Analytic Cohomology
on
Blown - Up Twistor Space

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Abstract

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A flat twistor space is a complex 3 - manifold having the property that every point of the manifold has a neighbourhood which is biholomorphic to a neighbourhood of a complex projective line in complex projective 3 - space. The Penrose transform provides an isomorphism between holomorphic structures on twistor spaces and certain field equations on (Riemannian or Lorentzian) space - times. The initial examples studied by Penrose were solutions to zero rest mass equations and, amongst these, the elementary states were of particular interest. These were elements of a sheaf cohomology group having a singularity on a particular complex projective line, with a codimension-2 structure similar, in some sense, to a Laurent series with a pole of finite order.

In this work we extend this idea to the notion of codimension-2 poles for analytic cohomology classes on a punctured flat twistor space, by which we mean a general, compact, flat, twistor space with a finite number of non-intersecting complex, projective lines removed. We define a holomorphic line bundle on the blow-up of the compact flat twistor space along these lines and show that elements of the first cohomology group with coefficients in the line bundle, when restricted to the punctured twistor space, are cohomology classes with singularities on the removed lines which have precisely the kind of codimension - 2 structure which we define as codimension-2 poles.

The dimension of this cohomology group on the blown-up manifold is then calculated for the twistor space of a compact, Riemannian, hyperbolic 4-manifold. The calculation uses the Hirzebruch - Riemann - Roch theorem to find the holomorphic Euler characteristic of the line bundle, (in chapter 3) together with vanishing theorems. In chapter 4 we show that it is sufficient to find vanishing theorems for the compact flat - twistor space. In chapter 5 we prove a number of vanishing theorems to be used. The technique uses the Penrose transform to convert the theorem to a vanishing theorem for spinor fields. These are then proved by using Penrose's Spinor calculus.

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Authors Declaration

At no time during the registration for the degree of Doctor of Philosophy has the author been registered for any other University award.

The following presentations were given:

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Signed. . Robin Horan . . . .

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Contents

Chapter 1 Introduction

Since its introduction by Penrose in [28], Twistor theory has provided a distinctive approach to problems in mathematical physics and pure mathematics. Many of the ideas go back much further, in particular to the late nineteenth century and the work of Felix Klein, who first formulated a correspondence between lines in complex projective 3 - space and a general quadric in complex projective 5-space. This correspondence was known to Penrose (see [34]) but his unique contribution, which initiated the whole of Twistor theory, was the observation that this classical geometry, together with a certain Radon type integral transform, provided a comprehensive set of solutions to many of the partial differential equations which arise naturally in modern physics and differential geometry. This transform, the Penrose transform, was the mathematical embodiment of Penrose's belief that "the actual space - time we inhabit might be significantly regarded as a secondary structure arising from a deeper Twistor - holomorphic reality." ([34], p.343).

The physical and mathematical theories most amenable to treatment by Twistor theoretic methods are those which are conformally invariant, such as the theory of spin - s massless fields (which include Maxwell electromagnetic fields, Dirac - Weyl Neutrino fields, and linearised gravitational fields)[8], [41], [42], Yang - Mills theory [23], [43], [47], Einstein - Weyl geometry [17], [29], [31], as well as the theory of 4-dimensional, self - dual conformal structures [33], [41] and quaternionic geometry [3], [25], [26], [28]. (A more comprehensive list of references may be found in the following [4], [5], [8], [27], [36], [43].) These theories are described by sets of differential equations which remain unchanged when acted upon by a conformal transformation of the particular space - time on which they are defined. The conformal geometry of the underlying space - time is then crucial to the understanding of the various physical phenomena described by those physical theories.

The first example of a 4 - dimensional space - time is real, affine Minkowski space, $\mathbf{M}_I$, which can be described as $\mathbf{R}^4$ equipped with a flat, Lorentz metric of signature (+,−,−,−). This affine space is the space - time for Einstein's Special Theory of Relativity, and much of modern physics has its roots in this setting. The group of transformations which preserve the metric is the Poincaré group.If a particular point **p** of $\mathbf{M}_I$ is identified as an origin for a set of co-ordinates then we can consider Minkowski vector space, **V** say, and the tangent space at **p** may be identified with $\mathbf{R}^4$. Any tangent vector to **p** can then be identified with a vector $\underline{x} = (x^0, x^1, x^2, x^3)$ in $\mathbf{R}^4$ and this vector is said to be null if

$$\|\underline{x}\|_p^2 = |x^0|^2 - |x^1|^2 - |x^2|^2 - |x^3|^2 = 0$$

Such a vector represents a light ray passing through **p**, and the set of all such light rays through **p** is the light cone $\mathbf{C}_p$ at **p**, i.e.

$$C_p = \left\{ \underline{x} \in \mathbf{R}^4 : \|\underline{x}\|_p^2 = 0 \right\}$$

Any linear transformation of the vector space **V** which preserves the Lorentz metric, is called a Lorentz transformation. The Lorentz transformations are then those elements of the Poincaré group which fix a point of $\mathbf{M}_I$; the Poincaré group is generated by the Lorentz transformations and translations in $\mathbf{M}_I$.

Earlier this century it was discovered that Maxwell's equations are invariant under a larger class of mappings than those in the Poincaré group. These mappings, together with the Poincaré group, generate the conformal group; it is the group generated by

(i) The Lorentz transformations,

(ii) translations,

(iii) dilations, i.e. $\underline{x} \rightarrow \alpha\underline{x}$ for some $\alpha \in \mathbf{R}$,

(iv) inversions, i.e. maps of type $\underline{x} \rightarrow \frac{-\underline{x}}{\|\underline{x}\|^2}$, where $\|\underline{x}\|$ is the Lorentz norm of $\underline{x}$, with respect to some co-ordinate system in $\mathbf{M}_I$.

The mappings which make up the conformal group are precisely those which preserve the Lorentz metric up to a (non-zero) scalar factor. Of course the latter transformations, i.e. the inversions, are not mappings on the whole of $\mathbf{M}_I$ since they will be singular on the null - cone of some point. It is possible, however, by adjoining a "light - cone at infinity" (see [36]) in a suitable manner, to compactify $\mathbf{M}_I$ in such a way that the conformal group becomes a bona-fide group of transformations on the compactification. This space, which we denote **M**, is compactified Minkowski space.

The complexification of **M**, written **CM**, is then obtained from **M** by simply replacing real numbers by complex numbers. This space, **CM**, is compactified, complexified Minkowski space and the Plücker embedding identifies this with a Klein quadric in $\mathbf{CP}^5$, complex projective 5 - space. Real compactified Minkowski space **M** is identified as a real quadric in $\mathbf{RP}^5$, and the real (complex) conformal group is induced by the real (complex) projective group on $\mathbf{RP}^5(\mathbf{CP}^5)$.

For our purposes it is important to consider the Euclidean version of these constructions. Instead of Minkowski space **M**, the starting point is now Euclidean 4 - space $\mathbf{E}^4$, which can be identified as $\mathbf{R}^4$ with the standard, flat, Euclidean metric. In this case the conformal compactification is $\mathbf{S}^4$, the 4 - sphere, and its complexification is also **CM**, i.e. compactified, complexified Minkowski space. The conformal structures of both Minkowski space and Euclidean space are thus encoded in the complex conformal structure of **CM**, and the symmetry group of the latter is precisely the subgroup of the projective group on $\mathbf{CP}^5$ which leaves the Klein quadric invariant. The Klein correspondence now relates the conformal geometry of **CM** to the holomorphic geometry of $\mathbf{CP}^3$. An accessible form of this correspondence can be obtained by way of the following construction,[8], [43].

Let **T** be a four - complex dimensional vector space equipped with an Hermitian form of signature (+,+,−,−), so that in a suitable co-ordinate system, if $\underline{Z} = (Z^0, Z^1, Z^2, Z^3)$ then $\Phi(\underline{Z}) = |Z^0|^2 + |Z^1|^2 - |Z^2|^2 - |Z^3|^2$. The group of invertible transformations on **T** which leave Φ invariant is written SU(2,2). Now define the following spaces:

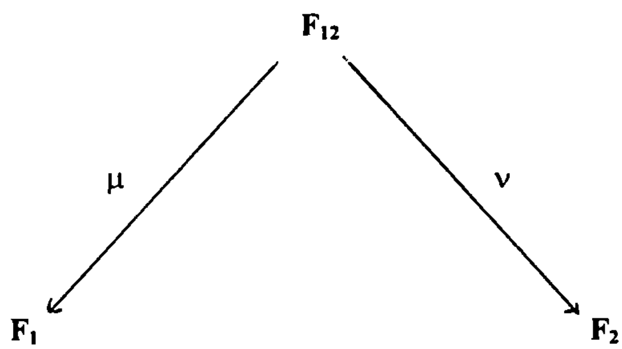
$$\mathbf{F}_1 = \{\mathbf{L}_1 : \mathbf{L}_1 \text{ is a 1 - dimensional subspace of } \mathbf{T}\},$$

$$\mathbf{F}_2 = \{\mathbf{L}_2 : \mathbf{L}_2 \text{ is a 2 - dimensional subspace of } \mathbf{T}\},$$

$$\mathbf{F}_{12} = \{(\mathbf{L}_1, \mathbf{L}_2) : \mathbf{L}_1 \text{ is a 1 - dimensional subspace of } \mathbf{T},$$

$$\mathbf{L}_2 \text{ is a 2 - dimensional subspace of } \mathbf{T},\ \mathbf{L}_1 \subseteq \mathbf{L}_2\}.$$

Of these manifolds, $\mathbf{F}_{12}$ is a flag - manifold, $\mathbf{F}_2$ is the Grassmanian of 2 - dimensional subspaces and $\mathbf{F}_1$ is the Grassmanian of 1-dimensional subspaces, i.e. complex projective 3-space, $\mathbf{CP}^3$. The Twistor correspondence is now obtained from the following double - fibbration.



where μ, ν are the respective "forgetful" maps, i.e. $\mu(L_1, L_2) = L_1, \nu(L_1, L_2) = L_2$

The Grassmanian $\mathbf{F}_2$ can be identified with complexified, compactified Minkowski space $\mathbf{CM}$, and $\mathbf{F}_1$, which is $\mathbf{CP}^3$, is called projective Twistor space. We now wish to show how to pick out $\mathbf{M}$, and how to recognise the conformal group from this picture.

The Hermitian form is defined on every subspace of $\mathbf{T}$. Thus Φ can be defined on each of the manifolds $\mathbf{F}_{12}$, $\mathbf{F}_1$, $\mathbf{F}_2$. If S is any subspace of dimension 2 then say

$$\begin{aligned} \Phi(S) > 0 \text{ if } \Phi(Z) > 0 \text{ for all } Z \in S \\ \Phi(S) = 0 \text{ if } \Phi(Z) = 0 \text{ for all } Z \in S \\ \Phi(S) < 0 \text{ if } \Phi(Z) < 0 \text{ for all } Z \in S \end{aligned}$$

Obviously if $\Phi(S) = 0$ then $\Phi(U) = 0$ for every 1 - dimensional subspace U of S, so Φ can be used to define 3 different subsets, depending upon the sign, (i.e. +, −, or 0) of the elements of the respective spaces. For example, we have

$$\begin{aligned} \mathbf{F}_{12}^{+} &= \{(L_1, L_2) \in \mathbf{F}_{12} : \Phi(L_2) > 0\} \\ \mathbf{F}_{12}^{0} &= \{(L_1, L_2) \in \mathbf{F}_{12} : \Phi(L_2) = 0\} \\ \mathbf{F}_{12}^{-} &= \{(L_1, L_2) \in \mathbf{F}_{12} : \Phi(L_2) < 0\} \end{aligned}$$

Similarly we may define $\mathbf{P}^+$, $\mathbf{P}^0$, $\mathbf{P}^-$ as subsets of $\mathbf{CP}^3$, and $\mathbf{CM}^+$, $\mathbf{CM}^0$, $\mathbf{CM}^-$ as subspaces of $\mathbf{CM}$. The action of SU(2,2) transfers to each of the manifolds $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_{12}$ in such a way that, in each case, the +,0,− subsets are SU(2,2) orbits. The subspace $\mathbf{CM}^0$ is a 4 - real dimensional manifold which is identified with $\mathbf{M}$, real compactified Minkowski space and the action of SU(2,2) on $\mathbf{CM}^0$ induces the action of the conformal group. The region $\mathbf{CM}^+$ can be identifed with those vectors of the form $x^a - iy^a$ where x^a, y^a are real and y^a is future pointing [36], [43].

This correspondence establishes the relationship between the two spaces. For example, any point $\mathbf{p}$ in $\mathbf{CM}$ corresponds to a complex projective line $\mathbf{CP}^1$ given by $\mu \circ \nu^{-1}(\mathbf{p})$. Similarly, any point $\mathbf{q}$ in $\mathbf{CP}^3$ corresponds to $\nu \circ \mu^{-1}(\mathbf{q})$ in $\mathbf{CM}$, which is a complex projective plane ($\mathbf{CP}^2$). If $\mathbf{p}_1$ and $\mathbf{p}_2$ are two points in $\mathbf{M}(\mathbf{CM}^0)$ which lie on a light ray (are null separated) then the two lines $\mu \circ \nu^{-1}(\mathbf{p}_1)$ and $\mu \circ \nu^{-1}(\mathbf{p}_2)$ which lie in $\mathbf{P}^0$, meet in a point. At each point of $\mathbf{M}$, the light rays through that point can be parameterised by a 2 - sphere $\mathbf{S}^2$, the celestial sphere, since each light ray emanating from that point must pass through exactly one point of this sphere. Since $\mathbf{S}^2$ is identical with $\mathbf{CP}^1$, the space $\mathbf{P}^0$ can be identified as the set of all lightrays in $\mathbf{M}$.

Maxwell's equations, the Dirac - Weyl neutrino equation and the linearised Einstein equations are all examples of massless free - field equations [45], [46], each of spin $s = 1, \frac{1}{2}, 2$ respectively. Any real analytic solution (in $\mathbf{M}$) to such an equation will have a complex analytic (i.e. analytic) extension into a neighbourhood of $\mathbf{M}$ in $\mathbf{CM}$. In particular the region $\mathbf{CM}^+$, called the forward tube, is the region in which ordinary space - time single - particle (positive - frequency) wave functions are holomorphically defined (p. 154, [36]).

One version of the Penrose transform is then

Theorem

Let U be an open subset of **CM** and $\hat{U} = \mu \circ \nu^{-1}(U)$ be its image in $\mathbf{CP}^3$ under the double fibration given above. For $n \geq 0$ let $\mathcal{O}(-n-2)$ be the sheaf of germs of holomorphic functions, homogeneous of degree $-n-2$, on $\mathbf{CP}^3$, and let Z'_n be the sheaf of holomorphic, right handed massless fields of spin $s = \frac{n}{2}$ on **CM**. Under mild geometric conditions on U, there is a canonical isomorphism

$$P: H^1(\hat{U}, \mathcal{O}(-n-2)) \rightarrow \Gamma(U, Z'_n)$$

There is a corresponding result for left - handed massless fields, [8], [43].

The massless free fields, which are solutions to systems of partial differential equations defined on U, have been transformed into elements of an analytic sheaf cohomology group.

A case of particular significance to our subsequent work is that of "massless fields based on a line" in $\mathbf{CP}^3$, that is the elements of the cohomology group $H^1(\mathbf{CP}^3\text{-}\mathbf{L}, \mathcal{O}(m))$, for some integer m, where **L** is a projective line in $\mathbf{CP}^3$.

Representatives of these cohomology classes will have singularities on the line **L** and will correspond, under the Penrose transform, to massless fields defined on the whole of **CM** <u>except</u> for those points on the null cone with vertex corresponding to the given line **L**. The other points on the null cone correspond to the projective lines in $\mathbf{CP}^3$ which intersect **L**.

Amongst these cohomology classes a particular subset may be distinguished, the elementary states based on **L**. As an example of this, take the line $\mathbf{L}^+$ in $\mathbf{CP}^3$ to be $Z^2 = Z^3 = 0$, where we are using the standard homogeneous co-ordinates for $\mathbf{CP}^3$, so that $\mathbf{L}^+$ is a line in $\mathbf{P}^+$, the top half of twistor space. The simplest example of an elementary state based on $\mathbf{L}^+$ is the element of $H^1(\mathbf{CP}^3 - \mathbf{L}^+, \mathcal{O}(-2))$, which has a representative of the form

$$\frac{1}{Z^2 Z^3}.$$

Under the Penrose transform this corresponds to a solution of the wave equation in Minkowski space - time. Similarly

$$\frac{Z^0}{Z^2 (Z^3)^2} + \frac{Z^1}{(Z^2)^2 Z^3}$$

is representative for a sum of elementary states in the same cohomology group. By writing the latter as

$$\frac{Z^0Z^2 + Z^1Z^3}{(Z^2Z^3)^2}$$

we can think of them both as having "codimension - 2 poles on L^+", the former of order 1, the latter of order 2. More generally, if $\ell \geq 1$ then one may consider an elementary state in $H^1(\mathbf{CP}^3 - \mathbf{L}^+, \mathcal{O}(m))$, as a cohomology class with a representative of the form.

$$\sum_{\substack{j,k>0 \\ j+k\leq \ell+1}} \frac{A_{jk}(Z^0,Z^1)}{(Z^2)^j(Z^3)^k}$$

where $\mathbf{A}_{jk}$ is a polynomial in (Z^0,Z^1) (i.e. defined on **L**) which is homogeneous of degree $j + k + m$. For a fixed positive integer ℓ, we shall say that such an elementary state has a <u>codimension - 2 pole of order ℓ on the line **L**</u> (providing at least one $\mathbf{A}_{jk} \neq 0$ for $j + k = \ell + 1$). The set of all elementary states of homogeneity m and with a codimension - 2 pole of order <u>at most</u> ℓ , is a finite dimensional subspace of $H^1(\mathbf{CP}^3 - \mathbf{L}^+, \mathcal{O}(m))$. We shall write this subspace as $H^1_\ell(\mathbf{CP}^3 - \mathbf{L}^+, \mathcal{O}(m))$ (see [7]). Obviously one can define the elementary states based on any distinguished line **L**, in an exactly similar way.

The massless spin-s fields on **M** can be identified with elements of $H^1(\mathbf{P}^0, O(-2s-2))$, where the cohomology is with respect to a natural definition of C - R cohomology of $\mathbf{P}^0$, this manifold being a 5-real dimensional hypersurface in $\mathbf{CP}^3$. There is then a decomposition of these fields into positive - frequency and negative - frequency parts given by

$$H^1(\mathbf{P}^0, O(-2s-2)) = H^1(\overline{\mathbf{P}}^+, O(-2s-2)) \oplus H^1(\overline{\mathbf{P}}^-, O(-2s-2))$$

where $\overline{\mathbf{P}}^\pm$ is the closure of $\mathbf{P}^\pm$ respectively.

It is shown in [9] that there are natural topologies on the relevant cohomology groups for which the elementary states in $H^1(\mathbf{CP}^3 - \mathbf{L}^-, O(-2s-2))$ form a dense subset of $H^1(\overline{\mathbf{P}}^+, O(-2s-2))$ and similarly the elementary states on $\mathbf{L}^+$ are dense in $H^1(\overline{\mathbf{P}}^-, O(-2s-2))$. The "massless fields" on $\mathbf{P}^0$ (corresponding to massless fields on **M**) then have a natural "codimension-2" Laurent series type expansion, into positive and negative frequency parts.

All of the preceeding ideas are based upon the original correspondence between the flat space-time, **CM**, and standard, flat Twistor - space, $\mathbf{CP}^3$. Subsequently this correspondence was generalised to curved space-times and curved twistor spaces, the only condition on the space-time being that its conformal structure be self-dual.

Associated with this correspondence was a Penrose transform similar to that for the flat case. This is the non-linear graviton construction [33]. There is also a generalisation of the positive - definite case, i.e. of the fibration of $\mathbf{CP}^3 \rightarrow \mathbf{S}^4$, which is the twistor - space corresponding to a (real) Riemannian 4 - manifold with self - dual conformal structure. There is a direct construction of this, which is described in [2], [15]. The Riemannian version of the Penrose transform will play an important part in our subsequent work and we shall return to it in more detail in chapter 5.

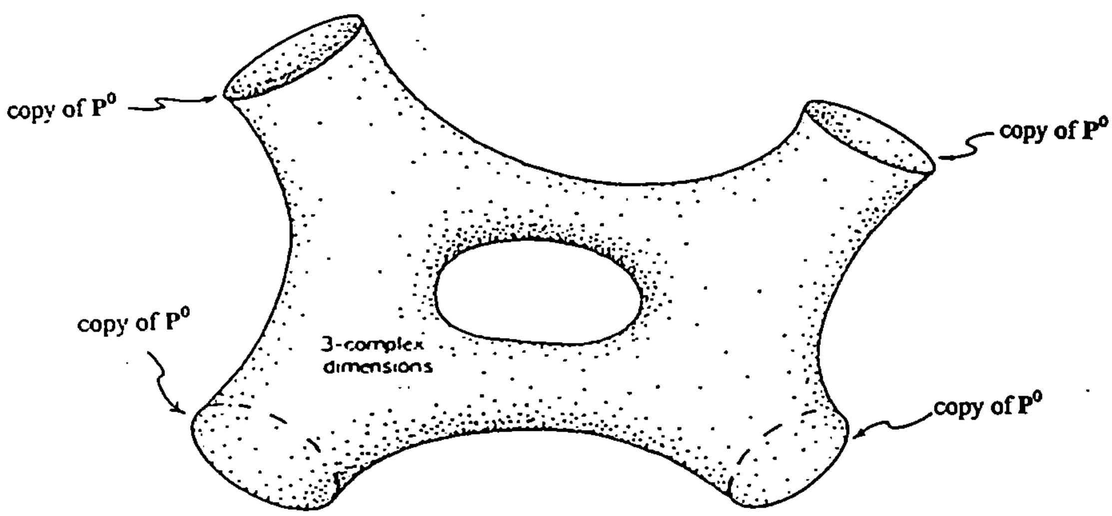
A complex - 3 - manifold **Z** is then a twistor - space if through each point of **Z** there is a complex projective line, with normal bundle $O(1) \oplus O(1)$. The twistor - space **Z** is real if it can be fibred by complex projective lines over a real 4 - manifold **X** with self - dual conformal structure, and **Z** is flat, if this conformal structure is conformally flat. This is equivalent to each point of **Z** having an open neighbourhood which is biholomorphic to a neighbourhood of a line in $\mathbf{CP}^3$.

These flat twistor spaces are fundamental to the construction of the 4 - dimensional conformal field theory proposed by Hodges, Penrose and Singer in [18]. (See also [38]), which is an attempt to generalise some important aspects of the axiomatic version of two dimensional conformal field theory, [48]. This programme for the construction of a 4 - dimensional conformal field theory might possibly provide a new model for the fundamental interactions of quantum fields in space - time.

This theory was successful in producing a picture of interacting quantum fields in which the interactions depend only on the intrinsic properties of a 2 - dimensional surface. One begins with a Riemann surface X, whose boundary ∂X is a disjoint union of n + m copies of the circle, each of which is parameterised, n being positively oriented and m negatively oriented, with respect to the orientation of X. A certain Hilbert space H is then identified, usually some subspace of the space of positive - frequency complex - valued functions on S^1, or a Hilbert space H modelled on it, together with its dual H* which arises from the space of negative - frequency complex - valued functions. One then has a rule for an amplitude: $\mathbf{H}^{\otimes n} \otimes \mathbf{H}^{*\otimes m} \to \mathbf{C}$ satisfying certain conditions, [38].

