

store

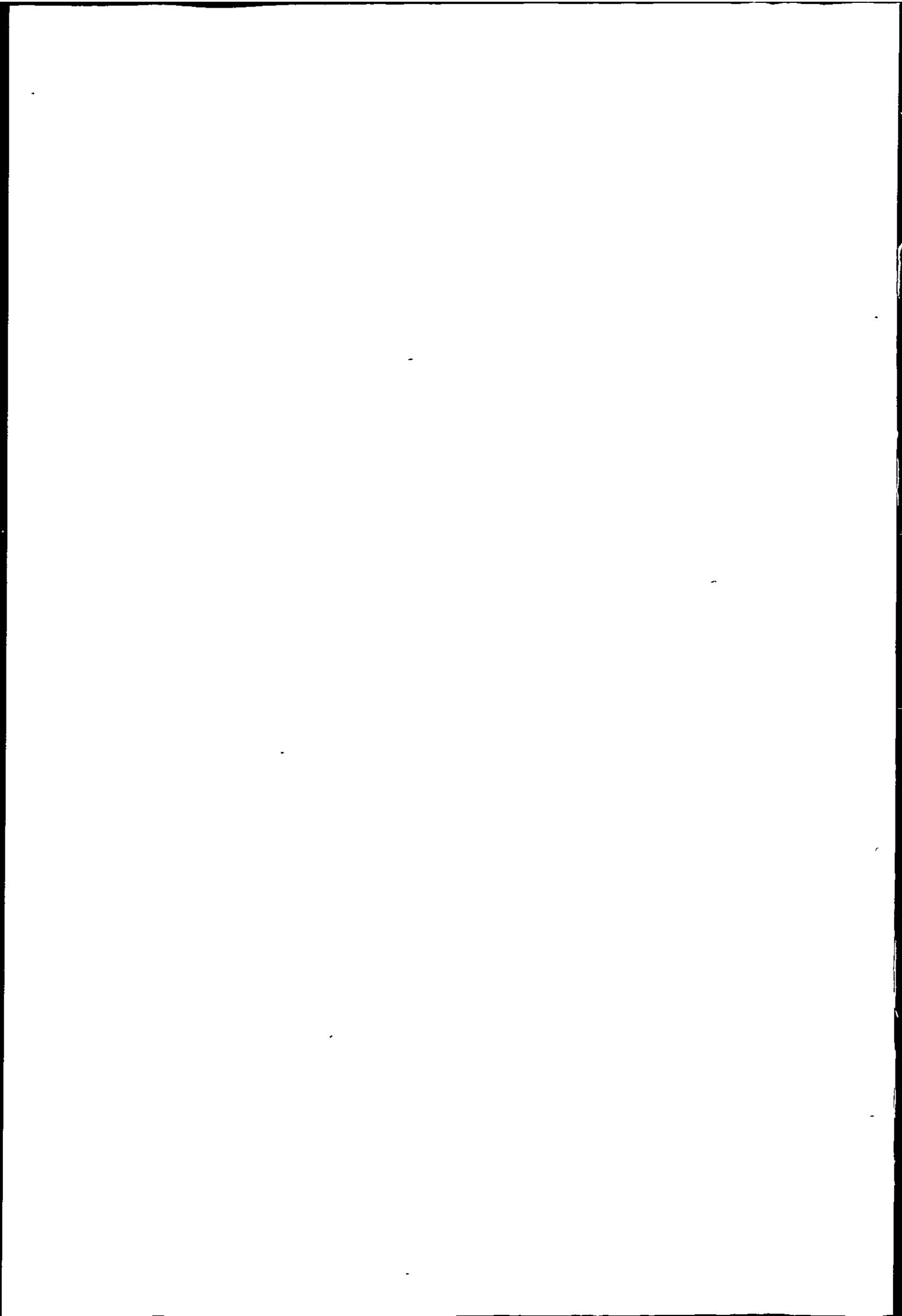
AN INVESTIGATION INTO THE
DEVELOPMENT OF ENGINEERING
STUDENTS' CONCEPTUAL UNDERSTANDING
OF MATHEMATICS

W. M. MAULL

Ph.D.

1998

This copy of the thesis has been supplied on condition that anyone who consults it is understood to recognise that its copyright rests with its author and that no quotation from the thesis and no information derived from it may be published without the author's consent.



AN INVESTIGATION INTO THE DEVELOPMENT OF ENGINEERING
STUDENTS' CONCEPTUAL UNDERSTANDING OF MATHEMATICS

by

WENDY MARY MAULL

A thesis submitted to the University of Plymouth
in partial fulfilment for the degree of

DOCTOR OF PHILOSOPHY

School of Mathematics and Statistics
Faculty of Technology

July 1998

LIBRARY STORE

REFERENCE ONLY

UNIVERSITY OF PLYMOUTH	
Item No.	900 3856050
Date	25 FEB 1999 T
Class No.	T 510.19 MAU.
Contl. No.	X703831355
LIBRARY SERVICES	

90 0385605 0



AN INVESTIGATION INTO THE DEVELOPMENT OF ENGINEERING STUDENTS' CONCEPTUAL UNDERSTANDING OF MATHEMATICS

by

WENDY MARY MAULL

Following widespread concern over an apparent decline in the mathematical skills of engineering students, this study employed survey and observation methods to investigate the ways in which engineering students understand mathematical concepts, and to compare these with the concepts held by students of mathematics.

It was found that the engineering students employ a different vocabulary from mathematics students in discussing mathematics, and that their understanding of mathematical concepts develops differently from mathematics students both in response to teaching (which appears to be a transitory effect) and as their experience gives meaning to the ideas in life outside study. These findings are important in two ways. We need to make the mathematics teachers of engineering students aware of the language and concepts of their students so that the possibility of mutual misunderstanding is reduced, and we as educators need to help engineering students to make these connections in order to ground their mathematics in reality and to use mathematics an instrument for understanding the world.

Compared with the classical mathematical modelling paradigm and the classical empirical modelling paradigm, the method used by engineering students was found to be a hybrid based on the identification of the type of problem and the application of a pre-existing law.

Some misconceptions concerning the behaviour of beams in bending were found to be widely held, by respondents with a range of levels of experience. Whereas the particular misconceptions are not important in themselves, it is salutary to realise that expertise in one area of study does not necessarily inoculate one against misconceptions in a closely related area.

A software package was written using the context of mathematical modelling to help students relate concepts in calculus to physical situations. This package was found not to engage the students sufficiently to provoke cognitive change, and suggests that a higher degree of interactivity is needed.

CONTENTS

1. An investigation into engineering students' conceptual understanding of mathematics.	1
1.1 Introduction: historical background	1
1.2 The study	4
1.3 Philosophical standpoint	5
1.4 Research question	8
2. What is the nature of mathematics?	9
2.1 Introduction	9
2.2 The nature of mathematics	9
2.3 Product or process?	10
2.4 Discovered or invented?	11
2.5 Social or individual?	12
2.6 Hirst and Forms of knowledge	14
2.7 Postmodernist mathematics	16
2.8 Summary	20
2.9 Mathematics for the engineer	20
2.9.1 According to writers of reports: a tool, a language, a competence	20
2.9.2 According to writer of textbooks	22
2.9.3 Mathematics as sensemaking: the making of meaning	27
2.9.4 Precision and approximation	28
2.10 Conclusions	29
2.10.1 On intellectual rigour	29
2.10.2 On the finality of mathematics	30
3. Observation of mathematical modelling in practice	32
3.1 Introduction	32
3.2 What is observation?	32
3.2.1 Strengths of observational methods	32
3.2.2 Weaknesses of observational methods	33
3.3 Observation within this study	34
3.4 Cold tea	34
3.4.1 Introduction	34
3.4.2 Observations	35
3.4.3 Analysis à la Perry	41
3.4.4 Interviews	41
3.5 The cascade problem	42
3.5.1 Use of written work to support discussion	43
3.5.2 Early insights	43
3.5.3 Focus	44
3.5.4 Vocabulary and concept set	44
3.5.5 Experimental technique	46
3.5.6 Starting to run the apparatus	47
3.5.7 Physical modelling	48
3.5.8 Spotting discrepancies between theory and practice	48

3.5.9 Other apparatus	49
3.5.10 Plotting results	50
3.6 Conclusions	50
3.6.1 On mathematical modelling	50
3.6.2 On mathematics and engineering students	50
3.6.3 On observation	52
4. Research method: questionnaire design and administration	53
4.1 Introduction	53
4.2 Social science	53
4.3 This study	55
4.4 Surveys	57
4.5 Aspects of experimental design in the questionnaire	57
4.5.1 Threats to internal validity:	58
4.5.2 Threats to external validity	59
4.6 Concept images and cognitive styles	61
4.7 Concept definition and concept image	62
4.8 Cognitive skill level discrimination: depth of representation	63
4.8.1 Encapsulation	63
4.8.2 Interconnectedness	65
4.8.3 Personal meaning	65
4.8.4 Skills and concepts	66
4.9 Cognitive style discrimination: mode of representation	67
4.9.1 Visual imagery in mathematics	67
4.9.2 Need for versatility	69
4.9.3 Engineering and mathematics students	69
4.10 Conclusions	70
5. The questionnaire: the practical process of design.	71
5.1 Introduction	71
5.2 Design of pilot questions	71
5.3 Pilot questionnaire	74
5.3.1 Administration methods	76
5.4 Main questionnaire	76
5.4.1 Administration methods	78
5.4.2 Confidentiality	79
5.4.3 Design faults in questionnaire	79
5.4.4 Other comments	80
5.5 Content of individual questions:	80
5.5.1 Question 1	81
5.5.2 Question 2	82
5.5.3 Question 3	84
5.5.4 Question 4	85
5.5.5 Question 5	87
5.5.6 Question 6	88
5.6 Conclusions	89

6. The questionnaire: responses and interpretation	91
6.1 Introduction	91
6.2 Interpretation of the responses	91
6.3 Revealing misconceptions	92
6.3.1 The beam will not bend at all, or whether it bends depends on its thickness	92
6.3.2 The point of greatest deflection must be under the load	93
6.3.3 Why do people think these things? Mental models of physical problems	94
6.3.4 Planks are, or may be, rigid	94
6.3.5 The deflection must be greatest under the load	95
6.3.6 Discussion	96
6.4 Preferences	97
6.4.1 Question 1	98
6.4.2 Question 2	100
6.4.3 Question 3	103
6.4.4 Question 4	104
6.4.5 Question 5	106
6.4.6 Question 6	107
6.5 Mode of representation	111
6.6 Attitude to mathematics	113
6.7 Depth of representation	115
6.7.1 Growth in meaning	115
6.7.2 Richness of association	116
6.7.3 Encapsulation	117
6.8 Conclusions	117
6.8.1 On the analysis of the results	117
6.8.2 On the mathematical representations of engineering students	118
6.8.3 On the questionnaire	119
7. Component and factor analysis	121
7.1 Introduction	121
7.2 Factors in statistics	121
7.3 Objectives of component and factor analysis	122
7.4 What is component analysis?	123
7.5 How it works: practical considerations	125
7.5.1 How many variables?	125
7.5.2 Sample size	125
7.5.3 Suitability of data	126
7.5.4 Correlations translated into angles between vectors	127
7.5.5 How many components?	128
7.5.6 Interpreting the components	129
7.5.7 From principal components to factors	130
7.6 Relationship between factor analysis and principal components analysis	130

8. Component analysis of the questionnaire data	133
8.1 Introduction	133
8.2 Typology and taxonomy	133
8.3 The analysis: in brief	134
8.4 What do these components represent?	137
8.4.1 Component 1	137
8.4.2 Component 2	138
8.4.3 Component 3	139
8.4.4 Component 4	141
8.4.5 Component 5	142
8.4.6 Component 6	143
8.4.7 Component 7	144
8.4.8 Component 8	145
8.4.9 Component 9	146
8.4.10 Component 10	147
8.5 Discussion	148
8.5.1 Canonical discriminant functions	148
8.5.2 The components	150
8.6 Conclusions	151
8.6.1 On the analysis	151
8.6.2 On the suitability of the data	152
8.6.3 On the questionnaire	152
9. Revised questionnaire	154
9.1 Introduction	154
9.2 Question 1: beam bending (statics)	155
9.3 Question 2: differentiation	155
9.4 Question 3: mass/spring/damper system (dynamics)	155
9.5 Question 4: integration	155
9.6 Question 5: pinball (unfamiliar kinematics)	156
9.7 Question 6: differential equation: exponential growth	156
9.8 Revised questions	156
9.8.1 Question 1	157
9.8.2 Question 2	158
9.8.3 Question 3	159
9.8.4 Question 4	160
9.8.5 Question 5	161
9.8.6 Question 6	162
10. Engineering students	163
10.1 Introduction	163
10.2 The engineering student	164
10.3 Learning style.	164
10.3.1 Kolb's Experiential Learning Model	165
10.3.2 Myers-Briggs Type Indicator	168
10.3.3 Approaches to Studying	170
10.4 Socialisation	171
10.5 Conclusions	173

11. Epistemological orientations, paradigms of curriculum, and learning theories	174
11.1 Introduction	174
11.2 Epistemological orientations, paradigms of curriculum, and learning theories	174
11.3 Epistemological orientations	175
11.3.1 Transmission model	175
11.3.2 Transaction model	177
11.3.3 Transformation model	179
11.4 Learning theories	180
11.5 Paradigms of curriculum	182
11.6 Applicability of the paradigms to engineering education	184
11.6.1 development of cognitive processes	184
11.6.2 curriculum as technology	184
11.6.3 curriculum for personal relevance	184
11.6.4 curriculum for social relevance- social adaptation	184
11.6.5 curriculum for social relevance- social reconstruction	185
11.6.6 academic rationalism.	185
11.7 Endnote	185
11.8 Conclusions	186
12. Computer aided learning (CAL) and computer aided instruction (CAI)	188
12.1 Introduction	188
12.2 Medium and message	188
12.3 Competing models: CAL and CAI	189
12.4 Lessons from Interactive Video	191
12.5 Freedom to roam	192
12.6 Learning theory-based research.	193
12.7 Avoiding impediments to learning	197
12.7.1 instructional design	197
12.7.2 screen design	197
12.7.3 human factors	198
12.7.4 Screen design	198
12.7.5 Graphics	199
12.7.6 Animations and sounds	201
12.7.7 Use of video	202
12.8 Matching instruction to learner	202
12.9 The role of the teacher/instructor/tutor	206
12.10 Computers in engineering education	207
12.11 Choice of authoring package	209
12.12 Conclusions	211
13. Mathematical modelling	213
13.1 Introduction	213
13.2 What is mathematical modelling?	213

13.3 Mathematical modelling and engineering	214
13.4 The process of mathematical modelling	215
13.5 Modelling paradigms	216
13.6 The theoretical paradigm	219
13.6.1 Two worlds	220
13.6.2 States and stages	221
13.6.3 Iteration	222
13.6.4 Eight-box diagram	223
13.7 Empirical modelling	224
13.8 Modelling behaviour of engineering students	225
13.9 Conclusions	231
14. Design of the courseware	233
14.1 Introduction	233
14.2 Overall principles	233
14.3 Structure and embedded metaphor	234
14.4 Navigation	235
14.5 Contents	236
14.5.1 Box 0. Reality	236
14.5.2 Box 1. Understand problem	237
14.5.3 Box 2. Simplify and make assumptions	237
14.5.4 Box 3. Set up mathematics	237
14.5.5 Box 4. Solve mathematics	237
14.5.6 Box 5. Investigate implications	237
14.5.7 Box 6. Compare with reality	239
14.5.8 Box 7. Write report	239
14.6 Case studies	239
14.6.1 Level zero: introduction to modelling	239
14.6.2 Level 1: the suspension bridge	239
14.6.3 Level 2: the cup of coffee	240
14.6.4 Level 3: the water tank	240
14.6.5 Level 4: the freely hanging chain	240
14.6.6 Level 5: the tank with a pipe	241
14.6.7 Level 6: the tank with pipe losses	241
14.7 Form: how the advantages of the medium were employed	241
14.7.1 Level zero	243
14.7.2 Level 1: the suspension bridge	243
14.7.3 Level 2: the cup of coffee	244
14.7.4 Level 3: the water tank	245
14.7.5 Level 4: the freely hanging chain	246
14.7.6 Level 5: the tank with a pipe	247
14.7.7 Level 6: the tank with pipe losses	247
14.8 Conclusion	248
15. Evaluation of the courseware	249
15.1 Introduction	249
15.2 Style	250
15.2.1 Overall impressions	250

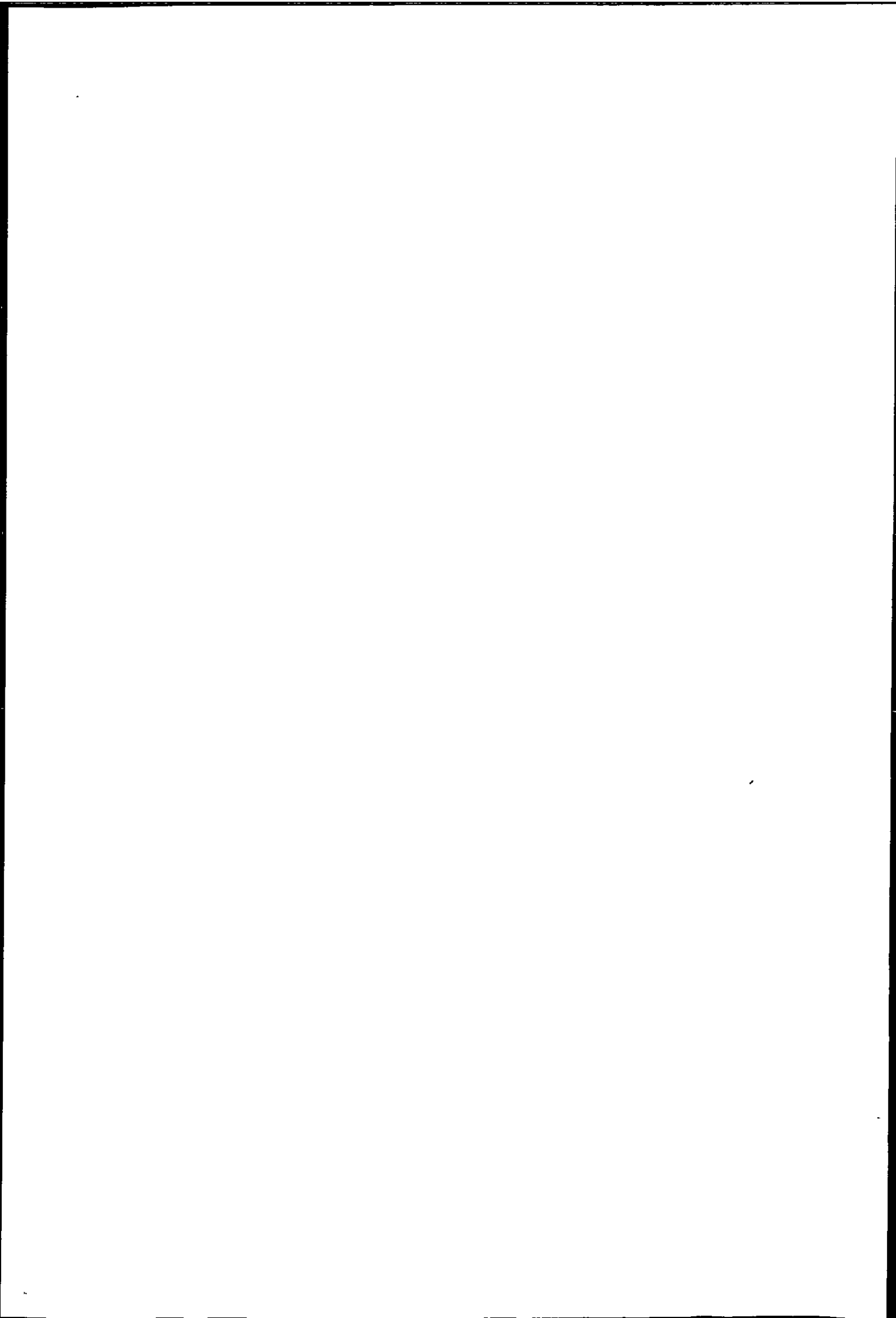
15.2.2 Commentary	250
15.2.3 Navigation: the home page structure	250
15.2.4 Pace	251
15.2.5 Help	251
15.2.6 Specific points	252
15.3 Content	252
15.3.1 The case studies	252
15.3.2 Level of mathematics	253
15.4 Discussion	253
15.4.1 Learning processes	253
15.4.2 On the effects of using the package	254
15.5 Conclusions	256
15.5.1 On the package	256
15.5.2 On engineering students	257
15.5.3 On a possible redesign	258
16. Conclusions and suggestions for further study	259
16.1 Introduction: the research questions	259
16.2 Concept images: mode and depth	259
16.2.1 Mode of image	260
16.2.2 Depth of image	260
16.2.3 Growth in meaning	261
16.2.4 Richness of association	261
16.2.5 Encapsulation	262
16.2.6 Comparison with mathematics students	262
16.2.7 Summary	263
16.3 Implications for teaching	263
16.4 Engineering students' attitudes to mathematics	264
16.4.1 Mathematics as external	264
16.4.2 Mathematics as not relevant to engineers	265
16.4.3 Mathematics as something you learn for exams and then forget	265
16.4.4 Mathematics has right and wrong answers	266
16.5 The courseware	268
16.6 Research methods	269
16.7 Misconceptions we all may have	269
16.8 Questions arising from the research	270
APPENDIX A: Transcriptions of recordings of students performing cascade modelling problem	271
APPENDIX B: The main questionnaire	308
APPENDIX C: Transcriptions of recordings of students evaluating courseware	315
REFERENCES	325

ILLUSTRATIONS AND TABLES

LIST OF ILLUSTRATIONS

Figure 2-1: Relationship between sign, concept and referent (Ogden and Richards, 1923)	17
Figure 3-1: The cold tea problem	35
Figure 3-2: OU flowchart analysing the process of mathematical modelling (Tunncliffe, 1981, p5)	35
Figure 3-3: The cascade problem	42
Figure 4-1: A scheme for analysing assumptions about the nature of social science (Burrell and Morgan, 1979)	54
Figure 5-1: Original DE question	72
Figure 5-2: DE question from pilot questionnaire	72
Figure 5-3: Original dynamics question	73
Figure 5-4: Question 1 from main questionnaire	82
Figure 5-5: Question 2 from main questionnaire	83
Figure 5-6: Question 3 from main questionnaire	85
Figure 5-7: Question 4 from main questionnaire	86
Figure 5-8: Question 5 from main questionnaire	88
Figure 5-9: Question 6 from main questionnaire	89
Figure 6-1: Modified diagram of shape of beam	94
Figure 6-2: Responses to question 1	98
Figure 6-3: Responses to question 2	101
Figure 6-4: Responses to question 3	103
Figure 6-5: Responses to question 4	104
Figure 6-6: Responses to question 5	106
Figure 6-7: Responses to question 6	107
Figure 8-1: Component Scree Plot	136
Figure 8-2: Scores on component 1	138
Figure 8-3: Scores on component 2	139
Figure 8-4: Scores on component 3	140
Figure 8-5: Scores on component 4	142
Figure 8-6: Scores on component 5	143
Figure 8-7: Scores on component 6	144
Figure 8-8: Scores on component 7	145
Figure 8-9: Scores on component 8	146
Figure 8-10: Scores on component 9	147
Figure 8-11: Scores on component 10	148
Figure 10-1: Kolb's Experiential Learning Model	165
Figure 12-1: Kolb's cyclical learning model (Kolb, 1981)	195
Figure 12-2: Romiszowski's learning model (Romiszowski, 1986)	195

Figure 13-1: McLone, 1984	214
Figure 13-2: Hart & Croft, 1988	216
Figure 13-3: MEI mathematical modelling flowchart (MEI, 1994)	217
Figure 13-4: Kolb's experiential learning diagram (Kolb, 1981)	219
Figure 13-5: OU flowchart analysing the process of mathematical modelling, (Tunnicliffe, 1981, p5)	220
Figure 13-6: Galbraith and Haines, 1997, after OU flowchart	221
Figure 13-7: Ikeda, 1997, after Burghes et al	221
Figure 13-8: Ikeda diagram transformed for comparison with OU diagram	222
Figure 13-9: Moscardini et al, 1984	223
Figure 13-10: Modified mathematical modelling flowchart	223
Figure 13-11: The NCTM Standards' (1989) characterisation of mathematical modelling (after Hodgson and Harpster, 1997)	224
Figure 13-12: The engineers' modelling cycle	227
Figure 14-1: A flowchart analysing the process of mathematical modelling (Tunnicliffe, 1981, p5)	234
Figure 14-2: Modified mathematical modelling flowchart	235
Figure 14-3: Home page	236
Figure 14-4: Transition page	242
Figure 14-5: Page with text and diagram	243
Figure 14-6: Photograph with measuring grid overlaid	244
Figure 14-7: Animated screen	245
Figure 14-8: "Hot words" help facility	246
Figure 14-9: Comparison of parabola and catenary	247
Figure 14-10: Commentary in right-hand column	248
Figure 16-1: Modification of OU modelling flowchart	267
Figure 16-1: The engineering modelling process	267



LIST OF TABLES

Table 2-1: Classified examples of text books	24
Table 2-2: Texts recommended to first year mechanical engineering undergraduates, February 1996	24-26
Table 5-1: Proposed questionnaire distribution	77
Table 5-2: Classification of questionnaire options	90
Table 6-1: Responses to question 1	99
Table 6-2: Responses to question 2	101
Table 6-3: Responses to question 3	104
Table 6-4: Responses to question 4	105
Table 6-5: Responses to question 5	107
Table 6-6: Responses to question 6	108
Table 7-1: Dependence of significance on sample size (Hair et al, 1984)	129
Table 8-1: Loadings of options on components	136
Table 8-2: Loadings on component 1	137
Table 8-3: Loadings on component 2	138
Table 8-4: Loadings on component 3	139
Table 8-5: Loadings on component 4	141
Table 8-6: Loadings on component 5	142
Table 8-7: Loadings on component 6	143
Table 8-8: Loadings on component 7	144
Table 8-9: Loadings on component 8	145
Table 8-10: Loadings on component 9	146
Table 8-11: Loadings on component 10	147
Table 8-12: Comparison of predicted with actual group membership	149
Table 8-13: Comparison of predicted with actual groups: subject studied only	150
Table 10-1: Kolb's classification of learning styles	166
Table 10-2: Engineering students' attitudes to studying (Ramsden & Entwistle, 1981)	170
Table 11-1: Epistemological orientations (from Brody, 1991, in Berry and Sahlberg)	176
Table 11-2: Learning theories (elaborated from Wilson et al, 1993)	181
Table 11-3: Paradigms of curriculum (elaborated from Helsel, 1987)	183
Table 12-1: Bartolomé's classification of degrees of interactivity (1992)	193
Table 12-2: Media comparison chart (Laurillard, 1993)	194
Table 12-3: Use of graphics in CBL packages (Clarke, 1992)	200
Table 12-4: Use of computers by engineering students (Smith, 1992)	209

ACKNOWLEDGEMENTS

Completing this thesis has taken a lot of help and co-operation from a wide variety of people, whom I unreservedly thank and acknowledge.

John Berry, Colin Stockel and Ted Graham have guided the study along the way and gently and patiently ushered me back when I was distracted by interesting side issues.

The whole team at the Centre for Teaching Mathematics have encouraged, and cheered, and generally made me want to come in to work on the project from day to day.

My family have borne the extra load with patience, and my husband Roger has spent many happy hours arguing paradigms and hermeneutic circles until we have come to some sort of shared understanding.

Finally, thanks to all the people who returned the questionnaire, and to the students who tested the courseware, without whom there would have been nothing to report.

AUTHOR'S DECLARATION

At no time during the registration for the degree of Doctor of Philosophy has the author been registered for any other University award.

This study was financed with the aid of a studentship from the University of Plymouth. A programme of study was undertaken which included two short courses on writing multimedia materials and a course in writing academic papers.

Publications and presentation of work:

Poster presentation: Mathematical knowledge of engineering undergraduates, ICME 8 (8th International Congress on Mathematical Education), Seville, 14-21 July, 1996

Conference papers: Teaching engineering mathematics through computer technology, W Maull, J Berry & C Stockel, 13th International Conference on Production Research, Jerusalem, 6-10 August, 1995

Modelling the mathematical ideas of engineering students, W Maull & J Berry, Mathematical Education of Engineers II, Loughborough, 7-9 April, 1997

Other Conferences attended:

Computer Aided Learning in Engineering, Sheffield, 5-7 September, 1994

Signed

Wendy Pawly

Date

30 Nov 98.

1. An investigation into engineering students' conceptual understanding of mathematics.

1.1 Introduction: historical background

The proceedings of the first IMA conference on the Mathematical Education of Engineers in Loughborough in 1995 (Mustoe & Hibberd, eds, 1995) were published with a chapter entitled "Conclusions after a decade of decline" (O'Carroll, 1995). This conference was followed by the publication of a series of reports from distinguished and interested bodies, having titles such as *The Changing Mathematical Background of Undergraduate Engineers* (Sutherland and Pozzi, 1995), *Tackling the Mathematics Problem*, (LMS/IMA/RSS, 1995), *Mathematics Matters in Engineering*, (IChemE/ICE/IEE/IMC/IMechE/LMS/IMA, 1995) and *A Mathematical Foundation*, (SEO/EC/SCSST, 1996). The overall opinion was that the mathematical skills of undergraduate engineers had declined over the previous 10 years. Sadly, there was no way of testing the hypothesis scientifically as the previous generation of undergraduate engineers had evolved into practising engineers, accountants, personnel managers, etc., and, in some cases, lecturers, and were not available for direct comparison with the current cohort.

Instead, the opinions of current lecturers were canvassed (Sutherland and Pozzi, 1995). They felt that standards had declined, and that there were multiple causes for this decline. Overall, they felt that the standard of mathematics taught in schools was lower. Secondly, as the number of university places in engineering subjects had increased, the demand for those places had declined, and so the overall ability of students entering engineering courses had been diluted. Thirdly, the proportion of students entering with qualifications other than A level mathematics and physics had increased as universities attempted to make up the shortfall in traditional entrants.

increased as universities attempted to make up the shortfall in traditional entrants. The question of the long-term stability of engineering lecturers' perceptions was neither raised nor investigated. (Crowther, 1997a, examines "the general opinion of university lecturer... that standards in mathematics have dropped substantially in recent years" and finds "very little empirical evidence in support of this allegation".)

Different solutions were suggested. For some, a change in the philosophy of teaching mathematics in schools was the answer. For others, more regulation of school mathematics and an insistence on the "gold standard" of the A level was called for.

The Engineering Institutions proposed to set a strict entry requirement to engineering degree courses (Engineering Council, Competence and Commitment, 1995, Engineering Council, SARTOR, 1990; IMarE, 1995), or else an externally set examination at the end of the first year. A mathematical core for the European Engineer was devised (Barry & Steele, 1992), building on an earlier proposal (OECD, 1966), and incorporating for the first time a summary of the mathematics the prospective student should have covered before embarking on undergraduate study.

An alternative strategy proposed by contributors to some of the reports was to accept the changes in mathematical skills of the entrants to degrees and to alter the emphasis in engineering degrees from mathematical skills to the appropriate use of computers, and the development of communications skills and other competences felt to form part of a new engineering core (for example, Challis & Gretton, 1997; Sutherland & Pozzi, 1995 p7 paras 19&23). The question of the effects of computing on engineering mathematics, both on its content and on teaching methods, was touched on in almost all the reports mentioned above, including the OECD report of 1966.

The computer aided education of engineers was the subject of another conference, (Eames & Johnson (eds), 1994) with delegates demonstrating their wares, or describing the ways in which they were using CAL materials, with claims for

improvements in performance, or reduction of time needed to cover a given slice of material. Because most engineering educators are engineers first and educators second, many brought a quasi-scientific approach to their research, while a few researchers from a primarily education or psychological background used a more phenomenological epistemology (for example, Brown, 1994).

All this takes place against the background of a decline in the number of students taking A level mathematics, a decline in the proportion of those taking mathematics also taking physics, and a decline in the proportion of those gaining A level mathematics who go on to study mathematics or technological subjects at university. (Hirst, 1996)

Before leaving this section I would like to quote from another report.

“This report discusses the results of an enquiry into the mathematical backgrounds and needs of engineering undergraduates and the methods universities adopt to meet these needs... The students in the enquiry have taken a large variety of A level mathematics syllabuses (61 identifiable ones)... Several themes recurred in discussions with university teachers of Mathematics or Engineering. The two major difficulties seemed to be (i) the diversity of students' mathematical backgrounds and attainments at the start of the university course; (ii) the general lack of confidence and accuracy in the routine processes of algebra, trigonometry and straightforward calculus... 10.6% of the students have Ordinary or Higher National Certificate or Diploma qualifications... The opinion at some universities was that the academic weaknesses of these students (especially in mathematics) were on the whole a handicap to them throughout their university courses. Others thought they had advantages in more applied skills, in design, and in awareness of the demands of the real world which compensated for these deficiencies, provided that they survived the first year of university.”

The Universities referred to here were all “old” universities since the report was written in 1978 when the changes anticipated were the results of the expansion of comprehensive schooling (Heard, 1978). The quotation could easily have been taken from any of the recent reports on the subject.

1.2 The study

In Plymouth, a small observation study appeared to indicate that engineering students were using mathematics differently from students of mathematics, particularly in a mathematical modelling context. One aspect of this difference was the vocabulary employed by the two groups of students. This was felt to be particularly interesting given that practising engineers use mathematics almost exclusively in a modelling context. Given that most of the lecturers who taught engineering mathematics were mathematicians by origin, the question arose whether the students and their lecturers were literally speaking different languages when it came to mathematics.

A questionnaire was developed which was applied to a range of respondents at different stages of an engineering career, and at different stages of a mathematics degree course. The responses showed up some patterns of ways of thinking about mathematical objects, and the written comments revealed some of the respondents' attitudes towards mathematics.

At the same time, the question of whether the use of computer software could be helpful in teaching engineering mathematics, which was raised in many of the reports, was addressed. It was quickly realised that writing computer courseware is highly intensive in effort. The package which was finally produced was considerably less ambitious than that which had at first been envisaged.

In the study a variety of research methods (participant observation, survey, component analysis, content analysis and interviews) was employed. The methods are described in the sections to which they apply. Finding research methods appropriate to a variety of situations was interesting and challenging, and the process of choosing appropriate methods helped to clarify the philosophical perspective of this study.

1.3 Philosophical standpoint

Experience with teaching had led to the conviction that a small-scale experimental procedure was unrealistic. No two groups of students are the same, and all groups of students react differently to what is nominally the same learning experience. Thus the scientific condition of repeatability is not fulfilled. At the same time the experimenter is intimately involved in the procedure, so the condition of independence of the observer is breached. It may be true that over large numbers the individual differences even one another out, and a scientific, experimental approach may be justified, but in this study the numbers were chosen to be small.

Checkland et al (1983) point out that the assumptions underlying classic scientific method include that the data should be independent of the observer, that the data and the research process should be mutually independent and that experimental conditions should always be controlled. The method becomes unsuitable when

- the processes in any one organisation are unique
- the facilities for controlled experimentation are unavailable
- the observer becomes an actor
- the causation is complex and interactive
- it is invalid to break down wholes into simple parts.

All these factors are present in the study of students' experience of learning. In addition, the students are human beings and are thus aware, as the apparatus of an experiment in physics is not. In medical experiments the placebo effect is well known. The technique of double blind trials, where neither the patient nor the local experimenter knows who is receiving the control or experimental treatment, has been evolved to take this into account. In other fields it is less easy to hide from subjects that they are receiving experimental treatment.

Thus the scientific paradigm was inadequate for this study. For the engineer in me, this was a serious blow. The determinism of the scientific paradigm, and the reductionism possible in dealing with the rational, physical world, are part of the underlying philosophical basis of engineering. Fortunately, another component of the engineer's philosophy is pragmatism: when one theory is inadequate, then another approach may be more appropriate. Various authors (Hirst, 1972, Burrell and Morgan, 1979, Sokal, 1997) point out that different epistemologies are appropriate when dealing with the physical world and with the social world, that the way we experience the social world is qualitatively different from the way in which we experience the physical world.

The work of Perry (e.g., 1981, 1988) was also helpful in finding an epistemological standpoint. He followed a group of students through their careers at Harvard, interviewing them at intervals and extracting from the interviews indications of their relationship to knowledge and the way it changed. On entry, many had a strongly positivist epistemology, and believed that they would learn the truth because they were going to be taught by the best experts there were. Gradually it dawned on students that their teachers did not necessarily agree with one another or with authority expressed in textbooks. This realisation caused the students to move through a series of positions including complete multiplicity (that everyone had their view and there was no way to decide what was right), via realising that some views were more defensible than others to a position of commitment in the face of ambiguity.

When an author (for example Laurillard, 1993, 43-46) describes this process in a treatise on university education, we suspect that there is a dual purpose: not only to open the mind of the reader to the processes of development in students, but also to provoke reflection and epistemological change in the reader.

This thesis concerns the teaching of mathematics to engineering students, their understanding of mathematical ideas, and ways of modifying that understanding. For this reason, epistemological questions arise at least three levels.

