# Random Effect Models For Repairable

# System Reliability

by

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# ABSTRACT

Random Effect Models for Repairable System Reliability

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The practical motivation for the work described in this thesis arose from the development of a new Jaguar car engine. Development tests on prototype engines led to multiple failure time data which are modelled as a non–homogeneous Poisson process in its log–linear form.

Initial analysis of the data using failure time plots showed considerable differences between prototype engines and suggested the use of models incorporating random effects for the engine effects. These models were fitted using the method of maximum likelihood. Two random effects have been considered: a proportional effect and a time dependent effect. In each case a simulation study showed the method of maximum likelihood to produce good estimates of the parameters and standard errors. There is also shown to be a bias in the estimate of the random effect, especially in smaller samples.

The likelihood ratio test has been shown to be valid in assessing the statistical significance of the random effect, and a simulation exercise has demonstrated this in practical terms. Applying this test to the models fitted to the Jaguar data gives the proportional random effect to be significant while the time dependent random effect is not found to be significantly different from zero. This test has also been demonstrated to be of use in distinguishing between the two models and again the proportional random effect model is found to be more suitable for the Jaguar data. Residual analysis is performed to aid model validation

Covariates are included, in various forms, in the proportional random effect model and the inclusion of these in the time dependent model is briefly discussed. The use of these models is demonstrated for the Jaguar data by including the type of test an engine performed as a covariate. The covariate models have also been used to compare engine phases. A framework for extending the models for interval censored data is developed.

Finally this thesis discusses possible extensions of the work summarised in the previous paragraphs. This includes work on alternative models, Bayesian methods and experimental design.

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# ACKNOWLEDGEMENTS

# Author's Declaration

At no time during the registration for the degree of Doctor of Philosophy has the author been registered for any other University award.

This study was jointly financed by Jaguar Cars Limited and QR funding at the University of Plymouth.

A programme of advanced study was undertaken and relevant scientific conferences and seminars were regularly attended. Papers are being prepared for publication.

Conferences and Courses Attended:

Royal Statistical Society Conference, University of Newcastle, 1994.

Computational Statistics (short course), University of Glasgow, 1996.

Royal Statistical Society Conference, University of Surrey, 1996.

Signed ....... J. Baker .......

Date ....... 21 / 11 / 97 .......

# CHAPTER 1

# Introduction

This project has been carried out in collaboration with Jaguar Cars Ltd. of Coventry, UK. Jaguar have developed a new engine, code named AJ26 which was released during September 1996 in the *XK8* sports car. The aim of this study is to provide statistical methodology to help development for future projects. The following sections in this chapter describe the current engine development processes and the data yielded by them, and conclude by showing where statistical methods could be used to improve them. Chapter 2 gives details of current methodology for reliability analysis while Chapter 3 details a non-homogeneous Poisson Process model with a random effect and its application to engine failure data. The basic random effect model is then expanded in Chapter 4 to include other random effects. The residuals for these models are discussed in Chapter 5, and the inclusion of covariate effects is described in Chapter 6. In all cases the methodology is applied to the Jaguar engine data. A hierarchical modelling procedure for the Jaguar data, based on the models from the three preceding chapters, is given in Chapter 7. A model for interval censored data is described in Chapter 8. The conclusions that are drawn from the modelling and their implications to the engine testing procedures are discussed in Chapter 9 along with suggestions for further study.

## 1.1 The Design and Verification Process

In order to secure a greater market share it has been recognised at Jaguar that the quality of their cars, in terms of reliability as well as performance, must be at least as good as their competitors. In recent years Jaguar have been working hard to improve the reliability of their cars and these improvements have led to increase in sales. The improvements are being achieved by changing the development methods for both the engine and the car as a whole. Traditionally, validation tests on a part were carried out only once, taking no account of the variation between parts. If the part passed the test then it was assumed to be suitable for use in the final product. The AJ26 uses a new development process, which incorporates statistically designed experiments to investigate problems, and Failure Mode and Effect Analysis (FMEA) to indicate where these problems lie. A flowchart representing this process is shown in Figure 1.1 (overleaf), and the explanation is as follows:

*New Design & Design Phase*    Engine development is carried out in phases. At the start of the project there will be a totally new design and the process enters the first design phase. As development progresses on this phase, the problems encountered will lead to changes in the design. After a certain time these changes will be incorporated into a new design and the next design phase will be entered. The number and timing of design phases is predetermined at the start of the project, although changes to the plan may occur.

*Failure Mode and Effect Analysis (FMEA)*    This is a tool used to highlight where problems are occurring, or may occur in the future. It can therefore be used to show where efforts should be concentrated to improve reliability. The engine is split into a number of systems, and for each of these systems a list of all the possible problems (Failure Modes) and their causes is compiled. A failure mode can have any number of causes, and each of these has a rating, on a scale of 1 to 10, for the occurrence, severity and detection assigned to it. The occurrence score is a rating of the frequency of the problem, with 1 attributed to the rarest problems and 10 attributed to the most common. The severity score is a rating of the seriousness of the problem if it occurs, where a severity of 1 has little risk to the user and a severity of 10 could be life threatening, for example brake failure. The severity rating remains unchanged for a failure
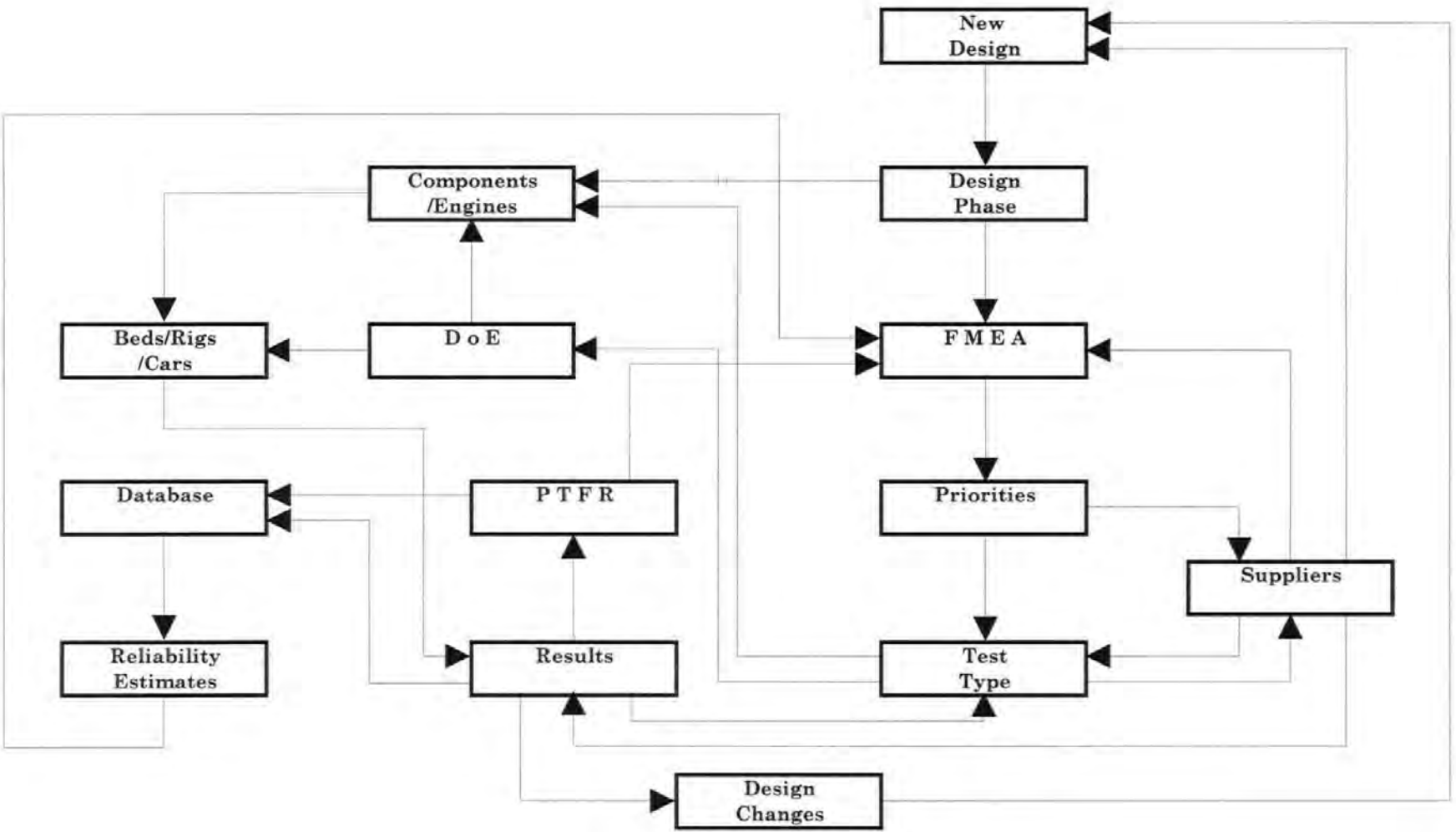
Figure 1.1: Flowchart for the Design and Verification Process.

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mode regardless of the cause. The detection score is a rating of the ease with which the problem may be highlighted during testing procedures, with 1 being easiest to detect and 10 being most difficult. These ratings are decided in a meeting attended by all people concerned with the relevant system, and although there are guidelines for assigning the ratings, the final attributes can be fairly subjective. When all the ratings have been assigned, a RPN (Risk Priority Number) is then found by multiplying the three ratings together.

*Priorities*   The priority actions for the development process can be decided by looking at the RPN values. The failure modes with the highest RPN's are those that require the most urgent action. Usually a cut-off point for the RPN's is assigned, with any failure modes that fall above this value being investigated. It is also recommended that any failure modes with a severity rating of 9 or 10 should be investigated, regardless of their RPN.

*Test Type*   The problems highlighted by the FMEA, or from previous testing, require investigation and an appropriate test needs to be carried out, either by Jaguar themselves or by a supplier. This may be done on a single component or on the engine as a whole. If the test is carried out on a component then that component is connected to a test rig, otherwise the test is carried out on the engine, which can be connected to a car or a test bed. When the engine is connected to a test bed there are a number of set simulations that can be run. These are computer controlled, and have been designed to simulate different driving conditions. Each of these simulations are accelerated from 'real-life' driving, although the acceleration factors are not known.

Initially the engines perform a Run & Rate test to check that the performance is acceptable before testing begins; the engine speed is increased in steps and the load at each of these is tested (the engine coolant temperature is kept constant). The different simulations carried out are: Reliability, Nardo, 400 hour, 180 High Speed and Hot & Cold. These tests comprise cycles that are repeated over the testing procedure. Plots of the engine speed, load and temperature for these tests, excluding the Hot & Cold test, are given in Appendix 1. Engine running that is not on any of these simulations is classed as General High or General Low according to the severity of the testing. The seven tests can then be categorised within

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two levels, High and Low, according to the demand on the engine. The demand is a combination of engine speed, load and temperature. General Low and Reliability tests are classed as low with all others classed as high.

*Suppliers*     Most of the development work is done in collaboration with the relevant part supplier. This is a sensible approach as the suppliers will have a better knowledge of available materials and processes than Jaguar.

*Design of Experiments (D o E)*  The AJ26 development process is making extensive use of statistically designed experiments for comparisons of different types of engine set-ups. Most of these experiments are $2^n$ factorial types although in some experiments one or two factors have 3 or 4 levels.

*Components/Engines*  When a new design phase is entered, there are plans to make a certain amount of new engines. Orders are placed for all the components (including spares) and after some delay the new parts will be delivered and the new engines built. Initially the engines are built by hand but as the development progresses the engine build becomes closer to the production process.

*Beds/Rigs/Cars*     All the testing that is carried out has to be done on either a test bed, a test rig or an actual car. A test rig uses only small parts of the engine and subjects them to testing whereas on a test bed the whole engine is connected to a computer and subjected to one of the running simulations listed above.

*Results*     This represents the flow of information gained from the testing procedures. Analysis of these results will help to decide the next actions. The results obtained will go forward to one or more of Design changes, Power Train Fault Report or Database.

*Powertrain Fault Reports (PTFR)*     This is a system which aims to ensure that every fault which occurs is noted, and appropriate action taken. For every fault a PTFR form is raised

and this remains in the system until any corrective action, if required, has been taken and the problem is shown to have been solved. All PTFR forms are entered onto the database.

*Design Changes*    The results from the testing may lead to a recommendation of a change to the design. The feasibility and cost of a number of alternative changes will be examined before they can go forward to the new design.

*Database*    All the information from the tests that have been run, and from the PTFR's, are entered onto a database. The data available (for each engine) is a full record of the engine's history to date, including how many hours the engine was run and on what test, and also any faults that occurred.

*Reliability Estimates*   Data can be extracted from the database in order to produce reliability estimates. At present the only estimates that have been made are the MTBF (Mean Time Between Failures) for each design phase. These are calculated simply by dividing the total number of hours run on a phase by the total number of faults recorded, for all engines taken together.

## 1.2  Jaguar Engine Data

The data used in this thesis have been obtained from the Powertrain section at Jaguar. The available data consists of engine running times and failure times from prototype test engines running on a test bed. For all data, the engines have been run on one of the seven simulations listed in Section 1.1.

The engines can be categorised by their design phase. The data set arises from the first four engine phases of the development project, and these are coded as A0, A1-1, A1-2 and A2-1. Data from later engine phases was not available in time to be included in this study. Engines from a given phase should all be of a similar build specification and quality, although

there will always be variability due to the quality of the build. It is therefore sensible to treat engines from the same phase as equivalent to each other, and to pool the data and perform the analysis appropriately.

The data, separated by phase, is given in Appendix 2. For each engine in a phase, identified by the engine number, the time of each failure is listed. These have been multiplied by a factor of $1/k$ ($k \neq 0$) for confidentiality reasons. The censoring code indicates whether the last observation represents a failure (*censor* takes the value of 0), or the end of the testing (*censor* takes the value of 1). The test code indicates whether the engine was run on high level (highly accelerated) test, when *test* takes the value of 1, or low level (slightly accelerated from normal running) test, when *test* takes the value of 0. The data have been classified in this way as on a given phase there are only one or two engines that have performed each of the tests listed previously. It is not possible from this to estimate the effect of each individual test, hence the two classifications have been made to allow some estimate of a test effect.

At the outset the only data available, for each engine, was the total number of hours run and the total number of failures occurring in that time interval, although this was initially inaccurate and difficult to access. Collection of the original data and transformation to its present, useful, form was achieved by searching through a large number of database files, the PTFR forms themselves and the records of running times for each engine. However, this proved to be an extremely time-consuming and difficult process. Before any further analysis could be carried out, the data originally entered had to be validated as the times were either not recorded on the database or did not agree with the times entered on the PTFR forms. In some cases the times of failures recorded were nonsensical; for example, a failure was reported at time x and days later a further failure would occur at time y with x > y. This necessitated an examination of the engine records in order to find the true running time on any given day. These records (each engine has its own A4 folder) contain written information on everything encountered by the engine and from this it was possible to calculate a reasonable estimate of the engine's running time. The process of validating the data (which involved spending a total of 7 months at Jaguar over the first 15 months of research) also required the following assumptions to be made:

1) A large number of failures were found to occur in the first 10 hours of running during the Run & Rate test. These have been attributed to faults in the build rather than failure of the components, and have been removed from the data.

2) For a number of engines multiple failures have been recorded at the same time. In most cases these observations relate to the same failure, and thus have been treated as a single failure.

3) A few engines in each phase have run for far longer than the others. At the suggestion of Jaguar engineers the data has been censored, at a coded time of 0.6, to avoid these engines having undue weight on the results of any analysis.

## 1.3 Analysis Requirements

The previous sections have described the development process used for the AJ26 engine and the data it yields. It can be seen that through the Failure Mode and Effect Analysis, Jaguar have a tool to highlight possible problems and also to help prioritise the testing procedures. They have also put into place data collection and storage procedures. Unfortunately, these collection and storage procedures have not been carried out as accurately as they might have been, and a large amount of time and effort has been required to correct and validate the data on file. It has also led to the data being of limited use in identifying the important effects of the different tests. On a given engine phase there are only one or two examples of each test, which does not yield enough data to estimate a test effect.

The aim of this research is to provide a methodology to model the reliability of the prototype engines. This will complement the other procedures already in use at Jaguar; the FMEA highlights possible problems, the testing and PTFR system discovers and monitors the problems and the reliability estimates will help to show how much more work will be required in order to make the engine competitive.

In order to estimate the reliability of the engine there are a number of factors which have to be taken into account. The data that is available come solely from engines running on a test bed, but ultimately any reliability estimate needs to be for the engine's performance in a car. As stated in Section 1.1, the test simulations are accelerated over normal running so any estimation technique should have a methodology that allows for this. At present the acceleration factors are not known, and this may remain the case until after the engine has been released into production and warranty data becomes available. The method, and its estimates, will still be of use however, as the warranty data will allow the test procedures to be correlated with actual driving conditions, and these can then be used for future development programmes. An updated version of the AJ26 is currently being developed and a comparison of its performance on tests with that of the current engine will be a valuable tool in assessing reliability.

# CHAPTER 2

# An Introduction to Reliability Theory

## 2.1 Introduction

The analysis of reliability data usually entails the use of time as the response variable, and the aim is generally to assess how long an item, or items, may last before failing. This chapter gives a description of the basic statistical concepts used in the analysis of reliability data. Section 2.2 introduces the notation and defines censoring, and Sections 2.3 and 2.4 describe non-parametric and parametric applications respectively. The later Sections, 2.5 to 2.7, introduce three important areas in survival analysis, namely regression models (including the proportional hazards model), Bayesian reliability theory, and the analysis of repairable systems.

## 2.2 Statistical Concepts and Notation

### 2.2.1 Inference Techniques

The notation and definitions are taken from Crowder $et$ $al.$ (1991). If $T$ is the random time to failure for a unit under test, where $T \geq 0$ and $T$ is a continuous random variable then the probability that a unit fails before time $t$ is given by

$$F(t) = P(T < t) \qquad (2.1)$$

where $F(t)$ is the failure distribution of $t$. In the parametric case, $F(t)$ is a cumulative distribution function. Conversely, the probability that a unit survives up to or beyond time $t$ is given by the survivor function $S(t)$, where

$$S(t) = P(T \geq t) = 1 - F(t). \tag{2.2}$$

The probability density function (pdf) of $T$, $f(t)$, can be found by differentiating $F(t)$, or the negative of $S(t)$, with respect to $t$. This pdf can be used to find the probability of a unit failing at time $t$, or more strictly the probability of a unit failing in the interval $[t, t+\delta t)$ where $\delta t$ tends towards zero is

$$P(t \leq T < t + \delta t) \approx f(t) \delta t.$$

The hazard of a unit at time $t$ is defined as the rate of failure at time $t$, given that the unit has survived until time $t$. The hazard function $h(t)$, is the hazard value across the entire time domain, and this is defined as

$$h(t) = \frac{f(t)}{S(t)} \tag{2.3}$$

The hazard function is a very important concept in reliability theory, as it gives information about the behaviour of the failure rate of a unit over time. If the hazard function is increasing over time then the unit is more likely to fail as it gets older, if it is decreasing then the unit is less likely to fail as it gets older, and if the hazard rate is constant then the unit's failures are not affected by time. If the hazard is constant then the statistical methodology is easier, and consequently many analyses are based on this assumption.

Integration of the hazard function gives the cumulative hazard function $H(t)$, defined by

$$H(t) = \int_0^t h(u)\, du \tag{2.4}$$

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If a plot of $H(t)$ with time is linear then the failure rate is constant, and if it curves up or down then the failure rate is increasing or decreasing respectively. Using equation (2.4), and previous results, an expression for the survivor function, $S(t)$, can be found in terms of $H(t)$.

$$S(t) = \exp\{-H(t)\} \tag{2.5}$$

A proof of (2.5) is given in Mann *et al.* (1974). This result is useful when using non-parametric methods of estimation, as described in Section 2.3.

### 2.2.2 Censoring

If at the end of a test the time to failure of a unit is unknown, then that unit is deemed to be censored at that time. If the test period was $x$ hours, and at the end of that period the unit has not failed, that is, the survival time of the unit is greater than or equal to $x$, then that unit is said to be right-censored. Similarly, if the survival time of a unit is less than or equal to $x$, then that unit is said to be left-censored. In practice the majority of censored observations are right-censored. There are a number of ways of obtaining censored data from an experiment and the most common of these are defined below. In each case the resulting data should be analysed according to the type of censoring present. Censored observations are discussed in many books on reliability, although the definitions below are taken from Lawless (1982) and Collett (1994).

1) Type I censoring - Occurs if $n$ items are placed on test, and a time $L$ is predetermined to be the length of the experiment. The experiment is then carried out for time $L$ and at the end of this period any unit that has not failed becomes a right-censored (Type I) observation with survival time greater or equal to $L$. Type I censoring is also called time censoring.

2) Type II censoring - Occurs if $n$ items are placed on test, and the experiment is carried out until $r$ items have failed, then at the end of the test there will be survival times for $r$ units,

and $n\text{-}r$ units will be right-censored (Type II). This method has the advantage of specifying in advance how much data the test will yield.

3) Random Censoring - This is the most common type of censoring. The end date of the experiment is predetermined but the units do not all enter the test at the same time, i.e. the entry to the test is a random process. Therefore, at the end of the test the number of censored observations will be random, and if there are $n$ censored observations then there can be up to $n$ different censoring times.

4) Interval Censoring - In some experiments the units on test are only observed at distinct times, rather than continuously. The data resulting from this method of collection will then be interval-censored since the actual time of the failure will not be known, only that a failure occurred between times $t_1$ and $t_2$. A model allowing for this type of censoring is discussed in Chapter 8.

## 2.3 Non-Parametric Methods of Estimation

### 2.3.1 Introduction

This section is based on the description given by Collett (1994). In the simplest case when there is no censoring the survivor function can be estimated by

$$\hat{S}(t) = \frac{\text{Number of units surviving} \geq t}{\text{Total number of units on test}} \tag{2.6}$$

that is, the ratio of the number of units surviving at time $t$ to the number of units starting the test. The survivor function is assumed to be constant over the periods between one failure and the next. This yields a decreasing step function. Using equations (2.2), (2.5) and (2.6), estimates of the failure function and the cumulative hazard function can be found. The latter can be used as a visual check to see whether the failure rate is constant, increasing, or decreasing over time.

An extension of equation (2.6), which allows for censoring, is the life-table estimate of the survivor function. This is calculated by dividing the data up into intervals. If there are $n$ intervals then the survivor function for the $k^{th}$ interval, where $k \leq n$, is estimated by

$$\hat{S}(t) = \prod_{j=1}^{k} \left( \frac{n'_j - d_j}{n'_j} \right)$$

(2.7)

where

$n'_j$ is given by $(n_j - c_j/2)$.

$c_j$ is the number of censored observations in the $j^{th}$ interval.

$n_j$ is the number of units at risk during the $j^{th}$ interval.

$d_j$ is the number of units that fail in the $j^{th}$ interval.

The calculation for $n'_j$ gives the average number of items on test during the $j^{th}$ interval, the underlying assumption being that the censoring occurs at equal periods of time over the interval. This estimate of the survivor function also gives a decreasing step function, with the probability of survival being constant over each interval. The hazard, cumulative hazard and failure functions can be estimated from (2.7) through equations (2.2) to (2.4) using (2.7) to estimate $S(t)$.

### 2.3.2 Kaplan-Meier Estimation of the Survivor Function

A more general non-parametric estimator of the survivor function was introduced by Kaplan and Meier (1958). The Kaplan-Meier estimator, also called the product limit estimator, is the most widely used non-parametric estimator of the survival function. The calculation is very similar to that of the life-table estimator. With the Kaplan-Meier estimator, the time intervals are split in such a way that each interval contains only one failure. More formally, if there are $n$ units on test and $r$ failures occur at times $0 \leq t_1 \leq t_2 \leq ... \leq t_r$ then the first interval will be from 0 to $t_1$, the second interval will be from $(t_1 + \delta t_1)$ to $t_2$, and so on. The survivor function for the $k^{th}$ interval, where $k \leq r$, is given by

$$\hat{S}(t) = \prod_{j=1}^{k} \left( \frac{n_j - d_j}{n_j} \right) \qquad (2.8)$$

where

$n_j$ is the number of units at risk at the start of the $j^{th}$ interval.

$d_j$ is the number of units that fail in the $j^{th}$ interval.

The value of $d_j$ will be 1 unless there are tied observations. If the largest observation is uncensored then the survivor function, $\hat{S}(t)$, will be a closed decreasing step function; that is $P(T \geq 0) = 1$ and $P(T \geq t_n) = 0$, for all times $T$ after the last failure, $t_n$. However, if there is a unit with a censored time $\leq t_r$ (the last observation is censored) then $\hat{S}(t)$ is undefined after $t_r$. The standard error of the Kaplan-Meier estimator for time $t$ where $t_k \leq t \leq t_{k+1}$ is given by

$$s.e\left\{\hat{S}(t)\right\} \approx \hat{S}(t) \sqrt{\sum_{j=1}^{k} \frac{d_j}{n_j \left( n_j - d_j \right)}} \qquad (2.9)$$

A proof of this is given by Collett (1994).

## 2.4 Parametric Models

### 2.4.1 Introduction

As with all other branches of statistics, parametric models feature extensively in reliability analysis. Unlike the non-parametric estimators of $S(t)$, parametric models yield a smooth function. Using the results from Section 2.2.1, if any one of $f(t)$, $F(t)$, $S(t)$, $h(t)$ or $H(t)$ has been estimated parametrically, all the others can be found. One major difference in reliability analysis is that the normal distribution does not feature highly. One reason for this is that survival times are all positive, and the parametric models are defined accordingly, but the normal distribution can always take negative values. The most commonly used models are described in the next sections, and the latter sections show how the parameters of the fitted models can be estimated and indicate how to decide whether a particular model is appropriate.

## 2.4.2 The Exponential Distribution

The simplest parametric model is the exponential model. It has one parameter to be estimated, and the survivor function is defined by

$$S(t) = \exp(-\lambda t) \qquad\qquad t > 0 \qquad\qquad (2.10)$$

Differentiating equation (2.10) gives the corresponding pdf, $f(t)$, where

$$f(t) = -\frac{dS(t)}{dt} = \lambda \exp(-\lambda t). \qquad\qquad (2.11)$$

Using equation (2.3), together with equations (2.10) and (2.11) gives the hazard function for an exponential distribution as

$$h(t) = \lambda \qquad\qquad (2.12)$$

The result in equation (2.12) shows that for the exponential model the hazard rate is constant over time. Referring back to the definition of the hazard function, this means that given a unit survives to time $t$, the likelihood of it failing instantaneously does not depend on the length of time $t$ that it has already survived. The mean and variance of an exponential distribution are $\lambda^{-1}$ and $\lambda^{-2}$ respectively.

A plot of $f(t)$, for an exponential distribution with $\lambda=1$, is given in Figure 2.1. The effect of changing $\lambda$ is to rescale the distribution, with the shape of $f(t)$ remaining the same. A plot of the hazard function for this distribution yields a straight line since $h(t)=1$, for $\lambda=1$. The exponential distribution is frequently used in reliability analysis because of its simplicity, although in many cases the assumption of constant failure rate is questionable.
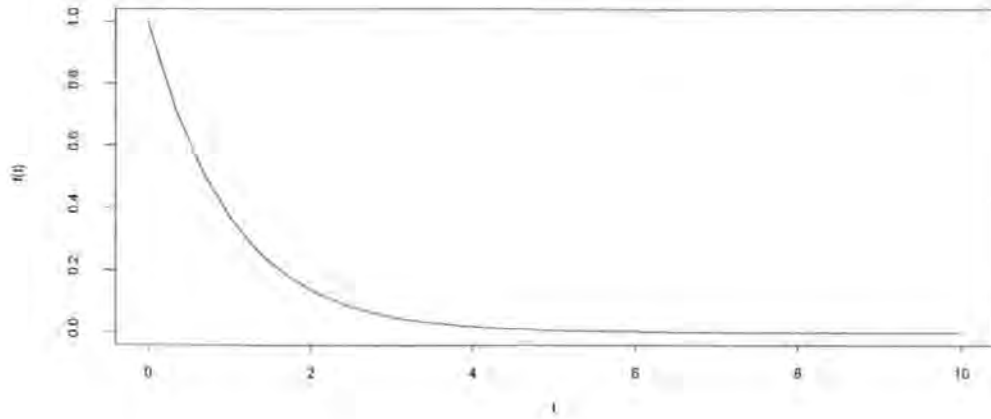
**Figure 2.1:** $f(t)$ for an exponential distribution with $\lambda=1$.

### 2.4.3 The Weibull Distribution

This distribution, which is a two parameter extension of the exponential distribution, was introduced by Weibull (1951). The survivor function of a Weibull distribution is given by

$$S(t) = \exp\left\{-\left(\frac{t}{\alpha}\right)^{\eta}\right\} \qquad\qquad t > 0 \qquad\qquad (2.13)$$

where

$\alpha$ and $\eta$ are both positive parameters.

$\alpha$ is the scale parameter.

$\eta$ is the shape parameter.

$E[T] = \alpha\Gamma(\eta^{-1}+1)$. $\qquad\qquad$ $Var[T] = \alpha^2\{\Gamma(2\eta^{-1}+1) - [\Gamma(\eta^{-1}+1)]^2\}$.

The Weibull distribution has the exponential distribution as a special case. If the shape parameter $\eta$ is 1, then equation (2.13) becomes equation (2.10) with $\lambda=1/\alpha$. The failure pdf and the hazard function for the Weibull distribution are given by equations (2.14) and (2.15) respectively.

$$f(t) = \eta\frac{1}{\alpha^{\eta}}t^{\eta-1}\exp\left\{-\left(\frac{t}{\alpha}\right)^{\eta}\right\} \qquad\qquad (2.14)$$

$$h(t) = \eta \frac{1}{\alpha^\eta} t^{\eta-1} \qquad\qquad\qquad (2.15)$$

Plots of the pdf's and hazard functions, for Weibull distributions with α=1 and η=0.5, 2, 4, are shown in Figures 2.2 and 2.3 respectively. These plots show the flexibility of the Weibull distribution. The parameters can be chosen to describe a constant, increasing or decreasing hazard rate. The hazard is constant for η = 1, decreasing for η < 1 and increasing for η > 1. The larger the value of the shape parameter, the steeper the slope. If η = 2 then the hazard is linear. The Weibull distribution is the most widely used distribution in the analysis of reliability data because of its flexibility.

### 2.4.4 The Gamma Distribution

A gamma distribution is a two parameter distribution with pdf, $f(t)$, given by

$$f(t) = \frac{\lambda^\rho t^{\rho-1} \exp(-\lambda t)}{\Gamma(\rho)} \quad t > 0 \qquad\qquad (2.16)$$

where

λ and $\rho$ are both positive parameters.

1/λ is a scale parameter.

ρ is a shape parameter.

$E[T] = \rho/\lambda$       .           $Var[T] = \rho/\lambda^2$.

As with the Weibull distribution, the gamma distribution has the exponential distribution as a special case. If ρ = 1 then equation (2.16) becomes equation (2.11). Gamma distributions result from the sum of independent and identically distributed exponential variables. It is generally used to model systems where redundancy occurs (where $n$ components comprise the system and all $n$ must fail to cause a system failure), although it appears to model many situations and can be used instead of a Weibull distribution. However, it is not as widely used as the Weibull because the survivor and hazard functions cannot be expressed in a closed form,
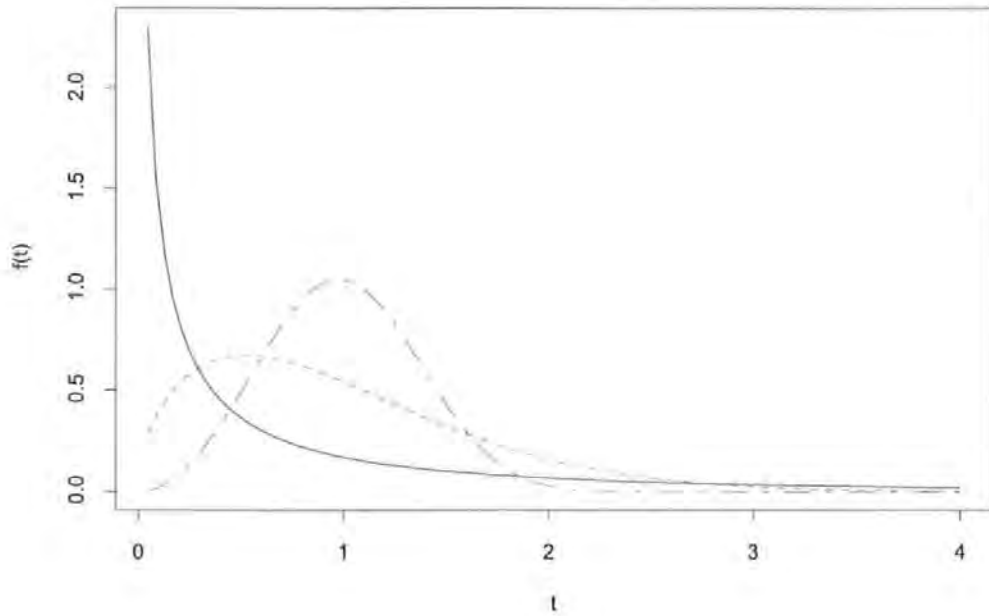
**Figure 2.2:** Pdf's for Weibull distributions with mean = 1 and shape
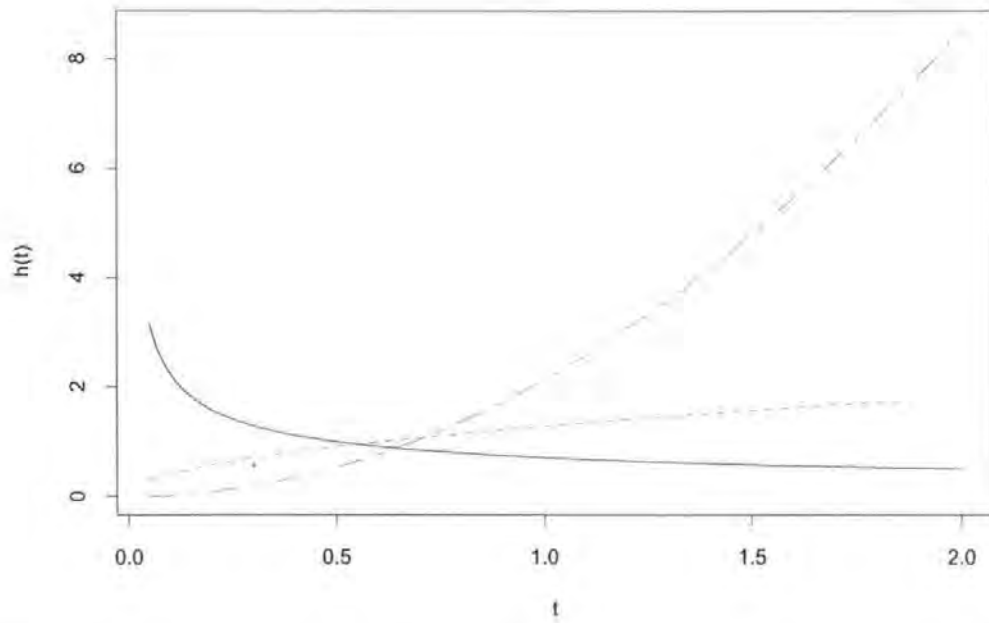
0.5( __ ), 1.5(----), 3( ._._.) .



**Figure 2.3:** Hazard functions for Weibull distributions with mean = 1 and shape

0.5( __ ), 1.5(-----), 3( ._._.) .

and are therefore difficult to interpret. When $\rho$ is an integer, the survivor function is given by equation (2.17)

$$S(t) = \exp\left[-\lambda t\left\{1 + \lambda t + \frac{(\lambda t)^2}{2!} + \ldots + \frac{(\lambda t)^{\rho-1}}{(\rho-1)!}\right\}\right]$$ (2.17)

Using equations (2.4) and (2.5), the hazard function when $\rho$ is an integer is

$$h(t) = \lambda + 2\lambda^2 t + \frac{3\lambda^3 t^2}{2!} + \ldots + \frac{\rho\lambda^\rho t^{\rho-1}}{(\rho-1)!}$$ (2.18)

For $\rho < 1$ the hazard function is decreasing, it is increasing for $\rho > 1$ and is constant for $\rho = 1$. The gamma distribution is right skewed, although as $\rho$ increases the skewness decreases and the function resembles a normal distribution. Figure 2.4 shows gamma pdf's with mean = 1 and $\rho = 0.5$, 2 and 10.
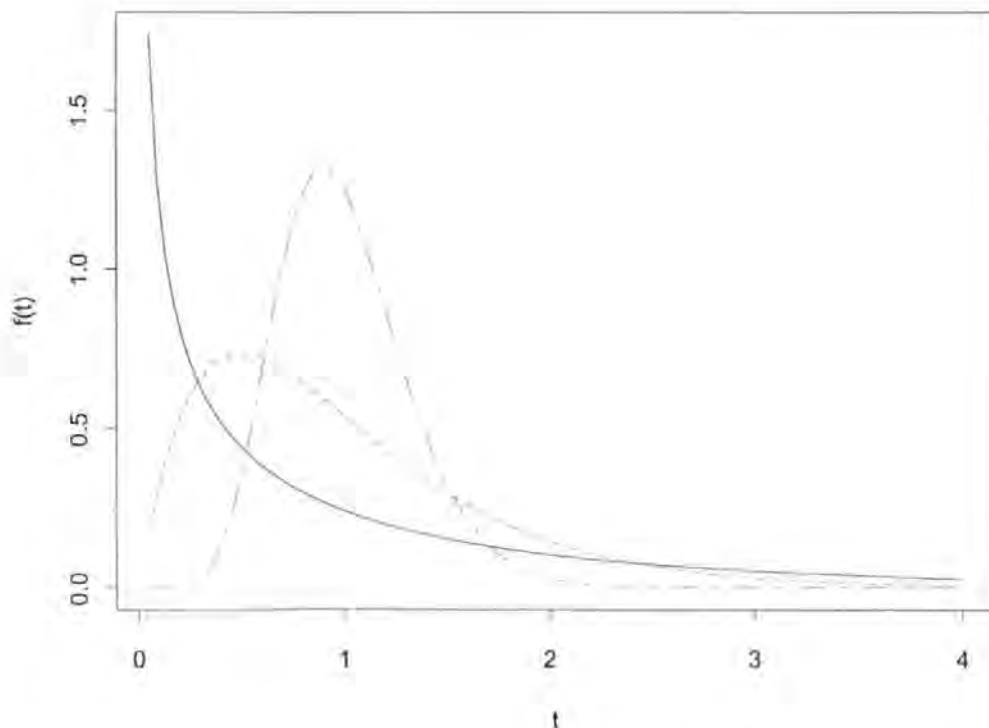


**Figure 2.4:** Pdf's of gamma distributions with mean = 1 and $\rho = 0.5(\underline{\quad})$, 2(....), 10(.\_.\_.).

## 2.4.5 The Lognormal Distribution

In Section 2.1.1 it was mentioned that the normal distribution is rarely used to model survival data because it allows the possibility of negative times. However, if $Y = \ln(T)$ can be assumed to be normally distributed, then this is equivalent to $T$ having a lognormal distribution. As $\ln(t)$ can take any value this distribution is then valid to model failure times. The pdf for a lognormal distribution is given by equation (2.19).

$$f(t) = \frac{1}{\sqrt{2\pi\sigma^2 t^2}} \exp\left\{ -\frac{\ln(t) - \mu}{2\sigma^2} \right\} \qquad t > 0 \qquad\qquad (2.19)$$

As for the gamma distribution, the survivor and hazard functions for a lognormal distribution cannot be expressed in closed form. Another problem is that the hazard functions initially increase and then decrease, approaching zero as $t\to\infty$. This is not a characteristic that many items would be expected to exhibit in practice.

## 2.4.6 Parameter Estimation

The parameters of any model can be estimated by a number of methods, but the most commonly used is the method of maximum likelihood. It can be used to obtain estimates for most reliability problems (Crowder et al. ,1991) and provides the minimum variance unbiased estimators. In the simplest case when no censoring occurs, the likelihood function is defined by the pdf of the distribution that is being fitted. If there are $n$ units on test, with failures occurring at $0 \leq t_1 \leq t_2 \leq ... \leq t_n$, and the density function is $f(t;\theta)$, where $\theta$ is a vector of parameters, then the likelihood function is

$$L(\theta) = \prod_{i=1}^{n} f(t_i;\theta) \,. \qquad\qquad (2.20)$$

The values in $\theta$ that maximise equation (2.20) are then the estimates of the parameters for the pdf. If censoring occurs then the likelihood given by equation (2.20) is split into two

parts: the likelihood for uncensored observations and the likelihood for censored observations. If $U$ denotes the set of uncensored observations and $C$ denotes the set of censored observations, then the appropriate likelihoods for when right-censoring and left-censoring occur are given by equations (2.21) and (2.22) respectively.

$$L(\theta) = \left\{ \prod_{i \in U} f(t_i : \theta) \right\} \left\{ \prod_{i \in C} S(t_i : \theta) \right\} \qquad (2.21)$$

$$L(\theta) = \left\{ \prod_{i \in U} f(t_i : \theta) \right\} \left\{ \prod_{i \in C} F(t_i : \theta) \right\} \qquad (2.22)$$

For interval-censored data, the likelihood for one unit is the probability that a failure occurs after the lower limit and before the upper limit. If there are $n$ units on test, with failures occurring between $t_{1L}$-$t_{1U}$, $t_{2L}$-$t_{2U}$, ..., $t_{nL}$-$t_{nU}$ then the likelihood is

$$L(\theta) = \prod_{i=1}^{n} \left\{ F(t_{iU} : \theta) - F(t_{iL} : \theta) \right\} \qquad (2.23)$$

Estimation of parameter values from the log-likelihood function, $\ell(\theta)$, gives the same results. This is usually easier to calculate than the actual likelihood since the log-likelihood becomes a summation rather than a product. The standard errors of the estimates can be found by inverting the matrix of second derivatives.

### 2.4.7 Goodness-of-Fit Tests

Part of the analysis when fitting a model to a set of data should include checks to test for a reasonable fit, and there are a number of ways of doing this. One way is to produce a probability-plot which provides a graphical check of the model. For a series of $k$ failure times $a_1 < a_2 < ... < a_k$ this is achieved by plotting $p_j$ against $F(a_j : \hat{\theta})$, where

$$p_j = 1 - \frac{\hat{S}(a_j) + \hat{S}(a_{j+1})}{2} \qquad (2.24)$$

$\hat{S}$ is from the Kaplan-Meier estimate of the survivor function and $F(a_j : \hat{\theta})$ is the fitted distribution. If the distribution provides a good fit, a plot of $p_j$ is against $F(a_j : \hat{\theta})$ will be approximately linear (Crowder *et al.*, 1991).

More formal hypothesis tests can also be conducted. These include the Pearson Chi-squared test and the likelihood ratio test. For a Chi-squared test the time scale is divided into $n$ intervals, and the number of failures expected in each interval, $E_i$, can be calculated from the fitted probability density. Let $O_i$ be the number of failures observed in the $i$th interval then if the null hypothesis (that the observed data does come from the fitted distribution) is true then the test statistic given by equation (2.25) will follow a $\chi^2$ distribution with degrees of freedom equal to $n - 1 -$ number of estimated parameters.

$$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i} \qquad (2.25)$$

The goodness-of-fit test is therefore used to check for evidence that the data do not follow the fitted distribution. Other goodness-of-fit tests, together with probability plotting techniques, are given by D'Agastino and Stephens (1986).

The likelihood ratio test is used to compare nested models. If model 2, with parameters $\theta_2$, is nested within model 1, with parameters $\theta_1$, then the statistic

$$W = 2\left\{ \ell(\hat{\theta}_1) - \ell(\hat{\theta}_2) \right\} \qquad (2.26)$$

arising from the difference in the maximised log-likelihoods, can be compared against a $\chi^2$ distribution with degrees of freedom equal to the difference in the number of parameters between the two models. This statistic looks for a significant change in the likelihood if

parameters are removed. The drawback with this test is that it only provides a relative test of which of the two models is preferable. It does not provide a test of the overall fit, and it cannot be used to compare non-nested models, for example the Weibull and lognormal distributions. One way around the latter problem is to use the generalised gamma distribution, discussed in Lawless (1982). This has the all the distributions mentioned above as a special case, and so a series of likelihood ratio tests can be conducted to compare all the different models. A test to compare the Weibull and lognormal distributions is given by Dumonceaux and Antle (1973).

## 2.5 Regression Models

### 2.5.1 Introduction

Regression models are a very important tool in reliability analysis as they incorporate the effect of covariates on the failure time. The models can then be used to identify factors that affect the survival time of a unit. There are two main classes of regression model in reliability analysis, the proportional hazards model and the accelerated life model, and these are described in the following sections. Traditionally, the proportional hazards model has been used in medical areas and the accelerated life in engineering applications although there is now more of a crossover, and there have been a number of papers comparing the two classes of model, for example, Gupta and Michalek (1985), Lawless (1986) and Wei (1992).

### 2.5.2 The Proportional Hazards Model

The proportional hazards model was introduced by Cox (1972), and since then has been widely used in medical applications (see for example Anderson, 1991). The proportional hazards model is defined by equation (2.27).

$$h(t;x) = h_0(t)\psi_x \qquad\qquad (2.27)$$

where $h(t;x)$ is the hazard function under covariate values $x$.

$h_0(t)$ is some baseline hazard function.

$\psi_x$ is a positive function of $x$, where $x$ denotes the covariate(s).

The baseline hazard function is defined as the underlying hazard if the covariate effects are zero, or are at their nominal values. The assumption with this model is that the covariate will have a proportional effect on the baseline hazard function; from equation (2.27) it can be seen that any change in $x$ will produce a constant multiplicative effect on the baseline hazard over the whole time scale. As $\psi_x$ has to be positive, it generally takes an exponential form, and equation (2.27) then becomes

$$h(t;x) = h_0(t)\exp\left(x^T\beta\right) \tag{2.28}$$

where

$x$ is a vector of covariate values.

$\beta$ is a vector of parameters.

Equation (2.28) gives the model due to Cox (1972). If there are $n$ observations, $r$ of which are uncensored, the likelihood function for equation (2.28) is

$$L(\beta) = \left\{\prod_{i=1}^{r} \frac{\exp\left(x_i^T\beta\right)}{\sum_{l\in R_i} \exp\left(x_l^T\beta\right)}\right\} \tag{2.29}$$

where $R_i$ is defined as the risk set of observation $i$. The risk set is then all observations, censored or uncensored, which have a time greater than or equal to $t_i$. A more rigorous proof of this likelihood is given by Kalbfleisch and Prentice (1973). As stated in Section 2.4.6., it is generally easier to work with log-likelihoods. The log of equation (2.29) is

$$\ell(\beta) = \sum_{i=1}^{r}\left\{x_i^T\beta - \ln\sum_{l\in R_i} \exp\left(x_l^T\beta\right)\right\} \tag{2.30}$$

To estimate the parameters of the model, the estimates of $\beta$ that maximise equation (2.30) have to be found. This is done by differentiating equation (2.30) with respect to $\beta$, producing the set of partial derivatives. These are then equated to zero and solved simultaneously to find

the estimates, $\hat{\beta}$. The standard errors of $\hat{\beta}$ can be found from the second derivatives. The partial derivative with respect to $\beta_p$ is

$$\frac{\partial l(\beta)}{\partial \beta_p} = \sum_{i=1}^{r} \left\{ \mathbf{x}_{ip} - \frac{\sum_{l \in R_i} \mathbf{x}_{lp} \exp\left(\mathbf{x}_l^T \beta\right)}{\sum_{l \in R_i} \exp\left(\mathbf{x}_l^T \beta\right)} \right\} \tag{2.31}$$

The likelihood in equation (2.29) does not allow for ties in the survival times. There have been a number of approximations to the likelihood when ties are included, but the most frequently used is that due to Breslow (1974). This likelihood, given by equation (2.32), is an extension of a likelihood for paired ties suggested by Peto (1972).

$$L(\beta) = \prod_{i=1}^{r} \left[ \frac{\exp\left(\mathbf{s}_i^T \beta\right)}{\left\{ \sum_{l \in R_i} \exp\left(\mathbf{x}_l^T \beta\right) \right\}^{d_i}} \right] \tag{2.32}$$

where

$s_i$ is the sum of the covariates for all observations at time $t_i$.

$d_i$ is the number of tied observations at time $t_i$.

The above methodology is discussed more fully by Collett (1994), and in more detail by Kalbfleisch and Prentice (1980). A FORTRAN program using a NAg routine to fit proportional hazard models, for the likelihood with ties (equation (2.32)) is given in Appendix 3. This program returns the estimates of the parameters and the value of the log-likelihood at these points. The likelihood ratio test, Section 2.4.7, can then be used to test for significance.

As with any statistical model a number of validation checks should be carried out in order to assess whether the model is appropriate. The significance of the fitted parameters, $\hat{\beta}$, can be checked either by using a likelihood ratio test, or by using the standard error of $\hat{\beta}$ to construct a hypothesis test. There are a number of types of residuals that can be plotted,

including those due to Cox and Snell (1968), Schoenfeld (1982) and Therneau *et al.* (1990). Collett (1994) discusses the definitions and uses of these residuals. There are also a number of tests to check the validity of the proportional hazards assumption, for example Schoenfeld (1980) and Gill and Schumacher (1987). Struthers and Kalbfleisch (1986) have investigated the results of using a proportional hazards model when it is not valid, that is when the covariates do not have a proportional effect on the hazard. They concluded that although the parameter estimates would be biased, the significant factors affecting the hazard would still be identified.

The interpretation of the fitted proportional hazards model is straightforward. If the estimates of the parameters are $\hat{\beta}$, for a given set of covariate values $x$ the hazard function is $\exp(x^T\hat{\beta})$ times the baseline. If the survivor function is required, given the covariates, it can be estimated from equation (2.33), where $S_0(t)$ is the baseline survivor function.

$$S(t;\mathbf{x}) = S_0(t)^{\exp(\mathbf{x}^T\hat{\beta})} \qquad (2.33)$$

It is not necessary in the analysis to specify the baseline hazard function. The model can be used simply as a tool to identify important factors, and to quantify their effects on the hazard. If however, predictions of the hazard rate or survival probabilities are required then the baseline hazard must be specified. The cumulative hazard is also needed for the residual analysis. If a parametric model has been found from a previous analysis then this can be used. Otherwise, the survivor function can be estimated from the data. A non-parametric method for doing this was introduced by Kalbfleisch and Prentice (1973), and explained in more detail in Kalbfleisch and Prentice (1980). For the case when no ties occur their formulation gives the baseline survivor function as equation (2.34). The estimate of the cumulative hazard is then the negative of the logarithm of the baseline survivor function, equation (2.5).

$$\hat{S}_0(t) = \prod_{j:t_{(j)}<t} \hat{\alpha}_j \qquad (2.34)$$

where $\alpha_j$ is

$$\hat{\alpha}_j = 1 - \left( \frac{\exp(x_j^T \hat{\beta})}{\sum_{l \in R_j} \exp(x_l^T \hat{\beta})} \right) \qquad (2.35)$$

If all the observations are from the same covariate level (that is there is no change in any of the covariates) then equation (2.34) becomes the Kaplan-Meier estimate of the survivor function, equation (2.8). A plot of the baseline survivor function may show a suitable parametric form for the model.

### 2.5.3 Accelerated Life Models

The majority of the literature on regression models for survival data is concerned with the proportional hazards model. However another important class of models is the accelerated life models. An accelerated life model is a model of the form

$$S(t;\mathbf{x}) = S_0(t\psi_x) \qquad (2.36)$$

where

$S_0(t)$ is a baseline survivor function.

$\psi_x$ is a positive function of $x$.

The hazard function for equation (2.36) is then given by

$$h(t;\mathbf{x}) = \psi_x h_0(t\psi_x) \qquad (2.37)$$

A comparison of equation (2.37) with (2.27), and equation (2.36) with (2.33), shows the difference between the two types of model. Whereas the Proportional Hazard model assumes a multiplicative effect on the hazard, the accelerated life model assumes a multiplicative effect on the time scale. If exponential or Weibull models are assumed for the data then the

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accelerated life model is equivalent to the proportional hazards model: see Kalbfleisch and Prentice (1980) for further details. An alternative form of the accelerated life model is expressed in terms of log lifetimes (Crowder *et al.*, 1991) by

$$\ln(T) = -\ln(\psi_x) + \ln(W) \tag{2.38}$$

where $W$ is the baseline variate. The log lifetimes are then distributed as $\ln(W)$ with a location shift.

## 2.6 Bayesian Methods in Reliability

### 2.6.1 Introduction

Up to this point fitted models have been considered from a classical viewpoint using maximum likelihood procedures. Over recent years Bayesian methods have been used very widely in most areas of applied statistics. Rapid developments in the areas of Bayesian computational statistics and in particular the use of Markov Chain Monte Carlo (MCMC) methods ( See Gilks *et al.*, 1996) have given impetus to these applications in situations where analytical methods are intractable. A very recent example is the work of Walker and Mallick (1997). A major benefit of the Bayesian approach in development projects such as the AJ26 is that it allows prior information to be incorporated into the analysis and it provides a natural way of modelling the development process as it proceeds from one stage to the next.

In Bayesian analysis, prior information on an unknown quantity $\theta$ is represented in terms of the prior probability density $\pi(\theta)$. Having observed data $x$, Bayes theorem is used to determine the distribution of $\theta$ given $x$, as

$$\pi(\theta|x) = \frac{\pi(\theta)\ell(x|\theta)}{\int \pi(\theta)\ell(x|\theta)d\theta} \quad . \tag{2.39}$$

This is termed the posterior distribution of θ, and is the object of all Bayesian inference.

In the phase to phase development process the posterior distribution from one phase forms the prior distribution for the next and the Bayesian paradigm provides a natural way of modelling the sequential development process. Moreover, at the start of the development the prior distribution provides a mechanism for designers and development engineers to express their prior knowledge about the engine or its components in a formal way. Bayesian methods thus provide a bridge between the engineers knowledge and subsequent data. This application of Bayesian methods in the context of design and development was illustrated by Whitmore *et al.* (1994). This is described in the following section.

## 2.6.2 A Bayesian Methodology for Technological Evolution

Whitmore *et al.* (1994) describe the use of Bayesian analysis to estimate the failure rate, or to find the probability of a certain number of failures, over a number of design updates in the development of a new device. The basic assumption with the method is that the failure rate is constant, that is the failure time follows an exponential distribution. Although the validity of this assumption is doubtful in practice, the analysis can be extended to allow for changing failure rates by using other distributions, for example the Weibull or gamma distributions. For the exponential case, the data required are the number of failures $r$ that occurred over the whole test time $t$.

Whitmore *et al.* (1994) consider the situation where a unit is run for time $t_1$ and yields $r_1$ failures, giving a failure rate of $\lambda_1 > 0$. This is assumed to follow a Poisson distribution with $\lambda$ equal to $\lambda_1 t_1$ and the likelihood is given by

$$\ell(r_1|\lambda_1) = \frac{(\lambda_1 t_1)^{r_1} \exp(-\lambda_1 t_1)}{r_1!}$$

(2.40)

The conjugate prior when the likelihood takes the form of a Poisson distribution is a gamma distribution. If the gamma prior has parameters $\alpha$ and $\beta$, $Ga(\alpha,\beta)$, then using equation (2.39) the posterior distribution $\pi(\lambda_1 \mid r_1)$ is

$$\pi(\lambda_1 \mid r_1) \propto \pi(\lambda_1) \ell(r_1 \mid \lambda_1)$$

giving

$$\pi(\lambda_1 \mid r_1) \propto \lambda_1^{r_1 + \alpha - 1} \exp\left\{-\lambda_1(t_1 + \beta)\right\}$$

The Posterior distribution is therefore given by

$$\pi(\lambda_1 \mid r_1) = Ga(\alpha',\beta')$$

where $\alpha' = r_1 + \alpha$ and $\beta' = t_1 + \beta$. If a second unit is made which is similar to the first, and testing yields $r_2$ failures in time $t_2$, the posterior distribution for the first unit, $\pi(\lambda_1 \mid r_1)$, can be modified to become the prior distribution for the second, $\pi(\lambda_2)$. The suggested modification is given by equation (2.41).

$$\pi(\lambda_2) = Ga\left(\frac{c}{d}\alpha', \frac{1}{d}\beta'\right) \tag{2.41}$$

The mean and variance of this distribution are given in Section 2.4.4 as $c\mu$ and $cd\sigma^2$ respectively, where $\mu$ and $\sigma^2$ are the mean and variance of the posterior distribution for the first unit. The effect of $c$ and $d$ is therefore to rescale the distribution, where $c$ gives the expected change in the mean and $cd$ gives the expected change in the variance. Using the previous results the posterior distribution for the second unit, $\pi(\lambda_2 \mid r_2)$, is given by equation (2.42). This analysis can be extended for any number of technological evolutions from the original unit.

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$$\pi(\lambda_2 | r_2) = Ga\left(\frac{c}{d}\alpha' + r_2, \frac{1}{d}\beta' + t_2\right) \tag{2.42}$$

The predictive distribution, $\pi(r_2 | t_2)$ can be found from equation (2.43), and is given by equation (2.44). This allows inferences to be made, for a given time period, about the expected number of failures, or the probability of obtaining $x$ number of failures.

$$\pi(r_2 | t_2) = \int_0^\infty \pi(r_2 | \lambda_2, t_2) \pi(\lambda_2) \, d\lambda_2 \tag{2.43}$$

$$\pi(r_2 | t_2) = \frac{\Gamma\left(r_2 + \frac{c}{d}\alpha'\right)}{\Gamma(r_2 + 1)\Gamma\left(\frac{c}{d}\alpha'\right)} t_2^{r_2} \left(\frac{\beta'}{d}\right)^{\frac{c}{d}\alpha'} \left(t_2 + \frac{\beta'}{d}\right)^{-\left(r_2 + \frac{c}{d}\alpha'\right)} \tag{2.44}$$

## 2.7 The Reliability of Repairable Systems

### 2.7.1 Introduction

This section introduces repairable system reliability. The methodology discussed so far is used to model single failure times, whereas repairable items can have multiple failure times. A model for repairable system reliability, the non-homogeneous Poisson process (NHPP), is discussed in Section 2.7.2. An introduction to these methods is given by Crowder *et al.* (1991), and they are discussed in further detail by Ascher and Feingold (1984).

A repairable system is defined as a system that, after a failure has occurred, can be restored to full working order by any means other than replacement of the whole system. In the analysis of repairable systems, it is the Rate of Occurrence of Failures (ROCOF), that is generally of interest. This is modelled by the function $v(t)$, defined as

$$v(t) = \frac{d}{dt} E\{N(t)\} \tag{2.45}$$

where $E\{N(t)\}$ is the expected number of failures at time $t$. A simple non-parametric estimator of $\upsilon(t)$ is obtained by equation (2.46), where the time scale is split into a number of intervals. These intervals are usually chosen to highlight the main features of the data.

$$\hat{\upsilon}(t) = \frac{no. \text{ of failures in } \left(t, t + \partial t\right]}{\partial t} \tag{2.46}$$

If the trend of $\upsilon(t)$ is linear then the ROCOF is said to be constant over time, otherwise it is either increasing or decreasing. A graph of the failure number against the cumulative failure time is an easy way to gain an indication of the trend. Figure 2.5 shows such a plot, for the failure of diesel engines on the USS Grampus (Crowder *et al.* (1991), page 161), from which it appears that the ROCOF is constant.
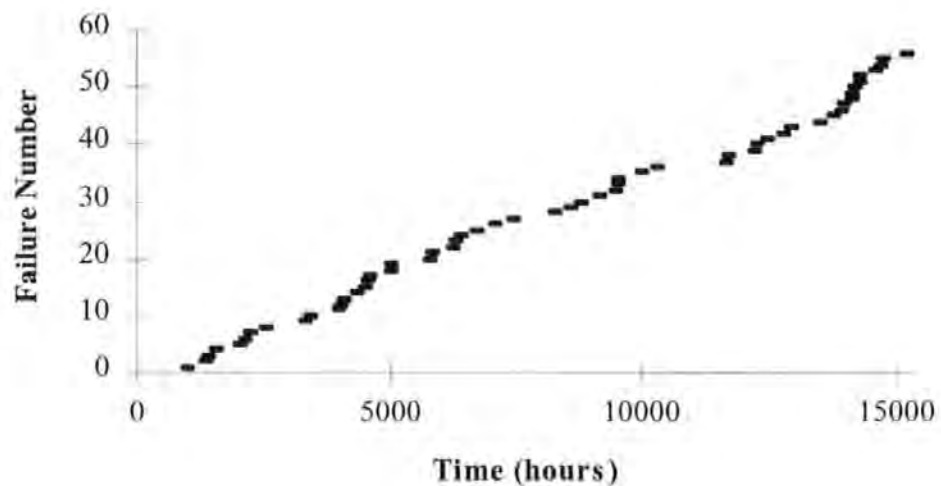


Figure 2.5 : Plot of failure number against cumulative time for USS Grampus data.