Central to this picture is the splitting property of smooth functions on S^1. Identifying S^1 with the equator of the Riemann sphere, any smooth function on S^1 is the sum of a 'positive - frequency' holomorphic function, which extends into the northern hemisphere, and a 'negative - frequency' holomorphic function which extends into the southern hemisphere. If $\mathcal{O}$(X) is the subspace of holomorphic functions on the Riemann surface X, then its (injective) restriction to ∂(X) is a subspace of $C^\infty(\partial X)$, and this splitting property will then apply seperately to each boundary circle component. The positive - (negative) frequency data on the n(m) bounding circles of the Riemann surface X is analogous to the in - (out -) states in a dynamical theory, so that the theory allows the intrinsic properties of X to determine the 'interaction' of these states.

The 4 - dimensional conformal field theory proposed in [18], [38], [39], replaces the Riemann surfaces with general flat - twistor spaces **Z**, having a boundary $\partial\mathbf{Z}$ consisting of n + m (disjoint, parameterised) copies of $\mathbf{P}^0$, n of which are positively oriented and m negatively oriented. The holomorphic functions $\mathcal{O}(X)$ are replaced by the elements of the sheaf - cohomology group $H^1(\mathbf{Z},(\mathcal{O}(-2s-2))$, these elements representing, via the Penrose transform, massless fields of helicity s on some 4 - dimensional space - time (hence the claim of a 4 - dimensional conformal field theory).



The general twistor space **Z** is a complex manifold of three dimensions, with a boundary consisting of disjoint copies of the five - real - dimensional space $\mathbf{P}^0$.

There is also an inclusion in this case, i.e. $H^1(\mathbf{Z},\mathcal{O}(-2s-2)) \subseteq H^1(\partial\mathbf{Z},\mathcal{O}(-2s-2))$ where the final group is 'CR - cohomology', and is a direct sum of m + n copies of

$H^1(\mathbf{P}^0, 0(-2s-2))$ (corresponding to the massless fields of helicity s on real, compactified Minkowski - space). The cohomology group $H^1(\mathbf{P}^0, 0(-2s-2))$ is itself the direct sum of $H^1(\overline{\mathbf{P}}^+, \mathcal{O}(-2s-2)) \oplus H^1(\overline{\mathbf{P}}^-, \mathcal{O}(-2s-2))$ where $\overline{\mathbf{P}}^\pm$ is the closure of $\mathbf{P}^\pm$, and this is the decomposition of cohomology into positive and negative - frequency parts, referred to earlier. This decomposition replaces the decomposition of holomorphic functions $\mathcal{O}(X)$ into positive and negative frequency parts in CFT2, and the roles of the Hilbert spaces H and H* are taken by $H^1(\mathbf{P}^+, 0(-2s-2))$ and $H^1(\mathbf{P}^-, 0(-2s-2))$ respectively.

The restriction placed on the boundary components of **Z**, that they are each a copy of $\mathbf{P}^0$, is a severe one but is met by a large class of twistor - spaces, including those which are twistor - spaces of Riemannian, conformally - flat 4 - manifolds, with spin - structure (corresponding to the existence of a fourth root of $\Lambda^3\mathbf{Z}$) and with a boundary consisting of a finite number of disjoint copies of the round sphere S^3.

In [39], Singer also gave an alternative possible twistor approach to conformal field theory, where the flat twistor spaces with $\mathbf{P}^0$ type boundaries are replaced by flat twistor spaces with punctures; a flat twistor space with punctures is a compact, boundaryless, flat twistor space from which a finite number of (non - intersecting) projective lines have been removed. This corresponds to an alternative possible approach to 2-dimensional conformal field theory which uses compact Riemann surfaces with a finite number of points removed, in place of the Riemann surfaces with circle - like boundary components. The Hilbert spaces H and H* are now considered to be attached to the punctures.

At the heart of the twistor conformal field theory programme then is the study of flat twistor spaces, and the associated 'massless - fields' that can be defined on them. In the case of standard flat twistor space $\mathbf{CP}^3$, information on $H^1(\overline{\mathbf{P}}^+, \mathcal{O}(-2s-2))$ could be obtained from knowledge of the elementary states in $H^1(\mathbf{CP}^3 - \mathbf{L}^-, \mathcal{O}(-2s-2))$, these being dense in the former space. One might then ask a similar question in the context of flat twistor spaces, i.e. if $\mathbf{Z}$ is a compact flat twistor space (without boundary) and $\mathbf{L}^-$ is a line on $\mathbf{Z}$ with a neighbourhood (biholomorphic to) $\mathbf{P}^-$, is there a relationship between the 'elementary states' in $H^1(\mathbf{Z} - \mathbf{L}^-, \mathcal{O}(-2s-2))$ and $H^1(\mathbf{Z} - \mathbf{P}^-, \mathcal{O}(-2s-2))$ where $\mathbf{Z} - \mathbf{P}^-$ has now a $\mathbf{P}^0$ boundary? This question is meaningless however, until the notion of an elementary state based on a line in flat twistor space is defined.

One possible approach is to try to extend an alternative definition of elementary states based on a line, which is implicit in the paper of Eastwood and Hughston, [7]. This definition arose as a result of the technique used by them in that work to give an alternative method of classifying the elmentary states based on a line $\mathbf{L}^+$ having a given homogeneity m and codimension - 2 pole of order at most ℓ. The collection of these elementary states is the group $H^1_\ell(\mathbf{CP}^3 - \mathbf{L}^+, \mathcal{O}(m))$ mentioned earlier in the discussion of elementary states. The techique they used was as follows: First blow up the line $\mathbf{L}^+$, to obtain the compact manifold $\mathbf{C}\widetilde{\mathbf{P}}^3$, and then put a certain holomorphic line bundle on this space. This line bundle, $\mathcal{O}(a,b)$, was the restriction to $\mathbf{C}\widetilde{\mathbf{P}}^3$ of the line bundle $\mathcal{O}(a) \otimes \mathcal{O}(b)$ on $\mathbf{CP}^3 \times \mathbf{CP}^1$. If $a \geq 0$ and $b \leq -2$ then it is shown in [7] that there is an injective restriction map: $H^1(\mathbf{C}\widetilde{\mathbf{P}}^3, \mathcal{O}(a,b)) \to H^1(\mathbf{CP}^3 - \mathbf{L}^+, \mathcal{O}(a+b))$, and the image is precisely the set of elementary states based on L^+ with a codimension - 2 pole of order at most $-b-1$ on L^+. This provides the alternative definition of elementary states, i.e. the elements in the image of the restriction map.

Blowing - up is a local construction which can be applied to any line **L** in a general flat twistor space **Z**, so that with a suitable choice of holomorphic line bundle $O(a,b)$ on the blown-up manifold $\tilde{\mathbf{Z}}$, we should have a good candidate for an element of $H^1(\mathbf{Z}-\mathbf{L}, O(a+b))$ with a codimension - 2 pole on **L** of given maximum order. One would then hope to extend this to the case of many lines, $\mathbf{L}_1, \ldots, \mathbf{L}_r$, so that we would have an interpretation of elements of $H^1(\mathbf{Z}-\mathbf{L}_1-\ldots-\mathbf{L}_r, O(-2s-2))$ which have codimension - 2 poles on $\mathbf{L}_1, \ldots, \mathbf{L}_r$ of prescribed maximum orders, say $\ell_1,\ldots,\ell_r$ on $\mathbf{L}_1, \ldots, \mathbf{L}_r$ respectively.

One can then ask the following question: if **Z** is a compact (boundaryless), flat twistor space and if $\mathbf{L}_1, \ldots, \mathbf{L}_r$ are (non-intersecting) projective lines in **Z**, what is the number of (linearly independent) elements of $H^1(\mathbf{Z}-\mathbf{L}_1-\ldots-\mathbf{L}_r, O(-2s-2))$ which have codimension-2 poles of order at most $\ell_1,\ldots,\ell_r$ on $\mathbf{L}_1, \ldots, \mathbf{L}_r$ respectively? **It is this question which is addressed in the following chapters, and we provide a solution when Z is the twistor space of a compact, boundaryless, hyperbolic 4-manifold, i.e. when this 4-manifold has constant scalar curvature -1, and no other curvature components.** We shall also give a partial answer in the case of positive scalar curvature conformally flat manifolds.

The question asked above has an analogue in Riemann surfaces and we shall discuss this briefly, since it provides a model for the method of solution which we employ in the twistor - space setting.

Let X be a compact Riemann surface with distinct points $P_1, \ldots, P_k$ and let $n_1, \ldots, n_k$ be arbitrary positve integers. How many linearly independent meromorphic functions are there on X, with poles exactly at $P_1, \ldots, P_k$, and with orders at most n_i on P_i respectively?

To answer the question one can adopt the following strategy:

(a) Convert the question to one involving global data.

This is achieved through the introduction of line bundles and divisors. The problem then becomes one of determining the dimension of the cohomology group $H^0(X,\mathcal{O}[D])$ where [D] is the line bundle of the divisor

$$D = \sum_{i=1}^{k} n_i P_i$$

(b) Use the Riemann - Roch theorem.

This enables holomorphic data to be calculated from topological data: specifically $\dim H^0(X,\mathcal{O}[D]) - \dim H^1(X,\mathcal{O}[D]) = \deg D + 1 - g$, where g is the genus of X.

(c) Use vanishing theorems for $H^1(X,\mathcal{O}[D])$ in order to eliminate the unwanted term.

One such is the Kodaira vanishing theorem; if $\deg(D) > 2g - 2$ then $H^1(X,\mathcal{O}[D]) = 0$.

It is essentially this strategy, modified to fit the case of flat twistor spaces, which is employed in the remainder of this work.

(A) In chapter 2 we shall show how to define a line bundle $0(a_1,...,a_r;b_1,...,b_r)$ on the blow - up $\tilde{\mathbf{Z}}$ of a flat twistor space $\mathbf{Z}$ along the non-intersecting lines $\mathbf{L}_1, \ldots, \mathbf{L}_r$, which is our candiate for the replacement of $O(a,b)$, used by Eastwood and Hughston (ibid). We shall show that the restriction map from $H^1(\tilde{\mathbf{Z}},O(a_1,...,a_r;b_1,...,b_r))$ to $H^1(\mathbf{Z}-\mathbf{L}_1-...-\mathbf{L}_r,O(a_1+b_1))$ (where $a_i + b_i = a_j + b_j$) is injective and give a characterisation of the image of this map. We shall demonstrate that the elements in this image have properties which make them a good choice for the cohomology classes with codimension - 2 poles on each $\mathbf{L}_i$ with a prescribed maximum order on each $\mathbf{L}_i$. As in (a) above, our question is now one involving global data, this time on $\tilde{\mathbf{Z}}$, i.e. what is the dimension of $H^1(\tilde{\mathbf{Z}},O(a_1,...,a_r;b_1,...,b_r))$?

(B) The Riemann - Roch theorem in (b) above is now replaced by the Hirzebruch - Riemann - Roch theorem. This enables the calculation of the holomorphic Euler characteristic of $O(a_1,...,a_r;b_1,...,b_r)$ on $\tilde{\mathbf{Z}}$ which is $\sum_{i=0}^{3}(-1)^i \dim H^i(\tilde{\mathbf{Z}},O(a_1,...,a_r;b_1,...,b_r))$, to be made in terms of topological data, given by certain Chern classes on $\tilde{\mathbf{Z}}$. This calculation is made in Chapter 3, using Poincaré duality and intersection of homology classes.

(C) Since we are interested in the dimension of the H^1 term, we need to find vanishing theorems for the other terms. In chapter 4 we show how the vanishing of these terms can be made dependent upon the vanishing of the cohomology classes $H^i(\mathbf{Z},O(m))$(where $m = a_j + b_j$), , on the flat twistor space **Z**. This simplifies the problem since vanishing theorems for flat twistor spaces are easier to find.

(D) In chapter 5 we shall then prove a number of vanishing theorems for $H^1(\mathbf{Z},O(m))$ for particular values of m, and for different twistor manifolds **Z**; in particular we shall prove one when **Z** is the twistor space of a self - dual, Einstein, compact, Riemannian 4 - manifold **X**, with negative scalar curvature. We shall also prove an extended version, which is for twistor spaces of compact quaternionic manifolds of negative scalar curvature. This will not be needed for our stated project, but is an interesting result which follows in a relatively painless manner, using a modification of the proof of the 4 - dimensional case. Both proofs use the Penrose transform to interpret the problem as a vanishing theorem on spinor fields.

(E) In chapter 6, we draw all of the information together to make our conclusions.

Chapter 2 Codimension-2 poles on flat twistor spaces.

2.1 Massless Fields based on a line.

We begin this chapter with an outline of the essential details of [7]. In that paper Eastwood and Hughston demonstrated that the elementary states based on a line **L** of $\mathbf{CP}^3$ could be neatly classified by regarding them as elements of a sheaf cohomology group defined on the blow-up along **L** of $\mathbf{CP}^3$; we denote this blow-up by $\mathbf{C\tilde{P}}^3$. An outline of this was given in chapter 1, but a more detailed description now follows.

The blow-up of $\mathbf{CP}^3$ along **L** is a subvariety of $\mathbf{CP}^3 \times \mathbf{CP}^1$ and the exceptional divisor is a quadric $\mathbf{CP}^1 \times \mathbf{CP}^1$. For integers a, b let $\mathcal{O}_{\mathbf{CP}^3}(a)$ be the sheaf of germs of holomorphic functions on $\mathbf{CP}^3$, which are homogeneous of degree a, and let $\mathcal{O}_{\mathbf{CP}^1}(b)$ be the corresponding sheaf on $\mathbf{CP}^1$. The sheaf $\mathcal{O}_{\mathbf{CP}^3 x \mathbf{CP}^1}(a,b)$ is then the tensor product of the pullbacks to $\mathbf{CP}^3 \times \mathbf{CP}^1$ of $\mathcal{O}_{\mathbf{CP}^3}(a)$ and $\mathcal{O}_{\mathbf{CP}^1}(b)$ respectively. The sheaf $\mathcal{O}_{\mathbf{C\tilde{P}}^3}(a,b)$ is then defined by the short exact sheaf sequence

$$0 \to \mathcal{O}_{\mathbf{CP}^3 x \mathbf{CP}^1}(a-1,b-1) \xrightarrow{\omega} \mathcal{O}_{\mathbf{CP}^3 x \mathbf{CP}^1}(a,b) \xrightarrow{\rho} \mathcal{O}_{\mathbf{C\tilde{P}}^3}(a,b) \to 0 \qquad (2.1.1)$$

where the first map is multiplication by the ideal sheaf of $\mathbf{C\tilde{P}}^3$, which is the subvariety of $\mathbf{CP}^3 x \mathbf{CP}^1$ given by $\omega = 0$, where $\omega \equiv W^2 Z^3 - Z^2 W^3$ and ρ is restricion to $\mathbf{C\tilde{P}}^3$.

This then leads to a long exact sequence in cohomology. If the parameters a, b have the restrictions $a \geq 0$, $b \leq -2$ imposed upon them, then part of this long exact sequence is

$$0 \to H^0(\mathbf{C\tilde{P}}^3, \mathcal{O}(a,b)) \to H^1(\mathbf{CP}^3 x \mathbf{CP}^1, \mathcal{O}(a-1,b-1)) \to H^1(\mathbf{CP}^3 x \mathbf{CP}^1, \mathcal{O}(a,b)) \to H^1(\mathbf{C\tilde{P}}^3, \mathcal{O}(a,b)) \to 0 \qquad (2.1.2)$$

where we have omitted the subscripts on the sheaves for notational convenience.
By examining the latter group the authors are able to conclude that its elements, when restricted away from the exceptional divisor E, to $\mathbf{C\tilde{P}} - \mathbf{E}$ (which is biholomorphic to $\mathbf{CP}^3 - \mathbf{L}$) are indeed representatives for elementary states based on **L**, of homogeneity a + b, and with a codimension - 2 pole on **L** of order at most $-b-1$, as defined in chapter 1.

The dimensions of the two intermediate terms of (2.1.2) can be calculated easily (using the Künneth formula) and the first term is isomorphic to $H^0(\mathbf{CP}^3, \mathcal{O}(a+b))$ (there is an easy direct proof of this). Using the rule that the alternating dimensions of an exact sequence sum to zero, they are then able to find the dimension of $H^1(\mathbf{C\tilde{P}}^3, \mathcal{O}(a,b))$ from (2.1.2).

Using a Leray cover of $\mathbf{CP}^3$ - **L** by two Stein open subsets, and a simple Mayer-Vietoris argument, they are able to obtain the dimension of the group $H^1_\ell(\mathbf{CP} - \mathbf{L}, \mathcal{O}(m))$ directly, where, as we saw in chapter 1, this is the group of all elementary states based on **L** of homogeneity m, and with a codimension - 2 pole on **L** of order at most ℓ. It turns out that when $m = a + b$ and $\ell = -b - 1$, with $a \geq 0$, $b \leq -2$, the dimension of this group agrees with that of $H^1(\mathbf{C\tilde{P}}^3, \mathcal{O}(a,b))$. The authors are then able to conclude that the restriciton map from $H^1(\mathbf{C\tilde{P}}^3, \mathcal{O}(a,b))$ to $H^1(\mathbf{CP}^3 - \mathbf{L}, \mathcal{O}(a+b))$ is injective, with image $H^1_\ell(\mathbf{CP}^3 - \mathbf{L}, \mathcal{O}(a+b))$.

There is also a simple characterisation of members of $H^1_\ell(\mathbf{CP}^3 - \mathbf{L}, \mathcal{O}(m))$ given in terms of a certain homogeneous ideal I(L). This can be seen most easily if we use co-ordinates, with **L** given by $Z^2 = Z^3 = 0$. An elementary state based on **L**, homogeneous of degree m, with a codimension - 2 pole of order at most ℓ on **L** has a representative of the form

$$\sum_{\substack{0<j,k \\ j+k\leq\ell+1}} \frac{A_{jk}}{(Z^2)^j (Z^3)^k} \qquad (2.1.3)$$

where A_{jk} is a polynomial on **L** (i.e. a polynomial in Z^0, Z^1)) which is homogeneous of degree $j + k + m$. The characterisation cited above is no more than the fact that every element of the sum (2.1.3) is annihilated in H^1 by $(Z^2)^j (Z^3)^k$ for some $j + k = \ell$.

Throughout the remainder of this chapter we shall take Z to be a compact, boundaryless, flat twistor space, so that every projective line (CP^1) in Z will have a neighbourhood base consisting of open sets which are biholomorphic to P^+.

The canonical bundle of **Z**, Λ^3**Z**, is identified with $0(-4)$ and there is always a square root bundle for this, though existence of a fourth - root bundle requires extra structure (see [16]) We shall ignore this restriction and simply assume that whenever we refer to the bundle $\mathcal{O}(m)$ on **Z**, the appropriate conditions for its existence are satisfied.

2.2 The definition of the bundle $\mathcal{O}(a_1, \ldots, a_r; b_1, \ldots, b_r)$ on $\widetilde{Z}$

In this section we shall define a line bundle on $\widetilde{Z}$, the blow-up of **Z** along a finite number of non-interecting lines in **Z**, which is our replacement for the bundle $\mathcal{O}(a,b)$ discussed in 2.1. We will also show that the restriction map from the first cohomology group on $\widetilde{Z}$ with coeffients in this line bundle, to the appropriate cohomology group on **Z** - **L**, is injective, where **L** is the union of those lines. The proof of this result will depend upon the results of [7], which were outlined in 2.1.

Let the distinguished lines be $L_1, \ldots, L_r$, and since they are non-intersecting in pairs, we can choose neighbourhoods N_i for each L_i, biholomorphic to P^+, and with $N_i \cap N_j$ empty if $i \neq j$. Since blowing up is a local process, we may blow-up each N_i to obtain $\widetilde{N}_i$, so that if $N = N_1 \cup \ldots \cup N_r$, then $\widetilde{N} = \widetilde{N}_1 \cup \ldots \cup \widetilde{N}_r$, and $\widetilde{Z} = (Z - L) \cup \widetilde{N}$. (For the definition of blow-ups, see [13].)

For the construction of the bundle let us consider just a single line $\mathbf{L}_i$, with neighbourhood $\mathbf{N}_i$. As in the case of $\mathbf{CP}^3$ (considering $\mathbf{N}_i$ as $\mathbf{P}^+$), $\tilde{\mathbf{N}}_i$ is a subvariety of $\mathbf{N}_i$ x $\mathbf{CP}^1$.

Given an integer m then, choose any integer b_i and let $a_i + b_i = m$. The construction of the line bundle $\mathcal{O}_{\tilde{CP}^3}(a,b)$ in 2.1 is dependent only upon the fact that $\tilde{\mathbf{CP}}^3$ is a subvariety of $\mathbf{CP}^3$ x $\mathbf{CP}^1$. Obviously $\tilde{\mathbf{P}}^+$ is also a subvariety of $\mathbf{P}^+$ x $\mathbf{CP}^1$, so that the same construction for a bundle $\mathcal{O}_{\tilde{P}^+}(a,b)$ on $\tilde{\mathbf{P}}^+$ will also hold. In particular $\mathcal{O}_{\tilde{N}_i}(-b_i, b_i)$ is well defined and, away from the exceptional divisor, it behaves like the trivial bundle. If we identify $\tilde{\mathbf{N}}_i$ with $\tilde{\mathbf{P}}^+$, this bundle can then be extended to the whole of $\tilde{\mathbf{Z}}$, by identifying it with the trivial bundle on $\mathbf{Z} - \mathbf{L}$ (which is biholomorphic to $\tilde{\mathbf{Z}}$ - $\mathbf{E}$). Call this bundle $\mathcal{O}(-b_1,\ldots,-b_r; b_1,\ldots,b_r)$.

The blow-up comes equipped with the usual map $g:\tilde{\mathbf{Z}}\rightarrow\mathbf{Z}$ and we define our bundle as

$$\mathcal{O}(a_1,\ldots,a_r;b_1,\ldots,b_r) = g^{-1}(\mathcal{O}(m))\otimes\mathcal{O}(-b_1,\ldots,b_r;b_1,\ldots,b_r) \qquad (2.2.1)$$

(with $a_i + b_i = m$ for each $i = 1,\ldots,r$).

The definition of this line bundle requires only that $a_i + b_i = m$ for each i, but we shall impose further restrictions for our use of it in defining codimension-2 poles. For this we shall impose the additional constraints that $a_i \geq 0$ and $b_i \leq -2$, corresponding to the identical conditions imposed on $\mathcal{O}(a,b)$ in [7].

The proof of the injectivity of the restriction map uses local cohomology and we recall the relevant facts.

2.2.2 Proposition

Let U denote the complement of the closed subset A of X. A sheaf F on X gives rise to a long exact sequence in cohomology.

$$\rightarrow H^p_A(X,F) \rightarrow H^p(X,F) \xrightarrow{r} H^p(U,F) \rightarrow$$

where $H^p_A(X,F)$ is the p'th local cohomology group of X with support in A and coefficients in F. The map r is the restriction map.

2.2.3 Proposition

If $A \subseteq V \subseteq X$ and if A is a closed and V an open subset of X, then we have the following (excision) isomorphism.

$$H^p_A(X,F) \cong H^p_A(V,F).$$

Both of these propositions may be found in [20]. They will allow us to prove the following.

2.2.4 Proposition

Let **Z** be a compact, boundaryless, flat, twistor space and let $\tilde{\mathbf{Z}}$ be the blow-up of **Z** along $\mathbf{L} = \mathbf{L}_1 \cup \ldots \cup \mathbf{L}_r$ as described above. Let $\mathcal{O}(a_1,\ldots,a_r;b_1,\ldots,b_r)$ be the line bundle whose construction has been given above, with $a_i + b_i = m$, $a_i \geq 0$, $b_i \leq -2$ for $i = 1,\ldots,r$. Then the restriction map

$$H^1(\widetilde{\mathbf{Z}}, O(a_1, \ldots a_r; b_1, \ldots, b_r)) \to H^1(\mathbf{Z} - \mathbf{L}, O(m))$$

is injective.

