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Brief paper

# Output feedback adaptive control of a class of nonlinear discrete-time systems with unknown control directions☆

Chenguang Yang, Shuzhi Sam Ge*, Tong Heng Lee

*Social Robotics Lab, Interactive Digital Media Institute, and Department of Electrical and Computer Engineering, National University of Singapore, 117576, Singapore*

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ABSTRACT

In this paper, output feedback adaptive control is investigated for a class of nonlinear systems in output-feedback form with unknown control gains. To construct output feedback control, the system is transformed into the form of the NARMA (nonlinear-auto-regressive-moving-average) model, based on which future output prediction is carried out. With employment of the predicted future output, a constructive output feedback adaptive control is given with the discrete Nussbaum gain exploited to overcome the difficulty due to unknown control directions. Under the global Lipschitz condition of the system functions, the boundedness of all the closed-loop signals and asymptotical output tracking are achieved by the proposed control. Simulation results are presented to show the effectiveness of the proposed approach.



## 1. Introduction

In the last two decades, the problem of controlling nonlinear systems with unknown control directions has received a great deal of attention for the continuous-time systems (Ge & Wang, 2003; Ge, Hong, & Lee, 2004; Kaloust & Qu, 1995; Lozano, Collado, & Mondie, 1990; Nussbaum, 1983; Ryan, 1994; Ye & Jiang, 1998). The control directions, defined as signs of the control gains, are normally required to be known *a priori* in adaptive control literature. When the signs of control gains are unknown, the adaptive control problem becomes much more difficult, since we cannot decide the direction along which the control operates. The unknown control directions problem had remained open till the Nussbaum-type gain was first introduced in Nussbaum (1983) for adaptive control of first order continuous-time systems. Later, the Nussbaum gain was adopted in the adaptive control of linear systems with nonlinear uncertainties (Ryan, 1994) to counteract the lack of *a prior* knowledge of control directions. Towards high order nonlinear systems, backstepping with Nussbaum function was then developed for general nonlinear systems in the triangular structure, with constant control gains (Ye & Jiang, 1998), and time varying control gains (Ge & Wang, 2003). Recently, the results have also been extended to nonlinear systems with general unknown nonlinear functions by using neural network parametrization techniques (Ge et al., 2004). It should be mentioned that besides Nussbaum gain, some other methods to deal with unknown control directions have also been developed in the literature (Kaloust & Qu, 1995; Lozano et al., 1990), but the application of these methods is restricted to certain systems and is not as general as Nussbaum gain.

In contrast to the aforementioned results for continuous-time systems, their discrete-time counterparts remain largely unexplored. In addition, many continuous-time control methods may be not suitable for discrete-time systems, e.g., the backstepping design proposed in Krstic, Kanellakopoulos, and Kokotovic (1995), a crucial ingredient for the development of solutions to many continuous-time adaptive nonlinear problems, may be not directly applicable to discrete-time systems (Ge, Li, & Lee, 2003). To develop a discrete-time counterpart of continuous-time adaptive backstepping, the approach that "looks ahead" and chooses the control law to force the states to acquire their desired values was proposed in Yeh and Kokotovic (1995). But the proposed adaptive control is not applicable to systems with unknown control gains. On the other hand, the discrete Nussbaum gain, a counterpart of the continuous-time Nussbaum gain, was less exploited since it was proposed in Lee and Narendra (1986) for adaptive control discrete-time linear systems. In this paper, we will employ discrete Nussbaum gain for adaptive output feedback control of a class of discrete-time nonlinear systems with finite unknown control gains. Under the global Lipschitz condition of the system functions, the proposed adaptive control guarantees asymptotical tracking performance.

Throughout this paper, the following notations are used in order.

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* Corresponding author. Tel.: +65 65166821; fax: +65 67791103.
*E-mail addresses:* cgyang82@gmail.com (C. Yang), samge@nus.edu.sg (S.S. Ge), eleleeth@nus.edu.sg (T.H. Lee).

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- $\|\cdot\|$ denotes the Euclidean norm of vectors and induced norm of matrices.
- $[\ ]^T$ represents the transpose of a vector or a matrix.
- $\mathbf{0}_{[p]}$ stands for a $p$-dimension zero vector.
- $\hat{()}$ and $\tilde{()}$ denote the estimate of parameters and estimation error, respectively.

## 2. Problem formulation and preliminaries

### 2.1. System representation

The systems we consider in this paper are in the output-feedback form with unknown control gains as follows:

$$\begin{cases} x_1(k+1) = \Theta_1^T \Phi_1(x_1(k)) + g_1 x_2(k) \\ x_2(k+1) = \Theta_2^T \Phi_2(x_1(k)) + g_2 x_3(k) \\ \vdots \\ x_n(k+1) = \Theta_n^T \Phi_n(x_1(k)) + g_n u(k) \\ y(k) = x_1(k) \end{cases} \tag{1}$$

where $\Theta_i \in R^{p_i}$ are the vectors of unknown constant parameters and $g_i \in R$ are unknown control gains, $\Phi_i(\cdot) : R \to R^{p_i}$, $i = 1, 2, \ldots, n$, are known nonlinear vector functions, $x_1(k), x_2(k), \ldots, x_n(k)$ are the system states, $n \geq 1$ is system order. It is noted that the nonlinearities that are multiplied by the unknown vector parameters depend only on the output $y(k) = x_1(k)$, which is the only measured state. This justifies the name of "output-feedback" form.

**Assumption 1.** The system functions $\Phi_i(\cdot)$ are Lipschitz functions, i.e., $\|\Phi_i(\varepsilon_1) - \Phi_i(\varepsilon_2)\| \leq L_i\|\varepsilon_1 - \varepsilon_2\|, \forall \varepsilon_1, \varepsilon_2 \in R$, $1 \leq i \leq n$, with finite constants $L_i$. The control gains $g_i \neq 0$.

