable denoting the limiting average cost per policy for a portfolio of n-year temporary assurance contracts.

Now, one could theoretically obtain the density function of ζ_n by integrating the joint density function of the y(i)'s over the appropriate domain. The expression would look like the following:

$$f_{\zeta_n}(z) = \int_{y_n} \cdots \int_{y_2} \int_{y_1} f_{\underline{Y}}(y_1, y_2, \ldots, y_n)\, dy_1 dy_2 \cdots dy_n \qquad 6.5$$

where $\underline{Y} = (y(1), y(2), \ldots, y(n))$ and is multivariate normal.

But this approach is not possible from a practical point of view as it is excessively laborious to evaluate 6.5 even for n as small as 5. In the next section, however, we derive a recursive equation from which one can approximate the distribution of ζ_n.

6.2 Recursive Equation.

Since ζ_n is a summation over the policy-years, it is easy to break it down into the sum of ζ_{n-1} and a term for the n^{th} policy year. The recursive equation for ζ_n is then given by:

$$\zeta_n = \sum_{i=0}^{n-1} {}_{i|}q_x \cdot e^{-y(i+1)} = \sum_{i=0}^{n-2} {}_{i|}q_x \cdot e^{-y(i+1)} + {}_{n-1|}q_x \cdot e^{-y(n)}$$

$$= \zeta_{n-1} + {}_{n-1|}q_x \cdot e^{-y(n)} \qquad 6.6$$

Let z_i be a possible realization of ζ_i and y_j be a possible realization of y(j).

Let the function $g_n(z_n, y_n)$, a somewhat unusual function based on the distribution function of ζ_n and the density function of y(n), be defined as:

$$g_n(z_n, y_n) = P\left(\zeta_n \le z_n\right) \cdot f_{y(n)}\left(y_n \middle| \zeta_n \le z_n\right) \qquad 6.7$$

or equivalently,

$$g_n(z_n, y_n) = f_{y(n)}(y_n) \cdot P\left(\zeta_n \le z_n \middle| y(n)=y_n\right) \qquad 6.8$$

From this last result, it follows immediately that the distribution function of ζ_n is given by:

$$F_{\zeta_n}(z_n) = \int_{-\infty}^{\infty} g_n(z_n, y_n) \cdot dy_n. \qquad 6.9$$

where the function $g_n(z_n, y_n)$ may be calculated with a high degree of accuracy from the recursive relationship contained in the following theorem.

<u>THEOREM</u> <u>6.2:</u> The function $g_n(z_n, y_n)$ may be obtained recursively from:

$$g_n(z_n, y_n) \cong \int_{y_{n-1}} f_{y(n)}\left(y_n \mid y(n-1)=y_{n-1}\right) \cdot g_{n-1}\left(z_n - z_{n-1} \mid q_x \cdot e^{-y_n}, y_{n-1}\right) dy_{n-1} \qquad 6.10$$

with the starting value:

$$g_1(z_1, y_1) = \begin{cases} \phi\left(\dfrac{y_1 - E[y(1)]}{sd[y(1)]}\right) & \text{if } z_1 \geq q_x \cdot e^{-y_1} \\ 0 & \text{otherwise} \end{cases} \qquad 6.11$$

We use the notation $\phi(\cdot)$ to denote the pdf of a zero mean and unit variance normal random variable. Note also that given that y(n-1) equal y_{n-1}, y(n) is normally distributed with mean:

$$E\left[y(n) \mid y(n-1)=y_{n-1}\right] =$$

$$E[y(n)] + \frac{cov(y(n), y(n-1))}{V[y(n)]} \cdot \left\{y_{n-1} - E[y(n-1)]\right\} \qquad 6.12$$

and with variance:

$$V\left[y(n)\,|\,y(n-1)=y_{n-1}\right] = V[y(n)] - \frac{\operatorname{cov}^2(y(n),y(n-1))}{V[y(n-1)]} \qquad 6.13$$

(see, for example, Morrison (1990, p.92))

<u>Proof:</u> To prove 6.10, we start by noting that from 6.6,

$$P\left(\zeta_n \le z_n \,|\, y(n)=y_n\right) = P\left(\zeta_{n-1} \le z_n -_{n-1} \,|\, q_x \cdot e^{-y_n} \,|\, y(n)=y_n\right) \qquad 6.14$$

Now using 6.7, 6.8 and 6.14, we have:

$$g_n(z_n,y_n)=P\left(\zeta_{n-1} \le z_n -_{n-1} \,|\, q_x \cdot e^{-y_n}\right) \cdot f_{y(n)}\left(y_n \,|\, \zeta_{n-1} \le z_n -_{n-1} \,|\, q_x \cdot e^{-y_n}\right) \qquad 6.15$$

but the conditional pdf of y(n) in 6.15 may be written as: (Melsa and Sage (1973, p.98))

$$f_{y(n)}\left(y_n \,|\, \zeta_{n-1} \le z_n -_{n-1} \,|\, q_x \cdot e^{-y_n}\right) =$$

$$\int_{-\infty}^{\infty} f_{y(n)}\left(y_n \,|\, y(n-1)=y_{n-1},\ \zeta_{n-1} \le z_n -_{n-1} \,|\, q_x \cdot e^{-y_n}\right) \cdot$$

$$f_{y(n-1)}\left(y_{n-1} \,|\, \zeta_{n-1} \le z_n -_{n-1} \,|\, q_x \cdot e^{-y_n}\right)\, dy_{n-1} \qquad 6.16$$

Now equation 6.7 implies that:

$$f_{y(n-1)}\left(y_{n-1} \middle| \zeta_{n-1} \leq z^{-}_{n\,n-1|q_x \cdot e^{-y_n}}\right) =$$

$$\frac{g_{n-1}\left(z^{-}_{n\,n-1|q_x \cdot e^{-y_n}},\, y_{n-1}\right)}{P\left(\zeta_{n-1} \leq z^{-}_{n\,n-1|q_x \cdot e^{-y_n}}\right)} \qquad 6.17$$

if we now make the approximation (see section 6.3 below)

$$f_{y(n)}\left(y_n \middle| y(n-1)=y_{n-1},\ \zeta_{n-1} \leq z^{-}_{n\,n-1|q_x \cdot e^{-y_n}}\right) \cong f_{y(n)}\left(y_n \middle| y(n-1)=y_{n-1}\right) \qquad 6.18$$

then equation 6.16 becomes:

$$f_{y(n)}\left(y_n \middle| \zeta_{n-1} \leq z^{-}_{n\,n-1|q_x \cdot e^{-y_n}}\right) \cong \int_{-\infty}^{\infty} f_{y(n)}\left(y_n \middle| y(n-1)=y_{n-1}\right) \cdot$$

$$\frac{g_{n-1}\left(z^{-}_{n\,n-1|q_x \cdot e^{-y_n}},\, y_{n-1}\right)}{P\left(\zeta_{n-1} \leq z^{-}_{n\,n-1|q_x \cdot e^{-y_n}}\right)}\, dy_{n-1} \qquad 6.19$$

Finally substituting this last expression 6.19 into 6.15, we obtain 6.10.

To obtain the starting value, we simply have to note that:

$$\zeta_1 = q_x \cdot e^{-y(1)} \qquad 6.20$$

and that

$$g_1(z_1, y_1) = P\left(\zeta_1 \le z_1 \mid y(1)=y_1\right) \cdot f_{y(1)}\left(y_1\right) \qquad 6.21$$

$$= P\left(\zeta_1 \le z_1 \mid y(1)=y_1\right) \cdot \phi\left(\frac{y_1 - E[y(1)]}{sd[y(1)]}\right) \qquad 6.22$$

Then, since

$$\zeta_1 = q_x \cdot e^{-y_1} \qquad \text{if } y(1)=y_1, \qquad 6.23$$

we have that

$$P\left(\zeta_1 \le z_1 \mid y(1)=y_1\right) = \begin{cases} 1 & \text{if } z_1 \ge q_x \cdot e^{-y_1} \\ 0 & \text{otherwise} \end{cases} \qquad 6.24$$

Finally, by combining 6.24 and 6.22, we obtain 6.11,

$$g_1(z_1, y_1) = \begin{cases} \phi\left(\frac{y_1 - E[y(1)]}{sd[y(1)]}\right) & \text{if } z_1 \ge q_x \cdot e^{-y_1} \\ 0 & \text{otherwise} \end{cases}$$

This completes the proof of theorem 6.2. □

Before looking at the problem of numerical evaluation of the results of theorem 6.2, it is important to study in greater detail and to justify the approximation involved here. This is done in the next section.

6.3 Justifications of the Approximation.

Looking at the proof of theorem 6.2, we note that the result is not exact due only to approximation 6.18 made in order to obtain a recursive equation involving only known quantities. This approximation may be justified theoretically by looking at two particular correlation coefficients, one of which validates the approximation for large values of n and the other for small values of n.

6.3.1 Correlation between y(n) and y(n-1).

From the subject of multivariate analysis, we know that the approximation 6.18 will be acceptable if y(n) and y(n-1) are highly correlated (see, for example, Mardia, Kent and Bibby (1979, section 6.5)). This is true since if they are highly correlated, knowing y(n-1) would explain much of y(n). Now if this is the case, introducing any other variable, correlated or not with y(n), in the regression model to further explain y(n) cannot improve the situation much.

Looking back at the definition of y(n) (see 2.37) it is clear that y(n-1) and y(n) must be highly correlated. Their correlation coefficient will be given by: (Ross (1988, p.280))

$$\rho(y(n),y(n-1)) = \frac{\mathrm{cov}(y(n),y(n-1))}{\mathrm{sd}[y(n)]\cdot \mathrm{sd}[y(n-1)]} \qquad 6.25$$

where cov(y(n),y(n-1)) is known from 2.39 and the standard deviation of y(n) is the square root of 2.40. Note that if the Ornstein-Uhlenbeck process is used to model the force of interest, the particular equations become 2.48 and 2.50 respectively.

The correlation coefficients between y(n) and y(n-1) for different values of n, when the force of interest is modelled by an Ornstein-Uhlenbeck process with parameter α=.1, .2 or .5 are presented in table 6.1.

Table 6.1 Correlation coefficient between y(n) and y(n-1)

Ornstein-Uhlenbeck α=.1, .2 and .5

n	$\rho(y(n),y(n-1))$ α=.1	$\rho(y(n),y(n-1))$ α=.2	$\rho(y(n),y(n-1))$ α=.5
2	.8773	.8707	.8516
3	.9474	.9423	.9270
4	.9701	.9659	.9535
5	.9804	.9769	.9664
6	.9860	.9829	.9739
7	.9894	.9867	.9788
8	.9916	.9891	.9821
9	.9931	.9909	.9846
10	.9942	.9922	.9865
20	.9980	.9969	.9940
40	.9992	.9987	.9972
60	.9995	.9991	.9981

It is interesting to note that the correlation coefficient between y(n) and y(n-1) is not influenced by the other parameters of the

Ornstein-Uhlenbeck process, namely, δ_0, δ and σ. As for δ_0 and δ, this is due to the fact that none of these two parameters appear in equation 2.48 or equation 2.50. As for the parameter σ, which appear in both 2.48 and 2.50, it is easily seen to cancel out in 6.25.

Table 6.1 clearly shows that y(n) and y(n-1) are very highly correlated, especially for large values of n. Therefore, the approximation made to obtain the recursive equation of theorem 6.2 should be acceptable.

Another correlation coefficient could also justify, independently of the one discussed here, the approximation 6.18. This is the subject of the next section.

6.3.2 Correlation between $e^{-y(n)}$ and ζ_n.

Again from the subject of multivariate analysis, we know that the approximation 6.18 would also be acceptable if y(n-1) and ζ_{n-1} contained about the same useful information to explain y(n) (see, for example, Mardia, Kent and Bibby (1979, section 6.5)). But since y(n-1) and $e^{-y(n-1)}$ contain exactly the same information, studying the relationship between y(n-1) and ζ_{n-1} has to be equivalent to studying the relationship between $e^{-y(n-1)}$ and ζ_{n-1}. The latter being much easier to work with.

In other words, if $e^{-y(n)}$ and ζ_n are highly correlated, the approximation would be reasonable. The correlation coefficient between these two random variables is: (Ross (1988, p.280))

$$\rho\left(e^{-y(n)}, \zeta_n\right) = \frac{\operatorname{cov}\left(e^{-y(n)}, \zeta_n\right)}{\operatorname{sd}\left[e^{-y(n)}\right] \cdot \operatorname{sd}\left[\zeta_n\right]} \qquad 6.26$$

$$= \frac{\operatorname{cov}\left(e^{-y(n)}, \sum_{i=0}^{n-1} {}_{i|}q_x \cdot e^{-y(i+1)}\right)}{\operatorname{sd}\left[e^{-y(n)}\right] \cdot \operatorname{sd}\left[\sum_{i=0}^{n-1} {}_{i|}q_x \cdot e^{-y(i+1)}\right]} \qquad 6.27$$

$$= \frac{\sum_{i=0}^{n-1} {}_{i|}q_x \cdot \operatorname{cov}\left(e^{-y(n)}, e^{-y(i+1)}\right)}{\operatorname{sd}\left[e^{-y(n)}\right] \cdot \left\{\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} {}_{i|}q_x \cdot {}_{j|}q_x \cdot \operatorname{cov}\left(e^{-y(i+1)}, e^{-y(j+1)}\right)\right\}^{.5}} \qquad 6.28$$

where $\operatorname{cov}\left(e^{-y(i)}, e^{-y(j)}\right)$ is given by:

$$\operatorname{cov}\left(e^{-y(i)}, e^{-y(j)}\right) = E\left[e^{-y(i)} \cdot e^{-y(j)}\right] - E\left[e^{-y(i)}\right] \cdot E\left[e^{-y(j)}\right] \qquad 6.29$$

The correlation coefficients between $e^{-y(n)}$ and ζ_n, for different values of n, when the force of interest is modelled by an Ornstein-Uhlenbeck process with particular parameters are presented in table 6.2.

Table 6.2 Correlation coefficient between $e^{-y(n)}$ and ζ_n

Ornstein-Uhlenbeck δ=.06, δ_0=.1, α=.1

n	$\rho(e^{-y(n)},\zeta_n)$ σ=.01 x=30	$\rho(e^{-y(n)},\zeta_n)$ σ=.02 x=30	$\rho(e^{-y(n)},\zeta_n)$ σ=.01 x=50
1	1.0000	1.0000	1.0000
2	.9899	.9899	.9912
3	.9824	.9824	.9849
4	..9770	.9770	.9802
5	.9728	.9727	.9765
6	.9693	.9692	.9735
7	.9665	.9663	.9708
8	.9642	.9638	.9684
9	.9622	.9617	.9662
10	.9605	.9599	.9641
20	.9535	.9518	.9455
40	.9368	.9321	.8693
60	.8730	.8494	—

Note that $\rho(e^{-y(1)},\zeta_1)$ is 1. This implies that approximation 6.18 is exact for n=2. The correlation coefficient of table 6.2 suggest that the approximation should be good especially for small values of n.

Combining the two conclusions drawn from the results presented in table 6.1 and table 6.2, we note that the approximation should be acceptable for all values of n. In effect, from table 6.1 we know that it is good for large values of n and from table 6.2 , we have just seen that it is good for small values of n.

Now that the results of theorem 6.2 appear to be justified, we need to solve the integral equations 6.9 and 6.10 in order to find the distribution of ζ_n. We may solve them analytically if possible, or else numerically.

6.4 Numerical Evaluation of the Distribution of ζ_n.

For non-parametric mortality tables such as the one used for our illustrations, it is certainly impossible to express the function g_n analytically. We are then confined to solve those equations numerically.

Note that even for parametric mortality models, it would be very surprising that analytical solutions exist to those equations.

In this section, two methods are proposed for finding numerically the distribution of ζ_n. They are numerical integration and discretization.

6.4.1 Numerical integration.

A first method that one might use to solve equations 6.9 and 6.10 is to integrate numerically. In this respect, one could use a composite integration method such as those presented in section 4.4 of Burden and Faires (1989).

For example, using the composite trapezoidal rule (Burden and Faires (1989, p.177)) with nby equally spaced values for y_{n-1}, the function g_n is approximately given by:

$$g_n(z_n, y_n[i]) \cong \left(\frac{y_{n-1}[nby] - y_{n-1}[1]}{2\cdot(nby - 1)}\right)\cdot$$

$$\left\{ f_{y(n)}\left(y_n[i] \,|\, y(n-1)=y_{n-1}[1]\right)\cdot g_{n-1}\left(z_n - _{n-1}|q_x\cdot e^{-y_n[i]}, y_{n-1}[1]\right) + \right.$$

$$f_{y(n)}\left(y_n[i] \,|\, y(n-1)=y_{n-1}[nby]\right)\cdot g_{n-1}\left(z_n - _{n-1}|q_x\cdot e^{-y_n[i]}, y_{n-1}[nby]\right) +$$

$$\left. 2\cdot\sum_{j=2}^{nby-1} f_{y(n)}\left(y_n[i] \,|\, y(n-1)=y_{n-1}[j]\right)\cdot g_{n-1}\left(z_n - _{n-1}|q_x\cdot e^{-y_n[i]}, y_{n-1}[j]\right)\right\} \qquad 6.30$$

where $y_n[1]$, $y_n[nby]$ are chosen to be:

$$y_n[1] = E[y(n)] - 4\cdot sd[y(n)] \qquad 6.31$$

$$y_n[nby] = E[y(n)] + 4\cdot sd[y(n)] \qquad 6.32$$

and where the particular values of the function g_{n-1} needed in 6.30 are obtained by linear interpolation. That is:

$$g_{n-1}\left(z_{n}^{-}{}_{n-1}\big|_{q_x}\cdot e^{-y_n[i]}, y_{n-1}[j]\right) \cong g_{n-1}\left(z^{1}_{n-1}, y_{n-1}[j]\right) +$$

$$\left\{g_{n-1}\left(z^{2}_{n-1}, y_{n-1}[j]\right) - g_{n-1}\left(z^{1}_{n-1}, y_{n-1}[j]\right)\right\} \cdot$$

$$\frac{\left(\left(z_{n}^{-}{}_{n-1}\big|_{q_x}\cdot e^{-y_n[i]}\right) - z^{1}_{n-1}\right)}{z^{2}_{n-1} - z^{1}_{n-1}} \qquad 6.33$$

with z^{2}_{n-1} being the smallest chosen value for z_{n-1} that is larger or equal to $z_{n}^{-}{}_{n-1}\big|_{q_x}\cdot e^{-y_n[i]}$ for which g_{n-1} is known. And z^{1}_{n-1} being the largest chosen value for z_{n-1} that is smaller or equal to $z_{n}^{-}{}_{n-1}\big|_{q_x}\cdot e^{-y_n[i]}$ for which g_{n-1} is known.

Finally, the distribution of ζ_n, using the composite trapezoidal rule with nby equally spaced values of y_n is approximately given by:

$$F_{\zeta_n}(z_n) \cong \left(\frac{y_n[nby] - y_n[1]}{2\cdot(nby - 1)}\right)\cdot\left(g_n(z_n, y_n[1]) + g_n(z_n, y_n[nby]) + 2\cdot\sum_{i=2}^{nby-1} g_n(z_n, y_n[i])\right) \qquad 6.34$$

This method has been tried without much success. The number of points required to obtain a reasonable accuracy was too large, making the method far too time consuming.

Another method of numerical integration (the composite Simpson's rule) has also been tried, but the improvement over the trapezoidal rule was not very significant. The number of points needed to obtain good results was still too large.

In the light of these results, a discretization has been studied. This is the subject of the next section.

6.4.2 Discretization.

As an alternative to numerical integration, one could consider using a discrete distribution for y(n) which reasonably approximates its continuous normal distribution. The effect of such a discretization on the results of theorem 6.2 would only be to replace the integrations by appropriate summations in 6.9 and 6.10. The specific results, using nby points, would be:

$$F_{\zeta_n}(z_n) \cong \sum_{i=1}^{nby} g_n(z_n, y_n[i]) \qquad 6.35$$

where $g_n(z_n, y_n[i])$ is obtained recursively from:

$$g_n(z_n, y_n[i]) \cong \sum_{j=1}^{nby} P\Big(y(n)=y_n[i] \,\big|\, y(n-1)=y_{n-1}[j]\Big) \cdot$$

$$g_{n-1}\Big(z_n - {}_{n-1|}q_x \cdot e^{-y_n[i]}, y_{n-1}[j]\Big) \qquad 6.36$$

The conditional probability in 6.36 may be chosen to be:

$$P\left(y(n)=y_n[i] \,|\, y(n-1)=y_{n-1}[j]\right) = \Phi(b) - \Phi(a) \qquad 6.37$$

where a and b are given by:

$$a = \frac{.5\left(y_n[i-1] + y_n[i]\right) - E\left[y(n) \,|\, y(n-1)=y_{n-1}[j]\right]}{sd\left[y(n) \,|\, y(n-1)=y_{n-1}[j]\right]} \qquad 6.38$$

$$b = \frac{.5\left(y_n[i+1] + y_n[i]\right) - E\left[y(n) \,|\, y(n-1)=y_{n-1}[j]\right]}{sd\left[y(n) \,|\, y(n-1)=y_{n-1}[j]\right]} \qquad 6.39$$

Note that $y_n[0]$ should preferably be a very small and negative value, such as -100, and $y_n[nby+1]$ be a very large value, such as 100. This would ensure that the probabilities given by 6.37, for i=1,2,...nby, sum to one, i.e.:

$$\sum_{i=1}^{nby} P\left(y(n)=y_n[i] \,|\, y(n-1)=y_{n-1}[j]\right) = 1 \qquad 6.40$$

Finally, the particular values of the function g_{n-1} needed in 6.36 may be obtained by linear interpolation as described in the preceding section (see 6.33).

