DEVELOPING EARLY ALGEBRAIC REASONING IN A MATHEMATICAL COMMUNITY OF INQUIRY

by

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Developing early algebraic reasoning in a mathematical community of inquiry

This study explores the development of early algebraic reasoning in mathematical communities of inquiry. Under consideration is the different pathways teachers take as they develop their own understanding of early algebra and then enact changes in their classroom to facilitate algebraic reasoning opportunities.

Teachers participated in a professional development intervention which focused on understanding of early algebraic concepts, task development, modification, and enactment, and classroom and mathematical practices.

Design research was employed to investigate both teaching and learning in the naturalistic setting of the schools and classrooms. The design approach supported the development of a model of professional development and the framework of teacher actions to facilitate algebraic reasoning. Data collection over the school year included participant observations, video recorded observations, documents, teacher interviews, and photo elicitation interviews with students. Retrospective data analysis drew the results together to be presented as cases of two teachers, their classrooms, and students.

The findings show that the integration of algebraic reasoning into classroom mathematical activity is a gradual process. It requires teachers to develop their own understanding of algebraic concepts which includes understanding of student reasoning, progression, and potential misconceptions. Task implementation and design, shifts in pedagogical actions, and the facilitation of new classroom and mathematical practices were also key elements of change. The important role which students have in the development of classrooms where algebraic reasoning is a focus was also highlighted.

These findings have significant implications for how teachers can be supported to develop their understanding of early algebra and use this understanding in their own classrooms to facilitate early algebraic reasoning.

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AUTHOR'S DECLARATION

At no time during the registration for the degree of Doctor of Philosophy has the author been registered for any other University award without prior agreement of the Graduate Committee.

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CHAPTER ONE

INTRODUCTION TO THE STUDY

1.1 INTRODUCTION

Important changes have been proposed for mathematics classrooms in recent years. These have been in response to a consideration of how mathematics education can best meet the needs of students in the 21st century and support them to participate in our developing "knowledge society". One aspect of the proposed changes is an increased emphasis on algebraic reasoning in primary school classrooms (Bastable & Schifter, 2008; Blanton & Kaput, 2005a; Carpenter, Franke, & Levi, 2003; National Mathematics Advisory Panel (NAEP), 2008). Within this frame, algebra is re-characterised as a strand which permeates through all levels of schooling and has links to a wide range of mathematical content areas. Developing such classrooms requires shifts both in the teachers' and learners' actions, engagement, and participation. This study, *Developing early algebraic reasoning in a mathematical community of inquiry,* was developed to investigate how such changes could be achieved in the United Kingdom and the British Isles. The main purpose of this study was to investigate how primary teachers can develop "algebra ears and eyes" (Blanton & Kaput, 2003) and then use this developing understanding to facilitate algebraic reasoning in their classroom.

This chapter identifies the central aim of this study. The background context outlines the international call for change in regards to teaching of early algebra and highlights the need for change within the context of the United Kingdom and the British Isles. Finally, the significance of this study and the initial stimulus for the development of the research project are both addressed.

1.2 RESEARCH AIM

This study aims to explore how teachers develop their students' early algebraic reasoning in a mathematical community of inquiry. The focus of the exploration is on the pathways teachers take as they develop their own algebra ears and eyes and then enact changes in their classroom to facilitate algebraic reasoning opportunities. The term algebra ears and eyes was coined by Blanton and Kaput (2003). Similar to their work, in this study algebra ears and eyes refers to teachers being able to recognise opportunities for algebraic reasoning in tasks when planning and also recognise and utilise spontaneous opportunities for algebraic reasoning during lessons. The study specifically investigates how teachers develop and implement tasks which involve opportunities to engage in early algebraic reasoning. It also seeks to understand the key pedagogical strategies and classroom and mathematical practices which teachers can use to facilitate algebraic reasoning. A final focus of the study is the implications of this type of change for student participation and engagement in classroom activity.

1.3 BACKGROUND CONTEXT OF THE STUDY

Significant changes have been proposed for Western mathematics classrooms of the 21st century in order to meet the needs of a "knowledge society". Developing learning communities where all students have opportunities to engage in mathematical practices which underlie algebraic reasoning has been an increasing focus in both national and international research and curricula reforms (e.g., Blanton & Kaput, 2003; National Council of Teachers of Mathematics (NCTM), 2000; Watson, 2009; National Mathematics Advisory Panel, 2008). Mason (2008) argues that learning algebra is an essential type of thinking for "participation in a democratic society" (p. 79). An important aspect of developing algebraic reasoning is the need to develop links to algebra in primary classrooms. Through expanding and integrating early algebraic reasoning into primary classrooms, Kaput (2008, p. 6) argues that four major goals will be achieved:

1) To add a degree of coherence, depth, and power typically missing in K-8 mathematics.
2) To ameliorate, if not eliminate the most pernicious and alienating curricular element of today's school mathematics: late, abrupt, isolated, and superficial high school algebra courses.
3) To democratise access to powerful ideas by transforming algebra from an inadvertent engine of inequity to a deliberate engine of mathematical power.
4) To build conceptual and institutional capacity and open curricular space for new 21st century mathematics desperately needed at the secondary level, space locked up by the 19th century high school curriculum now in place.

A major stimulus for the increasing emphasis on algebra is the growing acknowledgment of the insufficient algebraic understandings students develop during schooling and the way in which this denies them access to potential educational and employment prospects (Knuth, Stephens, McNeil, & Alibabi, 2006). Historically within Western schooling algebra has been situated as a gate-keeper—reserved for higher achieving students and introduced independently after a curriculum with a strong emphasis on computation and arithmetic (Kaput, 2008). The prevalent treatment and view of algebra in such school settings is as an abstract subject area characterised by symbolic manipulations (Chazan, 1996; Kaput, 2008; Stacey & Chick, 2004). Smith and Thompson (2008) link this narrow view of algebra with a fundamental problem in mathematics teaching and learning. They argue that for many teachers and their students, mathematics bears little useful relationship to their world. Initially, mathematics is a world of numbers and numerical procedures which then develops into a world of symbols and symbolic procedures. Consequently they argue that one aim of mathematics education should be to develop links between numbers and symbols with situations and problems.

In the context of the United Kingdom (UK), reports demonstrate that students have on-going difficulties with algebra. The 2007 Trends in International Mathematics and Science study (TIMMS) reported that the mathematical content area with the largest level of weakness for English students aged 13-years to 14-years old was algebra (Sturman et al., 2008). This finding was affirmed in a recent study by Hodgen, Kuchemann, Brown, and Coe (2009), who found that when comparing students aged 13-years to 14-years old understanding of algebra to results from 30 years ago there had been little change. These findings raise concerns for students' on-going participation in mathematics at higher levels.

In the UK and British Isles, both in primary and secondary schools, algebra is not delineated as a strand to be taught separately from other mathematical content areas. Number and algebra appear as a linked content area of mathematics in the national curriculum from Key Stage Two (Department for Education and Skills (DfES), 1999). This approach means that there are opportunities for integration of algebra across mathematics and for arithmetic to be seen as "particular instances of algebraic structures which have the added feature that they can be calculated" (Watson, 2009, p. 9). Integration of algebra with other mathematical content areas should begin at primary school, however this requires teacher understanding of the links between algebra, arithmetic, and other mathematical content areas.

This study considers how primary teachers can develop their algebra ears and eyes and use this understanding to more fully integrate early algebra into their mathematics lessons. As discussed in the following chapters, developing algebraic reasoning is more than acquiring knowledge of algebraic content areas or using algebraic tasks in the classroom. To effectively facilitate algebraic reasoning opportunities, teachers need to develop their own pedagogical content knowledge of algebra including an understanding of the development of student reasoning and potential misconceptions. Within the classroom, attention needs to be paid to task design and implementation and also the classroom and mathematical practices which support engagement in algebraic reasoning. As Watson (2009) argues with reference to a range of international research studies "the development of algebraic reasoning can happen in deliberately designed educational contexts" (p. 16). It is timely to explore how within the context of the UK and British Isles, teachers and researchers can collaborate to deliberately design primary classrooms which support student engagement with early algebra.

1.4 RATIONALE FOR THE STUDY

Many international studies (e.g., Bastable & Schifter, 2008; Blanton & Kaput, 2005a; Carpenter et al., 2003; Carraher, Schliemann, Brizuela, & Earnest, 2006) illustrate how primary teachers can develop aspects of algebraic reasoning in their classrooms. There are also some studies (e.g., Bills, Wilson, & Ainley, 2005; Brown & Coles, 1999; Hewitt, 2012; Jones, 2008; Jones & Pratt, 2012) based in the UK which investigate the development of specific areas of algebraic reasoning particularly with upper primary and lower secondary school students. However, we have a limited quantity of research directed towards exploring and examining how primary teachers in the UK and British Isles might develop algebraic reasoning through everyday mathematics lessons in their classrooms.

Watson (2009) calls for studies which address the experiences and educational environments that facilitate learners to shift from arithmetical to algebraic reasoning. As highlighted in the previous paragraph, there is a range of studies which focus on the development of early algebraic reasoning. Often these studies address an aspect of early algebra such as a particular content area or task development.

There are also many studies (e.g., Boaler & Brodie, 2004; Fosnot & Jacob, 2009; Kazemi, 1998; Khisty & Chval, 2002; McCrone, 2005; Monaghan, 2005; Schifter, 2009; Stein, Engle, Smith, & Hughes, 2008) which address productive classroom and mathematical practices in the mathematics classroom. However, there are few studies which specifically attend to algebraic content, task development and enactment, and the classroom and mathematical practices which facilitate primary students to engage in early algebraic reasoning. Understanding how these elements can be used as a focus both in professional development and within the classroom to facilitate algebraic reasoning is appropriate at this time.

For primary teachers to develop algebraic reasoning in their classrooms, it is essential that they have sound understanding of early algebra. A report by Ofsted (2008) identifies that many primary teachers in the United Kingdom have little experience with mathematics beyond what they studied within schooling; their view of mathematics and subsequent teaching is strongly influenced by this earlier learning and their own schooling experiences. A recommendation made in the Ofsted report is that an exploration is undertaken of approaches through which the subject expertise of teachers of mathematics can be developed. This is particularly pertinent for early algebra—a content area with which many teachers have had limited educational experience (Blanton & Kaput, 2005a) combined with limited exposure to research about teaching and learning algebra (Watson, 2009).

There are a number of international studies (e.g., Blanton & Kaput, 2003; Franke, Carpenter, & Battey, 2008; Koellner, Jacobs, Borko, Roberts, & Schneider, 2011; Warren, 2009) which address the professional development of teachers in the area of early algebra. Many of these studies focus on teacher participation during professional development or successful cases of teacher change in the classroom. It is timely to have a study which focuses on teacher development of algebra ears and eyes and examines potential shifts in classroom practice and pedagogical actions.

In recent years there has been increasing attention to ensuring all research participants including children can act as active participants in the research process and communicate their own views (Einarsdottir, 2007). This includes a growing range of studies (e.g., Boaler, Wiliam, & Zevenbergen, 2000; Cobb, Gresalfi, & Hodge, 2009; Franke & Carey, 2009; Hunter & Anthony, 2011; Pratt, 2006; Young-Loveridge, 2005) which investigate student perspectives in the mathematics classroom. However, there is limited research literature which addresses student perspectives in classrooms where early algebra is a focus or which attends to how students themselves may develop their own algebra ears and eyes. A direct focus is needed on student perspectives and shifts in their participation and engagement as algebraic reasoning is introduced into their classrooms.

1.5 STIMULUS FOR THIS STUDY

In 2008/2009 I undertook an observational research project which focused on teacher enactment of curriculum material from the Mathematics Enhancement Programme (MEP). The Mathematics Enhancement Programme was developed in order to improve mathematics teaching and learning in the United Kingdom by drawing on findings from the Kassel project (Burghes, 2004). This project investigated styles of teaching and attainment in mathematics teaching and learning internationally. Curriculum material was developed for MEP including resources (e.g., lesson plans, workbooks and online interactive resources). Many of the tasks in the curriculum material have implicit opportunities to facilitate students to engage in algebraic reasoning due to their structural basis. However, during the initial observational project, I noted that many of the opportunities for algebraic reasoning were missed due to the way in which the tasks were implemented and enacted in the classroom. Consequently a proposal for the current study was developed.

1.6 OVERVIEW OF THE THESIS

The thesis is presented in eight chapters. This chapter has provided the background and rationale for the study. The relevant research literature is reviewed in Chapters Two to Four. Chapter Two reviews research on the teaching and learning of early algebraic reasoning. Chapter Three examines elements of learning environments which support engagement in early algebra. This includes the types of tasks and their implementation, classroom practices, and mathematical practices. Chapter Four discusses how teachers may be supported to develop algebra ears and eyes and the implications of change in the classroom on student perspective and identity. Chapter Five outlines the use of a qualitative design approach and discusses the methods used to collect and analyse the data. Chapters Six and Seven present the results of the teachers' involvement in the professional development. Drawing on two cases, the findings are presented to illustrate how algebra was integrated into the classroom. Analysis of the teachers' developing classroom culture and student perspectives are discussed in relation to the literature. Chapter

Eight concludes the thesis with a discussion of the differing pathways the two teachers took. The conclusions and implications and recommendations for future research are presented.

CHAPTER TWO

THE BACKGROUND RESEARCH ON THE TEACHING AND LEARNING OF EARLY ALGEBRAIC REASONING

2.1 INTRODUCTION

The previous chapter outlined the call for change to place greater emphasis on algebraic reasoning in primary classrooms. For this reason, a close examination will be given in this chapter to research literature concerning student learning of algebraic concepts.

Section 2.2 provides a description of early algebraic reasoning. It outlines the historical treatment of algebra and its traditional role within the secondary school sector. Explanations are offered of the changing perspective with moves to re-define algebra and re-characterise it as a strand which permeates through all levels of schooling. This wider definition of algebraic reasoning includes both content and process.

Section 2.3 describes how early algebraic reasoning can be taught through generalised arithmetic. It examines those key areas which have links to early algebra:

- Equivalence, relational reasoning, and solving, using and representing equations.
- Understanding the properties of operations: The commutative, associative, and distributive properties
- Exploring ideas about operations and their relationships.
- Understanding the properties and relationships of numbers: Odd and even numbers, zero and one.

Key difficulties in each area are identified and a review of research studies which have addressed the teaching and learning of different aspects of early algebra are provided.

Section 2.4 describes how early algebraic reasoning can be taught through functional thinking. It provides a definition of functional thinking and research studies are drawn upon to illustrate how rich functional thinking can be developed with younger students.

Section 2.5 outlines the key role of tools and representations in facilitating students to engage in early algebraic reasoning. The support that can be offered by representational structures such as drawings, diagrams, and tables is described. Also highlighted is how notation such as variables allows students to work at a higher level of generality and examines how these may be introduced to young students.

2.2 DEFINING EARLY ALGEBRAIC REASONING

The perception and view of mathematics as a subject inclusive of notions of algebra, influences the ways in which it is presented and taught in classrooms. Historically, algebra was introduced in secondary school following on from a strong focus in primary schools on a computational curriculum. The exclusivity of algebra to the secondary sector was justified in terms of developmental theories. Younger students viewed as developmentally constrained by learning capability and cognitive development were considered unable to work algebraically. In particular, it was argued that young students were not ready to represent or operate on unknowns (Carraher et al., 2006). Similarly, research in this field during the 1980s had a strong emphasis on student errors and limitations to learning were often attributed to developmental constraints (Kaput, 2008; MacGregor & Stacey, 1997; Schliemann, Carraher, & Brizuela, 2007a). The dominant treatment of algebra in this context was as a narrow syntactical area of mathematics focused on abstract symbol manipulation (Chazan, 1996; Kaput, 2008; Stacey & Chick, 2004).

In recent years the view that student difficulties with algebra are related to cognitive development has been challenged. Links have been made with the specific difficulties attributed to the traditional approach of teaching computational arithmetic followed by abstract algebra when facilitating the development of algebraic understanding (Bills et al., 2005; Kaput, 2008; Smith & Thompson, 2008, Schliemann et al., 2007a). For example, Schliemann et al. (2007a) argue that children's difficulties stem largely from three key aspects: reliance on limited problem sets in early arithmetic, restricted use of notation to register computation rather than describing the known attributes of a problem, and a focus on computing sets of values rather than the relations across sets.

Advocates of mathematics curriculum reform (e.g., Blanton & Kaput, 2005a; Carpenter, Levi, Berman, & Pligge, 2005a; Warren & Cooper, 2001) have called for a move to re-characterise the nature of algebra and algebraic reasoning. They propose a wider definition of algebraic reasoning that includes both content and process and shares dual aspects with mathematics including a focus on generalising, expressing generalisations, and using symbols to reason with generalisations (Kaput, Blanton, & Moreno, 2008a; Mason, 2008; Watanabe, 2008). The definition of algebraic reasoning that is used in this study draws on the work of many researchers (e.g., Blanton & Kaput, 2005a; Ferrucci, Kaur, Carter, & Yeap, 2008; Smith & Thompson, 2008; Warren & Cooper, 2008). Algebraic reasoning is defined as thinking that is focused on identifying patterns and underlying mathematical relationships, establishing generalisations through the discourse of argumentation, and expressing these relationships and generalisations in age appropriate language and notational systems in ways that allow the relationship to become apparent. Within this definition, algebraic reasoning takes varying forms including:

> (a) the use of arithmetic as a domain for expressing and formalising generalisations (generalised arithmetic); (b) generalising numerical patterns to describe functional relationships (functional thinking); (c) modelling as a domain for expressing and formalising generalisations; and (d) generalising about mathematical systems abstracted from computations and relations. (Blanton & Kaput, 2005a, p. 413)

Shifting from arithmetic to algebraic reasoning is an important transition in mathematical reasoning. However, it is often a major hurdle for many students (Chazan, 1996; Stacey & Chick, 2004). A growing body of international research studies (e.g., Bastable & Schifter, 2008; Blanton & Kaput, 2005a; Carpenter et al., 2003; Carraher et al., 2006; Schifter, Bastable, Russell, Seyferth, & Riddle, 2008a) have investigated how early algebraic reasoning is fostered. Key findings of these studies include the importance of the content areas within the existing curriculum with which early algebra has connections, a focus on student thinking and reasoning including misconceptions, and the use of tasks and tools to promote algebraic reasoning.

The following sections address important concepts in students' learning of early algebra and provide a review of research studies in this area.

2.3 EARLY ALGEBRAIC REASONING AS GENERALISED ARITHMETIC

Development of algebraic reasoning occurs over a long period of time and has inter-related connections with other strands of mathematics, particularly arithmetic (Blanton & Kaput, 2005a). However, algebraic reasoning is differentiated from arithmetic which proceeds from the known to the unknown because algebraic reasoning begins by acknowledging the unknown and involves reasoning about unknown quantities or variables (Mason, 2008; van Ameron, 2003). Another difference is the use of letters which in arithmetic are often used as abbreviations while in algebra they are used to represent unknown numbers and variables (van Ameron, 2003). In order for students to think algebraically there is a need to develop a structural perspective focused on operations on mathematical objects rather than a procedural approach focused on operations on numbers and producing an outcome (Carpenter, Levi, Franke, & Zeringue, 2005b; Ferrucci et al., 2008; van Ameron, 2003).

Supporting primary students to develop ways of thinking about arithmetic which also support the development of algebraic reasoning has a dual outcome in both enhancing arithmetical and algebraic reasoning. In a classroom setting where there is a focus on algebraic reasoning students are supported to make sense of arithmetic rather than performing arithmetic instrumentally. This sense-making provides a bridge for developing conceptual understanding of algebra in later years (Blanton, 2008; Blanton & Kaput, 2005a; Carpenter et al., 2003; Lins & Kaput, 2004; Mason, 2008). For example, a New Zealand study by Irwin and Britt (2005) with students aged 11-years to 15-years old found that students who were encouraged to use a "flexible array of skills for manipulating arithmetical relations in ways that exhibit number sense as well as operational sense" (p. 182) developed foundations for their understanding of secondary school algebra.

Examination of the research literature reveals a range of areas within generalised arithmetic which are important in developing early algebraic reasoning. The following section highlights key findings of research studies related to the students' learning of algebraic reasoning within the following areas:

- Equivalence, relational reasoning, and solving, using and representing equations.
- Understanding the properties of operations: The commutative, associative, and distributive properties

- Exploring ideas about operations and their relationships.
- Understanding the properties and relationships of numbers: Odd and even numbers, zero and one.

2.3.1 Equivalence, relational reasoning, and solving, using and representing equations

Developing understanding of equality is a concept fundamental to algebraic reasoning (Jones, 2008; Knuth et al., 2006; Linsell & Tozer, 2010). The seminal research of Kieran (1981) illustrated that students often have an inadequate understanding of the equals sign. Recent studies (e.g., Carpenter et al., 2003; Jacobs, Franke, Carpenter, Levi, & Battey, 2007; Knuth et al., 2006; McNeil & Alibabi, 2005; Warren, 2003a) continue to provide evidence that many primary and middle school students lack deep understanding of the equals sign and that student understanding of the equals sign does not necessarily improve as students advance in year levels. Students with an inadequate understanding view the equals sign as an indicator of an operator rather than a symbol of a mathematically equivalent operation. This operational view equates the equals sign with a need to find a 'sum' or 'answer' or a left to right action of adding all the numbers to the left of the equals sign (Carpenter et al., 2005b; Knuth et al., 2006; McNeil & Alibabi, 2005).

Researchers have analysed errors made by students when solving open number equivalence problems in terms of the students' understanding of the equals sign. Freiman and Lee (2004) demonstrated that open number sentence problems in the form of $a + b = d + c$ involving a blank in the last two positions consistently caused difficulties across year levels. Carpenter et al. (2003) argue that students' errors in solving open number sentence problems are errors of syntax. Students erroneously interpret the rules for how the equals sign is utilised. For example, when solving 9 + 6 = __ + 5, students may put 15 in the blank space considering that the equals sign is an indication to put an answer. Alternatively other students who over-generalise the property of addition and assume the sequence of symbols in the number sentence is unimportant may put 20 in the blank space.

A limited understanding of the equals sign impedes students' mathematical development. It leads to a high computational burden with links to latter difficulties in solving symbolic expressions and equations (Jones, 2008; Kieran, 1981; Knuth et el., 2006). In contrast, understanding of the equals sign supports students to generate and interpret equations meaningfully and to operate on them (Jones, 2008; Linsell & Tozer, 2010). For example, younger students with sound understanding of the equals sign will be able to use either computational or relational forms of thinking to solve open number sentence equivalence problems such as 8 + 4 = __ + 5 successfully (Falkner, Levi, & Carpenter, 1999).

Relational reasoning has been identified as a foundation for developing algebraic reasoning (Carpenter et al., 2005b; Irwin & Britt, 2005; Stephens & Xu, 2009). Carpenter et al. (2005b) characterise this as the ability to examine expressions and equations in their entirety rather than as a set of procedural steps to be carried out. Using relational reasoning to solve problems requires that one is "able to see and use possibilities of variation between numbers in a number sentence" (Stephens, 2006, p. 479). Students who are able to use relational thinking to solve open number sentence problems invoke the fundamental properties of numbers and operations to consider the structure of the number sentence. They are able to solve the problem by using the relation between both expressions and by identifying the direction in which the missing number will change in order to maintain equivalence. In contrast, students who use computational thinking view the numbers on each side of the equals sign as representing separate calculations and depend on calculation to solve open number sentence problems (Carpenter et al., 2003; Molina, Castro, & Mason, 2008; Stephens, 2006; Stephens & Xu, 2009).

Facilitating understanding of equivalence and relational reasoning requires that students are provided with learning situations that enrich and expand their understanding of the equals sign and equations (Molina, Castro, & Castro, 2009). Findings of an interview based study with Brazilian students aged between 7-years to 11-years old carried out by Schliemann, Lessa, Lima, and Siqueira (2007b) indicated that students were able to understand the principle that an equation remained true if the same number was added or subtracted to each side. However, they noted that their results were dependent on the way the problem was presented. Similarly, a United States of America (USA) based study by NcNeil et al. (2006) investigated how the context within text-books affected student interpretation of the equals sign with students aged 11-years to 14-years old. These researchers found that when students were presented with non-standard representations more relational explanations were provided. Therefore they concluded that it was beneficial for students to see the equals sign in contexts with operations on both sides.

Findings from classroom research studies (e.g., Blanton & Kaput, 2005a; Warren & Cooper, 2003) suggest how a deeper understanding of the equals sign may be achieved. One approach advocated is the use of a balance scale model. This approach emphasises the need to consider the equation in its entirety and does not indicate a particular direction. In using the balance scale model with students aged 7-years to 8-years old, Warren and Cooper (2003) first began by developing links to language which represented equality (same) and inequality (different). Following this, the students applied the terms to kitchen objects before the formal equality and inequality symbols were introduced. Students were also encouraged to make links between physical quantities and number operations and then represent these using a balance scale. Finally, the students categorised the quantities as equal or not equal and recorded a representative number sentence (e.g., $4 + 3 = 6 + 1$). The researchers concluded that key elements in the success of the pedagogical approach were the development of a language framework centred upon the meaning of equivalent and non-equivalent situations and the utilisation of a physical model of balance. Blanton and Kaput (2005a) also describe a teacher working with students aged 8-years to 9-years old successfully using a balance scale to explore the notion of equivalence. However, Schliemann et al. (2007b) maintain that while balance scale models provide a meaningful context for students to develop initial understandings of equations, there are some challenges with this in providing a complete model. These researchers highlight the difficulties in representing subtraction as a relation rather than as an action on a scale and the difficulty in representing the concepts of multiplication, division, and variables on a balance scale model. Therefore it is also important to link equivalence with a range of alternative models such as comparisons between numbers of counters, verbal problems, and written equations.

An alternative approach to supporting student understanding of the equals sign is the use of true and false and open number sentences (e.g., Carpenter et al., 2005; Falkner et al., 1999; Molina et al., 2008). Falkner et al. (1999) describe an 18-month intervention with a class of students aged 6-years to 8-years old in the USA. In the initial assessment task most students gave incorrect responses to open number sentences indicative of understanding the equals sign as an operator or syntactic indicator. The teacher facilitated discussion of the equals sign by presenting the students with true and false number sentences, some of which were directly designed to confront misconceptions for example 8 + 2 = 10 + 4. Representational material in the form of unifix cubes was also used to link the number sentences with a physical representation. In the following year, assessment for their understanding of the equals sign indicated that many of the students were able to solve the number sentence problems correctly. Similarly, in a year-long intervention with students aged 11-years to 12-years old Carpenter et al. (2005a) found that relational understanding of the equals sign was developed by 84% of the students following classroom work with true and false number sentences.

Open number sentences involving a single value can be used to introduce students to finding unknown numbers; however, it is also important to extend beyond single-value open number sentences. Fujii and Stephens (2008) maintain that an emphasis on missing number sentences with a single unknown (invariable numbers) can create difficulties later when students are asked to solve problems involving variable quantities. Recent research studies (e.g., Blanton & Kaput, 2005a; Carpenter et al., 2003; Fujii & Stephens, 2008) involving open number sentence activities include situations with equations with several variables and single equations involving multiple repeated variables. Carpenter et al. (2003) suggest using word problems as a context to introduce multiple variables. In an example they provide from a classroom with students aged 7-years to 8-years old, a word problem was used which had the mathematical goal of showing all the possible ways that seven could be split into two groups of positive whole numbers. This supported students to think about generating combinations in an organised way, justifying they have all the solutions, and expressing the situation with notation. Similarly, these researchers and others (e.g., Blanton & Kaput, 2005a; Carpenter et al., 2003; Fujii & Stephens, 2008) have illustrated how number sentences involving two variables (e.g., $B + H = 9$) or repeated variables (e.g., $A + A + A = 21$) can be introduced and used with students to develop their understanding of solving equations involving varying or similar quantities. This provides opportunities for deepening relational reasoning.

2.3.2 Understanding the properties of operations: The commutative, associative, and distributive properties

The properties of operations, in particular, the commutative, associative, and distributive properties provide a critical foundation for school mathematics and for students to work with equations (Ding & Li, 2010). As students develop their own solution strategies for solving arithmetic problems they implicitly draw on the fundamental properties of arithmetic within their solutions. However, in many classrooms these properties remain implicit—they are not explicitly identified or examined (Carpenter et al., 2005a). In contrast, explicit exploration of number properties provides opportunities for students to engage in the practices of formulating, testing, and justifying generalisations (Ding & Li, 2010; Schifter et al., 2008a).

Commutative property

From a young age, many students will implicitly use the commutative property to support them to solve problems involving addition and multiplication. For example, Fosnot and Jacob (2010) describe an episode from a classroom with students aged 7-years to 8-years old in the USA where students were asked to solve an equivalence problem: $13 + 8 = 5 + 9 + 13 - 6$. One student's response was to change the problem to begin with the nine and take the six away the leaving him with $13 + 8 = 8 + 13$ which he justified through use of the commutative property stating, "I know that is equals because the numbers can be turned around" (p. 88). Similarly, in a New Zealand based study with students aged 8-years to 10-years old, Hunter (2010) describes how students were able to recognise quickly that number sentences such as $15 + 3 = 3 + 15$ were true through use of the commutative property.

Although students may use the commutative property to solve mathematics problems, results from research studies (e.g., Anthony & Walshaw, 2002; Warren, 2001a; 2001b) indicate that many students lack deep understanding of this operational law. One difficulty which students may experience is the assumption that the commutative and associative properties are interchangeable (Schifter, Monk, Russell, & Bastable, 2008b; Tent, 2006). This confusion arises as often the commutative and associative properties are used at the same time when solving equations. Another difficulty identified by Schifter et al. (2008b) is attributed to the form of written equations and students' lack of understanding of the equals sign. These researchers provide a vignette from a classroom episode where the students recognised the commutative property of addition however when the teacher asked whether 13 + 23 = 23 + 13 was true, many of the students disagreed stating that there was no answer. Schifter et al. (2008b) attribute this response to the students' interpretation of the equals sign.

Another common obstacle identified in research studies is the over-generalisation of the commutative property across different operations. Three studies, Anthony and Walshaw's (2002) study with students aged 7-years to 8-years old and older students aged 12-years to 13-years old and two Australian based studies by Warren (2001a; 2001b) involving students aged 8-years to 9-years old and 12-years to 14-years old demonstrated that while students recognised the commutative nature of addition and multiplication; many also thought that subtraction and division were also commutative. Schifter et al. (2008b) also report on an episode from a USA classroom study with students aged 8-years to 9-years old. The teacher engaged the students in a discussion of the commutative nature of addition which they termed the 'switch-around rule'. Following the discussion, when the students were asked to write reflections and give examples of when the rule applied it was evident many of the students over-generalised the commutative principle to include subtraction and division. These studies reflect that while students may superficially appear to understand the commutative property, deep generalised understanding is more difficult to achieve. This is particularly evident in the findings of Anthony and Walshaw

(2002) who asked students to use cubes to illustrate their conjectures about the commutative property. None offered generalised statements and few students were able to use materials to develop models of their conjectures.

Understanding of when the commutative property is applicable can be deepened through exploration involving making conjectures, justification, and generalisation. Opportunities for exploration can be utilised through carefully listening to student explanations and highlighting their explicit remarks about the regularities in the number system (Schifter, Russell, & Bastable, 2009). In an example provided by Bastable and Schifter (2008) from a classroom with students aged 8-years to 9-years old, students were asked to calculate a range of different amounts of money and share their solution strategies. From this task and the associated solution strategies, the question of whether the order of addends mattered arose and the teacher noted that a number of students were unsure whether if you added the amounts in a different order you would maintain the same sum. Using these observations, the teacher then planned further tasks which engaged the students in adding different amounts and discussing their solution strategies and observations of the commutative property. In reflective statements which the students wrote following the lesson it was evident that the additional activities supported the students to generalise the commutative property of addition.

Tasks may be specifically structured or sequenced to draw student attention to regularities such as the commutative property (Carpenter et al., 2005b; Schifter et al., 2009). For example, presenting students with a number sentence such as 47 + 56 = 56 + 47 commonly leads to explanations that the sentence is true because only the numbers have been swapped around. This can then be developed further into a discussion of whether this is true for all numbers and for different operations. Hunter's (2010) study with students aged 8-years to 11-years old used true and false number sentences to focus student attention on the correct application of the commutative property. The introduction of true and false number sentences involving subtraction and division caused conflict for many students and resulted in the development of clearer explanations of the non-commutative nature of subtraction and division.

Use of representational material to investigate conjectures and establish generalisations can also support students to develop further understanding of the commutative property (Schifter et al., 2009). Carpenter and Levi (2000) describe an instance from a lesson with students aged 8-years to 10-years old where they were asked to justify the generalisation of the commutative property of multiplication. Students initially began by calculating a lot of examples; however, following teacher questioning, a pair of students used linking cubes to illustrate a specific example. After some discussion, the students were then able to use the model to provide concrete justification and demonstrate how when rotated the product would remain the same. Similarly, Hunter (2010) provides examples of how students aged 8-years to 11-years old developed concrete forms of justification of the commutative property of multiplication through building arrays with counters. In this study the development of understandings of the commutative property of addition and multiplication provided a foundation for students to develop explanations of why the commutative property did not work for subtraction and division. However, it should be noted that a number of students continued to over-generalise the commutative property following the classroom work indicating the need for students to be provided with multiple opportunities to explore the properties of operations over an extended period of time.

Associative property

Developing understanding of the associative property and how this may be applied to multiplication supports students to work flexibly with the number system. However, Schifter et al. (2008b) argue that there are less frequent opportunities to explore this property than the commutative property in primary classrooms as students are often asked to only solve multiplication problems involving two factors rather than three or more factors. This is supported by an Australian study by Warren (2003b) with students aged 12-years to 14-years old which reported that many of the students found the associative property more difficult to recognise than the commutative property. While the majority of students correctly recognised the commutative nature of addition and multiplication, fewer correctly identified addition and multiplication number sentences involving the associative law as correct. Warren argues that this may also be due to the increased complexity of number sentences involving brackets.

Informal multiplication strategies such as doubling and halving or using base-ten strategies draw on the associative property (Schifter et al., 2008b). A study by Baek (2008) used problem-solving interviews to examine the varying multiplication strategies used by students aged 8-years to 11-years old. The results from the initial stage of the study demonstrated that students maintained informal understandings of the distributive and associative properties and were able to use these to construct efficient strategies for multi-digit multiplication. In an example provided from the study, a student aged 9-years old was asked to solve a word problem which involved finding the product of 30 multiplied by 40. The student directly modelled the problem using base ten blocks and was quickly able to apply the associative property to simplify the problem to 400 multiplied by three and solve it. This informal strategy use provides opportunities to deepen student understanding of the associative property through pressing students to explore factors and use representations of multiplication to justify their reasoning.

As noted in the previous example, facilitating classroom experiences where students investigate factors and varying representations of multiplication can support examination of the associative property of multiplication. Schifter et al. (2008b) provide classroom based examples from a class of students aged 8-years to 9-years old where a student generated investigation explored whether factors of a factor were also factors of the original number. A pair of students used an array model to demonstrate that eight was a factor of 120 and then by re-distributing the columns of eight into four, two and one, they developed a concrete form of proof to show that these were also factors of 120. Another classroom based example from these researchers demonstrated how students engaged in an investigation of the factors of different hundreds numbers (e.g., 100, 200, 300, 400...) and developed varying justification strategies for their observations of patterns. For example, one group of students proved that the factors of 200 were double the factors of 100 through using a rectangular area model. These examples show students using representations of multiplication to investigate the relationships within multiplication. Schifter et al. argue that these types of activities support students to attach meaning to the formal algebraic expression of the associative property when it is introduced in later years.

Distributive property

Developing understanding of the distributive property is fundamental in building both conceptual understanding of multiplication and for algebraic reasoning. Common errors in adding variables (e.g., $6f + 7k = 13fk$) occur for students who have not developed a deep understanding of the distributive property (Carpenter et al., 2003; Li & Ding, 2010). While students may implicitly draw on the distributive property within their solution strategies, it is often challenging for students to explicitly generalise and formalise their reasoning (Chick, 2009; Schifter et al., 2008b). An Australian based study by Chick (2009) examining secondary teachers' pedagogical content knowledge for teaching the distributive property found that a common approach was to use multiple numerical examples and link these to mental computation in order to facilitate students to generalise the distributive property. However, Quinlan (1994) maintains that students have difficulty in forming generalisations about the distributive property when taught with only arithmetical examples. Her study with Australian students aged 12-years to 13-years old compared student learning of the distributive property when taught either using arithmetic examples leading to generalisation or through an approach which drew on a concrete model. She found that those students whose lessons involved the concrete model performed better on items involving the distributive property in the post-test.

Deep understanding and application of the distributive property requires that students develop understanding of multiple models of multiplication. Results from an Australian study by Norton and Irvin (2007) involving students aged 14-years to 15-years old found that many of the students were unable to utilise understanding of the distributive property to successfully expand an expression with a variable in front. These researchers argue that this was due to a limited conception of multiplication as repeated addition. They maintain that these results reflect the need to facilitate students' understanding of arithmetic structures in early number learning experiences to develop algebraic reasoning.

Classroom based studies (e.g., Baek, 2008; Schifter et al., 2008b) with primary age students have investigated how tasks involving carefully structured multiplication problems can promote students to partition factors therefore drawing on the distributive property. The teachers involved in the study by Baek (2008) introduced word problems involving several related multiplication problems which the students were able to solve through using the relationships of the numbers in each problem. This facilitated the students to investigate and develop strategies based on the distributive property.

Using representations is one way of facilitating students to examine their use of the distributive property in informal solution strategies (Baek, 2008; Schifter et al., 2008b; Quinlan, 1994). Visual representations in the form of concrete models focus student attention on how the factors are split and regrouped through use of the distributive property (Ding & Li, 2010). For example, when finding the product of multiplication equations involving two digit numbers an array or grid model (see Figure 1) can be used to support students to find the solution strategy and as a form of justification.

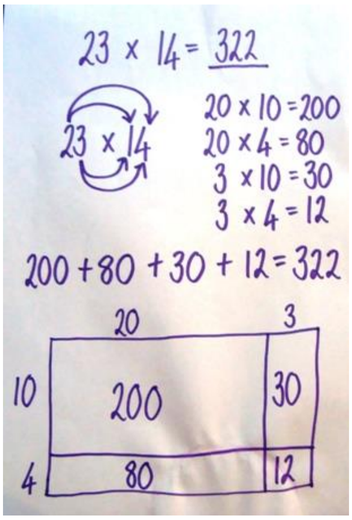

Figure 1. Representation of two-digit multiplication

Representations to support understanding of the distributive property may also include number sentences. In Baek's (2008) study, after observing students aged 8-years to 9-years old using solution strategies which involved partitioning, one of the teachers began recording their strategies in number sentences which explicitly represented the distributive and associative properties which they were drawing upon. These formal representations were later appropriated by the students and used to represent their partitioning strategies in such a way that the use of distributive property was highlighted. Teacher introduced representations of equivalent equations was also a feature in the classroom study of Schifter et al. (2008b). For example, after a student verbalised her solution strategy for 12×6 in which she grouped the twelve into 3 pairs the teacher recorded $12 \times 6 = (12 \times 2) + (12 \times 2) + (12 \times 2)$ and asked the students to analyse and justify whether this was a true statement. This provided the students with a clear method of representing their equivalent statements which re-grouped the factors from the initial equation through use of the distributive property.

Use of carefully structured true and false and open number sentences has also been found to support students to examine and generalise the distributive property. Baek (2008) maintains that the use of number sentences facilitates students to focus on the properties within the number sentences and shifts attention away from procedures for computation or the answer. In his study, when presented with true and false number sentences the students were able to recognise the equivalence of number sentences such as $4 \text{ x } 8 = (2 \text{ x } 8) + (2 \text{ x } 8)$ and justify the distributivity through use of partial products. Similarly, Carpenter et al. (2005b) describe a USA based interview study with students aged 8-years to 9-years old where the teacher utilised a series of carefully structured true and false number sentences to scaffold student understanding of the distributive property as related to multiplication facts. The initial examples used with students exemplified a model of multiplication as repeated addition, however further examples drew implicitly on the distributive property and explored different ways of representing multiplication facts. This facilitated the students to notice relations between the number facts and draw on these to solve unknown facts. Student provided explanations drew on relational reasoning and indicated an understanding of multiplication which drew on the distributive property.

2.3.4 Exploring ideas about operations and their relationships

Developing understanding of inverse relationships between operations supports students to develop flexible computation strategies. Opportunities to focus specifically on inverse relationships arise when students learn their basic number facts. Understanding the relationship between operations can support students to develop robust number fact knowledge more easily (Carpenter et al., 2003; Vance, 1998). Vance (1998) uses the example of $3 + 2 = 5$ and $5 - 2 = 3$ in which subtracting two is “a way to undo the result of adding two” (p. 284). Engaging students in investigating such relations can equip them to understand how expressions can be manipulated and equations solved in later years.

The use of word problems provides opportunities to provoke examination of the inverse relationship between operations. Often when students are asked to use an equation to represent and solve a word problem, their solutions utilise the inverse relationship. Facilitating the students to share their varying solution strategies during large group or whole class discussions can provide a context for discussing the relationships between operations (Bastable & Schifter, 2008; Carpenter et al., 2003). Specific teacher actions can also focus student attention on exploring inverse relationships. For example, Carpenter et al. (2003) provide a vignette from a classroom with students aged 8-years to 9-years old where the teacher noticed students solving a word problem using both subtraction and addition. She specifically structured the students to make a conjecture about the relationship between addition and subtraction and then provided them with a framework of ‘if...then’ to show that the conjecture would always work.

In addition to exploring the inverse relationships of operations, students should also be given opportunities to investigate the relationship between addition and multiplication and division and subtraction (Vance, 1998). Similar to investigating inverse relationships, this supports students in learning number facts as well as deepening their understanding of operations. Students will intuitively use their knowledge of addition to solve multiplication facts, for instance using knowledge of doubles to solve facts which involve multiplying by two. Carpenter et al. (2005b) provide an example from an interview study with a student aged 9-years old who was asked whether $3 \times 7 = 7 + 7 + 7$ was a true number sentence. The student response drew on her understanding of the relationship between addition and multiplication. Such tasks afford opportunities for students to deepen their understanding of the relationships between operations.

2.3.5 Understanding the properties of numbers: Odd and even numbers, zero and one

Odd and even numbers

Exploration of the structure of odd and even numbers provides a context which can support the development of early algebraic reasoning. Exploring and forming definitions of odd and even numbers and using representational material deepens student understanding of the structure of odd and even numbers and allows them to engage in learning to make and test conjectures and generalisations (Ball, 1993; Blanton & Kaput, 2005a; Carpenter et al., 2003; Hunter, 2010; Schifter et al., 2008a). Classroom research studies (e.g., Ball, 1993; Carpenter et al., 2003; Hunter, 2010) provide examples of how odd and even numbers can be used as a context for exploration. Ball (1993) describes a student developing an explanation of even numbers as having groups of two in them and odd numbers as having groups of two and one left over. In both Carpenter et al.'s (2003) study and the study by Hunter (2010) representational material along with the definitions of odd and even numbers supported the students to develop concrete justification of their conjectures.

Quasi-variables have been successfully used to support students' developing understanding of the structure of odd and even numbers. For example, Blanton and Kaput (2005a) illustrated how the introduction of large numbers as quasi-variables focused student attention on the structural features of odd and even numbers. Students were then able to draw on these structural features to connect arithmetic concepts in algebraic ways to justify their conjectures. Quasi-variables, also utilised in Hunter's (2010) study, facilitated the students to engage in mathematical argumentation which supported their developing understanding of the structure of the numbers. Some of the students in this study initially demonstrated difficulty in establishing which digit determined whether the numbers were odd or even, however, prolonged discussion and argumentation supported them to develop the understanding that the oddness or evenness of a number depended on the final digit. In subsequent lessons, the teacher was also able to press students to use their previously justified conjectures to determine whether sums would be odd or even.

Developing deep understanding of the structure of odd and even numbers also provides students with a means to justify their reasoning and conjectures about other mathematical areas. Schifter et al. (2009) describe an episode from a classroom with students aged 8-years to 9-years old where the students engaged in a discussion of factors. When the teacher questioned whether two was a factor of 156, a student began to provide reasoning based on a previously explored generalisation that the sum of two even numbers is even. She then recognised that the generalisation needed to be extended to include three even numbers and provided verbal justification for this. Another student then built on the reasoning through referring to the structure of even numbers and evoking a visual image as justification.

Properties of zero and one

The properties of zero and one are potentially rich areas for pressing students to develop and investigate conjectures and generalisations. By carefully structuring tasks, young students can be supported to make generalisations. For example, Carpenter et al. (2005a) provide an example from a case study with students aged 7-years to 8-years old. The students were initially provided with a false number sentence (e.g., 78 – 49 = 78) which provoked an in-depth discussion of the property of zero in relation to subtraction. The teacher introduced number sentences involving addition and subtraction with large numbers (e.g., 789 564 – 0 = 789 564 and 0 + 5869 = 5869) to press the students to articulate generalisations about the properties of zero in addition and subtraction. Student responses to the tasks and during the discussion demonstrated that they were able to determine whether number sentences were true through the application of generalisations.

2.4 EARLY ALGEBRAIC REASONING AS FUNCTIONAL THINKING

Functional thinking is described as "representational thinking that focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationships (individual incidences) to generalisations for that relationship across instances" (Smith, 2008, p. 143). Blanton and Kaput (2005b) maintain that definitions of functional reasoning should also include using diverse linguistic and representational tools to model and generalise both patterns and relationships. Using tasks which involve functional reasoning in primary classrooms provide further opportunities to integrate algebra into the existing curriculum and develop young learners' early algebraic reasoning (Blanton & Kaput, 2005a; Carraher et al., 2006).

Research studies (e.g., Barbosa, 2011; Lee, 1996; Schliemann et al., 2007a) highlight a range of difficulties that students may encounter when engaging with functional reasoning tasks. An initial difficulty is while students may easily notice patterns in sets of data these patterns may not always be useful or relevant or extend to other cases. Also, without specific teacher intervention many students use recursive strategies which are less sophisticated and a limited form of generalisation (Barbosa, 2011; Lannin, Barker, & Townsend, 2006).

Other difficulties are linked to representational forms such as tables of data. These are an important tool for recording functional patterns however, when examining number patterns in tabular forms students often focus on recursive relationships among the consecutive values or complete the tables of data without specifically focusing on the relations among the values (Bishop, 2000; McNab, 2006; Schliemann et al., 2007a). Other researchers (e.g., Barbosa, 2011; Beatty & Moss, 2006; Becker & Rivera, 2006) have highlighted further difficulties which are caused by students relying on numeric strategies. Findings of these studies indicate the importance of facilitating students to focus on the visual structure of the sequence or pattern to support them to move beyond pattern-spotting to generalising specific cases.

Evidence is provided in a range of research studies (e.g., Beatty & Moss, 2006; Blanton & Kaput, 2005a; Blanton & Kaput, 2005b; McNab, 2006) that young students are capable of engaging in functional thinking. Despite the difficulties outlined previously, tasks which involve working with functions provide students with opportunities to observe and describe patterns, develop rules and generalisations, interpret and test rules and formula, and develop their understanding of notation (Hunter, 2010; Lobato & Ellis, 2002; MacGregor & Stacey, 1997). Of key importance is the provision of opportunities for students to identify and develop verbal descriptions of numeric and geometric patterns and relationships which can then be used to scaffold symbolic representations of functional patterns (Bishop, 2000; Blanton & Kaput, 2005a). For younger students, providing solution strategies in everyday language gives the teacher an opportunity to act as a facilitator and ‘translate’ the students’ solution strategies into symbolic forms which represent their method (Bishop, 2000).

Using tasks which are carefully structured to facilitate students to use and connect a variety of representational forms to describe both numeric and geometric patterns support the communication, generalisation, and justification of functional rules (Beatty & Moss, 2006; Blanton & Kaput, 2005b; McNab, 2006). For example, McNab (2006) reports on a Canadian teaching intervention with students aged 7-years to 8-years old where students were facilitated to integrate different representations. For the initial tasks that included visual and spatial representations the integration of numeric data was encouraged by the placement of an ordinal position number under the pattern which was linked to the position of the pattern. Function machines with the results recorded on T-charts (see Figure 2) were then introduced to encourage exploration of numeric representations.

Number of tables	Number of people
1	5
2	8
3	11
4	14

Figure 2. T-chart

To avoid positioning students to focus on recursive patterns the use of non-sequential examples were used to ensure that student attention was drawn to the horizontal relationship or functional rule. The final tasks supported students to integrate all the aspects of the previous activities by developing their own geometric patterns and related secret rules. Assessment following the intervention indicated that many of the students were able to recognise functional rules and express these in multiple forms. Similarly, results from another Canadian study by Beatty and Moss (2006) with students aged 10-years to 11-years old found that students who integrated the use of representational context and numeric strategies to solve problems had more robust understanding of functions and were able to fluently use representations, including symbolic notation, for problem solving.

2.5 TOOLS AND REPRESENTATIONS TO SUPPORT EARLY ALGEBRAIC REASONING

Tools and representations play an important role in developing early algebraic reasoning. Both informal and formal representational forms can be used to support the development of early algebraic reasoning (Blanton & Kaput, 2005a; Carraher, Schliemann, & Schwartz, 2008). These may include drawings, diagrams, tables of data, and notation which support students to organise their reasoning as well as develop and express mathematical arguments and justification (Blanton & Kaput, 2005a). Often teachers need initially to encourage and scaffold student use of tools and representational forms, however, research studies (e.g., Blanton & Kaput, 2005a; Carraher et al., 2008) provide evidence that following teacher led introduction of representations, students can adopt and use tools for their own purposes without teacher prompting.

2.5.1 T-charts and tables of data

Introducing students to T-charts and tables of data can support them to organise the data within a problem situation. Moreover, they can facilitate the recognition of patterns and relationships in the data (Blanton & Kaput, 2005a; 2005b; Schliemann et al., 2007a). In their work with functional relationships, Blanton and Kaput (2005b) found that introduction of T-charts built an early representational infrastructure that supported algebraic reasoning. Tables of data may also be used to facilitate students to generalise patterns and interrelations among sets. For example, Carraher et al. (2008) used a table of data to scaffold students to recognise valid answers within two interrelated sets and to highlight the invariant features of the valid answers. Similarly, Blanton and Kaput (2005b) outline a classroom episode where a student developed a reasoned argument using a t-chart to refute the symbolic relationship her classmate had proposed.

2.5.2 Drawings and diagrams

One means of developing younger students' use of representational forms is to encourage the use of drawings to represent problem situations. Facilitating students' representation of their thinking and reasoning through drawings supports the students to develop informal representations which can later be bridged with more formal conventional representational forms (Carraher et al., 2008). It can also provide a means for teachers to investigate student thinking and understanding of problem situations. For example, Carraher et al. (2008) report on a two year teaching intervention with students aged 8-years to 10-years old where students were consistently asked to use drawings and explanations to demonstrate their understanding of mathematical problems. Towards the beginning of the intervention, students were asked to represent what they knew about a story problem involving unknown quantities through drawing. This provided the teacher with insight into student thinking which revealed two distinct foci: assigning the unknown quantity a specific value, or leaving the amount as indeterminate. The teacher was then able to use this to guide the lesson and as a means of introducing new notational forms. Later in the teaching intervention, the researchers demonstrate how students had begun to use drawings in a different role to convey general, algebraic representations of problem situations.

2.5.3 Algebraic notation and variables

One of the goals of early algebra is to support students to develop understanding of algebraic symbolisation. The notational system of algebra is a powerful tool which can allow students to work at a higher level of generality. However, without deep understanding many learners engage in meaningless symbol manipulation (Bastable & Schifter, 2008; Kaput et al., 2008a). Research studies (e.g., MacGregor & Stacey, 1997; Weinburg et al., 2004) illustrate that many young students have limited classroom experiences in exploring notation. MacGregor and Stacey's (1997) large-scale study using a pen and paper test with students aged 12-years to 16-years old found that the reason students often viewed letters as abbreviated words was due to inappropriate teaching methods where letters were used to represent objects. The lack of classroom opportunities to use and explore notation also led students to base their interpretation of notation on intuition, guessing, and false analogies. Another study by Weinburg et al. (2004) used a written assessment and semi-structured interviews to investigate students aged 12-years to 14-years old interpretation of algebraic notation. Their findings highlighted two common misconceptions among students, the notion that a single letter variable can only stand for a single number and that variables represented by different letters could not be the same number.

In primary classrooms focused on developing early algebraic reasoning students often begin expressing conjectures and generalisations with natural language. However, at times it can be difficult to state conjectures verbally in a precise, unambiguous way (Bastable & Schifter, 2008; Carpenter et al., 2003). Introducing algebraic notation to younger students provides them with opportunities both to explore notation and develop their understanding of the concept of variables as well as learn to express conjectures more precisely (Carpenter et al., 2003; Carraher et al., 2006; Schliemann et al., 2007). However, researchers note that it is important that formal notation is introduced gradually and students are provided with ample opportunity to develop, extend, and adjust their understanding (Carraher et al., 2008). Results of an interview based study carried out with students aged 8-years to 9-years old by Schliemann et al. (2007a) found that the students could develop consistent notations such as circles or shapes to "represent elements and relationships in problems involving known and unknown quantities" (p. 59). Provision of such classroom experiences where students develop their own symbolic representations can provide students with a scaffold for the introduction of more formal representations.

Research studies involving classroom interventions provide further evidence of young students' capability to use notation as a tool to understand and express arithmetical and functional relationships. Findings of research studies (e.g., Carpenter et al., 2003; Carpenter et al., 2005a) involving work with true and false and open number sentences indicate that these can provide students with an accessible notation to express generalisations with variables. Carpenter et al. (2005a) provide an example from a year-long classroom intervention with students aged 6-years to 8-years old. During one lesson students were asked to write an open number sentence that was true for any number. Following prolonged whole class discussion and teacher scaffolding, students were able to formulate generalisations with open number sentences such as $0 + m = m$. At the end of the year-long intervention the majority of students could use symbols to represent generalisations. These researchers replicated this work across the primary grades and in two other case studies they found that 80% of students aged 9-years to 12-years old were able to use variables to express generalisations.

Classroom work involving functional relationships provides students with opportunities to construct notations and to extend their understanding of variables. Students can also be facilitated to make sense of notation through connecting imagery and concrete experiences with symbolic notation (Blanton & Kaput, 2005b). For example, Hunter and Anthony (2008) report on a New Zealand based study with students aged 8-years to 11-years old where a pair of student specifically linked their symbolically represented notation to the geometric model to convince the other group members of their argument. The representation of functional rules also provides opportunities to deepen student understanding of the conventions of algebraic notation.

2.6 SUMMARY

This review provides research evidence to challenge the view that younger children constrained by cognitive ability cannot engage in algebraic reasoning. It highlights the difficulties caused by the artificial divide between arithmetic and algebra and gives a wider definition of early algebraic reasoning to include both content and process.

In order to make the opportunities to develop algebraic reasoning in primary classrooms explicit, a close examination of the key content areas which may be utilised to develop early algebraic reasoning was undertaken. The review recognises the complexity of developing rich algebraic understanding and offers examples of international classroom based case studies which highlight appropriate tasks and activities for engaging primary students in algebra. These studies provide exemplars of how the foundations may be developed for students to be prepared to learn formal algebra in later years. However, as outlined in Chapter One, there does appear to be a scarcity of primary classroom research studies situated within the UK which explicitly examine how early algebraic reasoning can be developed. The current study is designed to address this gap by investigating how teachers can support students within the primary schooling system in the UK and British Isles to develop rich forms of algebraic reasoning.

The development of algebraic reasoning goes beyond content areas and also requires specifically designed learning environments. Chapter Three will examine the aspects of the learning environment that support algebraic reasoning. The importance of task design and implementation will be explained. An analysis will be undertaken of the key classroom practices which support development of algebraic reasoning and the mathematical practices which are linked to this.

CHAPTER THREE

DEVELOPING LEARNING ENVIRONMENTS WHICH SUPPORT EARLY ALGEBRAIC REASONING

3.1 INTRODUCTION

In the previous chapter attention was drawn to the key mathematical content areas which offer opportunities for primary students to engage in algebraic reasoning. Critical aspects of early algebra reform include the 'what' and the 'how' teachers teach. This chapter examines those aspects of the learning environment that support engagement in early algebraic reasoning.

Section 3.2 discusses the importance of task design and implementation in developing early algebraic reasoning, focusing on those factors within task design and implementation which potentially offer greater affordances for algebraic activity.

Section 3.3 reviews research literature that describes the key classroom practices that support engagement with algebraic reasoning. It begins by identifying the importance of re-visiting algebraic concepts on multiple occasions and highlights how this facilitates students both to explore concepts deeply and reason in more complex ways. It then argues that specific types of discourse focused on using rich, conceptually embedded mathematical talk can support student construction of early algebraic reasoning. An overview is provided of how pedagogical actions can be used during small group work and whole class discussions to promote productive mathematical discourse. Illustrating how instruction focused on student thinking and reasoning can support early algebraic reasoning, the section highlights the changing ways that teachers must prepare and undertake their lessons. This section concludes with a discussion of the important role of the classroom practice of questioning for the development of the classroom environment. The argument is developed that through teacher modelling and scaffolding, students can themselves generate the types of questions which support engagement with algebraic reasoning

Section 3.4 outlines the key mathematical practices which are linked to the development of algebraic reasoning. It highlights the importance of supporting students to develop their use of mathematical language and construct reasoned mathematical explanations which are responsive to the listening audience. Research literature is drawn upon which illustrates how teachers can exploit opportunities in the classroom to engage students in the mathematical practices of justification and proof. Finally, the important role of the mathematical practice of making conjectures and developing these into generalisations is examined.

3.2 TASK DESIGN

The use of tasks is an important factor in developing early algebraic reasoning. As outlined in Chapter Two, there are a range of research studies which illustrate how early algebra can be taught through tasks involving generalised arithmetic and functional reasoning. However, research studies also highlight some general factors within task design which potentially provide greater affordances for algebraic activity. These factors will be outlined in the following section.
Facilitating algebraic reasoning can be accomplished through variation of the task parameters. Here the openness of the task, achieved by extending the number of answers from a closed single answer to multiple solutions, shifts the purpose of the task from computation to examining patterns or relationships (Blanton & Kaput, 2005b; Kaput & Blanton, 2005; Smith &

Thompson, 2008; Soares, Blanton & Kaput, 2005). For example, Blanton and Kaput (2005b) describe how the task parameters could be extended to develop what they term an 'algebrafied' problem by changing the single solution problem: *How many telephone calls could be made among 5 friends if each person spoke with each friend exactly once on the telephone?* to a sequential set of problems: *How many telephone calls would there be if there were 6 friends? Seven friends? Eight friends? Twenty friends? One hundred friends? Organize your data in a table. Describe any relationship you see between the number of phone calls and the number of friends in the group. How many phone calls would there be for n friends?* This set of problems provides opportunities for students to examine patterns and relationships. It also facilitates engagement with the mathematical practices of developing conjectures, justifying, using representations, and generalising.

Similarly, designing tasks which involve series of number sentences offer opportunities to engage students in algebraic reasoning (Carpenter et al., 2005b; Kaput & Blanton, 2005). Examples from research studies (e.g., Baek, 2008; Carpenter et al., 2005a; Carpenter et al., 2005b; Hunter, 2010) provided in Chapter Two showed how students could develop conjectures and generalisations from carefully chosen sequences of number sentences which illustrated patterns. When students were encouraged to examine sequences of unexecuted sums without computing answers, they began to engage in the types of structural analysis which supported early algebraic reasoning (Kaput & Blanton, 2005; Smith & Thompson, 2008).

Another method of facilitating students to attend to structure and generality is through tasks that require students to generate more problems of the same type (Mason, 2008; Soares et al., 2005). For example, students could be asked to offer solutions for the following question: *Nineteen divided by two equals nine with a remainder of one, what other numbers share this property?* In this way an arithmetic task can be developed into a task focused on generalisation. Also, asking students to generate more problems of a similar type can be used as a formative assessment task. For example, Carpenter et al. (2003) advocate asking students to generate their own true and false number sentences to provide an indication of their thinking about equality, relations, and other important mathematical ideas.

Another factor in designing or adapting tasks to facilitate algebraic reasoning is the use of contextual support within word problems to support sense-making of abstract concepts (Ding & Li, 2010; Koedinger & Nathan, 2004; Smith & Thompson, 2008). Ding and Li's (2010) study of text-books' presentation of the distributive property found that contextual support of word problems within USA text-books was limited when compared with Chinese text-books. For example, within the Chinese text-book the structure of the problem concerning finding the cost of 102 t-shirts priced at 32 yuan allowed students to make sense of distributive property by splitting 102 into 100 and 2 and then relating this to the context of the problem with 102 viewed as 100 t-shirts and 2 t-shirts. Careful construction of word problems allows students to solve problems through use of informal strategies and to make sense of different concepts within algebra (Carraher et al., 2008; Ding & Li, 2010; Koedinger & Nathan, 2004).

Tasks which offer students the opportunity to use multiple representations can also facilitate algebraic reasoning. Research studies (e.g., Beatty & Moss, 2006; Kaput & Blanton, 2005; McNab, 2006) demonstrate how tasks that offer opportunities for multiple representations can cultivate the practice of students using different forms of representation to communicate reasoning and to justify thinking. For example, Beatty and Moss (2006) describe the use of a problem where students were required to generate a functional rule that predicted the number of chairs that would fit around any number of tables. This required the students to use concrete materials, verbal explanations, and tables of data. Such tasks also support students to develop the ability to access different forms of representation including symbolic forms and move between these forms flexibly utilising the representation which provides the greatest affordance for the task (Schoenfeld, 2008).