Proof

We shall prove this when **L** is a single line, with $a \geq 0$, $b \leq -2$, $a + b = m$, and the extension to many lines will be obvious. The local cohomology exact sequence gives

$$\to H^1_E(\widetilde{\mathbf{Z}}, O(a,b)) \to H^1(\widetilde{\mathbf{Z}}, O(a,b)) \to H^1(\mathbf{Z} - \mathbf{L}, O(m)) \to \qquad (2.2.4)$$

where we identify $\widetilde{\mathbf{Z}}$ -**E** with **Z** - **L**, **E** being the exceptional divisor. We must show that the first group is zero.

For a flat twistor space the line **L** is contained in some neighbourhood biholomorphic to $\mathbf{P}^+$, so we have

$$\mathbf{L} \subseteq \mathbf{P}^+ \subseteq \mathbf{Z} \quad \text{and} \quad \mathbf{L} \subseteq \mathbf{P}^+ \subseteq \mathbf{CP}^3$$

so that (2.2.5)

$$\mathbf{E} \subseteq \widetilde{\mathbf{P}}^+ \subseteq \widetilde{\mathbf{Z}} \quad \text{and} \quad \mathbf{E} \subseteq \widetilde{\mathbf{P}}^+ \subseteq \mathbf{C}\widetilde{\mathbf{P}}^3$$

Using the excision theorem (2.2.3) we have

$$H^1_E(\widetilde{\mathbf{Z}}, O(a,b)) \cong H^1_E(\widetilde{\mathbf{P}}^+, O(a,b)) \cong H^1_E(\mathbf{C}\widetilde{\mathbf{P}}^3, O(a,b)) \qquad (2.2.6)$$

Now consider the local cohomology exact sequence for $\mathbf{C\widetilde{P}}^3$, i.e.

$$\begin{aligned} H^0(\mathbf{C\widetilde{P}}^3, O(a,b)) \xrightarrow{r_0} H^0(\mathbf{CP}^3 - \mathbf{L}, O(m)) \rightarrow H^1_E(\mathbf{C\widetilde{P}}^3, O(a,b)) \rightarrow \\ \rightarrow H^1(\mathbf{C\widetilde{P}}^3, O(a,b)) \xrightarrow{r_1} H^1(\mathbf{CP}^3 - \mathbf{L}, O(m)) \end{aligned} \tag{2.2.6}$$

where r_0, r_1 are the respective restriction maps. From the discussions in 2.1 we know that r_0 is an isomorphism, whilst r_1 is injective, so that $H^1_E(\mathbf{C\widetilde{P}}^3, O(a,b)) = 0$.

2.3 Codimension 2-poles.

We now wish to show that an element of the group $H^1(\widetilde{\mathbf{Z}}, O(a_1,\dots,a_r; b_1,\dots,b_r))$, when restricted away from the exceptional divisor, to an element of $H^1(\mathbf{Z} - \mathbf{L}, O(m))$, will indeed have a singularity structure on each line with a good claim to being a codimension 2 - pole. We shall display this singularity structure for Cech cocycles with respect to coverings of $\widetilde{\mathbf{Z}}$ and **Z** - **L** by certain Stein open sets, so that the cocycles will be a representative for the elements as members of the sheaf cohomology groups. (For the properties of Stein sets, see [12].).

Since singularity properties are local it will suffice to examine the case when **L** is a single line. We suppose that **L** has a neighbourhood (biholomorphic to) $\mathbf{P}^+$ and we shall take **L** to be given by $Z^2 = Z^3 = 0$ again. We first choose a cover for **Z** by Stein open sets with the property that if $\{U_i\}$ are the sets in this cover for which $\mathbf{L} \cap U_i \neq \phi$, then $U_i \subseteq \mathbf{P}^+$. (This is easy to arrange by first covering $\mathbf{P}^+$ by Stein open sets in $\mathbf{P}^+$, then covering **Z** - **L** by Stein open sets.).

Now think of $\mathbf{P}^+$ as a subset of $\mathbf{CP}^3$. In this setting the sets $V_j = \{[Z] \in \mathbf{CP}^3 : Z^j \neq 0\}$, $j = 2,3$, are Stein open sets. The sets $\{U_i \cap V_j\}$, together with the other sets in the cover for **Z**, is an open cover, by Stein open sets, of **Z** - **L**. Call this cover V.

In a similar fashion we can construct a Stein open cover for $\widetilde{\mathbf{Z}}$. Begin with the original open cover of **Z** given above, so that the $\{U_i\}$ for which $U_i \cap \mathbf{L} \neq \phi$ are all subsets of $\mathbf{P}^+$. The blow-up of $\mathbf{P}^+$ is a subvariety of $\mathbf{P}^+ \times \mathbf{CP}^1$ so that any Stein open cover for $\mathbf{P}^+ \times \mathbf{CP}^1$, when restricted to $\widetilde{\mathbf{P}}^+$ (which is a closed subset of $\mathbf{P}^+ \times \mathbf{CP}^1$) is a Stein open cover of $\widetilde{\mathbf{P}}^+$.

Let the (vertical) $\mathbf{CP}^1$ have homogeneous co-ordinates $[W^2 : W^3]$, so that $W_j = \{[W] \in \mathbf{CP}^1 : W^j \neq 0\}$, j = 2,3, is an open cover of $\mathbf{CP}^1$ by Stein sets. The restriction to $\widetilde{\mathbf{P}}^+$ of the Stein sets $\{U_i \times W_j\}$ is an open cover for $\widetilde{\mathbf{P}}^+$ and these, together with the Stein sets which cover **Z**- **L**, form an open cover of $\widetilde{\mathbf{Z}}$ by Stein sets. Call this cover $\mathcal{W}$. We note here that $\widetilde{\mathbf{P}}^+$ is the subvariety of $\mathbf{P}^+ \times \mathbf{CP}^1$ defined by the equation $Z^2 W^3 - Z^3 W^2 = 0$. More details of the blow - up may be found in chapter 3.

We are now in a position to display the singuarity structure on **L**, of the elements in the image of the restriction map which was investigated in Proposition 2.2.3. This will give a characterisation of the image of that map and provide two equivalent ways of defining codimension - 2 poles. The information required is contained in the following proposition.

2.3.1 Proposition

Let $\widetilde{\mathbf{Z}}$ be the blow-up of **Z** along the line **L**, and let $\mathcal{W}, \mathcal{V}$, be the covers for $\widetilde{\mathbf{Z}}$ and $\mathbf{Z} - \mathbf{L}$ respectively, which are defined above. Suppose that (a,b) are integers with $a \geq 0$ and $b \leq -2$.

(a) Let $\{\tilde{p}_{\alpha\beta}\}$ be a cech 1 - cocycle for the cover *W*, representing an element of $H^1(\tilde{\mathbf{Z}},\mathcal{O}(a,b))$ and let $\{p_{\alpha\beta}\}$ be its restriction to **Z** - **L**. Then $\{p_{\alpha\beta}\}$ is a Cech 1 - cocycle for the cover *V* and if $p_{\alpha\beta}$ is defined on $(U_i \cap V_j) \cap (U_k \cap V_l)$, with $U_i \cap U_j \cap \mathbf{L} \neq \phi$, then $p_{\alpha\beta}$ has the form

$$\frac{h_0 g_0}{(Z^2)^{-b}} + \frac{h_1 g_1}{(Z^2)^{-b-1}(Z^3)} + \ldots\ldots\ldots\ldots + \frac{h_{-b-1} g_{-b-1}}{(Z^2)(Z^3)^{-b-1}} + \frac{h_{-b} g_{-b}}{(Z^3)^{-b}} \tag{2.3.2}$$

where the g_i are homogeneous, holomorphic functions of $[Z^2, Z^3]$, with homogeneity zero, and defined on $V_j \cap V_l$, and the h_i are holomorphic homogeneous functions of degree a with holomorphic extension to $U_i \cap U_j$.

(b) Conversely if $\{p_{\alpha\beta}\}$ is a Cech 1-cocycle with respect to *V*, representing an element of $H^1(\mathbf{Z} - \mathbf{L}, \mathcal{O}(a+b))$, and if $p_{\alpha\beta}$ is defined on $(U_i \cap V_j) \cap (U_k \cap V_l)$ for which $U_i \cap U_k \cap \mathbf{L} \neq \phi$, and has the form (2.3.2), then there is a Cech 1 - cocycle $\{\tilde{p}_{\alpha\beta}\}$ for *W*, which represents an element of $H^1(\tilde{\mathbf{Z}},\mathcal{O}(a,b))$, whose restriction to **Z** - **L** is in the cohomology class of $\{p_{\alpha\beta}\}$.

Proof

(a) Let $\{\tilde{p}_{\alpha\beta}\}$ be defined on the (restrictions to $\tilde{\mathbf{P}}^+$ of the) sets $(U_i \times W_j) \cap (U_k \times W_l) = (U_i \cap U_k) \times (W_j \cap W_l)$, where $U_i \cap U_k \cap \mathbf{L} \neq \phi$. We need not concern ourselves with any other sets since they will be sufficiently far from **L** so as not to display any singularity. The element $\{\tilde{p}_{\alpha\beta}\}$ is then a sum of elements of the form $d(Z)e(W)\big|_{\tilde{\mathbf{P}}^+}$, where $d(Z) \in \mathcal{O}(a)(U_i \cap U_k)$ and $e(W) \in \mathcal{O}(b)(W_j \cap W_l)$. The function e(W) can be written as a Laurent series and will have the form

$$e(W) = \frac{1}{(W^2)^{-b}} \sum_{r=0}^{\infty} a_r \left(\frac{W^3}{W^2}\right)^r + \frac{1}{(W^3)^{-b}} \sum_{s=0}^{\infty} c_s \left(\frac{W^2}{W^3}\right)^s + \frac{e_1}{(W^2)(W^3)^{-b-1}} + \ldots + \frac{e_{-b-1}}{(W^2)^{-b-1}(W^3)}$$

When $d(Z)e(W)$ is restricted to $\widetilde{\mathbf{P}}^+$ and <u>away</u> from the exceptional divisor (so that $W^2 : W^3 = Z^2 : Z^3$) it will become

$$\frac{d(Z)}{(Z^2)^{-b}} \sum_{r=0}^{\infty} a_r \left(\frac{Z^3}{Z^2}\right)^r + \frac{d(Z)}{(Z^3)^{-b-1}} \sum_{s=0}^{\infty} c_s \left(\frac{Z^2}{Z^3}\right)^s + \frac{d(Z)e_1}{(Z^2)(Z^3)^{-b-1}} + \ldots + \frac{d(Z)e_{-b-1}}{(Z^2)^{-b-1}(Z^3)}$$

and this has the form described in (2.3.2).

(b) Conversely, suppose that $\{p_{\alpha\beta}\}$ is a 1 - cocycle for V with the stated property, i.e. $p_{\alpha\beta}$ is defined on $(U_i \cap V_j) \cap (U_k \cap V_l)$ with $U_i \cap U_j \cap L \neq \phi$, and

$$p_{\alpha\beta} = \sum_{t=0}^{-b} \frac{h_t g_t}{(Z^2)^{-b-t}(Z^3)^t} \qquad (2.3.3)$$

where h_t is holomorphic and homogeneous of degree a on the whole of $U_i \cap U_k$ and g_t is a holomorphic function of $[Z^2,Z^3]$ defined on $V_j \cap V_l$ and homogeneous of degree zero.

Now choose $\widetilde{p}_{\alpha\beta}$ on the (restriction to $\widetilde{\mathbf{P}}^+$ of the) set $(U_i \times W_j) \cap (U_k \times W_l)$ $= (U_i \cap U_k) \times (W_j \cap W_l)$ as follows. The g_t of (2.3.3) are holomorphic functions of (Z^2,Z^3) only, with homogeneity zero, so that $\frac{g_t(W)}{(W^2)^{-b-t}(W^3)^t} \in \mathcal{O}(b)(W_j \cap W_l)$ (since g_t is defined on $V_j \cap V_l$). The $h_t(z)$ are holomorphic on $U_i \cap U_k$ and homogeneous of degree a, so that if we take

$$\widetilde{p}_{\alpha\beta} = \left(\sum_{t=0}^{-b} h_t(z) . \frac{g_t(W)}{(W^2)^{-b-t}(W^3)^t}\right)\Bigg|_{\widetilde{P}^+}$$

then $\tilde{p}_{\alpha\beta}$, when restricted away from the exceptional divisor, to **Z** - **L**, is precisely $p_{\alpha\beta}$.

It remains to show that $\{\tilde{p}_{\alpha\beta}\}$ satisfy the cocycle conditions.

Suppose that $p_{\alpha\beta} - p_{\alpha\gamma} + p_{\beta\gamma} = 0$, and the common domain of these three elements is $(U_i \cap V_j) \cap (U_k \cap V_l) \cap (U_m \cap V_n) \neq \phi$ with $U_i \cap U_k \cap U_m \cap \mathbf{L} \neq \phi$. Then $\tilde{p}_{\alpha\beta} - \tilde{p}_{\alpha\gamma} + \tilde{p}_{\beta\gamma}$ is a holomorphic function defined on the (restriction to $\tilde{\mathbf{P}}^{+}$ of the) set $(U_i \cap U_k \cap U_m) x (W_j \cap W_l \cap W_n)$.

When the holomorphic function $\tilde{p}_{\alpha\beta} - \tilde{p}_{\alpha\gamma} + \tilde{p}_{\beta\gamma}$ is restricted away from the exceptional divisor, to the above set, it becomes $p_{\alpha\beta} - p_{\alpha\gamma} + p_{\beta\gamma}$, which vanishes there. Since $\tilde{p}_{\alpha\beta} - \tilde{p}_{\alpha\gamma} + \tilde{p}_{\beta\gamma}$ is holomorphic and zero on an open set, then it vanishes identically.

2.3.4 Remarks

(a) The form of the singularity given by (2.3.2) has a strong resemblance to that of the elementary state (2.1.3), though one way in which it differs is that (2.3.2) has the extra terms $\frac{h_0 g_0}{(Z^2)^{-b}}$ and $\frac{h_{-b} g_{-b}}{(Z^3)^{-b}}$. In the case of elementary states, the analogous terms would be coboundary terms. It seems likely that this is also true in the case considered here, though we have not been able to prove this.

(b) As mentioned after (2.1.3), elements of H^1 ($\mathbf{CP}^3 - \mathbf{L}$, $\mathcal{O}(m)$) which are elementary states on **L** with a singularity of order at most ℓ there, can be characterised as those members of the group which have representatives which can be annihilated by multiplication with $(Z^2)^j(Z^3)^k$ for some $j + k = \ell$, with $j > 0$ and $k > 0$. In the more general situation being considered here we have to allow this "annihilation" by a more general homogeneous term, i.e. by $\frac{(Z^2)^j(Z^3)^k}{g_k}$, where $j + k = \ell$, with $j \geq 0$, $k \geq 0$ and g_k a homogeneous function of degree zero in $[Z^2, Z^3]$.

(c) We can now say that an element of $H^1(\mathbf{Z} - \mathbf{L}, \mathcal{O}(m))$ has a codimension - 2 pole of order at most ℓ on the line **L**, if its singularity structure there is of the form given by (2.3.2). When **L** is the union of the disjoint lines $\mathbf{L}_1,\ldots,\mathbf{L}_r$, we can define codimension - 2 poles on each $\mathbf{L}_i$ by simply extending this to each line $\mathbf{L}_i$, i.e. we have $a_i \geq 0$, $b_i \leq -2$ and $a_i + b_i = m$ for $i = 1, \ldots, r$. These conditions correspond to those imposed by Eastwood and Hughston in the case of elementary states based on a line in $\mathbf{CP}^3$, and are sufficient to ensure that $H^1(\widetilde{\mathbf{CP}}^3, \mathcal{O}(a,b))$ is non-zero. Since we wish to have a genuine generalisation of this case, the conditions $a_i \geq 0$, $b_i \leq -2$ are natural ones. Thus we shall give the following definition:

2.3.5 Definition

An element of $H^1(\mathbf{Z} - \mathbf{L}, \mathcal{O}(m))$ will be said to have a pole of order at most ℓ_i on $\mathbf{L}_i$, for $i = 1, \ldots, r$, if there exist $a_i \geq 0$, $b_i \leq -2$, with $a_i + b_i = m$ and $\ell_i = -b_i - 1$, such that the element is in the image of the restriction map $H^1(\widetilde{\mathbf{Z}}, \mathcal{O}(a_1,\ldots,a_r;b_1,\ldots,b_r)) \to H^1(\mathbf{Z} - \mathbf{L}, \mathcal{O}(m))$.

As mentioned in the introductory chapter, our objective in this work is to obtain the dimension of $\mathbf{H}^1(\widetilde{\mathbf{Z}}, 0(\mathbf{a}_1,\ldots,\mathbf{a}_r;\mathbf{b}_1,\ldots,\mathbf{b}_r))$.

Chapter 3 Calculation of the holomorphic Euler characteristic of $\mathcal{O}(a_1,\dots,a_r;b_1,\dots,b_r)$ on $\tilde{Z}$

Throughout this chapter, Z will be the twistor space of X, where X is a (boundaryless) compact, Riemannian, self-dual 4 - manifold, and $\tilde{Z}$ will be its blow-up along L, which is the union of the non-intersecting lines $L_1,\dots,L_r$, which are fibres of Z over X. We shall take our bundle ξ to be

$$\xi = \mathcal{O}(a_1,\dots,a_r;b_1,\dots,b_r) \tag{3.1.1}$$

as defined in the previous chapter, subject only to the condition that $a_i + b_i = m$ for $i = 1,\dots,r$.

3.1 The strategy for the calculation.

In this section we outline the approach which is adopted for our calculation, together with some of the tools to be used. As mentioned above, the holomorphic Euler characteristic contains information on the dimensions of various cohomology groups. In fact, if ξ is a holomorphic vector bundle over a complex manifold, M, then

$$\chi(\xi) = \chi(M,\xi) = \sum_{q=o}^{n}(-1)^q \dim H^q(M,\xi) \tag{3.1.2}$$

is the holomorphic Euler characteristic of ξ, where n is the complex dimension of M. (See [14],[40]).

The calculation of $\chi(\xi)$ is made by the Hirzebruch-Riemann-Roch theorem, i.e.

$$\chi(\xi) = \big(ch(\xi)t(T(M))\big)[M] \tag{3.1.3}$$

where ch(ξ) is the Chern character of ξ and t(T(M)) is the Todd class of the holomorphic tangent bundle T(M), of M, ([14], [44]).

Determination of the Chern character and Todd class require knowledge of the individual Chern classes. We begin with the Chern classes of $T(\tilde{Z})$ and for these we use a formula, originally due to Porteous for algebraic manifolds but subsequently extended to the case of analytic manifolds ([14], pp 175, 176).

Let **L** be the disjoint union of the lines $L_1, \ldots, L_r$, in **Z** along which the blow-up of **Z** occurs and let **E** be the exceptional divisor. Then we have the following commutative diagram.

$$\begin{array}{ccc} \mathbf{E} & \xrightarrow{\ j\ } & \tilde{\mathbf{Z}} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \mathbf{L} & \xrightarrow{\ i\ } & \mathbf{Z} \end{array} \qquad (3.1.4)$$

where i, j are embeddings of submanifolds. The following holds:

$$g^*\left(ch(T(Z))\right) - ch\left(T(\tilde{Z})\right) = \frac{\left(1-e^{-h}\right)}{h} j_*\left(f^*[ch(\upsilon)] - j^*\left(e^h\right)\right) \qquad (3.1.5)$$

where υ is the normal bundle of L in Z, and $h \in H^2(\tilde{Z})$ is the Poincaré dual of the class represented by the cycle **E**. The map j_* is defined as follows: given a cohomology class in $H^*(E)$, find its Poincaré dual in $H_*(E)$, push forward into $H_*(\tilde{Z})$, then find its Poincaré dual in $H^*(\tilde{Z})$.

Since Chern classes play such a dominant role in this chapter we give a summary of the relevant facts. (See [6], [14]) Let ξ be a complex (differentiable) vector bundle of rank r, over a differentiable manifold M. The total Chern class of ξ is then

$$c(\xi) = 1 + c_1(\xi) + \ldots + c_r(\xi), \tag{3.1.6}$$

which is an element of H*(M,C). Now introduce the formal factorisation

$$c(\xi) = \prod_{i=1}^{r}(1 + x_i) \tag{3.1.7}$$

where $x_i \in H^*(M,C)$, so that $c_j(\xi)$ is identified with the elementary symmetric polynomial of order j in the x_i. Thus

$$c_1(\xi) = x_1 + \ldots + x_r \tag{3.1.8}$$

and

$$c_r(\xi) = x_1 \ldots x_r \tag{3.1.9}$$

Then the Todd class and Chern character of ξ are given, respectively by

$$t(\xi) = \prod_{i=1}^{r} \frac{x_i}{1 - e^{-x_i}} \tag{3.1.10}$$

$$ch(\xi) = \sum_{i=1}^{r} e^{x_i} \tag{3.1.11}$$

3.2 Calculation of some Chern classes.

In this section we shall calculate $j_*(f^*[ch(\upsilon)] - j^*(e^h))$. Before proceeding we make the following observation. As mentioned earlier, one of the characteristics of a twistor space is that the projective lines which fibre the space each have a normal bundle which is isomorphic to $H \oplus H = \mathcal{O}(1)\oplus\mathcal{O}(1)$. This normal bundle can then be identified with a tubular neighbourhood of the projective line, so that any two projective lines in distinct twistor spaces will have diffeomorphic tubular neighbourhoods. Since blowing-up a line is a topological process, and depends only on local properties, and since Chern classes are invariant under diffeomorphisms, any calculation using Chern classes of either the normal bundle of one of these projective lines in twistor space, or the normal bundle of the blow-up of the line, can be made on the assumption that the line is in $\mathbf{CP}^3$.

We begin by assuming that $\mathbf{L}$ is a single line. Once we have established the values of the various Chern classes in this case, the extension to the case where $\mathbf{L}$ is a disjoint union of the lines $\mathbf{L}_1,\ldots,\mathbf{L}_r$ is straightforward.

As noted the normal bundle of a projective line in twistor space is $H\oplus H$. The total Chern class of υ is thus

$$\begin{aligned} c(\upsilon) &= c(H\oplus H) = c(H).c(H) \\ &= (1 + \omega)^2 \qquad (3.2.1) \\ &= 1 + 2\omega \end{aligned}$$

where $\omega \in H^2(\mathbf{CP}^1)$ is a generator ([44], p.226), so that $c_1(\upsilon) = 2\omega$. Now $ch(\upsilon) = e^{x_1} + e^{x_2} = 2 + (x_1 + x_2)$, where $x_1 + x_2 = c_1(\upsilon)$. Thus $ch(\upsilon) = 2 + 2\omega$.

The exceptional divisor $\mathbf{E}$, is a product, $\mathbf{CP}^1 \times \mathbf{CP}^1$. Applying the Künneth formula to $\mathbf{E}$, we obtain

$$H^2(\mathbf{E}) = H^2(\mathbf{CP}^1 \times \mathbf{CP}^1) = H^2(\mathbf{CP}^1) \otimes H^0(\mathbf{CP}^1) \oplus H^0(\mathbf{CP}^1) \otimes H^2(\mathbf{CP}^1). \quad (3.2.2)$$

We shall take α to be a (horizontal) generator of $H^2(\mathbf{CP}^1 \times \mathbf{CP}^1)$ in $H^2(\mathbf{CP}^1) \otimes H^0(\mathbf{CP}^1)$, β to be a (vertical) generator of $H^2(\mathbf{CP}^1 \times \mathbf{CP}^1)$ in $H^0(\mathbf{CP}^1) \otimes H^2(\mathbf{CP}^1)$. We now have

$$f^*(\omega) = \alpha$$

and (3.2.3)

$$f^*(ch(\upsilon)) = 2 + 2\alpha$$

Our next step will be to find $j^*(h)$ and hence $j^*(e^h)$.