This completes the description of the discretization method considered to find the distribution of ζ_n. In the next section, we will briefly address some practical questions, such as the number of points to use in the discrete distribution, how to evaluate the normal distribution, Φ, in 6.37, etc...

6.5 Practical Considerations.

6.5.1 Evaluation of the normal distribution.

The normal distribution, Φ, needed in 6.37 has been approximated by the rational function: (Abramowitz and Stegun (1972, p.932))

$$\Phi(y) \cong 1 - \phi(y)\cdot(.31938153t - .356563782t^2 + 1.781477937t^3 - 1.821255978t^4 + 1.330274429t^5) \qquad 6.41$$

where

$$t = \frac{1}{1 + .2316419y} \qquad 6.42$$

Note that the absolute error in using this function to approximate the normal distribution is less than 7.5×10^{-8}.

6.5.2 Values of y_n.

Equations 6.35 and 6.36 were evaluated with nby values of y_n. The number of values of y_n was chosen to be an odd number in order to include the expected value of y(n) and to have an equal number of values each side of the expected value. Since y(n) is normally distributed, a symmetric distribution, it makes sense to choose the values between its expected value plus or minus a given multiple of its standard deviation. This multiple was chosen to be 4 and the nby values where chosen to be equally spaced between the expected value minus 4 standard deviations and the expected value plus 4 standard deviations. That is, between:

$$E[y(n)] - 4 \cdot sd[y(n)] \qquad 6.43$$

and

$$E[y(n)] + 4 \cdot sd[y(n)] \qquad 6.44$$

Other multiples of the standard deviation were tried but the results turned out to be less precise. For 3 standard deviations, we lose too much precision at the extremes of the distribution; with 5 standard deviations the discretization is fine at the extremes but we loose too much precision around the expected value.

Different choices for the values of y(n) were also tried, such as more values around the expected value of y(n) and less at the extremes, but equally spaced values seem to give better results in most situations as measured by the agreement between exact and approximated moments.

6.5.3 Values of z_n.

For each n, the discretization method described in section 6.4.2 has to be carried out for different values of z_n, in order to have different points of the cumulative distribution of ζ_n. Equations 6.35 and 6.36 were evaluated for nbz values of z_n.

The number of values of z_n was chosen to be an odd number in order to include the expected value of ζ_n and to have an equal number of values each side of the expected value. Even if ζ_n has an unknown, non-symmetric distribution, it makes sense to choose the values between its expected value plus or minus some multiple of its standard deviation. This multiple was chosen to be 5 and the nbz values were chosen to be between the greater of the expected value minus 5 standard deviations and zero, and the expected value plus 5 standard deviations. That is, between:

$$\max(0\ ,\ E[\zeta_n] - 5\cdot sd[\zeta_n]) \tag{6.45}$$

and

$$E[\zeta_n] + 5\cdot sd[\zeta_n] \tag{6.46}$$

Note that ζ_n is a non-negative random variable, so it is appropriate to limit the smallest value to zero. Other multiples of the standard deviation were tried but the results turned out to be less precise. For 3 or 4 standard deviations, we lose too much precision at the extremes of the distribution, especially at the right end. At 6 or more standard deviations, the approximation is good at the extremes but we loose too much precision around the expected value. Recall that since the approximation of the distribution of ζ_n depends on the entire distribution of ζ_{n-1}, it is important to balance the precision over the whole range of possible values for each ζ_n and not consider the precision only over a limited range.

Different choices of values were also tried. However equally spaced values to the left of the expected value and equally spaced values to the right of the expected value seem to give better results in most situations again as measured be the agreement between exact and approximated moments. Note that the distance between two consecutive values to the left of the expected value is not necessarily the same as the distance between two consecutive values to the right of the expected value. They are different if the smallest value of z_n is limited to zero.

6.5.4 Interpolation for g_{n-1}.

The particular values of g_{n-1} required in 6.36 are generally not known. Equation 6.33 uses linear interpolation between two consecutive values of ζ_{n-1} for which g_{n-1} is known. Although the function g is not linear, the linear interpolation has the advantage of preserving its strictly increasing characteristic.

Quadratic interpolation was also tried but this method does not guarantee a strictly increasing function. At the left end of the function, the interpolated value of g frequently happens to be negative, when we know that g is a non-negative function. Moreover, at the extreme right of the function, quadratic interpolation frequently gives larger values then the asymptotic maximum of the function g.

Other interpolation methods have been tried but all of them showed different inconsistencies.

Since it appears that preserving the strictly increasing nature of g is essential to obtain acceptable results, the linear interpolation is a very good method even though it will always overstate the function when its second derivative is positive and understate it when its second derivative is negative.

6.6 Illustration.

6.6.1 Distribution of ζ_n.

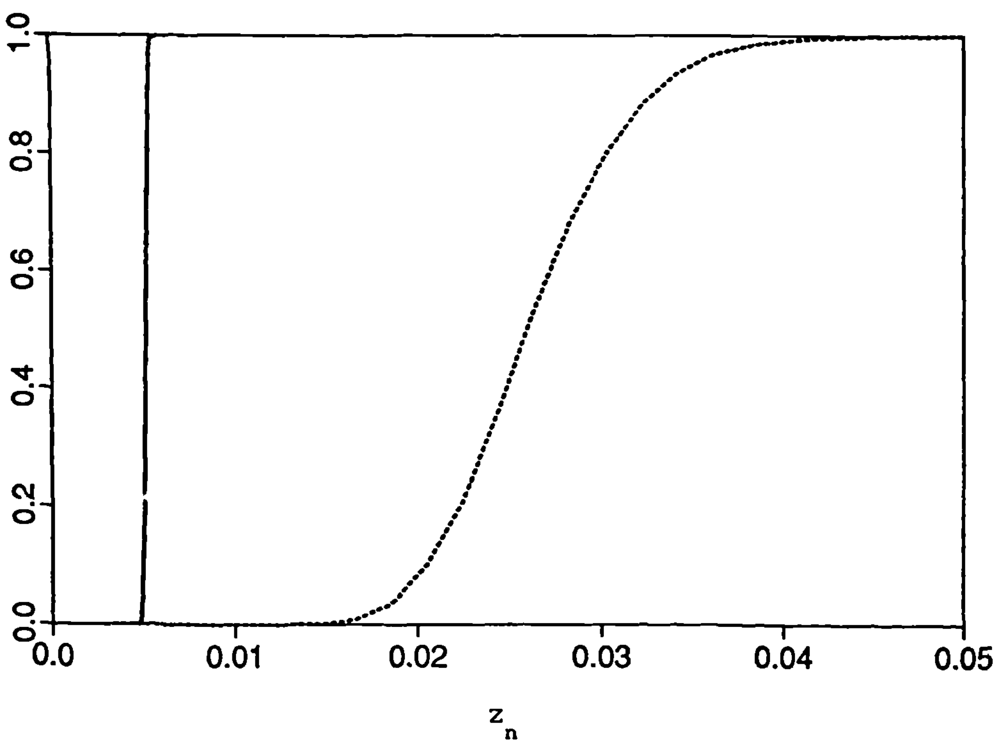
The discretization method described early, with the practical considerations of section 6.5, has been applied to specific situations. The number of values used for both y_n and z_n was 25, i.e. nby and nbz were set at 25.

Figure 6.1 illustrates the cumulative distribution function of ζ_5 and ζ_{25}, the limiting average cost per policy for 5 and 25 years temporary assurance contracts issued at age 30 and with the force of interest modelled by a Ornstein-Uhlenbeck process with parameters δ=.06, δ_0=.1, α=.1 and σ=.01.

Figure 6.1 Cdf of ζ_n

5 and 25 years temporary assurance issued at age 30

Ornstein-Uhlenbeck δ=.06 δ_0=.1 α=.1 σ=.01



—— 5 years

··· 25 years

The range of possible values for ζ_5 is much shorter than the one for ζ_{25}. This is due to the fact that with a limiting portfolio, there is no fluctuation due to mortality, and therefore, all the possible variations in the random variable ζ_n are caused by the force of interest. When there are only five years of fluctuating force of interest involved, it is clear that the results will be less spread than when there are 25 years of fluctuating force of interest. Finally, it should be obvious why ζ_{25} takes larger values than ζ_5.

6.6.2 Right tail of the distribution of ζ_n.

There is no doubt that the distribution of ζ_n contains very useful information in solvency problems. For example, one may be interested in using such information for pricing or valuation of a portfolio of assurance policies. In this regard, the relevant information is contained in the right tail of the distribution of ζ_n.

Since the graphs presented in figure 6.1 cannot provide sufficiently accurate information for this purpose, it is relevant to present some numerical values of the distribution.

Table 6.3 contains some values of the right tail of the distribution of ζ_n, when the force of interest is modelled by an Ornstein-Uhlenbeck process with parameters $\delta=.06$, $\delta_0=.1$, $\alpha=.1$ and $\sigma=.01$, for 5 and 25 years temporary assurance contracts issued to a life assured aged 30. The values of z_5 and z_{25} have been chosen from the set of values of z so that F has approximately the values .95, .975, .99, .995, .999.

Table 6.3 Right tail of the approximate distribution of ζ_n

5 and 25 years temporary assurance issued at age 30

Ornstein-Uhlenbeck $\delta=.06$ $\delta_0=.1$ $\alpha=.1$ $\sigma=.01$

5 years temporary		25 years temporary	
z_5	$F_{\zeta_5}(z_5)$	z_{25}	$F_{\zeta_{25}}(z_{25})$
.005381	.940609	.036136	.966095
.005436	.972183	.038092	.982494
.005547	.992830	.040048	.989498
.005602	.995229	.042004	.994551
.005823	.997927	.049827	.999505

From table 6.3, we know, for example, that a company charging a single premium of .005602 to each life assured of a very large portfolio of 5-year temporary contracts will meet its future liabilities with a probability of about .995.

6.7 Validations of the Approximation.

6.7.1 Comparison of moments.

A validation of the method of calculation described above has been done by comparing the exact first three moments of ζ_n with its estimated first three moments from the distribution obtained in this chapter.

A discretization of the variable ζ_n has been used to estimate the moments of the approximate distribution obtained by the method described in the preceding sections of this chapter. Algebraically, the m^{th} moment of ζ_n about the origin has been approximated by the following equation:

$$\hat{E}[\zeta_n^m] \cong \sum_{i=0}^{nbz} \left(\frac{z_n[i]+z_n[i+1]}{2}\right)^m \cdot \left(F_{\zeta_n}(z_n[i+1]) - F_{\zeta_n}(z_n[i]) \right) \qquad 6.47$$

where

$$z_n[0] = z_n[1] - \left(\frac{z_n[2] - z_n[1]}{2}\right) \qquad 6.48$$

$$z_n[nbz+1] = z_n[nbz] + \left(\frac{z_n[nbz] - z_n[nbz-1]}{2}\right) \qquad 6.49$$

$$F_{\zeta_n}(z_n[0]) = 0 \qquad 6.50$$

$$F_{\zeta_n}(z_n[nbz+1]) = 1 \qquad 6.51$$

The exact moments of ζ_n about the origin may be found in the following manner:

Using the definition of ζ_n given by 6.3, its m^{th} moment about the origin is given by:

$$E[\zeta_n^m] = E\left[\left(\sum_{i=0}^{n-1} {}_{i|}q_x \cdot e^{-y(i+1)}\right)^m\right] \quad 6.52$$

Now, with m equal 1, the first moment is:

$$E[\zeta_n] = \sum_{i=0}^{n-1} E\left[{}_{i|}q_x \cdot e^{-y(i+1)}\right] \quad 6.53$$

which is exactly equation 4.15.

With m equal 2, the second moment is:

$$E[\zeta_n^2] = E\left[\left(\sum_{i=0}^{n-1} {}_{i|}q_x \cdot e^{-y(i+1)}\right) \cdot \left(\sum_{j=0}^{n-1} {}_{j|}q_x \cdot e^{-y(j+1)}\right)\right] \quad 6.54$$

$$= E\left[\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} {}_{i|}q_x \cdot {}_{j|}q_x \cdot e^{-y(i+1)-y(j+1)}\right] \quad 6.55$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} {}_{i|}q_x \cdot {}_{j|}q_x \cdot E\left[e^{-y(i+1)-y(j+1)}\right] \quad 6.56$$

which is exactly 5.31.

With m equal 3, the third moment is:

$$E[\zeta_n^3] = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} {}_{i|}q_x \cdot {}_{j|}q_x \cdot {}_{k|}q_x \cdot E\left[e^{-y(i+1)-y(j+1)-y(k+1)}\right] \qquad 6.57$$

which is the same as 5.41.

The moments of ζ_n are exactly the limiting moments of the average cost per policy studied in chapter 5, as can be shown.

Table 6.4 presents, for temporary assurance contracts issued at age 30 with terms of 1 to 25 years , the exact moments of ζ_n, $E[\zeta_n^m]$, and the difference between the exact and the estimated moments (given by 6.47), i.e. $E[\zeta_n^m]-\hat{E}[\zeta_n^m]$, for m equal 1, 2 and 3. The force of interest is modelled by an Ornstein-Uhlenbeck process with parameters $\delta=.06$, $\delta_0=.1$, $\alpha=.1$ and $\sigma=.01$.

Note that, in order to present more significant digits, the first moment has been multiplied by 10, the second moment multiplied by 100 and the third moment multiplied by 1000.

From this table, we note that the exact and approximate first three moments of ζ_n agree to at least four, five and six decimal places respectively (for $n \le 25$). This is excellent, especially if one considers that many approximations were involved before obtaining the approximate moments of ζ_n.

Firstly, the results of theorem 6.2 have themselves been obtained by assuming that approximation 6.18 was acceptable. Secondly, the random variables y(n) and ζ_n were discretized, introducing another source of error. Thirdly, the values of the function g_{n-1} in 6.36 were obtained by linear interpolation and we know that this procedure

Table 6.4 Comparison of exact and approximate moments of ζ_n

n-year temporary assurance issued at age 30

Ornstein-Uhlenbeck $\delta=.06$ $\delta_0=.1$ $\alpha=.1$ $\sigma=.01$

n	$E[\zeta_n^m]$			$E[\zeta_n^m]-\hat{E}[\zeta_n^m]$		
	m=1 (x10)	m=2 (x100)	m=3 (x1000)	m=1 (x10)	m=2 (x100)	m=3 (x1000)
1	.01197	.00014	.00000	.00000	.00000	.00000
2	.02284	.00052	.00001	.00000	.00000	.00000
3	.03291	.00108	.00004	.00000	.00000	.00000
4	.04246	.00180	.00008	-.00001	.00000	.00000
5	.05160	.00266	.00014	-.00003	.00000	.00000
6	.06048	.00366	.00022	-.00005	-.00001	.00000
7	.06919	.00479	.00033	-.00008	-.00001	.00000
8	.07783	.00607	.00047	-.00011	-.00002	.00000
9	.08648	.00750	.00065	-.00014	-.00003	.00000
10	.09517	.00909	.00087	-.00017	-.00004	-.00001
11	.10395	.01085	.00114	-.00020	-.00005	-.00001
12	.11292	.01282	.00146	-.00023	-.00006	-.00001
13	.12216	.01501	.00186	-.00026	-.00008	-.00002
14	.13173	.01748	.00234	-.00028	-.00009	-.00002
15	.14163	.02023	.00292	-.00031	-.00011	-.00003
16	.15193	.02332	.00362	-.00033	-.00013	-.00004
17	.16263	.02677	.00446	-.00035	-.00015	-.00005
18	.17377	.03062	.00547	-.00037	-.00018	-.00006
19	.18533	.03490	.00668	-.00039	-.00021	-.00008
20	.19731	.03964	.00811	-.00041	-.00024	-.00009
21	.20971	.04489	.00981	-.00043	-.00028	-.00012
22	.22253	.05067	.01181	-.00045	-.00033	-.00015
23	.23580	.05704	.01416	-.00048	-.00038	-.00019
24	.24949	.06403	.01692	-.00051	-.00045	-.00024
25	.26356	.07167	.02013	-.00054	-.00053	-.00030

introduced some bias in the results. Finally, the moments of ζ_n were approximated from a discretization of the distribution obtained.

Let the relative error for the m^{th} moment of ζ_n be:

$$\frac{| E[\zeta_n^m] - \hat{E}[\zeta_n^m] |}{E[\zeta_n^m]} \qquad 6.58$$

Then, for any term, n, the relative error on the expected value of ζ_n is about .2% or less. For its second moment, it is about .7% or less. And for its third moment, it is about 1.5% or less.

The results for other parameters of the Ornstein-Uhlenbeck process and for other ages at issue, not illustrated here, were all excellent. The maximum relative error observed, generally for the third moment, being about 3%. Note that the error is sometimes positive, sometimes negative, and may even alternate over different ranges of values of the term, n. In all cases, however, the error is relatively small.

6.7.2 Simulation.

A second validation of the method has been done by studying the results of some simulations of ζ_n, as defined in 6.3, for n equals 5 and 25.

To simulate ζ_n, we simply need to evaluate 6.3 with a generated multivariate normal vector $\underline{y} = (y_1, y_2, \ldots , y_n)$, where y_i is a realization of y(i).

A multivariate normal with expected value μ and covariance matrix Σ, where element (i, j) of Σ is:

$$\Sigma_{ij} = cov(y(i), y(j)) \qquad 6.59$$

may be obtained by a linear transformation of independent normal variates of mean 0 and variance 1 (see, for example, Roussas (1973, p.405) or Morrison (1990, p.90)). The linear transformation is:

$$\underline{y} = \mu + L \cdot \underline{x} \qquad 6.60$$

where L is such that:

$$L \cdot L^t = \Sigma \qquad 6.61$$

Knowing Σ, one can find L by the Choleski factorization method. (see, for example, Burden and Faires (1989, p.370))

Finally, the independent normal variates $\underline{x}$ may be generated by the Box-Müller method. (see, for example, Ross (1988, p.394))

Table 6.5 presents the first three moments about the origin of 2000 simulations of ζ_5 and ζ_{25}.

Table 6.5 Moments about the origin of 2000 simulated values of ζ_n

5 and 25 years temporary assurance issued at age 30

Ornstein-Uhlenbeck $\delta=.06$ $\delta_0=.1$ $\alpha=.1$ $\sigma=.01$

n	m^{th} moment of ζ_n		
	m=1 (x10)	m=2 (x100)	m=3 (x1000)
5	.05158	.00266	.00014
25	.26239	.07089	.01974

The results of table 6.5 are to be compared with those of table 6.4 for n equals 5 and 25. They are all very close to each other, and the relative errors between the exact moments and the moments of the 2000 simulated values are usually not better than the relative errors between exact and approximate moments given by 6.47.

The empirical distribution functions constructed from the 2000 simulated values of ζ_5 and ζ_{25} have not been presented because they would almost be exactly juxtaposed on those appearing in figure 6.1.

Table 6.6 presents some percentiles of the 2000 simulated values of ζ_5 and ζ_{25}. They correspond to the proportion of values smaller or equal to the values of z_5 and z_{25} that we find in table 6.3. They were chosen accordingly in order to ease the comparison between the mentioned tables, i.e. the approximate (table 6.3) and simulated (table 6.6) results.

<u>Table 6.6 Estimated percentiles of 2000 simulated values of ζ_n</u>

<u>5 and 25 years temporary assurance issued at age 30</u>

<u>Ornstein-Uhlenbeck $\delta=.06$ $\delta_0=.1$ $\alpha=.1$ $\sigma=.01$</u>

5 years temporary		25 years temporary	
z_5	Percentile	z_{25}	Percentile
.005381	.9500	.036136	.9700
.005436	.9795	.038092	.9855
.005547	.9970	.040048	.9925
.005602	.9985	.042004	.9950
.005823	1.0000	.049827	.9995

Although 2000 simulations might be insufficient to estimate accurately the extreme values of a distribution, it appears that the simulated results agree very much with the results presented in table 6.3. In a sense, this may be interpreted as a control or a validation of the method proposed in earlier sections of this chapter.