A further design principle in developing tasks that support algebraic reasoning involves the utilisation of connections between different mathematical content areas. Tasks which exploit the connections within mathematics provide learners with opportunities to think about content in new ways. Also, a foundation is laid for the meaningful use of algebraic tools. For example, Schoenfeld (2008) describes how a rate and ratio problem could be extended through the introduction of a graphical representation. This then shifts the problem to include ideas about functional relationships. Extending tasks in this way supports students to view the connections within mathematics and algebra. It also allows students to be introduced to new representations in a way which links their interpretations with a meaningful context.

The careful design or extension of tasks within different mathematical areas to include algebra allows algebraic reasoning to become an everyday part of mathematics lessons (Kaput & Blanton, 2005; Schoenfeld, 2008). For example, Kaput and Blanton (2005) illustrate how a teacher with a class of students aged 6-years to 7-years old in adapting a combinatoric problem involving combinations of outfits of pants and shirts was able to support her students to build generalisations. The task was enacted in the classroom with students using coloured cut-outs of the pants and shirts and recording information onto chart paper. These researchers argue that many tasks can be extended and ‘algebrafied’ across the mathematics curriculum thereby allowing algebraic reasoning to permeate mathematics instruction.

It is necessary also to consider factors which support or inhibit successful task implementation. While the use of carefully designed algebraic tasks is important, there are also a range of classroom factors which may influence engagement with the task. Research studies (e.g., Henningsen & Stein, 1997; Sullivan, Mousley, & Zevenbergen, 2006) have investigated how to successfully maintain the cognitive demand of challenging tasks such as those used to develop algebraic reasoning with heterogeneous classes. Two of the key findings from Henningsen and Stein's (1997) study were the importance of how the task was implemented by the teacher and appropriate allocation of time to the task. They provide examples of teachers reducing the complexity of the task during the implementation phase by providing explicit procedures or solving the more difficult aspects of the task themselves. The demand of the task was also hindered by a shift towards a fast pace and focus on correct answers rather than meaning and understanding. Alternatively at other times, students were provided with too much time and the focus on mathematics was lost. Another factor identified by Sullivan et al. (2006) is the need for teachers to develop enabling prompts which facilitate all students to access high level tasks and extending prompts which can be used to extend thinking. Both of these studies highlight the need to move beyond task design to also consider how algebraic tasks are implemented within the classroom.

3.3 CLASSROOM PRACTICES TO SUPPORT ENGAGEMENT IN ALGEBRAIC REASONING

The previous section highlighted principles for designing tasks which support the facilitation of algebraic reasoning. However, the introduction of appropriate tasks alone does not necessitate student development of early algebraic reasoning. It is critical to establish a learning environment which supports children to engage in early algebra (Blanton, 2008; Schoenfeld, 2008; Smith & Thompson, 2008). Teachers have an important role in establishing the learning environment; ways of working which are valued and promoted by the teacher facilitate both the classroom environment and its ethos (Mason, 2008; Smith & Thompson, 2008). Studies provide evidence for the existence of a range of classroom practices that although not limited to early algebra afford opportunities for students to engage in algebraic reasoning. The following section will outline some of the key classroom practices which are identified in the research literature. These include: re-visiting key concepts; facilitating productive discourse; focusing on student thinking and reasoning; and developing questioning.

3.3.1 Re-visiting concepts

A key practice in developing early algebraic reasoning is creating a learning environment where students are re-exposed to important algebraic concepts on multiple occasions (Bastable & Schifter, 2008; Blanton, 2008; Ding & Li, 2010). A number of classroom research studies (e.g., Blanton & Kaput, 2005a; Falkner et al., 1999) demonstrate how teachers can integrate key concepts of early algebra effectively into their lessons. For example, Blanton and Kaput (2005a) reporting on the effective practices used by a teacher from their professional development group signalled that the practice of spiralling particular algebraic themes into lessons and re-visiting these over the school year was key. Similarly, the classroom based study by Falkner et al. (1999) investigating students aged 6-years to 8-years old understanding of the equals sign affirmed the need to re-visit the concepts of equality continually. In this study, the teacher integrated discussions of equality into the classroom over the whole school year through the use of open and true and false number sentences. As a result, the students were facilitated to build their understanding and reflect on the meaning of the equals sign. Both of these studies demonstrate how re-visiting key concepts results in students undertaking in-depth exploration of ideas and developing their ability to reason in more complex ways.

3.3.2 Facilitating productive discourse

Engaging in rich conceptually embedded mathematical talk is a critical component of classrooms with a focus on early algebra (Blanton, 2008). However, it is well-documented within research literature (e.g., Fisher, Frey, & Lapp, 2011; Mehan, 1979; Pape et al., 2010; Wood, Williams, & McNeal, 2006) that within traditional mathematics classrooms a typical pattern of classroom discourse follows a structure of IRE: initiate-respond-evaluate. In this type of discourse, the teacher initiates an interaction, often through questions which gather information and require immediate responses; the subsequent student response is then evaluated by the teacher. Consequently the teacher takes the role of the person who both holds and dispenses knowledge while the students receive the knowledge (Bell & Pape, 2012). Criticisms of this discourse pattern include that it lowers student engagement and offers limited opportunities for students to verbalise ideas, construct knowledge, and develop a sense of agency (Bell & Pape, 2012; Fisher et al., 2011; Nathan, Kim, & Grant, 2009).

Developing a learning community where students have the opportunity to engage in the type of mathematical discourse which underlies algebraic reasoning optimises students' engagement with early algebra. However, developing classroom communities which promote interactive mathematical talk is challenging for many teachers and their students, particularly because they may not have previously experienced learning and teaching in such classrooms (McCrone, 2005; Sherin, 2002a; Stein, 2007). This inexperience extends to environments which explicitly promote the type of discourse which supports student construction of early algebraic reasoning (Blanton, 2008). This section will outline how productive discourse can be facilitated during small group work and whole class discussions.

Teachers take a significant role in guiding the development of mathematical discourse in the classroom and ensuring active engagement by all students. Engagement in collaborative interaction necessitates a shift away from the more traditional role of students as passive receivers of instruction to active and constructively critical participants within the classroom community. Specific pedagogical practices can to support such a shift. For example, McCrone (2005) illustrated how a teacher in a classroom with students aged 10-years to 11-years old shifted students' participation in discourse from parallel conversations characterised by a lack of active listening to that of critical active participants through pedagogical actions such as modelling active listening and reflecting on the ideas of others. The teacher also initiated explicit discussions to emphasise the importance of active reflection and participation in mathematical discussions. Similarly, Reid and Zack (2009) described a classroom in which the students were expected to express their thinking and engage in analysing the reasoning of others. Changes in expectations by the teacher were accompanied by greater student awareness of how their peers could contribute to their understanding and how they could extend and reshape the ideas of others. Both of these studies demonstrate how teacher expectations and specific pedagogical actions can support the development of a classroom community where students' mathematical reasoning is facilitated through productive discourse.

To position students in interactive dialogue, teachers may also use key pedagogical tools—often referred to as talk moves (Chapin & O'Connor, 2007)—of rephrasing, repeating, and revoicing (O'Connor & Michaels, 1996; Stein, 2007). Through teacher revoicing, students learn to take a specific stance in the dialogue and develop the skills of inquiry and mathematical argumentation as they defend or challenge ideas (O'Connor & Michaels, 1996). Revoicing can also be used to build on student thinking, clarify reasoning, highlight specific aspects of the mathematical thinking, or extend, rephrase, and further develop it (Chapin & O'Connor, 2007; Lampert & Cobb, 2003; Stein, 2007). An example by Schifter et al. (2009) is shown in the vignette below. In this classroom episode, the teacher used revoicing to support the students to develop connections between their ideas and to facilitate them to take a specific stance in developing their generalisation.

Using revoicing to develop connections and position students to take a stance
Alice Royden's classroom of students aged between 8-years to 9-years old were investigating conjectures about the sum of even numbers. She had asked them to discuss whether the sum of even numbers would be even. A student begins by providing an explanation based on a previously agreed upon generalisation: Catherine: It's because even plus even equals even, and that even plus another even will equal even, and then you could just go on and on and on. *To encourage the students to hear different perspectives the teacher asked another student to present her way of understanding the idea.* Lark: I'm thinking maybe it's because of that two, because every even number is made up of twos. *The teacher then revoiced the student's idea as a question. In doing so she supported the class to develop the connections between Catherine and Lark's explanations. She also positioned Lark to clarify her generalisation and take a stance.* Teacher: Are you saying that another way of looking at Catherine's idea is to think of it as some count of two plus some other count of two plus, and so on, is always going to equal a number that has some count of two? *Lark then provided a further argument for the generalisation drawing on the definition the class had formulated of even numbers as a count of two.* Lark: So, since every even number is a count of two and I'm adding an even plus an even, what I'm doing is adding like three counts of two plus two counts of two. I'm always going to end up with some count of two, and I'm always adding a count of two together. And since counts of two plus counts of two always equal counts of two, it will always equal an even number, a count of two. *From Schifter, Russell, & Bastable (2009)*

In this vignette, the teacher used revoicing to support the students in developing generalisations. By rephrasing the explanation into a question, she positioned the students to take a stance and develop more specific explanations and arguments.

Many reform mathematics environments, including those which focus on early algebraic reasoning, feature the use of small group (2-4 children) mathematical activity followed by whole class discussions. During small group work learning opportunities arise from collaborative dialogue and the resolution of differing points of view (Whitenack & Yackel, 2002). Small group discussions can also provide opportunities for students to rehearse their explanations, justification, and analysis of their solution strategies (Hunter, 2009). But as in whole class discussion, explicit teacher scaffolding is important to structure small group interactions. Research studies (e.g., Hunter, 2009; Monaghan, 2005; Rojas-Drummond & Zapata, 2004) show how careful guidance can be provided by the teacher so that students can learn appropriate ways to work and talk within a group. These researchers built on the seminal work of Mercer (2000). Mercer outlined how young students use three different forms of talk—exploratory, disputational, and cumulative—during small group interaction. The three forms of talk involve different levels of engagement in the reasoning of peers. Mercer described disputational talk as characterised by students focusing on self-defence and holding control rather than trying to reach joint agreement. In using cumulative talk, students avoid questions and argument which results in a lack of evaluative examination of reasoning. These are both largely unproductive forms of talk. Exploratory talk in contrast is a productive form of talk in which the students explore and critically examine the shared reasoning. In many different studies Mercer et al. (e.g., Littleton et al., 2005; Mercer, Littleton, & Wegerif, 2004; Mercer & Sams, 2006) showed that constructing group interactions which use exploratory talk requires deliberate teacher attention, intervention, and scaffolding of group talk.

The use of whole class discussions extends mathematical reasoning beyond that of small groups. To develop algebraic reasoning, it is important that students are positioned to listen actively and make sense of a range of mathematical explanations. A range of research studies (e.g., Bastable & Schifter, 2008; Carraher et al., 2008; Fosnot & Jacob, 2009; Reid & Zack, 2009) have demonstrated how teachers are able to structure whole class discussions successfully to develop young students' algebraic reasoning. In these studies the teacher takes a central position in orchestrating and facilitating productive whole class discussions by leading shifts in the discussion to ensure that it is conceptually focused and reflective (Lampert & Cobb, 2003). Kazemi (1998) illustrated how discourse promoting conceptual reasoning was achieved through the use of specific pedagogical actions. These included questioning in sustained exchanges, pressing students to provide conceptually focused justification for mathematical actions, and facilitating student examination of similarities and differences across multiple strategies.

3.3.3 Focusing on student thinking and reasoning

Utilising student thinking and reasoning as a central element of instruction is key in classrooms with a focus on algebraic reasoning. In a role as facilitator the teacher can extend students' thinking into an exploration of generality (Bastable & Schifter, 2008; Blanton & Kaput, 2005b). As highlighted in the earlier section, developing early algebraic reasoning requires the use of well-structured instructional tasks which have varying solution pathways. However, this creates challenges for teachers, particularly because students may approach tasks in different and sometimes unanticipated ways (Sherin, 2002a; Stein et al., 2008). It is important that teachers are able to listen carefully and interpret student thinking (Blanton, 2008; Stein et al., 2008).

A key challenge described by Stein et al. (2008) is for teachers to use students' reasoning and responses to tasks in a way that advances the mathematical learning of the whole class. Stein et al. propose a model of five practices to successfully manage whole-class discussions where student reasoning is at the centre. These practices are: anticipating likely student responses; monitoring students' responses to the tasks; selecting particular students to present their mathematical responses; purposefully sequencing the student responses, and supporting students to make mathematical connections between different responses and key ideas.

Anticipating student responses involves thinking about the possibilities of both correct and incorrect strategies that students may use and how these relate to concepts, procedures, and representations (Stein et al., 2008). In the context of creating a classroom environment which focuses on early algebra this requires teachers to recognise both the opportunities for algebra within a task and student responses which demonstrate an algebraic way of thinking (Blanton & Kaput, 2003; Franke et al., 2008). For example, Franke et al. (2008) describe observing a lesson where students aged 8-years to 9-years old solved the problem: *If 6 cows have 4 legs each, how many legs are there altogether?* In this study the four different solution strategies that were elicited from the students by the teacher were not related in any productive ways to a discussion of commutativity. The researchers hypothesised that had the teacher anticipated potential responses and opportunities for early algebra within the task, more space may have been created for the integration of algebraic reasoning into lessons.

In Stein et al. (2008) five practices model, monitoring student responses while they are working on tasks allows teachers to assess and make sense of students' mathematical thinking. This can be achieved by carefully listening to student discussions and asking probing questions if necessary. Monitoring also provides opportunities to consider the solution strategies and representations which will be important to share during the whole class discussion (Stein et al., 2008). For example, Fosnot and Jacob (2009) describe an episode from a classroom with students aged 10-years to 11-years old where students were asked to solve a problem: *MT decides to hold a jumping contest. The three contestants are Cal, Sunny, and Legs. In this contest, all frog steps are the same size. Also when a frog jumps, he always travels the same distance. Your problem is to find out which frog has the biggest jump. When Cal jumps three times and takes six steps forward he lands in the same place if he jumps four times and takes two steps backwards. When Sunny jumps four times and takes 11 steps forward he lands in the same place if he jumps five times and takes four steps forward. When Legs jumps two times and takes 13 steps forward he lands in the same place if he jumps four times and take five steps backwards.* Solving this problem involved "cancellation of equivalent amounts with variables" (p. 113) using an open number line diagram as a tool. Careful monitoring of the students while they worked on the investigation revealed that a number of students were using arithmetic, many students had difficulty comparing the quantities due to the way in which they attempted to use the number lines, and some were beginning to make sense of how cancellation could be used to solve the problem. By monitoring these responses, the organisation of the subsequent discussion was able to be shaped effectively to enhance learning.

Careful monitoring of student observations and questions can also support the development of a 'conjecturing atmosphere' (Bastable & Schifter, 2008; Blanton, 2008; Mason, 2008). The generalisations that children engage with while explaining their reasoning during small group work can be identified and then explored during whole class discussions. An example of this is presented in a study by Schifter et al. (2008b) in which students aged 7-years to 8-years old were working to generate ways to make ten. As the teacher observed the students working, she noted that many of them were utilising the commutative principle which they had informally termed ‘turn arounds’. Noting also that some students were making statements such as: *turn arounds always work* prompted the teacher to use this to lead a whole group discussion and probe the students’ understanding of additive commutativity.

In addition to selecting and sequencing the student responses purposefully to advance particular mathematical concepts or focus on important mathematical ideas, the teacher can also ensure that common misconceptions are explored and examined or specific solution strategies introduced (Stein et al., 2008). For example, in the lesson described previously from the study by Fosnot and Jacob (2009), three students were selected purposefully to share their solution strategies during the whole class discussion. The first student selected had struggled but was beginning to make sense of cancellation through use of a diagram. This diagram was easily accessible for the other students and supported them to understand her reasoning. The second student was more confident in her explanation and also used a solution strategy which utilised a diagram. Finally, the third student presented a solution strategy to the class using a diagram and equations. His work included what could be considered a generalisation of the cancellation strategy. The selection and sequencing of the shared reasoning supported all the students to develop their understanding of cancellation strategies. The example provided shows how by carefully selecting and sequencing student responses, teachers are able to guide whole class discussions to advance the instructional agenda.

A key goal in orchestrating discussions advanced by Stein et al.. (2008) is that the teacher supports and facilitates students to draw mathematical connections between the strategies and representations that are shared. This may involve asking students to analyse the differences or similarities between solution strategies and assessing the efficiency of differing approaches (Kazemi, 1998) or to connect and analyse the types of representations, operations, or concepts that were used. For example, in the classroom episode reported by Fosnot and Jacob (2009), students were asked to analyse the first two strategies which were presented and discuss how they were similar and different. This led to the students connecting the similarities between the representations used and identifying that both solution strategies took away the unknown equivalent jumps. This was then used to promote reflection on the use of cancellation strategies with students being asked why equivalent jumps could be taken away and whether it mattered if the size of the jumps were unknown when they were taken away. When the third solution strategy was shared, the student was asked to connect his equation with the representational diagram he used and then the class was asked to analyse whether he also used the same rules as the previous solution strategies. This example showed the class working as a community to develop their understanding of the cancellation of equal amounts of an unknown. As illustrated, developing mathematical connections between different solution strategies and representations facilitated students to reflect on other students' ideas and the efficiency of their solution strategies. This also supports students to evaluate and revise their own ideas (Stein et al.., 2008).

The classroom practice of putting student reasoning at the centre of instruction gives students' authority over their mathematical work. However, students' work and reasoning must also be held accountable to the discipline of mathematics. An important role for the teacher is to facilitate students collectively to develop a set of ideas and processes which are accepted as worthwhile and important in mathematics. Through using the five practices described above, Stein et al. (2008) argue that accountability to the discipline may be maintained without undermining students' mathematical authority.

3.3.4 Developing questioning

Questioning takes an important role in the development of the classroom environment and ways in which students approach tasks. However, evidence from research studies (e.g., Boaler & Brodie, 2004, Graesser & Person, 1994; Hiebert & Wearne, 1993) highlights that the majority of teacher questions used in classrooms consist of low level questions which elicit short answer responses consisting of the recollection of facts, rules, and procedures. Other studies (e.g., Franke et al., 2009; Wood, 1998; Wood, Cobb, & Yackel, 1991) have illustrated how teacher questioning may be classified as leading or funnelling questions which direct students to complete a task in a specified way or to ensure correct answers. In this case, the teacher undertakes the mathematical thinking and the questioning relates to a teacher selected solution strategy rather than the students' mathematical thinking (Franke et al., 2009). The following vignette is used by Franke, Turrou, and Webb (2011) to illustrate how questions can be structured to focus students on the next step rather than elicit student reasoning.

Teacher questioning to lead students to a teacher selected solution strategy
Ms Gomez asked her students to solve the following number sentence: $14 \div 2 = 3 \times \_\_ + 1$. While the students are working in pairs, she steps in to help two students: Miguel: Three times one… *The teacher uses a leading question to direct the students towards her approach of solving one side.* Ms Gomez: Ok, but you are working on number one…So remember, we always ask ourselves, which of the two sides is complete. The left side or the right side? Miguel: Left side. (*pointing on paper with pencil)* *The teacher again uses leading questions to facilitate the students to complete the task in her chosen approach.* Ms Gomez: Ok, so can you solve the one that's complete? *(Miguel nods)* So, do you think that's a good idea if you solve that first? *(Miguel nods)* Ok. *(Miguel looks up at the teacher)* So what does that tell you? Miguel: That fourteen divided by two equals, is the same as seven?

Ms Gomez:	Ok, very well read. Now so, if this side equals seven, what does this side… Oh my, you've already solved it.
From Franke, Turrou, and Webb, (2011)	

In this vignette, the teacher used questioning to lead the student towards using her approach to solve the problem correctly. She did not elicit the student's understanding of the problem or provide opportunities for him to develop his understanding of what the task was asking him to do. As Franke et al. (2009) argue these types of questions can limit opportunities for students to build and develop their own mathematic understanding.

Transforming the types of questions asked in the classroom supports a shift in the classroom culture given that student provided explanations and reasoning are strongly influenced by the types of questions posed by the teacher (Blanton, 2008; Franke et al., 2009; Martino & Maher, 1999; Smith & Thompson, 2008). Specific questioning can be used both to focus student attention on the relationships and patterns within a task and to facilitate and scaffold students to approach tasks in different ways. In investigating the shift from arithmetic to algebraic reasoning, Smith and Thompson (2008) maintain that students are too frequently asked conceptually simple questions which focus on finding a single quantity from the given values of other quantities instead of questions which focus on the range of values, patterns, and relationships between the values. Furthermore, the types of questions asked can lead to students engaging in justification of their ideas (Blanton, 2008; Martino & Maher, 1999; Wood & McNeal, 2003). Through teacher modelling of questioning and scaffolding students to ask questions themselves, students can also be supported to begin generating the types of questions which support engagement with algebraic reasoning.

Developing algebraic reasoning requires students to approach tasks in a different way than arithmetical reasoning. As described above, there is a need to shift student attention from calculating answers towards developing a structural perspective which focuses on operations on mathematical objects—relational thinking (Carpenter et al., 2005b; Empson, Levi, & Carpenter, 2011; Ferrucci et al., 2008; Fosnot & Jacob, 2009). Examples of questioning that attends to relational thinking are particularly evident in research studies (e.g., Carpenter et al., 2003; Fosnot & Jacob, 2009) which focus on developing student understanding of equivalence. In Fosnot and Jacob's (2009) study, students aged 7-years to 8-years old working in pairs were asked to establish whether they had an equivalent amount of coins to each other. One pair began to complete the task procedurally by calculating the amounts to compare. However, when the teacher stepped in and asked 'is there another way to tell without adding it all up' (p. 104) there was evidence of one of the students beginning to compare the relations between the amounts to establish equivalence. Similarly, in the study by Carpenter et al. (2003), students aged 9-years to 10-years old were solving true and false number sentences. The teacher used questions such as 'can you do this without all the adding and subtracting?' (p. 32). These questions prompted the students to begin justifying their responses through relational rather than calculational reasoning. Both examples provide evidence of how teacher questioning can facilitate algebraic reasoning.

Teacher questioning can also be used to scaffold students to utilise representations and their underlying structure when enacting a task. The following classroom vignette offered by Bastable and Schifter (2008) illustrates how teacher questioning facilitated the students to draw on understanding of the structure of square numbers in developing their generalisations.

Teacher questioning to focus student attention on structural aspects
Jenny Richards has asked her students aged 11-years to 12-years old to work on the following task: *Yesterday we started a great debate about square numbers. We are thinking about 3 × 3 and 9 × 9 and 4 × 4 and 179 × 179. They create perfect squares on graph paper. Then we*

wondered about 0 x 0. Hmmm... what does that mean? A student begins by providing an explanation:

Lindsay: Zero times zero is zero. You can't make anything with it. You have to imagine it in your head and when I do that I imagine a square.

The teacher directs the students to think of the meaning of each expression rather than focusing on the answer. She uses a further question to facilitate them to visualise the representation.

Richards: What does this mean? Four times four. I don't want the answer. I want to know what it means.... What would it look like?

The student provided explanations draw on the visual representation of square numbers.

Carolyn: Four rows with four in each row

Richards: And nine times nine?

Chris: Nine rows with nine in each row.

The teacher again directs students to refer to the visual representation and structure of the expression.

Richards: And if you drew it on graph paper, you would get a perfect square? What does this mean, "zero times zero"?

Lesley: Zero rows with zero in each row.

From Bastable and Schifter (2008)

In this vignette, teacher questioning prompted students to use representations in their explanatory talk. This focus on using representations to support their explanations appeared to support further development of their understanding of the structure of square numbers.

Teacher questioning can also play an important role in supporting students to develop mathematical explanations (Franke et al., 2009; Franke et al., 2011; Wood, 1998). Franke et al. (2009) found that teacher questioning was a significant factor in whether students elaborated on the initial explanation provided. However, it is important to note that teacher questioning did not always lead to further elaboration of student thinking. Interestingly, these researchers found that sequences of specific questions led to students providing elaboration on their initial explanation. Specific questions alone or general questions not directly related to the explanation led to further elaboration in some instances. However, leading questions used to guide students towards a particular answer or strategy often did not lead to further elaboration. A key finding of this study was that teacher questioning in the form of multiple specific questions related to the initial explanation was necessary to press students to make their thinking explicit and to support connections between the explanation and the mathematics that was a focus of the task. These researchers argue that probing sequences of specific questions benefit all the students within the class. Firstly they help the student being questioned to “clarify, solidify, and correct his or her own thinking” (p. 390) but they also help the listening students to “connect their own thinking to what was being said, potentially enabling them to correct their own misconceptions” (p. 390).

Teacher questioning also facilitates the shift from students explaining their mathematical solution strategies to justifying and defending solution strategies within collaborative dialogue (Bell & Pape, 2012; Blanton, 2008; Franke et al., 2008; Martino & Maher, 2009; Wood & McNeal, 2003; Wood et al., 2006). Questioning can also be used to reposition all participants in the classroom to analyse shared solution strategies regardless of whether they are correct or not. Students can then use mathematical evidence to validate their agreement or disagreement (Bell & Pape, 2012; Martino & Maher, 2009). The following vignette by Bell and Pape (2012) shows how a teacher used questioning to press for explanation and justification and to engage her class in collective examination of a shared solution strategy.

Teacher questioning to press for explanation and justification

Without indicating whether it is correct or not Mrs Brenner asks her class to examine an incorrect response provided by a student:

Mrs Brenner:	What I’d like to discuss is this right here. I’m actually glad that someone answered this. Is this correct? Is the absolute value of 16, negative 16.

The teacher explicitly asks for justification of mathematical ideas.

Mrs Brenner:	And I need some justification. I want to hear what you’ve got to say. Lakina.
Lakina:	No
Mrs Brenner:	Why?…

Lakina:	I said when…if a number is positive, I mean negative, it comes out as a positive.
The teacher continues to press for justification and develops wider student involvement by asking for participation from other students.	
Mrs Brenner:	Why is that, though? Why is that? [pause] Romero.
Romero:	Because absolute value is the number of the counted spaces from zero to that number. So it should always be positive.
Lakina:	Yeah, it always should.
From Bell & Pape (2012)	

In this vignette, the teacher pressed for students to justify and defend their explanations. This led to the students collectively analysing the initial response and then building a mathematical justification by further developing each other's comments.

In classrooms with a focus on developing mathematical inquiry and algebraic reasoning, an important aim is for the responsibility of developing understanding to be spread across all participants. This necessitates that students begin to ask questions to support their own developing understanding and also to require justification from each other. Often in classrooms the role of questioning is taken by the teacher with only infrequent questioning by students (Franke et al., 2009; Graesser & Person, 1994; Martino & Maher, 1999). Research studies (e.g., Boaler & Brodie, 2009; Wood & McNeal, 2003) provide examples of how a specific teacher focus on questioning for understanding can support students to develop their questioning skills. Additionally they illustrate how teacher prompts for justification can be appropriated by students within the classroom community. In their study, Boaler and Brodie (2009) demonstrated a link between the growth in conceptual questions asked by the teacher and students beginning to ask conceptual questions themselves. They also found that in classrooms where conceptual and probing questions were used by the teacher that students were able to use these as a scaffold to challenge reasoning and prompt the construction of explanatory justification for the mathematical solution strategy during small group work. For example, in one instance the researchers overheard the students saying "She's going to ask us where we got the eight, where did we get it?" (p. 780). This demonstrates the link between teacher questioning and developing student questioning.

3.4 MATHEMATICAL PRACTICES TO SUPPORT ENGAGEMENT IN ALGEBRAIC REASONING

Many researchers (e.g., Bastable & Schifter, 2008; Carpenter et al., 2003; Kaput & Blanton, 2005a; 2005b; Mason, 2008; Schoenfeld, 2008; Smith & Thompson, 2008) maintain that optimal algebraic reasoning opportunities occur within classroom contexts which facilitate students to engage in mathematical practices linked to the development of algebraic reasoning. Construction of conceptual algebraic reasoning requires that students build understanding of key mathematical ideas and the connections between these and develop mathematical ways of thinking. This is achieved when students are provided with opportunities to engage in the process of "generating [mathematical] ideas, deciding how to express them …justifying that they are true, and in using them to justify the mathematical procedures they are learning" (Carpenter et al., 2003, p. 6). Moreover, as Kaput and Blanton (2005a) argue, frequent viable algebraic reasoning opportunities occur within classroom contexts where students are supported to make purposeful conjectures, construct mathematical arguments, justify, and generalise their ideas.

The following section highlights three key mathematical practices identified in the research literature which are linked to the development of early algebraic reasoning. These practices are: (1) developing mathematical explanations and using mathematical language; (2) justifying ideas and developing age appropriate proof; and (3) making conjectures and generalising. Discussion of the use of representations, another key mathematical practice linked to early algebra is integrated into each section.

3.4.1 Mathematical explanations and mathematical language

Engaging in the practice of developing mathematical explanations is an important foundation for early algebraic reasoning. Whilst using mathematical language is a central aspect of providing clear verbal explanations, students often begin by using informal or invented language to describe their observations which then become part of the shared language of the classroom community (Smith & Thompson, 2008). This invented terminology can draw on the characteristics of a representation (Fosnot & Jacob, 2009). For example, Schifter et al. (2008b) present a case in which students aged 7-years to 8-years old described the commutative nature of addition number sentences as 'turn-arounds'. A similar example is provided in Fosnot and Jacob's (2009) study. Students in this study were presented with the equation 13 + 8 = 5 + 9 + 13 – 6 represented on a double line. A student referring to the commutative relationship following cancellation stated "it's like a mirror on the number line ... like the symmetry we did" (p. 107). With this type of informal or invented language use, the meaning and explanations are often clear for the listening audience. Further use of mathematical language can evolve through discussions and teacher modelling.

During discussions within the classroom community, invented terminology may evolve and become a shared language between the participants which reflects their understanding of tasks. For example, in the Fosnot and Jacob (2009) study, students aged 10-years to 11-years old developed notions of a cancellation rule while using a double number line. A student provided a diagram to support her solution strategy which used a line to separate the quantities. This was described by another student as a ‘separator’. In the ensuing discussion other students drew on this invented terminology using the ‘separator’ line to break the problem into two parts and to ignore the equivalent amounts which were unnecessary to solve the problem. Despite the teacher introducing terms such as ‘ignore’ and ‘remove’, the students showed a preference for the terminology introduced by their peers and used this in their explanations. It appeared that use of these types of invented terms supported the students’ early understanding of cancellation more than formal terminology such as the cancellation law.

Teacher revoicing and modelling of mathematical language can facilitate students to develop their use of correct mathematical terminology and induct them into the specialised discourse of mathematics and algebra (Reid & Zack, 2009). Khisty and Chval (2002) present a case study of a teacher working with students aged 10-years to 11-years old from a predominantly Latino background. They describe her frequent modelling and use of mathematical terms which were then used by the students in their written and oral discourse. Teacher revoicing was another feature of the classroom which supported student use of mathematical terms. For example, in one lesson a student referred to the relationship between multiplication and division as opposite. The teacher responded to this by saying: *One is the inverse of the other, the opposite of the other* (p. 159). By attending to the mathematical discourse and populating her lessons with mathematical terminology, the teacher successfully facilitated her students to develop appropriate academic and mathematical discourse. Similarly, Reid and Zack (2009) present an example to illustrate the important role of the teacher in supporting students to connect their informal language with standard conventions. In this example, a student erroneously used the term square root instead of square number during her explanation. The teacher responded by repeating the explanation and then revoicing using the correct terminology before instigating a discussion of the terminology, concept, and notation associated with square numbers. Both of these examples illustrate the important role of the teacher in facilitating students' use of mathematical language.

The teacher also takes a significant role in facilitating students to provide clear explanations for others who are listening. The clarity of explanations can be developed through using verbal and written/drawn representations that are responsive to the listening audience (Bell & Pape, 2012). This entails clarifying and refining ideas if necessary and responding to the questions of those who are listening (Smith & Thompson, 2008). The following vignette is used by Bastable and Schifter (2008) as an example of students reconsidering generalisations about whole numbers when asked to reflect on operations in relation to rational numbers. It provides an example of how students can construct mathematical explanations which are responsive to the listening audience and their questions. It also illustrates the role of the teacher in facilitating students to give reasoned explanations and creating space for agreement and disagreement.

Developing mathematical explanations during a whole class discussion
Joanne Moynahan has asked her class of students aged 11-years to 12-years old to solve a problem involving fractions: *The Davis family attended a picnic. Their family made up 1/3 of the 15 people at the picnic. How many Davises were at the picnic?* The students are provided with time to work on the problem in pairs. Following the paired work and subsequent discussion, the teacher asks the students to consider which operation was involved in the problem. A student begins by stating which operation she thinks is appropriate: Mary: I think we should put division in there. *The teacher asks for further explanation of this and the student draws on her use of a concrete representation to support her argument.* Teacher: Why? Mary: Well, the problem said 1/3 of the people were Davises. I drew a circle and divided the circle into three parts—then I put the people in. Jeff: I agree. We divided our cubes into three groups... *The teacher then opens the discussion up to agreement and disagreement from the rest of the class. She also directs them to reflect on their understanding of division.* Teacher: Does everyone agree? Should I erase the "of" and put in "÷"? Think about what you know about division.

At this point, another student disagrees with the initial explanation and provides a reasoned argument which draws on the meaning of division

Rebecca: I don't think divide is right.

Teacher: Why do you think divide won't work?

Rebecca: *(comes to the board and writes 1/3 ÷ 15 = 5)* This *(points to the 15)* means how many 15s are in 1/3. I know that's not right. There aren't any!

The other students test whether the situation is represented by division and agree that it does not work. Rebecca then proposes an alternative argument and develops an explanation which draws on a symbolic representation but does not link to the context of the word problem.

Rebecca: I think it's times *(comes to the board and writes a row of 1/3s)*. That's 1/3 fifteen times. Now add them up.

Rebecca is aware that her argument has not convinced her classmates. She provides a second explanation which begins to refer to the context of the word problem.

Rebecca: I didn't multiply. I'm just trying to prove that you can. I divided the 15 people. She *(points to Mary)* says divide and I'm trying to show that multiply works.

At this point the class ends, however, the following day the teacher asks the students to again reflect on the previous discussion. After testing whether addition or subtraction would work for the problem, a student provides an argument which draws on the inverse relationship of multiplication and division.

Mary: Division is the opposite of multiplication. Take 12 × 2 = 24. Then 24 ÷ 2 = 12. So ... If 1/3 × 15 = 5, then 5 ÷ 15 = 1/3. Does that work?

Teacher: How many 15s are in 5?

Mark: There aren't any. You can't make any 15s if you only have 5. Wait. You could make part of a 15.

Jeff: I got it! You would have 1/3 of 15! It does work. Rebecca was right—it is multiply!

From Bastable and Schifter (2008)

In this vignette, the teacher maintained a consistent expectation that students will provide reasoned explanations. As a result, the students drew on varying representations to support their explanations and respond to the listening audience. Space was also created by the teacher for the students to agree or disagree and from this space students began engaging in mathematical argumentation.

3.4.2 Justification and proof

Developing mathematical explanations into sound, convincing arguments is an important mathematical practice related to algebraic reasoning (Stylianou, Blanton, & Knuth, 2009). Engaging in justifying mathematical ideas and developing age appropriate proof are key aspects of the generalisation process. However, a number of research studies (e.g., Bieda, Holden, & Knuth, 2006; Callingham, Falle, & Clark, 2004; Healy & Hoyles, 2000; Knuth, Choppin, & Bieda, 2009; Lannin, 2005; Maher, Powell, Weber, & Stohl Lee, 2006) document student difficulties with developing adequate justification and proof. These challenges have been attributed largely to a lack of understanding of generality along with difficulties with mathematical language and symbolism and a lack of problem-solving skills necessary to construct an argument. In order to support students to develop mathematical practices of justification and proof it is important that teachers create opportunities to engage students in justification and proof related activities appropriate to their age.

For primary students developing conceptions of appropriate justification for mathematical statements and reasoning is an important precursor to the development of proof in later years (Callingham et al., 2004). Research studies (e.g., Carpenter et al., 2003; Knuth et al., 2009; Schifter, 2009) which have investigated primary students' attempts to justify mathematical statements as true characterise these justifications into three broad categories: appeal to authority, justification by example, and generalisable arguments using representations. In the first case, students refer to an authority such as a teacher or parent or text-book to avoid having to provide a justification. An example is provided by Schifter (2009) whereupon a student in her study accepted the claim that the sum of two even numbers would be even because her older sister had told her this. The second category—justification by example—is well-documented in research studies (e.g., Bieda et al., 2006; Carpenter et al., 2005a; Healy & Hoyles, 2000; Lannin, 2005; Schifter, 2009). In this case, students perceive that providing specific examples or trying a number of cases is valid justification. The process of looking for examples and counter-examples is an important aspect of learning to justify and generalise. However, students require encouragement to develop these skills further and begin to use more refined techniques to justify (Mason, 2008; Schifter, 2009). There is a large body of research (e.g., Bieda et al., 2006; Carpenter et al., 2005a; Healy & Hoyles, 2000; Knuth et al., 2009) which shows that many students continue to use empirical methods to attempt to justify and prove throughout primary school, secondary school, and even tertiary studies unless specific attention is paid to facilitating their understanding of generality. In the third category, students use representations to develop reasoned general arguments. Although the representation may involve specific numbers, the argument is supported by the structure of the representation rather than the specifics of the instance which is used (Carpenter et al., 2003; Schifter, 2009). Through explicitly encouraging students to use this type of reasoning and justification, their work with proof in later years may be enhanced (Carpenter et al., 2003; Carpenter et al., 2005a; Schifter, 2009).

Supporting students to justify generalisations using arguments which are precursors of mathematical proof is challenging (Carpenter et al., 2005a). An initial step in developing students' use of justification strategies is to facilitate students to reflect on the adequacy of relying on empirical methods of justification. Both Carpenter et al. (2003) and Schifter (2009) provide examples of teachers using questioning to prompt reflection on the notion that empirical justification is sufficient to prove a claim is true for all numbers. A key challenge which was used by teachers in both studies was how the students could show the claim would be true for any number. Carpenter et al. (2003) provide a vignette where a student responded to this challenge by arguing that you could try different numbers. In this case, the teacher continued to prompt reflection by asking whether you could try all the numbers. Facilitating students to engage in this type of reflection can prompt them to begin thinking about forms of arguments which draw on representations.

Using concrete materials and representations to develop arguments is an accessible way in which students can establish general claims in primary classrooms (Blanton, 2008; Carpenter et al., 2003; Schifter, 2009). Schifter (2009) argues that when students justify general claims through the use of representation-based proofs, the following criteria needs to be met "(1) the meaning of the operation(s) involved is represented in diagrams, manipulatives, or story contexts; (2) the representation can accommodate a class of instances (for example, all whole numbers); and (3) the conclusion of the claim follows from the structure of the representation" (p. 76). As an example, students aged 8-years to 9-years old in Bastable and Schifter's (2008) study built arrays using unifix cubes to convince their classmates of the generality of the commutative principle for multiplication. Further examples are provided in research studies (e.g., Carpenter et al., 2003; Schifter, 2009; Schifter et al., 2008) which demonstrate how students can justify conjectures about operations involving odd and even numbers. In these studies, students represented even and odd number using blocks, cubes, or drawings. They showed the meaning of the operation (addition) through joining the sets which represented all even or odd numbers and were able to use the structure of the representation to show the claim would always work. Through engaging students in this type of process, they begin to privilege reasoning based on representations over appealing to an authority or example based justification (Schifter, 2009).

Within the classroom community, justification and proof may also draw on previously justified conjectures and generalisations (Carpenter et al., 2003; Fosnot & Jacob, 2009). In this way the process of justification can lead to claims being accepted as truths that can then be used to justify other conjectures or as a form of proof when solving tasks. For example, toward the end of Fosnot and Jacob's (2009) six month intervention with students aged 7-years to 8-years old, the students were asked to provide written proof for equations such as $N + 17 + 5 = 15 + 7 + N$. In response, students provided sequential arguments which drew on the previously justified generalisations which had become accepted truths within their classroom community. One student wrote: "There are two *N*s so those are out. If I switch the seven from the 17 and put in the five, then I have 15 + 7 = 15 + 7" (Fosnot & Jacob, 2009, p. 110). This example and the others provided previously show how primary students may engage in the processes of justification and proof.

3.4.3 Generalisation

Making conjectures and developing these into generalisations are key mathematical practices linked to the development of early algebraic reasoning (Bastable & Schifter, 2008; Blanton, 2008; Dooley, 2011; Mason, 2008; Yeap & Kaur, 2008). However, Mason (2008) maintains that too often it is the teacher who provides the examples, cases, and methods during mathematics—a practice which constrains the space for students to generalise. He argues for the need to develop classroom cultures where the expectation is that generalisations will be expressed and treated as conjectures and then justified. Creating a classroom culture which focuses on generalisation is not an easy task, particularly due to the difficulties students may encounter in both constructing and justifying generalisations. This is demonstrated in a number of international studies (e.g., Anthony & Walshaw, 2001; Chick, 2009; Schifter et al., 2008b; Warren, 2001a; 2003b).

However, there are a series of recent studies (e.g., Bastable & Schifter, 2008; Blanton & Kaput, 2005a; 2005b; Carpenter et al., 2003; Fosnot & Jacob, 2010; Reid & Zack, 2009; Russell, Schifter, & Bastable, 2011) that provide insights into how such a classroom culture may be created. As described in the earlier section of this chapter, the teacher takes an important role in facilitating students to generalise through choosing appropriate tasks and guiding the establishment of the key classroom practices such as focusing on student thinking and reasoning and developing questioning. Blanton (2008) proposes a model which characterises the five components of building a generalisation in the classroom. These include facilitating students to (1) explore a mathematical situation; (2) develop a conjecture or mathematical statement; (3) test the conjecture; (4) revise the conjecture if it is not true; and (5) develop the conjecture into a generalisation if there is sufficient evidence to show it is true. The role of the teacher in the initial phase is to develop an appropriate task for exploration and to facilitate the students through questioning. During this phase students need to be given time to explore mathematical ideas and then decide how to represent or model their reasoning. The development of a conjecture may occur incidentally while students are exploring the task (e.g., Bastable & Schifter, 2008). Alternatively tasks may specifically be designed to elicit conjectures from students (e.g., Carpenter et al., 2003). Following the development of conjectures, students need to engage in testing the conjectures to investigate whether they are true. It is important for students to have experiences in testing both true and false conjectures (Blanton, 2008). For example, Russell et al. (2011) describe a teacher asking her students aged 8-years to 9-years old to consider a general claim they had previously developed about addition in relation to subtraction: "when you're adding two numbers together, you can take some amount from one number and give it to the other, and if you add those up, it will still equal the same thing" (p. 51-52). In this case, the teacher did not evaluate the conjecture but asked the students to take a position to agree or disagree in response and provide convincing evidence. This increased student awareness of the need for justification and provided an opportunity for the conjecture to be revised by the students. After conjectures have been revised, and sufficient evidence has been provided, conjectures may be developed into generalisations. For example, following the discussion of the incorrect conjecture in the study by Russell et al., the students then developed a correct conjecture which was verbally developed into a generalisation about subtraction.

3.4 SUMMARY

This literature review identified that early algebra reform includes both what and how mathematics is taught. An examination was undertaken of the key elements in the learning environment which support students to engage in early algebraic reasoning. Exploration of factors within task design which offer the greatest affordances for engagement with early algebra provided the starting point to engage students in opportunities to develop algebraic reasoning. However, it was clear that more is needed than simply designing appropriate tasks. A review of key research literature identified the critical role of the teacher in developing the learning environment and classroom practices. It also highlighted the challenges for teachers when developing a learning environment which facilitates engagement with early algebra. Finally, the key mathematical practices which optimise algebraic reasoning opportunities in the classroom were reviewed.

The pedagogical practices and associated learning environment advocated in this chapter were derived from a review of a range of research studies that successfully engaged students in algebraic reasoning and led to desirable learning outcomes in early algebra. However, the review has also highlighted that there is a paucity of studies in the research literature which specifically address the interconnections between the role of task design, classroom practices, and mathematical practices in developing early algebraic reasoning. Whilst many studies have focused on the specific topic areas of early algebra, there appears to be less of an emphasis on the elements of the learning environment which support engagement with early algebra. The current study aims to integrate the focus on developing early algebraic reasoning both through content areas and the learning environment.

In both this chapter and Chapter Two, the important role of both the teacher and learner was made evident. The following chapter will examine ways in which teachers can develop their algebra ears and eyes which can then be used to support early algebraic reasoning in their classrooms. The implications of this type of change for students' perspectives and identity will also be addressed.

CHAPTER FOUR

DEVELOPING TEACHERS' AND LEARNERS' ALGEBRA EARS AND EYES

4.1 INTRODUCTION

Chapters Two and Three highlighted how algebraic reasoning can be developed through content areas and classroom and mathematical practices. An important aspect of these chapters was the key role of the teacher and learners. This chapter examines how teachers can be supported to develop algebra ears and eyes which can then be used to facilitate algebraic reasoning in their classrooms. Another important consideration is the implications of integrating algebra into the classroom on student participation and identity.

Reviewing the literature base, Section 4.2 identifies elements of professional development programs which are critical to supporting teachers to cultivate early algebraic reasoning within their classrooms. Two elements emerge as significant: (i) the development of a professional learning community and (ii) facilitating teachers to re-conceptualise their understanding of algebra.

Drawing on a group of key classroom studies focused on student perspectives and engagement in the mathematics classroom, Section 4.3 explores the role of students within the classroom community. It examines the impact of introducing new ways of working mathematically on students' beliefs, identities, and roles within the classroom.

4.2 DEVELOPING TEACHERS' ALGEBRA EARS AND EYES

Advocates of teaching early algebraic reasoning in primary classrooms emphasise the importance of early algebra as a strand which "tightly interweaves existing topics of early mathematics" (Carraher et al., 2008, p. 237). Within this frame, early algebra is not a new topic to be added to the curriculum but an area which already exists within the current curriculum and emerges as teachers and learners focus on the algebraic nature of mathematics. Creating a classroom environment with a focus on developing algebraic reasoning requires that teachers develop their algebra eyes and ears. They need to be able to recognise opportunities for both planned and spontaneous instances when algebraic reasoning can be facilitated in the classroom (Blanton & Kaput, 2003; Blanton & Kaput, 2005a). There is also a need for teachers to be aware of the classroom practices and pedagogical actions which can be utilised to support student engagement with algebraic reasoning (Blanton, 2008; Blanton & Kaput, 2005a; Franke et al., 2008).

Teachers take a critical role in reforming classroom practice. However, many teachers themselves have had little experience with the rich, connected types of algebra that support student development of algebraic reasoning (Blanton & Kaput, 2005a; Franke et al., 2008). Research studies (e.g., Even, 1993; Franke et al., 2008; McCrory, Floden, Ferrini-Mundy, Reckase, & Senk, 2012) report that many teachers have inadequate understanding of how to teach algebra successfully. For example, Franke et al. (2008) reported teacher difficulties with content knowledge when they worked with teachers on professional development focused on algebraic reasoning. These researchers contend that the stance both teachers and students take towards algebra is influenced by their conceptualisation of algebra and ideas about what it means to be doing mathematics. For example, they explain that often being good at mathematics is associated with providing quick answers or knowing rules and procedures.

Algebra is viewed as inaccessible and only appropriate for ‘smart’ students. These kinds of beliefs can lead to high levels of anxiety for teachers when they engage with work associated with algebra.

It is important to cultivate professional development experiences which both re-conceptualise algebraic reasoning for teachers and also support them to engage with the types of algebraic reasoning experiences which are relevant to primary school students (Franke et al., 2008). Effective professional learning is situated in practice. Ghousseini and Sleep (2011) note “learning in and from practice requires being able to see, hear and understand the many details of classrooms (e.g., the content, the students and the work of the teacher) and use this knowledge to analyse and improve one’s own teaching” (p. 148). These researchers maintain that effective practice based professional development uses representations which engage learners (the teachers) with a particular representation of practice, for example developing algebraic reasoning, and then supports the teachers as learners to become a deliberate user of this practice.

The findings of research studies which have investigated effective professional development (e.g., Back, De Geest, Hirst, & Joubert, 2009; Borko, 2004; Earley & Porritt, 2009; Wilson & Berne, 1999) and those studies which have specifically investigated teacher development programs focused on algebraic reasoning (e.g., Billings, 2008; Blanton & Kaput, 2003; Franke et al., 2008; Herbel-Eisenmann & Phillips, 2008; Koellner, Jacobs, Borko, Roberts, & Schneider, 2011; Schifter et al., 2008a) provide us with insight into the important elements of effective practice based professional development. These studies highlight the key indicators of effective continuing professional development to support the development of algebraic reasoning as: (1) opportunities to develop learning communities; (2) a focus on student learning and understanding; and (3) the facilitation of reflection on teaching practice.

4.2.1 Development of a professional learning community

A key aspect of effective professional development is offering teachers the opportunity and space to collaborate and build a strong professional community (Back et al., 2009; Earley & Porritt, 2009; Franke et al., 2008; Koellner et al., 2011). Within a professional learning community such as used to promote early algebra, the theoretical frameworks of communities of practice (Lave & Wenger, 1991; Wenger, 1998) and communities of inquiry (Jaworski, 2004; 2006) offer useful constructs.

Within the frame of Lave and Wenger's (1991) communities of practice there is a strong focus on the concepts of practice and identity and the relationship between these. The process of learning is viewed as developing participation in practice. This involves investigating how the participants learn to participate in the practices of the community and develop an identity within the community. Applying a community of practice approach to the analysis of teachers' professional development enables the exploration of how participants may identify themselves in multiple ways, in the service of more than one (social) purpose. This approach was used in Blanton and Kaput's (2008) depiction of teachers engaged in professional development focused on linking early algebra into the mathematics they taught. They claimed that some of the teachers positioned themselves as participants in the community and engaged with 'social' work associated with facilitating algebraic reasoning. This supported them to develop their professional identities within the group. However, other teachers remained isolated within the group exhibiting anxiety with demands to engage with algebraic reasoning. Blanton and Kaput maintain that this isolation meant these teachers found it difficult to develop their own algebraic understanding and integrate early algebra into their classrooms.

The notion of communities of practice (Lave & Wenger, 1991; Wenger, 1998) applied to professional learning is developed further by Jaworski (2006; 2008). She describes teaching as a social process in which teachers are practitioners. Teacher learning is conceptualised as the development of their identity as teachers through participation in a community of practice. Importantly, Jaworski notes that development does not necessitate change and may instead perpetuate similar practice within a group of teachers which conforms to existing classroom practices. For example, Franke, Carpenter, and Battey (2008) describe working with a group of teachers who had limited experience of eliciting student thinking within the classroom. They found that this meant their existing practices did not lend themselves to engaging in conversations about the ideas of algebraic reasoning. Franke et al. explain how an activity used with the group involving the equals sign and true and false number sentences was appropriated by the teachers and turned into a worksheet, an artifact of their already existing practice. In this way, the teachers used an existing practice which did not challenge student understanding of the equals sign or promote discussion of it.

For improved learning opportunities for students to occur, teachers need to have opportunities and reasons to question their practice through intentional inquiry (Ghousseini & Sleep, 2011; Jaworksi, 2006; 2008). The use of inquiry as a tool within a community can enable teachers and educators to explore key questions and issues in practice (Jaworski, 2008). However, it is through engagement in 'critical alignment' of practice, whereupon teachers seek "to develop, improve or enhance the status quo" (Jaworski, 2006, p. 191) that the move can be made from a community of practice to a community of inquiry. The use of inquiry then shifts from being a tool to becoming a 'way of being' through which the participants in a community develop their practice (Jaworski, 2006; 2008). Examples of how a community of inquiry can be developed are evident in research studies (e.g., Blanton & Kaput, 2008; Franke et al., 2008, Jacobs et al., 2007;

Koellner et al., 2011) that report on the principles of effective professional development in early algebra. Common features of these successful case studies are the facilitation of reflection on mathematical understandings, student thinking, and instructional practices.

4.2.2 Reconceptualising teacher understanding of algebraic reasoning

This section provides a review of studies which highlight the key aspects of professional development specifically focused on introducing algebraic reasoning into primary classrooms. Much of the research drawn upon within this section is from the USA. It is important to note that the scarcity of research studies which investigate teacher learning in the area of early algebra within the UK context.

A key feature of the reviewed studies is the act of engaging teachers in reconceptualising their understanding of algebraic reasoning. This requires that teachers themselves both make sense of algebraic ideas and further their understanding of students' thinking about algebraic ideas (Franke et al., 2008). The literature highlights three primary ways of achieving this. These are engaging teachers in: (i) solving, analysing, and adapting tasks, (ii) anticipating and examining student responses to tasks, and (iii) reflecting on teaching practice.

Solving, analysing and adapting tasks

Teacher knowledge is recognised as key to effective teaching, both within early algebra and in a wider context (Anthony & Walshaw, 2009; Askew, Brown, Rhodes, Johnson, & Wiliam, 1997; Warren, 2009). Engaging teachers in solving authentic mathematical tasks is a useful way to provide them with opportunities to develop content knowledge and make sense of important algebraic ideas (Blanton & Kaput, 2008; Jacobs et al., 2007; Warren, 2009). A number of research studies (e.g., Franke et al., 2008; Jacobs et al., 2007; Koellner et al., 2011; Ruopp, Cuoco, Rasala, & Kelemanik, 1997; Schifter et al., 2008; Stephens, Grandau, Asquith, Knuth, & Alibali, 2004) highlight how the key or 'big' mathematical ideas within early algebra can be explored by developing sets of tasks for professional development in order for teachers to explore algebraic ideas collaboratively.

The use of tasks gives teachers opportunities to engage with a common mathematical and pedagogical experience. This can give a structure for the development of a supportive community of inquiry (Koellner et al., 2011). This is important as often mathematics, and in particular algebra, can be a source of anxiety for primary teachers who may not view themselves as strong mathematically (Blanton & Kaput, 2005a; Franke et al., 2008; Jacobs et al., 2007). Within professional development, teachers can be supported to work through the problems and the processes that will be used with students (Jacobs et al., 2007). For example, Franke et al. (2008) report on an aspect of their professional development where teachers worked collaboratively to create a written conjecture about commutativity, edit, and justify it. By working through the process of justification, teachers were both able to consider the arguments that students may use but also reflect on their own proof schemes.

Solving and analysing tasks during professional development offers teachers the opportunity to develop pedagogical and specialised content knowledge required for teaching. A number of research studies (e.g., Franke et al., 2008; Koellner et al., 2011; Ruopp et al., 1997; Stephens et al., 2004) highlight the benefits of teachers collaboratively solving problems, discussing solution strategies, and making connections between these. For example, teachers involved in this kind of work are able to develop links between mathematical thinking at different grade levels and analyse how tasks may be adapted for students working at diverse levels (Blanton & Kaput, 2008; Ruopp et al., 1997). Other outcomes include teachers being able to make connections between differing solution strategies, representations, and mathematical ideas (e.g., Franke et al., 2008; Koellner et al., 2011; Ruopp et al., 1997; Stephens et al., 2004). Solving tasks can prompt reflection on how different types of thinking can be elicited, student progress supported, and how tasks may be effectively sequenced (Franke et al., 2008). For example, Stephens et al. (2004) report on a professional development initiative where teachers were asked to solve a task related to variables. They then shared their representation and discussed how the task related to student understanding of variables and potential misconceptions. Following this, the task was used by the teachers with their classes. Two specific examples are provided of teachers enacting the task in a way that facilitated students to think deeply about variables and their representations and develop connections. Stephens et al. contend that the teachers' recognition of the algebraic opportunities offered by the task shaped how it was enacted in the classroom.

The use of tasks during professional development also offers opportunities for teachers to understand how to develop aspects of algebraic reasoning (Billings, 2008; Stephens et al., 2004). For example, Billings (2008) used tasks involving pictorial growth patterns with teachers in order to encourage them to think algebraically and construct an image of how algebraic reasoning could be integrated into the existing curriculum. Teachers developed generalisations using a variety of tools including the physical representations, symbols, and an analysis of change in the pattern. The repeated problem-solving experiences and reflection on the mathematical concepts and tools inherent in solving the problem supported the teachers to develop understanding of how different tasks develop aspects of algebraic reasoning.

During professional development designed tasks and curricular material may be used as exemplars for teachers to analyse and adapt for their classrooms. This can support teachers to explore opportunities for algebra in current curricular material as well as prompt teachers to adapt tasks or design their own. For example, Stephens et al. (2004) engaged teachers with examining curriculum materials to find opportunities for algebra. The teachers were also asked to take a lesson from a 'non-algebra' strand, 'algebrafy it', teach it, and then discuss it with the group. Following professional development using designed tasks Blanton and Kaput (2003; 2008) reported growth in the teachers' practice from using only the resources the researchers supplied to both developing their own resources and identifying those which were apparent in the existing curricula. Franke et al. (2008) also worked with teachers during professional development to support them to begin developing their own tasks. Through facilitating teachers to develop a set of index cards of true and false number sentences, these researchers reported a shift in teachers' focus from correct answers to how to sequence tasks and develop productive conversations. A feature of all of these studies is that in the process of analysing tasks and transforming curriculum material, the teachers begin to see how early algebra can be integrated into everyday mathematics lessons. A habit of mind was formed which allowed the teachers to view the existing opportunities for algebra in the mathematics they taught (Blanton & Kaput, 2008).

Examining and anticipating student thinking and reasoning

The reviewed studies suggest that a focus on student thinking and reasoning during professional development can facilitate teacher learning. Researchers claim that this teacher learning is a key driver in supporting productive changes in pedagogical practice (Blanton & Kaput, 2008). Additionally Jacobs et al. (2007) argue that through a focus on student thinking, professional development becomes sustainable with teacher learning continuing after the development has finished. Research studies (e.g., Blanton & Kaput, 2008; Franke et al., 2008; Herbel-Eisenmann & Phillips, 2008; Jacobs et al., 2007; Jacobs, Lamb, & Phillip, 2010; Koellner et al., 2011; Stephens et al., 2004; Warren, 2009) provide insight into the range of ways that professional development can be structured to include a focus on examining and anticipating student thinking and reasoning. These include: specifically designed assessment; development of frameworks of student thinking; analysing video episodes or examples of students' written work; sharing and discussing anecdotes about classroom interactions; and trialling tasks in the classroom and sharing the results. These studies provide evidence that these strategies, individually or collectively implemented, have enhanced both teachers' mathematical and pedagogical understanding and support a focus on students' algebraic reasoning.

Building teacher knowledge of how children develop early algebraic reasoning including expected progression and potential misconceptions is another important component for teachers' development of algebra ears and eyes (Watson, 2009). Using assessment tasks to analyse student reasoning is one way in which teachers may increase understanding of students' algebraic reasoning and misconceptions. Studies by Stephens et al. (2004) and Franke et al. (2008) involved researchers undertaking initial assessments with students and then sharing the results with the teachers. For example, Stephens et al. asked students to solve problems involving variables and then presented the research data to the teachers to motivate teacher inquiry into student reasoning and solution strategies. Similarly, Franke et al. began the professional development experiences by sharing data on how students aged 6-years to 12-years old solved number sentences such as $8 + 5 = \_\_ + 9$. These researchers also created a mid-year assessment which could be used by the teachers to gain an understanding of their students' ability to engage with algebraic reasoning. While this facilitated teacher knowledge of student reasoning, it also helped them to generate possible next steps for instruction and begin to develop a structure to make sense of students' algebraic reasoning.

Developing teachers' insight of student reasoning may also be supported through the examination of artefacts such as students' written work and anecdotes or cases from classrooms. The use of artefacts has been a central feature of a number of successful professional learning studies involving early algebraic reasoning. In the study by Herbel-Eisenmann and Phillips (2008), teachers examined a set of work by students aged 13-years to 14-years old on an algebra problem which explored linear, exponential, and quadratic patterns of change. Discussion about the student thinking apparent in the solution and justification supported reflections on what student work showed about their understanding and the pedagogical actions which could be used to extend algebraic understanding further. Other studies (e.g., Blanton & Kaput, 2008; Jacobs et al., 2007; Schifter et al., 2008; Stephens et al., 2004) required the teacher participants to implement specified tasks in their own classrooms and then bring the student responses to a meeting for analysis and discussion. In Schifter et al.'s (2008) study analysing the tasks prior to using them with their own students encouraged the teachers to predict their student responses and misconceptions. Following this, the task was used in the classroom and the teachers wrote a case study including examples of student thinking, the representations used, and the student constructed solution strategies. This was then shared and discussed with others in the professional development community. These studies show that examining artefacts of student work provides teachers with insight into student thinking and opportunities for reflection on initial predictions.

Learning to notice and develop understanding of different aspects of student thinking can also be supported by the use of videoed episodes from classrooms. A range of research studies (e.g., Franke et al., 2008; Jacobs et al., 2010; Koellner et al., 2011; Warren, 2009) demonstrate how video excerpts can be used to develop teacher understanding of student reasoning and to prompt reflection. For example, Warren (2009) contends that the video observations which were shared during professional development sessions in her study facilitated the discussion of content and pedagogical knowledge and provided visual examples of student learning. As an alternative or in addition to video observations, research studies (Jacobs et al., 2007; Warren, 2009) have involved teachers and researchers working collaboratively in the classroom. In the study by Jacobs et al. (2007), on-site visits were used to support teachers to identify relational thinking by students. In Warren's (2009) study, along with the use of video excerpts, the researcher supported teacher development by initially modelling what the teaching of patterns and algebra may look like within a junior primary classroom. Following this, the teachers worked in pairs to plan and teach lessons with support via email from the researcher. In these studies, working with authentic classroom exemplars facilitated teachers to notice different types of student thinking as well as the opportunities for algebra which were present in the classroom and student conversations.