### 2.7.2 The Non-Homogeneous Poisson Process (NHPP)

A model for repairable systems has to be flexible to describe both increasing and decreasing failure rates, and the non-homogeneous Poisson process is able to do this. It states that for a repairable system with a ROCOF described by $\lambda(t)$, the number of failures over a time interval $(t_1, t_2]$ has a Poisson distribution with mean given by

$$\int_{t_1}^{t_2} \lambda(t)\, dt \tag{2.47}$$

This mean can then be used to make inferences about the data. For example, using the formula for a Poisson distribution, the probability of zero failures occurring in the time interval $(t_1,t_2]$ is

$$P\{\text{no failure in}(t_1,t_2]\} = \exp\left\{-\int_{t_1}^{t_2} \lambda(t)\, dt\right\} \tag{2.48}$$

Consider a repairable system monitored for time period 0 to $t_0$, during which n failures occurred at times $t_1,t_2, \ldots, t_n$ respectively (where $t_n \leq t_0$). The likelihood function, for a given failure rate $\lambda(t)$, is derived from the probabilities of failures at times $t_1, t_2, \ldots, t_n$ and the probabilities of no failures during the time intervals $(0, t_1], (t_1, t_2], \ldots, (t_n, t_0)$. The likelihood function is therefore

$$L = Pr\{\text{no failure in } (0,t_1]\} \times Pr\{\text{failure at } t_1\} \times Pr\{\text{no failure in } (t_1,t_2]\} \times$$

$$\ldots \times Pr\{\text{failure at } t_n\} \times Pr\{\text{no failure in } (t_n,t_0]\} \tag{2.49}$$

Using the results of equations (2.47) and (2.48) this becomes

$$L = \left[\exp\left\{-\int_0^{t_1}\lambda(t)\,dt\right\}\right]\left[\exp\left\{-\int_{t_1}^{t_1+\delta t_1}\lambda(t)\,dt\right\}\int_{t_1}^{t_1+\delta t_1}\lambda(t)\,dt\right]\left[\exp\left\{-\int_{t_1+\delta t_1}^{t_2}\lambda(t)\,dt\right\}\right] \times$$

$$\ldots \times \left[\exp\left\{-\int_{t_n}^{t_n+\delta t_n}\lambda(t)\,dt\right\}\int_{t_n}^{t_n+\delta t_n}\lambda(t)\,dt\right]\left[\exp\left\{-\int_{t_n+\delta t_n}^{t_0}\lambda(t)\,dt\right\}\right] \tag{2.50}$$

Simplifying this gives the likelihood function as

$$L = \left\{\prod_{i=1}^{n}\lambda(t_i)\right\}\exp\left\{-\int_0^{t_0}\lambda(t)\,dt\right\} \tag{2.51}$$

The likelihood can easily be extended for the case of interval-censoring; see Crowder *et al.* (1991) for further details. There are two most commonly used forms for $\lambda(t)$, and these are given by equations (2.52) and (2.53),

$$\lambda_1(t) = \exp(\beta_0 + \beta_1 t) \tag{2.52}$$

$$\lambda_2(t) = \gamma \delta t^{\delta-1} \qquad\qquad \gamma, \delta > 0 \tag{2.53}$$

The log-linear model, $\lambda_1(t)$, is discussed by Cox and Lewis (1966). If $\beta_1$ is less than 0 then the ROCOF is decreasing, if $\beta_1$ is greater than 0 then the ROCOF is increasing, and if $\beta_1$ is equal to 0 then the ROCOF is constant.

The power law model, $\lambda_2(t)$, is discussed by Ascher and Feingold (1984). This is equivalent to equation (2.15), the hazard function for a Weibull distribution, with $\delta$ substituted for $\eta$ and $\gamma$ replacing $1/\alpha^\eta$. Section 2.4.3 describes how the shape of this function changes with different values of $\delta$.

Once the form of $\lambda(t)$ has been specified this can be substituted into equation (2.51) and the log-likelihood, combined with the partial derivatives, can be used in order to obtain estimates of the parameters. For the log-linear model the likelihood function, equation (2.51), becomes

$$L = \left\{ \prod_{i=1}^{n} \exp(\beta_0 + \beta_1 t_i) \right\} \exp \left\{ - \int_0^{t_0} \exp(\beta_0 + \beta_1 t) \, dt \right\} \tag{2.54}$$

and the log-likelihood of equation (2.54) is

$$\ell = n\beta_0 + \beta_1 \sum_{i=1}^{n} t_i - \frac{\exp(\beta_0)\left\{ \exp(\beta_1 t_0) - 1 \right\}}{\beta_1} \tag{2.55}$$

The partial derivatives of equation (2.55) are given by equations (2.56) and (2.57).

$$\frac{\partial \ell}{\partial \beta_0} = n - \frac{\exp(\beta_0)\{\exp(\beta_1 t_0) - 1\}}{\beta_1}$$

(2.56)

$$\frac{\partial \ell}{\partial \beta_1} = \sum_{i=1}^{n} t_i - \left[ \frac{\beta_1 t_0 \exp(\beta_0 + \beta_1 t_0) - \exp(\beta_0)\{\exp(\beta_1 t_0) - 1\}}{\beta_1^2} \right]$$

(2.57)

By equating (2.56) to zero, $\exp(\beta_0)$ can be expressed in terms of $\beta_1$, $t_0$ and $n$ and this can be substituted into equation (2.57). Setting the resulting equation to zero and solving gives the maximum likelihood estimate of $\beta_1$. This estimate can then be used to find the maximum likelihood estimate of $\beta_0$.

It is important to check the significance of the gradient parameter, specifically to test for evidence that the ROCOF is not linear, that is testing for evidence that $\beta_1 \neq 0$. There are two ways in which this statistical test can be performed. A likelihood ratio test as described in Section 2.4.7 may be performed between the full model and a model with $\beta_1$ removed. For the latter case it is not necessary to fit the model; the maximised log-likelihood is given by

$$\ell_{max} = n\ln\left(\frac{n}{t_0}\right) - n$$

(2.58)

A significant difference between the maximised likelihoods for the two models indicates that the ROCOF is not constant. Alternatively, a hypothesis test using the parameter estimates and their standard errors may be performed. The covariance matrix can be estimated by the inverse of the negative of the matrix of second derivatives, and the standard errors are taken from the leading diagonal. The test statistic is given by the parameter estimate divided by its standard error and this is compared with a normal distribution.

## 2.8 Summary

This chapter has introduced the models and analyses that are most frequently used for reliability data. Although the different models have been discussed as separate topics, there is a great deal of overlap in the methodologies. For example, the proportional hazards model can be used to identify important covariate effects (in which case it is unnecessary to know the form of the baseline hazard) or it can be used as a predictive model. For the latter case the baseline hazard does need to be specified and may take either a parametric form (such as one of the models introduced in Section 2.4) or a non-parametric form. The non-parametric estimate of the baseline hazard uses a generalisation of the Kaplan-Meier estimate described in Section 2.3.3.

In Section 2.6, a Bayesian model for failure data from an exponential distribution was described. This analysis can be applied to other parametric models as well as the proportional hazards and repairable systems models. In effect, the Bayesian methodology can be applied to any suitable model where sequences of data due to a technological evolution of the design of the item(s), or data from similar items, are available.

# CHAPTER 3

# A Non-Homogeneous Poisson Process

# Model With Random Effect

## 3.1 A Repairable System Reliability Model for Multiple Units

The previous chapter introduced the Non-Homogeneous Poisson Process (NHPP) model for repairable system reliability through an example of reliability data obtained from a diesel engine on the USS Grampus as in Crowder *et al.*(1991). The Jaguar data (see Appendix 2) can be treated in the same way since all engines have a succession of failures, each of which is repaired before testing continues. In this, and the following chapters, the data from the four engine phases are treated separately. An alternative modelling procedure, which treats the phase differences as a covariate effect, is introduced in Chapter 7.

For the USS Grampus example the NHPP model was used to estimate the Rate of Occurrence of Failures (ROCOF) for data from a single engine. This allowed inferences about the trend of failures and the expected number of failures to be made. In the Jaguar data failures are recorded for a number of engines all of which are the same on a given phase. For this case the NHPP model in Section 2.7 can be extended to allow for data from multiple engines (units). This extended model is described below

For $m$ engines, where $n_i$ failures have occurred, at times $t_{1i}$, $t_{2i}$, .... $t_{in_i}$, on the $i^{th}$ engine and when the total time is $t_i$, the likelihood function given by equation (2.52) now becomes

$$L = \prod_{i=1}^{m}\left[\prod_{j=1}^{n_i} \lambda(t_{ij})\exp\left\{-\int_0^{t_i} \lambda(t)dt\right\}\right] \tag{3.1}$$

A log-linear model for the ROCOF $\lambda(t_{ij})$, equation (3.2), gives the log-likelihood function as equation (3.3)

$$\lambda(t_{ij}) = \exp\left(\beta_0 + \beta_1 t\right) \tag{3.2}$$

$$\ell = \sum_{i=1}^{m}\left[n_i\beta_0 + \beta_1\sum_{j=1}^{n_i} t_{ij} - \frac{\exp(\beta_0)\{\exp(\beta_1 t_i) - 1\}}{\beta_1}\right] \tag{3.3}$$

This function, equation (3.3), can be maximised to obtain the estimates of $\beta_0$ and $\beta_1$. To test for a constant ROCOF the maximised value of equation (3.3) can be compared with the maximised log-likelihood when $\beta_1$ is removed, given by

$$\ell_{max} = \sum_{i=1}^{m}\left\{n_i\ln\left(\frac{\sum_{j=1}^{m} n_j}{\sum_{j=1}^{m} t_j}\right) - n_i\frac{\sum_{j=1}^{m} n_j}{\sum_{j=1}^{m} t_j}\right\} \tag{3.4}$$

Alternatively, the standard errors can be used to construct a hypothesis test. This requires the matrix of second partial derivatives of $\ell$ to be evaluated at the maximum. The first partial derivatives are given by

$$\frac{\partial\ell}{\partial\beta_0} = \sum_{i=1}^{m}\left[n_i - \frac{\exp(\beta_0)\{\exp(\beta_1 t_i) - 1\}}{\beta_1}\right] \tag{3.5}$$

$$\frac{\partial \ell}{\partial \beta_1} = \sum_{i=1}^{m}\left[\sum_{j=1}^{n_i} t_{ij} - \frac{\beta_1 t_i \exp(\beta_0 + \beta_1 t_i) - \exp(\beta_0)\{\exp(\beta_1 t_i) - 1\}}{\beta_1^2}\right] \qquad (3.6)$$

The second derivatives are then

$$\frac{\partial^2 \ell}{\partial \beta_0^2} = \sum_{i=1}^{m} -\left[\frac{\exp(\beta_0)\{\exp(\beta_1 t_i) - 1\}}{\beta_1}\right] \qquad (3.7)$$

$$\frac{\partial^2 \ell}{\partial \beta_1 \partial \beta_0} = \sum_{i=1}^{m} -\left[\frac{\beta_1 t_i \exp(\beta_0 + \beta_1 t_i) - \exp(\beta_0)\{\exp(\beta_1 t_i) - 1\}}{\beta_1^2}\right] \qquad (3.8)$$

$$\frac{\partial^2 \ell}{\partial \beta_1^2} = \sum_{i=1}^{m} -\left[\frac{\exp(\beta_0)\{\beta_1 t_i \exp(\beta_1 t_i)(\beta_1 t_i - 2) + 2\exp(\beta_1 t_i) - 2\}}{\beta_1^3}\right] \qquad (3.9)$$

The information matrix **J** is the inverse of the negative of the matrix of second derivatives. The standard errors are estimated by the square root of the leading diagonal of **J**. The results of fitting the log-linear model for the ROCOF given by equation (3.2), to the Jaguar data in Appendix 2, where phases are treated separately, are given in Table 3.1. In order to fit the model to these data the failure times have been multiplied scaled by a factor of $1/k$ (for some $k \neq 0$). This is necessary for two reasons. First to maintain the confidentiality of the data and second to allow the computation to be performed, else due to the log-likelihood in equation (3.3) containing $\exp(t)$ the numbers become too large. This is especially true for the model introduced in Section 3.3 where the log-likelihood contains a double exponential.

This table shows the parameter estimates with their standard errors in brackets, the log-likelihood for the full and reduced model and the test statistic for a likelihood ratio test. The likelihood ratio test has one degree of freedom, since one parameter has been removed between the two models. The test statistic can be compared with 3.841 ($\chi^2_{1,0.05}$). The results show that there is evidence that the ROCOF is non-linear for phases A0, A1-1 and A2-1. This evidence is very strong for phases A0 and A2-1. However, for phase A1-2 the results suggest

that a linear ROCOF is reasonable. The results of a two-tailed Z-test using the standard errors are shown in Table 3.2. These results show that in all cases the intercept parameter, $\beta_0$, is highly significant. The tests for non-linearity give the same conclusions as before, with the exception of phase A1-1 which gives a marginally significant result on the likelihood ratio test and a marginally insignificant result on the Z-test.

**TABLE 3.1**

*Summary of Log-linear Models Fitted to Jaguar Data.*

| Engine Phase | $\lambda(t) = \exp(\beta_0 + \beta_1 t)$ | | | $\lambda(t) = \exp(\beta_0)$ | Likelihood Ratio Statistic |
|:---:|:---:|:---:|:---:|:---:|:---:|
| | $\hat{\beta}_0$ | $\hat{\beta}_1$ | $\ell_{max}$ | $\ell_{max}$ | |
| A0 | 4.048 (0.1519) | −6.401 (1.1554) | 219.76 | 198.60 | 42.32 |
| A1-1 | 3.683 (0.1855) | −1.789 (0.8671) | 169.59 | 167.30 | 4.58 |
| A1-2 | 3.154 (0.2354) | −1.626 (1.0896) | 91.88 | 90.70 | 2.36 |
| A2-1 | 3.421 (0.1401) | −2.891 (0.6375) | 227.41 | 215.64 | 23.54 |

**TABLE 3.2**

*Hypothesis tests for Fitted Models.*

| Engine Phase | $\hat{\beta}_0/\mathrm{SE}(\hat{\beta}_0)$ | $\hat{\beta}_1/\mathrm{SE}(\hat{\beta}_1)$ | $\beta_1$ *Significant at 5%* |
|:---:|:---:|:---:|:---:|
| A0 | 26.649 | −5.540 | ✓ |
| A1-1 | 19.854 | −2.063 | ✓ |
| A1-2 | 13.393 | −1.492 | ✗ |
| A2-1 | 24.418 | −4.3534 | ✓ |

Plots of the failure number against failure times for the first four engine phases, A0 to A2-1, are shown in Figures 3.1 to 3.4 respectively. These plots show that a large amount of

variability exists between engines on a given phase. This would be expected since there will always be some variability between engines, however small, due to the differences in the quality of parts and the engine build process. However, the models fitted to this data have only estimated the mean ROCOF for each of these plots, and have taken no account of this variability. Clearly it would be useful to be able to estimate the variability between engines. This would not only give confidence levels to be placed on any predictions, but would also allow comparisons of the variability between phases to be made. As an engine is developed a decrease in the engine variability would be expected. This would be due to the identification and removal of failure causes and the engine design and build process becoming more robust. Within a suitable model a random effect is required to allow analysis of the variability between engines. The next section introduces a random effect model appropriate for data from repairable systems, and the remainder of the chapter develops the methodology for one such model.

## 3.2 Random Effect Models for Repairable Systems

Lawless (1995) presents a number of approaches for analysing data from recurrent events (repairable systems). Specifically, conditional and marginal models are discussed for Poisson and renewal processes. It is the Poisson process which is of interest here and this section presents the models in this context. Lawless (1995) also discusses the inclusion of covariate effects and this is presented in Chapter 6.

A marginal analysis uses rates and means to describe the event process. As in Section 2.7 the rate of occurrence function is defined as

$$v(t) = \frac{dE\{N(t)\}}{dt} \tag{3.10}$$

The expected number of failures in the interval $(0,t_0]$ are then
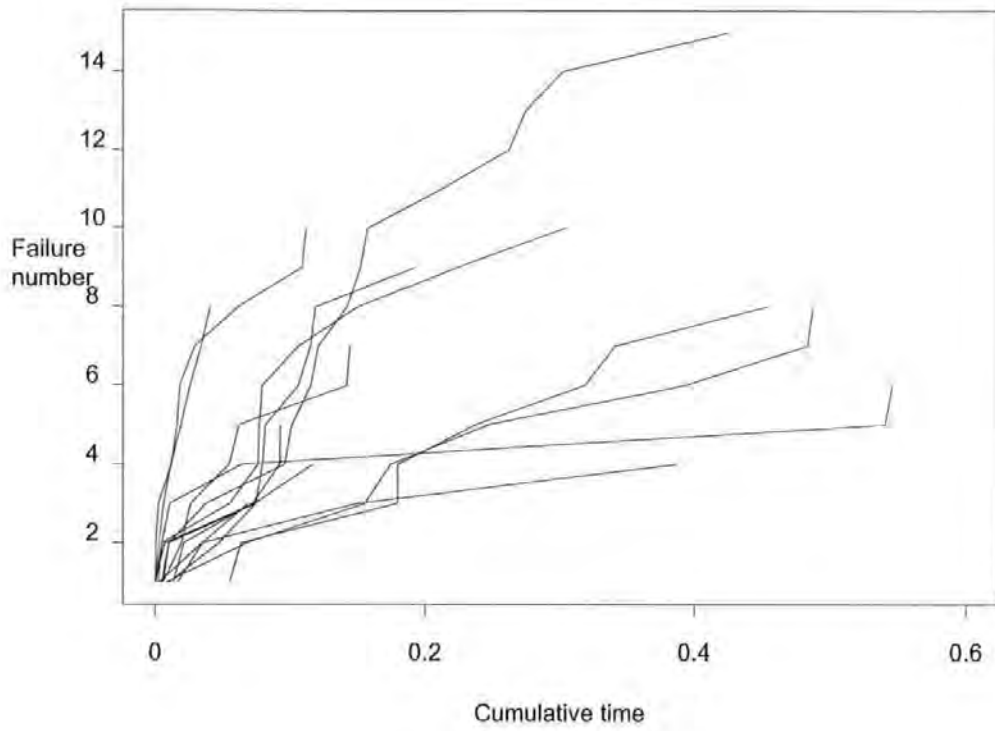
**Figure 3.1 :** Failure Time Plot for Engine Phase A0
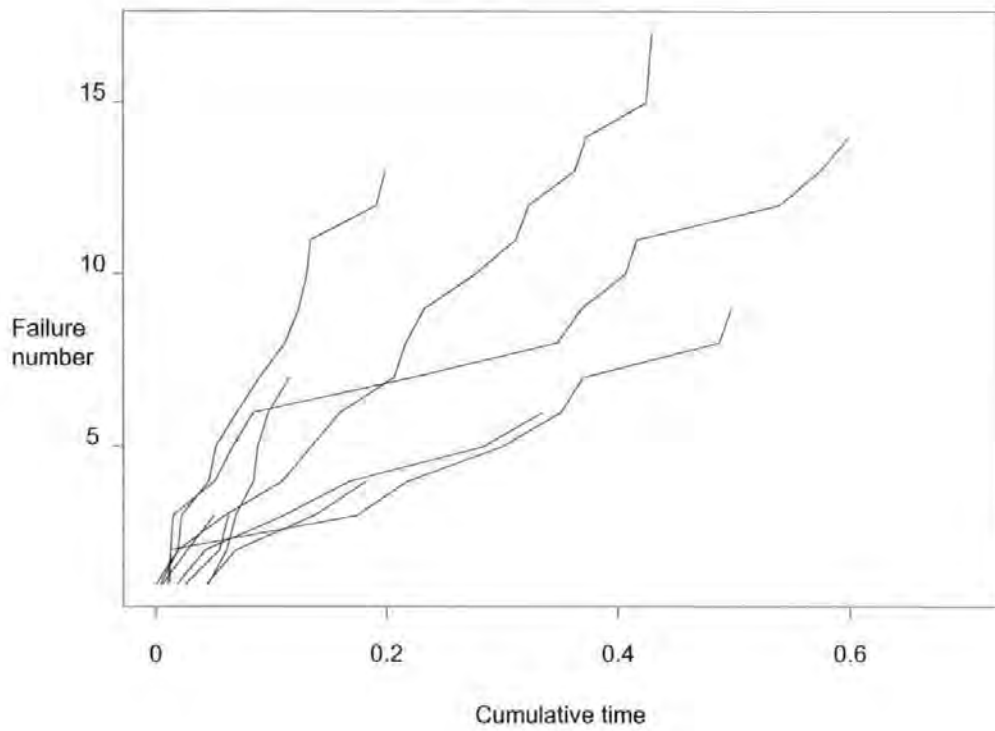


**Figure 3.2 :** Failure Time Plot for Engine Phase A1-1

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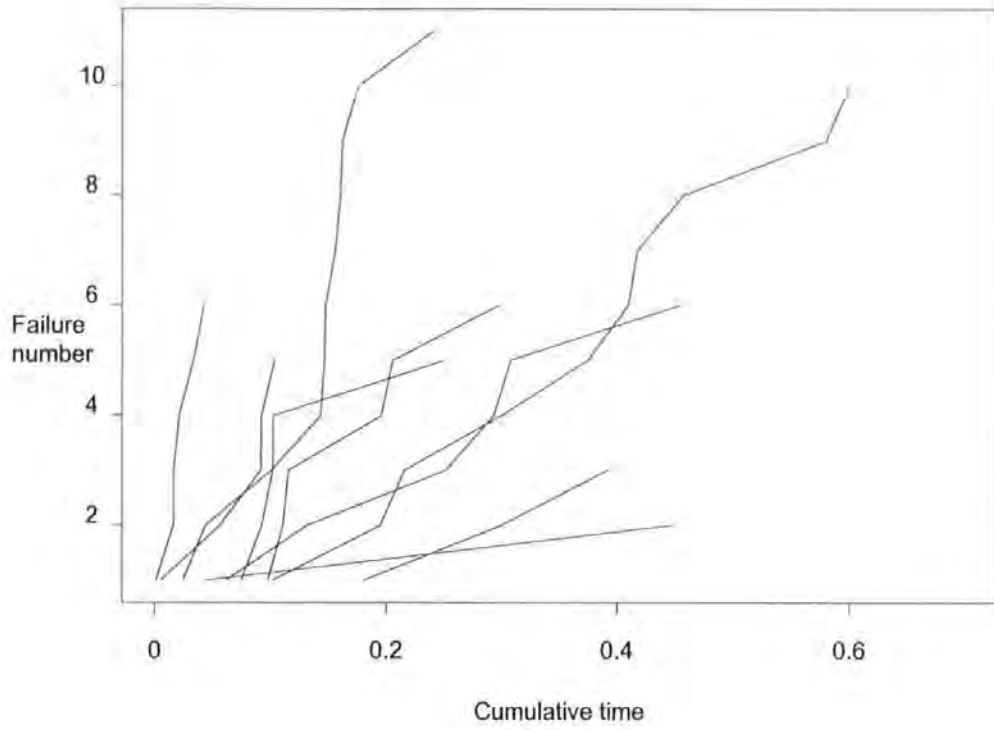
**Figure 3.3** : Failure Time Plot for Engine Phase A1-2
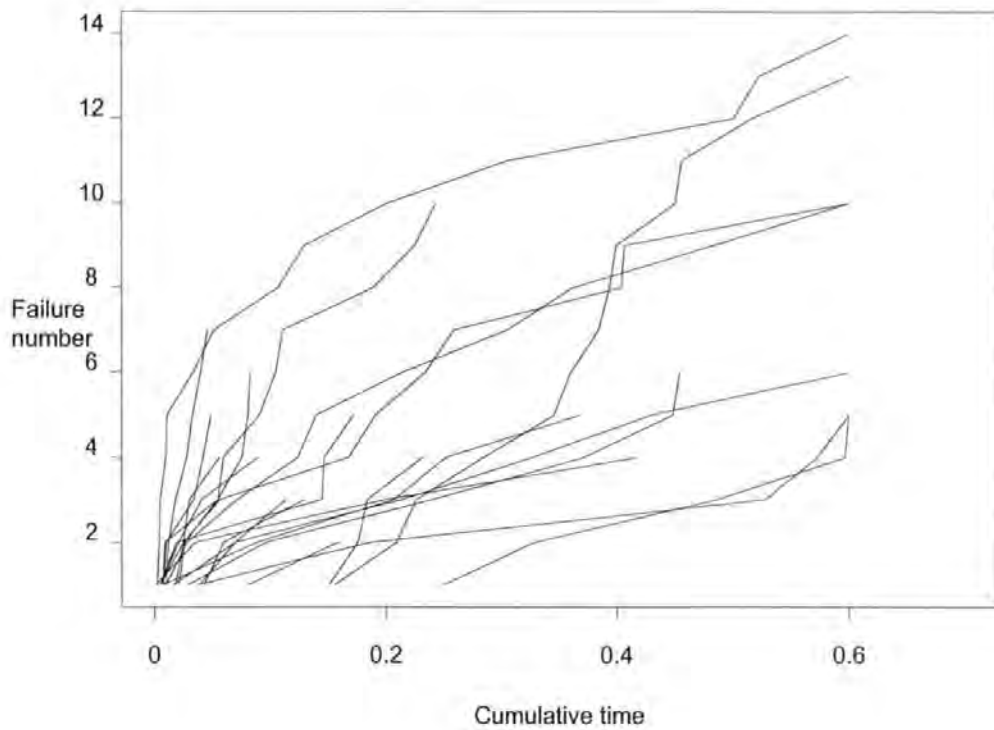


**Figure 3.4** : Failure Time Plot for Engine Phase A2-1

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$$\int_0^{t_0} v(t)\, dt. \tag{3.11}$$

Covariates can be included as a multiplicative function of the baseline rate function, $\rho(t)$. For covariate matrix $Z$ the model becomes

$$v(t;Z) = \rho(t)\phi(Z) \tag{3.12}$$

A conditional approach uses intensity functions to describe the event process over time. In the case of a single subject, let $H_t = \{N(s) : s < t\}$ describe the history of the process up to time $t$ and let $\partial N(t)$ denote the number of events in $(t, t + \partial t)$. The conditional intensity $\lambda(t; H_t)$ is

$$\lambda(t;H_t) = \lim_{\partial t \to 0}\left[\frac{P\{\partial N(t) = 1 \mid H_t\}}{\partial t}\right] \tag{3.13}$$

The probability density of $n$ events occurring at times $t_1, t_2, ..., t_n$ over interval $(0,T]$ is

$$\prod_{j=1}^{n} \lambda(t_j;H_t) exp\left\{-\int_0^T \lambda(t;H_t)\, dt\right\} \tag{3.14}$$

Covariates can be included in a multiplicative form as in the marginal model. The covariate intensity model is specified by

$$\lambda(t;H_t) = \rho(t)\phi(Z) \tag{3.15}$$

In general the intensity function is different from the rate function, one being defined in terms of probability of an event at time $t$ and the other in terms of the rate of events at time $t$. For a Poisson process, including the non-homogeneous Poisson process which is of interest here, the rate and intensity functions are the same. This is demonstrated by the likelihood function for the NHPP model with rate $\lambda(t)$, equation (2.51), being directly equivalent to the

pdf for the conditional intensity model in equation (3.14). A application of these models which is of interest in this study is given by Lawless (1995) and Kalbfleisch *et al.* (1991). This deals with the cost of warranty of automobiles in service, where there is a time lag between failures and reported failures.

Lawless (1995) also demonstrates the inclusion of a random effect on NHPP models of this form. This is an extension to the random effect model introduced by Lawless (1987). In both of these papers a Poisson process with an intensity of the form of $\alpha_i \rho(t)$ is used, where $\alpha_i$ is an unobservable random effect or 'frailty' variable for each subject. Covariate effects are also included in the same way as in equation (3.15). Lawless (1995, 1987) assumes the variability between subjects follows a gamma distribution with mean = 1 and variance = $\sigma_\alpha^2$. For a gamma distribution with parameters $\nu$ (shape) and $\tau$ (scale), the mean is given by $\nu/\tau$ and the variance by $\nu/\tau^2$. It then follows that for the assumption of subject variability following a gamma distribution with mean = 1, the shape and scale parameters are restricted to taking the same values. Therefore as the variability between subjects changes, the shape of the distribution will also change. This would not be expected in a realistic situation. Ciampi *et al.* (1992) provide a program for fitting this random effect model in a parametric and semi-parametric form.

An alternative model was proposed by Abu-Libdeh *et al.* (1990) in the context of the occurrence of skin cancers. Again the gamma distribution was used to model the variability between subjects. However, in this case both the shape and scale parameters are estimated with no restrictions on the mean and variance. This may lead to high levels of association between the parameters.

Both of these approaches use a gamma distribution to model the subject variability. The gamma distribution is used as it is analytically convenient, and the likelihood can be specified in order to obtain parameter estimates. The interpretation of these estimates however is not straightforward. This can be overcome by using a normal distribution for the random effect, which uses a single parameter to express the variability between subjects. In this case the likelihood cannot be expressed analytically, although computational techniques

such as numerical integration routines will allow a value of the likelihood to be estimated. Such a model is introduced in the following sections.

## 3.3 Log-Linear Model with Random Effect on the Intercept

The simplest random effect model introduces the random effect onto the intercept parameter $\beta_0$ of the log-linear model. This random effect allows each unit to have a different intercept. This implies that each unit will have a different starting reliability but the ROCOF will change at the same rate. The rate of occurrence of failure for the $i^{th}$ unit is now modelled by equation (3.16)

$$\lambda_i(t) = \exp(\beta_0 + \beta_1 t + \varepsilon_i)$$ (3.16)

where

$\varepsilon_i$ is a random effect due to unit $i$, and $\varepsilon \sim N(0, \sigma_\varepsilon^2)$

In this case the engine variation $\varepsilon$ is assumed to be normally distributed, with a mean of zero and constant variance. Placing the variance component on $\beta_0$ gives the $i^{th}$ unit a proportional effect on the number of failures; that is, the change in expected number of failures caused by the random effect is not dependent on time. Using equation (2.47) the expected number of failures in time interval $(a,b]$ are

$$E\{\text{failures}(a,b] \mid \varepsilon_i\} = \exp(\beta_o + \varepsilon_i) \int_a^b \exp(\beta_1 t) \, dt$$ (3.17)

The estimates ($\hat{\beta}_0$, $\hat{\beta}_1$ and $\hat{\sigma}_\varepsilon$) of the three parameters can be obtained by the method of maximum likelihood. Substituting equation (3.16) into equation (3.1), for $m$ units with $n_i$ failures on each, gives the likelihood function conditional on the random effect $\varepsilon$ as

$$P(\text{data}|\varepsilon_1,\ldots,\varepsilon_n,\beta) = \prod_{i=1}^{m}\left\{\prod_{j=1}^{n_i}\exp(\beta_0+\beta_1 t_{ij}+\varepsilon_i)\right\}\exp\left\{-\int_0^{t_i}\exp(\beta_0+\beta_1 t+\varepsilon_i)dt\right\} \tag{3.18}$$

where $\beta$ is the vector of parameters.

$$P(\text{data}|\beta) = \prod_{i=1}^{m}\int_{-\infty}^{\infty}P(\text{data}_i|\varepsilon_i,\beta)P(\varepsilon_i)\,d\varepsilon_i \tag{3.19}$$

$$L = \prod_{i=1}^{m}\left[\int_{-\infty}^{\infty}\left\{\prod_{j=1}^{n_i}\lambda_i(t_{ij})\right\}\exp\left\{-\int_0^{t_i}\lambda_i(t)\,dt\right\}\frac{1}{\sqrt{2\pi\sigma_\varepsilon^2}}\exp\left\{-\frac{(\varepsilon_i-\mu)^2}{2\sigma_\varepsilon^2}\right\}d\varepsilon_i\right] \tag{3.20}$$

In order to obtain the maximum likelihood estimates of the parameters, the condition on the random effect has to be removed. This is achieved by multiplying equation (3.18) by the probability density function of $\varepsilon$ and then integrating with respect to $\varepsilon$, as in equation (3.19). The likelihood function is now expressed by equation (3.20). Taking logarithms of equation (3.20) gives the log-likelihood function, which is easier to calculate than the likelihood, but produces the same estimates of the parameters. Using equation (3.16) for $\lambda_i(t)$, and the assumption of zero mean and constant variance ($\sigma_\varepsilon^2$) for the random effect, we obtain the log-likelihood after simplification as

$$\ell = \sum_{i=1}^{m}\ln\left(\exp\left(n_i\beta_0+\beta_1\sum_{j=1}^{n_i}t_{ij}\right)\int_{-\infty}^{\infty}\exp\left[n_i\varepsilon_i-\frac{\exp(\beta_0+\varepsilon_i)}{\beta_1}\left\{\exp(\beta_1 t_i)-1\right\}\right]\frac{1}{\sqrt{2\pi\sigma_\varepsilon^2}}\exp\left\{-\frac{\varepsilon_i^2}{2\sigma_\varepsilon^2}\right\}d\varepsilon_i\right) \tag{3.21}$$

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The integral in equation (3.21) cannot be evaluated analytically and must therefore be approximated numerically. Numerical methods of integration appropriate for estimating integrals of this type are discussed in Section 3.4.

In order to obtain the information matrix, $J$, and hence estimates of the standard errors, the matrix of second order partial derivatives is required. First we define the following quantities:

$$z_i = \exp\left[n_i\beta_0 + \beta_1\sum_{j=1}^{n_i}t_{ij} + n_i\varepsilon_i - \frac{\exp(\beta_0 + \varepsilon_i)}{\beta_1}\left\{\exp(\beta_1 t_i) - 1\right\}\right]\frac{1}{\sqrt{2\pi\sigma_\varepsilon^2}}\exp\left\{-\frac{\varepsilon_i^2}{2\sigma_\varepsilon^2}\right\} \qquad (3.22)$$

$$A_i = \int_{-\infty}^{\infty}z_i\,d\varepsilon_i \qquad (3.23)$$

$$B_i = n_i - \frac{\exp(\beta_0 + \varepsilon_i)}{\beta_1}\left\{\exp(\beta_1 t_i) - 1\right\} \qquad (3.24)$$

$$C_i = \sum_{j=1}^{n_i}t_{ij} - \frac{\exp(\beta_0 + \varepsilon_i)}{\beta_1^2}\left\{\exp(\beta_1 t_i)(\beta_1 t_i - 1) + 1\right\} \qquad (3.25)$$

$$D_i = \frac{\varepsilon_i^2 - \sigma_\varepsilon^2}{\sigma_\varepsilon^3} \qquad (3.26)$$

$$E_i = -\frac{\exp(\beta_0 + \varepsilon_i)}{\beta_1^3}\left\{\beta_1 t_i\exp(\beta_1 t_i)(\beta_1 t_i - 2) + 2\exp(\beta_1 t_i) - 2\right\}. \qquad (3.27)$$

The first order partial derivatives are now given by equations (3.28) to (3.30):

$$\frac{\partial \ell}{\partial \beta_0} = \sum_{i=1}^{m}\frac{1}{A_i}\int_{-\infty}^{\infty}B_i z_i\,d\varepsilon_i \qquad (3.28)$$

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$$\frac{\partial \ell}{\partial \beta_1} = \sum_{i=1}^{m} \frac{1}{A_i} \int_{-\infty}^{\infty} C_i z_i \, d\varepsilon_i \tag{3.29}$$

$$\frac{\partial \ell}{\partial \sigma_\varepsilon} = \sum_{i=1}^{m} \frac{1}{A_i} \int_{-\infty}^{\infty} D_i z_i \, d\varepsilon_i. \tag{3.30}$$

The second partial derivatives are given by equations (3.31) to (3.36) respectively:

$$\frac{\partial^2 \ell}{\partial \beta_0{}^2} = \sum_{i=1}^{m} \frac{1}{A_i{}^2} \left\{ A_i \int_{-\infty}^{\infty} \left( B_i - n_i + B_i{}^2 \right) z_i \, d\varepsilon_i - \left( \int_{-\infty}^{\infty} B_i z_i \, d\varepsilon_i \right)^2 \right\} \tag{3.31}$$

$$\frac{\partial^2 \ell}{\partial \beta_1 \partial \beta_0} = \sum_{i=1}^{m} \frac{1}{A_i{}^2} \left[ A_i \int_{-\infty}^{\infty} \left\{ \left( C_i - \sum_{j=1}^{ni} t_j \right) + B_i C_i \right\} z_i \, d\varepsilon_i - \int_{-\infty}^{\infty} B_i z_i \, d\varepsilon_i \int_{-\infty}^{\infty} C_i z_i \, d\varepsilon_i \right] \tag{3.32}$$

$$\frac{\partial^2 \ell}{\partial \sigma \partial \beta_0} = \sum_{i=1}^{m} \frac{1}{A_i{}^2} \left( A_i \int_{-\infty}^{\infty} B_i D_i z_i \, d\varepsilon_i - \int_{-\infty}^{\infty} B_i z_i \, d\varepsilon_i \int_{-\infty}^{\infty} D_i z_i \, d\varepsilon_i \right) \tag{3.33}$$

$$\frac{\partial^2 \ell}{\partial \beta_1{}^2} = \sum_{i=1}^{m} \frac{1}{A_i{}^2} \left\{ A_i \int_{-\infty}^{\infty} \left( E_i + C_i{}^2 \right) z_i \, d\varepsilon_i - \left( \int_{-\infty}^{\infty} C_i z_i \, d\varepsilon_i \right)^2 \right\} \tag{3.34}$$

$$\frac{\partial^2 \ell}{\partial \sigma_\varepsilon \partial \beta_1} = \sum_{i=1}^{m} \frac{1}{A_i{}^2} \left( A_i \int_{-\infty}^{\infty} C_i D_i z_i \, d\varepsilon_i - \int_{-\infty}^{\infty} C_i z_i \, d\varepsilon_i \int_{-\infty}^{\infty} D_i z_i \, d\varepsilon_i \right) \tag{3.35}$$

$$\frac{\partial^2 \ell}{\partial \sigma_\varepsilon{}^2} = \sum_{i=1}^{m} \frac{1}{A_i{}^2} \left[ A_i \int_{-\infty}^{\infty} \left\{ \left( \frac{\sigma_\varepsilon{}^2 - 3\varepsilon_i{}^2}{\sigma_\varepsilon{}^4} \right) + D_i{}^2 \right\} z_i \, d\varepsilon_i - \left( \int_{-\infty}^{\infty} D_i z_i \, d\varepsilon_i \right)^2 \right] \tag{3.36}$$

## 3.4 Numerical Approximation Methods of Integration

This section introduces and compares a number of techniques for approximating integrals. As stated previously, the integral within the log-likelihood function, equation (3.21), cannot be evaluated analytically and must be approximated by numerical methods. The aim of this section is to demonstrate a method that gives estimates of the value of the log-likelihood. These estimates must be both accurate to a number of significant figures and also computationally efficient. A number of rules exist, known as Quadrature rules, for approximating integrals and these take the following general form:

$$\int_a^b G(x)\, dx = \int_a^b \omega(x) f(x)\, dx = \sum_{j=1}^n H_j f(x_j) + E_n(f) \tag{3.37}$$

where

> $G(x)$ is the function to be integrated which decomposes to $f(x)$ and $\omega(x)$.
>
> $\omega(x)$ is a weight function.
>
> $x_j$'s are the abscissæ of the quadrature rule.
>
> $H_j$'s are the weights.
>
> $E_n(f)$ is the error term.

The weight function is a part of the integrand which is usually separated from $f(x)$. Quadrature rules can be divided into two classes, Newton-Cotes quadrature and Gaussian quadrature, and these are introduced in the following sections. For further details on these methods see Smith (1993) and Davis & Rabinowitz (1984).

In order to demonstrate the quadrature rules, they have been applied to the integral given in equation (3.38), the integral contained within the log-likelihood function, equation (3.21).

$$\int_{-\infty}^{\infty} \exp\left[ n_i \beta_0 + \beta_1 \sum_{j=1}^{n_i} t_{ij} n_i \varepsilon_i - \frac{\exp(\beta_0 + \varepsilon_i)}{\beta_1} \left\{ \exp(\beta_1 t_i) - 1 \right\} \right] \frac{1}{\sqrt{2\pi\sigma_\varepsilon^2}} \exp\left\{ -\frac{\varepsilon_i^2}{2\sigma_\varepsilon^2} \right\} d\varepsilon_i \tag{3.38}$$

In all cases the values of the parameters, $\beta_0$, $\beta_1$ and $\sigma_\varepsilon$, are 1 and the event times are given as t = 0.1, 0.3, 0.5, 0.7, 0.9, with no censoring occurring.

### 3.4.1 Newton-Cotes Quadrature

For Newton-Cotes quadrature the abscissæ, $x_j$'s, are equally spaced over the limits of integration. The weight function, $\omega(x)$, is always taken to be 1. The first and simplest Newton-Cotes rule is the Trapezoidal rule. If the interval of integration, $[a,b]$, is divided into $n$ equally spaced sub-intervals of width $h = (b-a)/n$, then the Trapezoidal rule is defined by:

$$\int_a^b f(x) \approx h\left\{\frac{1}{2}f(a) + \sum_{r=1}^{n-1} f(a + rh) + \frac{1}{2}f(b)\right\} \tag{3.39}$$

The second, and more complex Newton-Cotes rule is Simpson's rule. With $n$ and $h$ as for the Trapezoidal rule, Simpson's rule is given by equation (3.40). For this rule to be appropriate $n$ must be even. Simpson's rule is generally expected to be more accurate than the Trapezoidal rule.

$$\int_a^b f(x) \approx \frac{h}{3}\left\{f(a) + 4f(a+h) + 2f(a+2h) + 4f(a+3h) + \ldots + 2f(b-2h) + 4f(b-h) + f(b)\right\} \tag{3.40}$$

These two rules have been applied, with differing numbers of intervals, to the integral given in equation (3.38). Although this integral is defined between plus and minus infinity the integral has been calculated between plus and minus five standard deviations $\sigma_\varepsilon$ which gives a good approximation to the true value, and the results are given in Table 3.3. There are other higher order Newton-Cotes rules which are more complicated, for example the Romberg rule, although these have not been considered here.

The results in Table 3.3 show that for larger numbers of intervals over the limits of integration both of these methods give the value of the integral to be the same. For integrals of the type given by equation (3.38) the Trapezoidal rule appears to give more consistent

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approximations to the integral for smaller values of $n$. However, to ensure accuracy in the estimate, larger values of $n$ would be used, giving no clear preference in the choice of method.

**TABLE 3.3**

*Integral Approximations using Newton-Cotes Quadrature.*

| Number of | Newton-Cotes Quadrature Rule | |
|---|---|---|
| Intervals (n) | Trapezoidal | Simpson's |
| 2 | 40.93810 | 54.58413 |
| 4 | 20.56124 | 13.76895 |
| 6 | 15.67792 | 19.55103 |
| 10 | 15.81589 | 16.21551 |
| 20 | 15.81606 | 15.81612 |
| 50 | 15.81607 | 15.81607 |
| 100 | 15.81607 | 15.81607 |

### 3.4.2 Gaussian Quadrature

For Gaussian quadrature the abscissæ, $x_i$'s, are not equally spaced over the region of integration. The weight function, $\omega(x)$, varies for different Gaussian quadrature rules. The most common Gaussian rules, together with their weight functions and regions of integration, are given in Table 3.4.

The two rules that are most applicable to the integral given by equation (3.38) are the Gauss-Legendre and the Gauss-Hermite. The Gauss-Legendre quadrature rule can be applied to the integral without changing the form of equation (3.38), although some scaling is required to change the limits of integration to plus and minus one. To apply the Gauss-Hermite quadrature rule a factor of $\exp\left(-\varepsilon^2\right)$ must be extracted from $f(x)$ and used as the weight function. The quadrature rule is then applied to this revised $f(x)$. For both of these methods

**TABLE 3.4**

*Gaussian Quadrature Rules.*

| Gaussian Quadrature Rule | Weight Function $\omega(x)$ | Limits of Integration |
|---|---|---|
| Legendre | 1 | [-1,1] |
| Chebyshev | $(1-x^2)^{-\frac{1}{2}}$ | [-1,1] |
| Laguerre | $e^{-x}$ | $[0,\infty)$ |
| Hermite | $e^{-x^2}$ | $(-\infty,\infty)$ |

the abscissæ and weights ($x_j$'s and $H_j$'s) can be taken from published tables (see for example Tables 25.4 and 25.10 in Abramowitz and Stegun (1972) respectively for the Legendre and Hermite methods). In both cases the integral is approximated, with the appropriate limits, by

$$\int_a^b \omega(x)f(x)\,dx \approx \sum_{j=1}^n H_j f(x_j) \tag{3.41}$$

The results of applying these rules, for values of $n$ up to 20, are given in Table 3.5. The results show that the Gauss-Hermite approximation is a more consistent estimator than Gauss-Legendre. This would be expected since the Gauss-Hermite quadrature rule is more applicable to the integral of equation (3.38) due to its more specific weight function and limits of integration. For the values of $n$ given, neither of these rules appear to have converged to approximations given by the Newton-Cotes quadrature rules, and a further investigation of increased number of points is required. Alternatively, instead of increasing $n$, the Gauss-Legendre quadrature rule can be applied in composite form. This is achieved by dividing the region of integration into m equal intervals, and calculating the $n$-point Gauss-Legendre approximation for each interval. The overall value of the integral is then estimated by summing the $m$ approximations. The advantage of using this method is that the error term is reduced by a factor of $1/m^{2n}$ (see Ralston and Rabinowitz, 1978). The results of using this method for $m$ intervals when $n = 4$ are given in Table 3.6. These show that a good estimate is provided with small values of $m$ and that the approximation converges fairly quickly.

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**TABLE 3.5**

*Integral Approximations Using Gauss-Legendre and Hermite Rules.*

| Number of Points | Gaussian Quadrature Rule | |
|---|---|---|
| $n$ | Legendre | Hermite |
| 2 | 0.04199 | 12.39825 |
| 3 | 36.38991 | 16.71409 |
| 4 | 3.51534 | 15.89739 |
| 5 | 23.33083 | 15.45319 |
| 10 | 16.05028 | 15.87290 |
| 20 | 15.81593 | 15.81916 |

**TABLE 3.6**

*Results of Four Point Composite Gauss-Legendre Integration.*

| Number of Intervals ($m$) | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Approximation | 3.51534 | 15.17167 | 15.84944 | 15.84139 |

| Number of Intervals ($m$) | 5 | 10 | 20 | 50 |
|---|---|---|---|---|
| Approximation | 15.81481 | 15.81606 | 15.81607 | 15.81607 |

Another extension to the Gauss-Legendre quadrature is the Gauss-Kronrod scheme. For Gauss-Legendre quadrature, if the value of $n$ is increased the new approximation makes no use of the previous value. For Gauss-Kronrod, the weights and abscissæ are calculated so that they can be combined with the previous estimate, giving more accuracy for less computation. The *integrate* function in S-Plus uses an automatic Gauss-Kronrod scheme

that increases n at each step until a convergence criteria is reached. Using this method to estimate the integral given by equation (3.38) gives an answer of 15.81607.

### 3.4.3  Summary of Integration Results

The preceding sections have demonstrated a number of numerical integration techniques that allow the log-likelihood, equation (3.21), to be estimated. For the example given it can be seen that the value of the integral to 5 decimal places is 15.81607, and that a number of the methods converge to this figure.

S-Plus provides an integration function, using the Gauss-Kronrod method. However, this function is slow to produce an approximation and also uses large amounts of memory. For example, using a 486 dx2/66 machine takes 7.97 seconds to produce an estimate of the log-likelihood for the Jaguar A0 data in Appendix 2. Using the Composite Gauss-Legendre quadrature rule takes only 0.66 seconds to produce the same estimate. This rule has been demonstrated to estimate the integral as well as any of the other methods discussed, and requires smaller values of $n$ than other methods, and for these reasons the Composite Gauss-Legendre rule has been used, with $m = 10$, in all further analysis.

## 3.5  Fitted Models with Random Effect on the Intercept

The random effect log-linear model, given by equation (3.16), has been fitted to the Jaguar data in Appendix 2. The four phases have been treated separately, and as in Section 3.1 the failure times have been scaled by a factor of $1/k$. The fitted parameters and the maximised log-likelihoods are given in Table 3.7. The S-Plus functions for fitting these models, as well as a function for calculating the nodes for Gauss-Legendre integration, are given in Appendix 4 and the residuals for these models are considered in Chapter 5.

**TABLE 3.7**

*Parameter Estimates to Jaguar Data for Model with Random Effect on $\beta_0$.*

| Phase | $\hat{\beta}_0$ | $\hat{\beta}_1$ | $\hat{\sigma}_\varepsilon$ | $\ell_{max}$ |
|-------|-------|-------|-------|-------|
| A0 | 3.945 | -5.830 | 0.486 | 222.48 |
| A1-1 | 3.622 | -1.432 | 0.321 | 170.48 |
| A1-2 | 2.904 | 0.046 | 0.835 | 96.54 |
| A2-1 | 3.356 | -2.452 | 0.526 | 230.87 |

The significance of including the random effect can be tested by performing a likelihood ratio test between this model and the standard log-linear model, equation (3.2). The values of the likelihood ratio statistic for each of the four models are given in Table 3.8. These test statistics can be compared with 3.841 ($\chi^2_{1,0.05}$) and the results show, except for phase A1-1, a significant reduction in the log-likelihood function if the random effect is removed. Hence it can be concluded from this test that the random effect should be included in the model.

**TABLE 3.8**

*Likelihood Ratio Statistic for Random Effect Models Fitted to Jaguar Data.*

| Engine Phase | A0 | A1-1 | A1-2 | A2-1 |
|-------|-------|-------|-------|-------|
| Likelihood Ratio Statistic | 5.44 | 1.78 | 9.32 | 6.92 |

The standard errors and the resulting test statistic calculated from these to compare the estimated parameter with zero are given in Table 3.9. For the fixed effects a two tailed test is required, comparing the null hypothesis of $H_0 : \beta_i = 0$ against the alternative $H_0 : \beta_i \neq 0$. For the random effect the hypothesis is $H_0 : \sigma_\varepsilon = 0$ against the alternative $H_0 : \sigma_\varepsilon > 0$ since the standard deviation cannot take a negative value. The test statistics are compared with Z values of 1.64 and 1.96 for the one and two tailed tests respectively. The results show some agreement with the likelihood ratio test, although the random effect for phase A1-1 is now

found to significantly different from zero at the 5% level (p = 3.8%). The use of these two tests for the random effect is discussed in more detail in Section 3.9.

**TABLE 3.9**

*Standard Errors and Test Statistics for Random Effect Models Fitted to Jaguar Data.*

| Engine Phase | | $\hat{\beta}_0$ | $\hat{\beta}_1$ | $\hat{\sigma}_\varepsilon$ |
|---|---|---|---|---|
| A0 | SE | 0.2072 | 1.2417 | 0.1825 |
| | Test Statistic | 19.040 | -4.695 | 2.663 |
| A1-1 | SE | 0.2235 | 0.9612 | 0.1810 |
| | Test Statistic | 16.206 | -1.490 | 1.773 |
| A1-2 | SE | 0.3888 | 1.3219 | 0.3020 |
| | Test Statistic | 7.469 | 0.035 | 2.765 |
| A2-1 | SE | 0.1854 | 0.6999 | 0.1627 |
| | Test Statistic | 18.101 | -3.503 | 3.233 |

The tests for the intercept parameter $\beta_0$ show that the fitted values are highly significant (p < 0.001). The hypothesis tests for the gradient parameter $\beta_1$ yield the same results as the model without a random effect, see Section 3.1. For phases A1-1 and A1-2 there is no evidence to suggest that the ROCOF's are not constant over time. However, examination of the relevant failure time plots in Figures 3.2 and 3.3 shows that this is unlikely to be true. For some engines the ROCOF is increasing over time, whilst for others it is decreasing, so the constant ROCOF obtained could be due to a cancellation effect. These two phases also have a small sample size of engines ($m$ = 9) which will have an effect on the values and accuracy of the fitted parameters. These small sample sizes are also unlikely to yield good estimates of the variance component.

It should be noted that for three of the four fitted models, the gradient parameter $\beta_1$ is negative, the interpretation being that the ROCOF is decreasing over time. In other words, the reliability of an engine would be at its worst when new and would improve as time passes. This result is the opposite of what would be predicted, since an engine would be expected to be

at its most reliable at the early stages of its life and become less reliable as it gets older. As stated in Chapter 1, the testing procedures carried out by Jaguar on the engines are either simulations near to real-life engine running (the reliability test) or highly accelerated tests. The aim of the accelerated tests is to simulate the working lifetime of an engine over a shorter period of time. In practice therefore, the ROCOF should be increasing over time on these tests, yielding a positive estimate of the gradient parameter $\beta_1$. The contrary result obtained by fitting the models could be attributed to a number of causes. The data are currently censored at 0.6 units engine running time. However it may be that the engines will have to be tested for longer periods before engine fatigue is achieved. The decreasing ROCOF on the tests is due to a large number of failures early on in the testing, and a number of these will be infant mortalities associated with the engine build rather than component failure. At present all failures in the first few hours are classed as infant mortalities and are excluded from the analysis, but it may be that more failures need to be excluded before a true picture of the component failure is reached. Alternatively, the result could be attributed to the problems of recording the failure times accurately, as described in Sections 1.2 and 1.3.

The fitted models can be used to obtain predictions of the expected number of failures in a given time interval. Let us say that the number of failures up to 0.32 (rescaled time value) are of interest. Using equation (3.17) with this figure as the upper integration limit and zero as the lower limit, we may predict the expected number of failures in this interval. The random effect can be used to place an interval on the expected number of failures, allowing for the variation in the model. The fitted model gives the intercept parameter to be distributed normally with a mean of $\beta_0$ and variance $\sigma_\varepsilon^2$. A 95% Prediction Interval of the intercept parameter can be calculated from

$$\hat{\beta}_0 \pm 1.96\hat{\sigma}_\varepsilon \qquad\qquad (3.42)$$

By using the limits of this predicted interval in equation (3.17) an approximate interval on the expected number of failures can be obtained. The predicted number of failures for the fitted models on the four engine phases are given in Table 3.10.

**TABLE 3.10**

*Predicted Numbers of Failures for Model with Random Effect on $\beta_0$.*

| Engine Phase | Expected Number of Failures | | | Interval Width |
|---|---|---|---|---|
| | Mean | $\hat{\beta}_0 - 1.96\hat{\sigma}_\varepsilon$ | $\hat{\beta}_0 + 1.96\hat{\sigma}_\varepsilon$ | |
| A0 | 7.49 | 2.89 | 19.41 | 16.52 |
| A1-1 | 9.61 | 5.12 | 18.04 | 12.92 |
| A1-2 | 5.88 | 1.14 | 30.24 | 29.10 |
| A2-1 | 6.38 | 2.28 | 17.87 | 15.59 |

These results are actually much higher than the target figure for the engine. However, these failure data are from prototype engines which would not be expected to reach the target figure and therefore this difference is not necessarily due to an inappropriate model. Examination of the failure time plots in Figures 3.1 to 3.4 and the data in Appendix 2 show that the mean number of failures given in Table 3.10 appears reasonable. As stated in Section 2.7.2, the number of failures over a time interval follow a Poisson distribution with the mean given by equation (3.17). Using this result the probability of the test engines achieving the target reliability can be found. As the acceleration factors are unknown there is no way to predict field reliability from test data. This is a problem that needs to be addressed before accurate reliability predictions can be obtained.

Although the field reliability cannot be predicted from this test data, these estimates can still be used to monitor engine development. Since the engines from each phase have completed the same simulations under the same test conditions, comparisons can be made between the reliability of the development engine phases. It would be hoped that as the development progresses the reliability will improve and also that the variability between engines will decrease. A visual inspection of the results in Tables 3.7 and 3.10 show that this does not appear to be the case. A more rigorous approach to comparing the engine phases is given in Chapter 7.

## 3.6 Simulation Study on the Random Effect Model

### 3.6.1 Introduction

The previous sections have introduced the log-linear Non-Homogeneous Poisson Process model with a random effect on the intercept, and have given fitted models for the Jaguar data as well as appropriate tests of significance. These tests of significance for the fitted parameter values rely on the information matrix yielding good estimates of the standard errors. Although the true values of the standard errors are not known and so cannot be compared directly with the estimates, the validity of the estimates can be checked using simulated data. The following two sections describe how to simulate data from a log-linear NHPP and how this simulated data can be used to estimate the standard errors.

### 3.6.2 The Inversion Method for Simulating from a Log-Linear NHPP Model

To simulate a series of failure times from a Non-Homogeneous Poisson Process, the inversion method (Ripley, 1987 and Lewis & Shedler, 1976) is appropriate. This requires the inverse of the cumulative distribution function to be found. Random numbers generated from a standard uniform distribution can then be used with this inverse function to generate the event times.

For a single unit with Rate of Occurrence of Failures described by $\lambda(t)$, over the time interval $(0, t_0]$, the number of failures follows a Poisson distribution with parameter $\mu$ obtained from integrating equation (3.17) over the appropriate interval. The value of $\mu$ given the random effect can be expressed as a function of 0 and $t_0$, as

$$\mu = \Lambda(t_0) - \Lambda(0) = \int_0^{t_0} \exp(\beta_0 + \beta_1 t + \varepsilon_i) dt \qquad (3.43)$$

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where $\Lambda(t)$ is $\int \exp(\beta_0 + \beta_1 u + \varepsilon_i) du$ evaluated at $t$. The cumulative distribution function of $t$ on the interval $(0, t_0]$, is expressed by equation (3.44).

$$F(t) = \frac{\Lambda(t) - \Lambda(0)}{\Lambda(t_0) - \Lambda(0)} \qquad (3.44)$$

If there are $N$ failures/events, from Poisson($\mu$), in $(0, t_0)$ and these occur at times $T_1$, $T_2, ..., T_N$, then conditional on having observed $n$ ($>0$) events, the $T_i$'s are distributed as order statistics from $F(t)$, Ripley (1987).

To simulate for the log-linear model with the random effect on $\beta_0$, equation (3.16), the method is as follows. The cumulative distribution function is given by

$$F(t) = \frac{\exp(\beta_0 + \varepsilon_i)\{\exp(\beta_1 t) - 1\} / \beta_1}{\exp(\beta_0 + \varepsilon_i)\{\exp(\beta_1 t_0) - 1\} / \beta_1} = \frac{\exp(\beta_1 t) - 1}{\exp(\beta_1 t_0) - 1} \qquad (3.45)$$

For $\beta_1 \neq 0$ invert $\rho = F(t)$ to express the function in terms of $t$ as

$$t = \ln\left[\rho\{\exp(\beta_1 t_0) - 1\} + 1\right] / \beta_1 \qquad 0 \leq \rho \leq 1 \qquad (3.46)$$

Hence, for any value of $\rho$ within the range, the corresponding event time can be found. So a random observation generated from a standard uniform distribution can be substituted for $\rho$ in equation (3.46) to calculate an event time. These results can be combined as the following algorithm.

For the $i^{th}$ unit:

i. Generate a random variable $n_i$, the number of events that occur, from Poisson($\mu \mid \varepsilon_i$). If $n_i = 0$ then no events occur on this unit, so stop. Otherwise continue.

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ii. Generate $n_i$ uniform variates and order $U_1 \le U_2 \le ... \le U_{ni}$.

iii. Obtain $n_i$ event times by substituting $U_1,...,U_{ni}$ for $\rho$ into equation (3.46).

### 3.6.3 Validation of the Standard Errors

The previous section has shown how the inversion method can be used to simulate data from a given log-linear Non-Homogeneous Poisson Process model. This section details the use of simulation in validating the standard errors of the maximum likelihood estimates of the parameters.

The method for $n$ replications on $m$ simulated units is as follows. For each replication, $m$ series of event times, for a given time interval $(0,t]$, are simulated from the same parameter values. These $m$ series comprise the data set. The maximum likelihood estimates of the parameters for the fitted model to these data are obtained, leading to $n$ estimates of each parameter. The mean of these estimates should be approximately the parameter value that the data was simulated from and their standard deviation approximates the standard errors. Theoretically, as $n$ becomes very large the standard deviations of the parameter estimates approach the true value of the standard errors. Therefore, in order to validate the standard errors this method can be used to obtain approximations, for a suitable value of $n$, which can be compared to those calculated from the information matrix.