In general, the remarks made here about the moments and extreme values apply to all the simulations using different parameters and ages at issue that were done.

In conclusion, from the justifications made in section 6.3 and from the validations presented here, it appears that the approximation 6.18 suggested to obtain the results of theorem 6.2 has to be highly acceptable.

CHAPTER 7

LIMITING DISTRIBUTION FOR A n-YEAR ENDOWMENT ASSURANCE

7.1 Limiting Distribution.

In this chapter, we will study the limiting distribution of the average present value of the benefits for a portfolio of endowment assurances, each with sum assured 1, as the number of policies tends to infinity. We will closely follow the ideas presented in chapter 6 with appropriate adaptation for endowment assurances.

The definition of ${}_cZ$ involving a summation over the c contracts of the portfolio, see 5.1, is abandoned in favor of one summing the present value of the benefits in a given year over the n policy-years of the contract. Algebraically, instead of 5.1, one could use the following definition:

$$ {}_cZ = \sum_{i=0}^{n-1} c_i \cdot e^{-y(i+1)} + c_n \cdot e^{-y(n)} \qquad 7.1 $$

where c_i, i=0,1,...,n-1 is again the random variable denoting the number of policies where the death benefit is actually paid at time i+1. We let c_n be the number of lives assured surviving to the end of the term, n. Note that the sum of the c_i's from i equal 0 to n is c, the total number of policies in the portfolio. Thus,

$$ \sum_{i=0}^{n} c_i = c \qquad 7.2 $$

Using 7.1, one could intuitively derive that the average cost per policy, as the number of such policies tends to infinity, would simply be a weighted average of the present value functions from year 1 to year n. The weights being the expected proportion of contracts payable in each year, i.e. ${}_{i|}q_x$ for i=0,1,...,n-2 and $({}_{n-1|}q_x + {}_np_x)$ for the n^{th} policy year. The probabilistic version of this intuition is presented in theorem 7.1.

<u>THEOREM</u> <u>7.1:</u> As c tends to infinity, the average cost per policy for a n-year endowment assurance contract tends in distribution to:

$$\mathfrak{z}_n = \sum_{i=0}^{n-1} {}_{i|}q_x \cdot e^{-y(i+1)} + {}_np_x \cdot e^{-y(n)} \qquad 7.3$$

<u>Proof:</u> This result is true if

$$ {}_c\mathfrak{Z}/_c - \mathfrak{z}_n = \sum_{i=0}^{n-1}\left(c_{i/c} - {}_{i|}q_x\right)\cdot e^{-y(i+1)} + \left(c_{n/c} - {}_np_x\right)\cdot e^{-y(n)} \qquad 7.4$$

tends in probability to 0.

We know, from the proof of theorem 6.2, that the summation in 7.4 tends in probability to 0. Therefore theorem 7.1 will be proved if:

$$\left(c_{n/c} - {}_np_x\right)\cdot e^{-y(n)} \qquad 7.5$$

tends in probability to 0.

Now, since c_n is binomial(c, $_np_x$), $\left(c_n/c - {_np_x}\right)$ tends in probability to 0. And as $e^{-y(n)}$ is log-normally distributed with finite mean and variance, it follows that 7.5 also tends in probability to 0. (Chung (1974, p.92))

This completes the proof.□

Then one could think of $\mathfrak{Z}_n$ as the limiting average cost per policy of a portfolio of n-year endowment assurance contracts.

Theoretically one could obtain the density function of $\mathfrak{Z}_n$ by integrating the joint density function of the y(i)'s over the appropriate domain. That is:

$$f_{\mathfrak{Z}_n}(\mathfrak{z}) = \int_{y_n} \cdots \int_{y_2} \int_{y_1} f_{\underline{Y}}(y_1,\ y_2,\ \ldots, y_n)\ dy_1 dy_2 \cdots dy_n \qquad 7.6$$

where $\underline{Y}$ = (y(1), y(2), ...,y(n)) and is multivariate normal.

Unfortunately, this approach is inefficient from a practical point of view, as it would be very laboriuos to evaluate the integral, even for n as small as 5. In the next section, a recursive equation from which one can approximate the density of $\mathfrak{Z}_n$ is presented.

7.2 Recursive Equation.

Since $\mathfrak{Z}_n$ is a summation over the policy-years, it is possible to express 7.3 as the sum of $\mathfrak{Z}_{n-1}$ and a term for the n^{th} policy year. Starting with 7.3, the recursive equation for $\mathfrak{Z}_n$ is then obtained by:

$$\mathfrak{Z}_n = \sum_{i=0}^{n-1} {}_{i|}q_x \cdot e^{-y(i+1)} + {}_np_x \cdot e^{-y(n)}$$

$$= \sum_{i=0}^{n-2} {}_{i|}q_x \cdot e^{-y(i+1)} + {}_{n-1|}q_x \cdot e^{-y(n)} + {}_np_x \cdot e^{-y(n)} \qquad 7.7$$

$$= \mathfrak{Z}_{n-1} + \left({}_{n-1|}q_x + {}_np_x\right) \cdot e^{-y(n)} - {}_{n-1}p_x \cdot e^{-y(n-1)} \qquad 7.8$$

$$= \mathfrak{Z}_{n-1} + {}_{n-1}p_x \cdot \left(e^{-y(n)} - e^{-y(n-1)}\right) \qquad 7.9$$

since

$${}_{n-1|}q_x + {}_np_x = {}_{n-1}p_x \qquad 7.10$$

Let $\mathfrak{z}_i$ be a possible realization of $\mathfrak{Z}_i$ and y_j be a possible realization of y(j).

Let the somewhat unusual function $h_n(\mathfrak{z}_n, y_n)$ be defined as:

$$h_n(\mathfrak{z}_n, y_n) = P\left(\mathfrak{Z}_n \le \mathfrak{z}_n\right) \cdot f_{y(n)}\left(y_n \middle| \mathfrak{Z}_n \le \mathfrak{z}_n\right) \qquad 7.11$$

or equivalently,

$$h_n(\mathfrak{z}_n, y_n) = f_{y(n)}(y_n) \cdot P\left(\mathfrak{Z}_n \le \mathfrak{z}_n \middle| y(n) = y_n\right) \qquad 7.12$$

It is an immediate consequence of this definition that:

$$F_{\mathfrak{z}_n}(\mathfrak{z}_n) = \int_{-\infty}^{\infty} h_n(\mathfrak{z}_n, y_n) \cdot dy_n \tag{7.13}$$

The following theorem indicates how, subject to an approximation which we shall justify below, we may calculate the function $h_n(\mathfrak{z}_n, y_n)$ recursively.

<u>THEOREM</u> <u>7.2:</u> The function $h_n(\mathfrak{z}_n, y_n)$ may be obtained recursively from:

$$h_n(\mathfrak{z}_n, y_n) \cong \int_{y_{n-1}} f_{y(n)}\left(y_n \middle| y(n-1)=y_{n-1}\right) \cdot h_{n-1}\left(\mathfrak{z}_n - {}_{n-1}p_x \cdot (e^{-y_n} - e^{-y_{n-1}}), y_{n-1}\right) dy_{n-1} \tag{7.14}$$

with the starting value:

$$h_1(\mathfrak{z}_1, y_1) = \begin{cases} \phi\left(\dfrac{y_1 - E[y(1)]}{sd[y(1)]}\right) & \text{if } \mathfrak{z}_1 \geq e^{-y_1} \\ 0 & \text{otherwise} \end{cases} \tag{7.15}$$

Note that $\phi(\cdot)$ is the pdf of a zero mean and unit variance normal random variable. The distribution of y(n), given that y(n-1) equals y_{n-1}, is normal with mean and variance given by 6.12 and 6.13 respectively.

Proof: To prove 7.14, we start by noting that $h_n(\mathfrak{z}_n, y_n)$, from its definition given by 7.11 or 7.12, may also be expressed as:

$$h_n(\mathfrak{z}_n, y_n) = \int_0^{\mathfrak{z}_n} f_{\mathfrak{z}_n, y(n)}\left(z, y_n\right) \cdot dz \qquad 7.16$$

where the joint density of $\mathfrak{z}_n$ and y(n) may be obtained from: (see, for example, Melsa and Sage (1973, pp.95-97))

$$f_{\mathfrak{z}_n, y(n)}\left(z , y_n\right) = \int_{y_{n-1}} f_{y(n-1)}\left(y_{n-1}\right) \cdot f_{y(n)}\left(y_n \mid y(n-1)=y_{n-1}\right) \cdot$$

$$f_{\mathfrak{z}_n}\left(z \mid y(n)=y_n, y(n-1)=y_{n-1}\right) \cdot dy_{n-1} \qquad 7.17$$

Equation 7.9 above implies that, when $y(n) = y_n$ and $y(n-1) = y_{n-1}$, we have:

$$\mathfrak{z}_n = \mathfrak{z}_{n-1} + {}_{n-1}p_x \cdot \left(e^{-y_n} - e^{-y_{n-1}}\right) \qquad 7.18$$

and hence

$$f_{\mathfrak{z}_n}\left(z \mid y(n)=y_n, y(n-1)=y_{n-1}\right) =$$

$$f_{\mathfrak{z}_{n-1}}\left(z - {}_{n-1}p_x \cdot \left(e^{-y_n} - e^{-y_{n-1}}\right) \mid y(n)=y_n, y(n-1)=y_{n-1}\right) \qquad 7.19$$

If we then make the following approximation: (see section 7.3 below)

$$f_{\mathfrak{z}_{n-1}}\left(z - {}_{n-1}p_x \cdot \left(e^{-y_n} - e^{-y_{n-1}}\right) \mid y(n)=y_n, y(n-1)=y_{n-1}\right) \cong$$

$$f_{\mathfrak{z}_{n-1}}\left(z - {}_{n-1}p_x \cdot \left(e^{-y_n} - e^{-y_{n-1}}\right) \mid y(n-1)=y_{n-1}\right) \qquad 7.20$$

the joint density of $\mathfrak{z}_n$ and y(n) given by 7.17 may be approximated by:

$$f_{\mathfrak{z}_n, y(n)}\left(z , y_n\right) \cong \int_{y_{n-1}} f_{y(n-1)}\left(y_{n-1}\right) \cdot f_{y(n)}\left(y_n | y(n-1)=y_{n-1}\right) \cdot$$

$$f_{\mathfrak{z}_{n-1}}\left(z - {}_{n-1}p_x \cdot \left(e^{-y_n} - e^{-y_{n-1}}\right) \mid y(n-1)=y_{n-1}\right) \cdot dy_{n-1} \qquad 7.21$$

Now, substituting 7.21 in 7.16 we have:

$$h_n(\mathfrak{z}_n, y_n) = \int_0^{\mathfrak{z}_n} \int_{y_{n-1}} f_{y(n-1)}\left(y_{n-1}\right) \cdot f_{y(n)}\left(y_n | y(n-1)=y_{n-1}\right) \cdot$$

$$f_{\mathfrak{z}_{n-1}}\left(z - {}_{n-1}p_x \cdot \left(e^{-y_n} - e^{-y_{n-1}}\right) \mid y(n-1)=y_{n-1}\right) \cdot dy_{n-1}\, dz \qquad 7.22$$

which, after interchanging the order of integration, is equal to:

$$h_n(\zeta_n, y_n) = \int_{y_{n-1}} f_{y(n)}\left(y_n | y(n-1)=y_{n-1}\right) \cdot \left[\int_0^{\zeta_n} f_{y(n-1)}\left(y_{n-1}\right) \cdot \right.$$

$$\left. f_{\zeta_{n-1}}\left(z - {}_{n-1}p_x \cdot \left(e^{-y_n} - e^{-y_{n-1}}\right) \;\middle|\; y(n-1)=y_{n-1}\right) \cdot dz \right] dy_{n-1} \qquad 7.23$$

Finally, since the expression within the square brackets in 7.23 is equal to $h_{n-1}\left(\zeta_n - {}_{n-1}p_x \cdot (e^{-y_n} - e^{-y_{n-1}}), y_{n-1}\right)$, the proof of 7.14 is complete.

To prove the result for the starting value, we simply have to note that

$$\zeta_1 = e^{-y(1)} \qquad 7.24$$

and that

$$h_1(\zeta_1, y_1) = P\left(\zeta_1 \le \zeta_1 | y(1)=y_1\right) \cdot f_{y(1)}\left(y_1\right) \qquad 7.25$$

$$= P\left(\zeta_1 \le \zeta_1 | y(1)=y_1\right) \cdot \phi\left(\frac{y_1 - E[y(1)]}{sd[y(1)]}\right) \qquad 7.26$$

but since, if $y(1)=y_1$,

$$\zeta_1 = e^{-y_1} \qquad 7.27$$

we have

$$P\left(\mathfrak{z}_1 \le z_1 \mid y(1)=y_1\right) = \begin{cases} 1 & \text{if } z_1 \ge e^{-y_1} \\ 0 & \text{otherwise} \end{cases} \qquad 7.28$$

and finally 7.28 in 7.26 gives 7.15,

$$h_1(z_1, y_1) = \begin{cases} \phi\left(\dfrac{y_1 - E[y(1)]}{sd[y(1)]}\right) & \text{if } z_1 \ge e^{-y_1} \\ 0 & \text{otherwise} \end{cases}$$

This completes the proof of theorem 7.2. □

Before looking at the problem of numerical evaluation of the results of theorem 7.2, we will, in the next section, study in greater detail and justify the approximation on which the recursive formula is based.

7.3 Justifications of the Approximation.

Looking at the proof of theorem 7.2, we note that the result is not exact due only to approximation 7.20 made in order to obtain a recursive equation involving only known quantities. This approximation may be justified theoretically by considering two particular correlation coefficients, as was the case for approximation 6.18. Note that we will not unnecessarily repeat all the details of the justifications which were discussed in section 6.3.

7.3.1 Correlation between y(n) and y(n-1).

The justification of approximation 7.20 based on the correlation between the two conditioning random variables, here, y(n) and y(n-1), is the same as the justification of 6.18 discussed in section 6.3.2.

Recall that, from table 6.1, y(n) and y(n-1) are very highly correlated, especially for large values of n. Therefore, the approximation 7.20 should be acceptable.

Another correlation coefficient could also justify, independently of the one discussed here, the approximation 7.20. This is the subject of the next section.

7.3.2 Correlation between $e^{-y(n)}$ and $\mathfrak{Z}_n$.

From the justification presented in section 6.3.2, it should be clear that approximation 7.20 would also be acceptable if y(n-1) and $\mathfrak{Z}_{n-1}$ were highly correlated. But since y(n-1) and $e^{-y(n-1)}$ contain exactly the same information, studying the relationship between y(n-1) and $\mathfrak{Z}_{n-1}$ has to be equivalent to studying the relationship between $e^{-y(n-1)}$ and $\mathfrak{Z}_{n-1}$. The latter being much easier to work with.

In other words, if $e^{-y(n)}$ and $\mathfrak{Z}_n$ are highly correlated, the approximation would be reasonable. The correlation coefficient between these two random variables is: (Ross (1988, p.280))

$$\rho\left(e^{-y(n)}, \mathfrak{Z}_n\right) = \frac{\operatorname{cov}\left(e^{-y(n)}, \mathfrak{Z}_n\right)}{\operatorname{sd}\left[e^{-y(n)}\right]\cdot\operatorname{sd}\left[\mathfrak{Z}_n\right]} \qquad 7.29$$

$$= \frac{\operatorname{cov}\left(e^{-y(n)}, \sum_{i=0}^{n-1} {}_{i|}q_x \cdot e^{-y(i+1)} + {}_np_x \cdot e^{-y(n)}\right)}{\operatorname{sd}\left[e^{-y(n)}\right]\cdot\operatorname{sd}\left[\sum_{i=0}^{n-1} {}_{i|}q_x \cdot e^{-y(i+1)} + {}_np_x \cdot e^{-y(n)}\right]} \qquad 7.30$$

where

$$\text{cov}\left(e^{-y(n)},\ \sum_{i=0}^{n-1} {}_{i|}q_x \cdot e^{-y(i+1)} + {}_np_x \cdot e^{-y(n)}\right) =$$

$$\sum_{i=0}^{n-1} {}_{i|}q_x \cdot \text{cov}\left(e^{-y(n)}, e^{-y(i+1)}\right) + {}_np_x \cdot V\left[e^{-y(n)}\right] \qquad 7.31$$

$$\text{sd}\left[\sum_{i=0}^{n-1} {}_{i|}q_x \cdot e^{-y(i+1)} + {}_np_x \cdot e^{-y(n)}\right] =$$

$$\left\{\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} {}_{i|}q_x \cdot {}_{j|}q_x \cdot \text{cov}\left(e^{-y(i+1)}, e^{-y(j+1)}\right) + \right.$$

$$\left.\sum_{i=0}^{n-1} {}_{i|}q_x \cdot {}_np_x \cdot \text{cov}\left(e^{-y(i+1)}, e^{-y(n)}\right) + \left({}_np_x\right)^2 \cdot V\left[e^{-y(n)}\right]\right\}^{.5} \qquad 7.32$$

and where $\text{cov}\left(e^{-y(i)}, e^{-y(j)}\right)$ is given by 6.29.

The correlation coefficients between $e^{-y(n)}$ and 3_n, for different values of n, when the force of interest is modelled by an Ornstein-Uhlenbeck process with particular parameters are presented in table 7.1.

Since $\rho(e^{-y(1)}, 3_1) = 1$, this implies that approximation 7.20 is exact for n=2. The correlation coefficient of table 7.1 below suggest that in general the approximation should be very accurate especially for n small.

Note that, for $n \geq 2$, the other correlations presented as 1.000000 in table 7.1 are not exactly one; they have been rounded to six decimals.

Table 7.1 Correlation coefficient between $e^{-y(n)}$ and ζ_n

Ornstein-Uhlenbeck $\delta=.06$, $\delta_0=.1$, $\alpha=.1$

n	$\rho(e^{-y(n)},\zeta_n)$ $\sigma=.01$ x=30	$\rho(e^{-y(n)},\zeta_n)$ $\sigma=.02$ x=30	$\rho(e^{-y(n)},\zeta_n)$ $\sigma=.01$ x=50
1	1.000000	1.000000	1.000000
2	1.000000	1.000000	.999999
3	1.000000	1.000000	.999997
4	1.000000	1.000000	.999994
5	1.000000	1.000000	.999988
6	.999999	.999999	.999977
7	.999999	.999999	.999961
8	.999998	.999998	.999936
9	.999997	.999998	.999900
10	.999996	.999996	.999848
20	.999915	.999924	.996258
40	.990842	.992647	.887285
60	.884964	.871195	—

For example, the value of $\rho(e^{-y(2)},\zeta_2)$ in the first column (i.e. for σ = .01 and an age at issue, x, of 30) is 0.9999999673.

The two conclusions drawn from the results presented in table 6.1 and table 7.1, indicate that the approximation 7.20 should be acceptable for all values of n. In effect, from table 6.1 we know that it is good for large values of n and from table 7.1 , we have just seen that it is good for small values of n.

Having justified the results of theorem 7.2, we need to solve the integral equations 7.13 and 7.14 in order to find the distribution of $\mathfrak{z}_n$.

7.4 Numerical Evaluation of the Distribution of $\mathfrak{z}_n$.

For non-parametric mortality tables (such as the one used for our illustrations), it is certainly impossible to express the function h_n analytically. We must then solve those equations numerically.

Note that even for parametric mortality models, it would be very surprising that analytical solutions exist for the results of theorem 7.2.

In this section, the two numerical methods proposed in section 6.4 will be briefly discussed, they are numerical integration and discretization.

7.4.1 Numerical integration.

A first method that one could use to solve equations 7.13 and 7.14 is to integrate numerically.

Using the composite trapezoidal rule (Burden and Faires (1989, p.177)) with nby equally spaced values for y_{n-1}, the function h_n is approximately given by:

$$h_n(\mathfrak{z}_n, y_n[i]) \cong \left(\frac{y_{n-1}[nby] - y_{n-1}[1]}{2\cdot(nby - 1)}\right)\cdot\left(\xi + \beta + \gamma\right) \tag{7.33}$$

where ξ, β and γ are given by:

$$\xi = f_{y(n)}\left(y_n[i] \mid y(n-1)=y_{n-1}[1]\right)\cdot$$

$$h_{n-1}\left(\mathfrak{z}_n - {}_{n-1}p_x \cdot \left(e^{-y_n[i]} - e^{-y_{n-1}[1]}\right), y_{n-1}[1]\right) \qquad 7.34$$

$$\beta = f_{y(n)}\left(y_n[i] \mid y(n-1)=y_{n-1}[nby]\right)\cdot$$

$$h_{n-1}\left(\mathfrak{z}_n - {}_{n-1}p_x \cdot \left(e^{-y_n[i]} - e^{-y_{n-1}[nby]}\right), y_{n-1}[nby]\right) \qquad 7.35$$

$$\gamma = 2\cdot\left\{\sum_{j=2}^{nby-1} f_{y(n)}\left(y_n[i] \mid y(n-1)=y_{n-1}[j]\right)\cdot\right.$$

$$\left. h_{n-1}\left(\mathfrak{z}_n - {}_{n-1}p_x \cdot \left(e^{-y_n[i]} - e^{-y_{n-1}[j]}\right), y_{n-1}[j]\right)\right\} \qquad 7.36$$

where $y_n[1]$, $y_n[nby]$ are chosen to be:

$$y_n[1] = E[y(n)] - 4\cdot sd[y(n)] \qquad 7.37$$

$$y_n[nby] = E[y(n)] + 4\cdot sd[y(n)] \qquad 7.38$$

and where the particular values of the function h_{n-1} needed in 7.34, 7.35 and 7.36 are obtained by linear interpolation as before (see equation 6.33).