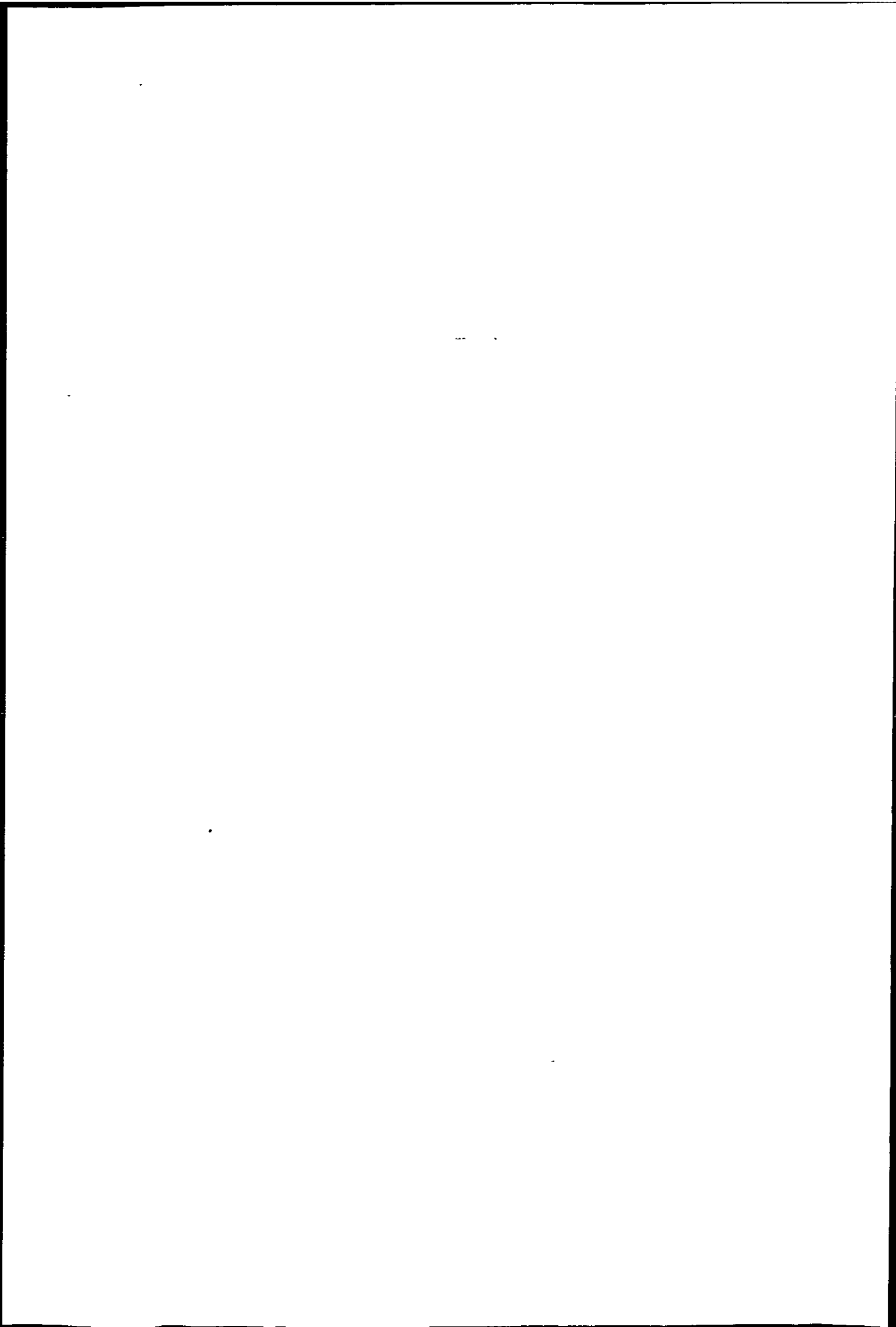
Firstly, the *nature of mathematics* itself. Is mathematics a pre-existing absolute entity, there to be discovered, or is it an evolving social invention, the product of human minds? A subsidiary point is the nature of engineering mathematics: are all kinds of mathematics essentially identical, or is engineering mathematics importantly different in some ways?

At a second level, because we are dealing with *learning* mathematics, we must ask how we believe individuals learn. Is knowledge accumulated by transmission: osmosis from a higher to a lower concentration, or built by the individual on the basis of accumulated interpreted experience, or is it formed by negotiation and discussion between peers as a way of making sense of shared problems? At the same time, we must also examine whether all individuals may be regarded as essentially similar, or whether important individual differences intervene.

Finally, as a *researcher*, how have I approached the process of extracting empirical material and organising it into a coherent thesis? Is that empirical data a reflection of a single objective truth, independent of the observer, and measurable with instruments, or is it material which might be interpreted in many ways, but out of which a consistent sense may be built?

These are the background questions which set the framework in which the research is carried out. In the end, the "both-and" approach of hermeneutic philosophy seemed to make more sense than an "either-or" strong objectivist or subjectivist position.

This approach emphasises the alternation between being part of the system studied (subjective understanding) and seeing the system as a separate entity upon which one



may operate (objective explanation). Understanding and explanation are continually evolving and developing as one meets obstructions which were not accounted for by one's former mental model. The model will never be a complete match with reality, but learning and experience refine the model and improve the match (Brown, 1997).

1.4 Research question

The result is that an interpretivist paradigm has been adopted as appropriate for this study. The research question has become "how do engineering students interpret mathematical entities?", which has been translated into Vinner's (1991) terms "what are their *concept images*?" through the subsidiary questions "what is the *mode* of the image?" and "what is the *depth* of the image?". This question is important because as Arzarello et al (1995) point out "it may happen that the teacher and the student use the same words which correspond to very different meanings in their heads; a genuine comedy of errors is thus generated: the pupil and the teachers enter into a vicious circle which is difficult to break".

Vinner points out that concept images are not stable over time, indeed that learning must involve changing images, so the next question is "how do the concept images of engineering students develop?" and "is this development particular to engineering students, or do, for example, students of mathematics, show the same development?"

Finally, we ask "what is the result of a particular intervention, the use of a specially written computer program, on that development?". Of course, all these questions are asked in the spirit, not of finding the correct answer, but of finding an interpretation consistent with experience, and on which future action may be based.

2. What is the nature of mathematics?

2.1 *Introduction*

In the introduction, we saw that we needed to look at background questions in three areas: the nature of mathematics, the process of learning mathematics and the approach to research. These questions frame the theoretical structure supporting the specific research questions. In this chapter, we address the nature of mathematics and particularly the nature of engineering mathematics, and the aspects of mathematics which are important to the engineer are underlined. These are not identical to the priorities of the mathematician.

Having examined some of these issues, we will be in a position to look more closely at students using mathematics in mathematical modelling of physical systems in chapter 3.

2.2 *The nature of mathematics*

I propose that mathematics, and particularly engineering mathematics, can be seen from two broadly opposed viewpoints: behaviourist and cognitivist.

Behaviourists believe that as the workings of the mind cannot be directly known, they cannot be meaningfully discussed. They frame learning as change in behaviour, and concentrate on the acquisition of skills. Learning outcomes are stated as sentences beginning with "The student shall...", and continuing with a verb such as recall, state, define (low level skills), apply, interpret, analyse (high level skills).

A behaviourist paradigm leads to the setting of specific learning objectives which may be tested, a formal curriculum and syllabus which may be regarded as stable for all time, the notion of standards and criterion referencing, and an absolute and Platonic

view of mathematics as pre-existing reality discovered by diligent investigation. This attitude tends to resist the adoption of new techniques such as the use of technology to aid in mathematical performance.

The cognitive paradigm proposes that although the mind may not directly be observed, there is some meaning in framing models or metaphors for the ways in which it works. As well as skill development, concept formation and the ways in which things are understood are interesting to the cognitivist. The title of this thesis, containing as it does the words "conceptual understanding", betrays that the author has more sympathy with the cognitivist viewpoint.

From this point of view mathematics is complex in nature, similar to a language with social and private aspects, constantly growing and changing, but stable enough for shared concepts to have shared meaning at least for a time. Some of the important skills are meta-mathematical skills: verbalising mathematics, convincing others, problem solving. These will be of lifetime use, even when the mathematical content alters. For the engineer, mathematical prostheses (devices which amplify the capacity of the individual to perform tasks) such as calculators and computer algebra systems may take over where the log tables and slide rules of earlier generations left off. Such devices are similar to Connell's (1997) notion of Intelligence Amplification (IA).

2.3 Product or process?

There is in many of the descriptions and definitions of mathematics a tension between the view of mathematics as a collection of skills and practices, the *product* of mathematical thought, and mathematics as a creative problem-solving activity, the *process* of mathematical thought (Tall, 1991). Tall regrets that the former tends to be taught, rather than the latter. The products of mathematical thought, the body of

mathematical knowledge appears to the student to be fixed and static, whereas a process is by its nature dynamic.

The idea of mathematics as a competence, a set of skills which can be listed and ticked off as they are mastered, belongs strongly to the behaviourist school of thought, but still seems to prevail in much engineering mathematics education. Challis and Gretton (1997) challenge the idea that the engineering mathematics syllabus can be presented as a list of mathematical topics to be mastered, and propose that broader skills such as the formulation of a problem, choosing the appropriate means to solve it, and convincing oneself and others about the results, should be developed. Brown (1997) argues that mathematics is intrinsically changing and developing, like a language, and that the use of criterion referenced assessment in mathematics is an attempt to freeze it into a static form based on a behaviourist paradigm which belies the true nature of the subject.

2.4 Discovered or invented?

The Platonic or realist school of thought regards mathematics as pre-existing truth which is there to be discovered (Godino and Batanero, 1996). On the other hand, a pragmatic theory of meaning takes the view that mathematical objects arise from the problem-solving activity of the community of mathematicians, and are thus inventions of human activity.

We may ask whether i , the square root of -1 , was discovered or invented. There was a period when its existence was debated. It turned out to be useful, and to behave according to a simple set of rules, and so it became part of the accepted mathematical structure.

Even if a mathematical entity does not obey all the standard rules, it may be so useful that new rules are invented to allow its existence. For example, zero does not obey the cancellation law ($3 \times 0 = 4 \times 0$, while $3 \neq 4$) but it is so useful that a special rule (you cannot divide through by zero) allows its continued existence.

If mathematics is pre-existing and discovered, then once found it is immutable and infallible. If invented then it is open to negotiation, revision and change.

Mathematical learning can be compared to mathematical research in that the participant in either activity is venturing out into unknown territory, and extending their own personal boundaries of experience. The difference lies in the type of terrain they encounter. For the researcher, it is truly *terra incognita*, where dragons or treasure may be revealed behind the next obstacle to be overcome. For the learner, there is the certainty that the land is well-trodden, and that signposted paths will exist, should they not stray from the way indicated by their guides. For the researcher, mathematics is there to be created: for the learner, it is already there and is to be discovered.

2.5 Social or individual?

“Clearly the acceptance of a theorem by practising mathematicians is a social process which is more a function of understanding and significance than of rigorous proof.”

(Hanna, 1991, p58)

The importance of social acceptance of mathematical ideas may easily be demonstrated.

In 1742 Goldbach conjectured that all even numbers may be expressed as the sum of two primes (taking 1 as prime where necessary)... Goldbach also stated that every odd number may be expressed as the sum of three primes. In the form given it by Edward Waring (which excludes 1 as prime) this assertion also remains an unproven conjecture.

(Mahoney, 1972)

The definition of a prime number has changed from a number which is only divisible by itself and unity (in which case 1 is a prime number) to a number which has exactly two factors, itself and unity (in which case 1 is not a prime number). The effect of this is to make the initial form of Goldbach's Conjecture untenable, at a stroke as it were. A hypothesis which may have been true has suddenly become palpably untrue by a social, mathematical decision.

The social nature of mathematics calls for the need for *relative* stability in the way mathematics is expressed, otherwise we would not be able to use common symbols to communicate meaning. However since mathematics is constantly being added to and changed, the meanings of the symbols (the referents corresponding to the signs) shift subtly over time and place, according to the context in which they are found. Thus $1+1=2$ in integer arithmetic; $1+1=0$ in modulus 2 arithmetic and $1+1=1$ in logical terms. (As Eddington, quoted in Rose, 1988, put it, "We used to think that if we knew one, we knew two, because one and one are two. We are finding that we must learn a great deal more about 'and' ".) Alternatively a single statement may be made in different symbolic terms, (the signs corresponding to a single referent) again depending on context, such as $A+B=C$, $A\cup B=C$, or $A\vee B=C$, and the same entity may be expressed as $\frac{1}{2}$, 0.5, 2^{-1} , or 50% depending on context.

Most statements in mathematics may be regarded as either true or false, and this is often seen as an intrinsic property of the statement itself. Ernest (e.g., 1991) argues that the objectivity of a mathematical statement arises from its acceptance by the community of mathematicians. This can only be so if the truth is not intrinsic within the statement itself but depends on whether rules may be agreed which make it either true or false. Thus a statement such as $p=\ln(-1)$ will be true if rules are

agreed which make it true, just as rules can be agreed which make $1 + 1 = 0$ true in context.

On the individual side, it is clear that mathematics is an activity carried out by individual humans. The questions they tackle may be either problems which are agreed by the community to be interesting, for example Fermat's last theorem, or subjects which they have arrived at themselves: why did rounding the last decimal places in the weather data cause the calculation to diverge so dramatically from the unrounded solution? Even in the most individual cases, mathematicians draw from the results of others and converse with others.

2.6 *Hirst and Forms of knowledge*

Hirst could probably be classified as a proponent of academic rationalism. He argues (e.g., 1972) for a liberal education in which the learner learns to think like a mathematician, a scientist, a moralist, etc., through a process of apprenticeship. This equips the civilised person to understand that there are different ways of thinking which are appropriate to the tackling of different types of question: that a moral question cannot sensibly be approached in a scientific mode of enquiry.

For Hirst, mathematics is a *form* of knowledge. He defines a form of knowledge as

a distinct way in which our experience becomes structured round the use of accepted public symbols. The symbols thus having public meaning, their use is in some way testable against experience and there is the progressive development of series of tested symbolic expressions.

The public meaning of the symbols acknowledges the social nature of knowledge; the testing against experience its private nature and the progressive development its mutability.

He describes certain distinguishing features which can be seen in the various forms of knowledge:

1. They each involve certain central concepts that are peculiar in character to the form. For example, those of ... number, integral and matrix in mathematics...
2. In a given form of knowledge these and other concepts that denote, if perhaps in a very complex way, certain aspects of experience, form a network of possible relationships in which experience can be understood. As a result the form has a distinctive logical structure. For example, the terms and statements of mechanics can be meaningfully related in certain strictly limited ways only...
3. The form, by virtue of its particular terms and logic, has expressions or statements (possibly answering a distinctive type of question) that in some way or other, however indirect it may be, are testable against experience... in accordance with particular criteria that are peculiar to the form... The sciences depend crucially upon empirical, experimental and observational tests: mathematics depends upon deductive demonstrations from certain sets of axioms...
4. The forms have developed particular techniques and skills for exploring experience and testing their distinctive expressions... The result has been the amassing of all the symbolically expressed knowledge that we now have in the arts and the sciences.

He distinguishes the *forms* of knowledge from *fields* of knowledge which may draw their content from different forms to inform a unifying subject matter. For example, geography "the study of man in relation to his environment" would be an example of a theoretical field of knowledge, and engineering a practical one.

In summary he proposes two types of classification of knowledge:

1. Distinct disciplines or forms of knowledge (subdivisible): mathematics, physical sciences, human sciences, history, religion, literature and the fine arts, philosophy, morals.
2. Fields of knowledge: theoretical, practical (these may or may not include elements of moral knowledge).

Mathematics is thus a form of knowledge which has a high internal consistency, a way of thinking and relating to experience. Moreover, because each form of knowledge involves the use of symbols and the making of judgements in ways which cannot be expressed in words and can only be learnt in a tradition, it must be learnt

from a master on the job. Note that in (4) above, Hirst distinguishes between the amassed body of symbolically expressed knowledge, which is the result of applying the techniques for exploration and testing, and the form of knowledge, which includes that amassed knowledge. This means that for instance, although a knowledge of mathematics includes factual knowledge, the essence of the discipline is not, for Hirst, contained therein.

Engineering will need to contain some mathematical knowledge, but Hirst argues that all students should study all the forms of knowledge, in order to know how and when to apply an appropriate way of thinking in context, and that engineering would be a field in which the forms of knowledge would be applied.

2.7 *Postmodernist mathematics*

Godino and Batanero (1996) take as a fundamental notion the type of problem that different people are trying to solve. Mathematics has a "triple nature... as an activity for solving socially shared problems, as a symbolic language, and as a logically organised conceptual system".

Another fundamental notion within their view is that of the *institution*.

"An institution is constituted by the people involved in similar problem-situations. The mutual commitments with the same problems imply the carrying out of shared social practices which are also linked to the institution whose characterisation is to be contributed."

"We call people within society who are engaged in solving new mathematical problems a *mathematical institution*. They are therefore the *producers of mathematical knowledge*. Other institutions (macro-institutions) involved in "mathematical situations" are the *users of mathematical knowledge* (applied mathematicians, technicians, scientists and other professionals), teachers and mathematics educators (teaching institutions)." (Authors' emphases)

(Godino and Batanero, 1996)

Mathematical activities are characterised by: "mathematical objects" (numbers, operations); symbolic representations in the statement of the question and in their

Handwritten scribbles or marks in the top left corner.

carrying out; symbolising, formulating, validating and generalising, described by Freudenthal (1991) as “mathematizing”. These correspond to Hirst’s central concepts, relationships and tests against experience, and techniques and skills (see 2.6 above)

The institution agrees meanings for mathematical objects, which may evolve over time, according to their usefulness in solving the mathematical problems on which the mathematical institution is working. (See for example the prime number, discussed above.) Mathematical objects are attributed signs, such as names or written symbols, which may also vary with time or place. Individuals also assign personal meanings to these signs, and may be said to understand the objects insofar as their personal meanings match the institutional meanings. This set of relationships can be mapped onto a classical semiological diagram. (Ogden and Richards, 1923):

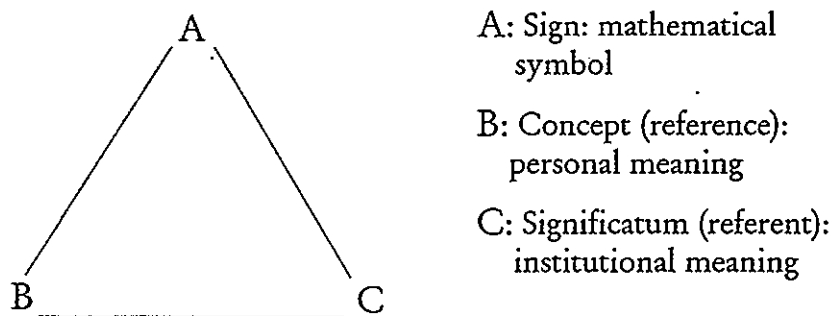


Figure 2-1: Relationship between sign, concept and referent, Ogden and Richards, 1923.

The relationship between the sign and the referent: the institutional or agreed meaning is moderated by a third entity: the concept or personal meaning.

For example, the plus sign $+$ is a sign. Its full mathematical meaning, that is all the things which the mathematical community understand by the plus sign, in all contexts, is the referent, and my own personal understanding of its meaning, which is a subset of that full meaning, with dominant and subordinate images, is a concept. In

fact I hope that my concept is a subset of the full meaning. It may contain elements which are not part of the referent, which could then be described as misconceptions. Arzarello et al (1995) describe how an inadequate relationship between a symbol (sign) and its sense leads to a pupil and teacher using the same words corresponding to different meanings "in their heads"

In addition, the institutional meaning of a given object in a teaching institution will be a subset or a sample of the full institutional meaning, and care must be taken in choosing the sample that the full institutional meaning is not distorted or lost in teaching.

Learning may be conceived as the process of construction and appropriation of viable conceptual networks by a progressive adjustment of a subject's cognitive structure to the structure of the institutional meanings.

(Arzarello et al, 1995)

The study of the teaching/learning processes in the mathematics classroom corresponds to the study of the effects on the personal meanings of "shocks" of didactic sequences carrying elements of meaning.

Within this analysis, "the centre of attention for didactic research should not be the student's mind, but the cultural and institutional contexts in which teaching takes place".

Brown (1997) (p70) suggests that "whilst there may be some over arching system of mathematics understood collectively by the community of mathematicians, we can never survey this holistically in a neutral way... Meaning is only created as signs are combined in stories that arise within the activities performed." That is, each person creates meaning according to the contexts in which they encounter signs and interpret them according to their experiences and interpretation of those experiences.

In the same way, he suggests, in language the relationship between signifier (word) and signified (thing to which the word refers) is unstable in the long term as both the signifier and the signified can change. In order for words to be useful in communicating meaning there has to be stability in the short term at least, so that the meaning of a word may be inferred from its context. However the context of a word is normally other words, each of which may be unstable. Like language, mathematics exists in a tension between *stability*, so that statements may continue to have meaning, and *instability*, as new meanings, relationships and entities are forged. Knowledge exists in the tension between *understanding*, where we are embedded in the experience, and *explaining*, where we are separate ourselves from the experience in order to articulate that which we have understood.

In the field of literary criticism, Culler (1982) explains that “all readings are misreadings”: that is that no reader can fully recapture the intended meaning of the author but imposes on the reading a unique set of experiences and interpretations to create a new meaning. Progress occurs when a “strong misreading” of an existing “text” takes place. That is, an existing entity is “misunderstood” creatively, or understood in a way not intended by the originator, such that a rich new reading is created. So the hieroglyphic “alphabet” of the Egyptians was adopted as a phonetic consonantal alphabet by the Phoenicians and Hebrews, and subsequently as a full vowel and consonant alphabet by the Greeks. Each step was a “misuse” which yielded greater functionality.

This “creative misunderstanding” is important in mathematical creativity:

Ironically for a discipline touted as precise, the student of mathematics has to develop a tolerance for ambiguity... Sometimes distinctions are better left blurred, e.g. the various roles of the minus sign and the use of $f(x)$ as both the function and the value of the function at x ... At the same time, when there is danger that genuine confusion might develop, the student must learn to become conscious of looseness and to apply the necessary amount of rigour. It

is this judgmental aspect of reasoning, so essential in mathematics education, that must be communicated to students.

Hanna (1991, p 61)

2.8 Summary

It was said of Gauss: he is like a fox who obliterates his tracks with his tail. He presented his conclusions, complete and perfected, without a clue as to the struggle which had led to them. This is how mathematics is often presented to the world in its public face, without ambiguity or uncertainty. Ambiguity however is a spring of creativity, as is unsatisfied need.

In analogy with language, mathematics has both stability and flow, a social and a private aspect.

2.9 Mathematics for the engineer

2.9.1 According to writers of reports: a tool, a language, a competence

The OECD Report "Mathematical Education of Engineers" (OECD, 1966), states that:

Mathematics is very important in the training of engineers for the following reasons:

- i) It provides a training in rational thinking and justifies confidence in the value of such thinking;
- ii) It is the principal tool for the derivation of quantitative information about natural systems;
- iii) It is the "second language" of human discourse and parallels natural language by providing a means of communication for ideas, as evidenced by the contents of technical papers;
- iv) It facilitates the analysis of natural phenomena;
- v) It is important in assisting the engineer to generalise from experience;
- vi) It trains the imagination and inquisitiveness of the student if properly taught;
- vii) It is a training for adaptation to the future.

(OECD, 1966, p11-12)

Barry and Steele (1992) reprint the list in a report for the Société Européenne pour la Formation des Ingénieurs (SEFI) (with the substitution of “an education” for “a training” in the last point) and add:

viii) Mathematics provides the language for formulating a model for computer analysis.

ix) Mathematics provides the means of understanding how a computer works and the computing process itself, and the means of assessing the accuracy of computer output.

Items *ii)*, *iv)*, and *vii)* relate to the development of professional communication and computational skills appropriate to an engineer. The remaining items identify mathematics as central to the intellectual formation process of the engineer.

(Barry and Steele, 1992, p15)

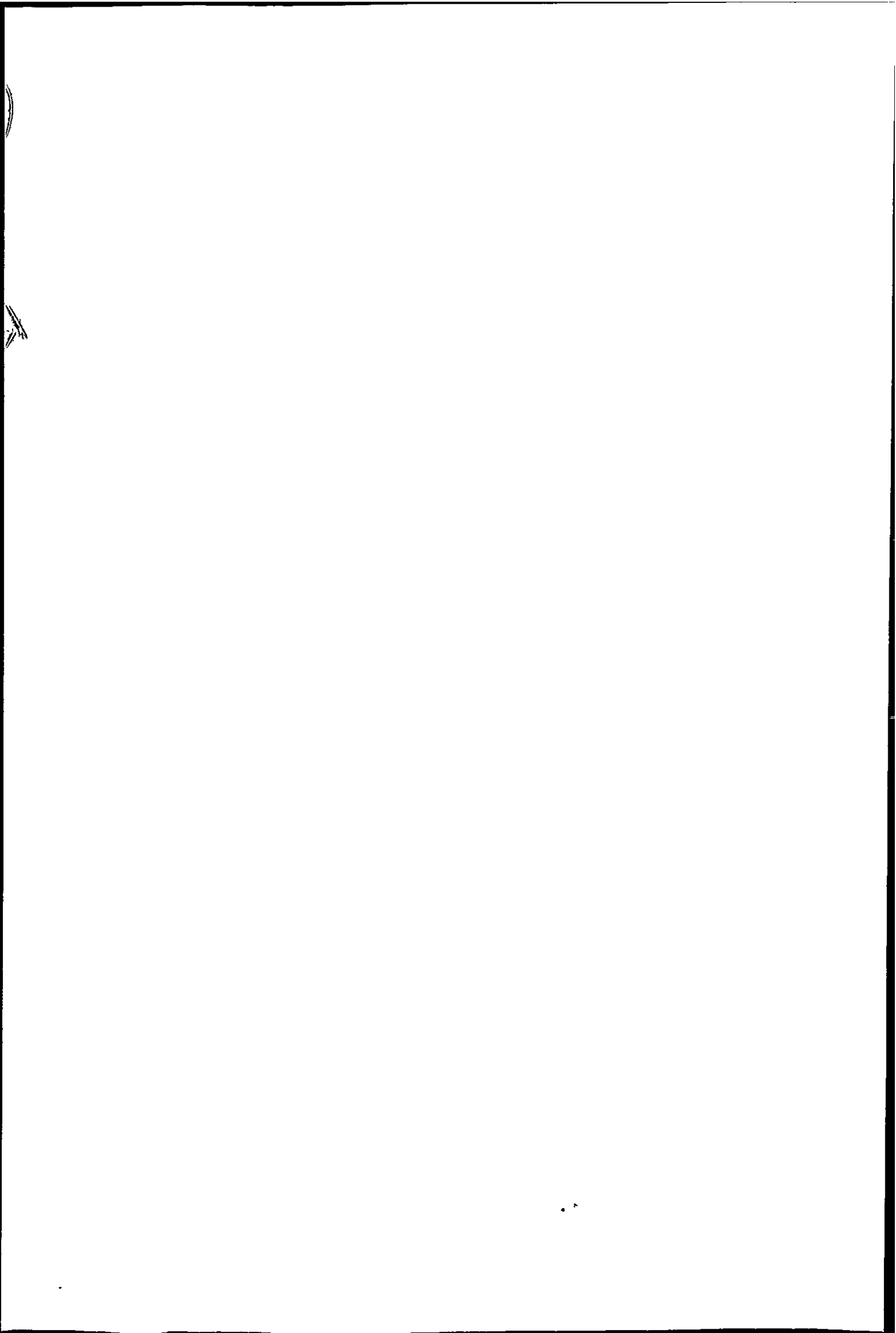
The SEFI analysis assembles the notions of “tool”, “language” and “training in rigorous rational thought” which we see in the prefaces quoted below.

Hermeneutical views are characterised by a circular movement encompassing a succession of alternative perspectives, for example between seeing language as embedded in what I am doing and seeing it as a separate labelling device. Differences in views of language held by such writers are essentially to do with the way they choose their home base on this spectrum and how far they stray from this base. Through seeing mathematics as functioning like a language, such a home base can similarly characterise the view held of mathematics.

(Brown, 1997, pp 218-9)

Brown’s view of the implications of seeing mathematics as a language has been discussed above.

Mathematics Matters in Engineering (IMA, 1995) states that “mathematics forms a key competence in engineering... One key area of competence required by most engineers is mathematics, for it is difficult to be innovative in engineering without such competence”. (p1) This recognises that engineering is an innovative discipline, but not that the mathematics of innovation is sometimes itself innovative. In order to innovate it may be more important for the engineer to be capable of learning and applying new mathematics than to have acquired all the mathematical competences



conceivably necessary for a career during the course of a degree. Cox et al (1995) consider that the attitudes of engineers towards mathematics are at least as important as the exact content of the engineering mathematics course, since there is simply not time to teach undergraduate engineers all the mathematics they will need to know in their careers.

2.9.2 According to writer of textbooks

Writers of mathematics textbooks sometimes allow a glimpse of their underlying assumptions in the prefaces they write. Here are a few samples gleaned from the prefaces of books written specifically for engineering users of mathematics. Without exception, writers of textbooks over the period refer to the student as *he*, which is defensible in a period when such students were overwhelmingly male, but jars a little on modern ears.

We would like to suggest two particular areas of concern which we feel should be reviewed from time to time by every analyst. The first is to maintain an awareness of the limitations of any mathematical model resulting from the various approximations imposed during the modelling process... Changes in engineering curricula and... improvements in teaching mathematics at the high school level... have increased the mathematical requirements for engineering students... [and] raised the level of mathematical *rigor* (sic). It was decided not to trade off the valuable physical applications for increased rigor in this edition.

(LA Pipes & LR Harvill, Applied Mathematics for Engineers and Physicists, 1970)

In a work of this nature, full *rigorous* proofs cannot be given, but the assumptions made have been carefully stated and wherever the existence of a rigorous proof is assumed, some indication of this assumption is given.

(BH Chirgwin & C Plumpton, A Course of Mathematics for Engineers and Scientists, 2nd edition, 1970)

A technologist who is taught mathematics purely as a series of techniques is on firm ground only as long as those techniques remain relevant; in his subsequent career he will encounter many quicksands where the new techniques he needs are out of his reach. The answer, in our opinion, is to stop treating the technologist as a second-class citizen, entitled to use mathematics but never to understand; we must allow him, indeed expect him, to come to terms with the *fundamentals* of the subject.

(M Bruckheimer et al, Mathematics for Technology: a New Approach, 1971)

The typical student for whom this book is intended is likely to look upon mathematics as a means to an end. We feel that it is nevertheless unfortunate if, as all too often happens, his mathematical armoury consists merely of a collection of unrelated techniques which he uses under appropriate (and possible inappropriate) circumstances.

(RJ Goult et al, *Applicable Mathematics: a Course for Scientists and Engineers*, 1973)

The emphasis is on the *practical* side of the subject and the more theoretical aspects have been omitted.... Although mathematical *rigour* has not always been emphasised in the programmes, they can serve as an introductory text for students [of mathematics]... giving... some idea of how mathematics is used in other subjects.

(AC Bajpai et al, *Mathematics for Engineers and Scientists*, 1973)

The text is primarily designed to assist engineering undergraduates and their teachers, but we hope it may also prove of value to students of other disciplines who use mathematics as a *tool*... We have tried to give equal emphasis to both the analytical and the numerical aspects of engineering mathematics, so that the reader is encouraged to make use of whatever mathematical tool is best for the problem he has in hand.

(AJM Spencer et al, *Engineering Mathematics Vol 1*, 1977)

Mathematics is an essential *tool* for the engineer and applied scientist and mathematics is often up to one third of an engineering student's curriculum in the third year.

(JS Berry and P Wainwright, *Foundation Mathematics for Engineers*, 1991)

While formal proofs are included where necessary to promote understanding, the emphasis throughout is on providing the student with sound mathematical *skills* and with a working knowledge and appreciation of the concepts involved.

(KA Stroud, *Engineering Mathematics*, 1995)

Mathematics is the language of engineering.

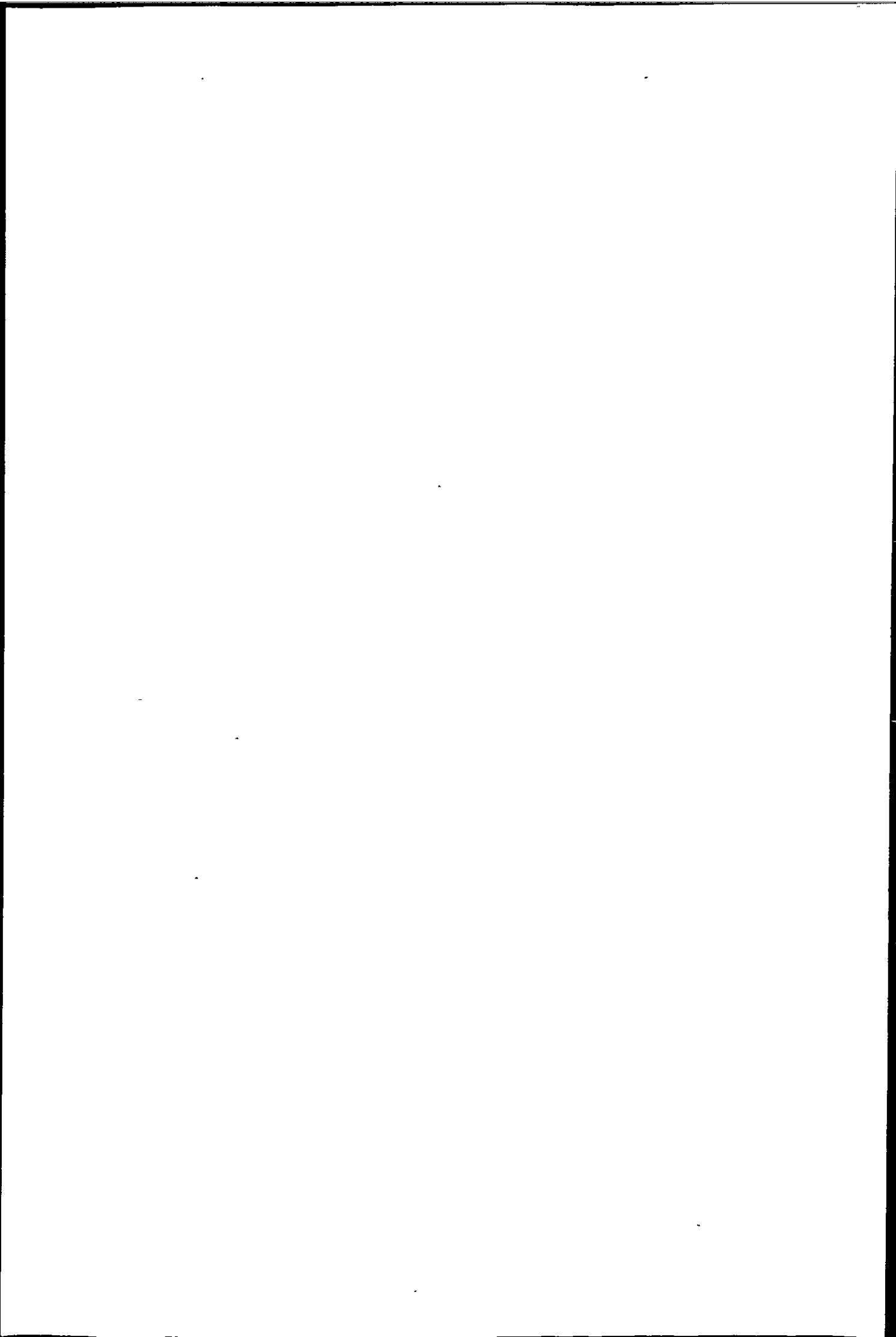
(A Croft et al, *Introduction to Engineering Mathematics*, 1995)

Mathematics is the language of engineering...

(L Mustoe, *Engineering Maths (sic)*, 1997)

Many authors have not been quoted, as they have simply listed the scope of the various chapters of the text in their preface, or stated the syllabuses to which the contained material corresponds, but among those who do make mention of their underlying approach, the ideas which emerge are the appropriate degree of mathematical rigour and mathematics as a tool for the technologist.

When we examine engineering mathematics text books, we discover they vary in two dimensions: those which are designed as an aid to learning, such as programmed texts,



and those which are designed primarily as a reference: and those which treat the mathematics by mathematical topic without reference to application versus those which work through extended examples.

	Programmed text	Traditional textbook	Reference work
Through applications		Berry, Northcliffe & Humble	Noble
With applications	Mustoe	Croft et al Bajpai et al James et al	
Application free	Stroud		Weltner

Table 2-1: classified examples of text books

A telephone survey of universities offering degrees in mechanical or general engineering was carried out in February 1996. Lecturers were asked what mathematics text books were recommended to their first year students. The replies are tabulated in Table 2-2. It will be seen that at the time Stroud and James et al dominated the market. These texts are instrumental in outlook, with Stroud regarding engineering mathematics as a collection of skills to be acquired by drill and practice.

University	Title	Author
Bath	Engineering Mathematics	Stroud (1995)
Birmingham	Engineering Mathematics Introduction to Engineering Mathematics	Stroud Croft, Davison & Hargreaves (1995)
Brighton	Engineering Mathematics	Stroud
Bristol	Modern Engineering Mathematics	James et al (1992)
Cambridge	Advanced Engineering Mathematics Modern Engineering Mathematics Advanced Modern Engineering Mathematics	Kreyszig (1993) James James et al (1993)

Table 2-2: Texts recommended to first year mechanical engineering

undergraduates, February 1996

Central England	Engineering Mathematics Modern Engineering Mathematics Introduction to Engineering Mathematics	Stroud James Croft et al
Central Lancashire	Engineering Mathematics	Stroud
City	Advanced Calculus Mathematical Methods for Science Students Engineering Mathematics Engineering Mathematics	Spiegel (1974) G Stephenson 1973) Stroud Bajpai, Mustoe & Walker (1989)
Coventry	Introduction to Engineering Mathematics	Croft et al
Durham	Advanced Mathematics for Engineers & Scientists Engineering Mathematics Further Engineering Mathematics	Spiegel Stroud Stroud (1990)
Greenwich	Modern Engineering Mathematics	James
Hertfordshire	Learning Mathematics through DERIVE Foundation Mathematics for Engineers Modern Engineering Mathematics	Berry, Graham & Watkins (1996) Berry & Wainwright (1991) James
Humberside	Engineering Mathematics	Stroud
Imperial	Mathematical Methods for Science Students Mathematical Techniques Engineering Mathematics	Stephenson Jordan & Smith (1994) Stroud
Kingston	Introduction to Engineering Mathematics Engineering Mathematics	Croft et al Stroud
Lancaster	Modern Engineering Mathematics	James
Liverpool John Moores	Engineering Mathematics	Stroud
Loughborough	Engineering Mathematics Mathematics for Engineers & Scientists Modern Engineering Mathematics	Stroud Weltner et al (1986) James
Luton	BTEC National M2 & M3 Mathematics for Technicians	Greer & Taylor (1989)
Manchester Metropolitan	Modern Engineering Mathematics	James

Table 2-2 (Continued): Texts recommended to first year mechanical engineering undergraduates, February 1996

Manchester	Modern Engineering Mathematics Engineering Mathematics	James Stroud
UMIST	Engineering Mathematics	Stroud
Middlesex	Engineering Mathematics DERIVE Manual	Stroud
Nottingham	Modern Engineering Mathematics Engineering Mathematics	James Evans (1992)
Oxford Brookes	Engineering Mathematics	Stroud
Oxford	Advanced Engineering Mathematics Mathematical Methods for Physical Science Engineering Mathematics Mathematical Methods for Science Students Modern Engineering Mathematics Mathematical Methods in Science & Engineering	Kreyszig Riley (1974) Spencer (1997) Stephenson James Heading (1970)
Plymouth	Engineering Mathematics	Stroud
Portsmouth	Engineering Mathematics	Stroud
Sheffield Hallam	Locally written material Foundation Mathematics Introduction to Engineering Mathematics	Booth Croft et al
Sheffield	Further Elementary Analysis Engineering Mathematics	Porter Stroud
South Bank	Mathematics for Technicians Engineering Mathematics	Greer & Taylor (1989) Stroud
Southampton	Mathematics for Engineers & Scientists (And for weak students)	Weltner et al (1986) Stroud
Sussex	Introduction to Engineering Mathematics Modern Engineering Mathematics Engineering Mathematics Advanced Engineering Mathematics Engineering Mathematics Further Engineering Mathematics	Croft et al James Evans Kreyszig Stroud Stroud
Warwick	Locally written material	
Westminster	Engineering Mathematics Calculus	Stroud J Stewart (1988)

Table 2-2 (Continued): Texts recommended to first year mechanical engineering undergraduates, February 1996

2.9.3 Mathematics as sensemaking: the making of meaning

Tall (in Tall, ed, 1991, p256) states "The evidence is that students of a wide range of abilities prosper when they can give meaning to ... ideas." Matos and Carreira (1997) describe learning as "an activity where students give meaning to ideas, problems, mathematical and non-mathematical concepts".