The first step in this exercise is to decide the number of replications, $n$, to be performed and also the number of units to be simulated within each set of data. Obviously, the larger the value of each of these, the more accurate the simulation exercise will be. There is more data available from which the parameters can be estimated. However, another important consideration is time, and larger values of the $n$ and $m$ can make the length of time required for the simulation exercise infeasible. For this reason a number of small studies were carried out to determine an optimum number of units to simulate in each data set. These simulations

use the parameter values from the fitted model phase A0. From Table 3.7 this gives the parameters for the simulation as $\beta_0 = 3.945$, $\beta_1 = -5.830$ and $\sigma_\varepsilon = 0.486$.

The initial studies showed that for all three model parameters, the maximum likelihood estimates were always below the true value, that is the true values were outside the distribution of the parameter estimates. The reason for this was traced to the censored observation. The original functions to simulate the data passed back only a matrix of event times without the last censored observation. However, if the data are simulated for a time interval $(0,t]$ and $n$ event times, occurring at times $t_1, t_2, ..., t_n$, are generated within this interval, then it is also known that no events occur in the interval $(t_n, t]$ and hence time $t$ must be included as a censored observation. With this extra information the error in the parameter estimation was corrected and the distributions of the parameter estimates became, in general, centred around the true parameter values. This result demonstrates the need to include all available data in the modelling procedure.

These small simulation studies also highlighted a bias in the estimation of the variance component of the random effect. This bias can be shown to be linked to the number of units within the data. In general, as would be expected, the smaller the sample size the more biased the estimates become. Figure 3.5, overleaf, shows the density estimates for the variance component $\sigma_\varepsilon$, with sample sizes of 10, 20 and 30 units, when the true value of $\sigma_\varepsilon$ is 0.486. The means, standard deviations and estimates of bias of the fitted parameters are shown in Table 3.11. This shows that for the fixed effects, $\beta_0$ and $\beta_1$, the parameter estimates are close to the true values ($\beta_0 = 3.945$ and $\beta_1 = -5.830$) and the bias is small. It can also be seen that the standard deviations become smaller as the sample size increases. This result is true for the standard deviation of $\sigma_\varepsilon$. However, the estimates of $\sigma_\varepsilon$ are biased since the mean of the estimates are less than the true value although this bias appears to decrease as the sample size increases. The bias in the variance component is discussed further in Section 3.7.
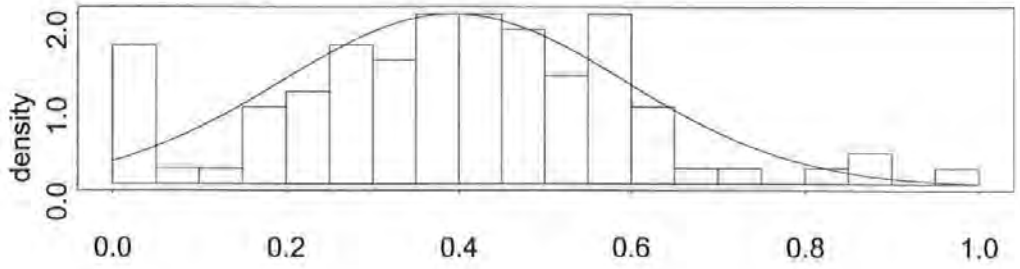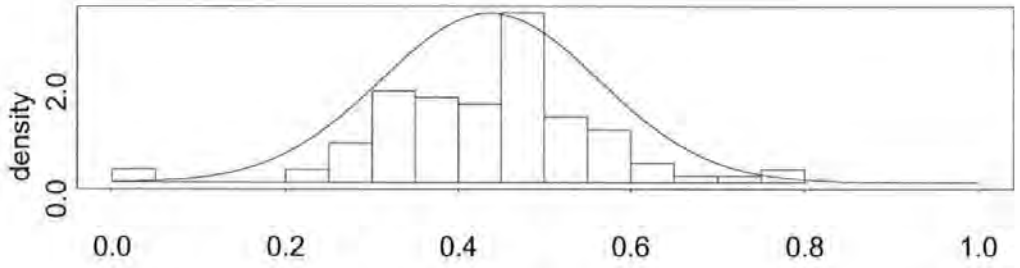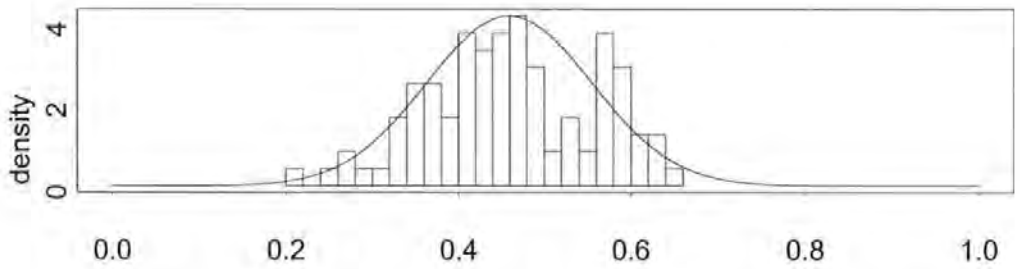
**Figure 3.5 :** Density Estimates of $\hat{\sigma}_\varepsilon$ for 100 simulations with sample size (a) 10 (b) 20 (c) 30

**TABLE 3.11**

*Mean and Standard Deviation of n = 100 Simulations with $\beta_0$=3.945,$\beta_1$ =–5.830 and $\sigma_\varepsilon$ =0.486.*

| Sample | $\hat{\beta}_0$ | | $\hat{\beta}_1$ | | $\hat{\sigma}_\varepsilon$ | |
|---|---|---|---|---|---|---|
| Size | Mean (Bias) | SD | Mean (Bias) | SD | Mean (Bias) | SD |
| 10 | 3.952 (0.007) | 0.2541 | -5.810 (–0.020) | 0.8662 | 0.394 (–0.092) | 0.2002 |
| 20 | 3.964 (0.019) | 0.1668 | -5.832 (0.020) | 0.5374 | 0.437 (–0.049) | 0.1261 |
| 30 | 3.943 (–0.020) | 0.1437 | -5.805 (–0.025) | 0.4798 | 0.459 (–0.027) | 0.0956 |

In order to reduce the bias shown above, a further simulation has been carried out with a sample size of $m$=40 engines. This sample size is sufficiently large for the bias to be small, yet the computational time required for the simulation is reasonable. This simulation allows the accuracy of the estimates of standard errors from the information matrix. The simulation has again been carried out for the parameter estimates of phase A0, and $n$=1000 replications have been performed. For each replication the maximum likelihood estimates of the parameters, and the estimates of the standard errors from the information matrix, have been recorded. If the information matrix yields good estimates of the standard errors then the mean of these standard errors (the mean of the asymptotic standard errors) and the standard deviation of the fitted parameters (the standard deviation of the sampling distribution) should be similar.

The results from the 1000 replications are summarised in Table 3.12. This shows that the mean of the asymptotic standard errors are approximately the same as the standard deviation of the sampling distribution of parameter estimates. From this it can be concluded that estimates of the standard errors obtained from the information matrix are valid. The histograms of the estimates of the three parameters for the 1000 replications, as well as the normal density function with the appropriate mean and standard deviation, are shown in

Figure 3.6 and these appear to be normal. The distribution of the variance component $\sigma_\varepsilon$ in Figure 3.6 (c) has improved with the larger sample size compared to those in Figure 3.5. The means of the parameters, Table 3.12, are close to the true values although $\hat{\sigma}_\varepsilon$ still shows a small amount of bias (the mean is 0.470 compared with the true value of 0.486) even with this sample size.

**TABLE 3.12**

*Results of n = 1000 Simulations with $\beta_0$ = 3.945, $\beta_1$ = −5.830 and $\sigma_\varepsilon$ = 0.486.*

|  | $\hat{\beta}_0$ | $\hat{\beta}_1$ | $\hat{\sigma}_\varepsilon$ |
|---|---|---|---|
| **True Parameter Values** | 3.945 | -5.830 | 0.486 |
| **Mean of Sampling Distribution of Estimates (and Bias)** | 3.944 (−0.001) | -5.867 (−0.037) | 0.470 (−0.016) |
| **Standard Deviation of Sampling Distribution of Estimates** | 0.1185 | 0.4160 | 0.0886 |
| **Mean of Asymptotic Standard Errors** | 0.1127 | 0.4064 | 0.0841 |

## 3.7 Investigating Bias in Parameter Estimates

### 3.7.1 Introduction

The simulation study performed in the previous section to validate the estimates of the standard errors highlighted a bias in the estimate of the random effect, and this requires further investigation. If the bias is very large the fitted models will give incorrect predictions and will be of limited use. Sections 3.7.2 and 3.7.3 introduce two methods for investigating bias, the Jackknife and the Bootstrap, and Section 3.7.4 gives the results of using these methods on both simulated data and the Jaguar data from Appendix 2.

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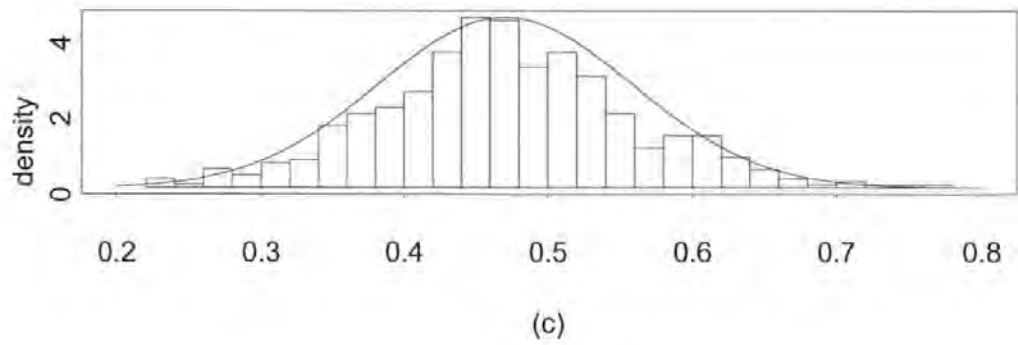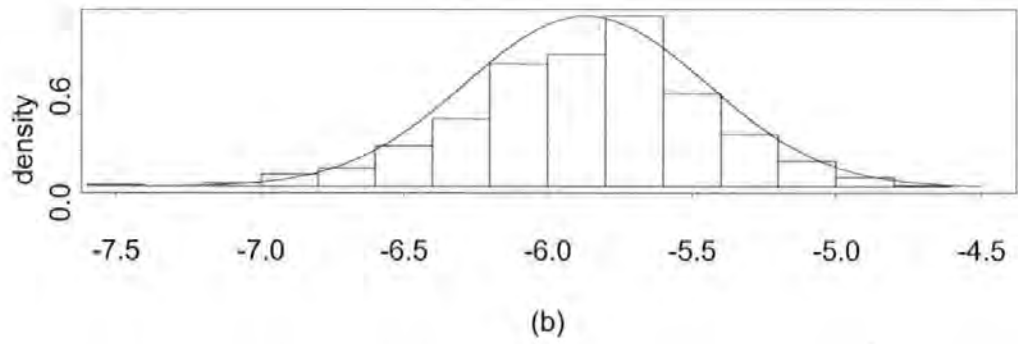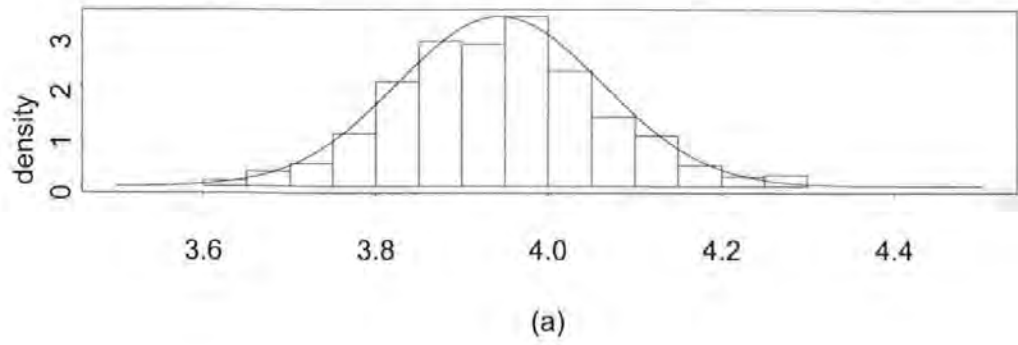
**Figure 3.6 :** Density Plots for 1000 Replications (a) $\hat{\beta}_0$, (b) $\hat{\beta}_1$, (c) $\hat{\sigma}_\varepsilon$

## 3.7.2 The Jackknife Method

The Jackknife was first formulated by Quenouille (1949), and Tukey (1958) derived estimates of the Jackknife standard errors. For a brief introduction to the Jackknife, see Kendall and Stuart (1973) and for further details see Efron and Tibshirani (1993). The Jackknife is derived in the following way. For a sample size $n$, let $\hat{\theta}$ be the estimated parameter values on data including all the observations. If the true parameter values are given by the vector $\theta$ then the bias of the estimate is given by

$$\text{bias} = \theta - \hat{\theta} \tag{3.47}$$

Let $\hat{\theta}_i$ be defined as the estimated parameter values for data excluding the $i^{\text{th}}$ observation. That is

$$\hat{\theta}_i = \hat{\theta}(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

Letting $\hat{\theta}_{(.)}$ be the mean of all the parameter estimates on the reduced data, that is the mean of all $\hat{\theta}_i$, enables the Jackknife estimate of the bias to be given as

$$\widehat{\text{bias}} = (n-1)(\hat{\theta}_{(.)} - \hat{\theta}) \tag{3.48}$$

The Jackknife corrected estimate of the bias, $\tilde{\theta}$, can then be given as

$$\tilde{\theta} = \hat{\theta} - \widehat{\text{bias}}$$

Therefore

$$\tilde{\theta} = n\hat{\theta} - (n-1)\hat{\theta}_{(.)} \tag{3.49}$$

Tukey (1958) derived the Jackknife standard errors of $\tilde{\theta}$ as

$$\hat{SE} = \sqrt{\frac{n-1}{n} \sum_{i=1}^{n} (\hat{\theta}_{(i)} - \hat{\theta}_{(.)})^2} \qquad (3.50)$$

Efron and Tibshirani (1993) state that the Jackknife provides a simple procedure for estimating the bias and the standard errors of parameter estimates. However, the Jackknife is only valid when the statistic $\hat{\theta}$ is smooth, that is small changes in the data cause only small changes in the estimates of the parameters.

### 3.7.3 The Bootstrap Method

The Bootstrap was first discussed by Efron (1977) and is detailed further by Efron and Tibshirani (1993). The Bootstrap algorithm, for a sample size of $n$, is as follows:

i.   Define a Bootstrap sample as a sample of size n drawn with replacement.

ii.  Repeat this $B$ times .

iii. Estimate parameters, $\hat{\theta}^*_{(b)}$, for each Bootstrap sample.     $b = 1, 2, .... B$.

The Bootstrap estimates of the standard errors are given by equation (3.51), where $\hat{\theta}^*_{(.)}$ is the mean of the $B$ Bootstrap estimates. Efron and Tibshirani(1993) suggest that the number of Bootstrap replicates, $B$, can normally be taken from the range 25 to 200.

$$\hat{SE} = \sqrt{\frac{1}{B-1} \sum_{b=1}^{B} (\hat{\theta}^*_{(b)} - \hat{\theta}^*_{(.)})^2} \qquad (3.51)$$

The Bootstrap estimate of the bias, when the parameter estimates are $\hat{\theta}$, is given by equation (3.52). This estimate requires $B$, the number of Bootstrap replicates, to be much larger than when estimating only the standard errors .

$$\hat{bias} = \hat{\theta}^{\ast}_{(.)} - \hat{\theta} \qquad (3.52)$$

### 3.7.4 Bias Correction

The simulations used to validate the standard errors in Section 3.6 showed there is considerable bias in the estimate of $\sigma_{\varepsilon}$ and that this bias appeared to be reduced when the sample size increased. In this section Jackknife and Bootstrap bias corrected estimates are assessed for both simulated data and the Jaguar data.

The inversion method of simulation, Section 3.6.2, has been used to simulate failure times for sample sizes of 10, 20, 30 for parameter values of $\beta_0 = 3.945$, $\beta_1 = -5.830$ and $\sigma_{\varepsilon} = 0.486$. The fitted models and standard errors for these data sets are consistent with the results obtained from the validation exercise in Section 3.6.3. The Jackknife and Bootstrap with $B=500$ have been used to produce corrected estimates of $\sigma_{\varepsilon}$, and these and the corresponding standard errors are given in Table 3.13. The maximum likelihood estimates from the simulation exercise in Section 3.6.3 are also given for comparison. It appears from these results that the Jackknife is effective in reducing the bias in the estimate of $\sigma_{\varepsilon}$.

The Jackknife has been used to produce bias corrected estimates for the Jaguar data. Efron and Tibshirani (1993) recommend that the bias on a parameter is ignored if the ratio of the bias to the standard error is less than 0.25. The ratios of the bias to the standard errors for the Jackknife on simulated data are given in Table 3.14. This table also gives estimates for the two fixed effects. These show that, in accordance with the results from the simulation exercise in Section 3.6.3 there is indeed no significant bias in the estimates of $\beta_0$ and $\beta_1$. However, there is a significant bias on the variance component. The results show that the ratio of the bias estimates to the standard errors are very high for the small sample sizes considered. In

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**TABLE 3.13**

*Maximum Likelihood and Bias Corrected Estimates of $\sigma_\varepsilon$ for Simulated Data.*

| Sample Size | Maximum Likelihood | | Jackknife | | Bootstrap | |
|:---:|:---:|:---:|:---:|:---:|:---:|:---:|
| | Mean (Bias) | SD | Mean (Bias) | SE | Mean (Bias) | SE |
| 10 | 0.394 (−0.092) | 0.2002 | 0.406 (−0.080) | 0.2028 | 0.411 (−0.075) | 0.1957 |
| 20 | 0.437 (−0.049) | 0.1261 | 0.450 (−0.036) | 0.1053 | 0.444 (−0.042) | 0.1187 |
| 30 | 0.459 (−0.027) | 0.0956 | 0.468 (−0.018) | 0.0639 | 0.459 (−0.027) | 0.0857 |

addition, both the bias and the ratio decrease as the sample size increases, as would be expected. For a sample size of 30 the ratio of bias to the standard error is just over 0.25 a further simulation shows it to be well below this figure for the sample size of 50. From this evidence it appears that a sample size of at least 30 is required to obtain good parameter estimates, and shows that the sample size of 40 used to validate the standard errors in Section 3.6.3 was reasonable.

**TABLE 3.14**

*Ratios of Bias to Standard Error for Jackknife Estimates.*

| Sample Size | $\beta_0$ | $\beta_1$ | $\sigma_\varepsilon$ |
|:---:|:---:|:---:|:---:|
| 10 | 0.009 | 0.008 | 0.390 |
| 20 | 0.011 | 0.026 | 0.348 |
| 30 | 0.041 | 0.079 | 0.281 |
| 50 | 0.010 | 0.010 | 0.15 |

The preceding investigation of the bias brings into question the estimates of the variance components for the Jaguar data. The four engine phases contain data from 15, 9, 9 and 21 engines respectively and all these sample sizes are much smaller than the minimum figure of 30 obtained from the simulated data. The Jackknife has been used to estimate the bias and hence produce corrected parameter estimates for the models in Section 3.5. The bias, standard errors and their ratios, for each of the four engine phases are given in Table 3.15. These show that in all cases the ratio of bias to the standard error on the variance component is close to or above the 0.25 threshold. For two of the phases the ratio of bias to standard error is high for the gradient parameter. These could be due to influential engines in the data, removal of which has a large effect on the parameter estimates. Influential engines are considered in the analysis of residuals in Chapter 5 These results agree with those from the simulated data in that a significant bias exists in the variance for sample sizes less than thirty.

TABLE 3.15

*Jackknife Bias Estimates for Jaguar Data.*

| Engine Phase | | $\beta_0$ | $\beta_1$ | $\sigma_\varepsilon$ |
|---|---|---|---|---|
| **A0** | **Bias** | 0.0096 | -0.3211 | -0.0471 |
| | **SE** | 0.2342 | 1.4550 | 0.2119 |
| | **Bias/SE** | 0.0410 | 0.2207 | 0.2223 |
| **A1-1** | **Bias** | -0.0120 | 0.0628 | -0.1019 |
| | **SE** | 0.2071 | 0.9478 | 0.1482 |
| | **Bias/SE** | 0.0579 | 0.0663 | 0.6876 |
| **A1-2** | **Bias** | 0.0384 | -0.3318 | -0.1128 |
| | **SE** | 0.3759 | 0.9600 | 0.2564 |
| | **Bias/SE** | 0.1022 | 0.3456 | 0.4399 |
| **A2-1** | **Bias** | -0.0044 | 0.0069 | -0.0426 |
| | **SE** | 0.2477 | 1.2134 | 0.2028 |
| | **Bias/SE** | 0.0177 | 0.0057 | 0.2101 |

The corrected parameter estimates are given in Table 3.16. As would be expected from previous results, the corrected parameter estimates of the fixed effects, $\beta_0$ and $\beta_1$, are similar to the maximum likelihood estimates and the estimate of the variance component, $\sigma_\varepsilon$, is larger.

**TABLE 3.16**

*Bias Corrected Estimates of Models Fitted to Jaguar Data.*

| Phase | $\beta_0$ | $\beta_1$ | $\sigma_\varepsilon$ |
|-------|-----------|-----------|------------|
| A0 | 3.936 | -5.509 | 0.533 |
| A1-1 | 3.634 | -1.495 | 0.4233 |
| A1-2 | 2.866 | 0.378 | 0.948 |
| A2-1 | 3.361 | -2.44 | 0.568 |

## 3.8 An Alternative Parameterisation of the Random Effect

So far the dispersion of the random effect has been parameterised in terms of $\sigma_\varepsilon$. However the sampling distribution of $\hat{\sigma}_\varepsilon$ may be positively skewed, particularly if $\sigma_\varepsilon$ is close to zero. Inferences based on the approximated standard errors, which assume normality, are in question in this case and a confidence interval for $\hat{\sigma}_\varepsilon$ may be produced with negative limits. This section uses the alternative parameterisation for the random effect of $\log(\sigma_\varepsilon)$ to improve the asymptotic approximations. The delta method may also be used to obtain estimates of the standard errors on the log scale (see Crowder *et al.*, 1991)

### 3.8.1 The Model

The log-linear NHPP model with a random effect on the intercept still takes the form of equation (3.16), namely

$$\lambda_i(t) = \exp\left(\beta_0 + \beta_1 t + \varepsilon_i\right)$$

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where

$\varepsilon_i$ is a random effect due to unit $i$, and $\varepsilon \sim N(0, \sigma_\varepsilon^2)$

In this case the log-likelihood is parameterised in terms of $\varphi_\varepsilon$, such that $\sigma_\varepsilon = \exp(\varphi_\varepsilon)$.

The log-likelihood now becomes

$$\ell = \sum_{i=1}^{m} \ln\left( \exp\left( n_i\beta_0 + \beta_1 \sum_{j=1}^{n_i} t_{ij} \right) \int_{-\infty}^{\infty} \exp\left[ n_i\varepsilon_i - \frac{\exp(\beta_0 + \varepsilon_i)}{\beta_1}\left\{ \exp(\beta_1 t_i) - 1 \right\} \right] \frac{1}{\sqrt{2\pi\exp(2\varphi_\varepsilon)}} exp\left\{ -\frac{\varepsilon_i^2}{2\exp(2\varphi_\varepsilon)} \right\} d\varepsilon_i \right)$$

(3.53)

Using the substitutions given by equations (3.54) to (3.59), the first and second partial derivatives are defined by the general equations for a single random effect in Appendix 5.

$$z_i = \exp\left[ n_i\beta_0 + \beta_1 \sum_{j=1}^{n_i} t_{ij} + n_i\varepsilon_i - \frac{\exp(\beta_0 + \varepsilon_i)}{\beta_1}\left\{ \exp(\beta_1 t_i) - 1 \right\} \right] \frac{1}{\sqrt{2\pi\exp(2\varphi_\varepsilon)}} exp\left\{ -\frac{\varepsilon_i^2}{2exp(2\varphi_\varepsilon)} \right\}$$

(3.54)

$$A_i = \int_{-\infty}^{\infty} z_i \, d\varepsilon_i$$

(3.55)

$$B_i = n_i - \frac{\exp(\beta_0 + \varepsilon_i)}{\beta_1}\left\{ \exp(\beta_1 t_i) - 1 \right\}$$

(3.56)

$$C_i = \sum_{j=1}^{n_i} t_{ij} - \frac{\exp(\beta_0 + \varepsilon_i)}{\beta_1^2}\left\{ \exp(\beta_1 t_i)(\beta_1 t_i - 1) + 1 \right\}$$

(3.57)

$$D_i = \frac{\varepsilon_i^2 - \exp(2\varphi_\varepsilon)}{\exp(2\varphi_\varepsilon)}$$

(3.58)

$$E_i = -\frac{\exp(\beta_0 + \varepsilon_i)}{\beta_1^3}\left\{ \beta_1 t_i \exp(\beta_1 t_i)(\beta_1 t_i - 2) + 2\exp(\beta_1 t_i) - 2 \right\}$$

(3.59)

## 3.8.2 Results

The results from the Jaguar data, showing the difference in using a log-likelihood of the form of equation (3.53) are given in Table 3.17. Of course the maximum likelihood estimates of the parameters (and therefore the bias) are the same as before and are not shown. However, the standard error is now given in terms of $\ln(\sigma_\varepsilon)$ and Table 3.17 gives a confidence interval for $\sigma_\varepsilon$ calculated from this, and also a confidence interval for $\sigma_\varepsilon$ calculated from the standard errors in Table 3.9. The results show a difference in the two confidence intervals suggesting that the distribution of $\sigma_\varepsilon$ is positively skewed. The lower confidence limit for $\sigma_\varepsilon$ from the A1-1 data is also negative. These results indicate that parameterising the standard deviation on a logarithmic scale will improve the asymptotic approximations.

**TABLE 3.17**

*Confidence Intervals of $\hat{\sigma}_\varepsilon$ for Jaguar Data from the Two Parameterisations of the Random Effect.*

| | | 95 % Confidence Interval Calculated From | |
|---|---|---|---|
| **Phase** | $ln(\hat{\sigma}_\varepsilon)$ | $\hat{\sigma}_\varepsilon$ | $ln(\hat{\sigma}_\varepsilon)$ |
| A0 | -0.722 (0.3757) | 0.088 to 0.883 | 0.214 to 1.102 |
| A1-1 | -1.135 (0.5632) | -0.122 to 0.764 | 0.081 to 1.1277 |
| A1-2 | -0.180 (0.3616) | 0.051 to 1.619 | 0.344 to 0.913 |
| A2-1 | -0.643 (0.3095) | 0.184 to 0.868 | 0.274 to 1.007 |

In Section 3.6.3 a simulation was carried out to validate the parameter estimates and their standard errors. Figure 3.6 (c) shows the sampling distribution of $\hat{\sigma}_\varepsilon$ from this

simulation to be approximately normal, and not exhibiting a positive skew. In this case confidence limits based on this standard error of $\hat{\sigma}_\varepsilon$ are valid. This can be demonstrated by comparing confidence limits calculated from both parameterisations. Table 3.18 shows the results for a repeat of the simulation in Section 3.6 when estimating $\ln(\sigma_\varepsilon)$. A 95% confidence interval for $\ln(\sigma_\varepsilon)$ derived from these results and re-expressed in terms of $\sigma_\varepsilon$ is (0.317 to 0.684) while the results in Section 3.6.3 give the confidence interval to be (0.290 to 0.650). This shows that if the distribution of $\sigma_\varepsilon$ is not skew then the two parameterisations give equivalent results.

**TABLE 3.18**

*Results of $n = 500$ Simulations on Logarithmic Scale with $\beta_0 = 3.945$, $\beta_1 = -5.830$ and $\sigma_\varepsilon = 0.486$.*

|  | $\hat{\beta}_0$ | $\hat{\beta}_1$ | $ln(\hat{\sigma}_\varepsilon)$ |
|---|---|---|---|
| **True Parameter Values** | 3.945 | −5.830 | −0.722 |
| **Mean of Sampling Distribution of Estimates (and \|Bias\|)** | 3.944 (0.001) | −5.817 (0.013) | −0.764 (0.042) |
| **Standard Deviation of Sampling Distribution of Estimates** | 0.1124 | 0.3904 | 0.1896 |
| **Mean of Asymptotic Standard Errors** | 0.1128 | 0.4032 | 0.1807 |

In summary, this section has shown that parameterising the random effect model on a logarithmic scale has no effect on the parameter estimates. In the case where the distribution of $\sigma_\varepsilon$ is not skewed when approaching zero confidence intervals based on the two parameterisations give the same result. However, if the distribution of $\sigma_\varepsilon$ is skewed, as is the case with the Jaguar data, the asymptotic properties are improved by using a logarithmic scale. This parameterisation will be used in all subsequent chapters.

## 3.9 Significance Tests on the Random Effect

As described in the previous section, the estimates of the fixed effects $\beta_0$ and $\beta_1$ and their standard errors do not depend on the parameterisation of the variance component. Hence the tests of significance on these parameters for the Jaguar data give the same results as in Section 3.5. This is not true for the tests of significance of the random effect. Difficulties arise in this situation because the null value is not an interior point of the parameter space since the equivalent of the null hypothesis of $\sigma_\epsilon = 0$ is given by a null hypothesis of $ln(\sigma_\epsilon) = -\infty$. It has, however, been established (see Cox and Hinkley, 1974, Ch 9) that, under appropriate conditions, the generalised likelihood ratio test is still valid asymptotically in certain special cases including the one presented here. The generalised likelihood ratio test can be motivated on the following grounds.

Adopting a similar argument to that presented in Cox and Hinkley (1974) the practical concern is not with the null hypothesis $H_0$: $\sigma_\epsilon = 0$ but with a null hypothesis of the form $H_0$: $\sigma_\epsilon \leq \delta$ where $\delta$ represents some small value below which the random effect is effectively zero. In terms of the log transformed parameter, this null hypothesis corresponds to $H_0$: $\log(\sigma_\epsilon) \leq \log(\delta)$. The null value $\log(\delta)$ is an interior point of the parameter space and the generalised likelihood ratio test applies asymptotically.

The argument so far requires the specification of $\delta$ however, as $\delta$ decreases the log-likelihood ratio statistic approaches the limit obtained by taking the null model with a variance parameter of zero. This is illustrated for the phase A0 model by fitting models with a range of $-10 < \log(\delta) < ln(\hat{\sigma}_\epsilon)$ and comparing them with the full model in Table 3.7. The value of $\log(\delta)$ at -10 is equivalent to a standard deviation of $4 \times 10^{-5}$ and is sufficiently small to allow a comparison with the values obtained from the Jaguar data. The likelihood ratio test statistic for these comparisons are shown in Figure 3.7. The solid line represents the likelihood ratio test statistic over the range of $\log(\delta)$ and the dotted line gives the likelihood ratio test when the random effect is removed from the model (as in Table 3.8). It can be seen that as $\delta$ (and hence $\log(\delta)$) decreases, the values of the likelihood ratio test converge to 5.44 which is the value

obtained previously. Thus the test for removal of the random effect is equivalent to a test where the null hypothesis is that the variance of the random effect is arbitrarily small and hence the test is valid.
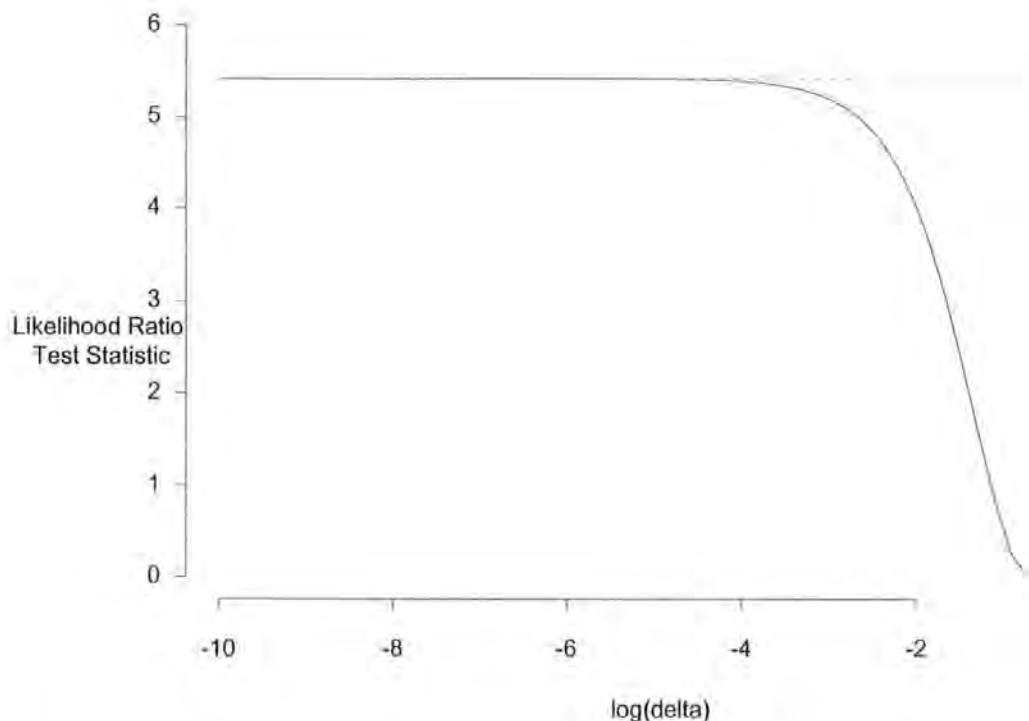


**Figure 3.7:** Likelihood Ratio Test Statistics for A0 Models with $-10 < \log(\delta) < ln(\hat{\sigma}_\varepsilon)$.

Simulations using a sample size of 40 and parameter values of $\beta_0 = 3.945$, $\beta_1 = -5.830$ and $\sigma_\kappa = 0.486$ have been performed to demonstrate the validity of the likelihood ratio test on the variance parameter. For comparison, the simulations have been repeated to test the significance using the parameter estimate divided by its standard error. In both cases the hypothesis test is $H_0$: $\sigma_\varepsilon = 0$ against $H_1$: $\sigma_\varepsilon > 0$.

For the case when $H_1$ is true both tests perform satisfactorily, out of 100 simulations the null hypothesis is correctly rejected 99 and 100 times for the likelihood ratio test and ratio of parameter estimate to its standard error respectively. This result does not hold when the null hypothesis is true. As demonstrated above, the null value is outside the parameter space and the asymptotic normality breaks down when the variance parameter is close to zero. In

this case a test using the ratio of the parameter estimate to its standard error would not be valid and the results of the simulation exercise confirm this. For 100 simulations the null hypothesis would be incorrectly rejected 52 times at the 5% significance level. In addition, a number of extreme test statistics are produced (for example, 126.19) which gives further evidence that a test of this type is not valid.

It has also been demonstrated that the likelihood ratio test is valid when the variance is close to zero. Figure 3.8 shows the histogram of the likelihood ratio test statistics for 100 simulations along with the corresponding $\chi^2$ distribution with 1 degree of freedom. From these results the null hypothesis was incorrectly rejected once at the 5% significance level. Although this is fewer than would be expected (compared with an expected number of 5) the graph shows the distribution of test statistics follows the $\chi^2$ distribution reasonably well and a simulation with more iterations should give a better fit.
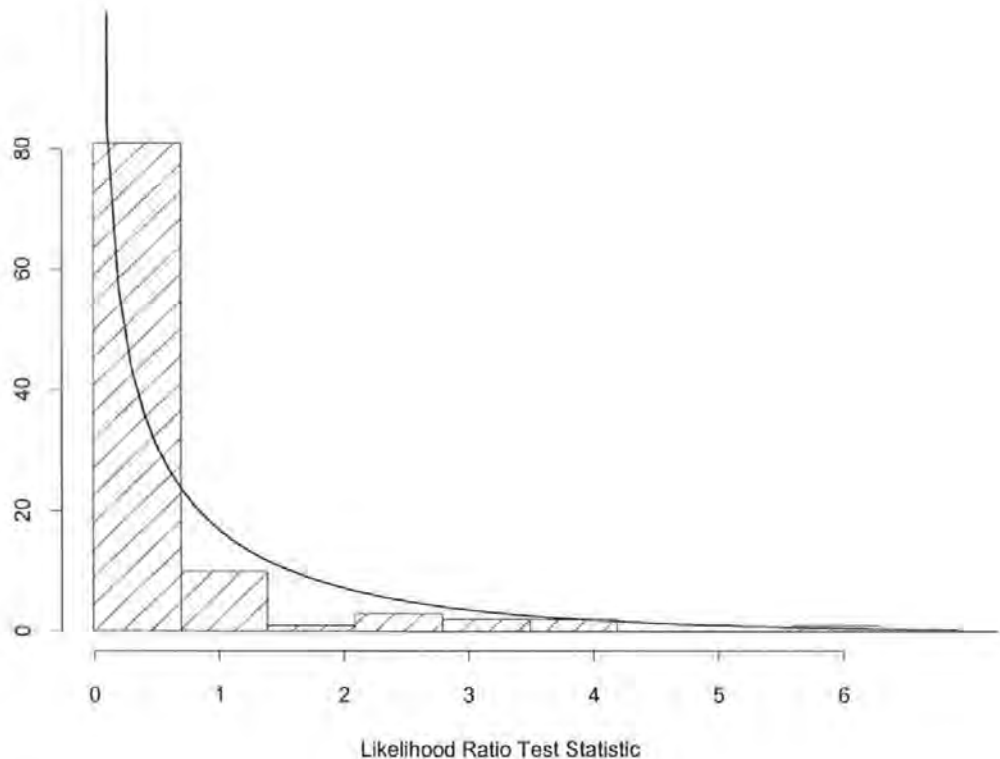


**Figure 3.8:** Likelihood Ratio Test Statistics when $H_0$: $\sigma_\xi = 0$ is true.

In Section 3.5, the two tests of significance were shown to give different results. The random effect parameter for phase A1-1 was found to be significantly different from zero using

the ratio of the estimated parameter to its standard error, which contradicted the result obtained from the likelihood ratio test. This section has shown the likelihood ratio test to be the more appropriate for assessing the significance of the random effect while the test using standard errors gives unreliable results. Hence, the conclusions for the random effect models fitted to the Jaguar data should be drawn from the likelihood ratio test.

## 3.10 Summary

This chapter has described a log-linear NHPP model with a random effect on the intercept parameter which may be useful when there is a variability in the ROCOF at time zero. The random effect has been assumed to follow a normal distribution, although the analysis can be extended for any other distribution such as the Gamma distribution used by Lawless (1995,1987) and Abu-Libdeh et al. (1990).

The random effect model has been fitted to the Jaguar data and it has been shown that the random effect is significant; that is there is a significant variation between the initial ROCOF of engines on a given phase. This variability can be attributed to differences in the build quality and the variation between parts. As the engine phases progress it would be hoped that this variability would decrease as the build process and engine parts become more like production parts. It is also the aim to reduce the ROCOF over this period. This model allows the variation of engines phases to be compared. The follow-up analysis conducted on these models has indicated that the models are of use, although unfortunately the sample size of the engines on a phase is small, leading to a significant bias in the variance component. A method for obtaining predictions of the reliability has also been demonstrated, but the data and test procedures are such that the results can only be used for illustrative purposes.

The model validation in this chapter has considered the bias in the parameter estimates and the estimates of the standard errors. The following chapter introduces another type of random effect and Chapter 5 describes the residual analysis for both of these random effect NHPP models.

# CHAPTER 4

# General Log-Linear Random Effect Models

## 4.1 Introduction

The previous chapter introduced a log-linear random effect model that allowed the variance between engines to be included in the analysis. It was found that a significant variation existed between the ROCOF of engines on a given phase. Placing the random effect on the intercept parameter, $\beta_0$, yields ROCOF's that are proportional for all engines on a given phase. In other words, from differing starting points the ROCOF's will all increase or decrease at the same rate. In order to illustrate the extent to which the proportional model allows for differences in the ROCOF between engines, Figure 4.1 shows the mean $\lambda(t)$ over the random effect plotted against time together with limits placed at $\beta_0 \pm 1.96\sigma_\varepsilon$. For each of these the gradient parameter is negative hence all the plots tend towards zero. Thus, this model allows for a large engine effect at time zero and smaller differences in ROCOF's between engines as time progresses. However, the assumption of proportional ROCOF's between engines on a given phase may not be valid and it could be the case that they change at different rates. Examination of the plots given in Figures 3.1 to 3.4 indicates that this may be the case for the Jaguar data. In the extreme, some engines within the same phase appear to have an increasing ROCOF whilst for others the ROCOF appears to be decreasing. This effect could
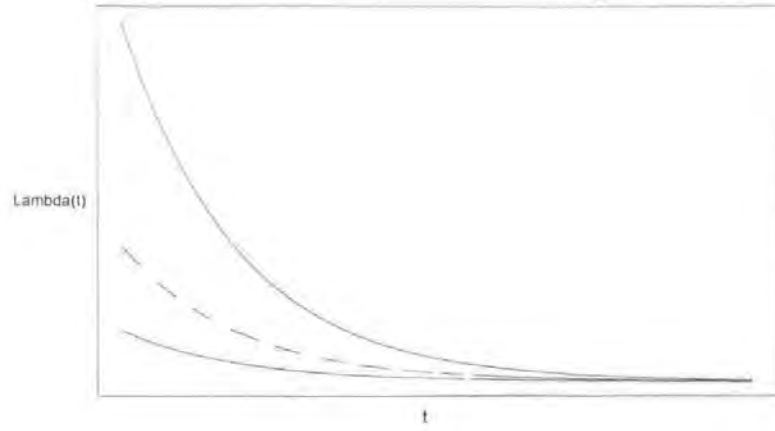
**Figure 4.1:** Mean $\lambda(t)$(- - -) and $\pm 1.96\sigma_\varepsilon$(———)against Time with Random Effect on $\beta_0$.
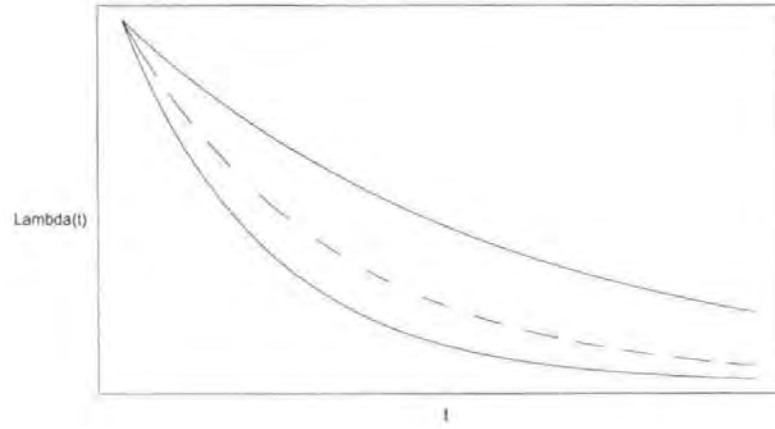


**Figure 4.2:** Mean $\lambda(t)$(- - -) and $\pm 1.96\sigma_\varepsilon$(———)against Time with Random Effect on $\beta_1$.
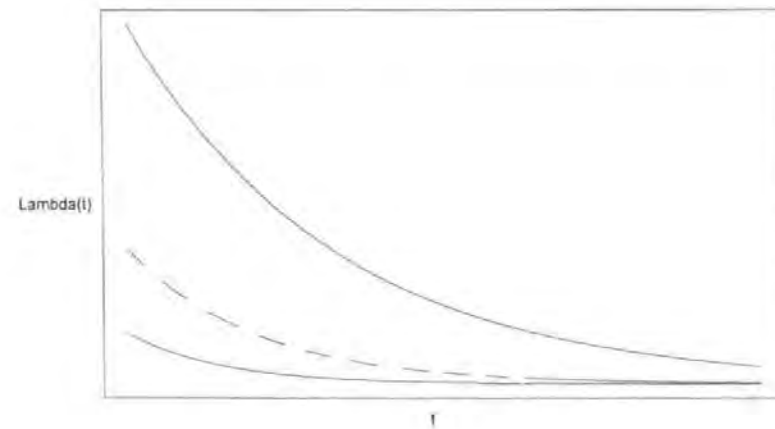


**Figure 4.3:** Mean(- - -) and interval(———) $\lambda(t)$ against Time with Both Random Effects.

lead to the conclusion for phases A1-1 and A1-2 that the gradient parameter $\beta_1$ is insignificant (due to increasing ROCOF's cancelling with those that are decreasing) although the small amount of data available on these two phases will also contribute to this result.

This chapter introduces two further random effect models which allow for non-proportional engine effects. One model introduces a random effect on the gradient parameter and this would be relevant when engine reliability is the same at time zero but the ROCOF's degrade at a different rate. A more flexible model combines both random effects to allow for a difference between engines at time zero and a difference in the rate of degradation. Figures 4.2 and 4.3 show the mean $\lambda(t)$ and its limits for each of these respectively and illustrate the differences between these two models and the proportional random effect model. The model illustrated in Figure 4.2 allows for no difference at time zero while the model with both random effects illustrated in Figure 4.3 is a combination of both single random effect models.

The next section describes the model with the random effect on the gradient. The variance component for this model has been parameterised on a logarithmic scale as in Section 3.8 to improve the asymptotic estimate of the standard error. The simulation and bias investigation of Chapter 3 have been extended for use with this model. A comparison is then made between this and the proportional random effect model, and methods to indicate which is more appropriate for the Jaguar data are introduced. The residuals for both these models are discussed in Chapter 5. A model with both random effects is described in Section 4.5.

## 4.2 Log-Linear Model with Random Effect on the Gradient

A different formulation of the random effect model $\lambda_i(t) = \exp(\beta_0 + \beta_1 t + \varepsilon_i)$ given in equation (3.10) is to introduce the random effect onto the gradient parameter, $\beta_1$, instead of the intercept parameter, $\beta_0$. This model is given by

$$\lambda_i(t) = \exp\{\beta_0 + (\beta_1 + \xi_i)t\} \tag{4.1}$$

where

$\xi_i$ is a random effect due to unit $i$, and $\xi \sim N(0, \sigma_\xi^2)$

Placing the random effect on $\beta_1$ means that variation between units can change the shape of $\lambda(t)$. As has been said in extreme cases, this variation can be so large that some engines have an increasing while others have a decreasing ROCOF. The expected number of failures in the time interval $(a,b]$ is no longer proportional between units. Instead,

$$E\{\text{failures}(a,b] \mid \xi_i\} = \exp(\beta_o) \int_a^b \exp\{(\beta_1 + \xi_i)t\}dt \tag{4.2}$$

The log-likelihood for model (4.1) follows from that of model (3.16) in Section 3.3, given by equation (3.21). In this case, where $\varphi_\xi = \ln(\sigma_\xi)$, the log-likelihood function is

$$\ell = \sum_{i=1}^m \ln\left\{ \exp\left(n_i\beta_0 + \beta_1 \sum_{j=1}^{n_i} t_{ij}\right) \int_{-\infty}^{\infty} \exp\left(\xi_i \sum_{j=1}^{n_i} t_{ij} - \frac{\exp(\beta_o)}{\beta_1 + \xi_i}\left[\exp\{(\beta_1 + \xi_i)t\} - 1\right]\right) \frac{\exp\left(\frac{-\xi_i^2}{2\exp(2\varphi_\xi)}\right)}{\sqrt{2\pi\exp(2\varphi_\xi)}} d\xi_i \right\} \tag{4.3}$$

Maximisation of the likelihood given by equation (4.3) enables parameter estimates to be found. In order to estimate the standard errors of the parameters, the first and second partial derivatives are required at the maximum. These values can be obtained by substituting equations (4.4) to (4.9) into the general expressions of the partial derivatives for a single random effect model in Appendix 5. These substitutions are equivalent to those for the random effect on the intercept given in equations (3.54) to (3.59). A comparison of these substitutions and the log-likelihood function, with those for the first model, show these to be of the same form with $\beta_0 + \varepsilon_i$ replaced by $\beta_0$, and $\beta_1$ replaced by $\beta_1 + \xi_i$.

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$$z_i = \exp\left\{ n_i\beta_0 + (\beta_1 + \xi_i)\sum_{j=1}^{n_i} t_{ij} - \frac{\exp(\beta_o)}{\beta_1 + \xi_i}\left[\exp\{(\beta_1 + \xi_i)t\} - 1\right]\right\} \frac{\exp\left(\dfrac{-\xi_i^2}{2\exp(2\varphi_\xi)}\right)}{\sqrt{2\pi\exp(2\varphi_\xi)}} \qquad (4.4)$$

$$A_i = \int_{-\infty}^{\infty} z_i \, d\xi_i \qquad (4.5)$$

$$B_i = n_i - \frac{\exp(\beta_o)}{\beta_1 + \xi_i}\left[\exp\{(\beta_1 + \xi_i)t\} - 1\right] \qquad (4.6)$$

$$C_i = \sum_{j=1}^{n_i} t_{ij} - \frac{\exp(\beta_o)}{(\beta_1 + \xi_i)^2}\left[\exp\{(\beta_1 + \xi_i)t\}\{(\beta_1 + \xi_i)t_i - 1\} + 1\right] \qquad (4.7)$$

$$D_i = \frac{\xi_i^2 - \exp(2\varphi_\xi)}{\exp(2\varphi_\xi)} \qquad (4.8)$$

$$E_i = -\frac{\exp(\beta_0)}{(\beta_1 + \xi_i)^3}\left[(\beta_1 + \xi_i)t_i\exp\{(\beta_1 + \xi_i)t_i\}\{(\beta_1 + \xi_i)t_i - 2\} + 2\exp\{(\beta_1 + \xi_i)t_i\} - 2\right] \qquad (4.9)$$

## 4.3 Simulation and Investigation of Bias in Estimates

### 4.3.1 Validation of the Standard Errors

In Section 3.6 the inversion method of simulating data from a log-linear NHPP was described. This method can be extended to simulate data when the random effect is placed on the gradient parameter. A simulation to validate the standard errors for this model has been performed using 500 replications on a sample size of 40. In Chapter 3 the simulations have been performed with the true parameter estimates the same as those from a fitted model. In this case the true parameter values have been taken as $\beta_0 = 4$, $\beta_1 = -3$ and $\sigma_\xi = 1$ ($\ln(\sigma_\xi) = 0$).

These values ensure the simulated data contains a reasonable number of failure times to enable good parameter estimation, but still allow the computation to be completed in a feasible time span. The results of this simulation are summarised in Table 4.1 and the sampling distributions are shown in Figure 4.4. These show that for the fixed effects, $\beta_0$ and $\beta_1$, the model is behaving well; the means of the sampling distribution of the parameter estimates are close to the true values, and the standard deviations of these sampling distributions are similar to the means of the asymptotic standard errors. This implies that the estimates of the standard errors from the information matrix are reasonable. However, the simulation exercise shows that the model is not providing good estimates for the random effect. Figure 4.4 (c) shows the sampling distribution of $\ln(\sigma_\xi)$ is not centred around the true value of 0. There is a large bias in the estimates of the parameter and the mean of the distribution is $-0.2322$. The standard deviation is also much greater than the mean of the asymptotic standard errors, so there can be little confidence in the estimate of the random effect or its standard error. Further evidence of this can be obtained by investigating bias of the parameter estimates, and this is described in the next section.

TABLE 4.1

*Results of $n =500$ Simulations for a random effect on the gradient with $\beta_0 =4$, $\beta_1 = -3$ and $\sigma_\xi = 1$.*

| | $\hat{\beta}_0$ | $\hat{\beta}_1$ | $\ln(\hat{\sigma}_\xi)$ |
|---|---|---|---|
| **True Parameter Values** | 4 | −3 | 0 |
| **Mean of Sampling Distribution of Estimates (and \|Bias\|)** | 3.997 (0.003) | −2.976 (0.024) | −0.2322 (0.2322) |
| **Standard Deviation of Sampling Distribution of Estimates** | 0.0740 | 0.3642 | 0.7197 |
| **Mean of Asymptotic Standard Errors** | 0.0718 | 0.3484 | 0.3841 |
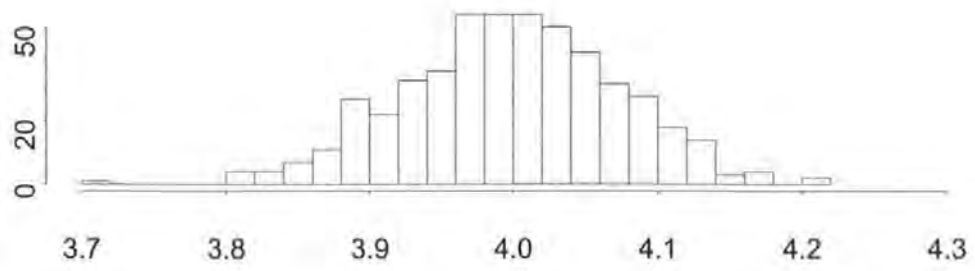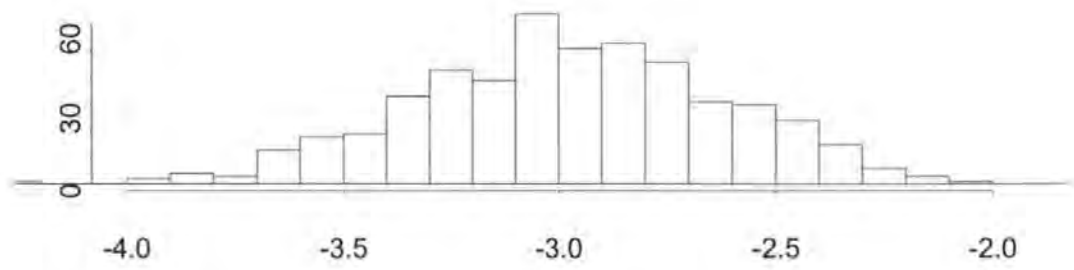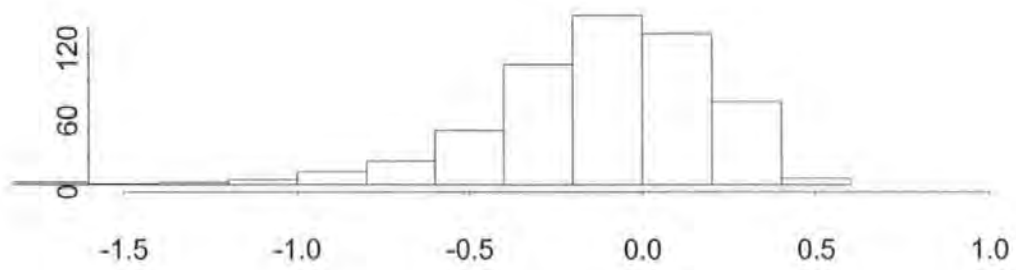
**Figure 4.4:** Estimated sampling distributions of (a) $\hat{\beta}_0$ (b) $\hat{\beta}_1$ and (c) $\ln\!\left(\hat{\sigma}_\xi\right)$

### 4.3.2 Investigating Bias in the Parameter Estimates

The Jackknife and Bootstrap methods of estimating bias in parameter estimates were described in Sections 3.7.2 and 3.7.3 respectively. The results in Section 3.7.4 showed that both methods gave comparable results and concluded that the Jackknife could be used in preference. Using these results, the Jackknife has been used to estimate the bias for simulated data (with the same parameter values as the previous simulation) for sample sizes 10, 20, 30 and 50. The Jackknife estimates of the bias, the standard errors and the ratio of the two are given in Table 4.2.

TABLE 4.2

*Estimates of Bias for Simulated Data from the Model with a Random Effect on the Gradient.*

| Sample Size | | $\beta_0$ | $\beta_1$ | $\ln(\sigma_\xi)$ |
|---|---|---|---|---|
| **10** | **Bias** | −0.0064 | −0.0136 | −5.8997 |
| | **SE** | 0.1837 | 1.4863 | 5.6775 |
| | **Bias/SE** | −0.034 | −0.009 | −1.047 |
| **20** | **Bias** | −0.0034 | −0.0177 | −0.0811 |
| | **SE** | 0.1683 | 1.2478 | 0.2073 |
| | **Bias/SE** | −0.021 | −0.014 | −0.391 |
| **30** | **Bias** | −0.0030 | 0.0238 | −0.0798 |
| | **SE** | 0.0969 | 0.7364 | 0.2171 |
| | **Bias/SE** | −0.031 | 0.032 | −0.367 |
| **50** | **Bias** | −0.0008 | −0.0053 | −0.1151 |
| | **SE** | 0.0663 | 0.5079 | 0.3024 |
| | **Bias/SE** | −0.012 | −0.010 | −0.381 |

A comparison of these results with those with the random effect on $\beta_0$ (see Tables 3.13 and 3.14) show that there is more bias on the variance component, especially on the smallest

sample size. However, in this case, except for a sample size of 10, the bias does not decrease over the increases of sample size. In all cases the ratio of the bias to its standard error is greater than 0.25 leading to the conclusion that there is a large bias in the parameter estimates (Efron & Tibshirani, 1993) of the variance component. For the model with the random effect on $\beta_1$ the sample size is required to be in excess of 50 to reduce this bias. The estimates of the fixed effects $\beta_0$ and $\beta_1$ again show no bias. The next section shows the results of fitting model (4.1) to the Jaguar data, although it is clear from these results that the parameter estimates of $\sigma_\xi$ will be biased.

## 4.4 An Application of a NHPP Model with a Random Effect on the Gradient

The model defined in Section 4.2 has been fitted to the four phases of Jaguar data to illustrate the methodology. The parameter estimates and the value of the maximised log-likelihood are shown in Table 4.3.

**TABLE 4.3**

*Parameter Estimates from the Jaguar Data for Model with Random Effect on $\beta_1$.*

| Phase | $\hat{\beta}_0$ | $\hat{\beta}_1$ | $\ln(\hat{\sigma}_\xi)$ | $\hat{\sigma}_\xi$ | $\ell_{max}$ |
|-------|------|------|-----------|------|------|
| A0 | 4.074 | −7.009 | 0.651 | 1.917 | 220.24 |
| A1-1 | 3.671 | −1.686 | 0.1411 | 1.152 | 169.81 |
| A1-2 | 3.140 | −1.524 | 0.982 | 2.670 | 92.81 |
| A2-1 | 3.434 | −3.048 | −0.188 | 0.829 | 227.73 |

A test of significance for the random effect of this form can be made by performing a Likelihood Ratio Test between the maximised log-likelihood in Table 4.3 and those for the model without a random effect in Table 3.1. The test statistics for the comparison of the

models for the four engine phases are given in Table 4.4. These statistics can be compared with 3.841 ($\chi^2_{1,0.05}$) and show that in all cases the random effect on $\beta_1$ is not significant.

**TABLE 4.4**

*Likelihood Ratio Statistic for Model with Random Effect on $\beta_1$ Fitted to Jaguar Data.*

| Engine Phase | A0 | A1-1 | A1-2 | A2-1 |
|:---:|:---:|:---:|:---:|:---:|
| Likelihood Ratio Statistic | 0.96 | 0.44 | 1.86 | 0.64 |

The standard errors of the parameter estimates, and the test statistics for comparing the fixed effects $\beta_0$ and $\beta_1$ with zero, are given in Table 4.5. The pattern of significance of the fixed effect parameters is the same as for the first random effect model (Section 3.5), namely the intercept parameter $\beta_0$ is always significant and the gradient parameter $\beta_1$ is significant for phases A0 and A2-1.

**TABLE 4.5**

*Standard Errors and Test Statistics for Model with Random Effect on $\beta_1$ Fitted to Jaguar Data.*

| Engine Phase | | $\hat{\beta}_0$ | $\hat{\beta}_1$ | $\ln\left(\hat{\sigma}_\xi\right)$ |
|:---:|:---:|:---:|:---:|:---:|
| A0 | SE | 0.1569 | 1.5901 | 0.7925 |
| | Test Statistic | 25.972 | −5.408 | - |
| A1-1 | SE | 0.1914 | 1.1218 | 0.1411 |
| | Test Statistic | 19.180 | −1.503 | - |
| A1-2 | SE | 0.2525 | 1.7324 | 0.5879 |
| | Test Statistic | 12.436 | −0.880 | - |
| A2-1 | SE | 0.1415 | 0.7313 | 0.7862 |
| | Test Statistic | 25.269 | −5.168 | - |

In all cases the value of the log-likelihood is higher for the model with the random effect on $\beta_0$, and for this model the random effect is found to be significant. Examination of the failure plots in Section 3.1 (Figures 3.1 to 3.4) suggests that there is variation between the engines. Combining this with the results of the likelihood ratio tests leads to the conclusion that the model with a random effect on the intercept parameter, given in equation (3.16), is able to incorporate the variation between engines more successfully than the model with the random effect on the gradient parameter, equation (4.1). This difference in the maximised value of the log-likelihoods is considered further in Section 4.5.