Supporting teachers to organise their ways of understanding student reasoning and thinking is important to the mathematical goal. An integral aspect of noticing and responding to students' thinking is to be able to connect their reasoning to key mathematical ideas and tasks. Research studies (e.g., Jacobs et al., 2007; Smith & Thompson, 2008; Stephens et al., 2004) suggest that a way of achieving this is to ask teachers to generate possible responses to a task and describe what the responses show about student understanding. Jacobs et al. (2007) provide an example of this type of work from a professional development workshop in which the teachers were asked to predict the possible student responses to $8 + 4 = \_\_ + 5$. Their responses included not only incorrect solutions but also a range of ways in which the correct response could be generated. Undertaking this type of work supports teachers in developing their understanding of student reasoning. It can also aid teachers to think about the pedagogical moves which will support students and plan appropriate questions to facilitate a productive class discussion (Smith & Thompson, 2008; Stein et al., 2008).

Reflecting on teaching practice

A key element of professional learning which facilitates change such as the cultivation of early algebraic reasoning in the classroom is teacher reflection on practice (Back et al., 2009; Ghousseini & Sleep, 2011; Jacobs et al., 2010; Sherin, 2002b). The first essential step in developing the capacity to reflect on practice is that of noticing. As Jacobs et al. note "noticing is a common act of teaching" (p. 169). However, to develop expertise within a profession it is important to learn to notice relevant phenomena in a particular way (Jacobs et al., 2010; Mason, 2002). This process of learning to notice is only developed through engaging with the act of noticing. It requires both knowledge of the relevant aspects to notice within a situation and the ability to be aware of them and respond appropriately while engaged in the act of teaching (Franke et al., 2008; Jacob et al., 2010; Sherin & van Es, 2003). Within the context of early algebra, professional development activities need to facilitate teachers to develop their understanding of the pedagogies they are using, and as outlined previously the mathematical content involved, and the ways in which students make sense of algebra. Furthermore, teachers also need to develop a disposition of inquiry and reflect how their own practices are aligned or are in contrast with the ideas of pedagogies from research into early algebra which they are seeking to adopt (Ghousseini & Sleep, 2011).

Developing reflection on practice is an integral aspect of a number of studies (e.g., Blanton & Kaput, 2008; Franke et al., 2008) which examine professional development to support the teaching of early algebra. Reflecting on practice involves teachers developing their tools and skills for noticing relevant aspects of practice. Researchers and those involved in professional development can support this by providing teachers with a lens for viewing (Franke et al., 2008; Ghousseini & Sleep, 2011). This can take the form of specific questions or frameworks. For example, Jacobs et al. (2007) focused on supporting teachers to develop their understanding of how to facilitate student participation in the types of mathematical conversations which support algebraic reasoning. Prior to watching examples of classroom conversation on video, they discussed with the teachers what to notice and asked them to make note of productive questions that moved classroom conversations forward. Another example is provided by Blanton and Kaput (2005a) who worked with teachers to reflect on whether a culture of inquiry was developing within their classroom. Focused on the classroom norms for argumentation and student questioning and justification, teachers successfully used a framework to support them to notice both relevant aspects and the more complex, subtle features of practice.

Developing a disposition of inquiry is a critical element of reflecting on practice. This involves "being curious about and open to alternative explanations while maintaining a critical stance" (Ghousseini & Sleep, 2011, p. 155). In professional development, the growth of a disposition of inquiry may be encouraged by eliciting multiple conjectures and alternative explanations about classroom practice and the impact of instruction decisions, cultivating opportunities to critique, and by pressing participants to make claims and support these with specific examples and evidence (Ghousseini & Sleep, 2011). For example, Franke et al. (2008) asked teachers in their professional development group to reflect upon and discuss an example they had observed in one of the classrooms where the teacher had missed an opportunity to facilitate a discussion of commutativity. The ensuing discussion enabled the teachers to analyse the variety of options and why they may or may not be pursued within the classroom.

Reflecting on practice and developing a disposition of inquiry is necessary for long-term changes in practice (Blanton & Kaput, 2005a). Koellner et al. (2011) provide an example where following professional development a teacher participant reflected on the need for a change in his practice and to develop a critical perspective of his actions in the classroom. Analysis of videoed lessons in the following year illustrated changes in his practice which supported his students to engage with algebra. Similarly, Blanton and Kaput (2005a) present a significant case study of the instructional practices of a teacher who successfully developed a culture of algebraic reasoning in her classroom following professional development focused on reflection and inquiry.

4.3 STUDENT ROLE AND PERSPECTIVES

The teacher has an important role in reforming classroom contexts to include opportunities for engagement in algebraic reasoning. However, it is also important to acknowledge the student role within the classroom. A number of studies (e.g., Boaler, Wiliam, & Zevenbergen, 2000; Cobb, Gresalfi, & Hodge, 2009; Franke & Carey, 1997; Pratt, 2006; Young-Loveridge, 2005; Young-Loveridge, Taylor, Sharma, & Hawera, 2006) have recognised and advocated for the need to consider student perspectives and acknowledge their voice when researching changes to practice in the mathematics classroom. There is, however, limited research literature which addresses student perspectives and identity within classrooms where the focus of reform is algebraic reasoning. Therefore the following review will draw on findings from general studies which have investigated student beliefs and perspectives about mathematics in changing classroom cultures.

Bringing about significant change in classrooms has on-going consequences for both students' learning of mathematical ideas, the development of their mathematical identity, and their epistemological beliefs of mathematics (Cobb et al., 2009; Franke & Carey, 1997; Hodge, 2008; Mason & Scrviani, 2004; Star & Hoffman, 2005). Drawing on a community of practice approach (Lave & Wenger, 1991; Wenger, 1998), identity is developed by an individual as they engage in the everyday activities within a community of practice (for example, the mathematics classroom). As an individual participates, they learn the ways of thinking and acting which are valued by the community and thereby develop both a sense of what it means to be a member of a specific community and a sense of self in relation to the community (Boaler et al., 2000; Hodge, 2008). From this perspective, identity is a function of participating in different communities; it is dynamic and situated rather than stable and consistent. For different students, there can be a greater or lesser sense of belonging in relation to the community. This is related to how students come to understand what it means to do mathematics in the classroom and to what extent they identify with this (Boaler et al., 2000; Cobb et al., 2009). Cobb et al. (2009) contend that there are generally three outcomes with students identifying with the classroom mathematics activity, merely cooperating with the teacher, or resisting engagement with the classroom activities and developing oppositional identities. These researchers call for the need to develop interpretive schemes which focus on the relationship between the microculture developed in the classroom and the identities students are developing in the classroom. They argue that this interpretive scheme needs to attend to the "nature of mathematical activity as it is realized in the classroom; to what the students come to think it means to know and do mathematics in the classroom; and to whether and why they come to identify with, merely comply or resist engaging in classroom mathematical activity" (p. 41).

Views of mathematics are shifting and dynamic and both the teacher and classroom environment influence how students view the subject of mathematics and what they think it means to do mathematics in the classroom. Reform classrooms advocated by research studies which focus on bringing early algebra into primary classrooms (e.g. Blanton & Kaput, 2005a; Carpenter et al., 2005a; Koellner et al., 2011; Schifter et al., 2008a) promote epistemological conceptions of mathematics that differ from traditional classrooms. A typical view of students from traditional classrooms is to conceive mathematics as a static body of knowledge which consists of rules and procedures (Boaler et al., 2000; Franke & Carey, 1997). For example, studies by Young-Loveridge et al. (2006) and by Cheeseman (2008) report that many students perceived mathematics as a body of knowledge to be learned or equated learning mathematics to remembering numbers. These epistemological beliefs can affect performance (Boaler et al., 2000; Colby, 2007; Franke & Carey, 1997; Mason & Scrivani, 2004; Star & Hoffman, 2005; Young-Loveridge et al., 2006). For instance, students who view mathematics as a set of isolated facts may have difficulty in developing relational reasoning or in making sense of others' solution strategies. In contrast, studies which investigate student beliefs about mathematics from reformed learning environments (e.g., Boaler et al., 2000; Franke & Carey, 1997; Mason & Scrivani, 2004; Star & Hoffman, 2005) report that students who have experienced different forms of mathematics teaching have different epistemological conceptions of mathematics. For example, the studies by Mason and Scrivani (2004) and by Star and Hoffman (2005) show the positive impact that innovative instruction had on students' mathematical beliefs and the sophistication of their conceptions of mathematics. Franke and Carey's (1997) interviews with students aged 6-years to 7-years old who had been taught mathematics in a problem-solving environment, likewise, revealed students' sophisticated conceptions of mathematics. Students in this study associated doing mathematics with solving problems, using materials, communicating ideas, using more than one solution strategy, and extended engagement with specific mathematical problems.

Students' views of their role are an important factor when considering change within the mathematics classroom. Pratt (2006) argues that there is a need for learners to understand their role, particularly in relation to social interactions such as talking, listening, and group work. His study used video stimulated reflective recall to investigate students aged 8-years to 11-years old perceptions of interactive whole class mathematics teaching in the UK. The findings showed that the students appeared to understand their own role in developing their mathematical understanding through listening to others. However, listening appeared to be privileged over talking as a form of meaning making. Another study from the UK by Edward and Jones (2003) investigated student attitudes towards collaborative group work. Interviews with students for whom collaborative group work was common practice indicated that students for the most part held positive beliefs about collaborative group-work. Responses included the benefits of working together, listening to others and respecting their ideas, sharing knowledge, confidence building, and motivation. An important aspect noted by these researchers was the need to teach students the skills to work productively in collaborative groups.

For students who are inducted into reform classroom communities, there are shifts in their role as a learner. Students begin to engage in ways of learning that privilege different forms of knowledge and participation (Hodge, 2008; Hunter & Anthony, 2011). For example, Cheeseman (2007) found that in classrooms where there was an expectation that mathematical thinking would be explicitly described, a high number (85%) of students were able to describe their mathematical thinking explicitly in post-interviews using video stimulated recall. Another study by Hodge (2008) investigated students aged 6-years to 8-years old perceptions in regards to their role and mathematical competence in two classrooms with significant differences in the forms of instruction. The students in the study had spent the first year in a classroom with reform inquiry based instruction methods before moving to more traditionally orientated instruction in the second year. These students gave markedly different responses in regards to what it meant to be a good mathematics student in each class. In the classroom with inquiry based practices, competence focused on talking, thinking and listening. In contrast, within the traditionally orientated classroom success was associated with steps and answers. A further study by Cobb et al. (2009) with students aged 13-years to 14-years old investigated perspectives of students from a traditionally taught algebra class and a reform oriented statistical investigation class. The students' descriptions of their obligations and roles within each class were significantly different. In the traditional algebra class the student role during discussion was to listen to the teacher and use her method. This contrasted with the statistical investigation class where the students reported an obligation during discussion to explain their own analyses and ask questions to help themselves understand the analyses of others. Researchers use this as evidence to illustrate the different roles which students are required to take within different classroom contexts.

4.4 SUMMARY

This review has highlighted the importance of facilitating teachers to recognise opportunities for early algebra within their classrooms. A range of studies were examined to identify key aspects of professional development which have been found to support teachers to develop algebra ears and eyes. Developing a professional learning community was described as an integral factor. The review highlights how teachers can be facilitated to reconceptualise their understanding of algebra. Three key elements of successful professional development were identified as engaging teachers in solving, analysing and adapting tasks, examining and anticipating student reasoning, and reflecting on practice. This review also highlights the scarcity of studies from the UK on professional development in algebraic reasoning for primary teachers.

Acknowledgement is made of the need to consider students' roles and perspectives when instigating and investigating change within the classroom. A review of studies investigating student role and perspective in regards to mathematical reform shows how changes in the classroom can influence student beliefs, identity, and their role and obligations within the classroom. However, the literature search also highlights the limited number of studies which have looked at the student perspective in relation to the introduction of early algebra.

The following chapter describes the methodology of design research which is used in the current study. This was selected as appropriate to the theoretical framework and the overall aim of developing a learning environment promoted by current theory as productive but not yet common practice (Design Based Research Collective (DBRC), 2003).

CHAPTER FIVE

METHODOLOGY

5.1 INTRODUCTION

This study aims to investigate how teachers develop algebraic reasoning in a mathematical community of inquiry. Innovative in nature, the project set within the school setting employed design research and case study methodology.

Following a recap of the research questions, Section 5.3 overviews the qualitative research approach to the study. In Section 5.4 an explanation is given of design research and justification for its use within this study. The professional development model for the teacher development is discussed. Section 5.5 provides a description of the case study approach employed within this study and the reasons for its use.

Section 5.6 outlines how the initial participation from the schools and teachers was established. It then provides details of the research setting including the context of the two schools, study group participants, and the two case study teachers. In Section 5.7 the ethical considerations concerning collaborative research in a school based setting such as the current study are discussed.

Section 5.8 outlines the multiple methods of data collection used in this study. Section 5.9 overviews the data analysis process both in the field and the retrospective analysis after data collection was complete. Finally, Section 5.10 describes how the findings are presented.

5.2 RESEARCH QUESTIONS

This study addresses the key question:

How can teachers develop early algebraic reasoning in a classroom community of mathematical inquiry?

To address this key overall question, the study focuses on three smaller research questions:

- How do teachers develop algebra ears and eyes?
- What pedagogical strategies and classroom practices support student engagement in early algebraic reasoning?
- What shifts occur in the way students engage in classroom activity as early algebraic reasoning is integrated into the everyday mathematics lessons?

These questions acknowledge the key role which the teacher takes in leading change within the classroom and the role of the students in engaging with this change. The focus of the exploration is on the different routes which teachers may take while developing their algebra eyes and ears. Specifically, the study focuses on how teachers adopt and adapt key pedagogical strategies and classroom practices in their collaborative and personal inquiry into ways to support students to engage in algebraic reasoning. Implicit in the final research question is the understanding that student engagement may change when new practices and foci are introduced into the classroom. In particular, this study seeks to examine how student engagement in classroom activity changes as algebraic reasoning is integrated into the everyday mathematics lessons.

5.3 THE QUALITATIVE RESEARCH PARADIGM

This research study is grounded in a sociocultural perspective and draws upon a qualitative, interpretive research paradigm (Bogdan & Biklen, 2003; Cohen, Manion & Morrison, 2007). This type of research is contextual, has a practical interest, and is conducted within a naturalistic setting (Bogdan & Biklen, 2003; Cohen et al., 2007; Denzin & Lincoln, 2005). Therefore it was appropriate to apply a qualitative, interpretive approach to this study which was situated within classrooms at two different schools.

Qualitative, interpretive research is described by Cohen et al. (2007) as focusing on micro-concepts. These they identify as “individual perspectives, constructs, negotiated meanings, definitions of situations” (p. 42). Through the use of multiple methods, descriptive data is collected and used to generate a rich description of the social world. Within this world, acknowledgement is made of participants’ multiple realities and differing ways of understanding a situation (Denzin & Lincoln, 2005). In this study both the ‘lived’ experiences of teachers and their students are drawn on as they develop algebra ears and eyes and shift their ways of participating within the mathematics classroom.

A key feature of qualitative, interpretive studies is that the researcher is personally involved in designing, implementing, and interpreting the research. This complements design research where practitioners and researcher work in concert to introduce innovative practice and develop change in the context of practice (DBRC, 2003). Within this frame, close relationships exist between those being researched and the researcher (Cohen et al., 2007; Denscombe, 2003; Denzin & Lincoln, 2003). In this study I drew on design-based methodology as described by Fishman et al. (2004), to work alongside both teachers and their students to design, develop, implement, and evaluate an innovation which focused on early algebra within the classrooms. As such, design research offered opportunities to develop learning environments which have been promoted by current theory as productive yet may not be common practice.

5.4 DESIGN RESEARCH

Design-based research is highly interventionist (Cobb et al., 2003) and is identified as a way to "create and extend knowledge about developing, enacting and sustaining innovative learning environments" (DBRC, 2003, p. 5). Both the type and scope of design experiments can vary according to the purpose and setting (Barab & Squire, 2004; Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Gorard, Roberts, & Taylor, 2004; Zhao & Cobb, 2006).

Design research methodology has been increasingly used in recent years in mathematics education to investigate innovations in teaching and learning in a wide range of naturalistic settings including pre-service education, in-service development, classrooms, schools, and school districts. The use of a naturalistic setting means that this type of research is intertwined with practice while also a means to account for the complex influence of context on teaching and learning (Fishman, Marx, Blumenfeld, Krajcik, & Soloway, 2004).

Participants in design research studies are regarded as co-participants in the design and at times the analysis (Barab & Squire, 2004, DBRC, 2003; Gorard et al., 2004). In order for the study to investigate how teachers can develop early algebraic reasoning in a mathematical community of inquiry, it was necessary to address teacher understanding of teaching which focused on and promoted students' understandings of early algebra concepts. To achieve this, a design similar to that described by Zhao and Cobb (2006) was used where activities were used in one setting—

professional development sessions—to support the overarching goal that the teachers would reorganise their activity in a different setting—the classroom. This included developing teacher understanding of types of tasks, pedagogical actions, classroom norms, and mathematical practices which support students to engage in early algebraic reasoning. A second layer of design research involved teacher collaboration as co-researchers and designers within their classroom to investigate how algebraic reasoning could be developed with their students.

Design research involves a strong relationship between theoretical research and practice. Theory is used as the basis for designing systems which use specifically planned teaching methods to facilitate learning (Walker, 2006). Gravemeijer and Cobb (2006) describe the necessity of researchers developing local instruction theories. These are formulated by drawing on existing research literature and then making conjectures about the trajectory of the possible learning process and how this learning process can be supported through tasks and activities, the development of a learning community, and the role of the instructor.

An iterative process of continuous cycles used within design research allows the design to be adapted and modified as necessary (Barab & Squire, 2004; Cobb et al., 2003; Walker, 2006). As the design is enacted, the researcher analyses participation and learning and uses this to assess the validity of the conjectures which form the local instruction theory and revise specific aspects of the design. In a professional learning design study this requires the researcher to observe lessons and undertake debriefing sessions with the teacher so a shared interpretation of lessons and changes being made may be developed (Gravemeijer & Cobb, 2006; Zhao & Cobb, 2006). In this way the extended nature of design research requires researchers to maintain on-going relationships with practitioners (Cobb et al., 2003).

Educational settings are complex and messy interacting systems. Many researchers (e.g., Barab & Squire, 2004; Cobb et al., 2003; Gravemeijer & Cobb, 2006) argue that design research is a means to address the complexity and messiness of educational settings. Gorard et al. (2004) extend the argument explaining that this type of methodology is also messier than traditional forms of research because it is required to "monitor many dependent variables, characterise the situation ethnographically, revise the procedures at will, allow participants to interact, develop profiles rather than hypotheses, involve users and practitioners in the design and generate copious amounts of data" (p. 580). As a result, design research requires specific use of interpretative frameworks to translate observations and the data collected into scientific interpretations (Gravemeijer & Cobb, 2006).

The overall model drawn upon in this thesis is shown in Figure 3. This model draws together the key elements which are proposed as critical if teachers are to support their students to construct early algebra concepts in mathematics classrooms. The model was developed as a result of a number of factors. The elements of classroom practices and mathematical practices were identified in part from a previous research study in a classroom (Hunter, 2007). This study examined the development of a classroom culture which provided opportunities for children to engage in algebraic reasoning. All four elements were supported from the review of the literature which is described in Chapter Two and Three. The element of tasks and enactment drew on the literature but the importance of enactment became more strongly apparent as the study progressed.

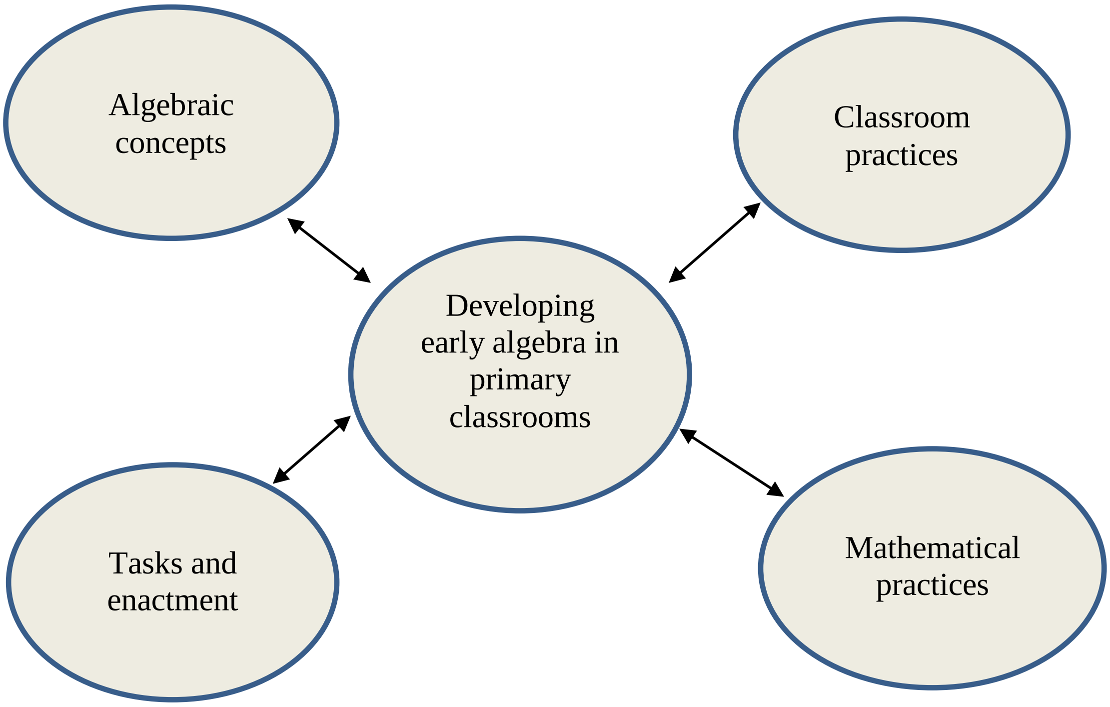

Figure 3. Key elements of developing early algebra in primary classrooms

5.4.1 Developing the professional development model

In the first instance the professional development model drew on findings from the literature chapters. The initial focus was placed on investigating teacher perceptions of early algebra and then to widen their understanding of this area by the introduction of an overview of early algebra concepts. The subsequent and on-going re-design of the model for professional development drew on researcher observations from the classrooms. For example, it was observed that the teachers needed professional development in facilitating students to generate and explore conjectures. In response, a task was designed to enable the teachers to explore possible conjectures which students would make and how these could be justified. The study group meetings, teacher interviews, and discussions also provided further information regarding the need for professional development activities. For example, both during study group meetings and interviews the teachers requested support in how to facilitate students to ask questions. Consequently in the following study group meetings research articles were used as the basis for a discussion on facilitating student questions. The focus for professional learning within the model comprised four separate but related components:

- Understanding of early algebraic concepts.
- Task development, modification, and enactment.
- Classroom practices.
- Mathematical practices

These foci of teacher learning/inquiry when combined with phases reflecting the possible trajectory of foci formed the conjectured framework of professional development as shown in Table 1. This was termed a conjectured framework because the focus of the study did not analyse whether each of the professional development activities was effective in regards to teacher development of algebra ears and eyes.

Table 1

Conjectured Framework of Professional Development to Develop Teachers' Algebra Ears and Eyes

	Understanding of early algebraic concepts	Task development, modification, and enactment	Classroom practices	Mathematical practices
PHASE ONE	Concept map of early algebra. Teacher analysis of student reasoning (task based interview results). Teacher examination of - Equals sign. - Relational reasoning - Commutative property. Overview of early algebra concepts.	Examination of MEP curriculum material for links to early algebra concepts.	Classroom development of - Collaborative discourse. - Mathematical explanations. Used framework to reflect on current classroom practice and set goals.	

<table>
<tr><td rowspan="3">PHASE TWO</td><td colspan="3">Used framework and classroom video-recorded lessons as reflective tools with a focus on
- Developing collaborative group work. - Developing whole class discussions.
- Strategies to integrate early algebra.</td></tr>
<tr><td>Developed new concept map of early algebra. Compared and contrasted previous understanding.
Examined a case of student reasoning to build understanding of relational reasoning.
Examination of common errors when using variables.</td><td>Used task to predict and plan for student responses.
Identified and adapted MEP tasks to develop algebraic reasoning.
Wrote number sentences using different numbers or properties which drew on relationships.</td><td>Orchestrating a productive whole class discussion.</td></tr>
<tr><td></td><td colspan="2">Classroom development of making conjectures and justification
- Generate possible student conjectures.
- Predict student justification of a conjecture.
- Justify a conjecture in three ways (using representational material, a verbal explanation, and symbolic form).</td></tr>
<tr><td rowspan="5">PHASE THREE</td><td></td><td>Collaborative lesson planning.</td><td>Developed overarching aim for the learning community.</td></tr>
<tr><td colspan="3">Lesson study post-lesson observation reflective meeting. Focus on
- Pedagogical strategies. - Student responses.
- Task design.</td></tr>
<tr><td></td><td>Collaborative lesson planning.
Identified and adapted MEP tasks to develop algebraic reasoning.</td><td>Classroom development of generalisation.
Used framework to reflect on current classroom practice and set goals.</td></tr>
<tr><td colspan="3">Reflective discussion. Focus on
- Understanding of early algebra. - Task design.
- Pedagogical strategies to support early algebra. - Student reasoning and participation.</td></tr>
<tr><td colspan="3">Lesson study post-lesson observation reflective meeting. Focus on
- Pedagogical strategies. - Student responses.
- Task design.</td></tr>
</table>

Initial instructional activities included in the professional development were selected to explore and challenge teachers' existing understanding and beliefs about early algebra. For example, an opening activity involved teachers drawing a concept map of their understanding of early algebra. This activity supported both the teachers and the researcher to reflect on their current understandings of early algebra. In subsequent meetings, it provided further opportunities for the teachers and researcher to continue reflecting on their developing understanding of early algebra.

Research articles were used as multi-purpose tools in the professional development. A range of articles and excerpts from research texts[1] were shared with the teachers during the study group meetings. These were used to extend teacher understanding of early algebra, to provide models of classrooms which would support early algebraic reasoning, and to promote reflection on current practice. Discussions held after reading each article required the teachers to respond to questions such as "what did you find interesting?" or "are there any ideas that you could bring to your classroom after reading that?" The articles also served the purpose of developing links between research and classroom practice as advocated by Watson (2009).

The selection, design, and enactment of tasks were a central focus for professional development. The current study built on previous studies (e.g., Blanton & Kaput, 2008; Franke et al., 2008; Jacobs et al., 2007; Schifter et al., 2008; Stephens et al., 2004) that successfully used mathematical tasks to engage teachers in reconceptualising their understanding of algebra. In these studies the use of algebra tasks provided teachers with multiple opportunities to reflect on their own understanding of algebraic concepts and the mathematical practices which support students' learning of early algebra. For example, in the current study the teachers were asked to

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Monaghan, F. (2005). Don't think it in your head, think aloud: ICT and exploratory talk in the primary mathematics classroom. *Research in Mathematics Education, 7,* 83-100.
Kazemi, E. (1998). Discourse that promotes conceptual understanding. *Teaching Children Mathematics, 4*(7), 410-414.
Carpenter, T. P., Levi, L., & Farnsworth, V. (2000). Building a foundation for learning algebra in the elementary grades. *In Brief, 1*(2), 1-8.
Carpenter, T., Franke, M., & Levi L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school.* Portsmouth: Heinemann.
Smith, M. S., Hughes, E. K., Engle, R. A., & Stein, M. K. (2009). Orchestrating discussions. *Mathematics Teaching in the Middle School, 14*(9), 548 – 556.

solve number sentences involving variables, develop their own number sentences, and at another time asked to develop different forms of justification for a conjecture. Specific tasks from the MEP curriculum material were used to provide ways for the teachers to identify opportunities for algebraic reasoning and also to investigate ways of modifying and further developing existing tasks.

Other research studies (e.g., Blanton & Kaput, 2008; Franke et al., 2008; Herbel-Eisenmann & Phillips, 2008; Jacobs et al., 2007; Koellner et al., 2011; Stephens et al., 2004) that highlight the usefulness of a focus on student thinking and reasoning to facilitate changes in practice also influenced the design of the professional development. In addition to designing and critiquing tasks, teachers in this study were encouraged to predict responses that students would give to algebraic tasks. They also engaged in activities which investigated student responses from task-based interviews conducted by the researcher and undertaken at the beginning and end of the study.

Prompting reflection on practice was a key element of the professional development. This required that the teachers were able to develop tools and skills for noticing relevant aspects of their practice (Franke et al., 2008; Ghousseini & Sleep, 2011). To support the teachers in this study to reflect on their own practice, they were provided with an adapted framework (see Appendix A) from Hunter (2009) which detailed classroom and mathematical practices linked to the development of algebraic reasoning. The framework provided the teachers with an objective lens to use when viewing video records from their own classrooms. It was also a useful tool to support them to reflect on their practices and set future goals.

As the study progressed, teachers had opportunities to engage in a lesson study cycle which drew different elements of the professional development together. Lesson study is based on a Japanese model of teacher development which emphasises student learning and reflection on practice (Lewis, 1995; Stigler & Hiebert, 1999). It aims to increase teachers' knowledge about mathematics, ways of teaching mathematics, and ways in which learners engage with and make sense of mathematics (Fernandez & Yoshida, 2004). Although a superficial view of lesson study is that it is focused on developing the 'perfect' lesson, the deeper intention of the lesson study cycle is to support teachers to engage with the processes of teaching and learning. Participating in a lesson study cycle can prompt teachers to reflect on their own approaches to the processes of teaching and learning and develop practices in ways which are meaningful within their working contexts (Burghes & Robinson, 2010; Stepanek & Appel, 2007).

In the lesson study process used in this study, each group of teachers worked as a community within their own school. The initial step involved the establishment of an over-arching aim which was relevant to each school. This collaboratively agreed goal established that the teachers wanted to develop creative and flexible problem-solvers. Following this, the teachers planned an area of focus for the study lessons. The foci corresponded to mathematical concepts their students had difficulties with or those which the teachers felt less confident about teaching. For example, at Hillview school the teachers wanted to address how their students over-generalised the commutative property to include subtraction and division. A lesson study cycle was devised which included lessons designed to facilitate student understanding and justification of the commutative property with a focus on the use of representations to model conjectures and justify reasoning. At Beaumont school the study lesson cycles aimed to develop student skill at solving multi-step word problems and part of the focus was placed on the equal sign. Through collaborative activity the 'study lessons' were planned and taught by one of the teachers and observed by the others. Teacher reflection was prompted from observation during the study lessons and the post discussion prompted by key questions (see Appendix B). The lesson was then re-planned and further developed on the basis of the student responses and consequently re-taught to a different group of students in another classroom while being observed by the other teachers and discussed again in a meeting following the observation.

5.5 CASE STUDY

Qualitative case studies are commonly used in educational research (Merriam, 1998). This is a holistic form of research methodology which uses multiple sources and methods of data collection to provide rich and detailed descriptions to illustrate findings and support theoretical conclusions (Cohen et al., 2007; Denscombe, 2003). Case studies may focus on a single case or involve multiple cases or sites. When examining multiple cases, the use of a comparative case study approach is used to develop understanding of both the unique and common factors between each case (Merriam, 1998). By undertaking a detailed examination of the site and group within the natural environment, the aim is to develop understanding and expand the range of interpretations (Scott & Usher, 1999).

Case study methodology was employed in this study due to the bounded nature of the schools, teachers, and students. The study included an in-depth examination of two distinct sites and groups within their natural environment and utilised multiple sources of data collection. A comparative case study approach was chosen as appropriate to investigate the development of two teacher’s algebra ears and eyes and the differing ways in which they used their developing understanding to implement changes within their classrooms. A detailed description of the reasons for the selection of the two case study teachers is provided in Section 5.6.2.

5.6 RESEARCH SETTING AND BACKGROUND TO THE STUDY

As outlined in Chapter One, this research was initiated as a follow-up to a previous observational research project focusing on teacher enactment of the Mathematics Enhancement Project (MEP). When the initial observational research project was completed, separate meetings were held with the senior management at Beaumont School and Hillview School[2] to discuss the possibility of staff participating in a collaborative research project focused on developing algebraic reasoning. An outline of the proposed research project was given to the teachers during a staff meeting and an invitation was made to those who wished to participate. Following this, three teachers from each school who were previously involved in the observational research project agreed to participate in the project.

In design-based research such as used in this study, teachers and researchers collaborate to develop meaningful changes (DBRC, 2003). Previous involvement with both schools meant that the foundation for future collaboration with the teachers was established. Moreover, my earlier role in the school as a researcher observer established my role as an ‘adopted’ staff member and enhanced both staff members and students’ acceptance of my presence within the classrooms.

5.6.1 Description of the schools

Beaumont School is a primary school in the British Isles. All classes are mixed ability and students have the same classroom teacher throughout the day. At the time of the study, it was the third year in which the school had been using the MEP curriculum material. Students at this school generally come from middle to high socio-economic home backgrounds. Most of the

[2] The names of both schools have been changed.

students are of white British ethnicity although a number of students also come from other ethnic backgrounds including African and Caribbean British descent, Asian, Polish, and Portuguese.

Hillview School is a primary school situated in a suburban area on the outskirts of London. Students in Key Stage Two are taught their daily mathematics lesson in ability groups with a lower and higher set for each year group. At the time of the study, it was the fourth year in which the school had been using the MEP curriculum material. Like Beaumont School, students at Hillview School generally come from middle to high socio-economic home backgrounds. Most of the students are of white British ethnicity although a number of students also come from other ethnic backgrounds including African and Caribbean British descent, Asian, and Indian descent.

5.6.2 Participants in the study groups

At both schools each study group had three teachers who had chosen to participate in the study—at Beaumont School a Year Two teacher and two Year Three teachers and at Hillview School two Year Three teachers and a Year Five teacher. The two Year Three teachers at Hillview School shared a class and all teachers in this study group taught the upper set mathematics class. All participant teachers had 6 to 18 years teaching experience. In Chapter Six and Seven, one teacher from Beaumont School and one teacher from Hillview School are featured as cases. These two teachers were selected due to the distinctly different pathways they took in developing their algebra ears and eyes and facilitating algebraic reasoning in their everyday mathematics lessons. Whilst both teachers had engaged in similar activity during the study group meetings they developed differing understanding of early algebra and at the end of the project there were distinct differences in opportunities available to the students in their respective classrooms for the development of algebraic reasoning. These two case teachers and the composition of their classes are described in the following section.

5.6.3 The case teachers and their students

To preserve the anonymity of the teachers and students involved in the two cases featured in the findings chapters, it is suffice to say that both teachers, each with at least 10 years of teaching were regarded as experienced classroom teachers. Both classrooms involved 25 students of mainly white British ethnic origin. Mrs Stuart[3] from Beaumont School (case reported in Chapter Six) taught Year Three students working at achievement levels between Level One and Level Four on the Primary National Curriculum (Department for Education and Skills (DfES), 1999). Mrs Willis[4] from Hillview School (case reported in Chapter Seven) taught upper set Year Five students working at achievement levels between Level Three and Level Five on the Primary National Curriculum (DfES, 1999).

5.7 ETHICAL CONSIDERATIONS

An important aspect of social research is ethical considerations. This study was designed and conducted in accord with Plymouth University's ethics guidelines and ethical approval was sought and obtained prior to data collection. It also built on guidance from the British Educational Research Association (BERA, 2011) which outlines the underlying ethical principles which guide the activity of educational researchers as including informed consent,

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[3] A pseudonym

[4] A pseudonym

openness, protection from harm, confidentiality, and privacy. The following section outlines how these principles were considered for this study.

5.7.1 Informed consent

Informed consent is regarded as an important principle of ethical research (Heath, Charles, Crow, & Wiles, 2007). Ethical guidelines from BERA (2011) provide a definition of informed consent as "the condition in which participants understand and agree to their participation without any duress, prior to the research getting underway" (p. 5). This requires that participation is voluntary and that individuals are provided with appropriate information to make a reasoned judgement about participation. In this project there were a number of reasons that it was important that the teachers had a thorough understanding of the research process. These included the lengthy on-going nature of the project, the collaborative aspects, and the significant time commitment involved in planning meetings, reflective interviews, and activities undertaken by the study group.

Due to the collaborative nature of this research informed consent was a key part of this study. However, in contexts such as primary schools, significant challenges may be faced in gaining informed consent. Heath et al. (2007) highlight these challenges as "inequalities in status between gate-keepers, researchers, and participants" (p. 404). For this study, discussions held with each of the teacher participants prior to the project included a detailed outline of the project, an overview of what participating would entail, and the opportunity for individuals to ask any questions. Prior to the discussions, areas which were identified as having the potential for harm were considered by the researcher so they could be discussed with the participants. These included embarrassment or discomfort from being observed both by the researcher and members of the study group, video-taping of classroom episodes and study group meetings, and reviewing video footage both individually and collectively within the study group. In accord with recommendations (e.g., David, Edwards, & Alldred, 2001; Valentine, 1999) that emphasise the need for consent to be re-negotiated throughout collaborative research designs, these areas were re-visited and discussed throughout the study. The benefits of being involved in the study were also highlighted to allow the teachers to consider both the risks and potential outcomes.

Following the initial discussions, the teacher participants were provided with information sheets about the project (see Appendix C). The information sheets re-iterated the right to refuse to answer particular questions, to ask for recording devices to be turned off and comments excluded, and the right to withdraw from the study at any time during the data collection phase. Teacher participants were asked to complete written consent forms after reading the information sheet.

Informed consent is also an important consideration when undertaking research with children. BERA (2011) advocates that children should be "facilitated to give fully informed consent" (p. 6) in collaboration with the approval of those who act in guardianship. Valentine (1999) describes the importance of both verbal and written information being provided to children and highlights asking children to sign a consent form as a means of giving them control, autonomy, and privacy. In this study, the project was first discussed with the students in their classroom including an opportunity for the students to ask any questions. They were then provided with parent and student information sheets and consent forms (see Appendix D). To maintain informed consent through the research project, students were always asked for verbal consent prior to any interviews and were also advised that they could choose whether to answer the questions or ask for the recording device to be turned off.

5.7.2 Anonymity and confidentiality

A norm within social and educational research is for participants to be given anonymity and data collected to be treated confidentially (BERA, 2011). However, anonymity between teacher participants and within the classroom was impossible as the participants were known to each other and the wider school community knew who was involved in the project. Participants were advised of this difficulty both verbally and on the information sheet prior to the beginning of the study. Participant schools, teacher, and student participants were assigned pseudonyms to ensure anonymity within the wider dissemination of the project. In order to preserve the anonymity further, only relevant information has been provided about the schools, the case teachers, and their students.

Confidentiality and anonymity are related to each other as they are both concerned with privacy of the participants. Tickle (2001) defines anonymity as preventing the identity of individuals being released and confidentiality as guaranteeing data will not be shared with others in any form. This means that confidentiality ensures specific contributions to the research data cannot be identified. In this study, although the study group participants both observed lessons, and video footage (selected by the teacher participants of their own practice), discussed instances from lessons, and engaged in focus group interviews, this involved no-one beyond the study group and teacher participants. Additionally, reflective interviews were held with individual participants to ensure that confidentiality was maintained.

5.8 DATA COLLECTION

The following section describes the timeline for data collection and the range of data collection methods used within the study. These included observational field-notes from the participant observations, video recorded observations, documents, teacher interviews, and photo elicitation interviews with students.

Table 2 and Table 3 outline the research programme during the course of the study from the initial observations to the point of withdrawal from each school. The schedule of observations and study group meetings are detailed.

Table 2

A Time-line of Data Collection from Beaumont School

2009 Term 3	
April -May	Observations prior to professional development at Beaumont school
2009 Term 1	
Week 4	Meeting with Senior management to discuss proposed research. Invitation to teachers to join the research project.
Week 6	Information sheets and consent forms provided for senior management and teachers.
Week 9	Teacher participation reconfirmed. Information sheets and consent forms provided for students and caregivers.
Week 12	Task-based interviews and semi-structured interviews with the students. Initial study group meeting. Data collection in classrooms. Photo-elicitation interviews with the students. Semi-structured interviews with the teachers.
2010 Term 2	
Week 7	Study group meeting. Data collection in classrooms. Photo-elicitation interviews with the students. Semi-structured interviews with the teachers.
Week 11	Study group meeting. Data collection in classrooms. Photo-elicitation interviews with the students. Semi-structured interviews with the teachers. Lesson study observation and reflective meeting.
2010 Term 3	
Week 4	Study group meeting. Data collection in classrooms. Photo-elicitation interviews with the students. Semi-structured interviews with the teachers.

Week 6	Data collection in classrooms. Photo-elicitation interviews with the students. Semi-structured interviews with the teachers. Lesson study observation and reflective meeting.
Week 8	Data collection in classrooms. Photo-elicitation interviews with the students. Semi-structured interviews with the teachers. Lesson study observation and reflective meeting. Task-based interviews and semi-structured interviews with the students.
Week 12	Lesson study observation and reflective meeting.

Table 3

A Time-line of Data Collection from Hillview School

2009 Term 3	
June July	Observations prior to professional development at Hillview school Meeting with Senior management to discuss proposed research. Invitation to teachers to join the research project.
2009 Term 1	
Week 2	Information sheets and consent forms provided for senior management and teachers.
Week 3	Teacher participation reconfirmed. Information sheets and consent forms provided for students and caregivers.
Week 5	Task-based interviews and semi-structured interviews with the students.
Week 6	Initial study group meeting.
Week 7	Data collection in classrooms. Photo-elicitation interviews with the students. Semi-structured interviews with the teachers.
Week 10	Study group meeting Data collection in classrooms. Photo-elicitation interviews with the students. Semi-structured interviews with the teachers.
Week 13	Data collection in classrooms. Photo-elicitation interviews with the students.
2010 Term 2	
Week 5	Study group meeting Data collection in classrooms. Photo-elicitation interviews with the students. Semi-structured interviews with the teachers.
Week 9	Data collection in classrooms. Photo-elicitation interviews with the students. Semi-structured interviews with the teachers. Lesson study observation and reflective meeting.
2010 Term 3	
Week 2	Study group meeting Data collection in classrooms. Photo-elicitation interviews with the students. Semi-structured interviews with the teachers. Lesson study observation and reflective meeting
Week 5	Data collection in classrooms. Photo-elicitation interviews with the students. Semi-structured interviews with the teachers.
Week 9	Data collection in classrooms. Photo-elicitation interviews with the students. Semi-structured interviews with the teachers. Task-based interviews and semi-structured interviews with the students.

5.8.1 Participant observation

Observation is a frequently used method of data collection in classroom studies. It can be classified broadly into two categories: systematic observation and participant observation. The observational methods used within these two categories differ and involve varying levels of structure from tightly formed observation schedules to less structured observation notes which emphasis depth rather than breadth of data (Cohen et al., 2007; Denscombe, 2003). In this research, I took the role of participant observer similar to that which is described by Clarke (1996) as a "participant in the actions, events and contexts being studied" (p. 4). Within this role, the degree of immersion in the research varied—from comprehensive involvement to complete detachment. During the study group meetings the role which I took was central in leading all activity and discussions. In contrast, the role which I took in the classrooms was largely as a passive observer. At no times did I intervene during the lesson or engage with the students unless requested by the teacher. At the same time, I acknowledge my presence within the classroom may have influenced some of the teacher and student actions during the observations. The role which was taken aligns with the role other researchers have taken while working with teachers and in classrooms in design-based research (e.g., Falkner et al., 1999; Franke et al., 2008; Jacobs et al., 2007; Koellner et al., 2011; Zhao & Cobb, 2006).

Undertaking a participant observational study over an extended period of time allows the researcher to develop a "holistic understanding of the phenomena under study" (DeWalt & DeWalt, 2002, p. 92). Sustained involvement and observations help to develop better understanding of the dynamics of a situation, the context, and the phenomenon under study (Cohen et al., 2007). Participant observation was an important element of this research as it helped to guide and define the relationships with the participants whilst also supporting the researcher to develop an understanding of the organisation of the mathematics classrooms and lessons. This included the relationships between the teacher participants as a learning community during the study group meetings, the teacher and student participants within the classroom, and what both the teacher and student participants valued as important related to mathematics teaching and learning and more specifically the developing algebraic nature of the classroom.

5.8.2 Video recorded observations

The opportunity that video recordings offer for close documentation and observation in naturalistic settings such as the classroom or teacher professional development is evident in their increasing utilisation in education research for teacher development and classroom design experiments (Clarke, 1996; Derry et al., 2010; Pirie, 1996; Powell, Francisco, & Maher, 2003). Video records offer both permanence and opportunities to re-visit and re-examine data that has been recorded.

The use of video-recording as a tool offers researchers the flexibility to gain detailed accounts of behaviour and interactions (Barron, Pea, & Engle, 2013; Powell et al., 2003). Derry et al. (2010) use the analogy of the camera as a microscope which increases the amount of interactional data that can be collected. An advantage of video recorded observations is that all that is in view of the eye of the camera is recorded and unlike participant observations this is not a selective process in regards to what is recorded (Rosenstein, 2002). However, as a number of researchers (e.g., Derry et al., 2010; Pirie, 1996; Powell et al., 2003; Rosenstein, 2002) highlight, video recordings are selective in regards to the positioning of the camera and the breadth of the lens. An initial challenge for researchers is to identify the aspect of the complex environment that they wish to record. In this study, each study group meeting was video-recorded with one camera placed in the corner of the room to capture the participants sitting around a table. Mathematics lesson observations were also recorded with one camera. Lessons were recorded in their entirety including the introduction of each task, group work, and the whole class discussion. The camera was placed to focus on the front of the classroom and whiteboards which were used both by the teacher and students during the task implementation and whole class discussion.

Video-recorded observations are a valuable tool for data-gathering; however there are also key methodological challenges in its use (Derry et al., 2010; Powell et al., 2003; Rosenstein, 2002). Issues of accurate representativeness can be raised when introducing video cameras to naturalistic environments. Both Roschelle (2000) and Rosenstein (2002) note that in front of a camera; individuals may change their behaviour and ways of interacting. This was addressed in the current study by explaining the purpose of taping the lessons to the students, by the teacher modelling natural behaviour, and by familiarising students with having a camera in their classroom with practice runs and also by its use regularly over a prolonged period.

The large amount of data captured in a video record creates its own challenges for analysis. Derry et al. (2010) note that for researchers to extract data and meaning from video records they need to use theoretical frameworks and research questions as a way to focus. However, one needs to be aware that one's use of theoretical interests to influence selection and focus can "both constrain and shape later analyses and presentation of results" (Powell et al., 2003, p. 408). Powell et al. (2003) argue that to develop more complete accounts, it is necessary to augment and triangulate video records with other forms of data. In this study the analysis drew on multiple data sources.

5.8.3 Documents

Documents support the development of a rich account of an event and in design research they may be used to track changes and also provide sources of information about how learning was generated and supported (Cobb et al., 2003; Cohen et al., 2007). In this study, documents gathered from the professional development included both the teacher participants' responses to activities undertaken during the study group meeting (e.g., concept maps, predictions of student responses, written and diagrammatic justification of conjectures, annotated lesson plans, and teacher reflections and goal setting annotations). Documents collected from the classroom work included work samples, tasks, lesson plans, and photographs from the lessons of small group work or student recorded solution strategies on the whiteboard.

5.8.4 Interviews with teachers

Interviewing can be used to develop insight into multiple perspectives and to gain deeper understanding of alternative meanings (Denscombe, 2003; Scott & Usher, 1999). While there appears to be many similarities between a conversation and an interview, Denscombe (2003) takes care to highlight that interviews are more than simply conversations. Described as an "interchange of views between two or more people on a topic of mutual interest" (Kvale, 1996, p. 14), a key consideration is that participants have the security to talk freely (Cohen et al., 2007). This was important in this study, as during the interviews I sought to develop my understanding of the perspectives of the teachers in regards to how they were developing their own understanding of early algebra and also how they were facilitating algebraic reasoning in their classrooms. Although I took an active role in developing and facilitating the study group activities, during the interviews I needed to empower the teacher participants as experts on their own developing practice.

The current study used two forms of recorded teacher interviews: semi-structured individual interviews and group interviews. During each school visit informal semi-structured individual interviews were held post the lesson observation. During these interviews the teacher and I briefly discussed the lesson including the tasks, student participation, and the development of algebraic reasoning. This was used both to promote reflection on practice and also as a way to support the development of future tasks.

In addition to post-lesson interviews each teacher participated in two formal semi-structured interviews with the researcher. The first interview, held in the second phase (approximately mid-way) of the study, explored the changes the teacher participants were noticing in their teaching and student engagement, pedagogical strategies that they were using to facilitate the development of algebraic reasoning, challenges they were facing, and the areas that they were planning to focus on in future lessons. The second interview, held towards the end of the study, investigated the teacher participants' reflections on their engagement in the research study. This included teacher reflections on task development and implementation, changes in the classroom and mathematical practices, and personal understanding of early algebra.

Records of group discussions that occurred during the study group meetings and the lesson study cycle were used to gain reflective data from the teacher participants on involvement in the research study and the perceived changes in their classrooms and teaching.

5.8.5 Photo elicitation interviews with students

Visual methods have been increasingly used in educational research to support interviewing children and provide insight into their worlds (e.g., Cappello, 2005; Clark-Ibanez, 2004; Meo, 2010). Using traditional interviews when working with children has been identified as problematic due to a number of factors including: level of linguistic communication, cognitive development, power relationships between adults and children, and the question and answer format (Cappello, 2005; Clark, 1999; Clark-Ibanez, 2004).

Photo elicitation interviews (PEI) are a form of research interview that involves the use of photographs. Images in PEI can be used in different ways as visual inventories, depictions of events or to document dimensions of the social (Harper, 2002). Use of photographs in interviews can help develop rapport between the child participant and researcher and also ease the awkwardness of the interview situation (Clark-Ibanez, 2004; Meo, 2010). In this research, I took photographs to capture images that depicted classroom events. Acting as "medium of communication between researcher and participant" (Clark-Ibanez, 2004 p. 1512) these images were used to provoke a participants' memory and prompt discussion.

A methodological issue when using PEI is who takes the photographs and what should be photographed to prompt reflection on an experience (Clark-Ibanez, 2004; Harper, 2002). In the current study, the researcher took photographs depicting instances during the whole class discussion featuring students sharing their solution strategies. For each of the lessons involved, between four to six students were selected on the basis of ensuring that all students participated in an approximately equal number of interviews spaced over the school year. The students were approached prior to the lesson to seek agreement for the post-lesson interview. In the post-lesson interview, students were shown between three to four photographs and asked to describe what was happening from their perspective and what they were thinking. This included asking them to recall solution strategies and explanations provided by their peers along with their own mathematical reasoning. They were also asked to respond to questions which focused on the newly developing classroom practices and mathematical practices such as engaging in collaboration with their peers, talking about their mathematical ideas, and listening to others share their responses (see Appendix E for the schedule of interview questions).

5.9 DATA ANALYSIS

A characteristic of both qualitative research and design research methodology is the close relationship between data collection and data analysis with each informing the other while the study is in progress (Cobb et al., 2003; Cresswell, 1994).

5.9.1 Data analysis in the field

To ensure that the design of the study was informed by evidence from the field, data collection and data analysis occurred simultaneously. With regards to the professional development the researcher watched the videos of the study group meetings after each session and wrote reflections which were used to assess assumptions about the study group activities. During observations in the classroom the researcher wrote reflective notes that were then used to test and inform the conjectures developed about teacher learning and to develop further conjectures which informed the design of the study group activity. They were also used to inform and shape the subsequent design of the framework of teacher actions to develop early algebraic reasoning (see Chapter Six) including feedback on whether aspects of the framework needed to be revised or extended.

5.9.2 Retrospective data analysis

Qualitative data is rich, complex, and messy. Analysing such data is a time-consuming process which requires the researcher to reduce a large amount of information to patterns and categories and then re-interpret this through the use of a schema. Characteristically, data analysis begins with the wide use of many different categories and with continual coding and review the major themes become apparent (Cresswell, 1994).

For large data collections, Derry et al. (2010) suggest that beginning with a set of guiding questions while also expecting unanticipated findings supports the initial data analysis process. They highlight that this does not exclude discovery oriented work. Similarly, Engle, Conant, and Greeno (2007) describe an approach which they refer to as progressive refinement of hypotheses. In this approach a general question is formed and data collected in response. After viewing the recorded data, hypotheses are formed. These are then examined in relation to other aspects of the data set and further evaluated and refined through multiple iterations of hypothesis generation and evaluation.

In this study, the research questions were used as guiding questions for the initial data analysis. Video observations were repeatedly viewed in their entirety and the transcripts were iteratively revised to ensure a reliable record was created (Barron et al., 2013; Derry et al., 2010). Coding began by examining the data collected from one teacher and her classroom. The initial codes were developed from a variety of sources including research literature, the initial viewing of the video records, and the observational and reflective field notes. Repeated viewing of the videos and re-reading of the transcripts and field notes confirmed or refuted the initial hypotheses and codes and other hypotheses and codes were developed as necessary. Multi-levels of coding were used to analyse the classroom video data through the use of QSR International's NVivo 10 qualitative software programme (2012). The first level of coding generated four codes (parent nodes): algebraic content, task design, teacher actions, and student actions. The derivation of sub-codes (child nodes) will be explained in the following section.

In the first instance, the video recorded observations from the classroom were divided into events (Derry et al., 2010). Each event segment consisted of the teacher implementing a task, students working on the task, and the subsequent whole class discussion. Those events that included algebraic content were further coded according to the nature of the content focus (see Table 4).

Table 4

Child Nodes for Algebraic Content

Associative property
Commutative property
Distributive property
Equivalence
Functions
Generalising a mathematical process
Inverse relationships
Odd and even numbers
Properties of rational numbers
Properties of zero and one
Relational reasoning
Using and solving equations
Using variables

Additionally all the event segments were analysed and coded if they contained a missed opportunity for algebraic reasoning. At times this included events which had an algebraic element when the implementation of the task or enactment resulted in a missed opportunity for algebraic reasoning for students.

The next iteration of coding focused on the task design as a parent node. Initially, event segments which had an algebraic element were classified as planned algebraic tasks or spontaneous algebraic tasks. Other elements of the task design were also coded and at the conclusion of this iteration, after testing, re-testing, discarding and confirming, there were five additional elements of task design coding (see Table 5).

Table 5

Child Nodes for Task Design

Independently developed task
Modified task from MEP curriculum
Open-ended task
Task focused on generalisation
Task involving misconception

Another level of coding focused on teacher actions. There were eight child nodes identified (see Table 6).

Table 6

Child Nodes for Teacher Actions

Developing classroom norms
Responding to errors
Facilitating productive discourse
Focusing on student thinking and reasoning
Developing generalisations
Supporting mathematical explanations
Use of questioning
Use of representations

Where appropriate, sub-codes were derived for some of the child nodes. For example, questioning was refined to include sub-codes listed in Table 7.

Table 7

Examples of Sub-codes for Use of Questioning

Models questioning
Provides space for students to ask questions
Probes or uses questioning to gain further reasoning from students
Uses questioning to position student
Establishing context
Exploring mathematical meaning or relationships
Facilitating generalisation
Gathering information or leading students through a procedure
Inserting terminology
Orienting focus
Linking and applying

The final parent code, student actions, included six child nodes (see Table 8).

Table 8

Examples of Child Nodes for Student Actions

Discourse
Errors
Questioning
Mathematical explanations and language
Generalisation
Justification and proof

Again many of these child nodes were further refined to include sub-nodes. For example, the coding for the child node student discourse is shown on Table 9.

Table 9

Sub-codes for Student Discourse

Provides mathematical reasoning
Provides answer with no reasoning
No response
Describes procedure
Gives answer as a question
Analyses other students' reasoning
Agrees with mathematical reasoning
Disagrees with mathematical reasoning
Revoices peers' reasoning or explanation
Unable to revoice peers' reasoning or explanation
Recognises peers' error during explanation

A portion of a transcript with coding through the use of Nvivo (2012) is shown in Appendix F. Examination of classroom artefacts, transcripts of teacher and student interviews, and the observation and reflective field-notes were used to triangulate the data analysis.

To augment the detailed analysis by codes, quantitative analysis was also completed (Barron et al., 2013). In the first instance, a table was created which examined the use of different types of algebra in each lesson and the task design through both the percentage of time and number of instances for each code for each teacher. Data analysis tables and graphs were also used to examine the teacher and student actions through quantification (see Appendix G for examples) and identify overarching patterns. Following this, close examination was undertaken of examples from within the codes. The identified patterns and themes were used to develop an understanding of how task development and enactment and teacher actions facilitated early algebra in the classroom and students' engagement with algebraic reasoning.

5.10 DATA PRESENTATION

The findings of the data analysis are illustrated through two cases of teachers engaged in professional development, their mathematics teaching within their classrooms, and their students' participation in the lessons. The cases provide a detailed and thick description (Barab & Squire, 2004; Gravemeijer & Cobb, 2006) of how each of these teachers developed algebra ears and eyes and then how effectively they facilitated algebraic reasoning in their classrooms. Direct quotations from the teachers are used in the findings from the study group meetings and interview data. The use of teacher voice provided a means to develop an understanding of their focus and reasoning as they worked to develop their own algebra ears and eyes and algebraic reasoning in their classroom. The discussion also draws on the teachers' task design and implementation and use of pedagogical strategies. Vignettes and examples from the classroom are provided to illustrate the actions of the teachers and students. Findings pertaining to corresponding shifts in student engagement and participation within each of the classrooms are also included with data sourced from classroom videos and photo elicitation interviews.

The findings are reported in four distinct phases beginning with an overview of classroom observations in the school year prior to the professional development. The following phases report on the findings at the beginning, middle, and end of the study.

5.10.1 Ecological validity, generalizability, and trustworthiness

A key aim of design research is to achieve ecological validity, that is "the results should provide a basis for adaptation to other situations" (Gravemeijer & Cobb, 2006, p. 46). In this way innovative instruction which is developed in the course of design research can then be used by others to support student learning productively in different classroom contexts. This is similar to how Barab and Squire (2004) describe generalisability within the context of design-based research with the findings from one study being able to be replicated to other contexts. Use of thick description is highlighted as a means of making replication possible and also supporting teachers and outsiders to think how the instructional design could be adjusted to their context appropriately (Barab & Squire, 2004; Gravemeijer & Cobb, 2006; McKenney, Nieveen & van den Akker, 2006). The reporting of this study aims to provide a comprehensive description of all elements of the research. The cases include insight into teacher participants' involvement in the professional development activities and a rich description of the teaching and learning context based on observation and participants' (teachers and learners) perspectives. Furthermore, the inclusion of teacher participants' feedback to develop and modify the framework of teacher actions (see Chapter Six) to facilitate algebraic reasoning significantly strengthens the ecological validity of the study (Gravemeijer & Cobb, 2006).

Trustworthiness and credibility are important elements of design research and may be seen as akin to notions of reliability and validity (Barab & Squire, 2004; Cobb et al., 2003). With the large amount of data generated in the course of a design experiment it is central that the resulting claims are trustworthy. Therefore there is strong link between trustworthiness and the credibility of the analysis. Cobb et al. (2003) argue that by explicitly describing the criteria and types of evidence used for inferences, other researchers can then "understand, monitor and critique the analysis" (p. 13). In the current study, a systematic approach was used to code the large amount of data generated through video records, interviews, and field-notes. There is a detailed description of the varied forms of data collection and also of the data analysis and coding given in this chapter.

The credibility and trustworthiness of a research study can also be enhanced through triangulation of different data sources. This is based on the premise that by using multiple data sources, weakness in a single data source is counterbalanced by strengths in another (Barron et al., 2013; McKenney et al., 2006). The current study drew on multiple sources of data including interviews, documents, and field-notes to triangulate the primary video data source. Extended engagement by the researcher with participants in the field also enhances credibility and trustworthiness of analysis and can reduce observer effects (McKenney et al., 2006). A key feature of this study was the sustained engagement with the teacher participants acting as co-researchers and the students in their classrooms over the duration of the study.

5.11 SUMMARY

This chapter outlined the key research question and guiding sub-questions for the study. Justification was provided for the selection of the qualitative research paradigm and use of design research methodology. A summary of the key characteristics of the design research and case study methodology was given

Rich descriptions were given of the research setting and data collection methods. It was shown how the data collection drew on multiple sources which provided opportunities to investigate perspectives from different participants. The study was shown to be cyclic and iterative and the complementary role of data collection and analysis was highlighted. Detailed descriptions of the multiple levels of coding were given which supported the development of the case studies.

Findings presented in the following chapters are based on two cases. Each case details a participant teacher's experiences during the professional development programme. They serve to illustrate the different pathways teachers take in developing algebra ears and eyes. They highlight the complex factors which are involved in facilitating algebraic reasoning in the classroom including task development and implementation and the development of classroom and mathematical practices which support engagement in early algebra. Subsequent shifts in student engagement and participation are documented.

CHAPTER SIX

DEVELOPING ALGEBRAIC REASONING IN A MATHEMATICAL COMMUNITY OF INQUIRY: MRS STUART

6.1 INTRODUCTION

Within this chapter, the case study is organised in four distinct phases which relate to when data collection occurred. The first section reports on the classroom context prior to the beginning of the professional development. Each of the following sections reports on involvement in the professional development; how algebra was integrated into the classroom; then the developing classroom culture; and finally the student perspectives. A commentary that links the changes in the classroom to the literature accompanies each section.