Finally, the distribution of $\mathfrak{z}_n$, using the composite trapezoidal rule with nby equally spaced values of y_n is approximately given by:

$$F_{\mathfrak{z}_n}(\mathfrak{z}_n) \cong \left(\frac{y_n[nby] - y_n[1]}{2\cdot(nby - 1)}\right)\cdot\left(h_n(\mathfrak{z}_n, y_n[1]) + h_n(\mathfrak{z}_n, y_n[nby]) + 2\cdot\sum_{i=2}^{nby-1} h_n(\mathfrak{z}_n, y_n[i])\right) \qquad 7.39$$

Again, this method and the composite Simpson's rule have been tried without much success. The number of points required to obtain a reasonable accuracy was too large, making the method far too time consuming.

In the light of these results, a discretization has been studied. This is the subject of the next section.

7.4.2 Discretization.

As an alternative to numerical integration, one could consider using a discrete distribution for y(n) which reasonably approximates its continuous normal distribution. The effect of such discretization on the results of theorem 7.2 would only be to replace the integrations by appropriate summations in 7.13 and 7.14. The specific results, using nby points, would be:

$$F_{\mathfrak{z}_n}(\mathfrak{z}_n) \cong \sum_{i=1}^{nby} h_n(\mathfrak{z}_n, y_n[i]) \qquad 7.40$$

where $h_n(\mathfrak{z}_n, y_n[i])$ is obtained recursively from:

$$h_n(\mathfrak{z}_n, y_n[i]) \cong \sum_{j=1}^{nby} P\Big(y(n)=y_n[i]\,\big|\,y(n-1)=y_{n-1}[j]\Big)\cdot$$

$$h_{n-1}\left(\mathfrak{z}_n - {}_{n-1}p_x\cdot\left(e^{-y_n[i]} - e^{-y_{n-1}[j]}\right), y_{n-1}[j]\right) \qquad 7.41$$

In the last expression, the conditional probability of y(n) given y(n-1) is obtained by 6.37.

The particular values of the function h_{n-1} needed in 7.41 may be obtained by linear interpolation.

This completes the description of the discretization method considered to find the distribution of $\mathfrak{z}_n$. In the next section, we will briefly address the same practical questions that were raised in section 6.5.

7.5 Practical Considerations.

7.5.1 Evaluation of the normal distribution.

In 6.37, the normal distribution, Φ, used to evaluate the conditional probability of y(n) given y(n-1) in 7.41, has been approximated by 6.41.

7.5.2 Values of y_n.

Equations 7.40 and 7.41 were evaluated using an odd number, nby, of values of y_n. The comments made in section 6.5.2 also apply here. Consequently, the nby values of y_n used in the discretization were equally spaced between the limits given in 6.43 and 6.44.

7.5.3 Values of $\mathfrak{z}_n$.

For each n, the discretization method described in section 7.4.2 has to be carried out for different values of $\mathfrak{z}_n$, in order to have different points of the cumulative distribution of $\mathfrak{Z}_n$. Equations 7.40 and 7.41 were evaluated for $nb\mathfrak{z}$ values of $\mathfrak{z}_n$.

Basically, the practical considerations concerning $\mathfrak{z}_n$ are the same as those described in section 6.5.3 for z_n.

Thus, an odd number, $nb\mathfrak{z}$, of values of $\mathfrak{z}_n$ were used. They were, $\left(\frac{nb\mathfrak{z} - 1}{2}\right)$ values equally spaced between

$$\max(0\ ,\ E[\mathfrak{Z}_n] - 5\cdot sd[\mathfrak{Z}_n]) \qquad 7.42$$

and the expected value of $\mathfrak{Z}_n$, $E[\mathfrak{Z}_n]$, the expected value itself, and the remaining $\left(\frac{nb\mathfrak{z} - 1}{2}\right)$ values between the expected value of $\mathfrak{Z}_n$ and

$$E[\mathfrak{Z}_n] + 5\cdot sd[\mathfrak{Z}_n] \qquad 7.43$$

7.5.4 Interpolation for h_{n-1}.

The particular values of h_{n-1} required in 7.34, 7.35 and 7.36 are generally not known. Equation 7.39 uses linear interpolation between two consecutive values of $\mathfrak{Z}_{n-1}$ for which h_{n-1} is known.

The function h, like the function g (see 6.7 or 6.8), is a strictly increasing function of $\mathfrak{z}_n$ (g being strictly increasing with z_n).

Again, it appears that preserving the strictly increasing nature of h is essential to obtain acceptable results. The linear interpolation of h, although h is not linear, is a very good method. This is so even though linear interpolation will always overstate the function when its second derivative is positive and understate it when its second derivative is negative.

7.6 Relationship between g_n and h_n.

The two functions g_n and h_n are of very similar nature, but are based on different random variables (ζ_n for the former and $\mathfrak{z}_n$ for the latter). These random variables are the limiting average cost per policy of portfolio of temporary and endowment assurance contracts respectively.

As the difference between endowment assurance and temporary assurance is the pure endowment benefit, it is possible to link the random variables, ζ_n and $\mathfrak{z}_n$.

We can also establish a relationship between g_n and h_n, which is given by the following theorem.

<u>THEOREM</u> <u>7.3:</u> The functions g_n and h_n are linked by the following equation:

$$h_n(\mathfrak{z}_n, y_n) = g_n(\mathfrak{z}_n - {}_np_x \cdot e^{-y_n}, y_n) \tag{7.44}$$

<u>Proof:</u> From the definitions, 6.3 for ζ_n and 7.3 for $\mathfrak{z}_n$, we have that:

$$\mathfrak{z}_n = \zeta_n + {}_np_x \cdot e^{-y(n)} \tag{7.45}$$

Since we condition on $y(n) = y_n$, the definition 7.12, i.e.

$$h_n(\zeta_n, y_n) = f_{y(n)}(y_n) \cdot P\left(\mathfrak{Z}_n \le \zeta_n \mid y(n)=y_n\right),$$

is equivalent to:

$$h_n(\zeta_n, y_n) =$$

$$f_{y(n)}(y_n) \cdot P\left(\mathfrak{Z}_n - {}_n p_x \cdot e^{-y(n)} \le \zeta_n - {}_n p_x \cdot e^{-y_n} \mid y(n)=y_n\right) \quad 7.46$$

Using 7.45, we have that:

$$h_n(\zeta_n, y_n) = f_{y(n)}(y_n) \cdot P\left(\zeta_n \le \zeta_n - {}_n p_x \cdot e^{-y_n} \mid y(n)=y_n\right) \quad 7.47$$

Finally, from 6.8, the right-hand side of 7.47 is simply $g_n(\zeta_n - {}_n p_x \cdot e^{-y_n}, y_n)$ and the proof of 7.44 is complete. □

Using 7.13 and 7.44, one can find the distribution of $\mathfrak{Z}_n$ from:

$$F_{\mathfrak{Z}_n}(\zeta_n) = \int_{-\infty}^{\infty} g_n(\zeta_n - {}_n p_x \cdot e^{-y_n}, y_n) \cdot dy_n \quad 7.48$$

Choosing the discretization method described in section 6.4.2 to find some values of the function g_n, the distribution of $\mathfrak{Z}_n$ could then be approximated by:

$$F_{\mathfrak{Z}_n}(\zeta_n) \cong \sum_{i=1}^{nby} g_n(\zeta_n - {}_n p_x \cdot e^{-y_n[i]}, y_n[i]) \quad 7.49$$

If the function g_n is known for different values of z_n, we can find each required value of g_n in 7.49 by linear interpolation between the two consecutive values of z_n (for which g_n is known) that are each side of $\mathfrak{z}_n - {}_n p_x \cdot e^{-y_n[i]}$.

This method introduces another source of error, by linearly interpolating the non-linear function g_n. This is due to the fact that we have to approximate the function for different values of $\mathfrak{z}_n - {}_n p_x \cdot e^{-y_n[i]}$ (i=1,2,...,nby) which do not correspond to the chosen values $z_n[i]$ (i=1,2,...,nbz) for which the function g_n has been recursively approximated. However, as will be seen in the next sections, the results obtained from this method are still very good.

Note that any attempt to match the values of $z_n[i]$ (i=1,2,...,nbz) with those of $\mathfrak{z}_n - {}_n p_x \cdot e^{-y_n[i]}$ (i=1,2,...,nby) would be of little help. Equation 7.49 requires that g_n be evaluated at values of $z_n[i]$ and $y_n[i]$ which give a constant value of $\mathfrak{z}_n$. This restriction makes it virtually impossible to avoid interpolation completely when evaluating 7.49 for different values of $\mathfrak{z}_n$ unless we repeat the entire recursive approximation of g_n with appropriate $z_n[i]$; i=1,2,...,nbz for each value of $\mathfrak{z}_n$. This last option is very time consuming and does not represent much improvement over the method suggested here or the one suggested in section 7.4.2.

7.7 Illustration.

7.7.1 Distribution of $\mathfrak{Z}_n$.

The discretization method using the relationship between g_n and h_n, described in section 7.6, has been applied to specific situations. The number of values used for both y_n and $\mathfrak{z}_n$ was 25, i.e. nby and nb$\mathfrak{z}$ were set at 25.

Figure 7.1 illustrates the cumulative distribution function of $\mathfrak{z}_5$ and $\mathfrak{z}_{25}$, the limiting average cost per policy for 5 and 25 years endowment assurance contracts issued at age 30, when the force of interest is modelled by a Ornstein-Uhlenbeck process with parameters δ=.06, δ_0=.1, α=.1 and σ=.01.

The range of possible values for $\mathfrak{z}_5$ is shorter than the one for $\mathfrak{z}_{25}$. This is due to the fact that with a limiting portfolio, there is no fluctuation due to mortality, and therefore, all the possible variations in the random variable $\mathfrak{z}_n$ are caused by the force of interest. When there are only five years of fluctuating force of interest involved, it is clear that the results will be less spread than when there are 25 years of fluctuating force of interest. Finally, it should be obvious why $\mathfrak{z}_{25}$ takes smaller values than $\mathfrak{z}_5$.

7.7.2 Right tail of the distribution of $\mathfrak{z}_n$.

The distribution of $\mathfrak{z}_n$ (like ζ_n) provides very useful information in solvency problems. For example, one may be interested in using such information for pricing or valuating a portfolio of assurance policies. In this regard, the relevant information is contained in the right tail of the distribution of $\mathfrak{z}_n$.

Since the graphs presented in figure 7.1 cannot provide sufficiently accurate information for this purpose, it is relevant to present some numerical values of the distribution.

Figure 7.1 Cdf of $\mathfrak{z}_n$

5 and 25 years endowment assurance issued at age 30

Ornstein-Uhlenbeck δ=.06 δ_0=.1 α=.1 σ=.01

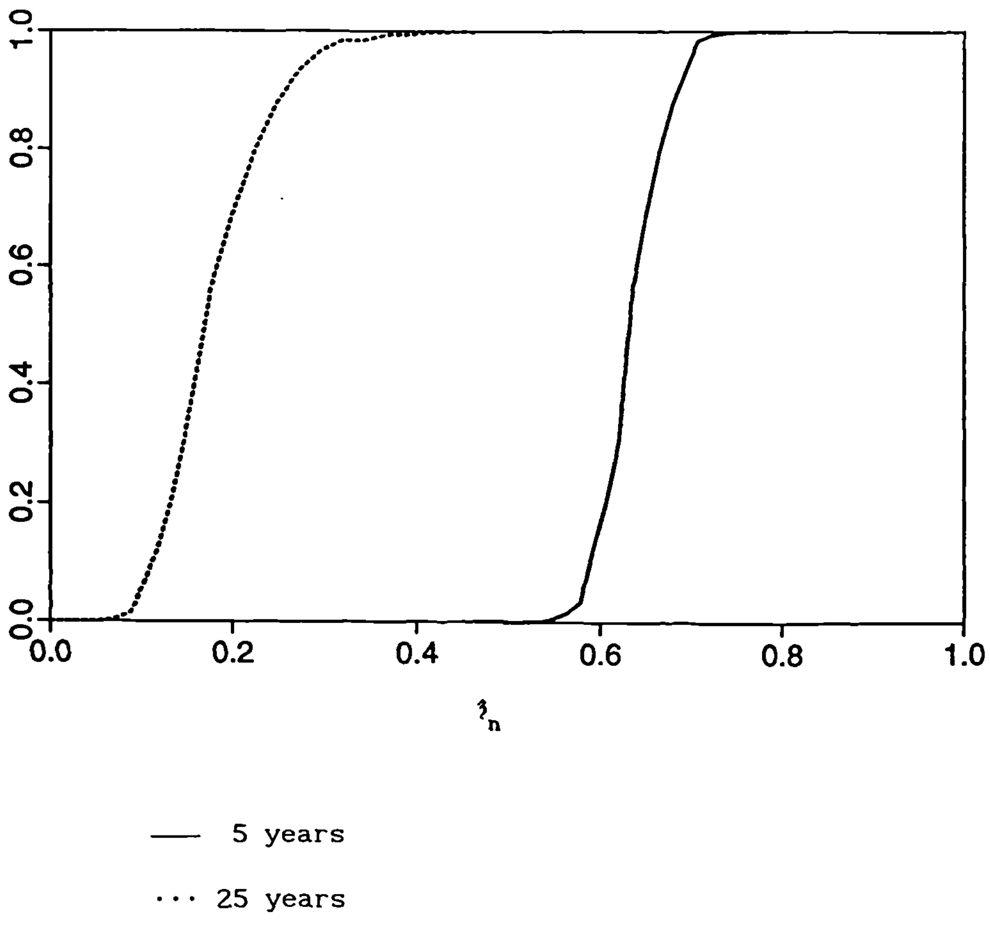


Table 7.2 contains some values of the right tail of the distribution of $\mathfrak{z}_5$ and $\mathfrak{z}_{25}$ for an age at issue of 30 and for a force of interest modelled by an Ornstein-Uhlenbeck process with parameters δ=.06, δ_0=.1, α=.1 and σ=.01.

From table 7.2, we know, for example, that a company charging a single premium of .734243 to each life assured of a very large portfolio of 5 years endowment contracts will meet its future engagements with a probability of about .9968.

Table 7.2 Right tail of the approximate distribution of $\mathfrak{Z}_n$:

5 and 25 years endowment assurance issued at age 30

Ornstein-Uhlenbeck $\delta=.06$ $\delta_0=.1$ $\alpha=.1$ $\sigma=.01$

5 years endowment		25 years endowment	
$\mathfrak{z}_5$	$F_{\mathfrak{Z}_5}(\mathfrak{z}_5)$	$\mathfrak{z}_{25}$	$F_{\mathfrak{Z}_{25}}(\mathfrak{z}_{25})$
.691584	.931803	.270676	.933177
.705804	.983629	.294392	.966609
.720023	.992748	.365540	.993782
.734243	.996828	.412971	.997693
.776901	.999589	.460403	.999229

7.8 Validations of the Approximation.

7.8.1 Comparison of moments.

Again, a validation of the method has been done by comparing the exact first three moments of $\mathfrak{Z}_n$ with the first three moments estimated from the distribution illustrated in section 7.7.

A discretization of the variable $\mathfrak{Z}_n$ has been used to estimate the moments of the approximate distribution obtained by the method described in section 7.6. Algebraically, the m^{th} moment about the origin of $\mathfrak{Z}_n$ has been approximated by the following equation:

$$\hat{E}[\mathfrak{Z}_n^m] \cong \sum_{i=0}^{nbz} \left(\frac{\mathfrak{z}_n[i]+\mathfrak{z}_n[i+1]}{2}\right)^m \cdot \left(F_{\mathfrak{Z}_n}(\mathfrak{z}_n[i+1]) - F_{\mathfrak{Z}_n}(\mathfrak{z}_n[i]) \right) \qquad 7.50$$

where

$$\mathfrak{z}_n[0] = \mathfrak{z}_n[1] - \left(\frac{\mathfrak{z}_n[2] - \mathfrak{z}_n[1]}{2}\right) \qquad 7.51$$

$$\mathfrak{z}_n[nb\mathfrak{z}+1] = \mathfrak{z}_n[nb\mathfrak{z}] + \left(\frac{\mathfrak{z}_n[nb\mathfrak{z}] - \mathfrak{z}_n[nb\mathfrak{z}-1]}{2}\right) \qquad 7.52$$

$$F_{\mathfrak{Z}_n}(\mathfrak{z}_n[0]) = 0 \qquad 7.53$$

$$F_{\mathfrak{Z}_n}(\mathfrak{z}_n[nb\mathfrak{z}+1]) = 1 \qquad 7.54$$

The exact moments about the origin of $\mathfrak{Z}_n$ may be found in the following manner:

The definition of $\mathfrak{Z}_n$ (7.3) implies that the m^{th} moment about the origin is:

$$E[\mathfrak{Z}_n^m] = E\left[\left(\sum_{i=0}^{n-1} {}_{i|}q_x \cdot e^{-y(i+1)} + {}_np_x \cdot e^{-y(n)}\right)^m\right] \qquad 7.55$$

With m equal 1, the first moment is:

$$E[\mathfrak{Z}_n] = \sum_{i=0}^{n-1} E\left[{}_{i|}q_x \cdot e^{-y(i+1)}\right] + E\left[{}_np_x \cdot e^{-y(n)}\right] \qquad 7.56$$

which is exactly equation 4.28.

With m equal 2, the second moment is:

$$E[\mathfrak{Z}_n^2] = E\left[\left(\sum_{i=0}^{n-1} {}_{i|}q_x \cdot e^{-y(i+1)} + {}_np_x \cdot e^{-y(n)}\right) \cdot \left(\sum_{j=0}^{n-1} {}_{j|}q_x \cdot e^{-y(j+1)} + {}_np_x \cdot e^{-y(n)}\right)\right] \qquad 7.57$$

$$= E\left[\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} {}_{i|}q_x \cdot {}_{j|}q_x \cdot e^{-y(i+1)-y(j+1)} + 2\sum_{i=0}^{n-1} {}_{i|}q_x \cdot {}_np_x \cdot e^{-y(i+1)-y(n)} + \left({}_np_x\right)^2 \cdot e^{-2y(n)}\right] \qquad 7.58$$

$$= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} {}_{i|}q_x \cdot {}_{j|}q_x \cdot E\left[e^{-y(i+1)-y(j+1)}\right] + 2\sum_{i=0}^{n-1} {}_{i|}q_x \cdot {}_np_x \cdot E\left[e^{-y(i+1)-y(n)}\right] + \left({}_np_x\right)^2 \cdot E\left[e^{-2y(n)}\right] \qquad 7.59$$

which is exactly 5.48, the limiting second moment about the origin of the average cost per policy. (See theorem 5.2 and its proof)

This last result is easily generalized for m equal 3, so that the third moment is:

$$E[\mathfrak{Z}_n^3] = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} {}_{i|}q_x \cdot {}_{j|}q_x \cdot {}_{k|}q_x \cdot E\left[e^{-y(i+1)-y(j+1)-y(k+1)}\right] + 3\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} {}_{i|}q_x \cdot {}_{j|}q_x \cdot {}_np_x \cdot E\left[e^{-y(i+1)-y(j+1)-y(n)}\right] + 3\sum_{i=0}^{n-1} {}_{i|}q_x \cdot \left({}_np_x\right)^2 \cdot E\left[e^{-y(i+1)-2y(n)}\right] + \left({}_np_x\right)^3 \cdot E\left[e^{-3y(n)}\right] \qquad 7.60$$

which is the same as 5.52, the limiting third moment about the origin of the average cost per policy. (See theorem 5.5 and its proof)

The moments of $\mathfrak{Z}_n$ are exactly the limiting moments of the average cost per policy studied in chapter 5, as can be shown.

Table 7.3 presents, for endowment assurance contracts issued at age 30 with terms of 1 to 25 years , the exact moments of $\mathfrak{Z}_n$, $E[\mathfrak{Z}_n^m]$, and the difference between them and the estimated moments given by 7.50, $E[\mathfrak{Z}_n^m]-\hat{E}[\mathfrak{Z}_n^m]$, for m equal 1, 2 and 3. The force of interest is modelled by an Ornstein-Uhlenbeck process with parameters $\delta=.06$, $\delta_0=.1$, $\alpha=.1$ and $\sigma=.01$.