Schoenfeld (1991, cited in Wilson et al, 1993) proposes that school mathematics concerns highly unrealistic situations, and that the main preoccupation is that the student solves the problem, rather than understanding it. Wilson's co-author Teslow gives a personal account of his experience of mathematics as sensemaking in the context of his engineering experience. This involves the application of mathematics to real problems whose solutions have practical applications and whose implementation will have ramifications in the real world.

In order to be useful in sensemaking, mathematics must be a unified body of knowledge, rather than neatly compartmentalised. It must be active and accessible rather than inert knowledge which can be recalled but is not applied. For the engineer, mathematics is a tool, consisting of both algorithms ("a sure-fire method that always leads to a solution of a particular problem") and heuristics ("rules-of-thumb that may solve a problem, but do not guarantee a solution") of when to apply the proper algorithm. Schoenfeld found it easy to teach algorithms, but difficult to teach the heuristics. It is possessing the appropriate heuristics which distinguishes the expert from the novice (Wilson et al, 1993).

Lave (1996) suggested that learning should be examined from the perspective of "becoming who we are going to be", a form of socialisation. Engineering students sometimes complain "I don't see the relevance of this" (Appleby, 1995, Coxhead, 1997). They are studying engineering in order to become engineers. It is not always

clear to them that the mathematics they study is relevant to being an engineer, as we shall see from some of the comments in the questionnaires in chapter 6.

2.9.4 Precision and approximation

Within engineering practice, calculations are made from two quite different methodological standpoints, and this is rarely made explicit to students. Engineering precision and approximation are different from their mathematical namesakes.

In the domain of approximate calculations a safety factor will generally be applied, to ensure the result falls on the desired side of a limit, either a maximum or a minimum. Classically the strength of a structure and its deflection under load fall into this category. The exact answer does not matter, since the whole structure will be over-designed to allow for unforeseen misuse and deterioration.

In precision applications, the calculation must eliminate uncertainty as far as possible, since either over- or under- estimate has undesirable consequences. Such calculations often concern the fit of components, or accurate timing. In an axial compressor, running at design speed, if there is a gap between the tips of the blades and the casing, air will leak through and the efficiency will be impaired. On the other hand, if the tips of the blades rub on the casing then the ensuing friction causes overheating and possibly fire. The calculation of the change of dimensions of the rotor and casing must be as accurate as possible.

The requirements for accurate calculation do not always coincide with where mathematics is exact. Numerical solutions are sometimes needed. In mathematical modelling applications the assumptions made to allow for mathematical exactness may be unjustified in the context of the application. In these cases, the aspirations of the engineer and the mathematician do not coincide. Mathematics becomes a tool,

rather than an end in itself, and the mathematical purist may wince at some of the things the engineer does.

Story: a professor of engineering ran into a colleague of mathematics one day. "So glad to see you, old chap," said the engineer, "I've been using some new theory in bridge design, and getting some really interesting results, but the maths is a bit beyond me. Would you cast your eye over it for me?" The mathematician frowned. "Can't possibly work," he muttered, "Assumptions all wrong", and away he shuffled. A while later, they met again. "Are you sure about my new theory?" asked the engineer plaintively, "I've been testing models, and they seem to hold up really well". "It's no good," growled the mathematician, "Only valid in the trivial case where all the variables are real and positive".

An important difference between mathematics *per se* and an engineer's mathematics is that in pure mathematics it is normal and permissible to add x to x^2 while in engineering mathematics adding (length) to (length²) is generally a sign of an error. The mathematics of the mathematician deals with dimensionless numbers or entities which stand proxy for numbers, while the engineer manipulates values of physical quantities with meaningful dimensions. There is thus a cultural problem for the student; in mathematics classes it is permissible to add x to x^2 , but elsewhere one may only add Kx to x^2 , where K has the dimension of x .

2.10 Conclusions

2.10.1 On intellectual rigour

It appears that one of the principal characteristics of mathematics is its intellectual rigour, and that in teaching mathematics to engineering students one of our aims is to teach them to think clearly. On the other hand, one of the complaints about

engineering students is that they lack feeling for mathematics and whether the answers they produce are correct or not. (Sutherland & Pozzi, 1995) We have to decide whether we regard mathematics for the engineer as a mental discipline (Hirst, 1972) or as a tool (for example, OECD, 1965, Barry & Steele, 1992, and others).

Given that engineering students suffer from crowded timetables, we have to be clear about the aspects of mathematics we want to develop in our students. This may be different from the mathematics we want to teach to mathematics students, and it will be important to make that explicit, so our expectations are realistic.

We may have to sacrifice rigour in mathematics and develop clear thinking in alternative ways, while attempting to enhance students' feeling for mathematics through the use of prostheses such as computer algebra and graphics calculators (Challis and Gretton, 1997), and the use of modelling from an early stage. (Cross, 1983)

2.10.2 On the finality of mathematics

Mathematicians are in the business of building mathematics. They are contributing to the growing and changing collection of mathematical structures. Mathematicians need to have the sense that mathematics is incomplete and may be challenged in order to motivate the making of new mathematics. I was told by a colleague that the most memorable point of his engineering degree course was when his lecturer reached a stage when he said "and that is as much as we know", and he realised that beyond was still inviting exploration.

There is a sense in mathematics that it is not open to negotiation: that to a given question there is one correct answer and any other is simply not right. That however is largely due to the sort of questions which are asked in "mathematics", and the rules of the game that are set in the "mathematics" which engineers are currently taught.

In mathematical modelling of real complex situations the room for negotiation is wide. It is more difficult to assess work in which there is not a correct answer.

Engineers are also driven by the motivation of satisfying incompleteness, to do things, to make things work, even if the gap to be filled is the provision a part for a machine rather than a new mathematical solution. (Shaw, 1989)

3. Observation of mathematical modelling in practice

3.1 *Introduction*

Having discussed the nature of mathematics in chapter 2, and suggested that there is a difference between engineering mathematics and that of mathematicians, we may now turn to observe some novices using mathematics and reflect upon the way they construct a mathematical model. We shall also compare the approaches taken by some mathematics and engineering students to the same problem, and conclude that there is some difference in the ways in which they are "doing mathematics". The students on whom we shall concentrate are final year students who have been well acculturated into their respective subject. In chapter 13 we shall examine two basic paradigms of mathematical modelling, and a hybrid third approach which appears to be adopted by engineering students.

3.2 *What is observation?*

Observation is a descriptive technique, that is, behaviour is observed and described. If behaviours are counted and then analysed statistically, it becomes a quantitative approach: if they are simply observed and described, it remains qualitative. In this case the observation is qualitative, as the sample in terms of numbers and of the time for which they were observed is small. The advantage of using a small sample is that it is possible to enter into greater depth and detail in the analysis, and this is gained at the expense of loss of breadth.

3.2.1 *Strengths of observational methods*

Observation is the first step in any investigation: to establish that there is a phenomenon to be investigated in the first place, and then to try to establish its nature. Before any

rules of gravitational attraction can be considered, it must first be observed that an object which is released will fall. Observation is first necessary to establish that theory derives from reality rather than fantasy. It is the guarantee of empiricism in the scientific method. (Cohen and Mannion, 1994, p20)

Observation in the social sciences is based on the case study: rather than manipulating variables to determine their causal significance in an experimental manner, the investigator observes the characteristics of a single unit (Cohen and Mannion, 1994, p106). The picture gained in this manner is richer than the measurements obtained from an experimental study, and it is argued that experimentation, in order to reduce the number of variables to a controllable level, impoverishes the study to something well below the norm of human social experience.

3.2.2 Weaknesses of observational methods

In participant observation, there is the danger that the observer becomes too immersed in the culture of the group being observed, and becomes "subjective, biased, impressionistic, idiosyncratic and lacking in the precise quantifiable measures that are the hallmark of survey research and experimentation". (Cohen and Mannion, 1994, p110)

There are questions of internal and external validity. Internally, to what extent were the observations coloured by the researcher's expectations? Externally, to what extent are any observations applicable to other cases? Phenomenological techniques refer to the *epoché* or bracketing of one's prejudices and understanding as far as possible what the subject says and does rather than what the observer expects that person to say and do, (Cohen and Mannion, p29, p293) and also to the circle of understanding where a sense of the whole (interview, case study, observation) informs the understanding of the parts, and at the same time the sense of the whole is composed of the totality of the

understanding of the parts. External validity may be checked by some form of triangulation, for example by verification against the more skeletal understanding gained from surveys.

The recording of observations is time-consuming and relies on the memory of the observer. If the observations are recorded on video or audio tape, they must still be transcribed or otherwise annotated.

3.3 Observation within this study

Two types of observation were used: non-participant observation, where the observer obtrudes as little as possible into the behaviour of the subjects, and participant observation, where the observer is a part of the behaviour observed.

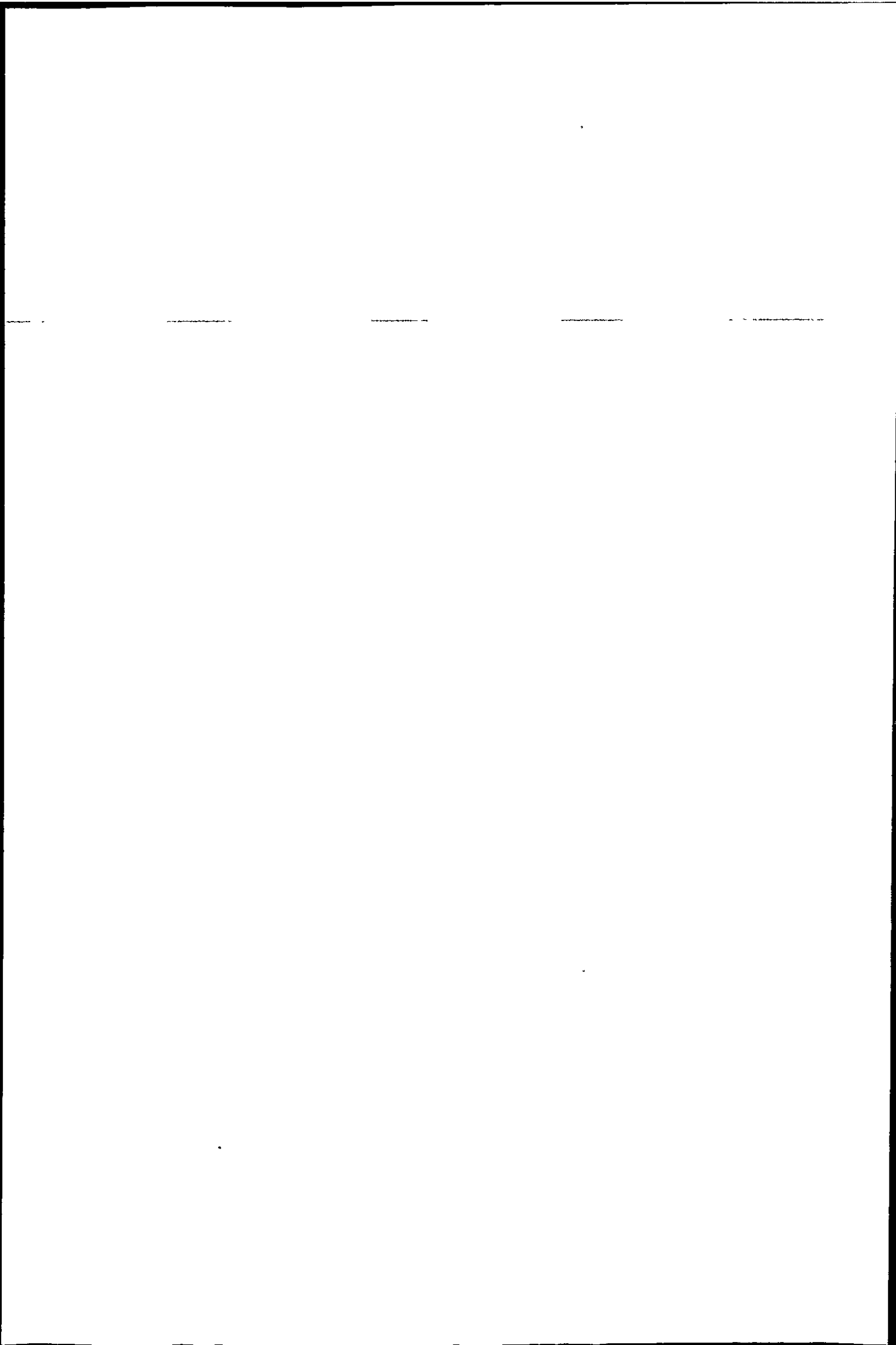
Three sets of observation were carried out in this study: the first was of a class of mathematics undergraduate students (mixed first and second years) modelling the cooling of a cup of hot water, the second was a comparison of final year mathematics and final year mechanical engineering students modelling the flow of water from a tank under gravity, and the third was the testing of the courseware written as part of this study. In this account, WMM is the author and researcher.

The testing of the courseware falls more conveniently into a later chapter, so I shall describe here the observations in the early part of the study.

3.4 Cold tea

3.4.1 Introduction

As the class of 18 students observed here were taking part in a scheduled class, observation was combined with some teaching, as the occasion arose. The technique could thus be described as participant observation. The class split themselves into four



groups, which I have labelled A, B, C and D, and interviews were carried out the following week with two of the groups of students who took part in the modelling class. The first session took place between 11am and 1 pm. Groups D and B participated in an interview session the following week.

The problem was stated as shown in Figure 3-1.

Modelling - cold tea

Joe drinks China tea. The Chinese method of making tea is to pour boiling water on tea leaves in a mug, either bone china or enamelled steel if you can afford it, or a jam jar if you can't. (You learn the skill of filtering out the tea leaves with your teeth.)

As you leave the tea to infuse, it cools down. Model the way the temperature varies with time.

Chinese tea mugs have little lids (and so do jam jars - you can screw the lid on and carry it around without spilling any). What difference does having a lid on make to the way the tea cools?

Figure 3-1: the cold tea problem

3.4.2 Observations

In describing the modelling process, I have adopted the stages described in the Open University Mathematical Modelling diagram, a structure which will appear at many points in this thesis.

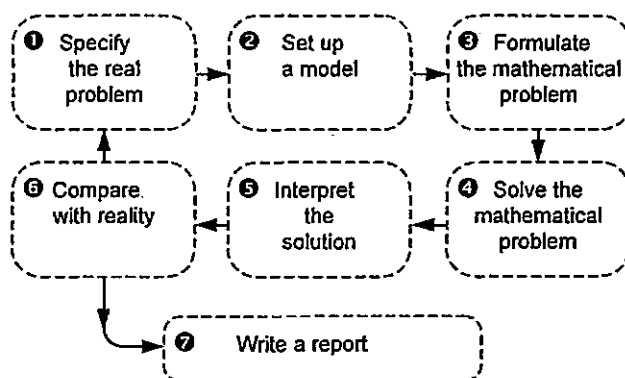


Figure 3-2: OU flowchart analysing the process of mathematical modelling, (Tunncliffe, 1981, p5)

There was of course a lot that happened in the room that I did not see, and a lot that I saw but did not have time to note down. The notes are therefore very selective.

3.4.2.1 Initial approach

Two different initial approaches were evident: three of the four groups (all first year students) began by finding hot water, cups and thermometers and obtaining some empirical data, whereas the group of second year students (who pointed out to me that they were second years) stated that they knew the answer, and wrote that they were assuming that $dT/dt = K(T - T_m)$ and that $T_m = 20^\circ\text{C}$ (having estimated the room temperature: they did not use a thermometer). Whereas the former groups may be regarded as working down the right hand side of the MEI modelling diagram (see Figure 13.3 in chapter 13), the latter may be interpreted as a *novice* approach to problem solving (Mestre, 1994), that is to “resort to formulaic approaches”.

3.4.2.2 Using technology

The data produced was plotted on graphics calculators. DERIVE and Omnigraph were available on computers in the room, and students also attempted to use these programs to plot data, one group to the extent of using Omnigraph to plot spline curves of temperature against time. Students from one group also played “Risk” on a nearby computer.

Some problems associated with using graphics calculators to plot the data were:

- the graphic definition is low
- the screen is very small
- the angle of view is narrow (so the straightness of a line cannot be judged by looking along it), and a group of students cannot all see the screen at once
- straight lines with small slopes are plotted as staircases

- students felt they were achieving something by performing regressions on the data when they did not know what it was they were doing.

The calculator did prove useful when a group used it to obtain the intercept and slope of a graph of $\ln(T-T_m)$ against time, once they had established that that was what they needed to do.

3.4.2.3 Making assumptions

In making assumptions, students did not appear to appreciate that making assumptions had implications for the mathematics they would set up, thus:

(Group C) S1: We must assume all the diameters are the same.

S2: It makes no odds.

S1: You can't have millions of arbitrary constants floating about.

But there was no discussion of what difference a change in diameter would have made.

(Group A) Assume boiling water poured into cold cups.

The model did not allow for a transient effect of the cup warming up.

It appeared more as though making assumptions were a part of the ritual of mathematical modelling, and so they were arbitrarily making some, rather than analysing the implicit assumptions they were making and how things may be changed if they were otherwise. This may have been due to the students not taking time to understand the problem, but instead diving into the experiment.

3.4.2.4 Set up mathematics : solve mathematics

Two possible models were considered by the students: linear and exponential.

"Is it linear?" Groups A, B and D

The first impulse of the students was to assume that the relationship would be, at least to a first approximation, linear. Some students were firmly wedded to the notion that linear is best and were prepared to sacrifice some of the empirical data to achieve this.

(Group B Time = 11.40) S3: The graph (on a graphic calculator) looks as though it would be linear if you lose the ends.

S4: You can't just lose the ends.

S5: Just do a graph and see what we get.

(Time = 12.15) S4: It looks not linear.

S5: Could be a bit exponential.

"If it isn't linear then it must be exponential."

For group B these two possible models existed side by side for a while (see below). Since an exponential model does fit the data quite well, that solution schema works on this occasion. On reflection, it is possible that for some students a misconception may have been reinforced by this experience.

Some students, once the notion of an exponential had been mooted, decided quickly on appropriate steps to check the model.

(Group A Time = 11.40) S6: You can just about see the curve (on calculator scatter plot)

S7: When shall we stop measuring?

(Group A Time = 12.00) S6: We think it isn't linear.

S7: It looks exponential.

WMM: How would you check to see if it's exponential?

S6: We could draw a log graph.

S7: Log both- no, just temperature against time.

S6: Do we use base e or 10?

Some students, although they clearly knew about exponential relations in theory, used inappropriate tools and needed more guidance towards checking their model.

(Group D Time = 12.12) S8: It cools faster without a lid.

WMM: What do you think about the shape of the graph?

S8: This one looks exponential.

WMM: How might you check if you think it's exponential?

S9: A log plot.

S8: Can we use Omnigraph?

(Time = 12.35) Students plotting spline curve of temperature against time on Omnigraph. They had also calculated linear and exponential regressions on a graphics calculator.

WMM: What is the form of an exponential equation?

S9: $y=e^x$

WMM: What happens if you take logs?

S9: $\ln(y)=x$

So what should you plot to get a straight line if it's an exponential?

S9: Natural log of y against x

WMM: What is your y?

S9: Temperature

WMM: And your x?

S9: Time. So we plot log temperature against time.

3.4.2.5 Investigate implications

An inappropriate model gives rise to a prediction of unlikely behaviour. The students are willing to be challenged and to modify their model, but it was not until 12.15 that they were ready to discuss how to see if data fitted an exponential model.

(Group B Time = 11.55) S3: Both (cooling curves for mugs with and without lids) look linear.

WMM suggests looking along line to see if there is a curve. In fact this is difficult on LCD display because of narrow angle of view.

WMM: What will happen at room temperature if it is linear?

S4: It will go on getting cooler.

WMM: Does water do that?

S4: I don't think so!

WMM: Do you think that "It looks linear over this range" is a good enough model?

3.4.2.6 Refining the model

This group might be regarded as having successfully produced a mathematical model for the cooling of the mug of water. They have produced a first model which they refine and from which they can now obtain numerical constants.

(Group A Time = 12.20) S7: The log thing looks nearly like a straight line.

S6: We stopped too early.

WMM: What does the temperature tend towards?

S6: Room temperature.

S7: So we could take away room temperature.

(Time = 12.40) S7: The graph of $\log(T-T_m)$ is straight. (data plotted on graphic calculator)

WMM: Did you find the constants from it?

S7: No - here they are. Gradient is -0.0245. Intercept is 4.184. So temperature is $4.184 - 0.0245t$. No. $\ln(T-T_m) = \text{something minus something times time. (some manipulation on paper)}$ $(T-T_m) = Ke^{-0.025t}$ where $K = e^{4.18}$

However, the model they have arrived at is more useful for prediction than for understanding, since they had not realised why it is important that the liquid is stirred, for example, although this is stated in their assumptions.

3.4.2.7 An alternative model

Group C: we know the answer. This group was unwilling to let anybody see the way they were working, but this seemed to be what they did.

11.10 Assuming that $dT/dt = K(T - T_m)$ and $T_m = 20C$

I think that the students separated the variables and integrated the differential equation to give $\ln(T - T_m) = Kt$, forgetting to add a constant. Taking antilogs then gave $(T - T_m) = e^{Kt}$. Rearranging would then give $T = e^{Kt} + T_m$.

11.35 (T_m measured as 24C.)

11.40 $T = e^{-0.05827t} + 25$

We could plot it on DERIVE

No, I'll use my calculator

The constant K was obtained by putting corresponding values of T and t into the equation. I do not know why T_m is now 25.

11.50 (loading experimental values into calculator) We'll work them all out and take the average of the lot.

For each pair of values for T and t , a corresponding value of K was calculated. The idea was to take the mean of all these values. The students were very resistant to a more conventional way of proceeding. Clearly the values they obtained did not fall within a narrow range.

11.55 We've got a duff model

But life isn't exact.

The exponents don't agree.

When the model was felt to be inadequate, the students blamed their assumptions.

12.05 We assume the room temperature was constant, but it wasn't- I'm feeling sweaty with all these people in here.

Anyway the readings aren't 100% accurate.

These, interestingly, were the students who had said early in the session:

We're all second years. Second years have learnt to argue.

Finally, they tried a completely different approach.

*(Time = 12.50) I'm trying to do this by dimensional analysis, but I don't seem to be getting anywhere. Could you show me how to do the last bit?
(Result obtained is $dT/dt = T/t f(VA^{3/2})$
Attempt to discuss why there is a better way to tackle it but they want to rush off (end of lesson).*

3.4.3 Analysis à la Perry

The first year students may be compared to Perry's (1981) early stages: looking for the correct answer, responding positively to authority, whereas group C was showing signs of "our way is as good as your way" relativism. From this point of view their thinking has progressed, although it makes working with them more difficult.

3.4.4 Interviews

Sadly neither group A nor group C came for interview the following week. In my notes on the session (31 May, after reflection) I have written "It will be interesting to see how reflective different groups are prepared to be and how this is a function of their perceived success in the exercise". Group A seem to have made the most direct progress through the exercise, groups B and D had moderate success, and group C did not appear to engage with it at all.

Both groups B and D stated that they had expected the curve to be exponential: students in group D had seen a similar curve in A level physics, and some in groups B and D had seen one in GCSE physics.

Both groups used a graphic calculator for logging the data and plotting a cooling curve, and both groups suggested that they could have used DERIVE, when asked what other resources they could have used. In group D's case, this was part of a list of possibilities, not all of which seemed realistic (e.g. Minitab).

Group D, when asked how they went about the modelling, assumed that I meant the experiment, and both groups, when asked how they could improve the model, made

comments on their experimental procedure. A student in group B, half joking, said “The maths model was perfect”. There was thus some confusion between the idea of a mathematical model and the system it describes.

An interesting final comment from group D, implying that they, having studied physics, “went right”.

You could try it with a group who haven't learnt about it in Physics and see how they go wrong.

3.5 The cascade problem

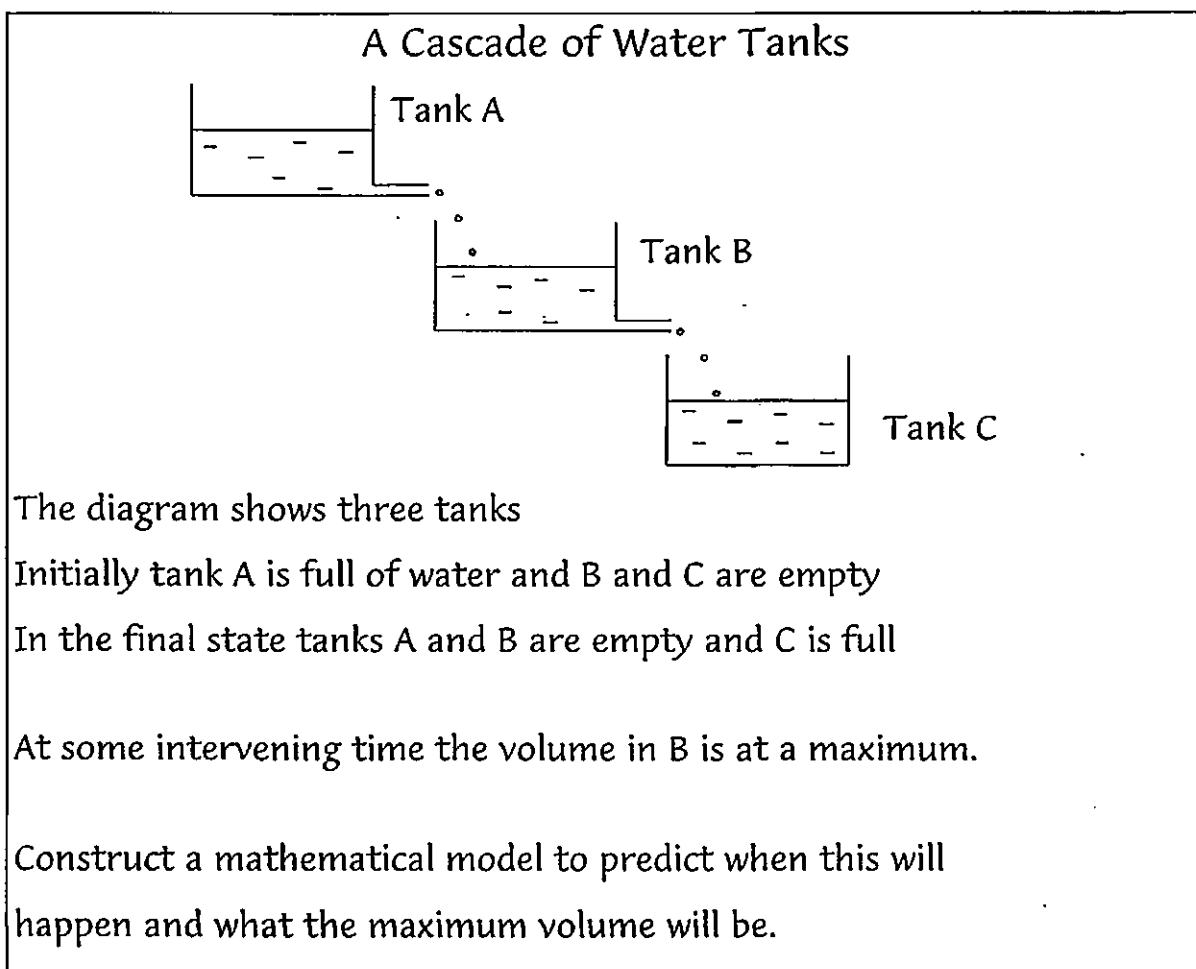


Figure 3-3: The cascade problem

The question was stated on paper (see Figure 3-3) and shown to 2 groups of students; final year mathematics and final year mechanical engineering. The students were filmed on video as they worked and the tapes transcribed. They showed striking differences of approach. (The transcriptions are included in Appendix A)

3.5.1 Use of written work to support discussion.

Both groups used their workings to share and discuss, that is there was a social aspect to the work in both groups and the documents they produced were used by both groups as a support for discussion. However the mathematics group produced pages of equations, while the engineering students produced pages of sketches of apparatus and curves.

Another difference apparent between the written notes produced is the use of numbers.

The mathematics students descended into numbers only to record the data from the experimental runs. The engineering students produced, as well as the results of two experimental runs, a page of numerical calculations to predict the time (or rather the mass flow rate, hence the time) when the maximum volume in B would occur.

It is argued by Osborne (1983) that although number appears to be abstract, they appear, in the minds of the pupils, to be tangible, or concrete. The use of numbers thus argues that the engineering students are happier to stay in the concrete type of thought.

3.5.2 Early insights.

Both groups at an early stage had the insight about the volume in B being a maximum when flow in = flow out, but while the engineering students linked this immediately to "head", the mathematics students did not mention that the height would be the same for another 13 minutes. It appears that the idea of flow rate being *dependent* on height (as opposed to being a *function* of height, a more abstract notion) is less firmly embedded in the mathematics students.

Both groups then went on to discuss whether the maximum height in B would be half the initial height in A, and then dismissed the idea, as during the time taken to reach the maximum, liquid would have flowed out of B. Neither group, apparently, used this to fix an upper bound for the value of the maximum volume in B.

3.5.3 Focus

The engineering students remained focused throughout on the quantity they had been asked to find: the time and level of the maximum in the middle tank. This fixation made it difficult for them to break the task into steps. Such a goal focus is characteristic of a novice problem-solving technique (Mestre, 1994).

The mathematics students were more focused on the task of building a mathematical model, which meant first modelling the flow in a simple system with two tanks and then extending the result, and although they did not finally build that model, they seemed satisfied in the end that they could have done it, and left the session fairly happy.

The engineering students were dissatisfied that they had not completed the task to their satisfaction, and the final comment caught on the tape was:

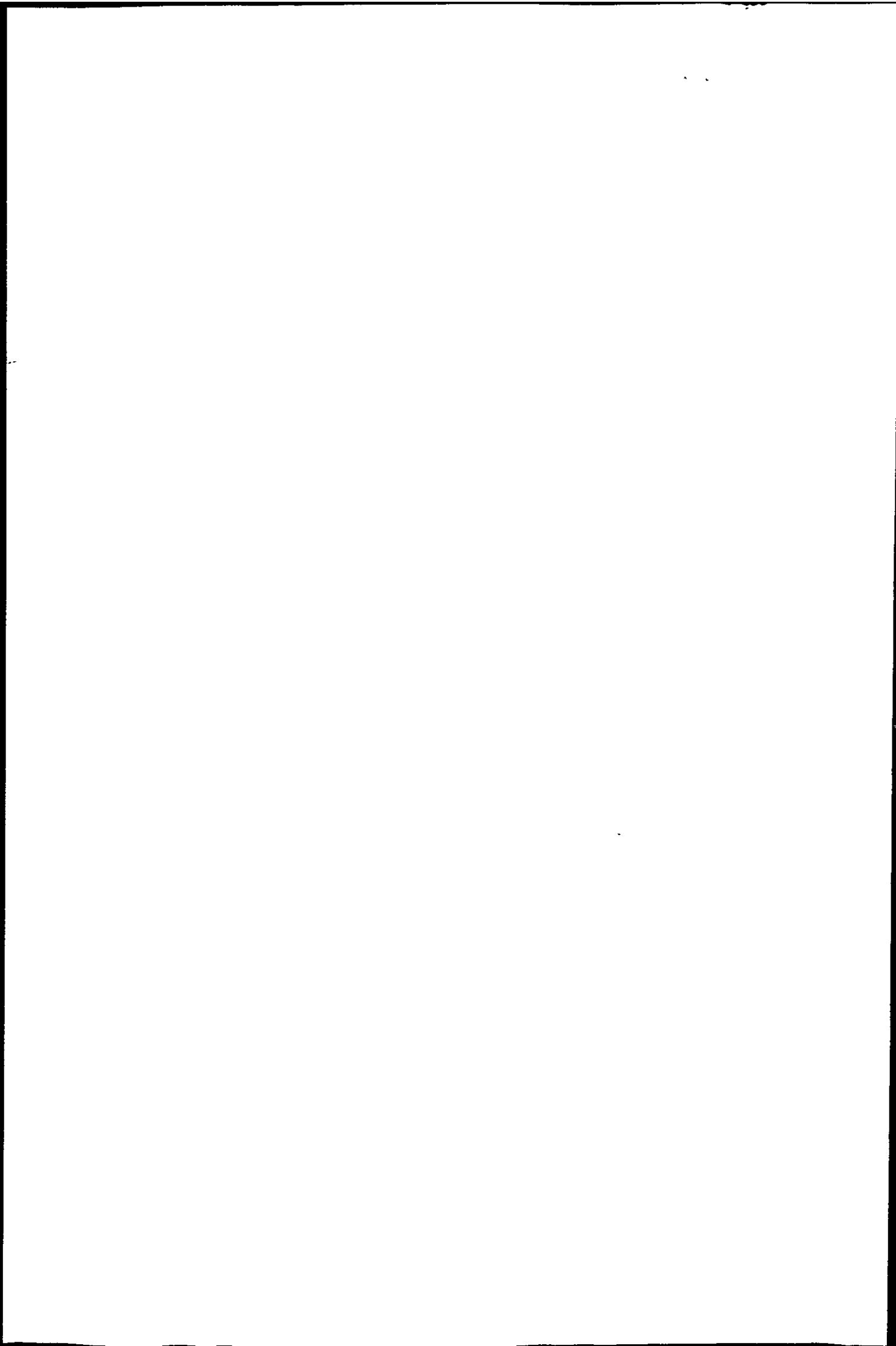
Adrian: Some of the others in our year would've sorted it.

3.5.4 Vocabulary and concept set.

It is clear from the tapes that the mathematics students "spoke mathematics" far more fluently than the engineering students. Their ability to read out mathematical expressions and to follow what was being read out was striking. The engineering students hardly spoke in mathematical terms and certainly not in mathematical expressions.

The mathematics students used the term "function of" freely, while the engineering students preferred "depends on". The engineering students used "head" and "mass flow rate" which separated the ideas of depth of liquid and rate of change of depth. Both groups seemed uncertain as to the propriety of regarding height, volume, and mass of liquid as interchangeable variables (to within a multiplying constant).

The mathematics students generated a wide range of suggestions to describe the flow:



Jason: Viscous, incompressible and irrotational. (laughter)
Ann: We've got a Newtonian flow, have we?
Jason: Yeh, I'm sure it doesn't matter.
Ann: Of course it matters. We've got to have those things written down.
Anthony: This is where we've got the Navier-Stokes equations.
Derek: Irrotational is it?
Anthony: Bernoulli's equation
Derek: Newtonian flow
Jason: Assume negligible viscosity.

The engineering students plotted a graph to describe the flow..

Jolyon: Should it be linear? Because it's proportional to v^2 isn't it? Dunno if that's..
Ordinate scale is volume.

After some silence:

WMM: What sort of relationship does it look like?
Adrian: There isn't any shape showing clearly there. I thought it tends to be linear.
WMM: What does the graph represent?
Jolyon: It relates the amount of volume to how long it's been going.
WMM: So what does the slope of the graph represent?
Adrian: The rate of flow
WMM: So..

After some circular argument:

WMM: Well, do you think it's- It's clearly not independent of the height, so you could write down an equation that says the flow rate is a function of the height. And what sort of function do you think that is? Do you think it might be?
Jolyon: Well, it's obviously not linear, from those results.
WMM: No- yes- if it were a straight line it would be independent of height. So you know it's some sort of function of the height.
Adrian: We thought it might be some sort of square.
WMM: How would you test what the relationship between flow rate and height is? If you're suggesting it's a quadratic, how would you test if it's a quadratic?
Jolyon: Surely you'd have that by seeing the results.
But we don't really know what's going on- we're not really sure what's happening between.. each container. So up to now we've only done experimental - and what we've got there - doesn't really show enough - doesn't really tell us enough about the flow rate against the height of the water.

Ironically, "some sort of square" is a good description of the shape of the graph, although the engineering students do not test this. In the Bernoulli equation, v represents velocity, which is proportional to flow rate, however that is measured.

The biggest difference was in the way students generated equations. The mathematics students wrote down their basic assumptions in equation form and continued from there. The engineering students found their equations from a formula sheet or card, and not by seeking to build new equations for themselves, although they were willing to manipulate and re-arrange the formulae thus obtained. In all the engineering students' written material there was no differential coefficient, although \dot{m} appeared as a variable. It is not clear whether they had made a strong connection between \dot{m} and dm/dt , or any differential coefficient.

In fact I think there is a basic difference here between the idea "is a function of", begging the question "what function?", and "depends on" which does not so clearly lead to the question "how?".

The mathematics students expressed the volumes in A and B as V_1 and V_2 , and the rate of change of volume in B as dV_2/dt , which enabled them to see the problem in terms of differential equations from the start. The engineering students used \dot{m} to express mass flow rate (which is standard engineering practice) and identified ρgh as the appropriate group to express pressure, so they measured the depth of the liquid. This choice of variables and notation does not immediately suggest a solution strategy.

Taken together, the words chosen to describe the relationship and the variables chosen by the two groups of students characterise their different approaches to the modelling problem, i.e. " dV_2/dt is a function of V_1 and V_2 " and " \dot{m} depends on ρgh " express different ways of seeing the situation which lead to different ways of dealing with it.

3.5.5 Experimental technique.

There appeared to be unspoken agreement between the engineering students that the initial conditions should be the same for each run. They were more experienced in running experiments. When it was pointed out to them that the rubber tubes on the

taps may have had an effect, they set up the whole apparatus to run without tubes. On the other hand they did not seek to verify the model they were using (Bernoulli's equation) for the simple case of a single tank, but remained focused on the overall task of predicting when the maximum would occur in the middle tank.

The mathematics students used the top two vessels, to verify the relationship between h and dh/dt . They appeared to have more direction in that they had an idea what they wanted to measure. The engineering students did not appear confident of the relationship between mass flow rate and dh/dt . They wanted to set up a constant head, steady flow apparatus, so they could collect and measure the volume (thus mass) flowing in a given time. The mathematics students used the distance dropped by the liquid level in equal time intervals as a proxy for flow rate, to plot against the mid-height in the interval- a much more sophisticated approach.

3.5.6 Starting to run the apparatus.

At the start of the session, a bottle of pink colouring was pointed out to the students as useful for making the liquid more visible. There was no water in the apparatus, so in order for the dye to be useful, the students would have to fill it with water themselves. It was felt that this would give implicit permission to the students to use the apparatus. Although one mathematics student wanted to run the apparatus at the start, she was strongly discouraged by her fellows, and they did not use the apparatus until they felt capable of making predictions of what would happen. Even when they did run it, it was at the insistence of this same student. The engineering students appeared to need explicit permission to try the apparatus in the context of what they felt to be a mathematical modelling exercise, but once having used it, they did so repeatedly (5 times).