Section 6.2 describes the classroom context in Mrs Stuart's classroom prior to the professional development commencing. Section 6.3 highlights the initial steps to introduce algebraic reasoning. It begins by outlining Mrs Stuart's active participation within the study group and her early reflection on practice. It shows how Mrs Stuart began to plan for algebraic reasoning opportunities although this was not supported by her existing classroom practices. Finally it portrays the implementation of changes to the classroom community and students' engagement with these.

Section 6.4 shows the continuing development towards integrating algebraic reasoning in the classroom. Focusing on task implementation, it describes how Mrs Stuart shifted to using inquiry as "a way of being" (Jaworksi, 2006; 2008). It also demonstrates the emergence of teacher requirement for collaborative interaction and shifts in students' learnt ways of thinking and acting mathematically.

Section 6.5 outlines the actions to embed algebraic reasoning in the classroom further. It details Mrs Stuart's re-conceptualisation of her understanding of algebra. It shows how this supported her to design and implement tasks which provided greater affordances for algebraic reasoning. Finally it illustrates the changing practices in the classroom and shifts in student roles.

6.2 PRIOR TO THE PROFESSIONAL DEVELOPMENT

6.2.1 Algebraic reasoning, classroom and mathematical practices, and student participation

Classroom observations prior to the professional development provided evidence that Mrs Stuart had some understanding of the links between arithmetic and algebra. Instantiations of the following types of algebra were evident during the five observed lessons: commutative property, functions, properties of zero and one, associative property, equivalence, inverse relationships, and odd and even numbers. However, Mrs Stuart did not facilitate the students to examine explicitly the properties of operations and numbers. For example, in a lesson involving the commutative property students were asked to share responses related to the following task (see Figure 4):

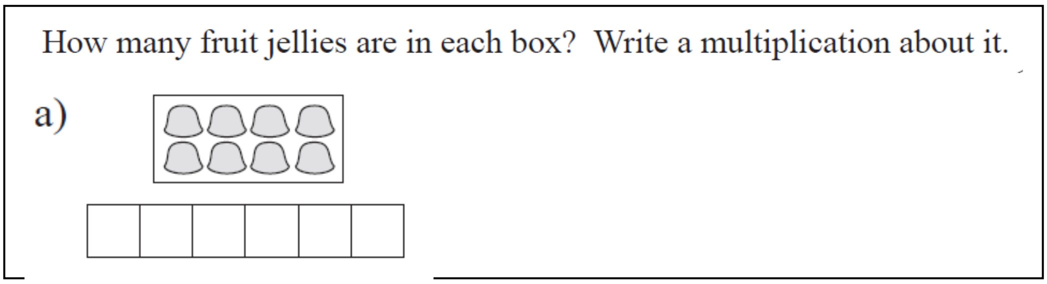


Figure 4. Multiplication problem. From *MEP practice book Y2b* (p. 107), by S. Hajdu, 1999, Budapest: Muszaki Publishing House.

The student constructed equations (e.g., 2 x 4 = 8 and 4 x 2 = 8) implicitly drew on the commutative property; however, there was no further examination of this. In a later lesson, Mrs Stuart began by asking the students what they noticed about the two alternative solutions that she had recorded on the whiteboard and then offered a brief explanation of the commutative property herself:

> Mrs Stuart: *What do you notice, Otto?*
> Otto: *It's the other way around?*
> Mrs Stuart: *What do you mean by it's the other way around?*
> Otto: *It's, it's the same but it's just changed around*
> Mrs Stuart: *And that's one of the really important things in multiplication, isn't it? It doesn't matter if we do two times five or five times two.*
> (Week Two, Term Three, 2008/2009)

The tasks used by Mrs Stuart with her students were taken directly from the MEP curriculum. Aside from one instance in the fifth lesson, the tasks were not modified.

Mrs Stuart spent a significant portion (between 16% and 27%) of the lessons introducing or orienting the students to the task. This involved students being carefully guided through the steps necessary to complete the task with a focus on a fast pace.

For almost half of the whole class discussions prior to task completion, Mrs Stuart used questioning characterised as leading or funnelling students towards correct responses or teacher chosen solution strategies.

An example of Mrs Stuart's task implementation prior to the professional development is provided in the following vignette where Mrs Stuart is introducing her class to a functional reasoning task (see Figure 5):

How many legs do several hens and cats have? Complete the table.

Number of each type of animal	0	1	2	3	4	5	6	7	8	9	10
Number of hens' legs			4								
Number of cats' legs								28			

Figure 5. Functional reasoning task. From *MEP practice book Y2b* (p. 107), by S. Hajdu, 1999, Budapest: Muszaki Publishing House.

Implementing a functional reasoning task

Mrs Stuart begins to introduce the students to the task by asking them the numbers of legs for each animal. She then asks them whether they think they could complete the table and most of the class put their hand up. She continues to carefully guide them through the completion of the first boxes on the table by acting each scenario out:

Mrs Stuart	We'll do one or two together and then we'll see altogether, so first column…how many of each animal are we talking about? Gabriel?
Gabriel	Zero
Mrs Stuart	Zero, so here we go. Watch I've bought them in especially for you today. Ready. Here you are, zero chickens (Mrs Stuart lifts imaginary chickens in her hands). How many legs on my zero amount of chickens Lorenzo?
Lorenzo	Two.
Mrs Stuart	How many legs can you see on this chicken?
Lorenzo	Two.
Mrs Stuart	Really?
Class	Zero.
Mrs Stuart	How many chickens are here?
Lorenzo	Zero.
Mrs Stuart	So how many legs are here?
Lorenzo	Zero
Mrs Stuart	Fill that in in the table…Right and here is my (lifts imaginary cat) how many cats am I holding?

She continues to lead them through filling in the table until the first three columns are completed. Then she asks if they think they can complete the table in a minute. After they have worked on the task for one minute she stops them.

Mrs Stuart	Okay, who managed to do some of the table? If you haven't done all of it, we'll fill it in as we go. So what do you notice about the middle row? What are those numbers called?
Aaron	Even.
Mrs Stuart	They're even numbers, good boy. What else are they, Elijah?
Elijah	They're multiples of two.
Mrs Stuart	They're multiples of two. So, are you ready?…Off we go…
Elijah	Zero, two, four, six, eight, ten, 12, 14, 16, 18, 20.
Mrs Stuart	Excellent. Well done. Now if you didn't do them all you can pop those in now. Can you see they're multiples of two?

Week Two, Term Three, 2008/2009

At this stage the questioning did not focus student attention on the relationships within tasks, and frequently guided them towards using computation to complete the table rather than to reason algebraically. Whole class discussions after the students had worked independently on a task were used to check that they had correct answers or to direct them to write the correct answer in their work book.

Mrs Stuart regularly used representational forms in her teaching. She drew on representations suggested in the MEP material including tables and concrete materials. Often she directly modelled how to use a representation to solve a problem. She also frequently asked students to use equations to show how they had solved a problem. However, the use of representation was typically limited to a single representational form.

Paired work was a feature of Mrs Stuart's lessons. After a task had been introduced, the students were requested to work with a partner. While some pairs worked cooperatively on the task, others simply sat next to each other rather than working together. Rather than complete tasks collaboratively, the partnerships appeared to be more used as a support mechanism when the students were stuck. As noted by Patrick, from the focus group discussion, having a partner helped him: *because if you don't know what to do, you have someone right there and the other* *person might know.* The expectation that students would record their work individually in their books matched the lack of expectation for students to use collaborative paired work to develop shared understanding of a joint solution strategy.

The teacher domination of discourse patterns during the whole class discussion limited opportunities for student engagement in mathematical reasoning. Frequently students gave answers with no mathematical reasoning or responded with an answer phrased as a question, for example: *is it nine?* There were no complete correct mathematical explanations provided by the students, however Mrs Stuart provided mathematical explanations at least twice and as many as seven times in a lesson. Often if a student provided a response with no reasoning, she would revoice it and provide a mathematical explanation herself.

The students perceived Mrs Stuart's role as giving them mathematical knowledge. For example, in a focus group interview when asked what their teacher did to help them, they stated:

Geoff:	*She explains it for you*
Researcher:	*How does she explain it for you?*
Geoff:	*Because in your book it says fill in the missing numbers and then if you want to do it with your partner she lets you and then you put your hand up if you don't and she explains it for you.*

When asked what it takes to be successful in the mathematics class, they responded:

Lorenzo	*To listen*
Researcher:	*To listen. Why does that help you?*
Lorenzo:	*Because you'll learn every single maths.*
Researcher:	*By listening?*
Lorenzo:	*Yeah*
Geoff:	*If you listen then she sometimes gives you sneaky information, if you listen carefully.*

These responses illustrate that the students privileged listening to the teacher over talking as a way of participating in the mathematics classroom.

6.2.2 Analysis of opportunities for algebraic reasoning, classroom and mathematical practices, and student participation prior to professional development

There was no explicit identification or examination of the properties of numbers or operations during lessons. This meant that for students, the properties remained implicit and they were not provided with opportunities to develop deep generalised understanding as advocated by many researchers (e.g., Anthony & Walshaw, 2002; Carpenter et al., 2003; Schifter et al., 2008b). Nor were they supported to develop algebraic reasoning through effectively utilising links between arithmetic and algebra. Blanton and Kaput (2005a) and Carpenter et al. (2003) contend that this is essential to develop early algebraic reasoning. Although Mrs Stuart showed some evidence of her own understanding of these links, her classroom practices did not support students to develop rich algebraic reasoning. As a result, lessons before the project began were characterised by missed opportunities.

The types of questions used limited the opportunities for the students to make sense of what the task was asking them to do or to develop their own mathematical understanding. Frequently, the task implementation guided the students through the necessary steps—all they had to do was fill in the answers on a table. As Henningsen and Stein (1997) explain, this results in reduced cognitive demands for students. Classroom discourse followed a structure of IRE with a focus on correct answers rather than student reasoning. This is a pattern of discourse within traditional classrooms where the teacher is the authority (Fisher et al., 2011; Mehan, 1979; Pape et al., 2010). While paired work was a feature of instruction in the classroom, the teacher scaffolding described by Monaghan (2005) and Rojas-Drummond and Zapata (2004) to structure productive small group interactions was not present.

Although tasks within the MEP curriculum offered opportunities for students to use multiple representational forms, students were often asked only for one representation. This limited their opportunities to make connections between different representations or to use these to communicate and justify (Beatty & Moss, 2006).

Classroom discourse patterns offered students limited opportunities to verbalise their ideas or to develop a sense of agency (Bell & Pape, 2012; Fisher et al., 2011; Nathan et al., 2009). Student perception of the teacher role was consistent with what Bell and Pape (2012) report as common in traditional mathematics classrooms with an IRE pattern of discourse. They viewed the teacher as the source of mathematical knowledge whose role was to transmit it to them.

6.3 PHASE ONE: INTRODUCING EARLY ALGEBRA

6.3.1 Teacher learning

From the onset of the project Mrs Stuart was an active member of the study group. She freely and confidently shared ideas with the group. Mrs Stuart drew a concept map (see Figure 6) to show what she considered early algebra was when enacted in a primary classroom.

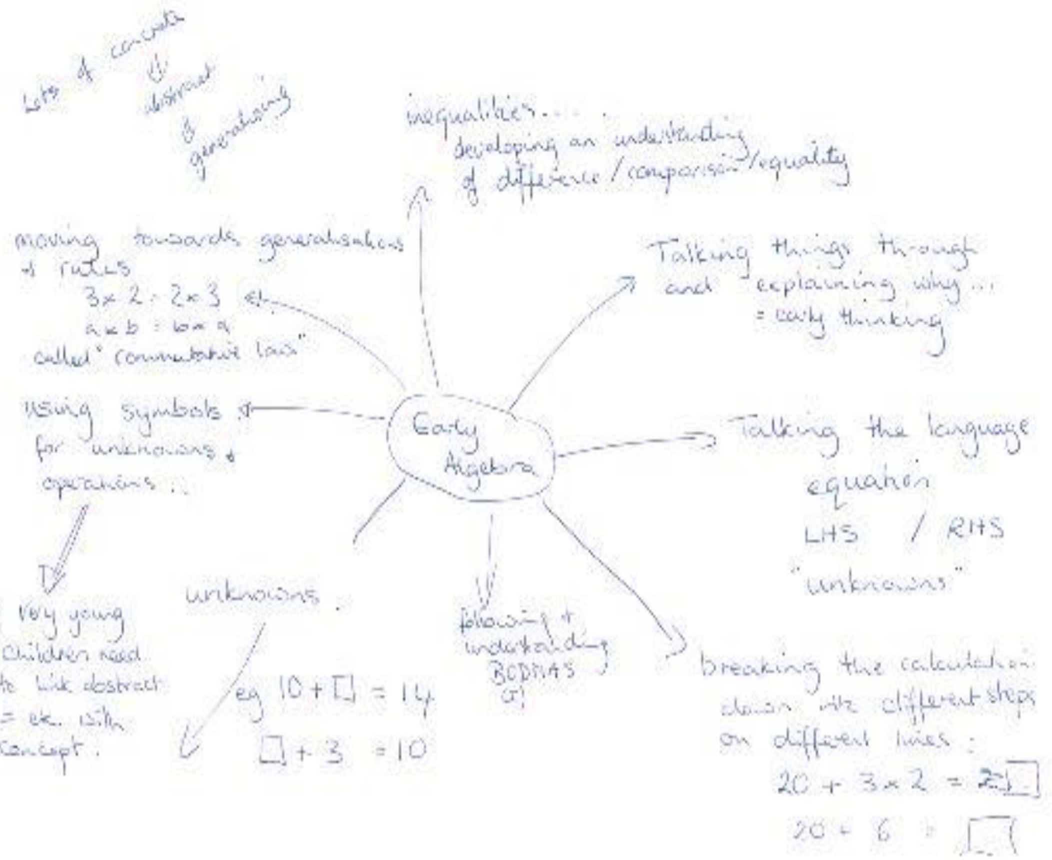


Figure 6. Mrs Stuart's concept map of early algebra at the start of the project

Her concept map suggested awareness of some aspects of the learning environment which can support students to engage in algebraic reasoning including discourse, mathematical language, and moving towards generalisations. She had begun the project with some understanding of the links between arithmetic and algebra but her pedagogical content knowledge of algebra did not readily extend to the expected progression of student learning or the potential misconceptions that her students may exhibit. Although she included on the concept map: *developing an understanding of equality,* her knowledge of potential student responses to open number sentences was incomplete and required researcher scaffolding. When asked to predict potential student responses to an open number sentence (e.g., 8 + 6 = __ + 5) her initial response drew on a previously taught correct computational strategy:

Mrs Stuart: *My class would look at the left hand side and the right hand side and then would say we will start with the left hand side because we could work that out and they would put the 14 above there and then write the 14 above that and work it out.*

Following extended discussion she identified a possible misconception related to the equal sign: *they might add them all*. Again after further pressing (by the researcher) she identified a possible relational solution: (indicating each part of the equation) *So six and the five and then make that nine.* She indicated strong interest in her students' responses to the task and on viewing student interview data she engaged in a prolonged analytical discussion with the researcher to understand what the varying responses indicated about her students' reasoning.

Research articles were introduced to the teachers to develop links between theory and practice. These were used as a tool by Mrs Stuart to begin reflecting on her existing practice. She noted her emphasis on listening rather than talking: *I have always spent a lot of time teaching children how to listen and what a good listener looks like, body language and eye contact but I don't think I have ever taught them the skills of talking.* Linked to this she recognised a need to shift the pattern of the classroom discourse including developing awareness of the need for an equitable classroom community: *It's getting those less able children to contribute too so that they can feel like they are part of the whole problem-solving thing.* She also noted the need to shift from an emphasis on teaching mathematics in a fun and exciting way to a focus on the deeper ideas: *they had the classroom community thing which we have been pushing and pushing but the really deep thing wasn't making everything fun and exciting, you know that is not enough, it is getting deeper into the ideas.*

Mrs Stuart's reflection on practice in the first meeting extended to her monitoring of student reasoning. Taking a critical stance she recognised that her emphasis while students worked independently was on supporting children identified as lower-achieving: *[I'll] go in and support those children who need that little bit extra so you are really focused on them but you are not tuning in, you are just aware that everybody else is busy and getting on.* Reflectively she stated that a key goal for her was to listen to students: *I am very aware of the need to be tuning in and listening to the children.* She also recognised that changing her practice and classroom norms would be an on-going process that required continual reflection: *it will be that reflective thing afterwards and that 'oh I could have done that' or thinking of a different way.*

6.3.2 Analysis of teacher learning

Mrs Stuart positioned herself as an active participant in the developing community of practice. Blanton and Kaput (2008) maintain that engaging with 'social' work related to facilitating algebraic reasoning and developing a professional identity within the group are important factors in developing links to algebra in classroom practice. Her responses during the study group meeting indicated that she was beginning to question her own practice and use inquiry as a tool. Similar to the findings of Jaworski (2008), this supported Mrs Stuart to explore key questions and issues in practice.

Initially Mrs Stuart showed limited insight into student thinking. She needed opportunities to develop a structure to make sense of students' algebraic reasoning. Teacher knowledge of expected student progressions and potential misconceptions are important factors in developing classrooms with a focus on early algebra (Franke et al., 2008; Watson, 2009). As previously illustrated in research studies (e.g., Franke et al., 2008; Jacobs et al., 2007; Smith & Thompson, 2008; Stephens et al., 2004), predicting student responses and using student assessment data to analyse reasoning proved useful to support Mrs Stuart to focus on students' algebraic reasoning and enhance her pedagogical understanding.

Additionally Mrs Stuart used research articles focused on classroom practices as a lens to view, notice, and reflect on her practice. Her reflections show her development of skills to notice relevant aspects of practice and thus develop a disposition of inquiry as described by Ghousseini and Sleep (2011).

6.3.3 Task design and implementation, integration of algebra, and development of the classroom community

An immediate change that Mrs Stuart made in the classroom involved her implementation of tasks as problem-solving opportunities. This included emphasising student effort to approach and complete cognitively challenging tasks. Mrs Stuart shifted student attention away from recording answers through implementing a new requirement that students first talk about the task with a partner. Initially the withdrawal of teacher guidance/instruction resulted in some students incorrectly approaching tasks or not completing within the time allocated. Commenting after a particularly challenging lesson, Mrs Stuart said: *last year I probably thought it was more successful but actually it wasn't…we were very much whole class together and it was very controlled, there were probably an awful lot of them who were sitting looking like learners but weren't actually engaging.*

In this phase Mrs Stuart, in line with design research, began intentionally developing and trialling ways of adapting her planning to integrate algebraic reasoning into her lessons. She examined MEP lesson plans and rather than asking students to complete the whole task, she presented them with parts of the task which focused their attention on an algebraic concept. However, as illustrated in the following vignette she did not notice or use opportunities for the spontaneous integration of algebra based on student responses, nor emphasise a deeper investigation of algebraic concepts. In this vignette Mrs Stuart planned to use a task (see Figure

7) to focus student attention explicitly on the inverse relationship between multiplication and division:

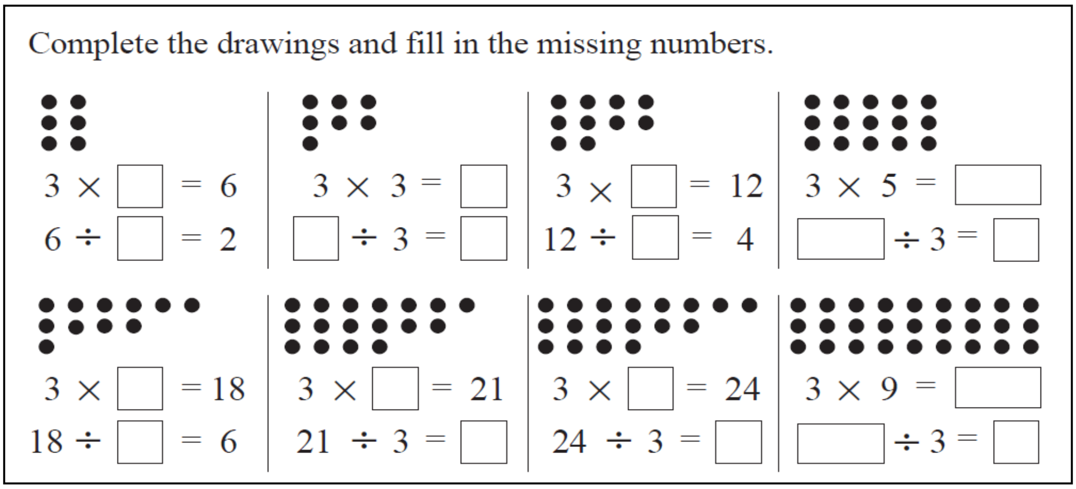


Figure 7. Inverse relationship between multiplication and division. From *MEP practice book Y2b* (p. 100), by S. Hajdu, 1999, Budapest: Muszaki Publishing House.

Developing links between arithmetic and algebra
Mrs Stuart asks the students to complete the first example on the task. After asking students to share their responses she records 3 × 2 = 6 and 6 ÷ 3 = 2 on the whiteboard:

Mrs Stuart — Let's have a look at those did anyone notice anything? Three times two equals six and six divided by three equals two. With your partner, what do you notice about those please?

The students talk with their partner, then she asks a student to say what he noticed:

Tristan — They're just the other way around.

Mrs Stuart asks Tristan to clarify his response:

Tristan — Because the three is in the middle and the six is at the beginning and at the end.

She revoices the response and directs the students to examine related equations where the position of the numerals has changed:

Mrs Stuart — So it's the same digits. Would it work if I put them in any order? If I did this *(writes on 2 ÷ 3 = 6 on the board)* two divided by three equals six because I've got the same numbers. Just talk that one through with your partner or what about this one, three divided by six equals two, is that true? Or six divided by three equals two *(writes the different equations on the board)* Are any of those true?

This was followed by further whole class discussion and the use of individual students modelling whether each equation is true through the use of magnetic counters on the whiteboard. Following this, she models writing the equation as $a \times b = c$. She then states her conjecture and asks them to generate equations for it:

Mrs Stuart	I have this theory that for every pair of factors and a product I can make two multiplications and two divisions let's see if that's right. With your partner at Planet X can you see if you can come up with equations for that?
Week 12, Term One, 2009/2010	

Here we see how Ms Stuart's initial questioning focused student attention on the general relationship between multiplication and division. However further questioning, shifting the focus to specific equations, limited the opportunities for students to explore the relationship. Then by asking the students to use magnetic counters to solve each equation, she shifted their attention specifically to calculating answers, thus the focus on the inverse relationship was lost. As shown, concrete material was introduced as a tool to solve the task rather than as a means of developing an argument and proving or justifying. Opportunities for the students to develop and explore their own conjectures and prove and justify their reasoning were missed by Mrs Stuart telling the students the conjecture that she had developed and then guiding them towards simply generating equations to match the conjecture.

To develop student participation, Mrs Stuart identified the need to focus student attention on developing new forms of productive talk. She drew on ideas from research articles introduced during the study group meetings and facilitated a discussion of how to talk together successfully and invited students to generate a set of rules. Agreed rules included the need for active participation, individual responsibility to listen to others, and also group responsibility to ensure that all class members understood. We see in the following vignette how Mrs Stuart affirmed productive shared discourse norms during paired or small group work:

Shaping ways to talk and work in a group	
Mrs Stuart drew attention to how a group ensured all participants understood their explanation and could explain it.	
Mrs Stuart:	Zanthe said to everybody 'do you get it?' And everyone nodded, but you didn't get it, did you? How did you know that Calvin hadn't got it?
Zanthe:	He nodded but he wasn't sure

Mrs Stuart:	How did you know he wasn't sure?...What did you ask him to do?
Zanthe:	I asked him to explain it.

Week 12, Term One, 2009/2010

The facilitation of discourse required constant and deliberate actions in this phase. In the next vignette, we see how Mrs Stuart facilitated students to explain and clarify their ideas. She used revoicing and questioning to introduce them to mathematical language. She also supported students to reflect on their peers' ideas through asking them whether they thought a response was correct. However, on this occasion she did not probe for further reasoning. She began developing an expectation that students analyse and make sense of reasoning by explicitly providing space for them to question their peers about their solution strategy and to agree or disagree. This practice of highlighting models of good practice was trialled extensively in this phase:

Developing collaborative interaction

Mrs Stuart asks the students to generate different two factor equations using the digits two, three and five. A student begins to provide her solution strategy:

Esme:	We think we should work out two times two first, then two times three and two times five.
Mrs Stuart:	Does anyone want to ask her a question?
Caleb:	If you were to do that, how would you be able to know whether you'd done the two and five, or two and three, or two and two, how would you know?
Mrs Stuart:	Do you think you would know Esme? Would you like to show us?
Esme:	(*records* $2 \times 2=$, $2 \times 3=$, $3 \times 2=$, $2 \times 5=$ *and* $5 \times 2=$)

Mrs Stuart asks the rest of the class to reflect on whether there are equations recorded that are not needed.

Alexis:	She doesn't need the ones that are the other way around.
Mrs Stuart:	Can you remember what it was called when it was the other way around?
Alexis:	The commutative law.
Mrs Stuart:	The commutative law. So which ones shall she rub out then Juliana?
Juliana:	The three times two and the five times two.

Week 12, Term One, 2009/2010

As a result of the shift in teacher focus and her introduction of different pedagogical actions which she tested and refined during lessons, the students' ways of participating changed. While many responses continued to lack reasoning, there was a shift towards students giving responses which included some mathematical reasoning. The data shows that mathematical explanations within discussions continued to originate mainly from Mrs Stuart (between four and seven times per lesson). Students were, however, beginning to develop some mathematical explanations (one to two provided in each lesson).

Most of the students indicated a positive disposition towards collaborative work during the photo elicitation interviews. An emphasis was placed on everyone having a go and sharing their different ideas. They viewed working with a partner as a way of helping them when they got stuck. However, it was clear in Phase One, that many students did not view the whole class discussion as favourably. Often in the photo elicitation interviews the students were unable to recall the explanation provided or their own mathematical thinking. Student recollections focused largely on either a description of the tasks or their own or others' actions. For example: *Esme was writing on the board* or: *we arranged the counters into groups of three.* Some perceived student explanations as aimed only at the teacher: *she was telling the teacher what it was.* Interactions between Mrs Stuart and an explaining student were seen as individual exchanges: *the teacher wrote some more on the board for her to understand.* Mrs Stuart was viewed as taking a central role in guiding the students towards a correct response: *that was when the teacher was showing us about thirds.*

6.3.4 Analysis of task design and implementation, integration of algebra, and development of the classroom community

Shifting the way in which tasks were implemented in the classroom resulted in higher cognitive demands being placed on the students than evidenced prior to the professional development. Ensuring that the high level tasks remained accessible to all students required that Mrs Stuart develop appropriate scaffolding such as the prompts described by Sullivan et al. (2006).

Mrs Stuart had begun to recognise existing opportunities within curricular material for algebraic reasoning but at this stage did not identify and use spontaneous opportunities during lessons. Carraher et al. (2008) argue the importance of teachers recognising the existing algebraic nature of primary mathematics rather than viewing early algebra as a new and additional topic. However, as Blanton and Kaput (2005a) note spontaneously integrating algebraic reasoning opportunities into lessons is key to developing classroom practices which characterise algebraic reasoning.

Mrs Stuart had begun to plan for algebraic reasoning but some existing classroom practices continued to limit opportunities for engagement with algebraic content. For example, her questioning focused on calculation of answers rather than attending to the general relationships, this limited opportunities to develop the structural perspectives which researchers (e.g., Carpenter et al., 2005b; Ferrucci et al., 2008; Fosnot & Jacob, 2009) argue are important aspects of algebraic reasoning.

Key mathematical practices such as making conjectures, developing generalisations, justification and proof (Bastable & Schifter, 2008; Carpenter et al., 2003; Kaput & Blanton, 2005a; Mason, 2008) were not established within the classroom during this phase. Mrs Stuart's practice of seeking examples and cases was promising, but her propensity to offer conjectures potentially reduced student opportunity to generalise (Mason, 2008).

Shifting discourse patterns is a lengthy and challenging process (McCrone, 2005; Reid & Zack, 2009). Leading the students in developing new shared discourse norms, the use of revoicing, questioning, and modelling are similar to pedagogical practices described by McCrone (2005) and Reid and Zack (2009) which their teachers used to develop collaborative interaction within the classroom. However, the new emphasis on trialling and developing different patterns of discourse resulted in some tensions for Mrs Stuart. In the post lesson discussions, she expressed concern about balancing coverage of lessons while focusing on student reasoning and discourse: *I was trying to get them talking more so we didn't get through as much of the lesson as expected…we perhaps should have moved on more quickly but I just wanted to see where they were.* She also noted frustration at what she perceived as a lack of engagement from the students at times and a shift to a lesson format which she described as: *question and answer.*

As Mrs Stuart initiated changes in the classroom, the students were required to participate differently, however at this stage many students did not understand the new role which was required. Most students were unable to describe their mathematical reasoning or the reasoning of others explicitly. While students viewed group work positively, they associated it with turn-taking which frequently resulted in non-productive cumulative talk (Mercer, 2000).

In summary, teacher actions evident in this section are illustrated below in Stage One of the Framework of Teacher Actions to Facilitate Algebraic Reasoning.

Table 10

Stage One of the Framework of Teacher Actions to Facilitate Algebraic Reasoning: Mrs Stuart

STAGE ONE	
Algebraic concepts	Address the following concepts in the classroom: understand the equal sign as representing equivalence; relational reasoning including whole numbers and rational numbers; commutative property; inverse relationships; odd and even numbers; properties of zero and one; distributive property; associative property; properties of rational numbers; using and solving equations; function
Teacher actions to develop and modify tasks and enact them in ways which facilitate algebraic reasoning	Implement tasks as problem-solving opportunities
	Emphasise student effort to approach and complete cognitively challenging tasks
	Extend or enact tasks to include opportunities for generalisation
	Interrogate tasks for opportunities to highlight structure and relationships
Teacher actions to develop classroom practices which provide opportunities for engagement in algebraic reasoning	Lead explicit discussion about classroom and discourse practices
	Ask students to apply their own reasoning to the reasoning of someone else
	Require students working in pairs or small groups to develop a collaborative solution strategy which all can explain
Teacher actions to develop mathematical practices which support the development of algebraic reasoning	Require students to explain their reasoning

6.4 PHASE TWO: DEVELOPING ALGEBRAIC REASONING

6.4.1 Teacher learning

Throughout this phase, Mrs Stuart actively sought further opportunities for her and the group to investigate their practice and develop their professional learning. After a study group meeting where the group had watched and discussed videos of their teaching she proposed that she approach the head teacher to arrange further viewing to reflect on how they related to research articles. Similarly, in another study group meeting she requested a copy of her original concept map of early algebra to support her analysis of her new one and her changes in thinking.

In line with design research Mrs Stuart used the time between researcher visits to further her professional learning and embed practices which aligned with early algebra in her classroom. For example in a study group meeting, she shared a video clip from her classroom which exemplified her intentional task adaption to focus student understanding on the equal sign. She described how she reviewed class data from the student task based interviews in which many students gave incorrect responses related to operational understanding of the equal sign. In another study group meeting as the group worked on developing and proving conjectures, she again volunteered information about how she was embedding the suggested new and innovative practices: *Duncan had one like that, it's like the odd and even thing…I have always got them to explore that just by finding lots of examples to sort of support it whereas this time we actually proved it because we got two little piles of two unifix and little piles of one and realised visually that if you were adding an even number to an even number you will always get an even number because you will not get any of the individuals so that was taking it on board to proof because you could visually see it.* In this meeting, she also noted the value of using physical representations as tools to facilitate students' understanding of structure and properties of numbers: *those ones are very powerful (points to a diagram in Figure 8) because we counted in twos, so two, four, six, eight so with the twos you can just keep going and how to make it odd, you can just put one in.*

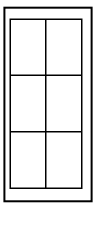


Figure 8. Odd and even numbers diagram

These activities in combination with those of the study group meetings collectively supported Mrs Stuart's understanding of early algebra. While engaged in an activity focused on making and proving conjectures during a study group meeting, she readily provided a range of conjectures which she had noticed her students making during mathematics lessons. Her expanding understanding of early algebra is shown in the concept map (see Figure 9) drawn during a meeting.

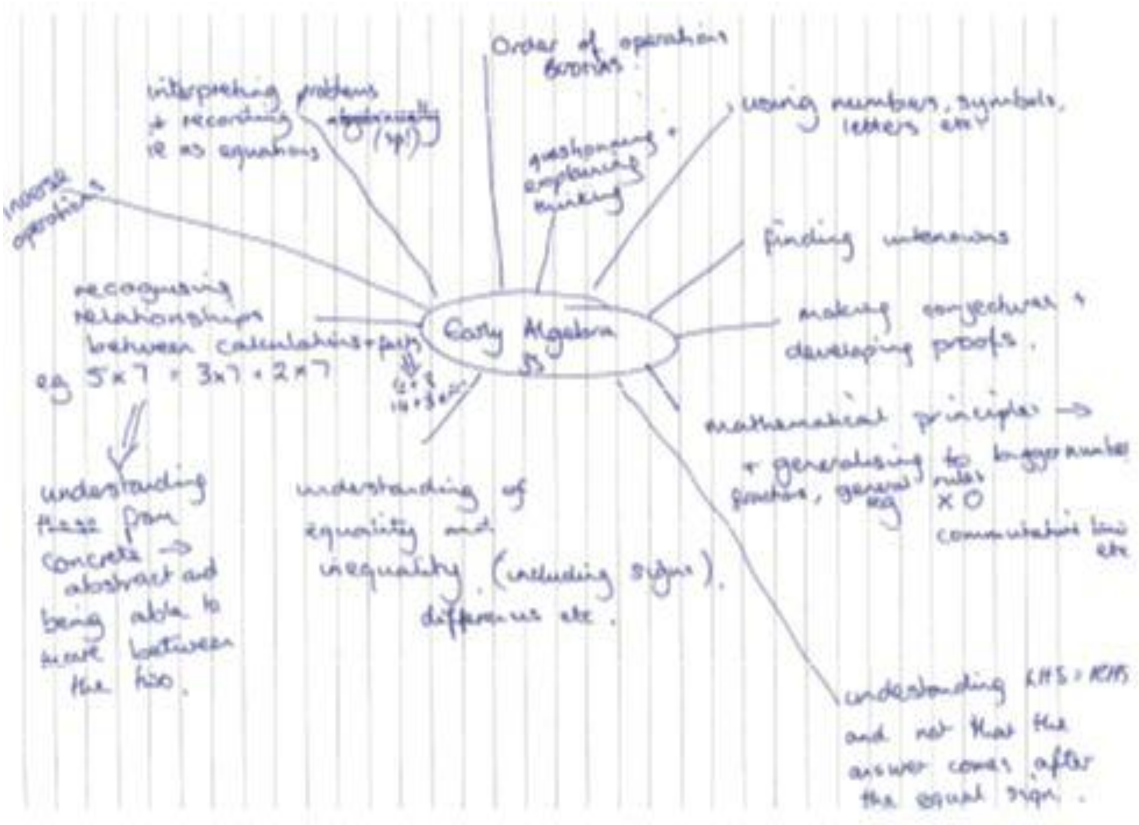


Figure 9. Mrs Stuart's concept map of early algebra in the second phase

This concept map signalled her growing awareness of a wider range of elements which support students to engage in algebraic reasoning. These include areas of content, the use of notation and representations to model problem situations, and classroom practices including explaining thinking, and important mathematical practices.

Reflecting and critiquing her practice was an important aspect of Mrs Stuart's professional learning. She identified key changes in her focus as noticing and understanding student reasoning and adapting her teaching in response:

Reflecting on changes to practice
Changes in task implementation Mrs Stuart: I have been trying to coach them a bit less, I suppose in that they have an opportunity to think about it and talk it through with their partner before we bring it together whereas I think I gave them steps, steps, steps before and then let them do Week 7, Term Two, 2009/2010
Changes in her task implementation Mrs Stuart: A big shift that's been a very conscious shift…the coaching and over coaching them. I think I did very much spoon feed them through and give them a lot of scaffolding until they got there. Week 11, Term Two 2009/2010
Monitoring student reasoning: Mrs Stuart: Before I was trying to be competitive and see who could do this in a minute and who couldn't but now I just give them a sheet and then rather than being worried about how much they do I am going around picking up issues. Before I would have said 'miss one out if you can't do it', now I want to work out why they can't do it. Week 7, Term Two, 2009/2010
Monitoring student reasoning and intervening: Mrs Stuart: Trying more to listen into their conversations as I'm going around and not take to intervening necessarily. Sometimes you need to intervene obviously but it's how I'm intervening with them…I think I'm trying to step back a little more. I don't think I've got it sorted yet because I talk too much but I think I'm trying to step back a little more and get them to talk and get me to listen a bit more. Week 7, Term Two, 2009/2010
Noticing and adapting in response to student reasoning: Mrs Stuart: Now I make more conscious decisions about which bits to go with and which bits to say I'll come back to you…Like today, Paul's idea was that dividing by four is the same as finding a quarter so I thought well I'll get the counters out and see if we can make that link…So that's been the shift because we wouldn't have done that in the past. We'd have talked about it and then I would have said 'dividing by four is the same as finding a quarter'. Week 11, Term Two 2009/2010

Mrs Stuart embraced reflective change as an on-going collaborative process. She actively sought potential areas to focus on via suggestions from the researcher and other study group members drawing on support from the learning community: *I just don't seem to be making any headway so I would like some input because I keep trying "could you ask any questions?" and then they are not getting it. I can't get my head around how to get them to ask the type of questions that I want them to.* She watched other member's videos and discussed the difficulties: *It's hard isn't it, because you're so used to doing the questioning. It's sort of mentally having to stop yourself.* Similarly, she commented critically on her own practice as the group watched an excerpt from her classroom in which a student recorded an incorrect response on the whiteboard and she stated: *I'm still coaching, aren't I?*

In Phase Two Mrs Stuart explicitly focused on facilitating students to develop their questioning skills. She trialled different strategies and carefully monitored their development as illustrated in a post lesson interview: *It's really been about getting them questioning each other and explaining their reasoning and it started off with the do you agree, do you disagree…Then getting them to explain and that was quite hard, there was a lot of yes, no, and then trying to get them to model for other people how to ask questions. So I used the more confident children, rather than saying can you ask a question so you can understand, to ask a question so that if anybody else didn't understand it would help them to understand. Because the likes of Hazel and that didn't have the questioning skills to actually be able to focus on what they needed. So putting it into, "imagine you didn't understand, what sort of question would you have to ask?" Then they really got into asking questions and then you suddenly become aware that they're just getting into that 'how did you get that?' type generic question, which actually for Hazel is okay. They are all asking questions now even if they get it so now I can move on to getting them if you don't get it, or which part, if somebody wasn't getting it.*

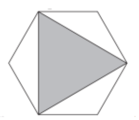
6.4.2 Analysis of teacher learning

Opportunities to develop, improve, and enhance current practice were actively sought by Mrs Stuart. Noticing and reflecting on practice were important aspects of Mrs Stuart's professional learning which enabled her to clearly identify the shifts she was making in her classroom practices. She had begun to develop what Ghousseini and Sleep (2011) term a disposition of inquiry; an ability to reflect on how her practices aligned or contrasted with the pedagogies that she was seeking to adopt. Jaworski (2006; 2008) identifies this as a hallmark in the shift of inquiry from being a tool to becoming a 'way of being'.

However, at this stage of the process it is important to recognise that Mrs Stuart identified the process of change as challenging and one in which she needed to question her own practice continually: *It's hard, isn't it to change a pattern? I've tried and I'm not too sure if I'm doing it. I'm reflecting more on what I'm doing and becoming a little bit more critical and that can be quite anxiety, raising your anxiety level higher because you think am I, you're almost over thinking what you're doing. And I think I'm doing that. You're thinking is this right or is this not right?*

6.4.3 Task design and implementation, integration of algebra, and development of the classroom community

The trialling and refinement of ways of task implementation continued to be a focus of development for Mrs Stuart. She no longer guided students through task procedures but instead used prompts to enable all the students to make sense of it. In the following vignette we see how Ms Stuart maintains the cognitive demands as the students work together to draw out the requirements of a task (see Figure 10):



Number of:

shapes	0	1	2	3	4	5	6	7	8	9	10
sides of △ s	0	3									
sides of ⬡ s	0	6									
sides altogether	0	9									

Figure 10. Shapes and sides problem. From *MEP practice book Y2b* (p. 126), by S. Hajdu, 1999, Budapest: Muszaki Publishing House.

Scaffolding student understanding of task requirements	
Mrs Stuart begins by asking the students to discuss the diagram in pairs. She then gives them the related chart and asks them to talk about it in pairs.	
Mrs Stuart:	Right what do you think this means? Those of you who haven’t got a clue what it means yet, you listen very carefully and if you don’t understand what someone is saying put your hand up and ask them a question. Jasia, what do you reckon it’s all about? Instead of looking at Jasia look at your chart while she’s talking.
Jasia:	It’s like counting in threes, like times-ing three because number two we can’t do.
Mrs Stuart:	What do you mean by number two?
Jasia:	Like three times two equals six and under that we can say six.
Mrs Stuart:	Okay so when you’ve filled it in you might think that, but just looking at it as it is at the moment without filling it in. Can somebody help to explain it to somebody who maybe can’t work out what’s happening and how to fill it in? Esme?
Esme:	It says, because it says we have on the top row it has the number zero...
Mrs Stuart:	What does that zero mean?
Esme:	Zero of the shapes
Mrs Stuart:	Zero of these {points at the board}, okay, none of these. Good.
Erin:	And then so if you have none of the triangles…so if you have zero triangles there won’t be any sides and if you have zero of those there won’t be any sides
Mrs Stuart:	So that first column is all zeros because if, this is what she is saying Martin, if you didn’t have any of these on the board at all how many triangles would you have?
Class:	None.
Week 7, Term Two, 2009/2010	

Shifts in task implementation also included progression towards use of questioning which oriented students to use a structural focus. With regard to patterning tasks, the MEP teaching guide material suggested asking students to calculate answers and then examine patterns at the end of the task. Mrs Stuart instead supported her students to develop structural perspectives by focusing their attention on the patterns and relationships within the task. For example, she introduced a task (see Figure 11) then said: *look at those questions and see if there is a pattern please, don't work out the answers yet, just have a look at it.*

Fill in the answers.

100 – 10 =	60 – 6 =	20 – 2 =
90 – 9 =	50 – 5 =	10 – 1 =
80 – 8 =	40 – 4 =	110 – 11 =
70 – 7 =	30 – 3 =	120 – 12 =

Figure 11. Number sentence problems. From *MEP practice book Y2b* (p. 126), by S. Hajdu, 1999, Budapest: Muszaki Publishing House.

After the students had talked, she asked them to share what they had noticed with the class. Then she drew their attention to patterns in the answers by asking: *as there is a pattern in the questions, do you think there might be a pattern in the answers?* Key to this shift was her asking the students to talk about what they noticed in the task rather than writing anything down. Reflectively she noted: *If they'd just worked out the answers for the first one, question one first of all, they'd have just said 'oh look it's the nine times table' but I think that the fact that we focused it in, are there any patterns in the questions, made it more explicit that if there are patterns in the question, there are likely to be, that is the reason why there are patterns in the answers.*

During Phase Two, Mrs Stuart continued to seek opportunities to develop and extend her planning to include early algebra in lessons. She worked collaboratively with the researcher to explore, modify, and adapt tasks within the curricular material so that they focused more clearly on aspects of algebra. For example, the researcher highlighted an activity (see Figure 12) which could be used to focus attention on relational reasoning.

Mental calculation

T has 12 cards stuck to side of BB. Ps come out to chose pairs of cards with same result, explaining reasoning.

Class agrees/disagrees.

Emphasise correct order of calculation.

$38 + 20 + 7 = 65$	$5 \times 4 \div 2 = 10$	$75 - 20 + 13 = 68$
$81 - 20 + 7 = 68$	$17 + 8 + 40 = 65$	$4 \div 4 \times 9 = 9$
$6 \times 6 \div 3 = 12$	$82 - 6 - 20 = 56$	$45 - 27 - 6 = 12$
$5 \times 9 \div 5 = 9$	$4 \times 7 \times 2 = 56$	$12 \div 6 \times 5 = 10$

Figure 12. Mental calculation problems. From *MEP lesson plans*, by S. Hajdu, 1999, retrieved from http://www.cimt.plymouth.ac.uk/projects/mepres/primary/y2blp_2.pdf

Mrs Stuart then independently modified the suggested equations as shown in Figure 13 so that the students could more readily solve them relationally.

17 + 18 + 30 =	5 x 4 ÷ 2 =	81 – 30 + 17 =
81 – 30 + 7 =	17 + 8 + 40 =	4 ÷ 4 x 9 =
6 x 6 ÷ 3 =	82 – 6 – 20 =	6 x 12 ÷ 6 =
9 ÷ 9 x 9 =	72 – 6 – 10 =	12 ÷ 6 x 5 =

Figure 13. Mrs Stuarts' adaptation of number sentences to involve relational reasoning.

However, while her growth in understanding different types of algebraic reasoning led to Mrs Stuart developing awareness to notice when students provided responses related to algebraic reasoning, this still did not extend to using responses to engage students spontaneously in a deeper investigation of an aspect of algebra. For example, when the students referred to odd and even numbers or other patterns they had noticed, she heard them but did not develop them further. In another example, two students shared a solution strategy for a word problem: six girls have seven apples each, how many apples do they have altogether? One recorded $6 \times 7 = 42$

and the other recorded $7 \times 6 = 42$. A student said: *It's the commutative law like it's just the other way around*. Later Mrs Stuart commented: *I was really impressed that they retained things from last term. You know Julio was like 'oh that's the commutative law'.* Although she noticed the statement, on this occasion she did not use the opportunity to engage students spontaneously in further investigation.

Mrs Stuart's developing understanding of early algebra included a greater awareness of potential student misconceptions. She was able to use this understanding to investigate how misconceptions could be more readily addressed through modification of the lessons. For example, in a lesson in which Mrs Stuart had asked the students to solve $36 - 6 = \_\_ + 20$ it became evident that some children had written 30. Drawing on her growing understanding of student misconceptions of the equal sign Mrs Stuart engaged the class in prolonged discussion of the open number sentence. She drew their attention to the need to have a balance on both sides of the equal sign. She then adapted the lesson and used scenarios involving fictional students to engage the students further in examining the equal sign. She began by asking the students to solve $24 + 4 = \_\_ - 2$. After they had successfully solved this, she said: *Somebody else came along and they put this answer in (writes $24 + 4 = \underline{28} - 2$). They're wrong. Can you work out why they're wrong and what they've done?* While modifications were aimed to support students' understanding of the equal sign, one could argue that in this lesson potential opportunities for students to use relational reasoning were hindered by the structure of the number sentences as they did not clearly exemplify relationships between each side.

In subsequent observations in March, Mrs Stuart began to recognise when spontaneous opportunities for algebra were present within enacted mathematical tasks. In the following vignette we see how Mrs Stuart began to trial the use of such opportunities to engage her students in algebra:

Integrating spontaneous opportunities for algebra into mathematics lessons
Mrs Stuart asked the students to use twelve counters and write number sentences related to these. She recorded two related number sentences (½ of 12 = 6 and 12 ÷ 2 = 6) on the board and asked the students what they noticed. A student (Paul) made a conjecture that to find a half you can divide by two. He then changed this to say that you could divide by two or divide by half. Mrs Stuart revoices this and asks the students to discuss it: After the students have talked to their partners, Mrs Stuart then facilitates a discussion. Mrs Stuart: Is Paul right there? Who agrees with Paul that if we divide by two it's like doing a half of something? Okay, who's not sure? Okay so what I'm going to put down here, that big idea, conjecture, Paul reckons that by dividing by two is like finding a half of something. Is that right Paul? And then he reckons we can just swap it into the equation. Caleb, what did you want to say? Caleb: If you are dividing 12 by a half then you are dividing 12 by half of one. Mrs Stuart: Right so let's have a think about that. That's like saying how many halves are in 12. Look at your counters. Now if I gave you an incredibly sharp knife and you divided your 12 counters into halves, you'd have to cut them all up wouldn't you? What would you end up with? How many bits of counter would you end up with? Talk that one through with your partner. After the students talked with their partners, Mrs Stuarts asks a student to share their answer. Willow: Twenty-four. Mrs Stuart: Twenty-four, oh, what do you notice about 12 and 24? Alec: If you times it by two it would be 24. Mrs Stuart: Wow, we are discovering things I hadn't planned this morning! So dividing by a half isn't the same as dividing by two Paul. Caleb: It's the same as timesing. Mrs Stuart: It's the same as timesing by two, dividing by a half is the same as multiplying by two...The idea is that dividing by a half didn't give us the same answer as dividing by two because 12 divided by two equals six...but didn't it give us the same answer as multiplying by two? I wonder if that works for everything, that is conjecture number two really, isn't it? If we divide by a half is that the same as multiplying by two? (writes the conjecture on the board) So let's just explore this first one for today. Week 11, Term Two, 2009/2010

Throughout this phase, Mrs Stuart maintained her focus on facilitating students to develop an ethos and expectation of talking and working together so that the diverse students in her class learnt to think mathematically. In particular, she wanted her students to develop shared understanding of a jointly constructed solution strategy. During a lesson she observed the students working in pairs and small groups and then told the class: *some people are still at the stage where they don't trust their partner and they think they've got to write themselves.* In another example in the same lesson, Mrs Stuart asked a pair of students to share their response. After Jasia said that she had written the equation by herself, she said: *No, you were working as a two so it shouldn't be 'I've got one. It should be we've got one'.*

To help build the discourse practices Mrs Stuart also engaged students in reflecting on the ways in which they were working. She asked them to think about what went well or what they could improve on in terms of group interactions. She explained in a reflective interview that she saw this as a useful way both to convince students of the benefit of working together and to make her expectations clear. Shown in the vignette below is how Mrs Stuart highlighted productive ways in which students had collectively approached tasks:

Reflecting on group work	
After the class had worked on an algebraic task involving unknowns, Mrs Stuart asked them to consider the ways their groups had approached the problem.	
Jacqueline:	First we tried to do add and then we found out that it wasn't add.
Mrs Stuart:	Okay so somebody had a theory, an idea, and they just didn't accept it…Caleb had an idea, "I think it's addition". Somebody else had an idea before then though, what did we think about all the little circles and the letters and things? Zanthe?
Zanthe:	Same letter same number.
Mrs Stuart:	Same letter same number…So same letter, same number and somebody else said the answer in the middle is the answer and the bits around the outside help us get to that answer. So those were important ways in. Caleb had a theory and he could have gone completely wrong. I overheard his theory and I thought 'they're going off on the wrong tangent', but you went to the next logical place where there was still only one unknown and you proved it.
Week 11, Term Two, 2009/2010	

Expectation of collaborative interaction extended into whole class discussions. Building on links between theory and practice, Mrs Stuart drew on pedagogical actions from research articles and the reflective framework outlined in Appendix A which were introduced during study group meetings and tested these actions in the classroom. She positioned students to listen actively to their peers' reasoning and explanations and make sense of these. During whole class discussions she intervened to provide space for other students to ask questions or modelled how to ask a question herself. Other pedagogical actions that supported collaborative participation included asking students to apply their own reasoning to the reasoning of their peers and then provide reasoned agreement or disagreement, asking students to add on and provide alternative solution strategies, and asking the class to analyse whether they were the same or different. Mrs Stuart regarded an important aspect of the collaborative process as: *looking at the ways that children represent it differently and making that more explicit.* The vignette below provides examples of the pedagogical actions that Mrs Stuart employed to develop collaborative interaction during whole class discussions:

Developing collaborative interaction during whole class discussions	
Students were asked to work together to represent and solve a word problem: *Each of 4 children has 3 matchboxes and each matchbox contains 3 marbles. How many marbles do they have altogether?* After students had worked on the problem in pairs, Mrs Stuart asked two students to share their solution strategy:	
Erin:	(writes 3 + 3 + 3 + 3 = 12) Three add three add three add three equals 12 because if you count in threes and you do four threes you will get 12.
Mrs Stuart:	So is your answer that they've got 12 marbles altogether?…Who agrees with them? Who disagrees with them? Okay right, would anyone like to ask them a question to try and get them to understand why you disagree with them? Okay, Esme?
Esme:	I disagree with you because they all had three boxes each so one person had three and another person had three and another person had three and another person had three so all of them had 12.
After further discussion a child says:	
Erin:	They are the match-boxes that each child has got.
Mrs Stuart models with counters to represent Erin's explanation and revoices it. After further exploration she asks for other strategy solutions:	
Caleb:	(records 12 × 3 =) Twelve times three.
Mrs Stuart:	Because, come on convince us.
Caleb:	Times three because there's three marbles in each matchbox and there's 12 matchboxes so you times the amount of matchboxes by the amount of marbles and you get 36.
Mrs Stuart:	Thirty-six. Who agrees that it's 36? {hands go up} Okay, there are probably

	different ways that you got to that answer. Did anyone do it a different way? Alec?
Alec:	We did nine times four.
Mrs Stuart:	You did nine times four…Why did you do nine times four? Come up here and tell us.
Tobias:	Because there's three, umm (pause) (Alec writes 4) there is four children and they've got three matchboxes each so umm (pause) and there is…
Mrs Stuart:	So the children and they've each got three matchboxes each (draws three match-boxes).
Tobias:	And there's three in each, (Alec writes 3 in each box Mrs Stuart has drawn) so and three threes are nine, and so we said that equals nine and so that is four times nine.
Alec:	(writes $4 \times 9 = 36$)
Mrs Stuart:	Four times nine is 36, did anyone else work it out that way? Okay did anyone have a different way of writing it? Esme and Jacqueline? So is there only one way of representing something with an equation or can we show it in different ways and get the same answer?
Esme:	Well we started off with doing four people (draws four people) and 3 matchboxes (draws arrows from each person to three matchboxes).
Mrs Stuart:	Stand to one side so everyone can see your picture.
Esme:	And they each had three marbles, (draws the marbles into the matchboxes) and so we did, we did, three times (pause) three add three add three is nine (points to the marbles in the matchboxes and writes 9), nine add nine add nine add nine (writes $9 + 9 + 9 + 9 =$) and we got (pause) added nine add nine add nine add nine and we got the answer (writes 36 at the end of the equation).
Mrs Stuart:	Is that the same way as doing it as Alec, or a different way? Talk to the person next to you.
After further partner talk, Mrs Stuart asked a student to share their idea:	
Misty:	Because there is nine...there's four nines, and it says, and he said four times nine, and so she had four nines and he had four times nine so it's basically the same.
Mrs Stuart:	Yeah that's where he got the nines from isn't it?
Week 7, Term Two, 2009/2010	

Use of representations was an important part of the developing classroom practices. For example, in the vignette above after a student shared a strategy solution where they had drawn pictures to help them solve it Mrs Stuart said: *you drew the pictures which were great. I remember when I was doing some maths that was quite tricky whenever it's too hard I always draw a picture because it really helps me see it.* Mrs Stuart encouraged her students to use representations to access the structure of number sentences and understand how they could be solved relationally. In an interview, she was able to recall an example readily from a lesson where a student had made up a true/false number sentence (e.g., 38 + 4 + 4 = 38 + 8) for the class to solve. She had asked students to visualise the number sentence as two jars of marbles and she said she had asked: *if I add four to that jar and another four to that jar, is that the same as adding eight to that jar?*

Mrs Stuart also promoted use of different representations (e.g., verbal, concrete materials such as counters and unifix cubes, and written such as words, equations, pictures) as a way of developing the clarity of explanations. Initially, she asked the students to share different equations to show how they had worked out the solution. As shown in the vignette below, she then began to ask the students to use equations to represent a problem situation and then relate this to the problem context supporting them to learn to explain and justify solution strategies by making connections between tasks and representations:

Using equations to represent and justify a word problem through making connections between the problem and the representation	
Mrs Stuart had asked the students to work with a partner to develop a representation for a word problem: *A farmer's wife had eight eggs. The farmer brought her nine boxes of eggs, six eggs in each box. How many eggs did the farmer's wife have then?* Hazel recorded 8 + 9 × 6 = on the board.	
Mrs Stuart:	Why should she write down eight? Convince us.
Paul:	Because it says there are eight eggs.
Mrs Stuart:	So the farmer's wife had eight eggs…Then along came the farmer…and he brought her nine boxes. Now that's really interesting. Why has she written an add sign?
Calvin:	Ah because, ah because ah, first they have to add eight and then it says what do you have altogether.
Mrs Stuart:	You know Calvin, I can see you're thinking about what's coming up next but why is this add sign here? You were half explaining it for me. Why is this add sign here?…Think about it. She had eight eggs then along came the farmer and brought along nine boxes. Why is this the add sign, Jasia?
Jasia:	Because he brought her nine boxes.
Mrs Stuart:	So what Calvin was saying, they're altogether, she had these nine boxes add the eight eggs. Right, what was special about the nine boxes?
Students:	They had six eggs.
Mrs Stuart:	They had six eggs each.
Week 7, Term Two, 2009/2010	

During this phase, a key shift for Mrs Stuart was her emphasis on facilitating student development of mathematical explanations rather than continuing to provide the majority of explanations herself. To achieve this, Mrs Stuart trialled the use of prompts such as: *I also want you to think because I'm sitting here and I'm dead confused, how you could explain it to me. So I'm not just interested in your answer, I'm interested in you explaining it.* The following vignette illustrates Ms Stuart's expectation for students to convince others in the classroom community:

Pressing students to develop a convincing explanation	
After students had worked on functional problem (see Figure 10) Mrs Stuart asked a student to share what they had noticed:	
Alec:	You count in threes, sixes, and nines.
Mrs Stuart:	What do you mean you count in threes, sixes, and nines?
Alec:	So at the start where the last one goes (pause) where it goes zero, one, and at the bottom it goes three, six, nine, and we reckon we're counting in threes, sixes, and nines.
Mrs Stuart:	Right I'm going to play confused here. I do know what you're saying but I'm going to play confused. Can anyone ask Alec a question to help me understand? Okay get him to explain it again. Right Duncan can you ask him a question?
Duncan:	I don't understand it. How can you say threes, sixes, and nines? It's not enough for me to understand.
Mrs Stuart:	Okay so you've got to convince us…Who knows what Alec's talking about when he says it's going in threes, sixes, and nines? Right here's a challenge, convince Duncan and I, what do you mean it's going in threes, sixes, and nines? Talk to each other how're you going to convince us?
Week 7, Term Two, 2009/2010	

Another shift in Mrs Stuart's classroom involved the introduction of the mathematical practices of generalisation, justification, and proof. Mrs Stuart purposefully planned an investigation of zero with the aim of students developing conjectures, justifying and generalising their thinking. She first drew student attention to a number sentence which had been constructed to reach the target number of 20 (e.g., 20 + 0 = 20) and asked them to discuss what they noticed. Then she directed them to conjecture and find examples which illustrated the conjecture. Following this, she pressed them beyond the use of examples as justification by requiring that they prove their conjectures using a range of concrete materials (e.g., acting out the scenario and using counters) before she asked them to symbolise it. These practices were new and so she requested researcher support. After the lesson she reflected: *I got a bit floundered to start with as to sort of thinking which way shall I take this and how far shall it go? I think I was over thinking it.* Interestingly, during the enactment of this task she reverted to using more leading and funnelling questions and did not probe students for their reasoning.

From these tentative beginnings, subsequent classroom observations toward the end of Phase Two revealed that students made conjectures more readily about the patterns they noticed. In the following vignette, we see how Mrs Stuart drew on these and then led them through the mathematical practices of generalisation, justification and proof.

Developing mathematical practices of generalisation	
A student had conjectured that dividing by two was the same as finding half of a set. Mrs Stuart asked her students to work in pairs to explore what happened when twelve counters were divided into different fractional parts including thirds and quarters.	
Jasia:	It is because one third is three and there is three here and you have divided them all by the same so the same as 12 and 12 divided by three equals four.
Mrs Stuart:	…Jasia's saying that the three there which is one third is that divided by three there (points the different parts of the equation as she speaks). So on this one Willow, there's the twelve, there's the sixes, divided by two there, and that two is part of the half in there. Okay so it looks like there is a pattern there as well, doesn't it? Paul wanted to know, his idea was: is dividing by two the same as finding a half. This time we've divided by three is that the same as finding a third?…Coming back to Paul's idea, dividing by two is the same as finding a half. Can anyone think what dividing by n would be the same as? Think that one through with your partner.
After further discussion Mrs Stuart returned to Paul.	
Mrs Stuart:	Any ideas? Is dividing by n the same as dividing by what do you reckon Paul?
Paul:	Finding an nth.

Week 11, Term Two, 2009/2010

A notable feature of this phase was the students' growing recognition of the benefits of collaborative work including support for their own mathematical understanding and learning different strategies. Many indicated that they developed solution strategies by talking and listening with others. Importantly, questioning was beginning to be viewed as a way of supporting peers: *if they don't understand, you could ask them questions and it will help them figure it out.*

In the latter part of phase two students remarked:

Justin: *It's okay to disagree because if something's wrong, someone has got to know.*

Other students referred to the necessity of convincing others.

Zanthe: *When somebody gets it wrong, we have to talk them into the right thing (...) because if you know it and your partner doesn't, you have to convince them to kind of say its right.*

Data from the photo elicitation interviews in this phase of the study confirmed that many of the students were now able to re-construct their peers' explanations and reasoning from whole class discussions. This included identifying their peers' errors and why the thinking was incorrect. Students were also able to recollect similarities between their peers' solutions strategies and explain this. For example, in the vignette example shown earlier in this chapter, students shared their solution strategies for the word problem: *Each of 4 children has 3 matchboxes and each matchbox contains 3 marbles. How many marbles do they have altogether?* In photo elicitation interviews undertaken after this lesson the students identified the addition and multiplication strategies as the same: *because nine add nine add nine add nine is the same as nine times four.*

Students were now beginning to describe explicitly their own mathematical thinking during the photo elicitation interviews. They referred to relational strategies which they used or patterns they noticed in tasks. At this stage most students were unable to provide justification for their thinking following the lesson. Many of the students were able to revoice conjectures that had been investigated in the classroom. However, some questioned whether these would always be true. Most students referred to specific examples to justify conjectures and were unable to relate them to different forms of representations.