It should be noted that neither the sign of $g_i$ (the control direction) nor the upper or lower bound of $g_i$ are assumed to be known in the paper. If the control gains $g_i$'s are all ones, the system becomes in the so called "parametric-output-feedback" form studied in Zhao and Kanellakopoulos (2002) where the authors proposed a parameter estimator that guarantees the estimates converge to the true values in finite steps. The control objective in this paper is to design an output feedback control $u(k)$ such that the output $y(k)$ tracks a bounded reference trajectory $y_d(k)$ and all the closed-loop signals are guaranteed to be bounded.

### 2.2. Preliminaries

**Definition 1** (*Chen & Narendra, 2001*). Let $x_1(k)$ and $x_2(k)$ be two discrete-time scalar or vector signals

- We denote $x_1(k) = O[x_2(k)]$, if there exist positive constants $m_1$, $m_2$ and $k_0$ such that $\|x_1(k)\| \leq m_1 \max_{k' \leq k} \|x_2(k')\| + m_2, \forall k > k_0$.
- We denote $x_1(k) = o[x_2(k)]$, if there exists a discrete-time function $\alpha(k)$ satisfying $\lim_{k\to\infty} \alpha(k) \to 0$ and a constant $k_0$ such that $\|x_1(k)\| \leq \alpha(k) \max_{k' \leq k} \|x_2(k')\|, \forall k > k_0$.
- We denote $x_1(k) \sim x_2(k)$ if they satisfy $x_1(k) = O[x_2(k)]$ and $x_2(k) = O[x_1(k)]$.

**Lemma 1** (*Ge, Yang, and Lee, 2008a*). *Under Assumption 1, for $i = 1, 2, \ldots, n$, the states and input of system (1) satisfy*

$$\xi_i(k) = O[y(k+i-1)], \qquad u(k) = O[y(k+n)].$$

**Definition 2** (*Yang, Ge, Xiang, Chai, and Lee, 2008*). Consider a discrete nonlinear function $N(x(k))$ defined on a sequence $x(k)$ with $x_s(k) = \sup_{k' \leq k}\{x(k')\}$. $N(x(k))$ is a discrete Nussbaum gain if and only if it satisfies the following two properties:

(i) If $x_s(k)$ increases without bound, then for any given constant $\delta_0$

$$\sup_{x_s(k)\geq\delta_0} \frac{S_N(x(k))}{x_s(k)} = +\infty, \qquad \inf_{x_s(k)\geq\delta_0} \frac{S_N(x(k))}{x_s(k)} = -\infty.$$

(ii) If $x_s(k) \leq \delta_1$, then $|S_N(x(k))| \leq \delta_2$ with some positive constants $\delta_1$ and $\delta_2$.

where $S_N(x(k))$ is defined with $\Delta x(k) = x(k+1) - x(k)$ as follows:

$$S_N(x(k)) = \sum_{k'=0}^{k} N(x(k'))\Delta x(k'). \tag{2}$$

In this paper, for adaptive control of system (1), the discrete Nussbaum gain $N(x(k))$ proposed in Lee and Narendra (1986) will be exploited, which requires the sequence $x(k)$ to satisfy

$$x(k) \geq 0, \quad \forall k, |\Delta x(k)| = |x(k+1) - x(k)| \leq \delta_0. \tag{3}$$

**Lemma 2** (*Ge, Yang, and Lee, 2008b*). *Let $V(k)$ be a positive definite function defined $\forall k$, $N(x(k))$ be a discrete Nussbaum gain, and $\theta$ be a nonzero constant. If the following inequality holds, $\forall k$*

$$V(k) \leq \sum_{k'=k_1}^{k} (c_1 + \theta N(x(k')))\Delta x(k') + c_2 x(k) + c_3 \tag{4}$$

*where $c_1$, $c_2$ and $c_3$ are some constants, $k_1$ is a positive integer, then $V(k)$, $x(k)$ and $\sum_{k'=k_1}^{k}(c_1 + \theta N(x(k')))\Delta x(k') + c_2 x(k) + c_3$ must be bounded, $\forall k$.*

## 3. System transformation

To facilitate the control design, let us consider a state transformation such that $\xi_i(k) = x_i(k)\prod_{j=0}^{i-1} g_j$ with $g_0 = 1$, which transforms system (1) into the following form:

$$\begin{cases} \xi_1(k+1) = \Theta_f^T \Phi_{f1}(\xi_1(k)) + \xi_2(k) \\ \xi_2(k+1) = \Theta_f^T \Phi_{f2}(\xi_1(k)) + \xi_3(k) \\ \vdots \\ \xi_n(k+1) = \Theta_f^T \Phi_{fn}(\xi_1(k)) + gu(k) \\ y(k) = \xi_1(k) \end{cases} \tag{5}$$

where the new parameters $\Theta_f$ and $g$ as well as new system functions $\Phi_{fi}(\cdot)$ are defined as

$$\Theta_f = [\Theta_{f1}^T, \ldots, \Theta_{fn}^T]^T \in R^p, \qquad \Theta_{fi} = \Theta_i \prod_{j=0}^{i-1} g_j$$

$$g = \prod_{j=0}^{n} g_j, \; \Phi_{fi}(\cdot) = \left[\mathbf{0}_{[M_i]}^T, \; \Phi_i^T(\cdot), \; \mathbf{0}_{[N_i]}^T\right]^T \in R^p$$

with $M_i = \sum_{j=1}^{i-1} p_j$, $N_i = \sum_{j=i+1}^{n} p_j$, $p = \sum_{j=1}^{n} p_j$. The transformed system (5) is very similar to the "parameter-output-feedback" form except that the control gain of $u(k)$ is an unknown constant $g$ rather than one. The existence of the unknown control gain makes it impossible to calculate future values of the outputs by the approach proposed in Zhao and Kanellakopoulos (2002).