3.5.7 Physical modelling.

The mathematics students were much more uncertain of the underlying physics than the engineering students, despite having a wide repertoire of possible ready-built models (Bernoulli, Navier-Stokes, Newtonian). They suggested that h (height of liquid) should be measured from the centre of the earth. They appeared to be using the concept of potential energy in a vague way to describe the problem, and relating this to the pressure at a given depth. Like the students in the cold tea example above, the mathematics students made the assumption there was a linear relationship between the variables (in this case volume and rate of change of volume) which led them to assume an exponential relationship between time and volume. Although this led them to implications which they could see were wrong, the students were not happy to challenge that initial assumption. It seems that the ideas of linear and exponential relationships are deeply ingrained in mathematics students.

The engineering students had the notion of "head", which is again an energy concept, but thought that the height should be measured from the minimum liquid level reached in the carboy, so did not relate measuring the depth from the point at which liquid was at atmospheric pressure.

In the diagram, the outlet is shown at the very bottom of the tank, so that the point at which there is no flow because there is no water and the point at which there is no flow because the pressure difference is zero coincide. In the apparatus, the flow stops because the water flow is cut off by a lip. The mathematics students had a concept ready to fit this (Heaviside), but did not incorporate it into their model.

3.5.8 Spotting discrepancies between theory and practice.

During the first session, with the mathematics students, the difficulty arose that the rubber tubes which had been to prevent splashing in fact effectively lowered the outlet

of the tanks by some 20 cm, and introduced pipe friction losses. This led to an unexpectedly near-linear relationship between flow rate and measured height of liquid. The engineering students saw straight away that this was not what they had expected, but had to be led gently to resolving the discrepancy. The mathematics students did not see the difference until they verified their assumptions about the relationship between dV/dt and V . They needed some leading to decide to see what happened when the tube was removed, and I am not sure that they understood why it made the difference it did. The mathematics students ran the apparatus twice as given, looking to see the height of the maximum. They did not notice that the flow out of the top vessel was not a "decreasing function", as they had predicted. The engineering students noticed after the first run that the flow rate was almost uniform, and not as they had expected. (This shows that they had stronger expectations than the mathematics students of the way the apparatus would behave.) They were quick at this point to seek outside help (i.e. from me). This behaviour Ramsden and Entwistle (1981) would regard as syllabus-bound. The mathematics students, having generated their own model, also checked out its implications and so discovered they had made, amongst other things, algebraic errors leading to nonsensical predictions. However they attributed all their nonsensical predictions to algebraic errors, rather than to their original assumption.

3.5.9 Other apparatus.

Neither group used the DERIVE set up on a nearby PC for them. Each group used a calculator; the mathematics students to calculate the log values needed to plot a graph, the engineering students to manipulate numbers to get a value out of a formula. Although the engineering students had a graphics calculator, they did not use its graphics facility.

WMM: Did you use the graphics facility on your calculator?

Adrian: No I didn't.. It's a bit of a mystery.. It's all right when you've got the equation to plot and you can pick off the minimums and maximums, but when you've got a set of results to put in I'm not too sure what to use.

3.5.10 Plotting results.

Neither group plotted graphs of results without prompting, although squared paper was available to (and indeed written on by) both groups. (The squared paper was provided as it could be used for plotting graphs, without making it obvious that that was what it was for, and so prompting the behaviour.) This was despite the numerous sketch curves drawn by the engineering students. The mathematics students, despite their more uncertain experimental technique practically, took scatter on a plot of results in their stride, and joked about trying to fit a curve to all the points. The engineering students plotted fewer points, and tried to draw curves which passed through all of them. They seemed to regard plotting a graph as more of a calibration exercise.

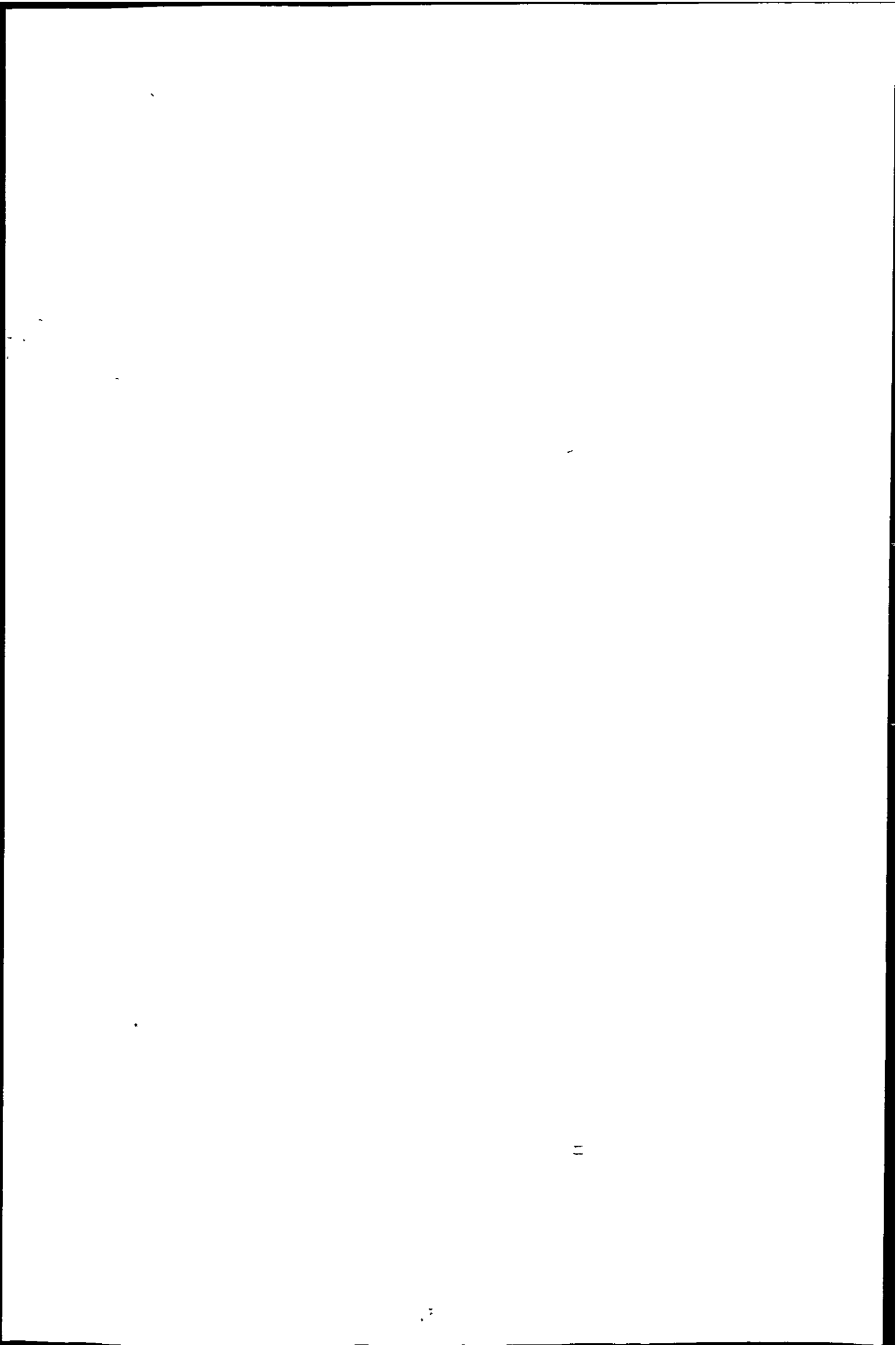
3.6 Conclusions

3.6.1 On mathematical modelling

Two types of mathematical modelling were seen in the first modelling exercise (cold tea). Most students used the empirical method, assuming they would fit the data to either a linear or an exponential model. One group started from a theoretical position, but did not manipulate the assumed relationship correctly or successfully match the experimental data to their model.

3.6.2 On mathematics and engineering students

The ways in which the two groups of students tackled the cascade problem were different enough to suggest it would be worth while designing and applying a questionnaire to try to trace the development of differences between students on the two



courses. In particular the differences in vocabulary suggested that in mathematics the engineering students might be speaking a different language from their lecturers, who are in general mathematicians. Arzarello et al (1995) suggest that students and teachers may be using the same words which correspond to different meanings in their respective heads, and that the invented meanings often have their own justifications. In this case we have different words being used to frame the same problem, which would compound the communication problem.

The engineering students' approach to modelling seemed biased towards empirical modelling, but slightly different, in that they relied on a ready-built model being available (Bernoulli). This reliance on the availability of models also appeared in the comments on the questionnaires, and is apparent when students ask "what is the formula?".

The engineering students were far more aware of the physical side of the model. When the apparatus did not behave as predicted, they were disturbed that something was wrong. They took care to set up the same initial conditions for each run. They suggested modifications to improve the apparatus, such as a height gauge and a flow meter. The theoretical world appeared more real to most of the mathematics students. Sadly, it seems that while the mathematics students had been taught mathematical modelling, the engineering students had managed to avoid it.

Despite software (DERIVE and Excel) being available and running on computers next to the experimental area, neither group used them. Neither group used advanced calculator functions, such as graphing functions. The mathematics students were even a little wary of using the clock function in MS Windows. Mathematical modelling and the use of technology were not, in other words, synonymous for either group of students.

3.6.3 On observation

As a method of collecting empirical material, observation proved to be rich if time-consuming.

Participant observation combined with taking notes was difficult and stressful. As noted in the chapter, much occurred which was not observed, and much was observed which was not recorded. The contemporaneous notes and subsequently recorded reflections are all the evidence which remains of the observations. Another researcher will have to depend upon the record of events translated through the observation of a third party.

The use of video recording meant that watching the sessions again after a period of time brought out new aspects, especially as the first session could be watched again from the perspective of comparison with the second. It was hard as an observer at the time to set aside one's preconceptions, and easier to do so as one was separated from the immediacy of the experience. However there is a cost in time, convenience and video tape in such a recording.

The transcripts of the recordings are less rich than the recordings themselves, but will be easier and quicker for another researcher to read than sitting through the recordings themselves, as well as more easily reproduced, more transportable and more accessible. As the transcripts have been word processed, they are available for such techniques as frequency analysis, a more quantitative approach. Again these advantages have been gained at the cost of the time taken in the processing.

4. Research method: questionnaire design and administration

4.1 Introduction

In chapter 3 I have described how engineering and mathematics students were observed carrying out a mathematical modelling task. I proposed that there was enough difference in the ways they were "doing mathematics" to make it interesting to carry out a wider survey into the mathematical ideas of engineering students and mathematics students, and if possible to compare how those ideas differed across subject studied and with the level of experience.

In this chapter, the theoretical background is surveyed: the practical process of design and administration, and the results are described and discussed in the subsequent chapters. This is a convenient point to review the third background issue mentioned in the introduction: the overall epistemology of the research project. We shall then discuss the dimensions of mathematical ideas which are addressed in the design of the questionnaire, namely the *depth* and *mode* of representation, which when combined make up the *concept image* (Vinner, 1991, 65-81)

4.2 Social science

Burrell and Morgan (1979) characterise two extreme positions of approach to social science, summarised in Figure 4-1.

The objectivist approach regards social science as essentially the same as the natural (physical) sciences, and holds that it is the task of the social scientist to determine the underlying laws governing social behaviour. The subjectivist view is that the social sciences are essentially different from the physical sciences, and that social reality is constructed by individuals. Social science is thus concerned with understanding the ways in which people explain the social world to themselves.

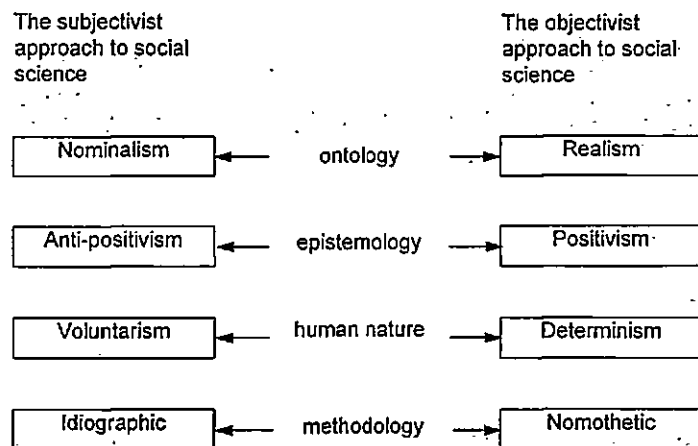


Figure 4-1: A scheme for analysing assumptions about the nature of social science. (Burrell and Morgan, 1979)

This difference naturally leads to differences in the ways in which investigators approach research. For the objectivist, *reality* is hard and quantifiable. The researcher is an observer, able to measure and determine *positive* relationships between real variables.

Human behaviour is *determined* by external circumstances. Research is concerned with determining the circumstances which control human behaviour and the relationships between them, and the laws which relate these. This is a *nomothetic* methodology.

From the subjectivist point of view, the social world consists of mental objects (words, hence *nominalist*), having no accessible counterpart in external reality. Knowledge about the social world is personal, so the researcher must become involved with the subjects as more than an observer (the *anti-positivist* view). Human behaviour is *voluntary*, and subject to the operation of free will, and research consists of understanding and describing the behaviour of the individual (*idiographic*).

These are two extreme and somewhat caricatured views, and like most studies, the present work adopts a stance somewhere between the two.

Thus an objectivist study would have been concerned with student behaviour, rather than students' mental constructs, and would have set up an experimental pre-test post-test design with randomised control and experimental groups where possible, and would

probably have been concentrated on the skills demonstrated by students in solving problems. The observation phase would have been characterised by counting manifestations of behaviours, and the statistical analysis of those behaviours would have been regarded as of great importance.

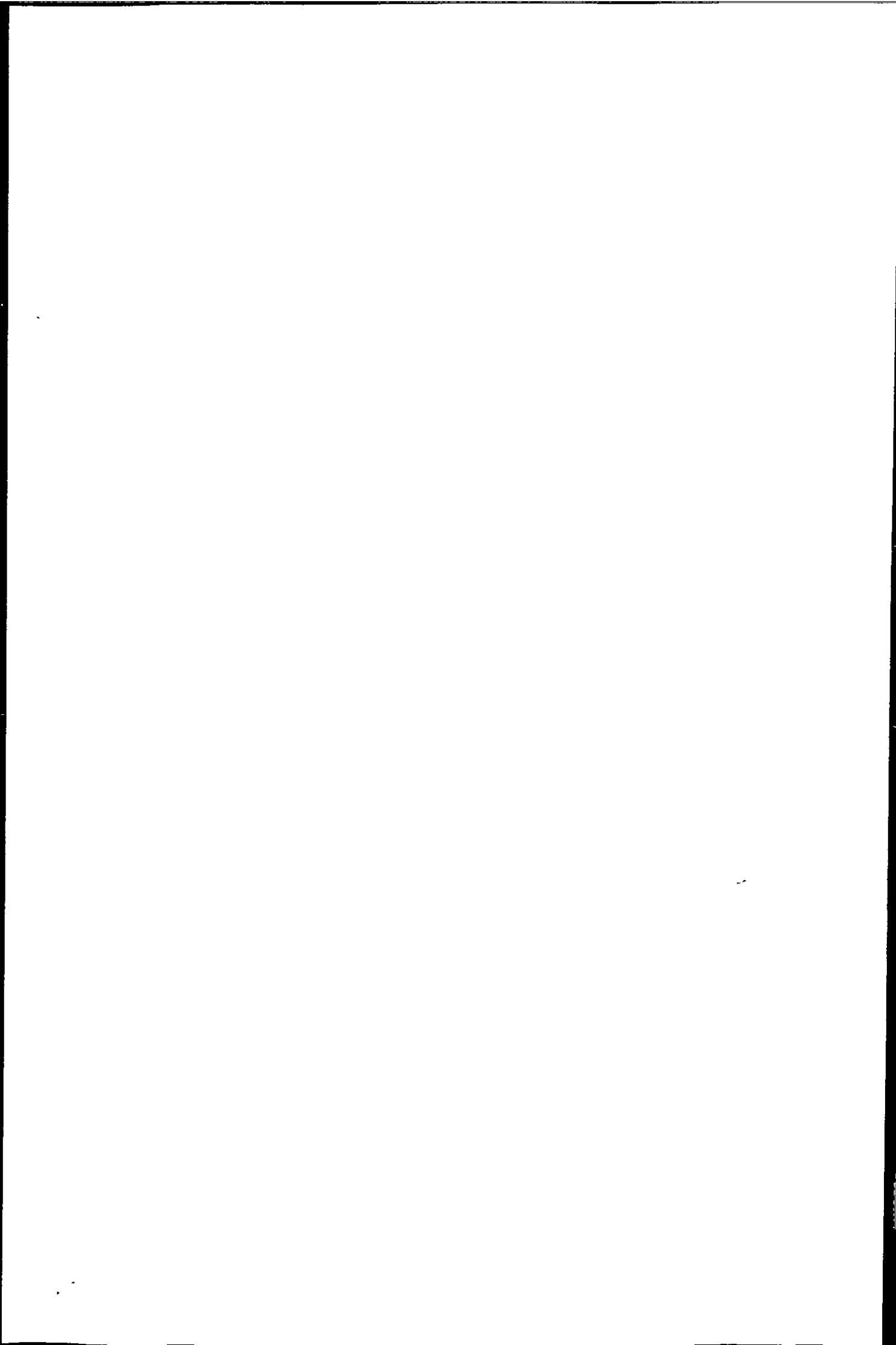
A purely subjectivist study would have involved few subjects, and would have been concerned with eliciting their constructs of the target concepts, through observation, interviews, and qualitative methods.

As mentioned in the introduction, the hermeneutic point of view as promoted by, for example, Brown (1997) allows for both of these sets of ideas to be considered in a "both-and" attitude. In a social system, such as a learning environment, we must spend time subjectively experiencing what it is like to be part of the system in order to understand the system and how it works. To explain the system, we must view it objectively, and see it as something separate from us on which we operate. A good example of this is riding a bicycle. No amount of explanation of how to ride a bicycle can take the place of actually riding the bicycle oneself in order to know what it feels like and how to do it. However in order to put the experience in place, to learn to do it better, and to connect it with other knowledge, such as reading road signs, one must get off the bicycle, and read the Highway Code.

4.3 This study

Models are useful for description, prediction or understanding. At a first stage, our model of engineering students' mathematical understanding is descriptive. We may then attempt to use this description to understand why engineering students differ from mathematics students, and finally predict the implications of these differences.

The first stage of description started in the comparison of the two groups of students carrying out the cascade exercise. That stage was purely qualitative. In order to



quantify the differences to some extent, we now turn to the questionnaire. In terms of the modelling flow chart, the observations fell into the box "Understand problem", and the process of designing the questionnaire is the "Simplify and make assumptions". In the objectivist conception of social reality, (Cohen and Mannion, 1994, p10), "abstraction of reality especially through mathematical models and quantitative analysis" is the paradigm of methodology. If this conception were held to in this study, then the next stages would be to set up mathematics and solve mathematics. The aim of the questionnaire would be to provide a statistically rigorous analysis of the differences between the ideas of mathematics and engineering students and the way these develop. In fact my basis for interpreting social reality is much more subjectivist: that the world exists, but that different people construe it in very different ways, and that we are in search of the ways which, overall, the different groups of students make sense of mathematics in particular in the context of their studies, their overall experiences and their aspirations.

Thus the questionnaire shows patterns of responses from different students. We may use analytic techniques to simplify these patterns (for example, factor analysis, statistical techniques), but in the end the reality is the students' beliefs, and not the model.

Haney (1984) is highly critical of the objectivist point of view in the context of discovering how individuals think and reason. He points out that the most useful tests were designed with a practical purpose in mind, rather than for generating statistics: Binet's "intelligence" test was originally intended to determine which pupils in a Parisian nursery school would benefit from remedial teaching. Tests measure an artefact of the process rather than the process itself. He suggests that for research purposes it would be preferable to use a more subjectivist paradigm, and talk to individuals about the way they solve given problems.

4.4 Surveys

Cohen and Mannion (1994) describe surveys as perhaps the most commonly used descriptive method in educational research. The data-gathering technique employed in this case was self-completion and postal questionnaires, using a form of attitude scales.

The sampling was non-random, using captive groups of students where possible, to maximise the return rate. The object was to survey the whole population of students at the appropriate stage of their studies on given courses at Plymouth. No students from other types of university were surveyed, and this would form the basis of an extension to the present project.

Surveys were carried out over groups of respondents, from mechanical engineering and mathematics backgrounds, at various stages in their academic careers, at the beginning and end of their first years, during their final years, and in the case of engineering, during postgraduate studies and after some 20 years of engineering experience.

4.5 Aspects of experimental design in the questionnaire

The present study differs from a classic experiment in that it relates to students' concepts, rather than their skills, or their attitudes. It was found during the observation stage that engineering and mathematics students appeared to be consulting different concepts to tackle the modelling task, which led to the questions of how these concepts differed, whether the differences could be measured or demonstrated, and how. The study did not allow for a complete cohort study to be carried out, particularly as many engineering students spend a year in industry, leading to a four-year degree course.

Cohen and Mannion (1994) describe various types of experimental and quasi-experimental design. The features of these research methods are the inclusion of pre-and post-tests, and the use of control groups.

The experimental approach in education reflects an objectivist paradigm which this researcher finds problematic. In scientific experiments, the control is intended to be identical to the experimental sample in all but the experimental variable. In educational research, no two groups can be regarded as identical: each individual has a different history and brings unique characteristics to the study. To make all the other conditions identical, the two groups would be taking the same course at the same institution with the same tutors. It is impossible to prevent transfer of experience between the two groups. Practice with one group would affect the tutors's treatment of the other group, transferring practices which work with one group to the other group. Students talk to each other and sometimes work together outside classes. Students have been known to compare coursework, and to attend sessions they were not supposed to, if they perceive some possible interest or benefit.

With these caveats in mind, the following construal may be made. The questionnaire applied to the first year students may be regarded as a quasi-experimental design: the mathematics students are a non-randomly selected control group and the questionnaire is given as a pre-and post-test. The experimental treatment is thus the first year of an engineering course, compared with the first year of a mathematics course.

The questions raised by this design concern internal validity and external validity.

(Cohen and Mannion, 170-171)

4.5.1 Threats to internal validity:

History: effects of external events. The use of a simultaneous control group is intended to minimise the possibility of external events producing effects which may be mistaken for the effects of the treatment. Since the control and experimental groups were not randomly assigned, it is possible that external events would affect members of the two groups differently.

Maturation: the natural maturing of the students over the passage of time is an integral part of the phenomenon being studied.

Statistical regression: since the scores in the questionnaires are preferences not test scores, regression is not an issue.

Testing: it is possible that the pretest sensitised the students to the issues raised by the questionnaire, but it was such a small part of their overall experience this was thought to be unlikely. The students in the postgraduate group who had reached a more introspective stage of development would be more likely to be influenced in this way.

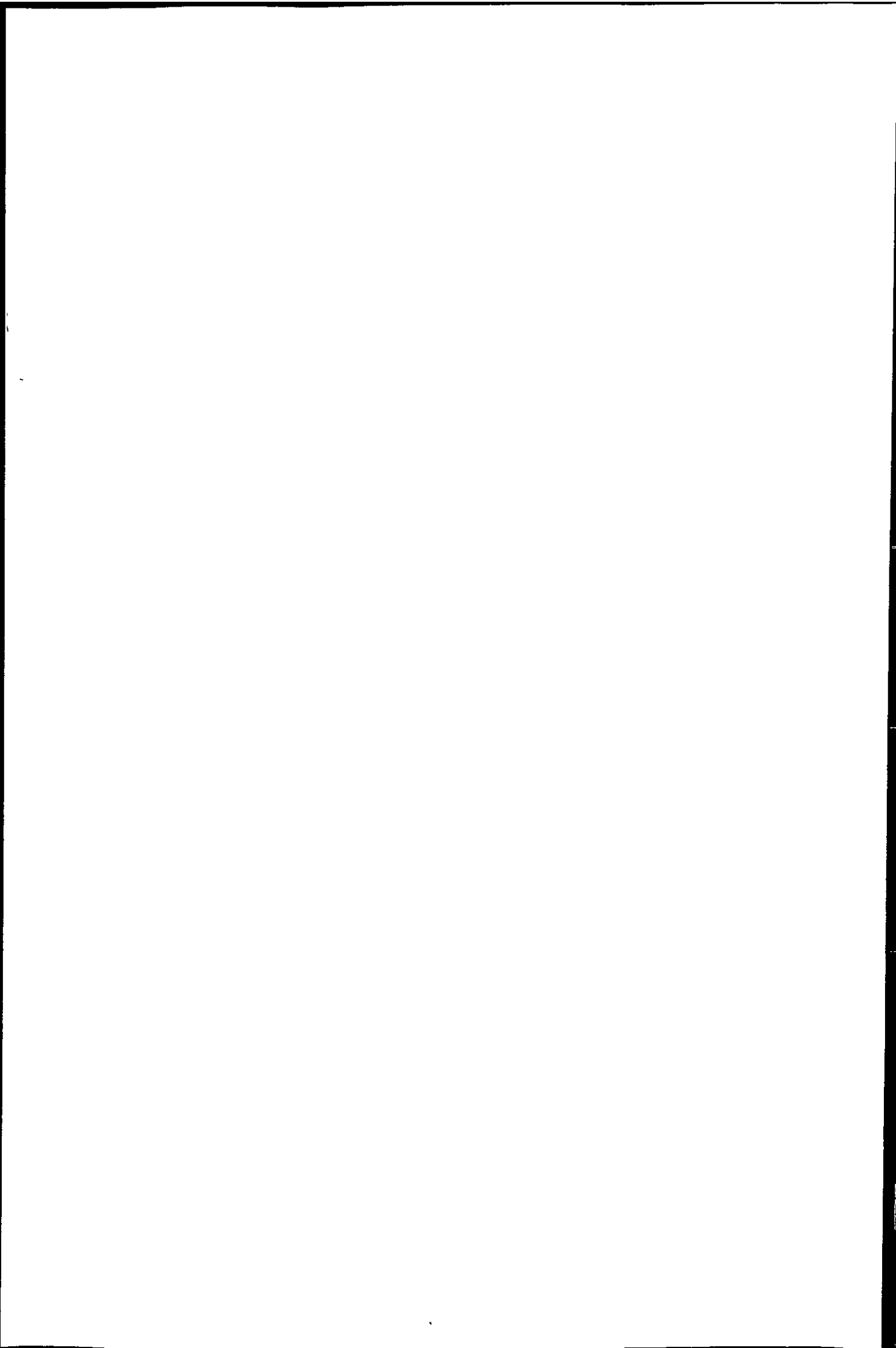
Instrumentation: the questionnaire is a novel instrument, and part of the purpose of the study was to test it.

Selection: the mathematics students may have been sufficiently different from the engineering students to begin with not to act adequately as a control group. On the other hand, since the aim of the study is to demonstrate to mathematicians who teach mathematics to engineering students that engineering students do have different ideas about mathematics, then mathematics students do act as an appropriate comparison group.

Experimental mortality: in both groups there were fewer respondents at the end of the first year than at the beginning. However it is not claimed that the two groups were randomly selected: in fact it is probably significant that the students do differ right from the start of their courses.

4.5.2 Threats to external validity

Failure to describe independent variables explicitly: the independent variables to be investigated were the mode and depth of students' concept images of mathematical target concepts, in the context of engineering mathematics applications.



Lack of representativeness of available and target populations: the students were sampled by availability. The results of the questionnaire indicate that the findings may not be applicable to people who graduated from "old" universities some 20 years ago. A further stage of the study would be to reapply the revised questionnaire across the current population of students in Plymouth and at other universities.

Hawthorne effect: the questionnaire impinged very slightly on the students' experience of their studies. It is unlikely that they were aware of being a group under study, or that any such awareness affected their attitude to their studies. Only the final year engineering students expressed curiosity about the outcomes of the research to the researcher.

Inadequate operationalizing of dependent variables: there is always a question as to how near responses to a questionnaire come to the actions of a respondent in a "live" context. However the questions were designed to look like the context of engineering mathematics problems, so that the images consulted by the students in answering the questionnaire would be like those consulted in tackling such problems.

Sensitization to experimental conditions: the students were not told that they would be retested, and the tests were far enough apart that the students should not have remembered their previous responses unless those responses were particularly vivid to them for some reason.

Interaction effects of extraneous factors and experimental treatments: such extraneous factors might include the staff of one school becoming involved in an IT initiative and changing their teaching style accordingly, industrial action, external examinations, epidemics, or extreme weather conditions. As far as we could ascertain, no events of this sort took place during the period of the experiment.

Overall it was felt that the threats to external validity were probably stronger than the threats to internal validity, and particularly that should an appropriate opportunity arise, once the questionnaire had been tested internally, groups of students from other universities similar to and different from Plymouth should be tested.

4.6 Concept images and cognitive styles

In designing the questionnaire, the aim was to try to elicit from the respondents the way they think of certain mathematical ideas to themselves: their *concept images*. The concept image differs from a formal definition, in that it is that which is constructed by the individual and may be held in the form of words, pictures, a set of rules or procedures, or any other form in which an idea can be held. For example, if one were simply to ask "What is differentiation?", the respondent may treat this as a test of memory, and try to recall the learnt definition. It is proposed by Vinner (1991) that the concept definition is rarely consulted when a concept is evoked in a cognitive process. Some indirect way of accessing the concept image must be used.

As a person learns more about a subject, and practises the associated skills, their concept images change and develop, and their *cognitive skills* mature. In particular, the relationships between concepts become richer and differently organised, and the concepts are understood at a deeper level.

The concept image may be held in a variety of modes, which may depend on the *cognitive style* of the respondent. It has been suggested by many researchers from Galton (1883) onwards that individuals vary in the extent to which they visualise or verbalise in their thinking. Tall (1991, p6) recounts his discovery of different modes of thinking: "It was some considerable time later that the realization dawned that not all students shared the geometric point of view."

4.7 *Concept definition and concept image*

A concept name when seen or heard is a stimulus to our memory. Something is evoked by the concept name in our memory. Usually, it is not the concept definition, even in the case where the concept does have a definition. It is what we call a "concept image".

(Vinner, p68)

Vinner equates the acquisition of a concept with the formation of a concept image for it.

"To understand, so we believe, means to have a concept image: Certain meaning should be associated with the words." (1991, p69) Everyday concepts such as cat, blue, table, have meaning without being easy to define. "The concept image is something non-verbal associated in our mind with the concept name. It can be a visual representation... a collection of impressions or experiences. [These representations] can be translated into verbal forms. But it is important to remember that these verbal forms were not the first things evoked in our memories." (1991, p68)

In the semiological terms of chapter 2, the concept name here corresponds to the sign, the definition to the significatum or referent, that is the institutional meaning, and the concept image to the concept or reference, the personal meaning.

The concept image is open to modification through experience, particularly when conflict occurs and as a wider variety of cases is encountered. Vinner explored students' concept images of "function", "tangent" and "limit", by asking them, for example, to draw the tangent to a curve at a non-typical point. It became clear that there was a conflict between the concept image and the concept definition which students had not attempted to resolve, since the definition had not been consulted in answering the questions. He proposes two didactic rules:

- (1) to avoid unnecessary conflicts with students.
- (2) to initiate cognitive conflicts with students when these conflicts are necessary to enhance the students to a higher intellectual stage. (This should be done only when the chance of reaching a higher intellectual stage is reasonably high.)

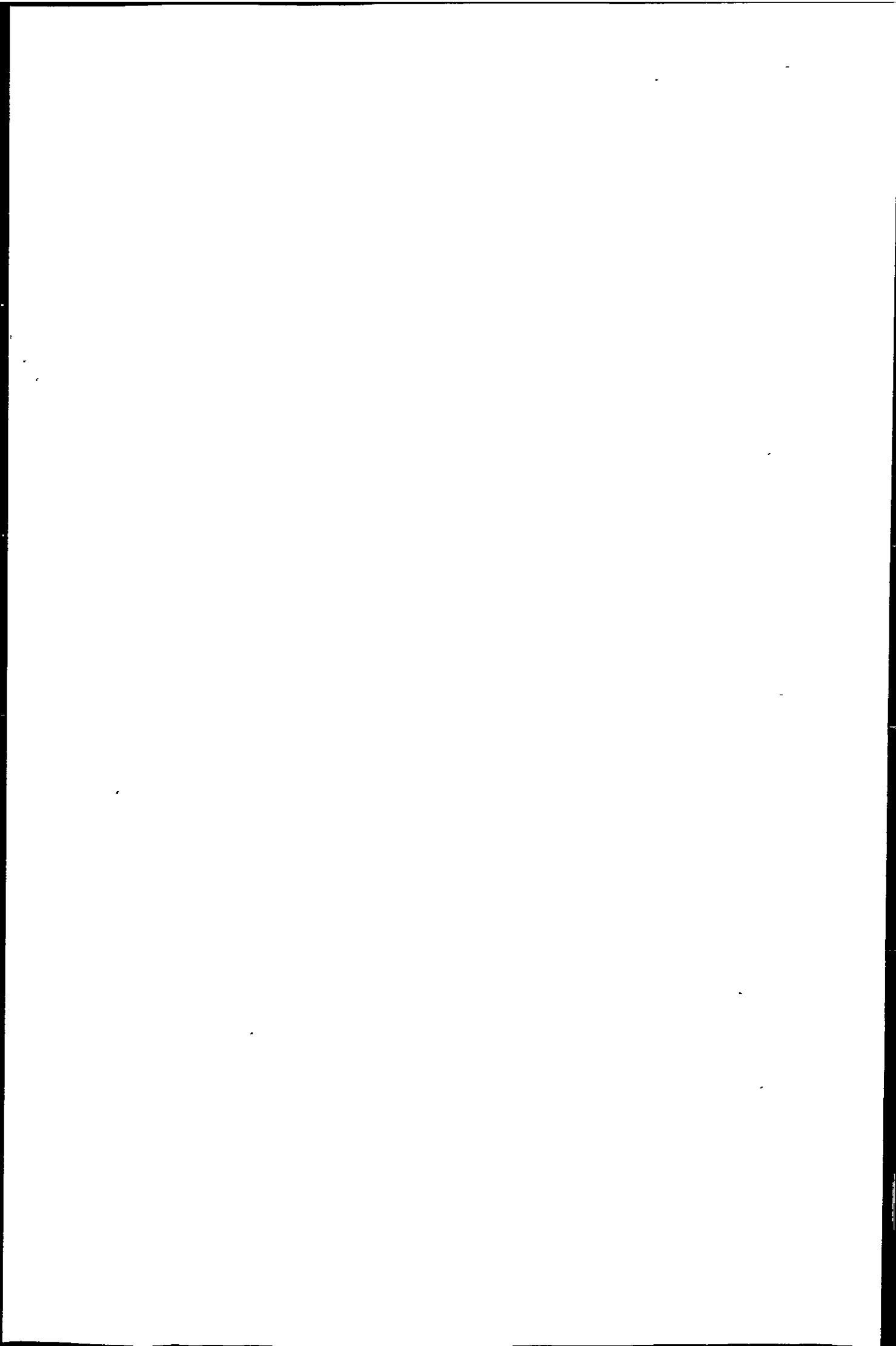
These rules presuppose that the teacher (or lecturer) is aware of (a) the concept image(s) held by the students, and (b) that this may not coincide with his or her own. He states "if ... the students are not candidates for higher mathematics then it is better to avoid the conflicts".

An example of an unnecessary conflict was given in the context of children learning multiplication. (Graham D, 1997) They had learnt that three boxes of two objects, making six objects, could be written as $3(2) \rightarrow 6$. Their teacher then went on to rewrite this as $3 \times 2 \rightarrow 6$. One day the teacher was absent and the school head took the class. The head developed $3(2) \rightarrow 6$ as $2 \times 3 \rightarrow 6$ (that is, two objects times three boxes make six objects). The children were devastated. This conflict may be found examined in Anghileri (1989).

4.8 Cognitive skill level discrimination: depth of representation

4.8.1 Encapsulation

The reasoning behind the design of the questionnaire was highly influenced by Royer et al (1993): Techniques and procedures for assessing cognitive skills. In this paper the development of a cognitive skill is described after Anderson (1982) as taking place in three stages: a declarative stage, a knowledge compilation stage and a procedural stage. A strong distinction is made between the behaviourist and cognitivist conceptions of a cognitive skill. For the behaviourist, a cognitive skill is a packet of information which may be acquired and demonstrated by performing a specified task. In contrast, the cognitivist view sees a cognitive skill as a capability which undergoes qualitative and quantitative change during its development. The novice may be able to demonstrate the skill, but the way in which it is performed is quantitatively and qualitatively different from the way in which an expert would work.



The *declarative stage* is the state of knowledge of a novice who can answer questions about the skill, and demonstrate it slowly, having to think consciously about each step. The novice uses fail-safe strategies, and transferable problem-solving techniques. A novice may be able to perform to a high enough degree of accuracy to fulfil a mastery learning criterion test, but the performance is inefficient, low in fluency, and requires a high level of concentration. It is proposed that the novice stores declarative knowledge in relatively small chunks, which are retrieved from memory and interpreted to carry out the task.

In the *knowledge compilation stage*, it is suggested that these chunks or steps are first collapsed into a single larger step, where one step leads into the next without time being needed for recall between each. This speeds up the performance and reduces the load on the memory for its performance. Secondly, these steps are proceduralised: their performance becomes automatic. The skill takes on the nature of a stimulus-response performance, without conscious processing being needed.

The *procedural stage* takes the automatically performed procedures and selects between good and poor rules: good rules are strengthened and poor ones weaken and fade away. The performance becomes fast, automatic, and efficient. This is the condition of the expert, who performs apparently without effort.

The development has taken place in two dimensions: the relationships between the steps of the skill and the structure of the skill have been internalised, and the whole skill has thus been consolidated so that the declarative or verbalised knowledge is squeezed out. The expert can imagine relationships between inputs and outputs to the system as a whole, without having to make a conscious effort to follow the transformations within the process.

The notion of encapsulation is expressed in different ways by different authors to express the way knowledge is chunked as an expertise is developed: for example, Tall's *procept* (Tall, 1995 specifically refers to procepts as encapsulation), Anderson's (1982) *knowledge compilation*, and Schoenfeld's (1985) *heuristics and algorithms*. Morgan (1990) found that engineering students were competent at routine (algorithmic) mathematics but weak at non-routine (heuristic) problem-solving.

4.8.2 Interconnectedness

Another idea which is mooted in Royer et al is the change in knowledge organisation and structure, and the depth of problem representation which occur as a skill is mastered. A novice holds knowledge corresponding to a skill as a set of unrelated or loosely related facts (at the information level). (Disessa, 1987, describes the nature of naive beliefs in physics as scattered islands or a patchwork, rather than a coherent theory.) As the skill is developed, these become highly interrelated (held as knowledge), and strong and weak relationships are differentiated. An expert would detect deep similarities between problems and reject superficial ones, while a novice would be distracted by the surface structure. These changes in particular would, it was hoped, be evidenced in the differences in responses between students at different stages in their learning careers.

4.8.3 Personal meaning

As a mathematical concept matures, it not only gains in interconnectedness with other mathematical concepts, but also with non-mathematical ideas: that is it becomes part of one's toolkit for interpreting reality, in Hirst's terms (Hirst, 1972). The mathematical concept may be used in making sense of one's world. This is referred to by Teslow (in Wilson et al, 1993) as sensemaking.