From this table, we note that the absolute difference between the exact and the approximate first three moments of $\mathfrak{Z}_n$ is always less than .003 (for $n \le 25$). This is excellent, especially if one considers the many approximations involved in estimating the approximate moments of $\mathfrak{Z}_n$. In addition to all the approximations mentioned in section 6.7.1, there is the linear interpolation in g_n in the evaluation of 7.49.

Let the relative error for the m^{th} moment of $\mathfrak{Z}_n$ be:

$$\frac{\left| E[\mathfrak{Z}_n^m] - \hat{E}[\mathfrak{Z}_n^m] \right|}{E[\mathfrak{Z}_n^m]} \qquad 7.61$$

Then, for any term, n, the relative error on the expected value of $\mathfrak{Z}_n$ is about 1.5% or less. For its second moment, it is about 2.9% or less. And for its third moment, it is about 4% or less.

The results for other sets of parameters for the Ornstein-Uhlenbeck process and for other ages at issue, not illustrated here, were all excellent. The maximum relative error observed, generally for the third moment, being about 10%. Note that the sign of the error generally alternates over different ranges of values of the term, n. In all cases, however, the error is relatively small.

Table 7.3 Comparison of exact and approximate moments of $\mathfrak{Z}_n$

n-year endowment assurance issued at age 30

Ornstein-Uhlenbeck $\delta=.06$ $\delta_0=.1$ $\alpha=.1$ $\sigma=.01$

n	$E[\mathfrak{Z}_n^m]$			$E[\mathfrak{Z}_n^m]-\hat{E}[\mathfrak{Z}_n^m]$		
	m=1	m=2	m=3	m=1	m=2	m=3
1	.90660	.82196	.74523	.00000	.00000	.00000
2	.82509	.68093	.56209	.00017	.00027	.00033
3	.75358	.56830	.42887	.00016	.00022	.00023
4	.69054	.47761	.33086	.00020	.00025	.00023
5	.63471	.40402	.25792	.00025	.00027	.00022
6	.58503	.34386	.20305	-.00007	-.00018	-.00025
7	.54065	.29432	.16133	-.00034	-.00056	-.00063
8	.50084	.25326	.12931	-.00036	-.00058	-.00062
9	.46501	.21901	.10448	-.00037	-.00059	-.00060
10	.43263	.19026	.08505	-.00054	-.00069	-.00061
11	.40328	.16599	.06973	-.00058	-.00069	-.00057
12	.37659	.14539	.05754	-.00061	-.00070	-.00054
13	.35226	.12782	.04777	-.00131	-.00108	-.00067
14	.33003	.11276	.03989	-.00192	-.00133	-.00070
15	.30965	.09980	.03348	-.00056	-.00052	-.00034
16	.29095	.08860	.02824	.00044	.00004	-.00009
17	.27374	.07889	.02394	-.00073	-.00049	-.00025
18	.25789	.07043	.02037	.00077	.00018	-.00001
19	.24327	.06305	.01742	-.00055	-.00045	-.00025
20	.22975	.05658	.01494	.00005	-.00027	-.00020
21	.21725	.05091	.01287	-.00060	-.00048	-.00023
22	.20568	.04591	.01113	-.00086	-.00056	-.00023
23	.19496	.04150	.00965	-.00157	-.00073	-.00024
24	.18503	.03760	.00840	-.00198	-.00085	-.00027
25	.17581	.03415	.00734	-.00263	-.00100	-.00029

7.8.2 Simulation.

A second validation of the method has been done by studying the results of some simulations of $\mathfrak{Z}_n$, as defined in 7.3, for n equal 5 and 25.

To simulate $\mathfrak{Z}_n$, we simply need to evaluate 7.3 with a generated multivariate normal vector $\underline{y} = (y_1, y_2, \ldots, y_n)$, where y_i is a realization of y(i).

(A method to generate a multivariate normal vector was presented in section 6.7.2).

Table 7.4 presents the first three moments about the origin of 2000 simulations of $\mathfrak{Z}_5$ and $\mathfrak{Z}_{25}$.

<u>Table 7.4 Moments about the origin of 2000 simulated values of</u> $\mathfrak{Z}_n$

<u>5 and 25 years endowment assurance issued at age 30</u>

<u>Ornstein-Uhlenbeck</u> $\delta=.06$ $\delta_0=.1$ $\alpha=.1$ $\sigma=.01$

n	m^{th} moment of $\mathfrak{Z}_n$		
	m=1	m=2	m=3
5	.63448	.40412	.25804
25	.17343	.03321	.00702

The results of table 7.4 are to be compared with those of table 7.3 when n equals 5 and 25. They are all very close to each other, and the relative errors between the exact moments and the moments of the simulated values are neither better nor worse than the relative errors between exact and approximate moments, discussed in section 7.8.1.

The empirical distribution functions constructed from the 2000 simulated values of $\mathfrak{Z}_5$ and $\mathfrak{Z}_{25}$ have not been presented, because they would almost be exactly juxtaposed on those appearing in figure 7.1.

Table 7.5 presents some percentiles of the 2000 simulated values of $\mathfrak{Z}_5$ and $\mathfrak{Z}_{25}$. They correspond to the proportion of values smaller or equal to the values of $\mathfrak{z}_5$ and $\mathfrak{z}_{25}$ that we find in table 7.2. They were chosen accordingly in order to ease the comparison between the mentioned tables, i.e. the approximate (table 7.2) and simulated (table 7.5) results.

<u>Table 7.5 Estimated percentiles of 2000 simulated values of</u> $\mathfrak{Z}_n$

<u>5 and 25 years endowment assurance issued at age 30</u>

<u>Ornstein-Uhlenbeck</u> $\delta=.06$ $\delta_0=.1$ $\alpha=.1$ $\sigma=.01$

5 years temporary		25 years temporary	
$\mathfrak{z}_5$	Percentile	$\mathfrak{z}_{25}$	Percentile
.691584	.9465	.270676	.9425
.705804	.9740	.294392	.9675
.720023	.9860	.365540	.9945
.734243	.9965	.412971	.9990
.776901	1.0000	.460403	.9995

Although 2000 simulations might be insufficient to accurately estimate the extreme values of a distribution, it appears that the simulated results agree very much with the results presented in table 7.2. In a sense, this may be interpreted as a control or a validation of the method proposed in section 7.6.

In general, the remarks made here about the moments and extreme values apply to all the simulations using different parameters and ages at issue that were done.

Note that the other two methods suggested in this chapter to approximate the distribution of $\mathfrak{Z}_n$, namely the numerical integration of section 7.4.1 and the discretization of section 7.4.2, have also been tried. The numerical integration did not appear to be very good measured by the accuracy of the calculated moments.

The discretization of section 7.4.2, generally gave very similar results to those presented here, but on some occasions, the results were worse.

It is interesting to note that although the discretization method of section 7.4.2 has one less source of error due to the extra interpolation required by the discretization method of section 7.6, the former is sometimes worse than the latter.

Part of the explanation might be found in the fact that when evaluating 7.41, the round-off error involved in performing the subtraction

$$e^{-y_n[i]} - e^{-y_{n-1}[j]} \qquad 7.62$$

might be very significant if $y_n[i]$ and $y_{n-1}[j]$ are close. (see, for example, Burden and Faires (1989, p.16))

This was confirmed by tracing the execution of the computer program for 7.41, which showed that significant round-off error did in fact frequently occur.

We conclude this chapter by observing that on the basis of both the justification made in section 7.3 and the further validations presented above, it appears that the approximation 7.20 suggested to obtain the results of theorem 7.2 is highly acceptable.

CHAPTER 8

DISTRIBUTION OF A FINITE PORTFOLIO OF POLICIES

8.1 Temporary Assurance.

8.1.1 Introduction.

In this section, we will study the distribution of the present value of the benefits for the portfolio of a finite number, c, of temporary assurance policies described in chapter 5.

We will use the definition of $_c\mathcal{Z}$ involving a summation of the present value of the benefits in a given year over the n policy-years of the contract. That is definition 6.1.

$$_c\mathcal{Z} = \sum_{i=0}^{n-1} c_i \cdot e^{-y(i+1)}$$

where c_i, i=0,1,...,n-1 is the random variable denoting the number of policies where the death benefit is actually paid at time i+1. We let c_n be the number of lives assured surviving to the end of the term, n. Note that the sum of the c_i's from i equal 0 to n is c, the total number of policies in the portfolio. Thus, equation 6.2 holds, that is:

$$\sum_{i=0}^{n} c_i = c$$

We will assume that the c_i's are independent of the y(j)'s. Then, one could theoretically obtain the density function of $_c\mathcal{Z}/_c$ by summing and integrating the product of the joint density function of the y(i)'s with the probability function of the c_i's over the appropriate domain. The expression would look like the following:

$$f_{{}_c\mathcal{Z}/_c}(z) =$$

$$\sum_{c_0}\sum_{c_1}\cdots\sum_{c_n}\int_{y_1}\int_{y_2}\cdots\int_{y_n} f_{\underline{Y}}(y_1,y_2,\ldots,y_n)\cdot P(c_0,c_1,\cdots,c_n)\, dy_n\cdots dy_2 dy_1 \qquad 8.1$$

where $\underline{Y}$= (y(1), y(2), ...,y(n)) and is multivariate normal. The first summation over c_0 ranges from 0 to c, the next summation over c_1 ranges from 0 to $c-c_0$ and in general the summation over c_i; i=0,1,...,n, ranges from 0 to $c-(c_0+c_1...+c_{i-1})$.

The vector $(c_0, c_1, \cdots, c_n)$ has a multinomial distribution with parameters c, ${}_{0|}q_x$, ${}_{1|}q_x$, $\cdots$, ${}_{n-1|}q_x$ and ${}_np_x$. Hence the probability that $c_0=c_0$, $c_1=c_1$,...and $c_n=c_n$ is given by: (Ross (1988, p.203))

$$P(c_0,c_1,\cdots,c_n) = \frac{c!}{c_0!\ c_1!\ \cdots\ c_n!}\cdot\left[\prod_{i=0}^{n-1}\left({}_{i|}q_x\right)^{c_i}\right]\cdot\left({}_np_x\right)^{c_n} \qquad 8.2$$

Unfortunately, as before, this approach is inefficient from a practical point of view as it would be very laborious to evaluate 8.2 even for n and c as small as 5. In the next section, we develop a recursive equation from which one can approximate the distribution of ${}_c\mathcal{Z}/_c$.

8.1.2 Recursive equation.

In order to avoid confusion when establishing and using a recursive equation, we will introduce a more specific notation. So far, we have used ${}_{c}\mathcal{Z}/_{c}$ to denote the average cost per policy of a portfolio of c identical policies. Now, let ACPP(T,n,c) denote the average cost per policy of a portfolio of c n-year temporary policies. ACPP(T,n,c) is a random variable and is given by:

$$\text{ACPP}(T,n,c) = \sum_{i=0}^{n-1} \frac{c_i}{c} \cdot e^{-y(i+1)} \qquad 8.3$$

Since ACPP(T,n,c) is a summation over n policy-years, it is easy to break it down into the sum over the first n-1 policy-years plus a term for the n^{th} policy year. The recursive equation for ACPP(T,n,c) is then expressed in the form:

$$\text{ACPP}(T,n,c) = \sum_{i=0}^{n-1} \frac{c_i}{c} \cdot e^{-y(i+1)} = \sum_{i=0}^{n-2} \frac{c_i}{c} \cdot e^{-y(i+1)} + \frac{c_{n-1}}{c} \cdot e^{-y(n)}$$

$$= \text{ACPP}(T,n-1,c) + \frac{c_{n-1}}{c} \cdot e^{-y(n)} \qquad 8.4$$

Note that c_{n-1} in 8.4 is the random variable of the number of policies with a death benefit payable at time n (and not the number of survivors to time n-1).

We now introduce a new random variable to simplify the notation. For i=1,2,...,n, let $\mathfrak{C}_i$ be the random variable denoting the total number of policies paid between time 0 and time i.

Thus

$$\mathfrak{C}_i = \sum_{j=0}^{i-1} c_j \qquad 8.5$$

Let $\mathfrak{c}_i$ (a gothic c) be a realization of $\mathfrak{C}_i$. Then:

$$\mathfrak{c}_i = \sum_{j=0}^{i-1} c_j \qquad 8.6$$

Finally, let z_i be a realization of ACPP(T,n,c), let y_j be a realization of y(j), and define the function $\mathcal{q}_n(z_n, y_n, \mathfrak{c}_n)$ as:

$$\mathcal{q}_n(z_n, y_n, \mathfrak{c}_n) =$$

$$f_{y(n)}(y_n)\cdot P\left(\mathfrak{C}_n=\mathfrak{c}_n \middle| y(n)=y_n\right)\cdot P\left(\text{ACPP}(T,n,c)\le z_n \middle| \mathfrak{C}_n=\mathfrak{c}_n, y(n)=y_n\right) \qquad 8.7$$

This function $\mathcal{q}_n(z_n, y_n, \mathfrak{c}_n)$ is a mixture of distribution and density functions for the three events $\text{ACPP}(T,n,c)\le z_n$, $\mathfrak{C}_n=\mathfrak{c}_n$ and $y(n)=y_n$ simultaneously.

Since we assume that the c_i's are independent of the y(j)'s, we have that:

$$P\left(\mathfrak{C}_n=\mathfrak{c}_n \middle| y(n)=y_n\right) = P\left(\mathfrak{C}_n=\mathfrak{c}_n\right) \qquad 8.8$$

Hence 8.7 becomes:

$$\mathcal{q}_n(z_n, y_n, \mathfrak{c}_n)=f_{y(n)}(y_n)\cdot P\left(\mathfrak{C}_n=\mathfrak{c}_n\right)\cdot P\left(\text{ACPP}(T,n,c)\le z_n \middle| \mathfrak{C}_n=\mathfrak{c}_n, y(n)=y_n\right) \qquad 8.9$$

From this last equation it follows that the distribution of ACPP(T,n,c) is given by:

$$F_{ACPP(T,n,c)}(z_n) = \sum_{c_n=0}^{C} \int_{-\infty}^{\infty} \varphi_n(z_n, y_n, c_n) \cdot dy_n \qquad 8.10$$

The following theorem now gives a recursive method to approximate the function $\varphi_n(z_n, y_n, c_n)$.

<u>THEOREM</u> <u>8.1:</u> The function $\varphi_n(z_n, y_n, c_n)$ may be obtained recursively from:

$$\varphi_n(z_n, y_n, c_n) \cong \sum_{j=0}^{c_n} \int_{-\infty}^{\infty} f_{y(n)}\left(y_n \middle| y(n-1)=y_{n-1}\right) \cdot$$

$$P\left(c_{n-1}=j \middle| \mathfrak{C}_{n-1}=c_n-j\right) \cdot \varphi_{n-1}\left(z_n - \frac{j}{C}\cdot e^{-y_n}, y_{n-1}, c_n-j\right)\, dy_{n-1} \qquad 8.11$$

with the starting value:

$$\varphi_1(z_1, y_1, c_1) = \begin{cases} \left(q_x\right)^{c_1} \cdot \phi\left(\dfrac{y_1 - E[y(1)]}{sd[y(1)]}\right) & \text{if } z_1 \geq c_1 \cdot e^{-y_1} \\ 0 & \text{otherwise} \end{cases} \qquad 8.12$$

Note that $\phi(\cdot)$ is the pdf of a zero mean and unit variance normal random variable. The random variable y(n) given that y(n-1) equal y_{n-1}

is normally distributed with mean given by 6.12 and variance given by 6.13, so the conditional density of y(n) given y(n−1) in 8.11 is known. The conditional probability that c_{n-1} equals j in 8.11 is given by: (Ross (1988, pp.203,219))

$$P\left(c_{n-1}=j \,\middle|\, \mathfrak{C}_{n-1}=c_n-j\right) = \frac{P\left(c_{n-1}=j,\ \mathfrak{C}_{n-1}=c_n-j\right)}{P\left(\mathfrak{C}_{n-1}=c_n-j\right)}$$

$$= \frac{\dfrac{c!}{j!\ (c_n-j)!\ (c-c_n)!}\cdot\left({}_{n-1}q_x\right)^{c_n-j}\cdot\left({}_{n-1|}q_x\right)^{j}\cdot\left({}_{n}p_x\right)^{c-c_n}}{\dfrac{c!}{(c_n-j)!\ (c-c_n+j)!}\cdot\left({}_{n-1}q_x\right)^{c_n-j}\cdot\left({}_{n}p_x\right)^{c-c_n+j}}$$

$$= \frac{(c-c_n+j)!}{j!\ (c-c_n)!}\cdot\left(q_{x+n-1}\right)^{j}\cdot\left(p_{x+n-1}\right)^{c-c_n} \quad ;\ j=0,1,\ldots,c_n \qquad 8.13$$

Intuitively, equation 8.13 is not surprising, since, given that $\mathfrak{C}_{n-1} = c_n-j$, c_{n-1} should be binomially distributed with parameters $c-(c_n-j)$ and q_{x+n-1}.

<u>Proof:</u>

To prove 8.11, we start by noting that, from the definition 8.7, $\mathcal{q}_n(z_n,y_n,c_n)$ may also be expressed as:

$$\mathcal{q}_n(z_n,y_n,c_n) = \int_0^{z_n} f_{ACPP(T,n,c),\,y(n),\,\mathfrak{C}_n}\left(z,\ y_n,\ c_n\right)\cdot dz \qquad 8.14$$

where $f_{ACPP(T,n,c),y(n),\mathfrak{C}_n}$, the joint density function of ACPP(T, n, c), y(n) and $\mathfrak{C}_n$, may be obtained from Melsa and Sage (1973, pp. 95-97) as:

$$f_{ACPP(T,n,c),y(n),\mathfrak{C}_n}\left(z,\ y_n,\ c_n\right) =$$

$$\int_{y_{n-1}} f_{y(n-1)}\left(y_{n-1}\right) \cdot f_{y(n)}\left(y_n \middle| y(n-1)=y_{n-1}\right) \cdot P\left(\mathfrak{C}_n=c_n \middle| y(n)=y_n, y(n-1)=y_{n-1}\right) \cdot$$

$$f_{ACPP(T,n,c)}\left(z \middle| y(n)=y_n, y(n-1)=y_{n-1}, \mathfrak{C}_n=c_n\right) \cdot\ dy_{n-1} \qquad 8.15$$

Now, since the c_i's are independent of the y(j)'s, we have that:

$$P\left(\mathfrak{C}_n=c_n \middle| y(n)=y_n, y(n-1)=y_{n-1}\right) = P\left(\mathfrak{C}_n=c_n\right) \qquad 8.16$$

Thus 8.15 becomes:

$$f_{ACPP(T,n,c),y(n),\mathfrak{C}_n}\left(z, y_n, c_n\right) = \int_{y_{n-1}} f_{y(n-1)}\left(y_{n-1}\right) \cdot f_{y(n)}\left(y_n \middle| y(n-1)=y_{n-1}\right) \cdot$$

$$P\left(\mathfrak{C}_n=c_n\right) \cdot f_{ACPP(T,n,c)}\left(z \middle| y(n)=y_n, y(n-1)=y_{n-1}, \mathfrak{C}_n=c_n\right) \cdot\ dy_{n-1} \qquad 8.17$$

Again from Melsa and Sage (1973, pp. 95-97), we may express the product of the last two terms in 8.17 as:

$$P\left(\mathfrak{C}_n=c_n\right)\cdot f_{ACPP(T,n,c)}\left(z\,|\,y(n)=y_n, y(n-1)=y_{n-1}, \mathfrak{C}_n=c_n\right) = \sum_{j=0}^{c_n} P\left(c_{n-1}=j\right)\cdot$$

$$P\left(\mathfrak{C}_n=c_n\,|\,c_{n-1}=j\right)\cdot f_{ACPP(T,n,c)}\left(z\,|\,y(n)=y_n, y(n-1)=y_{n-1}, \mathfrak{C}_n=c_n, c_{n-1}=j\right) \qquad 8.18$$

From 8.5, we know that, given $c_{n-1} = j$, the event $\mathfrak{C}_n = c_n$ is the same as the event $\mathfrak{C}_{n-1} = c_n - j$. Therefore, we have that:

$$P\left(\mathfrak{C}_n=c_n\,|\,c_{n-1}=j\right) = P\left(\mathfrak{C}_{n-1}=c_n-j\,|\,c_{n-1}=j\right) \qquad 8.19$$