We note first that $h = \Phi$, where Φ is the Thom class of υ_E, the normal bundle of **E** in $\widetilde{\mathbf{Z}}$ (see [6], p.67).

We may consider $j : \mathbf{E} \rightarrow \upsilon_E$ (identified as a tubular neighbourhood of **E** in $\widetilde{\mathbf{Z}}$) as the zero section and by Prop 6.41 of ([6] p.74), $j^*(h) = j^*(\Phi)$ is the Euler class (in this case the first Chern class) of υ_E. Since **E** has complex codimension 1 in $\widetilde{\mathbf{Z}}$, υ_E is a line bundle and there is an explicit formula for the first Chern class (see e.g. [6], p.73, (6.38)), viz

$$c_1(\upsilon_E) = -\frac{1}{2\pi i}\sum_{\gamma} d\left(p_\gamma d\log g_{\gamma\alpha}\right) \qquad \text{on } U_\alpha \qquad (3.2.4)$$

where $\{U_\gamma : \gamma \in I\}$ is an open cover for the manifold **E**, $\{p_\gamma : \gamma \in I\}$a partition of unity subordinate to U_γ, and $(g_{\gamma\alpha})$ are the transition functions for the line bundle υ_E over **E**.

Our next step is to find the transition functions for υ_E. The normal bundle of **E** in $\tilde{\mathbf{Z}}$ is given by

$$\upsilon_E = \frac{\left(T'_{\tilde{Z}}\right)\Big|_E}{T'_E}$$

i.e. the quotient of the holomorphic tangent bundle of $\tilde{\mathbf{Z}}$, restricted to **E**, by the holomorphic tangent bundle of **E**. We shall therefore need to find the transition functions for $T'_{\tilde{Z}}$, and the required transition functions may easily be extracted from this data. Since we are considering the case of $\mathbf{Z} = \mathbf{CP}^3$, the manifold $\tilde{\mathbf{Z}}$ is defined as a particular subvariety of $\mathbf{CP}^3 \times \mathbf{CP}^1$, so that a covering of $\tilde{\mathbf{Z}}$ can be formed from the obvious covering of $\mathbf{CP}^3 \times \mathbf{CP}^1$.

On $\mathbf{CP}^3 \times \mathbf{CP}^1$ we have the following cover:

Let $\quad U_i = \{\underline{z} : z^i \neq 0\}, \quad i = 0, 1, 2, 3$

$$V_j = \{\underline{w} : w^j \neq 0\}, \quad j = 2, 3 \tag{3.2.5}$$

where $\{U_i : i = 0, 1, 2, 3\}$ covers $\mathbf{CP}^3$ and $\{V_j : j = 2, 3\}$ covers $\mathbf{CP}^1$.

Then $\{U_i \times V_j : i,j\}$ covers $\mathbf{CP}^3 \times \mathbf{CP}^1$. Take the $\mathbf{CP}^1$ to be blown up in $\mathbf{CP}^3$ as $z^2 = z^3 = 0$. In this case neither U_2 nor U_3 intersect the distinguished line, so we need only consider the covering given by

$$\mathcal{U}_0 = U_0 \times V_2 \qquad \mathcal{U}_1 = U_0 \times V_3 \tag{3.2.6}$$
$$\mathcal{U}_2 = U_1 \times V_2 \qquad \mathcal{U}_3 = U_1 \times V_3.$$

The manifold $\widetilde{\mathbf{Z}}$ is then given by

$$\widetilde{\mathbf{Z}} = \{(\underline{z}\ ,\ \underline{w}) \in \mathbf{CP}^3 \times \mathbf{CP}^1 : z^2w^3 - z^3w^2 = 0\}. \tag{3.2.7}$$

Local coordinates for $\widetilde{\mathbf{Z}}$ can then easily be seen to be:

(i) in U_0 $\left\{\left[\frac{z^1}{z^0}, \frac{z^2}{z^0}, \frac{w^3}{w^2}\right] z^0w^2 \neq 0\right\}$

(ii) in U_1 $\left\{\left[\frac{z^1}{z^0}, \frac{z^3}{z^0}, \frac{w^2}{w^3}\right] z^0w^3 \neq 0\right\}$

(iii) in U_2 $\left\{\left[\frac{z^0}{z^1}, \frac{z^2}{z^1}, \frac{w^3}{w^2}\right] z^1w^2 \neq 0\right\}$

(iv) in U_3 $\left\{\left[\frac{z^0}{z^1}, \frac{z^3}{z^1}, \frac{w^2}{w^3}\right] z^1w^3 \neq 0\right\}$

The change of variable functions are then, with the obvious meanings

$$\varphi_{10}(u,v,w) = \left[u, vw, \frac{1}{w}\right]$$

$$\varphi_{20}(u,v,w) = \left[\frac{1}{u}, \frac{v}{u}, w\right]$$

$$\varphi_{30}(u,v,w) = \left[\frac{1}{u}, \frac{vw}{u}, \frac{1}{w}\right]$$

$$\varphi_{21}(u,v,w) = \left[\frac{1}{u}, \frac{vw}{u}, \frac{1}{w}\right]$$

$$\varphi_{31}(u,v,w) = \left[\frac{1}{u}, \frac{v}{u}, w\right]$$

$$\varphi_{32}(u,v,w) = \left[u, vw, \frac{1}{w}\right]$$

The transition functions for the holomorphic tangent bundle of $\widetilde{\mathbf{Z}}$ are then, when restricted to **E**,

$$\psi_{10} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & w & 0 \\ 0 & 0 & -\frac{1}{w^2} \end{bmatrix}$$

$$\psi_{20} = \begin{bmatrix} -\frac{1}{u^2} & 0 & 0 \\ 0 & \frac{1}{u} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\psi_{30} = \begin{bmatrix} -\frac{1}{u^2} & 0 & 0 \\ 0 & \frac{w}{u} & 0 \\ 0 & 0 & -\frac{1}{w^2} \end{bmatrix}$$

$$\psi_{21} = \begin{bmatrix} -\frac{1}{u^2} & 0 & 0 \\ 0 & \frac{w}{u} & 0 \\ 0 & 0 & -\frac{1}{w^2} \end{bmatrix}$$

$$\psi_{31} = \begin{bmatrix} -\frac{1}{u^2} & 0 & 0 \\ 0 & \frac{1}{u} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\psi_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & w & 0 \\ 0 & 0 & -\frac{1}{w^2} \end{bmatrix}$$

The transition functions for the normal bundle to E are thus

$$\psi'_{10} = w \qquad \psi'_{21} = \frac{w}{u}$$

$$\psi'_{20} = \frac{1}{u} \qquad \psi'_{31} = \frac{1}{u}$$

$$\psi'_{30} = \frac{w}{u} \qquad \psi'_{32} = w$$

Let ρ_0, ρ_1 be a partition of unity subordinate to U_0, U_1 and σ_2, σ_3 a partition of unity subordinate to V_2, V_3. Then $\rho_i\sigma_j$ is a partition of unity subordinate to U_i x V_j.

Now using (3.2.4) we see that, on U_0,

$$c_1(\upsilon_E) = -\frac{1}{2\pi i}\left[d(\rho_0\sigma_3 d\log w) + d\left[\rho_1\sigma_2 d\log\left[\frac{1}{u}\right]\right] + d\left[\rho_1\sigma_3 d\log\left[\frac{w}{u}\right]\right]\right] \qquad (3.2.8)$$

which, when simplified, becomes

$$c_1(\upsilon_E) = -\frac{1}{2\pi i}\left[d\sigma_3 d\log w - d\rho_1 d\log u\right] \qquad (3.2.9)$$

and exactly analogous results can be obtained on U_1, U_2, U_3. We note that, by definition, the universal bundle of CP^1 has transition function

$$g_{01} = \frac{z_0}{z_1} = w \ , \tag{3.2.10}$$

and the Chern class of this bundle is $c_1 = -\omega$, where $\omega \in H^2(CP^1)$ is as before, a generator. (See e.g. [11], p.119 - 120).

It is now easy to see that

$$c_1(\upsilon_E) = \alpha - \beta \tag{3.2.11}$$

where α, β are the cohomology classes referred to earlier (see (3.2.3) and preceding paragraph).

This completes the calculation for $j^*(\Phi) = j^*(h)$.

In the general case since **L** is a union of pairwise non-intersecting lines $\mathbf{L}_i$, the exceptional divisor **E** is a disjoint union of the quadrics $\mathbf{E}_i$. The normal bundle of **L** (**E**) is then a disjoint union of the normal bundles of the $\mathbf{L}_i$ ($\mathbf{E}_i$) and these may be identified with (pairwise non-intersecting) tubular neighbourhoods of the $\mathbf{L}_i$ ($\mathbf{E}_i$). Now for each $\mathbf{E}_i$, take α_i, β_i corresponding to the α, β for the case when **E** is a single component. If we assume that the supports of these α_i, β_i are in disjoint tubular neighbourhoods of $\mathbf{E}_i$, then in the general case we have

$$\alpha = \sum_{i=1}^{r} \alpha_i \, , \ \beta = \sum_{i=1}^{r} \beta_i \, .$$

Referring to (3.1.5), using (3.2.3) and (3.2.11) we see that

$$f^*(ch(\upsilon)) - j^*(e^h) = (2+2\alpha) - \left[1 + j^*(h) + \frac{j^*(h)^2}{2}\right]$$

$$= (2+2\alpha) - \left[1 + (\alpha-\beta) + \frac{(\alpha-\beta)^2}{2}\right] \qquad (3.2.12)$$

$$= 1 + \alpha + \beta + \alpha\beta.$$

i.e. $$f^*(ch(\upsilon) - j^*(e^h) = 1 + \sum_{i=1}^{r}(\alpha_i + \beta_i + \alpha_i\beta_i) \qquad (3.2.13)$$

Our final step in this section is to obtain a characterisation of the map j_* defined in the paragraph following (3.1.5). We note that this result is implicit in [1], p.4, but prefer to present here a direct proof, which is a modification of the proof of 6.24 of [6].

3.2.14 Proposition

Let $j: E \to \tilde{Z}$ be the inclusion of E in $\tilde{Z}$. Let υ_E be the normal bundle of E in $\tilde{Z}$, identified as a tubular neighbourhood of E in $\tilde{Z}$, and let $\pi: \upsilon_E \to E$ be the usual bundle map. Then if $\psi \in H^*(E)$,

$$j_*(\psi) = \pi^*(\psi) \wedge \Phi$$

where Φ is the Thom class of υ_E.

(Note: strictly speaking we should have $\pi^*(\psi) \wedge k_*(\Phi)$ where $k: \upsilon_E \to \tilde{Z}$ is inclusion and k_* is extension by zero, but we omit this for notional convenience).

Proof

We may consider $j: E \to \upsilon_E$ as the zero section. Let $\psi \in H^*(E)$ and let $S \subseteq E$ be its Poincaré dual in $H_*(E)$. Let $i_1 : S \to E$, $i: S \to \upsilon_E \subseteq \widetilde{Z}$ be embeddings, so that $i = j \circ i_1$. We must show that $\pi^*(\psi) \wedge \Phi$ is (a representative for) the Poincaré dual of $i(S)$ in $\widetilde{Z}$.

Let θ be any closed form in $\widetilde{Z}$ of complementary dimension to $\pi^*(\psi) \wedge \Phi$. Since $\pi: \upsilon_E \to E$ is a deformation retract of υ_E onto E (with υ_E identified as a tubular neighbourhood of E), π^* and j^* are inverse isomorphisms in cohomology.

Since $\theta \wedge \pi^*(\psi)$ is a closed form on $\widetilde{Z}$, there is an exact form $d\tau$ such that

$$\begin{aligned}\theta \wedge \pi^*(\psi) &= \pi^* j^*\big(\theta \wedge \pi^*(\psi)\big) + d\tau \\ &= \pi^* j^*(\theta) \wedge \pi^*(\psi) + d\tau\end{aligned}$$

Thus
$$\begin{aligned}\int_{\widetilde{Z}} \theta \wedge \pi^*(\psi) \wedge \Phi &= \int_{\upsilon_E} \theta \wedge \pi^*(\psi) \wedge \Phi \qquad \text{as } \Phi \text{ has support in } \upsilon_E. \\ &= \int_{\upsilon_E} \big(\pi^* j^*(\theta) \wedge \pi^*(\psi) + d\tau\big) \wedge \Phi \\ &= \int_{\upsilon_E} \pi^* j^*(\theta) \wedge \pi^*(\psi) \wedge \Phi\end{aligned}$$

since $\int_{\upsilon_E} d\tau \wedge \Phi = \int_{\upsilon_E} d(\tau \wedge \Phi) = 0$ by Stokes' theorem, and the vanishing of Φ near the boundary of υ_E.

Hence
$$\begin{aligned}\int_{\widetilde{Z}} \theta \wedge \pi^*(\psi) \wedge \Phi &= \int_{\upsilon_E} \pi^*\big(j^*(\theta) \wedge \psi\big) \wedge \Phi \\ &= \int_E \big(j^*(\theta) \wedge \psi\big) \wedge \pi_*\Phi \qquad \text{by 6.15 of [6], where } \pi_* \text{ is integration along the fibres} \\ &= \int_E j^*(\theta) \wedge \psi \qquad \text{as } \pi_*\Phi = 1 \text{ (p.64 of [6])}\end{aligned}$$

Since ψ is the Poincaré dual of S in E, we have

$$\int_E j^*(\theta) \wedge \psi = \int_S i_1^* j^*(\theta) = \int_S i^*(\theta)$$

Thus, for any closed form θ of suitable dimension, we have

$$\int_{\tilde{Z}} \theta \wedge \pi^*(\psi) \wedge \Phi = \int_S i^*(\theta)$$

so that, by definition, $\pi^*(\psi) \wedge \Phi$ is the Poincaré dual of S in $\widetilde{Z}$, which is $j_*(\psi)$.

This completes the proof.

The Thom class of the normal bundle of E can be written as

$$\Phi = \Phi_1 + \ldots + \Phi_r$$

where Φ_i is the Thom class of the normal bundle of E_i. Since Φ_i, α_i, β_i are all represented by closed forms with support in a tubular neighbourhood of E_i, we have

$$\begin{aligned} j_*(f^*ch(\upsilon) - j^*(e^h) &= j_*(1 + \sum_{i=1}^{r}(\alpha_i + \beta_i + \alpha_i\beta_i)) \\ &= \sum_{i=1}^{r}\left(\Phi_i + \Phi_i \wedge \pi^*(\alpha_i) + \Phi_i \wedge \pi^*(\beta_i) + \Phi_i \wedge \pi^*(\alpha_i\beta_i)\right) \quad (3.2.15) \end{aligned}$$

3.3 The Poincaré duals of the cohomology classes in (3.2.15)

Our next objective is to obtain sufficient information on the homology of $\widetilde{Z}$ to enable all calculations to be made in terms of homology and the Poincaré duals of the cohomology classes involved. (For the relationship between cohomology classes, wedge products and integrals with Poincaré duals in homology, intersections and intersection numbers, see e.g. [6] , [13]).

Let ω be a generator of the cohomology of **Z**, i.e. $H^*(\mathbf{Z},\mathbf{R})$, so that $H^*(\mathbf{Z},\mathbf{R})$ is a free $H^*(\mathbf{X},\mathbf{R})$ module, with generator ω, where **Z** is the twistor space of **X**. Since **Z** is an S^2 bundle over **X**

$$H^2(\mathbf{Z},\mathbf{R}) = H^2(\mathbf{CP}^1) \otimes H^0(\mathbf{X}) \oplus H^0(\mathbf{CP}^1) \otimes H^2(\mathbf{X})$$

and we may consider ω to be the generator of $H^2(\mathbf{CP}^1) \otimes H^0(\mathbf{X})$. We can then choose a basis for $H^2(\mathbf{Z})$ consisting of ω, together with elements from the other direct summand. Since integration is a non-degenerate pairing between homology and cohomology, we can choose a dual basis in homology for $H_2(\mathbf{Z})$ via this pairing. Thus

$$\int_{\mathbf{P}^1} \omega = 1 \quad \text{and} \quad \int_{A} \omega = 0 \tag{3.3.1}$$

for any other member, A, of the dual basis of $H_2(\mathbf{Z})$.

Since **L** is a disjoint union of the lines $\mathbf{L}_i$, and since the $\mathbf{L}_i$ are in the same homology class in **Z**, we must have

$$\int_{\mathbf{L}} \omega = \sum_{i=1}^{r} \int_{\mathbf{L}_i} \omega = r \qquad \text{since} \quad \int_{\mathbf{L}_i} \omega = 1 \qquad \text{is clearly true.} \tag{3.3.2}$$

The Poincaré dual k_1 of ω, meets every $\mathbf{L}_i$ transversely in a single point. Thus it meets every normal direction to $\mathbf{L}_i$, so its pre-image K_1 in $\tilde{\mathbf{Z}}$, will have the property that $\mathbf{K}_1 \cap \mathbf{E}_i$ is a vertical $\mathbf{CP}^1$, for each $i = 1,\ldots,r$.

No other linearly independent member of $H_4(\mathbf{Z})$ meets **L**, (strictly, has an intersection number with **L** which is zero), so that no such homology class will have a pre-image in $H_4(\tilde{\mathbf{Z}})$ which intersects **E**.

We shall now determine the self intersection $\mathbf{E} \cap \mathbf{E}$, since this will play a significant part in the calculation of §3.4.

Take a section of the normal bundle of E (identified as a tubular neighbourhood in $\tilde{\mathbf{Z}}$) which intersects the base transversely. Since such a section is homotopic to the base, it must belong to the same homology class. By proposition 12.8 of [6], the zero locus of this section, which is $\mathbf{E} \cap \mathbf{E}$, is Poincaré dual to the Euler class of the normal bundle υ_E, of E. In this case the Euler class of υ_E is $c_1(\upsilon_E)$ and by (2.4),

$$c_1(\upsilon_E) = \sum_{i=1}^{r}(\alpha_i - \beta_i).$$

In E, which is a disjoint union of the $\mathbf{E}_i$, each α_i is Poincaré dual to a vertical $\mathbf{CP}^1$ in $\mathbf{E}_i$ and each β_i is Poincaré dual to a horizontal $\mathbf{CP}^1$ in $\mathbf{E}_i$. It follows that

$$\mathbf{E} \cap \mathbf{E} = \sum_{i=1}^{r}(\mathbf{E}_i \cap \mathbf{E}_i) = \sum_{i=1}^{r}\left(\text{vertical } \mathbf{CP}^1 \text{ in } \mathbf{E}_i - \text{horizontal } \mathbf{CP}^1 \text{ in } \mathbf{E}_i\right). \qquad (3.3.3)$$

We shall not label the individual lines in $\mathbf{E}_i$ but will rely upon the context to make clear which are being used.

We now define the homology class in $H_4(\tilde{\mathbf{Z}})$ given by

$$\mathbf{K}_2 = \mathbf{K}_1 - \mathbf{E} \qquad (3.3.4)$$

Then

$$K_2 \cap E = K_1 \cap E - E \cap E$$

$$= \sum_{i=1}^{r}\left[\left(\text{vertical } \mathbf{CP}^1\right) - \left(\text{vertical } \mathbf{CP}^1 - \text{horizontal } \mathbf{CP}^1\right)\right]$$

$$= \sum_{i=1}^{r}\left(\text{horizontal } \mathbf{CP}^1\right).$$

We thus have:

$$K_1 \cap E = \sum_{i=1}^{r}\left(\text{vertical } \mathbf{CP}^1\right)$$

$$K_2 \cap E = \sum_{i=1}^{r}\left(\text{horizontal } \mathbf{CP}^1\right)$$

$$E \cap E = \sum_{i=1}^{r}\left(\text{vertical } \mathbf{CP}^1 - \text{horizontal } \mathbf{CP}^1\right)$$

As mentioned earlier, the calculation of $\chi(\tilde{\mathbf{Z}},\xi)$ is performed by using the intersection properties of appropriate members of $H_*(\tilde{\mathbf{Z}},\mathbf{R})$. Before we can begin this calculation however, we shall need to find the Poincaré duals in $H_*(\tilde{\mathbf{Z}},\mathbf{R})$, of each of the cohomology classes which occur in (3.2.15). To this end, we prove the following:

3.3.6 Proposition

With the above notation, if $\alpha = \sum_{i=1}^{r}\alpha_i$ and $\beta = \sum_{i=1}^{r}\beta_i$, (where the α_i, β_i are defined prior to (3.2.12)), then

$$\int_{K_i}\pi^*(\alpha) \wedge \Phi = \begin{cases} r & \text{if } i = 2 \\ 0 & \text{if } i = 1 \end{cases}, \qquad \int_{K_i}\pi^*(\beta) \wedge \Phi = \begin{cases} r & \text{if } i = 1 \\ 0 & \text{if } i = 2 \end{cases}$$

Proof

Let ψ be the Poincaré dual of $\mathbf{K}_1$ in $\widetilde{\mathbf{Z}}$. Then

$$\int_{K_1}\pi^*(\alpha)\wedge\Phi = \int_{\widetilde{Z}}\pi^*(\alpha)\wedge\Phi\wedge\psi$$

$$= \int_E j^*\big(\pi^*(\alpha)\wedge\psi\big)$$

$$= \int_E \alpha\wedge j^*\psi$$

where $j : \mathbf{E}\to\widetilde{\mathbf{Z}}$ is the embedding, and $j^*\pi^*$ is the identity on cohomology, as in Proposition (3.2.14).

The homology classes $\mathbf{K}_1$ and $\mathbf{E}$ intersect transversely, so that $j^*\psi = \eta_{j^{-1}(K_1)}$, where η_A represents the Poincaré dual of A. (See p.69 of [6], or p.59 of [13]). Now $j^{-1}(\mathbf{K}_1) = \mathbf{K}_1\cap\mathbf{E}$ so that

$$\int_{K_1}\pi^*(\alpha)\wedge\Phi = \int_E \alpha\wedge\eta_{K_1\cap E} = \int_{K_1\cap E}\alpha = 0$$

The other results follow in an analogous manner, and we have established the proposition.

Since Φ is the Poincaré dual of $\mathbf{E}$, it is easy to see that

$$\int_{\widetilde{Z}}\pi^*(\alpha\beta)\wedge\Phi = \int_E \alpha\beta = \sum_{i=1}^{r}\int_{E_i}\alpha_i\beta_i = r.$$

We have seen that $\mathbf{K}_1$ meets $\mathbf{E}$ in r vertical lines, so that, in the generic case, $\mathbf{K}_1\cap(\mathbf{K}_1\cap\mathbf{E})$ is empty, as $\mathbf{K}_1 \cap \mathbf{E}_i$ can be any one of $\mathbf{CP}^1$ distinct vertical lines. The intersection number is then zero, i.e. $\mathbf{K}_1\mathbf{K}_1\mathbf{E} = 0$. For a similar reason, we clearly have

$K_1K_2E = K_2K_1E = r$ and $K_2K_2E = 0$. Since any other member of the homology basis for $H_4(Z)$ has intersection number zero with L, none of the other members of this basis, after blowing-up, will have non-zero intersection numbers with $K_1 \cap E$, or $K_2 \cap E$. With the aid of Proposition 3.3.6, we are then able to conclude that $K_1 \cap E$ is Poincaré dual to $\pi^*(\alpha) \wedge \Phi$ and $K_2 \cap E$ is Poincaré dual to $\pi^*(\beta) \wedge \Phi$.