The Jackknife has also been performed on the models from the Jaguar data. The sample sizes for this data are all small and from the results gained for the simulated data (Section 4.3.2) a large amount of bias in the estimate of the variance component would be expected for these models. The results from the Jackknife are given in Table 4.6. These indeed show a significant bias on the variance component, and also on the fixed effects for phase A1-2. However, for these models the standard errors of the parameter estimates have become much larger, hence nothing is gained from obtaining the Jackknife corrections.

Using the models in Table 4.3 to obtain predictions for the number of failures in the time interval of 0 to 0.32 unit time gives the results shown in Table 4.7. As for the random effect on the intercept a prediction interval has been calculated using the 95% limits of $\sigma_\xi$. The expected number of failures, for all engine phases, do not differ from those given in Table 3.10. This shows that the two models perform similarly for the mean Rate of Occurrence of Failures, despite differing parameter values for $\beta_0$ and $\beta_1$. This would be expected as both functions are modelling the same mean ROCOF and the influence of the different random effects would be on the prediction interval. In this case the width of the interval obtained from the variance parameter changes considerably between the two models. The model with the random effect on $\beta_1$ (which was not found to be significant in the model) produces a narrower prediction interval. This could be due to the known bias in the estimates of the random effect.

**TABLE 4.6**

*Jackknife Bias Estimates for Model with Random Effect on $\beta_1$ Fitted to Jaguar Data.*

| Engine Phase | | $\beta_0$ | $\beta_1$ | $\ln(\sigma_\xi)$ |
|---|---|---|---|---|
| **A0** | Bias | 0.0048 | -0.4206 | -1.0979 |
| | SE | 0.2345 | 2.3003 | 1.7492 |
| | Bias/SE | 0.020 | -0.183 | -0.746 |
| **A1-1** | Bias | -0.0299 | 0.1879 | -3.1214 |
| | SE | 0.2163 | 1.0574 | 3.2953 |
| | Bias/SE | -0.138 | 0.178 | -0.947 |
| **A1-2** | Bias | -0.1177 | 0.8127 | -3.6518 |
| | SE | 0.3866 | 1.3725 | 3.9751 |
| | Bias/SE | -0.305 | 0.592 | -0.919 |
| **A2-1** | Bias | -0.0267 | 0.2159 | -10.4322 |
| | SE | 0.2401 | 1.0638 | 6.0536 |
| | Bias/SE | -0.111 | 0.203 | -1.723 |

**TABLE 4.7**

*Predicted Numbers of Failures for Model with Random Effect on $\beta_1$ Fitted to Jaguar Data*

| Engine Phase | Expected Number of Failures | | | Interval Width |
|---|---|---|---|---|
| | Mean | $\hat{\beta}_1 - 1.96\hat{\sigma}_\xi$ | $\hat{\beta}_1 + 1.96\hat{\sigma}_\xi$ | |
| A0 | 7.50 | 5.29 | 11.71 | 6.42 |
| A1-1 | 9.72 | 7.14 | 13.80 | 6.66 |
| A1-2 | 5.84 | 3.02 | 15.19 | 11.17 |
| A2-1 | 6.34 | 5.15 | 7.97 | 2.82 |

A comparison of the influence of the two types of random effect can be made

graphically. The first comparison is made by dividing the time scale into a number of

intervals. The expected number of failures in each of these intervals , as well as the corresponding prediction interval, for the models fitted to the Jaguar data can be found using equations (3.17) and (4.2). Plots of the trendlines of the expected number of failures and the upper and lower intervals from the two models fitted to the phase A0 data are given in Figures 4.5 and 4.6, and plots for the other engine phases are given in Appendix 6. These plots also show the mean number of failures observed in each time interval (represented by the dots). If the random effect is modelling the engine variation well, then the observed values will be within the prediction limits. Conversely if the random effect is not modelling the engine variation well then a number of observations would be expected outside these limits. In all cases when the random effect is placed on the intercept parameter, $\beta_0$, the observed mean numbers of failures fall within or close to the upper and lower limits, whereas when the random effect is placed on the gradient parameter, $\beta_1$, a number of observations fall outside the limits. This suggests the random effect on $\beta_0$ is more representative of the engine variability.
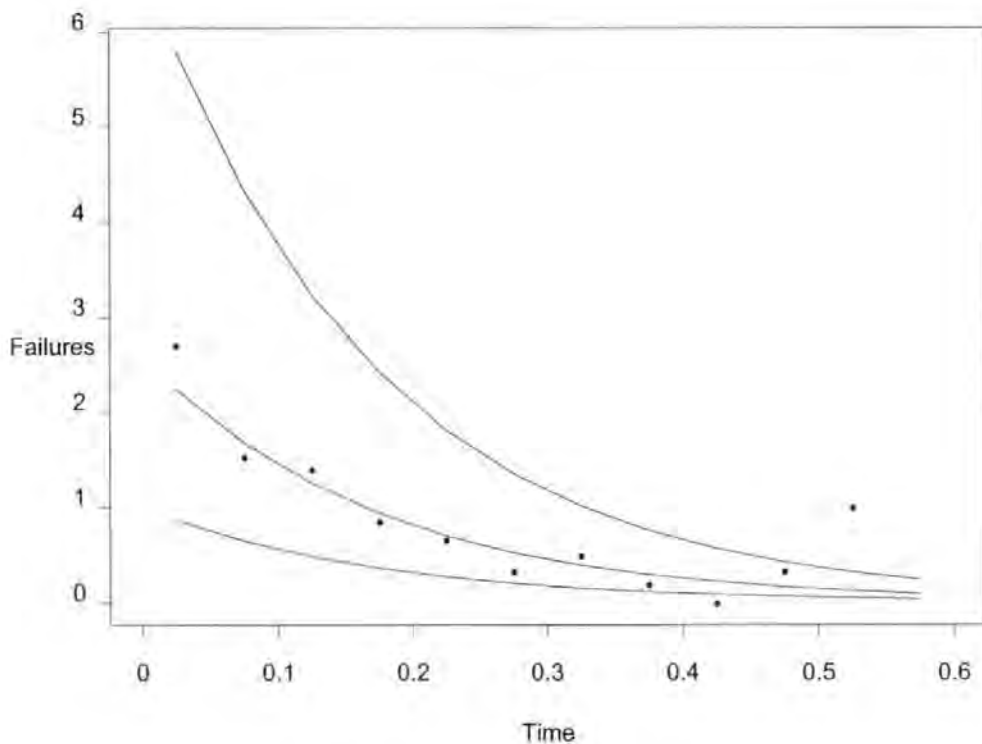


**Figure 4.5** : Mean Failures against Predicted for A0 Model with Random Effect on $\beta_0$.
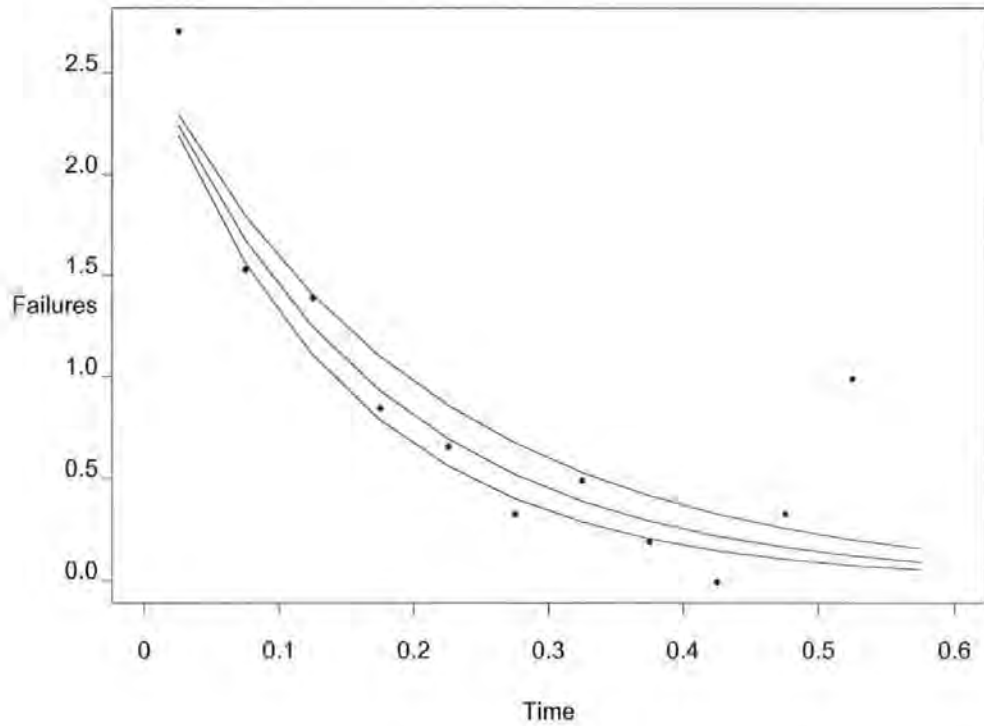
**Figure 4.6 :** Mean Failures against Predicted for A0 Model with Random Effect on $\beta_1$.

Examination of the trendlines of the prediction intervals illustrates the characteristics of the two random effects. The random effect on $\beta_0$ allows for a large difference in engines at time zero and has less influence as time increases, whereas the random effect on $\beta_1$ has no effect at time zero and allows for a difference in engines as time increases. A further comparison can be made by plotting simulated data for the two models. Data has been simulated $n = 15$ for both models estimated from phase A0, and plots of these are given in Figures 4.7 and 4.8. If the model provides a good level of fit to the data, a failure time plot of a set of data simulated from the estimated parameters should resemble the original data. A comparison of these plots with that of the original data, Figure 3.1, shows that out of the two models, the simulated data from the model with the random effect on the intercept resembles the plot of the original data more closely. The variability between the early failure times is better represented by the random effect on $\beta_0$. Although this result is demonstrated on only one set of data for each model, further analysis shows the repeatability of this result for all the engine phases. Therefore on the evidence thus far (significance tests and graphical
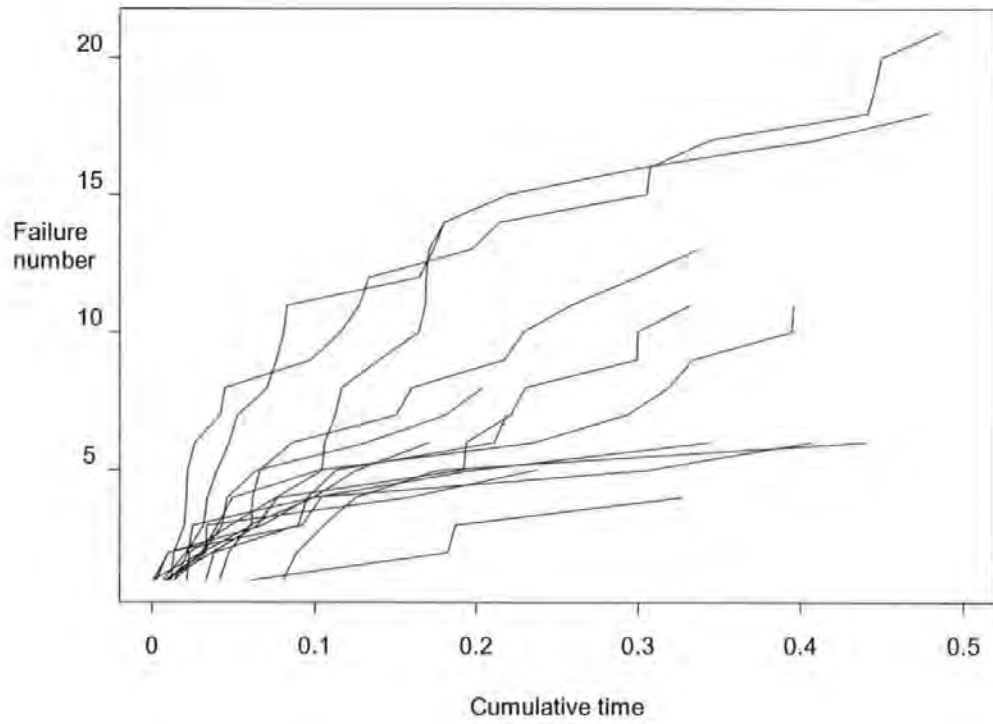
**Figure 4.7 :** Failure Time Plot for Simulated A0 Data from Random Effect on $\beta_0$
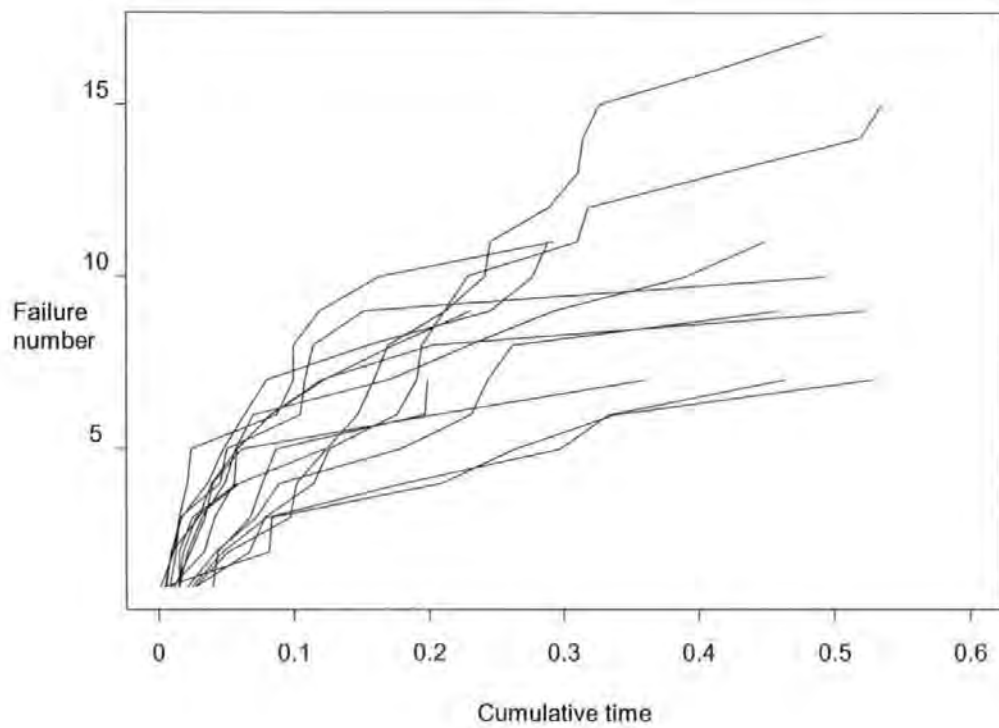


**Figure 4.8 :** Failure Time Plot for Simulated A0 Data from Random Effect on $\beta_1$

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comparisons) it would appear that a random effect on $\beta_0$ is more appropriate for the Jaguar data. This gives evidence that the model with the random effect on the intercept is the more appropriate model.

## 4.5 A Log-Linear NHPP Model with Two Random Effects

### 4.5.1 The Double Random Effect Model

The log-linear Non-Homogeneous Poisson Process model can also be formulated to include both of the random effects, and is therefore a combination of the models defined in Sections 3.3 and 4.2. The rate of occurrence of failure for the $i^{th}$ unit in this case is

$$\lambda_i(t) = \exp\{\beta_0 + (\beta_1 + \xi_i)t + \varepsilon_i\} \tag{4.10}$$

where

$\varepsilon_i$ is a random effect on $\beta_0$ due to unit $i$, and $\varepsilon \sim N(0, \sigma_\varepsilon^2)$

$\xi_i$ is a random effect on $\beta_1$ due to unit $i$, and $\xi \sim N(0, \sigma_\xi^2)$

The random effects in this case are assumed to be independent and the effect of these random components is unchanged from previous models and is described in Sections 3.3 and 4.2. Hence, equation (4.10) is modelling the situation where the ROCOF's are different at time zero and change at a different rate. It follows that the expected number of failures in time interval (a,b] is given by

$$E\{\text{failures}(a,b] \,|\, \varepsilon_i, \xi_i\} = \exp(\beta_o + \varepsilon_i) \int_a^b \exp\{(\beta_1 + \xi_i)t\}dt \tag{4.11}$$

The usefulness of this model for predictive purposes is questionable as it is essentially allowing for totally random data. However, it can be used to distinguish between the two other random

effect models. Both of the previous models, given by equations (3.16) and (4.1), are nested within equation (4.10). This allows a Likelihood Ratio Test, see Section 2.4.7, to be performed between the model of equation (4.10) and the two models nested within it. This procedure checks for a significant decrease in the likelihood if either of the random effects is removed and will provide evidence as to which one of the random effects is required to model the data.

The parameters are estimated by the method of maximum likelihood, and the likelihood function can be formulated in the same way as before. However, for this model the likelihood is conditional on two random effects and both of these must be integrated over (as in equation (4.12)) to give the likelihood function, equation (4.13).

$$P(\text{data}|\underline{\beta}) = \prod_{i=1}^{m} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(\text{data}_i|\varepsilon_i,\xi_i,\underline{\beta})P(\varepsilon_i)\,\mathrm{d}\varepsilon_i\,P(\xi_i)\,d\xi_i \tag{4.12}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \prod_{j=1}^{n_i} \left\{\lambda_i(t_{ij})\right\} \exp\left\{-\int_0^{t_i}\lambda_i(t)dt\right\} \frac{\exp\left\{-\dfrac{(\mu_1-\varepsilon_i)^2}{2\exp(2\varphi_\varepsilon)}\right\}}{\sqrt{2\pi\exp(2\varphi_\varepsilon)}} d\varepsilon_i \frac{\exp\left\{-\dfrac{(\mu_0-\xi_i)^2}{\exp(2\varphi_\xi)}\right\}}{\sqrt{2\pi\exp(2\varphi_\xi)}} \, d\xi_i \tag{4.13}$$

The log-likelihood function, using equation (4.10) as the form of $\lambda_i(t)$ and assuming zero mean and constant variance for each random effect, is then given by

$$\ell = \sum_{i=1}^{m} \ln\left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left( n_i(\beta_0+\varepsilon_i)\beta_0 + (\beta_1+\xi_i)\beta_1 \sum_{j=1}^{n_i} t_{ij} - \frac{\exp(\beta_o+\varepsilon_i)}{\beta_1+\xi_i}\left[\exp\{(\beta_1+\xi_i)t\}-1\right]\right) \frac{\exp\left\{\dfrac{-\varepsilon_i^2}{2\exp(2\varphi_\varepsilon)}\right\}}{\sqrt{2\pi\exp(2\varphi_\varepsilon)}} d\varepsilon_i \frac{\exp\left\{\dfrac{-\xi_i^2}{2\exp(2\varphi_\xi)}\right\}}{\sqrt{2\pi\exp(2\varphi_\xi)}} d\xi_i \right) \tag{4.14}$$

The composite Gauss-Legendre method of integration, described in Section 3.4, can be applied over a double integral of this form. To do this, the weights and nodes are calculated for the outer integral, and for each of these the value of the inner integral is calculated using a second set of weights and nodes. The final value of the integral is then given by summing the

weights multiplied by the corresponding value of the inner integral. The weights and nodes are calculated as before.

In order to estimate the standard errors of the maximum likelihood estimates, and also to improve computational efficiency, the first and second derivatives are required. The first and second partial derivatives are defined as in Appendix 5, together with the substitutions in equations (4.15) to (4.23).

$$
z_i = \exp\left( n_i(\beta_0 + \varepsilon_i)\beta_0 + (\beta_1 + \xi_i)\beta_1 \sum_{j=1}^{n_i} t_{ij} - \frac{\exp(\beta_o + \varepsilon_i)}{\beta_1 + \xi_i}\Big[\exp\{(\beta_1 + \xi_i)t\} - 1\Big]\right) \frac{\exp\left\{\dfrac{-\varepsilon_i^2}{2\exp(2\varphi_\varepsilon)}\right\}}{\sqrt{2\pi\exp(2\varphi_\varepsilon)}} \frac{\exp\left\{\dfrac{-\xi_i^2}{2\exp(2\varphi_\xi)}\right\}}{\sqrt{2\pi\exp(2\varphi_\xi)}}
$$

$$(4.15)$$

$$
A_i = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z_i \, d\varepsilon_i d\xi_i
$$

$$(4.16)$$

$$
B_i = n_i - \frac{\exp(\beta_o + \varepsilon_i)}{\beta_1 + \xi_i}\Big[\exp\{(\beta_1 + \xi_i)t\} - 1\Big]
$$

$$(4.17)$$

$$
C_i = \sum_{j=1}^{n_i} t_{ij} - \frac{\exp(\beta_o + \varepsilon_i)}{(\beta_1 + \xi_i)^2}\Big[\exp\{(\beta_1 + \xi_i)t\}\{(\beta_1 + \xi_i)t_i - 1\} + 1\Big]
$$

$$(4.18)$$

$$
D_i = \frac{\varepsilon_i^2 - \exp(2\varphi_\varepsilon)}{\exp(2\varphi_\varepsilon)}
$$

$$(4.19)$$

$$
E_i = \frac{\xi_i^2 - \exp(2\varphi_\xi)}{\exp(2\varphi_\xi)}
$$

$$(4.20)$$

$$
F_i = -\frac{\exp(\beta_0 + \varepsilon_i)}{(\beta_1 + \xi_i)^3}\Big[(\beta_1 + \xi_i)t_i\exp\{(\beta_1 + \xi_i)t_i\}\{(\beta_1 + \xi_i)t_i - 2\} + 2\exp\{(\beta_1 + \xi_i)t_i\} - 2\Big]
$$

$$(4.21)$$

$$G_i = \frac{-2\varepsilon_i^2}{\exp(2\varphi_\varepsilon)}$$

(4.22)

$$H_i = \frac{-2\xi_i^2}{\exp(2\varphi_\xi)}$$

(4.23)

### 4.5.2 Results for Jaguar Data

The model has been fitted to the four phases of Jaguar data given in Appendix 2. The parameter estimates, their standard errors and the value of the maximised log-likelihood are given in Table 4.8. A comparison between these fitted models and those for the models with a single random effect (Tables 3.7 and 4.1) shows that the parameter values are the same as those for the models with a random effect on the intercept, with a very small value for the random effect on the gradient. For example, the random effect on the gradient for phase A0 has a normal distribution with an estimated standard deviation of 0.05. The exception to this is the fitted model to the phase A2-1 data. This data set has a larger sample size which enables the two variance components to be estimated, although from Sections 3.7 and 4.3 these estimates are biased.

**TABLE 4.8**

*Parameter Estimates(Standard Errors) to Jaguar Data for Model with Both Random Effects.*

| Phase | $\hat{\beta}_0$ | $\hat{\beta}_1$ | $ln(\hat{\sigma}_\varepsilon)$ | $\hat{\sigma}_\varepsilon$ | $ln(\hat{\sigma}_\xi)$ | $\hat{\sigma}_\xi$ | $\ell_{max}$ |
|---|---|---|---|---|---|---|---|
| A0 | 3.945 (0.2072) | -5.830 (1.2421) | -0.722 (0.3757) | 0.486 | -2.901 (13.1011) | 0.055 | 222.48 |
| A1-1 | 3.622 (0.2235) | -1.432 (0.9612) | -1.135 (0.5623) | 0.321 | -3.854 (16.7772) | 0.021 | 170.48 |
| A1-2 | 2.904 (0.3888) | 0.046 (1.3219) | -0.180 (0.3616) | 0.835 | -5.273 (17.1033) | 0.014 | 96.54 |
| A2-1 | 3.357 (0.1910) | -2.493 (0.8027) | -0.595 (0.3128) | 0.552 | 0.029 (0.8609) | 1.029 | 231.67 |

The significance of the random effects can be tested using likelihood ratios, and as both the single random effect models are nested within the double random effect model this gives further evidence to suggest which is the appropriate model. For example, a hypothesis test for $\sigma_\xi$ can be constructed using the difference in the maximum log-likelihoods of the double random effect model (Table 4.8) and the model with the random effect on $\beta_0$ (Table 3.7). Similarly, a hypothesis test for $\sigma_\varepsilon$ uses the maximised log-likelihood from the model with a random effect on $\beta_1$ (Table 4.3). Table 4.9 gives a summary of the Likelihood Ratio Test statistics for these models, and these values can be compared with 3.841 ($\chi^2_{1,0.05}$). These show that the test statistic when the random effect on the gradient is removed is either zero (2 dp), or very close to it, and lead to the conclusion that there is not a significant change in the likelihood when this parameter is absent. Therefore this parameter should not be included in the model. In all but one case the test statistic when the random effect on the intercept is removed from the model is greater than the critical value, and hence this parameter should be included in the model. For phase A1-1 where $\varphi_0$ is not significant the test statistic is greater than when $\sigma_\xi$ is not included, and therefore $\sigma_\xi$ would be removed first. These results agree with the graphical checks that the model with the random effect on the intercept parameter is more appropriate for the Jaguar data.

TABLE 4.9

*Likelihood Ratio Statistic for Testing Significance of Random Effects.*

| Random Effect | Engine Phase | | | |
|:---:|:---:|:---:|:---:|:---:|
| Removed | A0 | A1-1 | A1-2 | A2-1 |
| $\sigma_\varepsilon$ | 5.48 | 1.34 | 7.46 | 6.84 |
| $\sigma_\xi$ | 0.00 | 0.00 | 0.00 | 0.56 |

### 4.5.3 Simulated Likelihood Ratio Tests

As already stated, the double random effects model, equation (4.10), can be used to differentiate between the two single random effects models using likelihood ratio tests. To

demonstrate this, simulations using 100 replications and a sample size of 40 were performed. In these, data is simulated from each of the two single random effect models and then all three models were fitted. The likelihood ratio test statistic for removing each of the random effects was then calculated for each replication.

Consider first data generated from equation (3.16), that is with a random effect on the intercept. Figure 4.9 (a) shows the histogram of the Likelihood Ratio test statistics for removing $\varepsilon$ from equation (4.10), which we would *not* want to do, while Figure 4.9 (b) shows the histogram for removing $\xi$, which we *would* want to do. These show that most of the time there is a large change in the likelihood when $\varepsilon$ is incorrectly removed, but a small change when $\xi$ is correctly removed. In fact, in this study, comparison with 3.841 ($\chi^2_{1,0.05}$) would lead to the correct inclusion of $\varepsilon$ on 99% of occasions and the correct exclusion of $\xi$ on 98% of occasions. Thus, for data simulated with a random effect on the intercept, the test of $H_0 : \varepsilon$ *not needed* against $H_1 : \varepsilon$ *is needed* has high power (as we would wish) while the test of $H_0 : \xi$ *not needed* against $H_1 : \xi$ *is needed* is conservative (also as we would wish).

The reverse of this can be observed in Figure 4.40 when data are generated from equation (4.1) with a random effect on the gradient. In this case, small values of the Likelihood Ratio test statistic result when $\varepsilon$ is correctly dropped from the model (98% of these in Figure 4.40 (a) are below 3.84) while large values tend to result when $\xi$ is incorrectly removed as shown in Figure 4.40 (b). However, the distinction is not as clear in the case of $\xi$ with only 42% of test statistics being significant at the 5% level. Thus, for data simulated with a random effect on the gradient, the test of $H_0 : \varepsilon$ *not needed* is conservative (as we would wish) but the test of $H_0 : \xi$ *not needed* has low power reflecting the inadequacy of a sample size of 40 when dealing with the model given by equation (4.1). Overall though, these results serve to demonstrate that the double random effect model is a useful tool and suggest that the modelling process is a reasonable one.
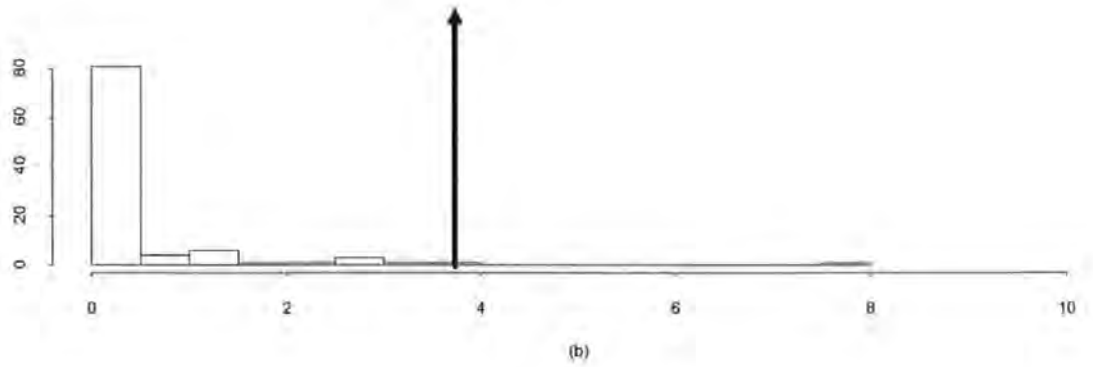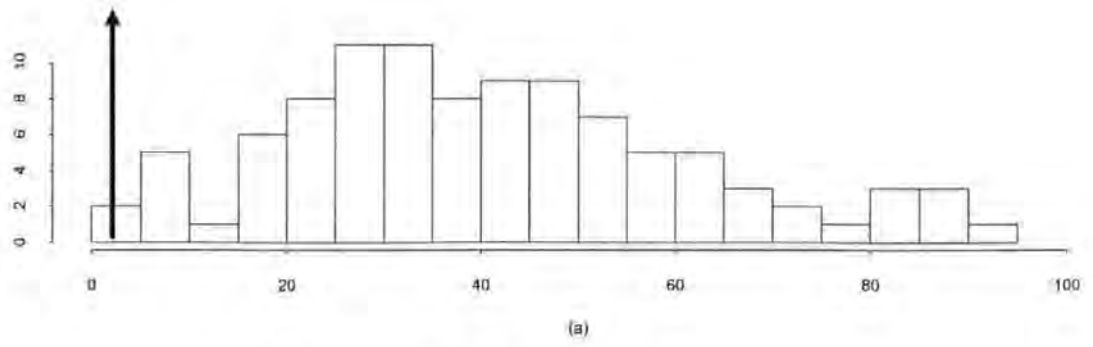
**Figure 4.9**: Data generated with a single random effect on the intercept ($\varepsilon$); histograms of L-R test statistics (a) for removing $\varepsilon$ (b) for removing $\xi$ ( $\uparrow$ shows $\chi^2_{1,0.05}$).
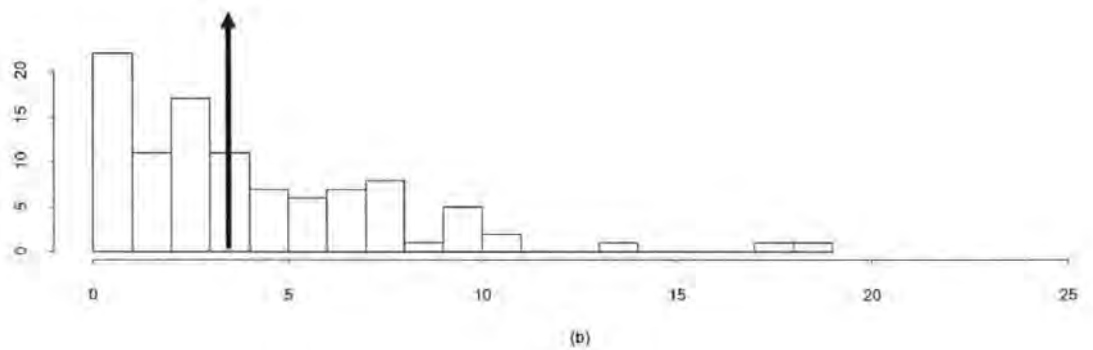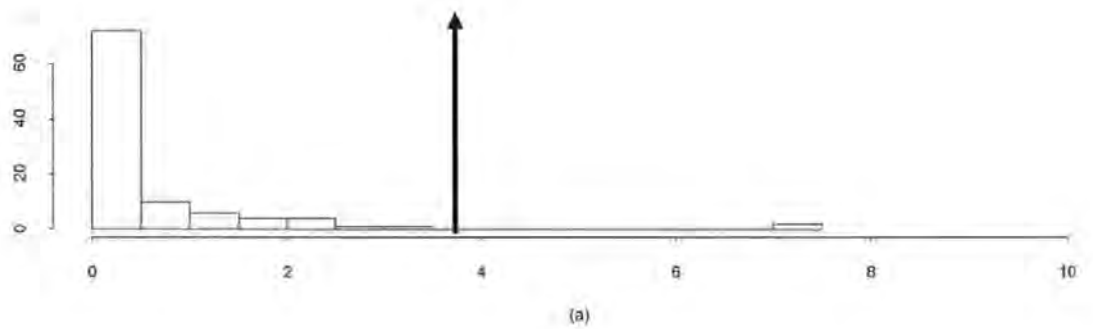


**Figure 4.10:** Data generated with a single random effect on the gradient ($\xi$); histograms of L-R test statistics (a) for removing $\varepsilon$ (b) for removing $\xi$ ( $\uparrow$ shows $\chi^2_{1,0.05}$).

## 4.6 Summary

This chapter has described a log-linear NHPP model with a random effect placed on the gradient parameter $\beta_1$. This model has been fitted to the Jaguar data given in Appendix 2 and the random effect of this type was found not to be significant. A number of methods were introduced to help differentiate between this model and the model from Chapter 3. These methods indicated that in this case the model with the random effect on the intercept parameter $\beta_0$ is more appropriate. The residuals for these two models are considered in Chapter 5.

Placing the random effect on the gradient rather than the intercept allows greater flexibility in the modelling, but requires more data than the previous model in order to obtain unbiased parameter estimates. An investigation into the bias in the parameter estimates showed that sample sizes much larger than those available with the Jaguar data are required before unbiased estimates can be obtained. This limits the practical use of this model as the large sample sizes required are difficult to obtain in practice.

Due to the problems of obtaining unbiased parameter estimates, and the variance component of the random effect on the gradient parameter not being significant, the models described in the later chapters only consider the case for a random effect on the intercept parameter. However, using the definitions in Section 4.2 these models can be extended to place the random effect on the gradient parameter.

# CHAPTER 5

# Residual Analysis

## 5.1 Introduction

The preceding chapters have introduced the random effect log-linear NHPP models, and fitted these to the Jaguar data given in Appendix 2. The hypothesis tests have enabled the significance of the fitted parameters to be assessed, and a nested model approach has given a comparison between the two models. The conclusion from these tests of significance leads to the variation between engines being modelled with a random effect on $\beta_0$. However, these tests give no indication of how well the models fit the actual data. In order to do this a residual analysis must be performed. This chapter introduces two approaches to calculating the residuals based on the definitions made by Lawless (1995). In this paper Lawless gives formulas for residuals from the marginal and conditional models defined in Section 3.2. That section states that for the NHPP model, the rate function for the marginal model and the intensity function for the conditional model are equivalent. The following sections give the residual analysis based on each approach.

## 5.2 Residuals For A Marginal Model

Lawless (1995) defines the residual for the $i^{th}$ unit from the marginal model (see Section 3.2) in terms of $\hat{\rho}(t)$ and $\phi(Z_i,\beta)$, as

$$e_i(t) = N_i(t) - \int_0^{t_i} \hat{\rho}(u)\phi(Z_i,\beta)du \tag{5.1}$$

where $\hat{\lambda}_i(u)$ represents $\hat{\rho}(u)$ and $\phi(Z_i,\beta)$ in our case. This gives the residual to be the difference between the expected and actual number of failures in the interval $(0,t]$. Another approach is to split the time scale into a series of intervals and to define the residual for the $j^{\text{th}}$ interval as

$$e(t_{j-1},t_j) = \frac{N(t_{j-1},t_j)}{n(t_{j-1},t_j)} - \int_{t_{j-1}}^{t_j} \hat{\lambda}_i(u)\,du \tag{5.2}$$

where $n(t_{j-1},t_j)$ is the number of units on test during the interval $(t_{j-1},t_j]$. The residuals are then the differences between the actual and predicted mean number of failures. The residuals defined by equation (5.2) allow the model to be assessed over time rather than just using a residual for each unit. In general the model will be used to predict failures in a given interval as well as looking at failure trends and it is felt that the residuals defined by equation (5.2) are more appropriate for this case.

Figure 5.1 shows three residual plots for the model with the random effect on $\beta_0$ fitted to phase A0, and Figure 5.2 shows the corresponding plots for the random effect on $\beta_1$. The residual plots for the other phases are given in Appendix 7. These plots all use the residuals defined by equation (5.2). Plot (a) gives the residual values against time to check for a time dependent effect. This effect is not in evidence for any of these models. Plot (b) gives a residual plot of residuals against fitted values , which will show a random scatter around zero if the model is appropriate. Plot (c) shows a normal probability plot of the residuals which will be an approximate straight line if the residuals are from a normal distribution.

For the random effect on $\beta_0$ the residual plots in Figure 5.1, and those given in Appendix 7, give further support to the conclusions drawn from the hypothesis tests on the fitted parameter values. For phases A0 and A2-1 (the two phases where all the fitted
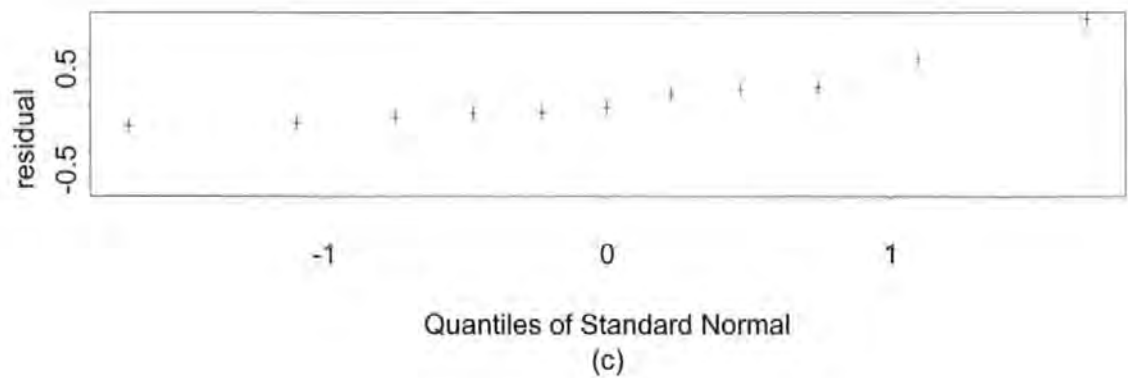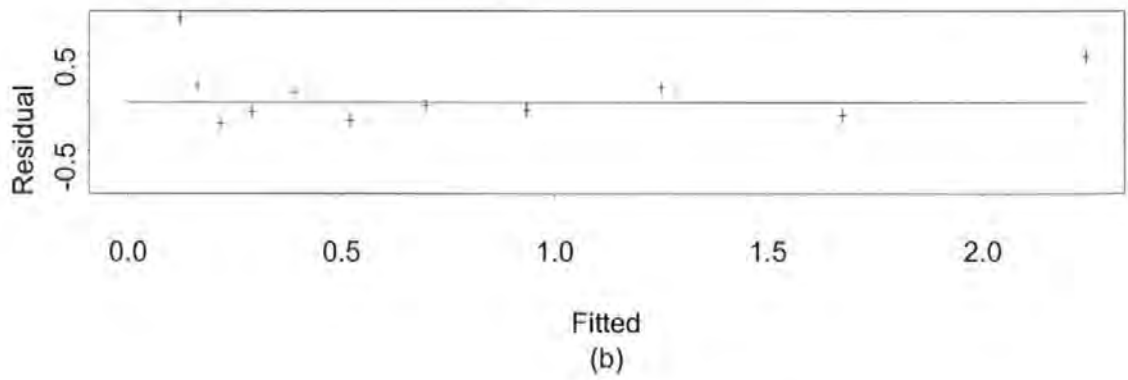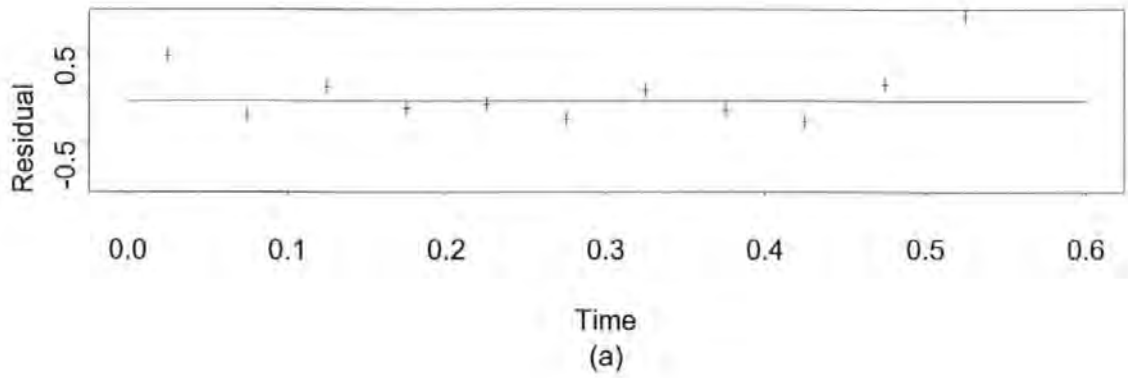
**Figure 5.1:** Plots of Marginal Residuals for a random effect on $\beta_0$ from A0 data.
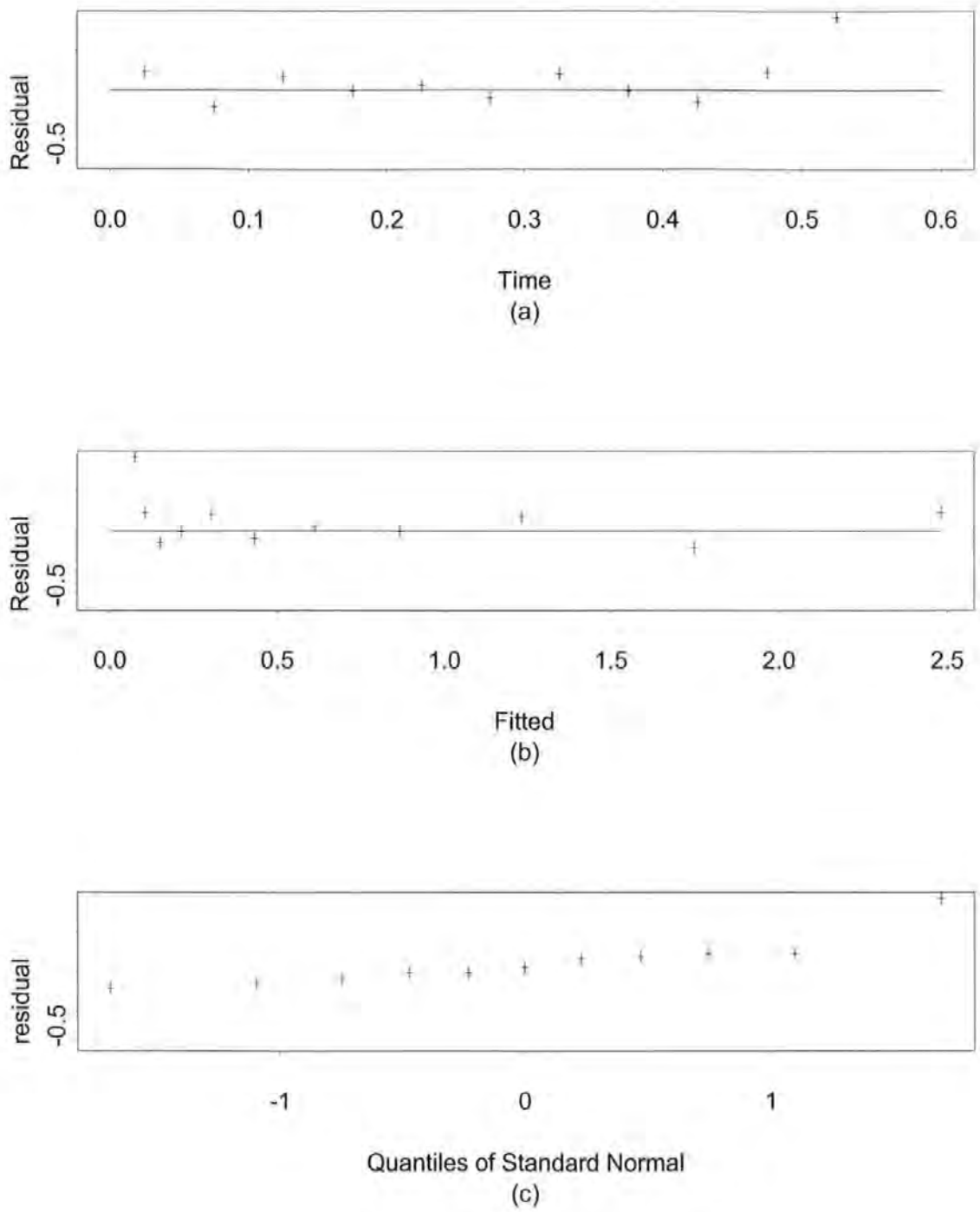
**Figure 5.2:** Plots of Marginal Residuals for a random effect on $\beta_1$ from A0 data.

parameters were found to be significantly different from zero at the 5% level) the residual plots all appear as expected. There appears to be no residual trend over time, the residuals plotted against fitted values gives a random scatter, and the normal probability plots show reasonably straight lines. For phase A0 there appears to be an unusually large residual value after 0.5 unit time and this can be attributed to the fact that data from only one engine is available after this time. For phase A2-1 where a number of engines ran up to the censoring time of 0.6 this outlier is not in evidence. The plots for both phases also show that the residual value for the first time interval (0 to 0.05 unit time) is very high. That is there are many more failures in this interval than would be expected from the fitted models. This supports the view that the testing procedures are recording a number of infant mortalities as well as other failures.

The hypothesis tests for phases A1-1 and A1-2 led to the conclusion that the estimate of $\beta_1$ is not significantly different from zero (the ROCOF is constant over time) and should be removed from the model. The residual plots for these two phases reflect this; as the gradient parameter is small there is little range in the fitted values, especially for phase A1-2 where the maximum likelihood estimate of $\beta_1$ is very close to zero.

Examination of the residual plots for the two models show little difference. From the results in Section 4.4 this would be expected as it was stated that predictions of the number of failures would be similar. The exception to this is the models for phase A1-2 where the fitted values differ considerably. However, in both cases $\beta_1$ was not found to be significantly different from zero and hence should be removed. This would result in constant predictions which would be similar for both models. As the residual plots for the random effect on $\beta_1$ are the same as for the random effect on $\beta_0$ (with the noted exception), the interpretations of these plots are the same as for those above.

The analysis of marginal residuals in this section has considered the models in terms of the mean ROCOF of the engines. Residual plots for each phase suggest the models fit reasonably well. This analysis has not given any evidence to suggest which of the random effect models is more appropriate as only the means are considered. The conditional residuals

described in the next section calculate residual values for each failure time, which does allow a comparison to be made.

## 5.3 Residuals For A Conditional Model

Lawless (1995) defines a residual for a conditional model with intensity function $\lambda(t;H_{it})$ as

$$\hat{e}_{ij} = \int_{t_{ij-1}}^{t_{ij}} \hat{\lambda}(t;H_{it}) \, dt \qquad (5.3)$$

That is, the residuals for the $i^{th}$ unit are given by the integral of the intensity function between successive failure times. If the fitted model is valid, the residuals $e_{ij}$ will follow a standard exponential distribution. To demonstrate this, consider the case for one unit with intensity function $\lambda(t;H_t)$ and with first failure at time $T$. From equation (5.3) it follows that the residual $e_1$ is

$$e_1 = \int_0^T \lambda(t;H_t) \, dt \qquad (5.4)$$

Let $\varepsilon$ be defined as a nominal point for the residual such that

$$P[e_1 > \varepsilon] = P\left[ \int_0^T \lambda(t;H_t) \, dt > \varepsilon \right] \qquad (5.5)$$

If $u(\varepsilon)$ is the function of $\varepsilon$ such that $\int_0^{u(\varepsilon)} \lambda(t;H_t) \, dt = \varepsilon$ then equation (5.5) becomes

$$P[e_1 > \varepsilon] = P[T > u(\varepsilon)] \qquad (5.6)$$

By definition time $T$ will be greater than $u(\varepsilon)$ if no failures occur in the interval $(0, u(\varepsilon)]$. The probability of this is given as

$$P[T > u(\varepsilon)] = \exp\left\{-\int_0^{u(\varepsilon)} \lambda(t; H_t)\, dt\right\}$$

$$= \exp(-\varepsilon) \tag{5.7}$$

Therefore the residual for time $T$ will come from a standard exponential distribution. Due to the memoryless property of the exponential distribution this will also hold for any time $T_l$ such that $T_l > T$. Hence for any time $t$, for multiple units and failures, the residuals will come from a standard exponential distribution.

For the log-linear random effect model introduced in Chapter 3, the value of the coefficient of the random effect is required for the $i$th unit in order to estimate the residuals defined by equation (5.3). Abu-Libdeh *et al.* (1990) and Lawless (1987) have achieved this for their respective models by obtaining the posterior distribution for the random effect and calculating the expected values. The prior distribution of the random effect is specified in the definition of the model, equation (3.16), namely

$$P[\varepsilon] \sim N(0, \sigma_\varepsilon^2) \tag{5.8}$$

For a series of times $t$, the posterior distribution of $\varepsilon$ is derived from Bayes' Theorem as

$$P[\varepsilon | t, \beta] = \frac{P[t | \varepsilon, \beta] P[\varepsilon]}{\int P[t | \varepsilon, \beta] P[\varepsilon]\, d\varepsilon} \tag{5.9}$$

The denominator of equation (5.9) is the likelihood for the model as given in equation (3.20). The expected value of $\varepsilon_i$, using the parameter estimates for the fitted model is then

$$E\left[\varepsilon_i | t, \hat{\beta}\right] = \int \varepsilon_i P\left[\varepsilon_i | t, \hat{\beta}\right] d\varepsilon \tag{5.10}$$

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To illustrate the method of calculating conditional residuals consider the model with a random effect on $\beta_0$ for the phase A0 engines. Equation (5.10) has been used to obtain the estimates of the random coefficient parameter using the parameter estimates from the fitted model ($\hat{\beta}_0 = 3.945$, $\hat{\beta}_1 = -5.830$, $\hat{\sigma}_\varepsilon = 0.486$). These estimates are given in Table 5.1 along with the total test time and total number of failures for each engine. From this and the data in Appendix 2 it can be noted that the most negative estimates of $\varepsilon_i$ (which indicates a lower starting failure rate) correspond to the engines with high test times and few failures. Conversely, the positive estimates (which indicate a higher starting failure rate) correspond to those engines with a high number of failures in a short test time. These estimates are therefore meaningful when compared to the data and their mean ($2.1 \times 10^{-6}$) is very close to zero and consistent with the model specification.

**TABLE 5.1**

*Estimates of the Random Coefficients for the Random Effect on $\beta_0$ from Phase A0 Engines*

| Engine | $\hat{\varepsilon}_i$ | Total Test Time | Number of Failures |
|--------|------|-----------------|--------------------|
| 1 | 0.159 | 0.193 | 9 |
| 2 | 0.389 | 0.425 | 14 |
| 3 | 0.549 | 0.112 | 10 |
| 4 | −0.161 | 0.069 | 2 |
| 5 | −0.330 | 0.546 | 5 |
| 6 | −0.495 | 0.388 | 3 |
| 7 | −0.130 | 0.455 | 7 |
| 8 | 0.705 | 0.041 | 7 |
| 9 | −0.047 | 0.075 | 3 |
| 10 | −0.074 | 0.117 | 4 |
| 11 | 0.071 | 0.144 | 6 |
| 12 | −0.731 | 0.195 | 0 |
| 13 | 0.109 | 0.305 | 9 |
| 14 | −0.138 | 0.487 | 7 |
| 15 | 0.127 | 0.093 | 5 |

The values of the random component in Table 5.1 have been used to calculate the conditional residuals for the fitted A0 model. A quantile-quantile plot (QQ-plot) for these residuals is shown in Figure 5.3. A QQ-plot can be used to compare data against a hypothesised distribution; the quantiles for the hypothesised distribution are plotted against the sorted data. If the data follow the distribution the resulting QQ-plot will be a straight line. In Figure 5.3 (and other figures in this chapter) the straight line represents a unit exponential distribution and the points are the QQ-plot for the residual values. The conclusion from this plot is that the residuals do not come from a unit exponential, although the majority of values do form a reasonable straight line suggesting an exponential with a smaller $\lambda$ (the estimate of $\lambda$ for these residuals is 0.84). However the tail values do not follow a straight line suggesting that the exponential assumption does not hold in this region. For comparison Figure 5.4 shows the quantile plots of four simulated data sets. These are simulated for 15 units with the same parameter estimates as the fitted A0 model. These show a similar pattern to the previous graph. The majority of values follow a straight line just below the unit exponential (albeit
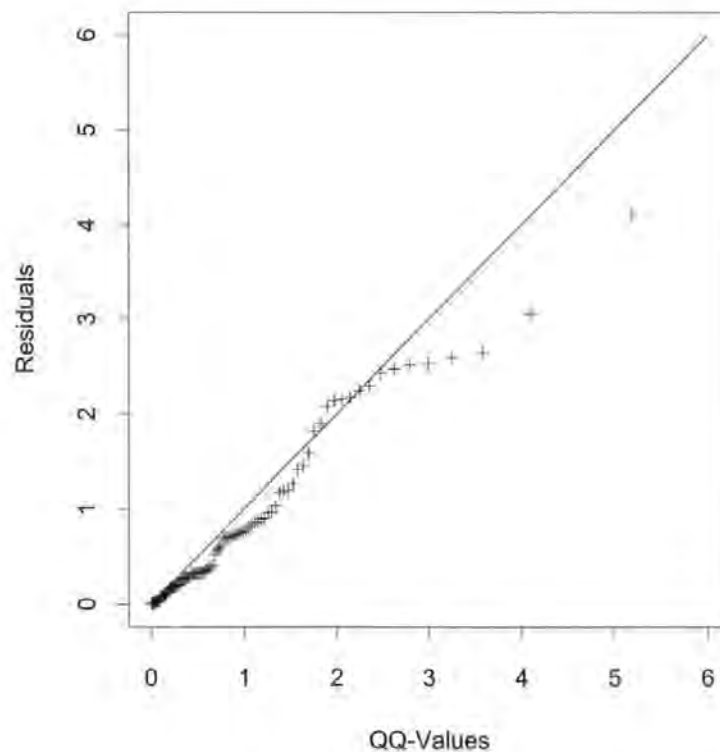


**Figure 5.3:** Conditional Residuals from Phase A0 for Model with Random Effect on $\beta_0$.
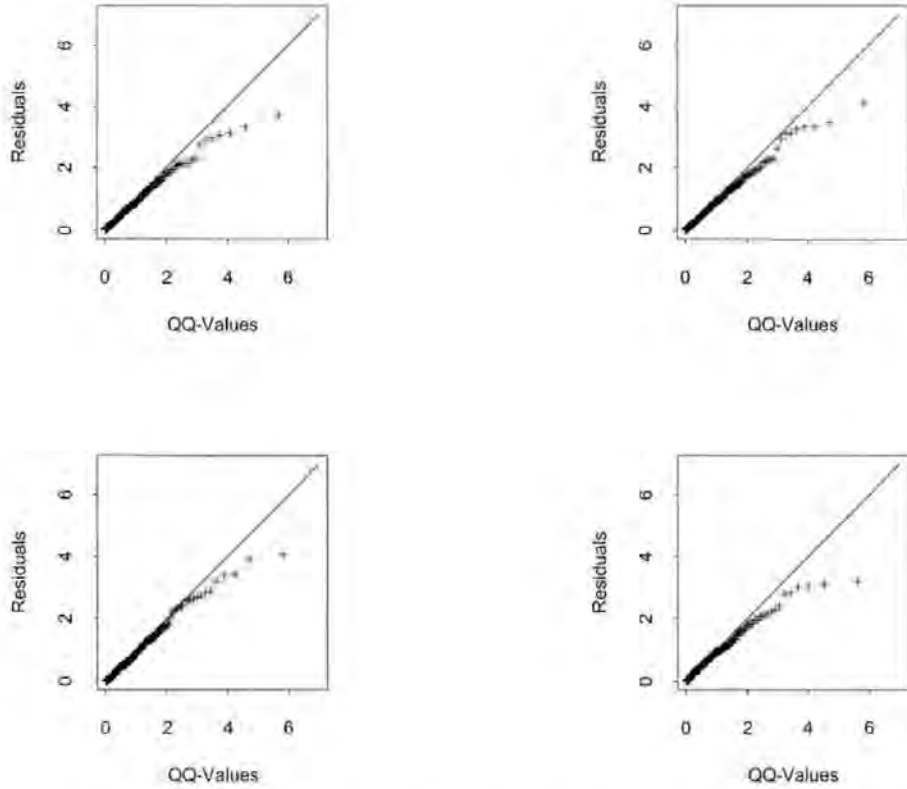
**Figure 5.4:** Conditional Residuals from Four Sets of Simulated Data for the Model with Random Effect on $\beta_0$.

closer than before) and the distribution deviates at the tail. A Kolmogorov-Smirnov goodness of fit test (see Daniel, 1990) can be used to assess the null hypothesis that the residuals follow a unit exponential. With the random effect on the intercept model fitted to A0 data the test statistic is 0.155 with an associated p-value of 2.5%, leading to the rejection of the null hypothesis. A test against any exponential distribution gives a p-value of 12%. The conclusion is that although the residuals do not follow a unit exponential there is no evidence against an exponential distribution with some other mean.

In Figure 5.4 it can be seen that the residuals for the simulated data do not follow the unit exponential. A simulation can be performed to compare the Kolmogorov-Smirnov test statistic from the A0 model with those for data where the model is true. Figure 5.5 shows the histogram of test statistics for 1000 simulated data sets. The test statistic for the A0 model is in the tail of this histogram and only one simulated value is more extreme. The conclusion is

that the A0 residuals, when the random effect is on the intercept, differ from those generated from the hypothesised model. Thus the conditional residual analysis suggests that the model with a random effect on the intercept is not suitable for the A0 data.
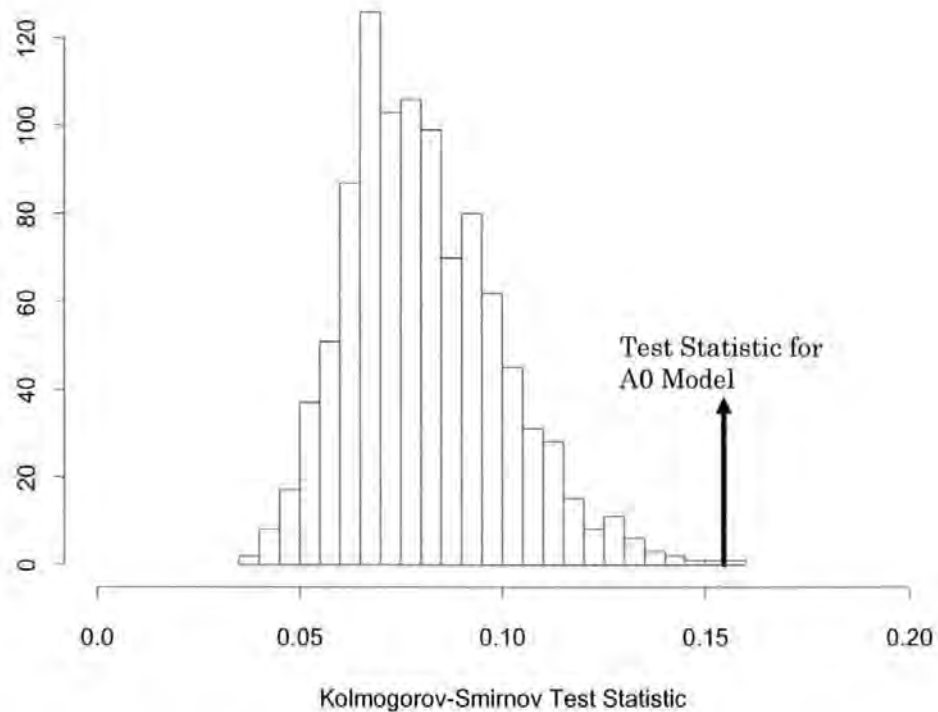


**Figure 5.5:** Histogram for simulated Kolmogorov-Smirnov Test for A0 Model.

Alternatively, it may be that the deviation in the residuals is due to an outlier engine having undue influence on the fitted model. The residual plots in Figures 5.3 and 5.4 can be split by engine to investigate this. Figures 5.6 and 5.7 show these for the A0 residuals and simulated residuals respectively. For the simulated data there does not appear to be any line standing out from the others in any of the plots. For the A0 model there is one line which could be considered an outlier. This is due to engine 8 which has many failures in a short period of time. This engine also has the most positive random coefficient. Another engine which would appear to be an outlier in the model is engine 12 which had no failures in 0.195 unit time on test (this has the most negative random coefficient). Although it does not have a residual associated with it as no failures occurred and does not therefore stand out on any of the plots, it could have a large influence on the fitted parameters. In order to test the effects of

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these, engines have been removed both separately and together, and the model refitted. The resulting models and the value of the Kolmogorov-Smirnov test statistic for unit exponentiality are given in Table 5.2 and plots of the residuals associated with these model are given in Figures 5.8 to 5.10. The fitted parameters for the gradient and variance show a relatively large change with the removal of these engines and the test statistic is also still very extreme compared to the simulated values. However, these plots compare more favourably with the unit exponential distribution. Hence there does appear to be an outlier effect due to these engines although even after their removal the residuals do not follow a standard exponential distribution. The conditional residuals from the A0 model with the random effect on the intercept have also been produced for the bias corrected parameter estimates in Table 3.16. These show no difference compared to the conditional residual plot for the original parameter estimates.