Also, from 8.4, the event $ACPP(T,n,c) = z$ given $y(n) = y_n$, $y(n-1) = y_{n-1}$, $\mathfrak{C}_n = c_n$ and $c_{n-1} = j$ implies that $ACPP(T,n-1,c) = z - \frac{j}{c}\cdot e^{-y_n}$ under the same conditions. In appendix C we show that $ACPP(T,n-1,c)$ given $\mathfrak{C}_{n-1}$ is independent of c_{n-1}. Hence,

$$f_{ACPP(T,n,c)}\left(z\,|\,y(n)=y_n, y(n-1)=y_{n-1}, \mathfrak{C}_n=c_n, c_{n-1}=j\right) =$$

$$f_{ACPP(T,n-1,c)}\left(z - \frac{j}{c}\cdot e^{-y_n}\,|\,y(n)=y_n, y(n-1)=y_{n-1}, \mathfrak{C}_{n-1}=c_n-j\right) \qquad 8.20$$

If, now, we make the approximation

$$f_{ACPP(T,n-1,c)}\left(z - \frac{j}{c}\cdot e^{-y_n} \,\middle|\, y(n)=y_n, y(n-1)=y_{n-1}, \mathfrak{C}_{n-1}=c_n-j\right) \cong$$

$$f_{ACPP(T,n-1,c)}\left(z - \frac{j}{c}\cdot e^{-y_n} \,\middle|\, y(n-1)=y_{n-1}, \mathfrak{C}_{n-1}=c_n-j\right) \qquad 8.21$$

then, using 8.19, 8.20 and 8.21, we may express 8.18 in the form:

$$P\left(\mathfrak{C}_n=c_n\right)\cdot f_{ACPP(T,n,c)}\left(z \,\middle|\, y(n)=y_n, y(n-1)=y_{n-1}, \mathfrak{C}_n=c_n\right) \cong$$

$$\sum_{j=0}^{c_n} P\left(c_{n-1}=j\right)\cdot P\left(\mathfrak{C}_{n-1}=c_n-j \,\middle|\, c_{n-1}=j\right)\cdot$$

$$f_{ACPP(T,n-1,c)}\left(z - \frac{j}{c}\cdot e^{-y_n} \,\middle|\, y(n-1)=y_{n-1}, \mathfrak{C}_{n-1}=c_n-j\right) \qquad 8.22$$

Now using the fact that:

$$P\left(c_{n-1}=j\right)\cdot P\left(\mathfrak{C}_{n-1}=c_n-j \,\middle|\, c_{n-1}=j\right) = P\left(\mathfrak{C}_{n-1}=c_n-j\right)\cdot P\left(c_{n-1}=j \,\middle|\, \mathfrak{C}_{n-1}=c_n-j\right) \qquad 8.23$$

and by substituting the right-hand side of this last equation in equation 8.22, then combining the resulting equation with 8.17 above and rearranging the terms, we obtain:

$$f_{ACPP(T,n,c),y(n),\mathfrak{C}_n}\left(z,y_n,c_n\right) \cong \sum_{j=0}^{c_n} P\left(c_{n-1}=j \,|\, \mathfrak{C}_{n-1}=c_n-j\right)\cdot$$

$$\int_{y_{n-1}} f_{y(n)}\left(y_n|y(n-1)=y_{n-1}\right)\cdot\left[f_{y(n-1)}\left(y_{n-1}\right)\cdot P\left(\mathfrak{C}_{n-1}=c_n-j\right)\cdot\right.$$

$$\left.f_{ACPP(T,n-1,c)}\left(z-\frac{j}{c}\cdot e^{-y_n}\,|\,y(n-1)=y_{n-1},\mathfrak{C}_{n-1}=c_n-j\right)\right]\cdot dy_{n-1} \qquad 8.24$$

Finally, since the expression in square brackets in 8.24 is

$$f_{ACPP(T,n-1,c),y(n-1),\mathfrak{C}_{n-1}}\left(z-\frac{j}{c}\cdot e^{-y_n},y_{n-1},c_n-j\right) \qquad 8.25$$

by integrating both sides of 8.24 with respect to z from 0 to z_n we obtain 8.11.

To obtain the starting value, we simply have to note that

$$ACPP(T,1,c) = \frac{\mathfrak{C}_1}{c}\cdot e^{-y(1)} \qquad 8.26$$

and that, assuming independence between $\mathfrak{C}_1$ and y(1),

$$\mathcal{q}_1(z_1,y_1,c_1) =$$

$$P\left(ACPP(T,1,c)\le z_1 \,\middle|\, y(1)=y_1,\ \mathbb{C}_1=c_1\right)\cdot f_{y(1)}\left(y_1\right)\cdot P\left(\mathbb{C}_1=c_1\right) \qquad 8.27$$

$$= P\left(ACPP(T,1,c)\le z_1 \,\middle|\, y(1)=y_1,\ \mathbb{C}_1=c_1\right)\cdot \phi\left(\frac{y_1 - E[y(1)]}{sd[y(1)]}\right)\cdot P\left(\mathbb{C}_1=c_1\right) \qquad 8.28$$

Since

$$P\left(\mathbb{C}_1=c_1\right) = \left(q_x\right)^{c_1} \qquad 8.29$$

and, if $y(1)=y_1$ and $\mathbb{C}_1=c_1$,

$$ACPP(T,1,c)= \frac{c_1}{c}\cdot e^{-y_1} \qquad 8.30$$

we have that

$$P\left(ACPP(T,1,c)\le z_1 \,\middle|\, y(1)=y_1,\ \mathbb{C}_1=c_1\right) = \begin{cases} 1 & \text{if } z_1 \ge \frac{c_1}{c}\cdot e^{-y_1} \\ 0 & \text{otherwise} \end{cases} \qquad 8.31$$

and finally 8.31 in 8.28 gives 8.12,

$$\mathcal{q}_1(z_1,y_1,c_1) = \begin{cases} \left(q_x\right)^{c_1}\cdot \phi\left(\frac{y_1 - E[y(1)]}{sd[y(1)]}\right) & \text{if } z_1 \ge c_1\cdot e^{-y_1} \\ 0 & \text{otherwise} \end{cases}$$

Subject to justifying the approximation 8.21 above, this completes the proof of theorem 8.1. □

8.1.3 Justifications of the approximation.

The approximation 8.21 is a key step in obtaining a recursive equation involving only known quantities. This approximation may be justified theoretically by considering two particular correlation coefficients.

First, the high correlation between y(n) and y(n-1) does justify approximation 8.21 in the same way that it justified approximations 6.18 and 7.20. The reader is referred to table 6.1 for numerical illustrations of the correlation between y(n) and y(n-1).

Secondly, from Mardia, Kent and Bibby (1979, section 6.5), the approximation would be good if the difference between the two multiple correlation coefficients ρ_1 and ρ_2 is small, where:

$$\rho_1 = \rho_{ACPP(T,n-1,c)\cdot e^{-y(n)},\, e^{-y(n-1)},\, \mathfrak{C}_{n-1}} \qquad 8.32$$

and

$$\rho_2 = \rho_{ACPP(T,n-1,c)\cdot e^{-y(n-1)},\, \mathfrak{C}_{n-1}} \qquad 8.33$$

In words, ρ_1 is the multiple correlation coefficient between ACPP(T,n-1,c) and the three random variables $e^{-y(n)}$, $e^{-y(n-1)}$ and $\mathfrak{C}_{n-1}$. And ρ_2 is the multiple correlation coefficient between ACPP(T,n-1,c) and the two random variables $e^{-y(n-1)}$ and $\mathfrak{C}_{n-1}$. If these two multiple correlation coefficients are close, it implies that the random variable $e^{-y(n)}$ is not useful in explaining ACPP(T,n-1,c) when $e^{-y(n-1)}$ and $\mathfrak{C}_{n-1}$ are already considered. (see, for example, Mardia, Kent and Bibby, (1979, example 6.5.2 p.170.))

A multiple correlation coefficient between Y and a vector X may be obtained from the general expression: (using the notation of Mardia, Kent and Bibby (1979, p. 168))

$$\rho_{Y\cdot X} = \left\{ \frac{S_{12}\cdot S_{22}^{-1}\cdot S_{21}}{s_{11}} \right\}^{.5} \qquad 8.34$$

where

$$s_{11} = V[Y] \qquad 8.35$$

So, for ρ_1, we will have:

$$s_{11} = V[ACPP(T,n-1,c)] \qquad 8.36$$

$$S_{12}' = S_{21} = \begin{pmatrix} \mathrm{cov}\left(ACPP(T,n-1,c)\ ,\ e^{-y(n)}\right) \\ \mathrm{cov}\left(ACPP(T,n-1,c)\ ,\ e^{-y(n-1)}\right) \\ \mathrm{cov}\left(ACPP(T,n-1,c)\ ,\ \mathfrak{C}_{n-1}\right) \end{pmatrix} \qquad 8.37$$

and

$$S_{22} = \begin{pmatrix} V\left[e^{-y(n)}\right] & \mathrm{cov}\left(e^{-y(n)}, e^{-y(n-1)}\right) & \mathrm{cov}\left(\mathfrak{C}_{n-1}, e^{-y(n)}\right) \\ \mathrm{cov}\left(e^{-y(n)}, e^{-y(n-1)}\right) & V\left[e^{-y(n-1)}\right] & \mathrm{cov}\left(\mathfrak{C}_{n-1}, e^{-y(n-1)}\right) \\ \mathrm{cov}\left(\mathfrak{C}_{n-1}, e^{-y(n)}\right) & \mathrm{cov}\left(\mathfrak{C}_{n-1}, e^{-y(n-1)}\right) & V\left[\mathfrak{C}_{n-1}\right] \end{pmatrix} \qquad 8.38$$

Now the elements of S_{22} (8.38) are known. Note that $\mathrm{cov}\left(e^{-y(i)}, e^{-y(j)}\right)$ is given by 6.29, that $\mathrm{cov}\left(e^{-y(i)}, \mathfrak{C}_{n-1}\right)$ is zero for all i because y(i)'s and c_j's are assumed to be independent for all i and j, and the variance of $\mathfrak{C}_{n-1}$ is given by:

$$V\left[\mathfrak{C}_{n-1}\right] = V\left[\sum_{i=0}^{n-2} c_i\right] = \sum_{i=0}^{n-2}\sum_{j=0}^{n-2} \text{cov}\left(c_i, c_j\right) \qquad 8.39$$

where (see, for example, Larson (1982, p.307))

$$\text{cov}\left(c_i, c_j\right) = \begin{cases} -c\cdot {}_{i|}q_x \cdot {}_{j|}q_x & i\neq j \\ c\cdot {}_{i|}q_x\cdot(1-{}_{i|}q_x) & i=j \end{cases} \qquad 8.40$$

As for the elements of the vectors S_{12} and S_{21} (8.37), they are:

$$\text{cov}\left(\text{ACPP}(T, n-1, c), e^{-y(n)}\right) =$$

$$E\left[\sum_{i=0}^{n-2}\frac{c_i}{c}\cdot e^{-y(i+1)}\cdot e^{-y(n)}\right] - E\left[\sum_{i=0}^{n-2}\frac{c_i}{c}\cdot e^{-y(i+1)}\right]\cdot E\left[e^{-y(n)}\right] \qquad 8.41$$

by independence of the c_i's and y(j)'s for all i and j, we have:

$$= \sum_{i=0}^{n-2} E\left[\frac{c_i}{c}\right]\cdot\left\{E\left[e^{-y(i+1)-y(n)}\right] - E\left[e^{-y(i+1)}\right]\cdot E\left[e^{-y(n)}\right]\right\} \qquad 8.42$$

$$= \sum_{i=0}^{n-2} {}_{i|}q_x\cdot \text{cov}\left(e^{-y(i+1)}, e^{-y(n)}\right) \qquad 8.43$$

which is exactly the expression for $\text{cov}\left(e^{-y(n)}, \zeta_n\right)$ (See the numerators of 6.26 and 6.28).

Similarly, we have:

$$\mathrm{cov}\left(\mathrm{ACPP}(T,n-1,c), e^{-y(n-1)}\right) = \sum_{i=0}^{n-2} {}_{i|}q_x \cdot \mathrm{cov}\left(e^{-y(i+1)}, e^{-y(n-1)}\right) \qquad 8.44$$

And finally, the third covariance of S_{12} and S_{21} is:

$$\mathrm{cov}\left(\mathrm{ACPP}(T,n-1,c), \mathfrak{C}_{n-1}\right) =$$

$$E\left[\sum_{i=0}^{n-2}\frac{c_i}{c}\cdot e^{-y(i+1)}\cdot\sum_{j=0}^{n-2}c_j\right] - E\left[\sum_{i=0}^{n-2}\frac{c_i}{c}\cdot e^{-y(i+1)}\right]\cdot E\left[\sum_{j=0}^{n-2}c_j\right] \qquad 8.45$$

$$= \sum_{i=0}^{n-2}\sum_{j=0}^{n-2}\left\{E\left[\frac{c_i}{c}\cdot e^{-y(i+1)}\cdot c_j\right] - E\left[\frac{c_i}{c}\cdot e^{-y(i+1)}\right]\cdot E\left[c_j\right]\right\} \qquad 8.46$$

by independence of the c_i's and $y(j)$'s for all i and j, we have:

$$= \sum_{i=0}^{n-2}\sum_{j=0}^{n-2} E\left[\frac{e^{-y(i+1)}}{c}\right]\cdot\left\{E\left[c_i\cdot c_j\right] - E\left[c_i\right]\cdot E\left[c_j\right]\right\} \qquad 8.47$$

$$= \sum_{i=0}^{n-2}\sum_{j=0}^{n-2} E\left[\frac{e^{-y(i+1)}}{c}\right]\cdot \mathrm{cov}\left(c_i, c_j\right) \qquad 8.48$$

where $\mathrm{cov}\left(c_i, c_j\right)$ is given by 8.40.

To obtain ρ_2 we simply need to use 8.34 with 8.36 and:

$$S'_{12} = S_{21} = \begin{pmatrix} cov\left(ACPP(T,n-1,c)\ ,\ e^{-y(n-1)}\right) \\ cov\left(ACPP(T,n-1,c)\ ,\ \mathfrak{C}_{n-1}\right) \end{pmatrix} \qquad 8.49$$

and

$$S_{22} = \begin{pmatrix} V\left[e^{-y(n-1)}\right] & cov\left(\mathfrak{C}_{n-1}, e^{-y(n-1)}\right) \\ cov\left(\mathfrak{C}_{n-1}, e^{-y(n-1)}\right) & V\left[\mathfrak{C}_{n-1}\right] \end{pmatrix} \qquad 8.50$$

Note that the elements of S_{21} and S_{22} defined as in 8.49 and 8.50 (for ρ_2) are the same as those of S_{21} and S_{22} defined by 8.37 and 8.38 (for ρ_1) except that in 8.49 and 8.50 all the elements involving the random variable $e^{-y(n)}$ have been deleted.

In table 8.1 we show some multiple correlation coefficients, ρ_1 and ρ_2, for portfolios of n-year temporary contracts issued at age 30 of size 10 and 100000, for different values of n, when the force of interest is modelled by a Ornstein-Uhlenbeck process with parameters $\delta=.06$, $\delta_0=.1$, $\alpha=.1$ and $\sigma=.01$.

The results of table 8.1 clearly show that, for any value of n, the multiple correlation coefficients ρ_1 and ρ_2 take values that are very close. Moreover, the two coefficients are just about identical (at least to five decimals) for small values of n.

Table 8.1 Multiple correlation coefficients ρ_1 and ρ_2

Ornstein-Uhlenbeck $\delta=.06$, $\delta_0=.1$, $\alpha=.1$ and $\sigma=.01$

n	Portfolio of 10 contracts		Portfolio of 100000 contracts	
	ρ_1	ρ_2	ρ_1	ρ_2
1	0.99998	0.99998	0.99998	0.99998
2	0.99883	0.99883	0.99867	0.99860
3	0.99699	0.99699	0.99628	0.99584
4	0.99451	0.99451	0.99287	0.99144
5	0.99147	0.99146	0.98888	0.98576
6	0.98788	0.98787	0.98492	0.97960
7	0.98378	0.98378	0.98149	0.97383
8	0.97920	0.97920	0.97878	0.96892
9	0.97418	0.97418	0.97675	0.96498
10	0.96875	0.96874	0.97525	0.96189
20	0.89640	0.89624	0.97153	0.95279
40	0.73908	0.73413	0.95872	0.93667
60	0.65408	0.63413	0.90274	0.87293

It is important to appreciate that it is not the fact that the correlation coefficients are high which make the approximation 8.21 a good one or not. Rather, it is the fact that ρ_1 and ρ_2 are close. Therefore, the lower correlation coefficients for a portfolio of size 10 (particularly when n is large) do not imply that the approximation will be worse than for a larger portfolio of size 100000.

In order to apply the results of theorem 8.1, it is necessary to do some discretization. Compared to the limiting portfolio case, the finite portfolio has a higher level of complexity in using the procedure herein described to find the distribution function of the average cost per policy. This is due to the random number of benefits that will become payable in a given year, instead of a fixed proportion for the limiting portfolio. Accordingly, more numerical analysis and tests have to be done to illustrate successfully the results of theorem 8.1. This is certainly an interesting subject for further research.

For the purpose of comparison of the finite distributions with the limiting one, we will use a simulation approach. The next section presents the main steps of the simulatior that was used to estimate the distribution function of portfolios of n-year temporary assurance contracts of finite size.

8.1.4 Simulation of ACPP(T,n,c) and illustrations.

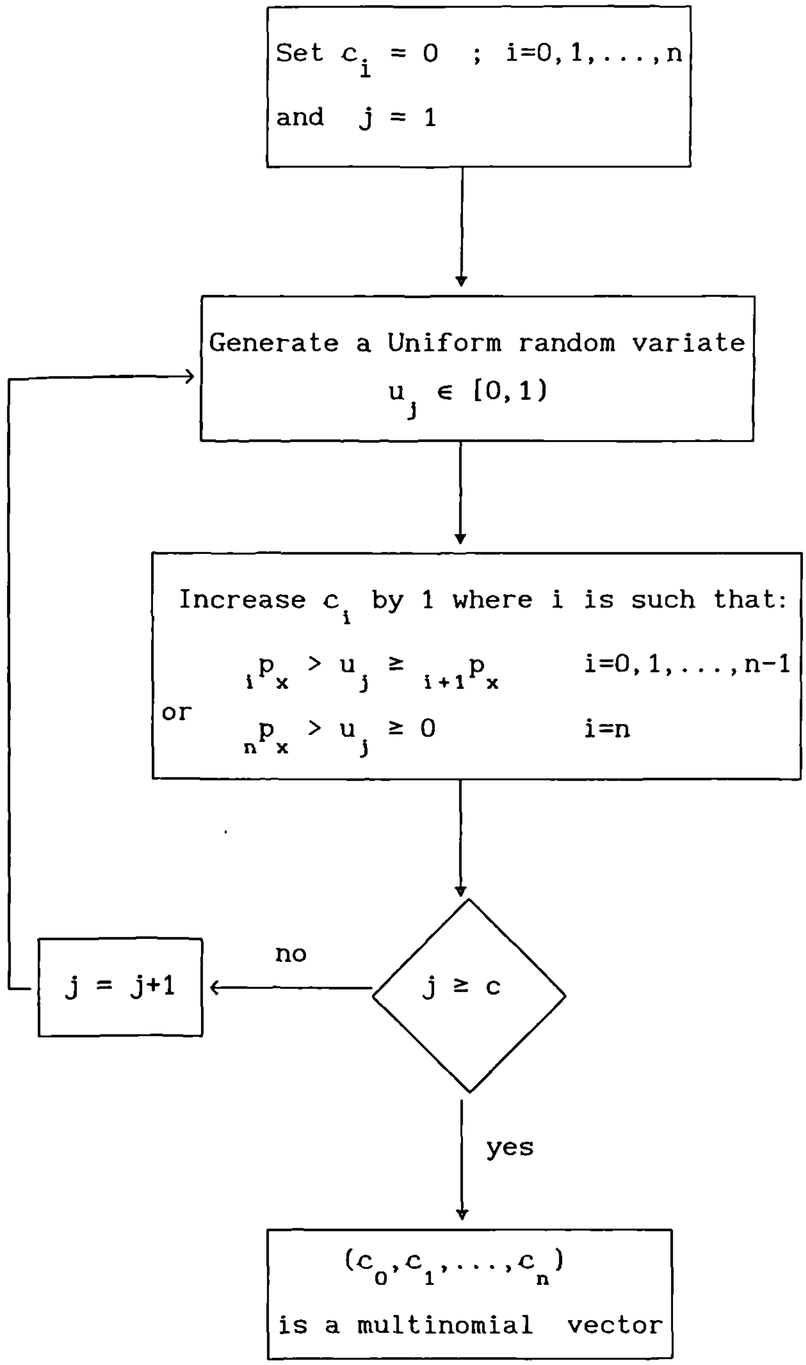
To simulate a realization of ACPP(T,n,c), we can evaluate 8.3 with a generated multivariate normal vector $\underline{y} = (y_1, y_2, \ldots, y_n)$ where y_i is a realization of y(i) and with a generated multinomial vector $(c_0, c_1, \ldots, c_n)$ where c_i is a realization of c_i.