We shall also require $g^*(\omega)$ in our calculations, where ω is the generator for the cohomology of $H^*(Z,R)$, mentioned at the beginning of this section. To find its Poincaré dual we note that

$$\int_{K_1 \cap E} g^*(\omega) = \int_{g_*(K_1 \cap E)} \omega = \int_{r\,pts} \omega = 0,$$

$$\int_{K_2 \cap E} g^*(\omega) = \int_{g_*(K_2 \cap E)} \omega = \sum_{i=1}^{r} \int_{P^1} \omega = r,$$

and for the reasons outlined at the beginning of this section, integrating $g^*(\omega)$ over any other member of a basis for $H_2(\tilde{Z})$ will give zero. The Poincaré dual of $g^*(\omega)$ is then K_1.

We summarise the above information in the following table.

3.3.7

Cohomology Class	Poincaré duals in homology
Φ	$E = K_1 - K_2$
$g^*(\omega)$	K_1
$\pi^*(\alpha) \wedge \Phi$	$K_1 \cap E$
$\pi^*(\beta) \wedge \Phi$	$K_2 \cap E$
$\pi^*(\alpha\beta) \wedge \Phi$	r points

3.4 The calculation of $X(\widetilde{Z},\xi)$.

We are now able to begin the calculations necessary for $\chi(\xi)$. For simplicity we shall not distinguish between the intersection of cohomology classes and their intersection number, so that, for example, K_1K_2E is either r points, as the intersection of sets, or r as an intersection number. It should be clear from the context which sense is intended. We note the following intersection properties:

$$\left.\begin{array}{l} K_1K_1E = 0 \\ K_2K_2E = 0 \\ K_1K_2E = r \\ K_1EE = -r \\ K_2EE = \ \ r \end{array}\right\} \qquad (3.4.1)$$

where the latter two are calculated by using the fact that $E = K_1 - K_2$.

Referring to (1.3), we have

$$\frac{1-e^{-h}}{h} = 1 - \frac{h}{2} + \frac{h^2}{6}$$

and since $h = \Phi$, the Poincaré dual of this is

$$\widetilde{Z} - \frac{1}{2}(K_1 - K_2) + \frac{1}{6}(EK_1 - EK_2).$$

Also, the Poincaré dual of $j_*[f^*(ch(\upsilon)) - j^*e^h]$ is $[E + K_1E + K_2E + r \text{ pts}]$ so that the right hand side of (3.1.5), in Poincaré dual form, is

$$[\widetilde{Z} - \frac{1}{2}(K_1 - K_2) + \frac{1}{6}(K_1E - K_2E)][E + K_1E + K_2E + r \text{ pts}]$$
$$= E + \frac{1}{2}(K_1E + 3K_2E) + \frac{2}{3} r \text{ pts}. \qquad (3.4.2)$$

Referring to (3.1.5) we see that we are left with the problem of determining ch(T(Z)).

For a general 3-complex dimensional manifold M_3, we have

$$ch(M_3) = 3 + c_1 + \left[\frac{1}{2}c_1^2 - c_2\right] + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) \tag{3.4.3}$$

where c_1, c_2, c_3 are the Chern classes of M_3.

In [16], Hitchin has calculated the Chern classes we require. If **Z** is the twistor space of X, ω the generator of $H^*(Z)$, as described at the beginning of §3, and e(X) is the Euler class of X, then $c_1(Z) = 4\omega$, $c_2(Z) = e(X) + 4\omega^2$, $c_3(Z) = 2\omega e(X)$. The intersection properties of the Poincaré dual of $g^*(e(X))$, which we denote e_1, are easily found. We shall only need K_1e_1 and E_ie_1. Now

$$\int_{E_i} g^*(e(X)) = \int_{g_*(E_i)} e(X) = 0$$

since $g_*(E_i)$ is a line, whereas $e(X) \in H^4(Z)$. [Strictly speaking $e(X) \in H^4(X)$, but $H^*(Z)$ is a free $H^*(X)$ module generated by ω, so we can identify e(X) as a cohomology class in Z]. Thus $Ee_1 = 0$.

The value of K_1e_1 can be found as follows:

$$K_1e_1 = \int_{\tilde{Z}} g^*(\omega)g^*(e(X)) = \int_{g_*(\tilde{Z})} \omega e(X) = \int_Z \omega e(X) = \chi(X)$$

since $c_3 = 2\omega e(X)$ has Chern number 2χ, where χ is the Euler characteristic of X (p.135 of [16]). In similar fashion the intersection number of K_1^3 is

$$K_1^3 = \int_{\tilde{Z}} g^*(\omega^3) = \int_{g_*(\tilde{Z})} \omega^3 = \frac{1}{4}(2\chi - 3\tau)$$

where τ is the signature of **X**. (See [16]).

To summarise this information, we have found that the Poincaré duals of the pullbacks by g, of the Chern classes of **Z** are

$$\begin{aligned} g^*(c_1) &= 4\mathbf{K}_1 \\ g^*(c_2) &= e_1 + 4\mathbf{K}_1^2 \\ g^*(c_3) &= 2\mathbf{K}_1 e_1 \end{aligned}$$

and the intersection properties we shall require, are

$$\left.\begin{aligned} \mathbf{E}_i e_1 &= 0 \\ \mathbf{K}_1 e_1 &= \chi(\mathbf{X}) \\ \mathbf{K}_1{}^3 &= \frac{1}{4}(2\chi - 3\tau) \end{aligned}\right\} \qquad (3.4.4)$$

It is now easy to see that

$$g^*(\mathrm{ch}(\mathbf{Z})) = 3\widetilde{\mathbf{Z}} + 4\mathbf{K}_1 + \left(4\mathbf{K}_1{}^2 - e_1\right) + \frac{1}{6}\left[16\mathbf{K}_1{}^3 - 6\mathbf{K}_1 e_1\right]$$

is the Poincaré dual form of the pull-back by g of (3.4.3). Combining this result with (3.4.2) in (3.1.3), and rearranging, we obtain

$$\begin{aligned} \mathrm{ch}\left(T(\widetilde{\mathbf{Z}})\right) = 3\widetilde{\mathbf{Z}} + \left(4\mathbf{K}_1 - \mathbf{E}\right) + \frac{1}{2}\left[8\mathbf{K}_1{}^2 - \mathbf{K}_1\mathbf{E} - 3\mathbf{K}_2\mathbf{E} - 2e_1\right] \\ + \frac{1}{6}\left[16\mathbf{K}_1{}^3 - 6\mathbf{K}_1 e_1 - 4r\ \mathrm{pts}\right] \end{aligned} \qquad (3.4.5)$$

Using (3.4.3) again, this time for $\widetilde{\mathbf{Z}}$, we can show that

$$\left.\begin{aligned} c_1(\widetilde{\mathbf{Z}}) &= 4\mathbf{K}_1 - \mathbf{E} \\ c_2(\widetilde{\mathbf{Z}}) &= 4\mathbf{K}_1{}^2 - 3\mathbf{K}_1\mathbf{E} + \mathbf{K}_2\mathbf{E} + e_1 \end{aligned}\right\} \qquad (3.4.6)$$

Let $\in_1$ be the Chern class of the line bundle ξ on $\tilde{\mathbf{Z}}$. The expressions for the Chern character of ξ and the Todd class of (the holomorphic tangent bundle) $T(\tilde{\mathbf{Z}})$, can be found in terms of $\in_1$, and the Chern classes c_1, c_2 of $T(\tilde{\mathbf{Z}})$. An elementary calculation shows that the Hirebruch-Riemann-Roch theorem is, in this case,

$$\chi(\tilde{\mathbf{Z}},\xi) = \left\{ \frac{c_1 c_2}{24} + \frac{\in_1 (c_1{}^2 + c_2)}{12} + \frac{\in_1{}^2 c_1}{4} + \frac{\in_1{}^3}{6} \right\} [\tilde{\mathbf{Z}}] \qquad (3.4.7)$$

which can easily be obtained using (3.1.10) and (3.1.11).

We are left with the problem of determining $\in_1$, the Chern class of the bundle ξ. The construction of the bundle was dealt with in chapter 2 and we recall that

$$\xi = g^{-1}(\mathcal{O}(m)) \otimes \mathcal{O}_{\tilde{Z}}(-b_1, \cdots, -b_r; b_1, \cdots, b_r)$$

where, for each i, $m = a_i + b_i$.

Since $\mathcal{O}_{\tilde{Z}}(-b_1, \cdots, -b_r; b_1, \cdots, b_r)$ is trivial away from the exceptional divisor, the transition functions will be constant there. One can then see from the explicit formula (3.2.4) for example, that the only contributions to the Chern class of this bundle are from neighbourhoods of the $\mathbf{E}_i$.

Let N_i be a neighbourhood of $\mathbf{L}_i$, with $\tilde{N}_i$ a subvariety of N_ix $\mathbf{CP}^1$. The bundle $\mathcal{O}(1,-1)$ on N_i x $\mathbf{CP}^1$, when restricted to $\tilde{N}_i$ will have Chern class $w_i - v_i$, where w_i is a representative for ω but restricted to N_i, and v_i is a generator for $H^2(\mathbf{CP}^1)$, for the vertical $\mathbf{CP}^1$ in $\mathbf{E}_i$. The Chern class for $\mathcal{O}_{\tilde{Z}}(-b_1, \cdots, -b_r; b_1, \cdots, b_r)$ is then $\sum_{i=1}^{r} -b_i(w_i - v_i)$.

In Poincaré dual form this can be written as $-\sum_{i=1}^{r} b_i(\mathbf{H}_i - \mathbf{G}_i)$ where $\mathbf{H}_i$, $\mathbf{G}_i$ are elements of $H^4(\widetilde{\mathbf{Z}})$ having the intersection properties

$$\mathbf{H}_i \cap \mathbf{E}_j = \begin{cases} \text{vertical } \mathbf{CP}^1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{G}_i \cap \mathbf{E}_j = \begin{cases} \text{horizontal } \mathbf{CP}^1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Thus $\mathbf{H}_i$ - $\mathbf{G}_i$ = $\mathbf{E}_i$, so that the Chern class of ξ is

$$\in_1 = m\mathbf{K}_1 - \sum_{i=1}^{r} b_i \mathbf{E}_i \tag{3.4.8}$$

We now calculate the elements required for the evaluation of (3.4.7).

We have

$$\begin{aligned} c_1 c_2[\widetilde{\mathbf{Z}}] &= (4\mathbf{K}_1 - \mathbf{E})(4\mathbf{K}_1{}^2 - 3\mathbf{K}_1\mathbf{E} + \mathbf{K}_2\mathbf{E} + e_1)[\widetilde{\mathbf{Z}}] \\ &= 12\chi - 12\tau \end{aligned} \tag{3.4.9}$$

$$\begin{aligned} c_1{}^2 + c_2 &= (16\mathbf{K}_1{}^2 - 8\mathbf{K}_1\mathbf{E} + \mathbf{E}^2) + (4\mathbf{K}_1{}^2 - 3\mathbf{K}_1\mathbf{E} + \mathbf{K}_2\mathbf{E} + e_1) \\ &= 20\mathbf{K}_1{}^2 - 10\mathbf{K}_1\mathbf{E} + e_1. \end{aligned} \tag{3.4.10}$$

Using this, we can show that

$$\in_1 (c_1{}^2 + c_2)[\widetilde{\mathbf{Z}}] = 11m\chi - 15m\tau - 10\sum_{i=1}^{r} b_i \tag{3.4.11}$$

Similarly

$$\epsilon_1^{\,2} = m^2\mathbf{K}_1^2 - 2m\sum_{i=1}^{r} b_i\mathbf{K}_1\mathbf{E}_i + \sum_{i=1}^{r} b_i^2\mathbf{E}_i^2 \tag{3.4.11}$$

so that

$$\epsilon_1^{\,2}\, c_1[\tilde{Z}] = 2m^2\chi - 3m^2\tau - 2\sum_{i=1}^{r} b_i(m+b_i). \tag{3.4.12}$$

Finally

$$\begin{aligned} \epsilon_1^{\,3} &= \left[m\mathbf{K}_1 - \sum_{i=1}^{r} b_i\mathbf{E}_i \right]^3 \\ &= m^3\mathbf{K}_1^3 + 3m\sum_{i=1}^{r} b_i^2\mathbf{K}_1\mathbf{E}_i^2 - \sum_{i=1}^{r} b_i^3\mathbf{E}_i^3 \end{aligned} \tag{3.4.13}$$

so that

$$\epsilon_1^{\,3}\,[\tilde{Z}] = \frac{1}{4}\left(2m^3\chi - 3m^3\tau\right) - \sum_{i=1}^{r} b_i^2(3m - 2b_i) \tag{3.4.14}$$

On substituting these values into (3.4.7) and rearranging, we find that we have proved the following.

3.4.15 Theorem

Let **Z** be the twistor space of a (boundaryless) compact, Riemannian, self - dual, 4 - manifold **X**, with Euler characterisitc χ and signature τ, and let $\widetilde{\mathbf{Z}}$ be the blow-up of **Z** along the non-intersecting lines $\mathbf{L}_1, \ldots, \mathbf{L}_r$. Let the bundle

$$\xi = \mathcal{O}(a_1, \ldots, a_r; b_1, \ldots, b_r)$$

be the line bundle on **Z** whose construction was given in 2.2, with $a_i + b_i = m$ for $i = 1, \ldots, r$. Then the holomorphic Euler characteristic of ξ is given by

$$\begin{aligned}\chi(\widetilde{\mathbf{Z}}, \xi) = &\frac{1}{12}(m+1)(m+2)(m+3)\chi \\ &-\frac{1}{8}(m+2)[(m+1)(m+3)-1]\tau \\ &-\frac{1}{6}\sum_{i=1}^{r} b_i(b_i+1)(3m+5-2b_i).\end{aligned}$$

From the expression for $\chi(\widetilde{\mathbf{Z}}, \xi)$ we see that the first two terms in the sum are dependent only upon properties intrinsic to **Z**, while the final term is intimately related to the values of the codimension-2 poles on the $\mathbf{L}_i$, as defined in chapter 2.

We conclude with a couple of special cases, where the holomorphic Euler characteristic has already been calculated.

In the conformally flat case the signature is zero so that

$$\chi(\widetilde{\mathbf{Z}}, \xi) = \frac{1}{12}(m+1)(m+2)(m+3)\chi - \frac{1}{6}\sum_{i=1}^{r} b_i(b_i+1)(3m+5-2b_i). \quad (3.4.15)$$

Standard twistor space, $\mathbf{CP}^3$, is the twistor space of S^4 and $\chi(S^4) = 2$. Substituting this into the expression (3.4.15) and rearranging, assuming we have only one blown-up line,

$$\chi\left(\mathbf{C}\widetilde{\mathbf{P}}^3,\xi\right) = \frac{1}{6}(a+1)(a+2)(a+3b+3). \tag{3.4.16}$$

This coincides with the value of χ which can be obtained from the calculations in [7].

Taking all of the $b_i = 0$ in (4.9) is equivalent to finding $\chi(\mathbf{Z},O(a))$, so that

$$\chi\left(\mathbf{Z},O(a)\right) = \frac{(a+1)(a+2)(a+3)}{12}\chi. \tag{3.4.17}$$

Chapter 4 The relationship between analytic cohomologies on blown-up twistor space and analytic cohomologies on flat twistor space.

4.1 Introduction

In the previous chapter an expression was found for the holomorphic Euler characteristic of the line bundle $\mathcal{O}(a_1,\ldots,a_r;\ b_1,\ldots,b_r)$ on the compact, complex 3 manifold $\tilde{\mathbf{Z}}$. In order to isolate the dimension of the H^1 term from the alternating series making up the holomorphic Euler characteristic, some means must be found of eliminating the other terms.

One way of achieving this is by using vanishing theorems, where they exist, and it is essentially this strategy which is adopted here. This is in keeping with the direction outlined in the introduction.

The difficult term to eliminate is the H^2 term, and in this chapter we shall show how this can be achieved whenever certain vanishing theorems exist for the flat-twistor space **Z**. Using Serre duality, the H^2 term is shown to be isomorphic to $H^1(\tilde{\mathbf{Z}},\ \mathcal{O}(c_1,\ldots,c_r;\ d_1,\ldots,d_r))$ for certain coefficients c_i, d_i on $\tilde{\mathbf{Z}}$. In the case giving rise to co-dimension - two poles, i.e. when $a_i \geq 0$, $b_i \leq -2$, the corresponding coefficients c_i, d_i have the important property that $d_i \geq 1$.

A cursory examination of the H^1's with these coefficients will show that each of these elements appear to have 'zeros' of order d_i on the line $\mathbf{L}_i$, and thus bear a striking resemblance to elements of $H^1(\mathbf{Z},\ \mathcal{O}(c_i+d_i))$ which vanish on each line $\mathbf{L}_i$, where $c_i + d_i = c_j + d_j$ is the order of homogeneity.

Since we are now fixing attention on flat twistor spaces, each line $\mathbf{L}_i$ will be contained in an open neighbourhood which is biholomorphic to $\mathbf{P}^+$, so that each $\mathbf{E}_i$ has a neighbourhood which is biholomorphic to $\widetilde{\mathbf{P}}^+$, i.e. $\mathbf{P}^+$ blown-up along $\mathbf{L}_i$.

We shall show that $H^1(\widetilde{\mathbf{P}}^+, \mathcal{O}(c_i,d_i))$ can be identified, up to isomorphism, with a subgroup of $H^1(\mathbf{P}^+,\mathcal{O}(c_i + d_i))$. By employing a Mayer-Vietoris argument we shall then show that the vanishing of $H^0(\mathbf{Z},\mathcal{O}(c_i+d_i))$ and $H^1(\mathbf{Z},\mathcal{O}(c_i+d_i))$ will enable the dimension of $H^1(\widetilde{\mathbf{Z}}, \mathcal{O}(c_1,\ldots,c_r; d_1,\ldots,d_r))$ to be calculated. The solution to the problem will then depend upon the existence of vanishing theorems for $\mathbf{Z}$, which is a much simpler space to deal with, and these will be discussed in a later chapter.

4.2 The Serre dual of $H^2(\widetilde{\mathbf{Z}},\mathcal{O}(a_1,\ldots,a_r;b_1,\ldots,b_r))$.

We shall begin by assuming that $\widetilde{\mathbf{Z}}$ contains only a single blown-up line. Since most of the calculations to find the Serre dual take place in the neighbourhood of the exceptional divisor, the extension to the general case will be immediately obvious.

Serre duality [44] in this case gives

$$H^2\left(\widetilde{Z},\mathcal{O}(a,b)\right)^* = H^1\left(\widetilde{Z}, k \otimes \mathcal{O}(a,b)^*\right)$$

where k is the canonical bundle on $\widetilde{\mathbf{Z}}$ and $\mathcal{O}(a,b)^*$ is the dual bundle. Plainly this latter is $\mathcal{O}(-a,-b)$.

To find the canonical bundle of $\widetilde{\mathbf{Z}}$ we need only consider holomorphic 3-forms in a neighbourhood of the exceptional divisor $\mathbf{E}$, which is the line $\mathbf{L}$ blown-up. In the case of a flat twistor space, this line is contained in a neighbourhood which is biholomorphic to $\mathbf{P}^+$, so it will suffice to find this bundle for $\mathbf{C}\widetilde{\mathbf{P}}^3$.

We use the co-ordinate system for $C\widetilde{P}^3$ introduced in 3.2 of the previous chapter.

We let $\underline{z} = (z_0, z_1, z_2, z_3)$ be homogeneous coordinates for CP^3. The line L is taken as $z_2 = z_3 = 0$ and if $\underline{w} = (w_2, w_3)$ gives homogeneous co-ordinates for CP^1, then $C\widetilde{P}^3$ is the subvariety of $CP^3 \times CP^1$ defined by the equation

$$w_2 z_3 - w_3 z_2 = 0 \tag{4.2.1}$$

In the open set $z_0 \neq 0$, $w_2 \neq 0$, local co-ordinates for $C\widetilde{P}^3$ are then $\left[\frac{z_1}{z_0}, \frac{z_2}{z_0}, \frac{w_3}{w_2}\right]$ and the exceptional divisor is given by the equation $z_2 = 0$.

A basis for the holomorphic 3- forms is then

$$\eta = d\left[\frac{z_1}{z_0}\right] \wedge d\left[\frac{z_2}{z_0}\right] \wedge d\left[\frac{w_3}{w_2}\right]$$

$$= \left[\frac{z_0 dz_1 - z_1 dz_0}{z_0^2}\right] \wedge \left[\frac{z_0 dz_2 - z_2 dz_0}{z_0^2}\right] \wedge \left[\frac{w_2 dw_3 - w_3 dw_2}{w_2^2}\right]$$

$$\text{i.e. } \eta = \left[\frac{z_0 dz_1 dz_2 - z_1 dz_0 dz_2 + z_2 dz_0 dz_1}{z_0^3}\right] \wedge \left[\frac{w_2 dw_3 - w_3 dw_2}{w_2^2}\right] \tag{4.2.2}$$

From the equation of the variety $w_2 z_3 = w_3 z_2$ we obtain

$$w_2 dz_3 + z_3 dw_2 = w_3 dz_2 + z_2 dw_3$$

so that

$$w_2 dw_3 = \frac{w_2}{z_2}(z_3 dw_2 - w_3 dz_2 + w_2 dz_3).$$

Hence

$$\frac{w_2 dw_3 - w_3 dw_2}{w_2^2} = \frac{w_2 z_3 dw_2 - w_2 w_3 dz_2 + w_2^2 dz_3 - z_2 w_3 dw_2}{z_2 w_2^2}$$
$$= \frac{w_2 dz_3 - w_3 dz_2}{z_2 w_2} \tag{4.2.3}$$

since $w_2 z_3$ - $z_2 w_3 = 0$, being the equation of the subvariety, (4.2.1).

Substituting this expression (4.2.3) into the expression for η in (4.2.2), and after some elementary algebra which we omit, we obtain the expression for η as

$$\eta = \frac{z_0 dz_1 dz_2 dz_3 - z_1 dz_0 dz_2 dz_3 + z_2 dz_0 dz_1 dz_3 - z_3 dz_0 dz_1 dz_2}{z_0^3 z_2}.$$

This has a pole of order 1 on the exceptional divisor, i.e. along $z_2 = 0$, and a pole of order 3 in the horizontal direction, on the hyperplane $z_0 = 0$. Thus we see that the canonical bundle is isomorphic to $\mathcal{O}(-3,-1)$.

By Serre duality we now have

$$H^2(\tilde{Z}, \mathcal{O}(a,b))^* = H^1(\tilde{Z}, \mathcal{O}(c,d))$$

where $c = -a-3$ and $d = -b-1$. Since $a \geq 0$ and $b \leq -2$, we have $c \leq -3$ and $d \geq 1$.

In the general case, with $a_i \geq 0$, $b_i \leq -2$ and $a_i + b_i = m$, the resulting Serre dual is

$$H^2(\tilde{Z}, \mathcal{O}(a_1,\ldots,a_r;b_1,\ldots,r_r))^* = H^1(\tilde{Z}, \mathcal{O}(c_1,\ldots,c_r;d_1,\ldots,d_r))$$

where $c_i = -a_i - 3$, $d_i = -b_i - 1$ and we shall take $c_i + d_i = n$, so $n = -m - 4$.

4.3 Cohomologies on $\widetilde{\mathbf{P}}^+$ and $\mathbf{P}^+$.

We shall now establish some of the properties of the cohomology classes H^1 $(\widetilde{\mathbf{P}}^+, \mathcal{O}(c,d))$ and H^1 $(\mathbf{P}^+, \mathcal{O}(c+d))$ which will be required in the final section of this chapter. Here we are considering **L** to be a single line with neighbourhood $\mathbf{P}^+$, so that $\widetilde{\mathbf{P}}^+$ is $\mathbf{P}^+$ blown-up along a single line. The major contribution of this section will be to prove the existence of a monomorphism from H^1
$0 \to \mathcal{O}_{CP^3 \times CP^1}(c-1, d-1) \to \mathcal{O}_{CP^3 \times CP^1}(c,d) \to \mathcal{O}_{\widetilde{CP}^3}(c,d) \to 0$ into H^1 $(\mathbf{P}^+, \mathcal{O}(c+d))$, when $c \le -3$, $d \ge 1$.