6.4.4 Analysis of task design and implementation, integration of algebra, and development of the classroom community

During the second phase of the study there were two notable shifts in regards to task implementation. Firstly, enabling prompts as described by Sullivan et al. (2006) were used by Mrs Stuart to ensure the diverse range of students could successfully approach tasks. Secondly, a change in her use of questioning focused student attention on the structure of the task rather than on calculating answers.

Mrs Stuart continued to recognise opportunities within the curricular material to integrate algebra. This now also extended to noticing spontaneous opportunities to develop algebraic reasoning during lessons and in the later observations during this phase Mrs Stuart was able to realise the possibilities within enacted tasks to integrate algebraic reasoning spontaneously into her lessons. This indicates that she was now beginning to develop pedagogical moves described by Smith and Thompson (2008) and Blanton and Kaput (2005a) to develop algebraic reasoning.

Mrs Stuart's developing ability to notice student reasoning and understand different aspects of student thinking encompassed an awareness of misconceptions connected to algebra such as the misunderstanding of the equals sign described by Carpenter et al. (2005b) and Knuth et al. (2006). Mrs Stuart attended to student misconceptions in the moment and re-designed her lessons to address these by supporting students to reflect on their understanding of the equals sign. However the numbers and structures she used encouraged use of computational strategies rather than also addressing relational reasoning. Relational reasoning is described by a range of researchers (Carpenter et al. 2005b; Irwin & Britt, 2005; Stephens & Xu, 2009) as a foundation for developing algebraic reasoning.

In this phase there was now a strong focus within Mrs Stuart's classroom on promoting interactive mathematical talk within the learning community. As other researchers (e.g., Kazemi, 1998; McCrone, 2005; Reid & Zack, 2009) have shown, specific pedagogical actions were required to support a shift in student participation so they became critical participants within the classroom community. Collectively, the vignettes highlighted teacher actions that promoted: engaging students in explicit discussions about the need for collaboration and asking them to reflect on their ways of working; positioning students to listen actively; and asking students to agree and disagree using mathematical reasoning and examine similarity and difference across solution strategies. However, within this phase, although students were supported to make mathematical connections between different responses Mrs Stuart was not yet confident in the process of monitoring their responses or purposefully sequencing them. This resulted in what Stein et al. (2008) describe as missed opportunities to advance specific mathematical concepts or focus on important mathematical ideas.

What was particularly noticeable across this phase was the changing role of representations. Acting as important tools in the classroom, a range of informal and formal representational forms were introduced as a way of organising and exploring, explaining, and justifying reasoning. Similar to Blanton and Kaput (2005a) and Carraher et al. (2008), Mrs Stuart initially needed to encourage and scaffold student use of these tools.

Embedded within these changes was an expectation that students would consistently provide mathematical reasoning for their solution strategies and develop mathematical explanations. Like the teachers in Franke, Turrou, and Webb's (2011) study, Mrs Stuart used follow-up questions which required students to provide further explanation of their ideas. She drew on specific instances of student noticing of patterns to engage the whole class in developing reasoned explanations. Alongside mathematical explanations Mrs Stuart initiated a growing expectation that generalisations would be expressed and treated as conjectures. In doing this, Mrs Stuart facilitated a 'conjecturing atmosphere' such as described by Mason (2008). Exploration of properties of zero was a rich area to scaffold students to develop and investigate conjectures and generalisations. The richness of this context is recognised also by Carpenter et al. (2003; 2005a). These researchers and Schifter (2009) note that providing students with opportunities to use concrete materials and representations as a means to develop an argument and establish a general claim is an important aspect of learning to justify algebraic reasoning.

Initially, enacting these new mathematical practices in the classroom caused uncertainty and anxiety for Mrs Stuart and on occasions she was observed to fold back to more teacher directive practices. Franke, Carpenter, and Battey (2008) explain how teachers may appropriate an activity and develop it into an artefact resembling their current practice when unfamiliar with a new practice.

Similar to the findings of Edwards and Jones (2003), students in Mrs Stuart's class increasingly were able to identify a range of positive outcomes of collaborative group-work. Students privileged both talking and listening as forms of learning mathematics. Questioning, disagreeing, and developing convincing explanations were noted as important aspects of collaborative work.

As Mercer (2000) describes, these practices support students to explore and critically examine shared reasoning through exploratory talk.

As the students participated in the changing practices of the classroom, they learnt the ways of thinking and acting which were valued in the community. Boaler et al. (2000) and Hodge (2008) explain that this helps develop a sense of what it means to be a member of a specific community. There was a significant shift in students' participation and their understanding of their role as a learner. With the increased emphasis on describing mathematical thinking, students developed competence at explicitly describing their thinking after lessons.

In summary, teacher actions evident in this section are illustrated below in Stage Two of the Framework of Teacher Actions to Facilitate Algebraic Reasoning.

Table 11

Stage Two of the Framework of Teacher Actions to Facilitate Algebraic Reasoning: Mrs Stuart

STAGE TWO	
Algebraic concepts	Address the following concepts in the classroom: understand the equal sign as representing equivalence; relational reasoning including whole numbers and rational numbers; commutative property; inverse relationships; odd and even numbers; properties of zero and one; distributive property; associative property; properties of rational numbers; using and solving equations; function
Teacher actions to develop and modify tasks and enact them in ways which facilitate algebraic reasoning	Adapt tasks to highlight structure and relationships. This may include changing numbers or extending to multiple solutions
	Structure tasks to address potential misconceptions
	Use enabling prompts to facilitate all students to access tasks
	Implement tasks by focusing attention on patterns and structure
	Recognise and use spontaneous opportunities for algebraic reasoning during task enactment
Teacher actions to develop classroom practices which provide opportunities for engagement in algebraic reasoning	Require that students indicate agreement or disagreement with part of an explanation or a whole explanation and provide mathematical reasons for this
	Lead explicit discussions about ways of reasoning
	Provide space for students to ask questions for clarification

	Request students to add on to a previous contribution
	Ask students to repeat previous contributions
	Use student reasoning as the basis of the lesson
	Facilitate students to examine solution strategies for similarities or differences
Teacher actions to develop mathematical practices which support the development of algebraic reasoning	Require students to develop mathematical explanations which refer to the task and its context
	Facilitate students to use representations to develop understanding of algebraic concepts
	Ask students to develop connections between tasks and representations
	Provide opportunities for students to formulate conjectures and generalisations in natural language. Leads students in examining and refining these conjectures and generalisations
	Listen for conjectures during discussions. Facilitates students to examine these
	Require students to use different representations to develop the clarity of their explanation

6.5 PHASE THREE: EMBEDDING ALGEBRAIC REASONING

6.5.1 Teacher learning

During the last phase of the professional development research project Mrs Stuart continued to identify opportunities for collaboration and discussion within the study group as important elements for her development. In the group meetings the teachers began to plan a lesson study cycle. First they developed an overarching aim collaboratively through focused discussion on developing norms for effective mathematics teaching. The following vignette captures the shared understanding of what they valued as a group.

Study group reflection on shifts in focus of effective mathematics teaching
The study group is discussing and writing an overarching aim for their lesson study cycles. Rebecca: To have a positive attitude, to challenge and extend their own and others. You see that's what's different now... Michelle: Yeah it is. Rebecca: Is getting those others thinking.

Michelle:	It's good. We've changed quite a lot haven't we, when you look at that?
Rebecca:	You see the reflective we would have had before, the actively engaged we would have had before.
Michelle:	But that's not enough without the other.
Rebecca:	That's not enough without the other, and it's the other that we've missed. The challenge and extend.

Week 4, Term Three, 2009/2010

Within this collegial community, Mrs Stuart identified conversations about classroom culture as a significant driver of changes in her classroom practice. In the final meeting she linked changes in practice to: *the amount of time we've had working together as a three, because it's been sustained over the year and it's been backed up with the reading and the on-going discussion between the three of us for the year.*

Reflection had become a core practice for Mrs Stuart. In this phase she identified the challenge of closely monitoring and selecting student responses as they worked collaboratively for later inclusion in whole class discussion. She described how her fascination with how students worked together meant she sometimes missed opportunities to know what other students were doing. This resulted in a need for her: *tune in on what to pick out to move the class forward.*

In this phase Mrs Stuart was able to identify how the pedagogical content knowledge she was developing related to early algebra. In particular, she recognised the growth in her understanding of relational reasoning. At the same time, she knew her growth was on-going because she told the group that while confident with addition type problems, she still found using relational reasoning to solve number sentences involving subtraction challenging. To solve these she had either to draw a picture or think about concrete material.

In planning the lesson study, Mrs Stuart used her developing pedagogical content knowledge to critique the structure of tasks planned to develop algebraic reasoning. For example, a task was designed which aimed to develop student understanding of the equals sign by asking students to record equations for a target number (e.g., 27 = 10 + 10 + 7 = 40 – 13 = 27 + 5 – 5). Following this, in another task students were asked to solve true or false number sentences. While solving the true and false number sentences it was evident that some students still did not view the equals sign as representing equivalence. After the lesson Mrs Stuart critiqued the tasks telling the group: *I think maybe because we historically present children with a lot of things with the answer just being one box that sort of one where they had to look maybe provoked that thinking a little bit more. You know at the beginning where they said something, something equals and then the next child does equals, I don't know, when I look at it now I think it is a fantastic activity and a fantastic assessment...but maybe they are just seeing and the next one, and the next one, and now it's my turn and they don't actually see the equal sign whereas this question here and that one here in particular really made them think about the idea of balance.*

Shifts in Mrs Stuart's understanding included changes to the way in which she viewed algebra and mathematics in general. She described viewing mathematics for most of her teaching career as being: *little pockets of knowledge* and viewing algebra as: *the missing number and shoving in an X here.* She related to the group that her school experiences of mathematics had focused on computation and procedures. She spoke of encountering algebra at teachers college: *I got to the first maths tutorial in the first week and it was that little problem, you know how many moves to get those three people and they all have to swap places. My maths lady said 'why does that work? Show me with algebra' and I was like 'oh my god I'm on the wrong course'.* In contrast, Mrs Stuart stated in the final interview that she now viewed algebra as much bigger than she previously perceived: *it's the way they think and it's the way they take something and can take it* *to a wider context, and the opportunities for them to really discover things and make them their own.* She described how she now viewed mathematics as a creative endeavour and saw her students making connections across big ideas within mathematics. She also showed her understanding about the way in which mathematics is a process of on-going construction when she said: *keep being cyclical with all our teaching…don't assume that they've got it and then leave it. Keep dipping in so they don't lose it.*

Within this broadening perspective it was clear that Mrs Stuart's perception of algebra had also expanded and she now viewed facilitating algebra in the classroom as encompassing more than just content. She described her greater awareness of what she viewed as social constructs in the classroom including ways in which students collaborated and talked together and she attributed these to her deliberate pedagogical actions which focused on explanations, questioning, and discussion. Her own lack of experience with algebra during schooling was her motivation to ensure her students engaged in mathematical practices related to algebra and her increasing expectation that students would explain and justify was a personal response: *if he doesn't learn to explain and justify, he will be like me in his first tutorial and think nobody has ever asked me to justify that before.*

Mrs Stuart identified changes in how her students worked within the classroom. This included significant shifts in the way they talked and worked together and how they engaged in making conjectures and generalising and the key role she took in developing these practices. Now she identified differences between traditional classrooms and where she had taken her classroom to in noting concerns over their next transition in a different classroom.

Reflecting on the shifts in the way students work	
Mrs Stuart:	They talk more mathematically, they come up with conjectures, but if they

> weren't asked the same sort of questions, you know if the language of conjecture and generalisation suddenly stops then that's going to filter further away from them and I want them to be able to build on what they've got because…they see things algebraically…We're using the word algebra, we're talking about relationships and they're just taking it in their stride…At least they are being exposed to what maths really is, rather than a series of calculations so I'm really excited about it.
>
> Week 4, Term Three, 2009/2010

6.5.2 Analysis of teacher learning

Blanton and Kaput (2008) describe how the way teachers position themselves when participating within a community can influence the development of a professional identity. Mrs Stuart positioned herself as an active participant within the study group and in this way developed her professional identity. Similar to the findings from other successful case studies in early algebra (e.g., Blanton & Kaput, 2005a; Franke et al., Jacobs et al., 2007) involvement in the community of inquiry facilitated Mrs Stuart to reflect on mathematical understandings, student thinking, and instructional practices. Her developing disposition of inquiry led to changes in her instructional practices.

An important factor in the shift in Mrs Stuart's understanding and practice was the re-conceptualisation of her understanding of algebra. Within the professional development, activities which facilitated engagement in mathematical practices such as generalisation and justification led to Mrs Stuart widening her conceptualisation of algebra. This was evidenced in the opportunities and support for students to engage in these mathematical practices. As shown in other studies (e.g., Blanton & Kaput, 2008; Franke et al., 2008; Jacobs et al., 2007; Koellner et al., 2011; Ruopp et al., 1997; Stephens et al., 2004; Warren, 2009) solving and analysing a range of tasks during professional development supported Mrs Stuart to develop her own personal knowledge and make sense of mathematical ideas. This meant that she began to be able to critique tasks, consider effective sequencing, and reflect on how different types of algebraic reasoning are elicited.

6.5.3 Task design and implementation, integration of algebra, and development of the classroom community

Mrs Stuart continued to use enabling prompts and collaborative work to ensure that her students could approach challenging tasks without their cognitive load being reduced. She also affirmed the requirement/expectation that students listen to their peers' strategy solutions during whole class discussions and reflect on how they had used information from the task to develop a solution strategy.

Shifts in the way that Mrs Stuart implemented tasks led to changes in the ways students approached them. She provided an example of when her students were working in a more logical manner when solving a puzzle involving missing numbers as follows: *this group had all gone down the line of assuming that it was addition but had taken it to a point where they'd worked out that it couldn't be addition because such and such didn't become the sum. So they were explaining; they were able to fold it back for the class as to how they knew it wasn't addition.* These new ways of participating in mathematics learning, Mrs Stuart claimed, meant that students were more independent: *less likely to think what is it she wants us to do? But more what do I have to do with the maths?* We see evidence of this in the following vignette based on a task (see Figure 14) which was presented as a closed problem in the MEP material. Rather than coaching students through it she gave it to them in their groups.

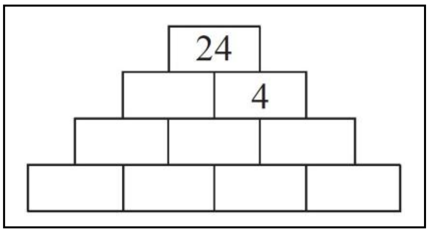


Figure 14. Missing number puzzle. From *MEP copy masters*, by S. Hajdu, 1999, retrieved from http://www.cimt.plymouth.ac.uk/projects/mepres/primary/y2bcm_3.pdf

Students approaching a closed task as an open-ended problem

Mrs Stuart asks a group that she had noted as recording multiple solution strategies to share their response with the class.

Mrs Stuart:	What did you write in here Erin?
Erin:	Six.
Mrs Stuart:	…You also had something else here didn't you?
Erin:	Yes, twenty.
Mrs Stuart:	(records 6/20) Come to the front and talk me through why you did this and then where you went from there? Anyone else think why they did that?…
Juliana:	We put the six and the four there to make 24 but then we thought that it could be addition or multiplication so we thought it could be six times four equals 24 or 20 add four
Mrs Stuart:	So where did you go from there? Do you see what they did there?...
Juliana:	Then we decided to think of the six and try and fill in the box and then we used 20 to see if that worked as well.
Mrs Stuart:	Have you tried both of them?
Tobias:	No, we tried one.
Mrs Stuart:	Which one have you tried?
Erin:	Six.
Mrs Stuart:	Six. So if they were using the six, and looking round quite a few of you were using the six there. What are you saying that 24 is?
Juliana:	The product.
Mrs Stuart:	So they're saying 24 is the product of the two numbers underneath it. So if we're following that rule for our pyramid, what two numbers are going to go there? Do we have a choice again? What could we have there? The rule is that the number on top is the product of those.
Juliana:	Two times two or one times four.
Mrs Stuart:	You could have two times two, or we could have one times four. What about six, what's going to happen there?
Juliana:	One times six.

Mrs Stuart records the different solution strategies in the pyramid and then asks the students to try the other rule and see if it works for addition.

Week 6, Term Three, 2009/2010

In a post lesson interview, Mrs Stuart stated that students had developed: *a natural understanding that there is more than one way to go about a problem…They opened it up and they had different solutions and they were buzzing with natural curiosity about it.*

Mrs Stuart now planned classroom activities in a way which focused on opportunities for early algebra. She described previous planning as looking through activities to make sense of them and decide which she would cover or leave out due to time constraints. In contrast, her current planning involved careful consideration of how to enact tasks, and deliberate thinking about grouping and opportunities for discussion, as well as identifying areas of content with links to algebra and which types of representations could be used. She described herself thinking as she planned about how to: *draw out the commutative law from this one, or this could be a great discussion point for, like the other week when we were doing timesing by one, or dividing by zero, get them to come out with conjectures*. These changes resulted in a clear focus on algebraic reasoning integrated into all lessons and included coverage of a broad range of algebraic concepts.

A point of difference in this phase was Mrs Stuart's propensity to engage in anticipating the outcomes of the task enactment. She continued to develop her use of monitoring, noticing and sequencing student responses which could be used to further investigate algebraic concepts. There were now two to three instances in each lesson when she spontaneously integrated algebraic reasoning. The following vignette features Mrs Stuart using a student response to integrate algebra spontaneously into the lesson:

Integrating spontaneous opportunities for algebraic reasoning
Mrs Stuart asked her students to think about an efficient method to solve 26 – 8 = Misty: You could break it down into six and two

Mrs Stuart: Break what down into six and two?
Misty: Eight.
Mrs Stuart: Why have you chosen to break eight down into six and two? Not one and seven or four and four?
Misty: Because then it makes it easy to do 26 and it is not 27.
Mrs Stuart: Did anyone else do it that way? Well done. Okay does everyone understand what she was doing? Does anyone want to ask her a question?…Let’s just go off track a bit, if you were doing 34 take away seven, with your partner can you just talk about how Misty and the other children would tackle that?

Week 12, Term Three 2009/2010

With the changes in the classroom, students were observed to engage more frequently with algebraic reasoning. When asked to write number sentences to make a certain number, they used relational patterns. They drew on relational reasoning to justify how they knew the number sentence was true: *It is true because you just put two on the 21 and then take it off again.* The following vignette illustrates how students independently used structural aspects and patterns to help them solve a task (see Figure 15).

A school was taking its pupils on a trip on a steam railway.

The carriages in the train were so small that they could seat only 6 people.

Complete the table to show how many carriages were needed.

Number of Children	24	25	26	27	28	29	30
Full carriages							
Children remaining							

Figure 15. Train and carriage problem. From *MEP practice book Y2b* (p. 149), by S. Hajdu, 1999, Budapest: Muszaki Publishing House.

Using patterns to solve tasks

During whole class discussion Zanthe shared a solution strategy for the task (if there were 25 children). She drew a box with six written in it then drew an extra line to show how she would solve the task. Mrs Stuart asked her to show how to solve it for 26 children and 27 children. For each of these Zanthe drew another line.

Mrs Stuart: Now this is like Duncan and Ferdinand you were using some kind of pattern to

	do yours weren't you? Right 28 children (*Zanthe draws another stick figure*). Why has Zanthe carried on drawing stick men instead of drawing a carriage with a number six in it? Why is that working? She's up to 28 children in the class now and every time she just draws a stick man why doesn't she draw a carriage? Talk to your partner please.

After the students talk to their partners, Mrs Stuart asks them to share their ideas.

Mrs Stuart:	Okay Sabrina can you explain to me why she's just drawing stick men and not another carriage?
Sabrina:	Because you would have to have another six before they go in a carriage
Mrs Stuart:	So what's the next number when she's going to stop drawing stick men?
Sabrina:	Thirty
Mrs Stuart	It'll be 30, because 30 is, what's special about 30?
Sabrina:	It's a multiple of six.

Week 4, Term Three, 2009/2010

Increasingly discourse was highlighted as a way of learning. Students were expected to explain their reasoning and an emphasis on developing collaborative explanations was maintained. A consistent expectation was established that students would work as a collaborative community. When students explained their strategy solutions during whole class discussions, Mrs Stuart emphasised that their partners or group needed to listen carefully and support them when necessary. She made the speaker aware of peer support and facilitated the rest of the class to listen to the explanation and make sense of it while supporting everyone in the class to understand it. The vignette below shows the range of prompts which Mrs Stuart developed and trialled to further facilitate collaboration of the students in her classroom:

Developing a collaborative classroom community

After students have developed solution strategies for the first part of a functional reasoning task (see Figure 15). Mrs Stuart then asks them to develop an explanation for the next part.

Mrs Stuart:	Can you think of a way to explain that on the board? Practise with your partner please. Now I've got 25 children and I end up with four carriages and one left over. Practise it with your partner.

After the students have talked with their partners, Mrs Stuart calls upon Jasmine to explain.

Mrs Stuart:	Saffron, are you watching very carefully because she is your partner and you

	can help her out if you need to. Think of a way that makes sense to you. Talk us through as you go Jasmine.
Jasmine:	Twenty-five
Mrs Stuart:	Why 25?
Jasmine	Because there are 25 children in the class.
Mrs Stuart	Because there are 25 children in the class this time. Are you listening, Jasia? Because she might need some support.

Following Jasmine's explanation, Mrs Stuart then asks another student to develop an explanation using a picture as a representation.

Mrs Stuart:	Right, Zanthe talk us through as you go. What have you drawn us?
Zanthe:	A box.
Mrs Stuart:	And what's the box meant to be?
Zanthe:	A carriage.
Mrs Stuart:	A carriage, okay, and what's going to happen with that carriage?
Zanthe:	It's going to have six people in it
Mrs Stuart:	Well I'm asking all the questions, see if you can ask some questions to help Zanthe. Keep going, and if you wobble they're ready, they're listening, and they're ready to ask you a question to help you through it.

Week 4, Term Three, 2009/2010

Although an emphasis was placed on developing a collaborative community, Mrs Stuart continued to use pedagogical actions to ensure that students did not view this as always needing to agree with their peers. She emphasised mathematical argumentation when working with partners saying: *I was really impressed with the discussion that was going on when you didn't agree with your partner.* This focus led to students attending both to their own thinking and the thinking of others and using mathematical reasoning to agree or disagree. The vignette below illustrates how students had begun to use questioning to clarify ideas and increasingly to probe for justification for reasoning:

Using questioning to probe for clarification and justification

Mrs Stuart had asked the students to find two sixths of twelve. Two students share their solution strategy. They begin by separating 12 counters into six piles of two.

Tristan:	We made it into six piles and in each there were two and then we pointed at two of them and it made four.
Mrs Stuart:	Okay, there seems to be lots of hands up, Jacqueline, what would you like to say?

Jacqueline:	Why did you divide them into two?
Tristan:	Because then it made six groups.
Mrs Stuart:	Any other questions, Willow?
Willow:	Why did you have to get 12 altogether?
Tristan:	Because it says 12, two sixths of 12 so we had to get twelve counters.

Week 8, Term Three, 2009/2010

Another important shift during this phase was students' recognition of erroneous explanations. As the vignette below shows, students now questioned or challenged errors which supported the student explaining to reflect upon their thinking and re-construct this if necessary:

Challenging explanations

Mrs Stuart has asked her class to examine a set of number sentences to establish which are true or false. A student is sharing her response to 12 – 8 ÷ 4 = 1. She begins by identifying that the division needs to be completed first but then records:

Zanthe:	[12 – 8 = 4] Twelve take away eight equals four.
Mrs Stuart:	Lots of hands are popping up. Julio?
Julio:	Zanthe just did eight divided by four, and she did now twelve take away eight.
Mrs Stuart:	What are you saying then Julio?
Zanthe:	I know.
Mrs Stuart:	Has he helped you to think? Well done Julio. Keep going then Zanthe.

Week 12, Term Three, 2009/2010

Mrs Stuart continued to encourage student use of multiple representations. But more than just using a selected representation, she now developed an expectation that the students would translate between different representations. Shown in the vignette is how this included both asking students to draw on multiple representations in relation to a task (see Figure 16) and to listen to explanations by their peers and then to use an alternative representation for the explanation. This meant that students readily drew on a range of representations to support their explanations.

In Baby Jane's cot there are 12 soft toys and a third as many rattles.
How many rattles are there? How many toys are in the cot altogether?

Figure 16. Word problem. From *MEP lesson plans*, by S. Hajdu, 1999, retrieved from http://www.cimt.plymouth.ac.uk/projects/mepres/primary/y2blp_3.pdf

Using representations to support an explanation

Students have been working on a task. Mrs Stuart asks a pair to share their explanation.

Tobias:	One third of 12 equals four [records ⅓ of 12 = 4 and 4 + 12 = 16] because if you get three fours that equals 12 and four add twelve equals 16.
Juliana:	[arranges 12 counters into three groups of four].

Mrs Stuart asks the class for questions. A student asks them to explain about the one third.

Tobias:	Because in the story it said that there is one third of (pause) well it said that there are 12…soft toys [points to the 12] and it said there was a third as many rattles.
Mrs Stuart:	Does that help you? Is it the third of 12 that you don't understand or is it how they got the answer four which you don't understand?
Ferdinand:	How they got the answer four
Tobias:	Because if you are thinking about your four times table, it goes four, eight, 12 so that is three fours equals 12.
Mrs Stuart:	Juliana, what have you been doing with the counters there?
Juliana:	Well I got 12 counters and divided them into three groups and then we got them in fours and then that one is one third (covers one group of four) so that is one third of twelve.

Mrs Stuart asks for further questions.

Jacqueline:	Why did you add twelve to four?
Tobias:	Because it said how many toys were there altogether so there is 12 soft toys like there [points to the 12 in 12 + 4 = 16] and four rattles [points to the four] so you add it to get 16.

Week 8, Term 3, 2009/2010

Mrs Stuart maintained the expectation that conjectures would be expressed and proved while facilitating a consistent expectation for generalisation. She used questioning such as: *would it work for different numbers?* Or: *can I change that into something that would work for any number?* She was able to seamlessly use student reasoning to focus on generalisations. As the vignette displays she noted a student response to a fraction task (see Figure 17) and used this as an opportunity to engage the students in generalising:

1) What is one fifth of 15? 2) What is three fifths of 15? 3) What is one eighth of 16?

Figure 17. Finding fractions of a set (independently developed by Mrs Stuart)

Pressing for generalisation
As students were solving fraction tasks, Mrs Stuart noted that some students recorded this as $15 \div 5 \times 3 = 9$. Mrs Stuart: Is three fifths of 15 the same as doing 15 divided by five times three? Then she asked them to use a similar division strategy to find one eighth of 16 before she asked them to generalise: Mrs Stuart: What if I wanted to write one *n*th of 715. How could I write that as a division? After students talked with their partners, she asked a student to share: Alec: Seven hundred and fifteen divided by *n* equals *x*. Mrs Stuart: Can I change that into something that would work for any number? After students worked with their partners, she asked them to share: Hazel: One *h*th of *x* (long pause) Mrs Stuart: Come on Ferdinand, you're her partner. Ferdinand: [writes $x \div h =$] For one *h*th of *x*, you can do *x* divided by *h*. Week 8, Term Three, 2009/2010

Increasingly, representations were introduced as a way of providing a concrete justification for conjectures and generalisations. Mrs Stuart built on the earlier norms which she had developed and expected the students to justify their conjectures by using concrete material. For example, a student made a conjecture about dividing by one: *it's just like you're getting one group and dividing it by one group so you have already done it. If you've got a number and you divide it by one, it ends up that number.* Mrs Stuart responded by asking: *show what you mean with counters on the board.* As the students gained more experience in justification, they more readily drew on material to prove their reasoning. In the vignette below, Mrs Stuart facilitates the students to draw on representations to justify their reasoning regarding a task (see Figure 18) involving the distributive property:

Compare the results. Write the corrrect sign between them (<, >, =)

a)	14×6	$\square$	$10 \times 6 + 4 \times 6$	b)	32×3	$\square$	$30 \times 3 + 2$
	9×14	$\square$	$9 \times 7 + 9 \times 7$		17×4	$\square$	$8 \times 4 + 8 \times 4$

Figure 18. Number sentence problems. From *MEP practice book Y2b* (p. 158), by S. Hajdu, 1999, Budapest: Muszaki Publishing House.

Drawing on representations to justify reasoning

Over a number of lessons, students had been investigating how relational reasoning could be used to solve tasks involving the distributive property. Many students began to generalise the distributive property to solve the tasks. Mrs Stuart asks a student to share her explanation:

Misty: Seven add seven is 14 [*notates an arrow from each seven and writes 14 underneath*] and there is a 14 there [*indicates 14 on the left-hand side*] and they are both times nine so you have got 14 times nine and 14 times nine.

Mrs Stuart then asks the students to work in pairs using Misty's reasoning to prove whether $9 \times 6 = 9 \times 3 + 9 \times 3$. A student begins by building an array to represent 9×6, Misty then develops this further.

Misty: Because there is three there [*indicates splitting the six rows into three by drawing a line*]. There is three rows there and three rows there and that is just the same as those [*points to $3 \times 9 + 3 \times 9$ in the equation*] and then it is times nine [*points across the rows*].

Week 8, Term Three, 2009/2010

The consistent focus on the mathematical practices of justification, generalisation, and proof also led to students drawing on previously examined conjectures and generalisations in their explanations. For example, Mrs Stuart asked the students to investigate what numbers would have a remainder of one when you divided them by two. Juliana drew on her understanding of odd and even numbers from previous discussions: *fifteen because if you divided it into twos and it is an odd number so you have one left over.* Similarly, in a later lesson many students used a generalisation from an earlier lesson that anything multiplied by zero was zero to argue that $6 \times 0 + 5 = 7 + 4$ was a false number sentence. This illustrated the development of student awareness that one can draw on ones' own thinking.

During this phase it was evident from the photo elicitation interviews that for students in this classroom, collaboration with peers had come to be viewed as an important aspect of mathematical learning. During group work the students explored and investigated each other's perspectives while also reflecting on their own. Misty described: *once you've got all the ideas, you test one of the ideas then use the other idea if it doesn't work.* Explaining ideas to partners was viewed as beneficial both in supporting personal understanding and understanding of peers. Students were also able to identify clearly how they reflected on different solution strategies than their own.

Juliana: *We tried the half and we got it right, but someone else got it right in a different way so then we tried to think it through on the next one.*

They also described how they used their peers' ideas from previous lessons.

Paul: *I used Caleb's ideas from yesterday for the hedgehogs, to use the double bit.*

Both talking and listening were emphasised as key tools. Discussing mathematical ideas was highlighted as a way of approaching challenging tasks. Students described how if a task was less challenging, they only talked a little but if it was difficult they talked a lot. Talking about how they were going to do it was identified as making it easier. Clear links were made to the reciprocal nature of talking and listening. Students explained how both explaining your ideas and listening to explanations were reflective tools. For example, the researcher asked whether talking helps to learn mathematics.

Jacqueline: *It helps me because by teaching other people, it will make yourself get it more.*

Similarly, another student described how listening to explanations during whole class discussions was helpful.

Jasia: *It helps me learn because I'm saying is that right and then I'm thinking it in my brain.*

Students had developed an understanding of the role which was required within the interactive nature of their classroom. For example, Juliana described listening to Alec's erroneous solution strategy and stated: *I was a bit confused and I should have thought it through and asked a question.*

In contrast to the students' classroom participation practices in Phase One, in the photo elicitation interviews the students were now able to describe their own mathematical reasoning explicitly and re-explain solution strategies provided by their peers. Often they justified these explanations by referring to the context of the problem. For example, the students gave explanations of strategy solutions from the whole class discussions and compared this with their own.

Juliana: *She said that they had 28 apples and they divided it into four baskets for each child and then they did four, counted in fours and they had seven fours so they worked out that the answer was seven.*
Researcher: *Is that the same as what your group did or did your group do it differently?*
Juliana: *We did it the same way but instead of doing it in baskets we did it in, well we just drew the 28 apples and then we drew the four children and we just wrote an equation about 28 divided by four equals seven.*

Also in the photo elicitation interviews, the students were able to describe clearly patterns which they used to solve problems, how they identified these, and provide justification for the pattern. Often they explained how the pattern would be true for all numbers and generalised this to a different example: *because no matter what you take away, even if it's a really high number you can still add it on again and you equal the same.* Other times they drew on known number properties to justify verbally a specific case: *in multiplication you have commutative law so you can swap them round either way and five lots of three is the same as three lots of five.*

6.5.4 Analysis of task design and implementation, integration of algebra, and development of the classroom community

As outlined in the section above, Mrs Stuart reconceptualised her understanding of algebra. Using Blanton and Kaput's (2003) term, she had developed her algebra ears and eyes. She was increasingly able to recognise opportunities for algebra in the classroom including both planned and spontaneous instances.

As discussed in the literature review (see Section 3.2) specific factors in task design and implementation lead to greater affordances for algebraic activity. Drawing on learning from the professional development, Mrs Stuart carefully planned tasks to include opportunities for algebra. As Kaput and Blanton (2005) contend this meant that algebraic reasoning became an everyday part of the mathematics classroom.

With shifts in the classroom in the way that the tasks were implemented and the focus on algebraic reasoning there were resulting shifts in how the students engaged in the classroom mathematics activity. Along with Mrs Stuart, the students had also developed their own algebra ears and eyes and more readily drew upon algebraic reasoning, using structural aspects and patterns to solve tasks. Similar to what has been described by a number of researchers (e.g., Blanton & Kaput, 2005a; Cobb et al., 2009; Hodge, 2008) following reform in mathematics classrooms, Mrs Stuart described the new roles that her students were taking which included a different understanding of their obligations within the classroom.

As Mason (2008) and Smith and Thompson (2008) highlight through the teacher valuing and promoting specific ways of working, the classroom environment and its ethos is shaped. In Mrs Stuart's classroom, the emphasis was placed on developing mathematical discourse and collaborative interaction. During small group work, she emphasised the learning opportunities that arose from resolving different points of view. Similar to other research studies (e.g., Bastable & Schifter, 2008; Carraher et al., 2008; Fosnot & Jacob, 2009; Reid & Zack, 2009), Mrs Stuart structured the whole class discussion in a way that developed her young students' algebraic reasoning. She achieved this by positioning the students to listen actively and make sense of a range of explanations. As developing understanding became a shared responsibility there was evidence that students cultivated new questioning skills. They asked questions both to support their own developing understanding but also to probe for justification from their peers. This extends previous work by Boaler and Brodie (2009).

As shown in other research studies (e.g., Beatty & Moss, 2006; McNab, 2006) the use of tasks which involved multiple representations provided students with opportunities to use different forms to communicate reasoning and justify thinking. Mrs Stuart required the students to move between different forms of representation flexibly. As Schoenfeld (2008) contends, this supported students to begin utilising the representation which allowed the greatest affordance for the task. The focus on using representations led to the students consistently drawing on these to support their explanations.

By carefully monitoring and using student reasoning, Mrs Stuart developed a 'conjecturing atmosphere' as advocated by a number of researchers (e.g., Bastable & Schifter, 2008; Blanton, 2008; Mason, 2008). Using a similar model to the one that Blanton (2008) describes, Mrs Stuart engaged her students in building generalisations in the classroom. She achieved this through noting the conjectures that students made and then facilitating the whole class to investigate these. This involved testing and revising the conjecture and developing it into a generalisation.

A new expectation that was developed was that students would justify their conjectures using concrete materials. In this way, the students began to use representations to develop reasoned, general arguments. Schifter (2009) outlines specific criteria for student justification of general claims through the use of representation-based proofs. Examples from the current study meet Schifter's criteria as the meaning of the operation (e.g., multiplication) was represented in the manipulative and structure used and showed that the claim (e.g., distributive nature of multiplication) would work for all cases. A number of researchers (e.g., Carpenter et al., 2003; Carpenter et al., 2005a; Schifter) argue that facilitating students to use concrete material to justify conjectures and explanations enhances students' work with proof in later years.

Students in this classroom were able to describe clearly their different roles and obligations within this classroom. As Cobb et al. (2009) propose they developed identities which were related to micro-culture developed in the classroom. Their descriptions of collaborative discourse were aligned with what Mercer (2000) describes as exploratory talk. Similar to the students in Franke and Carey's (1997) study, these students now perceived doing mathematics as testing ideas, communicating thinking, and using differing solution strategies.

In summary, teacher actions evident in this section are illustrated below in Stage Three of the Framework of Teacher Actions to Facilitate Algebraic Reasoning.

Table 12

Stage Three of the Framework of Teacher Actions to Facilitate Algebraic Reasoning: Mrs Stuart

STAGE THREE	
Algebraic concepts	Address the following concepts in the classroom: understand the equal sign as representing equivalence; relational reasoning including whole numbers and rational numbers; commutative property; inverse relationships; odd and even numbers; properties of zero and one; distributive property; associative property; properties of rational numbers; using and solving equations; function
Teacher actions to develop and modify tasks and enact them in ways which facilitate algebraic reasoning	Recognise and use links to algebra in tasks across mathematical areas
	Implement tasks as open-ended problems
	Anticipate student responses which could provide opportunities for algebra
	Recognise and use spontaneous opportunities for algebraic reasoning from student responses
Teacher actions to develop classroom practices which provide opportunities for engagement in algebraic reasoning	Sequence solution strategies to advance mathematical thinking and reasoning
	Provide space for students to question for justification
Teacher actions to develop mathematical practices which support the development of algebraic reasoning	Lead explicit discussion about mathematical practices
	Listen for implicit use of number or operational properties. Uses these as a platform for students to make conjectures and generalise
	Facilitate students to represent conjectures and generalisations in number sentences using symbols
	Ask students to consider if the rule or solution strategy they have used will work for other numbers. Consider if they can use the same process for a more general case
	Promote use of concrete forms of justification
	Require students to translate between different representations

6.6 SUMMARY

This chapter has documented the journey that Mrs Stuart took as she developed her algebra ears and eyes and used this learning to facilitate algebraic reasoning in a mathematical community of inquiry. The development of Mrs Stuarts' understanding of early algebra was supported by the professional development tool (see Table 1). The learning culture of her classroom was transformed as she used the actions described in the Framework of Teacher Actions to Facilitate Algebraic Reasoning (see Table 10, 11, 12).

Initially algebra was not a part of the everyday mathematics lessons and many of the classroom practices were aligned with what is commonly described from studies of traditional mathematics classrooms (e.g., Bell & Pape, 2012; Fisher et al., 2011; Mehan, 1979; Pape et al., 2010). Classroom observations from Phase One provide evidence of the shifts in task implementation and the way Mrs Stuart began consciously to plan to integrate algebra into lessons. At this point some of the existing classroom practices limited opportunities for engagement with algebra. Evidence of Mrs Stuart's professional learning is shown in Phase Two and Three both during study group meetings and researcher visits. It was clear that she viewed her development as a personal responsibility and for her inquiry became a way of being (Jaworksi, 2006; 2008). In the classroom, shifts in task implementation saw students being facilitated to draw on mathematical structure and relationships to approach tasks. Mrs Stuart continued to extend her planning for algebraic reasoning and also began to notice and respond to spontaneous opportunities during the task enactment. Increasingly, the classroom practices and mathematical practices supported the students to engage with algebraic reasoning. These changes meant that not only did Mrs Stuart develop her algebra ears and eyes but the students in her class also increasingly approached tasks in an algebraic way and engaged in the key mathematical practices which are linked with algebra.

The following chapter describes a contrasting case of a second teacher Mrs Willis. Description is provided of the challenges that this teacher faced in developing her own algebra ears and eyes and instigating changes in her classroom.

CHAPTER SEVEN

CHALLENGES IN DEVELOPING ALGEBRAIC REASONING AND A MATHEMATICAL COMMUNITY OF INQUIRY: MRS WILLIS

7.1 INTRODUCTION

The previous chapter described the transformation of a classroom learning environment into one which focused on developing early algebraic reasoning within a mathematical community of inquiry. This chapter documents the contrasting case of change within Mrs Willis's classroom where although the teacher engaged in some change this became largely absorbed into her previous forms of practice. This meant that the creation of algebraic reasoning opportunities was not supported in the classroom.

Section 7.2 describes the Mrs Willis's classroom context prior to professional development commencing. Section 7.3 highlights the initial steps taken to introduce algebraic reasoning. Within the constraints of Mrs Willis's limited understanding of algebra, the discussion outlines the difficulties she had in planning and integrating algebraic reasoning opportunities. Finally, it highlights the small shifts that did take place and the main obstacles to the initial implementation of changes to the classroom community.

Section 7.4 describes the continuing challenges in developing algebraic reasoning in this classroom. It shows how anxiety about subject knowledge led to Mrs Willis positioning herself as a peripheral member of the study group. Drawing on classroom episodes it illustrates the missed opportunities for engagement with algebra due to obstacles in task implementation, attempts to develop collaborative work, and a lack of student understanding of the new role they were being expected to take.

Section 7.5 outlines Mrs Willis's shift in practice and the return of the classroom back to earlier forms of instruction. It shows that for Mrs Willis reflection on practice and the use of inquiry did not become a part of her everyday practice. Tasks, enacted with a computational focus, meant that students did not engage in investigation of structure and relationships.

7.2 PRIOR TO THE PROFESSIONAL DEVELOPMENT

7.2.1 Algebraic reasoning, classroom and mathematical practices, and student participation

Prior to the beginning of the research project, it appeared that there was limited integration of arithmetic and algebra within Mrs Willis's classroom. Aside from tasks involving algebra taken directly from the MEP curriculum, there were no types of algebra evident in the three lessons observed prior to the project. Enactment of the algebra tasks from the MEP curriculum involved a procedural and calculational focus. This is shown in the vignette below where the students were asked to solve a series of number sentences (see Figure 19):

Practise mental division.

a)	i)	$36 \div 9 =$	ii)	$3.6 \div 9 =$	iii)	$0.36 \div 9 =$
b)	i)	$56 \div 7 =$	ii)	$5.6 \div 7 =$	iii)	$0.56 \div 7 =$
c)	i)	$48 \div 6 =$	ii)	$4.8 \div 6 =$	iii)	$0.48 \div 6 =$
d)	i)	$96 \div 8 =$	ii)	$9.6 \div 8 =$	iii)	$0.96 \div 8 =$

Figure 19. Number sentences. From *MEP practice book Y5b* (p. 126), by T. Szalontai, 2003, Budapest: Muszaki Publishing House.

Enacting a relational reasoning task

Students were asked to solve number sentences which drew on a relational structure. Mrs Willis asked the students to share their answers and then provided them with an explanation of the process.

Mrs Willis: We started off with 36 divided by nine. Daniella, 36 divided by nine is…
Daniella: Four.
Mrs Willis: Four. What, therefore, is three point six divided by nine, Mandy?
Mandy: Three point six divided by nine is zero point four.
Mrs Willis: Is zero point four. We start off with, started with 36, we had our multiplication which is ten times smaller. We divide 36 by ten, so our answer is also going to be divided by ten as well. Ten times smaller. This time we've got zero point three six divided by nine, Albert, which is what?
Albert: Zero point, zero four.
Mrs Willis: Zero point, zero four. Again, our zero point three six is ten times smaller than three point six, our answer will be ten times smaller as well.

Week Seven, Term Three, 2008/2009

Whole class discussions were used to check or correct student answers. When students gave erroneous responses Mrs Willis would try to use questioning to lead them to a correct response. Alternatively, she would correct the response herself or ask another student in an attempt to get the correct answer.

Mrs Willis: What's my remainder when I divide one by two? So I've decided there are no twos go into one, but my remainder is…
Mandy: Five?
Mrs Willis: If I divided one by two. Twos into one don't go at all. What's left over?
Mandy: Half?
Mrs Willis: (Laughs)
Mandy: Two.
Mrs Willis: What did you say to me before? I'm just confusing you here. Anyone help her out?
Mandy: Zero point five
Mrs Willis: What is my remainder? If I'm dividing two by one I get nothing. What's left over? Meredith?
Meredith: Uh, one whole?

On this occasion, Mrs Willis then continued to use directive questions to funnel the students to a correct response.

In the lesson observations, Mrs Willis was regularly seen to make tasks easy for her students to avoid any confusion. For example students had attempted to explain how to solve a division task (see Figure 20) using an informal strategy.

I have 81 pounds and 70 pence. I want to divide that money between five lucky children. How much money does each child get?

Figure 20. Word problem developed by Mrs Willis

Mrs Willis interrupted the explanation stating: *I think I'm going to leave that now, because I think you're just getting more and more confused. We will go and do it straight into the long division which I think will be much, much easier.* An overall expectation in the class was that not all students could successfully solve the tasks or access explanations. Often students did not respond to questions posed or provided an incorrect answer. In response Mrs Willis would ask an alternate student for a response.

There was limited use of representations in the lessons. For the most part students were directed to use equations to illustrate their thinking.

Collaborative or paired work was infrequently used. Most of the lessons were in a whole class format with Mrs Willis predominantly using funnelling questions and calling on individual students to answer. When students worked independently on tasks, they worked individually with an emphasis on a fast pace and correct answer. For example, Jaime said she liked working by herself because: *I can work quicker when I am by myself.* Talking with those sitting in close proximity was viewed as a way ensuring a correct answer: *you can talk with the people next to you to check if you got the same answers and you can change them so there is more chance of getting them right.*

At the start of the project the discourse patterns in this classroom were similar to what is described in Chapter Six in Mrs Stuart's classroom. They were largely teacher dominated and as shown in the examples student responses were generally limited to one word answers or answers phased as questions. Mrs Willis spent a significant portion of the lessons restating student responses and adding an explanation or justification for the students' reasoning. During the three lessons observations there were no complete, clear explanations provided by students.

The students perceived Mrs Willis as the mathematical authority in the classroom. They described how she explained what they were going to do. They stated that: *when we are doing the answers, she does the method so I know what to do.* However, contrary to this norm, a desire to construct their own understanding was expressed by some students. For example, they referred to liking maths more when: *we can actually come up to the board and do stuff without the teacher just telling us what it is.* Overall, they viewed listening to the teacher, practice, tests, and homework as important ways to learn mathematics and succeed in the subject.

7.2.2 Analysis of algebraic reasoning, classroom and mathematical practices, and student participation

Opportunities to engage students in algebraic reasoning were not recognised or used by Mrs Willis. An emphasis in this classroom was on a fast pace, generating correct answers, and avoiding challenge or confusion for students. To achieve this, the majority of teacher questioning consisted of low level, funnelling questions. As highlighted in a range of research studies (e.g., Boaler & Brodie, 2004, Franke et al., 2009; Graesser & Person, 1994; Hiebert & Wearne, 1993; Wood, 1998; Wood et al., 1991) these types of questions aim to elicit correct short answer responses. The cognitive demand of tasks was lowered by Mrs Willis providing explicit procedures and solving more difficult aspects herself. This is similar to that described by Henningsen and Stein (1997).

Discourse in this classroom followed a traditional IRE pattern as described within research literature (e.g., Fisher et al., 2011; Mehan, 1979; Pape et al., 2010). This limited the opportunities that students had to construct understanding and to develop a sense of agency. Similar to the role Cobb et al. (2009) describe of students in a traditionally taught algebra class, Mrs Willis's students described their obligations within the context of listening to the teacher and using her method. As a range of researchers (e.g., Colby, 2007; Franke & Carey, 1997; Mason & Scrivani, 2004; Star & Hoffman, 2005) show epistemological beliefs can affect mathematical performance. These students viewed mathematics as a body of knowledge which was learnt through teacher directed instruction, tests, practice, and homework.

7.3 PHASE ONE: INTRODUCING EARLY ALGEBRA

7.3.1 Teacher learning

At the start of the project although Mrs Willis shared her thoughts and ideas freely, she was a quieter member of the study group. Her expressed view that she did not have a clear idea of what early algebra was matched her concept map contribution depicting the use of formula and equations involving unknowns (Figure 21).

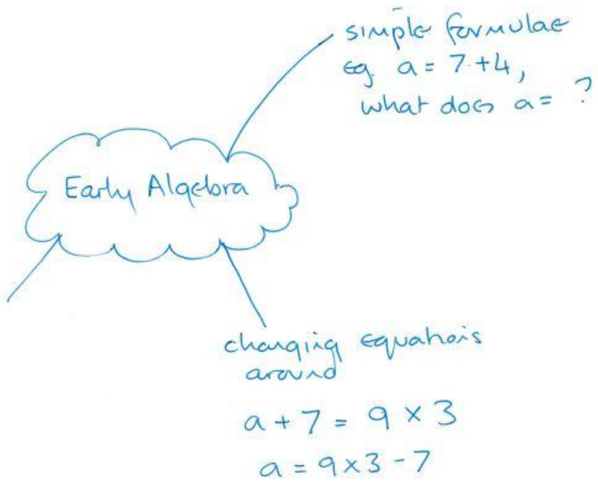


Figure 21. Mrs Willis's concept map of early algebraic reasoning at the start of the project

Although Mrs Willis felt she had a limited understanding of algebra, she was able to identify potential incorrect responses to open number sentences. Her prediction for a possible student response to an open number sentence (e.g., 8 + 6 = __ + 5) was 14. She also described observations from her classroom of students recording incorrectly when they were asked to solve number sentence problems (e.g., 7 + 3 + 2 =): *initially they would do seven plus two equals nine plus three [records 7 + 3 + 2 = 7 + 2 = 9 + 3 = 12] and I keep saying to them 'but does seven plus two equal nine plus three?'* Interestingly, although Mrs Willis explained how she had tried to address this misconception, many of her students (68%) provided responses in the initial task-based interview which reflected their continued misunderstanding of the equals sign. While Mrs Willis was able to identify errors related to the equals sign, this did not extend to potential use of relational strategies to solve open number sentences.

Similar to Mrs Stuart (see Chapter Six), Mrs Willis used a research article by Monaghan (2005) to reflect on her own practice. This prompted her to begin to question whether she guided students too much: *I think maybe we do too much from the front and direct it too much rather than giving every pair a chance to discuss things first.* She also considered her role when students made an error: *I wonder…how much I sort of say "no that is not right" rather than* *letting them work it through themselves and it can just become a discussion between the teacher and one or two children leaving out the rest.*

Mrs Willis's reflection on practice led to her identifying specific areas in which she wanted to develop her practice. Important areas of focus included: facilitating students to question each other; providing students with time for paired discussions; and strategies to ensure her students actively listened when another student explained their reasoning to the class.

7.3.2 Analysis of teacher learning

Mrs Willis's narrow view of algebra as abstract symbol manipulation is a common perception described by a range of researchers (e.g., Chazan, 1996; Kaput, 2008; Stacey & Chick, 2004). She expressed some anxiety over what she perceived as her lack of understanding and knowledge of early algebra. As Blanton and Kaput (2008) highlight, such anxiety can lead to teachers remaining isolated within a community of practice. Despite this, Mrs Willis was able to anticipate some student reasoning in the area of early algebra and shared her recognition that many students had inadequate understanding of the equals sign such as described by research studies (e.g., Carpenter et al., 2005b; Knuth et al., 2006; McNeil & Alibabi, 2005).

Similar to Mrs Stuart research articles were used by Mrs Willis to begin questioning and reflecting on her own practice. As highlighted by Ghousseini and Sleep (2011) and Koellner et al (2011) taking a critical view of actions within the classroom can lead to changes in instructional practice.

7.3.3 Task design and implementation, integration of algebra and development of the classroom community

Initially Mrs Willis implemented the tasks from the MEP curriculum by carefully guiding her students through the steps necessary to complete them. Her use of funnelling questions typically elicited short answer responses. She expressed to the researcher her unfamiliarity with implementing tasks differently and asked for support as to how to introduce curriculum material in a way which led to students independently solving problems in pairs or small groups. Following discussion with the researcher she gradually trialled the introduction of some small changes. In the first instance she began to ask her students to continue to work on tasks in pairs or groups after her whole-class introduction.

During this initial phase Mrs Willis either used tasks as presented in the MEP material or with researcher support adapted tasks based on curricular material. For the initial lessons Mrs Willis, with the intent of supporting students to make generalisations about fractions, adapted the implementation of a task (see Figure 22) involving students comparing improper fractions to one whole.

Comparison with 1

Study these fractions. Let's circle the fractions which are greater than 1 and write each as a mixed number. What is a mixed number?
(a number containing a whole number and a fraction)

Ps come to BB to circle a fraction and rewrite it, explaining reasoning. Class points out errors. Elicit that fractions greater than one have a numerator greater than the denominator.

BB: $\frac{2}{3}$, $\frac{4}{5}$, $\frac{9}{2}$, $\frac{3}{8}$, $\frac{11}{37}$, $\frac{18}{14}$, $\frac{33}{33}$, $\frac{35}{33}$, $\frac{8}{8}$, $\frac{5}{4}$

$4\frac{1}{2}$ $\quad 1\frac{4}{14} = 1\frac{2}{7}$ $\quad 1\frac{2}{33}$ $\quad 1\frac{1}{4}$

We have circled the numbers more than 1, but what could you say about the other numbers? (numbers are not more than 1, or less than or equal to 1). Ps point out numbers equal to 1.

T summarises:

- If the numerator is less than the denominator, a positive fraction is less than 1.
- If the numerator is equal to the denominator, a positive fraction is equal to 1.
- If the numerator is greater than the denominator, a positive fraction is greater than 1.

Figure 22. Comparison with one fraction task. From *MEP lesson plans*, by T. Szalontai, 2003, retrieved from http://www.cimt.plymouth.ac.uk/projects/mepres/primary/y5alp_1.pdf

Mrs Willis introduced this by stating: *I'd like you to tell me what you can tell about the fractions comparing the numerator and the denominator and I want you in your groups to come up with statements that you can make about these fractions.* In other examples, she asked students to generalise what fractions would be equivalent to one whole and the process of finding equivalent fractions.

While Mrs Willis used fraction tasks as a basis for the students to develop conjectures about fractions the discussion focused on specific examples. The vignette below illustrates how Mrs Willis missed an opportunity to shift a discussion of a conjecture to a generalisation and instead focused on specific examples which were examined procedurally.

Missed opportunity to develop a generalisation
A student provides a diagram to illustrate the conjecture that a fraction with the same numerator and denominator would equal one whole Maggie: *(draws a long rectangular shape and shades in five fifths)* At the start I only shaded in two fifths because, well, but that didn't equal five fifths and the five fifths would be the whole of that. Mrs Willis: Okay so you divided it up into five equal parts and you then have to shade in five of those five equal parts which gives you the same numerator and denominator *(notates 5/5)*. Okay thank you. What does this fraction mean? *(points to 5/5)*. What is it telling us to do? Heera? Heera: You have five pieces and you have to divide them by five. Mrs Willis: So five out of five means that you are looking at the numerator five and you divide it by five okay *(writes 5 ÷ 5)*. What does five divided by five give us? What is five divided by five, Terence? *(pause)* If I had five sweets and I had to divide them among five children what would they each get? Terence: One. Mrs Willis: They would get one each *(notates 5/5 = 5 ÷ 5 = 1)*. Well done. So five fifths is equivalent to one whole unit - does that work for every number? *(pause)* If we had seven sevenths *(writes 7/7)* - what is that telling us to do, Todd? Todd: Divide the whole unit by seven. Mrs Willis: Okay and then how many parts of that unit are we looking at? Todd: Seven. Week Seven, Term One, 200/2010

In the example above, Mrs Willis missed an opportunity to highlight a diagram as a form of concrete justification. Instead she used it as a specific example. While she did ask students whether the pattern would be true for every number, she then validated use of examples by using a specific example.

Although Mrs Willis attempted to adapt task activity to include algebra, a number of other factors inhibited student engagement with algebraic reasoning. In the first instance, Mrs Willis did not anticipate the integration of any algebraic activity within the adaptations. For example, she described the focus of these lessons as: *getting them used to fractions that was the main focus, to really establish what a fraction is and so they understand what the denominator and numerator is.* Lack of a clear focus meant that at times the purpose of the activity was unclear and often it shifted to a procedural focus. Linked to this limitation in planning, was the evidence that Mrs Willis did not reflect on possible student responses. This meant that she was frequently confronted with an unexpected response and found it difficult to appropriately respond as seen in the following vignette where students were examining fractions of shapes (see Figure 23).

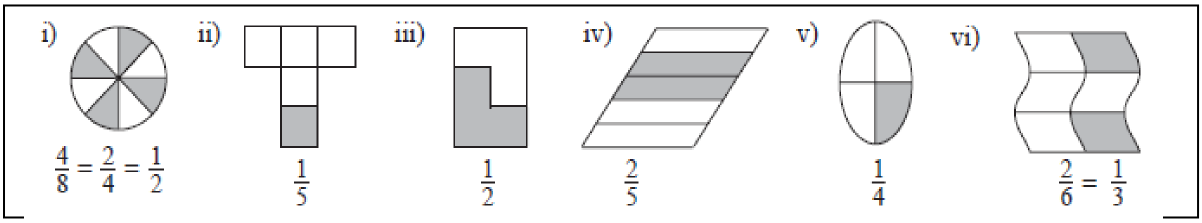


Figure 23. Fractions of shapes. From *MEP lesson plans*, by T. Szalontai, 2003, retrieved from http://www.cimt.plymouth.ac.uk/projects/mepres/primary/y5alp_1.pdf

Challenges in engaging students in algebraic reasoning

After examining fractions of shapes and identifying possible equivalent fractions, a student made a conjecture that you would need to divide the numerator and the denominator by the same number to maintain equivalence. He followed this up by giving an example of in-equivalence stating that if you had six sixths and only divided the numerator by two it would not be equivalent as the result would be three sixths. Mrs Willis asked another student to re-explain this.

Amber:	Because say if you did *(pause)* three *(pause)*
Mrs Willis:	Why would you have to divide six sixths by two over two to get a simplified fraction?
Amber:	Because otherwise you wouldn't get an answer.
Mrs Willis:	Well you would get an answer, I have got an answer here but it is not correct. It is not an equivalent fraction because I have got my unit divided up into *(draws rectangle divided into six)* six parts, I should have shaded in the whole thing and that would give me six sixths but I am only going to do one sixth now so it is not the same size as the whole shape.
Amber:	Because if you *(pause)* well that would be one *(pause)* it wouldn't be one sixth because *(pause).*
Mrs Willis:	Try again, try again, why do I need to divide by a numerator and the denominator that are the same? Why am I dividing six by two and dividing six by two on the bottom as well? Why am I dividing by two over two? What is two over two equivalent to? Jasmine what is two over two equivalent to?
Jasmine:	One whole.
Mrs Willis:	One whole, it is equivalent to one whole. If I divide any number by one whole what does that give me, Maggie?
Maggie:	The number that you were dividing by.
Mrs Willis:	If I divide any number by one, what does that leave me with? If I had 24 and I divided by one what would I end up with?
Maggie:	Twenty-four.

Mrs Willis continued to ask different students in the class to divide different numbers by one. She then returns to the original student who she had asked to explain.

Amber:	Because it is a whole number and you can't divide a fraction by a whole number.
Mrs Willis:	How do you mean?
Amber:	Well if you just had a plain six and you divided it by six it would be six but if you had six sixths divided by six (pause) it would be one whole.
Mrs Willis:	I don't quite follow, I'm not sure if I am following. Could you explain again? Come up to the board and write what you are trying to explain to see if we can get it clear because I am not following maybe as well as I should be.
Amber:	*(writes 6/6 ÷ 6 = 1)* Well if you had six sixths and you divided it by six, it will give you one *(pauses and looks at Mrs Willis).*
Mrs Willis:	Why would it give you one?
Amber:	Because *(pause)* if you do six divided by six you will get one *(pause).*

Mrs Willis then uses funnelling questioning to lead the student to a correct response.

Mrs Willis:	Are you dividing both the numerator and the denominator by six? Is that what you are saying to do?
Amber:	Yeah.
Mrs Willis:	So effectively what we're doing is dividing six over six by six over six *(writes 6/6 ÷ 6/6)* so why can we divide it by six over six, what are we actually dividing by?
Amber:	*(long pause)* One.
Mrs Willis:	We're dividing by one, fantastic, well done. You did know what you were talking about and I was just not following you.

Week Seven, Term One, 200/2010

Referring to this exchange during a reflective interview Mrs Willis described herself as panicking in the moment. Although she felt that: *it's quite a straightforward concept* she described herself as getting: *all muddled in it and I think that by the time I got Amber up there at the board, I felt, gosh, where am I going with this and I lost track. I think I got bogged down in other things rather than actually having that clear focus.* She noted that if she repeated the lesson: *I would think more clearly about what I was wanting from them, because I don't think I did. I don't think I thought about it clearly enough. I was just thinking 'how can I do this one and how can I change that bit around?'*

Listening to and building on student reasoning proved challenging for Mrs Willis. As shown in the vignette above, she was frequently unable to make sense of and revoice student reasoning. Due to this difficulty Mrs Willis expressed some doubt during interviews with the researcher about the efficiency of having students explain their reasoning: *I suddenly lose my thread when they're starting to explain, I think well what was our focus and I find that I lose my way. I wonder if that doesn't really help the children.* Despite these doubts she could identify some positive aspects of using student reasoning: *their explanations are completely different from how I would have explained it and I think that's actually something quite important to hold on to that, that there is no one explanation that is correct and you can explain things at different levels.*

Before the project began collaborative work was an uncommon practice within Mrs Willis's classroom. In the initial lessons in Phase One Mrs Willis organised the students into groups and then asked them to complete tasks. However, she did not address group norms nor did she engage the students in discussion about the new expectations. Mrs Willis's observations in the post lesson interviews were: *it was very much one person doing it and then not listening to any of the others.* This was also evident in the large group discussion. For example, when Mrs Willis asked a student to share back what his group had discussed he responded: *Well Amber was the one who pretty much discussed it all.* These observations led Mrs Willis to consider how to further develop collaborative work.