**TABLE 5.2**

*Parameter Estimates for A0 data with Extreme Engines Removed.*

| Engine Removed | $\hat{\beta}_0$ | $\hat{\beta}_1$ | $\hat{\sigma}_\varepsilon$ | Test Statistic |
|---|---|---|---|---|
| 8 | 3.849 | −5.524 | 0.389 | 0.174 |
| 12 | 4.082 | −6.132 | 0.328 | 0.155 |
| 8 & 12 | 3.988 | −5.797 | 0.240 | 0.170 |
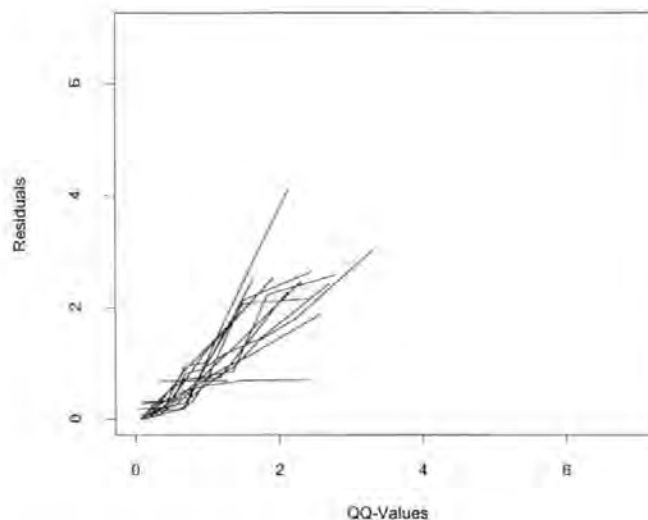


**Figure 5.6**: Conditional Residuals for A0 data by engine.

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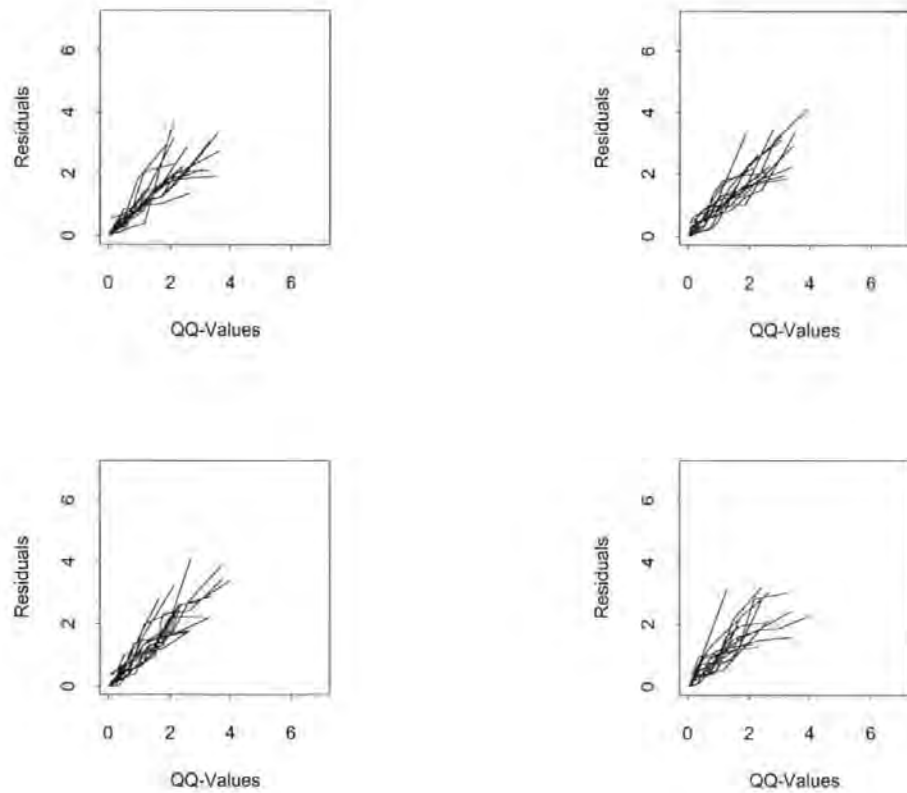
**Figure 5.7:** Conditional Residuals for simulated A0 data by engine.
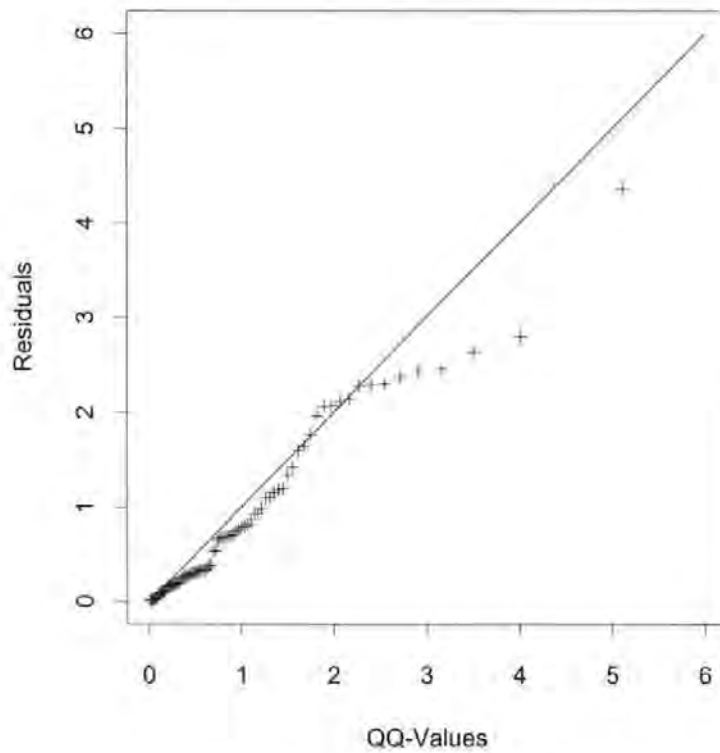


**Figure 5.8:** Conditional Residuals for A0 data with engine 8 removed.

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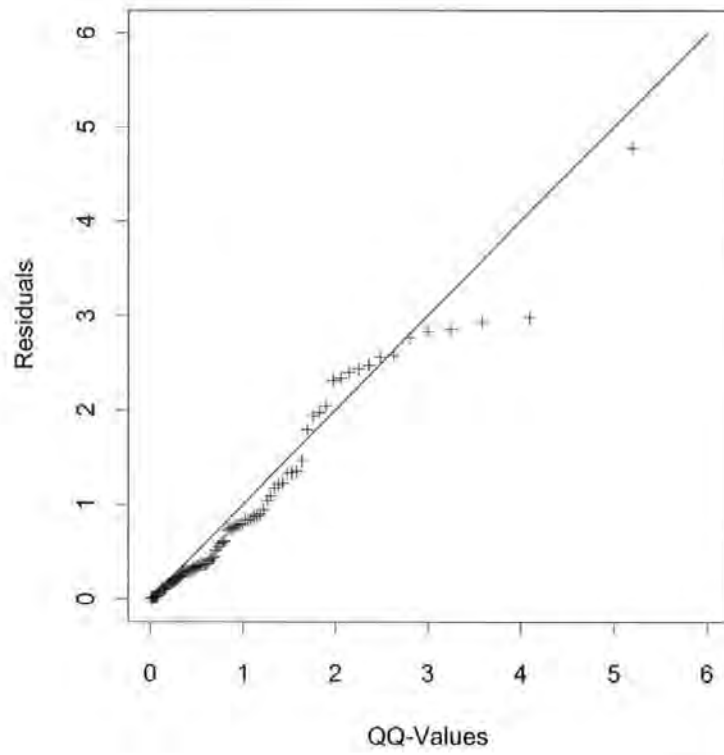
**Figure 5.9:** Conditional Residuals for A0 data with engine 12 removed.
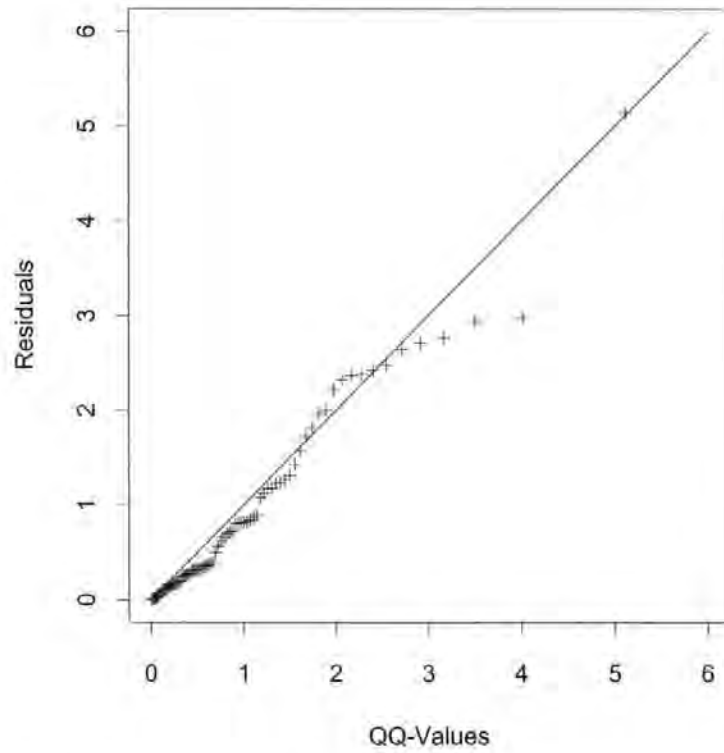


**Figure 5.10:** Conditional Residuals for A0 data with engine 8 and 12 removed.

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This section has so far defined the conditional residuals and has investigated those produced by the random effect on the intercept model using the A0 data. Figure 5.11 shows the QQ-plot for the corresponding residuals from the model with a random effect on the gradient. Appendix 8 gives the QQ-plots for conditional residuals from both models for the remaining engine phases. A comparison of these plots for the two different models show that the residuals from the random effect on the gradient model follow a unit exponential distribution more closely than those from the random effect on the intercept model. From the results of the significance tests in Chapter 4 this is the reverse of the expected result. The random effect parameter $\sigma_\xi$ is not significant for any of the phases, hence the residuals when the random effect is on the intercept would be expected to compare more favourably with the unit exponential. However, the Kolmogorov-Smirnov test statistic produced from the model with a random effect on the gradient, for phase A0, is 0.184 which is larger than the test statistic for the proportional random effect model. The Kolmogorov-Smirnov test statistics and p-values for the engine phases are given in Table 5.3. This shows the p-value to be smaller when the random effect is on $\beta_1$ for all but one phase, concluding that there is less evidence against the null hypothesis of unit exponentiality for the proportional random effect model. The exception to this occurs for phase A1-1 where the random effect is not significant in either model (the proportional random effect is significant for all other phases).

TABLE 5.3

*Kolmogorov-Smirnov Test Statistics (P-Values) for Conditional Residuals Compared to a Unit Exponential Distribution.*

| Phase | Random Effect On | |
|---|---|---|
| | $\beta_0$ | $\beta_1$ |
| A0 | 0.155(0.033) | 0.184(0.004) |
| A1-1 | 0.092(0.582) | 0.066(0.921) |
| A1-2 | 0.133(0.327) | 0.14(0.251) |
| A2-1 | 0.09(0.267) | 0.11(0.105) |
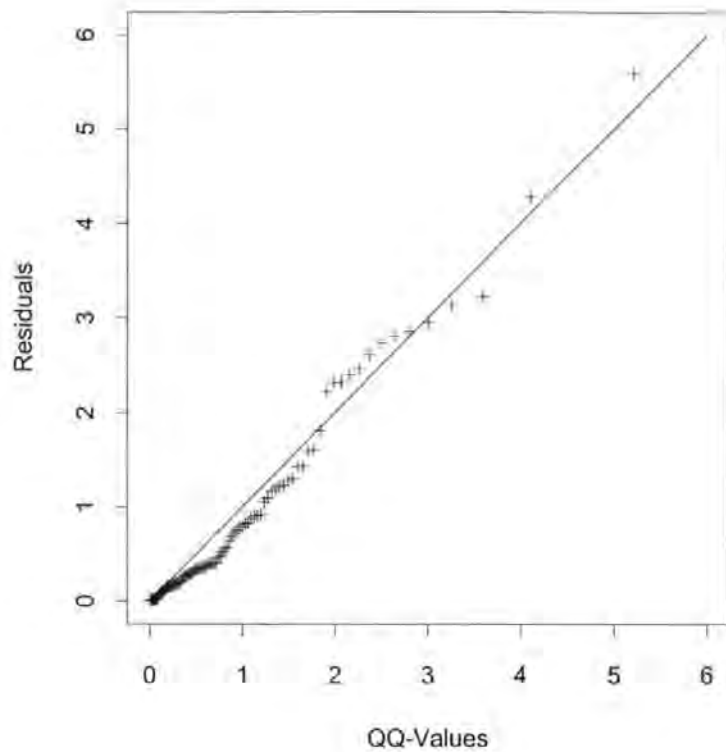
**Figure 5.11:** Conditional Residuals from Phase A0 for Model with Random Effect on $\beta_1$.

It is noted from the conditional residual plots for the random effect on $\beta_1$ that the values in the tail of the distribution have been bought closer to the line of the unit exponential. In contrast, the conditional residuals from the two models show little difference for the majority of values.

## 5.4 Summary

This chapter has introduced two types of residuals for model checking and comparison. The marginal residuals in Section 5.2 indicate the performance of the model against the mean ROCOF, whereas the conditional residuals in Section 5.3 provide an analysis against the individual failure times. In the latter case, the random effect is included when calculating the residual value.

The conclusion from the two types of residuals is that the models fit the mean number of failures reasonably well. The Kolmogorov-Smirnov goodness of fit test favours the proportional random effect model. Incorporating the results of this chapter with those in Chapter 4 would suggest that the model with a random effect on the intercept performs better with the Jaguar data. In addition, the estimate of this random effect is not subjected to the levels of bias in the alternative model. Subsequent chapters use these results and only a model with a random effect on the intercept is considered, although the analysis can be extended to include the random effect on the gradient as in Chapter 4.

# CHAPTER 6

# Random Effect Models With Covariates

## 6.1 Introduction

This chapter introduces extensions to the basic log-linear model with a random effect

on the intercept (equation (3.16)) which allow covariates to be included by means of indicator

variables. As with the random effect itself there are a number of ways in which the covariates

can be included, namely as a proportional effect, a time multiplicative effect or both of these.

The interpretation of the proportional covariate model is that the ROCOF's of the units

between the covariate groups are different at time zero but change at the same rate. For the

time multiplicative covariate model, the ROCOF's between the covariate groups are the same

at time zero but change at a different rate. It follows from this that including both types of

covariate models leads to a situation where the ROCOF's differ at time zero and change at

different rates. The three covariate models are described, for $k$ covariates, in Sections 6.2, 6.3

and 6.4 respectively. A simulation study has been conducted to compare the three covariate

models and to assess the power of the tests on the fitted parameters. The results of this are

given in Section 6.5.

In Chapter 1 it was stated that the tests conducted on the engines at Jaguar can be

classified into two groups. These indicate whether the engine was subjected to high or low

demand on test. The failure data in Appendix 2 gives this information coded as *test* (0 denotes

low, 1 denotes high). Using the test type as a covariate, each of the random effect covariate

models introduced in this chapter are fitted to the Jaguar data. As in previous chapters the engine phases have been treated separately. Examination of the data for the four engine phases shows that there is little data available for low level testing. For example, on phases A1-1 and A1-2 there are only two engines where testing has been carried out at a low level. As a result, there can be little confidence in the estimated values of the covariate parameter. However, fitting the models to this data is still useful for illustrative purposes, and also as a formal method for assessing a covariate effect due to test type.

Failure time plots for the data, split into low and high levels, are given in Figures 6.1 to 6.4. The solid lines represent engines run at a high level, and the dotted lines represent engines run at low level. From these graphs it appears that there is no effect due to the level of testing. The rate of occurrence of failures for the low level tested engines does not stand out on the graphs as being different from those engines tested at a high level. The only exception to this is for the data on phase A1-2 where the low level tested engines differ from the majority of other engines, but with only two observations for low level this would not be expected to be a significant difference.

## 6.2 Proportional Covariate Effects

### 6.2.1 The General Model

For $k$ covariates $x_1, x_2, \ldots, x_k$ the model with a random effect on the intercept parameter is:

$$\lambda_i(t) = \exp\left( \beta_0 + \beta_1 t + \sum_{j=1}^{k} \beta_{2j} x_{ij} + \varepsilon_i \right) \tag{6.1}$$

where

$x_{ij}$ is an indicator variable for the $j^{\text{th}}$ covariate level of unit $i$.

$\beta_{2j}$ is the covariate parameter for the $j^{\text{th}}$ covariate level of unit $i$.

$\varepsilon_i$ is a random effect due to unit $i$, and $\varepsilon \sim N(0, \sigma_\varepsilon^2)$.

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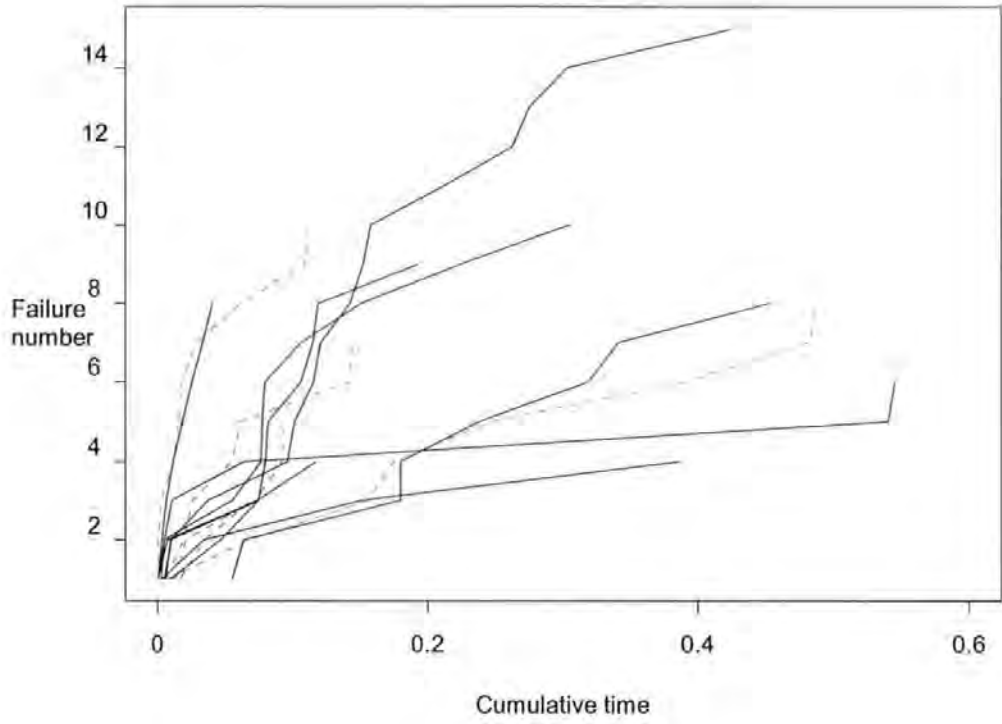
**Figure 6.1 :** Failure Time Plot for Engine Phase A0 at Low (- - - -) and High (——) Levels.
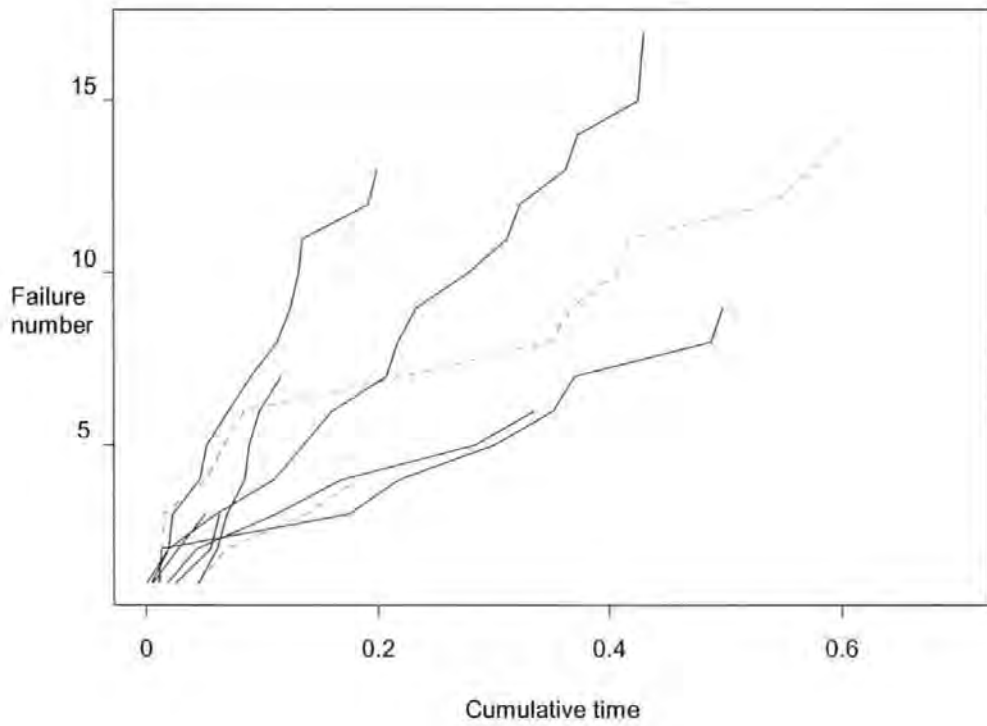


**Figure 6.2 :** Failure Time Plot for Engine Phase A1-1 at Low (- - - -) and High (——) Levels.
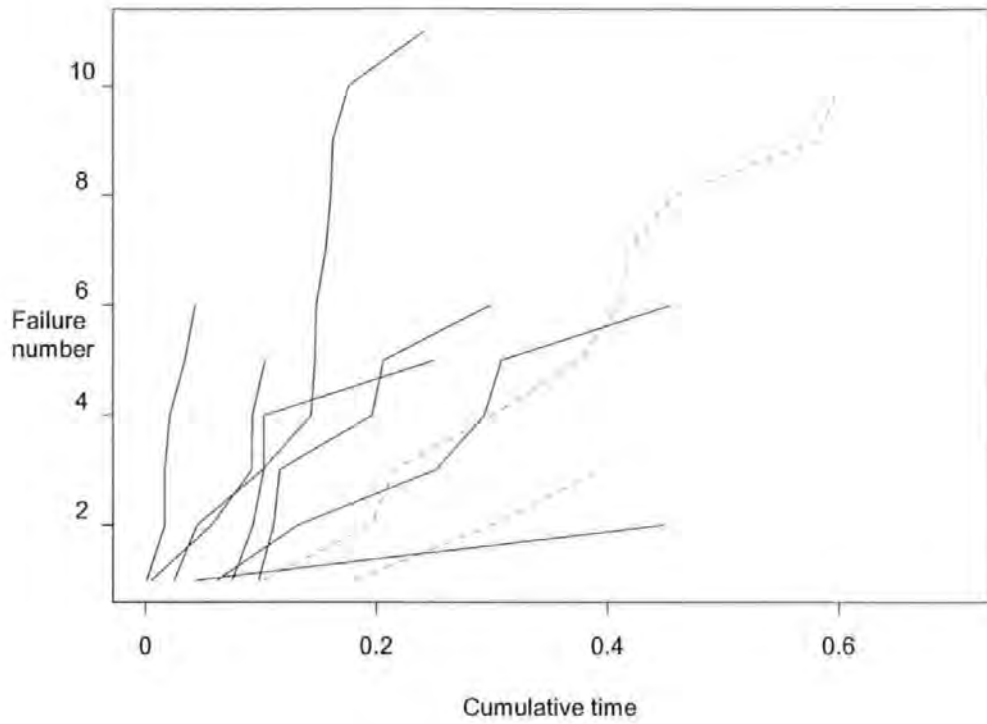
**Figure 6.3 :** Failure Time Plot for Engine Phase A1-2 at Low (- - - -) and High (———) Levels.
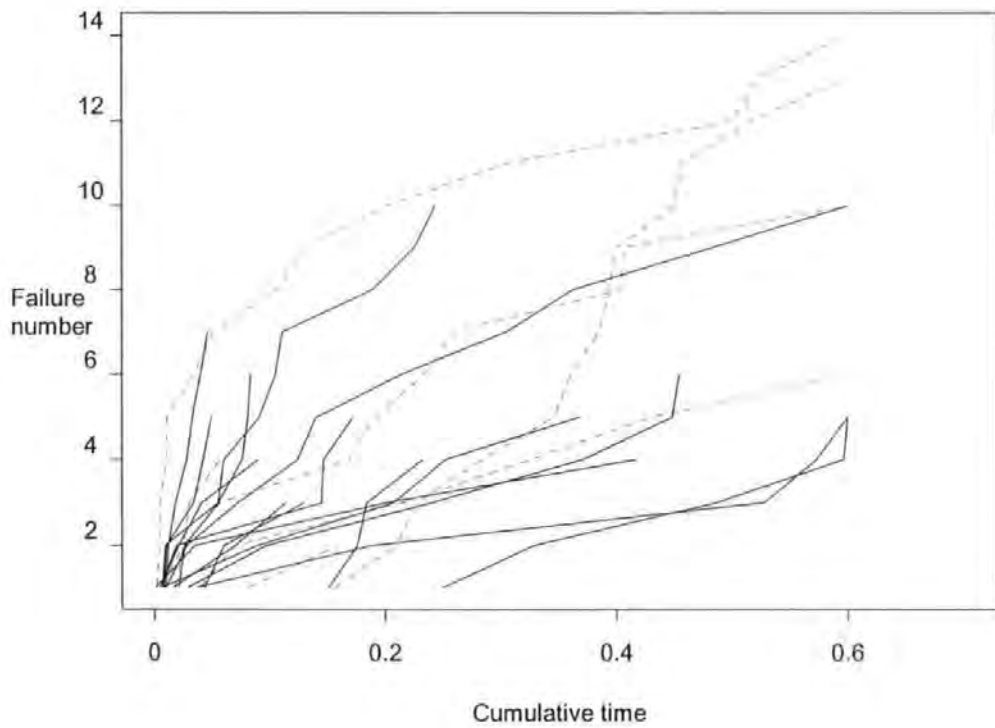


**Figure 6.4 :** Failure Time Plot for Engine Phase A2-1 at Low (- - - -) and High (———) Levels.

This model is of the same form as the Proportional Intensity Model from Lawless (1987) but with a different parameterisation of the random effect. It has been further discussed by Ciampi *et al.* (1992) who describe a computer program to fit a proportional intensity model, and Abu-Libdeh *et al.* (1990) where a Power Law baseline function, equation (2.53), has been used.

Expressing the model in this form gives the covariate a proportional effect on the expected number of failures. If all $k$ covariate levels of unit $i$ are 0 then the model is of the form of equation (3.16), otherwise the intercept parameter becomes $\beta_0 + \sum_{j=1}^{k} \beta_{2j} x_{ij}$. The expected number of failures in time interval $(a,b]$ then becomes

$$\text{E}\{\text{failures}(a,b] \mid \varepsilon_i\} = \exp\left(\beta_o + \sum_{j=1}^{k}\beta_{2j}x_{ij} + \varepsilon_i\right)\int_a^b \exp(\beta_1 t)dt \qquad (6.2)$$

The log-likelihood function is derived as in Section 3.3. Assuming a mean of zero and a constant variance for the random effect the log-likelihood function is given by

$$\ell = \sum_{i=1}^{m}\ln\left(\exp\left(n_i\left[\beta_0 + \sum_{j=1}^{k}\beta_{2j}x_{ij}\right] + \beta_1\sum_{j=1}^{n_i}t_{ij}\right)\int_{-\infty}^{\infty}\exp\left[n_i\varepsilon_i - \frac{\exp\left(\beta_0 + \sum_{j=1}^{k}\beta_{2j}x_{ij} + \varepsilon_i\right)}{\beta_1}\{\exp(\beta_1 t_i)-1\}\right]\frac{\exp\left\{-\frac{\varepsilon_i^2}{2\exp(2\varphi_\varepsilon)}\right\}}{\sqrt{2\pi\exp(2\varphi_\varepsilon)}}d\varepsilon_i\right)$$

$$(6.3)$$

Substituting equations (6.4) to (6.9) given below into the expressions for a single random effect (given in Appendix 5) gives first and second partial derivatives with respect to $\beta_0$, $\beta_1$ and $\sigma_\varepsilon$.

$$z_i = \exp\left[n_i\left(\beta_o + \sum_{j=1}^{k}\beta_{2j}x_{ij}\right) + \beta_1\sum_{j=1}^{n_i}t_{ij} + n_i\varepsilon_i - \frac{\exp\left(\beta_o + \sum_{j=1}^{k}\beta_{2j}x_{ij} + \varepsilon_i\right)}{\beta_1}\{\exp(\beta_1 t_i)-1\}\right]\frac{\exp\left\{-\frac{\varepsilon_i^2}{2\exp(2\varphi_\varepsilon)}\right\}}{\sqrt{2\pi\exp(2\varphi_\varepsilon)}}$$

$$(6.4)$$

$$A_i = \int_{-\infty}^{\infty} z_i \, d\varepsilon_i \tag{6.5}$$

$$B_i = n_i - \frac{\exp\left(\beta_0 + \sum_{j=1}^{k} \beta_{2j} x_{ij} + \varepsilon_i\right)}{\beta_1} \left\{\exp(\beta_1 t_i) - 1\right\} \tag{6.6}$$

$$C_i = \sum_{j=1}^{n_i} t_{ij} - \frac{\exp\left(\beta_0 + \sum_{j=1}^{k} \beta_{2j} x_{ij} + \varepsilon_i\right)}{\beta_1^{\,2}} \left\{\exp(\beta_1 t_i)(\beta_1 t_i - 1) + 1\right\} \tag{6.7}$$

$$D_i = \frac{\varepsilon_i^{\,2} - \exp(2\varphi_\varepsilon)}{\exp(2\varphi_\varepsilon)} \tag{6.8}$$

$$E_i = -\frac{\exp\left(\beta_0 + \sum_{j=1}^{k} \beta_{2j} x_{ij} + \varepsilon_i\right)}{\beta_1^{\,3}} \left\{\beta_1 t_i \exp(\beta_1 t_i)(\beta_1 t_i - 2) + 2\exp(\beta_1 t_i) - 2\right\} \tag{6.9}$$

Let $\left(\dfrac{\partial \ell}{\partial \beta_0}\right)_i$ be defined as the value of the first order partial derivative with respect to $\beta_0$

due to the $i$th unit, such that $\dfrac{\partial \ell}{\partial \beta_0} = \sum_{i=1}^{n} \left(\dfrac{\partial \ell}{\partial \beta_0}\right)_i$ . It then follows that the first and second partial

derivatives with respect to $\beta_{2j}$ ($j$th covariate) can be shown to take the form of equations (6.10)

to (6.15), namely

$$\frac{\partial \ell}{\partial \beta_{2j}} = \sum_{i=1}^{m} x_{ij} \left(\frac{\partial \ell}{\partial \beta_0}\right)_i \tag{6.10}$$

$$\frac{\partial^2 \ell}{\partial \beta_0 \partial \beta_{2j}} = \sum_{i=1}^{m} x_{ij} \left(\frac{\partial^2 \ell}{\partial \beta_0^{\,2}}\right)_i \tag{6.11}$$

$$\frac{\partial^2 \ell}{\partial \beta_1 \partial \beta_{2j}} = \sum_{i=1}^{m} x_{ij} \left(\frac{\partial^2 \ell}{\partial \beta_0 \partial \beta_1}\right)_i \tag{6.12}$$

$$\frac{\partial^2 \ell}{\partial \beta_{2j}^2} = \sum_{i=1}^{m} x_{ij} \left( \frac{\partial^2 \ell}{\partial \beta_0^2} \right)_i \tag{6.13}$$

$$\frac{\partial^2 \ell}{\partial \beta_{2j} \partial \sigma_\varepsilon} = \sum_{i=1}^{m} x_{ij} \left( \frac{\partial^2 \ell}{\partial \beta_0 \partial \sigma_\varepsilon} \right)_i \tag{6.14}$$

The second partial derivative with respect to $\beta_{2j}$ and $\beta_{2k}$ (the $j^{th}$ and $k^{th}$ covariates) is then given by

$$\frac{\partial^2 \ell}{\partial \beta_{2j} \partial \beta_{2k}} = \sum_{i=1}^{m} x_{ij} x_{ik} \left( \frac{\partial^2 \ell}{\partial \beta_0^2} \right)_i \tag{6.15}$$

A comparison of the log-likelihood and derivatives, given in equations (6.1) to (6.9), for the proportional covariate model, with those for the random effect model defined by equation (3.16) shows them to be of the same form with $\beta_0 + \sum_{j=1}^{k} \beta_{2j} x_{ij}$ replacing $\beta_0$ in the latter.

### 6.2.2 Simulation on Proportional Covariates

The inversion method of simulation described in Section 3.6 is easily extended to include observations from a covariate model. In this section the results of a simulation exercise from a model with a single covariate are given to validate the estimates of the parameters and their standard errors. In this case, from equation (6.1) $k=1$ and the model now becomes

$$\lambda_i(t) = \exp\left(\beta_0 + \beta_1 t + \beta_2 x_i + \varepsilon_i\right) \tag{6.16}$$

The simulation uses parameter values of $\beta_0 = 4$, $\beta_1 = -3$, $\beta_2 = -1$ and $\sigma_\varepsilon = 0.5$. This is equivalent to simulating a number of units with either $\beta_0 = 4$ or $\beta_0 = 3$, with $\beta_1$ and $\sigma_\varepsilon$ remaining constant. Figure 6.5 shows a plot of simulated observations from these parameter

values with 10 units in each covariate group, where the solid lines represent the units where $x_i = 0$. In general, the units with the covariate equal to one have a smaller rate of occurrence of failure and have less failures, especially in the latter part of the time interval. Although there is an overlap between the two groups, this suggests that the covariate effect can be identified by the model.
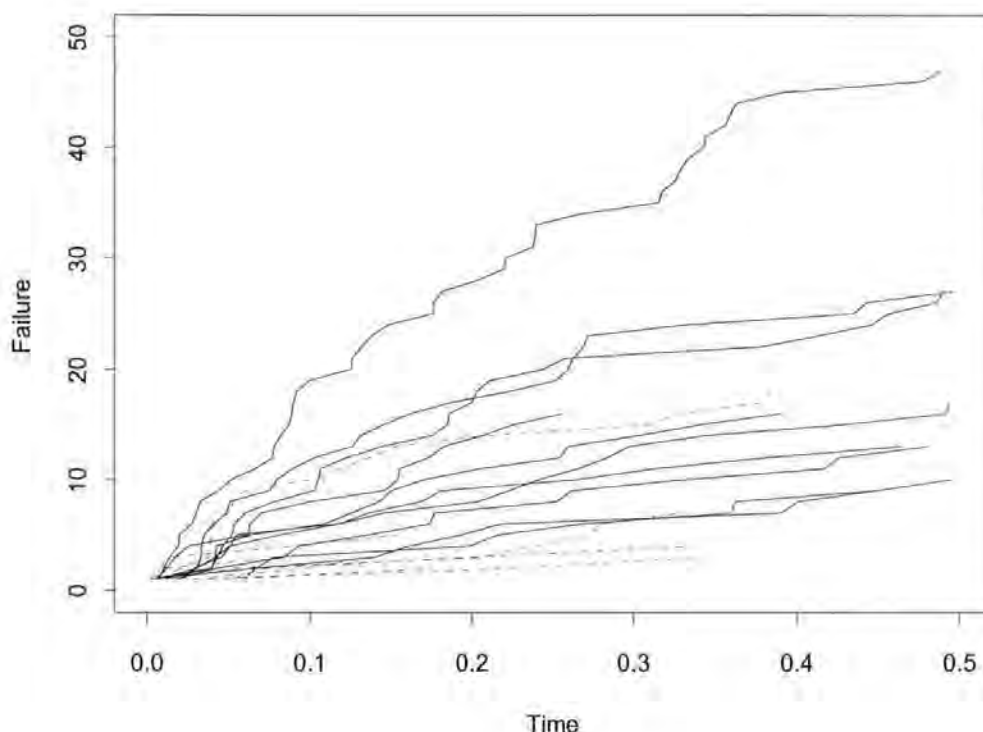


**Figure 6.5 :** Failure Plot of Simulated Proportional Covariate Data at Low (- · - ·)

and High (———) Levels.

A simulation using the above values as the true parameters has been conducted. In this case a sample size of 30 has been used and 100 replications have been performed. The results of this simulation are summarised in Table 6.1. These show that the parameter estimates and their standard errors are close to the true values for the fixed effects. The variance component is biased and the estimate of its standard error is poor. The ratio of the mean estimate of the covariate parameter to its standard error is 4.4 which suggests the model is able to identify the covariate effect. The significance of the covariate is discussed further in Section 6.5.

**TABLE 6.1**

*Results of Simulation Exercise With Proportional Covariate Effects.*

| | $\hat{\beta}_0$ | $\hat{\beta}_1$ | $\hat{\beta}_2$ | $\ln(\hat{\sigma}_\varepsilon)$ |
|---|---|---|---|---|
| **True Parameter Values** | 4 | −3 | −1 | -0.693 |
| **Mean of Sampling Distribution of Estimates** | 4.031 | −3.033 | −1.070 | -0.819 |
| **Standard Deviation of Sampling Distribution of Estimates** | 0.1797 | 0.4713 | 0.2402 | 0.3526 |
| **Mean of Asymptotic Standard Errors** | 0.1597 | 0.4055 | 0.2174 | 0.2217 |

## 6.2.3 Proportional Covariate Model for Jaguar Data

The model for a single proportional covariate effect, equation (6.16), has been fitted to the four engine phases of the Jaguar data given in Appendix 2. The parameter estimates, their standard errors and the maximised log-likelihood are summarised in Table 6.2. Examination of the difference between the two test types on the failure plots for the four phases, Figures 6.1 to 6.4, suggested that a covariate effect would not be significant. The results from the fitted models support this. In all cases the estimate of $\beta_2$ (the covariate effect) is of the same order as, or smaller than its standard error. The conclusion from this is that the covariate effect is not significantly different from zero. The Likelihood ratio test can also be used to compare these models and those without the covariate, given in Table 3.7. There is little difference between the maximised log-likelihoods of the two models, and all the test statistics are much smaller than the critical value of 3.84 ($\chi^2_{1,0.05}$). This means the log-likelihood does not change significantly when the covariate is removed. In other words, the models in Chapter 3 would be fitted in preference to the proportional covariate effect models.

**TABLE 6.2**

*Parameter Estimates (Standard Errors) of Fitted Proportional Covariate Effect Models.*

| Phase | $\hat{\beta}_0$ | $\hat{\beta}_1$ | $\hat{\beta}_2$ | $\ln(\hat{\sigma}_\varepsilon)$ | $\hat{\sigma}_\varepsilon$ | $\ell_{\max}$ |
|-------|-----------------|-----------------|-----------------|----------------------------------|-----------------------------|---------------|
| A0 | 4.105 (0.3007) | −5.768 (1.2407) | −0.243 (0.3515) | −0.752 (0.3880) | 0.417 | 222.71 |
| A1-1 | 3.343 (0.3877) | −1.340 (0.9613) | 0.354 (0.3852) | −1.216 (0.5910) | 0.324 | 170.90 |
| A1-2 | 2.129 (0.7361) | 0.250 (1.3257) | 0.961 (0.7423) | −0.285 (0.3615) | 0.752 | 97.37 |
| A2-1 | 3.349 (0.3090) | −2.428 (0.7075) | 0.010 (0.3346) | −0.642 (0.3099) | 0.526 | 230.87 |

## 6.3 Time Multiplicative Covariate Effects

### 6.3.1 The General Model

This section describes an alternative covariate model in which the effect is dependent upon time. For $k$ covariates $x_1, x_2, \ldots, x_k$ the model with a random effect on the intercept parameter given in equation (3.16) then becomes

$$\lambda_i(t) = \exp\left\{\beta_0 + \left(\beta_1 + \sum_{j=1}^{k}\beta_{3j}x_{ij}\right)t + \varepsilon_i\right\}$$
(6.17)

where

$x_{ij}$ is an indicator variable for the $j^{\text{th}}$ covariate level of unit $i$.

$\beta_{3j}$ is the covariate parameter for the $j^{\text{th}}$ covariate level of unit $i$.

$\varepsilon_i$ is a random effect due to unit $i$, and $\varepsilon \sim N(0, \sigma_\varepsilon^2)$.

As is the case with the random effect on $\beta_1$ described in Chapter 4 where the gradient parameter is not fixed, the gradient parameter here depends on the level of the covariate. If

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all the covariate levels for unit $i$ are 0 then the gradient parameter is $\beta_1$, otherwise the

gradient is $\beta_1 + \sum_{j=1}^{k} \beta_{3j} x_{ij}$. The expected number of failures in the time interval $(a,b]$ is given by

$$E\{\text{failures}(a,b] \mid \varepsilon_i\} = \exp(\beta_o + \varepsilon_i) \int_a^b \exp\left\{\left(\beta_1 + \sum_{j=1}^{k} \beta_{3j} x_{ij}\right) t\right\} dt \tag{6.18}$$

If we assume a mean of zero and a constant variance for the random effect, the

log-likelihood function is given by

$$\ell = \sum_{i=1}^{m} \ln\left( \exp\left( n_i \beta_0 + \left[ \beta_1 + \sum_{j=1}^{k} \beta_{3j} x_{ij} \right] \sum_{j=1}^{n_i} t_{ij} \right) \int_{-\infty}^{\infty} \exp\left[ n_i \varepsilon_i - \frac{\exp(\beta_0 + \varepsilon_i)}{\beta_1 + \sum_{j=1}^{k} \beta_{3j} x_{ij}} \left\{ \exp\left( \left[ \beta_1 + \sum_{j=1}^{k} \beta_{3j} x_{ij} \right] t_i \right) - 1 \right\} \right] \frac{\exp\left\{ -\frac{\varepsilon_i^2}{2\exp(2\varphi_\varepsilon)} \right\}}{\sqrt{2\pi\exp(2\varphi_\varepsilon)}} d\varepsilon_i \right)$$

$$\tag{6.19}$$

Substituting equations (6.20) to (6.25) given below into the expressions for a single

random effect (given in Appendix 5) gives first and second partial derivatives with respect to

$\beta_0$, $\beta_1$ and $\sigma_\varepsilon$.

$$z_i = \exp\left[ n_i \beta_o + \left( \beta_1 + \sum_{j=1}^{k} \beta_{3j} x_{ij} \right) \sum_{j=1}^{n_i} t_{ij} + n_i \varepsilon_i - \frac{\exp(\beta_o + \varepsilon_i)}{\left( \beta_1 + \sum_{j=1}^{k} \beta_{3j} x_{ij} \right)} \left\{ \exp\left( \left( \beta_1 + \sum_{j=1}^{k} \beta_{3j} x_{ij} \right) t_i \right) - 1 \right\} \right] \frac{\exp\left\{ -\frac{\varepsilon_i^2}{2\exp(2\varphi_\varepsilon)} \right\}}{\sqrt{2\pi\exp(2\varphi_\varepsilon)}}$$

$$\tag{6.20}$$

$$A_i = \int_{-\infty}^{\infty} z_i \, d\varepsilon_i \tag{6.21}$$

$$B_i = n_i - \frac{\exp(\beta_0 + \varepsilon_i)}{\beta_1 + \sum_{j=1}^{k} \beta_{3j} x_{ij}} \left\{ \exp\left( \left( \beta_1 + \sum_{j=1}^{k} \beta_{3j} x_{ij} \right) t_i \right) - 1 \right\} \tag{6.22}$$

$$C_i = \sum_{j=1}^{n_i} t_{ij} - \frac{\exp(\beta_0 + \varepsilon_i)}{\left(\beta_1 + \sum_{j=1}^{k}\beta_{3j}x_{ij}\right)^2}\left\{\exp\left(\left(\beta_1 + \sum_{j=1}^{k}\beta_{3j}x_{ij}\right)t_i\right)\left(\left(\beta_1 + \sum_{j=1}^{k}\beta_{3j}x_{ij}\right)t_i - 1\right) + 1\right\} \tag{6.23}$$

$$D_i = \frac{\varepsilon_i^{\,2} - \exp(2\varphi_\varepsilon)}{\exp(2\varphi_\varepsilon)} \tag{6.24}$$

$$E_i = -\frac{\exp(\beta_0+\varepsilon_i)}{\left(\beta_1+\sum_{j=1}^{k}\beta_{3j}x_{ij}\right)^3}\left\{\left(\beta_1+\sum_{j=1}^{k}\beta_{3j}x_{ij}\right)t_i\exp\left(\left(\beta_1+\sum_{j=1}^{k}\beta_{3j}x_{ij}\right)t_i\right)\left(\left(\beta_1+\sum_{j=1}^{k}\beta_{3j}x_{ij}\right)t_i-2\right)+2\exp\left(\left(\beta_1+\sum_{j=1}^{k}\beta_{3j}x_{ij}\right)t_i\right)-2\right\}$$

$$\tag{6.25}$$

The first and second partial derivatives with respect to $\beta_{3j}$ ($j$th covariate) can be shown to take the form of equations (6.26) to (6.30):

$$\frac{\partial \ell}{\partial \beta_{3j}} = \sum_{i=1}^{m} x_{ij}\left(\frac{\partial \ell}{\partial \beta_1}\right)_i \tag{6.26}$$

$$\frac{\partial^2 \ell}{\partial \beta_0 \partial \beta_{3j}} = \sum_{i=1}^{m} x_{ij}\left(\frac{\partial^2 \ell}{\partial \beta_0 \partial \beta_1}\right)_i \tag{6.27}$$

$$\frac{\partial^2 \ell}{\partial \beta_1 \partial \beta_{3j}} = \sum_{i=1}^{m} x_{ij}\left(\frac{\partial^2 \ell}{\partial \beta_1^{\,2}}\right)_i \tag{6.28}$$

$$\frac{\partial^2 \ell}{\partial \beta_{3j}^{\,2}} = \sum_{i=1}^{m} x_{ij}\left(\frac{\partial^2 \ell}{\partial \beta_1^{\,2}}\right)_i \tag{6.29}$$

$$\frac{\partial^2 \ell}{\partial \beta_{3j} \partial \varphi_\varepsilon} = \sum_{i=1}^{m} x_{ij}\left(\frac{\partial^2 \ell}{\partial \beta_1 \partial \varphi_\varepsilon}\right)_i \tag{6.30}$$

The second partial derivative with respect to $\beta_{3j}$ and $\beta_{3k}$ (the $j^{th}$ and $k^{th}$ covariates) is then given by

$$\frac{\partial^2 \ell}{\partial \beta_{3j} \partial \beta_{3k}} = \sum_{i=1}^{m} x_{ij} x_{ik} \left( \frac{\partial^2 \ell}{\partial \beta_1^2} \right)_i \qquad (6.31)$$

As was the case with the proportional covariate model, the forms of the log-likelihood and partial derivatives for the time multiplicative model can be adapted from the equivalent functions for the model with a random effect on the intercept. These can be found by replacing the gradient parameter $\beta_1$ by $\left( \beta_1 + \sum_{j=1}^{k} \beta_{3j} x_{ij} \right)$ in equations (3.53) to (3.59).

## 6.3.2 Simulation on Time multiplicative Covariates

This section gives the results of a number of simulation exercises carried out to validate the estimates of the parameters and standard errors for a time multiplicative covariate model. These simulations use a single covariate giving the model to be

$$\lambda_i(t) = exp\{\beta_0 + (\beta_1 + \beta_3 x_i)t + \varepsilon_i\} \qquad (6.32)$$

The first simulation uses the same parameter values as in Section 6.2.2, namely $\beta_0 = 4$, $\beta_1 = -3$, $\beta_3 = -1$ and $\sigma_\varepsilon = 0.5$. A failure plot of a simulated data set from these parameters, with 10 units in each covariate group, is given in Figure 6.6. This plot shows no real difference between the two covariate groups; the variability in the rate of occurrence of failures is dominated by the random effect.

The results of a simulation based on 100 replications using the above parameter values on a sample size of 30 are given in Table 6.3. The estimates of the parameters and standard errors are close to their true values, although as with the previous models there is a bias on the variance component. Figure 6.6 showed that a time multiplicative covariate value of $-1$

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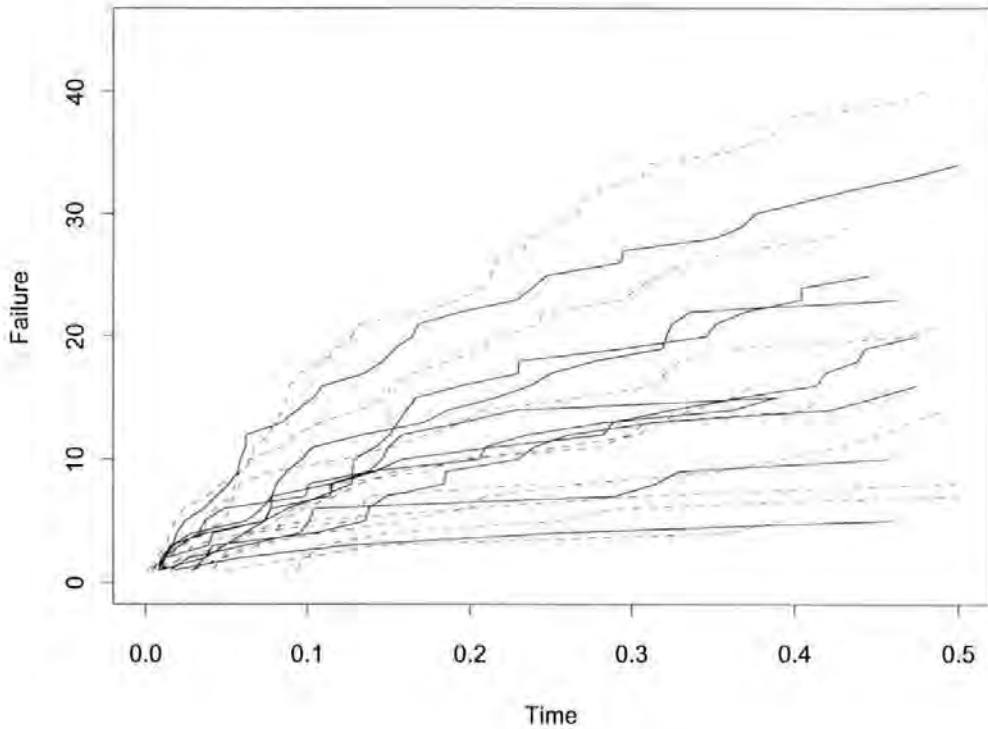
**Figure 6.6 :** Failure Plot of Simulated Time multiplicative Covariate Data at Low (- - - -) and High (———) Levels ($\beta_0 = 4$, $\beta_1 = -3$, $\beta_3 = -1$ and $\sigma_\varepsilon = 0.5$).

has less impact on the resulting ROCOF's than the variance due to the random effect. The covariate estimate is close to the true value, but the standard error is such that a test would not reject the hypothesis that $\beta_3 = 0$.

In order to obtain results from a statistically significant model, the simulation was repeated for a different set of parameter values. The first simulation showed that a covariate value of −1 is not a large enough effect for the power of the test, so the simulation has been carried out with $\beta_3 = -2$ (all other parameter values remain the same). A failure plot for 20 units is given in Figure 6.7. As with Figure 6.6 the ROCOF of the two covariate levels are not distinctly grouped. However, in this case the ROCOF when $x_{ij} = 1$ (the dotted lines) tend to level off more quickly. This suggests that the ROCOF in this case is decreasing at a faster rate and that the covariate effect may be identifiable. The results of 100 simulations are summarised in Table 6.4. As with the previous simulation, this gives good estimates of the

parameters and their standard errors. With the larger covariate effect the estimate of $\beta_3$ is now significantly different from

**TABLE 6.3**

*Results of Simulation Exercise With Time multiplicative Covariate Effects*

$(\beta_0 = 4, \beta_1 = -3, \beta_3 = -1 \text{ and } \sigma_\varepsilon = 0.5).$

| | $\hat{\beta}_0$ | $\hat{\beta}_1$ | $\hat{\beta}_3$ | $\ln(\hat{\sigma}_\varepsilon)$ |
|---|---|---|---|---|
| **True Parameter Values** | 4 | −3 | −1 | −0.693 |
| **Mean of Sampling Distribution of Estimates** | 4.012 | −3.033 | −1.026 | −0.758 |
| **Standard Deviation of Sampling Distribution of Estimates** | 0.1352 | 0.4677 | 0.6149 | 0.1917 |
| **Mean of Asymptotic Standard Errors** | 0.1220 | 0.4417 | 0.6112 | 0.1848 |
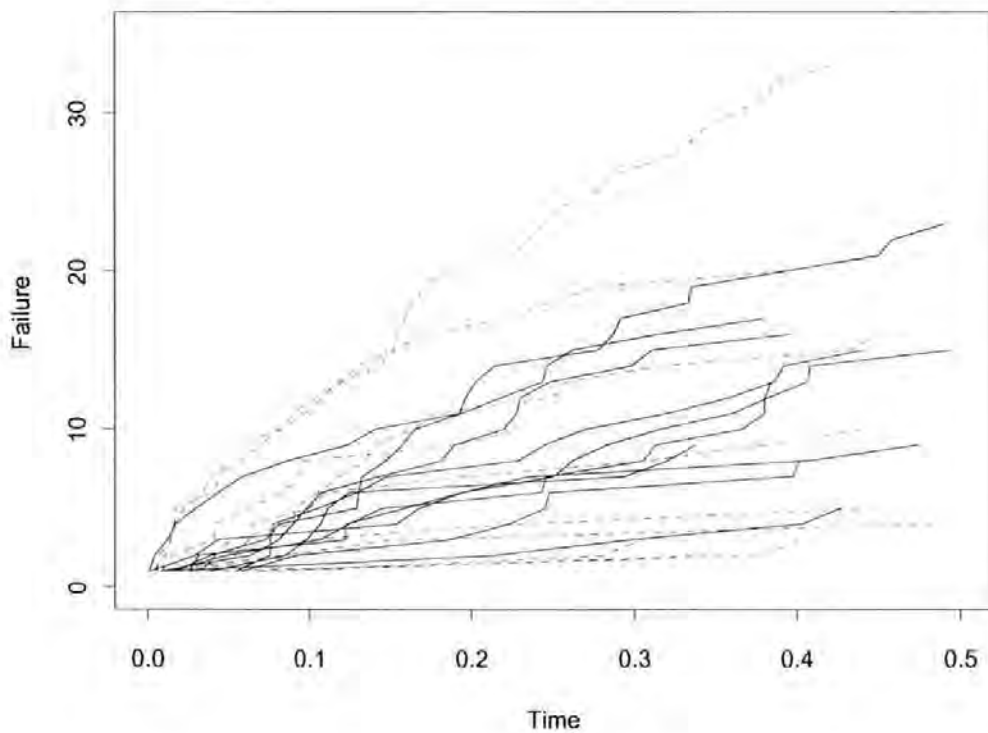


**Figure 6.7 :** Failure Plot of Simulated Time multiplicative Covariate Data at Low (- - - -) and High (———) Levels ($\beta_0 = 4$, $\beta_1 = -3$, $\beta_3 = -2$ and $\sigma_\varepsilon = 0.5$).

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zero. The ratio of the parameter estimate to its standard error is approximately 2.9 which is greater than the corresponding critical $Z$-value.

**TABLE 6.4**

*Results of Simulation Exercise With Time multiplicative Covariate Effects*

$(\beta_0 = 4, \ \beta_1 = -3, \ \beta_3 = -2 \ and \ \sigma_\varepsilon = 0.5)$.

| | $\hat{\beta}_0$ | $\hat{\beta}_1$ | $\hat{\beta}_3$ | $\ln(\hat{\sigma}_\varepsilon)$ |
|---|---|---|---|---|
| **True Parameter Values** | 4 | −3 | −2 | −0.693 |
| **Mean of Sampling Distribution of Estimates** | 4.007 | −3.050 | −1.985 | −0.786 |
| **Standard Deviation of Sampling Distribution of Estimates** | 0.1442 | 0.4514 | 0.6922 | 0.1899 |
| **Mean of Asymptotic Standard Errors** | 0.1219 | 0.4434 | 0.6596 | 0.1917 |

### 6.3.3 Time multiplicative Covariate Model for Jaguar Data

To illustrate the use of the time multiplicative covariate effect, equation (6.32) for a single covariate has been fitted to the four engine phases of the Jaguar data given in Appendix 2. The parameter estimates, their standard errors and the maximised log-likelihood are summarised in Table 6.5. The results from fitting the time multiplicative models are similar to those obtained for the proportional model in Table 6.2. In all cases the estimate of the covariate effect $\beta_3$ is of the same order as, or smaller than its standard error. There is also no significant difference between the maximised log-likelihoods of this model and the model without the random effect. The conclusion from this is that, as with the proportional model, the covariate effect is not significantly different from zero and should not be included in the model.