The manifold $\widetilde{\mathbf{Z}}$ is the blow-up of **Z** along $\mathbf{L}_1 \cup \ldots \cup \mathbf{L}_r$, where the $\mathbf{L}_i$ are pairwise non-intersecting projective lines. Each $\mathbf{L}_i$ is contained in a neighbourhood $\mathbf{N}_i$ which is biholomorphic to $\mathbf{P}^+$ and these $\mathbf{N}_i$ can also be chosen to be pairwise disjoint. If $\widetilde{\mathbf{N}}_i$ represents the blow-up of $\mathbf{N}_i$ along $\mathbf{L}_i$ then with $\mathbf{N} = \mathbf{N}_1 \cup \ldots \cup \mathbf{N}_r$, the cohomology group $H^1(\widetilde{\mathbf{N}}, \mathcal{O}(c_1, \ldots, c_r; d_1, \ldots, d_r))$ is the direct sum of the groups $H^1(\widetilde{\mathbf{N}}_i, \mathcal{O}(c_i, d_i))$, and similarly $H^1(\mathbf{N}, \mathcal{O}(c_i + d_i))$ is the direct sum of the $H^1(\mathbf{N}_i, \mathcal{O}(c_i + d_i))$, where $c_i + d_i = m$.

The properties established for $H^1(\widetilde{\mathbf{P}}^+, \mathcal{O}(c,d))$ and $H^1(\mathbf{P}^+, \mathcal{O}(c+d))$ will then extend to the corresponding cohomology groups of $\widetilde{\mathrm{N}}$ and N respectively.

We first display the structure of the elements of $H^1(\widetilde{\mathbf{P}}^+, \mathcal{O}(c,d))$.

4.3.1 Lemma

If $c \le -3$ and $d \ge 1$ then every element of $H^1(\widetilde{\mathbf{P}}^+, \mathcal{O}(c,d))$ can be represented as the restriction to $\widetilde{\mathbf{P}}^+$ of an element of the form f(z)g(w) for some $f \in H^1(\mathbf{P}^+, \mathcal{O}(c))$ and $g \in H^0(\mathbf{P}^1, \mathcal{O}(d))$.

Proof

The method is precisely that of [7]. The bundle $\mathcal{O}(c,d)$ is defined on $\tilde{\mathbf{P}}^+$ by the short exact sequence of sheaves

$$0 \to \mathcal{O}_{P^+ \times CP^1}(c-1, d-1) \xrightarrow{\ \sigma\ } \mathcal{O}_{P^+ \times CP^1}(c, d) \xrightarrow{\ \rho\ } \mathcal{O}_{\tilde{P}^+}(c, d) \to 0 \qquad (4.3.2)$$

where the first map is multiplication by the ideal sheaf of $\tilde{\mathbf{P}}^+$, and the second map ρ is restriction to this subvariety. This leads to a long exact sequence in cohomology which, for $c \le -3$ and $d \ge 1$, yields

$$\begin{aligned} &0 \to H^0\big(\tilde{\mathbf{P}}^+, \mathcal{O}(c,d)\big) \to H^1(\mathbf{P}^+ \times \mathbf{CP}^1, \mathcal{O}(c-1,d-1)) \to \\ &\to H^1\big(\mathbf{P}^+ \times \mathbf{CP}^1, \mathcal{O}(c,d)\big) \to H^1\big(\tilde{\mathbf{P}}^+, \mathcal{O}(c,d)\big) \to 0, \end{aligned} \qquad (4.3.3)$$

the penultimate map being restriction.

This is a result of the following facts. By the Künneth formula for sheaf cohomology

$$H^0(\mathbf{P}^+ \times \mathbf{CP}^1, \mathcal{O}(c,d)) = H^0(\mathbf{P}^+, \mathcal{O}(c)) \otimes H^0(\mathbf{CP}^1, \mathcal{O}(d))$$

$$\begin{aligned} H^2(\mathbf{P}^+ \times \mathbf{CP}^1, \mathcal{O}(c-1,d-1)) = {} & H^2(\mathbf{P}^+, \mathcal{O}(c-1)) \otimes H^0(\mathbf{CP}^1, \mathcal{O}(d-1)) \\ & \oplus H^1(\mathbf{P}^+, \mathcal{O}(c-1)) \otimes H^1(\mathbf{CP}^1, \mathcal{O}(d-1)) \\ & \oplus H^0(\mathbf{P}^+, \mathcal{O}(c-1)) \otimes H^2(\mathbf{CP}^1, \mathcal{O}(d-1)). \end{aligned}$$

The first vanishes since $H^0(\mathbf{P}^+, \mathcal{O}(c)) = H^0(\mathbf{CP}^3, \mathcal{O}(c)) = 0$ if $c < 0$.

The second vanishes since both $\mathbf{P}^+$ and $\mathbf{CP}^1$ have vanishing second cohomology and $H^1(\mathbf{CP}^1, \mathcal{O}(d-1)) = 0$ for $d-1 \geq 0$, i.e. $d \geq 1$. These results also imply that

$$H^1(\mathbf{P}^+ \times \mathbf{CP}^1, \mathcal{O}(c,d)) = H^1(\mathbf{P}^+, \mathcal{O}(c)) \otimes H^0(\mathbf{CP}^1, \mathcal{O}(d)) \tag{4.3.4}$$

and hence we have proved the lemma.

Having such a clear representation for elements of $H^1(\widetilde{\mathbf{P}}^+, \mathcal{O}(c,d))$ is advantageous. Indeed it is precisely this representation which gives the clue to the relationship between elements of $H^1(\widetilde{\mathbf{Z}}, \mathcal{O}(c,d))$ and elements of $H^1(\mathbf{Z}, \mathcal{O}(c+d))$. The $g(w)$ in (4.3.1) is an element of $H^0(\mathbf{P}^+, \mathcal{O}(d))$ and for $d \geq 1$, this is a homogeneous polynomial of degree d. We can use this to prove the following.

4.3.5 Lemma

Let $c \leq -3$ and $d \geq 1$. If $f_1 \in H^1(\widetilde{\mathbf{P}}^+, \mathcal{O}(c,d))$ then there exists $f_2 \in H^1(\mathbf{P}^+, \mathcal{O}(c+d))$ such that

$$f_1\big|_{\widetilde{\mathbf{P}}^+ - \mathbf{E}} = f_2\big|_{\mathbf{P}^+ - \mathbf{L}}$$

where we identify $\widetilde{\mathbf{P}}^+ - \mathbf{E}$ with $\mathbf{P}^+ - \mathbf{L}$.

Proof

If $f_1 \in H^1(\widetilde{\mathbf{P}}^+, \mathcal{O}(c,d))$ then by 4.3.1 we can find $f(z) \in H^1(\mathbf{P}^+, \mathcal{O}(c))$ and $g(w) \in H^0(\mathbf{CP}^1, \mathcal{O}(d))$ such that $f_1 = f(z)g(w)\big|_{\widetilde{\mathbf{P}}^+}$. Taking $[z^0, z^1, z^2, z^3]$ as coordinates for $\mathbf{P}^+$ and $[w^2, w^3]$ for $\mathbf{CP}^1$, then in $\widetilde{\mathbf{P}}^+$, away from the exceptional divisor, we have $z^2:z^3 = w^2:w^3$. Thus f_1, when restricted away from the exceptional divisor, becomes

$f(z)g(z)|_{\tilde{P}^{+}-L}$, where we have identified $g(z) = g(w)$ with an element of $H^0(\mathbf{P}^+, \mathcal{O}(d))$. The result is now obvious.

The above lemma shows that the image of the restriction map from $H^1(\tilde{\mathbf{P}}^+, \mathcal{O}(c,d))$ to $H^1(\tilde{\mathbf{P}}^+ - \mathbf{E}, \mathcal{O}(c,d))$ is contained in the image of the restriction map from $H^1(\mathbf{P}^+, \mathcal{O}(c+d))$ to $H^1(\mathbf{P}^+ - \mathbf{L}, \mathcal{O}(c+d))$. If these restriction maps were both injective, then we would have our monomorphism of $H^1(\tilde{\mathbf{P}}^+, \mathcal{O}(c,d))$ into $H^1(\mathbf{P}^+, \mathcal{O}(c+d))$. This is what we shall prove below.

4.3.6 Lemma

Let s_1, s_2 be the restriction maps

$$s_1 : H^1(\tilde{\mathbf{P}}^+, \mathcal{O}(c,d)) \to H^1(\tilde{\mathbf{P}}^+ - \mathbf{E}, \mathcal{O}(c+d))$$
$$s_2 : H^1(\mathbf{P}^+, \mathcal{O}(c+d)) \to H^1(\mathbf{P}^+ - \mathbf{L}, \mathcal{O}(c+d))$$

Then s_1 and s_2 are both monomorphisms.

Proof

For $\tilde{\mathbf{P}}^+$ the local cohomology exact sequence is

$$\to H^0(\tilde{\mathbf{P}}^+, \mathcal{O}(c,d)) \to$$

$$\to H^0(\mathbf{P}^+ - \mathbf{L}, \mathcal{O}(c+d)) \to H^1_E(\tilde{\mathbf{P}}^+, \mathcal{O}(c,d)) \xrightarrow{\gamma} H^1(\tilde{\mathbf{P}}^+, \mathcal{O}(c,d)) \to H^1(\mathbf{P}^+ - \mathbf{L}, \mathcal{O}(c+d)) \tag{4.3.7}$$

and we shall show that γ is the zero map, so that the restriction map $s_1 : H^1(\tilde{\mathbf{P}}^+, \mathcal{O}(c,d)) \to H^1(\mathbf{P}^+ - \mathbf{L}, \mathcal{O}(c+d))$ is injective.

We note first that $H^0(\widetilde{\mathbf{P}}^+, \mathcal{O}(c,d)) = H^0(\mathbf{C}\widetilde{\mathbf{P}}^3, \mathcal{O}(c,d))$ since any element of the first group, in a neighbourhood of the exceptional divisor and away from it, is a holomorphic function of degree c+d, and this can be automatically extended to the rest of $\mathbf{CP}^3 - \mathbf{L}$. It is well known that $H^0(\mathbf{CP}^3, \mathcal{O}(c+d)) \cong H^0(\mathbf{P}^+, \mathcal{O}(c+d))$.

The bundle $\mathcal{O}$(c,d) on $\mathbf{C}\widetilde{\mathbf{P}}^3$ is defined in the same way as for $\widetilde{\mathbf{P}}^+$, i.e. we have the short exact sequence

$$0 \to \mathcal{O}_{\mathbf{CP}^3 \times \mathbf{CP}^1}(c-1, d-1) \to \mathcal{O}_{\mathbf{CP}^3 \times \mathbf{CP}^1}(c,d) \to \mathcal{O}_{\mathbf{C}\widetilde{\mathbf{P}}^3}(c,d) \to 0$$

and this gives rise to the long exact cohomology sequence

$$\to H^0(\mathbf{CP}^3 \times \mathbf{CP}^1, \mathcal{O}(c,d)) \to H^0(\mathbf{C}\widetilde{\mathbf{P}}^3, \mathcal{O}(c,d)) \to H^1(\mathbf{CP}^3 \times \mathbf{CP}^1, \mathcal{O}(c-1, d-1)) \to$$
$$\to H^1(\mathbf{CP}^3 \times \mathbf{CP}^1, \mathcal{O}(c,d)) \to H^1(\mathbf{C}\widetilde{\mathbf{P}}^3, \mathcal{O}(c,d)) \to H^2(\mathbf{CP}^3 \times \mathbf{CP}^1, \mathcal{O}(c-1, d-1)) \to$$

Using the Künneth formula with $c \le -3, \quad d \ge 1,$ one can easily show that

$$H^0(\mathbf{C}\widetilde{\mathbf{P}}^3, \mathcal{O}(c,d)) = H^1(\mathbf{C}\widetilde{\mathbf{P}}^3, \mathcal{O}(c,d)) = 0 \qquad (4.3.8)$$

Thus (4.3.7) becomes

$$0 \to H^0(\mathbf{P}^+ - \mathbf{L}, \mathcal{O}(c+d)) \to H^1_E(\widetilde{\mathbf{P}}^+, \mathcal{O}(c,d)) \xrightarrow{\gamma} H^1(\widetilde{\mathbf{P}}^+, \mathcal{O}(c,d)) \to H^1(\mathbf{P}^+ - \mathbf{L}, \mathcal{O}(c+d))$$
$$(4.3.9)$$

Since $\mathbf{E} \subseteq \widetilde{\mathbf{P}}^+ \subseteq \mathbf{C}\widetilde{\mathbf{P}}^3$, with **E** closed and $\widetilde{\mathbf{P}}^+$ open in $\mathbf{C}\widetilde{\mathbf{P}}^3$, we may use (2.2.3) and (2.2.4) to determine $H^1_E(\widetilde{\mathbf{P}}^+, \mathcal{O}(c,d)) \cong H^1_E(\mathbf{C}\widetilde{\mathbf{P}}^3, \mathcal{O}(c,d))$ from the local cohomology exact sequence

$$H^0(\mathbf{C}\widetilde{\mathbf{P}}^3, \mathcal{O}(c,d)) \to H^0(\mathbf{CP}^3 - \mathbf{L}, \mathcal{O}(c+d)) \to H^1_E(\mathbf{C}\widetilde{\mathbf{P}}^3, \mathcal{O}(c,d)) \to H^1(\mathbf{C}\widetilde{\mathbf{P}}^3, \mathcal{O}(c,d)).$$

Using (4.3.8) we deduce that $H^0(\mathbf{CP}^3 - \mathbf{L}, O(c+d)) \cong H^1_E(\mathbf{C\widetilde{P}}^3, O(c,d))$.

We note that the restriction map $H^0(\mathbf{P}^+, O(n)) \to H^0(\mathbf{P}^+ - \mathbf{L}, O(n))$ is injective, since any holomorphic section of $O(n)$ on $\mathbf{P}^-$ which is zero on $\mathbf{P}^+ - \mathbf{L}$, an open subset of $\mathbf{P}^+$, must be identically zero on $\mathbf{P}^+$. Further, the restriction map is also surjective, since if not then there is a section (locally given by a holomorphic function) which is defined in $\mathbf{P}^+ - \mathbf{L}$, with a singularity on $\mathbf{L}$. This is impossible since such a singularity would have codimension 1, whereas $\mathbf{L}$ is of codimension 2.

It is not difficult to see now that

$$H^0(\mathbf{P}^+ - \mathbf{L}, O(c+d)) \cong H^0(\mathbf{P}^+, O(c+d)) \cong H^0(\mathbf{CP}^3, O(c+d)) \cong H^0(\mathbf{CP}^3 - \mathbf{L}, O(c+d)).$$

Since all of these are finite dimensional, the map γ in (4.3.9) is indeed the zero map.

For the restriction map $s_2 : H^1(\mathbf{P}^+, O(n)) \to H^1(\mathbf{P}^+ - \mathbf{L}, O(n))$ we have the local cohomology exact sequence

$$\to H^1_L(\mathbf{P}^+, O(n)) \to H^1(\mathbf{P}^+, O(n)) \to H^1(\mathbf{P}^+ - \mathbf{L}, O(n)) \to$$

and using the excision isomorphism we can evaluate $H^1_L(\mathbf{P}^+, O(n)) = H^1_L(\mathbf{CP}^3, O(n))$ from the local cohomology exact sequence

$$H^0(\mathbf{CP}^3, O(n)) \to H^0(\mathbf{CP}^3 - L, O(n)) \to H^1_L(\mathbf{CP}^3, O(n)) \to H^1(\mathbf{CP}^3, O(n)).$$

Since $H^1(\mathbf{CP}^3, O(n)) = 0$ and $H^0(\mathbf{CP}^3, O(n)) \to H^0(\mathbf{CP}^3 - \mathbf{L}, O(n))$ is an isomorphism, $H^1_L(\mathbf{CP}^3, O(n)) = H^1_L(\mathbf{P}^+, O(n)) = 0$.

This completes the proof of Lemma 4.3.6.

As a consequence of this lemma we have the following

4.3.10 Proposition

For $c \le -3$ and $d \ge 1$ there are monomorphisms

$$r_1 : H^1(\tilde{\mathbf{P}}^+, O(c,d)) \to H^1(\mathbf{P}^+, O(c+d))$$
$$r_0 : H^0(\tilde{\mathbf{P}}^+, O(c,d)) \to H^0(\mathbf{P}^+, O(c+d)).$$

Proof

For the former simply take $r_1 = s_2^{-1}s_1$ where the s_2, s_1 are as in 4.3.6. For the latter, the restriction map $H^0(\tilde{\mathbf{P}}^+, O(c,d)) \to H^0(\mathbf{P}^+ - \mathbf{L}, O(c+d))$ is injective, and the restriction map $H^0(\mathbf{P}^+, O(c+d)) \to H^0(\mathbf{P}^+ - \mathbf{L}, O(c+d))$ is an isomorphism.

4.3.11 Corollary

If $\mathbf{N} = \bigcup_{i=1}^{r} \mathbf{N}_i$ and $\tilde{\mathbf{N}}$ are as defined at the beginning of §2, and if $c_i \le -3$, $d_i \ge 1$ with $c_i + d_i = n$, then there exist monomorphisms

$$r_1 : H^1(\tilde{\mathbf{N}}, O(c_1, \ldots, c_r; d_1, \ldots, d_r)) \to H^1(\mathbf{N}, O(n))$$
$$r_0 : H^0(\tilde{\mathbf{N}}, O(c_1, \ldots, c_r; d_1, \ldots, d_r)) \to H^0(\mathbf{N}, O(n)).$$

Proof

These are the obvious extensions of r_1 and r_0 in (4.3.10) to the relevant direct sums.

4.4 The effect of vanishing theorems for $H^1(Z,\mathcal{O}(n))$ on the cohomology of $\tilde{Z}$

With Z,N,L as above, we can write

$$Z = (Z-L)\cup N \quad \text{and} \quad \tilde{Z} = (\tilde{Z}-E)\cup\tilde{N} = (Z-L)\cup\tilde{N}$$

where the latter decomposition uses the fact that Z - L and $\tilde{Z}-E$ are biholomorphic. Omitting the sheaves, which are $\mathcal{O}(c_1,\dots,c_r;d_1,\dots,d_r)$ on the blown-up space, or $\mathcal{O}(m)$ on the flat space, whichever is appropriate, we have the following two Mayer-Vietoris sequences for sheaf cohomology, (see [20]).

$$\begin{matrix} H^0(\tilde{N}) \\ \oplus \\ H^0(Z-L) \end{matrix} \xrightarrow{\;-h^*,\ \ell^*\;} H^0(N-L) \xrightarrow{\;\alpha\;} H^1(\tilde{Z}) \xrightarrow{\;j^*,\ k^*\;} \begin{matrix} H^1(\tilde{N}) \\ \oplus \\ H^1(Z-L) \end{matrix} \xrightarrow{\;-h^*,\ \ell^*\;} H^1(N-L) \qquad (4.4.1)$$

$$\begin{matrix} H^0(N) \\ \oplus \\ H^0(Z-L) \end{matrix} \xrightarrow{\;-m^*,\ \ell^*\;} H^0(N-L) \xrightarrow{\;\beta\;} H^1(Z) \xrightarrow{\;s^*,\ t^*\;} \begin{matrix} H^1(N) \\ \oplus \\ H^1(Z-L) \end{matrix} \xrightarrow{\;-m^*,\ \ell^*\;} H^1(N-L) \qquad (4.4.2)$$

The relationship between $H^i(\tilde{N})$ and $H^i(N)$ for i = 0, 1, which is given by corollary 4.3.10, together with (4.4.1) and (4.4.2), suggest that a close examination of these two sequences might yield valuable information on the interdependence of $H^1(\tilde{Z})$ and $H^1(Z)$. A detailed analysis of this arrangement leads to the main result of this chapter, which we now state and prove.

4.4.3 Theorem

Let $\mathbf{Z}$ be a compact, flat, twistor-space and let $\mathbf{L} = \bigcup_{i=1}^{r} \mathbf{L}_i$ where the $\mathbf{L}_i$ are pairwise non-intersecting complex projective lines. Let $\mathbf{Z}$ be the blow-up of $\mathbf{Z}$ along $\mathbf{L}$ and let $O(c_1,\ldots,c_r;d_1,\ldots,d_r)$ be the line bundle on $\mathbf{Z}$ described above, with $c_i \leq -3$, $d_i \geq 1$, and $c_i + d_i = n$.

If
$$H^0(\mathbf{Z}, O(n)) = H^1(\mathbf{Z}, O(n)) = 0$$
then

$$\dim H^1(\widetilde{\mathbf{Z}}, O(c_1,\ldots,c_r;d_1,\ldots,d_r)) = r.\dim H^0(\mathbf{P}^+, O(n))$$

$$= \begin{cases} \dfrac{r(n+1)(n+2)(n+3)}{6} & \text{if } n \geq 0 \\ 0 & \text{if } n < 0. \end{cases}$$

Proof

Referring to (4.4.1) and (4.4.2) we establish the following facts.

(4.4.4) β is the zero map.

As in the proof of (2.8) the restriction map $H^0(\mathbf{P}^+, O(n)) \to H^0(\mathbf{P}^+ - \mathbf{L}, O(n))$, where $\mathbf{L}$ is a single line, is an isomorphism. This clearly extends to the more general case of direct sums, so that $\ell^* - m^*$ is surjective.

(4.4.5) There are monomorphisms

$$r_0 \oplus i_0 : H^0(\tilde{N}) \oplus H^0(Z-L) \to H^0(N) \oplus H^0(Z-L)$$
$$r_1 \oplus i_1 : H^1(\tilde{N}) \oplus H^1(Z-L) \to H^1(N) \oplus H^1(Z-L)$$

Simply take r_0, r_1 of (4.3.10) and i_0, i_1 the relevant identity maps.

(4.4.6) The kernel of $\ell^* - h^*$ in (4.4.1) is mapped injectively by $r_1 \oplus i_1$, into the kernel of $\ell^* - m^*$.

In the setting of (4.4.1) and (4.4.2) the map r_1 is given by $(m^*)^{-1} h^*$.

(4.4.6) There is a monomorphism of $\mathrm{im}(j^* \oplus k^*)$ into $H^1(Z)$.

The map $r_1 \oplus i_1$ maps the kernel of $\ell^* - h^*$, which is the image of $j^* \oplus k^*$, injectively into the kernel of $\ell^* - m^*$, which is the image of $s^* \oplus t^*$. Since β is the zero map by (3.3.1), $s^* \oplus t^*$ is injective.