Combining all the equations in (5) together by iterative substitution, we obtain the following equation

$$y(k+n) = \Theta_f^{\mathrm{T}} \sum_{i=1}^{n} \Phi_{fi}(y(k+n-i)) + gu(k). \tag{6}$$

To control system (6), one difficulty lies in the unknown control gain $g$. Because to avoid the singularity problem, the sign and upper bound of the gain $g$ are usually required to be known but no *a priori* information on the control gain is assumed in this paper. The other difficulty lies in the future outputs involved in (6). In order to avoid noncausal problem in the control law, we will consider future output prediction in the next section.

## 4. Future output prediction

In this section, future output prediction is proposed to facilitate the control design. Let $\hat{\Theta}_f(k)$ denote the estimate of $\Theta_f$ and $\hat{g}(k)$ the estimate of $g$. For convenience, let us define

$$\Psi_l(k) = \left[\sum_{i=1}^{l-1} \Phi_{fi}^{\mathrm{T}}(\hat{y}(k+l-i|k)) + \sum_{i=l}^{n} \Phi_{fi}^{\mathrm{T}}(y(k-(i-l))),\right.$$
$$\left. u(k-(n-l))\right]^{\mathrm{T}} \in R^{p+1}, l = 1, 2, \ldots, n-1$$
$$\bar{\hat{\Theta}}(k) = [\hat{\Theta}_f^{\mathrm{T}}(k), \hat{g}(k)]^{\mathrm{T}} \in R^{p+1} \tag{7}$$

where $\hat{y}(k+l-i|k)$ is the predicted $(i-1)$th future output at the $k$-th step that will be defined later.

**Step** 1: According to (6), we define one-step ahead prediction $\hat{y}(k+1|k)$, the estimation of $y(k+1)$ at the $k$-th step as follows:

$$\hat{y}(k+1|k) = \bar{\hat{\Theta}}^{\mathrm{T}}(k-n+2)\Psi_1(k). \tag{8}$$

**Step** 2: Based on the one-step prediction, $\hat{y}(k+1|k)$, defined in (8), we define two-step ahead prediction, $\hat{y}(k+2|k)$, the estimation of $y(k+2)$ at the $k$-th step as follows:

$$\hat{y}(k+2|k) = \bar{\hat{\Theta}}^{\mathrm{T}}(k-n+3)\Psi_2(k). \tag{9}$$

**Step** $l$: Based on the previous steps predictions $\hat{y}(k+1|k)$, $\hat{y}(k+2|k)$, ..., $\hat{y}(k+l-1|k)$, we define the $l$-step prediction $\hat{y}(k+1|k)$, $l = 1, 2, \ldots, n-1$, the estimation of $y(k+l)$ at the $k$-th step as follows:

$$\hat{y}(k+l|k) = \bar{\hat{\Theta}}^{\mathrm{T}}(k-n+l+1)\Psi_l(k). \tag{10}$$

We see that the prediction procedure is defined in such a way that the $l$-step prediction is based the the predictions in previous steps. The parameter estimates in output prediction are obtained from the following update law

$$\bar{\hat{\Theta}}(k+1) = \bar{\hat{\Theta}}(k-n+2) - \frac{\tilde{y}(k+1|k)\Psi_1(k)}{1+\Psi_1^{\mathrm{T}}(k)\Psi_1(k)}$$
$$\bar{\hat{\Theta}}(0) = \mathbf{0}_{[p+1]}, \tilde{y}(k+1|k) = \hat{y}(k+1|k) - y(k+1). \tag{11}$$

The following lemma will be used for stability analysis later.

**Lemma 3.** *Consider the outputs prediction given in (8)–(10) with parameter estimator defined in (11). Define the $l$-step prediction error as $\tilde{y}(k+l|k) = \hat{y}(k+l|k) - y(k+l)$, $l = 1, 2, \ldots, n-1$, then the prediction errors satisfy $\tilde{y}(k+l|k) = o[O[y(k+l-1)]]$. In addition, the parameter estimate $\bar{\hat{\Theta}}(k)$ is globally bounded.*

**Proof.** See Appendix. ■

## 5. Adaptive control design

### 5.1. Control and parameter estimation

Using the predicted future outputs in Section 4, the noncausal problem mentioned in Section 3 can be solved. It is noted in the following control design that we estimate $\Theta_{fg} = g^{-1}\Theta_f$ and $g^{-1}$ instead of $\Theta_f$ and $g$ because the potential controller singularity problem exists if the estimation of $g$ appear in the denominator of control law.

The adaptive control for system (1) is given as follows:

$$u(k) = -\hat{\Theta}_{fg}^{\mathrm{T}}(k) \sum_{i=1}^{n-1} \Phi_{fi}(\hat{y}(k+n-i|k))$$
$$- \hat{\Theta}_{fg}^{\mathrm{T}}(k)\Phi_{fn}(y(k)) + \hat{g}_l(k)y_d(k+n) \tag{12}$$

where $\hat{\Theta}_{fg}^{\mathrm{T}}(k)$, $\hat{g}_l(k)$ are the estimates of $\Theta_{fg} = g^{-1}\Theta_f$ and $g^{-1}$. Substituting the adaptive control (12) into (6) which is equivalent to (1), the following tracking error equation can be obtained:

$$\begin{aligned} e(k+n) &= y(k+n) - y_d(k+n) \\ &= g\Theta_{fg}^{\mathrm{T}} \sum_{i=1}^{n} \Phi_{fi}(y(k+n-i)) - g\hat{\Theta}_{fg}^{\mathrm{T}}(k)\Phi_{fn}(y(k)) \\ &\quad - g\hat{\Theta}_{fg}^{\mathrm{T}}(k) \sum_{i=1}^{n-1} \Phi_{fi}(\hat{y}(k+n-i|k)) \\ &\quad + g\hat{g}_l(k)y_d(k+n) - gg^{-1}y_d(k+n) \\ &= -g\tilde{\Theta}_{fg}^{\mathrm{T}}(k)\Psi(k+n-1) + g\tilde{g}_l(k)y_d(k+n) \\ &\quad - g\beta(k+n-1) \end{aligned} \tag{13}$$