At the same time the real world interpretation makes sense of the mathematical concept.
(Schliemann, 1985, Lave, 1996)

Mathematics as sensemaking would predict the increasing use of the descriptions of differentiation and integration.

4.8.4 Skills and concepts

Royer et al deal both with knowledge as exemplified by skills, and as embodied in concepts. Skills are relatively accessible by asking people to perform tasks which employ the skills and observing (and even measuring) the outcomes. Concepts on the other hand remain very private and may only be deduced from the ways their use informs an individual's interpretation of the world. If asked directly what so-and-so means the individual may (a) lie, either to try to please the questioner, or because the private world is being invaded, (b) produce the definition as the learned response to that question, while not consulting the definition in normal use of the concept, (c) be unable to verbalise the concept, through lack of appropriate vocabulary or because the concept is held in non-verbal form, or (d) answer accurately. It is difficult to tell from a response what type it is.

Concepts are the objects on which skills operate, and skills may themselves become collapsed into concepts, just as a mathematical function may become the object of another operation such as integration or the variable in a differential equation. This duality, and the mental versatility and tolerance for ambiguity it implies is explored in the book *Advanced Mathematical Thinking* (Tall, ed, 1991) which is referred to in several places in this chapter and chapter 2.

4.9 Cognitive style discrimination: mode of representation

Three modes of representation were thought appropriate: visual, verbal and algebraic. The verbal and visual representations correspond to the axes of a well-known research instrument, the Verbaliser-Visualiser Questionnaire (VVQ). (See, for example, Kirby et al, 1988)

4.9.1 Visual imagery in mathematics

The usefulness of visual imagery in mathematics is unresolved. Tall (1991, p18) suggests that:

Visual ideas without links to the sequential processes of computation and proof are insights which lack mathematical fulfilment. On the other hand, logical sequential processes without a vision of the total picture, are blinkered and limiting. It is therefore a worthy goal to seek the fruitful interaction of these very different modes of thought.

We use the metaphor of vision to describe a holistic appreciation: "seeing the big picture", "a snapshot view", or an "overview" of a situation. Tall's vocabulary in the above quotation is very visual: "insights", "vision", "total picture", "blinkered", and even "seek" represent a visual metaphor, that of the different modes of representation being integrated in a holistic notion (view) of mathematics.

Poincaré (cited in Tall, ed, 1991, Chapter 1) suggested that there are two types of mathematical minds: one kind preoccupied by logic, the other guided by intuition. He observes the same differences in his students:

Some prefer to treat their problems 'by analysis', others 'by geometry'. The first are incapable of seeing in space, the others are quickly tired of long calculations and become perplexed.

This distinction would be interpreted in the Jungian framework of the Myers-Briggs analysis as the difference between the strictly logical Thinking style and the more intuitive Feeling style. (see MacCaulley, 1976, and chapter 10 of this thesis)

At the same time, Poincaré recognises that both types of thinking are needed in solving problems: in the *Foundations of Science* (1924), he describes how mathematical creativity for him consists often of a first period of conscious effort, followed by an unconscious stage in which the intuition works, then a second conscious time of work in which the intuitive insight is cleaned, polished, verified and made presentable.

On numerical applications of visual imagery, Galton suggested that about 5% of people have a mental image of a number line. Ernest (1983) found that 65% of teacher training college staff had an internalised image of a number line, but few had non-straight line patterns. These few, he speculated, were Galton's 5% whose internal number lines were spontaneously generated: the others had been taught to use such imagery by the use of physical number lines in teaching.

Presmeg (1986) found that visualisers were under-represented among high mathematical achievers, and suggested that the vividness and particularity of a visual image affected students' ability to generalise mathematically. She quotes Galton as saying in 1880

An over-readiness to perceive clear mental pictures is antagonistic to the acquirement of habits of highly generalised and abstract thought and if the faculty of producing them was ever possessed by men who think hard, it is apt to be lost by disuse. The highest minds are probably those in which it is not lost, but subordinated, and is ready for use on suitable occasions.

Thompson (1990) discusses visual imagery and the ways that individuals differ in the extent to which they use such imagery. Even when learner and teacher both visualise vividly, if they have different images for a given concept then the teacher's use of visual imagery in teaching may not help the learner. He raises the question whether strong visualisers or verbalisers should be encouraged to stick to their strengths or whether all students should be encouraged to become versatile in their use of thinking style.

4.9.2 Need for versatility

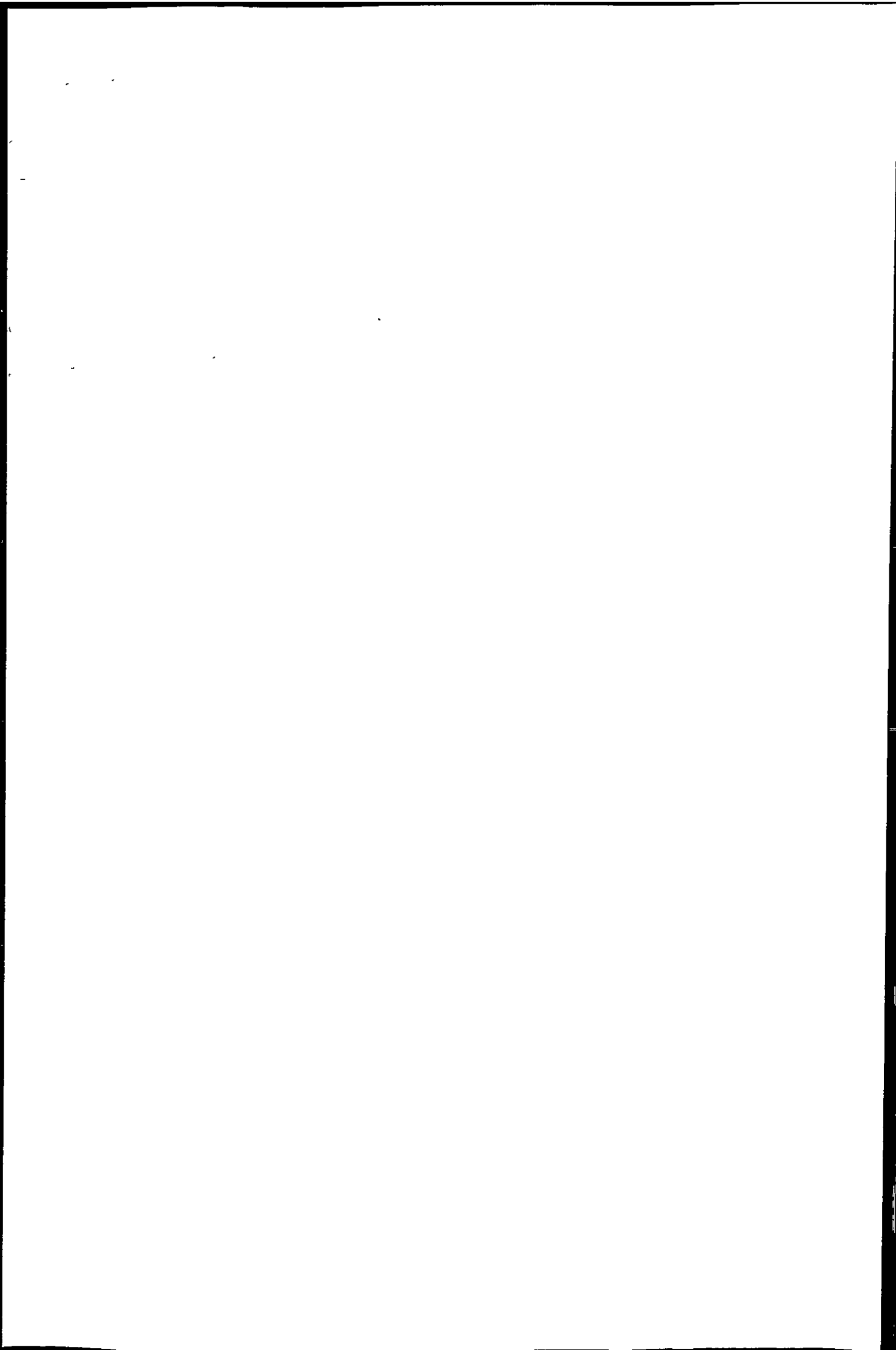
Several writers suggest that it is necessary under certain circumstances to challenge learners' preferred styles of thinking. The circumstances under which students' ways of thinking should be challenged may be regarded as similar to Vinner's criteria for challenging students' concept images.

Perry (1981, 1988) finds that cognitive development through tertiary education involves a shift in ways of thinking, often initiated through conflict, and Kolb (1981) also emphasises the need to work in ways which may run against our preferences in order to learn. Laurillard (1979) feels that students are versatile in their styles of learning, and Hirst's view of a liberal education (Hirst, 1972) is one in which the student learns to think like a mathematician, a historian, a moralist, etc., in order to gain a rounded perspective of knowledge.

4.9.3 Engineering and mathematics students

It was suggested by Crowther (1997b) that engineering students see themselves as visual people, and the engineering and mathematics students in the mathematical modelling exercise seemed to differ in the use of visual material, in the form of sketches, as a means of communication. The engineering students made sketches, for example of the expected shape of the graph of height against time: the mathematics students also passed around pieces of paper, but theirs had equations written on them.

It was thought probable that the engineering students would show a preference for pictorial or diagrammatic representations over algebraic ones: it would be interesting to see whether mathematics students would also show the same preferences, or whether either would prefer word descriptions and explanations.



4.10 Conclusions

The aspects of the concept image which the questionnaire will be designed to investigate will be

- preferred modes of representation (verbal, visual or algebraic);
- depth of representation (novice versus expert, through encapsulation, interconnectedness and personal meaning);
- preferred representation in abstract “mathematical” context and applied “mechanics” context.

The survey will investigate

- evolution of the concept image with experience
- differences between engineers’ and mathematicians’ concept images.

5. The questionnaire: the practical process of design.

5.1 Introduction

The questionnaire was designed in stages, with each stage being tested. As the design became elaborated, the testing became more extensive. The concepts described in the previous chapter, that is *depth* and *mode* of representation were taken into account in designing the questions and the options for response. In this chapter the specific design decisions are outlined and conclusions on the process of questionnaire design and administration are drawn.

5.2 Design of pilot questions

The first question to be written (Figure 5-1) involved a differential equation (DE), with four responses being suggested, each of which bore a similarity to the target at a different level. These were: appearance, method of approach (separable variables), simple function of x on RHS, exponential result. The aim was to test the respondents' depth of problem representation. Respondents were asked to choose the most similar option to the target. The question was tested on a number of colleagues and students, and as all options were chosen, all were retained as feasible responses.

It was realised that by asking respondents to put the options in order rather than to choose just one the question would yield four data points with three degrees of freedom rather than just one. Some ambiguities in the question were also eliminated by altering its wording and layout (Figure 5-2).

The first mechanics-type question to be written (Figure 5-3) concerned a mass bouncing in damped harmonic motion. The aim here was to try to elicit the mode in which respondents represented the motion to themselves: visual, verbal or algebraic; whether an unfamiliar graphical representation would be acceptable, simply because it was visual,

given that the engineering students in the modelling exercise had so used sketches to communicate with one another; and how familiarity with the analysis of the problem would change the acceptability of the algebraic responses as the engineering course proceeded.

$$\frac{dy}{dx} = e^x$$

$$y^2 \frac{dy}{dx} = e^x$$

$$\frac{dy}{dx} = e^y$$

$$\frac{dy}{dx} = mx + c$$

$$\frac{dy}{dx} = my$$

Which of the equations in the right-hand column is most like the differential equation in the box?

Why did you choose that one?

Figure 5-1: Original DE question

$$\frac{dy}{dx} = e^x$$

Arrange the equations below according to how like you think they are to the differential equation above.

a) $y^2 \frac{dy}{dx} = e^x$

b) $\frac{dy}{dx} = e^y$

c) $\frac{dy}{dx} = mx + c$

d) $\frac{dy}{dx} = my$

Figure 5-2: DE question from pilot questionnaire

A mass suspended from a spring and dashpot is pulled down from its equilibrium position and released. Which of the following best describes to you what happens next?

Why did you choose that one?

Damped harmonic response

$$\ddot{y} + k\dot{y} + \omega^2 y = 0$$

$$y = Ae^{-\lambda t} \cos \omega t$$

The mass bounces up and down, going less far each time, until it settles back to its original position.

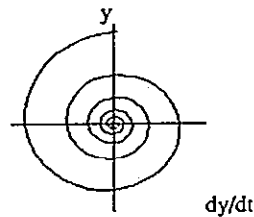
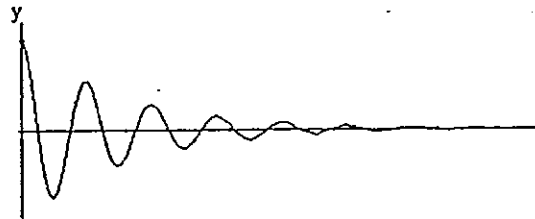
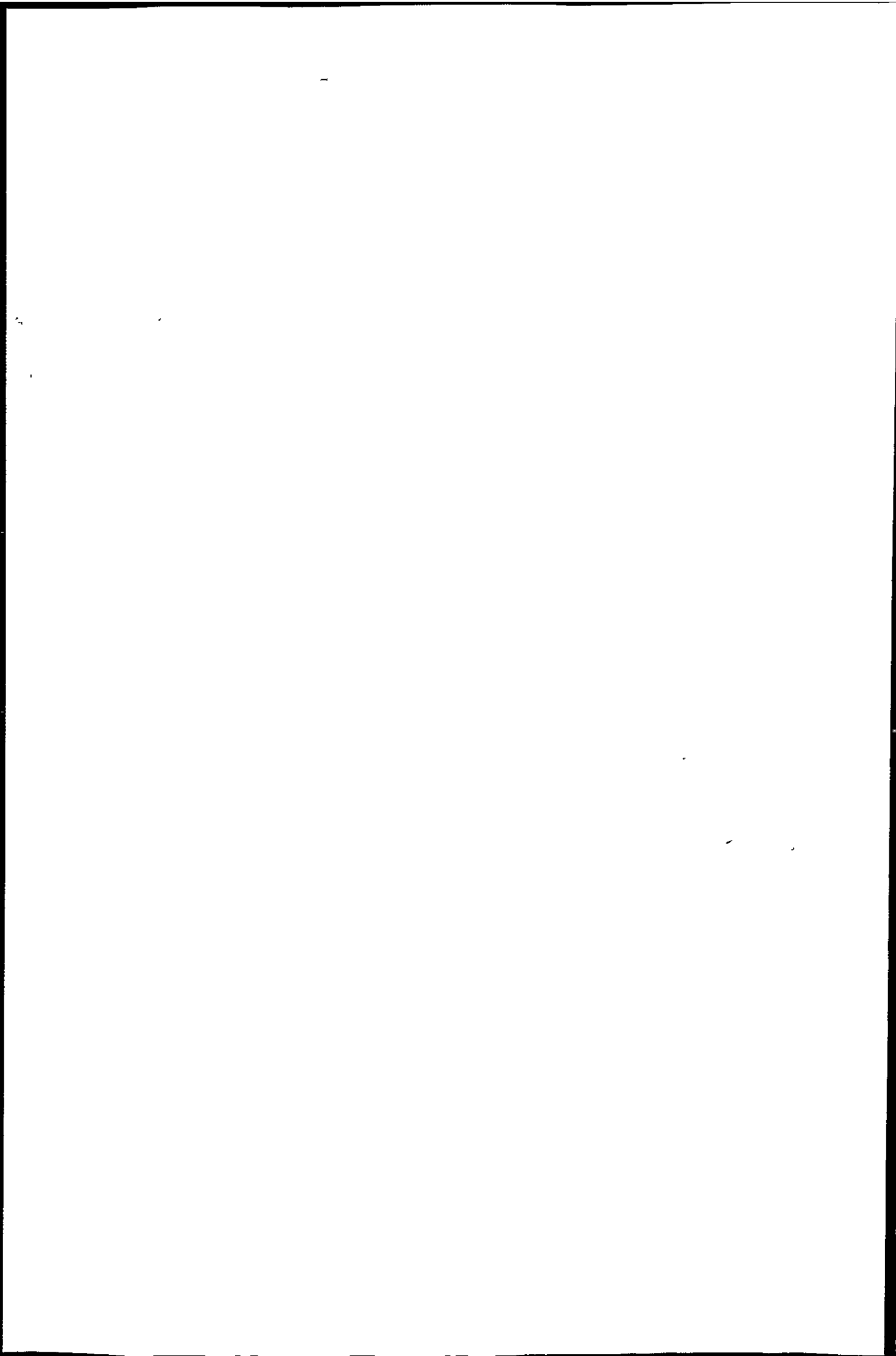


Figure 5-3: Original dynamics question

6 possible ways of representing the motion were offered: 2 diagrammatic, 2 algebraic and 2 verbal: and respondents were asked to rank them in order of preference. This gave the advantage of increasing the number of data items per question, from one to a possible maximum of five (because when the first five have been chosen the last one is fixed) where all six options were put in order. As it will be seen, not all respondents did always manage to put all the options in order.

This question was also tested on colleagues. Again, some constructive comments led to modifications in the layout and wording of the question and the options, but overall the concept of the question seemed successful, and it was adopted.



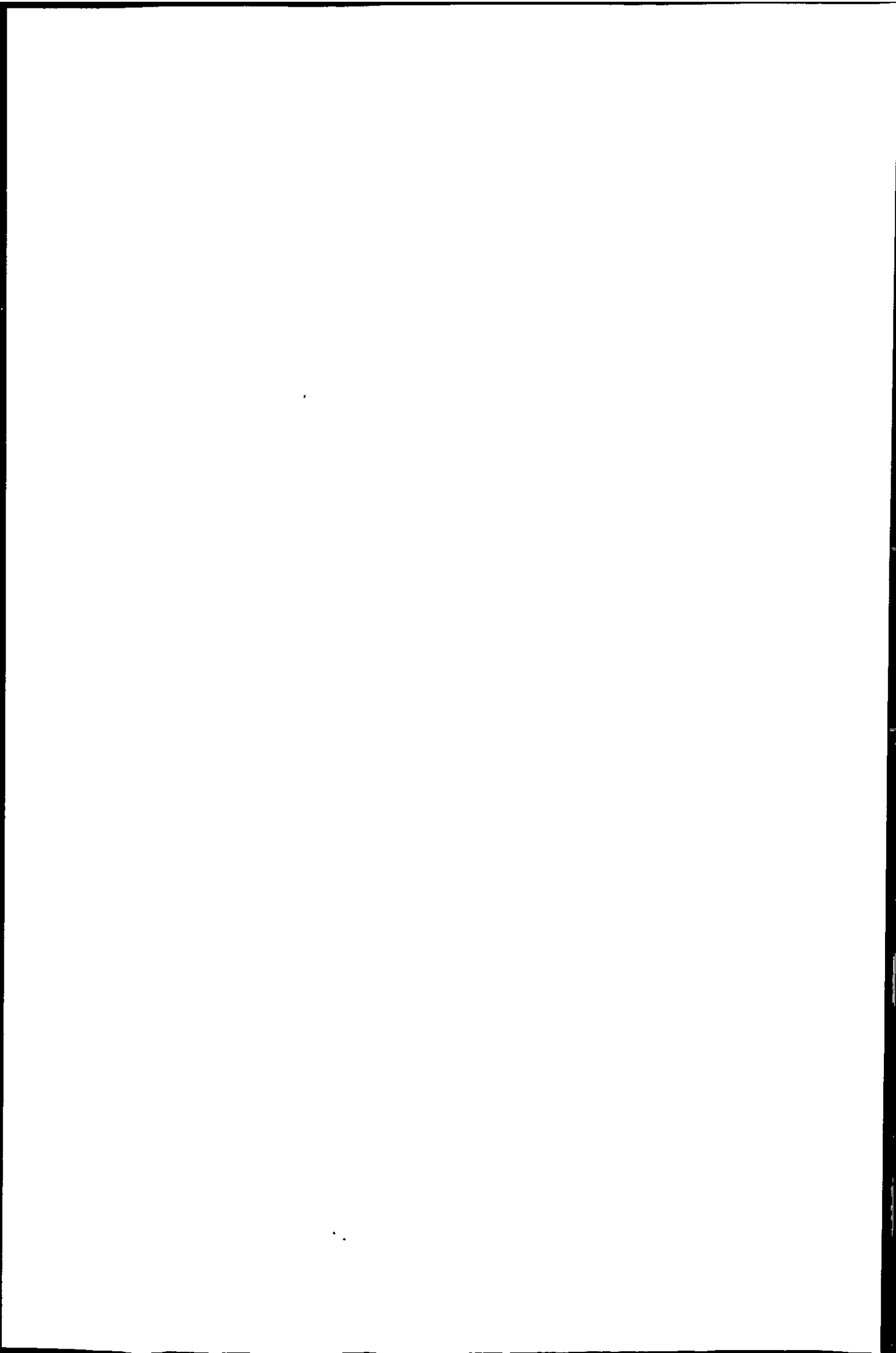
5.3 Pilot questionnaire

Based on these questions, four others were designed, to give a total of three on “mathematical” topics: differential equations (DEs), integration, and differentiation, and three on “mechanics” topics: beam bending (a statics standard case, which should be met during the first year of a mechanical engineering degree), damped oscillation (a standard case in dynamics, normally met after the first year in a mechanical engineering degree), and acceleration of a pinball (a non-standard case in dynamics, which should however be a familiar experience to most students).

Care was taken in framing the questions to try to avoid language which might suggest a “correct” answer mode: e.g. visual metaphor, (“picture”, “envisage”, “show”) which would perhaps prejudice students towards a pictorial representation; “describe” or “tell” which suggest a verbal response, etc. This led to an impoverished vocabulary for setting the questions and it was a challenge to try to find ways of setting the questions to minimise repetitiveness but at the same time to ask respondents to carry out a very similar task each time.

In setting out the responses, an attempt was made to put the options suspected to be most popular away from middle positions which are those most apt to draw random choices.

Pictorial and diagrammatic representations and word descriptions were used whenever they could be used without violating the sense of the question, to increase the apparent “friendliness” and accessibility to people who had not studied the subject, for example, students in the early years of their course, and to accommodate those with a strong preference for that mode of response.



The rubric on the front page refers to the question posed on p46 of "Know your own IQ" (Eysenck, 1962), which runs as follows:

Underline the odd-man-out.

house igloo bungalow office hut

Ans: Office (People don't live in an office.)

Although there is officially a correct answer to the question, each of the others could defensibly be chosen, depending on the particular classification system privately adopted by the respondent.

The questionnaire was laid out as a booklet of eight pages. The cover which contained the personal details could be detached completely and stored separately from the questions and responses.

The information requested about the student was:

Name: so that responses from the same student at the beginning and end of the first year could be matched, without warning the students at the start of their first year that a retest was planned.

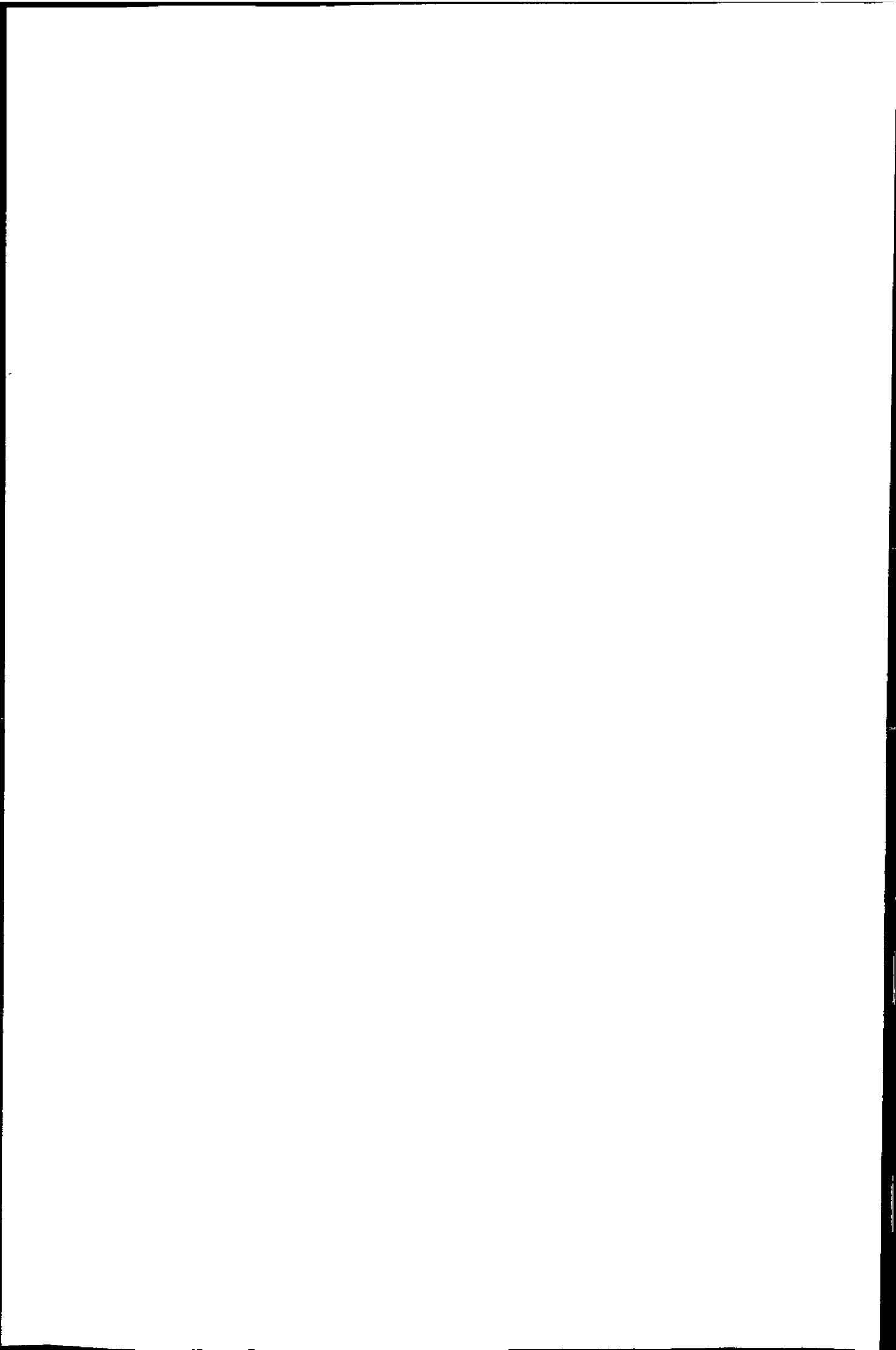
Course: to distinguish between engineering and mathematics students.

Year: to distinguish between first and final year students

Date: to distinguish between the papers from the start of the first year and those from the end.

The questions were printed so that only one was visible at a time, to prevent visual comparisons between one set of options and the next, so that the response to one question was not coloured by expectations from the pattern of responses to the last.

The pilot questionnaire was tested on various groups of students: second year Computer Systems and Networks (CSN) degree students ($n=4$), final year mechanical engineering students ($n=13$), Teaching Company Associates (TCAs) (engineering graduates working



in local firms while studying for a higher degree at the University) (n=12), mathematics degree final year students (n=5), and anybody else who was willing to try it.

These students were selected so as not to contaminate the sample for the main test, but to be similar groups to the ones it was hoped to test.

5.3.1 Administration methods

Three methods of administration were used:

- a) in class group, or at a gathering for an end-of-year photograph: the questionnaire was given and collected in presence of experimenter. The response rate was high, as may be expected, but the responses were poor in comments.
- b) by mailshot. The response rate was lower, but those returned were rich in comments. The questionnaire had stated it was a pilot study and explicitly asked for comments. As only well motivated students replied, they were less typical than the first group.
- c) distributed to TCAs via the Teaching Company Centre. The response rate was high, and the responses were richer in comments than the class groups.

From the responses to the pilot questionnaire, it was concluded that the respondents had understood most of the questions, and that there were no other options which respondents had seen as important but missing.

5.4 *Main questionnaire*

As a result of the responses to the pilot questionnaire, two changes were made. The questionnaire is included as Appendix B of this thesis.

Two options were added to the differential equation question so that it had the same number as the others (i.e. six). This then meant that a first choice option was scored 5, as in the other questions.

The questions were re-ordered to put the differential equation question last, as some respondents had indicated that the question had put them off answering the rest. The questions now alternate mechanics/mathematics, and increase in sophistication of the concepts addressed.

It was planned to administer the questionnaire to a matrix of groups:

	Mechanical engineering	Mathematics
1a: start of 1st year		
1b: end of 1st year		
3: final year		
4: staff		

Table 5-1: Proposed questionnaire distribution

Ideally, one would follow a cohort of mathematics and engineering students through their courses, seeing how each individual evolved in his or her views over time.

Unfortunately the length of a degree course is as much as, or even more than the length of a PhD project, especially given that engineering students often spend a year working on a project in industry, so this was not practical. It was possible to test the same group at the beginning and the end of their first year, and other groups were used as a proxy for the final year.

It was hoped to test the final year students in their final semester, but the course structure made access to them difficult, and so it was decided that the best compromise was to test as late as possible before Christmas in the final year.

Two other groups were also tested: Teaching Company Associates (TCAs), who are graduates (principally engineering graduates) now working in industry but following a

MSc course in Management of Technology, and a small number of second year mathematics students who were studying a course module with a majority of third year mathematics students. All of this latter group had also been surveyed at the start of their first year, but not all at the end of their first year.

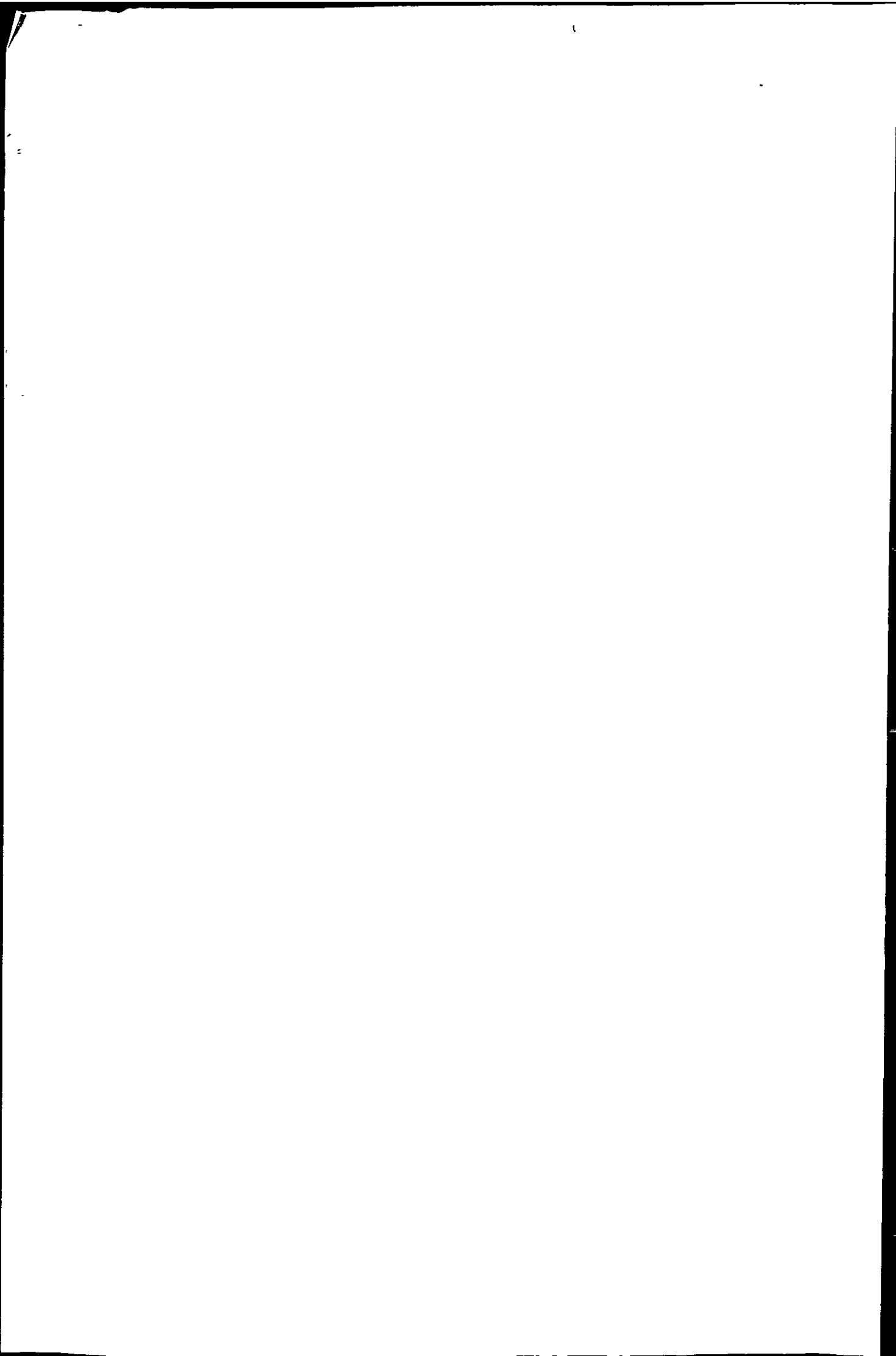
5.4.1 Administration methods

Three administration methods were again used: for first year students of both mathematics and engineering, a class group was tested at the beginning and end of the academic year, and for the final year students both mailshot and class group methods were used. It would have been ideal to use the same administration method for all groups, so as not to select for the most highly motivated group members by using a mailshot. However the TCAs do not generally meet in class groups, they were again contacted via the Teaching Company Centre.

Difficulties were encountered in finding an appropriate opportunity for administering the questionnaire to a class group of final year students: the modular system meant the whole year group was together for very few sessions, and the workload of the final year students was felt by their teaching staff to be too high to allow the researcher to administer the questionnaire during one of these sessions.

As the response to a mailshot was poor, some teaching staff responsible for modules taken by a subset of the mathematics and the mechanical engineering final year students kindly gave access to their class groups.

Administering the questionnaire to a class group takes about 20 minutes, including an explanation of what it is about. The introduction emphasises to the respondents that a) while they may feel that one option is obviously best, others may find a different option obviously best: b) it is helpful to teachers to discover that students don't all think the same way they do.



5.4.2 Confidentiality

Students were asked to write their names on the cover, which was removed and stored separately from the responses. The booklets were numbered as they were returned in an arbitrary, non-alphabetical, order. A code was used in which the last two digits referred to an individual within a group, and the first one or two denoted the group. Where a group of students was tested twice, the students responding twice were assigned the same pair of final digits at each session, so questionnaires from the same respondent could be compared.

Again student names are on the cover which can be removed and stored separately. For students seen more than once, the names need to be traceable. The 3-digit code then has the appropriate 1st digit for the series with the last 2 digits the same on both occasions for the student.

5.4.3 Design faults in questionnaire

Larger numbers of responses were now involved, and a design fault in the questionnaire became evident: it was difficult to translate a list of letters in order into a numerical ranking. In any future studies, it is recommended that the options be given Likert-style scales for marking, with points ranging from "exactly matches my idea" to "not remotely like my idea". This would have the advantage of evoking scores for the unpopular options in Question 6, for example, where many respondents chose one response only.

Other design faults also revealed themselves:

- a) The two new responses to DE question may be directly derived from the target equation, leading students to make comments that these are "correct" and the others are wrong.

- b) The set of options to the pinball question is weak, as may be seen from the table in section 5.6. The depth of representation is not addressed explicitly and the choice of modes of expression is not clear. The question could be improved by including, for example, a velocity-time or velocity-displacement graph, and a velocity-time or velocity-displacement expression.
- c) The beam bending question may be made more accessible to mathematics students by including an expression making it more explicitly a boundary value problem in differential equations.
- d) It may also be better to create diagrams in EXCEL which produces drawing objects, which print more clearly than bitmapped images created in DERIVE.

5.4.4 Other comments

Most of the class groups tested gave few comments, as they had little time for reflection.

It was hoped that it would be possible to compare the overall responses of the mathematics staff and the engineering staff with those of the students to pick up, in particular, the match or mismatch between mathematics staff and engineering students.

Unfortunately the rate of staff response was very poor, and given that the group size was small, useful comparisons could not be made.

5.5 *Content of individual questions:*

When the context allowed, a variety of modes of representation, verbal, diagrammatic and algebraic, was presented. Options were also chosen to represent different depths of representation, as described below.

5.5.1 Question 1


The question concerns the bending of a beam (Figure 5-4). This is a standard case in the mechanics of solids and is often used as an application to illustrate the use of end conditions in the solution of differential equations. The mechanical analysis forms part of the first year engineering syllabus. No indication of the process of arriving at the shape of the bent beam was given.

- (a) is a statement in words of what may be expected to happen. It is completely non-technical, and would be comprehensible to anyone who has never studied mechanics.
- (b) is the slightly unexpected result of a calculation of the deflected shape. The point under the load may be expected to be that with the greatest deflection: in fact it is the point where the rate of change of slope, that is the curvature, is greatest. It would be recognisable after an engineering analysis of beam bending.
- (c) is a statement in algebraic terms that the bending moment at any point is proportional to the curvature of the beam: the curvature however is expressed as the second derivative of displacement.
- (d) is again a statement in words, but it is the technical abstraction of the case, as it might be described in a beam-bending problem.
- (e) contains two statements in integral form about the changes in shear force and bending moment along the beam. They should also be recognisable after an engineering analysis of the topic.
- (f) is a diagram of the type students would draw in solving the idealised case.

A plank 1.5 m long is placed on two bricks very near its ends. A bar of gold is placed across it 0.5m from one end. Rank the following according to how well they represent this to you.

(a) The beam bends under the weight of the gold bar.

(b) Deflected shape



(c) Bending Moment $M = k \frac{d^2 y}{dx^2}$

(d) A simply supported beam with a point load at one-third span.

(e) Shear Force $S = \int F dx$
Bending moment $M = \int S dx$

(f)

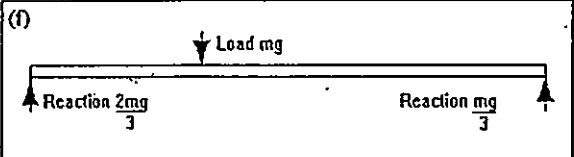


Figure 5-4: Question 1 from main questionnaire

5.5.2 Question 2

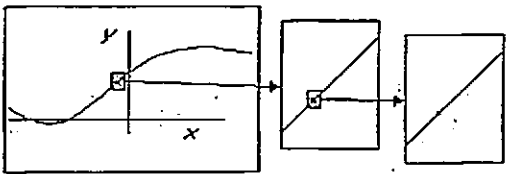
Differentiation is the lowest level of calculus concept, and the first to be taught. It is hoped that if there is a maturing of the concepts it will be shown first in the ideas connected with the derivative.

However the notion of differentiation depends on the philosophically difficult idea of the limit or infinitesimal which teachers themselves may have trouble in dealing with. Hence the idea may be taught as the “slope of the tangent” and this, being a vivid image couched in convincingly mathematical terms, may well be hard to shift.

$$\frac{dy}{dx} = f'(x)$$

All of (a)-(f) can be associated with the statement above. Please arrange them in order of how closely they are linked to it in your mind.