(4.4.7) $\mathrm{im}\alpha \cong \dfrac{H^0(N) + \mathrm{im}\ell^*}{H^0(\tilde{N}) + \mathrm{im}\ell^*}$

The image of α is the cokernel of $\ell^* - h^*$, which is $H^0(N-L)/(\mathrm{im}\ell^* + \mathrm{im}h^*)$. Now $r_0 = (m^*)^{-1} h^*$ or $h^* = m^* r_0$ where r_0 is a monomorphism and m^* is an isomorphism. This means that we can identify $h^*(H^0(\tilde{N}))$ as a subgroup of $H^0(N - L)$ and also $r_0(H^0(\tilde{N}))$ as a subgroup of $H^0(N)$, so that, up to isomorphism, we have

$$H^0(\tilde{N}) + \mathrm{im}\ell^* \subseteq H^0(N) + \mathrm{im}\ell^* \subseteq H^0(N-L)$$

Using the second isomorphism theorem for groups we see that

$$\operatorname{im}\beta = \operatorname{coker}(\ell^* - m^*) = \frac{H^0(N-L)}{H^0(N)+\operatorname{im}\ell^*}$$

$$\cong \left(\frac{H^0(N-L)}{H^0(\tilde{N})+\operatorname{im}\ell^*}\right) \Big/ \left(\frac{H^0(N)+\operatorname{im}\ell^*}{H^0(\tilde{N})+\operatorname{im}\ell^*}\right).$$

Since β is the zero map from (4.4.4), we see that

$$\operatorname{im}\alpha \cong \frac{H^0(N)+\operatorname{im}\ell^*}{H^0(\tilde{N})+\operatorname{im}\ell^*}$$

As shown in the proof of (4.3.8), if $c \le -3$, $d \ge 1$, then $H^0(\tilde{\mathbf{P}}^+, \mathcal{O}(c,d)) = H^0(\mathbf{C}\tilde{\mathbf{P}}^3, \mathcal{O}(c,d)) = 0$. Since $H^0(\tilde{N})$ is a direct sum of such groups we obtain

$$\operatorname{im}\alpha \cong \frac{H^0(N)+\operatorname{im}\ell^*}{\operatorname{im}\ell^*}$$

Now $\mathbf{N} - \mathbf{L}$ is an open subset of $\mathbf{Z} - \mathbf{L}$ so ℓ^* is injective.

We can now complete the theorem since from (3.3.4), the vanishing of $H^1(\mathbf{Z})$ implies the vanishing of $\operatorname{im}(j^* \oplus k^*)$ in (3.1). The dimension of $H^1(\tilde{\mathbf{Z}})$ will then be the dimension of $\operatorname{im}\alpha$. If $H^0(\mathbf{Z})$ is zero then so is $H^0(\mathbf{Z} - \mathbf{L})$, since they are isomorphic. This means that $\operatorname{im}\ell^* = 0$, and so we arrive at our result.

Theorem 4.4.3 allows the dimension of $H^2(\tilde{\mathbf{Z}}, \mathcal{O}(a_1,\ldots,a_r; b_1,\ldots,b_r))$ to be calculated when it is known that $H^0(\mathbf{Z},\mathcal{O}(n))$ and $H^1(\mathbf{Z},\mathcal{O}(n))$ both vanish (where $n = -(a_i + b_i) - 4$, $i = 1,\ldots,r$). Another elementary result which we shall require is the following.

4.4.8 Theorem

Let $\widetilde{\mathbf{Z}}, \mathbf{Z}, O\,(c_1,\ldots,c_r;d_1,\ldots,d_r)$, $O(n)$ be given as above. If $H^0(\mathbf{Z},O(n)) = 0$ then $H^0\left(\widetilde{\mathbf{Z}},O(c_1,\ldots,c_r;d_1,\ldots,d_r\right) = 0$.

Proof

The restriction map $H^0(\widetilde{\mathbf{Z}}) \to H^0(\mathbf{Z}-\mathbf{L})$ is a monomorphism, since $\mathbf{Z} - \mathbf{L}$ is an open subset of $\widetilde{\mathbf{Z}}$, so any holomorphic section of the sheaf which vanishes on $\mathbf{Z} - \mathbf{L}$ must be identically zero.

This reasoning also proves that $H^0(\mathbf{Z}) \to H^0(\mathbf{Z}-\mathbf{L})$ is a monomorphism. A simple application of Hartog's theorem [13], also proves that any holomorphic section of the sheaf over **Z** - **L** may be holomorphically extended to **Z**, so that this mapping is also surjective.

Thus there is a monomorphism from the latter group into the former group, and the result is now obvious.

4.4.9 Corollary

If $H^3(\mathbf{Z},O(n)) = 0$ then $H^3(\widetilde{\mathbf{Z}},O(c_1,\ldots,c_r;d_1,\ldots,d_r)) = 0$.

Proof

This follows immediately on taking the Serre duals of both groups and applying 4.4.8.

Chapter 5 Vanishing Theorems

5.1 In this chapter we prove some vanishing theorems which will be used for the calculation of the dimension of the group $H^1(\tilde{Z}, \mathcal{O}(a_1,\ldots,a_r;b_1,\ldots,b_r))$ using the methods outlined in chapter 1. Both of the vanishing theorems of 5.3 are for cohomologies on the twistor space of a 4-dimensional, Riemannian, self-dual manifold **X**, with negative scalar curvature. In 5.4 we prove a vanishing theorem for the twistor space of a compact, quaternionic-Kähler manifold of negative scalar curvature.

This vanishing theorem was first proved by Thornber [40], a student of LeBrun, using Hermitian geometry in the twistor space. This theorem was then used by him to prove a rigidity theorem (See 5.5 of this chapter). In our proof we first use the Penrose transform to convert the theorem into one involving the vanishing of certain spinor fields on **X**. The game is then to construct a Weitzenbock formulae for the spinor fields involved.

In 5.2 we begin by outlining the abstract index notation of Penrose [35], which is used throughout this chapter.

5.2 Background and notation

If **X** is an oriented, Riemannian 4-manifold then the structure group is SO(4) and locally this is isomorphic to SU(2) × SU(2). Each of these factors defines, at least locally, two spin-bundles, V_+ and V_- with fibre $\mathbf{C}^2$. The projective (negative) spin-bundle $P(V_-)$ is globally well defined and has an almost complex structure [2] , which is integrable if **X** is also self-dual. The total space of this bundle is then the twistor-space **Z** of **X**.

Each spin bundle is equipped with a hermitian metric $< , >$, together with a quaternionic conjugation and a symplectic form ε which are compatible in the obvious way.

The identification of cotangent vectors with spinors is taken as

$$\mathbf{T^*X} = \mathbf{V}_+^* \otimes \mathbf{V}_-^* \tag{5.2.1}$$

and the metric on **X** is then

$$g = \varepsilon \otimes \bar{\varepsilon} \in \Gamma\left(\wedge^2 \mathbf{V}_+^* \otimes \wedge^2 \mathbf{V}_-^*\right) \subseteq \Gamma\left(S^2\left(\mathbf{T^*X}\right)\right) \tag{5.2.2}$$

where S^k is the k-th symmetric tensor product and ε and $\bar{\varepsilon}$ are the symplectic forms on V_+, V_- respectively.

If **X** is a 4k dimensional quaternionic-Kähler manifold, where $k > 1$, then the situation is similar. In this setting the bundle V_+ will have fibre $\mathbf{C}^{2k}$ and the projective bundle $P(V_-)$ will automatically have a complex structure, since the definition of quaternionic-Kähler implies the right-flat (corresponding to self-dual in 4 dimensions) and Einstein conditions [3].

We shall also denote by **Z**, the total space of this bundle, which is then the twistor space of **X**.

The two bundles V_+ and V_- will have quaternionic conjugations, and compatible symplectic forms, which we denote by e in the case of V_+.

Both cases can be elegantly described using the formalism developed by Bailey and Eastwood in [3], which in turn uses the abstract index notation of Penrose, [35]. In this notation indices are used simply as 'place markers' and do not imply a choice of basis. This permits an explicit description of intricate tensor operations while maintaining an approach which is completely basis free.

Lower case Roman letters are used for the tangent bundle **TX**, so that $\mathbf{X}^a$ indicates a tangent vector and Y_a is a one-form. The bundle indices for V_+ and V_- are upper case Roman letters, primed for V_- and unprimed for V_+. The tangent bundle indices a, b, ... can then be changed for pairs of indices **AA'**, BB',... by using the isomorphism of (5.2.1). This notation may then be used on bundles, e.g.

$$\mathbf{V}_+ = \mathbf{V}_A, \quad \wedge^1 = \mathbf{V}_a = \mathbf{V}_{AA'} \quad \text{etc} \tag{5.2.3}$$

In this notation, the ε, $\bar{\varepsilon}$ of (5.2.2) are given respectively by e_{AB} and $\varepsilon_{A'B'}$. Their inverses are e^{AB} and $\varepsilon^{A'B'}$, and they may be used to raise and lower indices. When $k = 1$ we may replace e_{AB} by ε_{AB}.

We shall take the inner product on V^A to be given by

$$u^A \bar{v}_A = e_{BA} u^A \bar{v}^B \tag{5.2.4}$$

so that if $u \neq 0$, then

$$u^A \bar{u}_A = |u|^2 > 0 \tag{5.2.5}$$

and similarly for $V^{A'}$, where $\bar{u}_A$ is the quaternionic conjugate of u_A.

The L^2-norm of any spinor v is obtained by raising all free indices of v, contracting with its conjugate $\bar{v}$ with all free indices lowered, and integrating over the manifold. Thus if $v = v^{AB'}{}_{CD'}$ then

$$\|v\|^2 = \int v^{AB'CD'}\,\bar{v}_{AB'CD}\,. \tag{5.2.6}$$

Symmetry operations will play an important role in the work of this chapter. The notation and convention used again follows that of [35]. Thus

$$\begin{aligned} S_{(AB)} &= \frac{1}{2}(S_{AB} + S_{BA}) \\ S_{[AB]} &= \frac{1}{2}(S_{AB} - S_{BA}) \end{aligned} \tag{5.2.7}$$

are the symmetric and skew-symmetric parts of S_{AB}, respectively. With this convention we have

$$e_{AB} \in \mathbf{V}_{[AB]}, \quad \varepsilon_{A'B'} \in \mathbf{V}_{[A'B']}\,. \tag{5.2.8}$$

<u>Connection and curvature conventions</u>

We shall write $\nabla = \nabla_a$ for our (torsion-free) metric, covariant derivative operator on **X**. The Riemann tensor is defined by the identity

$$2\nabla_{[a}\nabla_{b]}u_c = \mathbf{R}_{abc}{}^{d}u_d \tag{5.2.9}$$

This means that the sign convention differs from that of [35], so that in our convention the standard metric on the sphere has positive curvature.

The commutator $2\nabla_{[a}\nabla_{b]}$ can be decomposed as

$$2\nabla_{[a}\nabla_{b]} = \square_{A'B'AB} + \square_{AB}\varepsilon_{A'B'} \tag{5.2.10}$$

where

$$\square_{A'B'AB} = \square_{(A'B')[AB]} = 2\nabla_{(A'|A}\nabla_{B|B')} \tag{5.2.11}$$

and

$$\square_{AB} = \square_{(AB)} = \nabla_{X'(A}\nabla^{X'}_{B)} \tag{5.2.12}$$

We note that in the 4-dimensional case, (5.2.11) can be rewritten as

$$\square_{A'B'AB} = \square_{A'B'}\varepsilon_{AB}$$

where

$$\square_{A'B'} = \nabla_{C(A'}\nabla^{C}_{B')} \tag{5.2.13}$$

thus conforming to the notation of [35].

Since we shall be concerned here mainly with the case of self-dual, Einstein manifolds in 4 dimensions, or quaternionic-Kähler manifolds in 4k dimensions (k > 1), we shall give the action of the curvature operators in precisely those cases.

$$\Box_{AB}\alpha^{C} = \left(-\Psi_{ABQ}{}^{C} - 2\Lambda\delta_{(A}{}^{C}e_{B)Q}\right)\alpha^{Q} \tag{5.2.14}$$

$$\Box_{AB}\beta^{C'} = 0 \tag{5.2.15}$$

$$\Box_{A'B'AB}\alpha^{C} = 0 \tag{5.2.16}$$

$$\Box_{A'B'AB}\beta^{C'} = -2\Lambda e_{AB}\varepsilon_{(A'}{}^{C'}\varepsilon_{B')Q'}\beta^{Q'} \tag{5.2.17}$$

The curvature quantity $\Psi_{ABC}{}^{D}$ has the properties

$$\Psi_{ABC}{}^{D} = \Psi_{(ABC)}{}^{D}, \quad \Psi_{ABC}{}^{A} = 0. \tag{5.2.18}$$

The scalar field Λ is a positive multiple of the scalar curvature R, where $R = R_{ab}{}^{ab}$.

In fact
$$\Lambda = \frac{R}{8k(k+2)}. \tag{5.2.19}$$

We note also that

$$\begin{aligned} \delta_{A}{}^{B} &= e_{A}{}^{B} = e_{AC}e^{BC} \\ \delta_{D}{}^{D} &= 2k. \end{aligned} \tag{5.2.20}$$

(5.2.21) Remarks

(a) We note that for self-dual Einstein and quaternionic-Kähler manifolds, the scalar curvature, and hence Λ, is constant.

(b) Although the actions of the curvature operators given in (5.2.14)-(5.2.17) are for the case of self-dual, Einstein manifolds in four dimensions, equation (5.2.17) is actually true for <u>any</u> self-dual 4-manifold, not necessarily Einstein.

5.3 Vanishing theorems I : the case of 4 dimensions.

The Penrose transform for left-handed fields [15], [49], identifies certain analytic data on **Z** with Dirac fields on **X**. In particular, if $n \geq 2$ it identifies the cohomology group $H^i(\mathbf{Z}, \mathcal{O}(n-2))$ with the i'th cohomology of the elliptic complex

$$\Gamma\left(\mathbf{X}, V^{(A'_3 \ldots A'_n)}\right) \xrightarrow[D_0]{\nabla_A^{A'_2}} \Gamma\left(\mathbf{X}, V_A{}^{(A'_2 \ldots A'_n)}\right) \xrightarrow[D_1]{\nabla^{AA'_1}} \Gamma\left(\mathbf{X}, V^{(A'_1 \ldots A'_n)}\right) \to 0 \qquad (5.3.1)$$

We shall begin by showing that there are no H^0's if the scalar curvature is negative. This fact is already well known but we shall need this result later, so we include the proof for completeness. The proof is valid for the case of a self-dual but not necessarily Einstein manifold **X**, since the action of the curvature operator given in (5.2.17), is true in this more general setting.

5.3.2 Proposition

Let **X** be a compact, self-dual, Riemannian 4 - manifold with negative scalar curvature, and let **Z** be its twistor space. Then $H^0(\mathbf{Z}, \mathcal{O}(n-2)) = 0$ if $n > 2$.

Proof

This is equivalent to proving that if

$$\text{(a)} \qquad g \in V^{(A_3' \ldots A_n')}$$

$$\text{(b)} \qquad \nabla_A^{(A_3'} g^{A_3' \ldots A_n')} = 0 \tag{5.3.3}$$

then $g = 0$ for $\Lambda < 0$ and $n > 2$. Now

$$\nabla_{AA_1'} \nabla^{A}_{A_2'} = \square_{A_1'A_2'} + \frac{1}{2}\varepsilon_{A_1'A_2'} \nabla_b \nabla^b \tag{5.3.4}$$

Using (5.2.17), we have

$$\begin{aligned} \square_{A_1'A_2'} g_{A_3' \ldots A_n'} &= \Lambda \sum_{i=3}^{n} \left(\varepsilon_{A_1'A_i'} \varepsilon_{Q'A_2'} + \varepsilon_{A_2'A_i'} \varepsilon_{Q'A_1'} \right) g^{Q'}_{A_3' \ldots \hat{A}_i' \ldots A_n'} \\ &= \Lambda \sum_{i=3}^{n} \left(\varepsilon_{A_1'A_i'} g_{A_2' \ldots \hat{A}_i' \ldots A_n'} + \varepsilon_{A_2'A_i'} g_{A_1'A_3' \ldots \hat{A}_i' \ldots A_n'} \right) \end{aligned} \tag{5.3.5}$$

On symmetrising we obtain

$$\square_{A_1'(A_2'} g_{A_3' \ldots A_n')} = \Lambda(n-2)\varepsilon_{A_1'(A_2'} g_{A_3' \ldots A_n')} \tag{5.3.6}$$

where the latter term has vanished from symmetrising over the (skew) ε.Using the decomposition (5.3.4), together with the cocycle condition (5.3.3) (b), we have

$$\Lambda(n-2)\varepsilon_{A'_1(A'_2}g_{A'_3\ldots A'_n)} + \frac{1}{2}\varepsilon_{A'_1(A'_2}\nabla_{|b|}\nabla^b g_{A'_3\ldots A'_n)} = 0 \qquad (5.3.7)$$

After contracting both sides with $\varepsilon^{A'_1A'_2}$ and simplifying, it is easy to show that

$$\Lambda(n-2)g_{A'_3\ldots A'_n} + \frac{1}{2}\nabla_b\nabla^b g_{A'_3\ldots A'_n} = 0 \qquad (5.3.8)$$

Hence

$$\begin{aligned}\int \Lambda(n-2)g^{A'_3\ldots A'_n}\bar{g}_{A'_3\ldots A'_n} &= -\frac{1}{2}\int \bar{g}_{A'_3\ldots A'_n}\nabla_b\nabla^b g^{A'_3\ldots A'_n} \\ &= \frac{1}{2}\int(\nabla^b g^{A'_3\ldots A'_n})(\nabla_b\bar{g}_{A'_3\ldots A'_n})\end{aligned} \qquad (5.3.9)$$

where the latter term has been obtained from above by using integration by parts over the compact manifold **X**. Now if $n > 2$ and $\Lambda < 0$, the usual Weitzenbock type argument forces $g = 0$. **This completes the proof of (5.3.2).**

Since the complex given in (5.3.1) is an elliptic complex [44], in the case of the H^1's we may use Hodge theory to simplify the problem. Every element of $H^1(\mathbf{Z}, \mathcal{O}(n-2))$ can then be identified with a unique harmonic representative for the corresponding element in the first cohomology group of the complex. Such a representative f will have the following three properties:

(a) $f \in \Gamma(\mathbf{X}, V_A^{(A'_2 ... A'_n)})$

(b) $D_1 f = 0$ (5.3.10)

(c) $D_0^* f = 0$

where D_0^* is the adjoint map of D_0 relative to the L^2 - inner product. In abstract index notation this is

(a) $f \in \Gamma\left(X, V_A^{(A'_2 ... A'_n)}\right)$

(b) $\nabla_A^{(A'_1} f^{A'_2 ... A'_n)A} = 0$ (5.3.11)

(c) $\nabla_{AA'_2} f^{AA'_2 A'_3 ... A'_n} = 0$

The vanishing of $H^1(Z, \mathcal{O}(n-2))$ is thus equivalent to the vanishing of those f satisfying the conditions of (5.3.11). We shall first prove a simple lemma which will enable the two field equations (b), (c) to be replaced by a single field equation. This lemma is also important in the proof of the quaternionic-Kähler version of the vanishing theorem, in the next section of this chapter.

5.3.12 Lemma

If $S^{A'_1 ... A'_n}$ is symmetric in its final (n - 1) indices and if $S^{(A'_1 A'_2 ... A'_n)} = 0$ then

$$nS^{A'_1 ... A'_n} = (n-1)\varepsilon^{A'_1(A'_2} S_{C'}{}^{A'_3 ... A'_n)C'}$$

Proof

This is a simple exercise in combinatorics. We have

$$S^{(A'_1 \ldots A'_n)} = 0 = \frac{1}{n}\left(S^{A'_1 \ldots A'_n} + \ldots + S^{A'_i A'_1 \ldots \hat{A}'_i \ldots A'_n} + \ldots + S^{A'_n A'_1 \ldots A'_{n-1}}\right)$$

Since

$$S^{A'_1 \ldots A'_n} - S^{A'_i A'_1 \ldots \hat{A}'_i \ldots A'_n} = \varepsilon^{A'_1 A'_i} S_{C'}{}^{C' A'_2 \ldots \hat{A}'_i \ldots A'_n}$$

we obtain

$$S^{A'_i A'_1 \ldots \hat{A}'_i \ldots A'_n} = S^{A'_1 \ldots A'_n} - \varepsilon^{A'_1 A'_i} S_{C'}{}^{C' A'_2 \ldots \hat{A}'_i \ldots A'_n}$$

Substituting this into the above bracket and rearranging, we quickly obtain the result.

Putting $\nabla_A^{A'_1} f^{A'_2 \ldots A'_n A} = S^{A'_1 \ldots A'_n}$ in the lemma shows that (a), (b), (c) of (5.3.11) can be replaced by the equivalent conditions.

(a) $\quad f \in \Gamma(\mathbf{X}, V_A^{(A'_2 \ldots A'_n)})$

$$(5.3.12)$$

(b) $\quad \nabla_A^{A'_1} f^{A'_2 \ldots A'_n A} = 0$

Furthermore (b) is equivalent to saying that $\nabla^{\dot{A}_1 A} f^{B\dot{A}_2 \ldots \dot{A}_n}$ is symmetric in AB. This turns out to be a crucial element in the proof of the vanishing theorem, and we now have all of the information necessary for the proof. We give a full statement of the theorem since we shall need to refer it in the final chapter.

5.3.13 Theorem

Let X be a compact, oriented, Riemannian, self-dual, Einstein 4-manifold with negative scalar curvature, and let Z be its twistor-space. Then if $n > 2$, $H^1(Z, \mathcal{O}(n-2)) = 0$.

Proof

The above discussion of the Penrose transform and Hodge theory show that the vanishing of $H^1(Z, \mathcal{O}(n-2))$ for $n > 2$, is equivalent to the vanishing of those f satisfying (a) and (b) of (5.3.12). What we shall actually prove is that, for such f,

$$\left\| \nabla^{A}_{C'} f^{BC'\dot{A}_3 \ldots \dot{A}_n} \right\| = \Lambda(n-2)\|f\| \qquad (5.3.14)$$

and the vanishing theorem will follow if $\Lambda < 0$ and $n > 2$.

We have

$$\left\|\nabla^{A}_{C'}f^{BC'A'_1\ldots A'_n}\right\| = \int\left(\nabla^{A}_{C'}f^{BC'A'_1\ldots A'_n}\right)\left(\nabla_{AD'}\bar{f}^{D'}_{BA'_1\ldots A'_n}\right)$$

$$= -\int\bar{f}^{D'}_{BA'_1\ldots A'_n}\left(\nabla_{AD'}\nabla^{A}_{C'}f^{BC'A'_1\ldots A'_n}\right)$$

$$= -\int\bar{f}^{D'}_{BA'_1\ldots A'_n}\left(\nabla_{AD'}\nabla^{B}_{C'}f^{AC'A'_1\ldots A'_n}\right)$$

$$= \int\bar{f}^{BD'}_{A'_1\ldots A'_n}\left(\nabla_{AD'}\nabla_{BC'}f^{AC'A'_1\ldots A'_n}\right). \qquad (5.3.15)$$

Where we have used, in turn, integration by parts over the manifold, the symmetry in AB given by (5.3.12), and the spinor see-saw.

If we examine the action of the commutator on f we see that

$$\left(\nabla_{AD'}\nabla_{BC'} - \nabla_{BC'}\nabla_{AD'}\right)f^{AC'A'_1\ldots A'_n} = \nabla_{AD'}\nabla_{BC'}f^{AC'A'_1\ldots A'_n}$$

where the second term in brackets vanishes by (b) of (5.3.12).

Hence

$$\nabla_{AD'}\nabla_{BC'}f^{AC'A'_1\ldots A'_n} = (\varepsilon_{AB}\Box_{D'C'}+\varepsilon_{D'C'}\Box_{AB})f^{AC'A'_1\ldots A'_n}$$

Using the properties of the operators $\Box_{D'C'}$ and $\Box_{AB}$ given in (5.2.14), (5.2.15), (5.2.16) and (5.2.17), one can easily obtain

$$\varepsilon_{D'C'}\square_{AB}f^{AC'A'_3\ldots A'_n} = -3\Lambda f_{BD'}^{A'_3\ldots A'_n}$$

$$\varepsilon_{AB}\square_{D'C'}f^{AC'A'_3\ldots A'_n} = (n+1)\Lambda f_{BD'}^{A'_3\ldots A'_n}$$

so that

$$\nabla_{AD'}\nabla_{BC'}f^{AC'A'_3\ldots A'_n} = (n-2)\Lambda f_{BD'}^{A'_3\ldots A'_n}$$

Substituting this into the above equation now yields

$$\begin{aligned}\left\|\nabla^{A}_{C'}f^{BC'A'_3\ldots A'_n}\right\| &= \int \bar{f}^{BD'}_{A'_3\ldots A'_n}\Lambda(n-2)f_{BD'}^{A'_3\ldots A'_n}\\ &= \int \Lambda(n-2)f^{BD'A'_3\ldots A'_n}\bar{f}_{BD'A'_3\ldots A'_n}\end{aligned}$$

where we have raised and lowered BD', resulting in no overall sign change. Since the scalar curvature, and hence Λ, is constant for an Einstein manifold, equation (5.3.14), and hence the theorem, is proved.