where $\tilde{\Theta}_{fg}(k)$, $\tilde{g}_l(k)$, $\beta(k)$ and $\Psi(k)$ are defined as

$$\tilde{\Theta}_{fg}(k) = \hat{\Theta}_{fg}(k) - \Theta_{fg}, \qquad \tilde{g}_l(k) = \hat{g}_l(k) - g^{-1}$$
$$\beta(k+n-1) = \hat{\Theta}_{fg}^{\mathrm{T}}(k) \sum_{i=1}^{n-1} \tilde{\Phi}_{fi}(k+n-i)$$
$$\Psi(k+n-1) = \sum_{i=1}^{n} \Phi_{fi}(y(k+n-i))$$
$$\tilde{\Phi}_{fi}(k+n-i) = \Phi_{fi}(\hat{y}(k+n-i|k)) - \Phi_{fi}(y(k+n-i)).$$

It is noted that in (13) there is an unknown control gain $g$, which will make parameter estimation difficult, because without knowledge of the sign of $g$, we cannot decide to which direction shall we update the parameter estimates. Thus, the discrete Nussbaum gain is introduced into the update law to make it insensitive to the control direction as follows:

$$\epsilon(k) = \frac{\gamma e(k) + N(x(k))\psi(k)\beta(k-1)}{G(k)}$$
$$\hat{\Theta}_{fg}(k) = \hat{\Theta}_{fg}(k-n) + \gamma \frac{N(x(k))}{D(k)}\Psi(k-1)\epsilon(k)$$
$$\hat{g}_l(k) = \hat{g}_l(k-n) - \gamma \frac{N(x(k))}{D(k)}y_d(k)\epsilon(k)$$
$$\Delta\psi(k) = \psi(k+1) - \psi(k) = \frac{-N(x(k))\beta(k-1)\epsilon(k)}{D(k)}$$
$$\Delta z(k) = z(k+1) - z(k) = \frac{G(k)\epsilon^2(k)}{D(k)}$$
$$x(k) = z(k) + \frac{\psi^2(k)}{2}, \qquad G(k) = 1 + |N(x(k))| \tag{14}$$

where

$$\beta(k-1) = \hat{\Theta}_{fg}^{\mathrm{T}}(k-n)\sum_{i=1}^{n-1}\tilde{\Phi}_{fi}(k-i)$$

$$\Psi(k-1) = \sum_{i=1}^{n}\Phi_{fi}(y(k-i))$$

and the normalization term $D(k)$ is defined as

$$D(k) = (1+|\psi(k)|)(1+|N^3(x(k))|) \times (1+\|\Psi(k-1)\|^2 + y_d^2(k) + \beta^2(k-1) + \epsilon^2(k))$$

and the initial estimates are chosen as $\hat{\Theta}_{fg}(j) = \mathbf{0}_{[p]}$, $\hat{g}_l(j) = 0$, $j = 0, -1, -2, -n+1$, $z(0) = \psi(0) = 0$, $\epsilon(k)$ is introduced as an augmented error and $\gamma > 0$ can be arbitrary positive constant, $N(x(k))$ is the discrete Nussbaum gain defined in Lee and Narendra (1986). From the definition of $x(k)$ in (14), we see $x(k) \geq 0$ and we have

$$\begin{aligned}\Delta x(k) &= \Delta z(k) + \frac{\psi^2(k+1)}{2} - \frac{\psi^2(k)}{2}\\ &= \Delta z(k) + \frac{1}{2}[\Delta\psi(k)(\Delta\psi(k) + 2\psi(k))]\\ &= \frac{G(k)\epsilon^2(k)}{D(k)} + \frac{N^2(x(k))\beta^2(k-1)\epsilon^2(k)}{2D^2(k)}\\ &\quad - \frac{N(x(k))\beta(k-1)\epsilon(k)\psi(k)}{D(k)}.\end{aligned}$$

Noting the definition of $D(k)$ in (14), we see that $\Delta x(k)$ is bounded, such that the requirements of sequence $x(k)$ in (3) are well satisfied.

### 5.2. Stability analysis

In this subsection, the detailed stability analysis of the closed-loop system using the proposed adaptive control will be given. First of all, let us summarize the stability results when applying the designed adaptive control to system (1).

**Theorem 1.** *Consider the adaptive closed-loop system consisting of system (1) under Assumption 1, control (12) with parameter estimates adaptation law (14), predicted future outputs defined in from (8) to (10) with parameter estimator (11). All the signals in the closed-loop system are bounded and the tracking error $e(k)$ will converge to zero.*

**Proof.** Substituting the error dynamics (13) into the augmented error $\epsilon(k)$, we have

$$\begin{aligned}G(k)\epsilon(k) &= -\gamma g\tilde{\Theta}_{fg}^{\mathrm{T}}(k-n)\Psi(k-1) + \gamma g\tilde{g}_l(k-n)y_d(k)\\ &\quad -\gamma g\beta(k-1) + N(x(k))\psi(k)\beta(k-1) \qquad (15)\end{aligned}$$

from which it can be obtained that

$$\begin{aligned}&\gamma\tilde{\Theta}_{fg}^{\mathrm{T}}(k-n)\Psi(k-1) - \gamma\tilde{g}_l(k-n)y_d(k)\\ &\quad = -\frac{1}{g}G(k)\epsilon(k) - \gamma\beta(k-1) + \frac{1}{g}N(x(k))\psi(k)\beta(k-1). \qquad (16)\end{aligned}$$

Consider a positive definite function $V(k)$ as

$$V(k) = \sum_{j=1}^{n}\|\tilde{\Theta}_{fg}(k-n+j)\|^2 + \sum_{j=1}^{n}\tilde{g}_l^{\,2}(k-n+j). \qquad (17)$$