To generate the vector $\underline{y}$ we used the method that has already been presented in section 6.7.2.

To generate the multinomial vector $(c_0, c_1, \ldots, c_n)$ we used the algorithm presented in figure 8.1. There are many different ways of generating a multinomial vector.

Table 8.2 presents the expected value, standard deviation and skewness, both theoretical and from 2000 simulations, of the average cost per policy of portfolios of size 10 for 5 and 25 years temporary assurance contracts issued at age 30.

Figure 8.1 Algorithm to generate a multinomial vector with parameters $(c;\ q_x,\ {}_{1|}q_x,\ {}_{2|}q_x,\ \ldots,\ {}_{n-1|}q_x,\ {}_np_x)$



Note: ${}_ip_x - {}_{i+1}p_x = {}_{i|}q_x$

Table 8.2 Expected value, standard deviation and skewness of ACPP(T,n,c)

Theoretical and from 2000 simulations for c=10

5 and 25 years temporary assurance issued at age 30

Ornstein-Uhlenbeck $\delta=.06$ $\delta_0=.1$ $\alpha=.1$ $\sigma=.01$

n	Theoretical			2000 Simulations		
	E.V.	sd	sk	E.V.	sd	sk
5	.00516	.01989	3.8934	.00504	.02005	4.3337
25	.02636	.03195	1.5732	.02551	.03142	1.5193

Table 8.3 presents equivalent results for the average cost per policy of similar portfolios but this time of size 100.

Table 8.3 Expected value, standard deviation and skewness of ACPP(T,n,c)

Theoretical and from 2000 simulations for c=100

5 and 25 years temporary assurance issued at age 30

Ornstein-Uhlenbeck $\delta=.06$ $\delta_0=.1$ $\alpha=.1$ $\sigma=.01$

n	Theoretical			2000 Simulations		
	E.V.	sd	sk	E.V.	sd	sk
5	.00516	.00629	1.2331	.00504	.00621	1.3972
25	.02636	.01104	0.6666	.02645	.01108	0.6525

Note, however, that knowledge of the moments of the distribution of the average cost per policy will not in general be sufficient to answer all questions about the solvency of a portfolio. The right tail of the distribution is definitely valuable, and possibly essential, information when dealing with solvency problems.

Tables 8.4 and 8.5 present some percentiles for 2000 simulated values of the average cost per policy for two portfolios of temporary assurance contracts, with terms of 5 and 25 years, issued at age 30. One portfolio has 10 policies, the other 100.

Note that each value presented under the headings ACPP(T,n,c) represents the value of z for which the estimated cdf of ACPP(T,n,c) at z, i.e. $F_{ACPP(T,n,c)}(z)$, equals the given percentile.

Table 8.4 Estimated percentiles of 2000 simulated values of ACPP(T,n,10)

5 and 25 years temporary assurance issued at age 30

Ornstein-Uhlenbeck $\delta=.06$ $\delta_0=.1$ $\alpha=.1$ $\sigma=.01$

5 years temporary		25 years temporary	
ACPP(T,5,10)	Percentile	ACPP(T,25,10)	Percentile
.065764	.95	.090608	.95
.080648	.975	.105477	.975
.090210	.99	.125712	.99
.090863	.995	.138444	.995
.165801	.9995	.170409	.9995

Table 8.5 Estimated percentiles of 2000 simulated values of ACPP(T,n,100)

5 and 25 years temporary assurance issued at age 30

Ornstein-Uhlenbeck $\delta=.06$ $\delta_0=.1$ $\alpha=.1$ $\sigma=.01$

5 years temporary		25 years temporary	
ACPP(T,5,100)	Percentile	ACPP(T,25,100)	Percentile
.016282	.95	.046575	.95
.020250	.975	.051104	.975
.024326	.99	.057575	.99
.029645	.995	.061161	.995
.040541	.9995	.073184	.9995

From tables 6.3, 8.4 and 8.5 we can see how the size of the portfolio affects its probability of solvency. Or, if one prefers, one could look at the net single premium that should be charged to each policyholder of a portfolio of a given size in order to have a given probability of solvency.

For example, if one wants a probability of solvency of at least .995, the net single premium per policyholder should be about .0908 or .0296 for portfolios of 5 years temporary assurance contracts of size 10 and 100 respectively, and about .0056 for a limiting portfolio. For 25 years temporary assurance contracts the net single premium should be about .1384, .0612 and .0420 for portfolios of size 10, 100 and a limiting portfolio respectively.

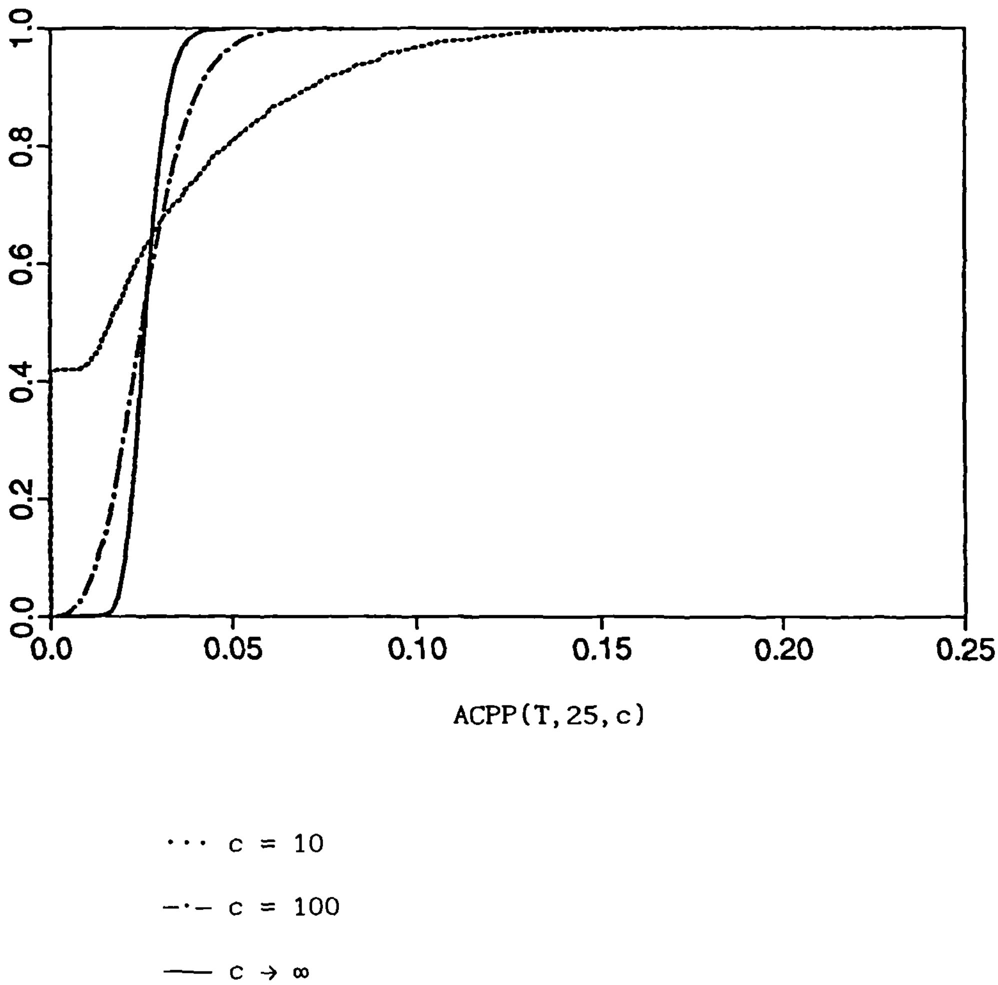
From this example, we see that the right tail of the distribution of ACPP(T,25,100) is closer to the right tail of the limiting distribution (table 6.3) than ACPP(T,5,100) is. This is due to the fact that the uncertainty due to mortality is greater for 5 years temporary contracts than for 25 years contracts, therefore, requiring more policies to eliminate this uncertainty.

For a better illustration of how the size of the portfolio affects the distribution of the average cost per policy, figure 8.2 presents the cdf of ACPP(T,n,c) for 25 years temporary contracts issued at age 30 for the three values of c considered here, namely, 10, 100 and ∞.

Figure 8.2 Cdf of ACPP(T,25,c)

25 years temporary assurance issued at age 30

Ornstein-Uhlenbeck $\delta=.06$ $\delta_0=.1$ $\alpha=.1$ $\sigma=.01$



From figure 8.2, we can see that even for 25-year temporary assurance contracts, 10 policies is clearly insufficient to eliminate the uncertainty due to mortality. There is still a probability of paying no benefits at all of more than .4 and the distribution is far from the one corresponding to no uncertainty due to mortality (i.e. the solid curve). Even 100 such policies is not enough. Although the distribution is much closer to the limiting one, the mortality risk is still relatively important, hence reinsurance.

8.2 Endowment Assurance.

8.2.1 Introduction

In this section we will study the distribution of the present value of the benefits for the portfolio of a finite number of endowment assurance policies described in chapter 5.

Again, we will use definition 7.1 of ${}_c\mathcal{Z}$, the one involving a summation of the present value of the benefits in a given year over the n policy-years of the contract. That is:

$$ {}_c\mathcal{Z} = \sum_{i=0}^{n-1} c_i \cdot e^{-y(i+1)} + c_n \cdot e^{-y(n)} $$

where c_i is defined as in chapter 7 and in section 8.1.

We will still assume that the c_i's are independent of the y(j)'s for all i and j.

As with the temporary contracts, we theoretically could obtain the density function of ${}_c\mathcal{Z}/_C$ by summing and integrating the product of the joint density function of the y(i)'s with the probability function of the c_i's over the appropriate domain. The expression would still look like 8.1 but the domain of integration would be different.

This method is, however, impracticable and we to have consider other approaches. One method would be to adapt the results of section 8.1.2 to endowment contracts, just as we did in section 7.2 (adapting the results of section 6.2) for the limiting portfolio.

Another approach would be to use the relationship which exists between the average cost per policy for portfolios of temporary and endowment contracts. This is the approach that is suggested and it is the subject of the next section.

8.2.2 Relationship between ACPP(T,n,c) and ACPP(E,n,c).

Let ACPP(E,n,c) denote the average cost per policy of a portfolio of c n-year endowment policies. The random variable ACPP(E,n,c) is then given by:

$$ACPP(E,n,c) = \sum_{i=0}^{n-1} \frac{c_i}{c} \cdot e^{-y(i+1)} + \frac{c_n}{c} \cdot e^{-y(n)} \qquad 8.51$$

Define the function $h_n(z_n, y_n, c_n)$ to be a mixture of distribution and density functions for the three events $ACPP(E,n,c) \le z_n$, $\mathfrak{C}_n = c_n$ and $y(n) = y_n$ simultaneously, by the equation

$$h_n(z_n, y_n, c_n) =$$

$$f_{y(n)}(y_n) \cdot P\left(\mathfrak{C}_n = c_n \middle| y(n) = y_n\right) \cdot P\left(ACPP(E,n,c) \le z_n \middle| \mathfrak{C}_n = c_n, y(n) = y_n\right) \qquad 8.52$$

Since we assume that the c_i's are independent of the y(j)'s, we have that 8.8 holds. Therefore, 8.52 becomes:

$$h_n(z_n, y_n, c_n) = f_{y(n)}(y_n) \cdot P\left(\mathfrak{C}_n = c_n\right) \cdot P\left(ACPP(E,n,c) \le z_n \,\middle|\, \mathfrak{C}_n = c_n, y(n) = y_n\right) \qquad 8.53$$

From this last equation it follows that the distribution of ACPP(E,n,c) is given by:

$$F_{ACPP(E,n,c)}(z_n) = \sum_{c_n=0}^{C} \int_{-\infty}^{\infty} h_n(z_n, y_n, c_n) \cdot dy_n \qquad 8.54$$

By combining equations 8.3 and 8.51, we note that

$$ACPP(E,n,c) = ACPP(T,n,c) + \frac{c_n}{C} \cdot e^{-y(n)} \qquad 8.55$$

This last equation makes it possible to link the functions $g_n(z_n, y_n, c_n)$ and $h_n(z_n, y_n, c_n)$ and therefore to find the distribution of ACPP(E,n,c).

The following theorem gives the existing relationship between the function $h_n(z_n, y_n, c_n)$ just defined and the function $g_n(z_n, y_n, c_n)$, discussed in section 8.1.2. Note that the latter can be obtained from the result of theorem 8.1.

THEOREM 8.2: The functions $g_n(z_n, y_n, c_n)$ and $h_n(z_n, y_n, c_n)$ are linked by the following equation:

$$h_n(z_n, y_n, c_n) = g_n\left(z_n - \frac{(c-c_n)}{c} \cdot e^{-y_n}, y_n, c_n\right) \tag{8.56}$$

Proof: Since we condition on $y(n) = y_n$ and on $\mathfrak{C}_n = c_n$, definition 8.53, is equivalent to:

$$h_n(z_n, y_n, c_n) = f_{y(n)}(y_n) \cdot P\left(\mathfrak{C}_n = c_n\right) \cdot$$

$$P\left(ACPP(E, n, c) - \frac{c_n}{c} \cdot e^{-y(n)} \le z_n - \frac{c_n}{c} \cdot e^{-y_n} \mid \mathfrak{C}_n = c_n, y(n) = y_n\right) \tag{8.57}$$

Note that if $\mathfrak{C}_n = c_n$, then from 6.2, 8.5 and 8.6,

$$c_n = c_n = c - c_n \tag{8.58}$$

And from 8.55 and 8.58, 8.57 becomes:

$$h_n(z_n, y_n, c_n) = f_{y(n)}(y_n) \cdot P\left(\mathfrak{C}_n = c_n\right) \cdot$$

$$P\left(ACPP(T, n, c) \le z_n - \frac{(c-c_n)}{c} \cdot e^{-y_n} \mid \mathfrak{C}_n = c_n, y(n) = y_n\right) \tag{8.59}$$

Finally, from 8.9, the right-hand side of 8.59 is simply $q_n(z_n - \frac{(c-c_n)}{c} \cdot e^{-y_n}, y_n, c_n)$. This observation completes the proof of 8.56. □

To summarize, we could use theorem 8.1 to find $q_n(z_n, y_n, c_n)$ then find $h_n(z_n, y_n, c_n)$ from 8.56 and finally we find the cdf of ACPP(E,n,c) by using 8.54.

Of course, to obtain numerical illustrations from 8.54, we would need to do some discretization and first of all, evaluate the function $q_n(z_n, y_n, c_n)$. Since this was not done in section 8.1, we cannot present numerical illustrations when using theorem 8.2. This could also be the subject of a further research.

As for finite portfolios of temporary contracts, for endowment contracts we use a simulation approach to compare finite distributions with the limiting one. The main steps of the simulation that was carried out to estimate the distribution function of portfolios of n-year endowment contracts of finite size are presented in the next section. Some numerical illustrations for small portfolios are also presented.

8.2.3 Simulation of ACPP(E,n,c) and illustrations.

To simulate a realization of ACPP(E,n,c), we can evaluate 8.51 with a generated multivariate normal vector $\underline{y} = (y_1, y_2, \ldots, y_n)$ where y_i is a realization of y(i) and with a generated multinomial vector $(c_0, c_1, \ldots, c_n)$ where c_i is a realization of c_i.

To generate the vector $\underline{y}$ we used the method that has already been presented in section 6.7.2.

To generate the multinomial vector $(c_0, c_1, \ldots, c_n)$ we used the algorithm presented in figure 8.1.

Table 8.6 presents the expected value, standard deviation and skewness, both theoretical and from 2000 simulations, of the average cost per policy of portfolios of size 10 for 5 and 25 years endowment assurance contracts issued at age 30.

Table 8.6 Expected value, standard deviation and skewness of ACPP(E,n,c)

Theoretical and from 2000 simulations for c=10

5 and 25 years endowment assurance issued at age 30

Ornstein-Uhlenbeck δ=.06 δ_0=.1 α=.1 σ=.01

n	Theoretical			2000 Simulations		
	E.V.	sd	sk	E.V.	sd	sk
5	.63471	.03438	.15815	.63545	.03447	0.1541
25	.17581	.06037	.94358	.17626	.06042	0.9183

In table 8.7, we find the same results for portfolios with exactly the same characteristics, except that they are now of size 100 instead of size 10.

Table 8.7 Expected value, standard deviation and skewness of ACPP(E,n,c)

Theoretical and from 2000 simulations for c=100

5 and 25 years endowment assurance issued at age 30

Ornstein-Uhlenbeck $\delta=.06$ $\delta_0=.1$ $\alpha=.1$ $\sigma=.01$

n	Theoretical			2000 Simulations		
	E.V.	sd	sk	E.V.	sd	sk
5	.63471	.03415	0.1607	.63448	.03313	0.1687
25	.17581	.05727	1.0459	.17537	.05702	1.1195

Again, for solvency purposes, the right tail of the distribution is of great importance.

Tables 8.8 and 8.9 present some percentiles from 2000 simulated values of the average cost per policy for portfolios of endowment assurance contracts, with terms of 5 and 25 years, issued at age 30, of size 10 and 100 respectively.

Note that each value presented under the headings ACPP(E,n,c) represents the value of z for which the estimated cdf of ACPP(E,n,c) at z, i.e. $F_{ACPP(E,n,c)}(z)$, equals the given percentile.

From tables 7.2, 8.8 and 8.9 we can see the small effect (at least for 10 or more policies) that the size of the portfolio has on its probability of solvency. This contrasts with the situation discussed earlier for temporary assurance contracts. Alternatively, we may determine the net single premium that should be charged to each policyholder of a portfolio of a given size in order to have a given probability of solvency.

Table 8.8 Estimated percentiles of 2000 simulated values of ACPP(E,n,10)

5 and 25 years endowment assurance issued at age 30

Ornstein-Uhlenbeck $\delta=.06$ $\delta_0=.1$ $\alpha=.1$ $\sigma=.01$

5 years endowment		25 years endowment	
ACPP(E,5,10)	Percentile	ACPP(E,25,10)	Percentile
.693402	.95	.280414	.95
.706170	.975	.314682	.975
.719300	.99	.355580	.99
.726813	.995	.391021	.995
.755820	.9995	.464104	.9995

Table 8.9 Estimated percentiles of 2000 simulated values of ACPP(E,n,100)

5 and 25 years endowment assurance issued at age 30

Ornstein-Uhlenbeck $\delta=.06$ $\delta_0=.1$ $\alpha=.1$ $\sigma=.01$

5 years endowment		25 years endowment	
ACPP(E,5,100)	Percentile	ACPP(E,25,100)	Percentile
.689475	.95	.278319	.95
.702747	.975	.312318	.975
.717362	.99	.345953	.99
.725402	.995	.401141	.995
.742427	.9995	.478923	.9995

For example, if one wants a probability of solvency of at least .995, the net single premium per policyholder should be about .7268 or .7254 for portfolios of 5 years endowment assurance contracts of size 10 and 100 respectively, and about the same value for a limiting portfolio.

For 25 years temporary assurance contracts the net single premium should be about .3910 or .4011 for portfolios of size 10, 100 respectively and about the same for a limiting portfolio.

From this example, we see that the right tail of the distribution of ACPP(E,25,c) or ACPP(E,5,c) is close to the right tail of the limiting distribution (table 7.2), at least for c greater than 10. This is because the uncertainty due to mortality for endowment contracts is relatively small, and therefore, only a few policies are required to almost completely eliminate this uncertainty.

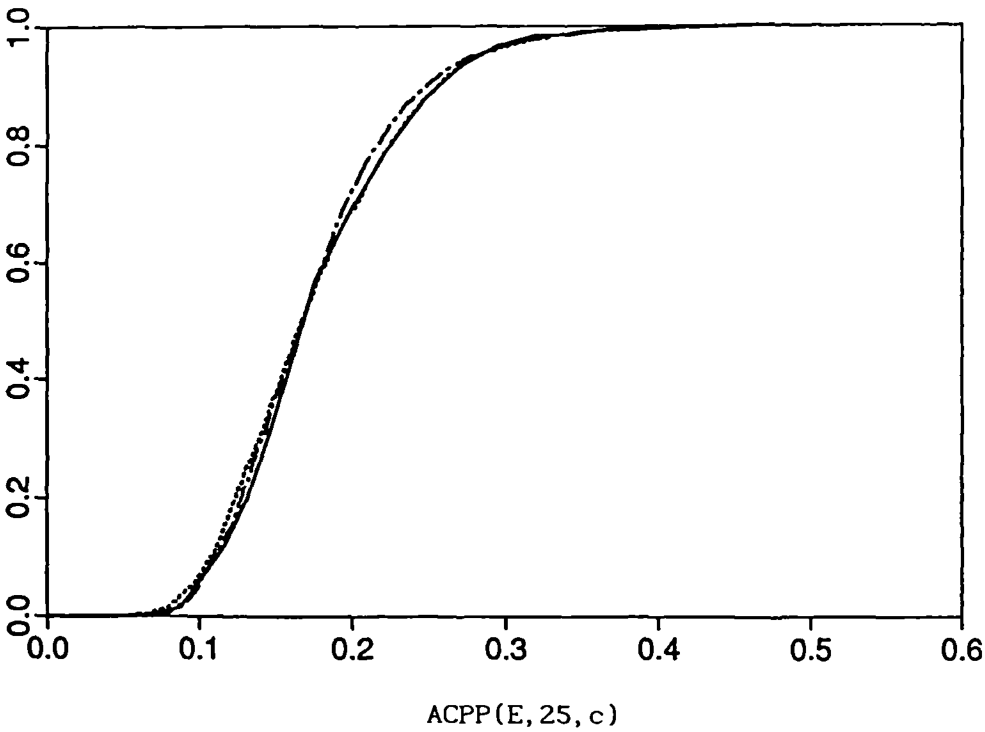
For a better illustration of how the size of the portfolio affects the distribution of the average cost per policy, figure 8.3 presents the cdf of ACPP(E,n,c) for 25 years endowment contracts issued at age 30 for the three values of c considered here, namely, 10, 100 and ∞.