(a)
 $f'(x)$ is the slope of the tangent to a graph of y against x .

(b)


(c)
 dy/dx tells you how quickly something is changing.

(d)

$$f'(x) = \lim_{(x_2 - x_1) \rightarrow 0} \frac{y_2 - y_1}{x_2 - x_1}$$

(e)
 As you zoom in more and more closely to a small section of the curve, it seems to straighten out. The slope of the tiny straight section is dy/dx at that point.

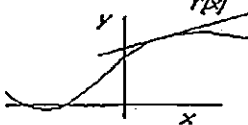
(f)


Figure 5-5: Question 2 from main questionnaire

(a) is the expression in words of a common view of the meaning of the derivative, avoiding the problem of the “vanishingly small”.

(b) is the idea of the derivative as the gradient of a locally straight curve, expressed pictorially. This is Tall’s (1990, cited by Robert and Schwartzenberger, in Tall (ed) p136) archetypal example of a mathematical idea which is meaningful to students at their current state of development yet contains the potential to grow into a fully fledged mathematical concept.

(c) is a statement in words of the practical significance of the derivative. (depth of problem representation)

(d) is the derivative expressed as a limit in algebraic terms: it may be regarded as a mathematical definition.

(e) contains the same idea as (b) but expressed in words.

(f) contains the same idea as (a) but as a diagram.

5.5.3 Question 3

The second "mechanics" question concerned a standard case of a mass bouncing on a linear spring with linear viscous damping. This would be covered by second year mechanics teaching. Moreover, the problem represents a common application of the harmonic form of differential equation, which should be familiar to students of mathematics. The situation is commonly experienced in applications such as vehicle suspensions, so should be accessible to non-engineers and non-mathematicians as well. The phase plane diagram was included as an unusual example of a diagrammatic representation which would not be appealing at a superficial level.

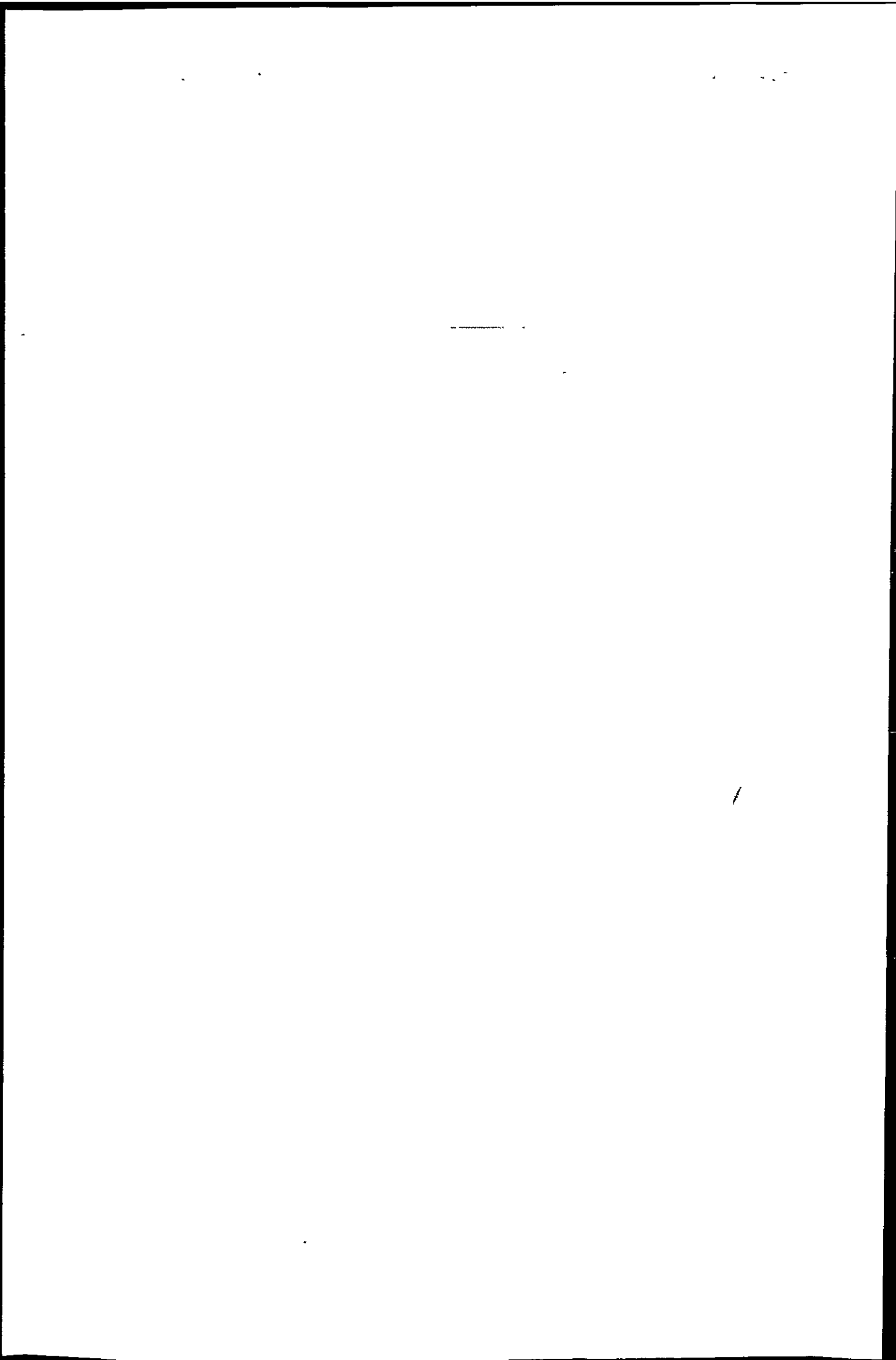
(a) is a non-technical, verbal description.

(b) is a standard description of the motion as a differential equation using dot notation, which would be met in an engineering analysis of the case.

(c) is a phase plane diagram of the motion. It gives a vivid visual depiction of the motion to one who can read it. Phase plane diagrams tend to be more familiar to electrical, control and robotics engineers than mechanical engineering students. (unfamiliar visual representation)

(d) is the solution to the differential equation in (b). It is the algebraic expression of the curve plotted in (f).

(e) is a general technical term for the class of cases of which this is a member.

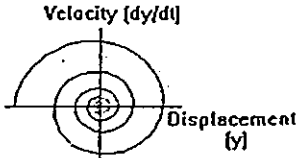


(f) is a graphical depiction of the position of the mass as it varies with time.

A mass suspended from a spring and dashpot is pulled down from its equilibrium position and released. Which of the following do you think best describes what happens next?
Please arrange the answers in order of how well you think they describe the movement of the mass (best first, worst last).

(a) The mass bounces up and down, going less far each time, until it settles back to its original position.

(b) $\ddot{y} + k\dot{y} + \omega^2 y = 0$

(c) 

(d) $y = Ae^{-\lambda t} \cos \omega t$

(e) Damped harmonic response

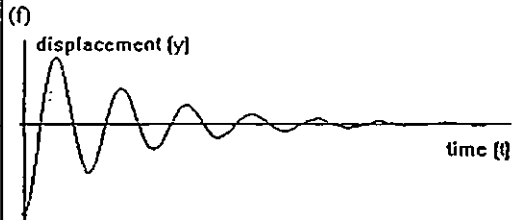
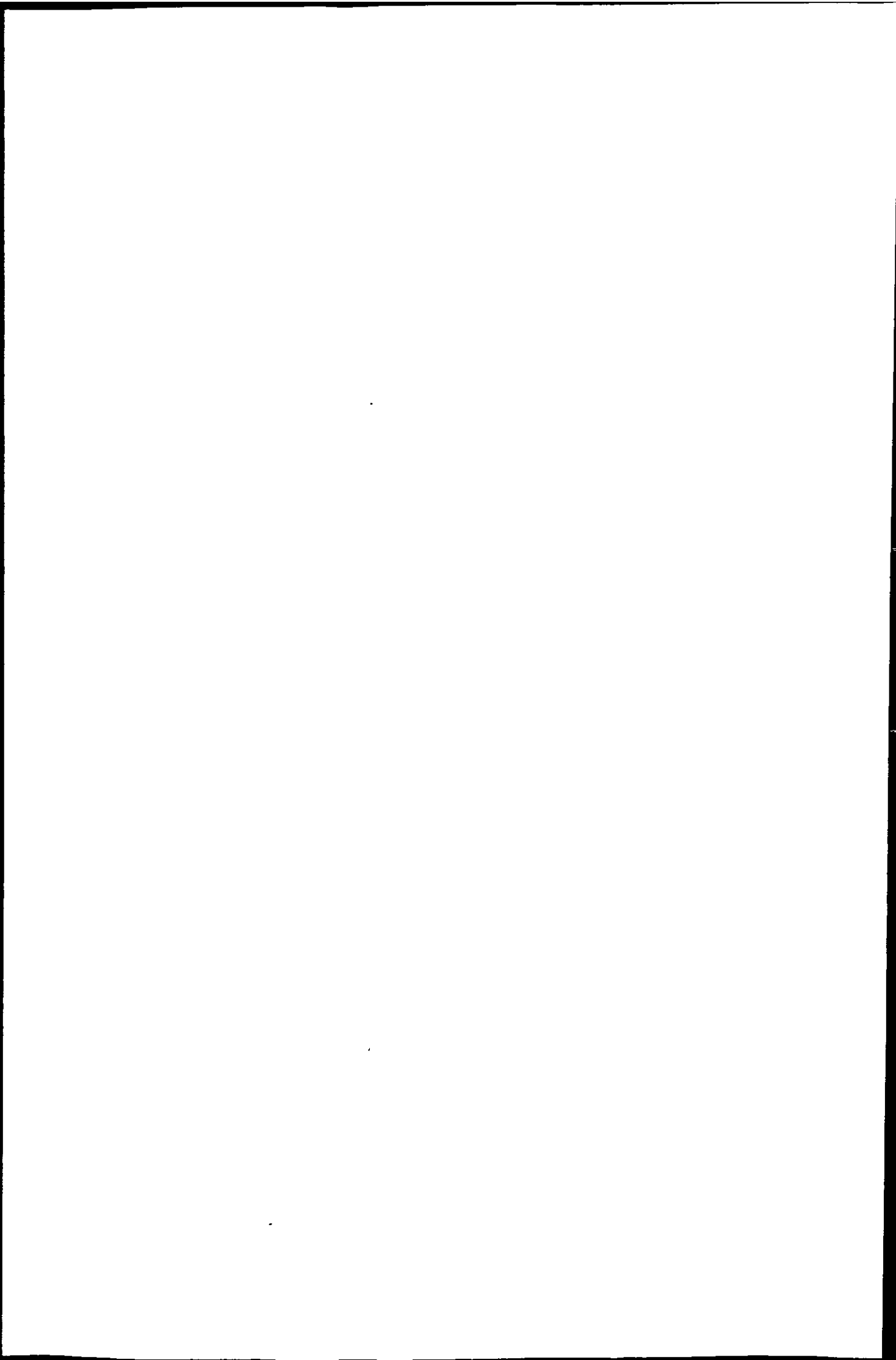
(f) 

Figure 5-6: Question 3 from main questionnaire

5.5.4 Question 4

Integration is a higher level concept in calculus, and usually taught later than differentiation. It was thought that mathematics students may have a more mature image of the concept than engineering students, as it is used more intensively in mathematics than in engineering courses.

Again the notion of the limit may be avoided by use of a graphical interpretation: that of the area under the curve. In itself this is not unproblematic, because the area between the x-axis and the function when its value falls below zero must be interpreted as a negative area, a beast never encountered in nature. However this approach follows



almost inevitably from describing the derivative as the slope of the tangent to the curve, and does have the advantage of being susceptible to a concrete interpretation in the sense that it may be drawn on the black- (or white-) board, and gives rise to word problems of the type “What is the area of a lay-by...?”.

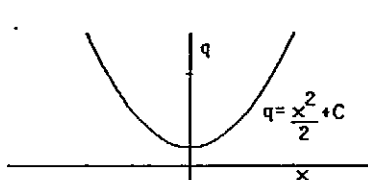
The options were intended to sort the responses at two levels: firstly by giving the choice of different modes of representation, and secondly by giving a variety of depths of representation.

$q = \int x dx$

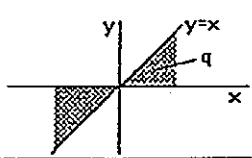
(a)-(f) may all be associated with this statement. Please arrange them in order of how closely they fit the way you think of it.

(a) q is the area under the curve $y=x$.

(b) $q = \frac{x^2}{2} + C$

(c) 

(d) The integral tells you how things build up.

(e) 

(f) $\frac{dq}{dx} = x$

Figure 5-7: Question 4 from main questionnaire

(a) is the expression in words of a common view of the meaning of integration in general. (verbal-visual preference)

(b) is the solution of the integration expressed algebraically. It suggests that integration is a process to be performed.

(c) is the solution in (b) expressed graphically.

(d) is a statement in words of a practical significance of the integral. (depth of problem representation)

(e) is the graphical expression of the same idea as in (a) (verbal-visual preference)

(f) is an algebraic expression of the integral as the inverse of differentiation. (depth of problem representation. Relatedness to differentiation)

5.5.5 Question 5

The question concerns an application of mechanics (dynamics) which, although it is not a standard case, should be a familiar physical situation to most students. It was designed to test whether there was any change in students' views of the physical world with their increased mathematical knowledge. In particular, it would be interesting to see if engineering students interpreted the world in a different applied mathematical way to mathematics students.

(a) is a statement about potential energy as an equation in words.

(b) is another energy statement using a well-known integral expression.

(c) is a statement about energy in algebraic terms, supposed to be equivalent to

$$\frac{1}{2}mv^2 \text{ (that is kinetic energy)} = \frac{1}{2}kx^2 \text{ (that is potential energy stored in the spring).}$$

On further reflection, it became clear that the equation is meaningless, since the instantaneous x in (dx/dt) is not the same as the instantaneous x in $\frac{1}{2}kx^2$. As questionnaire had already been administered it was left to stand.

(d) is a standard statement about the change in momentum, in form similar to (b).

(e) is a platitude in mechanics, expressed algebraically.

(f) is a non-technical statement in words expressing what should be a common experience of pinball machines.

In a pinball game, a ball is fired by releasing a taut spring behind it, propelling the ball out at speed. Arrange the following in order of how well they describe this to you.

(a) Energy stored in spring = $\frac{1}{2}$ Force \times Extension

(b) Energy imparted to ball = $\int F dx$

(c)
$$\frac{1}{2} m \left(\frac{dx}{dt} \right)^2 = \frac{1}{2} k x^2$$

(d) Change in momentum = $\int F dt$

(e)
$$F = ma$$

(f) The further you pull back the spring, the faster the ball will go

Figure 5-8: Question 5 from main questionnaire

5.5.6 Question 6

These options were designed to try to distinguish between respondents' *depths of representation* of a simple differential equation.

(a) is similarly solved by separating variables, and its solution is also an exponential growth.

(b) is the next stage in solving the equation given: it is an equivalent statement.

(c) has a superficial similarity of appearance, but its solution is an exponential decay.

(d) like the given equation, has a linear function in x on the RHS.

$$\frac{dy}{dx} = e^x$$

Arrange the differential equations below according to how similar you think they are to the one above.

(a) $y^2 \frac{dy}{dx} = e^x$

(b) $\int dy = \int e^x dx$

(c) $\frac{dy}{dx} = e^y$

(d) $\frac{dy}{dx} = mx + c$

(e) $\frac{d^2y}{dx^2} = e^x$

(f) $\frac{dy}{dx} = my$

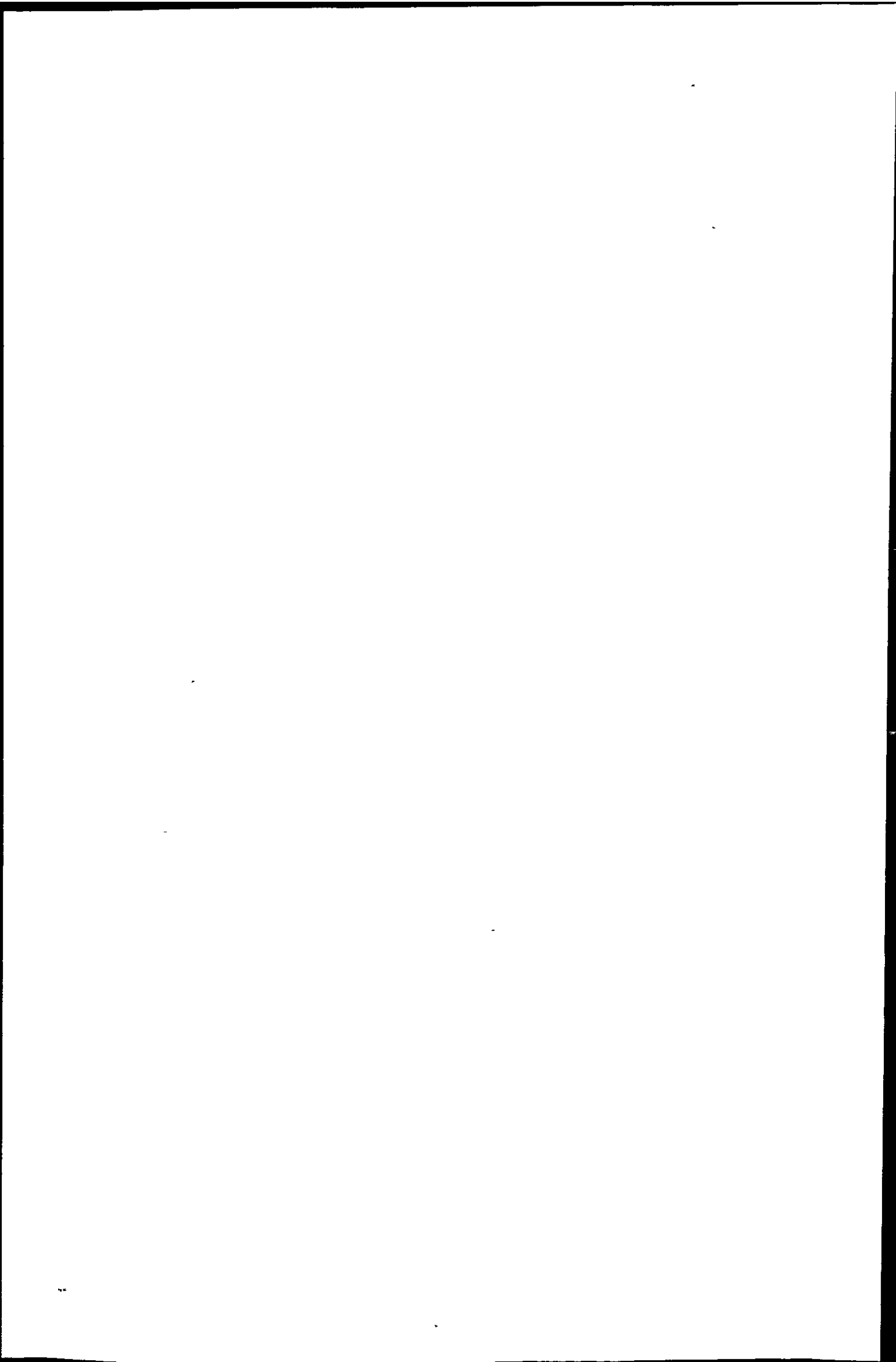
Figure 5-9: Question 6 from main questionnaire

(e) is the differential of the given equation, but has an identical RHS. Its solution will have an extra arbitrary constant.

(f) looks unlike the given equation, but its solution is also an exponential growth.

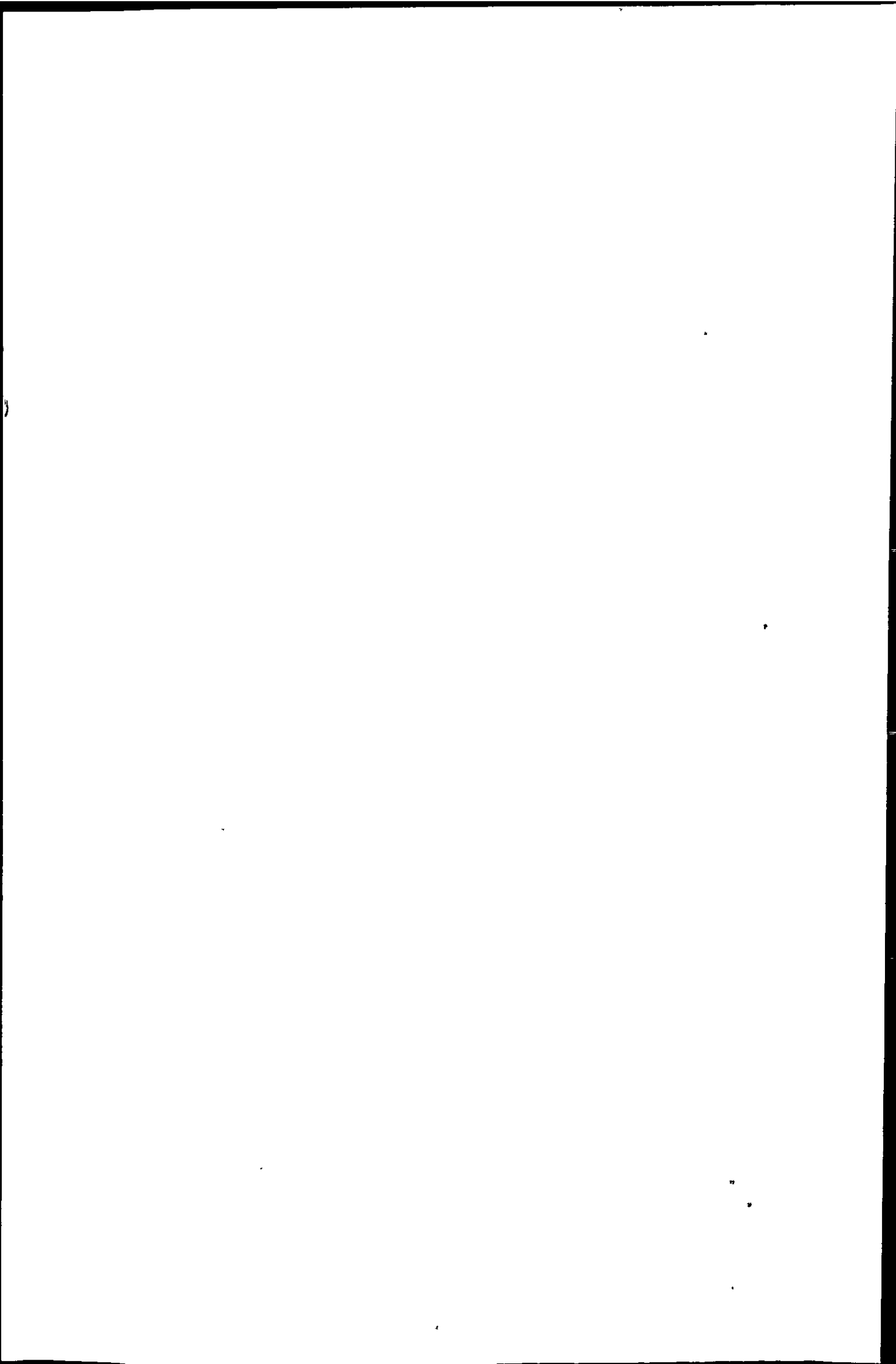
5.6 Conclusions

The issues mentioned in the conclusions to chapter 4 are addressed in these questions as follows. The depth of representation of the target concepts has been explored in various ways according to the topic.



Option	Mode of representation	Depth of representation
1a	verbal	novice
1b	visual	output of expert mental model
1c	algebraic	deeper than 1e
1d	verbal	encapsulated
1e	algebraic	superficial
1f	visual	input to mental model
2a	verbal	novice
2b	visual	"scientific"
2c	verbal	personal meaning
2d	algebraic	"mathematical"
2e	verbal	"scientific"
2f	visual	novice
3a	verbal	novice
3b	algebraic	input of mental model
3c	visual	deeper than 3f
3d	algebraic	output to mental model
3e	verbal	encapsulated
3f	visual	superficial
4a	verbal	novice
4b	algebraic	process (superficial)
4c	visual	process (superficial)
4d	verbal	personal meaning
4e	visual	novice
4f	algebraic	relatedness to differentiation
5a	verbal	formula/superficial
5b	verbal/algebraic	input to mental model
5c	algebraic	superficial
5d	verbal/algebraic	formula/irrelevant
5e	algebraic	novice
5f	verbal	novice
6a	algebraic	method and outcome (depth)
6b	algebraic	process
6c	algebraic	superficial (appearance)
6d	algebraic	structural (depth)
6e	algebraic	similar appearance
6f	algebraic	depth

Table 5-2: Classification of questionnaire options



6. The Questionnaire: responses and interpretation

6.1 Introduction

Having examined the theory behind the design of the questionnaire and the practical process of designing and administering it, in this chapter the responses to each question are summarised and compared according to the subject studied by the respondents and their level of experience. As well as the numerical data of the scores accorded to each option, we have the comments of the respondents to help in interpreting the ideas which are being expressed, and some themes which emerge from those comments are explored. The responses to the first two questions in particular are examined in some detail as they produced some interesting and unexpected results.

6.2 Interpretation of the responses

This questionnaire cannot be regarded as a precise instrument determining a scientific truth: although the analysis appears quantitative, asking respondents to rank the options in order of preference is more qualitative in spirit. As Vinner points out, concept image is context-dependent and changing. The results of the questionnaire must be regarded as indicative. Ideally, the same group of people should have been followed in a cohort study, but by surveying different groups at different stages of their careers a range of experience was sampled. Some of the changes in preferences must be attributed to the non-matching of the groups of respondents.

The results show some evolution of ideas over a period of learning, some of which are relatively slow and seem to relate to a maturing process, and others which are more rapid and may be related directly to teaching. An unexpected result was that some misconceptions concerning the bending of beams were revealed, which raise some interesting questions about the effect of holding misconceptions in general.

In general the ordered responses give an indication of the relative popularity of the options, and the comments reveal more about why respondents made those choices.

Although there are many interesting comments which may be made about the responses of the mathematics students, in this chapter as in others I shall restrict remarks on them to a comparison with engineering students, given that the reason for including them in this study was as a comparison group.

The comments made by respondents have been quoted where they illustrate or illuminate a point. In some cases common themes emerge in the comments, either on a particular question or on the questionnaire overall.

6.3 Revealing misconceptions

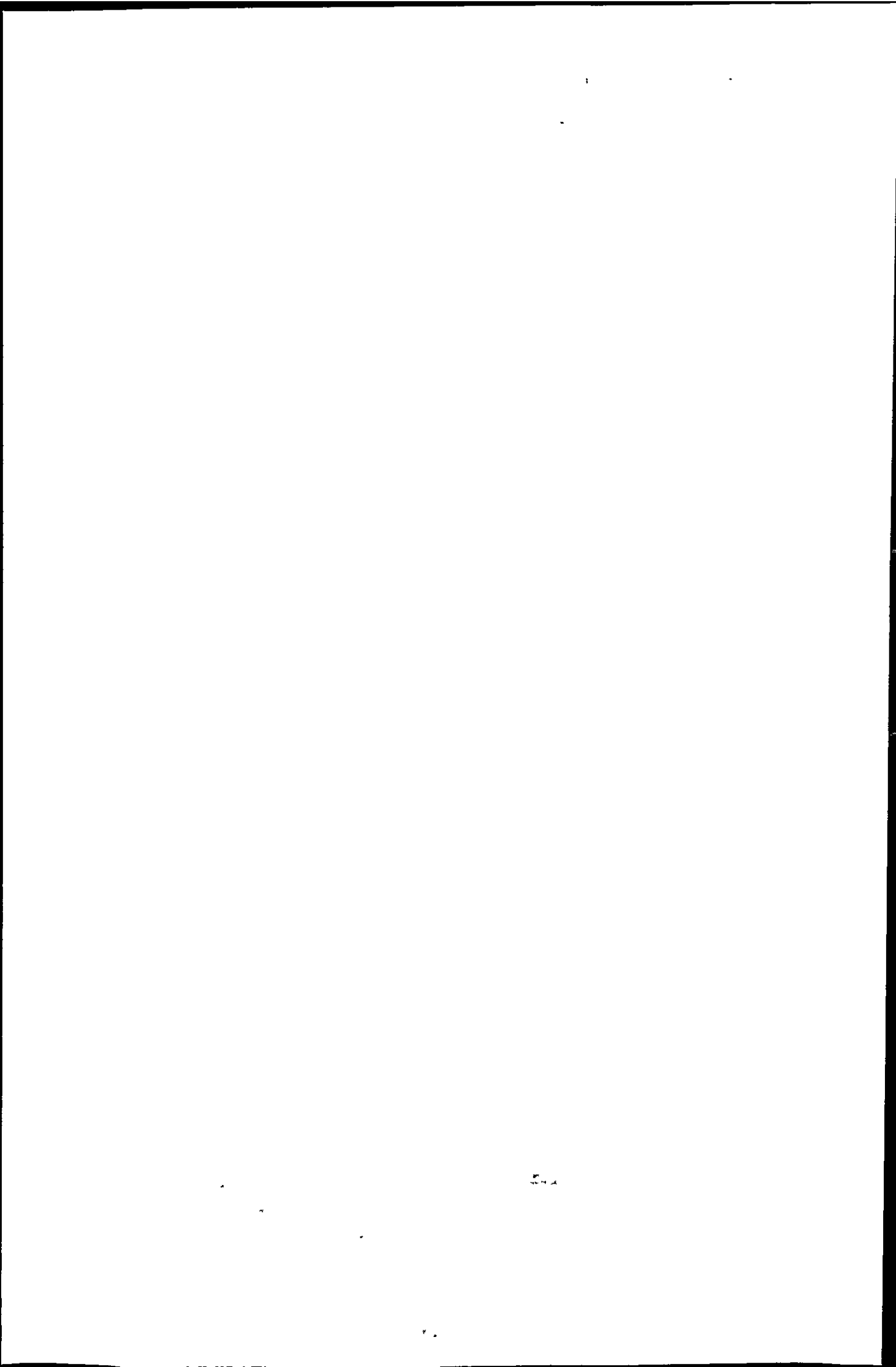
The questionnaire was not designed with the object of revealing misconceptions (or mental models which do not match the institutionally accepted version), but assuming that most respondents' concept images would lie within the range of essentially acceptable, but naive to more sophisticated options given.

These models may be revealed when people make statements which do not coincide with the predictions of the accepted or institutional meaning of the concept.

It was not expected that any of the questions would arouse particularly strong feelings in respondents. The rubric to the questionnaire explicitly stated that there were no trick questions, but some respondents still objected so strongly to two of the given choices in the question on bending that they wrote comments about them.

6.3.1 The beam will not bend at all, or whether it bends depends on its thickness

Option (a) stated "The beam bends under the weight of the gold bar". This was included particularly so that respondents who had never seen an analysis of the case



would not feel that the questionnaire was dealing with matters above their heads, and it was thought it would be popular with first year students at the start of their first year.

Some respondents made comments such as the following.

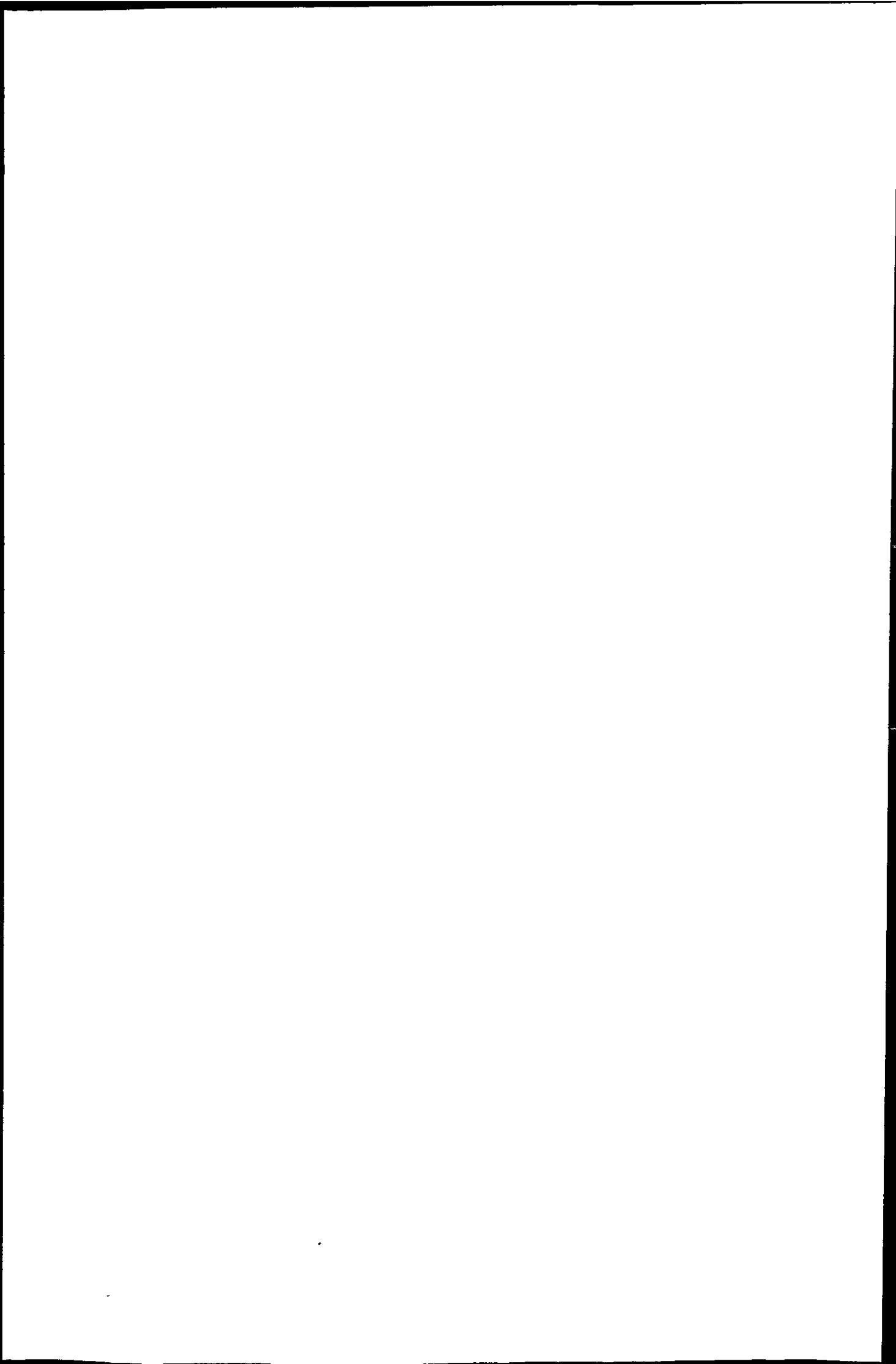
- (a) Nobody says it actually bends, so automatically assume rigidity. (final year maths student)
- (b) Not $a \propto \cos$ depends on thickness of plank (mathematics student, start of first year)
- (c) It depends on how thick the plank is (a). (mechanical engineering student, start of first year)
- (d) How thick is the plank? How heavy is the bar of gold? (second year computer systems engineer, pilot study)
- (e) I feel a bit uncomfortable not knowing the weight of the gold or the thickness & width of the plank. (engineering lecturer)
- (f) I assume the deflection is minimal. (practising engineer)
- (g) 'a' may not be very valid- The deflection may be so small as to be negligible. (practising engineer)

There is a graduation from assuming absolute rigidity to wondering whether the assumption is valid under the circumstances.

6.3.2 The point of greatest deflection must be under the load

Option (b) was a diagram of the deflected shape.

- (a) b looks like the bar would be in the middle. (engineering student, start of first year)
- (b) b is wrong (final year mechanical engineering student)
- (c) I would rather have a drawing but (b) looks wrong. (final year mechanical engineering student)
- (d) not keen on (a) (too simplistic) and (b) (wrong?) (mathematics student, start of first year)
- (e) b is useless! (mathematics student, end of first year)
- (f) b isn't quite right, but I've assumed poetic license with the artist! (practising engineer)
- (g) b (slightly changed) (see Figure 6-1) (experienced mathematics and mechanics teacher)
- (h) I don't recognise any of the equations and (b) doesn't look quite like what I'd expect! (postgraduate, A level maths, degree in Business Administration)
- (i) My first thought is 'how can I get the gold bar'! My second is that the diagram at b is not drawn correctly. (practising engineer)



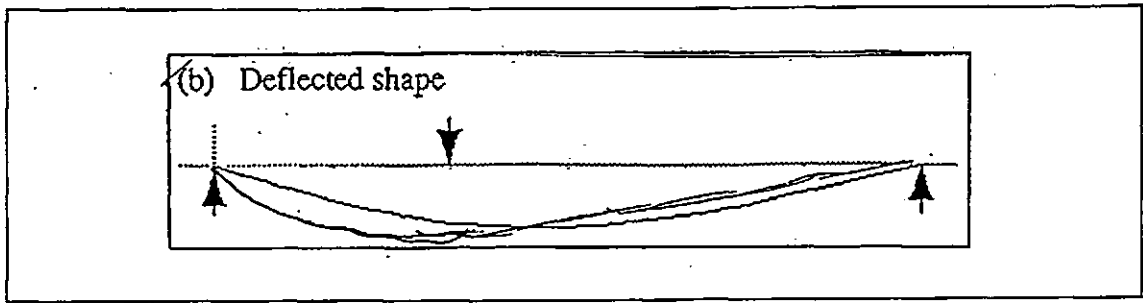


Figure 6-1: Modified diagram of shape of beam

6.3.3 Why do people think these things? Mental models of physical problems

These ideas do not come out of thin air, but are based on the mental models that the respondents hold. These models are not directly accessible to investigators, but the comments that have been given are predictions these respondents have made of the behaviour of the system according to their mental models. Given the predictions, it is possible to deduce the nature of the models. Anzai and Yokoyama (cited in Royer et al, 1993) classify models as *experiential*, *correct scientific* or *false scientific*. Experiential models, which are derived directly from experience, do not have any technical or scientific content. The statement "The beam bends..." was intended to appeal to this type of model. A correct scientific model is a set of scientific concepts and relations that are correct and sufficient to capture problem information. Such a model would characterise the bending in terms of bending moment and shear force, loads and reactions, displacements, stresses and strains. False scientific models are those which contain scientific concepts and relations, but incorrectly characterise the problem. It is this type of model which is shown in the comments quoted above.

6.3.4 Planks are, or may be, rigid.

The first set of comments represent the view of rigidity as the natural state of a beam, given that this is a frequently made assumption in statics problems. This is sometimes held at the same time as the concept that the deflection of a beam does depend on its

dimensions, its loading, and, not specifically mentioned by our respondents, the material stiffness (Young's modulus) of the beam, which we see in 6.3.1 (b)-(e) above. It is perfectly possible to hold two opposite views on a physical phenomenon as long as they are not brought into direct conflict. The point is that these quantities do not affect *whether* a beam will bend, but *how much* it will bend: as comment 6.3.1 (f) points out, the bending may be negligible, but negligible is still not the same as non-existent.