By using (14) and (16) and noting that

$$\Delta x(k) = \Delta z(k) + \psi(k)\Delta\psi(k) + \frac{[\Delta\psi(k)]^2}{2}$$

$$|N(x(k))|[\Delta\psi(k)]^2 = \frac{|N(x(k))|^3\beta^2(k-1)\epsilon^2(k)}{D^2(k)} \leq \Delta z(k)$$

we have the difference of $V(k)$ as follows:

$$\begin{aligned}\Delta V(k) &= V(k) - V(k-1)\\ &= [\tilde{\Theta}_{fg}(k) - \tilde{\Theta}_{fg}(k-n)]^{\mathrm{T}}[\tilde{\Theta}_{fg}(k) - \tilde{\Theta}_{fg}(k-n)]\\ &\quad + 2\tilde{\Theta}_{fg}^{\mathrm{T}}(k-n)[\tilde{\Theta}_{fg}(k) - \tilde{\Theta}_{fg}(k-n)]\\ &\quad + (\tilde{g}_l(k) - \tilde{g}_l(k-n))^2 + 2\tilde{g}_l(k-n)(\tilde{g}_l(k) - \tilde{g}_l(k-n))\\ &= \gamma^2\frac{N^2(x(k))(\Psi^{\mathrm{T}}(k-1)\Psi(k-1) + y_d^2(k))}{D^2(k)}\epsilon^2(k)\\ &\quad + 2N(x(k))\frac{\gamma\tilde{\Theta}_{fg}^{\mathrm{T}}(k-n)\Psi(k-1) - \gamma\tilde{g}_l(k-n)y_d(k)}{D(k)}\epsilon(k)\\ &\leq \gamma^2\frac{G(k)\epsilon^2(k)}{D(k)} - 2\gamma\frac{N(x(k))\beta(k-1)\epsilon(k)}{D(k)}\\ &\quad - \frac{2}{g}N(x(k))\frac{G(k)\epsilon^2(k)}{D(k)}\\ &\quad + \frac{2}{g}N(x(k))\psi(k)\frac{N(x(k))\beta(k-1)\epsilon(k)}{D(k)}\\ &\leq \gamma^2\Delta z(k) + 2\gamma\Delta\psi(k) - \frac{2}{g}N(x(k))\Big(\Delta z(k)\\ &\quad + \psi(k)\Delta\psi(k) + \frac{[\Delta\psi(k)]^2}{2}\Big) + \frac{1}{|g|}|N(x(k))|[\Delta\psi(k)]^2\\ &\leq c_1\Delta z(k) + 2\gamma\Delta\psi(k) - \frac{2}{g}N(x(k))\Delta x(k) \qquad (18)\end{aligned}$$

where $c_1 = \gamma^2 + \frac{1}{|g|}$.

Noting $0 \leq \Delta z(k) \leq 1$, $0 \leq \Delta\psi(k) < 1$ and $x(k) = z(k) + \frac{\psi^2(k)}{2}$, then taking summation on both hand sides of (18) results

$$\begin{aligned}V(k) &\leq -\frac{2}{g}\sum_{k'=0}^{k}N(x(k'))\Delta x(k') + c_1 z(k') + 2\gamma\psi(k') + c_2\\ &\leq -\frac{2}{g}\sum_{k'=0}^{k}N(x(k'))\Delta x(k') + c_3 x(k') + c_4 \qquad (19)\end{aligned}$$

where $c_2$, $c_3$ and $c_4$ are finite constants. Applying Lemma 2 to (19) yields the boundedness of $V(k)$, $N(x(k))$ and $x(k)$, which further implies the boundedness of $\hat{\Theta}_{fg}(k)$, $\hat{g}_l(k)$, and $G(k)$. Considering that both $\psi^2(k)$ and $z(k)$ are nonnegative, the boundedness of $x(k)$ implies the boundedness of both of them.

Notice that $e(k) = y(k) - y_d(k)$, where the reference signal $y_d(k)$ is bounded and thus we obtain $y(k) = O[e(k)]$. Noting the Lipschitz condition of function $\Psi(\cdot)$, we have $\Psi(k-1) = O[e(k)]$.

Consider the boundedness of $\hat{\Theta}_{fg}(k)$ and Lemma 3. We have the following equation for $\beta(k)$:

$$\begin{aligned}\|\beta(k+n-1)\| &\leq \|\hat{\Theta}_{fg}^{\mathrm{T}}(k)\|\sum_{i=1}^{n-1}L_i|\tilde{y}(k+n-i|k)|\\ &= o[O[e(k+n)]]. \qquad (20)\end{aligned}$$

Next, let us sort the order between augmented error $\epsilon(k)$ and tracking error $e(k)$. According to the boundedness of $N(x(k))$, $\psi(k)$, $G(k)$ and equation (20), it is easy to establish that $\epsilon(k) = c_{e1}e(k) + c_{e2}o[O[e(k)]]$ and $\epsilon(k) \sim e(k)$, where $c_{e1}$ and $c_{e2}$ are some finite constants. Moreover, due to the Lipschiz condition on the system functions $\Psi(\cdot)$, one can easily obtain the relation between the normalization term $D(k)$ and augmented error $\epsilon(k)$ as $D(k) = O[\epsilon^2(k)]$.