It was evident that the lack of established norms for collaborative work extended into whole class discussion. Within this classroom the students frequently recorded their ideas on the whiteboard and shared responses with the class during discussions. However, when students came to the board to share, they would speak facing the whiteboard with their back to other students or alternatively they would direct their responses solely to Mrs Willis. After researcher input Mrs Willis began to address this by asking students to turn around and speak to the class when explaining their ideas.

To encourage more productive student participation during whole class discussions, Mrs Willis drew on design research and began to trial ideas and test and refine these during lessons. She first emphasised new expectations involving explanation of reasoning rather than just provision of answers. She began to ask listening students to participate more actively. This included asking them to add on to ideas, agree or disagree with reasons, and ask questions. Initially students were hesitant to ask their peers questions about their solution strategies. By increasing the expectation for all students to revoice the shared solution strategy, Mrs Willis began to develop space for listening students to question: *I am going to ask any one of you to explain what Brad has just explained so if you don't understand you need to question him now.* This was particularly important in these early lessons as students were generally unable to revoice or repeat mathematical reasoning provided by their peers.

Some students were hesitant to provide their thinking and reasoning. At other times, as shown in the vignette above, Mrs Willis had difficulty following their reasoning. She would respond by asking other students to explain the idea rather than expecting the original student to further develop their reasoning. Often the students she chose to take over the explanation were those who Mrs Willis identified in discussion with the researcher as: *very bright and capable.* For example after a student had attempted to explain his reasoning she said: *I think you are doing the same thing as Todd but I am not quite following how you worked it out but I can see exactly how Todd did. So Todd can you come up to the front?* Similarly, in another lesson she asked a student to agree or disagree with an explanation and provide her reasoning for this as illustrated in the example below.

Mrs Willis:	*Okay right, Brittany can you say why you agree with her?*
Brittany:	*Because if you have three quarters of, or if you have two and you add one more. (Pause)*
Mrs Willis:	*How do you find, how do you find just one quarter?*
Brittany:	*(pauses and does not respond)*
Mrs Willis:	*Haileigh, how does she find one quarter?*
Haileigh:	*Ummmm (pause)*
Mrs Willis:	*…I want to know why you drew it, you drew the line that long there must have been a reason that you drew it. Todd why did you draw it?*

Selecting another student to complete the explanation had the consequence of positioning students as less or more competent within the classroom.

Students continued to predominantly provide answers with no reasoning. However, with increased pressing from Mrs Willis they did begin to provide reasoning for some of their responses. However these typically lacked clarity—no complete, clear explanations were provided in these lessons.

During Phase One a notable change was observed in the nature of collaboration during whole class discussions. For the first time there were instances where students agreed or disagreed with justification with a peer’s reasoning. They also began to ask their peers questions for clarification.

Working together in a group was commented on positively by the students. However, during the photo elicitation interviews the students did not refer to any mathematical benefits of group work, instead they made statements such as: *it was kind of fun because we could all talk* or: *it was alright…we all just had a bit of a laugh.* Comments which referred to specific benefits mainly focused on other people helping them. Likewise, students did not view the whole class discussion as a way of developing their understanding. In contrast they referred to difficulties they had in listening to and understanding their peers' solution strategies. Many of these difficulties could be attributed to the lack of established norms for large group discussion. Students made statements which indicated that they had difficulty hearing what the explainer was saying or that the person explaining was talking too fast. Interestingly, in the photo elicitation interviews after Amber incorrectly stated that six sixths divided by six was equal to one whole, more than half of the six interviewed students thought that this was correct. For example, Millie stated: *six sixths divided by six sixths is just the same as doing six sixths divided by six.* Rather than being viewed as opportunities to learn, for those students who recognised the error in Amber's explanation, this was identified as a negative event for Amber. For example Haileigh described herself as thinking: *Oh my god, you got it wrong.*

7.3.4 Analysis of task design and implementation, integration of algebra and development of the classroom community

A range of researchers (e.g., Anthony & Walshaw, 2009; Askew et al., 1997; Warren, 2009) highlight the role of sound teacher knowledge to develop effective teaching of mathematics including early algebra. Specifically, Franke et al. (2008) highlight the importance of teachers making sense of algebraic ideas and furthering their understanding of student reasoning about algebra. Mrs Willis was unable to draw on a sound understanding of algebra to identify algebraic concepts clearly within a lesson. She did not anticipate student reasoning or potential areas of difficulty which meant that she was unable to generate possible next steps for instruction as the tasks were enacted.

Research studies (e.g., McCrone, 2005; Reid & Zack, 2009) highlight the significant role which teachers take in guiding the development of active classroom discourse. The findings of this study support Mercer et al. (2004) and Edward and Jones (2003) contention that productive group interactions require teacher intervention and scaffolding. In initiating change within the classroom and trying to shift discourse away from an IRE pattern, Mrs Willis attempted to move away from a role which Bell and Pape (2012) describe as the mathematical authority. However, in some instances she positioned her students as less or more competent. Thus although student reasoning was beginning to be more of a focus, this was only the case for some students. Therefore in contrast to what is described by Stein et al. (2008), many of her students were not given authority over their mathematical work.

Hodge (2008) highlights the shifts in students' role as learners when there are changes in the classroom. Although Mrs Willis began to initiate change within the classroom for most students the new practices contrasted with their prior experiences of what it means to do mathematics in the classroom. Students in change as Cobb et al. (2009) note may choose to either co-operate or resist engagement with the classroom activities.

In summary, Mrs Willis did not follow the same trajectory as Mrs Stuart in her use of teacher actions to facilitate algebraic reasoning. Teacher actions evident in Phase One for Mrs Willis are those highlighted below (Table 13) in *Stage One of the Framework of Teacher Actions to Facilitate Algebraic Reasoning*. Those actions which she consistently used successfully are highlighted in yellow. Other actions which she took but only implemented partially are highlighted in blue. Those actions not shaded indicate productive instructional strategies that were evident in Mrs Stuart's classroom (see Table 10) but not so in Mrs Willis class during the similar time period.

Table 13

Stage One of the Framework of Teacher Actions to Facilitate Algebraic Reasoning: Mrs Willis

STAGE ONE	
Algebraic concepts	Address the following concepts in the classroom: understand the equals sign as representing equivalence; relational reasoning including whole numbers and rational numbers; commutative property; inverse relationships; odd and even numbers; properties of zero and one; distributive property; associative property; properties of rational numbers; using and solving equations; function
Teacher actions to develop and modify tasks and enact them in ways which facilitate algebraic reasoning	Implement tasks as problem-solving opportunities
	Emphasise student effort to approach and complete cognitively challenging tasks
	Extend or enact tasks to include opportunities for generalisation
	Interrogate tasks for opportunities to highlight structure and relationships
Teacher actions to develop classroom practices which provide opportunities for engagement in algebraic reasoning	Lead explicit discussion about classroom and discourse practices
	Ask students to apply their own reasoning to the reasoning of someone else
	Require students working in pairs or small groups to develop a collaborative solution strategy which all can explain
Teacher actions to develop mathematical practices which support the development of algebraic reasoning	Require students to explain their reasoning

7.4 PHASE TWO: CHALLENGES IN DEVELOPING ALGEBRAIC REASONING

7.4.1 Teacher learning

As the project continued Mrs Willis remained a more reticent member of the study group. While she shared ideas and examples in meetings this was to a lesser degree than the other two members of the study group. Additionally, in between research meetings, she frequently did not reply to emails from the researcher. She remained anxious about her personal understanding of early algebra. For example, in a study group meeting the teachers were asked to bring lesson plans from the MEP curriculum which could be used or modified to include early algebra. She expressed that she found it difficult to identify relevant material and described herself prior to the meeting as skimming through the material: *a bit in a panic like help what can I do that's going to be relevant?*

This anxiety was also reflected in Mrs Willis's reaction to some of the activities during the study group meeting. When the study group worked on justifying the conjecture that two odd numbers added together resulted in an even number, she was most comfortable using specific examples as a form of justification. The researcher then asked the teachers to experiment with forms of concrete justification. Mrs Willis stated: *it's quite difficult though using a diagram for odd numbers, am I being really thick here?* In another example the researcher asked the teachers to consider possible solutions for the number sentence: $a + b = 10$. She then asked whether both a and b could be five. Mrs Willis argued: *they couldn't both be five…logically they can't be because they have to look the same to be the same.* In the subsequent discussion of this Mrs Willis described herself and another teacher as: *we both failed.*

It appeared that Mrs Willis maintained a largely procedural, rule oriented understanding of mathematics. For example, in a study group meeting the teachers constructed a range of possible true or false conjectures that students may generate during lessons. One possible conjecture provided was: *if you multiply by ten you can just stick a zero or add a zero to the end*. In the ensuing discussion Mrs Willis argued that this conjecture was true aside from when decimal numbers were involved. This procedural, rule-oriented view appeared to influence her mathematics teaching. She described how she favoured teaching traditional algorithms such as long multiplication arguing that they were preferable because students experienced difficulties when using methods that involved number properties: *when we're doing multiplication, they still* *can't necessarily see that 49 is the same as 40 and a nine and you're multiplying both sides by…five.*

Despite her on-going anxiety about her knowledge Mrs Willis noted improvements in her understanding of early algebra. Her concept map (see Figure 24) illustrates an extended view of algebraic concepts including equivalence and relationships.

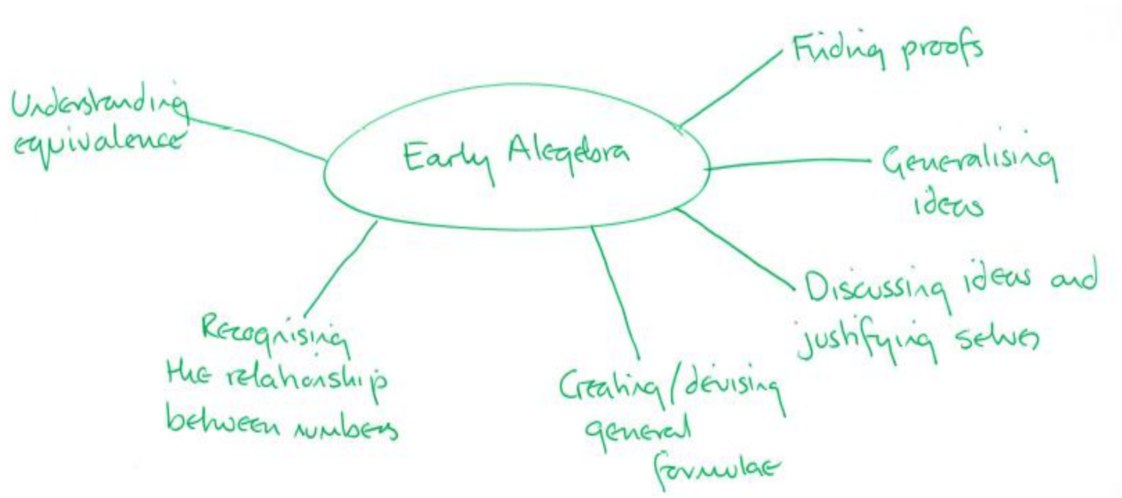


Figure 24. Mrs Willis's concept map of early algebra in the second phase

Moreover, her concept map now included both classroom practices and mathematical practices including justification of ideas, generalisation, and the use of proof.

During the study group meetings Mrs Willis continued to be able to make predictions of possible student responses to tasks. For example, she was able to identify and describe a range of conjectures which she had heard students make in the classroom. These included conjectures about the properties of zero and one and extended to other areas such as measurement and area. In another example during a meeting the study group was asked to predict student responses to a task (see Figure 25). Mrs Willis was able to suggest a range of incorrect and correct solution strategies.

In a race, the runners are started 1 minute after each other. The first runner covers 174 m each minute and the second runner covers 182 m each minute.

What distance will be between the two runners:

a) 10 minutes after the first runner started

b) 30 minutes after the first runner started?

Figure 25. Runners word problem. From *MEP practice book Y5b* (p. 154), by T. Szalontai, 2003, Budapest: Muszaki Publishing House.

In study group meetings and interviews with the researcher Mrs Willis identified shifts in the way her students participated in the classroom. She could also describe pedagogical actions that she was testing and refining to facilitate these shifts. Key shifts included the way in which students worked in collaborative small groups, their explanations during whole class discussions, and how the rest of the class participated during an explanation. She attributed improvements in their small group work to increased opportunities to work in this way but also: *I'm trying to let them actually think it through more for themselves which I think has actually made quite a difference rather than being reliant on me they have to be much more reliant on themselves.* She also noted that her students were more confident to explain their ideas to the class: *rather than to explain to me.* In facilitating changes in student focus she remarked that: *I think it helps if I'm standing at the back as well because then they, rather than…directing their explanation to me, they are then addressing the whole class.* Mrs Willis also highlighted her realisation of the need to facilitate students to question for understanding when they were listening to their peers' explanations. Through focusing on this she described how students had become: *very good at saying "no, I don't follow you".* She described her longer term aim as scaffolding students to identify the specific part of the explanation which they did not understand.

Although Mrs Willis highlighted these changes following the initial research visits, these changes were not sustained between later research visits. It appeared that Mrs Willis viewed these ways of working as only appropriate for some types of mathematical work. Therefore, in interviews during the end of the second phase she explained to the researcher that because they had been doing geometry: *it hasn't been structured in quite the same way…They haven't been working in groups.*

7.4.2 Analysis of teacher learning

While Mrs Willis participated to some degree in the emerging community of practice it appeared that anxiety over her mathematical understanding led her to position herself as a weaker, peripheral member of the community. This contrasts with Mrs Stuart who positioned herself as engaged with 'social' work associated with facilitating algebraic reasoning.

Despite improvements in algebraic understanding Mrs Willis was aware of the limitations concerning her own mathematical understanding. For example, during a study group meeting she recounted an example from a lesson where she accidentally gave the students a problem to solve which was $2 \times \_\_ = 1$: *A lot of them said that's impossible, you can't do it and yet there were the rational ones who said 'well just think about it, how does a number become smaller if you are multiplying?'…I found myself saying 'well but if you multiply two numbers together they always get bigger' but I thought well actually, no they don't, well what do I do now?*

Despite these challenges Mrs Willis was beginning to reconceptualise her understanding of algebraic reasoning. She was beginning to think about the types of pedagogy that she used in the classroom and learning to notice elements of her practice and reflect on the strategies which she used that led to shifts in classroom practices. As a range of researchers highlight (Blanton & Kaput, 2008; Franke et al., 2008; Jacobs et al., 2007; Koellner et al., 2011; Schifter et al., 2008) these are important aspects of professional learning which support the teaching of early algebra.

7.4.3 Task design and implementation, integration of algebra, and development of the classroom community

Engaging students with cognitively challenging tasks was an unfamiliar practice for Mrs Willis. During a study group meeting she described her reaction when she came to what she perceived as a challenging task in the curriculum material: *I don't do that with the children because some of them picked it up straight away but the ones who didn't, it just threw them completely so I just avoid it.*

Following the initial study group meetings there was evidence that Mrs Willis had begun to question her previous practice of carefully guiding the students through tasks and teaching by telling. During her mathematics lessons in Phase Two of the study she further trialed new ways of task implementation. She began to read the task to students and then ask them to solve it in pairs or small groups. This shift in task implementation meant that many students were increasingly challenged by the tasks. Mrs Willis referred to this in an interview after a lesson stating: *they struggled a lot more than I thought they would and I find it quite frustrating because I feel, if I jumped in, but I didn't want to jump in, I wanted them to really think it through for themselves.* She continued on to critique her previous method of task implementation of telling the students what to do: *where's the learning in that? They just follow and what you've said, some will remember and the majority won't.*

As Mrs Willis moved away from carefully guiding the whole class through each task step, she had new challenges in regards to evaluating the time required to complete a task. She struggled to assess the appropriate amount of time for students to complete tasks and described: *I get a bit bogged down.* Aiming to ensure that each group reached a correct solution meant that other groups had longer on the task than they required. She acknowledged that often she gave students too much time on one task and then would have limited time for other tasks she had planned: *we ended up spending a lot more time, I think I shouldn't have…I think I carried on with it for too long…We then didn't have enough time to go through the other problem properly.*

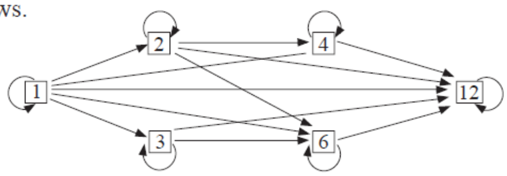
Mrs Willis continued to attempt to integrate algebra into her lessons by using the MEP curriculum tasks as a basis for students to develop and investigate conjectures. For example, in December after asking students to complete the task (see Figure 26), she then asked them to develop division statements drawing on inverse relationships.

Multiples 1

T has the numbers, 1, 2, 3, 4, 6 and 12 written on BB as in diagram below. Let's draw arrows from each number towards its multiples. What is a multiple of a number? (e.g. The result of multiplying that number by another number.)

Ps come to BB to draw arrows, explaining reasoning. e.g. 'I draw an arrow from 2 to 6, as 6 is 3 times two.' Class points out errors or missed arrows.

BB:



If the arrows pointed in the opposite direction, what would it mean? (The arrows would be pointing towards the factors of each number.)

Elicit that a factor of a number divides into that number exactly.)

Figure 26. Multiples 1. From *MEP lesson plans*, by T. Szalontai, 2003, retrieved from http://www.cimt.plymouth.ac.uk/projects/mepres/primary/y5alp_3.pdf

Following this, students were asked to develop if and then statements (e.g., If 2 × 3 = 6 then 6 ÷ 2 = 3 or 6 ÷ 3 = 2), test whether this always worked, and develop if and then statements using variables. Research visit lessons also included existing algebraic tasks from the curriculum and some algebraic tasks which she had developed with support from the researcher based from what she had learnt during the study group meetings.

However although there was some integration of algebra into lessons there were several observed instances where Mrs Willis did not change or adapt tasks to optimise algebraic reasoning opportunities. For example, in November, she used the following tasks (see Figure 27) without taking the opportunities to include algebra through variation of the task parameters.

Combinatorics 1

In a box there are 1 *red*, 2 *white* and 3 *green* marbles. Draw a diagram in your *Ex Bks*. First P finished comes to BB to draw a diagram on BB.

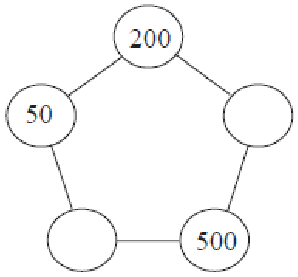
If you took 3 marbles out of the box with your eyes shut, what colours could they be? List all the possiblities in your *Ex. Bk.*

Combinatorics 2

Four children, A, B, C and D are spending the night in a tent in a field. They want to keep a 2-man watch. In how many ways could they do it?

Figure 27. Combinatorics problems. From *MEP lesson plans*, by T. Szalontai, 2003, retrieved from http://www.cimt.plymouth.ac.uk/projects/mepres/primary/y5alp_2.pdf

Likewise, in other tasks within the MEP curriculum that included patterns and associated relationships between these patterns observations revealed frequent missed opportunities to focus student attention on structural features of the task. This meant that tasks which could have potentially been used to develop algebraic reasoning stayed within a computational context. An example is provided in the vignette below where a task (see Figure 28) had relational patterns:



Write 2 more numbers so that the total of all the numbers is 1000.

Figure 28. Number puzzle. From *MEP lesson plans*, by T. Szalontai, 2003, retrieved from http://www.cimt.plymouth.ac.uk/projects/mepres/primary/y5blp_3.pdf

Missed opportunity to develop relational reasoning

Mrs Willis provides her students with a task with links to relational patterns

Mrs Willis: Right, you have got five connecting numbers. We have 200, 50, 500. We have two circles which are blank. You need to write in two more numbers so that the total of the numbers is 1000. Take 20 seconds to have a think about how you will work it out. Look carefully, without discussion, you are thinking. No pencils, just thinking. Okay work with your partner. You've got 30 seconds.

Mrs Willis asks a student to share their response.

Haileigh: Fifty add 50 makes 100, 200 and 500 make 700 so add 100…Add them all together so far, it would make 800. Add 200 will make 1000.
Mrs Willis: Fantastic. So why did you add 50 first of all?
Haileigh: Because then it would make 100.
Mrs Willis: Excellent. So you'd already spotted, to make it up to 100, you've got a 50, you would need to add another 50 on. So then you added them up together, gave you 800.
Haileigh: And then add 200 equals 1000.

Mrs Willis asks for students to share their different answers.

Jacinda: I did 500 add 200 is 700, then 50 equals 750.

Mrs Willis asks another student to repeat this and then asks Jacinda to explain what she did next.

Jacinda: Um, then 250
Mrs Willis: Why do you need 250?
Jacinda: Because it makes 1000.
Mrs Willis: Okay so we had 750 plus something equals 1000. Thank you. And then so you knew that there had to be 250… So how did you decide 150 and 100?
Jacinda: Um, cause 150 plus 100 equals 250.

Mrs Willis asks for other different answers and continues to ask students to share their calculations with the class.

Week Five, Term Two, 200/2010

Here we see that missed opportunities to focus attention on relationships within the task resulted in discussion limited to computational strategies. During this phase there were no observed instances when Mrs Willis drew on a spontaneous opportunity during a lesson to integrate algebraic reasoning.

Missed opportunities for developing students' algebraic reasoning also occurred during group work. A lack of careful monitoring of student responses during paired or small group work meant that Mrs Willis was unable to sequence student responses purposefully in a way that promoted algebraic reasoning. For example, in the following vignette students were chosen randomly to share. This meant their strategy solutions did not move the whole class discussion towards the goal of using relational reasoning. Also students in this classroom did not readily draw upon structural aspects or algebraic reasoning to solve tasks. This in turn limited opportunities that Mrs Willis had to integrate algebra into lessons spontaneously.

Attempting to develop relational reasoning	
Mrs Willis asks her class to solve 83 + 77 = 103 + __ using a relational strategy.	
Mrs Willis:	I don't want you to work out what 83 and 77 equal together. You need to think about the relationship between the numbers on the left hand side of the equation with the relationship of the numbers on the right hand side of the equation. Talk to your partners and see if you can work out what the missing number would be. Talk to your partner, talk to your partner.
Mrs Willis asks a student to share their strategy solution with the class.	
Terence:	I did 83 add seven to 90 and then did 80 add 70 which gives 160. So basically you can find out with 57 plus 103 equals 160.
Mrs Willis revoices the strategy solution and records this on the board. She then asks another student to share her strategy solution with the class.	
Millie:	What we did was, we realised that in the above equation, we did 83 add 77 which was 160. So we used the method up there. And then I knew that um the,

	if you add 50 to 153, you would get, if you added 50 to 103, you'd get 153, then you add the seven which would give you 160.
Mrs Willis:	So in effect, you've done a similar thing to Terence, haven't you? You've added the two together so you know one side of the equation here gives you 160, the other side of the equation added gives you 160. Okay, anybody do it differently? Elsie, can you come on up and show us what you've done?
Elsie:	That and 160, the difference would by fif- no it wouldn't. The difference would be a hu- 30…
Mrs Willis:	What, the difference between, 103 and 83 is 30? Are you sure?
Elsie:	No, 20.
Mrs Willis:	Twenty. Okay. Fantastic. So you looked to see what the difference was between 83 and 103. The difference is 20, well done. So what did you have to do next?
Elsie:	Well, add the difference of 77 and 160.

Students begin to raise their hands to ask Elsie a question. Elsie is unable to answer. Although Mrs Willis asks Elsie to re-explain Elsie is unable to complete her explanation.

Week Five, Term Two, 200/2010

Mrs Willis recognised the need to facilitate students to work collaboratively in pairs or small groups. In the first instance she addressed this by introducing new group norms to the class and providing them with specific roles to take on during collaborative work. For example, in November she began the lesson by stating*: I need one person in the group to be leading the discussion, to be using the pencil and to be making notes. Others can chip in, others can say if they don't understand or if they disagree then you are going to put your point of view across but I want one person to be doing the leading of the team. We will swap over according to which tasks we are doing. Remember also when you feed back with your ideas, everybody in the group needs to be able to explain their thinking, the whole group needs to be able to explain how to do everything so if you are not sure when somebody is explaining in your group, you need to ask, you need to take responsibility so you know what you as a group are doing*.

Establishing the expectation and capacity that all group members could explain and justify their group strategy required on-going teacher support. However as the vignette below illustrates students were not always able to reach group understanding of the task in hand.

Sharing a group solution strategy
Mrs Willis had asked her students to solve the equations below during paired work: If 83 + 77 = 160 Then 73 + __ = 160 And __ + 75 = 160 She asks a student to share their solution for the third equation. Brittany: *(writes 85 in the blank space)* Um, we got 85 but I don't know why because I asked Heera why it was and I couldn't understand. Mrs Willis: Right, I'd like to see if you can work it out. Think about what we did previously. We added, if we look at this one. What Sarah did, she added ten to 77 to get her 87 and so she needed to balance it out by taking away ten from 73. So what have you done here to your 87 to give you 85? Brittany: Um, take away two. Mrs Willis: Okay so if you've taken away two from 87 to give you 85, how do you still keep this equation balanced? Brittany: You add two. Mrs Willis: And you told me you didn't know what to do. You are brilliant. Exactly right. Well done. Week Five, Term Two, 200/2010

Moreover, Mrs Willis use of questioning to lead the student towards using relational reasoning to demonstrate why the answer was correct did not maintain the expectation that a collaborative strategy would be developed during paired work. Instead she directed the student towards a solution and then praised her for this. The example highlights the on-going need to require collaboration consistently.

The emphasis on collaboration that Mrs Willis hoped to achieve also extended to developing a collaborative classroom community. In line with design research, Mrs Willis developed and tested strategies in the classroom to enhance collaboration. There were two key aspects to this related to responsibilities when explaining a solution strategy and conversely listening to an explanation. She now frequently probed students to provide mathematical reasoning for their solution strategies. When sharing their reasoning students were expected to be aware of the listening audience and in particular questions that they may have. Secondly, while listening to an explanation there was a new expectation that students would take responsibility for making sense of their peers' reasoning. This included students revoicing explanations by others, questioning for understanding, and agreeing or disagreeing using mathematical reasoning. The vignette below illustrates how Mrs Willis both placed responsibility on the explaining student to share their reasoning about a task (see Figure 29) in a clear way but also emphasised the responsibility of the listening audience to make sense of the explanation.

Figure 29. Envelopes and pencils problem. From *MEP lesson plans*, by T. Szalontai, 2003, retrieved from http://www.cimt.plymouth.ac.uk/projects/mepres/primary/y5alp_2.pdf

Developing a collaborative community during a whole class discussion

Mrs Willis asked her students to work in groups to solve a task. After some time she recognises that many groups are struggling with the task. She stops them and asks a student to share his groups' solution strategy.

Marlon:	Well um we took, we took one envelope and one pencil from each side. And um that was three pencils on the left side um and four, uh, five envelopes on the other side.
Mrs Willis:	…Remember if you've got any questions, you put your hands up so that he knows that there are some of you who don't understand what he's done and why he's done it.
Marlon:	Um well, well um *(long pause)* if we, if we took um one pencil and one envelope from each side, yeah um it equalled three pencils on this side and just five envelopes on this side.

Mrs Willis asks the listening students whether they understand his explanation.

Shreya:	I don't understand.
Mrs Willis:	Tell him what you're not happy about.
Shreya:	I don't understand.
Mrs Willis:	What don't you understand? Tell me a bit more, what exactly is it you're struggling with?
Shreya:	Um, the whole part that he just did.

Marlon:	Well um, if there's four um pencils and one envelope on the left, and one pencil and six envelopes on the right, um, if you take something away from one side, you have to take the same away from the other. And because there's only one pencil on the right side, that's all we can take. So we took one pencil and one envelope from each side…and one, that envelope, one. So that's it. Then it left us with three pencils and five envelopes.
Mrs Willis:	Are you happy?
Shreya:	Yep.
Mrs Willis:	…Okay, right. Alan are you happy with it?
Alan:	Yeah.
Mrs Willis:	Can you explain where, or how he ended up with three pencils equalling five envelopes?
Alan:	No.
Mrs Willis:	So what aren't you happy about then?
Alan:	I don't really understand what he's done.
Mrs Willis:	Tell me what you don't understand with what he's done so that he can explain himself.
Alan:	I don't get how he can just cancel those um envelopes and pencils out.
Marlon:	Because um you can cancel, you can cancel anything out as long as it's the same each side because it would be equal the same. So I cancelled out the same on each side.
Mrs Willis:	You happy? Yeah? Alan, how about you?
Alan:	Um, yeah?
Mrs Willis:	You don't sound convinced. Explain to me then, what's Marlon done and how has he been able to do it?
Alan:	I don't know.
Mrs Willis:	Then you need to ask Marlon, don't you?
Alan:	I still don't get it.
Mrs Willis:	Then you need to explain to him what you don't understand so that he can explain…
Alan:	The same thing, I still don't understand why he can cancel it.
Marlon:	Well because um you can just cancel the same amount from each side if you have enough, so it basically equals the same.
Week Ten, Term One, 2009/2010	

The vignette highlights a range of pedagogical strategies Mrs Willis used to develop a collaborative community within her classroom. She ensured that the explaining student developed a clear explanation for the class which was supported by use of representations. She also positioned this student to be aware of questions from the listening audience. Space was created during discussion for questions and students were directed to question each other for understanding. There was a further expectation to identify particular areas of the explanation which were difficult to understand and develop specific questions focused on this.

During whole class discussions students who were explaining continued to need to be reminded to face the class when speaking. They also most commonly provided answers with no reasoning and often phrased answers as questions. However, they increasingly provided reasoning when probed and also began to develop mathematical explanations. The more significant shifts were when the students listened to an explanation as shown in the vignette above. Frequently students would now recognise if their peer made an error and would question or challenge this. They would use mathematical reasoning to agree or disagree with a solution strategy. They also began independently to ask questions of the explaining student. These questions were mainly used to clarify an explanation, however, some questions were also used to request justification.

In this phase another notable change was the increased use of representations as a way of making explanations clearer. Initially Mrs Willis modelled how to represent a student provided solution strategy for a task (e.g., $4 \times x + 40 = 200$) which involved solving an equation with an unknown.

Mrs Willis: *One way that might make things a little bit clearer is if you think about what happens to your equation each time. If we start off with four multiplied by X plus 40 equals 200 (writes 4 × X + 40 = 200) then the first thing that Lydia did was to take away 40 that gives us four times X take away that 40 equals 200 minus 40 (re-writes 4 × X = 200 – 40). Therefore four times X equals 160 (writes 4 × X = 160) then we go on to do exactly the same as Lydia has done. She divided both sides of the equation by four (writes ÷ 4 underneath) if we divide four times X by four it leaves us with X, divide 160 by four, it leaves us with X equals 40 (writes X = 40).*

She also began to ask the students to support their explanations with a written representation. For example, after a student gave a verbal explanation Mrs Willis said: *Can you show us on the board because I'm not sure whether everybody follows that? Can you write up on the board what you've done?* However as in Phase One, the use of representations was largely limited to a single representation, generally an equation. Students were not encouraged to make use of multiple forms of representation or make connections between representations.

As shown in the previous section Mrs Willis attempted to facilitate her students to engage in algebraic reasoning through extending tasks to support the development of conjectures. Initially students relied on specific examples to show the validity of conjectures they developed. In response Mrs Willis facilitated students to reflect on using specific cases to test a conjecture or as proof. For example, Mrs Willis asked students whether they agreed that you could always change multiplication and division equations by drawing on the inverse property.

Elsie:	*I worked out that it does work for every number because I did a really big one.*
Mrs Willis:	*Have you written down every single number to see if it works? Every single number?*
Elsie:	*Not every single number but I did a really massive one.*
Mrs Willis:	*…The way you've tried to prove it is you've looked at lots and lots of different numbers, haven't you? Some very large numbers, some very small numbers and each time you found out that it works, is that right?…Can you actually prove something by looking at every single number? Talk to your partner…*
Jasmine:	*No, because numbers never stop, because there's an infinite amount of numbers, it would take you…a while. An infinite…*
Mrs Willis:	*It would take you more than a while. It would take an infinite amount of time. So we couldn't prove it that way but we can see a general trend and we can make an assumption from that general trend. That would be our proof until it is disproved. Until we can show categorically that it is wrong, then we can assume that, that the proof is there.*

Later in the lesson she asked her students to develop diagrams to prove or disprove conjectures. However this was not emphasised as a concrete form of justification but instead as another way to show specific examples which the students had generated. Following the lesson, the four interviewed students maintained the thinking that conjectures could be proven by use of specific cases. None of these students were able to describe any alternative forms of justification which could be used.

For students there was a mixed reaction to new expectations of collaborative group work. In the photo elicitation interviews in November a number of students expressed negative dispositions to group work. This was mainly attributed to other group members being disruptive and failing to contribute ideas or listen to those who were speaking. Other students stated that they preferred to work by themselves as they could work quicker. There was also evidence that some children were being positioned as the mathematical authority within the group. For example, Maggie stated: *In our group Jasmine is clever and she wrote it down for all of us and then we all understood it.* In later photo elicitation interviews in December and February, the students had shifted to mainly positive dispositions towards group work. However, as in Phase One, the students provided a limited range of benefits of group work. Their responses mainly referred to benefits of having someone else to help you and explain things to you while others referred to their peers giving them answers or correcting errors. For example, Peyton stated: *Alan helped me a lot…firstly I did this bit and then when I was trying to work out that one he gave me the answer and he's helped me.* These responses indicated that they did not necessarily view group work as a collaborative activity.

Moving from the initial state where most students found it difficult to describe their own mathematical thinking in the photo elicitation interviews, half of the students could now recall their own solution strategies and mathematical thinking during these interviews. Students also often recalled errors in their own thinking and how their peers helped them to resolve these. However, the ability to re-construct explanations given by their peers during whole class discussions remained undeveloped unless the person explaining had used a similar strategy to their own. In such instances they often described the task that they had been asked to solve rather than recalling the solution strategy or mathematical reasoning for this.

7.4.4 Analysis of task design and implementation, integration of algebra, and development of the classroom community

In this phase Mrs Willis made a deliberate effort to reduce her provision of explicit guidance about how to solve the task. As Henningsen and Stein (1997) illustrate, this can support greater maintenance of cognitive demands of tasks. However in Mrs Willis's classroom this also resulted in some students having difficulty in accessing the tasks. Mrs Willis responded to this by attempting to provide enough time to ensure all groups solved the tasks correctly. This meant that for some students too much time was provided and as described by Henningsen and Stein, they lost their focus on mathematics.

While Mrs Willis attempted to integrate algebraic reasoning into her lessons through the use of number sentences and some task extensions to facilitate development of conjectures, she continued to have difficulty identifying opportunities for algebra in the curriculum material. This meant that unlike the examples provided in a range of research studies (e.g., Blanton & Kaput, 2005b; Kaput & Blanton, 2005; Smith & Thompson, 2008; Soares, Blanton & Kaput, 2005), she did not modify tasks to offer greater affordances for algebraic activity nor recognise spontaneous instances which could be used as opportunities for algebraic reasoning.

Everyday activities in the mathematics classroom influence how students understand what it means to do mathematics (Boaler et al., 2000; Hodge, 2008). As a number of researchers (e.g., Boaler et al., 2000; Colby, 2007; Franke & Carey; Mason & Scrivani, 2004; Star & Hoffman, 2005; Young-Loveridge et al., 2006) have highlighted epistemological beliefs can affect how students approach tasks and perform in the classroom. Students in this classroom continued to draw on their computational experiences to solve problems and were not developing their own algebra ears and eyes.

Mrs Willis began addressing how her students engaged in collaborative work within the small groups. Reid and Zack (2009) describe how teacher expectations shape participation in the classroom. Mrs Willis began by sharing her new expectations for group work with her students. However these expectations were not clearly maintained and there was no further focus on developing appropriate ways to work and talk in a group.

Pedagogical strategies were used by Mrs Willis to develop collaborative interaction during whole class discussions. Teacher questioning is highlighted by Franke et al. (2009) as an important aspect of supporting students to develop mathematical explanations. Mrs Willis both probed the explaining student for reasoning and also positioned other students in the class to also probe for clarification and reasoning. Despite Mrs Willis using pedagogical strategies to try and support collaborative interaction during whole class discussions, her students did not appear to understand the new role required when explaining. They only infrequently provided reasoning unless prompted and continued to phrase questions as answers.

Mrs Willis facilitated students to link their explanations with written representations. She first modelled their use, then in later lessons asked students to support their verbal explanation with a written representation. As Fosnot and Jacob (2009) and Yackel and Cobb (1996) explain written representations can help develop the clarity of an explanation. However, as the use of representations was generally limited to a single representation usually in the form of an equation, there were missed opportunities for students to use multiple representations. As a range of researchers (e.g., Beatty & Moss, 2006; McNab, 2006; Schoenfeld, 2008) show the use of different forms of representations can support students to communicate reasoning and also move between forms flexibly.

Despite an increase focus on tasks that would support student to develop conjectures the students in this classroom did not readily express conjectures or make generalisations. In testing those conjectures that were developed, students would frequently try a number of cases. Schifter (2009) and Mason (2008) both describe looking for examples and counter-examples as an important part of learning to justify. However these researchers also note that students need to be encouraged to develop more refined skills to justify. Similar to the teacher in Carpenter et al.'s (2003) study, Mrs Willis attempted to engage her students in reflecting on how you could show a claim is true for any numbers. She also introduced the use of diagrams to justify, however her use of these was not structured in a way such as Schifter (2009) describes where representational based proof justifies a general claim through encapsulating the meaning of an operation, accommodating a class of instances and having the conclusion of the claim following from the structure of the representation.

Students initially had negative dispositions towards group work and their responses reflected the need for specific teacher scaffolding to structure effective small group interactions. This is highlighted in a range of research studies (e.g., Edward & Jones, 2003; Monaghan, 2005; Rojas-Drummond & Zapata, 2004). While students shifted to more positive dispositions towards group work they were only able to identify limited benefits of such work.

In summary, the teacher actions evident in Phase Two for Mrs Willis are those highlighted below in *Stage One and Stage Two of the Framework of Teacher Actions to Facilitate Algebraic Reasoning*. As detailed earlier, Mrs Willis followed a differing trajectory than Mrs Stuart (see Table 11) which meant that some teacher actions from Stage One were introduced during Phase Two. Those actions which she consistently used successfully are highlighted in yellow. Other actions which she took but only implemented partially are highlighted in blue.

Table 14

Stage One and Two of the Framework of Teacher Actions to Facilitate Algebraic Reasoning: Mrs Willis

	STAGE ONE	STAGE TWO
Algebraic concepts	Address the following concepts in the classroom: understand the equal sign as representing equivalence; relational reasoning including whole numbers and rational numbers; commutative property; inverse relationships; odd and even numbers; properties of zero and one; distributive property; associative property; properties of rational numbers; using and solving equations; function	
	STAGE ONE	STAGE TWO
Teacher actions to develop and modify tasks and enact them in ways which facilitate algebraic reasoning	Implement tasks as problem-solving opportunities	
	Emphasise student effort to approach and complete cognitively challenging tasks	
	Extend or enact tasks to include opportunities for generalisation	
	Interrogate tasks for opportunities to highlight structure and relationships	
		Adapt tasks to highlight structure and relationships. This may include changing numbers or extending to multiple solutions
		Structure tasks to address potential misconceptions
		Use enabling prompts to facilitate all students to access tasks
		Implement tasks by focusing attention on patterns and structure
		Recognise and use spontaneous opportunities for algebraic reasoning during task enactment
	STAGE ONE	STAGE TWO
Teacher actions to develop classroom practices which provide opportunities for engagement in algebraic reasoning	Lead explicit discussion about classroom and discourse practices	
	Ask students to apply their own reasoning to the reasoning of someone else	
	Require students working in pairs or small groups to develop a collaborative solution strategy which all can explain	
		Require that students indicate agreement or disagreement with part of an explanation or a whole explanation and provide mathematical reasons for this
		Lead explicit discussions about ways of reasoning
		Provide space for students to ask questions for clarification
		Request students to add on to a previous contribution
		Ask students to repeat previous contributions

		Use student reasoning as the basis of the lesson
		Facilitate students to examine solution strategies for similarities or differences
	STAGE ONE	STAGE TWO
Teacher actions to develop mathematical practices which support the development of algebraic reasoning	Require students to explain their reasoning	
		Require students to develop mathematical explanations which refer to the task and its context
		Facilitate students to use representations to develop understanding of algebraic concepts
		Ask students to develop connections between tasks and representations
		Provide opportunities for students to formulate conjectures and generalisations in natural language. Leads students in examining and refining these conjectures and generalisations
		Listen for conjectures during discussions. Facilitates students to examine these
		Require students to use different representations to develop the clarity of their explanation
		Model and support the use of questions which lead to generalisations like 'Does it always work?', ' Can you see any patterns?', Would that work with all numbers'

7.5 PHASE THREE: CONTINUED CHALLENGES IN DEVELOPING ALGEBRAIC REASONING

7.5.1 Teacher learning

Collaboration with other teachers in the group was identified by Mrs Willis as a useful means to develop further her reflection on teaching and learning mathematics. In the study group meetings in this phase, the teachers were involved in undertaking a lesson study cycle. For both Mrs Willis and the other teachers at Hillview School being involved in collaboratively planning and then observing a lesson was a new experience. In contrast to her previous experiences of being observed for appraisal purposes Mrs Willis described collaboration in the lesson study cycle as: *much more valuable because we are all working together to try and help each other and that is the main thing rather than me being judged.* In particular, Mrs Willis highlighted the usefulness of the collaborative planning process. She outlined how the group often came to what she described as: *brick walls* where they had difficulty in agreeing on a suitable activity to meet the lesson objective. However, she viewed these as useful as: *it made us really focus on what we were trying to look at.*

Mrs Willis highlighted the value of observation activity during the lesson study cycle: *we could take a step back and look at what somebody else is doing…I think that then makes you reflect on what you are doing yourself.* She also noted the usefulness of using resources and concrete representations to support learning. For example, with reference to a pair of students who were unsure of whether division would be commutative she observed: *when they had the pegs in front of them they could argue it but they couldn't argue it just on paper. They needed to be able to see the five pegs and they can't divide them amongst ten people.* Reflecting on her current utilisation of resources, she developed a goal which she shared with the group to allow her students access to: *supplies of resources which the children could then choose to use.*

The process of lesson observation also led her to engage in critiquing practice regarding whole-class discussion. For example, in observing a lesson focused on the commutative property Mrs Willis challenged Mrs Magri's selection of students who had focused on the inverse relationship between addition and subtraction to share their ideas. Mrs Willis said she thought it would be better to avoid this and instead select students who were investigating the commutative law.

Similarly, Mrs Willis described the process of being observed herself as also facilitating her to reflect on her practice. After she taught the study lesson, she stated: *when other people are watching you then you have to reflect on what you are doing… I know I will look back and say "I should have done this and I should have done this" and then I will take that thought before I do something else at another time and then taking on board the views of other people as well.* The vignettes below show the key aspects of her own practice which she critiqued.

Reflecting on the shifts in the way students work
Developing the pace of lessons. Mrs Willis: I spent far too long on the first section which they knew…I think they could tell what the commutative law was for addition, it wasn't a problem and I should have just cut that bit short much more quickly. Week 2, Term Three, 2009/2010
Monitoring small group work. Mrs Willis: I think my aim is to go around and get a good idea of what they are all doing but then I still just get drawn in and I have got to make more of an effort to avoid that. Week 2, Term Three, 2009/2010
Sequencing and connecting student responses during a whole class discussion. Mrs Willis: When we came to reviewing it all, I didn't do it in a logical way at all and it would have helped if we had gone through systematically but I didn't do that. They didn't delve as deeply as I thought they would have done, some children went to negative numbers and were working with those but I didn't, I suppose I didn't really push them either and I think that's what was limited…I am not sure that they were all clear on the justification. They found that harder, again I don't think I followed it through well enough and because the way I reviewed it with them was all muddled up and we didn't go through systematically. Week 2, Term Three, 2009/2010

Anticipating possible student responses was not a practice which Mrs Willis regularly engaged in prior to lessons. She noted: *you don't really think about the problems that are going to arise which is why then suddenly a child will say something and you think "do I go off in that* *direction or not?"* Following the first lesson study cycle, she described herself realising: *the importance of thinking about what could happen and what tangents you might be presented with and whether they are worth going off on or not.* However, in the subsequent group meeting when the group began to re-plan the lesson to be taught in her classroom, Mrs Willis shared that she still found it difficult to anticipate what the potential difficulties or misconceptions might arise. Previously she had observed and noted students' difficulties in using representations to justify: *they were using the blocks to try and create the numbers.* However in the later planning she did not draw on this observation to anticipate that her students may have similar difficulties. Following the lesson Mrs Willis described her observation of a student attempting to use counters to show that multiplication was commutative: *she was doing three times four and she did a group of three and another group of four alongside it…and I hadn't expected that at all.* Rather than considering pedagogical actions which could be used to facilitate students to use equipment appropriately she instead took the stance that the physical material hindered student learning: *because of the way some of them were working with it…like when they tried to do arrays and they weren't actually managing and they weren't using the equipment in a way that actually illuminated what they were trying to say, I think it actually hindered them.*

Mrs Willis used the framework (see Appendix A) provided in the study group meetings to reflect on her own practice. In the first instance she identified that she did not: *put enough emphasis on them finding different ways to explain what they're doing. They tend to have… right this is my explanation and I can't veer from it…I need to start trying to wheedle it out, a bit more…how they can explain things in different ways.* She also recognised the need to develop further the collaborative norms. She stated: *when I've got somebody explaining at the front, I don't get them to go back to their group and get the group to help in the justification and I think I need to do more of that so it's more of a collaborative effort…It's very much that child in* *isolation so when there are other children who don't understand then I should be getting that child to go back to the group to help them explain.*

Although Mrs Willis identified pedagogical actions to further develop collaborative group work in the whole class context, she did not recognise the need for specific teacher actions during the small group work time. In reference to the continuing difficulties many of her students had in working collaboratively—*I keep finding every combination has a problem…I'm trying to get some people more involved…but they just won't*—her over-riding perception was that some children were unable to ever work collaboratively: *I know obviously people are still, like Genevievie who won't ever function in a group.* Rather than addressing the group norms, her strategy was to: *keep changing the groups around trying to find the appropriate groups for them to work in and some groups work well and others I don't think ever will.*

What became evident toward the latter part of Phase Three was that although Mrs Willis reflected on her practice and identified areas of improvement within the study group meetings, these were not being enacted with effect in her classroom. For example, following the initial study lesson Mrs Willis stated that a goal for her was to incorporate the use of physical representations and equipment into her teaching and classroom. However it was evident in both classroom observations and in the second study lesson that this had not been accomplished. In response to another teacher's observation of difficulties she observed with the Mrs Willis's students using representations to develop concrete justification, Mrs Willis responded: *I think it is partly that it is a novelty in that they don't use material much.*

In Phase Three, it was evident that there continued to be limitations in Mrs Willis's understanding of algebra and the practices underlying it. For example, in the second study lesson a student attempted to prove that multiplication was commutative by referring to colours: *red times blue equals purple and blue times red equals purple.* Mrs Willis accepted this example without further discussion. In the reflective discussion following the lesson she did not recognise it as a non-mathematical example. Similarly Mrs Willis herself provided non-mathematical reasons for her choice of the equipment (e.g., plastic teddy bears and flat people shaped figures) given to the students to use to justify in this lesson. This equipment did not necessarily easily lend itself to justification of the commutative nature of multiplication. When questioned about this Mrs Willis responded by saying that if she did the lesson again she would use the people: *no question, I would have taken the people because they were tactile.* In this case, the reason for the choice of the representation was not linked to the mathematical purpose of the lesson.

During the study group meetings in Phase Three, the teachers engaged in discussion related to their changing perceptions of algebra. Mrs Willis remained largely silent during these discussions. She also did not respond to group email questions investigating participants' changed perceptions of algebra. Her view of algebra appeared to be similar to what was reported in the previous section (see 7.4.1). In discussion of what changes the teachers noted in regards to the way their classes worked algebraically, Mrs Willis noted only that she thought her students now made connections between different solution strategies.

Although Mrs Willis did not clearly identify changes related specifically to algebraic reasoning, she did identify a number of changes within her practice. She described the overall change in her teaching as looking at much smaller areas in her lessons rather than trying to cover everything in the MEP lesson plan: *I am only picking out a few things to focus on which before I wasn't.* She also described some changes the way her students worked. These were linked to the students having greater confidence mathematically and also to being more open to different solution strategies: *they understand that they are going to get there, one way or another and their method might be very different from someone else's but it's equally valid…They're much more accepting of other children's ideas than they were previously and they're much more open to listening to others and also for putting their own point of view even if it conflicts with other people which I think can be quite hard.* She also described how her students would now question others when they explained their ideas. While she was able to identify these changes, she was unable to identify specific pedagogical strategies which she used to facilitate these. For example, when she was asked how she facilitated the students to question each other, she stated: *it's just been encouraging them to do it*.

7.5.2 Analysis of teacher learning

During the study group meetings, Mrs Willis engaged in reflecting upon and questioning her practice. However, for the most part the goals that she set from this reflection process were not maintained in-between the meetings. While Mrs Willis did engage in some reflection on practice, it did not appear that the use of inquiry shifted to become a 'way of being' through which practice was continually developed (Jaworski, 2006; 2008).

Mrs Willis's overall understanding of early algebra appeared to remain limited. She developed her understanding of some of the classroom practices and pedagogical actions described by Blanton and Kaput (2005a) and Franke et al. (2008) to support student engagement with algebraic reasoning. However in many cases she did not recognise the need for specific teacher actions to sustain or continue to facilitate change.

7.5.3 Task design and implementation, integration of algebra, and development of the classroom community

For a significant portion of lessons observed in Phase Three of the study there was a shift back to Mrs Willis's previous form of task implementation. That is, she used a whole class approach whereupon she would read the task to the students and then they would systematically complete the answers. Students were provided with limited time to think about their responses or talk to their peers. The vignette below shows an example of the shift back to this type of task implementation.

Task implementation	
Mrs Willis began by telling her class the scenario that there are eight boys and 12 girls. She asks the students to solve a series of ratio problems by going through each question and answer with the class.	
Mrs Willis:	What is the ratio of girls to boys?…Have you got it, Maria? Peyton, have you written the answer down?…Right and…Maria, what is the ratio of girls to boys?
Maria:	Eight to 12.
Mrs Willis:	Okay, we had for every 12 girls, there are eight boys. I think that's the way it should be written, don't you? For every 12 girls there should be eight boys and what can I do with this ratio? What do you notice about it? What do you notice, Brittany, about the ratio? 12 to eight.
Brittany:	Um, it's more than, um, it's the other way around?
Mrs Willis:	It's the correct way around because we've got, we're looking at the number of girls in relation to the number of boys. So there are for every 12 girls, there are eight boys. What can we do with our ratio though? What can we do with it, Lydia?
Lydia:	Um…
Mrs Willis:	Think about what we did previously. We had, we had eight to 12…
Lydia:	Divide it by two?
Mrs Willis:	Okay which gives us a ratio of…
Lydia:	Six colon four.
Mrs Willis:	So for every six girls there are four boys. Are we happy with that? Or can we do anything else with it?
Lydia:	And… divide it again?
Mrs Willis:	We can divide it again which gives us…
Lydia:	Three…
Mrs Willis:	Sorry I didn't hear the second…
Lydia:	Two.
Mrs Willis:	Three colon two. So for every three girls there are two boys.

Week 9, Term 3, 2009/2010

As the vignette highlights, Mrs Willis used questioning to provide the students with procedures to approach the task. Students often responded with short answers which were phrased as questions and did not readily provide reasoning for their responses.

When Mrs Willis did ask her students to work on tasks in small groups she would read the problem to the class and then ask them to solve it in a group. However enabling prompts were not used to ensure all students could access the tasks. This meant that not all members of the group were able to access the task or solution strategy. After the groups had completed their task the whole class discussion was used as a way of checking and correcting individual answers. This pattern of instruction and participation meant that there were missed opportunities for recording and connecting a series of answers. Furthermore, the links and patterns which could be made across the whole task were not apparent.

There were few links made to algebra during these lessons. All lessons aside from that planned by the study group for the lesson study cycle used tasks taken directly from the MEP curriculum. No modifications or extensions to these tasks to develop links with algebra were evident. When tasks did have existing links to algebra often their implementation was structured in such a way that these links were not evident or highlighted. Furthermore, pedagogical actions used during task implementation did not facilitate the students to approach tasks algebraically. For example, a lesson began with a sequence task (see Figure 30) which was linked to functional reasoning.

1.2, 2.4, 4.8 What are the next numbers in the sequence?

Figure 30. Sequence problem. From *MEP lesson plans*, by T. Szalontai, 2003, retrieved from http://www.cimt.plymouth.ac.uk/projects/mepres/primary/y5blp_2.pdf

Mrs Willis read out the first three terms, she then asked the students individually to say the next numbers. If they gave an incorrect response she told them to sit down and asked someone else. None of the responses were recorded. After a number of responses had been given, she then asked one child to explain what they had been doing. He stated that they were doubling and there was no further discussion of this. In this example the task remained a computation task; the patterns were not explicitly examined and students were not facilitated to engage in generalisation.

Although algebraic themes were a feature of earlier lessons in Phase Two, there were few connections made between these and the tasks in the current lessons. For example, in Section 7.3.3 we saw how some tasks involving fractions were extended to facilitate conjectures and generalisations. In a later lesson in Phase Three students were asked to find equivalent fractions which included fractions that were equal to one whole. Rather than linking this to a conjecture the students had previously explored (e.g., $a/a = 1$), Mrs Willis instead led them procedurally through simplifying a fraction.

Mrs Willis: *So what about six sixths?*
Amber: *You can make it into one whole.*
Mrs Willis: *Into one whole. What would you do to both the numerator and the denominator to get it to one whole? What's a factor of both six and six?*
Amber: *Six?*
Mrs Willis: *Six. We divide by six.*

This meant that there were missed opportunities for students to revisit algebraic concepts and further develop their reasoning.

Mrs Willis continued to use pedagogical actions which she had trialled and introduced in Phase One and Two to shape how her students participated during whole class discussions. These included asking students to share alternative solution strategies, providing space for those listening to question the person explaining, and facilitating students to apply their own reasoning to the reasoning of others. For example, before a student explained their solution strategy she stated: *I'd like the rest of you to be checking that he is correct in the way that he has done it. If you're not sure what he's done, make sure to put your hands up and ask him.* Other actions included asking students to repeat contributions by other students with an emphasis on understanding each other's reasoning. For example, a pair of students explained their solution strategy for finding two thirds of 100. After they had completed their explanation Mrs Willis said: *I just want to go back the beginning, because I'm still not convinced that everybody understands why they divided 100 by three and then multiplied by two. What were they actually trying to find, Alan?* Mrs Willis also continued to probe her students to provide mathematical reasoning for their answers; however this happened to a lesser extent in the final lessons.

Although Mrs Willis used pedagogical actions to shape student participation in whole class discussions, it was apparent that norms for these discussions were not clearly established. Students continued to face the board and speak with their back facing the class. Alternatively, they would direct their explanation to Mrs Willis. The vignette below highlights a typical example of a student sharing their solution strategy for a task (see Figure 31) after small group work.

a) *If 6 kg of potatoes cost £*$2\frac{2}{5}$*, what does 1 kg of potatoes cost?*

Figure 31. Word problem. From *MEP lesson plans*, by T. Szalontai, 2003, retrieved from http://www.cimt.plymouth.ac.uk/projects/mepres/primary/y5blp_2.pdf

Task implementation	
Following group work on a task, Mrs Willis asks a student to share her groups' solution strategy.	
Rachel:	*(goes to whiteboard writes* $6\overline{)240}$ *and begins to speak with her back to the class and reading from her)* We did six into 240.
Mrs Willis:	Rachel, turn around. There are still other hands up, it's not just Amber's hand.
Millie:	Why have you got 240 there?
Rachel:	Um… (long pause)
Mrs Willis:	You need to explain a little bit more because you're right. Explain it a little more.
Rachel:	*(turns to speak to Mrs Willis)* Two fifths equals 40.
Mrs Willis:	Equal to 40? Equal to 40 what? 40 chickens?
Rachel:	Forty pence.
Mrs Willis	Okay so one kg would cost 40 pence. Thank you, Rachel.
Week 2, Term 3, 2009/2010	

As shown in the vignette, Mrs Willis needed to remind Rachel to speak to the class rather than facing the board or directing the solution strategy at her. Similarly although Mrs Willis was seen in the lesson observations asking students to repeat their peers' contributions, her students consistently had difficulty in achieving this.

As the previous section highlighted, Mrs Willis now used a mixture of small group work in combination with implementing tasks through a whole class approach. When tasks were implemented using a whole class approach the interaction patterns frequently followed an IRE pattern with students giving a short response which Mrs Willis evaluated. When students provided incorrect responses (or responses she perceived to be incorrect as in the following episode), she would ask other students to try and elicit a correct response or attempt to correct the response herself. For example, Mrs Willis asked a student to turn the ratio for circles to squares (see Figure 32) into a fraction.

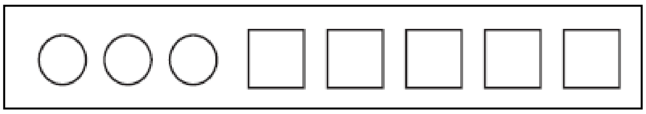


Figure 32. Ratio problem. From *MEP lesson plans*, by T. Szalontai, 2003, retrieved from http://www.cimt.plymouth.ac.uk/projects/mepres/primary/y5blp_2.pdf

Genevieve:	*Three eighths to five eighths.*
Mrs Willis:	*No, nice try, but it's not correct. Elsie?*
Elsie:	*Three fifteenths to five fifteenths?*
Mrs Willis:	*Nope. Our ratio is three to five so for every three circles we have five squares. What do you think the ratio would be as a fraction?…It would actually be three fifths.*

In the example above Mrs Willis's response reflected her difficulties with subject knowledge. Although the researcher was aware of these errors, they were not addressed in discussion with Mrs Willis due to two reasons. Firstly the focus of the study was on developing understanding of early algebra; therefore addressing content knowledge beyond this was not within the remit of the study. Secondly, the researcher wished to maintain a positive relationship with the teacher participant throughout the duration of the study and it was viewed that highlighting Mrs Willis's difficulties with subject knowledge may threaten the established relationship.

In Phase Three, Mrs Willis continued to revoice and re-state student responses. The vignette examples below highlight how she used revoicing to correct answers, provide mathematical reasoning or to evaluate responses:

Use of revoicing
Revoicing to correct a response. Mrs Willis: Ninety-one thirteenths. Can you explain why, what it is equal to and why it is equal?… Alan: Um, because it's, the numerator is bigger than the denominator so that's equal to one whole? Mrs Willis: Okay so numerator is bigger than the denominator. This improper fraction is more than one whole. Week 2, Term Three, 2009/2010
Revoicing to show a correct response had been given. Mrs Willis asks her students the ratio of squares to circles. Peyton: Five thirds. Mrs Willis: It would be five thirds. Well done. For every five squares we have, there are three circles. What would that be as a ratio with a colon in the centre? How

	would we write it down as a ratio? Millie.
Millie:	Five colon three.
Mrs Willis:	Brilliant, fantastic, five colon three.
Week 9, Term 3, 2009/2010	

As shown in the vignettes throughout this section, students most commonly continued to provide answers with no reasoning or phrase their responses as questions. When students did attempt to construct mathematical explanations these were often unclear, based on procedural understanding or incomplete. For example, a student attempted to explain why they had thought that addition would not be commutative with negative numbers.

Maggie:	*Um, well we thought that if we had it the two minuses would come together and make a plus sign, it would make um the amount different but it didn't really work.*
Mrs Willis:	*Can you give me an example?*
Maggie:	*Um say if you were to add minus three um add minus three, add um three, um we thought that ummm would be zero.*

Furthermore students frequently did not respond to questions from Mrs Willis or their peers. Overall shifts in student participation by the end of the project were those related to how they participated when listening to explanations. As described in the previous section, they would ask questions for clarification and less frequently justification. They would also use mathematical reasoning to disagree with a peers' explanation.

The mathematical practices of making conjectures, generalising, and justifying were not observed in any of the Phase Three lessons aside from the study lesson for the lesson study cycle. During this lesson Mrs Willis used similar practices to those outlined in the earlier sections. The students mainly attempted to justify by the use of examples and Mrs Willis frequently shifted their attention from the general to the specific. In Phase Three there were no shifts in the use of representations with these being largely limited to the use of equations.