Perkins & Simmons (1988) regard this as a defect of priority among concepts: the novice treats "rigidity" as a more important concept than "springiness", while the expert sees "springiness" as the more powerful explanatory tool.

6.3.5 The deflection must be greatest under the load

This idea may come from one of several sources:

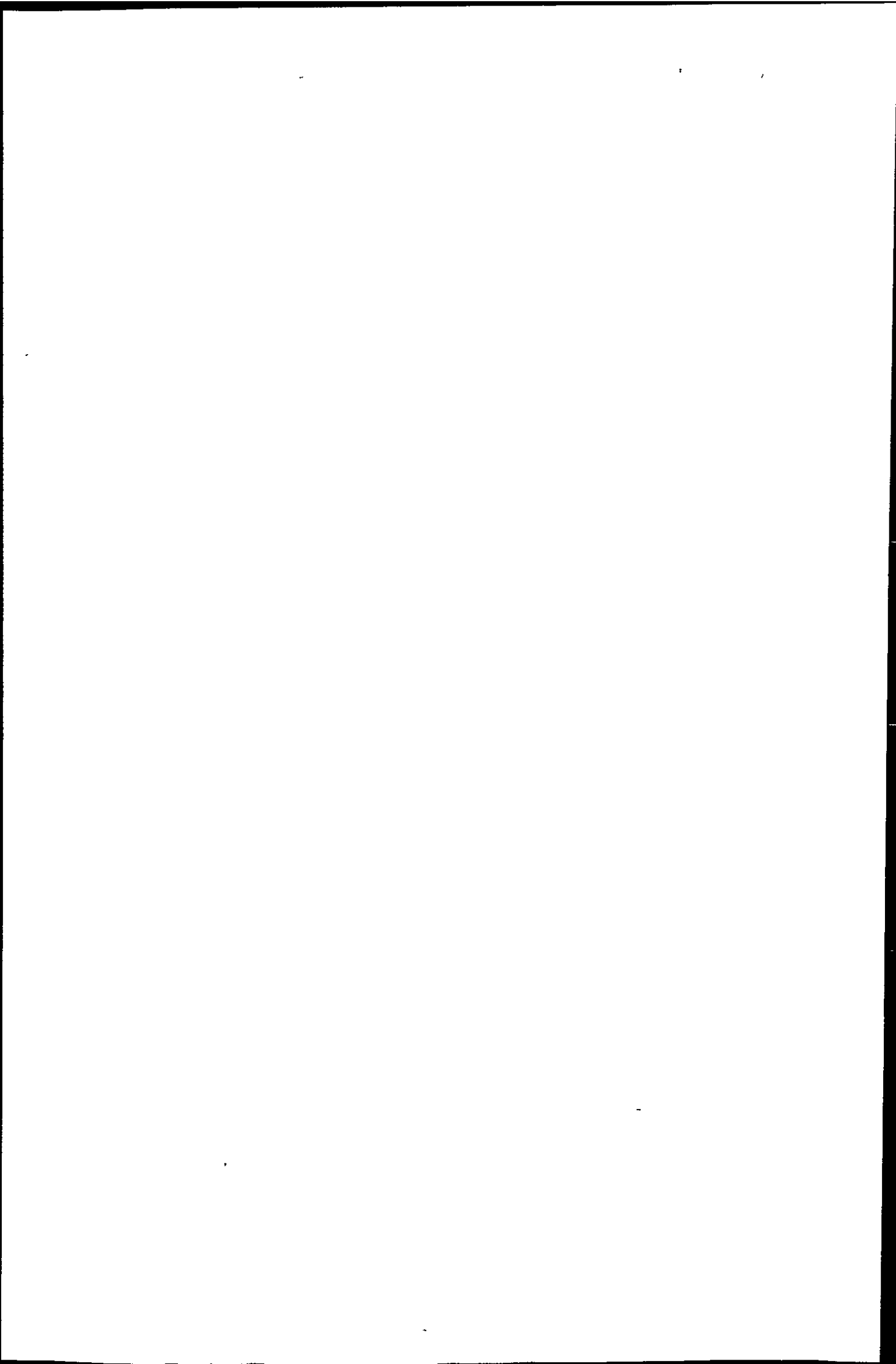
- (a) Weightless strings and point masses
- (b) The lowest point is the lowest (potential) energy position
- (c) Shear dominated deflection

6.3.5.1 *Point masses versus solid bodies*

The first stage of modelling that students encounter in mechanics is of the idealised world of point masses, weightless strings and infinite bodies of infinite stiffness. In such a world, the nearest approximation to our weight on a beam is a weight hung on a loose horizontal string, one-third of the way between its points of suspension. For horizontal equilibrium, the weight would have to fall so that both parts of the string are under tension, pulling the string into an asymmetrical V-shape.

6.3.5.2 *Potential energy*

The powerful idea of potential energy being minimised would seem to mean that the



weight must be at the lowest possible point, which must be the lowest part of the beam.

The lowest part of the beam must thus be under the weight.

6.3.5.3 *Shear dominated deflection*

When beams are designed to use material to perform as efficiently as possible in bending, the notion of putting as much as possible into top and bottom flanges connected by a thin web emerges, and we have an I-beam. The stiffness of the I-beam in bending is greatly enhanced, but its stiffness in shear is related simply to the cross-sectional area. In extreme cases, the deflection due to shear, normally negligible, can dominate, so that the load is close to the lowest point of the beam. This would not happen in the case of a plank lying between two bricks.

It would be speculative to suggest which of these is the principal source of error, but it is suspected from experience that for the students at least the notion of weightless strings and point masses is the most important.

6.3.6 Discussion

The questionnaire was not designed to pick up incorrect mental models, but rather to tease out how people were holding mental representations of some engineering and mathematical concepts. Nevertheless it appears to have brought out into the open some alternative representations which we may not have discovered in teaching or discussion.

We should ask ourselves how important these misconceptions are in the scheme of things. To most people they are probably never going to matter. To those to whom they will make a difference, they will probably discover in time that they have been mistaken. However, particularly for those people who objected to the shape drawn in (b), the revelation comes as a shock. A comfortable assumption has been shaken, and it is unpleasant.

These results were presented at a conference (Mathematical Education of Engineers, Loughborough, 1997, see Maull & Berry, 1997). One delegate commented that aero modelling with balsa developed an intuitive understanding of this type of bending. It was also pointed out that in most cases the objective in engineering is to reduce deflections to the negligible.

6.4 Preferences

Every option in every question was placed first by at least one respondent. This reflects (a) the diversity of the responses and (b) either that every option proposed was a possible first preference or that some people were answering at random.

An individual response to a single question consisted of a list of up to six letters, each representing an option, in order of decreasing preference. These lists were converted into scores, with the first being given a score of 5, the second 4, and so on, with the sixth choice and any unchosen options being given a score of zero. For a group of respondents, the scores for each option for each question were summed and normalised so that a score of 5 would mean that every respondent in a group had put a given option first, and zero would mean that everyone in that group had either put that option last or nor placed it.

These results were then summarised as shown in the accompanying charts. From these we can see the relative popularity of each option in any group, and the ways these change as people progress through their course or through life. The results were also tested using one-way ANOVA, to find the significance of the differences at different experience levels within each subject group and between the subject groups as a whole. For each group the mean score for each option is shown with the standard deviation in parentheses. Many of the options tested inconclusively, but those which yield results at a confidence of better than 5% are remarked upon. For a variable to distinguish

significantly between two groups, the difference between the means has to be large compared to the variance within the groups. Because the spread of popularity of each option is wide, even when the means shown on the charts are different, the statistical significance may be low.

As Hair et al (1984) point out, statistical and practical significance are not the same thing. In a class of nine pupils, four may be boys, and there is no statistical significance. If in a class of thirty the same is true the fact is statistically significant. However the practical significance is the same when it comes to making up a boys' football team.

6.4.1 Question 1

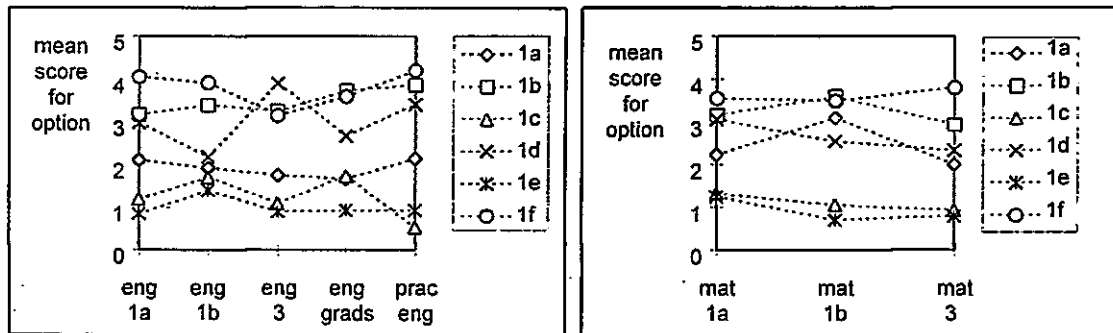
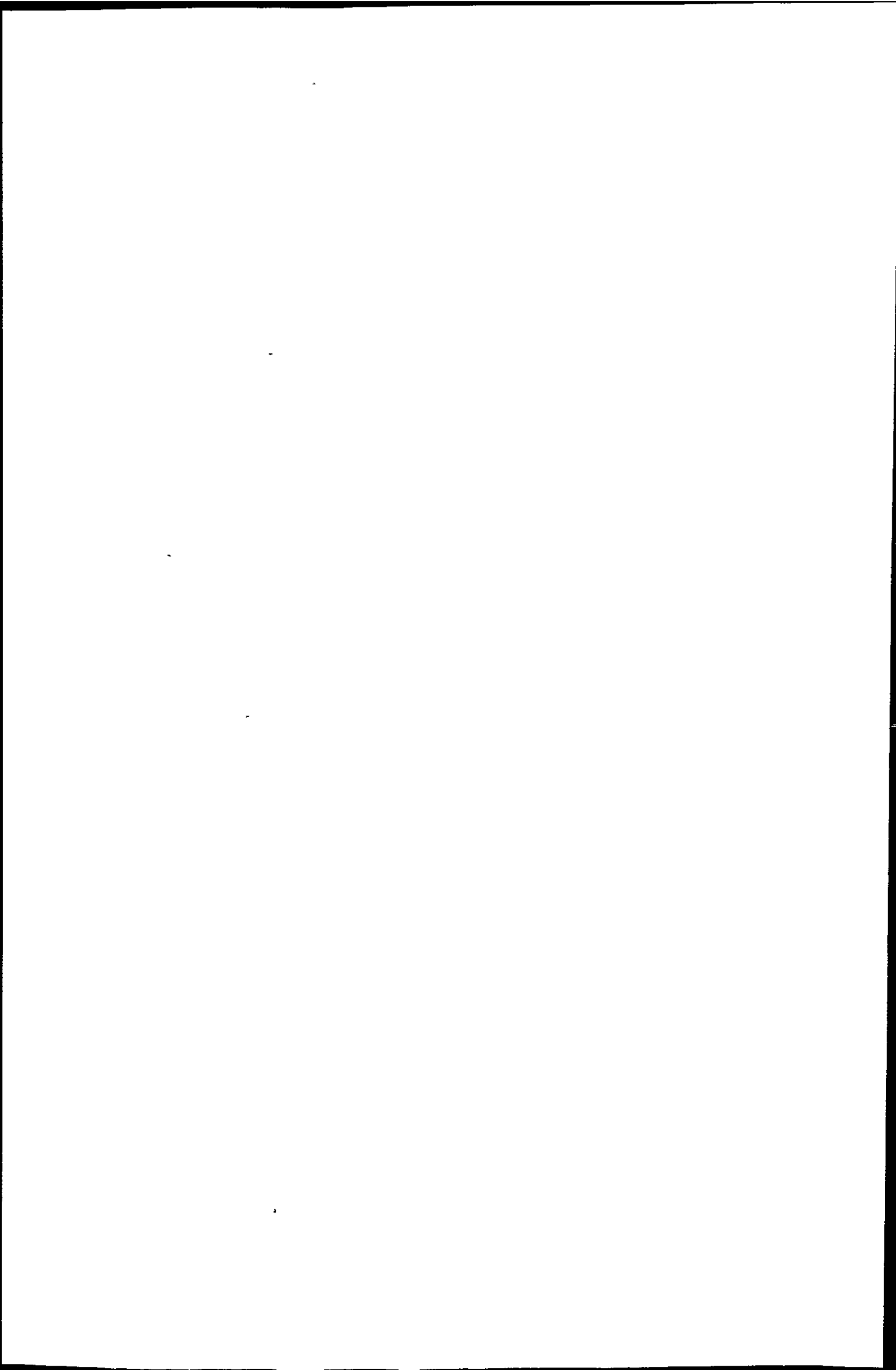


Figure 6-2: Responses to question 1

The most interesting aspect of Question 1 was the misconceptions which it uncovered, which are discussed in the first part of this chapter. The changing popularity of some of the options may reveal a response to teaching.



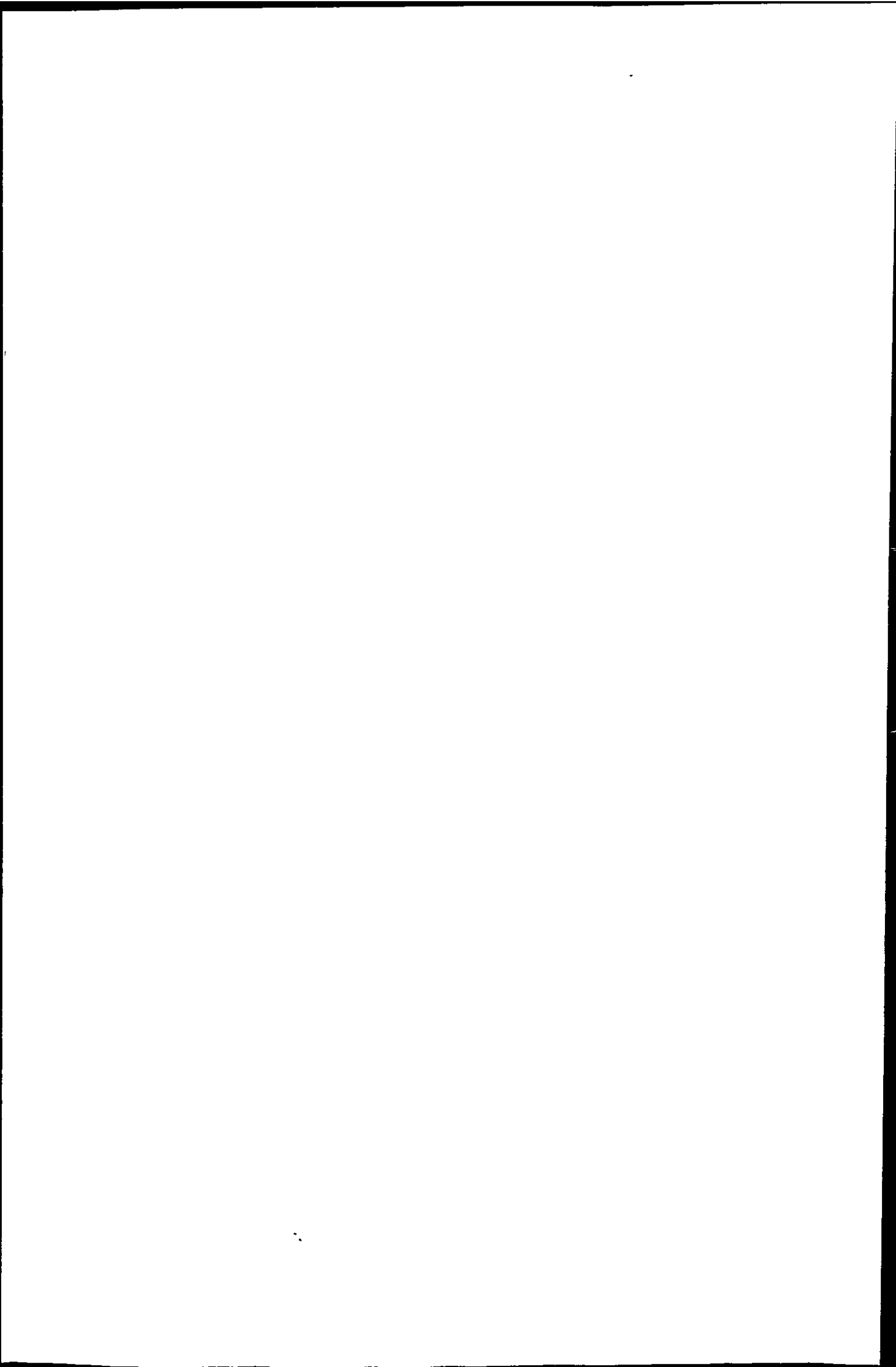
	1a The beam bends	1b Deflected shape	1c Bending moment equation	1d Simply supported beam	1e Shear force =, BM =	1f Loading diagram
Mean Rank						
Engineering at entry	1.95 (1.65)	3.18 (1.65)	1.20 (1.16)	2.98 (1.37)	0.85 (1.10)	4.05 (1.36)
Engineering end of 1st year	1.91 (1.98)	3.39 (1.27)	1.70 (1.61)	2.17 (1.56)	1.39 (1.34)	3.91 (1.38)
Engineering final year	1.75 (1.77)	3.25 (1.94)	1.10 (1.07)	3.90 (1.21)	0.90 (0.97)	3.15 (1.38)
Engineering postgrads	1.67 (1.80)	3.73 (1.53)	1.73 (1.28)	2.67 (1.72)	0.93 (1.03)	3.60 (1.40)
Practising engineers	2.13 (0.99)	3.87 (1.25)	0.53 (0.92)	3.40 (0.74)	0.93 (0.96)	4.20 (1.08)
Engineers overall	1.89 (1.67)	3.40 (1.57)	1.27 (1.27)	2.99 (1.46)	0.99 (1.11)	3.82 (1.36)
Significance within engineers	.9433	.5618	.0386	.0014	.4277	.1040
Mathematics at entry	2.35 (1.75)	2.95 (1.74)	1.24 (1.26)	3.20 (1.66)	1.12 (1.44)	3.31 (1.51)
Mathematics end of 1st year	2.81 (1.82)	3.26 (1.58)	1.41 (1.37)	2.44 (1.67)	1.04 (1.26)	3.56 (1.45)
Mathematics final year	2.14 (1.88)	2.79 (1.93)	0.86 (1.03)	2.27 (2.02)	0.79 (1.25)	3.79 (1.48)
Mathematicians overall	2.45 (1.78)	3.01 (1.71)	1.23 (1.26)	2.85 (1.75)	1.05 (1.36)	3.45 (1.49)
Significance within mathematicians	.4241	.6458	.4189	.0760	.7047	.5152
Significance between subject groups	.0215	.0892	.8368	.5377	.7209	.0587

Table 6-1: Responses to question 1

Overall, options 1a (The beam bends..) was preferred significantly more by the mathematicians than by the engineers as a group. Options 1c (Bending moment equation) and 1d (A simply supported beam...) distinguish between the levels of experience of the engineers, with the practising engineers rejecting option 1c more than other engineers and the final year engineers having a stronger preference for option 1d than any other group.

6.4.1.1 Responses to teaching

In the responses from the engineering students in particular, some options show jumps in popularity, either up or down, during the course of the undergraduate degree. These jumps coincide with intervals in which teaching on particular subjects occurs, and these



jumps may be interpreted as evidence of responses to teaching. Thus in Q1, option (d), "A simply supported beam with a point load at one-third span" represents technical language with which the students would have become familiar between the end of the first and the early part of the final year.

We find comments on Question 1 such as:

- The equations in c and e mean nothing to me as I have not studied them yet. (engineering student, start of first year)
- ..the mathematical definitions are starting to make more sense. (engineering student, start of first year)
- We've only just started. Ask again in a few weeks! (engineering student, start of first year)
- No idea, may be true, tell ya later. (engineering student, start of first year)
- b, f & d are the only ones that mean anything to me as I've not come across the others yet. (engineering student, start of first year)

and after studying simple bending:

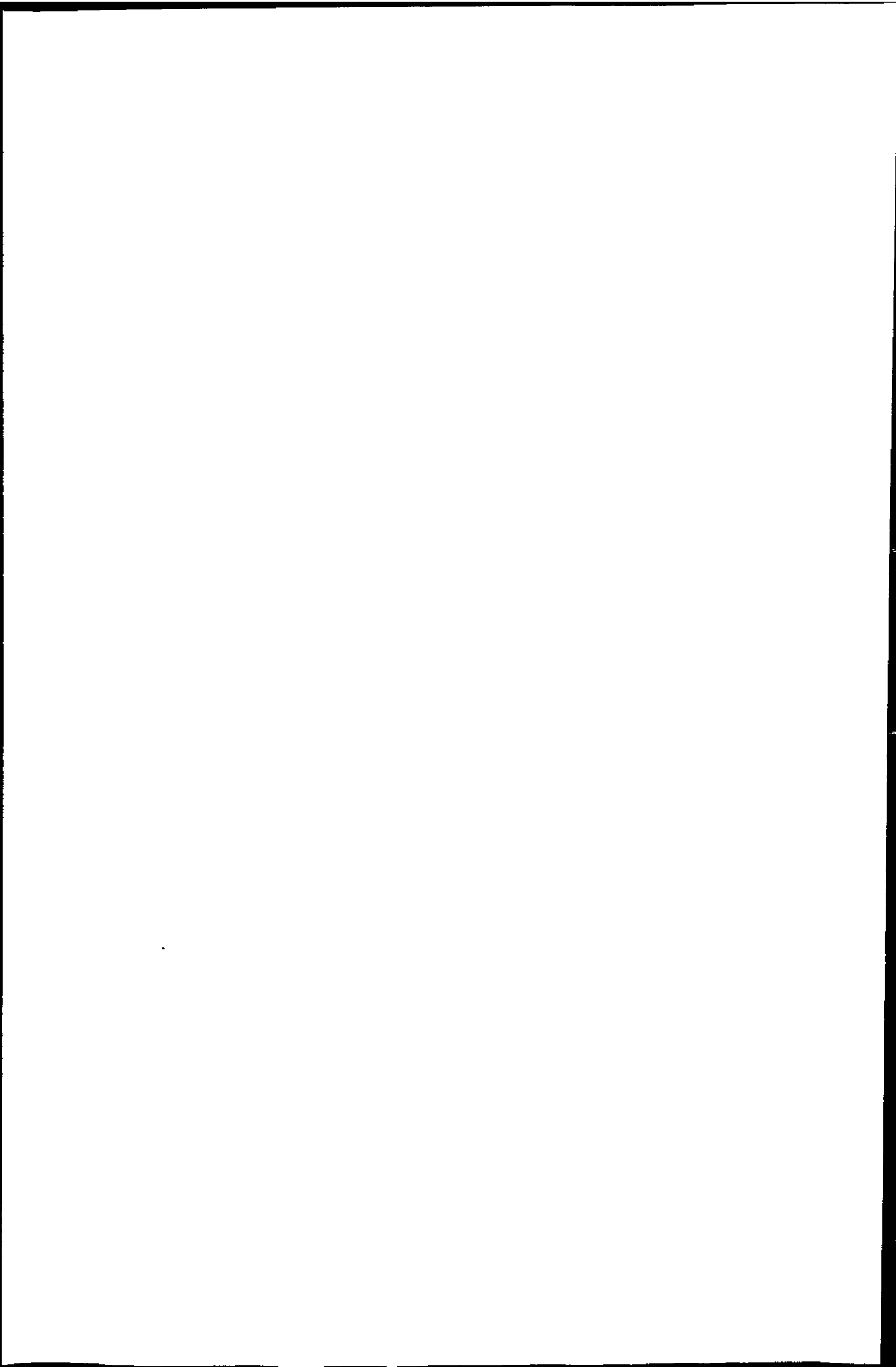
- Due to the intensity of learning mechanics during the first year (engineering student, end of first year)

Other examples are: 3(b), the differential equation $\ddot{y} + k\dot{y} + \omega^2 y = 0$, which also increases in popularity over the course of the second year of teaching, and 6(b) which changes place with 6(e) as the students gain in confidence in integration.

6.4.2 Question 2

In general, the mathematics students prefer the statement about the tangent at all three stages, and their next favourite option is the corresponding diagram.

For the mathematics students, the third favourite option is the statement about "things changing", whereas for the engineering students, the "things changing" statement overtakes the other two. Although the last set of responses (practising engineers) seems to contradict this trend, the respondents in this group were different in a number of ways from the respondents in the first four: they were all from old universities and had



been trained by the Ministry of Defence, and they were considerably older than the rest of the respondents, so their education had been conducted under a different regime.

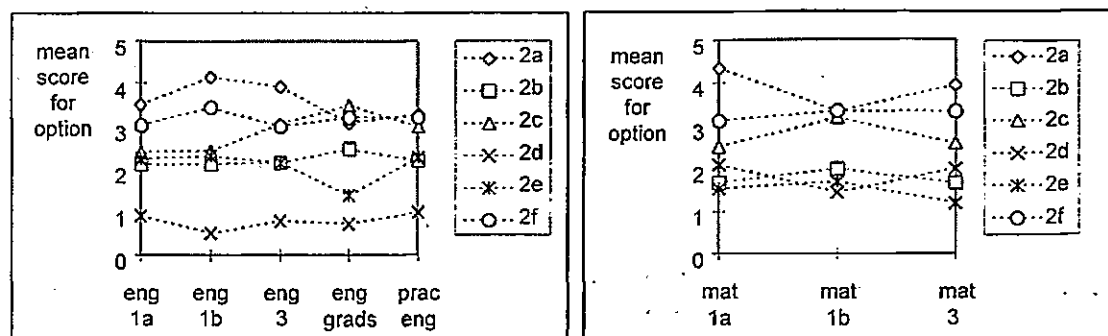


Figure 6-3: Responses to question 2

Mean Rank	2a Slope of tangent	2b Zoom in diagram	2c How quickly changing	2d Limit expression	2e As you zoom in	2f Tangent diagram
Engineering at entry	3.50 (1.66)	2.13 (1.26)	2.40 (1.86)	0.93 (1.42)	2.25 (1.63)	3.03 (1.67)
Engineering end of 1st year	4.13 (1.10)	2.13 (1.55)	2.43 (1.56)	0.52 (1.08)	2.30 (1.06)	3.43 (1.47)
Engineering final year	3.90 (1.25)	2.15 (1.46)	3.05 (1.70)	0.80 (1.32)	2.15 (1.42)	3.00 (1.45)
Engineering postgrads	3.06 (1.91)	2.47 (1.51)	3.47 (1.81)	0.73 (1.10)	1.40 (1.24)	3.20 (1.32)
Practising engineers	3.26 (1.79)	2.20 (1.86)	3.00 (1.46)	1.00 (1.51)	2.27 (1.49)	3.20 (1.32)
Engineers overall	3.61 (1.57)	2.19 (1.46)	2.74 (1.74)	0.81 (1.30)	2.13 (1.43)	3.15 (1.49)
Significance within engineers	.2033	.9562	.2111	.7765	.3256	.8561
Mathematics at entry	4.36 (1.01)	1.51 (1.45)	2.56 (1.58)	2.00 (1.59)	1.45 (1.23)	3.05 (1.46)
Mathematics end of 1st year	3.41 (1.47)	1.89 (1.53)	2.96 (1.51)	2.07 (2.04)	1.44 (1.48)	3.22 (1.28)
Mathematics final year	3.86 (1.66)	1.79 (1.25)	2.64 (1.91)	1.93 (1.59)	1.29 (1.38)	3.29 (1.43)
Mathematicians overall	4.02 (1.31)	1.66 (1.44)	2.69 (1.60)	2.01 (1.71)	1.43 (1.31)	3.15 (1.40)
Significance within mathematicians	.0062	.5045	.5721	.9655	.9105	.8015
Significance between subject groups	.0437	.0091	.8106	.0000	.0003	.9405

Table 6-2: Responses to question 2

The engineers prefer options 2b (Zoom in diagram) and 2e (As you zoom in...) more strongly than the mathematicians and dislike options 2a (The slope of the tangent...) and

2d (Limit expression) considerably more than the mathematicians do. No options distinguish significantly between the different levels of experience of the engineers. On Option 2b (Zoom in diagram) the differences between the engineering groups are small enough to say that they their preference is the same to $p > 95\%$.

Some interesting comments were made in response to this question:

- An engineering graduate commented that he had really only understood the idea of the differential as the rate of change in his “year out” in industry.
- My extra learning has taught me that $dy/dx =$ gradient which I now understand as a rate of change in bus[iness] environment. (Engineering postgraduate)

The above comments tend to confirm that the greater popularity of option 2c among engineering final year and postgraduate students is a matter of development and maturing, and not a statistical quirk of the group of students responding.

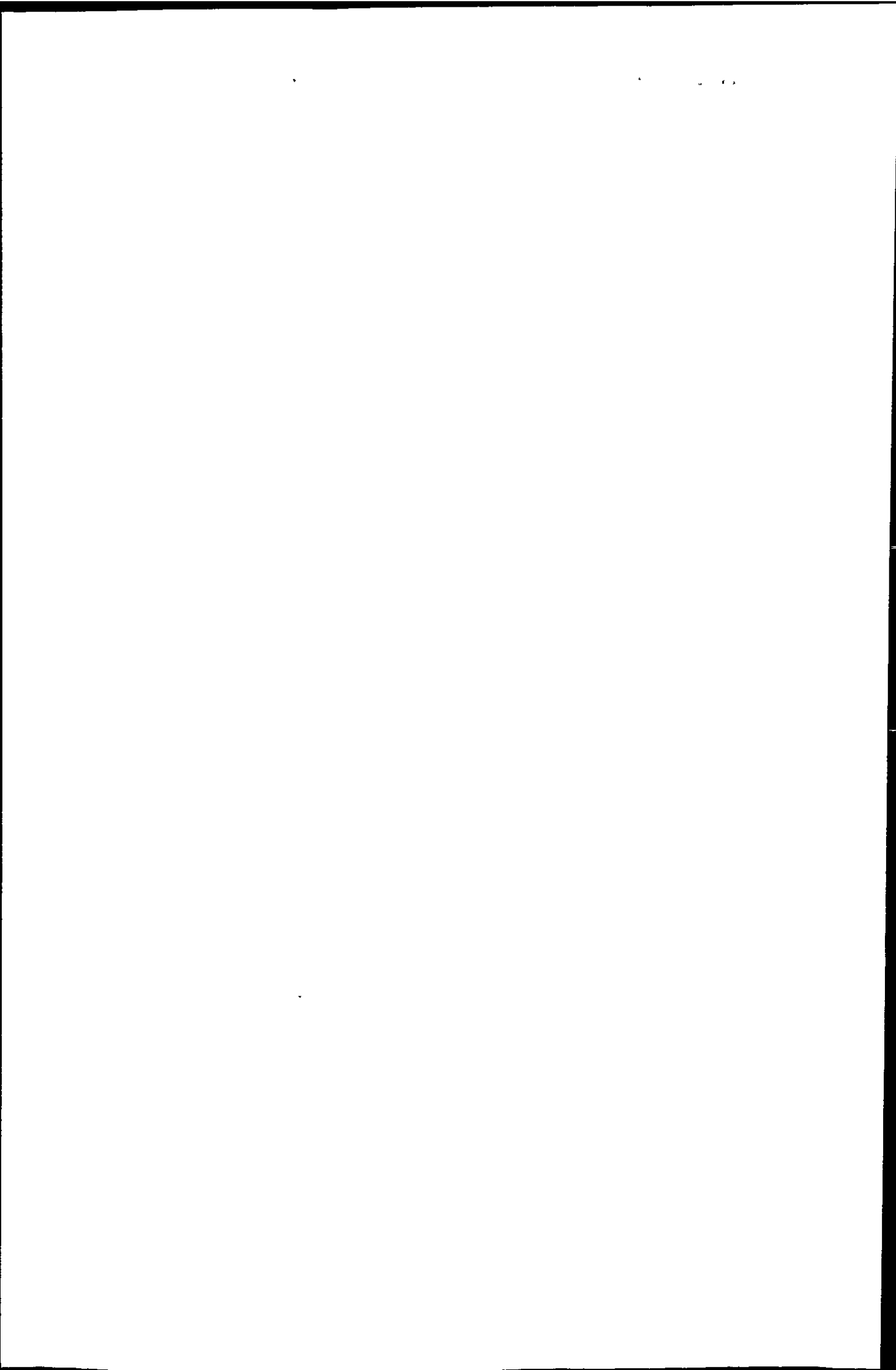
- I seem to use both (c) and (f) as models for dy/dx . I see them as 2 equally closely associated but quite different ideas. (Engineering postgraduate)

As Vinner points out, the concept image depends on context. This respondent shows he recognises two of the options as equally closely related to the target concept.

- Words first to get an idea of the problem. Then a “diagram”. Then some maths \equiv Greek! (Engineering postgraduate)
- As an engineer I tend to represent problems like this first verbally, then graphically, and as a last resort mathematically... This is because my mathematics skills are not brilliant and I need to reference (*sic*) back to old notes for these types of problems. (Engineering postgraduate)

Engineering students told Crowther (1997b) that they like to visualise. This questionnaire showed all the groups of students surveyed preferred diagrammatic representation in the “mechanics” questions, but in general they preferred verbal representations in the “mathematics” questions.

The theme of needing to or being able to refer back to notes or a textbook for mathematics is also one which recurs in the comments of engineering students, and is discussed in the chapter on mathematical modelling.



In the integration question, there is a corresponding option to the “things changing” statement, which is “The integral tells you how things build up”. This remains the least popular option, but grows steadily across the first four groups of engineering respondents.

6.4.3 Question 3

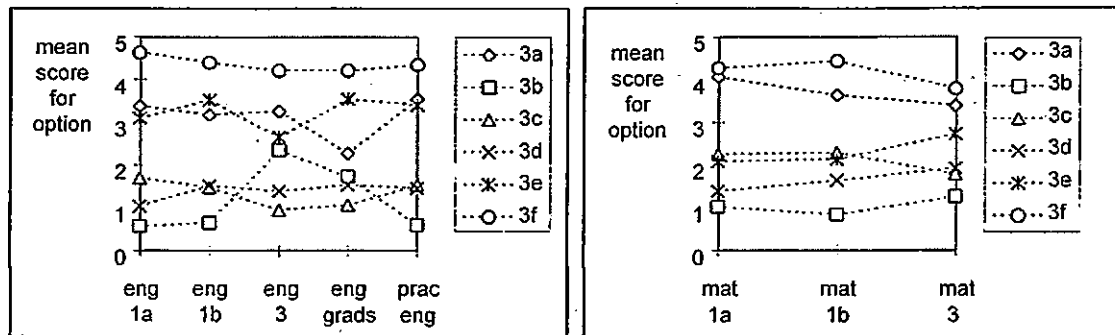


Figure 6-4: Responses to question 3

Options 3a (The mass bounces..) and 3c (Phase plane diagram) are preferred by the mathematicians as a group significantly more than by the engineers. Options 3e (Damped harmonic response), and 3f (Displacement diagram) are preferred significantly more by the engineers. Option 3b (Differential equation) is preferred very significantly more by final year engineers than by first year and practising engineers.

Option 3b shows a “learning jump” between the end of the first and the final year in the engineering students. This is particularly interesting given that the differential equation looks very mathematical, and students in their final year often claim to be out of touch with mathematics. I am beginning to suspect, given this and some comments quoted later in this chapter, that engineers cease to regard something as mathematics once they have incorporated it into their “engineering” knowledge.