**TABLE 6.5**

*Parameter Estimates (Standard Errors) of Fitted Time multiplicative Covariate Effect Models.*

| Phase | $\hat{\beta}_0$ | $\hat{\beta}_1$ | $\hat{\beta}_3$ | $\ln(\hat{\sigma}_\varepsilon)$ | $\hat{\sigma}_\varepsilon$ | $\ell_{\max}$ |
|---|---|---|---|---|---|---|
| A0 | 3.936 (0.2073) | −3.831 (2.0297) | −2.614 (2.2803) | −0.726 (0.3742) | 0.484 | 223.10 |
| A1-1 | 3.599 (0.2274) | −2.041 (1.2883) | 1.107 (1.5105) | −1.127 (0.5589) | 0.324 | 170.75 |
| A1-2 | 2.916 (0.3949) | 0.202 (1.6068) | −0.364 (2.1306) | −0.179 (0.3615) | 0.836 | 96.55 |
| A2-1 | 3.356 (0.1896) | −2.436 (0.8396) | 0.012 (1.1978) | −0.642 (0.3241) | 0.526 | 230.87 |

## 6.4 A Combined Covariate Model

### 6.4.1 The General Model

The models can be extended to include both proportional and time multiplicative covariate effects. In this case the model for $k$ covariates becomes

$$\lambda_i(t) = \exp\left\{\beta_0 + \sum_{j=1}^{k}\beta_{2j}x_{ij} + \left(\beta_1 + \sum_{j=1}^{k}\beta_{3j}x_{ij}\right)t + \varepsilon_i\right\} \tag{6.33}$$

In this formulation each covariate is assumed to have both a proportional and time multiplicative effect on the ROCOF. This model could be reparameterised such that some or all of the parameters have only one covariate effect, and can therefore reduce into equations (6.1) or (6.17). For equation (6.33), the expected number of failures in the time interval $(a,b]$ is

$$E\{\text{failures}(a,b]\,|\,\varepsilon_i\} = \exp\left(\beta_0 + \sum_{j=1}^{k}\beta_{2j}x_{ij} + \varepsilon_i\right)\int_a^b \exp\left\{\left(\beta_1 + \sum_{j=1}^{k}\beta_{3j}x_{ij}\right)t\right\}dt \tag{6.34}$$

In Section 6.2.1 it was demonstrated that the log-likelihood, and the first and second partial derivatives for the proportional covariate model take the same form as for the model with a random effect on the intercept described in Chapter 3. To obtain all these expressions for the proportional covariate model $\beta_0$ should be replaced by $\beta_0 + \sum_{j=1}^{k} \beta_{2j} x_{ij}$ in equations (3.53) to (3.59). These can then be substituted into the expressions for the first and second partial derivatives in Appendix 5. Likewise, from Section 6.3.1, $\beta_1$ can be replaced by $\beta_1 + \sum_{j=1}^{k} \beta_{3j} x_{ij}$ for the time multiplicative covariate model. For the combined model, in equation (6.33), both of these substitutions can be made to obtain the log-likelihood function, and the first and second partial derivatives with respect to $\beta_0$, $\beta_1$ and $\sigma_\varepsilon$. The first and second derivatives with respect to the covariates are then obtained by applying the rules given by equations (6.10) to (6.15) and (6.26) to (6.31). The second partial derivative with respect to $\beta_{2j}$ and $\beta_{3k}$ (the $j^{th}$ proportional covariate and the $k^{th}$ time multiplicative covariate) is then given by:

$$\frac{\partial^2 \ell}{\partial \beta_{2j} \partial \beta_{3k}} = \sum_{i=1}^{m} x_{ij} x_{ik} \left( \frac{\partial^2 \ell}{\partial \beta_0 \partial \beta_1} \right)_i \tag{6.35}$$

### 6.4.2 Simulation on Combined Covariate Models

As with the other covariate models a simulation exercise has been performed for validation purposes. The single covariate random effect model for this exercise is given by

$$\lambda_i(t) = \exp\left\{ \beta_0 + \beta_2 x_i + \left( \beta_1 + \beta_3 x_i \right) t + \varepsilon_i \right\} \tag{6.36}$$

The simulations for the time multiplicative covariate model showed that the true value of $\beta_3$ has to be reasonably large to enable its estimate to be significantly different from zero. In line with these results and those for the proportional covariate model, the true parameter values for the simulation on the combined model were first chosen to be $\beta_0 = 4$, $\beta_1 = -3$, $\beta_2 = -1$,

$\beta_3 = -2$ and $\sigma_\varepsilon = 0.5$. A failure plot of data simulated from these values with 10 units in each covariate level is given in Figure 6.8. This shows a clear difference between the ROCOF's of the two covariate levels. The results of 100 simulations on a sample size of 30 are given in Table 6.6. As would be expected from the previous simulations, this demonstrates that the model produces good estimates of the parameters and their standard errors  However the inclusion of the proportional covariate in the model has increased the standard error of $\beta_3$ to 1.15 from its previous value of 0.69 in Section 6.3.2. The ratio of the estimate of the time multiplicative covariate parameter to its standard error is less than the critical $Z$-value. Hence it would be concluded that the time multiplicative covariate parameter is not significantly different from zero.
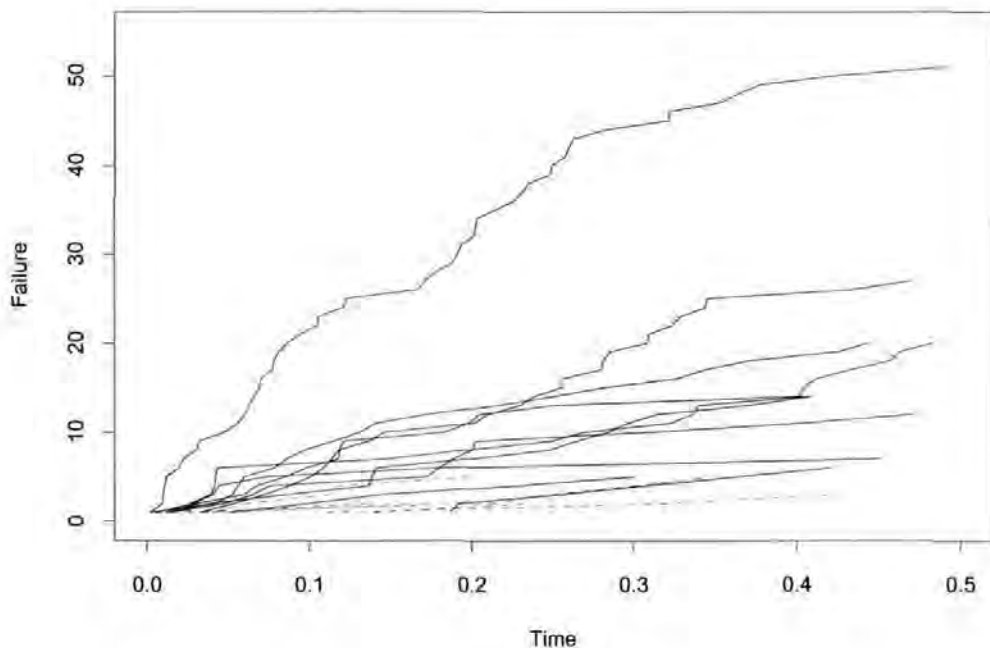


**Figure 6.8 :** Failure Plot of Simulated Combined Covariate Effect Data at Low (- - - -) and High (——) Levels ($\beta_0 = 4$, $\beta_1 = -3$, $\beta_2 = -1$, $\beta_3 = -2$ and $\sigma_\varepsilon = 0.5$).

The simulation was repeated with the true parameter values taken to be $\beta_0 = 4$, $\beta_1 = -2$, $\beta_2 = -1$, $\beta_3 = -4$ and $\sigma_\varepsilon = 0.5$. Figure 6.9 gives a failure plot for a simulated data set of 20 units, and the results of 100 simulations on a sample size of 30 are given in Table 6.7. This gives

good estimates of the parameters and standard errors; for this larger time multiplicative covariate effect the mean parameter estimate is much larger than the standard error.

**TABLE 6.6**

*Results of Simulation Exercise With Combined Covariate Effects*

$(\beta_0 = 4, \beta_1 = -3, \beta_2 = -1, \beta_3 = -2$ and $\sigma_\varepsilon = 0.5)$.

|  | $\hat{\beta}_0$ | $\hat{\beta}_1$ | $\hat{\beta}_2$ | $\hat{\beta}_3$ | $\ln(\hat{\sigma}_\varepsilon)$ |
|---|---|---|---|---|---|
| **True Parameter Values** | 4 | −3 | −1 | −2 | −0.693 |
| **Mean of Sampling Distribution of Estimates** | 4.000 | −2.958 | −0.994 | −1.928 | −0.848 |
| **Standard Deviation of Sampling Distribution of Estimates** | 0.1771 | 0.5076 | 0.3442 | 1.1460 | 0.3225 |
| **Mean of Asymptotic Standard Errors** | 0.1656 | 0.4757 | 0.2901 | 1.1126 | 0.2454 |

**TABLE 6.7**

*Results of Simulation Exercise With Combined Covariate Effects*

$(\beta_0 = 4, \beta_1 = -2, \beta_2 = -1, \beta_3 = -4$ and $\sigma_\varepsilon = 0.5)$.

|  | $\hat{\beta}_0$ | $\hat{\beta}_1$ | $\hat{\beta}_2$ | $\hat{\beta}_3$ | $\ln(\hat{\sigma}_\varepsilon)$ |
|---|---|---|---|---|---|
| **True Parameter Values** | 4 | −2 | −1 | −4 | −0.693 |
| **Mean of Sampling Distribution of Estimates** | 3.993 | −1.950 | −0.964 | −4.215 | −0.794 |
| **Standard Deviation of Sampling Distribution of Estimates** | 0.1598 | 0.4054 | 0.2627 | 1.1743 | 0.2381 |
| **Mean of Asymptotic Standard Errors** | 0.1635 | 0.4169 | 0.2973 | 1.2461 | 0.2199 |
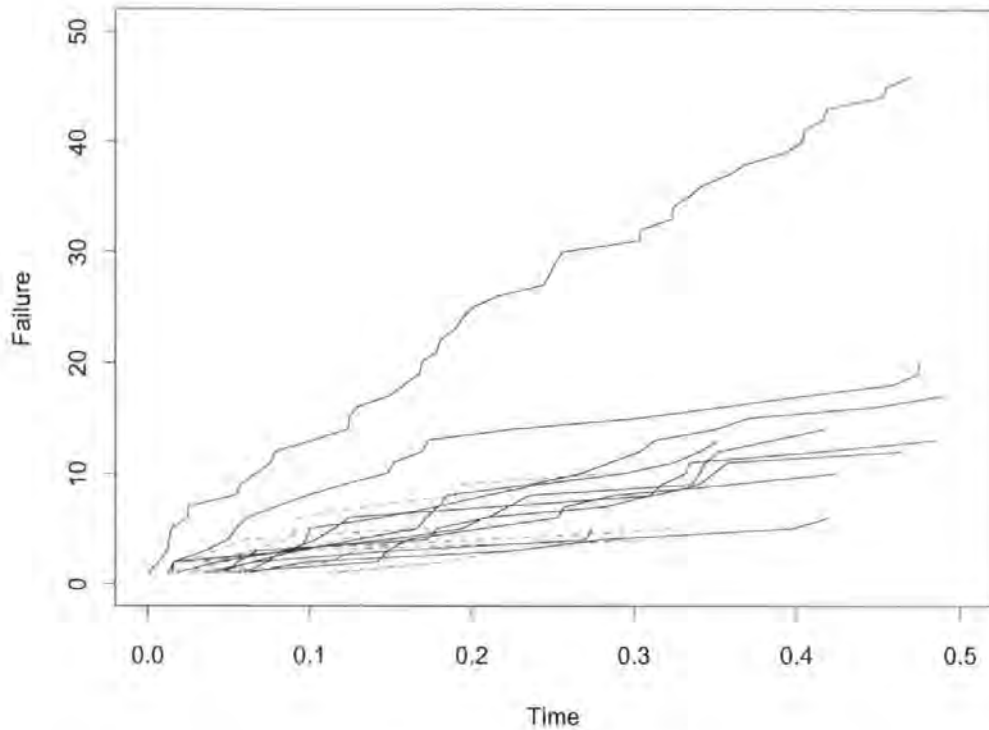
**Figure 6.9 :** Failure Plot of Simulated Combined Covariate Effect Data Low (- - - -)

and High (——) Levels ($\beta_0 = 4$, $\beta_1 = -2$, $\beta_2 = -1$, $\beta_3 = -4$ and $\sigma_\varepsilon = 0.5$).

### 6.4.3 Combined Covariate Effect Model for Jaguar Data

The combined covariate model has been fitted to the four engine phases of the Jaguar data in Appendix 2. As neither covariate was found to be significant when estimated individually, and because of the small sample size available for low testing, these models would not be expected to offer an advantage over the standard model in Chapter 3. The fitted models are given in Table 6.8. These results show that there is not a significant reduction in the log-likelihood when both covariates are removed from the model. It can therefore be concluded that, with the data available, the ROCOF does not change significantly between the two test type classifications.

**TABLE 6.8**

*Parameter Estimates (Standard Errors) of Fitted Combined Covariate Effect Models.*

| Phase | $\hat{\beta}_0$ | $\hat{\beta}_1$ | $\hat{\beta}_2$ | $\hat{\beta}_3$ | $\ln(\hat{\sigma}_\varepsilon)$ | $\hat{\sigma}_\varepsilon$ | $\ell_{max}$ |
|---|---|---|---|---|---|---|---|
| A0 | 3.968 (0.3488) | −3.944 (2.3033) | −0.049 (0.4193) | −2.447 (2.6956) | −0.727 (0.3757) | 0.483 | 223.10 |
| A1-1 | 3.383 (0.4482) | −1.547 (1.5330) | 0.298 (0.5146) | 0.350 (2.0071) | −1.189 (0.5967) | 0.305 | 170.75 |
| A1-2 | 1.713 (0.7908) | 1.848 (1.8886) | 1.607 (0.8744) | −3.325 (2.7608) | −0.485 (0.4531) | 0.616 | 96.55 |
| A2-1 | 3.348 (0.3286) | −2.423 (0.9401) | 0.012 (0.3978) | −0.012 (1.4267) | −0.643 (0.3261) | 0.526 | 230.87 |

## 6.5 Likelihood Ratio Tests for Covariate Effects

### 6.5.1 Introduction

This chapter has introduced three methods for incorporating covariate effects into a NHPP model. These have taken the form of proportional and time multiplicative effects, with the final model comprising a combination of the two. It can be noted that the proportional covariate effect model, equation (6.1), and the time multiplicative covariate effect model, equation (6.17), are both nested within the combined model, equation (6.33). In order to compare these models likelihood ratio tests can therefore be used to indicate if either of these covariate effects is appropriate. Simulation exercises have been performed to demonstrate and validate this procedure. The method of simulation is as follows:

I)    Simulate data for one of the covariate models using the inversion technique.

II)    Fit all covariate models to this data.

III)    Use likelihood ratio tests to indicate which covariate model is appropriate.

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Obviously it would be hoped that the likelihood ratio tests would indicate the appropriate covariate model to be the one corresponding to the model used to simulate the data.

This procedure has been carried out in conjunction with the validation exercises on the estimates of the parameters and standard errors. The following three sections show the results of these simulations for each of the covariate models, based on 100 replications and a sample size of 30. In Section 6.5.5 the effects of increasing the sample size are examined.

### 6.5.2 The Proportional Model

The data in this section have been simulated from the same parameter values as in the simulation exercise on the proportional covariate model in Section 6.2.1, namely $\beta_0 = 4$, $\beta_1 = -3$, $\beta_2 = -1$ and $\sigma_\varepsilon = 0.5$. Each of the random effect covariate models has been fitted to this data. The likelihood ratio tests show a significant change in the log-likelihood when $\beta_2$ (associated with a proportional covariate) is removed from the combined model, and no significant change in the log-likelihood when $\beta_3$ (associated with a time multiplicative covariate) is removed. Table 6.9 gives the probabilities of concluding that each of the covariates are not significant (at the 5% significance level). This confirms that the likelihood ratio test is useful in identifying the true covariate model since, in general, the time multiplicative covariate parameter is correctly removed from the model, and the proportional covariate parameter has a small probability of being removed incorrectly.

TABLE 6.9

*Likelihood Ratio Tests on Proportional Covariate Data.*

| Parameter | P(Removed from Model) | |
|:---:|:---:|:---:|
| | Actual | Ideal |
| $\beta_2$ | 0.08 | Low |
| $\beta_3$ | 0.90 | High |

### 6.5.3 The Time multiplicative Model

The validation exercises on the time multiplicative model showed that the covariate parameter was not significant unless it was of sufficient magnitude. For a sample size of 30 it appeared that a covariate value of $-1$ did not have a significant effect on the ROCOF, but a value of $-2$ was found to be statistically significant. Table 6.10 gives the results from likelihood ratio tests for three values of the covariate parameter $\beta_3$, $-1$, $-2$ and $-4$. For the smallest covariate effect $-1$ it can be seen that although the proportional covariate is correctly identified as being insignificant, the probability of incorrectly removing the time multiplicative effect from the model is high. From the mean standard error for the parameter estimates, shown in Table 6.3, this result would be expected. However, as the covariate effect is increased the probability of incorrectly removing the time multiplicative parameter $\beta_3$ reduces. In all cases there is a very high probability of correctly concluding that the proportional parameter $\beta_2$ is not required in the model. For the final simulation the value of $\beta_1$ changed from $-3$ to $-2$. Without this change the covariate effect had a gradient parameter equivalent to $-7$ and this was found to produce very little failure data.

**TABLE 6.10**

*Likelihood Ratio Tests on Time multiplicative Covariate Data.*

| Model | Parameter | P(Removed from Model) | |
|---|---|---|---|
| | | Actual | Ideal |
| $\beta_0 = 4$, $\beta_1 = -3$, | $\beta_2$ | 0.96 | High |
| $\beta_3 = -1$, $\sigma_\varepsilon = 0.5$ | $\beta_3$ | 0.73 | Low |
| $\beta_0 = 4$, $\beta_1 = -3$, | $\beta_2$ | 0.94 | High |
| $\beta_3 = -2$, $\sigma_\varepsilon = 0.5$ | $\beta_3$ | 0.28 | Low |
| $\beta_0 = 4$, $\beta_1 = -2$, | $\beta_2$ | 0.98 | High |
| $\beta_3 = -4$, $\sigma_\varepsilon = 0.5$ | $\beta_3$ | 0.01 | Low |

## 6.5.4 The Combined Effect Model

The likelihood ratio tests on the combined model should indicate a significant change in the log-likelihood if either of the covariate parameters are removed. The results of the simulated likelihood ratio tests are given in Table 6.11. As with the time multiplicative model the probability of falsely removing the time multiplicative parameter $\beta_3$ is very high when the covariate effect is as small as $-1$. However, with the inclusion of the proportional covariate effect the probability of falsely removing $\beta_3$ still remains high when the time multiplicative covariate effect is changed to $-2$, as indicated by the magnitude of the standard error from the validation exercise in Section 6.3.2. When $\beta_3$ is increased further however, both covariate effects are generally found to be significant and the probability of incorrectly removing either of them becomes small (0.05 and 0.02 for $\beta_2$ and $\beta_3$ respectively).

**TABLE 6.11**

*Likelihood Ratio Tests on Combined Covariate Effect Data.*

| Model | Parameter | P(Removed from Model) | |
| --- | --- | --- | --- |
| | | Actual | Ideal |
| $\beta_0=4$, $\beta_1=-3$, $\beta_2=-1$ $\beta_3 = -1$, $\sigma_\varepsilon = 0.5$ | $\beta_2$ | 0.04 | Low |
| | $\beta_3$ | 0.89 | Low |
| $\beta_0=4$, $\beta_1=-3$, $\beta_2=-1$ $\beta_3 = -2$, $\sigma_\varepsilon = 0.5$ | $\beta_2$ | 0.11 | Low |
| | $\beta_3$ | 0.54 | Low |
| $\beta_0=4$, $\beta_1=-2$, $\beta_2=-1$ $\beta_3 = -4$, $\sigma_\varepsilon = 0.5$ | $\beta_2$ | 0.05 | Low |
| | $\beta_3$ | 0.02 | Low |

## 6.5.5 Likelihood Ratio Test with Larger Samples

The previous sections have shown the probabilities of removing the covariate parameters from the three models. In all cases a sample size of 30 has been used. This was chosen to enable reasonable estimates of the variance component to be made and also to enable

the simulations to be completed in a reasonable time. Table 6.12 gives the results of the likelihood ratio tests for all three models, with 100 replications and the sample size increased to 50. In this case the true parameter values are $\beta_0 = 4$, $\beta_1 = -3$, $\beta_2 = -1$, $\beta_3 = -2$ and $\sigma_\varepsilon = 0.5$, and these results can be compared with the results for the same models in Sections 6.5.2 to 6.5.4. For the proportional covariate model the likelihood ratio test performed well in identifying the correct effect for a sample size of 30 and these corresponding results show no real difference. With the two models including the time multiplicative covariate there is a large change in the probabilities of falsely removing $\beta_3$, although this probability is still relatively high for the combined effect model. As is expected, increasing the sample size also increases the precision of the parameter estimates; the mean standard error of $\beta_3$ decreases from 0.6922 to 0.5794 for the time multiplicative model and from 1.146 to 0.9017 for the combined effect model.

**TABLE 6.12**

*Likelihood Ratio Tests on Large Sample Size Data.*

| Model | Parameter | P(Removed from Model) | |
|---|---|---|---|
| | | Actual | Ideal |
| $\beta_0 = 4$, $\beta_1 = -3$, | $\beta_2$ | 0.01 | Low |
| $\beta_2 = -1$, $\sigma_\varepsilon = 0.5$ | $\beta_3$ | 0.89 | High |
| $\beta_0 = 4$, $\beta_1 = -3$, | $\beta_2$ | 0.95 | High |
| $\beta_3 = -2$, $\sigma_\varepsilon = 0.5$ | $\beta_3$ | 0.13 | Low |
| $\beta_0 = 4$, $\beta_1 = -3$, $\beta_2 = -1$ | $\beta_2$ | 0.01 | Low |
| $\beta_3 = -2$, $\sigma_\varepsilon = 0.5$ | $\beta_3$ | 0.35 | Low |

## 6.6 Summary

This chapter has described three formulations for including covariate effects in a log-likelihood random effect NHPP model. Through the use of simulations it has been shown that the method of maximum likelihood provides good estimates of the parameters and their

standard errors. It has also been shown that the likelihood ratio test is useful in indicating which of the models is appropriate. In reality, knowledge of the units under test and of the testing conditions will suggest which of these models should be used. The data itself should also provide an indication as to which model should be fitted.

To illustrate the use of these models in a practical situation they were fitted to the Jaguar data in Appendix 2 but, as suggested by Figures 6.1 to 6.4, no significant covariate effects were found. The data for the covariate in this case indicated whether an engine has been tested under high or low conditions. As stated in Chapter 1 each of these conditions encompasses a number of different tests whose effects cannot be individually estimated (due to lack of data). If the effects of the tests grouped together are very different (and at present these effects are unknown) then the standard error of the parameter estimate will be large. This will obviously decrease the probability of rejecting the null hypothesis of there being no covariate effect.

For the Jaguar data all engines perform a Run and Rate test in their first few hours of running as described in Chapter 1. It is after this time that engines are exposed to high or low level testing. Before this testing begins, the ROCOF's of engines at both levels of the *test* covariate should be the same. Thereafter any effect of test type will manifest itself. The high test type is a more demanding test and is highly accelerated over time whereas the low test is similar to actual engine usage. It would therefore be expected that the ROCOF's at the two levels of the covariate should change at a different rate, with the ROCOF's for the high level engines increasing faster than those for the low level engines. This suggests that the time multiplicative covariate effect would be the more appropriate model for the Jaguar data although at present it is noted that no covariate effect can be identified.

# CHAPTER 7

# Hierarchical Covariate Models:

# A Practical Application

## 7.1 Introduction

The analysis of the Jaguar data described in the previous chapters has not made use of the fact that there may be a relationship between the engines from the four phases of development. Fitting a separate non-homogeneous Poisson process random effect model to each phase is allowing for a different intercept, gradient and variance in each phase. However this is possibly an overparameterisation. Although design changes are made for each phase, the majority of components remain the same. An obvious point of interest is what effect, if any, these component changes are having. This chapter demonstrates how effects due to the engine phase can be modelled.

In Chapter 6 a number of models for including covariate effects were introduced. These models were used to estimate the effects of the high and low test types. Another approach is to treat the engine phase as a covariate and these models can then be used to examine differences between phases. This chapter describes a hierarchical modelling approach. Let the *Full* model be defined as that which enables estimates of the parameters under the assumption that there is no relationship between phases. In this case the intensity function for the *Full* model with a random effect on the intercept parameter for $n$ phases is

$$\lambda_{ji}(t) = \exp\left(\underline{x}_i^T \underline{\beta_0} + \underline{x}_i^T \underline{\beta_1} t + \varepsilon_{ji}\right) \tag{7.1}$$

where

$\underline{x}_i$ is a vector containing 0's and 1's to indicate the engine phase of the $i^{th}$ engine

$\underline{\beta_0}$, $\underline{\beta_1}$ are vectors of parameters for each engine phase

$\varepsilon_{ji}$ is the random effect of the $i^{th}$ engine and the $j^{th}$ phase. $\varepsilon_{ji} \sim N(0, \sigma_{\varepsilon j}^2)$

For the Jaguar data $n = 4$ and equation (7.1) is a reparameterisation of the four separate models fitted in Chapter 3. The parameter estimates for the *Full* model will be the same as the parameter estimates found for each phase separately. The maximised log-likelihood of the *Full* model is then the sum of the log-likelihoods of the separate models. In Section 7.3 various covariate models to estimate the true phase effect are investigated. These models are nested within the full model. In the next section the three random effect models described in Chapters 3 and 4 are further compared using the combined data for the engine phases.

## 7.2  A Comparison of Random Effect Models

The analysis in previous chapters has concluded that for the Jaguar data placing a random effect on the intercept parameter is more appropriate than placing it on the gradient. It has also been shown that the estimates of the variance component are biased, especially when the sample size is small. For the model with the random effect on the intercept, it has been demonstrated that a sample size > 30 is sufficient to overcome this but for the other models the sample size required to reduce the bias is considerably larger. The engine phases in the Jaguar data have sample sizes of 15, 9, 9 and 21 respectively, with the consequence that all estimates of the variance component for these are biased. The comparisons between the three random effect models made in Chapter 5 have used these biased estimates.

The full model in equation (7.1) assumes that the engine phases are all different and that they have no values of parameter estimates in common. Let the *null* model be defined

such that the ROCOF's are identical for all phases. In this case, all the parameters in the *Full* model will be the same in the sense that for any two phases $k$ and $l$, $\beta_{0k} = \beta_{0l}$, $\beta_{1k} = \beta_{1l}$ and $\sigma_{\varepsilon k} = \sigma_{\varepsilon l}$. Under this assumption the data can be combined to a single set with $m = 54$ engines. With this larger sample size the bias on the estimated variance component will be greatly reduced.

The results of fitting the three random effect models, and a model with no random effect, to the combined data are given in Table 7.1. A comparison of the maximised log-likelihoods for the double and single random effect models show a large change when $\varepsilon_i$ is removed ( likelihood ratio test statistic = 23.52) and a small change when $\xi_i$ is removed ( likelihood ratio test statistic = 0.7). These test statistics are both compared with $\chi^2_{1,0.05}$ (3.84) and this confirms the conclusion from the previous chapters that the random effect on the intercept is the more appropriate. This random effect model can also be compared with the fixed effects models. The change in the maximised log-likelihood is very large in this case and a likelihood ratio test is again significant when compared with $\chi^2_{1,0.05}$. In other words a random effect model is required for the Jaguar data and this effect is best included on the intercept parameter.

**TABLE 7.1**

*Parameter Estimates (Standard Errors) of Models Fitted to Jaguar Data Assuming No Phase Effect.*

| Random Effect on | $\hat{\beta}_0$ | $\hat{\beta}_1$ | $\ln(\hat{\sigma}_\varepsilon)$ | $\hat{\sigma}_\varepsilon$ | $\ln(\hat{\sigma}_\xi)$ | $\hat{\sigma}_\xi$ | $\ell_{max}$ |
|---|---|---|---|---|---|---|---|
| - | 3.595 (0.0836) | -3.171 (0.4274) | - | - | - | - | 696.54 |
| $\beta_0$ | 3.480 (0.1182) | -2.449 (0.4781) | -0.573 (0.1731) | 0.564 | - | - | 711.42 |
| $\beta_1$ | 3.616 (0.0854) | -3.484 (0.5737) | - | - | 0.491 (0.3401) | 1.634 | 700.01 |
| $\beta_0, \beta_1$ | 3.485 (0.1195) | -2.562 (0.5464) | -0.574 (0.1781) | 0.563 | -0.042 (0.7176) | 0.959 | 711.77 |

## 7.3  Hierarchical Models for Engine Phase Effect

The previous section has indicated that the model containing the random effect on the intercept should be used for the Jaguar data. In this section a number of covariate models are fitted to investigate the phase effect, all containing the random effect on the intercept parameter. The general form of the covariate models of this type is given by

$$\lambda_{ji}(t) = \exp\left\{\beta_0 + \underline{x}_i^T \underline{\beta_2} + \left(\beta_1 + \underline{x}_i^T \underline{\beta_3}\right) t + \varepsilon_{ji}\right\} \qquad (7.2)$$

where

$\beta_0$ is the intercept parameter for the reference phase

$\beta_1$ is the gradient parameter for the reference phase

$\underline{x}_i$ is a vector of indicator variables for other phases

$\underline{\beta_2}$ is a vector containing the differences in the intercept parameter for other phases

$\underline{\beta_3}$ is a vector containing the differences in the gradient parameter for other phases

$\varepsilon_{ji}$ is the random effect of the $i^{th}$ engine and the $j^{th}$ phase, $\varepsilon_{ji} \sim N(0, \sigma_d^2)$

Let equation (7.2) define the *Full* model for the engine phases. This form of the *Full* model is a reparameterisation of equation (7.1). For the Jaguar data, which has four phases, the covariate effects will be given by $\underline{\beta_2} = \left(\beta_{21} \quad \beta_{22} \quad \beta_{23}\right)^T$ and $\underline{\beta_3} = \left(\beta_{31} \quad \beta_{32} \quad \beta_{33}\right)^T$. Phase A0 has been used as the reference phase, hence if engine $i$ is from this phase $\underline{x}_i = \left(0 \quad 0 \quad 0\right)^T$. Other covariate models to describe the phase effect can be derived by removing various terms from the *Full* model. There are a number of models containing common intercepts and/or gradients and/or variance components which can be fitted to the four phases. The eight possible combinations are defined in Table 7.2 where the blanks indicate that a common parameter is fitted and ✓indicates the different parameters required in a model for a number of phase effects.

These models will all be nested within the *Full* model. Two of the models (6 and 7) are not appropriate for the Jaguar data. The gradient parameters differ but the intercept parameter will be common in both of these models which means that the starting ROCOF's will be the same for all phases. Clearly this is not the case when considering the difference between engine phases. At each phase a large number of changes are made and if they do affect the ROCOF the difference would not manifest itself in this way. A difference would be expected between the phases at time zero. Figure 7.1 gives the hierarchical structure of the models fitted to the Jaguar data and the interpretation of these models is given below.

TABLE 7.2

*Parameters required for Phase Effect Models.*

| Model | Parameters Required For Phase Effect | | |
| | Intercept | Gradient | Variance |
|---|:---:|:---:|:---:|
| *Full* | ✓ | ✓ | ✓ |
| 1 | ✓ | ✓ | |
| 2 | ✓ | | ✓ |
| 3 | ✓ | | |
| 4 | | | ✓ |
| 5 | | | |
| 6 | | ✓ | ✓ |
| 7 | | ✓ | |

*Full Model* $\quad \lambda_{ji}(t) = \exp\left\{ \beta_0 + \underline{x}_i^T \underline{\beta_2} + \left( \beta_1 + \underline{x}_i^T \underline{\beta_3} \right) t + \varepsilon_{ji} \right\} \quad$ (number of parameters = 12)

This is defined by equation (7.2) and introduced in Section 7.1. It assumes no relationship between the engine phases and includes different parameters for each phase.
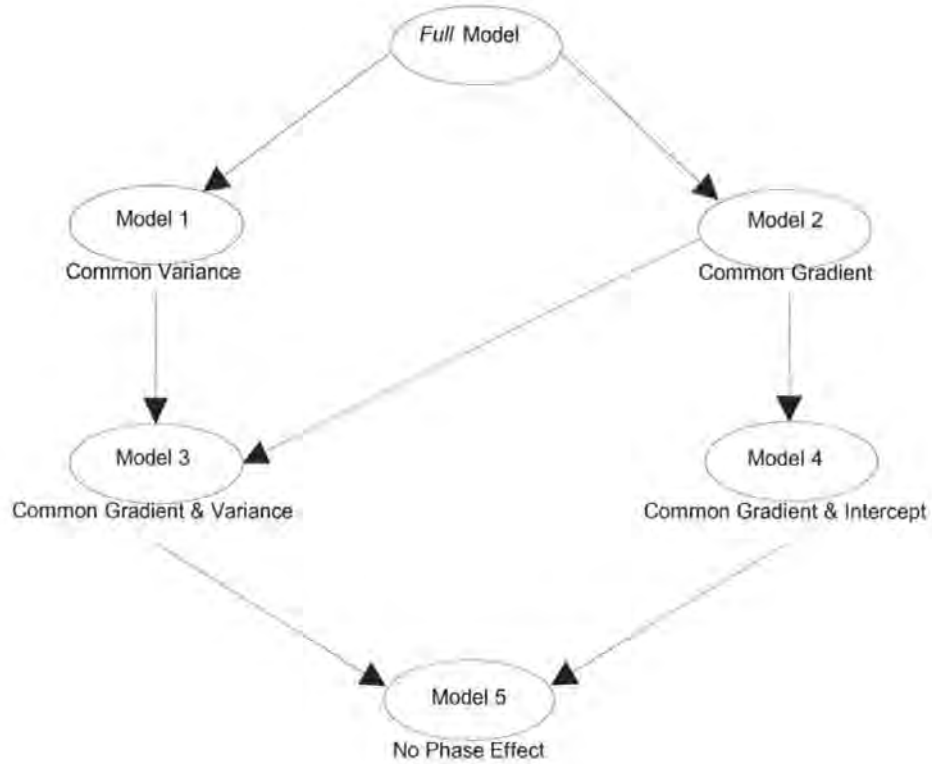
**Figure 7.1:** Hierarchical Structure for Modelling Engine Phase Effect

*Model 1* $\qquad \lambda_{ji}(t) = \exp\left\{\beta_0 + \underline{x}_i^T \underline{\beta_2} + \left(\beta_1 + \underline{x}_i^T \underline{\beta_3}\right) t + \varepsilon_i\right\}$ $\qquad$ (number of parameters = 9)

This model is parameterised for a difference in starting ROCOF and also a different gradient parameter for each phase. It contains both $\underline{\beta_2}$ and $\underline{\beta_3}$ covariates to model the phase effect. In this case the variance of engines within phases is assumed to be constant across phases. In other words only one variance component is included in the model.

*Model 2* $\qquad \lambda_{ji}(t) = \exp\left\{\beta_0 + \underline{x}_i^T \underline{\beta_2} + \beta_1 t + \varepsilon_{ji}\right\}$ $\qquad$ (number of parameters = 9)

This allows for a difference in the starting ROCOF between phases and also a difference in variation between phases. Hence, the covariate included in the model is now $\underline{\beta_2}$ and $j = 1, 2, 3, 4$ for the random effects.

*Model 3*        $\lambda_{ji}(t) = \exp\{\beta_0 + \underline{x}_i{}^T \underline{\beta_2} + \beta_1 t + \varepsilon_i\}$        (number of parameters = 6)

The covariate effects are the same as in *Model 2*. In this case there is assumed to be no difference in engine variation between phases.

*Model 4*        $\lambda_{ji}(t) = \exp\{\beta_0 + \beta_1 t + \varepsilon_{ji}\}$        (number of parameters = 6)

There are assumed to be no covariate effects between the engine phases. The difference between phases is modelled by the variance component ($j = 1, 2, 3, 4$).

*Model 5*        $\lambda_{ji}(t) = \exp\{\beta_0 + \beta_1 t + \varepsilon_i\}$        (number of parameters = 3)

This model assumes there to be no phase effect and is the model fitted in Section 7.2. With respect to equation (7.2) $\underline{\beta_2}$ and $\underline{\beta_3}$ have been removed and there is a single random effect.

These models have been fitted to the Jaguar data and the maximised log-likelihoods of each are given in Table 7.3. From a visual inspection of these values *Model 1* and the *Full Model* stand out from the others and likelihood ratio tests performed on these values confirm this. Following the structure in Figure 7.1 no significant difference is found between the maximised log-likelihoods of *Models 2* to *5*, but there is a difference between *Models* 1 and 3 and between *Models* 2 and *Full*. The conclusion here being that the log-likelihood is significantly higher when covariates for the phase effect are included on both the intercept and gradient parameters. A comparison between *Model 1* and the *Full Model* yields a likelihood ratio test statistic of 2.48 against a tabulated value of 7.815 ($\chi^2_{3,0.05}$). Hence the final conclusion to be drawn from these hierarchical models is that whilst engines on a given phase differ from engines in other phases in both their starting reliability and in the gradient of the ROCOF, the variability between engines is not affected by phase. The parameter estimates for the final common variance model are given in Table 7.4.

**TABLE 7.3**

*Maximised Log-Likelihoods for the Covariate Models.*

| Model | | $-\ell_{max}$ |
|---|---|---|
| *Full* | $\lambda_{ji}(t) = \exp\left\{\beta_0 + \underline{x}_i^T\underline{\beta_2} + \left(\beta_1 + \underline{x}_i^T\underline{\beta_3}\right)t + \varepsilon_{ji}\right\}$ | 720.37 |
| *1* | $\lambda_{ji}(t) = \exp\left\{\beta_0 + \underline{x}_i^T\underline{\beta_2} + \left(\beta_1 + \underline{x}_i^T\underline{\beta_3}\right)t + \varepsilon_i\right\}$ | 719.13 |
| *2* | $\lambda_{ji}(t) = \exp\left\{\beta_0 + \underline{x}_i^T\underline{\beta_2} + \beta_1 t + \varepsilon_{ji}\right\}$ | 713.89 |
| *3* | $\lambda_{ji}(t) = \exp\left\{\beta_0 + \underline{x}_i^T\underline{\beta_2} + \beta_1 t + \varepsilon_i\right\}$ | 712.89 |
| *4* | $\lambda_{ji}(t) = \exp\left\{\beta_0 + \beta_1 t + \varepsilon_{ji}\right\}$ | 711.98 |
| *5* | $\lambda_{ji}(t) = \exp\left\{\beta_0 + \beta_1 t + \varepsilon_i\right\}$ | 711.42 |

**TABLE 7.4**

*Parameter Estimates for Engine Phase Effect Model 1.*

| Parameter | $\beta_0$ | $\beta_1$ | $\beta_{21}$ | $\beta_{22}$ | $\beta_{23}$ |
|---|---|---|---|---|---|
| Estimate | 3.934 | −5.783 | −0.363 | −0.912 | −0.577 |

| Parameter | $\beta_{31}$ | $\beta_{32}$ | $\beta_{33}$ | $\sigma_\varepsilon$ |
|---|---|---|---|---|
| Estimate | 4.591 | 5.179 | 3.347 | 0.651 |

As in Section 7.2 the estimate of the variance component is calculated from a sample size of 54, and hence, from the results in Section 3.7, will have negligible bias. To make a comparison between the separate models for each phase and the combined phase effect model, the predictions of the number of failures per engine in 0.32 units of time are given in Table 7.5. As would be expected, when these are compared with the predictions in Table 3.10 the mean number of failures show little difference. The differences in the prediction are for the lower and upper intervals due to the change in the estimate of the standard deviation.

**TABLE 7.5**

*Predicted Numbers of Failures for the Engine Phase Effect Model.*

| Engine Phase | Expected Number of Failures | | | Interval Width |
|:---:|:---:|:---:|:---:|:---:|
| | Mean | $\left(\hat{\beta}_0 + \underline{x}_i \hat{\underline{\beta}}_2\right) - 1.96 \hat{\sigma}_\varepsilon$ | $\left(\hat{\beta}_0 + \underline{x}_i \hat{\underline{\beta}}_2\right) + 1.96 \hat{\sigma}_\varepsilon$ | |
| A0 | 7.45 | 2.68 | 20.70 | 18.02 |
| A1-1 | 9.46 | 3.40 | 26.28 | 22.88 |
| A1-2 | 5.97 | 2.15 | 16.60 | 14.45 |
| A2-1 | 6.38 | 2.30 | 17.73 | 15.43 |

With the data for the four engine phases combined, and covariates for the phase effect included, there is a total of 15 engines that have been run on the low test type. For the separate models there is very little data for each phase from the low tests, but the combined data with the covariate effects is more appropriate to investigate a test effect. In Chapter 6 it was stated that a time multiplicative covariate effect model is the more logical model for incorporating test type. The final model, *Model 1*, can be reparameterised to include this covariate. The resulting maximised log-likelihood of this fitted model is 720.07 and the estimate of the parameter for the test effect is -1.096 with a standard error of 0.8005. The log-likelihood changes very little when this extra effect is included and the parameter estimate is similar in magnitude to its standard error. Hence, as in Chapter 6, it can be concluded that there is no evidence from this data of a change in ROCOF due to the test type.

## 7.4 Summary

This chapter has shown how the covariate models from Chapter 6 can be used to model effects due to engine phase. A hierarchical approach to fitting these covariate models, and a comparison with the separate models fitted in Chapter 3, has shown that there is no evidence of a difference in engine variation between the engine phases. Likelihood ratio tests have been used to compare the various models. The appropriate model has combined proportional and

time multiplicative covariate effects. In other words, this model gives the phases to have different intercept and gradient parameters but the variance component remains the same. The mean predictions from this model have been shown to be approximately the same as before.

Using the engine phase as a covariate also increases the sample size, which substantially reduces the bias on the estimated standard deviation as well as giving more data for each of the test types. Using the covariate model with the larger sample sizes to investigate an effect due to test type leads to the same conclusion found in Chapter 6 that there is no evidence of a test effect. From the information about the test types given in Chapter 1 this result is unexpected, and it must be concluded that the engine testing procedures and the data collection techniques currently used do not allow this test effect to be identified.

# CHAPTER 8

# A Log-Linear Interval Censoring Random Effect NHPP Model

## 8.1 Introduction

The Jaguar data used for this study, given in Appendix 2, consists of a series of failure times for each engine with a censoring indicator depending on whether the last observation indicates a failure or the end of the test. In some cases, through the nature of the testing procedures the data will come from a series of time intervals When an engine is undergoing a long test it is thoroughly checked at certain times, every two hundred hours say, and a number of faults may be found at this stage. These faults are obviously interval censored between this time and the previous check and it can be argued as to whether these failures should be included in the analysis. The fault is only discovered when the engine is taken apart and does not affect performance in any way. In our case it has not been possible to obtain data for these failures but such situations can arise and the following section provides a methodology to incorporate them into the analysis.

## 8.2 Interval Censoring Models

The likelihood function for an interval censored non-homogeneous Poisson process model, from Crowder *et al.* (1991), is given by equation (8.1) below. This is for $n$

non-overlapping intervals at times $(T_0, T_1], (T_1, T_2],...,(T_{n-1}, T_n]$ with $m_1, m_2,...,m_n$ interval censored failures respectively.

$$L = \exp\left\{-\sum_{i=1}^{n}\int_{T_{i-1}}^{T_i}\lambda(t)dt\right\}\prod_{i=1}^{n}\frac{1}{m_i!}\left(\int_{T_{i-1}}^{T_i}\lambda(t)dt\right)^{m_i}$$ (8.1)

A model of this form could be used for the Jaguar data but this would classify a number of failure times which are actually known as interval censored, resulting in a loss of information and an increase in the uncertainty of the parameter estimates. A more appropriate model will allow a mixture of known and interval censored observations. Consider such a case in its simplest form, that is when there is only one time interval. If the test is run from times $T_0$ to $T_1$ with $n$ failures occurring of which $d$ failure times are known (giving $n - d$ interval censored observations) then the likelihood function becomes

$$L = \exp\left\{-\int_{T_0}^{T_1}\lambda(t)dt\right\}\prod_{k=1}^{d}\lambda(t_k)\frac{1}{(n-d)!}\left(\int_{T_0}^{T_1}\lambda(t)dt\right)^{n-d} \qquad \text{where } d > 0$$ (8.2)

If $d = 0$ (all observations are censored) equation (8.1) is appropriate and if $d = n$ (all failure times are known) equation (8.2) reduces to the likelihood function of the non-interval censored model given by equation (2.51). In the more general case for $p$ time intervals at $(T_0, T_1], (T_1, T_2],...,(T_{p-1}, T_p]$, with $n_j$ total failures and $d_j$ known failure times in the $j^{th}$ interval, the likelihood function becomes

$$L = \prod_{j=1}^{p}\left[\exp\left\{-\int_{T_{j-1}}^{T_j}\lambda(t)dt\right\}\left(\prod_{k=1}^{d_j}\lambda(t_{jk})\right)\frac{1}{(n_j-d_j)!}\left(\int_{T_{j-1}}^{T_j}\lambda(t)dt\right)^{n_j-d_j}\right]$$ (8.3)

As before, this reduces to equation (8.1) or equation (2.51) if complete or no censoring occur respectively. Repeating the analysis in Sections 3.1 and 3.3 allows this model to be extended for $m$ units and to include a random effect. For the $i^{th}$ unit let there be $p_i$ intervals at

times $(T_0, T_1], (T_1, T_2],..., (T_{pi-1}, T_{pi}]$, with $n_{ij}$ failures and $d_{ij}$ uncensored observations in each interval $(j = 1, 2,...,p_i)$. The likelihood function in this case is

$$L = \prod_{i=1}^{m} \prod_{j=1}^{p_i} \left[ \exp\left\{ -\int_{T_{i(j-1)}}^{T_{ij}} \lambda_i(t)dt \right\} \left( \prod_{k=1}^{d_{ij}} \lambda_i(t_{ijk}) \right) \frac{1}{(n_{ij}-d_{ij})!} \left( \int_{T_{i(j-1)}}^{T_{ij}} \lambda_i(t)dt \right)^{n_{ij}-d_{ij}} \right] \qquad (8.4)$$

Substituting into equation (8.4) the model with the random effect on the intercept for $\lambda_i(t)$, given by equation (3.16), and integrating over the random effect gives the log-likelihood function to be

$$\ell = \sum_{i=1}^{m} \ln \left[ \int_{-\infty}^{\infty} \left( \prod_{j=1}^{p_i} Z_{ij} \right) \frac{1}{\sqrt{2\pi \exp(2\varphi_\varepsilon)}} \exp\left\{ -\frac{\varepsilon_i^2}{2\exp(2\varphi_\varepsilon)} \right\} d\varepsilon_i \right] \qquad (8.5)$$

where

$$Z_{ij} = \frac{1}{(n_{ij}-d_{ij})!} \exp\left[ d_{ij}(\beta_0+\varepsilon_i) + \beta_1 \sum_{k=1}^{d_{ij}} t_{ijk} - \frac{\exp(\beta_0+\varepsilon_i)}{\beta_1} \left\{ \exp(\beta_1 T_{ij}) - \exp(\beta_1 T_{i(j-1)}) \right\} \right]$$
$$\left[ \frac{\exp(\beta_0+\varepsilon_i)}{\beta_1} \left\{ \exp(\beta_1 T_{ij}) - \exp(\beta_1 T_{i(j-1)}) \right\} \right]^{n_{ij}-d_{ij}} \qquad (8.6)$$

Using this and the substitutions given by equations (8.7) to (8.13), namely

$$A_i = \frac{1}{\sqrt{2\pi \exp(2\varphi_\varepsilon)}} \exp\left( -\frac{\varepsilon_i^2}{\exp(2\varphi_\varepsilon)} \right) \qquad (8.7)$$

$$B_{ij} = n_{ij} - \frac{\exp(\beta_0+\varepsilon_i)}{\beta_1} \left\{ \exp(\beta_1 T_{ij}) - \exp(\beta_1 T_{i(j-1)}) \right\} \qquad (8.8)$$

$$C_{ij} = \sum_{k=1}^{d_{ij}} t_{ijk} - \frac{\exp(\beta_0+\varepsilon_i)}{\beta_1^2} \left\{ \exp(\beta_1 T_{ij})(\beta_1 T_{ij} - 1) + \exp(\beta_1 T_{i(j-1)})(1 - \beta_1 T_{i(j-1)}) \right\} \qquad (8.9)$$

$$D_i = \frac{\varepsilon_i^2 - \exp(2\varphi_\varepsilon)}{\exp(2\varphi_\varepsilon)} \tag{8.10}$$

$$E_{ij} = -\frac{\exp(\beta_0 + \varepsilon_i)}{\beta_1^2}\left\{\exp(\beta_1 T_{ij})(\beta_1 T_{ij} - 1) + \exp(\beta_1 T_{i(j-1)})(1 - \beta_1 T_{i(j-1)})\right\} \tag{8.11}$$

$$F_{ij} = \left[\frac{\exp(\beta_0 + \varepsilon_i)}{\beta_1^3}\left\{\exp(\beta_1 T_{ij})(\beta_1^2 T_{ij}^2 - 2\beta_1 T_{ij} + 2) + \exp(\beta_1 T_{i(j-1)})(\beta_1^2 T^2_{i(j-1)} - 2\beta_1 T_{i(j-1)} + 2)\right\}\right]$$
$$\left(\frac{n_{ij} - d_{ij}}{B_{ij}} - 1\right) + \left(\frac{n_{ij} - d_{ij}}{B_{ij}^2}\right)E_{ij}^2$$

$$\tag{8.12}$$

$$G_i = \frac{\varepsilon_i^2}{\exp(2\varphi_\varepsilon)} \tag{8.13}$$

we may obtain the first order partial derivatives

$$\frac{\partial \ell}{\partial \beta_0} = \sum_{i=1}^{m}\left\{\frac{\displaystyle\int_{-\infty}^{\infty}\left(\prod_{j=1}^{p_i} Z_{ij}\right)\sum_{j=1}^{p_i} B_{ij} A_i \, d\varepsilon_i}{\displaystyle\int_{-\infty}^{\infty}\left(\prod_{j=1}^{p_i} Z_{ij}\right)A_i \, d\varepsilon_i}\right\} \tag{8.14}$$

$$\frac{\partial \ell}{\partial \beta_1} = \sum_{i=1}^{m}\left\{\frac{\displaystyle\int_{-\infty}^{\infty}\left(\prod_{j=1}^{p_i} Z_{ij}\right)\sum_{j=1}^{p_i} C_{ij}\left(\frac{n_{ij} - d_{ij}}{B_{ij}} - 1\right)A_i \, d\varepsilon_i}{\displaystyle\int_{-\infty}^{\infty}\left(\prod_{j=1}^{p_i} Z_{ij}\right)A_i \, d\varepsilon_i}\right\} \tag{8.15}$$

$$\frac{\partial \ell}{\partial \varphi_\varepsilon} = \sum_{i=1}^{m}\left\{\frac{\displaystyle\int_{-\infty}^{\infty}\left(\prod_{j=1}^{p_i} Z_{ij}\right)D_i A_i \, d\varepsilon_i}{\displaystyle\int_{-\infty}^{\infty}\left(\prod_{j=1}^{p_i} Z_{ij}\right)A_i \, d\varepsilon_i}\right\} \tag{8.16}$$

The second order partial derivatives are then given by equations (8.17) to (8.22) respectively.

$$\frac{\partial^2 \ell}{\partial \beta_0{}^2} = \sum_{i=1}^{m} \left\{ \int_{-\infty}^{\infty} \left( \prod_{j=1}^{p_i} Z_{ij} \right) A_i \, d\varepsilon_i \int_{-\infty}^{\infty} \left( \prod_{j=1}^{p_i} Z_{ij} \right) \left( \left( \sum_{j=1}^{p_i} B_{ij} \right)^2 + \left( \sum_{j=1}^{p_i} B_{ij} - n_{ij} \right) \right) A_i \, d\varepsilon_i - \right.$$

$$\left. \left( \int_{-\infty}^{\infty} \left( \prod_{j=1}^{p_i} Z_{ij} \right) \sum_{j=1}^{p_i} B_{ij} A_i \, d\varepsilon_i \right)^2 \right\} \bigg/ \left\{ \int_{-\infty}^{\infty} \left( \prod_{j=1}^{p_i} Z_{ij} \right) A_i \, d\varepsilon_i \right\}^2 \tag{8.17}$$

$$\frac{\partial^2 \ell}{\partial \beta_0 \beta_1} = \sum_{i=1}^{m} \left\{ \int_{-\infty}^{\infty} \left( \prod_{j=1}^{p_i} Z_{ij} \right) A_i \, d\varepsilon_i \int_{-\infty}^{\infty} \left( \prod_{j=1}^{p_i} Z_{ij} \right) \left( \sum_{j=1}^{p_i} B_{ij} \sum_{j=1}^{p_i} C_{ij} + \sum_{j=1}^{p_i} E_{ij} \right) A_i \, d\varepsilon_i - \int_{-\infty}^{\infty} \left( \prod_{j=1}^{p_i} Z_{ij} \right) \sum_{j=1}^{p_i} B_{ij} A_i \, d\varepsilon_i \right.$$

$$\left. \int_{-\infty}^{\infty} \left( \prod_{j=1}^{p_i} Z_{ij} \right) \sum_{j=1}^{p_i} C_{ij} A_i \, d\varepsilon_i \right\} \bigg/ \left\{ \int_{-\infty}^{\infty} \left( \prod_{j=1}^{p_i} Z_{ij} \right) A_i \, d\varepsilon_i \right\}^2 \tag{8.18}$$

$$\frac{\partial^2 \ell}{\partial \beta_0 \varphi_\varepsilon} = \sum_{i=1}^{m} \left\{ \int_{-\infty}^{\infty} \left( \prod_{j=1}^{p_i} Z_{ij} \right) A_i \, d\varepsilon_i \int_{-\infty}^{\infty} \left( \prod_{j=1}^{p_i} Z_{ij} \right) \left( \sum_{j=1}^{p_i} B_{ij} \right) D_i A_i \, d\varepsilon_i - \int_{-\infty}^{\infty} \left( \prod_{j=1}^{p_i} Z_{ij} \right) \sum_{j=1}^{p_i} B_{ij} A_i \, d\varepsilon_i \right.$$

$$\left. \int_{-\infty}^{\infty} \left( \prod_{j=1}^{p_i} Z_{ij} \right) D_i A_i \, d\varepsilon_i \right\} \bigg/ \left\{ \int_{-\infty}^{\infty} \left( \prod_{j=1}^{p_i} Z_{ij} \right) A_i \, d\varepsilon_i \right\}^2 \tag{8.19}$$

$$\frac{\partial^2 \ell}{\partial \beta_1{}^2} = \sum_{i=1}^{m} \left\{ \int_{-\infty}^{\infty} \left( \prod_{j=1}^{p_i} Z_{ij} \right) A_i \, d\varepsilon_i \int_{-\infty}^{\infty} \left( \prod_{j=1}^{p_i} Z_{ij} \right) \left( (\sum_{j=1}^{p_i} C_{ij})^2 + \sum_{j=1}^{p_i} F_{ij} \right) A_i \, d\varepsilon_i - \right.$$

$$\left. \left( \int_{-\infty}^{\infty} \left( \prod_{j=1}^{p_i} Z_{ij} \right) \sum_{j=1}^{p_i} C_{ij} A_i \, d\varepsilon_i \right)^2 \right\} \bigg/ \left\{ \int_{-\infty}^{\infty} \left( \prod_{j=1}^{p_i} Z_{ij} \right) A_i \, d\varepsilon_i \right\}^2 \tag{8.20}$$

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$$\frac{\partial^2 \ell}{\partial \beta_1 \varphi_\varepsilon} = \sum_{i=1}^{m}\left\{ \int_{-\infty}^{\infty}\left(\prod_{j=1}^{p_i} Z_{ij}\right)A_i\, d\varepsilon_i \int_{-\infty}^{\infty}\left(\prod_{j=1}^{p_i} Z_{ij}\right)\left(\sum_{j=1}^{p_i} C_{ij}\right)D_i A_i\, d\varepsilon_i - \int_{-\infty}^{\infty}\left(\prod_{j=1}^{p_i} Z_{ij}\right)\sum_{j=1}^{p_i} C_{ij} A_i\, d\varepsilon_i \right.$$

$$\left. \int_{-\infty}^{\infty}\left(\prod_{j=1}^{p_i} Z_{ij}\right)D_i A_i\, d\varepsilon_i \right\} \Bigg/ \left\{ \int_{-\infty}^{\infty}\left(\prod_{j=1}^{p_i} Z_{ij}\right)A_i\, d\varepsilon_i \right\}^2 \tag{8.21}$$

$$\frac{\partial^2 \ell}{\partial \varphi_\varepsilon^{\,2}} = \sum_{i=1}^{m}\left\{ \int_{-\infty}^{\infty}\left(\prod_{j=1}^{p_i} Z_{ij}\right)A_i\, d\varepsilon_i \int_{-\infty}^{\infty}\left(\prod_{j=1}^{p_i} Z_{ij}\right)\left(D_i^{\,2} + G_i\right)A_i\, d\varepsilon_i - \right.$$

$$\left. \left(\int_{-\infty}^{\infty}\left(\prod_{j=1}^{p_i} Z_{ij}\right)D_i A_i\, d\varepsilon_i\right)^2 \right\} \Bigg/ \left\{ \int_{-\infty}^{\infty}\left(\prod_{j=1}^{p_i} Z_{ij}\right)A_i\, d\varepsilon_i \right\}^2 \tag{8.22}$$

This enables the information matrix and hence the standard errors of the parameters to be estimated.

## 8.3 Summary

The calculations in the previous section show how the mixed interval censoring method can be applied to the log-linear model with a random effect on the intercept parameter. Any of the other random effect and covariate models demonstrated in Chapters 4 and 6, and therefore the hierarchical modelling procedure of Chapter 7, can also be fitted into this framework. Unfortunately, as stated in Section 8.1, there is no data presently available in this form and consequently the interval censored models have not been used. However, the data is likely to be produced in this form in the future and such a model formulation will be invaluable in this case.

# CHAPTER 9

# Further Developments and Conclusions

The aims of this study, as outlined in the introductory chapter, were to provide a statistical methodology to aid the physical development of engines at Jaguar. In order to accomplish this the reliability of the engines must be modelled and predictions of failures obtained. The model must also be able to include covariate effects, such as test type and engine phase.

In Chapter 2 the basic definitions of survival analysis are given, as well as a review of the most commonly used models. From this chapter it follows that a repairable system reliability model, namely the non-homogeneous Poisson process model, is appropriate to analyse the Jaguar failure data. Using a log-linear form of this model has provided the basis of the work carried out.

Initially the model defined in Section 2.7.2 for a single engine was extended to allow for data from multiple engines. Failure time plots of each phase of the Jaguar failure data show there to be large variability between the ROCOF's of engines of the same type. This variability has been included into the model in the form of a random effect, and this enables the differences between engines to be estimated and accounted for in the predictions of reliability. The random effect has been incorporated into the model in three ways – on the intercept parameter, on the gradient parameter and on both of these.

The parameter estimates for all of the random effect models are obtained by the method of maximum likelihood, and the standard errors are estimated from the corresponding information matrices. For each of these, an integral is required to be estimated numerically. A composite form of the Gauss-Legendre technique has been demonstrated to be appropriate for integrals of this type. Simulation exercises have shown the estimates of the parameters and their standard errors to be valid, although there is a bias associated with the estimate of the variance parameter when the sample size is small. The results of the simulation studies indicate an appropriate sample size for each of the random effect models such that this bias becomes negligible. In the case when the random effect is placed on the gradient parameter this recommended sample size is very large and restricts the use of this model in practical terms. Such a sample size is unlikely to be available for engineering applications due to the high costs of prototype components and testing procedures, although other application areas such as medical studies may produce sufficient data.

In the analysis of the Jaguar engine data in Chapters 3 and 4, the significance testing and a nested model approach to the fitted models indicated that the model with the random effect on the intercept parameter is more appropriate. Chapter 5 introduced marginal and conditional residuals (Lawless, 1995) for model checking and these also suggest the proportional random effect model. In these chapters the four engine phases have been treated as separate data sets, resulting in an estimated model for each phase. In effect this assumes that ROCOF's of engines from different phases are independent. However, engines from separate phases contain many of the same parts so this assumption is not necessarily true and needs to be checked. A practical issue for the engineers on the development project is to test for a difference between the engine phases. As development progresses the engine design is modified either to improve reliability or to maintain the current reliability whilst reducing total engine cost, and the effects of these changes need to be monitored. Modelling the phases separately may produce a point (and interval) estimate of the expected number of failures but it does not address this issue. A statistical test for differences between the phases is required and a hierarchical modelling approach is demonstrated in Chapter 7. This method incorporates the covariate modelling procedures introduced in Chapter 6.

The hierarchical modelling procedure performed on the Jaguar engine data concluded that a separate model for each engine phase is not necessary. The final model has significant covariate effects for phases on the intercept (proportional effect) and gradient (time multiplicative effect) parameters and a single variance component. The interpretation of this is that the ROCOF's between phases are different at time zero and also change at varying rates, but that the variance between engines is constant over the phases. In other words, there is a significant difference between the engine phases, and the expected number of failures for each phase show how these differences translate in real terms. A number of issues on the variance component arise from this final model. Firstly, the sample size of engines when the data sets are combined increases such that the estimate of the standard deviation will have negligible bias. This is in contrast to the separate models where the estimates are based on sample sizes as small as nine and have been shown to have a large bias. The second is a more practical issue. As the engine development progresses, as well as improving reliability the aim is to make the engines more consistent, that is to reduce the variation between them. Obviously, with a conclusion of constant variance between phases there is no evidence that this is happening. However, the data have been obtained from engines that have all been hand built, and part of the variance between them can be attributed to the variability of the build. This variability will be the same throughout the four engine phases and may be the dominating factor in the differences in engine reliability. If this is true then the conclusion of constant variance across the four phases would be expected. In subsequent phases (which were not available for this study) development engines are built in a manner much more like production engines. In this case a reduction in variability between engines would be expected as the phases progress towards the final manufacturing line.

For each model an equation has been shown which gives the mean number of failures in time interval $(a,b]$, and figures found for the Jaguar engines in the first 0.32 unit time of running. The failures in this interval follow a Poisson distribution with the consequence that the probability of achieving a target reliability can be found. For the Jaguar data there are more failures than would be expected under normal engine running conditions and this can be attributed to the test conditions of engines under development. First, there is a more rigorous inclusion of faults compared to those reported by customers. Indeed, a number of faults are

included of which a customer would be unaware as discussed in Chapter 8. Second, in Chapter 1 the development processes and the tests performed on the engines have been described. It was stated that the seven test types are performed on engines and these are accelerated to varying degrees over real driving conditions. At this stage the acceleration factors for each test are not known. Due to the small sample sizes on each phase there is little data available for each test performed, and comparisons between the tests cannot be made. In order to attempt a comparison between engines on different tests the simulations have categorised into two groups, low and high. The tests classified as low are thought to be only marginally accelerated over real driving conditions. In theory using only these engines for the modelling would allow more realistic predictions to be made. Unfortunately, apart from ignoring a large number of engines, there is very little data available for this grouping and it cannot be used with any confidence for this purpose. Indeed, as demonstrated in Chapter 6 and Section 7.3, the reliability of the engines on the low tests cannot be differentiated from those in the high test group. In order to address this issue more data would be required.