Student perceptions of small group work reported in the photo elicitation interviews continued to be mixed depending largely on whether the group had worked successfully together. Some students continued to engage in off-task behaviour during group work. For example, Maggie described her group interaction as: *no one was listening to others' ideas and everyone kept talking about other stuff, not maths.* In other instances students referred positively to group-work stating that everyone was working together or describing it as an opportunity to get help from others. Some students continued to position others as more mathematically competent and attribute positive experiences with group work to this. For example, Jason described small group work during the lesson as positive because: *I was working with Sarah and Marlon who are very good at maths. I wouldn't say I'm good at maths.*

In the photo elicitation interviews when asked to recall explanations given during the whole class discussion, a growing number of students were able to recall the solution strategy that was used by the explaining student. However they were often not able to link the solution strategy to mathematical reasoning. Many students continued to either describe the task, student actions, or were only able to recall a solution strategy if it was the same as their own. For example, Peyton stated that he could not recall an explanation because: *they'd done something different to our group so I can't remember.* Students more easily recalled their own groups' solution strategy and some could also provide mathematical reasoning for this.

Talking about mathematical ideas was viewed positively by students. Discussing ideas in a small group was valued as a way of gaining confidence. Other students described how talking helped you explain your answers and reflect on your ideas. For example, Maggie stated: *they might have another idea and you can learn more from the partner because like you can bounce ideas off each other*. Many also referred to how talking about your ideas allowed others in the class could help you. However some students in the class viewed talking about your ideas as only helpful if you had an incorrect answer or did not know what to do. For example, Jasmine stated that talking about your ideas was only helpful: *when you have a wrong answer and then your partner has a right answer, you learn something from that.*

Listening to others explaining their ideas was also described positively by the students. They predominantly described listening to explanations as a way of supporting them to find different ways to solve problems. Others described student provided explanations as showing them what to do. Many of the students viewed listening to an explanation helpful if you were unsure of what to do, didn't know the answer or were wrong. For example, Marlon stated: *say I had a different answer to someone else at the board, if I didn't get it I'd put my hand up and ask them to explain it again and then I'd be able to know the right answer to do it next time the right way.*

7.5.4 Analysis of task design and implementation, integration of algebra, and development of the classroom community

Both task design and implementation are highlighted as factors that can provide greater affordances for algebraic activity by a range of researchers (e.g., Bastable & Schifter, 2008; Blanton & Kaput, 2005b; Carpenter et al., 2005b; Carraher et al., 2008; Mason, 2008). In this phase, however, there was limited evidence that Mrs Willis designed, extended, or implemented tasks in a way that intentionally facilitated algebraic reasoning opportunities. Links between particular algebraic concepts which were a focus of earlier lessons were not drawn on. Mrs Willis shifted back to her previous practice of implementing tasks with a focus on correct answers. Tasks were enacted with a computational focus and students focused on calculating answers rather than investigating structure and relationships.

Although small changes were evident in how students participated in group work and whole class discussions the change process appeared to have stalled. The newly introduced norms and practices did not appear to be consistently maintained and therefore they had not become habitual. Data from the classroom indicated that whilst Mrs Willis had appropriated some practices to develop productive discourse and collaborative interaction at this stage of the project their effectiveness was diluted due to the reemergence of previous less effective communication and participation practices. Most notably, during whole class discussions there was a shift back to an IRE pattern. As a range of researchers (e.g., Bell & Pape, 2012; Fisher, Frey, & Lapp, 2011; Nathan, Kim & Grant, 2009) explain this lowers students' engagement and opportunities to verbalise ideas and construct knowledge.

Similarly, mathematical practices were not introduced into the classroom in a way that assured they would became part of the everyday classroom context. Research studies (e.g., Bastable & Schifter, 2008; Carpenter et al., 2003; Kaput & Blanton, 2005a; 2005b; Mason, 2008; Schoenfeld, 2008; Smith & Thompson, 2008) highlight that rich algebraic reasoning opportunities occur within classroom contexts which facilitate students to engage in mathematical practices linked to the development of algebraic reasoning. The students in this classroom were not facilitated to view representations as important tools which could be used to explain and justify reasoning. In the final phase of the study, attention was not drawn to students generating conjectures and there was a continued lack of pressing of students to move beyond exploring conjectures through specific examples. This limited opportunities for student engagement in mathematical practices associated with algebraic reasoning.

Some shifts in students' role as learners were noted. Pratt (2006) contends that it is important students understand their role in the classroom and there was evidence that many of these students did develop some understanding of the importance of both speaking and listening to develop their mathematical understanding. However, this appreciation of discourse centred on individual learning and knowledge acquisition rather than views about doing and learning mathematics as part of a community of learners. These students did not appear to develop epistemological conceptions of mathematics that would support them to be active agents is developing and using algebraic reasoning within their mathematics activity.

In summary the teacher actions evident in Phase Three for Mrs Willis are those highlighted below in *Stage One, Stage Two and Stage Three of the Framework of Teacher Actions to Facilitate Algebraic Reasoning.* Those actions which she consistently used successfully are highlighted in yellow. Other actions which she took but only implemented partially are highlighted in blue. The lack of instances in Stage 3, compared to Mrs Stuart's trajectory of actions (see Table 12), serve to highlight the different trajectories of teacher actions evidenced in the two teachers' classrooms.

Table 15

Stage One, Two and Three of the Framework of Teacher Actions to Facilitate Algebraic Reasoning: Mrs Willis

	STAGE ONE	STAGE TWO	STAGE THREE
Algebraic concepts	Address the following concepts in the classroom: understand the equal sign as representing equivalence; relational reasoning including whole numbers and rational numbers; commutative property; inverse relationships; odd and even numbers; properties of zero and one; distributive property; associative property; properties of rational numbers; using and solving equations; function		

	STAGE ONE	STAGE TWO	STAGE THREE
Teacher actions to develop and modify tasks and enact them in ways which facilitate algebraic reasoning	Implement tasks as problem-solving opportunities		
	Emphasise student effort to approach and complete cognitively challenging tasks		
	Extend or enact tasks to include opportunities for generalisation		
	Interrogate tasks for opportunities to highlight structure and relationships		
		Adapt tasks to highlight structure and relationships. This may include changing numbers or extending to multiple solutions	
		Structure tasks to address potential misconceptions	
		Use enabling prompts to facilitate all students to access tasks	
		Implement tasks by focusing attention on patterns and structure	
		Recognise and use spontaneous opportunities for algebraic reasoning during task enactment	
			Recognise and use links to algebra in tasks across mathematical areas
			Implement tasks as open-ended problems
			Anticipate student responses which could provide opportunities for algebra
			Recognise and use spontaneous opportunities for algebraic reasoning from student responses

	STAGE ONE	STAGE TWO	STAGE THREE
Teacher actions to develop classroom practices which provide opportunities for engagement in algebraic reasoning	Lead explicit discussion about classroom and discourse practices		
	Ask students to apply their own reasoning to the reasoning of someone else		
	Require students working in pairs or small groups to develop a collaborative solution strategy which all can explain		
		Require that students indicate agreement or disagreement with part of an explanation or a whole explanation and provide mathematical reasons for this	
		Lead explicit discussions about ways of reasoning	
		Provide space for students to ask questions for clarification	
		Request students to add on to a previous contribution	
		Ask students to repeat previous contributions	
		Use student reasoning as the basis of the lesson	
		Facilitate students to examine solution strategies for similarities or differences	
			Lead explicit discussion about mathematical practices
			Sequence solution strategies to advance mathematical thinking and reasoning
			Provide space for students to question for justification

	STAGE ONE	STAGE TWO	STAGE THREE
Teacher actions to develop mathematical practices which support the development of algebraic reasoning	Require students to explain their reasoning		
		Require students to develop mathematical explanations which refer to the task and its context	
		Facilitate students to use representations to develop understanding of algebraic concepts	
		Ask students to develop connections between tasks and representations	
		Provide opportunities for students to formulate conjectures and generalisations in natural language. Leads students in examining and refining these conjectures and generalisations	
		Listen for conjectures during discussions. Facilitates students to examine these	
		Require students to use different representations to develop the clarity of their	

		explanation	
		Model and support the use of questions which lead to generalisations like ‘Does it always work?’, ‘ Can you see any patterns?’, Would that work with all numbers’	
			Listen for implicit use of number or operational properties. Uses these as a platform for students to make conjectures and generalise
			Facilitate students to represent conjectures and generalisations in number sentences using symbols
			Ask students to consider if the rule or solution strategy they have used will work for other numbers. Consider if they can use the same process for a more general case
			Promote use of concrete forms of justification
			Require students to translate between different representations

7.6 SUMMARY

This chapter has mapped out the challenges that can be encountered when attempting to develop teachers’ algebra ears and eyes and instigate pedagogical changes to develop algebraic reasoning in a mathematical community of inquiry. The initial classroom culture described did not support student engagement with algebraic reasoning. Professional development activities were used to attempt to support Mrs Willis to develop algebra ears and eyes. However, for Mrs Willis, the professional development activities did not appear to challenge her beliefs about algebra and mathematics in general. Her participation in the study was associated with continued anxiety and at times self-doubt and importantly there was limited engagement with new practices between research visits.

In Phase One, classroom observations provide evidence of shifts in task implementation and integration of algebra with researcher support. However, in later phases Mrs Willis faced difficulties in clearly identifying algebraic concepts which were a focus of the lesson and adapting material appropriately. Integration of algebra into lessons throughout the different phases of the study was not consistent and there were many missed opportunities when tasks were not adapted or implemented in a way which highlighted structure or relationships. This also meant that students themselves did not readily draw upon structure or relationships which in turn limited the spontaneous opportunities available to engage in early algebraic reasoning.

In Phase One and Two there were small shifts in the development of the classroom community. However, by Phase Three it was evident that many of these shifts did not become embedded practices and instead were absorbed into her previous ways of practice.

The following chapter draws together the findings from this chapter and Chapter Six. It will address the initial research questions and will highlight the different journeys which the two teachers took and suggest possible reasons for this. It will examine the contributions that this research makes to the field and the limitations and implications of the study.

CHAPTER EIGHT

CONCLUSIONS AND IMPLICATIONS

8.1 INTRODUCTION

The purpose of this thesis was to gain insight into how teachers can develop early algebraic reasoning in a classroom community of mathematical inquiry. The literature review drew together three important elements when considering how early algebraic reasoning can be developed within a classroom community of mathematical inquiry. These were: (1) student learning of algebraic concepts; (2) aspects of the learning environment which support engagement with algebraic reasoning; and (3) how teachers can be supported to develop instructional practices to support algebraic reasoning and implications of this type of change for student participation within their classrooms.

Teacher change and enactment of changes within the classroom is a complex and challenging process. The methodology of design research was chosen as a means to create and extend understandings of how innovative learning environments such as those focused on developing early algebraic reasoning can be developed, enacted, and maintained (DBRC, 2003).

The findings presented in this thesis concern two contrasting cases detailing teacher engagement in professional development, the subsequent shifts within the teachers' classrooms, and changes in their students' participation in mathematical activity. In both cases, the integration of algebraic reasoning into classroom mathematical activity was a gradual process. It involved changes to task implementation and design, shifts in pedagogical actions and the facilitation of new classroom and mathematical practices. These two cases illuminated significant differences in how the respective teachers engaged in the professional development and subsequently enacted changes in their classrooms.

This research had an overall focus on how teachers can develop early algebraic reasoning in classroom communities of mathematical inquiry. To address this wider focus, there were three key research questions:

- How do teachers develop algebra ears and eyes?
- What pedagogical strategies and classroom and mathematical practices support student engagement in early algebraic reasoning?
- What shifts occur in the way students engage in classroom activity as early algebraic reasoning is integrated into the everyday mathematics lessons?

Section 8.2 begins by summarising the journey the two case teachers in this study took in developing their algebra ears and eyes. It then examines the pedagogical strategies and classroom and mathematical practices which supported student engagement in algebraic reasoning in their classrooms and the shifts that occurred in their students' participation as the changes were implemented.

Section 8.3 examines the limitations of the study. In Section 8.4 and 8.5 the contributions this research has made to the research field are presented along with the implications of the study and suggestions for further research. Section 8.6 provides a final conclusion to this research.

8.2 DEVELOPING EARLY ALGEBRAIC REASONING IN CLASSROOM COMMUNITIES OF MATHEMATICAL INQUIRY

This research documented the journeys two teachers took to develop early algebraic reasoning with their students. Within the research study both the teachers were provided with the same professional development activities and resources. The findings presented in Chapter Six and Seven, however, illustrate that each travelled a unique pathway as they strove to develop their own understanding of algebra, facilitate their students' understanding and integrate algebra into their everyday mathematics lessons. Initially there were many similarities in their understanding of algebra, use of tasks, and the pedagogical strategies they used in their classrooms. At the end of the research study there were significant differences in the teachers' development in regards to both their own thinking and their pedagogical and classroom practices. There were also notable differences in both the level and nature of student participation during mathematics lessons in the two classrooms.

8.2.1 Teacher development of algebra ears and eyes

At the beginning of the study, both the teachers had a limited understanding of algebra and the links between algebra and other mathematical content areas. This is similar to what is reported in many previous studies investigating teacher understanding of how to teach algebra successfully (e.g., Even, 1993; Franke et al., 2008; McCrory et al., 2012). For these two teachers their mathematical and pedagogical content knowledge did not extend to a deep understanding of the progression in student learning related to algebraic concepts or how to address common misconceptions. As the project progressed it was evident that the two teachers took differing pathways and approaches in developing their understanding.

Mrs Stuart used her interest in understanding student reasoning to develop her knowledge of the expected progression in student understanding and become aware of potential common misconceptions. Throughout the project there were two key ways in which she continued to build her understanding of student reasoning and develop her pedagogical content knowledge. Firstly within the classroom, she took care to listen to student responses and used these to further her understanding. Secondly in the final phase of the study, using what she had learnt from listening to her students, she began to explicitly anticipate possible student responses prior to lessons. This, in turn, deepened her understanding of student reasoning.

Throughout the project Mrs Willis demonstrated anxiety over her mathematical content knowledge. At times this anxiety appeared to inhibit her growth of understanding of early algebra. Teacher anxiety in regards to algebraic reasoning and its negative effect on engagement in professional development is also reported in studies by Franke et al. (2008) and Blanton and Kaput (2008). During the study meetings, Mrs Willis engaged in activities that involved anticipating student responses, however she did not translate these practices to extend her planning prior to classroom lessons. In the classroom, evidence of Mrs Willis's engagement with students' thinking was sporadic. During small group work she frequently intervened to attempt to lead students to a correct solution strategy. This meant that she had difficulty following and understanding student reasoning during the whole class discussions and resulted in missed opportunities for her to extend her understanding of student reasoning. This also reduced her ability to further develop pedagogical content knowledge.

An indication of teacher development of algebra ears and eyes is their ability to draw on spontaneous opportunities to integrate algebra within lessons (Blanton & Kaput, 2005a). In this study planning for algebraic opportunities was an important factor in the teachers' development and there were notable shifts in the way algebra was integrated into the mathematics lessons over the duration of the research project. Initially both teachers worked with the researcher or independently to recognise opportunities to plan to integrate algebra into lessons. Of key importance was for the teachers to recognise the inherent algebraic structure of number. In the first instance Mrs Stuart selected and used only parts of the tasks from the MEP curriculum to focus on algebra. She developed rich understandings of different types of algebraic reasoning through her engagement in study group activities as she investigated and critiqued tasks. She also carefully listened and reflected on her students' reasoning. Although initially student statements were not used to engage students in spontaneous investigation, as the study progressed she demonstrated a clear shift in her ability to use the spontaneous opportunities arising from student responses to engage all students in algebraic investigation. She was able to draw on her classroom interactions with students to exemplify areas of development of both her content knowledge and pedagogical content knowledge related to algebra.

Initially Mrs Willis was keen to modify and extend existing tasks in ways that would engage her students in making conjectures and generalising. She utilised support from the researcher and also independently developed tasks based on activities from the study group meeting. However, as the study progressed and researcher support was withdrawn, Mrs Willis did not continue to adapt or modify existing tasks from the MEP curriculum to facilitate algebraic reasoning. In addition, her attempts to integrate algebra into lessons were inhibited by a lack of clear focus on the lesson goals or algebraic purpose as well as a lack of anticipation of student responses. In turn, this meant that Mrs Willis's own understanding of early algebra was not extended in any significant depth.

Understanding of the classroom and mathematical practices which link to the development of algebraic reasoning are key to teachers' development of algebra ears and eyes. Initially both teachers lacked sound understanding of the processes of generalising and justifying. Lack of understanding of these mathematical practices resulted in them shifting student focus from general cases to specific examples. Representational material was used to support students to solve tasks rather than as a tool to support concrete justification. Mrs Stuart developed her understanding of how to facilitate students to engage in the practices of generalisation and justification through actively engaging in activities during the study group meeting. She also used opportunities to develop her understanding of these practices further during her lessons. Her active inquiry into her own practice resulted in a new appreciation of the value of physical representations as tools to facilitate student understanding of the structure and properties of numbers and to develop forms of proof. During the professional development activities focused on generalisation and justification, Mrs Willis restricted her participation. For example, as shown in Section 7.4.1 she chose to use specific examples to illustrate conjectures about odd and even numbers and displayed vocal anxiety when asked to experiment with other forms of justification.

Research studies which focus on effective professional development (e.g., Back et al., 2009; Earley & Porritt, 2009; Franke et al., 2008) note the importance of teachers having time and space to collaborate. This was also important in the development of teachers' algebra ears and eyes. Both teachers viewed engagement in the lesson study cycle as an important opportunity to collaborate with colleagues and engage in reflection. However, they took contrasting roles during study group meetings. Mrs Stuart keenly participated in all meetings. She actively sought opportunities to investigate her practice and develop her professional learning both during the professional development and beyond. She pushed for the activities of the group (for example, watching and reflecting on videoed classroom episodes) to be extended beyond the meetings. An example of this was when she arranged for the group to meet in addition to the set meeting time. She used the time between researcher visits to explore ideas in her lessons and further embed algebra into her practice. Her actions illustrated that she viewed change as an on-going collaborative process as she sought feedback from both the researcher and other study group members. Mrs Stuart identified the collegiality established through working collaboratively over a long period of time as a significant factor in the changes in her practice. In contrast, Mrs Willis was a quiet participant and frequently positioned herself as a weaker member of the group. From her reflective comments, it appeared that she viewed her obligations to engage in the project as largely confined to the study group meetings and researcher visits. Beyond her initial trialling of some tasks in the early stages of the project, she did not engage in further activity between the meetings and changes which she initiated during the research visits were not sustained between visits.

Facilitating the development of algebra ears and eyes and promoting change to pedagogical and classroom practices required reflection on practice and for the teachers to engage in a cycle of inquiry. Common tools which both teachers used to prompt and support reflection on their pedagogical and classroom practice included research articles, frameworks of teaching, video recorded classroom episodes, and observation of lessons. Some of these tools (e.g., frameworks, video and classroom observations) have also been identified as useful tools in in other studies (e.g., Blanton & Kaput, 2005a; Jacobs et al., 2007) investigating teacher learning of early algebra. Mrs Stuart recognised the need for reflection to be an on-going process to develop change. She consistently critiqued her own practice and trialled different strategies in her classroom. She clearly identified shifts in her classroom practice and the pedagogical strategies she used to support such shifts. For her, reflection had become a core tool which she used to analyse practice and identify areas to focus on. Her actions highlight the importance of teachers questioning their own practice to facilitate change and engaging in a continual cycle of inquiry so that it becomes a way of being. Mrs Willis engaged in reflective activity during the meetings; but this did not appear to extend beyond these meetings as an on-going, continual process. When prompted in workshop discussions and research interviews she was able to identify some shifts in her classroom and the pedagogical actions she used to facilitate change. However, when confronted with difficulties in making shifts in her classroom practice, rather than critique her current pedagogical strategies or reflect on alternative strategies, she fell back to a position where she raised doubts about the efficacy of the changes she was attempting to initiate.

An integral part of the development of algebra ears and eyes is the way in which teachers view both algebra and mathematics overall. During the study, Mrs Stuart shifted from a view of mathematics which focused on computation and procedures to viewing it as a creative endeavour which required a process of on-going construction. Similarly, her view of algebra widened from perceiving algebra as simply about content to encompass classroom culture and mathematical practices as well. Mrs Stuart was motivated by her own lack of experience with rich connected types of algebra during her schooling to facilitate her own students to have different experiences of algebra. Mrs Willis's perception of algebra was extended to include a wider range of content areas and some classroom and mathematical practices. However, engagement in the study did not appear to challenge Mrs Willis's view of mathematics which was largely procedural and rule oriented.

In summary, teachers develop their algebra ears and eyes through engagement in professional development activities which challenges their conceptions of both algebra and mathematics. It is important that teachers view algebra as encompassing classroom culture. This means that both pedagogical content knowledge of algebra and a focus on classroom and mathematical practices which facilitate algebraic reasoning opportunities needs to be incorporated into professional learning and development. Of importance is that teachers view their professional learning as a personal responsibility which is not confined to professional development or researcher visits. This includes engaging teachers in learning beyond the study group meetings to include the classroom setting. It is important for teachers to inquire actively and reflect on their own practice and the developing classroom and mathematical practices. Carefully listening to student responses is a key factor in both developing pedagogical content knowledge and extending understanding of student reasoning. In providing opportunities to learn and practise algebraic reasoning, teachers need to be able to adapt existing tasks to include algebra. Enacting the task successfully also requires that they identify the focus of the task, the purpose of the adaptation, and anticipate the possibilities which may happen in the task enactment.

8.2.2 Pedagogical strategies and classroom and mathematical practices to support student engagement in early algebraic reasoning

Before the professional development started, the two teachers used similar pedagogical strategies and there were similarities in their established classroom practices. In both classrooms, task implementation involved the students being carefully guided through the procedures required to complete the task successfully with the teachers taking the role of mathematical authority and dominating the classroom discourse. Whole class discussions were used to check answers and the teachers used leading and funnelling questions to guide the students to reach correct answers. A range of studies (e.g., Bell & Pape, 2012; Fisher et al., 2011; Mehan, 1979; Pape et al., 2010; Wood et al., 2006) report these types of pedagogical actions and classroom practices as common within traditional classrooms and associated with lowered student engagement and development of agency. Additionally, in both classrooms there was limited use of representations and associated productive mathematical practices. The pedagogical strategies used and the classroom practices did not support student engagement in early algebraic reasoning and the lessons were characterised by missed opportunities for algebra.

Developing new methods of task implementation was an important pedagogical strategy to facilitate algebraic reasoning in the classroom. In both classrooms withdrawing step-by-step teacher guidance when implementing tasks resulted in students initially approaching tasks incorrectly. As part of efforts to change their practice and support their students to approach tasks independently, both teachers asked students to work in pairs. Mrs Stuart took a structured approach to this and facilitated her students to discuss the task requirements with a partner. She successfully scaffolded students to access the tasks, without lowering the cognitive demand, through the use of enabling prompts such as described by Sullivan et al. (2006). Further changes to task implementation were achieved through her use of questioning which focused student attention on patterns and relationships within tasks. A key aspect of this was asking students to talk about what they noticed rather than recording answers. In contrast, enabling prompts were not a feature of Mrs Willis's practice and she continued to face challenges both with her students accessing cognitively challenging tasks and in evaluating an appropriate amount of time to complete the tasks. As a result, in the latter part of the study, Mrs Willis had largely reverted to her previous form of task implementation. The lack of focused questioning or introduction of tasks in a manner which highlighted structural features meant that tasks remained in a computational context.

To support student engagement in algebraic reasoning it was also necessary to address the ways in which students worked collaboratively and the forms of talk which were used in the classroom. Mrs Stuart explicitly discussed with her students how to successfully talk together and facilitated them to generate rules for productive talk. She drew on student models to develop understanding of the new expectations in the classroom. This included the important new expectation that students would explain and clarify their ideas and reasoning. Key to shifting and embedding these participation practices was engaging students in reflecting on their participation. Throughout the study, Mrs Stuart continued to increase the expectation on her students to talk and work collaboratively. This collaborative work included developing shared understanding of a jointly constructed solution strategy. A final key emphasis in Mrs Stuart's classroom was on student development of mathematical explanations.

Mrs Willis's change began by organising her students into groups and asking them to complete tasks together. However, at this stage she did not address group norms with the class and this meant that although they worked *in* a group they did not collaborate. In later lessons, she introduced new expectations for group work but there was some inconsistency between her voiced expectations for collaboration and her teacher actions (see Section 7.4.2). This meant that students were not consistently encouraged to collaborate in group situations.

To advance all students' opportunities to engage in algebraic reasoning it was important to extend collaboration to whole class discussions and structure these in a productive way. Initially, in both classrooms, the students did not perceive whole class discussions as a way of collectively developing understanding. To effect change, both teachers needed to emphasise to their students the requirement to explain their reasoning rather than simply providing answers during whole class discussions. They also needed to facilitate students to be aware of the listening audience. These norms did not become established in Mrs Willis's classroom; her students did not habitually provide reasoning and often spoke with their backs to the other students.

The teachers also addressed student participation during discussions and there were a number of common pedagogical strategies which they used to achieve this. These included making space for student questioning during explanations, facilitating students to apply their own reasoning to the reasoning of others and to agree or disagree with mathematical reasoning. Other key aspects of practice to develop collaborative interaction were to position students to actively listen and sense-make while others explained their reasoning and developed explanations. Pedagogical actions which focused on developing interactive mathematical talk shifted students in Mrs Stuart's classroom to become critical, active participants within the classroom community. Although Mrs Willis used similar pedagogical actions, there were fewer shifts in her students' ways of participating. Without a consistent expectation for them to provide reasoning both Mrs Wills and her students were unable to access and build on each other's thinking – knowledge building therefore, remained largely the responsibility of the individual with the authority continuing to reside with the teacher.

The introduction of key mathematical practices associated with algebraic reasoning were important aspects to support student engagement with algebraic reasoning. Both teachers introduced students to the mathematical practice of using representations and expected them to support their explanations with a representation. Mrs Stuart further facilitated her students' use of representations as a key way for them to support their own reasoning and to access the structure of tasks and develop understanding. She also asked students to draw on and connect different representations and this helped them to develop explanations, link both tasks and representational forms, and to engage in concrete justification. Both teachers attempted to introduce their students to the mathematical practices of generalisation, justification, and proof. Mrs Stuart began by purposefully planning an investigation of the properties of zero. This familiarised students with the processes of making conjectures and finding examples to illustrate these. In latter lessons, representations were introduced as a powerful form of concrete justification and Mrs Stuart readily drew on conjectures she heard her students making. She then used these to engage them in mathematical practices of generalisation, justification and proof. In contrast, when Mrs Willis extended tasks to include opportunities for generalisations, she frequently shifted the discussion from a general context back to specific examples. Failure to use concrete forms of justification hindered student engagement with the mathematical practices.

In summary, there are a number of key pedagogical strategies and classroom and mathematical practices that support student engagement in algebraic reasoning. The first pedagogical strategy which requires attention is implementing tasks in ways which focus on structural and relational aspects. This may be achieved by shifting student attention from getting an answer towards identifying useful patterns and relationships. Also of importance is addressing collaboration and forms of productive talk which facilitate algebraic reasoning. While teachers lead the introduction of new classroom practices, students also need to be part of the development of the new expectations and also develop their own understanding of why it is important to engage in the new practices. This also includes understanding the requirements of the differing roles when explaining or listening. Finally attention also needs to be given to the development of key mathematical practices. Teachers need to draw on student generated conjectures and use these to engage their students in justifying and generalising. Regarded as thinking tools (Anthony & Walshaw, 2007), representations should be introduced as a way of supporting an explanation but also as a form of concrete justification.

8.2.3 Shifts in student engagement in classroom activity

Before the professional development there was minimal evidence of students engaging in algebraic reasoning in either of the classrooms. Student attention focused on using computational strategies to find answers rather than drawing on the relationships within tasks. Although some students worked cooperatively with a partner, most worked individually unless they were stuck on a task and needed help. During whole class discussions students took a passive role and frequently provided answers with no reasoning.

Classroom practices and pedagogical actions influenced whether students engaged in classroom activity in a way which developed their algebraic reasoning. Initially the practices in both classrooms limited student opportunities to develop structural perspectives. In Mrs Stuart's classroom a shift in the type of teacher questioning focused student attention on patterns and relationships within tasks which supported them to approach tasks structurally. In Mrs Willis's class a continuing use of funnelling questions and a procedural focus meant that students were not provided with rich opportunities to develop algebraic perspectives. The lack of emphasis on relationships and structure meant that students continued to use computation to solve tasks instead of utilising relationships. Therefore they were not developing their algebra ears and eyes.

Early in the project changes to Mrs Stuart's task implementation involved a move from providing specific directions to using enabling prompts and scaffolds. This led to her students becoming more independent when approaching tasks, taking a more logical approach, and also developing an understanding that there is more than one way to solve a problem. As the study progressed, students in Mrs Stuart's class engaged more readily with algebraic reasoning and began to recognise and use structural aspects and patterns independently to solve tasks.

Student engagement and use of mathematical practices shifted with changes in the classroom. Initially in both classrooms students did not regularly use representations nor did they engage in generalisation and justification. In Mrs Stuart's classroom attention to engaging students in mathematical practices such as making conjectures, justifying, and generalising led to students more readily making conjectures about patterns which they noticed. Mrs Stuart also increasingly expected her students to use representations. They began to draw on different representations to support their explanations and also began to translate between the representations that either they had used or those used by their peers. As the project progressed Mrs Stuart introduced representations as a way of providing concrete justification when generalising. With further classroom experiences focused on justification, students more readily drew on material to prove reasoning. They also began to refer to conjectures and generalisations which had been previously examined in their explanations.

Mrs Willis supported her students to develop conjectures by extending already existing tasks (see Section 7.3.2). Investigation of these conjectures mostly focused on developing specific examples rather than the development of a generalised claim. Hence, a 'conjecturing atmosphere' (Mason, 2008) where students would readily express conjectures was not forthcoming. While Mrs Willis also required her students to support their explanations with a representation, this was limited to the use of a single representation, most often an equation. Moreover students were not facilitated to reflect on alternative ways of justifying beyond use of specific examples. This meant that they were not engaged in developing a range of more refined justification skills.

In both classrooms the introduction of pedagogical actions focused on developing productive talk led to shifts in student participation. Similar to the findings of other studies (e.g., McCrone, 2005; Reid & Zack, 2009) the changes in the ways students participated took time to develop. Initially in both classrooms during whole class discussions the students continued mainly to provide answers with no reasoning. However with both teachers increasingly emphasising the need to provide reasoning, the frequency of responses with reasoning and attempts to develop mathematical explanations increased. Mrs Stuart's consistent and explicit expectations that mathematical reasoning would be provided supported students' on-going development of reasoned explanations. In contrast, in Mrs Willis's classroom the expectation for reasoning was not consistently sustained. As the study progressed, it was evident that norms for the whole class discussion had not been firmly established; students consistently faced the board when speaking unless reminded to turn around and they did not habitually provide reasoning. They also frequently phrased their answers as questions and did not respond to questions both from their peers or the teacher.

The focus on collaborative interaction during whole class discussions led to shifts in student participation. Initially whole class discussions were not viewed by students as an opportunity to develop understanding in either classroom. This meant that they were unable to recall their peers' explanations or explain their own mathematical thinking. For both teachers the use of pedagogical actions which focused on interactive mathematics talk shifted students to become more active participants who asked questions to clarify ideas and agree and disagree with their peers' ideas. In both classrooms the students would recognise erroneous explanations and question or challenge these.

Mrs Stuart further facilitated her students to analyse and attend to both their own thinking and the thinking of others. As a result, her students increasingly used questioning to probe for justification and used mathematical reasoning to agree or disagree. They were now able to re-construct their peers' explanations and reasoning from whole class discussions both during and after the lesson. They also began to describe explicitly their own mathematical thinking although at this point this did not extend to justification for their thinking. Her continuing focus on interactive talk meant that towards the end of the study, students had begun to justify both their own and their peers explanations by referring to the context of the problem.

Mrs Willis continued to engage her students in questioning for understanding; however she did not require them to provide reasoning for agreement and disagreement. Initially, her students were unable to reconstruct their peers' explanations following the lessons and also had difficulty in describing their own mathematical thinking. With further facilitation from Mrs Willis for active participation while listening to the reasoning of others, the students began to ask questions for clarification more frequently and use mathematical reasoning to disagree. They were now able to recall their own solution strategies and a growing number could recall their peers' solution strategies although this did not extend to the reasoning which supported this.

Collaborative interaction during small-group work was another teacher focus which led to changes in the way students participated. In both classrooms, students initially had a positive view of group work. However, their emphasis was on turn-taking or assistance when stuck rather than collaborating to construct understanding jointly. Mrs Stuart worked with her class to develop a set of group norms. Importantly, she also engaged her students in reflection on the ways in which they were working; this supported them to gain understanding of the benefits of collaboration beyond the emphasis on turn-taking. Mrs Stuart's students maintained a positive disposition to group-work throughout the study with their emphasis moving from co-operating to collaborating and developing joint solution strategies. They identified ways of supporting collaboration such as questioning, disagreeing with reasoning, and convincing others. Collaborating with others and both talking and listening were identified by the students as key reflective tools for learning.

Although Mrs Willis asked her students to work in groups and shared some group rules with them, there was also some inconsistency in Mrs Willis's requirement for collaboration. The students did not work with Mrs Willis to develop these rules for group work or to reflect on the benefits of working in such a way. The students reported mixed feelings about group-work. Whilst many students regarded both talking about mathematical ideas and listening to others as beneficial to their mathematical learning, benefits concerned getting answers or help rather than collaboration. Negative reports could be attributed to the lack of developed group norms (see Section 7.4.3 and 7.4.5).

In summary, there are a number of key shifts that occur in the way students engage in classroom activity as early algebraic reasoning is successfully integrated into the everyday mathematics lessons. An important shift is for students themselves to develop their algebra ears and eyes. This means that they approach tasks in algebraic ways through drawing on structure, relationships, and patterns. As tasks are enacted students will readily make conjectures about patterns which they notice and then draw on materials both to prove and justify their reasoning. Another important change in student participation is the shift to view whole class discussions as a way of developing both personal and collective mathematical thinking and understanding. Developing an appreciation of the collective will enhance students' sense of obligation to provide mathematical reasoning, develop explanations, and justification for all members of the classroom (both other students and the teacher).

8.3 LIMITATIONS OF THE STUDY

This study contributes new knowledge and findings to the field in a number of ways. However it is also important to acknowledge that any research has potential limitations. This research involved the empirical analysis of a small sample of teachers and students from two schools in the UK and the British Isles. The generalisability of the findings for teachers in different contexts and classroom settings may be limited. However inclusion of the conjectured framework of professional development (see Table 1), the Framework of Teacher Actions to Facilitate Algebraic Reasoning (see Table 16), and the detailed descriptions of both the teacher learning and pedagogical practices provides opportunities for others to trial a similar study.

Due to the complex nature of teacher learning, schools, and classrooms, interpretation of the results of this study can only provide an emerging understanding of how teachers develop algebra ears and eyes and then use their developing understanding to facilitate algebraic reasoning in their classrooms. Although a range of triangulation methods were used, consideration should be given to the possibility of bias in the results as the findings are based on one researcher's interpretation of the data and other interpretations are possible. Although the interpretation in this study was strengthened by the use of wide range of data sources and the multiple iterations of coding, the findings need to be read in terms of the educational and cultural context in which they were situated. Although there are widespread international calls for increased focus on algebra in primary classrooms, not all countries would recognise the forms of discourse-based teaching promoted within the professional development model as aligned to their educational values (Clarke, 2013).

8.4 CONTRIBUTION TO THE RESEARCH FIELD

Within this study clear implications are provided for thinking about ways in which early algebraic reasoning can be integrated into primary mathematics classrooms. It adds to the research field with its broad perspective of algebra to include both areas of content and classroom and mathematical practices which support student engagement in algebraic reasoning. This study is unique within the UK with its simultaneous focus on how primary teachers develop algebra ears and eyes and the changes within the classroom community and student engagement as algebra is integrated within everyday mathematics lessons. The focus on primary students' perspectives in classrooms where algebra is being facilitated provides a significant addition to the research literature.

The *Framework of Teacher Actions to Facilitate Algebraic Reasoning* (see Table 16 below) is offered as contribution to the field. Importantly this framework integrates four separate, interlinked components which this study identifies as key to the development of early algebraic reasoning. These include:

- Teacher awareness of and a purposeful focus on algebraic concepts;
- Teacher actions to develop and modify tasks and enact them in ways which facilitate algebraic reasoning;
- Teacher actions to develop classroom practices which provide opportunities for engagement in algebraic reasoning; and
- Teacher actions to develop mathematical practices which support the development of algebraic reasoning.

Table 16

Framework of Teacher Actions to Facilitate Algebraic Reasoning

<table>
<tr><th></th><th>STAGE ONE</th><th>STAGE TWO</th><th>STAGE THREE</th></tr>
<tr><td>Algebraic concepts</td><td colspan="3">Address the following concepts in the classroom: understand the equal sign as representing equivalence; relational reasoning including whole numbers and rational numbers; commutative property; inverse relationships; odd and even numbers; properties of zero and one; distributive property; associative property; properties of rational numbers; using and solving equations; function</td></tr>
<tr><td></td><td>STAGE ONE</td><td>STAGE TWO</td><td>STAGE THREE</td></tr>
<tr><td rowspan="13">Teacher actions to develop and modify tasks and enact them in ways which facilitate algebraic reasoning</td><td colspan="3">Implement tasks as problem-solving opportunities</td></tr>
<tr><td colspan="3">Emphasise student effort to approach and complete cognitively challenging tasks</td></tr>
<tr><td colspan="3">Extend or enact tasks to include opportunities for generalisation</td></tr>
<tr><td colspan="3">Interrogate tasks for opportunities to highlight structure and relationships</td></tr>
<tr><td></td><td colspan="2">Adapt tasks to highlight structure and relationships. This may include changing numbers or extending to multiple solutions</td></tr>
<tr><td></td><td colspan="2">Structure tasks to address potential misconceptions</td></tr>
<tr><td></td><td colspan="2">Use enabling prompts to facilitate all students to access tasks</td></tr>
<tr><td></td><td colspan="2">Implement tasks by focusing attention on patterns and structure</td></tr>
<tr><td></td><td colspan="2">Recognise and use spontaneous opportunities for algebraic reasoning during task enactment</td></tr>
<tr><td></td><td></td><td>Recognise and use links to algebra in tasks across mathematical areas</td></tr>
<tr><td></td><td></td><td>Implement tasks as open-ended problems</td></tr>
<tr><td></td><td></td><td>Anticipate student responses which could provide opportunities for algebra</td></tr>
<tr><td></td><td></td><td>Recognise and use spontaneous opportunities for algebraic reasoning from student responses</td></tr>
</table>

	STAGE ONE	STAGE TWO	STAGE THREE
Teacher actions to develop classroom practices which provide opportunities for engagement in algebraic reasoning	Lead explicit discussion about classroom and discourse practices		
	Ask students to apply their own reasoning to the reasoning of someone else		
	Require students working in pairs or small groups to develop a collaborative solution strategy which all can explain		
		Require that students indicate agreement or disagreement with part of an explanation or a whole explanation and provide mathematical reasons for this	
		Lead explicit discussions about ways of reasoning	
		Provide space for students to ask questions for clarification	
		Request students to add on to a previous contribution	
		Ask students to repeat previous contributions	
		Use student reasoning as the basis of the lesson	
		Facilitate students to examine solution strategies for similarities or differences	
			Lead explicit discussion about mathematical practices
			Sequence solution strategies to advance mathematical thinking and reasoning
			Provide space for students to question for justification

	STAGE ONE	STAGE TWO	STAGE THREE
Teacher actions to develop mathematical practices which support the development of algebraic reasoning	Require students to explain their reasoning		
		Require students to develop mathematical explanations which refer to the task and its context	
		Facilitate students to use representations to develop understanding of algebraic concepts	
		Ask students to develop connections between tasks and representations	
		Provide opportunities for students to formulate conjectures and generalisations in natural language. Leads students in examining and refining these conjectures and generalisations	
		Listen for conjectures during discussions. Facilitates students to examine these	
		Require students to use different representations to develop the clarity of their	

		explanation	
		Model and support the use of questions which lead to generalisations like 'Does it always work?', ' Can you see any patterns?', Would that work with all numbers'	
			Listen for implicit use of number or operational properties. Uses these as a platform for students to make conjectures and generalise
			Facilitate students to represent conjectures and generalisations in number sentences using symbols
			Ask students to consider if the rule or solution strategy they have used will work for other numbers. Consider if they can use the same process for a more general case
			Promote use of concrete forms of justification
			Require students to translate between different representations

Each of the four key aspects integrated within the framework have been linked with specific supportive teacher actions. Based on evidence of 'what works' in terms of teacher practice, this is an important contribution that will enhance professional learning and development opportunities to build capacity to enact reforms in early algebra teaching and learning.

The significant role which students have in the development of classrooms where algebraic reasoning is a focus is a key finding of the study. Findings affirm the importance of developing a learning community which embraces both teacher and the students' learning simultaneously.

Many previous studies have focused on teachers' learning to develop algebra ears or eyes or using pedagogical actions to develop algebraic reasoning in the classroom. This study extends work in this field by providing insight into the contribution that students make. In responding to the opportunities afforded by the teacher's algebra eyes and ears students contribute their own algebraic reasoning, and in the process develop their own algebra ears and eyes and strengthen the collective algebra ears and eyes of the community.

8.5 IMPLICATIONS AND FURTHER RESEARCH

This study was undertaken in two different schools where students were generally white British with only a few in each class from different ethnic groupings. They came from mainly middle to high socio-economic backgrounds. Teachers who participated in the study were experienced teachers from white British backgrounds. Further studies which extended this research beyond these contexts would be useful. In particular, it would be beneficial to extend understanding to how algebra ears and eyes can be developed with teachers of different ethnic backgrounds and levels of experience, in different types of schools in varying locations. It would also be useful to investigate how algebraic reasoning can be integrated into everyday mathematics lessons with students from varying year levels, different ethnic groupings and socio-economic backgrounds within the UK.

It is evident that it is the teacher who makes the integration of algebraic reasoning into the learning community possible and the study highlights the key role which the teacher takes in implementing and leading change within the classroom. Initially the teachers in this study held understandings of algebra which were grounded in their own schooling experiences. This involved more traditional approaches where computational arithmetic was taught in primary school followed by the introduction of abstract algebra in secondary school. They appeared to lack experience with the rich, connected types of algebra which support the development of early algebraic reasoning. Teacher knowledge was an important aspect of the study; however this study did not specifically assess content or pedagogical content knowledge of the participating teachers in relation to early algebra. Further research is needed in the UK which investigates the pedagogical content knowledge of primary teachers in relation to early algebra and how to further develop teachers' pedagogical content knowledge effectively.

Within research literature continuing difficulties which primary students have with specific algebraic concepts such as the equals sign or over-generalising the commutative property are well documented (e.g., Davis, 1984; Kieran, 1981). However, these findings were not known by the teachers in this study. Therefore an important implication of this study is to consider how research findings can be communicated to teachers through pre-service education or professional development experiences.

An unintended focus which resulted from the research study was how we can account for differences in change between teachers. For example, it became evident in the data analysis that while there were similarities in the nature and depth of teachers understanding of algebra at the onset of the study, a key difference that appeared as the study progressed was in how they sought to develop their understanding further. Similar to working in classrooms with students, working with teachers to develop their algebra ears and eyes held many challenges. Further research which captures the teachers' learning and perspectives as participants in professional development would be useful. It would also be productive to examine more fully how research-based tools and activities provide opportunities for teacher learning.

Related to the previous paragraph, teacher disposition and attitudes and belief were not a focus of the current study, however it appears that they were a significant factor in whether the teachers developed their own algebra ears and eyes and facilitated the development of early algebraic reasoning in a classroom of mathematical inquiry. Further research is needed in the UK to explore specific attitudes and beliefs teachers hold towards the value of early algebra and the use of classroom and mathematical practices which support students to develop algebraic reasoning. Further trialling of interventions such as those used in this study need to be undertaken to assess factors which are successful in shifting teachers' attitudes and beliefs.

Whilst professional development activities to support teachers to develop algebra ears and eyes were a key part of this study design, the research focus did not include explicit evaluation of their effectiveness. Evaluation in terms of movement within the design framework was more to do with the placement of activities in relation to teachers' learning/change. However, it was clear in the analysis of the respective teachers' journey (Chapter Six and Seven) that professional learning/change was influenced by more than the provision of activities. Some important factors appeared to be the sustained focus over the school year, opportunities to collaborate with other teachers, the facilitation of reflection through observation, watching video recordings, using frameworks of effective teaching, and access to relevant research material. These findings have implications for the way in which professional development for teachers may be advanced. Further investigation is required into the effectiveness of particular activities to develop pedagogical content knowledge in the area of early algebra.

8.6 FINAL THOUGHTS

Algebra has been recognised as a gatekeeper and there is widespread acknowledgement of how insufficient development of algebraic understanding during schooling denies individuals access to potential educational and employment opportunities (Cai & Knuth, 2011; Chazan, 1996; Kaput, 2008). The key focus of this research was to investigate how teachers can develop early algebraic reasoning in classroom communities of mathematical inquiry. Teacher development of algebra eyes and ears through active engagement in professional development and implementing what they were learning in the classroom was shown to be an important factor in whether algebra was successfully integrated into the classroom. A key finding of the study was that the integration of algebra into everyday lessons within the mathematics classroom required more than the introduction of algebraic concepts. It was necessary for the teachers to also attend to the development of the classroom community and to facilitate the growth of classroom practices and mathematical practices which supported collective student participation and engagement with algebraic reasoning.

This design study methodology generated a significantly rich data set concerning teacher learning and change in practice. The two contrasting cases exemplify the complexities and the challenges for both teachers and students to develop algebra ears and eyes. Evidence from the research study shows that despite the complex challenges teachers such as Mrs Stuart can facilitate early algebraic reasoning. Key to the success is the development of a community of mathematical inquiry in which all participants, both the teacher and students, are able to co-construct a range of mathematical practices and participatory practices that explicitly enhance algebraic ways of thinking and reasoning.

It is hoped that the findings of this research provide a productive model for researchers and designers of professional development to use to develop algebraic reasoning with primary teachers. However, we need to be mindful in using the model with teachers to develop early algebraic reasoning with students in primary classrooms that “changing teaching is risky and that risk taking means trusting that the outcomes will be worth the risk” (Askew, 2012, p. 138). In this case, having students develop algebra eyes and ears.

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APPENDICES

APPENDIX A: ADAPTED FRAMEWORK FROM HUNTER (2009)

Developing conceptual explanations (includes using the problem context to make explanations experientially real)				
Model providing a mathematical explanation. Use the context of the problem not just the numbers.	Re-voice, and extend an explanation using the problem context. Expect mathematical reasons (e.g., 19+7=20+6 because 7-1=6 and 19+1=20)	Question to scaffold students to extend their explanations to include the problem context and what they did to the numbers mathematically.	Model and support the use of questions which clarify an explanation. What do you mean by? What did you do in that bit? Can you show us what you mean by? Could you draw a picture of what you are thinking?	
Have the students develop two or more ways to explain a strategy solution which may include using materials	Have students examine their explanation, predict the questions they will be asked and prepare explanations.	Have the students read the question as a pair or group, and discuss, interpret and reinterpret problems collectively.	Have students describe their different starting points when solving a problem. Reinforce acceptability of multiple ways. Support them to make connections to other or previous problems.	
Active listening and questioning for sense-making of an explanation.				
Discuss and role-play active listening. Use inclusive language "show us", "we want to know", "tell us".	Structure the students explaining and sense making section by section.	Emphasise need for individual responsibility for sense-making	Provide space in explanations for thinking and questioning	Affirm models of students actively engaged and questioning to clarify sections or gain further information
Collaborative support and responsibility for the reasoning of all group members				
Provide students with problem and think-time then discussion and sharing before recording	Establish use of one piece of paper and one pen.	Establish an expectation that students will agree on the construction of one solution strategy that all members can explain.	Have students examine their explanation, predict the questions they will be asked and prepare explanations.	

Developing justification and mathematical argumentation

Require that students indicate agreement or disagreement with part of an explanation or a whole explanation	Ask the students to provide mathematical reasons for agreeing or disagreeing with an explanation.	Model ways to justify an explanation "I know 3 + 4 = 7 because 3 + 3 = 6 and one more is 7".	Model and support the use of questions which lead to justification like 'How do you know it works?', 'Can you convince us'.	Provide range of materials for students to use to justify their explanations and conjectures.
Ask the students to be prepared to justify sections of their solution strategy in response to questions.	Expect that pairs/group members will support each other when explaining and justifying to a larger group	Require that the students prepare ways to re-explain in a different way an explanation to justify it.	Provide wait time to allow students to prepare questions which lead to justification	Encourage the use of 'so if', 'then', 'because' to make justifications. Use this format to validate an explanation

Developing representations of reasoning

Expect the use of a range of representations including acting it out, drawing a picture or diagram, visualising, making a model, using symbols, verbalising or putting into words, using materials.	Expect the students to explain and justify using the representation as actions on quantities not manipulation of symbols (use context)	Require that the students compare and contrast representations and evaluate for efficiency	Ask students to re-represent their thinking in different forms in response to questions or for clarification

Developing generalisations

Model and support the use of questions which lead to generalisations like 'Does it always work?',' Can you see any patterns?' Would that work with all numbers'.	Ask the students to consider if the rule or solution strategy they have used will work for other numbers. Consider if they can use the same process for a more general case. (e.g. what happens if you multiply any number by 2)	Listen for implicit use of number or operational rules. Encourage the explicit description of these as a platform for students to make conjectures and generalise.	Facilitate students to represent conjectures and generalisations written in natural language in open number sentences using symbols they are comfortable with and linking this back to the written form.
Provide opportunities for students to formulate conjectures and generalisations in their own natural language. Have the class examine the recorded conjectures and generalisations and collectively refine these.			

APPENDIX B: LESSON STUDY REFLECTION QUESTIONS

Clear objectives to enhance students' learning:
Did the planned lesson activities support the children in reaching the learning objectives? How?

Did the lesson build on prior learning?

Did the lesson activities provoke disagreement and mathematical argument? How? If not how could that have been achieved?

Did the students understand why the topic/concepts/activity was important mathematically?

Tasks and learning aids that help students accomplish learning objectives:
Was the degree of challenge appropriate for the students at the time? How? If not how could it have been made more challenging?

Were there any unanticipated responses?

Were the students given opportunities to express mathematical ideas individually, in a group, in pairs, in whole class discussion?

What changes could be made to promote more exchanges of mathematical ideas?

Did the lesson incorporate appropriate use of visual and communicative tools?

Did the lesson provide students with opportunities to extend or secure their knowledge understanding or skills?

Teacher questioning and support for student learning
How did (teacher's and peers) questions and guidance enhance students' learning?

Which questions from peers or teacher appear to facilitate the students' learning?

How were students provided with appropriate support to overcome misconceptions or misunderstanding?

How were students supported in making use of the resources available for the lesson?
How was grouping used to maximise student learning?

Effective integration of assessment
How was formative assessment used in decisions to modify or adjust the plan?

Did the students accomplish the learning objectives? What evidence do you have of this?

APPENDIX C: TEACHER INFORMATION SHEET AND CONSENT FORM

ETHICS PROTOCOL
Facilitating the development of early algebraic reasoning

1. **Who I am**

My name is Jodie Hunter. I am a Research Fellow in Mathematics and Statistics at the University of Plymouth.

2. **What this research is about**

This study aims to investigate how the classroom environment and instructional tasks can support children in developing early algebraic reasoning. It will also investigate student and teacher perspectives of classroom events during mathematics lessons.

I will be investigating and exploring:

- How professional development meetings can support teachers to facilitate early algebraic reasoning opportunities in their classrooms.
- Classroom organisation and practice of teachers participating in this study.
- Teacher views of student learning and participation in mathematics lessons.
- Student views of their learning and participation in mathematics lessons.
- Student achievement in early algebra in classrooms in this study.

I am inviting you to be part of a collaborative design experiment research process in which we will look at some of the ways children construct algebraic understanding as they participate in mathematical activity in classroom. We will also examine how the instructional environment and tasks support children to develop early algebraic understandings.

I anticipate that the project will yield findings that will lead to conference papers and research papers submitted for publication. Some excerpts from the recordings made during interviews or in the classroom may be used as part of conference presentations.

3. **Data Collection**

Permission to participate in the study will be sought from both the parents of the pupils in your class and the pupils themselves. The pupils and their parents/caregivers will be given full information and consent will be requested in due course.

During the study I plan to:

- Conduct interviews with you which will be audio-recorded.
- Audio record six professional development and planning meetings over the school year.

- Assess student learning of early algebraic concepts through the use of individual task based assessment.
- Interview your students to gain their perspectives on the learning of algebra.
- Observe classroom mathematics lessons including the use of audio and video recording and still photography. Collect examples of student work from the lessons. Conduct follow-up interviews involving video and photo stimulated recall to gain both teacher and student perspectives on the mathematical learning during the lesson.
- Collect an audio-recorded reflective log from you.

Your involvement in this project is entirely voluntary. If you are willing to be involved, the interview and classroom observations will involve the use of video and audio recording. During the interview or classroom mathematics activities you may ask at any time that the audio or video recorder be turned off and any comments you have made be excluded from the study. All project data will be stored in a locked drawer at the University of Plymouth, with no public access and used only for this research and any publications arising from this research. After completion of ten years, all data pertaining to this study will be destroyed in a secure manner.

4. Confidentiality

All efforts will be taken to maximize confidentiality and anonymity for participants. The school name and names of all participants will be assigned pseudonyms to support their anonymity. However complete anonymity cannot be guaranteed due to the nature of this study as classroom and school based. At the end of the study a summary will be provided to the school and teachers involved.

5. Consent and your rights

Please note you have the following rights in response to my request for you to participate in this study. To:

- refuse to participate in the study
- decline to answer any particular question;
- withdraw from the study at any time;
- ask any questions about the study at any time during participation;
- provide information on the understanding that your name will not be used unless you give permission to the researcher;
- ask for the audio or video recorder to be turned off and any comments you have made be excluded from the study;
- be given access to a summary of the project findings when it is concluded.

If you have further questions about this project you are welcome to discuss them with me personally:

Jodie Hunter
Phone: 01752 585347 email: jodie.hunter@plymouth.ac.uk

Developing early algebraic reasoning

CONSENT FORM: TEACHER PARTICIPANTS

THIS CONSENT FORM WILL BE HELD FOR A PERIOD OF TEN (10) YEARS

I have read the Information Sheet and have had the details of the study explained. My questions have been answered to my satisfaction, and I understand that I may ask further questions at any time or withdraw completely or in part from the research.

I agree/do not agree to being audio-taped

I agree/do not agree to being videotaped

I agree/do not agree to participating in this study under the conditions set out in the Information Sheet.

Signature: .... **Date:** ....

Full Name - printed ....

APPENDIX D: STUDENT AND CAREGIVER INFORMATION SHEET AND CONSENT FORM

ETHICS PROTOCOL
Developing early algebraic reasoning with primary school students

6. Who I am
My name is Jodie Hunter. I am a Research Fellow in Mathematics and Statistics at the University of Plymouth.

7. What this research is about
I am interested in finding out how children can develop their number and algebra understandings. I would like to find out:

- How you learn algebra in your classroom?
- What do you think about learning algebra?

Your teacher has agreed to participate in this study. I would like to invite you with your parent's permission to be involved in this study. I plan to write conference papers and research papers from the information that I find out, but I will not mention your name or the school in anything that is published. Some excerpts from the recordings made during interviews or in the classroom may be used as part of conference presentations.

8. Data Collection
If you agree to be involved, I will interview you about your number and algebra knowledge and your teacher and I will use this to plan activities to be used in the classroom. The interview will be tape-recorded and you may ask at any time that the tape recorder be turned off. If you change your mind or are not happy about what you said, you can request that any comments you have made are not used in the study.

Some of the normal mathematics lessons that are taught in your classroom will be audio and video recorded and photographed. During classroom mathematics activities you may at any time ask that the audio or video recorder be turned off. If you change your mind or are not happy about what you said, you can request that any comments you have made are not used in the study. With your permission sometimes I might collect copies of your mathematics written work. You have the right to refuse to allow the copies to be taken and it will not count against you in any way. Following the mathematics lessons, I may ask you to talk about your learning during the lesson and this will be audio-recorded.

The mathematics activities you do in class will be the same whether you agree to be in the study or not. The interview and observations will take place in the classroom and be part of the normal mathematics programme. It is possible that talking about your learning may help you clarify what you know about number and algebra and what you need to know next.

9. Confidentiality

All the information gathered will be stored in a locked drawer at the University of Plymouth and used only for this research. After completion of ten years all the information will be destroyed. All efforts will be taken to maximise your confidentiality and anonymity which means that your name will not be used in this study and only non-identifying information will be used in reporting. However total anonymity cannot be guaranteed due to the research taking place in a classroom and school where others may overhear.

10. Consent and your rights

I ask that you discuss all the information in this letter fully with your parents before you give your consent to participate.
Please note that you have the following rights:

- To say you do not want to participate in the study
- To refuse to answer any questions that have been asked
- To withdraw from the study at any time
- To ask for any comments you have made to be excluded from the study
- To refuse to allow copies of your written work to be taken
- To ask questions about the study at any time
- To participate knowing that you will not intentionally be identified at any time
- To be given a summary of what is found at the end of the study

If you have further questions about this project you are welcome to discuss them with me personally:

Jodie Hunter
Phone: 01752 585347 email: jodie.hunter@plymouth.ac.uk

Developing early algebraic reasoning

CONSENT FORM: STUDENT PARTICIPANTS

THIS CONSENT FORM WILL BE HELD FOR A PERIOD OF TEN (10) YEARS

I have read the Information Sheet and have had the details of the study explained. My questions have been answered to my satisfaction, and I understand that I may ask further questions at any time or withdraw completely or in part from the research.

I agree/do not agree to being audio-taped

I agree/do not agree to being videotaped

I agree/do not agree to participating in this study under the conditions set out in the Information Sheet.

Child's Signature: .... **Date:** ....

Full Name - printed ....

CONSENT FORM: PARENT/CAREGIVERS OF STUDENT PARTICIPANTS

THIS CONSENT FORM WILL BE HELD FOR A PERIOD OF TEN (10) YEARS

I have read the Information Sheet and have had the details of the study explained. My questions have been answered to my satisfaction, and I understand that I may ask further questions at any time or withdraw my child completely or in part from the research.

I agree/do not agree to________________________________ being audio-taped

I agree/do not agree to________________________________ being videotaped

I agree to ____ participating in this study under the conditions set out in the Information Sheet.

Parent/Caregivers Signature: .... **Date:** ....

Full Name - printed ....

APPENDIX E: SCHEDULE OF STUDENT INTERVIEW QUESTIONS

Questions which were asked throughout the duration of the research study

- What were you learning about in maths today?
- As each photo was shown - What was happening in that part of the lesson?
- You were working with a partner and in a group today, how did that go for you?
- Did anybody help you or did you learn anything from anybody today?

Additional questions asked during Phase One interviews

- Is it important for you to be able to explain your ideas in maths? Why or why not?
- Is it important for you to listen to other children explain or talk about how they solved problems in maths? Why or why not?

Additional questions during Phase Three interviews

- When you explain your ideas or talk about them with other people in the class does it help you? Why or why not?
- When people talk about their ideas or give explanations does that help you learn maths? Why or why not?

APPENDIX F: EXAMPLE OF TRANSCRIPT OF CODING FROM NVIVO

Mrs Stuart:

Before I put any of this black writing I said how many different arrays can you make? Do you all know what an array is? Yep. What shape is it going to end up in? Rectangle, yep? Okay, draw each one, so lots of you have drawn looking around, if this was the one on your board, which it isn't because that's only 6 counters, you've drawn that. And then I've just added write as many equations as you can for each drawing. So what could I write for this one please? Saffron?

00:39

Saffron:

Three add three

Mrs Stuart:

Three add three (pause) three add three equals six (...). Zanthe?

Zanthe:

Two times three equals six

Mrs Stuart:

Two times three equals six. What else could you write? Take your time (pause) Hazel?

Hazel:

Three times two equals six

Mrs Stuart:

Good, three times two equals six. What about a repeated addition? Multiplication and division we're going for because we're doing arrays so just...you can have repeated addition and repeated subtraction. Paul?