5.4 Vanishing Theorems II: the quaternionic-Kähler case.

In the case of quaternionic-Kähler manifolds **X**, the Penrose transform has a structure similar to that in the 4-dimensional case. For $n \geq 2$ the cohomology group $H^1(\mathbf{Z}, \mathcal{O}(n-2))$ on the twistor-space **Z**, of **X**, is isomorphic to the first cohomology of the complex.

$$0 \to \Gamma\left(\mathbf{X}, V^{(A'_3\ldots A'_n)}\right) \xrightarrow[D_0]{\nabla^{A'_2}_{B}} \Gamma\left(\mathbf{X}, V_B{}^{(A'_2\ldots A'_n)}\right) \xrightarrow[D_1]{\nabla^{A'_1}_{A}} \Gamma\left(\mathbf{X}, V_{[AB]}^{(A'_1\ldots A'_n)}\right) \qquad (5.4.1)$$

Since this is also an elliptic complex the remarks on Hodge theory, given in the context of 4-dimensions, are equally valid. Each element of $H^1(Z,\mathcal{O}(n-2))$ can be identified with a unique representative f satisfying the conditions:

$$\text{(a)} \qquad f \in V_B^{(A'_2 \ldots A'_n)}$$

$$\text{(b)} \qquad \nabla_{[A}^{(A'_1} f_{B]}^{A'_2 \ldots A'_n)} = 0 \qquad (5.4.2)$$

$$\text{(c)} \qquad \nabla_{AC'} f^{AC'A'_3 \ldots A'_n} = 0.$$

Here (b) is the abstract index version of $D_1 f = 0$ and (c) is that of $D_0^* f = 0$. In the 4 - dimensional case condition (b) above is equivalent to condition (b) of (5.3.11), but in the 4k- dimensional case it is stronger. However using e^{AB} to raise B and contract with A, condition (b) implies $\nabla_A^{(A'_1} f^{A'_2 \ldots A'_n)A} = 0$. This, together with the same reasoning as in the case of 4-dimensions proves the following.

5.4.3 Lemma

If f satisfies the conditions of (5.4.2), then $\nabla_A^{A'_1} f^{A'_2 \ldots A'_n A} = 0$.

To prove the vanishing theorem in the quaternionic-Kähler case we shall construct a Weitzenbock formula for $\nabla_{C'}^{[A} f^{B]C'A'_3 \ldots A'_n}$, but we need a subsidiary result.

5.4.4 Lemma

If f satisfies the conditions of (5.4.2), then

$$\left\| \nabla^{A'_1[A} f^{B]A'_2 \ldots A'_n} \right\| = \frac{(n-1)}{n} \left\| \nabla^{[A}_{C'} f^{B]C'A'_2 \ldots A'_n} \right\|$$

Proof

By putting $S^{A'_1 \ldots A'_n} = \nabla^{A'_1[A} f^{B]A'_2 \ldots A'_n}$ we can see that S satisfies the conditions of lemma (5.3.12), so that

$$n \nabla^{A'_1[A} f^{B]A'_2 \ldots A'_n} = (n-1) \varepsilon^{A'_1(A'_2} \nabla^{[A}_{C'} f^{B]|A'_3 \ldots A'_n)C'}$$

Then

$$\begin{aligned} n^2 \left| \nabla^{A'_1[A} f^{B]A'_2 \ldots A'_n} \right|^2 &= (n-1)^2 \varepsilon_{A'_1 A'_2} \left(\nabla_{D[A} \bar{f}^{D'}_{B]A'_3 \ldots A'_n} \right) \left(\varepsilon^{A'_1(A'_2} \nabla^{[A}_{C'} f^{B]|A'_3 \ldots A'_n)C'} \right) \\ &= n(n-1) \left| \nabla^{[A}_{D'} f^{B]D'A'_3 \ldots A'_n} \right|^2 \end{aligned}$$

and the result follows by integrating over the manifold.

We can now prove the vanishing theorem for the quaternionic-Kähler case. We state the theorem in full.

5.4.5 Theorem

Let **X** be a compact quaternionic-Kähler manifold of dimension 4k, for $k > 1$, with negative scalar curvature and let **Z** be its twistor-space. Then $H^1(\mathbf{Z}, \mathcal{O}(n-2) = 0$ if $n \geq 2$.

Proof

Following the above discussions, proving the theorem is equivalent to proving the vanishing of those f satisfying the conditions of (5.4.2). For such f, we shall establish that

$$\left\| \nabla_{C'}^{[A} f^{B]C'A'_3 \ldots A'_n} \right\| = 2\Lambda n(k-1) \|f\| \tag{5.4.6}$$

and the vanishing of f for $k > 1$, $n \geq 2$ and $\Lambda < 0$ will then be obvious.

Now

$$\begin{aligned} \left\| \nabla_{C'}^{[A} f^{B]C'A'_3 \ldots A'_n} \right\| &= \int \left(\nabla_{C'}^{[A} f^{B]C'A'_3 \ldots A'_n} \right) \left(\nabla_{D'[A} \bar{f}^{D'}_{B]A'_3 \ldots A'_n} \right) \\ &= -\int \bar{f}^{D'}_{BA'_3 \ldots A'_n} \left(\nabla_{D'A} \nabla_{C'}^{[A} f^{B]C'A'_3 \ldots A'_n} \right) \\ &= -\int \bar{f}^{D'}_{BA'_3 \ldots A'_n} \left(\nabla_{A(D'} \nabla_{C')}^{[A} + \nabla_{A[D'} \nabla_{C']}^{[A} \right) f^{B]C'A'_3 \ldots A'_n} \end{aligned} \tag{5.4.7}$$

Examining the second integral on the right hand side of the above, we see that this may be written as

$$-\int \bar{f}^{D'}_{BA'_1\ldots A'_n}\frac{1}{2}\varepsilon_{D'C'}\nabla_{AK'}\nabla^{K'[A}f^{B]C'A'_1\ldots A'_n} = -\frac{1}{2}\int \bar{f}_{BC'A'_1\ldots A'_n}\nabla_{AK'}\nabla^{K'[A}f^{B]C'A'_1\ldots A'_n}$$

$$= \frac{1}{2}\int\left(\nabla_{K'[A}\;\bar{f}_{B]C'A'_1\ldots A'_n}\right)\left(\nabla^{K'[A}f^{B]C'A'_1\ldots A'_n}\right)$$

$$= \frac{(n-1)}{2n}\left\|\nabla^{[A}_{C'}f^{B]C'A'_1\ldots A'_n}\right\|$$

on using the results of lemma (5.4.2). Substituting this into (5.4.7) and rearranging, we obtain

$$\frac{(n+1)}{2n}\left\|\nabla^{[A}_{C'}f^{B]C'A'_1\ldots A'_n}\right\| = -\int \bar{f}^{D'}_{BA'_1\ldots A'_n}\nabla_{A(D'}\nabla^{[A}_{C')}f^{B]C'A'_1\ldots A'_n}$$

$$= \int \bar{f}^{D'}_{BA'_1\ldots A'_n}\left(\nabla_{A(D'}\nabla^{B}_{C')}f^{AC'A'_1\ldots A'_n} - \nabla_{A(D'}\nabla^{A}_{C')}f^{BC'A'_1\ldots A'_n}\right) \tag{5.4.8}$$

From (5.2.11), we have

$$\square_{D'C'AB} = \nabla_{A(D'}\nabla_{C')B} - \nabla_{B(D'}\nabla_{C')A}$$

and the final operator annihilates $f^{AC'A'_1\ldots A'_n}$ by lemma (5.4.3). Thus

$$\begin{aligned}\nabla_{A(D'}\nabla^{B}_{C')}f^{AC'A'_1\ldots A'_n} &= \square_{D'C'A}{}^{B}f^{AC'A'_1\ldots A'_n}\\ &= \Lambda(n+1)f^{BA'_1\ldots A'_n}_{D'}\end{aligned}$$

as can be found by using (5.2.16) and (5.2.17). We can also use these to prove that

$$\nabla_{A(D'}\nabla^{A}_{C')}f^{BC'A'_3\ldots A'_n} = \frac{1}{2}\Box_{D'C'A}{}^{A}f^{BC'A'_3\ldots A'_n}$$

$$= k\Lambda(n+1)f^{BA'_3\ldots A'_n}_{D'}.$$

Substituting these values into (5.4.8) above, we obtain

$$\frac{(n+1)}{2n}\left\|\nabla^{[A}_{C'}f^{B]C'A'_3\ldots A'_n}\right\| = \int \Lambda(n+1)(1-k)f^{BA'_3\ldots A'_n}_{D'}\bar{f}^{D'}{}_{BA'_3\ldots A'_n}$$

Raising and lowering the contracted index D', together with the appropriate sign change, and a little elementary algebra, yields the proposed Weitzenbock formula.

5.5 Remarks

5.5.1

For our purpose, a vanishing theorem for the case when the 4 - manifold **X** is conformally flat, non-Einstein and with negative scalar curvature would have been very useful, but we were not able to prove such a result.

5.5.2

Combining this vanishing theorem with le Brun's arguments given in [24], one immediately infers that quaternionic - Kähler structures with non - zero scalar curvature on a compact manifold have no non - trivial deformations through quaternionic - Kähler structures, i.e. they are rigid. (See also [40].)

Chapter 6 The dimension of $H^1(\tilde{Z},O(a_1,\ldots,a_r;b_1,\ldots,b_r))$.

6.1 Introduction

In this final chapter we shall gather together the information contained in the previous chapters and use the results that have been obtained to calculate the dimension of $H^1(\tilde{\mathbf{Z}}, O(a_1,\ldots,a_r;b_1,\ldots,b_r))$, for those manifolds **Z** which are subject to the various conditions required, and with the conditions necessary for codimension - 2 poles, i.e. $a_i \geq 0$, $b_i \leq -2$ and $a_i + b_i = m$.

Throughout the chapter we shall be considering the twistor space Z of a compact, Riemannian, self - dual, 4 - manifold X.

We split the discussion into two parts: in 6.2 we consider the case when the scalar curvature of X is negative; in 6.3 we consider the case when the scalar curvature of X is positive.

6.2 The case of negative scalar curvature.

We begin by looking at vanishing theorems on **Z**. The simplest cases are for the H^0's and H^3's. Referring to the elliptic complex of (5.3.1), the Penrose transform immediately produces $H^3(\mathbf{Z},O(n-2)) = 0$ for $n \geq 2$, whilst Proposition 5.3.2 gives $H^0(\mathbf{Z}, O(n-2)) = 0$

for $n > 2$. The Serre dual of $H^0(\mathbf{Z}, \mathcal{O}(n-2))$ is $H^3(\mathbf{Z}, \mathcal{O}(-n-2))$, and the Serre dual of H3 $(\mathbf{Z}, \mathcal{O}(n-2))$ is $H^0(\mathbf{Z}, \mathcal{O}(n-2))$ that these vanish if $n > 2$ and $n \geq 2$ respectively.

For H^1's, we have 5.3.13, which requires **X** to have the additional property of being Einstein . Putting all of this together we have the following.

6.2.1 Proposition

Let **X** be a compact, Riemannian, self - dual, 4 - manifold, with negative scalar curvature, and let **Z** be its twistor space. If $k > 0$ or $k < -4$, then $H^0(\mathbf{Z}, \mathcal{O}(k)) = H^3(\mathbf{Z}, \mathcal{O}(k)) = 0$. If **X** is also Einstein and $k > 0$, then $H^1(\mathbf{Z}, \mathcal{O}(k)) = 0$.

Let **Z** be a compact, flat twistor space, and let $\mathbf{L} = \mathbf{L}_1 \cup \ldots \cup \mathbf{L}_r$ be the union of non-intersecting (complex projective) lines in **Z**. In chapter 2 we defined the line bundle $\mathcal{O}(a_1,\ldots,a_r; b_1,\ldots, b_r)$ on $\widetilde{\mathbf{Z}}$, the blow - up of **Z** along **L**, subject only to the condition that $a_i + b_i = k$ for $i = 1, \ldots, r$. If $a_i \geq 0$, $b_i \leq -2$ then we also showed that the restriction map from $H^1(\widetilde{\mathbf{Z}}, \mathcal{O}(a_1,\ldots,a_r;b_1,\ldots,b_r))$ to $H^1(\mathbf{Z} - \mathbf{L}, \mathcal{O}(k))$ was injective, and elements in the image of this map were, by definition, precisely those elements of the latter group with a codimension - 2 pole on $\mathbf{L}_i$ of order at most $-b_i - 1$. Our objective is to calculate the dimension of the former group, i.e. $H^1(\widetilde{\mathbf{Z}}, \mathcal{O}(a_1,\ldots,a_r;b_1,\ldots,b_r))$, when $a_i + b_i = k$, $a_i \geq 0$, $b_i \leq -2$. We extract this information from the holomorphic Euler characteristic, which was calculated in chapter 3, by using the vanishing theorems of 6.2.1 to determine the unwanted terms.

Our first result in this direction is provided by theorem 4.4.8 and its corollary. We state this in full, as a theorem.

6.2.2 Theorem

Let **X** be a compact, Riemannian, conformally flat, 4 - manifold with negative scalar curvature, and let **Z** be its twistor space (which is a flat twistor space). Let **L** be the union of the non-intersecting lines $\mathbf{L}_i$ and let $\mathcal{O}(a_1,\dots,a_r;b_1,\dots,b_r)$ be the bundle on the blow-up $\widetilde{\mathbf{Z}}$, subject only to $a_i + b_i = k$, $i = 1, \dots, r$.

Then

$$H^0(\widetilde{\mathbf{Z}},\mathcal{O}(a_1,\dots,a_r;b_1,\dots,b_r)) = H^3(\widetilde{\mathbf{Z}},\mathcal{O}(a_1,\dots,a_r;b_1,\dots,b_r)) = 0$$

if $k > 0$ or $k < -4$.

Proof

Theorem 4.4.8 and its corollary, together with 6.2.1, make this obvious.

It remains to account for the H^2 term and for this we recall, from 4.2, that if $a_i \geq 0$ and $b_i \leq -2$ then the Serre dual of $H^2(\widetilde{\mathbf{Z}}, \mathcal{O}(a_1,\ldots,a_r;b_1,\ldots,b_r))$ is $H^2(\widetilde{\mathbf{Z}}, \mathcal{O}(c_1,\ldots,c_r;d_1,\ldots,d_r))$, where $c_i = -a_i - 3$, $d_i = -b_i - 1$ so that if $a_i + b_i = m$ $(i = 1, \ldots, r)$ then $c_i + d_i = n = -m-4$, for $i = 1, \ldots, r$. Theorem 4.4.3 may then be used whenever $H^1(\mathbf{Z}, \mathcal{O}(n)) = H^0(\mathbf{Z}, \mathcal{O}(n)) = 0$.

In 6.2.1 we saw that a vanishing theorem for H^1's existed when **X** was, in particular, self - dual and Einstein. We also required **X** to be conformally flat, so that putting all of these conditions together, we can then prove the following result, which we state in full, since it is the major conclusion of this work.

6.2.3 Theorem

Let **X** be a compact, Riemannian, conformally flat, Einstein, 4 - manifold, having negative scalar curvature, and let **Z** be its twistor space. Let $\mathbf{L} = \mathbf{L}_1 \cup \ldots \cup \mathbf{L}_r$ be a union of the non - intersecting (complex projective) lines $\mathbf{L}_i$ and let $\mathcal{O}(a_1,\ldots,a_r;b_1,\ldots,b_r)$ be the bundle on the blow - up $\widetilde{\mathbf{Z}}$, of **Z**, along **L**, as defined in 2.2. of chapter 2, with $a_i + b_i = m$, $a_i \geq 0$, $b_i \leq -2$ for $i = 1, \ldots, r$.

Then if $m < -4$ we have

$$\dim H^1(\widetilde{\mathbf{Z}},\mathcal{O}(a_1,\ldots,a_r;b_1,\ldots,b_r)) = \frac{r(-m-3)(-m-2)(-m,-1)}{6}$$
$$-\frac{1}{12}(m+1)(m+2)(m+3)\chi$$
$$+\frac{1}{6}\sum_{i=1}^{r} b_i(b_i+1)(3m+5-2b_i)$$

where χ is the Euler characteristic of $\mathbf{X}$.

Proof

This is a simple matter of collecting together the various results. Firstly, for $m < -4$, both the H^0 and H^3 terms of the holomorphic Euler characteristic are zero, by 6.2.2. Under the conditions given above, the Serre dual of $H^2(\widetilde{\mathbf{Z}},\mathcal{O}(a_1,\ldots,a_r;b_1,\ldots,b_r))$ is $H^1(\widetilde{\mathbf{Z}},\mathcal{O}(c_1,\ldots,c_r;d_1,\ldots,d_r))$, with $c_i \leq -3$, $d_i \geq 1$ and $c_i + d_i = -m-4 > 0$.

Since $H^0(\mathbf{Z}, \mathcal{O}(-m-4)) = H^1(\mathbf{Z},\mathcal{O}(-m-4)) = 0$ for this case, by 6.2 1, we may use the result of 4.4.3 to conclude that

$$\dim H^2(\widetilde{\mathbf{Z}},0(a_1,\ldots,a_r;b_1,\ldots,b_r)) = \frac{r(-m-3)(-m-2)(-m-1)}{6}$$

The proof is now immediate on using (3.4.15) and a little arithmetic.

6.2.4 Remarks

(a) The conditions imposed upon the manifold **X** in the above theorem are quite severe, and the question arises as to whether there are any such manifolds. Fortunately we are able to answer this in the affirmative since this class of manifolds has been well studied. It is precisely the class of compact, hyperbolic 4-manifolds. These manifolds are formed from hyperbolic 4 - space by taking compact quotients using discrete subgroups of the symmetry group. For a discussion of such matters, see e.g. [1], [22], [29], [37].

(b) Having obtained a solution to our problem when $m < -4$, we might ask if these methods could be used to provide a similar result when $m > 0$, since the H^0 and H^3 terms of the holomorphic Euler characteristic are both zero in this case. The answer is negative, for the following reason.

To obtain the result of 6.2.3 required a vanishing theorem for $H^1(\mathbf{Z}, \mathcal{O}(n))$ and when $n = -m-4$, with $m < -4$, such a theorem existed, i.e. 5.3.13. If we wished to use the same methods when $m > 0$, then this would require a vanishing theorem for $H^1(\mathbf{Z}, \mathcal{O}(n))$ when $n = -m-4 < -4$, and this would have to be true at least when the manifold **X** is compact conformally flat and with negative, scalar curvature. In this case it would be true for the manifold **X** of 6.2.3. But this is unlikely since it would imply that its Serre dual, $H^2(\mathbf{Z}, \mathcal{O}(-n-4)) = H^2(\mathbf{Z}, \mathcal{O}(m)) = 0$ for $m > 0$.

In the case of such **X** therefore, we would have H^i (**Z**,$\mathcal{O}$(m)) = 0 for m > 0 and i = 0, 1, 2, 3, which would mean that the holomorphic Euler characteristic of $\mathcal{O}$(m) on **Z** is zero. (see 3.4.16).

6.3 The case of positive scalar curvature.

In the case of positive scalar curvature we are unable to give a definitive answer but are able to make some useful observations. As with the case of negative scalar curvature, in order to use the techniques developed in the earlier chapters we shall need to find vanishing theorems for the cohomology of **Z**, when **X** has positive scalar curvature. Referring to the complex (5.3.1) we see immediately that H^3 (**Z**, $\mathcal{O}$(n − 2) = 0 for n ≥ 2. There is also a vanishing theorem for H^1's when **X** has positive scalar curvature, which is well known. There is a Penrose transform for 'right - handed' fields, which shows that for n ≥ 0, H^1 (**Z**, $\mathcal{O}$(−n−2)) is isomorphic to the spinor fields f on **X** satisfying

(a) $f \in \Gamma(\mathbf{X}, V^{(A'_1 \ldots A'_n)})$

(b) $\nabla^{A}_{A'_1} f^{A'_1 A'_2 \ldots A'_n} = 0.$

[ ], and a simple Weitzenbock argument, similar to that of 5.3.2, can be employed to show that these are all zero when the scalar curvature is positive and n > 0. We collect these facts together in the following.

6.3.1 Proposition

Let **X** be a compact, Riemannian, self - dual, 4 - manifold having positive scalar curvature and let **Z** be its twistor space. If $k \leq -4$ we have

$$H^3(\mathbf{Z}, \mathcal{O}(-k-4)) = H^0(\mathbf{Z}, \mathcal{O}(k)) = H^1(\mathbf{Z}, \mathcal{O}(k)) = 0.$$

The vanishing of the above cohomology groups enable us to prove the following theorem.

6.3.2 Theorem

Let **X** be a compact, Riemannian, conformally flat 4 - manifold, with positive scalar curvature, and let **Z** be its twistor space. Let $\mathcal{O}(a_1,\ldots,a_r;b_1,\ldots,b_r))$ be defined on $\widetilde{\mathbf{Z}}$, the blow - up of **Z** along the non -intersecting complex projective lines $\mathbf{L}_1, \ldots, \mathbf{L}_r$, , as in chapter 2, with $a_i + b_i = m$, $a_i \geq 0$, $b_i \leq -2$. Then if $m \geq 0$, we have

$$\begin{aligned}\dim H^1(\widetilde{\mathbf{Z}},\mathcal{O}(a_1,\ldots,a_r;b_1,\ldots,b_r)) = \dim H^0(\widetilde{\mathbf{Z}},\mathcal{O}(a_1,\ldots,a_r;b_1,\ldots,b_r))\\ -\frac{1}{12}(m+1)(m+2)(m+3)\chi\\ +\frac{1}{6}\sum_{i=1}^{r} b_i(b_i+1)(3m+5-2b_i)\end{aligned}$$

where χ is the Euler characterisitc of **X**.

Proof

From 6.3.1 we have $H^0(\mathbf{Z}, \mathcal{O}(-m-4)) = H^1(\mathbf{Z}, \mathcal{O}(-m-4)) = 0$. Using 4.4.3 this implies that (the Serre dual of) $H^2(\tilde{\mathbf{Z}}, \mathcal{O}(a_1, \ldots, a_r; b_1, \ldots, b_r))$ is zero. Since $H^3(\mathbf{Z}, \mathcal{O}(m)) = 0$ for $m \geq 0$, we can deduce that $H^3(\tilde{\mathbf{Z}}, \mathcal{O}(a_1, \ldots, a_r; b_1, \ldots, b_r))$ is zero by the corollary to 4.4.8. This proves the theorem.

6.3.3 Remarks

(a) As we have seen, a sufficient condition for the vanishing of the H^0 term in the above is the vanishing of $H^0(\mathbf{Z}, \mathcal{O}(m))$. If $m = n - 2 \geq 0$ then this group is isomorphic to those spinor fields g, on **X**, satisfying the conditions of 5.3.3, but this time with **X** having positive scalar curvature. The conditions imposed on g are heavily over - determined so that one would expect them to be zero, except when very special conditions are put on **X**.

(b) The observations made in 6.2.4 (b) are also valid in the case of positive scalar curvature.

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