These unknown acceleration factors would seem to restrict the usefulness of the non-homogeneous Poisson process approach to modelling the reliability of Jaguar test engines. At present these models cannot be used to predict actual engine reliability. However, they can be used to predict the reliability of engines under test conditions and can therefore be used to monitor the effect of design changes. If a design change causes an improvement in test reliability it follows that an improvement to engine reliability has been made. Being able to monitor this is a valuable tool that has been previously unavailable to the design engineers at Jaguar.

The non-homogeneous Poisson process random effect model may become more useful in future engine development projects. After a period of time, warranty data for the AJ26 will become available and estimates of actual engine reliability can then be made. This will enable comparisons to be made with the reliability predicted from test engines so that the acceleration factors can be estimated. If the assumption is made that these acceleration factors remain constant for different engine types (which appears reasonable) then the test reliability of engines under development in the future can be translated to a more realistic operational

reliability. However, the correlation of the test and true reliability will require a large amount of data if this translation is to be successful.

This study has demonstrated the use of log-linear random effect non-homogeneous Poisson process models for assessing reliability characteristics of prototype engines. The suitability of these models for repairable systems in other engineering applications is clear, but these models are not just restricted to this area. Another major application can be found in medicine for modelling the re-occurrence of diseases, see for example Abu–Libdeh *et al.* (1990). This final chapter has considered some practical issues relating to the modelling and prediction process. Further developments to the described models include:

- *Experimental Design*

  Presently the design of the engine testing on each phase is mainly between engines with each engine performing only one type of test. One implication of the large variability between engines is that such experiments involving between engine comparisons are likely to require large numbers of engines to achieve reasonable statistical power. Within engine designs based on random effects models provide a very attractive alternative and these designs need to be investigated further. The extensive work on crossover designs in the literature on clinical trials is relevant here. See for example Senn (1992).

- *Warranty Data*

  As stated in previously warranty data will soon be available for the AJ26 engine. Further work carried out to model and correlate this to the test data will be invaluable in future engine projects. It will allow the expected 'field' reliability of prototype engines to be predicted before being released for production.

- *Interval Censoring*

  Chapter 8 introduced the mixed censoring random effect model (which can be reduced to a fully interval censored model). Data from the testing procedures is more likely to be in this form as it can be difficult to record the exact failure time. The form of the likelihood and

the first and second partial derivatives are given but further work is required to investigate their use in parameter estimation.

- *Other NHPP Models*

The work carried out has used the log-linear form of $\lambda(t)$ exclusively, and has investigated the addition of different covariate and random effects. There are other forms of $\lambda(t)$, such as the power law model of equation (2.53), and these will also be of use in some applications. Lawless (1987) and Abu-Libdeh (1990) have used the power law model with a random effect. Ascher and Feingold (1980) list some other forms for $\lambda(t)$.

- *Bayesian Analysis*

The random effects models presented in Chapters 3 and 4 are amenable to Bayesian analysis using MCMC in which samples of unknown parameters and random effects are sampled from their posterior distribution by setting up a Markov Chain which has the posterior distribution as its stationary distribution. These samples can be used to obtain summaries of posterior distributions and predictive distributions concerning future engines. The work on Frailty models presented in Walker and Mallick (1997) would provide a good starting point for further work in this direction.

A further extension would be to try to model sequentially the different engine phases by setting up the random effects models in Chapters 3 and 4 within a sequential framework similar to that presented by Whitmore *et al.* (1994) and described here in Section 2.6.2. The posterior distribution from one engine phase can be used to form the basis of the prior distribution for the next engine phase. The development of a sequential Bayesian analysis based on the non-homogeneous Poisson process random effect models described in this study would form the next step in improving the methodology, and ultimately the reliability predictions.

In conclusion, a methodology has been presented which allows the reliability of prototype engines to be modelled in terms of the number and trend of the failures. The

methodology also allows estimates to be made of the variability between engines. The extensions listed above will improve this process further. In the present economic climate customers are demanding more reliable products at reasonable prices and, to remain competitive, manufacturers are reducing product lead times in order to minimise development costs. Under these conditions it is important to make maximum use of available data in order that an early indication of a product's reliability can be obtained. The random effect NHPP models go some way towards achieving this.

# APPENDIX 1

# Jaguar Engine Test Simulations

# Jaguar AJ26

## Run and Rate



Engine Speed ······ Engine Load ········ Coolant Temp

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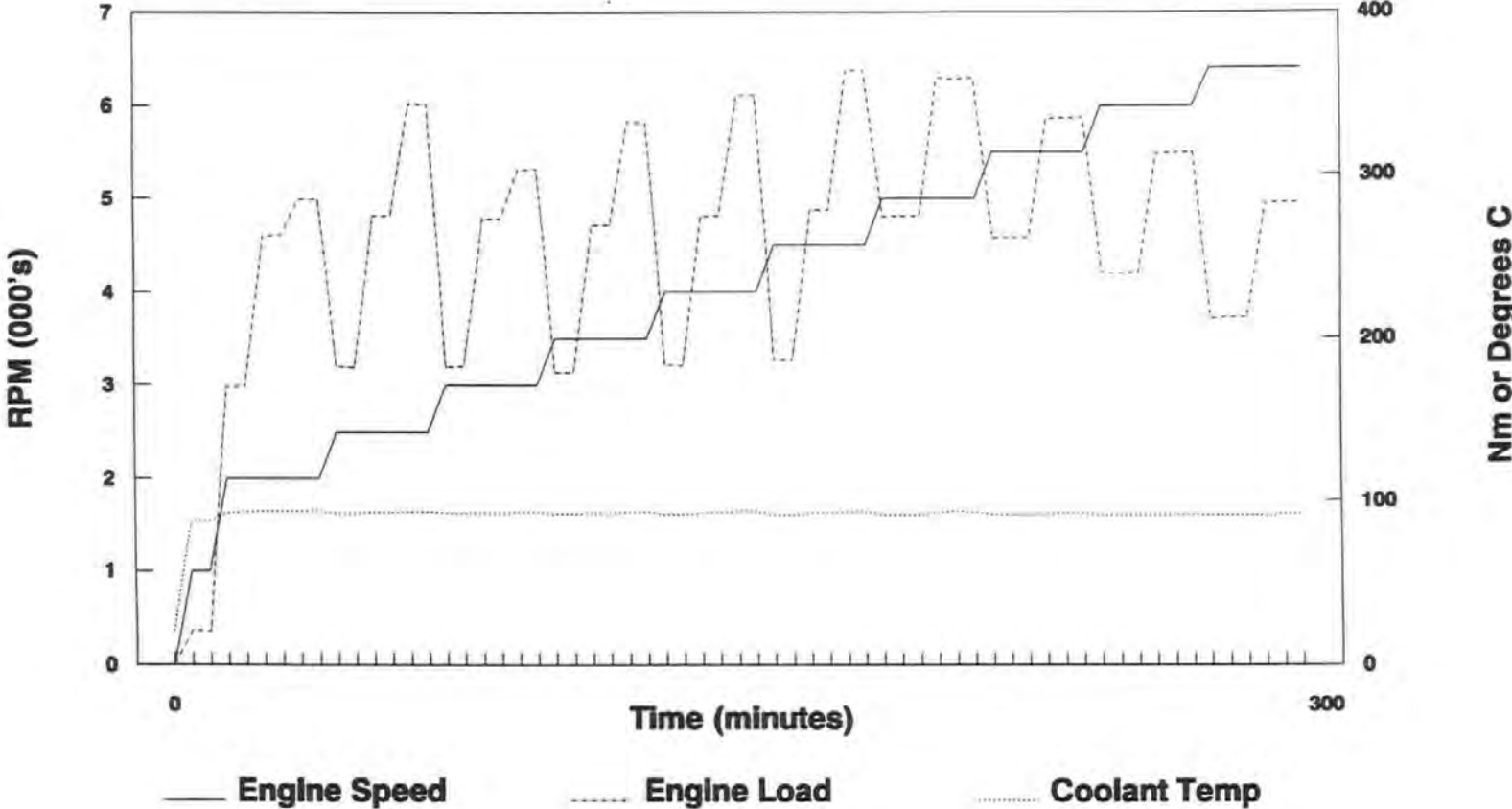
# Jaguar AJ26

**Reliability Cycle**



Engine Speed — — — Engine Load ········· Coolant Temp
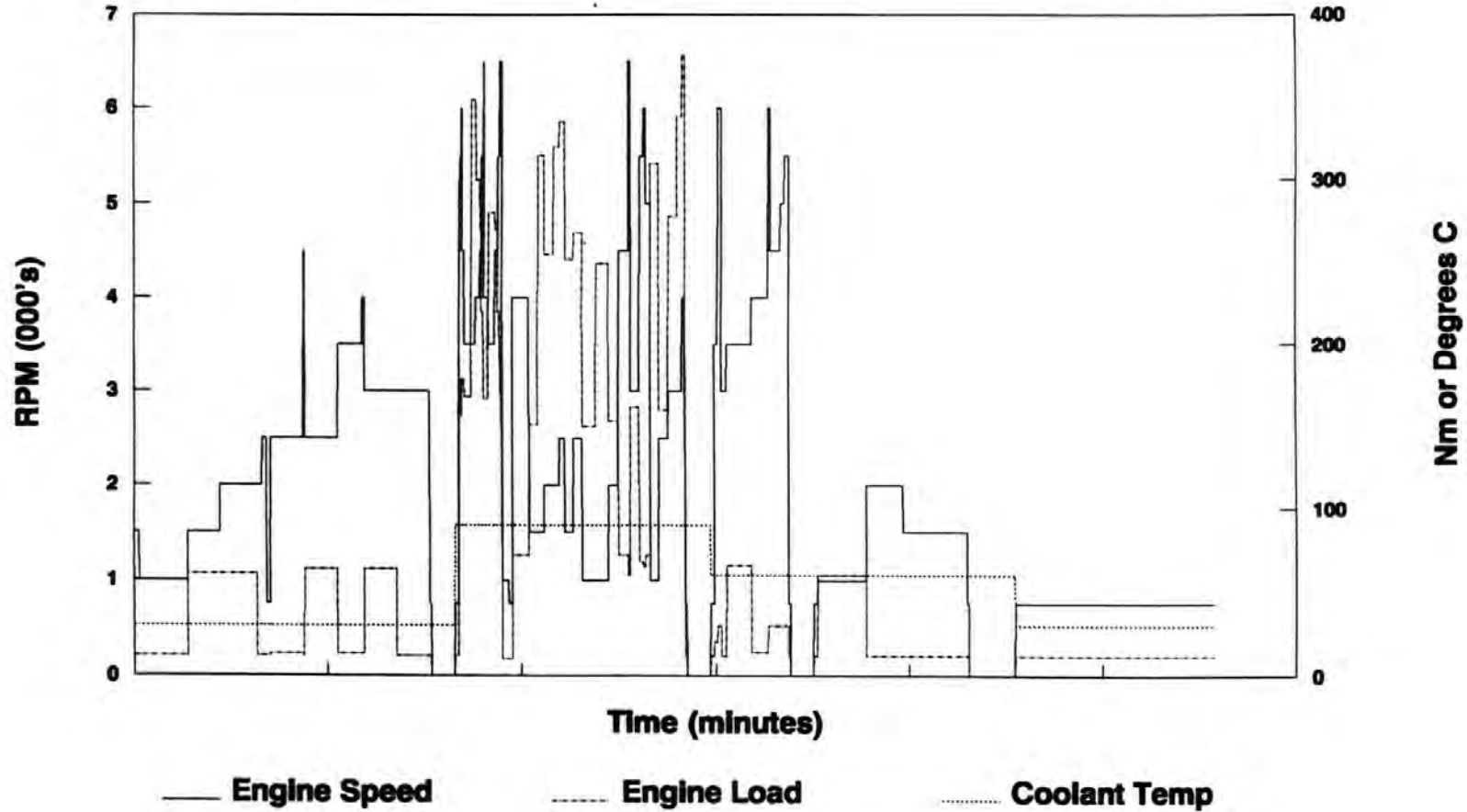
# Jaguar AJ26

## 400 Hour Cycle
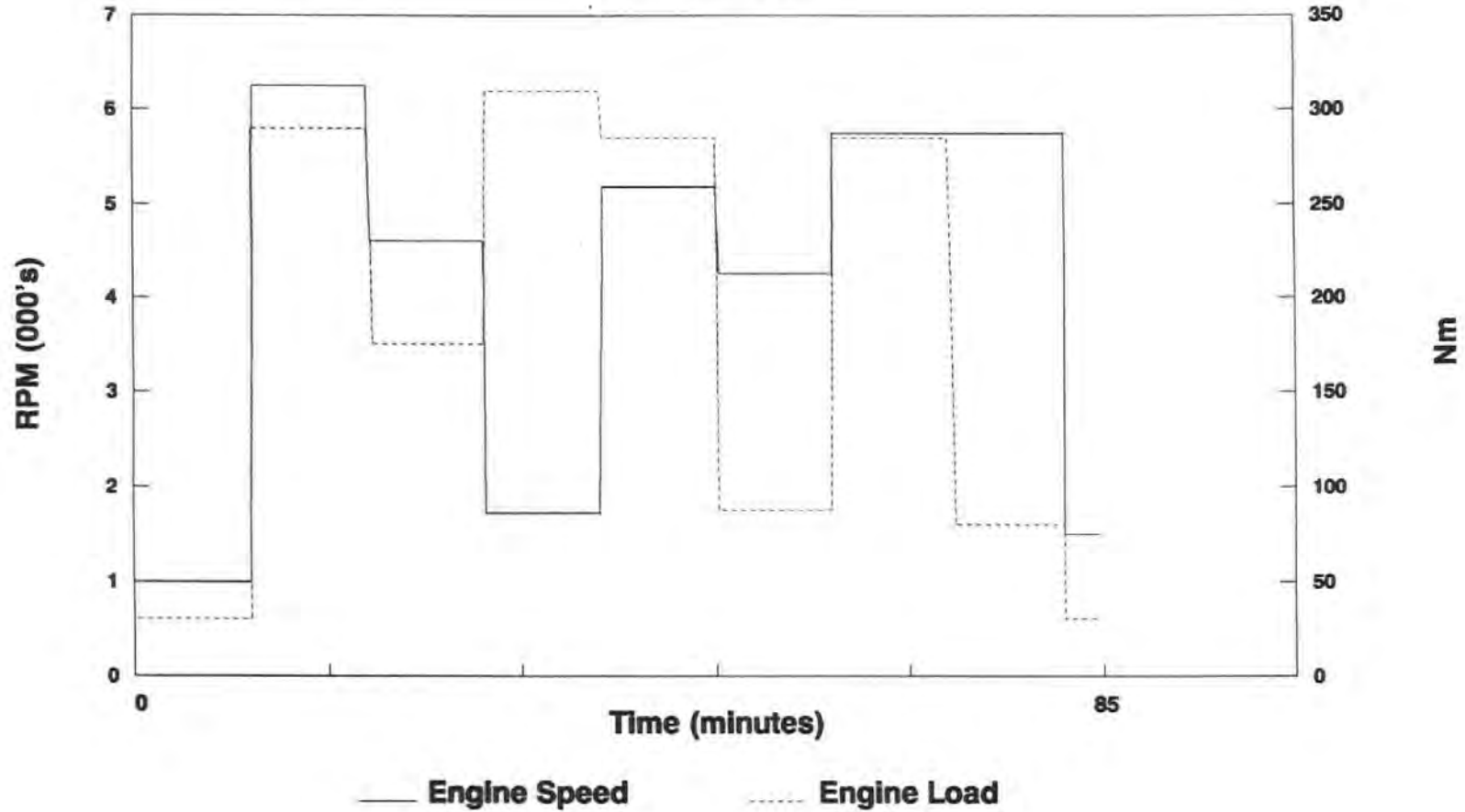
# Jaguar AJ26

**180hr High Speed**
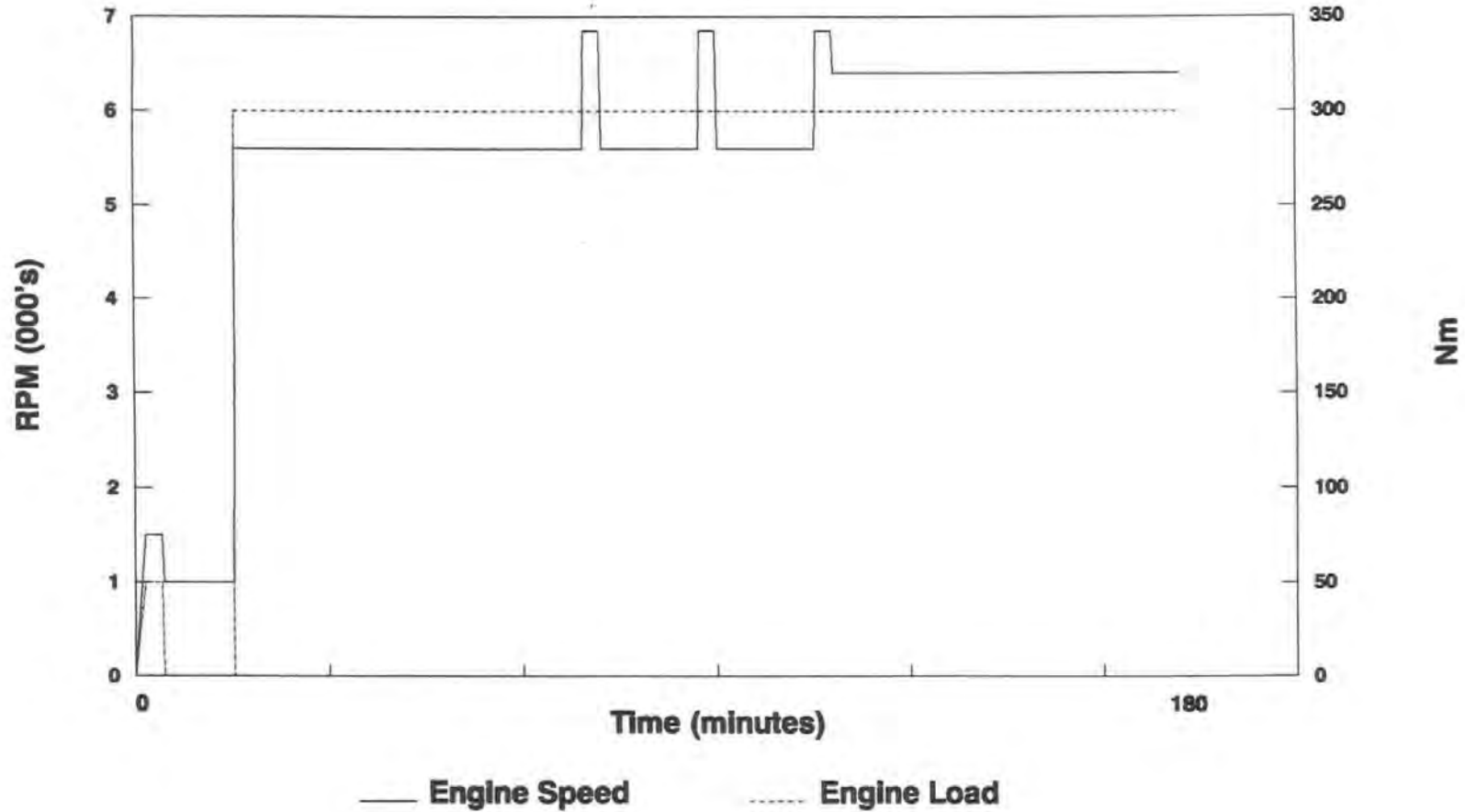
# Jaguar AJ26

**Nardo Simulation Cycle**
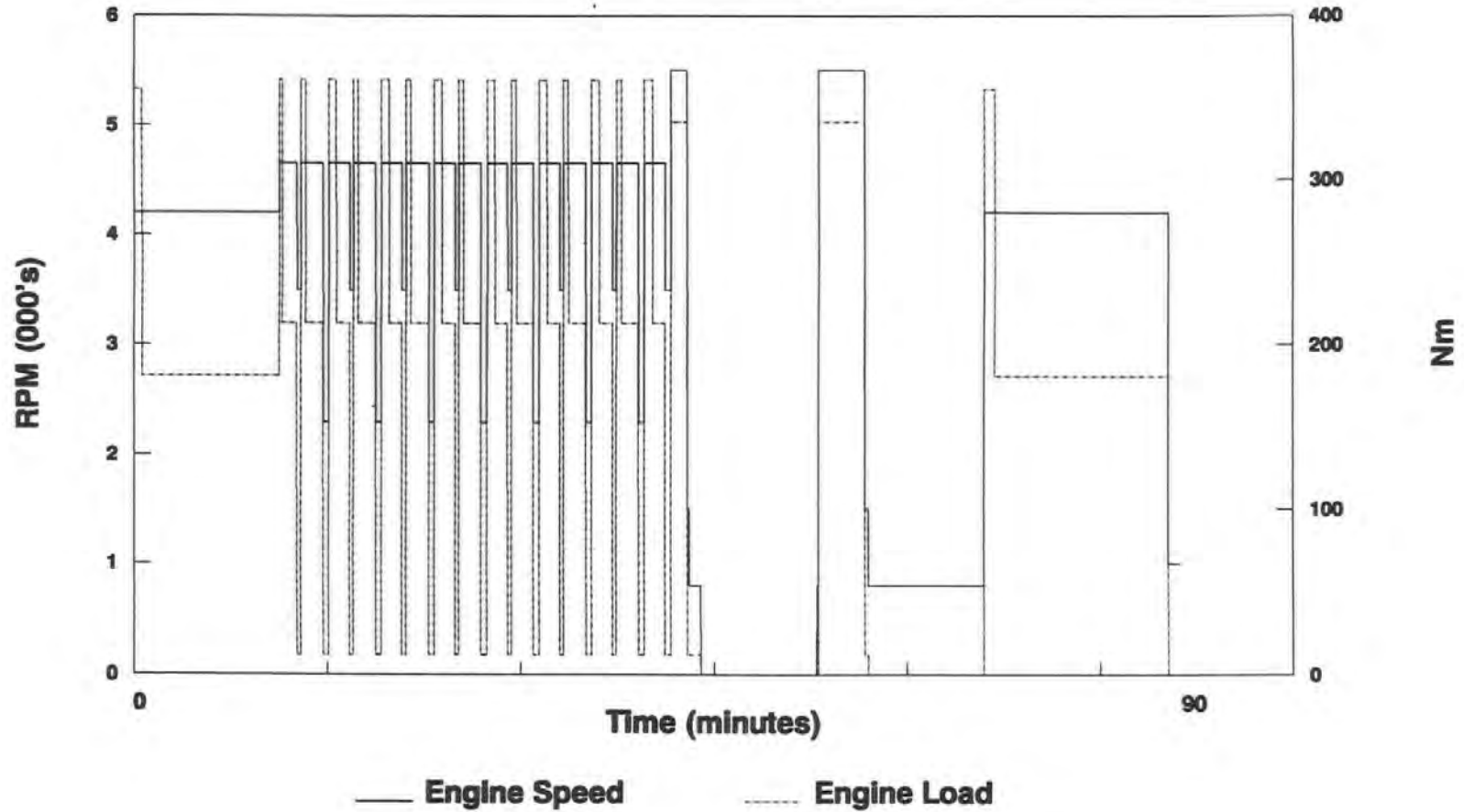
# Jaguar AJ26

**Hot and Cold Module 1: Gasket Initiation**
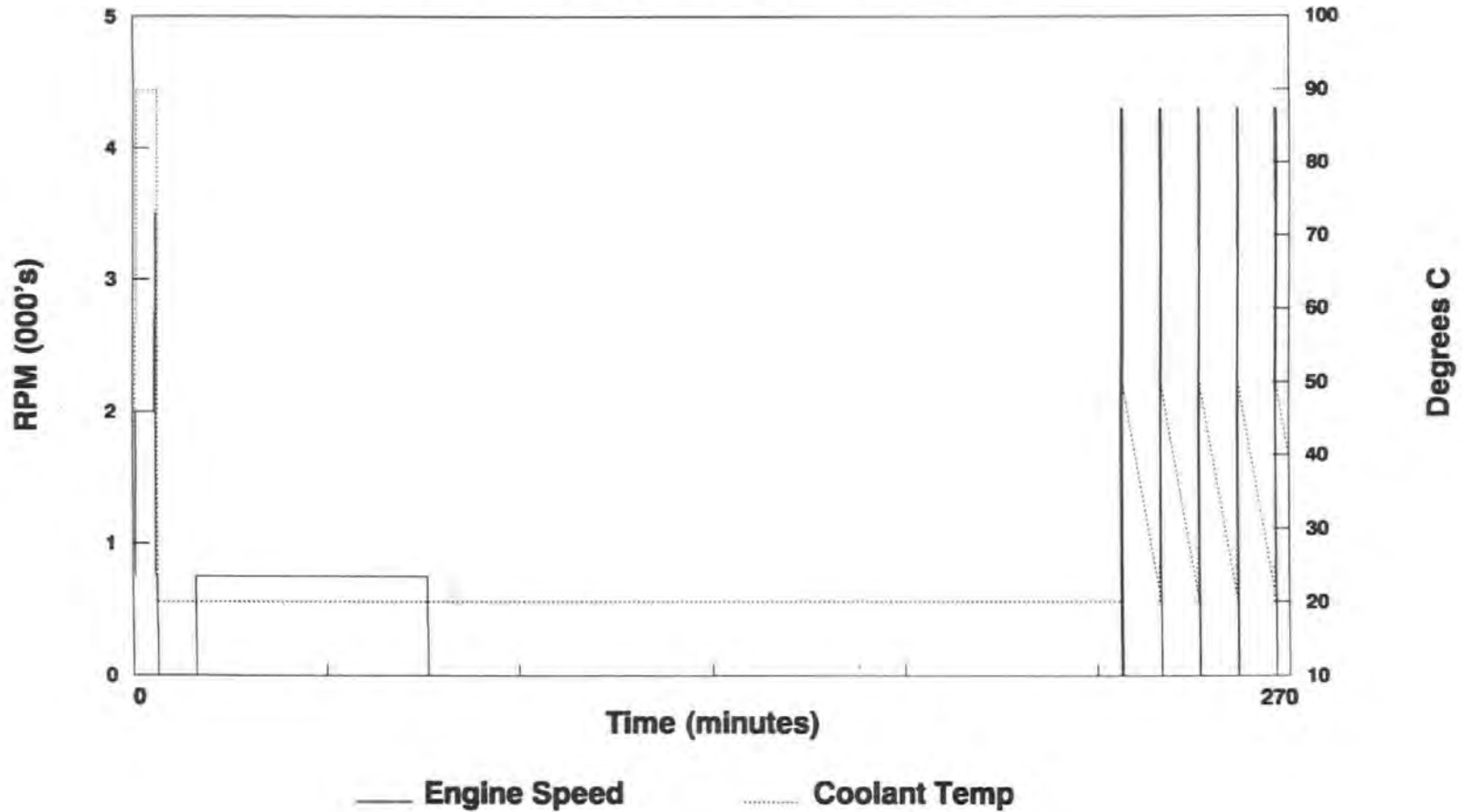
# Jaguar AJ26

**Hot and Cold Module 2: Leakage Test**
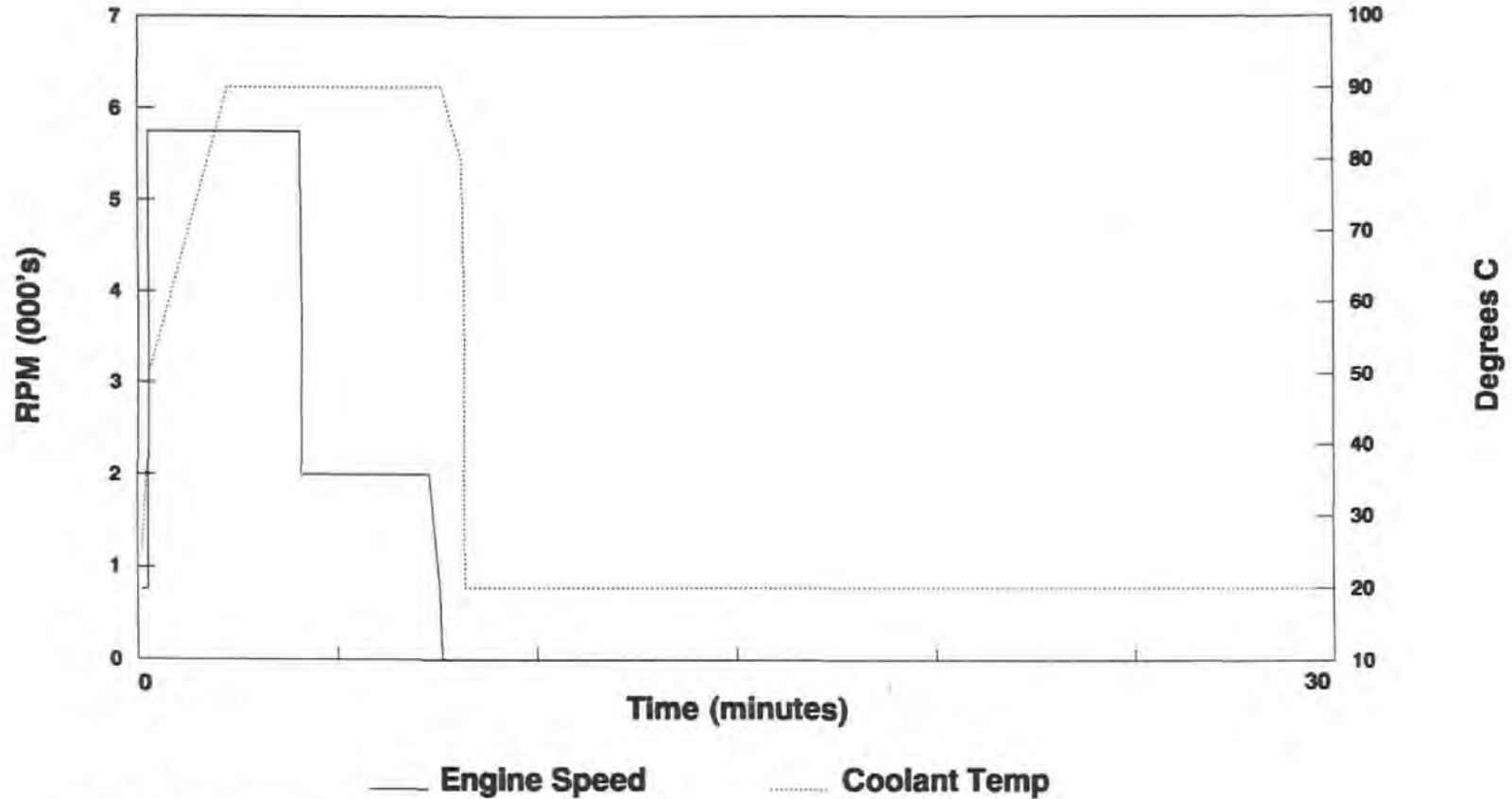


Repeated 20x (each cycle 30 mins)

# Jaguar AJ26

## Hot and Cold Module 3: Power Curve
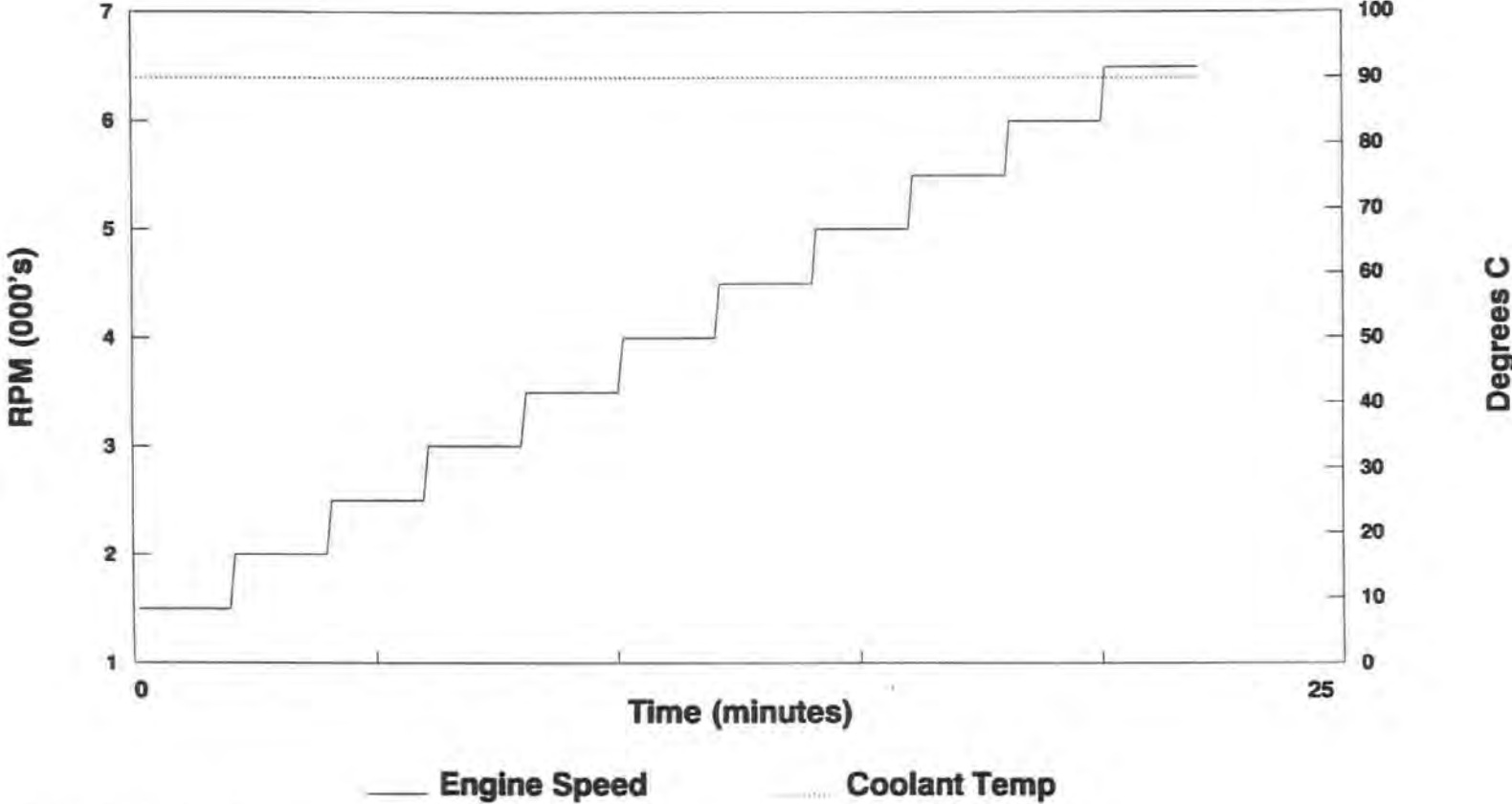


Engine at max load

# Jaguar AJ26

## Hot and Cold Module 4: Ice Start
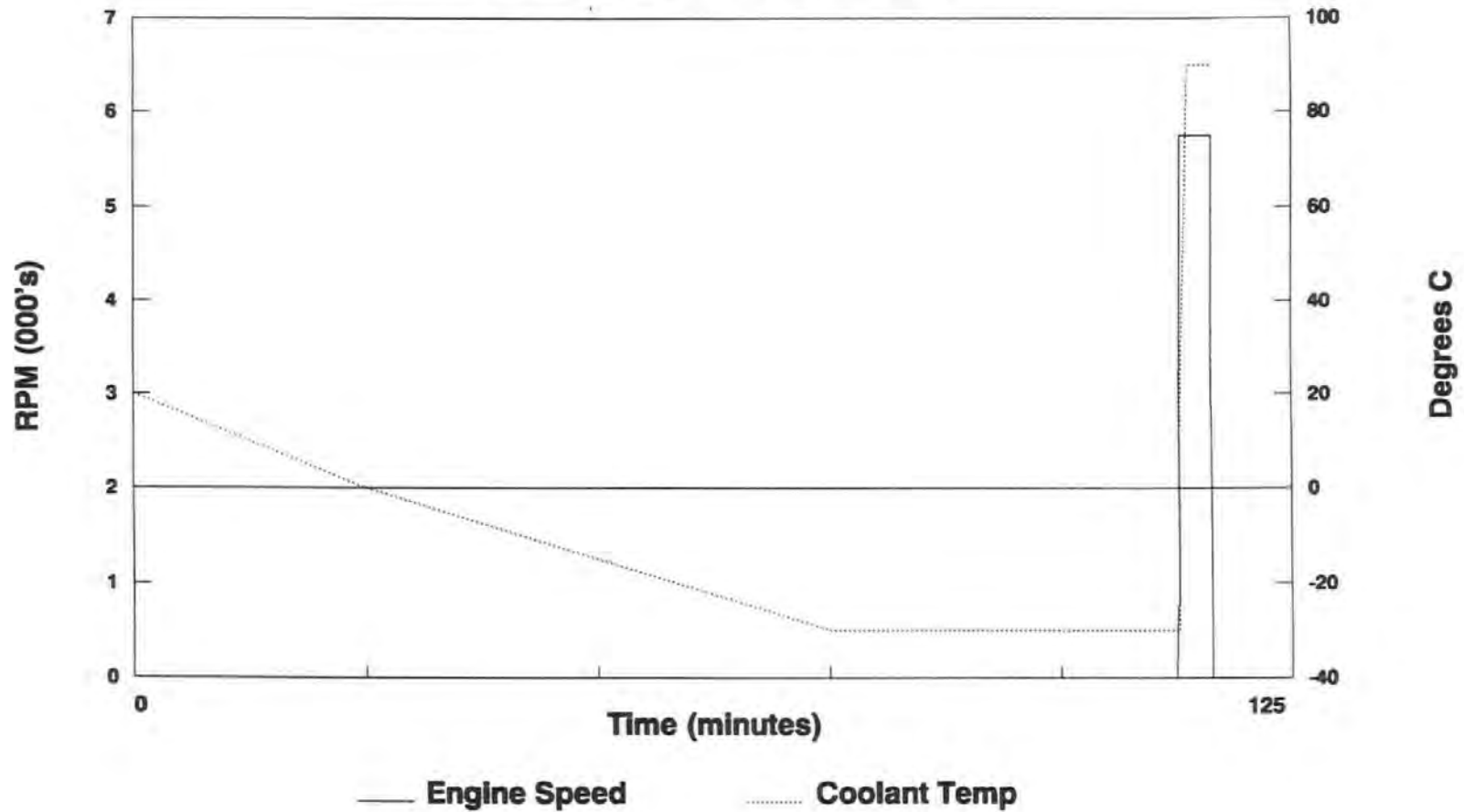
# Jaguar AJ26

## Hot and Cold Module 5: Hot Test
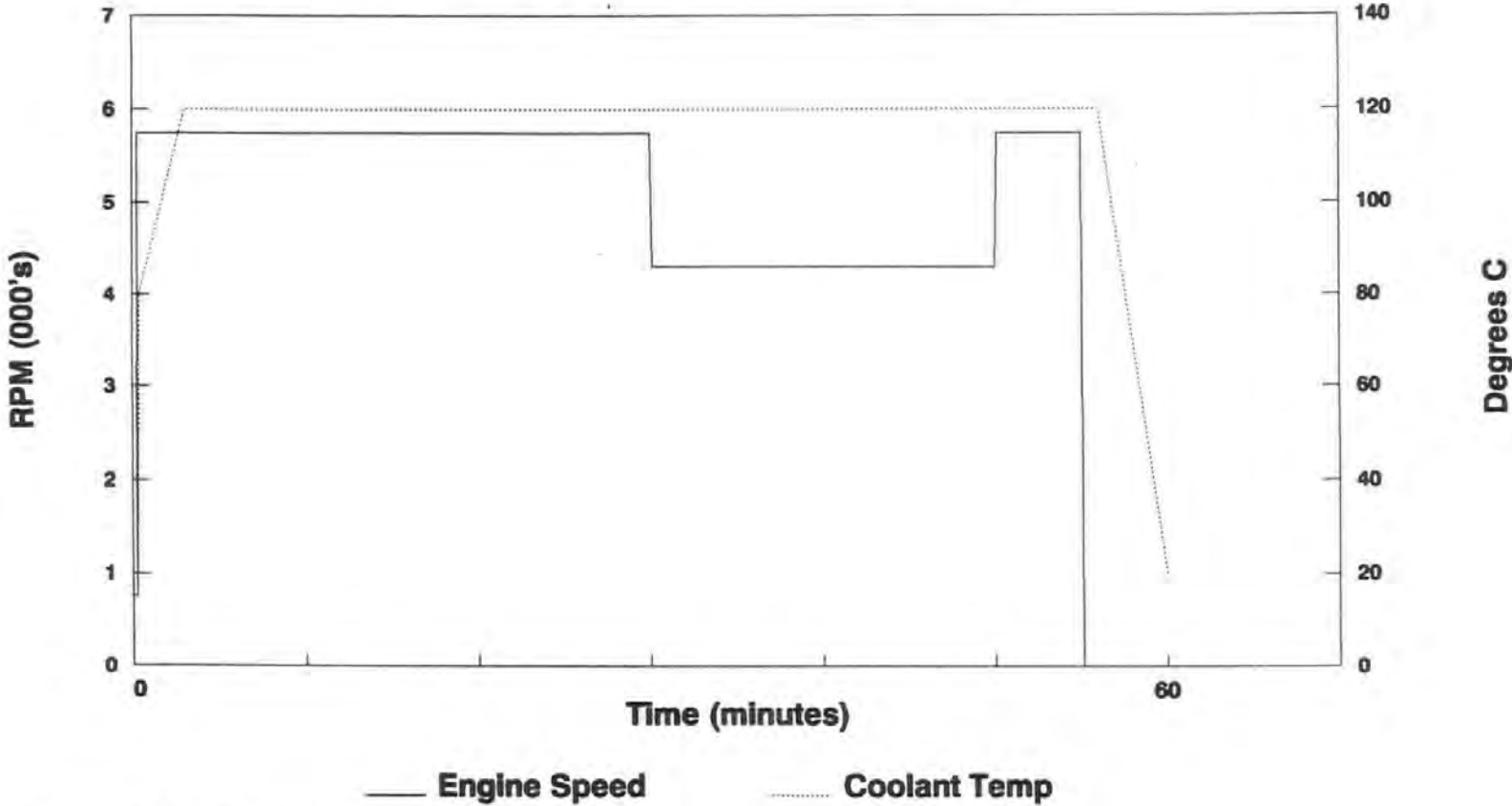
# Jaguar AJ26

**Hot and Cold Module 6: Deep Thermal Shock**

RPM (000's) — vertical axis: 0, 1, 2, 3, 4, 5, 6, 7

Degrees C — vertical axis: -50, 0, 50, 100, 150

Time (minutes): 0 ... 51

x4

x2

——— Engine Speed ............ Coolant Temp
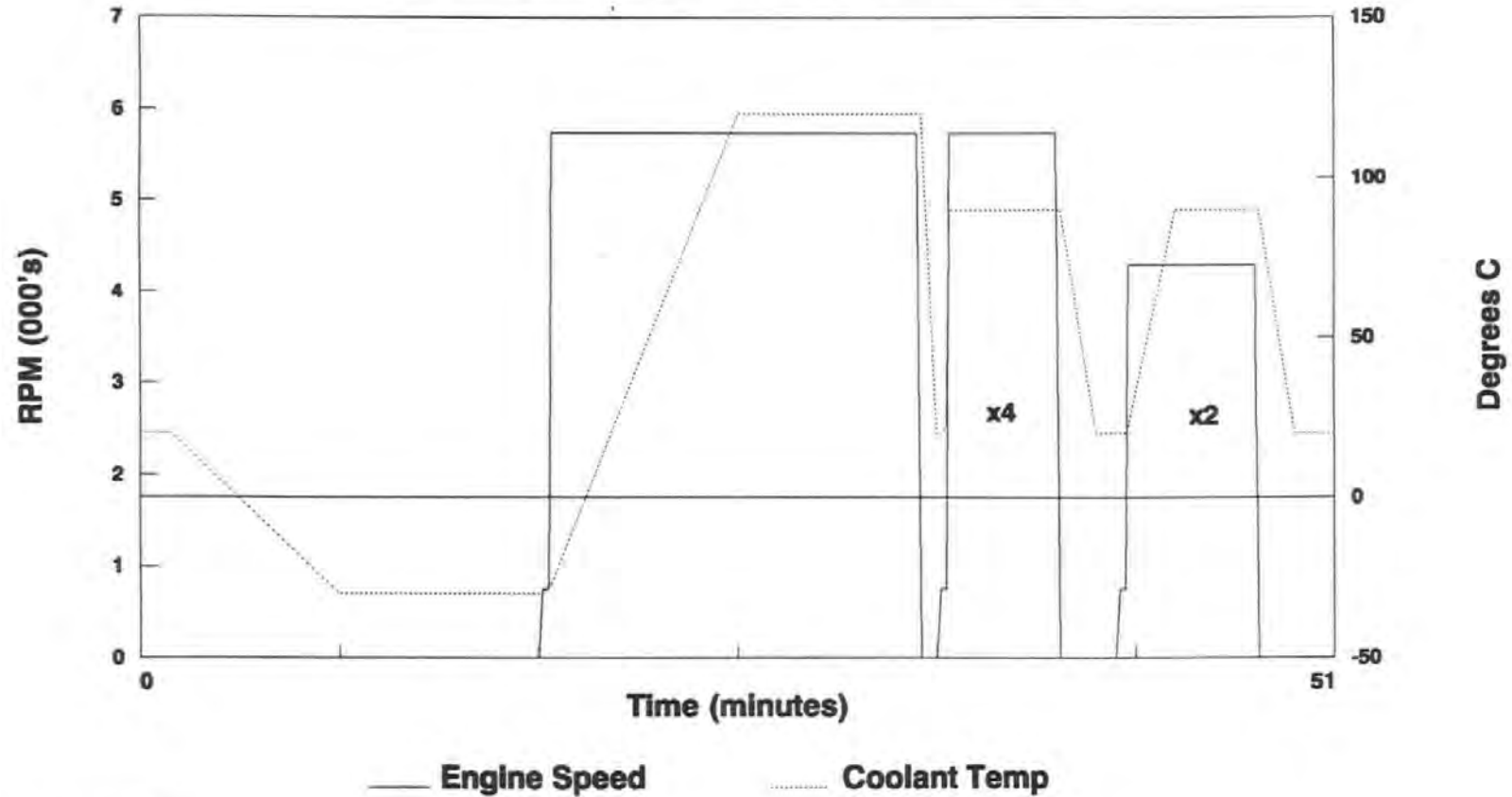
Repeated 80x

# APPENDIX 2

# Jaguar Engine Test Data (Coded)

Censor:     0 not censored      1 censored.

test:      0 low test level      1 high test level.

## Phase A0

| Failure | Engine 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.01 | 0.005 | 0.00025 | 0.01656 | 0.00253 | 0.003 | 0.05514 | 0.0002 |
| 2 | 0.047 | 0.01 | 0.0005 | 0.035 | 0.005 | 0.03475 | 0.0636 | 0.0029 |
| 3 | 0.07464 | 0.03734 | 0.00259 | 0.069 | 0.0108 | 0.15 | 0.17994 | 0.0062 |
| 4 | 0.07964 | 0.09622 | 0.012 | | 0.0638 | 0.38762 | 0.18 | 0.0115 |
| 5 | 0.08164 | 0.1013 | 0.01592 | | 0.5408 | | 0.23784 | 0.01883 |
| 6 | 0.10625 | 0.11547 | 0.018 | | 0.54585 | | 0.31877 | 0.02552 |
| 7 | 0.11514 | 0.12074 | 0.029 | | | | 0.34134 | 0.0337 |
| 8 | 0.1187 | 0.14226 | 0.06159 | | | | 0.45463 | 0.04053 |
| 9 | 0.19294 | 0.15201 | 0.10918 | | | | | |
| 10 | | 0.15774 | 0.11201 | | | | | |
| 11 | | 0.212 | | | | | | |
| 12 | | 0.2625 | | | | | | |
| 13 | | 0.275 | | | | | | |
| 14 | | 0.303 | | | | | | |
| 15 | | 0.4245 | | | | | | |
| Censor | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 |
| Test | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 1 |

| Failure | Engine 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|
| 1 | 0.006 | 0.00025 | 0.006 | 0.195 | 0.0005 | 0.008 | 0.01335 |
| 2 | 0.00988 | 0.00718 | 0.01942 | | 0.005 | 0.068 | 0.021 |
| 3 | 0.075 | 0.07362 | 0.02605 | | 0.05517 | 0.15574 | 0.07221 |
| 4 | | 0.11723 | 0.05497 | | 0.07635 | 0.1749 | 0.09244 |
| 5 | | | 0.062 | | 0.0773 | 0.2478 | 0.09297 |
| 6 | | | 0.142 | | 0.07922 | 0.39309 | |
| 7 | | | 0.14436 | | 0.10623 | 0.4836 | |
| 8 | | | | | 0.15 | 0.4876 | |
| 9 | | | | | 0.2246 | | |
| 10 | | | | | 0.305 | | |
| 11 | | | | | | | |
| 12 | | | | | | | |
| 13 | | | | | | | |
| 14 | | | | | | | |
| 15 | | | | | | | |
| Censor | 0 | 0 | 1 | 1 | 1 | 1 | 0 |
| Test | 1 | 1 | 0 | 1 | 1 | 0 | 0 |

## Phase A1-1

| Failure | Engine 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.0005 | 0.025 | 0.004 | 0.0181 | 0.01 | 0.012 | 0.00542 | 0.04507 | 0.04345 |
| 2 | 0.019 | 0.055 | 0.01929 | 0.043 | 0.013 | 0.013 | 0.027 | 0.0612 | 0.06945 |
| 3 | 0.022 | 0.063 | 0.06 | 0.1115 | 0.17437 | 0.015 | 0.05067 | 0.06927 | 0.1375 |
| 4 | 0.046 | | 0.11 | 0.168 | 0.218 | 0.0515 | | 0.08495 | 0.1825 |
| 5 | 0.052 | | 0.135 | 0.284 | 0.3 | 0.067 | | 0.08847 | |
| 6 | 0.0705 | | 0.16 | 0.335 | 0.351 | 0.085 | | 0.09765 | |
| 7 | 0.09 | | 0.207 | | 0.37015 | 0.223 | | 0.11627 | |
| 8 | 0.1125 | | 0.217 | | 0.488 | 0.348 | | | |
| 9 | 0.1245 | | 0.233 | | 0.49844 | 0.369 | | | |
| 10 | 0.1315 | | 0.277 | | | 0.407 | | | |
| 11 | 0.1345 | | 0.312 | | | 0.417 | | | |
| 12 | 0.1915 | | 0.323 | | | 0.54 | | | |
| 13 | 0.1995 | | 0.363 | | | 0.575 | | | |
| 14 | | | 0.373 | | | 0.6 | | | |
| 15 | | | 0.425 | | | | | | |
| 16 | | | 0.4275 | | | | | | |
| 17 | | | 0.43 | | | | | | |
| Censor | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 |
| Test | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 0 |

## Phase A1-2

| Failure | Engine 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.075 | 0.0245 | 0.04336 | 0.09806 | 0.102 | 0.001 | 0.005 | 0.06189 | 0.181 |
| 2 | 0.093 | 0.044 | 0.44893 | 0.111 | 0.195 | 0.016 | 0.056833 | 0.13172 | 0.3 |
| 3 | 0.10225 | 0.101 | | 0.116 | 0.216 | 0.0165 | 0.092 | 0.25169 | 0.39248 |
| 4 | 0.10258 | 0.14347 | | 0.196 | 0.3 | 0.021 | 0.09275 | 0.2931 | |
| 5 | 0.25001 | 0.14681 | | 0.206 | 0.375 | 0.03355 | 0.1035 | 0.30783 | |
| 6 | | 0.148 | | 0.298 | 0.41 | 0.04262 | | 0.455 | |
| 7 | | 0.15619 | | | 0.418 | | | | |
| 8 | | 0.16068 | | | 0.458 | | | | |
| 9 | | 0.16221 | | | 0.58 | | | | |
| 10 | | 0.176 | | | 0.6 | | | | |
| 11 | | 0.241 | | | | | | | |
| Censor | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 1 |
| Test | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 |

# Phase A2-1

| Failure | | | | Engine | | | |
|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 1 | 0.005 | 0.007 | 0.001 | 0.002 | 0.018 | 0.037 | 0.01 |
| 2 | 0.025 | 0.089 | 0.035 | 0.004 | 0.025 | 0.1871 | 0.028 |
| 3 | 0.057 | 0.207 | 0.2 | 0.005 | 0.03 | 0.5281 | 0.072 |
| 4 | 0.168 | 0.252 | 0.417 | 0.01 | 0.05675 | 0.5731 | 0.124 |
| 5 | 0.191 | 0.368 | | 0.011 | | 0.6 | 0.14 |
| 6 | 0.234 | | | 0.0335 | | | 0.214 |
| 7 | 0.259 | | | 0.0515 | | | 0.305 |
| 8 | 0.404 | | | 0.107 | | | 0.362 |
| 9 | 0.407 | | | 0.13 | | | 0.484 |
| 10 | 0.6 | | | 0.201 | | | 0.6 |
| 11 | | | | 0.305 | | | |
| 12 | | | | 0.501 | | | |
| 13 | | | | 0.523 | | | |
| 14 | | | | 0.6 | | | |
| Censor | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| Test | 0 | 1 | 1 | 0 | 0 | 1 | 1 |

| Failure | | | | Engine | | | |
|---|---|---|---|---|---|---|---|
| | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| 1 | 0.082 | 0.1556 | 0.029 | 0.008 | 0.016 | 0.021 | 0.01 |
| 2 | 0.159 | 0.2095 | 0.097 | 0.01 | 0.07 | 0.026 | 0.012 |
| 3 | | 0.2265 | 0.24 | 0.034 | 0.114 | 0.055 | 0.018 |
| 4 | | 0.2845 | 0.371 | 0.042 | | 0.076 | 0.028 |
| 5 | | 0.3455 | 0.449 | 0.049 | | 0.081 | 0.033 |
| 6 | | 0.3605 | 0.455 | | | 0.083 | 0.04 |
| 7 | | 0.3845 | | | | | 0.046 |
| 8 | | 0.394 | | | | | |
| 9 | | 0.4 | | | | | |
| 10 | | 0.451 | | | | | |
| 11 | | 0.4565 | | | | | |
| 12 | | 0.5165 | | | | | |
| 13 | | 0.6 | | | | | |
| 14 | | | | | | | |
| Censor | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| Test | 0 | 0 | 1 | 1 | 1 | 1 | 1 |

| Failure | | | | Engine | | | |
|---|---|---|---|---|---|---|---|
| | 15 | 16 | 17 | 18 | 19 | 20 | 21 |
| 1 | 0.007 | 0.005 | 0.005 | 0.151 | 0.039 | 0.25 | 0.043 |
| 2 | 0.009 | 0.021 | 0.019 | 0.176 | 0.07 | 0.328 | 0.06 |
| 3 | 0.055 | 0.041 | 0.145 | 0.184 | 0.214 | 0.487 | 0.13 |
| 4 | 0.06 | 0.09 | 0.147 | 0.232 | 0.326 | 0.597 | |
| 5 | 0.091 | | 0.172 | | 0.43 | 0.6 | |
| 6 | 0.105 | | | | 0.6 | | |
| 7 | 0.111 | | | | | | |
| 8 | 0.189 | | | | | | |
| 9 | 0.225 | | | | | | |
| 10 | 0.243 | | | | | | |
| 11 | | | | | | | |
| 12 | | | | | | | |
| 13 | | | | | | | |
| 14 | | | | | | | |
| Censor | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| Test | 1 | 1 | 1 | 1 | 0 | 1 | 1 |

# APPENDIX 3

# Fortran Program for a Proportional

# Hazards Models

```
              program hazard

c
c   This program will calculate a proportional hazard model for
c   the inputed data.
c
c   The data needs to be of the form:
c                1)  First column with survival/censored times.
c                2)  Second column with status code
c                        0 censored 1 non-censored.
c                3)  next n columns containing n covariates.
c                4)  the data must be in ascending order.
c                5)  the file length and the no. of covariates must
c                    be specified.
c
c   Assign Arrays
         implicit real*8 (a-h,o-z)
         character*64 cin,cout
         integer istat(200),ncount,id(200),np(20)
         dimension time(200),data(200,20),cova(200,20)
         dimension work(1800),est(20),start(20)
         common/ar/istat,time,cova,id
         common m,ncount,n
         external fcn

         ncount=0
c
c   Input summary information
c
         print*,'This program will calculate a proportional Hazards model'
         print*,'for up to 20 covariates with 1000 data rows'
         write(*, '(A/)') 'Please type the input file name'
         read(*, '(A)') cin
         write(*, '(A/)') 'Please type the output file name'
         read(*, '(A)') cout
         print*,'How many covariates?'
         read*,nit
         print*,'How many Rows?'
         read*,m
c
c   Define the input and output files
c
         open(unit=7,file=cin,status='unknown')
         open(unit=8,file=cout,status='unknown')
c
c   Read in the data file
c
         do 1,i=1,m
         read(7,*) time(i),istat(i),(data(i,j),j=1,nit)
    1    continue
c
c       calculate id array for ties
c
         mm=m-1
         do 2,i=1,mm
         id(i)=-1
    2    continue
```

```
            id(m)=1
            do 51,i=1,mm
            if (id(i).ne.0) then
                id(i)=1
                if (istat(i).eq.1) then
                    ii=i+1
                    do 52,j=ii,m
                    if (time(i).eq.time(j).AND.istat(j).eq.1) then
                        id(i)=id(i)+1
                        id(j)=0
                    else
                        goto 51
                    end if
52                  continue
                end if
            end if
51          continue

777         print*,'How many parameters to be removed from the data'
            read*,nremov
            if (nremov.eq.0) goto 90
            print*,'Which covariate numbers?'
            do 91,k=1,nremov
                read*,np(k)
91          continue

90          nsub=0
            do 92,nrem=1,nit
                kremov=0
                do 93,ncheck=1,nremov
                    if (nrem.eq.np(ncheck)) kremov=1
93              continue
                if (kremov.eq.0) then
                    do 94,nj=1,m
                        cova(nj,(nrem-nsub))=data(nj,nrem)
94                  continue
                else
                    nsub=nsub+1
                end if
92          continue

            n=nit-nremov

            if (n.ne.0) then
c
c   Read in the initial starting values
c
            do 3,i=1,n
            print*,'Please type the starting value for parameter ',i
            read*,start(i)
  3         continue
            tol= sqrt(x02aaf(0.0))
            lwa=1800
            ifail=0
            call c05nbf(fcn,n,start,est,tol,work,lwa,ifail)

            else
                start(1)=0
                do 800,i=1,m
                    cova(m,1)=data(m,1)
800             continue
            end if
```