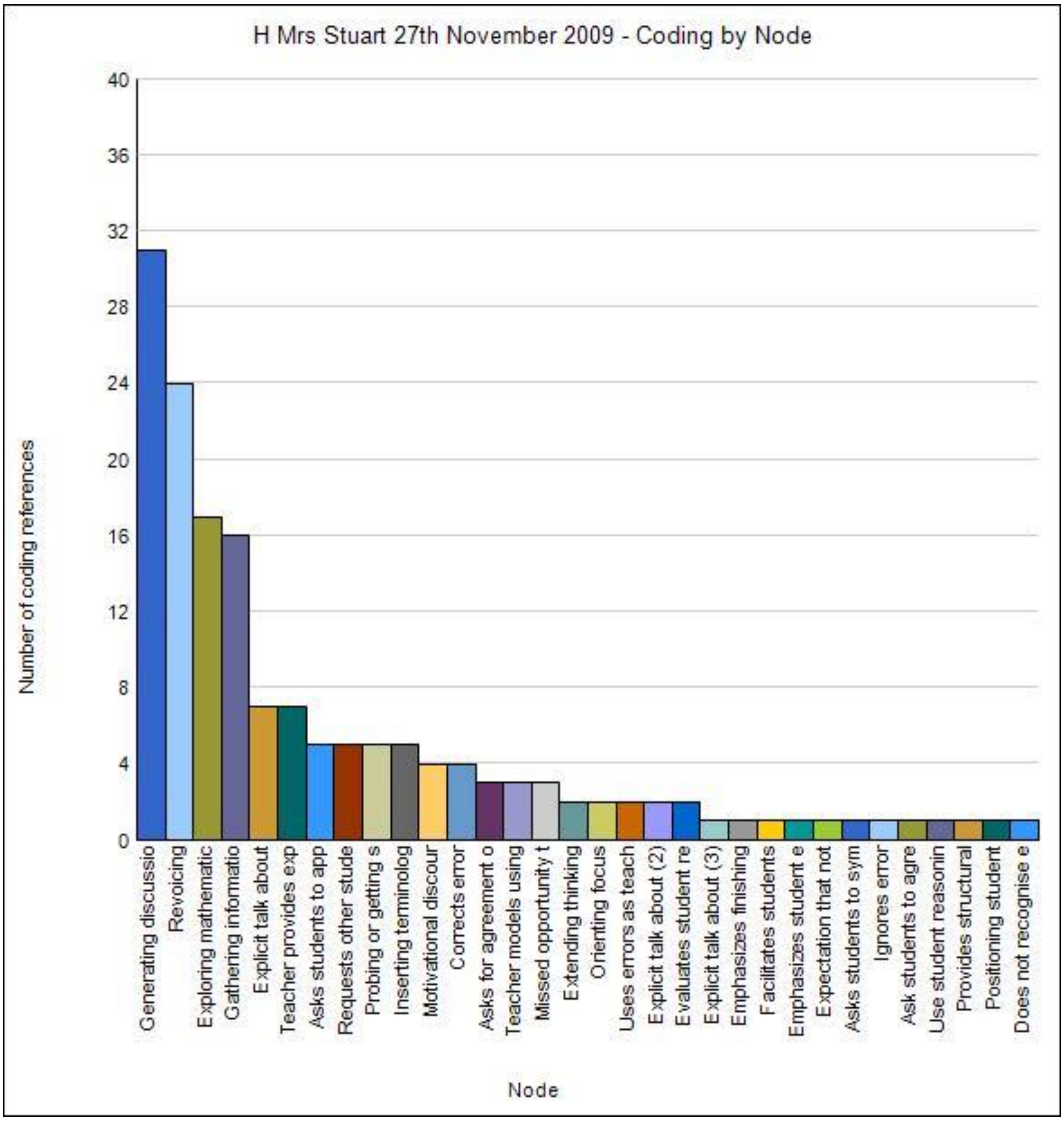
Paul:

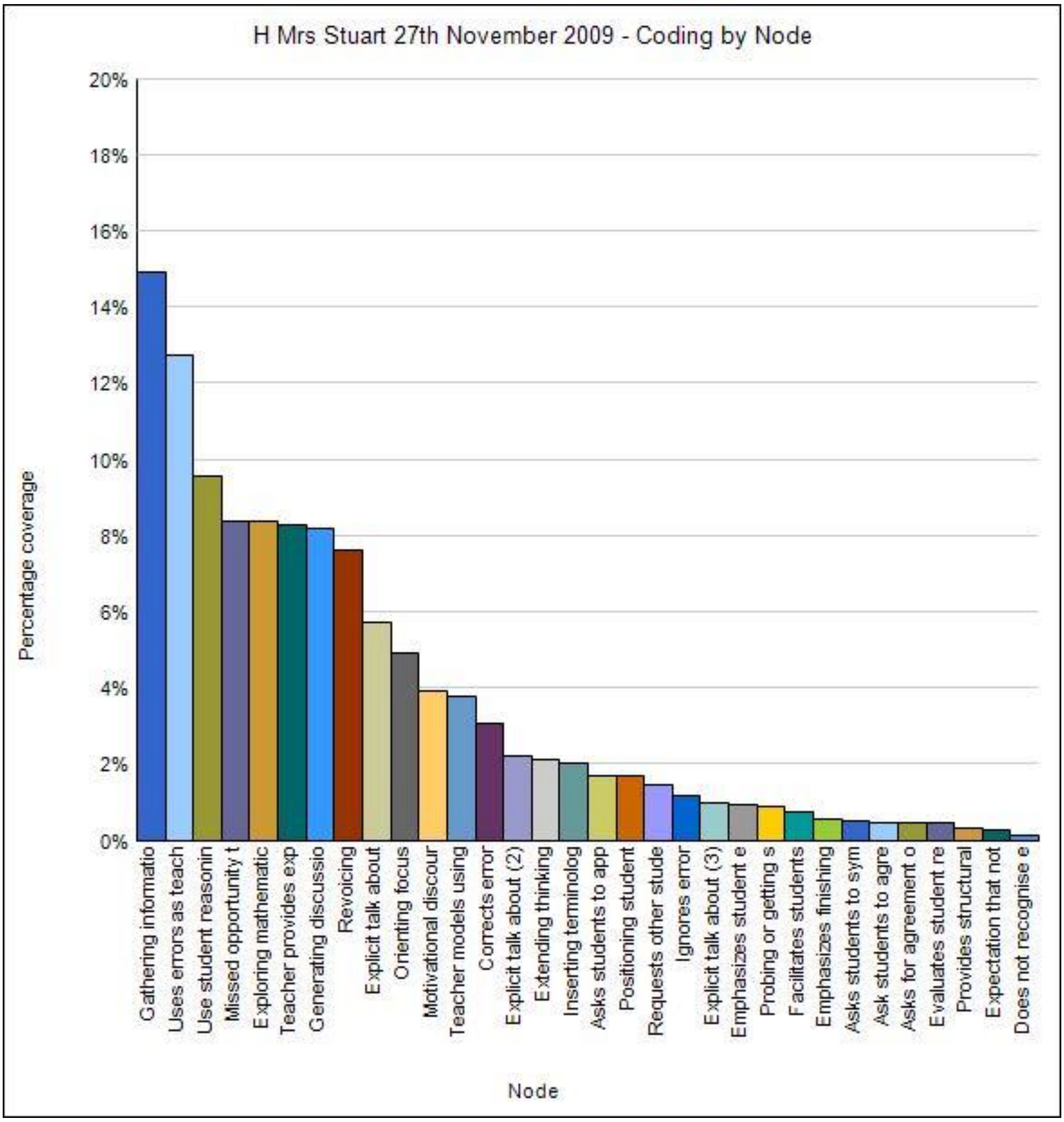
Twelve divided by two equals six

Mrs Stuart:

{pointing to the board} Have I got 12 counters here?

APPENDIX G: EXAMPLES OF DATA ANALYSIS GRAPHS





H Mrs Stuart 27th November 2009 - Coding by Node
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Positioning student
Requests other stude
Ignores error
Explicit talk about (3)
Emphasizes student e
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Asks for agreement o
Evaluates student re
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Developing Teacher Understanding of Early Algebraic Concepts Using Lesson Study

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This paper reports on the use of lesson study as a professional development tool. In particular the paper focuses on the way in which the teachers increased their understanding of how tasks, classroom activity and teacher actions scaffolded student learning of early algebraic reasoning of equivalence and the commutative principle. Teacher voice is used to illustrate how lesson study cycles caused the teachers to reflect and review their own understandings of early algebraic concepts and how their students considered the concepts.

Introduction

Algebra is often provided as a reason for both the difficulties individuals encounter learning and making sense of mathematics, and the disaffection many people hold towards it. Given the position algebra holds in the educational and economical future of all individuals, Knuth and his colleagues (Knuth, Stephens, McNeil, & Alibabi, 2006) describe a growing consensus between researchers and educators that algebra be introduced at a much younger age with a focus on the integration of teaching and learning arithmetic and algebra in classrooms. This emphasis is confirmed in policy documents which describe a unified curricula strand (e.g., Department for Education and Employment, 1999; National Council of Teachers of Mathematics 2000). Teachers, within this changing context are required to find ways to make algebra accessible to all their students, through the use of rich learning tasks in environments which provide all students opportunities to learn algebra with rich, conceptual understanding (Chazan, 1996). The focus of this paper is on how a group of teachers used lesson study to explore how some designed tasks could be used to better support student development of key early algebraic concepts.

Teachers have a key role in reforming classroom practice and activities which integrate arithmetic and algebra. But, we know that for many this poses considerable challenges; they may not have understandings of how to make links between arithmetic and algebra, nor may they have had experience constructing and using rich connected types of integrated (arithmetic/algebra) problems. As Blanton and Kaput (2003) suggest, they may not have developed their algebra ‘ears and eyes’ when working with the patterns and relationship in number which promote rich connected conceptual understandings. Blanton and Kaput suggest a remedy for this situation could be a form of professional development in which opportunities are structured so teachers identify numerical patterns and relationships which connect to early algebraic reasoning. In this paper lesson study was used as a form of professional development to facilitate a group of teachers enhanced algebraic ‘eyes and ears’. The aim of this paper is to explore how professional development in the form of lesson study supported the teachers to ‘notice’ opportunities for developing student’s early algebraic reasoning. The questions asked in this study were: How did the use of lesson study support teachers to comprehend how their students understood the key concepts of equivalence and the commutative principle, and; how did the repeated cycles in lesson study

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facilitate the teachers to identify the challenges involved in students' constructing conceptual understandings of equivalence and the commutative principle.

Lesson study, developed in Japan, is one form of professional development which aims to increase teachers' knowledge about mathematics, knowledge about ways of teaching mathematics, and knowledge about the ways in which learners engage with and make sense of mathematics (Fernandez & Yoshida, 2004). In the lesson study format a group of teachers collaboratively plan a lesson (termed the 'study lesson') over a series of meetings. A cycle is developed. The 'study lesson' is taught by a team member, observed by other members with particular focus on student responses. Then in subsequent meetings the observed 'study lesson' is discussed, analyzed, reconstructed in line with student responses, then re-taught to a different group of students. This cycle may be repeated or different lessons developed. In this paper lesson study was used as a form of continuing professional development (CPD), which focused on enhancing specific aspects of teacher knowledge but with a particular emphasis on student learning.

Developing Early Algebraic Reasoning

Students constructing rich conceptual understandings of algebraic reasoning takes a long time and requires that their attention is placed on the inter-related connections across all other types of mathematics, and particularly arithmetic (Blanton & Kaput, 2005). The students' intuitive knowledge of patterns and numerical reasoning are used to provide a foundation for transition to early algebraic thinking (Carpenter, Franke, & Levi 2003). Carpenter and his colleagues explain that for students to justify, and generalise their mathematical reasoning about the properties of numbers they also need to be provided with opportunities to make conjectures in the classroom environment. Research studies investigating young children's development of early algebraic reasoning covers a wide field including those which focus on classroom practices which scaffold student justification and generalisations. However, in this paper to explore how lesson study supported a group of teachers to develop understanding of how their students constructed early algebraic reasoning the focus is narrowed to two areas of early algebra, equivalence (equality) and the commutative principle. The next section makes a brief examination of the literature related to equivalence, the commutative property and lesson study as a professional development process.

Equivalence

Developing understanding of equality is a concept fundamental to algebraic reasoning. Kieran (1981) in her seminal studies illustrated that many elementary school students have an inadequate understanding of the equal sign. Other studies (e.g., Carpenter et al., 2003; Knuth et al., 2006) concur. The difficulties these students encounter are caused because they view the equal sign as an indicator of an operator rather than a symbol of a mathematically equivalent operation. This limits the strategies they have available to solve equivalence problems and in later years symbolic equations (Knuth et al.). To address this problem, teachers need to be aware of how many students view the equal sign and construct and use activities in the classroom which expand student understandings of the equal sign and ensure that the misconceptions are identified and addressed. A range of successful classroom interventions (e.g., Carpenter, et al., 2005; Molina, Castro, & Castro, 2009) which enriched student understanding of equivalence have included non-standard representations such as true and false number sentences and balance scale representations.

Commutative Principle

Opportunities to explore the properties of numbers and operations provide a rich platform for developing algebraic reasoning. However, many exploratory studies (e.g., Anthony & Walshaw 2002, Warren 2001) illustrate that elementary students often have limited classroom experiences in exploring the properties of numbers and operations. As a result the students lack understanding of the operational laws and are unable to construct correct generalisations of the commutativity principle. Anthony and Walshaw illustrated that many students generalised the commutative nature of addition and multiplication, but over-generalised the relationship to include subtraction and division. They showed that while some students could explain the commutative property they could not construct generalised statements nor use materials to model their conjectures. However, studies by Blanton and Kaput (2003) and Carpenter and his colleagues (2003) provided clear evidence that when young children are provided with opportunities in the classroom they learn to construct and justify generalisations about the fundamental structure and properties of numbers. Importantly, these studies demonstrated that when classroom activity targeted students' numerical reasoning they explored, constructed and validated conjectures using appropriate generalisations and justifications

Theoretical Framework

The theoretical framing of this paper is based within a socio-cultural perspective. In this view the processes of teaching and learning hold a reciprocal relationship. The teaching is integrally connected to student learning as manifested through the changing competencies and disposition of the students. In turn, the teachers' professional development is interrelated and identified through evidence of their actions in the classroom, and changes in their professional competencies and attitudes.

Methodology

This paper reports on episodes drawn from a larger study which involved a year-long continuing professional development classroom-based intervention. The participants included two separate groups of elementary teachers (one group from England the other from the Channel Islands). The sample was an opportunistic one of teachers who wanted to extend understandings of ways to facilitate young students' development of early algebraic reasoning. This paper specifically reports on one section of the larger study. In this section the teachers engaged in lesson study for the first time although the Beaumont School teachers had engaged in a paired collaborative observation approach the previous school year, teachers at Hillview School had no experience using collaborative approaches to planning or teaching. The schools were a mixture of rural and suburban contexts and the students came from a range of socio-economic and ethnic backgrounds. The teachers had varying levels of experience.

In the lesson study process used in this study each group of teachers worked as a professional learning community within their own school. Over-arching aims relevant to each school were established immediately. These collaboratively agreed goals broadly established that the teachers wanted to develop creative and flexible problem solvers. Then all members of the research team (the teacher groups and researchers) planned an area of focus for the study lessons. The foci corresponded to mathematical concepts their students had difficulties with or those which the teachers felt less confident about teaching. Through collaborative activity 'study lessons' were planned and taught in one classroom and observed by the research group In-depth analysis and discussion followed observations of the study lesson and subsequent iterations as it was re-planned, re-taught and re-observed in different classrooms as part of the lesson study cycle.

The lesson study cycles in the two settings differed. At Hillview the teachers wanted to address how their students over-generalised the commutative principle to include subtraction and division. A lesson study cycle was devised which included lessons designed to facilitate student understanding and justification of the commutative property with a focus on the use of representations to model conjectures and justify reasoning. The students were given the following statement made by a student in an earlier lesson: 'If you have two numbers and you are adding them it does not matter which number you add first the answer will still be the same.' The students worked in small groups of four and explored with equipment whether the statement held when applied to the different operations. They were required to model their reasoning with equipment, as well as represent it verbally, symbolically and solve problems which involved multiple operations. At Beaumont the study lesson cycles aimed to develop students' skill at solving multi-step word problems and part of the focus was placed on the equal sign. The students were asked to make a specific number using a number sentence which was then represented as equivalent to another number sentence (for example $45 = 20 \times 2 + 5 = 20 + 25 = 45 – 0$) and included some incorrect multi-step equations.

Data gathering included detailed field notes, video and audio records of the planning meetings and classroom lessons and artefacts. The video and audio recordings were wholly transcribed and through an iterative process using a grounded approach, patterns, and themes were identified. The on-going and retrospective data analysis supported the development and construction of case studies of the two study groups. Evidence was triangulated using classroom observations, artefacts and analytical discussion.

Findings

The first section outlines how the use of the lesson study process facilitated the teachers to notice key aspects of early algebraic reasoning and included both planned opportunities for student learning and spontaneous opportunities which arose as tasks were enacted in classrooms.

Developing Understanding of Student Approaches to the Tasks

Discussion and analysis of student responses in the study group illustrated that opportunities to closely observe student responses during the lesson provided a foundation for the teachers to build understandings of how students approach tasks which challenge their understandings of the commutative principle. The teachers expressed surprise that many students began with the use of counter-examples to show that the commutative property did not apply to subtraction. For example Ellen commented:

> Iris and her partner were looking at subtraction without even being prompted to do it because they said straight away "it doesn't work for subtraction but it is working for addition.

Similarly, in further discussion another teacher noted that the students initiated their investigation with the development of a counter-example:

> They were doing it with the subtraction. They did four minus one equals three and one minus four and Lauren said '"so that is subtraction done then, that doesn't work" and she did it for one, if it doesn't work, it doesn't work whereas she then said "actually five times three and three times five works hmm". Then they did something with twos and then she said "does it only work with twos though". So then they tried with a different number. That was Lauren who said that so she had got the

idea that if with one it didn't work, she just discarded that straight away and went straight onto the next one.

Through their observations in the lesson study they had observed how the students intuitively realised that a counter-example disproved the conjecture. However, the students would explore further with other numbers if the conjecture appeared to be correct; in that situation they were not satisfied that one example proved a conjecture.

Developing Understanding of the Role of Materials in Sense-making

Lesson study provided opportunities to develop teacher knowledge of how students could justify their conjectures using materials. For example, in a study group discussion it was evident that a teacher lacked understanding of how the children could justify their conjectures through use of an array. During the observed lesson two groups of students justified an explanation that multiplication was commutative through use of an array. The teacher did not use their explanation to extend the other students' reasoning. Then when another student had difficulties articulating the same concept the teacher stopped her explanation. In the post lesson discussion the researcher stated what the student was explaining:

Researcher: What Andrea was trying to say but she couldn't quite articulate it was if you just kept making it longer it could be any number because you could just keep adding on and it is still the same amount multiplied by the same amount.

Monica: I was conscious of the time, the bell was going to go and I wasn't sure of what she was trying to say from where I was standing.

The teacher's response illustrates that she did not understand how an array supported the explanation nor could she build on and extend the students' explanations of the commutative property. This was reiterated during further analysis in the follow-up discussion. As the other teachers explained and analyzed the student responses the teacher clarified her own actions:

Monica: I didn't know what he was saying about the two numbers.

Melissa: He was moving the rows.

Ellen: Yes, he was saying to turn them around.

Melissa: He moved the rows, he said look you don't have to…

The teacher explains from her own point of confusion her response:

Monica: I thought actually that might have confused everybody else.

Ellen: But he knew what he meant so he could explain it the other way.

At this point Monica acknowledged that because she was confused by what the representation showed she assumed many students would also be.

The teachers also became aware of how important it is that students have access to equipment to scaffold their understanding. After observing that a group of students encountered difficulties investigating whether division was commutative Ellen commented:

When they had the pegs in front of them then they could argue it but they couldn't argue it just on paper. They needed to be able to see the five pegs and they can't divide them amongst the ten people and quite a few groups were like that.

The teachers also observed the way in which the students used equipment to link to real-life situations to model their reasoning:

> I think it was Iris and Andrea, they were talking about the objects and they suddenly became sweets, "if we have got three sweets we can't divide them between seven people" so they were then jumping ahead and moving that relationship on, that was good. I think it was the resources that prompted that.

During further discussion the teachers illustrated how they now understood how physical representations supported students to work at higher levels of generality:

> Monica: I think even with John if he hadn't seen it on the grid [referring to an array constructed on a pegboard] he probably wouldn't have got it as quickly as he did.
>
> Melissa: Because he had really got it in his head, hadn't he? Because he wasn't even really sure if six times four what it equaled, he just knew that it was the same.
>
> Monica: Originally he was convinced that it didn't work so it was only after Sridatta disagreed and showed him it on the grid.
>
> Melissa: The fascinating part is he didn't even work out what the answer was. It didn't matter, it was irrelevant [indicates turning array with hands].

However, the teachers were surprised at the difficulties students had using equipment to model and justify conjectures. During the first lesson cycle their attention was drawn to how the procedural use of symbols dominated how the students responded to tasks:

> Melissa: The thing is them trying to use them as symbols and they got fixated on the idea, like that group over, they even had the scissors as an equals sign
>
> Ellen: And using the blocks to try and create the numbers... I think making it explicit that the objects are representative of a proper number and that they are not to then start creating equations out of them. We don't want to see them as numbers but as objects.

Again the next lesson cycle drew their attention to the student attempts to use materials as symbols:

> Melissa: I think that's the same thing again, they wanted to start putting in the signs and symbols...so they had three colours then a white peg, then one peg and then a white peg and then four pegs and she said "we've put that peg to mean add" so they were doing the same thing. It's like they need to have the symbols there rather than just having like their array as a justification.

Developing Knowledge of How Understanding of the Commutative and Equal Sign is Constructed Over Time and through Specific Teacher Actions

The study group discussion provided many opportunities for the teachers to reflect on what happened in the study lessons and identify missed teaching opportunities. For example in one section of a lesson the discussion focused on examining the structure of multiplication operations and the teacher shifted the children from the general to the specific by guiding them to solve the equations to show the answers were the same. In the analytical discussion with colleagues she recognized that by directing the students towards answers rather than the general structure of multiplication some students became focused on specific equations rather than generalized understandings of the commutative nature of multiplication:

> Monica: I shot myself in the foot because I did that because I knew that some of them hadn't got it so I wanted to show them that actually you know you could tell if you worked them out separately. You could ascertain they had the same answer but then it kind of made other people get stuck at that stage.

In the continuing conversation she saw how her actions caused many students to use procedural rather than conceptual understandings.

The lesson cycles also provided a foundation for the teachers to recognise the need to press students beyond specific examples to generalised reasoning. After a second lesson cycle a teacher observed:

> Melissa: It is almost as though because they had chosen numbers that were simple enough that they knew that two times four made eight so they weren't looking at them as an array, they were looking at them as if that is the numbers that we are dealing with.

This statement led to further discussion in the group, of actions they could use to scaffold the students towards more generalised reasoning.

The post-lesson discussions also provided evidence of the teachers' growing ability to notice student misconceptions of early algebraic reasoning beyond that of the lesson foci. One example occurred when the teachers discussed the difficulties the students had in representing the commutative principle as a number sentence (for example $6 + 5 = 5 + 6$).

> Ellen: They seem to find it really hard to write one continuous number sentence.

In response Monica drew the groups' attention to the on-going difficulties the students had with the equal sign as a concept of equivalence:

> They are still not understanding the proper meaning of the equal sign or perhaps they are but when it comes to applying it in a context then they're not.

In the teacher discussions evidence was provided that they became aware that constructing understanding of equivalence is a lengthy and difficult process which requires a press from the teacher and a lot of student discussion and exploration:

> Zara: We still had to keep coming back to that, that the two sides of the equation had to balance. How much time we have done that, and even given that they had done that in the first part of the lesson. They don't seem to see that as the same.

Within this discussion on-going analysis of the observation and how the activities caused students to think about equivalence led to further analysis and reflection from another teacher.

> Rebecca: I think maybe because we historically present children with a lot of things with the answer just being one box that sort of one where they had to look maybe provoked that thinking a little bit more. You know at the beginning where they said something, something equals and then the next child does equals, I don't know, when I look at it now I think it is a fantastic activity and a fantastic assessment...but maybe they are just seeing and the next one, and the next one, and now it's my turn and they don't actually see the equal sign whereas this question here and that one here in particular really made them think about the idea of balance.

In this statement the teacher has voiced her growing awareness of why students develop misconceptions around the equal sign and the importance of considering how teacher actions coupled with rich tasks structure how students make links between arithmetical and algebraic reasoning.

Discussion and Conclusions

The use of teacher voice in the study group discussions sheds light on the many learning opportunities the teachers encountered as they observed and listened to student activity during the lesson study. Clearly they observed the pivotal role the teacher had, in the study lessons, in making links between the early arithmetical and algebraic reasoning and pressing the students towards situations of generality. Their algebra 'eyes and ears' (Blanton & Kaput, 2003) became more attuned to recognizing common misconceptions as the teachers worked together in the lesson cycles. They also developed cognizance of the need to better match their actions to the classroom discussions and activity.

Of importance in this study was the teachers' recognition of the many challenges they face in developing students' rich connected learning about equivalence and the commutative principle. As previous researchers (e.g., Carpenter et al., 2003; Blanton & Kaput, 2005)

note, students have common misconceptions but for the teachers recognition of these caused reflective and analytic discussions. The findings of this paper suggest that lesson study has considerable promise as a learning tool for teachers to support them reforming their practices to integrate arithmetic and algebra.

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Facilitating Sustainable Professional Development through Lesson Study

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Developing sustainable professional development which facilitates teachers of mathematics to develop effective mathematics pedagogy has been a key aim in recent years. This paper examines how lesson study can be used with networks of teachers as a vehicle to promote and sustain professional development. Drawing on findings from a year-long study involving four schools, the paper highlights how through the process of collaboratively planning a lesson, observing and discussing it, teachers were facilitated to adopt approaches to mathematics teaching that are aligned with the factors identified as effective mathematics pedagogy. It also illustrates how lesson study can support teachers to engage in a collaborative network, develop their professional knowledge, and reflect on their teaching practice.

A key aim of professional development in recent years both in the United Kingdom and internationally has been to develop sustainable networks of teachers of mathematics who engage in developing effective mathematics pedagogy (Askew & Burns, 2005; Jaworski, 2006; Muir & Beswick, 2007). According to the findings of the report by Back, De Geest, Hirst, and Joubert (2009), some key indicators of the effectiveness of continuing professional development are opportunities to develop networks, a focus on student learning, and the facilitation of reflection on teaching practice. This paper considers the essential elements of lesson study as a vehicle for sustainable professional development with networks of teachers. It addresses the ways in which the post-teaching discussion draws on classroom observations of teaching practices and evidence of student learning. It illustrates how the ongoing relationships developed in the networks support the participants in sustaining their development as expert users of effective pedagogical practices in teaching mathematics. The specific questions addressed in this paper are: does involvement in the process of lesson study sustain professional development? How does that involvement sustain professional development?

Background and Theoretical Framework

The theoretical framing of this paper is based within a socio-cultural perspective on the processes of teaching and learning. In this perspective teaching and learning have a reciprocal relationship with effective teaching identified through evidence of the response to it in terms of student learning. This learning can be manifested through the changing identities, dispositions and competencies of the students involved. In addition, teachers' professional development can be identified through evidence of changes in professional identity, attitudes, and their actions in their classrooms.

There are four important elements that underlie the research described in this paper and underpin the theoretical stance. These elements are the process of lesson study, the notion of professional learning communities, the nature of effective mathematics pedagogy, and reflection on practice. The first three key elements are addressed in turn before the fourth section highlights how reflection on practice may be facilitated through the process of lesson study which integrates both professional learning communities and an examination of effective mathematics pedagogy.

Lesson Study

The research study reported in this paper involved groups of teachers participating in communities of inquiry with the methodology known as lesson study as their foci. A brief account of the lesson study process and its use as a vehicle for professional development are provided in the following section.

The process of lesson study is complex consisting of many variations. As a form of professional development, it pays attention to the key aspects of teachers' mathematical knowledge for teaching identified in the literature (Joubert & Sutherland, 2008). These key aspects include: knowledge about mathematics, knowledge about ways of teaching mathematics, and knowledge about the ways in which learners engage with and make sense of mathematics.

Lesson study as a form of continuing professional development (CPD) is to a greater or lesser degree based on Japanese models of development (Burghes & Robinson, 2010; Fernandez & Yoshida, 2004; Lewis, 1995) that particularly emphasise student learning. The process of lesson study involves a group of teachers in collaboratively planning a lesson called the 'study lesson' over a series of meetings. This lesson is then taught by one of the teachers in the group and videoed or observed by the whole team, with a particular emphasis on the student responses to the lesson. The observed lesson is then discussed at a meeting of the group where it is further developed on the basis of the student responses and consequently re-taught to a different group of students. This may then be repeated or a different lesson developed. The size of a lesson study group can vary considerably but generally involves up to five teachers with a minimum of three. In Japan, lesson study is the main form of professional development for teachers and each teacher would expect to be involved in several lesson study cycles during the course of an academic year.

A superficial view of lesson study is that it is a process centred on developing the 'perfect' lesson on a particular topic for students of a specific age; however, this is not the intention of the process. In contrast, a number of subtle aspects support the development of a process in which teachers are led to a deep engagement with the processes of teaching and learning. Consequently, engaging in a lesson study cycle prompts teachers to reflect on their own approaches to the processes of teaching and learning and to develop their own practices in ways that are meaningful to them in their working contexts. A key aspect of lesson study identified by Stepanek and Appel (2007) is the identification of an overarching aim. This aim should be closely aligned with the broad aims and intentions of the school and may be linked to mission statements or plans for improvement. An example might be: 'Our students will become independent thinkers (learners) who enjoy working together to produce creative solutions in unfamiliar situations'. This over-arching aim would then be evident in each study lesson that the group collaboratively develops with all of the lessons contributing to its achievement in some way. In a wider sense, this may influence other lessons that are taught and therefore can lead to a wider adoption of the strategies and approaches developed for the study lesson. Another key aspect of lesson study is the development of a lesson which makes the students' learning visible to the observers (Burghes & Robinson, 2010). This can be described as an 'open approach' and involves developing tasks for the students which stimulate responses that reveal their engagement with the problem presented and their thinking as they present their solutions. Typically these open tasks may provoke a range of responses and the teacher will observe the students' responses during the course of the lesson and build an appropriate sequence in which they can be presented to the class. During the course of the presentation of the responses, the students are expected to interrogate their fellow students' work and to validate its mathematical integrity and truth.

This summary of lesson study constitutes the basic pattern of the approach; however, there are many variations on this both in Japan and in other contexts. These variations result in different levels of engagement with the principles underpinning lesson study. Engagement with the principles of lesson study is neither straightforward nor easy, and learning and change through engaging with this process takes prolonged time periods.

Participating in professional learning communities

To consider the nature of the professional learning community in which participants were involved through the process of engaging with lesson study, the theoretical frameworks of communities of practice developed by Wenger (1998) and communities of inquiry developed by Jaworski (2004, 2006) are used. Wenger's notion of communities of practice focuses on the concepts of practice and identity and the interrelationships between them. Learning is seen as developing participation in practice and involves investigating how the participants learn to participate in the practices of the community. Application of a community of practice approach to the analysis of teachers' professional development allows exploration of the possibility that any one participant might aim to identify himself or herself in multiple ways, in the service of more than one (social) purpose. Teachers engaged in professional development within a lesson study group are involved in positioning themselves as a participant in the lesson study community in which they are engaged with social 'work' associated with developing the study lesson and developing their professional identities within the group. Use of this framework does not imply that participants are consciously aware of this work as it may be entirely tacit and embodied in the act of participation itself. Instead, it involves the teachers in positioning themselves 'as' someone in the lesson study group which can be conceived of as a community of practice. Wenger, from a sociocultural perspective, notes that practice "is a process by which we can experience the world and our engagement with it as meaningful" (1998, p. 51). From this perspective, in the case of the teachers in the study, the focus of their participation in lesson study was on the practice of lesson study as a vehicle for developing their practice as effective teachers of mathematics. They were seeking to develop their practice as teachers of mathematics and develop meaningful approaches to that practice that increasingly developed their identities as users of effective mathematics pedagogies.

Jaworski (2004, 2006) develops Wenger's (1998) notion of communities of practice and applies it to professional development work with teachers. She sees teaching as a social process in which teachers are practitioners and in which their learning is conceptualised as developing their identities as teachers through their participation in a community of practice. In addition, Jaworski suggests that this development can perpetuate the status quo of practice within a group of teachers that settles down to conformity with classroom practices that are essentially harmonious but which are 'not necessarily providing effective learning opportunities for all students' (p. 190). Jaworski argues that for practice to develop in ways that offer improved learning opportunities for students, teachers need to be viewed as learning in practice or "learning-to-develop-learning" (2006, p. 191). This involves a conception of teachers being engaged in 'critical alignment' of practice in which they seek "to develop, improve or enhance the status quo" (2006, p. 191) through their involvement in a community of inquiry. Therefore, inquiry can develop from being used as a tool to enable teachers and educators to explore key questions and issues in practice to becoming a 'way of being' through which participants in a community develop their practice (Jaworski, 2006).

Effective Mathematics Pedagogy

Mathematics holds a key role both in terms of individuals' abilities to function in society and in the future educational and employment opportunities which are available for them (Ernest, 2010). However, many students continue both to struggle and become disaffected with mathematics. Therefore in recent years, both in the United Kingdom and internationally, research has sought to define the pedagogical practices which lead to effective mathematical learning. A large scale study by Askew and his colleagues (1997) based in the UK identified and investigated different aspects of effective teachers of numeracy. The study aimed to understand effective teachers by developing a model of their classroom practices, individual beliefs and knowledge of mathematics and mathematics pedagogy. Data were gathered through questionnaires, observations and interviews. Askew et al. identified three types of mathematics teaching and characterised 'connectionist' teaching as the most effective.

Another model for effective mathematics pedagogy was developed by Swan (2006) in conjunction with a design research project that sought to improve learning in mathematics for students at tertiary colleges who had a history of finding success in mathematics problematic. Within this study, Swan developed a professional development course for teachers which supported them in developing pedagogies valued by their students. Other research (e.g., Anthony & Walshaw, 2009; Hiebert & Grouws, 2007) on effective mathematics pedagogy includes research syntheses that draw together the findings of international research studies to develop a rich knowledge base of pedagogical practices that contribute to positive outcomes for students. Anthony and Walshaw use their findings from the synthesis to develop a framework including ten principles of effective mathematics pedagogy. These different explorations of effective mathematics pedagogy reveal some commonalities and these are used as a theoretical base for this study. The following section summarises these commonalities.

An important aspect of effective mathematics teaching is developing tasks and classroom activities which focus on key mathematical ideas (Anthony & Walshaw, 2009; Askew et al., 1997). In particular, tasks should be designed to engage students in furthering their understanding of important mathematical concepts and relationships. It is also important that tasks are posed in such a way that learners are able to access their prior learning and make connections between their previous experiences and important mathematical ideas (Anthony & Walshaw, 2009; Askew et al., 1997; Swan, 2006). In such a way, instruction builds on the learners' thinking, and misconceptions and mistakes are addressed and used as learning opportunities. Providing opportunities for students to explore the connections between mathematical ideas, differing strategy solutions and multiple representations also supports effective mathematics learning (Anthony & Walshaw, 2009; Askew et al., 1997; Muir, 2006; Swan, 2006).

Effective mathematics pedagogy requires both a focus on developing learners' mathematical knowledge and the development of an effective classroom community (Anthony & Walshaw, 2009; Hunter, 2009; Muir, 2006). This involves developing a learning community which is responsive to the learners' needs. Anthony and Walshaw highlight factors such as carefully structured mixed attainment grouping, providing opportunities for individual, paired and group work, and the provision of a supportive environment that develops student autonomy. Classroom discourse also has an important role in effective mathematics pedagogy. This includes the facilitation of purposeful discussion which challenges children's thinking and a focus on developing student use of explanatory justification (Anthony & Walshaw, 2009; Askew et al., 1997; Muir, 2006). Children also need opportunities to learn how to agree and disagree and how to question their peers to make sense of student provided explanations during small group work and whole class discussions (Hunter, 2009).

A critical factor in developing effective mathematics pedagogy is teacher knowledge and learning. As Anthony and Walshaw (2009) state, "how teachers organize classroom instruction is very much dependent on what they know and believe about mathematics and on what they understand about mathematics teaching and learning" (p. 157). Sound mathematical subject knowledge is a key factor in supporting teachers in identifying the connections between different areas of mathematics. This in turn supports the teachers in assessing students' understanding of mathematical topics (Anthony & Walshaw, 2009; Askew et al., 1997; Muir, 2006). Also important is knowledge of how students learn mathematics including their expected progression and an understanding of potential obstacles to learning or misconceptions. This supports teachers to make sense of student explanations and use questioning to facilitate learning (Anthony & Walshaw, 2009; Askew et al., 1997). Another key aspect of teacher knowledge is the knowledge of teaching approaches that support students to develop rich conceptual understanding of mathematics (Askew et al., 1997). These three key factors – mathematical subject knowledge, knowledge of the ways in which students make sense of mathematics, and knowledge of ways of teaching mathematics – have also been identified in the literature as central to effective professional development for teachers of mathematics (Joubert & Sutherland, 2008).

Developing Reflective Practice

Developing reflection on practice is a key component of sustainable professional development (Back et al., 2009). This section draws together the three elements described above as central to effective professional development and examines how they may support teachers to reflect on their practice. In developing the capacity to reflect on practice the first and essential step is that of noticing and being aware of relevant phenomena. This process of noticing only develops through engaging with it and involves both knowledge of relevant aspects of a given situation and also an increasing ability to be aware of them and reflect on them in the immediate context of the classroom. In developing reflection on aspects of teaching and learning mathematics, teachers in a lesson study group need to develop understanding of the pedagogies that they are using, to have knowledge of the mathematics involved and to be aware of the ways in which the children make sense of the mathematics. Furthermore, they also need to notice how their practices resonate with, or are in conflict with, the ideas of effective mathematics pedagogies which they are seeking to adopt. This idea of noticing is captured in John Mason, Leone Burton, and Kaye Stacey's (2010) seminal text on developing mathematical thinking now in its second edition:

> None of the processes or activities I have mentioned is unusual or new. They happen spontaneously inside everyone to varying degrees, often below the level of awareness. By becoming aware of them, and seeing how effective they can be in appropriate circumstances, they should begin to happen more frequently and more intensely than before. (p. 106)

These authors equate the process of noticing thinking as similar to developing an internalised tutor who monitors that thinking for you. Involvement in the lesson study process, in particular the reflective discussion with colleagues about the study lessons, can work in a similar way for the teachers.

To summarise, this paper draws on research literature to examine how engaging in lesson study may facilitate sustainable professional development through the development of learning communities in which both awareness of effective mathematics pedagogy and reflection on practice are promoted.

Methodology

Research Context

The overarching aims of the project were to investigate and evaluate whether lesson study could be used with teachers as a form of research-based professional development and as a form of classroom-based research. The study involved four groups of primary teachers within England and the Channel Islands who were interested in implementing the lesson study process within their schools during the 2009/2010 school year. All schools and participants involved in the study were assigned pseudonyms to ensure anonymity. The schools included a mixture of urban, rural and suburban contexts with students from a range of socio-economic and ethnic backgrounds, and the teachers had varying levels of experience. Specific details of the participants from each school are shown in Table 1.

Table 1
Schools and teachers at each year level

	EY	R	Y1	Y2	Y3	Y4	Y5	Y6	HT
Beaumont				1	2				
Hillview					2		1		
Kingsland					2	2	2	1	
Hamilton	1			1	2		1		1

The sample was an opportunistic one of willing schools drawn from the group of schools with which the research team was involved in research and development activities.

The lesson study approach had not been used previously at any of the schools; however, two of the schools had some degree of experience in using models of collaborative practice. The teachers at Beaumont School had engaged in a paired collaborative observation approach during the previous school year. Additionally, the teachers from Hamilton School had previously worked collaboratively and across year groups in a variety of subject areas. At the other two schools, Hillview School and Kingsland School, collaborative approaches to planning or teaching had not been previously used.

Within the study, during the lesson study process each group of teachers worked as a professional learning community within their own school. The

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initial step for each group was to agree on an over-arching aim which was relevant to their school context. Developing this aim was a goal of the first meeting and collaborative agreement from the group was sought. Following the initial meeting the teachers decided on an area of focus for the study lessons. This area was chosen typically as either an area in which the children at the school had difficulties, or alternatively an area which the teachers felt less confident about teaching. The group then collaboratively planned the study lesson. Following this, the lesson was taught in one classroom while the rest of the group and researchers observed. The group then engaged in an in-depth analysis and discussion of the study lesson and consequently the lesson was re-planned based on the observations from the lesson and re-taught and observed in a different classroom. The second lesson was then the subject of another in-depth analysis and discussion and the overall engagement of the teachers in the lesson study process was reflected on by them. Members of the research team were present at the majority of the planning meetings, the presentation of the study lessons, and the follow-up meetings.

Evidence of the practice that formed the focus of the study was gathered through observations of the meetings held and the study lessons taught by the teachers involved. Data were collected through field-notes and video and audio recordings. The findings of the case studies were developed through on-going and retrospective data analysis. The video and audio recordings were wholly transcribed and through an iterative process using a grounded approach, patterns and themes were identified.

The findings are presented as accounts of the learning and development in collaboration with the teachers involved and their personal reports of changes in their attitudes, identities and actions in their classrooms. Evidence of this has been triangulated from both our observations of lessons they have taught and developed and the discussion following the study lesson.

Findings and Discussion

In this section, a description of how engaging in the lesson study process is facilitated the teachers to notice key aspects of mathematics pedagogy is provided. Analysis of the data revealed five central themes that relate to the development of effective mathematics pedagogy. These were: key mathematical ideas; prior learning and misconceptions; developing connections; facilitating an effective learning community; and classroom discourse. These themes are presented as three sections in the findings due to their interwoven nature. For example, the discussions which focused on the key mathematical idea of the equal sign as equivalence were also linked to discussion of children's prior learning and misconceptions. The findings are presented in relation to the inter-linked five themes.

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Focusing on Key Mathematical Ideas, Prior Learning and Misconceptions

During the post-lesson discussion it was evident that the in-depth observation of the students' responses to the classroom activity facilitated the teachers to reflect on student understanding of key mathematical ideas. During some of the discussions, the key mathematical ideas that were highlighted had been the specific focus of the lesson. For example, at Hillview School the lessons were designed to facilitate student understanding and justification of the commutative property. Following the first lesson with children aged seven and eight years, the teachers provided detailed observations of the children's responses to the task:

> Ellen: Iris and her partner were looking at subtraction without even being prompted to do it because they said straightaway 'it doesn't work for subtraction but it is working for addition'. They were very confident in the addition and less confident with the division. This group in particular were happy with the multiplying but they couldn't then..., they weren't so comfortable with the division.

Melissa also offered a specific example of how a group of students responded to the task focusing on the key mathematical idea of the commutative principle:

> Athena, Christopher and Lauren, they were doing it with the subtraction. They did four minus one equals three and one minus four [puts hand up to indicate question mark] and then they said, and Lauren said 'so that is subtraction done then, that doesn't work' and she did it for one, if it doesn't work, it doesn't work whereas she then said 'actually five times three and three times five works hmm' and then they did something with twos and then she said 'does it only work with twos though?' So then they tried with a different number, that was Lauren who said that so she had got the idea that if with one it didn't work [indicates throwing something away] she just discarded that straightaway and went straight on to the, on to the next one.

In other instances during the discussions, the key mathematical ideas were not the specific focus of the lesson but were perceived by the teachers to be influencing the children's learning. During the discussion, there was also evidence of the teachers' developing ability to notice children's misconceptions. For example, in the second study lesson at Hillview in a class of nine and ten year olds, the teachers commented on the difficulties the children had in representing the commutative principle as a number sentence (for example, $6 + 5 = 5 + 6$) due to their limited understanding of the equal sign. Ellen stated:

> They seem to find it really hard to write one continuous number sentence.

Monica further explored this commenting:

> They are still not understanding the proper meaning of the equal sign, I would say, or perhaps they are but when it comes to applying it in a context then they're not.

In some instances the discussions of children's learning of key mathematical ideas supported the teachers to reflect on how tasks could be structured to promote better learning. This also included reflection on the reasons for children's misconceptions related to classroom activity. At Beaumont School the study lessons were focused on developing children's ability to solve multi-step word problems. However, two activities again highlighted children's understanding of the equal sign. During the first activity, students were asked to make a specific number using a number sentence which was then represented as equivalent to another number sentence (for example, $45 = 20 \times 2 + 5 = 20 + 25 = 45 - 0$). The second activity prompted the children to correct incorrect multi-step equations some of which included balance equations (for example, $8 + 9 = 7 + \_\_$). The teachers observed the students' initial difficulties in correcting the balance equations and subsequent discussion between the students that arose. Early in the post study-lesson discussion, Zara highlighted the activities, both of which were related to understanding the equal sign:

> We still had to keep coming back to that, that the two sides of the equation had to balance, how much time we have done that, and even given that they had done that in the first part of the lesson [referring to the first activity]. They don't seem to see that as the same as being presented with one, you know because that first activity is doing the same thing, isn't it, and they see it there but they don't seem to see if you write it down for them but I think they got some good discussions out of all the correcting the mistakes.

This initial analysis of how the activities were structured to facilitate children's thinking about the key mathematical idea of equivalence led to further analysis and reflection from a different member of the group.

> Rebecca: I think maybe because we historically present children with a lot of things with the answer just being one box that sort of one where they had to look maybe provoked that thinking a little bit more. You know at the beginning where they said something, something equals and then the next child does equals, I don't know, when I look at it now I think it is a fantastic activity and a fantastic assessment thing to see where they are coming from (...) but maybe they are just seeing and the next one, and the next one, and the next one and now it's my turn and it's my turn and they don't actually see the equal sign whereas this question here and that one here in particular really made them think about the idea of balance.

These discussions highlight how observing children's responses to tasks through the lesson study process supported the teachers to notice aspects of effective mathematics pedagogy such as key mathematical ideas, the role of prior learning, and misconceptions.

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Developing Connections

The teachers in this study were supported in noticing the connections between mathematical ideas, differing strategy solutions and multiple representations through the lesson study process. A common feature across all the lesson study groups was the identified need to facilitate children to make connections between physical representations or concrete materials and mathematical ideas. For example, Mark commented:

> They need lots of different representations and practical experiences of concepts.

The connections which the children developed between the concrete materials and mathematical ideas were also highlighted as supporting and deepening the children's understanding. For example, in discussion of the children's developing understanding of the commutative principle, specific examples were provided of the physical representations supporting the children's understanding:

> Ellen: I think it was Iris and Andrea, they were talking about the objects and they suddenly became sweets, if we have got three sweets we can't divide them between seven people so they were then jumping ahead and moving that relationship on, that was good.
>
> Melissa: That is really good.
>
> Ellen: I think it was the resources that prompted that.
>
> Monica: I think even with John if he hadn't seen it on the grid [referring to an array], he probably wouldn't have got it as quickly as he did.

Close observation of the students by the teachers during the study lesson also supported them to notice varying connections in student provided solution strategies. For example, in one lesson during the whole class discussion a student provided a correct solution; however, the representation and method was not linked to the problem context. This was identified by the teacher and she prompted the student to reflect on this. Another teacher in the group highlighted this as an example of good practice in terms of facilitating student learning:

> Rebecca: Afonso, you know he had done it a different way than to the actual problem because you could have just said 'yeah that is great well done' but you said 'but does that actually model what the problem was about?'

Similarly, missed opportunities in developing connections between student-provided solution strategies were also a feature of the discussions. For example, a student provided a solution that involved drawing individual flowers to solve a division problem which was critiqued by another student as inefficient. During the following discussion between the teachers this was highlighted as a possible learning opportunity:

Rebecca: Bethany when she said ‘oh that took ages’, it would be nice to sort of pick up on that so you have got the halfway house, you know because it is a very inefficient method really. It’s something they go through but maybe they just realise that they can just write five in each box rather than drawing each thing because that would have been a nice opportunity to pick up on what Bethany said. You know maybe some sort of race I put here, where you have someone drawing and someone finding a quicker way.

These examples suggest that through involvement in the lesson study process, teachers can be supported to notice a range of connections within the mathematics classroom.

Facilitating an Effective Learning Community and Focusing on Classroom Discourse

Discussions of effective grouping strategies were a common feature across the lesson study groups. However, there were differing levels of engagement with ideas about how to group children effectively. For example, at Hillview School the discussion touched on grouping but did not engage deeply with how children could be facilitated to work effectively within a group:

Ellen: We keep changing the groups around trying to find the appropriate groups for them to work in and some groups work well and others I don't think will.

Monica: Some people never work well in a group anyway, do they? So whichever group you put them in, they are always going to struggle.

In contrast, at Beaumont School the discussion involved consideration of how to facilitate the children to work effectively in their groups. For example, Michelle described her inclination to put the higher attaining children in a group together although she was unsure of this in terms of inclusion:

> I would be so tempted to put, because there is four in there who really do get it, to put them in one group, but I know that that really wouldn't spread it then.

Rebecca responded to this comment by suggesting pedagogical actions to support effective group work using mixed attainment groupings:

> It is how you train those other children to question, the more able so they are drawing the learning out of the less able child and that is tricky.

This was followed by an exchange which focused on how social norms could be developed for collaboration during group work and class discussions:

Rebecca: I think it is trying to get everybody having that chance in terms of the discussion and to try and get your brighter ones, because you have got a lot of bright ones there, and how do you train them to get everybody in their group responding so you shift their role which broadens them wider because then they have to think okay how do I explain this in a different way? So they are trying to get others in the group of four, say Harper, to explain it and they are trying to work out what is going on in her head so they can ask the questions to draw out her explanation and giving them that as 'oh it is not sitting back and waiting for the slow person to do it' but giving them that important role.

Michelle: You can see within the whole class, it is Shaun and it is Fraser, and they are there because they are just full of it and they just want to but it is just trying to get them to hold back.

Rebecca: It is 'I know you know the answer but how can you get them to explain it really clearly, so you have to listen really carefully and work out what they are doing' so you are sort of shifting the emphasis.

This exchange exemplifies how engaging with the lesson study process can support teachers in considering factors of effective mathematics pedagogy such as carefully structuring mixed attainment grouping and facilitating the development of social norms for effective group work.

Another significant aspect during the teachers' discussion was the focus on students' developing questioning skills. The teachers' observations included questioning both during small group work and in whole class discussions. For example, Ellen identified an instance where questioning was used during small group work to resolve a mathematical disagreement:

> With Alan's group and they were working with the negative numbers, they got their addition wrong which then provoked, it certainly provoked the argument between them, well does it actually work? Does it not work? And they had to really delve deeply to work out whether it was an exception or whether it was the norm.

As the teachers were working in groups across year levels, another discussion centred on the development in questioning skills across the age groups. For example, Melissa contrasted the use of questioning from her younger seven and eight-year-old students with the questioning she observed in the class of nine and ten-year-old children during a whole-class discussion:

> When Julia was talking some of the others put their hands up and they questioned her directly and she explained her thinking back to them. I think that worked really well. (…) A number of them did that and I don't think ours have got that language, sometimes they need a bit more modelling and structuring of how to question the person who is explaining because they do sometimes say 'I don't understand' rather than asking them a specific question.

The excerpt highlights how the teachers were able to use the observations during the study lesson to reflect on their own classroom practice and further analyse how specific areas could be developed.

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Another area of significance, which the teachers focused on during the post study lesson discussion, was the structuring of explanations to develop an effective whole-class discussion. This meant that the teacher had to listen carefully to children's responses during small group work and carefully select the groups to contribute to the discussion so that it would be structured in a logical manner in order to develop the children's mathematical reasoning and thinking. Through engaging in the lesson study process the teachers analysed how the whole-class discussion facilitated the children's development of mathematical concepts. For example, Ellen critiqued the choice of a group to share during the whole-class discussion as the group had concentrated on the inverse property rather than the focus of the lesson which was the commutative property:

> I think avoiding the inverse could be something to do next time so to be specific when the example came up with the group from the front there, they weren't investigating the commutative law, and they were investigating the inverse and I think what I would have tried to have done is just said that's fine and try to direct them but not bring them up to the front to make their explanation.

This also led to the teachers considering the need to think of children's possible responses to tasks in order to best facilitate a whole class discussion.

Discussion of the choice of children to share during the whole-class discussion was also an opportunity to examine how the discussion might be used to scaffold all the learners' understanding. Examination by the teachers of who was selected to share during a whole-class discussion and the reason behind this also provided opportunities to share the pedagogical actions that were used to develop inclusion and student autonomy. For example, Zara highlighted her selection of a lower attaining student to share a solution strategy and how she supported her to rehearse prior to explaining to the whole class:

> Rebecca: There is still an awful lot of them doing ding, ding, ding [indicates counting on fingers].
>
> Zara: But then that is also why I brought out Madeira because she did the five, ten, fifteen.
>
> Rebecca: Yes and how confident was Madeira because she was one of the ones who used to need a lot of support, didn't she? It was excellent.
>
> Zara: Yes and she still does. We rehearsed that in her place because I caught her doing it and said 'oh I want to share with everyone what you have just done'.
>
> Michelle: So you were rehearsing it to give her the confidence?

In this way, engaging in the lesson study process offered opportunities for the teachers to notice pedagogical actions that developed a classroom environment which was inclusive and responsive to children working at different attainment levels.

Developing Reflection on Practice and Professional Knowledge

In this section, a description is provided of the teachers' awareness of their professional development as users of effective mathematics pedagogies through the process of involvement with their lesson study group. Analysis of the data revealed two central themes: firstly the teachers commented on how lesson study supported them in working collegially and reflectively and secondly they identified ways in which their professional knowledge had developed. The findings are presented in relation to these two themes.

Collegiality and Reflection

Teachers from all the groups commented on how the process of lesson study gave them opportunities to work together with other teachers. This broadened their insights into the topic, its teaching, and the sense that children might make of the learning opportunities involved. For instance, Monica commented:

> I think it gave me an insight as well and I think, you know when you are planning a lesson on your own and it is all in your head, it is one thing but when you are verbalizing it in a group and discussing it, there is so much. It gives you a depth that I don't feel I achieve when I am planning a lesson on my own necessarily.

And also:

> Michelle: We are also having more of an understanding of the development of learning these concepts which has been good, it has been great to see that development today and it can only be beneficial.
>
> Zara: It shows you more of that journey and you are picking up things from other people which you think I could take that, I could adapt that, and that is another way of doing what I am doing or it's another idea of something I have not tried.

These features of collaboration often led them to reflect on their own teaching as Zara is beginning to say in the quote above. Once again, all the groups made reference to how involvement in the lesson study process facilitated them to reflect on their own teaching. For example:

> Orla: I think it definitely makes you look at other aspects of maths, not just fractions [the topic of the study lesson] and it has made me think every time I've planned lessons since like how could I do this differently this time, like look into different ways of exploring a lesson that I've done before so again maybe looking into research based on that concept.

Similarly, one of the teachers from Hillview commented:

> Ellen: We are constantly analysing what we do anyway and I think it has just given more of a focus for that because Melissa and me we could

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take a step back and look at what somebody else is doing. I think that always then makes you reflect on what you are doing yourself.

Other comments indicated that the reflective process facilitated by engagement in the lesson study prompted the teachers to engage in deep reflection on their practice and begin to develop teaching practices which align more closely with effective mathematics pedagogy:

Monica: It has made me think about my practice, I have to say and it has made me more aware of what I can improve and it has made me more aware of what I need to be including in my lessons and kind of working within a broader structure. Whereas I was always good at questioning and pitching different questions to different children and whereas now I think I am doing it in a deeper and broader way. It has kind of widened out a lot and I am much more receiving, although I always wanted to get things out of the children but now it is a bit more different. Whereas before I saw things as right or wrong, I am much more focused on the process now rather than whether it is right or wrong.

This shows a teacher who is giving considerable thought to her practice and ways in which to change it in order to support her students to learn mathematics.

Professional Knowledge

As previously stated, the literature identifies three aspects of professional knowledge that support teachers in their endeavour to teach mathematics effectively. These are the teachers' own knowledge of mathematics as a subject, their knowledge of ways of teaching mathematics, and their knowledge of the ways in which children make sense of mathematics (Joubert & Sutherland, 2008). All three aspects are crucial to effective mathematics pedagogies and all three were mentioned by the teachers in the course of their discussions after the lessons they observed.

Teachers' knowledge of mathematics is not dealt with directly in the lesson study approach to professional development; nonetheless, each of the groups mentioned how much they felt they had learnt about the mathematics through the process of their engagement. As one of the teachers said:

Judith: It certainly raises your awareness about the importance of mathematical subject knowledge and relevant terminology and how much you should be using and what you should be using and why. It means that instead of making some glib comment about something that you think is mathematically correct, you would take the time to make sure and get your maths dictionaries out, check those, build that in.

This was supported further on in the discussion by another member of the group who said:

Mark: It's raising your awareness that you might not know, there are things

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> you don't know that you don't know, that you might teach without even knowing you're teaching them which means you can teach misconceptions which can be tricky … so if you've got a different attitude to think maybe I don't know then that immediately opens it up.

The focus of lesson study on the teaching of a specific lesson on a specific topic results in an in-depth consideration of the ways of teaching mathematics. This was an aspect highlighted and discussed by all the groups within the study in relation to their developing professional knowledge. In many cases this led them to more general discussion about how to teach mathematics. For example, Ellen reached some general principles about using resources to support the teaching and learning of mathematics from her group's use of resources in the study lesson:

> How much the resources are very useful and how I don't generally think I use them enough. … I would like to have supplies of resources which the children could then choose to use, not to impose them on them but to at least have access.

In another group, one of the teachers became interested in the connections that were being made between the work they did with the children on fractions and other mathematical topics, especially measures. As she said:

> Judith: And it shows you more of that journey and you are picking up things from other people which you think I could take that, I could adapt that, and that is another way of doing what I am doing or it's another idea of something I have not tried. For me making sure that we had a range… that we looked at grams, that we looked at millilitres and we looked at centimetres and so we were trying to draw together lots of areas of mathematics to build on the previous knowledge, to really think about the application, … skills that are notoriously quite difficult and our children find most challenging.

This statement is also linked to developing teacher knowledge of how children make sense of mathematics. This was also a feature that was apparent across the groups during the discussions. As the groups of teachers were involved in teaching children of different age groups, this led to some discussion about the progression of understanding of mathematical concepts. This is illustrated by the following excerpt which looks at the differences in understandings of multiplication between a class of 7-year-old children and another class of 9-year-old children:

> Melissa: I think I would discuss with them what they actually thought multiplication was to get them to go right back and kind of …
>
> Monica: Going back to 'lots of'…
>
> Melissa: Seeing it as lots of and seeing it as repeated addition and spending a little bit of time looking and representing multiplication in different ways.

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> Monica: I think there is a bit of a similar problem as Ellen's class that they probably at some level have an understanding that it's 'lots of' and at some level because we have done arrays and they can see it somehow but when it comes to applying it at an everyday level, it's not secure enough to solve the problems that they need to solve.

However, this was most evident in the teachers' focus on predicting the children's responses to the learning opportunities that were offered in the study lessons. As Melissa said:

> Melissa: We spent some time trying to predict how the children were going to perform and true to form they always perform. You know we did anticipate a lot of it but there is always something that you haven't anticipated which then you have to think on your feet and look through and change your original plan. For me it was just fascinating to see somebody else do it.

In addition to predicting the children's responses to the tasks, the teachers developed knowledge of the learners from in-depth observation of the children's work on the mathematical tasks during the study lessons. Another common feature highlighted by each group during the discussion was the value of observing the learners. As Ruth said:

> The thing with observing which is really helpful is that you kind of put yourself in the position of the child whereas when you're teaching, yeah, you are thinking about their learning but you've also got to think about what you're doing and how you're sort of delivering it. With observing you listen to the teacher as if you were that child and so you really see how they are learning.

Ruth also commented on the value of having other teachers as observers when she taught the study lesson:

> It's useful though having people at the tables listening because you obviously can't listen to every single child and then having that feedback, I mean that amazing talk in your classroom to learn from.

In response to this, Mark added:

> Very useful I know. You've got one person in every group, no child can escape, the ideas they have are going to be captured which is fantastic.

In several of the groups this level of observation proved very encouraging to the teachers who were impressed by the focus of the children's talk on the mathematics of the lesson.

This section has illustrated how through the process of engaging with lesson study, teachers are provided with opportunities to work collegially to reflect on their own practice and develop their professional knowledge.

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Conclusion and Implications

The results discussed in this paper support the argument that engaging teachers in lesson study is an effective way to support them in sustaining their professional development through facilitating awareness of effective mathematical pedagogies and the teachers' use of these pedagogies in their teaching practice. Research studies (Anthony & Walshaw, 2009; Askew et al., 1997; Swan, 2006) investigating effective mathematics pedagogy highlight the key factors of tasks which: focus on important mathematical ideas; link to prior learning and draw out misconceptions; develop connections; focus on effective discourse; and establish a learning community which is responsive to learners' needs. The teachers in this study demonstrated increasing understanding of strategies that supported their students' learning of mathematics, both in the lessons that they prepared as a group and in their reflections on the lessons that they taught and observed. The discussion that formed an important part of the lesson study process gave the teachers opportunities to articulate their observations about these issues and so helped them to notice relevant phenomena. They became the drivers of their own development as they increasingly sought to adopt strategies that encouraged more of their pupils to engage meaningfully with mathematics. This led them to try to engage all the children with mathematics whatever their attainment levels and to adopt strategies that encouraged discussion and mathematical reasoning. In this way, they began to adopt more approaches to their mathematics teaching that reflected the common attributes identified in the literature as effective mathematics pedagogies.

A key indicator of effective sustainable professional development is the development of networks of teachers that focus on student learning and facilitate reflection on practice (Back et al., 2009). Evidence from the study indicates that engaging in the lesson study process supported the teachers to work in a collegial manner through developing the lesson plan, observing the lesson and engaging in in-depth discussion which followed the study lesson. This collaboration also supported them to reflect on their own teaching practice. In this way, as Jaworski (2006) argues, inquiry developed from being used as a tool to enable the teachers and educators to explore key questions and issues in practice to being a 'way of being' through which participants in a community developed their practice. As the communities of practitioners engaged in lesson study through the process of developing the study lesson and its following evaluation, the members of the community addressed: issues related to the mathematics involved in the lesson; ways of teaching mathematics; ways of developing their practices; and ways of developing their identities as mathematics teachers in order to address the issues they had identified. Therefore, as the teachers engaged with the practice of lesson study in order to develop the meaning of the processes of teaching and learning mathematics, they also developed their own identities as teachers of mathematics and members of a community of inquiry engaged with lesson study.

Engaging with the process of lesson study is time intensive and requires significant contribution from the teachers involved in the groups. However, the findings of this study indicate the potential benefits of engaging in lesson study through the facilitation of awareness of effective mathematics pedagogy and the development of professional knowledge and reflection on teaching practice. Analysis of further cases of teachers engaged in the lesson study process would support further investigation into its potential as a vehicle for professional development.

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