Because $z(k)$ is a nondecreasing positive sequence, the boundedness of $z(k)$ implies that $\Delta z(k) = \frac{G(k)\epsilon^2(k)}{D(k)} \to 0$. Noting the boundedness of $G(k)$ and applying the Key Technical Lemma in Goodwin and Sin (1984), we will conclude that $\epsilon(k) \to 0$ and thus $e(k) \to 0$ according to $\epsilon(k) = c_{e1}e(k) + c_{e2}o[O[e(k)]]$ and then the boundedness of outputs is obvious. In addition, according to Lemma 1, the bounedness of control $u(k)$ and states $\xi_i(k)$, $i = 1, 2, \ldots, n$ is obtained. This complete the proof of the boundedness of all the closed-loop signals and the asymptotical tracking performance. ■

## 6. Simulation results

The following second order nonlinear plant will be used for simulation.

$$\begin{cases} \xi_1(k) = a_1\xi_1(k)\cos(\xi_1(k)) + a_2\dfrac{\xi_1^2(k)}{1+\xi_1^2(k)} + g_1\xi_2(k) \\ \xi_2(k) = b_1\sin(\xi_1(k)) + b_2\dfrac{\xi_1^3(k)}{2+\xi_1^2(k)} + g_2u(k) + d(k) \\ y(k) = \xi_1(k) \end{cases} \tag{21}$$

where $a_1 = 0.2, a_2 = 0.1, g_1 = 1, b_1 = 0.3, b_2 = -0.6$ and $g_2 = \mp 0.2$. The small additive term $d(k) = 0.1\cos(0.1k)\cos(\xi_2(k))$ can be regarded as an external disturbance. The control objective is to make the output $y(k)$ track the desired reference trajectory $y_d(k) = 1.5\sin(\frac{\pi}{5}kT) + 1.5\cos(\frac{\pi}{10}kT), T = 0.05$. The initial system states are $\bar{\xi}_2(0) = [0,\ 1]^{\mathrm{T}}$.

To illustrate that the proposed adaptive control is insensitive to the control direction, the simulation is carried out twice in such a way that the control is fixed while the simulated system (21) assumes negative control gain parameter $g_2$ in the first time and assumes positive value in the second time. The simulation results are presented in Figs. 1–3. Fig. 1 shows the output $y(k)$ and the reference signal $y_d(k)$. Fig. 2 illustrates the boundedness of the control input $u(k)$, the estimated parameters $\hat{g}_l(k)$ and $\hat{\Theta}_{fg}(k)$. Fig. 3 demonstrates the discrete Nussbaum gain $N(x(k))$ and the sequence $\psi(k)$ and the augmented error $\epsilon(k)$. In the simulation results, it is seen that the proposed adaptive control works well with either negative or positive control gains, and even in the presence of small additive disturbance and noise. It demonstrates some degree of robustness of the proposed adaptive control law. As illustrated in Fig. 3, to detect the control direction, the discrete Nussbaum gain adapts by searching alternately in the two directions: when the control gain $g$ is negative, the sign of $N(x(k))$ changes from positive to negative and remains so for good; when the control gain is positive, the sign of $N(x(k))$ keeps positive without any switch.

## 7. Conclusion

In this paper, a systematic control method has been developed for global tracking of a class of nonlinear discrete-time systems with unknown control directions. The system is transformed to a class of NARMA system and the unknown control gains are lumped together. A certainty equivalent control has been constructed by using predicted future outputs. To counter the effect of prediction error on closed-loop system stability, an augmented error is introduced in the parameter estimator, in which the discrete Nussbaum gain is also employed to make the output-feedback adaptive control design feasible in the presence of unknown control direction. The proposed adaptive control guarantees the boundedness of all the closed-loop signals and achieves asymptotical tracking performance.

## Appendix

Let us introduce the notation of parameter estimation error $\tilde{\Theta}_f(k) = \hat{\Theta}_f(k) - \Theta_f, \tilde{g}(k) = \hat{g}(k) - g$, and $\bar{\tilde{\Theta}}(k) = [\tilde{\Theta}_f^{\mathrm{T}}(k), \tilde{g}(k)]^{\mathrm{T}}$. Then, the one step prediction error, $\tilde{y}(k+1|k) = \hat{y}(k+1|k) - y(k+1)$, can be written as

$$\tilde{y}(k+1|k) = \bar{\tilde{\Theta}}^{\mathrm{T}}(k-n+2)\Psi_1(k) \tag{22}$$

where $\Psi_1(k)$ is defined in (7). Define a Lyapunov function $V(k) = \sum_{j=k-n+2}^{k}\|\bar{\tilde{\Theta}}(j)\|^2$. It is easy to derive that

$$\begin{aligned} V(k+1) - V(k) &= \frac{\tilde{y}^2(k+1|k)\|\Psi_1(k)\|^2}{[1+\|\Psi_1(k)\|^2]^2} \\ &\quad - 2\frac{\tilde{y}(k+1|k)\bar{\tilde{\Theta}}^{\mathrm{T}}(k-n+2)\Psi_1(k)}{1+\|\Psi_1(k)\|^2} \\ &\le \frac{\tilde{y}^2(k+1|k)}{1+\|\Psi_1(k)\|^2} - 2\frac{\tilde{y}^2(k+1|k)}{1+\|\Psi_1(k)\|^2} \\ &= -\frac{\tilde{y}^2(k+1|k)}{1+\|\Psi_1(k)\|^2} \le 0 \end{aligned} \tag{23}$$

which yields the following results Goodwin and Sin (1984):

(i) $\frac{\tilde{y}(k+1|k)}{D_0(k)} \in L^2[0,\infty),\ \ D_0(k) = [1+\|\Psi_1(k)\|^2]^{1/2}$

(ii) $\bar{\tilde{\Theta}}(k)$ is globally bounded.

According to Lemma 1 and Assumption 1, we have

$$\Psi_1(k) = O[y(k)], \qquad D_0(k) = O[y(k)]. \tag{24}$$

Denote $\alpha(k) = \frac{\tilde{y}(k+1|k)}{D_0(k)}$. The conclusion (i) $\alpha(k) \in L^2[0,\infty)$ implies $\alpha(k) \to 0$ as $k \to \infty$. Then, we have

$$\tilde{y}(k+1|k) = o[D_0(k)] = o[O[y(k)]]. \tag{25}$$

Now, let us analyze the two-step prediction error $\tilde{y}(k+2|k) = \hat{y}(k+2|k) - y(k+2)$. Define

$$\check{y}(k+2|k) = \hat{y}(k+2|k) - \hat{y}(k+2|k+1) \tag{26}$$

then we have $\tilde{y}(k+2|k) = \check{y}(k+2|k) + \tilde{y}(k+2|k+1)$.