This last figure confirms that as long as a portfolios counts at least 10 policies, the uncertainty that is left in its average cost per policy is virtually entirely due to the random force of interest. The uncertainty due to mortality being relatively small.

Figure 8.3 Cdf of ACPP(E,25,c)

25 years endowment assurance issued at age 30

Ornstein-Uhlenbeck δ=.06 δ_0=.1 α=.1 σ=.01



··· c = 10

—·— c = 100

—— c → ∞

CHAPTER 9

CONCLUSION

9.1 Summary.

With regard to the problems of setting contingency reserves and assessing the solvency of life assurance companies, this thesis presents some fundamental results. It was not our intention to determine specific contingency reserves or solvency margins but to show how one can obtain useful information to take appropriate decisions concerning these. Among the main results, there is, firstly, a general and efficient recursive method to find the first three moments of the present value of the benefits payable for a portfolio of identical policies. Three particular cases have been considered and illustrated, namely, the temporary assurance, the endowment assurance and the whole-life assurance. We found that it was possible to spread the mortality risk but not the interest rates risk. Also, the number of policies required to spread most of the mortality risk varies with the type and term of the assurance contract.

Secondly, an accurate recursive method is developed to find the distribution function of the average cost per policy (or average of the present value of benefits payable) for a limiting (i.e. when the number of policies tend to infinity) portfolio of temporary assurance contracts. The method involves an approximation which has been justified theoretically and validated in different ways.

Thirdly, it is shown how one can obtain the distribution of the average cost per policy for a limiting portfolio of endowment assurance contracts from that of a limiting portfolio of temporary assurance contracts.

The development of a method to obtain the distribution of the average cost per policy for limiting portfolios might suggest that the recursive method used to find the moments is no longer needed. One could always approximate the moments from the distribution. This is however not the case as those moments are essential information to find the limiting distributions efficiently.

Finally, we derive a method to find recursively the average cost per policy for portfolios of finite size for temporary assurance and endowment assurance contracts.

The earlier chapters of this thesis set out the basic results which are used extensively in later chapters to obtain the main results given above. The moments and distributions of the present value function are considered in chapter 3 and those of the present value of assurance benefits are considered in chapter 4.

Although all the illustrations are using a continuous-time first order stochastic differential equation, the Ornstein-Uhlenbeck process, the validity of the results is not restricted to this process. The main results are very general in terms of the stochastic process that may be used to model the interest rates (or the force of interest). Nevertheless, based on the criteria discussed in chapter 2, the Ornstein-Uhlenbeck process is believed to be an adequate model for the force of interest.

9.2 Applications.

Although the results have all been presented in relation to assurance contracts, they would also find applications in other fields. For example, when finding the moments of the present value of an assurance benefit, one may simply think of the benefits as future cash flows (positive or negative) and the mortality rates as probabilities of receiving or paying the cash flows; the formula presented would provide a way of finding the moments of the present value of any cash flow.

Replacing the probabilities of dying by cash flows in the results of chapter 6 would provide a way of approximating the distribution of the present value of the cash flows. This would find some applications in project evaluation, pension funds, etc...

The results are presented for net single premium contracts but they would just as well apply to net level premium contracts or any contract with premiums spread over part or all of the coverage period.

Only portfolios of identical policies were considered but the results for limiting portfolios consisting of different types of assurance contracts are easily obtainable by adapting the results of chapter 6. Instead of using the expected number of policies where the benefit is due at a given time, one would need to use the expected total amount of benefits payable at a given time. Note that although the results for limiting portfolios of identical contracts are easily adaptable for limiting mixed portfolios, the adaptation is not so simple when the portfolios are of finite size.

The results of chapter 8 assumed that the random variables c_i; $i=0,1,\ldots,n-1$, had a multinomial distribution, which is the case for the particular portfolio considered. These results could be extended to cases where the c_i's would have a different distribution function. Given appropriate modelling of the c_i's, the method of chapter 8 could find some applications in casualty assurance and disability and sickness assurance.

9.3 Suggestions for further research.

In addition to the different applications discussed above, there exist numerous closely related areas for further research. For example, it would be of interest to study in detail the moments and distribution of the average cost per policy (profit or loss) for portfolios of annuity contracts.

It would also be interesting to compare the distributions for different processes used to model the force of interest (independent and identically distributed, Ornstein-Uhlenbeck process, second order SDE, etc...) and for different discrete processes for the interest rates (random walk, ARIMA(p,d,q)).

Trying to fit certain known distributions like translated Gamma, lognormal, Pearson, etc.. to the approximated distributions (with particular attention given to the tails of the distributions) obtained from the methods suggested here is another possibility for further research.

Note that this research has set forth some fundamental results for analyzing the problems of contingency reserve and solvency. A thorough analysis of these problems would be a worthwhile project with possibly immediate applications in the assurance business.

The results for finite portfolios are presented theoretically but they have not been carried out in practice. Therefore, the practical considerations involved in finding the distribution of the present value of benefits for portfolios of finite size would deserve some attention.

As for solvency purposes, it would be interesting to investigate, in greater detail, the number of contracts necessary in order to spread as much of the risk as possible, bearing in mind that not all of it may be spread.

A very interesting problem, closer to the kind of business actually carried out by assurance companies, would be to consider portfolios of finite size consisting of different types of assurance contracts.

One could also consider a generalization of the model which would allow for expenses and withdrawals possibly linked in some way to the interest rates model.

We hope that this research will be helpful to people who need to consider actuarial functions with stochastic mortality and interest rates. A further hope is that it will generate some new research aimed at studying some of the unanswered questions raised here.

APPENDIX A

Alternative proof of 2.48

The result obtained in 2.48 can also be obtained by using the results appearing in Arnold (1974, p.131).

First let us find the derivative of y(t) as defined in 2.37. Using Itô's lemma (see, for example, Arnold (1974, p.91)) we have:

$$dy(t) = \frac{\delta y(t)}{\delta t}\cdot dt + \frac{\delta y(t)}{\delta\delta_t}\cdot d\delta_t + .5\cdot\frac{\delta^2 y(t)}{\delta\delta_t\cdot\delta\delta_t}\cdot d\delta_t d\delta_t \qquad \text{A.1}$$

$$= \delta_t\cdot dt + 0\cdot d\delta_t + .5\cdot 0\cdot d\delta_t d\delta_t = \delta_t\cdot dt \qquad \text{A.2}$$

Now combining this result with 2.6 for δ=0 and writing both in matrix form, we get:

$$d\begin{pmatrix}\delta_t\\ y(t)\end{pmatrix} = \begin{pmatrix}-\alpha & 0\\ 1 & 0\end{pmatrix}\cdot\begin{pmatrix}\delta_t\\ y(t)\end{pmatrix}\cdot dt + \begin{pmatrix}\sigma\\ 0\end{pmatrix}\cdot dW_t \qquad \text{A.3}$$

(Note that the autocovariance function does not depend on the long term mean δ, so in order to simplify this proof we can assume without loss of generality that δ is equal to 0.)

Then from Arnold (1974, p.131, 8.2.6(a)) we know that the expectation of such a system is given by:

$$E\begin{bmatrix}\delta_t\\ y(t)\end{bmatrix} = e^{Mt}\cdot\begin{pmatrix}\delta_0\\ y(0)\end{pmatrix} \qquad \text{A.4}$$

where

$$M = \begin{pmatrix} -\alpha & 0 \\ 1 & 0 \end{pmatrix} \qquad \text{A.5}$$

Expanding e^{Mt} we get:

$$e^{Mt} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -\alpha & 0 \\ 1 & 0 \end{pmatrix} \cdot t + \begin{pmatrix} -\alpha & 0 \\ 1 & 0 \end{pmatrix}^2 \cdot \frac{t^2}{2} + \begin{pmatrix} -\alpha & 0 \\ 1 & 0 \end{pmatrix}^3 \cdot \frac{t^3}{6} + \ldots \qquad \text{A.6}$$

$$= \begin{pmatrix} \exp\{-\alpha t\} & 0 \\ \frac{1-\exp\{-\alpha t\}}{\alpha} & 1 \end{pmatrix} \qquad \text{A.7}$$

And the expectation would be (note that δ=0 and y(0)=0):

$$E\begin{bmatrix} \delta_t \\ y(t) \end{bmatrix} = e^{Mt} \cdot \begin{pmatrix} \delta_0 \\ 0 \end{pmatrix} = \begin{pmatrix} \delta_0 \cdot \exp\{-\alpha t\} \\ \delta_0 \cdot \frac{1-\exp\{-\alpha t\}}{\alpha} \end{pmatrix} \qquad \text{A.8}$$

And now the autocovariance function in the present situation becomes by Arnold (1974, p.131, 8.2.6(b)) assuming $s \leq t$:

$$\operatorname{cov}\left(\begin{pmatrix} \delta_s \\ y(s) \end{pmatrix}, \begin{pmatrix} \delta_t \\ y(t) \end{pmatrix}\right) = e^{Ms}\left[\int_0^s (e^{Mu})^{-1} \cdot \begin{pmatrix} \sigma \\ 0 \end{pmatrix} \cdot (\sigma \;\; 0) \cdot ((e^{Mu})^{-1})^T du\right] \cdot (e^{Mt})^T \qquad \text{A.9}$$

$$= e^{Ms} \cdot \left[\int_0^s \begin{pmatrix} \exp\{\alpha u\} & 0 \\ \frac{1-\exp\{\alpha u\}}{\alpha} & 1 \end{pmatrix} \cdot \begin{pmatrix} \sigma^2 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \exp\{\alpha u\} & 0 \\ \frac{1-\exp\{\alpha u\}}{\alpha} & 1 \end{pmatrix}^T du\right] \cdot (e^{Mt})^T \qquad \text{A.10}$$

$$= \begin{pmatrix} \exp\{-\alpha s\} & 0 \\ \frac{1-\exp\{-\alpha s\}}{\alpha} & 1 \end{pmatrix} \cdot \frac{\sigma^2}{2\alpha^2} \begin{pmatrix} \alpha(e^{2\alpha s}-1) & 2e^{\alpha s}-e^{2\alpha s}-1 \\ 2e^{\alpha s}-e^{2\alpha s}-1 & 2s-\frac{4}{\alpha}e^{\alpha s}+\frac{e^{2\alpha s}}{\alpha}+\frac{3}{\alpha} \end{pmatrix} \cdot \begin{pmatrix} \exp\{-\alpha t\} & 0 \\ \frac{1-\exp\{-\alpha t\}}{\alpha} & 1 \end{pmatrix}^T \qquad \text{A.11}$$

Here we are interested in the element (1,1) of A.11 which is:

$$\text{cov}(\delta_s, \delta_t) = \frac{\sigma^2}{2\alpha}\cdot\left(e^{\alpha(s-t)} - e^{-\alpha(t+s)}\right) \qquad s \le t \qquad \text{A.12}$$

and it is equal to the expressions 2.33 and 2.9.

We are also interested in the element (2,2) of A.11 which after simplification is found to be, again assuming that $s \le t$:

$$\text{cov}(y(s), y(t)) = \frac{\sigma^2}{\alpha^2}\cdot s + \frac{\sigma^2}{2\alpha^3}\left[-2+2e^{-\alpha s}+2e^{-\alpha t}-e^{-\alpha(t-s)}-e^{-\alpha(t+s)}\right] \qquad \text{A.13}$$

Finally going through exactly the same steps but with $t \le s$ and combining the two situations we get the expression 2.48 and this completes the proof.

APPENDIX B

Mortality Table CA 80-82 Male

x	q_x	x	q_x	x	q_x	x	q_x
0	.01092						
1	.00081	26	.00143	51	.00694	76	.06442
2	.00063	27	.00139	52	.00768	77	.07002
3	.00048	28	.00136	53	.00848	78	.07607
4	.00047	29	.00134	54	.00933	79	.08251
5	.00039	30	.00132	55	.01026	80	.08941
6	.00030	31	.00132	56	.01127	81	.09683
7	.00022	32	.00134	57	.01239	82	.10483
8	.00019	33	.00139	58	.01360	83	.11338
9	.00019	34	.00145	59	.01488	84	.12243
10	.00022	35	.00153	60	.01628	85	.13203
11	.00027	36	.00163	61	.01781	86	.14227
12	.00035	37	.00175	62	.01951	87	.15319
13	.00049	38	.00189	63	.02138	88	.16475
14	.00069	39	.00205	64	.02339	89	.17692
15	.00092	40	.00223	65	.02556	90	.18975
16	.00112	41	.00245	66	.02790	91	.20332
17	.00128	42	.00271	67	.03046	92	.21767
18	.00139	43	.00301	68	.03317	93	.22325
19	.00147	44	.00334	69	.03601	94	.22003
20	.00153	45	.00372	70	.03907	95	.22234
21	.00157	46	.00414	71	.04243	96	.24450
22	.00158	47	.00461	72	.04617	97	.30086
23	.00157	48	.00512	73	.05024	98	.41245
24	.00153	49	.00567	74	.05460	99	.56973
25	.00148	50	.00628	75	.05930	100	.74112
						101	.89506
						102	1.00000

APPENDIX C

Proof of independence between c_{n-1} and ACPP(T,n-1,c) given $\mathfrak{C}_{n-1}$

We want to prove that the conditional random variables ACPP(T,n-1,c) and c_{n-1} given $\mathfrak{C}_{n-1}$ are independent. This is necessary for equation 8.20 to be true.

It should be remembered here that c_{n-1} is the rand·m variable for the number of policies (in a portfolio of size c) for which there is a death benefit payable at time n.

Therefore, we want to show that knowing the number of deaths between time 0 and time n-1, the average cost per policy of a portfolio of (n-1) year temporary assurance contracts of size c is independent of the number of deaths occurring between time n-1 and time n.

Intuitively, although c_i, i=0,1,2,...,n-2 and c_{n-1} are dependent, it makes sense that the latter does not influence ACPP(T,n-1,c) if we already know the sum of the c_i's from i=0 to i=n-2.

To prove this independence, we start with the definition of ACPP(T,n-1,c). From 8.3, ACPP(T,n-1,c) is given by:

$$\text{ACPP}(T,n-1,c) = \sum_{i=0}^{n-2} \frac{c_i}{c} \cdot e^{-y(i+1)} \qquad \text{C.1}$$

Since (by assumption) c_{n-1} is independent of the y(i)'s, it follows from the definition C.1 that, ACPP(T,n-1,c) given $\mathfrak{C}_{n-1}$ will be independent of c_{n-1} if, given $\mathfrak{C}_{n-1}$, c_{n-1} is independent of c_i for any i=0,1,...,n-2.

Now, from Melsa and Sage (1973, p.101, eq 3.9-3), the random variable $c_{n-1} | \mathfrak{C}_{n-1}$ is independent of $c_k | \mathfrak{C}_{n-1}$, k=0,1,...,n-2 if:

$$P\left(c_0=c_0,\ c_1=c_1,\dots,c_{n-2}=c_{n-2},\ c_{n-1}=c_{n-1} \middle| \mathfrak{C}_{n-1}\right) =$$

$$P\left(c_0=c_0,\ c_1=c_1,\dots,c_{n-2}=c_{n-2} \middle| \mathfrak{C}_{n-1}\right) \cdot P\left(c_{n-1}=c_{n-1} \middle| \mathfrak{C}_{n-1}\right) \quad \text{C.2}$$

Using equation 3.8-2 of Melsa and Sage (1973, p.94), we may find the three probabilities in C.2 as follows:

$$P\left(c_0=c_0,\ c_1=c_1,\dots,c_{n-2}=c_{n-2},\ c_{n-1}=c_{n-1} \middle| \mathfrak{C}_{n-1}=i\right) =$$

$$\frac{P\left(c_0=c_0,\ c_1=c_1,\dots,c_{n-2}=c_{n-2},\ c_{n-1}=c_{n-1},\ \mathfrak{C}_{n-1}=i\right)}{P\left(\mathfrak{C}_{n-1}=i\right)} \quad \text{C.3}$$

$$P\left(c_0=c_0,\ c_1=c_1,\dots,c_{n-2}=c_{n-2} \middle| \mathfrak{C}_{n-1}=i\right) =$$

$$\frac{P\left(c_0=c_0,\ c_1=c_1,\dots,c_{n-2}=c_{n-2},\ \mathfrak{C}_{n-1}=i\right)}{P\left(\mathfrak{C}_{n-1}=i\right)} \quad \text{C.4}$$

and

$$P\left(c_{n-1}=c_{n-1} \middle| \mathfrak{C}_{n-1}=i\right) = \frac{P\left(c_{n-1}=c_{n-1},\ \mathfrak{C}_{n-1}=i\right)}{P\left(\mathfrak{C}_{n-1}=i\right)} \quad \text{C.5}$$

Finally, as the c_k's are mutinomial, the required probabilities in C.3, C.4 and C.5 in order to check that C.2 holds are:

$$P\left(c_0=c_0,\ c_1=c_1, \ldots, c_{n-2}=c_{n-2},\ c_{n-1}=c_{n-1},\ \mathfrak{C}_{n-1}=i\right) =$$

$$\begin{cases} \binom{c}{c_0, c_1, \ldots c_{n-1}} \cdot \prod_{k=0}^{n-1} \left({}_{k|}q_x\right)^{c_k} \cdot \left({}_{n}p_x\right)^{c-i-c_{n-1}} & \text{if } \sum_{k=0}^{n-2} c_k = i \\ 0 & \text{Otherwise} \end{cases} \qquad \text{C.6}$$

$$P\left(c_0=c_0,\ c_1=c_1, \ldots, c_{n-2}=c_{n-2},\ \mathfrak{C}_{n-1}=i\right) =$$

$$\begin{cases} \binom{c}{c_0, c_1, \ldots c_{n-2}} \cdot \prod_{k=0}^{n-2} \left({}_{k|}q_x\right)^{c_k} \cdot \left({}_{n-1}p_x\right)^{c-} & \text{if } \sum_{k=0}^{n-2} c_k = i \\ 0 & \text{Otherwise} \end{cases} \qquad \text{C.7}$$

$$P\left(c_{n-1}=c_{n-1}, \mathfrak{C}_{n-1}=i\right) = \binom{c}{c_{n-1}, i} \cdot \left({}_{n-1|}q_x\right)^{c_{n-1}} \cdot \left({}_{n-1}q_x\right)^{i} \cdot \left({}_{n}p_x\right)^{c-i-c_{n-1}} \qquad \text{C.8}$$

and

$$P\left(\mathfrak{C}_{n-1}=i\right) = \binom{c}{i} \cdot \left({}_{n-1}q_x\right)^{i} \cdot \left({}_{n-1}p_x\right)^{c-i} \qquad \text{C.9}$$

It is then a straightforward exercise to check that equation C.2 is valid and therefore that the conditional random variables ACPP(T,n-1,c) and c_{n-1} given $\mathfrak{C}_{n-1}$ are independent.□

It is interesting to note that although c_i, i=0,1,2,...,n-2 and c_{n-1} are dependent, on conditioning on $\mathfrak{C}_{n-1}$ they become independent.

The following is a simple example of this situation:

Let X_i, i=1,2,...,5 be five random variables such that:

$$X_3 = X_1 + X_2 \tag{C.10}$$

and

$$X_5 = X_3 + X_4 \tag{C.11}$$

where X_1, X_2 and X_4 are independent.

Then

$$P\left(X_5=x|X_3,\ X_2\right) = P\left(X_5=x|X_3\right) \tag{C.12}$$

which means that given X_3, X_5 and X_2 are independent. But as X_5 equals the sum $X_1+X_2+X_4$, X_5 and X_2 are clearly not independent.

REFERENCES

JIA: Journal of the Institute of Actuaries.

JRI: Journal of Risk and Insurance.

SAJ: Scandinavian Actuarial Journal.

SOA: Society of Actuaries.

TFA: Transactions of the Faculty of actuaries.

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