190

```
      vlik=0
      do 401,nrow=1,m
         if (istat(nrow).ne.0.AND.id(nrow).ne.0) then
            call llike(nrow,start,a,b)
            vlik=vlik+(a-b)
         end if
  401 continue


      write(8,1000)

      do 998,k=1,n
      write(8,1001) k,start(k)
  998 continue

      write(8,1002)

      write(8,1003) vlik
      write(8,1004) nremov
      write(8,1005)

      do 888,i=1,nremov
         write(8,*) np(i)
  888 continue

      print*,'Type 1 to run for different parameters'
      read*,ncheck
      if (ncheck.eq.1) goto 777

 1000 format (///,10X,'Proportional Hazards Model Results',//
     * ,5X,'Final Parameter Estimates',/)
 1001 format (2X,'Parameter ',i2,5X,'Program estimate = ',e14.6)
 1002 format (//,5X,'The Log-Likelihood is evaluated as:',/)
 1003 format (2X,'l =',3x,e14.6,//)
 1004 format ('Number of Covariates removed=',5X,i2,//)
 1005 format ('The Covariates removed were:',/)

      stop
      end

c
*********************************************************************
c  Subroutine called from within the NAg routine to evaluate the partial
c  derivatives at the current estimates.
c
c  n is the number of parameters/partial derivatives
c  x is the vector containing the current parameter estimates
c  fvec is the vector containing the evaluated f(x)'s on exit
c  iflag can be set to a neg value in order to exit NAg routine (not
used)
c
*********************************************************************

      subroutine fcn(nn,x,fvec,iflag)
      implicit real*8 (a-h,o-z)
      integer nn,iflag,istat(200),m,id(200)
      dimension time(200),cova(200,20),x(nn),fvec(nn)

      common/ar/istat,time,cova,id
      common m,ncount,n
      ncount=ncount+1
```

```
c
c   This first loop defines each of the derivatives w.r.t i
c
      do 101,ii=1,nn
         fvec(ii)=0
c
c   This second loop goes through the rows of the covariate matrix as long
c   as the observation is non-censored
c
         do 201,jj=1,m
           if (istat(jj).eq.1.AND.id(jj).ne.0) then
              call risk(ii,jj,x,nn,a,b,c)
              fvec(ii)=fvec(ii)+(a-(b/c))
           end if
 201     continue
 101  continue

      return
      end


c
************************************************************************
c This subroutine is called from the fcn subroutine.
c It calculates and sums the risk sets, required in the partial
derivatives,
c for the current observation in fcn
c
c p gives the vector identifier for the current partial derivative
c srow is the row identifier for the current observation in fcn
c y is the vector of current parameter estimates
c l is the number of parameters/derivatives
c On exit:
c    s contains x(i,p)
c    t contains sum x(l,p)*exp(xl*b)
c    u contains sum exp(xl*b)
c
************************************************************************

      subroutine risk(p,srow,y,l,s,t,u)
      implicit real*8 (a-h,o-z)
      integer p,srow,l
      integer istat(200),m,id(200)
      dimension time(200),cova(200,20),y(l)

      common/ar/istat,time,cova,id
      common m,ncount,n

      s=0
      do 302,isum=1,id(srow)
      icorr=srow+isum-1
      s=s+cova(icorr,p)
 302  continue
      t=0
      u=0
c     sum2=0

      do 102,jjj=srow,m
      sum2=0
      do 202,iii=1,l
      sum1=cova(jjj,iii)*y(iii)
```

192

```
      sum2=sum2+sum1
202   continue

      u=u+exp(sum2)
      t=t+(cova(jjj,p)*exp(sum2))

102   continue

      t=t*id(srow)

      return
      end

      subroutine llike(nrow,param,a,b)
      implicit real*8 (a-h,o-z)
      integer nrow
      integer istat(200),m,id(200)
      dimension time(200),cova(200,20),s(20),param(20)

      common/ar/istat,time,cova,id
      common m,ncount,n

      do 403,icount=1,n
      s(icount)=0
403   continue

      a=0
      b=0

      do 404,icount=1,n
         do 405,mcount=1,id(nrow)
            itemp=nrow-1+mcount
            s(icount)=s(icount)+cova(itemp,icount)
405      continue
404   continue

      do 406,icount=1,n
         a=a+(s(icount)*param(icount))
406   continue

      do 407,mcount=nrow,m
         tcalc=1
         do 408,icount=1,n
            tcalc=tcalc*exp(cova(mcount,icount)*param(icount))
408      continue
         b=b+tcalc
407   continue

      b=id(nrow)*log(b)

      return
      end
```

# APPENDIX 4

# S-Plus Functions

## Log-Likelihood for Proportional Random Effect Model

randb0

function(x, data, censor)

{

        int1 <- ncol(data)

\# Define empty matrix and parameter values from x

        mlike <- matrix(nrow = int1, ncol = 1)

        b0 <- x[1]

        b1 <- x[2]

        sdev <- x[3]

\# m1 is assigned the Gauss-Legendre Weights and Nodes for 5×Standard Deviations

        j <- sdev * 5

        m1 <- integ( - j, j, 20)

\# Loop to calculate values for each unit

        for(i in 1:int1) {

                ind <- censor[i]

                tdat <- c(na.omit(data[, i]))

                n <- length(tdat)

\# Calculates number of failures and sum of failures allowing for censored value

                ni <- n - ind

                v1 <- m1[, 2] * (exp((ni * m1[, 1]) - (1/b1) * exp(b0 + m1[, 1]) * (

                        exp(b1 * tdat[n]) - 1)) * dnorm(m1[, 1], mean = 0, sd = sdev))

                x <- sum(v1)

\# The Likelihood due to the $i^{th}$ engine

                mlike[i, 1] <- exp(ni * b0 + b1 * (sum(tdat) - (ind * tdat[n]))) * x

        }

        lmlike <- log(mlike)

        value <- -1 * sum(lmlike)

        value

}


## 1st and 2nd Partial Derivatives for Proportional Random Effect Model

\> randb0.hes

function(x, data, censor)

\# provides values of 1st derivatives and the upper triangle of the hessian for input into the S-Plus function *NLMINB*

{

        int1 <- ncol(data)

        sc <- 1e+100

        b0 <- x[1]

        b1 <- x[2]

```
sdev <- x[3]
j <- sdev * 5
j1 <- max(1, j)
ans <- matrix(nrow = int1, ncol = 9)
m1 <- integ( - j1, j1, 20)
for(i in 1:int1) {
        ind <- censor[i]
        tdat <- na.omit(data[, i])
        n <- length(tdat)
        ti <- tdat[n]
        t <- sum(tdat) - (ind * tdat[n])
        ni <- n - ind
```
# Calculates number of failures and sum of failures allowing for censored value
# See Equations (3.22) to (3.36) in Thesis
```
        aa <- exp(ni * b0 + b1 * t + ni * m1[, 1] - (1/b1) * exp(b0 + m1[, 1])
        *(exp(b1 * ti) - 1)) * dnorm(m1[, 1], mean = 0, sd = sdev)
        bb <- ni - (1/b1) * (exp(b0 + m1[, 1]) * (exp(b1 * ti) - 1))
        cc <- t - exp(b0 + m1[, 1]) * ((exp(b1 * ti) * (b1 * ti - 1) + 1)/b1^2)
        dd <- (m1[, 1]^2 - sdev^2)/sdev^3
        temp <- m1[, 2] * aa
        a <- sum(temp)
        temp <- m1[, 2] * bb * aa
        b <- sum(temp)
        temp <- m1[, 2] * cc * aa
        c1 <- sum(temp)
        temp <- m1[, 2] * dd * aa
        d <- sum(temp)
        temp <- m1[, 2] * ((bb - ni) + bb^2) * aa
        se1 <- (a * (sum(temp)/sc) - b * (b/sc))
        temp <- (m1[, 2] * ((cc - t) + (bb * cc)) * aa)
        se2 <- a * (sum(temp)/sc) - b * (c1/sc)
        temp <- (m1[, 2] * bb * dd * aa)
        se3 <- (a * (sum(temp)/sc) - b * (d/sc))
        temp <- (m1[, 2] * ((-1/b1^3) * exp(b0 + m1[, 1]) * (b1 * ti * exp(b1 * ti)
        * (ti * b1 - 2) + 2 * exp(b1 * ti) - 2) + cc^2) * aa)
        se4 <- (a * (sum(temp)/sc) - c1 * (c1/sc))
        temp <- m1[, 2] * cc * dd * aa
        se5 <- (a * (sum(temp)/sc) - c1 * (d/sc))
        temp <- m1[, 2] * ((m1[, 1]^4 - 5 * m1[, 1]^2 * sdev^2 + 2 * sdev^4)/
                sdev^6) * aa
        se6 <- (a * (sum(temp)/sc) - d * (d/sc))
```

```
                vec1 <- c(se1, se2, se3, se4, se5, se6)

                vec1 <- (vec1/(a * (a/sc)))

                vec2 <- (c(b, c1, d))/a

                ans[i, ] <- c(vec1, vec2)

        }

        info1 <- -1 * (c(sum(ans[, 7]), sum(ans[, 8]), sum(ans[, 9])))

        info2 <- -1 * (c(sum(ans[, 1]), sum(ans[, 2]), sum(ans[, 4]), sum(ans[, 3]),

                sum(ans[, 5]), sum(ans[, 6])))

        list(gradient = info1, hessian = info2)               }
```

<u>Simulation of Data from Proportional Random Effect Model</u>

```
randb0.sim

function(x, y, sdev, t, n)

#  Provides Simulated data for n units up to time t using the inversion method

{

#  Generate the values of the random component and hence the resulting intercept parameter
for each unit

        x1 <- rnorm(n, mean = 0, sd = sdev)

        x <- x + x1

#  Calculate the resulting mean failures in time t

        con <- exp(y * t) - 1

        mo <- (exp(x) * con)/y

#  Then generate a simulated number of failures using the Poisson distribution

        no <- rpois(n, mo)

        sum1 <- sum(no)

#  Set the empty matrix to store the simulated times

        data <- matrix(nrow = (max(no) + 1), ncol = n)

        censor <- matrix(1, nrow = 1, ncol = n)

        times <- matrix(nrow = (max(no) + 1), ncol = n)

#  Generate the required number of observations from a standard uniform distribution

        randno <- runif(sum1)

        count <- 1

#  Loop calculates simulated failure times for each unit

#  If no failures have been simulated for unit i then only the censored time is given

#  else the failure times are generated by the inversion method. See Section 3.6.2

        for(i in 1:n) {

                if(no[i] == 0) {

                        times[1, i] <- t

                }

                if(no[i] != 0) {

                        data[1:no[i], i] <- sort(randno[count:(count - 1 + no[i])], na.last = T)
```

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```
                    count <- count + no[i]
                    times[, i] <- log(data[, i] * con + 1)/y
                    times[(no[i] + 1), i] <- t
            }
    }
    list(dat = times, cen = censor)
}
```

## Functions for Marginal Residuals from Proportional Random Effect Model

```
margres
function(x, data, cen, model, divs = seq(0, 600, length = 13), sc = 1)
{
#  calculate the actual number of failures in each interval
        faildat <- resd(data, cen, divs)
        mfail <- apply(faildat, 1, mean, na.rm = T)
        #  calculate the expected number of failures in each interval
        #  and the intervals due to the random effects
        if(model == 1) {
                efails <- expt(divs/sc, x[1], x[2], x[3])
        }
        if(model == 2) {
                efails <- expt1(divs/sc, x[1], x[2], x[3])
        }
#  calculate the mid-points of the intervals
        n <- length(divs)
        divs2 <- matrix(ncol = 1, nrow = (n - 1))
        for(i in 1:(n - 1)) {
                divs2[i, 1] <- (divs[i] + divs[i + 1])/2
        }
        res <- mfail - efails[, 1]
        list(mid = divs2[, 1], observed = mfail, expected = efails, residual = res)
}
resd
function(x, z, y)
{
#  Caluates the actual number of failures in division z for the data in x and the censoring codes
in y
        t <- ncol(x)
        temp <- matrix(ncol = t, nrow = (length(y) - 1))
        for(i in 1:t) {
                temp1 <- na.omit(x[, i])
```

```
                    n <- length(temp1)
                    m <- temp1[n]
                    if(z[i] == 1) {
                            temp1[n] <- NA
                            temp1 <- na.omit(temp1)
                    }
                    if(length(temp1) != 0) {
                            temp2 <- hist(temp1, breaks = y, plot = F)
                            temp[, i] <- temp2$counts
                            n <- length(y)
                            for(j in 1:(n - 1)) {
                                    if(y[j] > m) {
                                      temp[j, i] <- NA
                                    }
                            }
                    }
            }
            temp
}




expt
function(x, b0, b1, sdev)
{
# Calculates the expected number of failures to time t giving the mean value and ± 1.96
*the estimate of the standard deviation
        t <- length(x)
        temp <- matrix(ncol = 3, nrow = (t - 1))
        for(i in 1:(t - 1)) {
                temp[i, 1] <- 1/b1 * exp(b0) * (exp(b1 * x[i + 1]) - exp(b1 * x[i]))
                b <- b0 - 1.96 * sdev
                temp[i, 2] <- 1/b1 * exp(b) * (exp(b1 * x[i + 1]) - exp(b1 * x[i]))
                b <- b0 + 1.96 * sdev
                temp[i, 3] <- 1/b1 * exp(b) * (exp(b1 * x[i + 1]) - exp(b1 * x[i]))
        }
        temp
}
```

## Estimate of Random Coefficients for Proportional Random Effect Model

randcpb0

function(x, data, censor)

# Calculates estimates of the random coefficients using the posterior distribution

# See Equations (5.9) and (5.10)

```
{
        int1 <- ncol(data)
        comp <- matrix(nrow = int1, ncol = 1)
        b0 <- x[1]
        b1 <- x[2]
        sdev <- x[3]
        j <- sdev * 5
        j1 <- max(1, j)
        m1 <- integ( - j1, j1, 20)
        for(i in 1:int1) {
                ind <- censor[i]
                tdat <- c(na.omit(data[, i]))
                n <- length(tdat)
                ni <- n - ind
                v1 <- m1[, 2] * (exp((ni * m1[, 1]) - (1/b1) * exp(b0 + m1[, 1]) *
                (exp(b1 * tdat[n]) - 1)) * dnorm(m1[, 1], mean = 0, sd = sdev))
                x1 <- sum(v1)
                x2 <- sum(m1[, 1] * v1)
                comp[i, 1] <- x2/x1
        }
        list(comp = comp)
}
```

## Conditional Residuals for Proportional Random Effect Model

condrsb0

function(x, data, censor, rcomp)

# Uses estimates of random coefficients to calculate conditional residuals

```
{
        int1 <- ncol(data)
        int2 <- nrow(data)
        resids <- matrix(nrow = int2, ncol = int1)
        b0 <- x[1]
        b1 <- x[2]
        for(i in 1:int1) {
                ind <- censor[i]
                tdat <- c(na.omit(data[, i]))
```

```
                    n <- length(tdat)
                    ni <- n - ind
                    dat1 <- tdat[1:ni]
                    dat2 <- c(0, dat1)
                    b0e <- b0 + rcomp[i, 1]
                    if(ni >= 1) {
                            for(j in 1:ni) {
# Residual from definition in equation (5.3)
                                    resids[j, i] <- 1/b1 * exp(b0e) * (exp(b1 * dat2[j + 1])
                                    - exp(b1 * dat2[j]))
                            }
                    }
            }
# Outputs Residuals by unit and also all residuals in a single column
            resids2 <- matrix(resids, ncol = 1)
            resids3 <- c(na.omit(resids2))
            list(R1 = resids, R2 = resids3)
}


Composite Gauss-Legendre Integration
integ
function(a, b, p)
{
# Nodes and Weights for Interval -1 to 1
            pnode <- c(-0.861136311594053, -0.339981043584856, 0.339981043584856,
                    0.861136311594053)
            pwei <- c(0.652145154862546, 0.347854845137454)
            ans <- matrix(nrow = (4 * p), ncol = 2)
            node <- rep(pnode, p)
# Calculate Weights for the New Interval size
            pw <- pwei * ((b - a)/(2 * p))
            ans[, 2] <- rep(c(pw[2], pw[1], pw[1], pw[2]), p)
            i <- 0:(p - 1)
# Calculation of nodes for the Each New Interval Size
            x1 <- a + ((b - a) * i)/p
            x2 <- x1 + (b - a)/p
            x1a <- rep(x1, rep(4, p))
            x2a <- rep(x2, rep(4, p))
            y <- (x1a + x2a) + (x2a - x1a) * node
            ans[, 1] <- y/2
            ans      }
```

# APPENDIX 5


# Partial Derivatives

# Model with a Single Random Effect

For time dependent random effect substitute $\xi$ for $\varepsilon$.

## First Order Partial Derivatives

$$\frac{\partial \ell}{\partial \beta_0} = \sum_{i=1}^{m} \frac{1}{A_i} \int_{-\infty}^{\infty} B_i z_i \, d\varepsilon_i$$

$$\frac{\partial \ell}{\partial \beta_1} = \sum_{i=1}^{m} \frac{1}{A_i} \int_{-\infty}^{\infty} C_i z_i \, d\varepsilon_i$$

$$\frac{\partial \ell}{\partial \varphi_\varepsilon} = \sum_{i=1}^{m} \frac{1}{A_i} \int_{-\infty}^{\infty} D_i z_i \, d\varepsilon_i$$

## Second Order Partial Derivatives

$$\frac{\partial^2 \ell}{\partial \beta_0{}^2} = \sum_{i=1}^{m} \frac{1}{A_i{}^2} \left\{ A_i \int_{-\infty}^{\infty} (B_i - n_i + B_i{}^2) z_i \, d\varepsilon_i - \left( \int_{-\infty}^{\infty} B_i z_i \, d\varepsilon_i \right)^2 \right\}$$

$$\frac{\partial^2 \ell}{\partial \beta_1 \beta_0} = \sum_{i=1}^{m} \frac{1}{A_i{}^2} \left\{ A_i \int_{-\infty}^{\infty} ((C_i - \sum_{j=1}^{ni} t_j) + B_i C_i) z_i \, d\varepsilon_i - \int_{-\infty}^{\infty} B_i z_i \, d\varepsilon_i \int_{-\infty}^{\infty} C_i z_i \, d\varepsilon_i \right\}$$

$$\frac{\partial^2 \ell}{\partial \varphi_\varepsilon \beta_0} = \sum_{i=1}^{m} \frac{1}{A_i{}^2} \left\{ A_i \int_{-\infty}^{\infty} B_i D_i z_i \, d\varepsilon_i - \int_{-\infty}^{\infty} B_i z_i \, d\varepsilon_i \int_{-\infty}^{\infty} D_i z_i \, d\varepsilon_i \right\}$$

$$\frac{\partial^2 \ell}{\partial \beta_1{}^2} = \sum_{i=1}^{m} \frac{1}{A_i{}^2} \left\{ A_i \int_{-\infty}^{\infty} (E_i + C_i{}^2) z_i \, d\varepsilon_i - \left( \int_{-\infty}^{\infty} C_i z_i \, d\varepsilon_i \right)^2 \right\}$$

$$\frac{\partial^2 \ell}{\partial \varphi_\varepsilon \beta_1} = \sum_{i=1}^{m} \frac{1}{A_i{}^2}\left\{ A_i \int_{-\infty}^{\infty} C_i D_i z_i \ d\varepsilon_i - \int_{-\infty}^{\infty} C_i z_i \ d\varepsilon_i \int_{-\infty}^{\infty} D_i z_i \ d\varepsilon_i \right\}$$

$$\frac{\partial^2 \ell}{\partial \varphi_\varepsilon{}^2} = \sum_{i=1}^{m} \frac{1}{A_i{}^2}\left\{ A_i \int_{-\infty}^{\infty}\left( \frac{2\varepsilon_i{}^2}{exp(2\varphi_\varepsilon)} + D_i{}^2 \right) z_i \ d\varepsilon_i - \left( \int_{-\infty}^{\infty} D_i z_i \ d\varepsilon_i \right)^2 \right\}$$

# Model with a Double Random Effect

**First Order Partial Derivatives**

$$\frac{\partial \ell}{\partial \beta_0} = \sum_{i=1}^{m} \frac{1}{A_i} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_i z_i \ d\varepsilon_i d\xi_i$$

$$\frac{\partial \ell}{\partial \beta_1} = \sum_{i=1}^{m} \frac{1}{A_i} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_i z_i \ d\varepsilon_i d\xi_i$$

$$\frac{\partial \ell}{\partial \varphi_\varepsilon} = \sum_{i=1}^{m} \frac{1}{A_i} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} D_i z_i \ d\varepsilon_i d\xi_i$$

$$\frac{\partial \ell}{\partial \varphi_\xi} = \sum_{i=1}^{m} \frac{1}{A_i} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_i z_i \ d\varepsilon_i d\xi_i$$

**Second Order Partial Derivatives**

$$\frac{\partial^2 \ell}{\partial \beta_0{}^2} = \sum_{i=1}^{m} \frac{1}{A_i{}^2}\left\{ A_i \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (B_i - n_i + B_i{}^2) z_i \ d\varepsilon_i d\xi_i - \left( \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_i z_i \ d\varepsilon_i d\xi_i \right)^2 \right\}$$

$$\frac{\partial^2 \ell}{\partial \beta_1 \beta_0} = \sum_{i=1}^{m} \frac{1}{A_i{}^2}\left\{ A_i \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left( (C_i - \sum_{j=1}^{ni} t_j) + B_i C_i \right) z_i \ d\varepsilon_i d\xi_i - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_i z_i \ d\varepsilon_i d\xi_i \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_i z_i \ d\varepsilon_i d\xi_i \right\}$$

$$\frac{\partial^2 \ell}{\partial \varphi_\varepsilon \beta_0} = \sum_{i=1}^{m} \frac{1}{A_i^2} \left\{ A_i \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_i D_i z_i \, d\varepsilon_i d\xi_i - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_i z_i \, d\varepsilon_i d\xi_i \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} D_i z_i \, d\varepsilon_i d\xi_i \right\}$$

$$\frac{\partial^2 \ell}{\partial \varphi_\xi \beta_0} = \sum_{i=1}^{m} \frac{1}{A_i^2} \left\{ A_i \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_i E_i z_i \, d\varepsilon_i d\xi_i - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_i z_i \, d\varepsilon_i d\xi_i \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_i z_i \, d\varepsilon_i d\xi_i \right\}$$

$$\frac{\partial^2 \ell}{\partial \beta_1^2} = \sum_{i=1}^{m} \frac{1}{A_i^2} \left\{ A_i \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (F_i + C_i^2) z_i \, d\varepsilon_i d\xi_i - \left( \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_i z_i \, d\varepsilon_i d\xi_i \right)^2 \right\}$$

$$\frac{\partial^2 \ell}{\partial \varphi_\varepsilon \beta_1} = \sum_{i=1}^{m} \frac{1}{A_i^2} \left\{ A_i \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_i D_i z_i \, d\varepsilon_i d\xi_i - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_i z_i \, d\varepsilon_i d\xi_i \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} D_i z_i \, d\varepsilon_i d\xi_i \right\}$$

$$\frac{\partial^2 \ell}{\partial \varphi_\xi \beta_1} = \sum_{i=1}^{m} \frac{1}{A_i^2} \left\{ A_i \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_i E_i z_i \, d\varepsilon_i d\xi_i - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_i z_i \, d\varepsilon_i d\xi_i \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_i z_i \, d\varepsilon_i d\xi_i \right\}$$

$$\frac{\partial^2 \ell}{\partial \varphi_\varepsilon^2} = \sum_{i=1}^{m} \frac{1}{A_i^2} \left\{ A_i \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (G_i + D_i^2) z_i \, d\varepsilon_i d\xi_i - \left( \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} D_i z_i \, d\varepsilon_i d\xi_i \right)^2 \right\}$$

$$\frac{\partial^2 \ell}{\partial \varphi_\varepsilon \varphi_\xi} = \sum_{i=1}^{m} \frac{1}{A_i^2} \left\{ A_i \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} D_i E_i z_i \, d\varepsilon_i d\xi_i - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} D_i z_i \, d\varepsilon_i d\xi_i \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_i z_i \, d\varepsilon_i d\xi_i \right\}$$

$$\frac{\partial^2 \ell}{\partial \varphi_\xi^2} = \sum_{i=1}^{m} \frac{1}{A_i^2} \left\{ A_i \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (H_i + E_i^2) z_i \, d\varepsilon_i d\xi_i - \left( \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_i z_i \, d\varepsilon_i d\xi_i \right)^2 \right\}$$
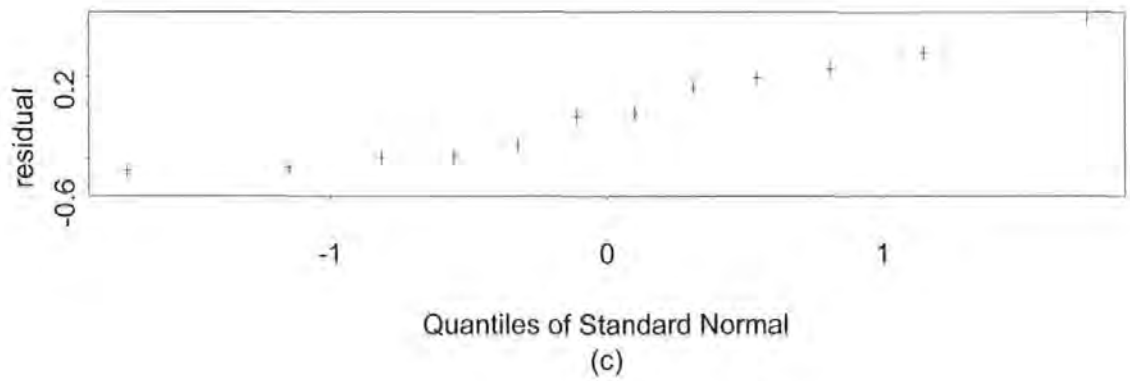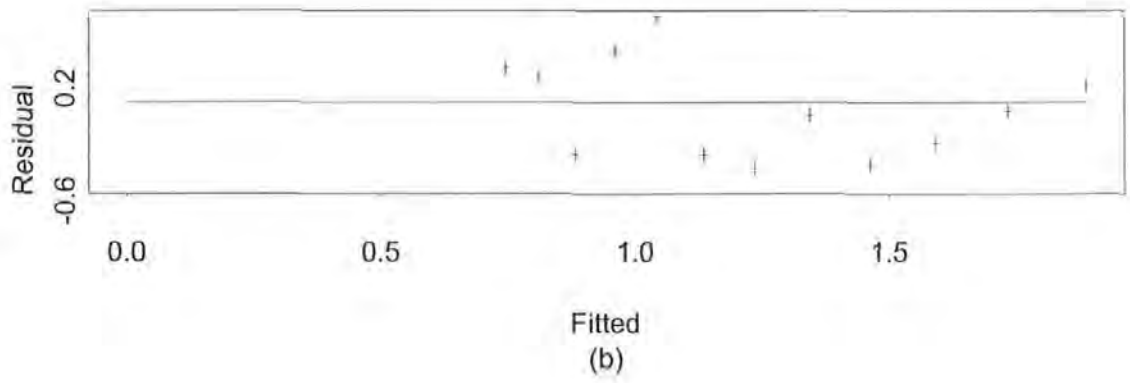
# APPENDIX 6

# Observed Mean Numbers of Failures

# Plotted Against the Predicted Values

# Phase A1-1

Random Effect on $\beta_0$
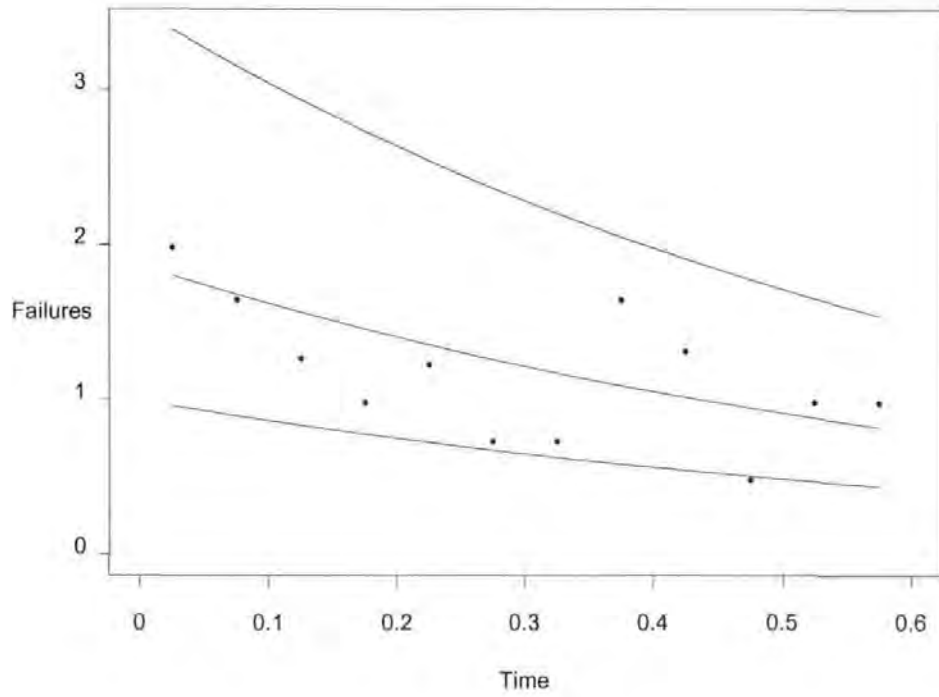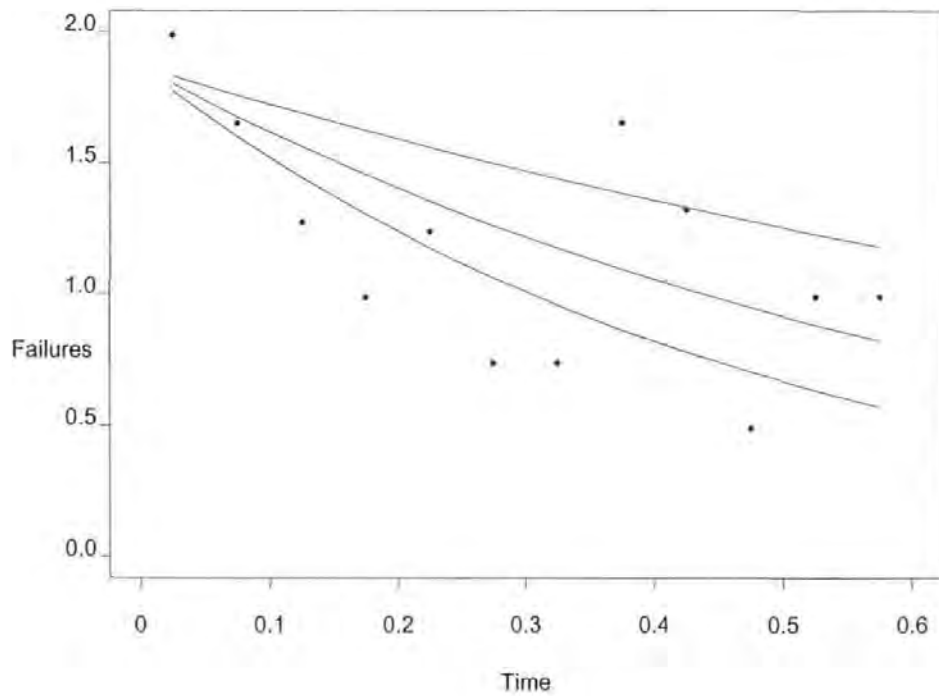


Random Effect on $\beta_1$

# Phase A1-2
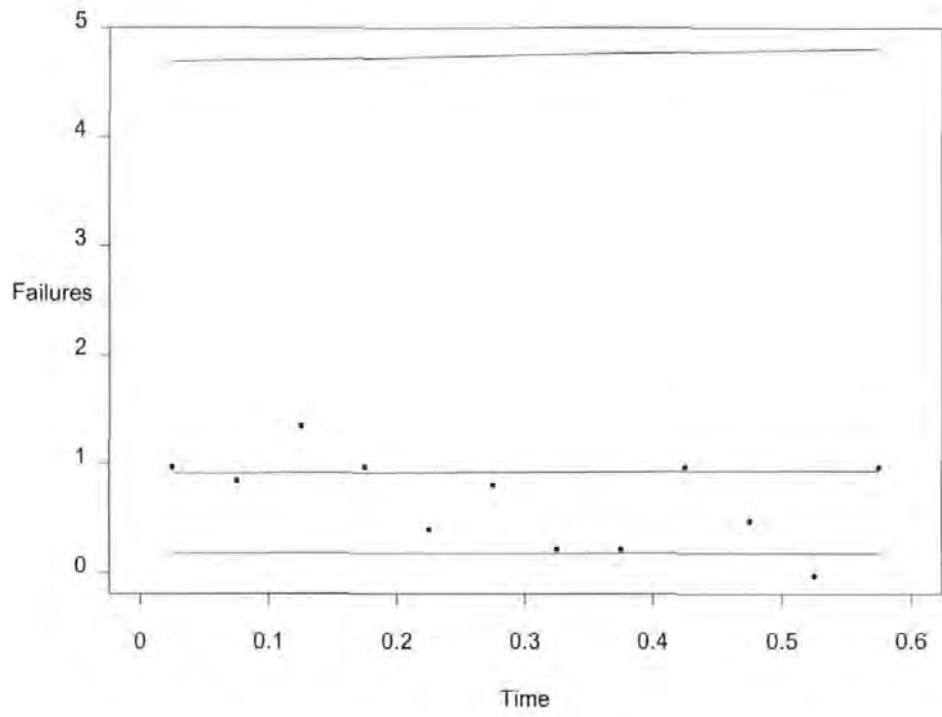
Random Effect on $\beta_0$
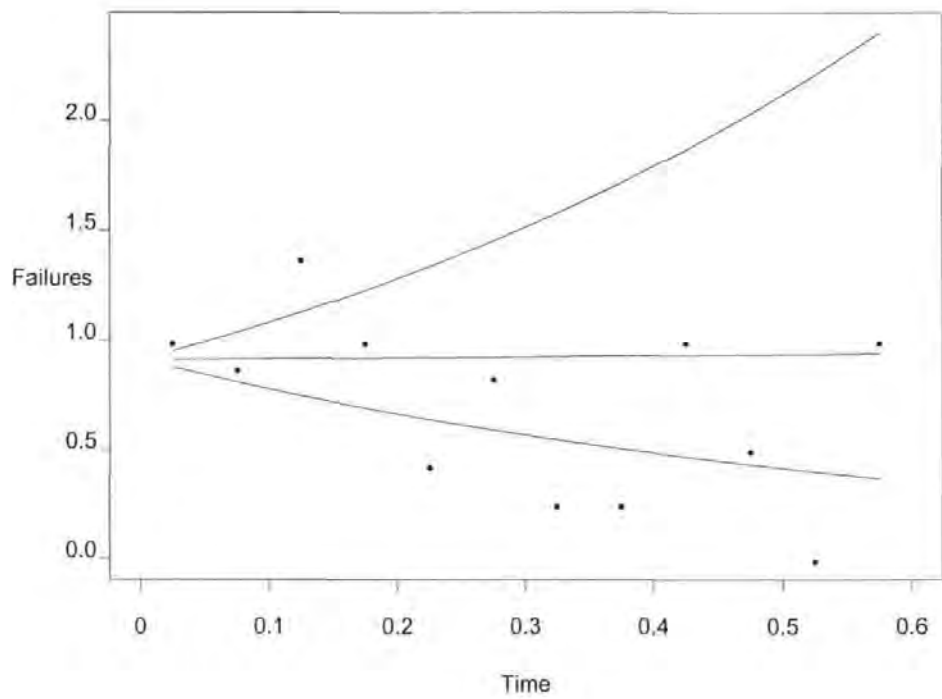


Random Effect on $\beta_1$

# Phase A2-1
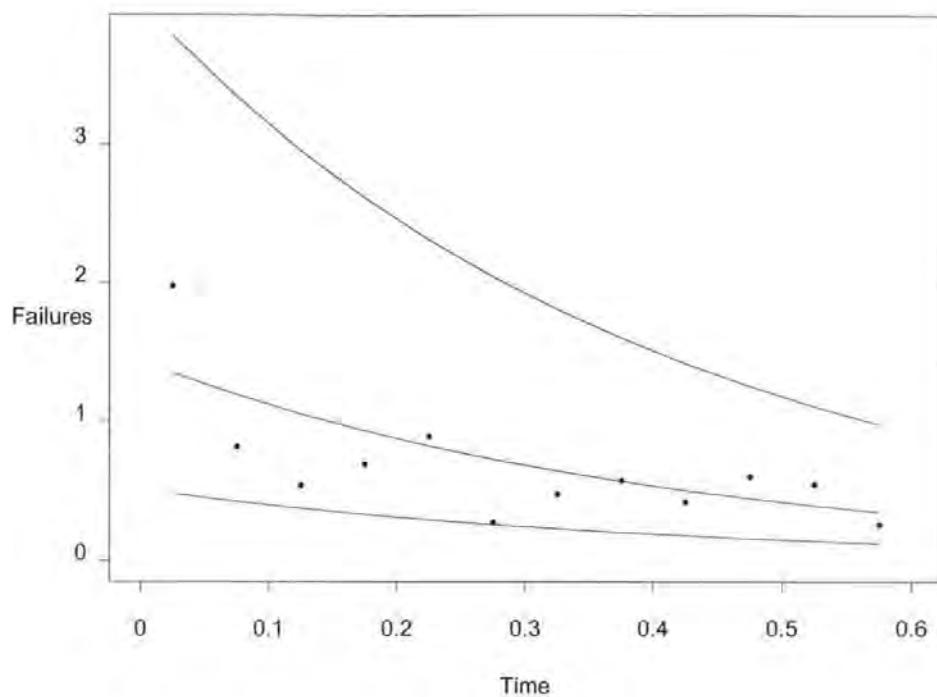
Random Effect on $\beta_0$
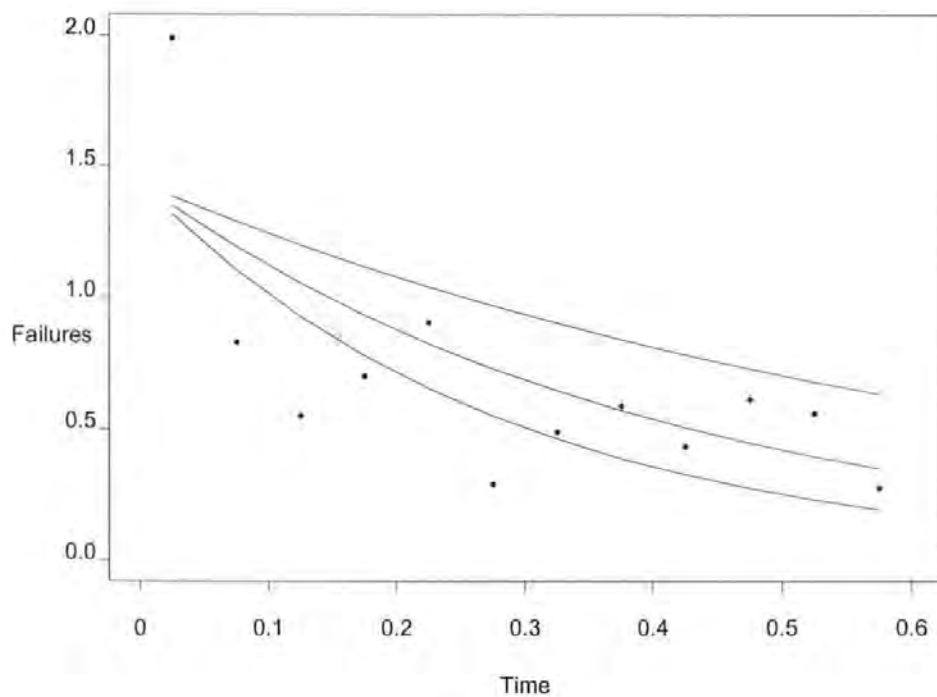


Random Effect on $\beta_1$

# APPENDIX 7
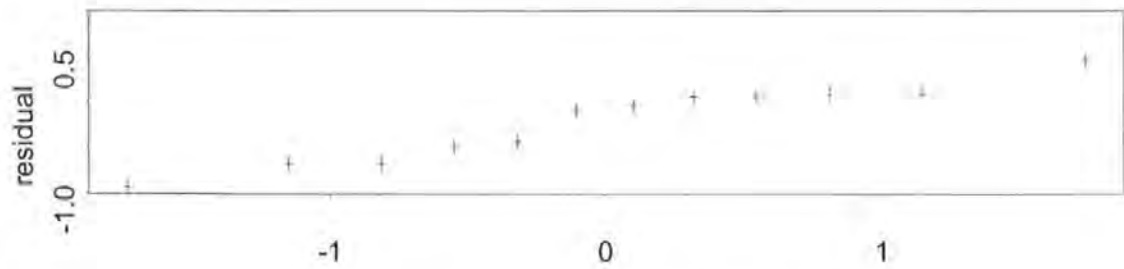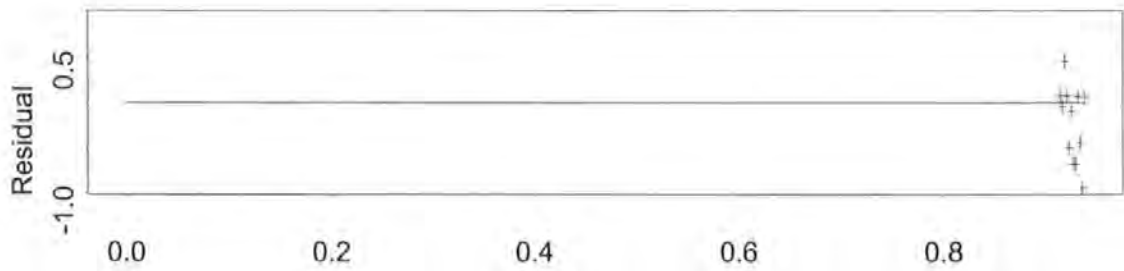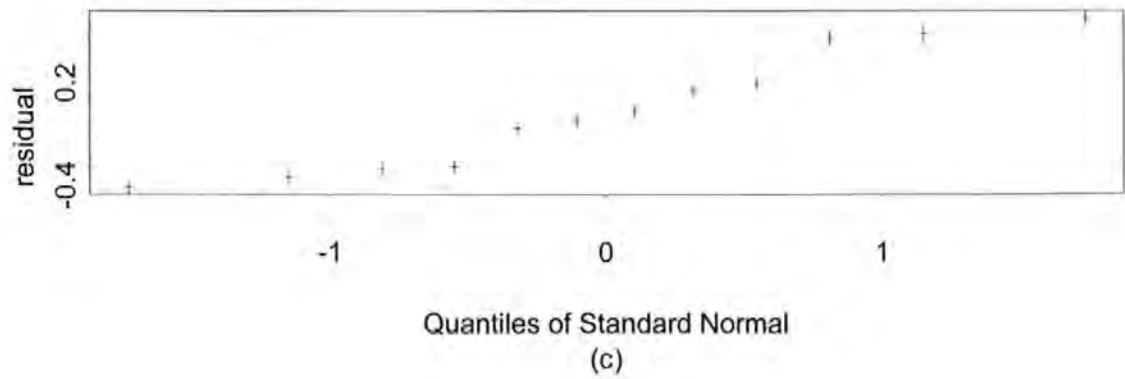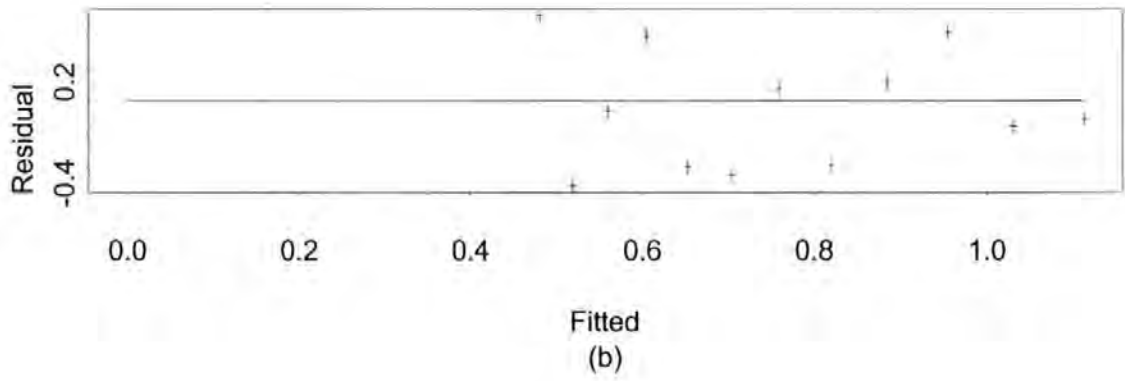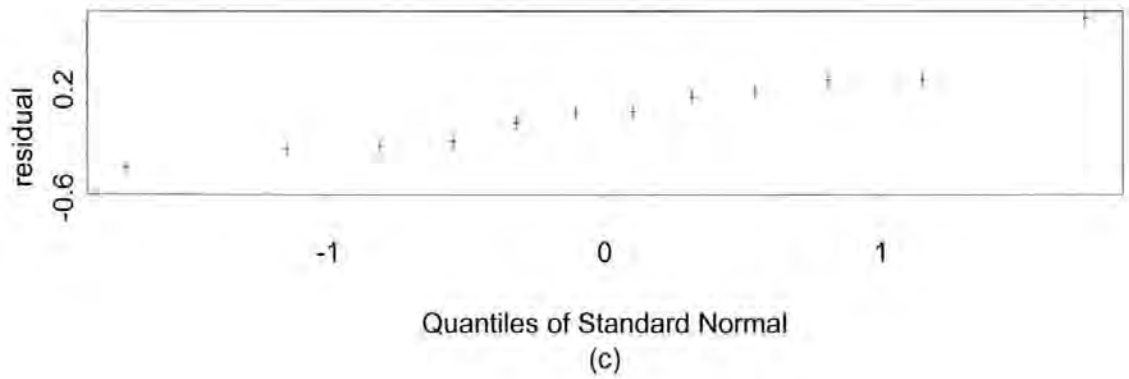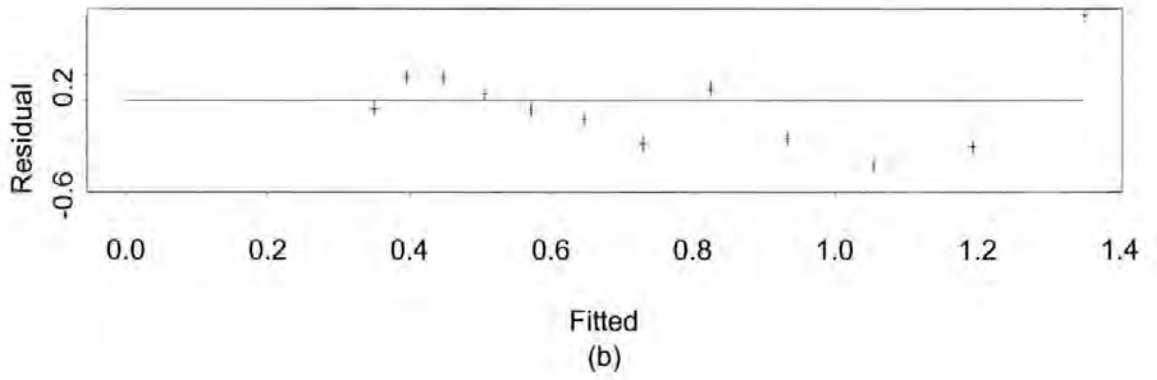
# Marginal Residuals

# Phase A1-1

Random Effect on $\beta_0$



Time

(a)



Fitted

(b)



Quantiles of Standard Normal

(c)

# Phase A1-1

Random Effect on $\beta_1$



(a)



(b)



(c)

# Phase A1-2

Random Effect on $\beta_0$



Time
(a)



Fitted
(b)



Quantiles of Standard Normal
(c)

# Phase A1-2

Random Effect on $\beta_1$



(a)



(b)



(c)

214

# Phase A2-1

Random Effect on $\beta_0$



Time
(a)



Fitted
(b)



Quantiles of Standard Normal
(c)

# Phase A2-1

Random Effect on $\beta_1$



Time
(a)



Fitted
(b)



Quantiles of Standard Normal
(c)

216
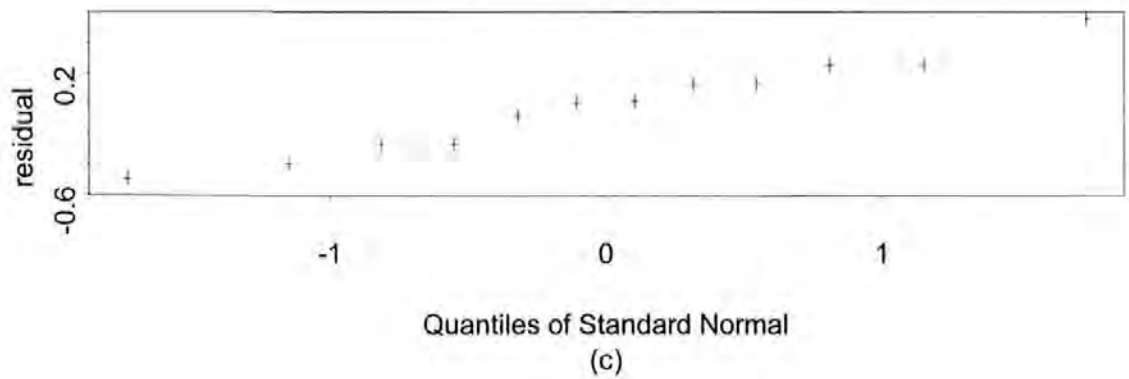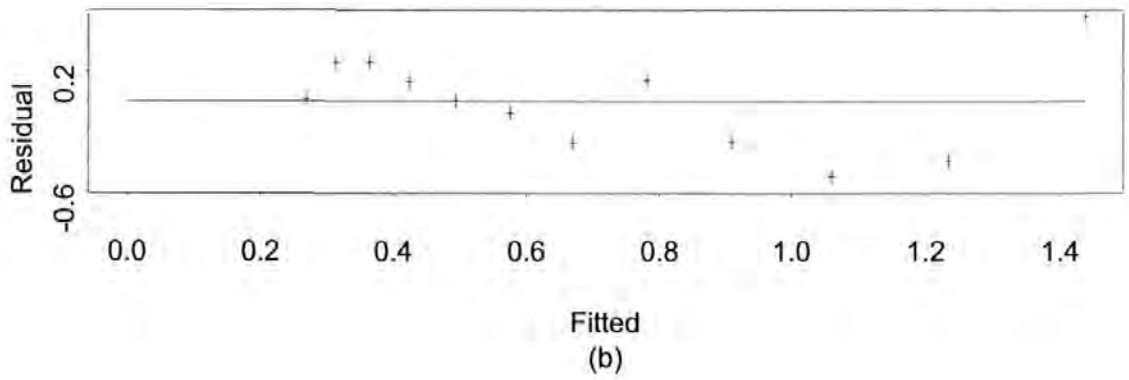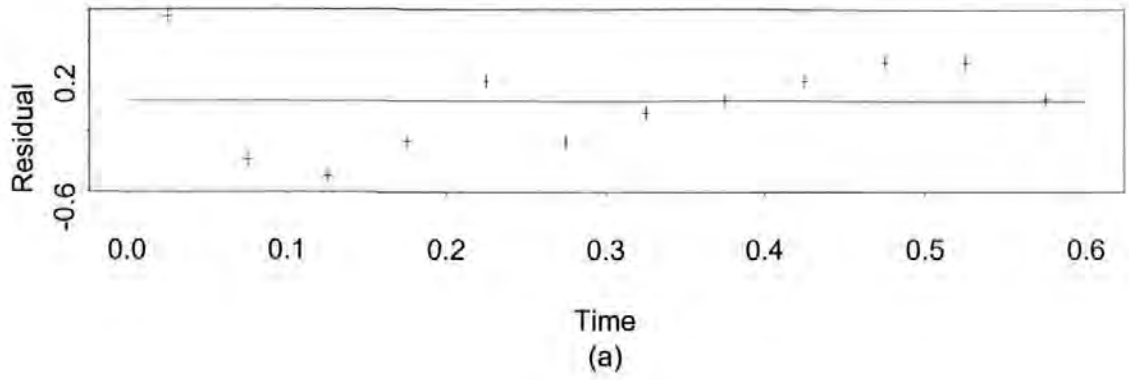
# APPENDIX 8

# Conditional Residuals

# Phase A1-1
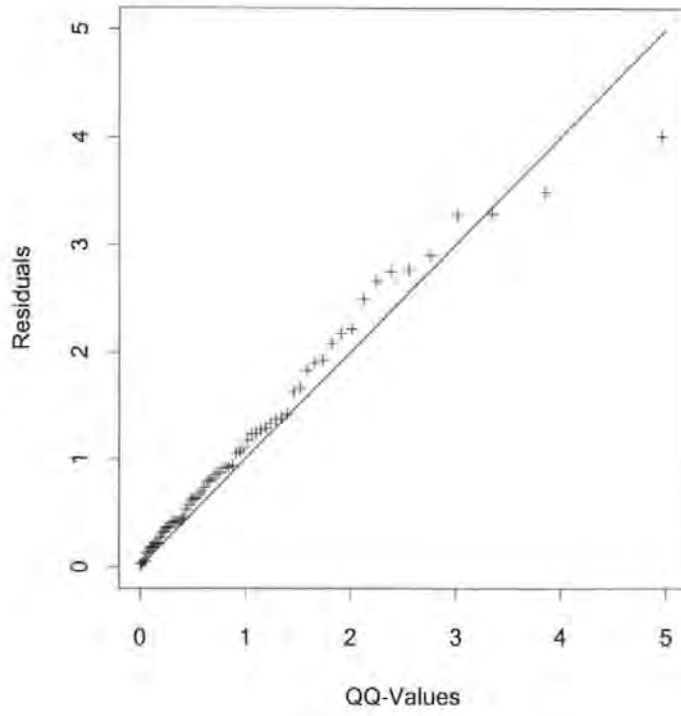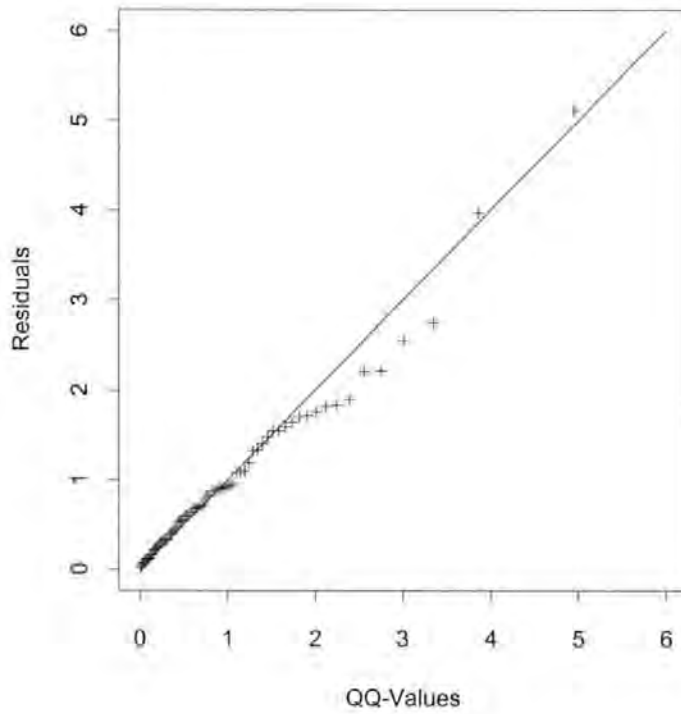
Random Effect on $\beta_0$



Random Effect on $\beta_1$

# Phase A1-2
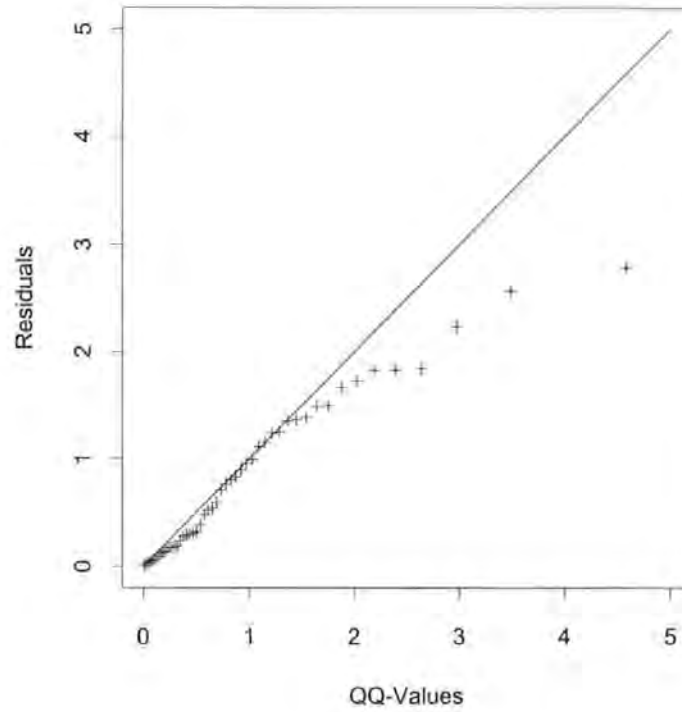
Random Effect on $\beta_0$
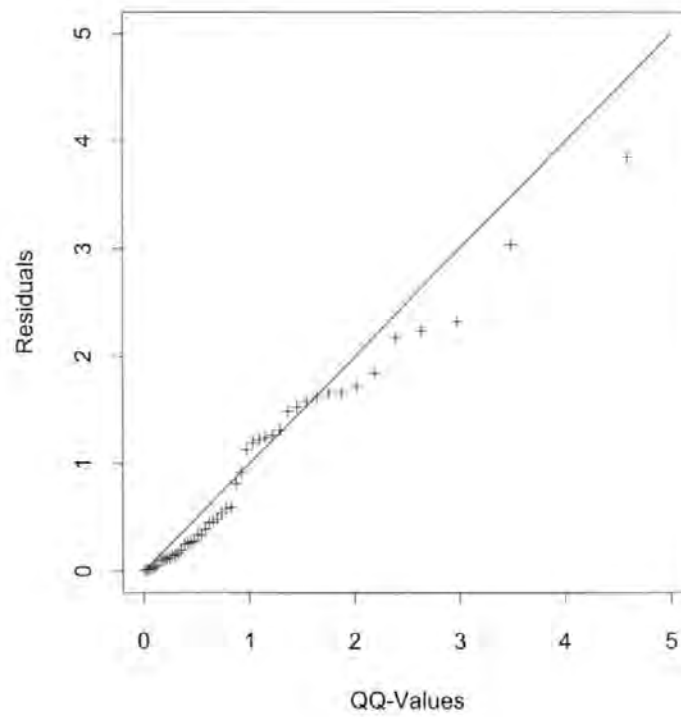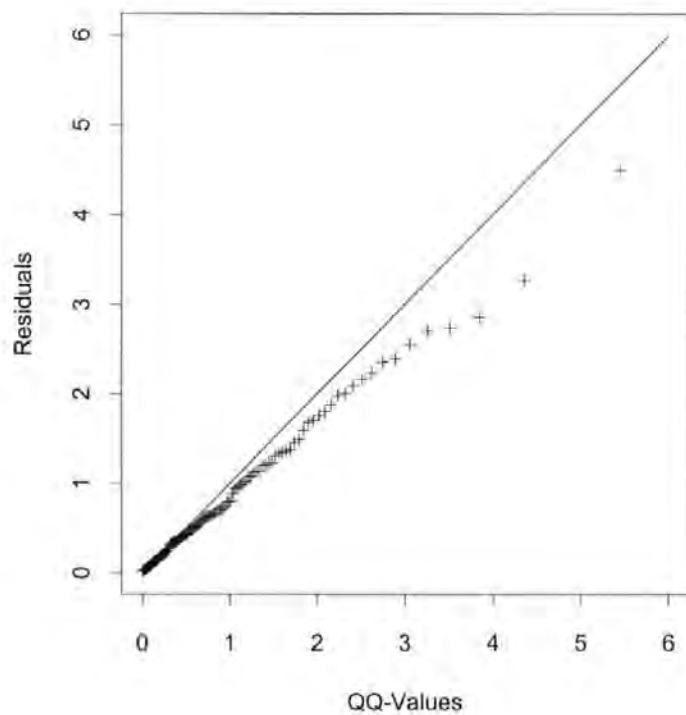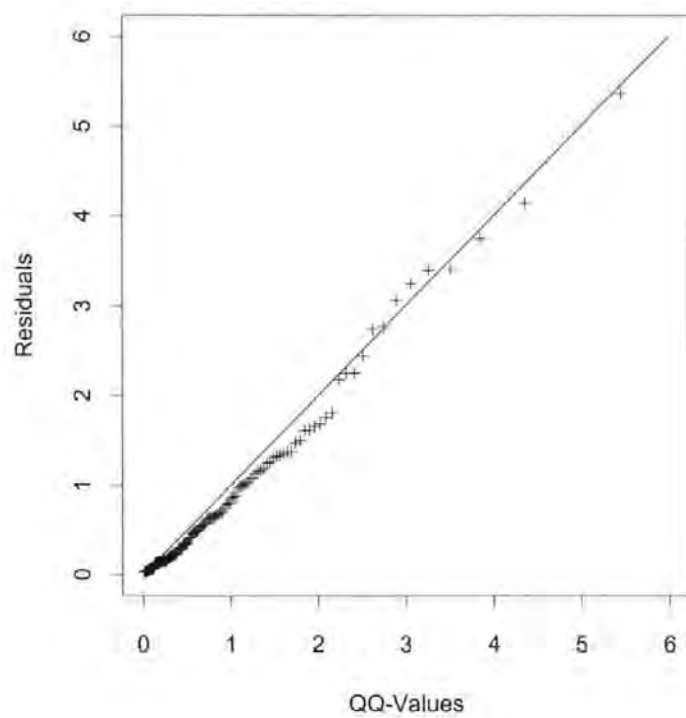


Random Effect on $\beta_1$

# Phase A2-1

Random Effect on $\beta_0$



Random Effect on $\beta_1$

# References

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