According to the prediction laws in (8) and (9), and the boundedness of $\bar{\tilde{\Theta}}(k)$, we have

$$\begin{aligned} \|\check{y}(k+2|k)\| &= \|\hat{y}(k+2|k) - \hat{y}(k+2|k+1)\| \\ &= \|\hat{\Theta}_f^{\mathrm{T}}(k-n+3)[\Phi_{f1}(\hat{y}(k+1|k)) - \Phi_{f1}(y(k+1))]\| \\ &\le \|\hat{\Theta}_f^{\mathrm{T}}(k-n+3)\|L_1\|\tilde{y}(k+1|k)\| = o[O[y(k)]]. \end{aligned} \tag{27}$$

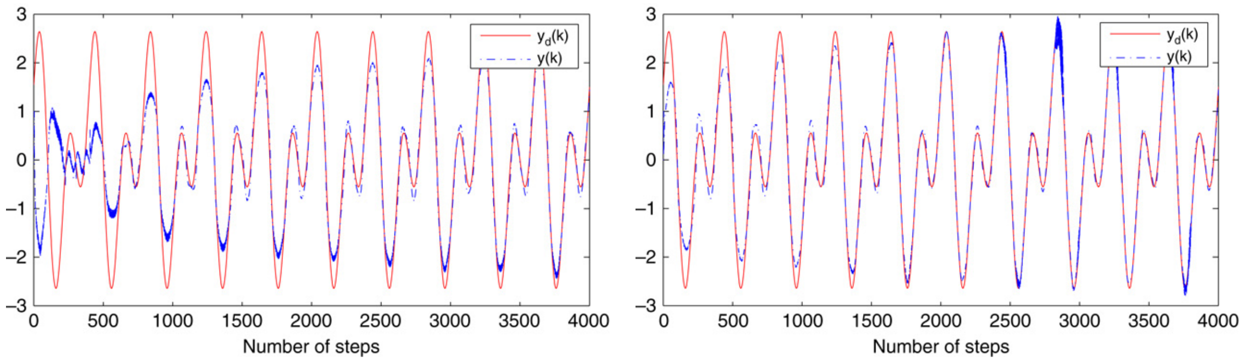
Furthermore, we have

$$\begin{aligned} \tilde{y}(k+2|k) &= \check{y}(k+2|k) + \tilde{y}(k+2|k+1) \\ &= o[O[y(k)]] + o[O[y(k+1)]] \\ &= o[O[y(k+1)]]. \end{aligned} \tag{28}$$

In the similar manner, let us analyze the $l$-step prediction, $\tilde{y}(k+l|k) = \hat{y}(k+l|k) - y(k+l)$. For convenience, let us introduce the following definition

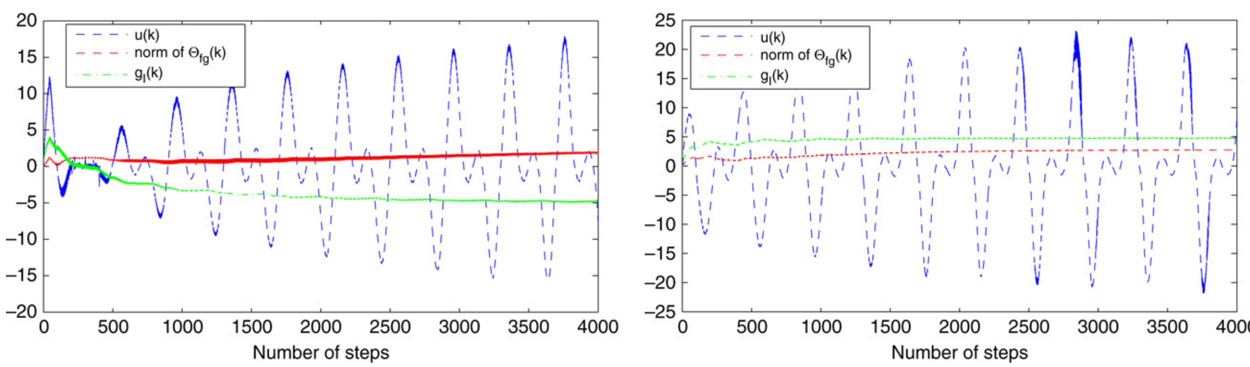
$$\check{y}(k+l|k) = \hat{y}(k+l|k) - \hat{y}(k+l|k+1) \quad l = 3, 4, \ldots, n-1.$$

Consistently, the following derivation can be obtained



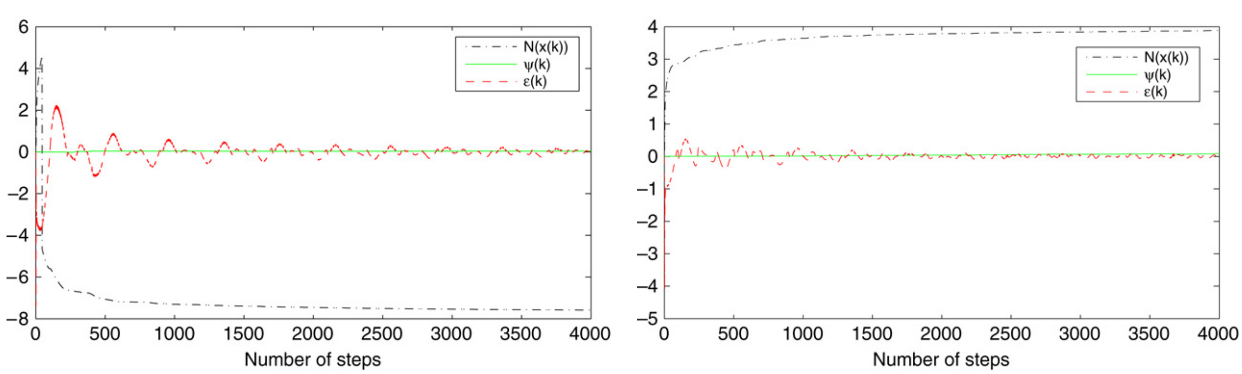

(a) $g_2 = -0.2$. (b) $g_2 = +0.2$.

**Fig. 1.** Tracking performance.




(a) $g_2 = -0.2$. (b) $g_2 = +0.2$.

**Fig. 2.** Control and estimations.




(a) $g_2 = -0.2$. (b) $g_2 = +0.2$.

**Fig. 3.** Discrete Nussbaum gain $N(x(k))$ and $\psi(k)$ and $\epsilon(k)$.

$$\begin{aligned}
\|\check{y}(k+l|k)\| &= \|\hat{y}(k+l|k) - \hat{y}(k+l|k+1)\| \\
&= \Bigg\| \hat{\Theta}_f^{\mathrm{T}}(k-n+l+1) \\
&\quad \times \left\{ \sum_{i=1}^{l-2} [\Phi_{fi}(\hat{y}(k+l-i|k)) - \Phi_{fi}(\hat{y}(k+l-i|k+1)) \right. \\
&\quad \left. + \Phi_{f(l-1)}(\hat{y}(k+1|k)) - \Phi_{f(l-1)}(y(k+1))] \right\} \Bigg\| \\
&\le \|\hat{\Theta}_f(k-n+l+1)\| \\
&\quad \times \left\{ \sum_{i=1}^{l-2} \|\Phi_{fi}(\hat{y}(k+l-i|k)) - \Phi_{fi}(\hat{y}(k+l-i|k+1))\| \right. \\
&\quad \left. + \|\Phi_{f(l-1)}(\hat{y}(k+1|k)) - \Phi_{f(l-1)}(y(k+1))\| \right\} \\
&\le \|\hat{\Theta}_f^{\mathrm{T}}(k-n+l+1)\| \\
&\quad \times \left[ \sum_{i=1}^{l-2} L_i \|\check{y}(k+l-i|k)\| + L_{l-1}\|\tilde{y}(k+1|k)\| \right] \\
&= o[O[y(k)]]. 
\end{aligned} \tag{29}$$

To proceed, we have

$$\begin{aligned}\tilde{y}(k+l|k) &= \check{y}(k+l|k) + \tilde{y}(k+l|k+1)\\ &= o[O[y(k)]] + o[O[y(k+l-1)]]\\ &= o[O[y(k+l-1)]]. \end{aligned} \quad (30)$$

This completes the proof. ■

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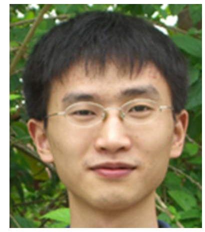


**Chenguang Yang** received the B.E. degree in measurement and control technologies and devices from College of Automation of Northwestern Polytechnical University, Xi'an, China, in 2005. He has been with the Department of Electrical and Computer Engineering, the National University of Singapore, as a Ph.D candidate since August, 2005. His current research interests lie in adaptive control, intelligent control and robotics.



**Shuzhi Sam Ge**, IEEE Fellow, P.Eng. is the founding Director of Social Robotics Lab, Interactive Digital Media Institute and Full Professor in the Department of Electrical and Computer Engineering, the National University of Singapore. He received his B.Sc. degree in control engineering from Beijing University of Aeronautics and Astronautics (BUAA) in 1986, and the Ph.D. degree in robotics and the Diploma of Imperial College (DIC) from Imperial College of Science, Technology and Medicine, in 1993.

He has (co)-authored three books: Adaptive Neural Network Control of Robotic Manipulators (World Scientific, 1998), Stable Adaptive Neural Network Control (Kluwer, 2001) and Switched Linear Systems: Control and Design (Springer-Verlag, 2005), edited a book: Autonomous Mobile Robots: Sensing, Control, Decision Making and Applications (Taylor and Francis, 2006), and over 300 international journal and conference papers. He has served/been serving as an Associate Editor for a number of flagship journals including IEEE Transactions on Automatic Control, IEEE Transactions on Control Systems Technology, IEEE Transactions on Neural Networks, and Automatica. He also serves as an Editor of the Taylor & Francis Automation and Control Engineering Series. He has been elected to as Vice President of Technical Activities, 2009–2010, member of Board of Governors, 2007–2009, and Chair of Technical Committee on Intelligent Control, 2005–2008, IEEE Control Systems Society. He provides technical consultancy to industrial and government agencies. He is the Editor-in-Chief of the International Journal of Social Robotics. His current research interests include social robotics, multimedia fusion, adaptive control, and intelligent systems.



**Tong Heng Lee** received the B.A. degree (with first class honors) in engineering tripos from Cambridge University, Cambridge, UK, in 1980 and the Ph.D. degree in electrical engineering from Yale University, New Haven, CT, in 1987. He is a Professor in the Department of Electrical and Computer Engineering, National University of Singapore. He is also currently Head of the Drives, Power and Control Systems Group in this Department. His research interests are in the areas of adaptive systems, knowledge-based control, intelligent mechatronics, and computational intelligence. He has also coauthored three research monographs, and holds four patents (two of which are in the technology area of adaptive systems, and the other two are in the area of intelligent mechatronics). He is an Associate Editor of Control Engineering Practice, the International Journal of Systems Science, the Mechatronics Journal, and the IEEE Transactions in Systems,Man and Cybernetics. Dr. Lee received the Cambridge University Charles Baker Prize in Engineering.