Mean Rank	3a The mass bounces	3b Differential equation	3c Phase plane diagram	3d Solution equation	3e Damped harmonic response	3f Displacement diagram
Engineering at entry	3.38 (1.51)	0.58 (0.71)	1.70 (1.09)	1.05 (1.01)	3.10 (1.32)	4.63 (0.49)
Engineering end of 1st year	3.17 (1.53)	0.65 (0.83)	1.48 (1.24)	1.52 (1.56)	3.52 (0.90)	4.39 (0.66)
Engineering final year	3.25 (1.74)	2.35 (1.50)	0.95 (1.00)	1.40 (1.70)	2.65 (1.23)	4.20 (0.95)
Engineering postgrads	2.27 (1.87)	1.77 (1.33)	1.07 (1.16)	1.53 (1.51)	3.53 (1.51)	4.20 (0.94)
Practising engineers	3.53 (1.36)	0.60 (1.12)	1.53 (1.19)	1.47 (1.30)	3.40 (0.83)	4.33 (0.82)
Engineers overall	3.19 (1.61)	1.06 (1.26)	1.42 (1.15)	1.33 (1.37)	3.20 (1.22)	4.41 (0.74)
Significance within engineers	.1843	.0000	.1156	.6216	.1106	.1720
Mathematics at entry	4.20 (1.39)	0.87 (0.98)	2.18 (1.20)	1.20 (1.30)	1.96 (1.50)	4.13 (0.70)
Mathematics end of 1st year	3.56 (1.87)	1.07 (1.24)	2.26 (1.10)	1.74 (1.38)	2.04 (1.53)	4.26 (0.81)
Mathematics final year	3.29 (1.94)	1.21 (1.58)	1.93 (1.21)	2.00 (1.61)	2.64 (1.74)	3.86 (0.86)
Mathematicians overall	3.89 (1.65)	0.98 (1.15)	2.17 (1.17)	1.47 (1.39)	2.08 (1.55)	4.13 (0.76)
Significance within mathematicians	.0833	.5436	.6876	.0759	.3389	.2752
Significance between subject groups	.0022	.6223	.0000	.4607	.0000	.0071

Table 6-3: Responses to question 3

6.4.4 Question 4

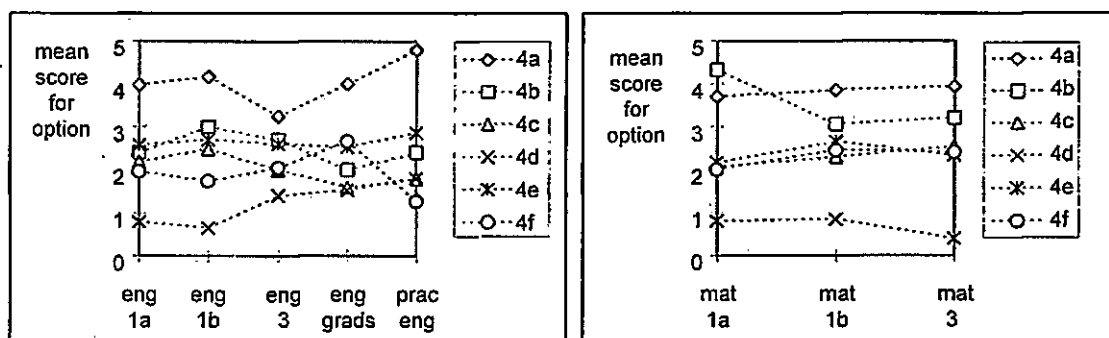


Figure 6-5: Responses to question 4

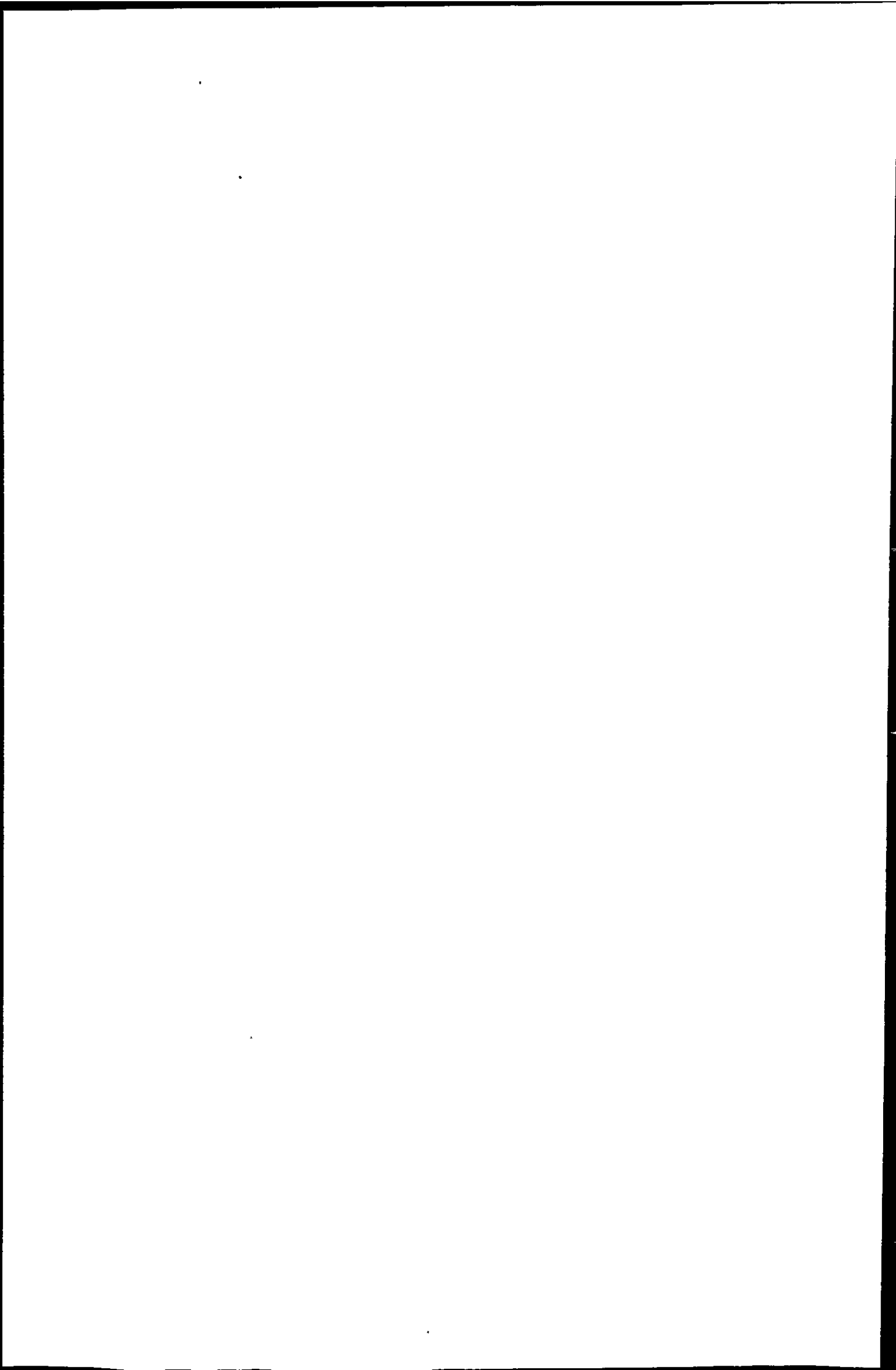
Mean Rank	4a Area under curve	4b Integration done	4c Integration done diagram	4d How things build up	4e Area under curve diagram	4f Opposite of differentiation
Engineering at entry	3.98 (1.53)	2.40 (1.66)	2.20 (1.52)	0.80 (0.94)	2.58 (1.71)	1.98 (1.72)
Engineering end of 1st year	4.17 (1.11)	3.00 (1.68)	2.48 (1.34)	0.65 (0.88)	2.70 (1.58)	1.74 (1.63)
Engineering final year	3.25 (1.62)	2.70 (1.89)	2.00 (1.56)	1.40 (1.57)	2.60 (1.60)	2.05 (1.70)
Engineering postgrads	4.00 (1.36)	2.00 (1.77)	1.60 (1.55)	1.53 (1.68)	2.53 (1.77)	2.67 (1.50)
Practising engineers	4.80 (0.56)	2.40 (1.30)	1.80 (1.32)	1.80 (1.42)	2.87 (1.46)	1.27 (1.53)
Engineers overall	4.00 (1.40)	2.52 (1.68)	2.09 (1.47)	1.11 (1.29)	2.64 (1.62)	1.94 (1.66)
Significance within engineers	.0239	.4358	.3926	.0142	.9779	.2161
Mathematics at entry	3.53 (1.15)	4.25 (1.28)	1.09 (1.47)	0.85 (1.08)	2.25 (1.54)	1.98 (1.47)
Mathematics end of 1st year	3.63 (1.45)	3.11 (1.78)	2.33 (1.52)	0.93 (1.21)	2.48 (1.65)	2.52 (1.50)
Mathematics final year	3.86 (1.03)	3.21 (1.67)	2.71 (1.82)	0.43 (0.65)	2.21 (1.22)	2.43 (1.65)
Mathematicians overall	3.60 (1.22)	3.78 (1.58)	2.15 (1.55)	0.81 (1.07)	2.31 (1.51)	2.20 (1.51)
Significance within mathematicians	.6634	.0023	.1688	.3375	.7908	.2666
Significance between subject groups	.0320	.0000	.7845	.0780	.1372	.2415

Table 6-4: Responses to question 4

The engineers as a group prefer option 4a (The area under the curve..) significantly more than the mathematicians, and 4b (Integration carried out) significantly less. The growth in popularity of option 4e (How things build up) among engineers is statistically significant, but on option 4f (Area under the curve diagram) the preferences of the engineers at different stages are significantly identical.

As mentioned above, option 4d (The integral tells you how things build up) increases slowly but steadily across all the levels of experience of the engineering respondents, including the practising engineers. Option 4f, that integration is the opposite of differentiation, has a similar popularity profile to option 2c (how things build up), with growth across the first four sets of engineers, and a drop in the practising engineers.

These two options were intended to indicate (4f) an increase in the inter-relatedness of concepts with maturity, and (4d) an increase in the meaning of concepts with maturity.



The profiles of option 4b (integration done) is rather different in the two sets of respondents. There is an interesting difference of phase between engineering and mathematics students. Most of the mathematics students entered university fresh from their A level studies, having recently been introduced to integration. Its high popularity with the first group of mathematics students reflects this. The greatest popularity of this option with engineers falls at the end of their first year, reflecting that many engineering students are first acquainted or are reacquainted with integration over the course of that year.

6.4.5 Question 5

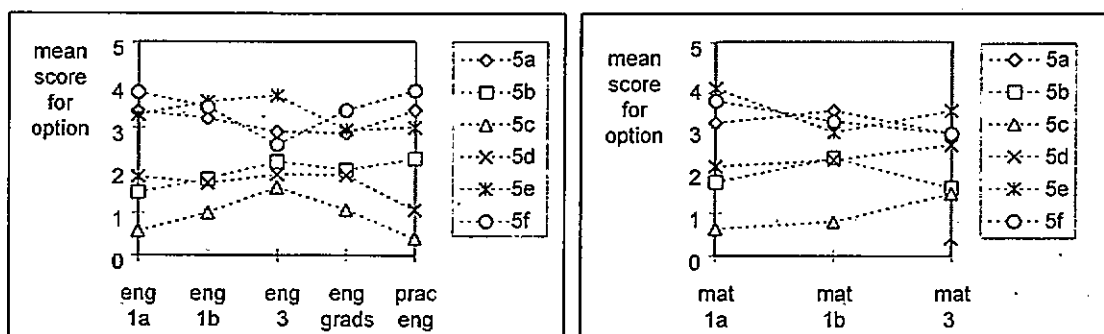
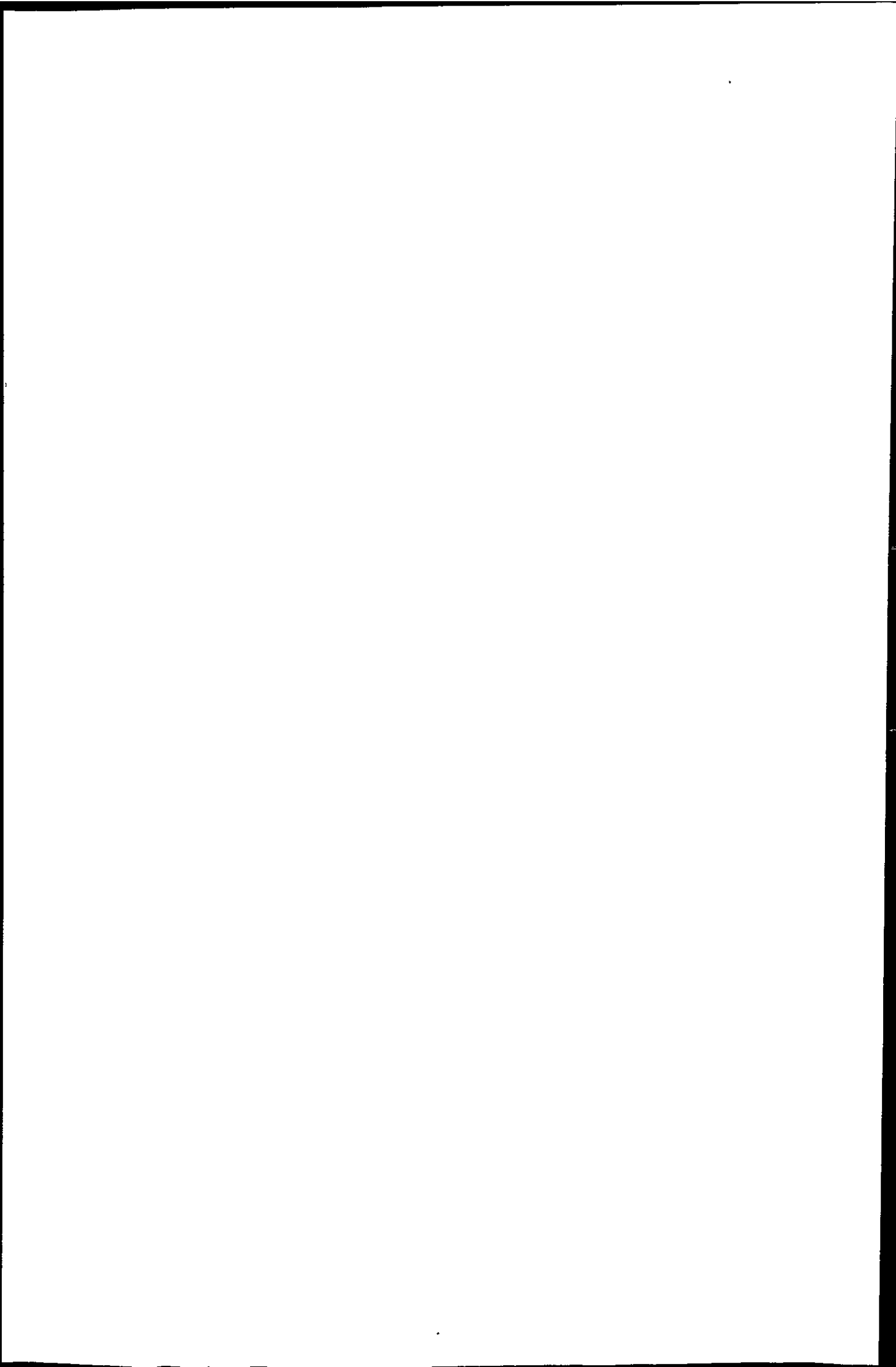


Figure 6-6: Responses to question 5

On reflection, it was felt that questions 5 and 6 were not well designed and that the responses to these questions were not in general very revealing.

Option 5c distinguishes significantly statistically between the levels of experience of the engineers. The popularity of this option rise in the first three groups then falls again. The shape of this polygon looks a little like a learning peak, but it is difficult to see the practical significance.



Mean Rank	5a Energy stored equation	5b Energy imparted to ball = $\int F dx$	5c K.E. = P.E. equation	5d Change in momentum = $\int F dt$	5e $F=ma$	5f The further you pull back...
Engineering at entry	3.38 (1.46)	1.48 (1.15)	0.58 (1.20)	1.85 (1.27)	3.30 (1.45)	3.83 (1.52)
Engineering end of 1st year	3.22 (1.09)	1.78 (1.41)	1.00 (1.24)	1.70 (1.49)	3.61 (1.78)	3.48 (1.62)
Engineering final year	2.90 (1.52)	2.20 (1.15)	1.60 (1.57)	1.90 (1.44)	3.75 (1.59)	2.60 (2.09)
Engineering postgrads	2.87 (1.19)	2.00 (1.36)	1.07 (1.58)	1.97 (1.55)	2.93 (1.94)	3.40 (2.20)
Practising engineers	3.40 (1.76)	2.27 (1.33)	0.40 (1.30)	1.07 (0.80)	3.00 (1.56)	3.87 (1.36)
Engineers- overall	3.20 (1.41)	1.84 (1.28)	0.89 (1.38)	1.73 (1.34)	3.35 (1.63)	3.49 (1.76)
Significance within engineers	.6207	.1456	.0436	.3513	.4819	.1185
Mathematics at entry	2.89 (1.37)	1.6364 (1.28)	0.60 (1.01)	1.84 (1.42)	3.82 (1.33)	3.65 (1.61)
Mathematics end of 1st year	3.37 (1.44)	2.22 (1.22)	1.22 (1.53)	2.33 (1.41)	2.96 (1.74)	2.74 (2.19)
Mathematics final year	2.79 (1.19)	1.64 (1.15)	1.57 (1.91)	2.57 (1.45)	3.50 (1.83)	2.71 (2.20)
Mathematicians overall	3.01 (1.37)	1.8 (1.26)	0.92 (1.37)	2.08 (1.44)	3.53 (1.56)	3.26 (1.92)
Significance within mathematicians	.2685	.1243	.0216	.1330	.0647	.0639
Significance between subject groups	.3411	.8269	.8681	.0651	.4247	.3745

Table 6-5: Responses to question 5

6.4.6 Question 6

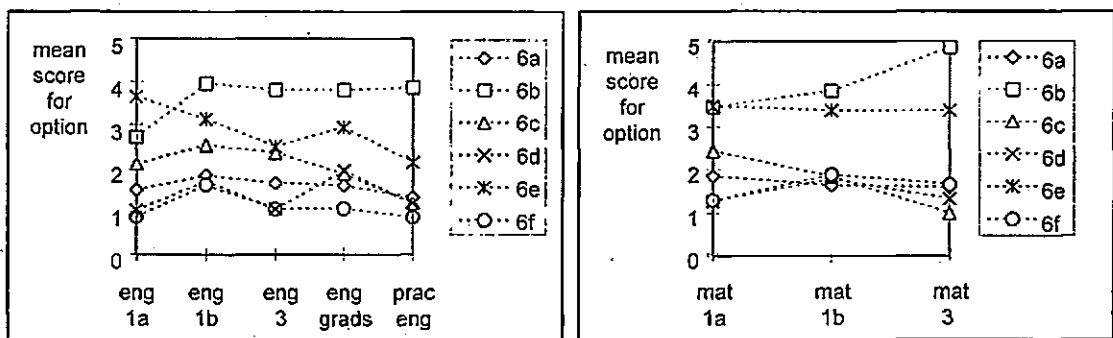
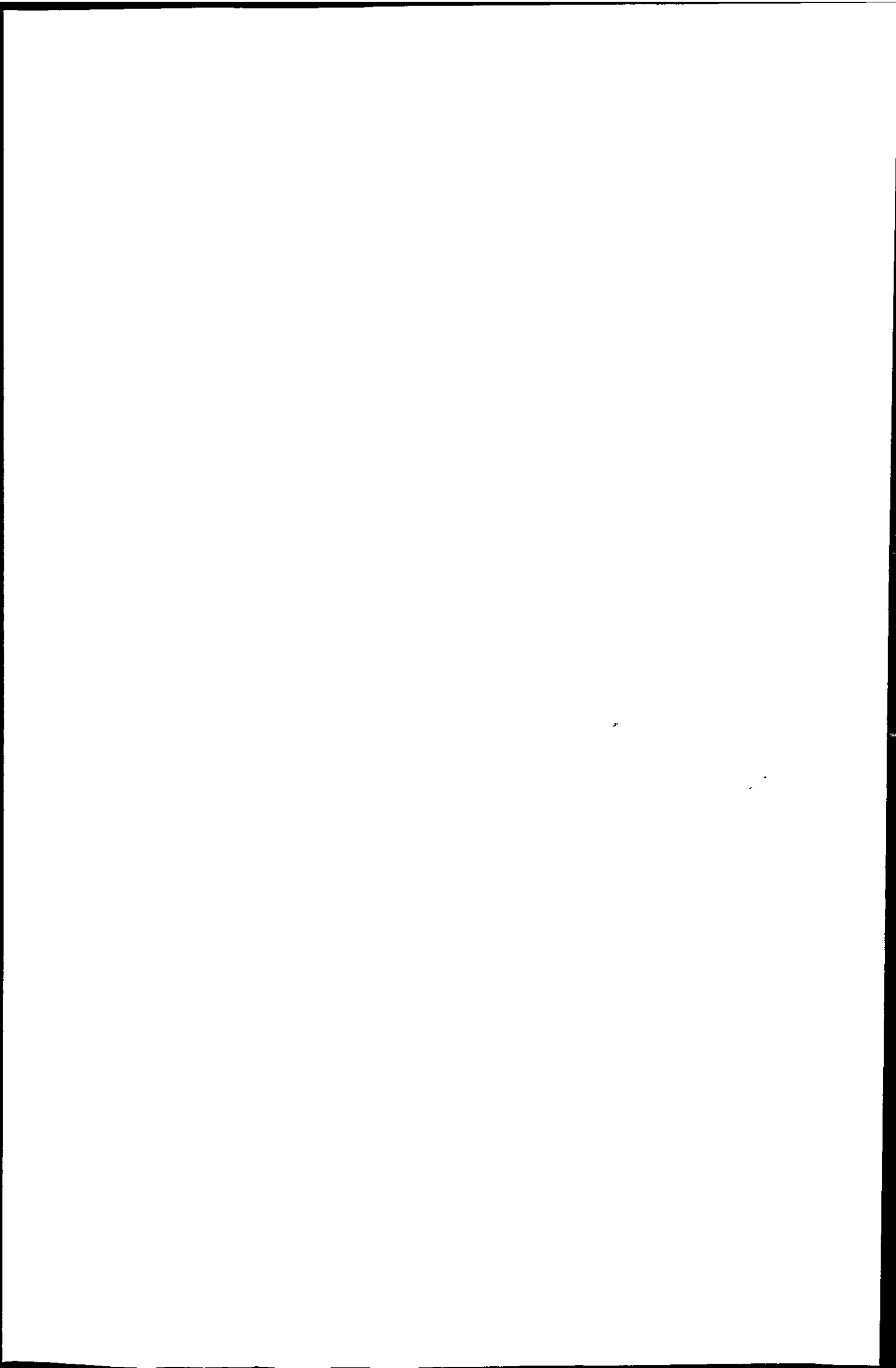


Figure 6-7: Responses to question 6

This question was the most closely related to those used by Chi et al (1981, cited by Royer et al, 1993) in their investigations of the differences between expert and novice



concepts. Apart from the rich crop of comments it evoked from respondents the

question did not appear to produce very useful results.

Mean Rank	6a $y^2 \frac{dy}{dx} = e^x$	6b $\int dy = \int e^x dx$	6c $\frac{dy}{dx} = e^y$	6d $\frac{dy}{dx} = mx+c$	6e $\frac{d^2y}{dx^2} = e^x$	6f $\frac{dy}{dx} = my$
Engineering at entry	1.50 (1.36)	2.70 (1.90)	2.10 (1.89)	1.03 (1.42)	3.65 (1.92)	0.88 (1.14)
Engineering end of 1st year	1.83 (1.59)	3.87 (1.39)	2.52 (1.81)	1.70 (1.33)	3.13 (1.82)	1.61 (1.53)
Engineering final year	1.65 (1.32)	3.80 (1.91)	2.35 (1.81)	1.05 (1.67)	2.50 (1.85)	1.05 (1.23)
Engineering postgrads	1.60 (1.80)	3.80 (1.74)	1.87 (1.60)	1.93 (1.62)	2.93 (1.94)	1.07 (1.16)
Practising engineers	1.33 (1.54)	3.87 (2.03)	1.20 (1.32)	1.20 (1.26)	2.13 (2.20)	0.87 (1.51)
Engineers overall	1.58 (1.46)	3.43 (1.86)	2.08 (1.77)	1.31 (1.48)	3.04 (1.97)	1.08 (1.30)
Significance within engineers	.8732	.0426	.2145	.1679	.0687	.2704
Mathematics at entry	2.00 (1.36)	3.18 (1.63)	2.51 (1.87)	1.35 (1.43)	3.53 (1.90)	1.25 (1.28)
Mathematics end of 1st year	1.37 (1.39)	4.00 (1.24)	1.96 (1.60)	1.59 (1.50)	3.52 (1.63)	1.78 (1.63)
Mathematics final year	1.50 (1.51)	4.86 (0.36)	1.07 (1.27)	1.43 (1.16)	3.36 (1.60)	1.79 (1.72)
Mathematicians overall	1.75 (1.41)	3.66 (1.53)	2.15 (1.78)	1.43 (1.40)	3.50 (1.77)	1.48 (1.46)
Significance within mathematicians	.1256	.0003	.0201	.7595	.9489	.2185
Significance between subject groups	.4053	.3502	.7888	.5589	.0824	.0378

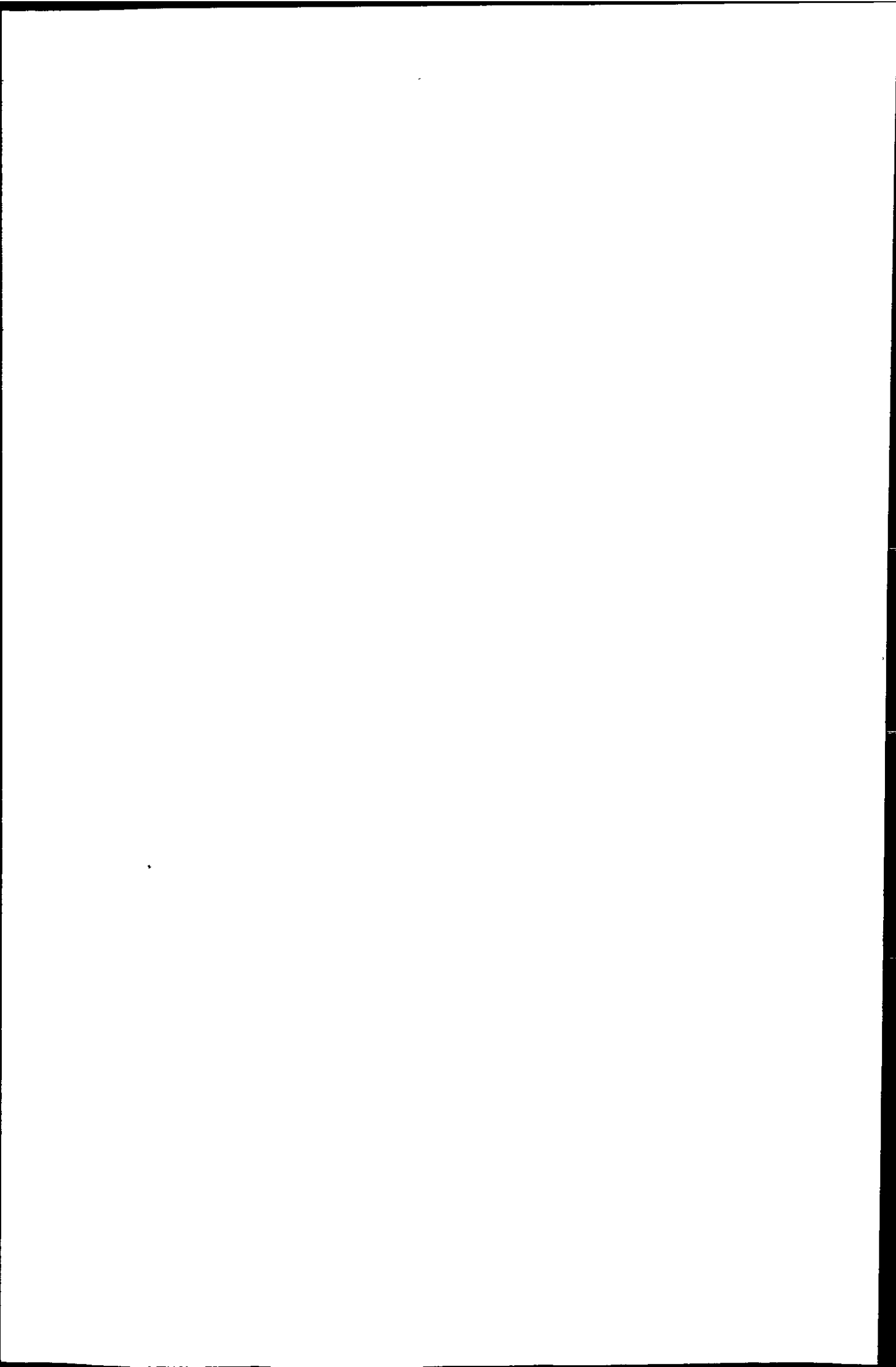
Table 6-6: Response to question 6

The engineers at entry prefer option 6b significantly less than the other engineers, and the engineers as a group dislike option 6f more than the mathematicians.

The two options (b and e) which were added to this question to bring the number of options up to six like the other questions proved to be the most popular with all groups of respondents. Because they were obtainable from the target expression some respondents felt that the other options were "incorrect", despite the strong statement in the rubric, and in the introductory talk, that there were no wrong answers and that the questionnaire was not a test of any kind.

Apart from the strength of some of the comments, it is notable that there is not complete consistency in the options felt to be “correct”. This is a good example of where a group of students may be holding concept images different from one another and from their lecturer, and for communication to be affected by the different meanings ascribed by the various parties to supposedly public symbols.

- (a, c, d, f) are all wrong. (Practising engineer)
- This one I found most difficult- only b seemed “right” (Practising engineer)
- beyond e, no obvious similarity. (Practising engineer)
- b is sol. of example, f has got right form, c is wrong (I think!), e is next derivative, d & a are probably wrong as well. (Practising engineer)
- b = ok. f → impossible. (Final year engineering student)
- The last 3 [d, f, c] have nothing to do with it. (Final year engineering student)
- Latter ones [f, d, e, c, a] are not related (Final year engineering student)
- a, c and d are not ranked- they’re all very different to the given equation. (Final year mathematics student)
- e, d, c have no relationship with the original, although as pictures it is possible to see a likeness. (Final year mathematics student)
- b, e - rest false! (engineering lecturer)
- c, b. Rest don’t figure (in my eyes) (mathematics lecturer)
- Only b looks good. All the others look equally wrong. (Engineering postgraduate)
- f, b. Only those! (mathematics student, start of first year)
- f is wrong, surely? (a) opposite to above question. (mathematics student, start of first year)
- e. Don’t like the others. (mathematics student, start of first year)
- e. I can’t immediately see any similarity in any of the others. (engineering student, start of first year)
- [first preference c] No real link in many cases (engineering student, start of first year)
- d is rubbish. rest are nothing like above. (mathematics student, start of first year)
- b is the only correct one. (engineering student, start of first year)
- e, b are the only true ones. The others don’t even rank. (engineering student, start of first year)
- e, b. Both true statements. [implies that the other options, unlisted, are untrue] (engineering student, start of first year)
- e first as it is a true statement. [implies that the other options, unlisted, are untrue] (engineering student, start of first year)



Some respondents did not distinguish between “wrong” and “right” answers- all the responses were “wrong” for them.

- None are similar. (Final year engineering student)
- Don't really understand the question as all the answers are different from the sample. (Final year engineering student)
- All dissimilar (Engineering postgraduate)
- none really (mathematics student, start of first year)
- None of them look like the top one. They have all got a similar amount of dissimilarity. (final year engineering student, pilot study)
- None equivalent. (mathematics postgraduate, pilot study)

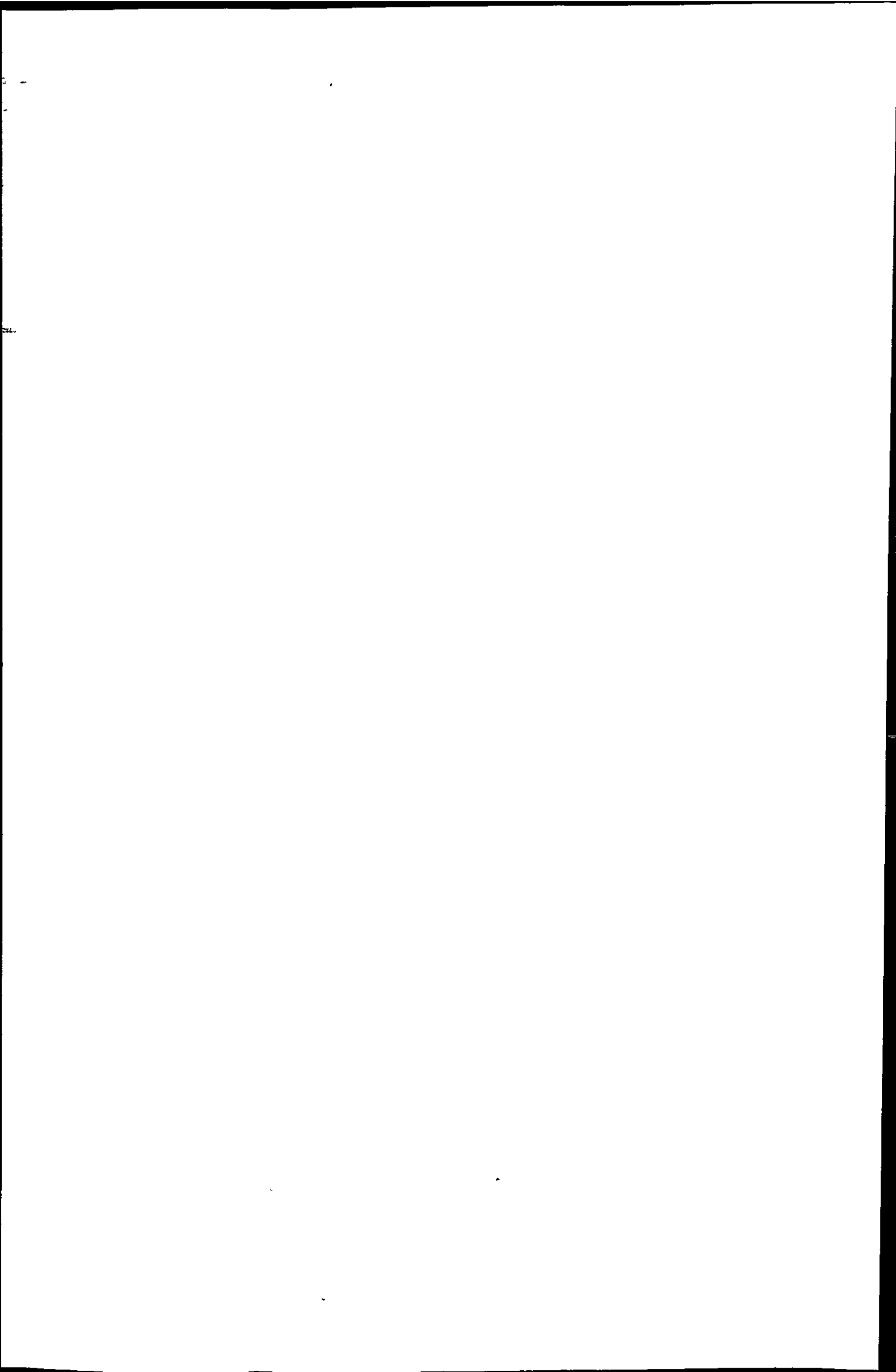
Some respondents found the word ‘similar’ (or ‘like’ in the pilot study) problematic.

- Nasty one- what do you mean by similar?! (Mathematics lecturer)
- Not a particularly clear one this- what does ‘similar’ mean? (Engineering lecturer)
- This question doesn't make much sense to me... I guess that ‘similar’ is not well defined, but perhaps that's the point! (Engineering lecturer)
- What do you mean by similar, the same result, played with, what? (engineering student, start of first year)
- By ‘like’ do you mean form or value? (Engineering postgraduate, pilot study)
- As an engineer, I find the word ‘like’ in the question confuses the issue. (Engineering postgraduate, pilot study)
- ‘Like’ what is that, is it exactly the same relationship i.e. the same equation or does it mean the same order of differential equation. Confusing. (final year engineering student, pilot study)

Some students early in their careers are tentative, but seem to look forward to learning more. The engineering students at the start of their first year were the only group to prefer option 6e over 6b overall. The change is probably because the first year of the degree is the first time some students encounter differential equations.

- slightly beyond me; only sure about e. (mathematics student, start of first year)
- after e, all the rest are guesses. (mathematics student, start of first year)
- Not familiar with logs etc so mainly guess. (engineering student, start of first year)
- e first. can't really say about the others yet. (engineering student, start of first year)
- only just started differentiation so do not fully understand it. (engineering student, start of first year)

Some engineering respondents were reminded of how much mathematics they had forgotten, or how little they had used. This point was also made in the OECD report



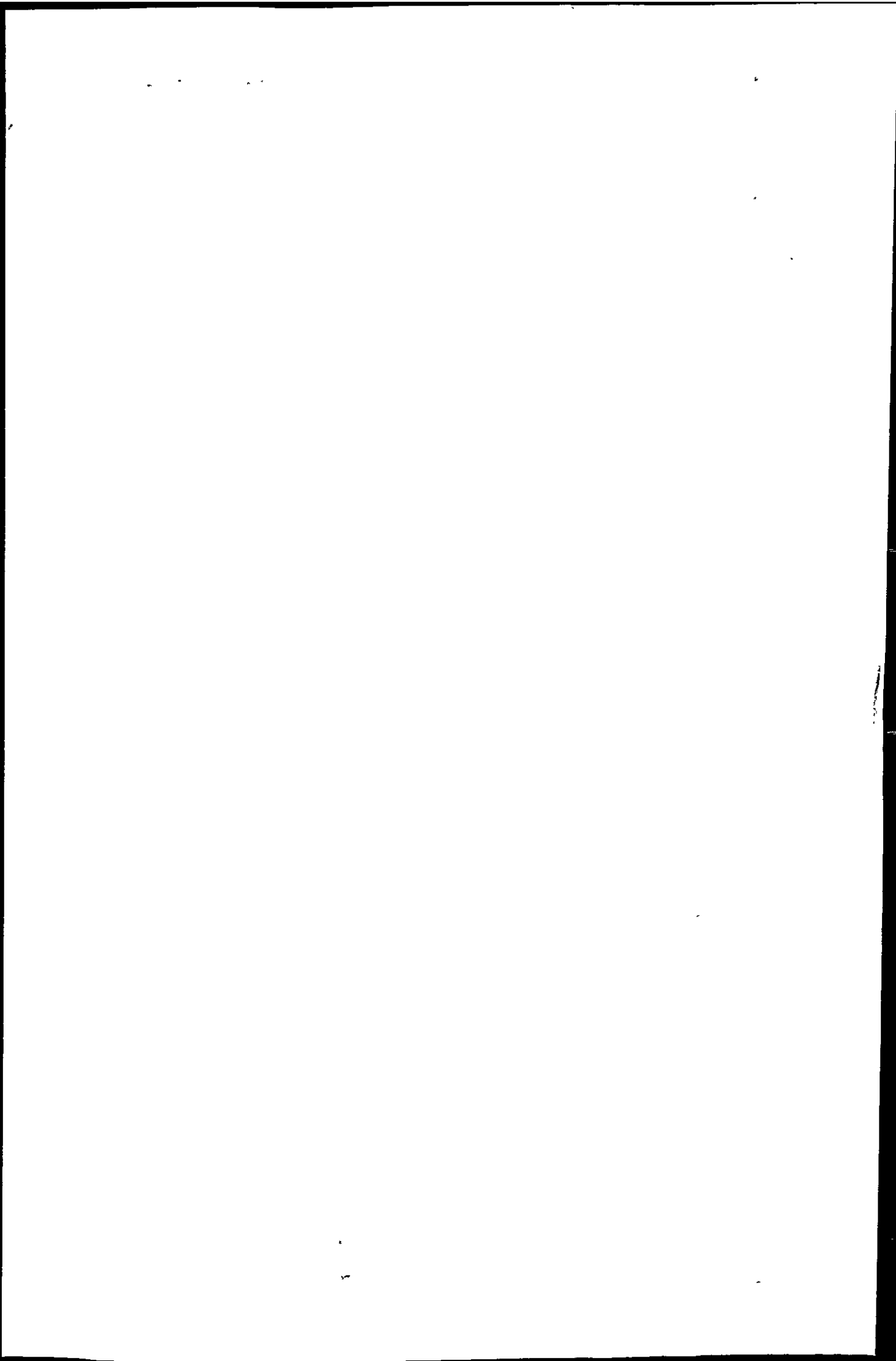
(1966) where most of the respondents reported using nothing more than simple algebra in their engineering careers.

- This makes me realise how long ago I did my engineering maths course and how little I've used it since!!! (Engineering postgraduate)
- I can not remember much about my degree only 1½ years ago. I can remember most of the chatty explanation or lab work. (Engineering postgraduate)
- I can't remember how differentiation works so I have guessed based on what I can remember. (Engineering postgraduate)
- I think I've forgotten everything! This is sadly honest but I hope it helps. (Final year engineering student)
- I have managed to forget differential equations for the past 40 years with great success! (Practising engineer)
- Having not used calculus for 18 years, I'm guessing here. (Practising engineer)
- b,? I would be guessing the remainder. (Practising engineer)
- It's not easy to admit it but it's so long since I even looked at simple equations such as this there is no rationale behind my list. (Mathematics lecturer)
- Differential Equations for engineers are on a need to know basis. For exams I needed to know: now I don't. (Engineering postgraduate, pilot study).
- I've not done anything like this since my first year, hence I've forgotten it all. (final year engineering student, pilot study)

6.5 Mode of representation

Crowther (1997b), after interviewing some eighty engineering students, found them to feel that they learn best from visual or previously understood concrete mathematical examples. The preferences expressed by the students in this investigation tend to agree with this. Diagrams such as 1b, 1f, 2f and 3f were consistently popular choices with both engineers and mathematicians. Comments such as :

- Pictures represent a thousand formulae! (physics graduate, pilot study),
- Representation is preferably visual for me (practising engineer),
- I prefer to visualise the effect and then calculate the how and why. (engineering graduate, pilot study)
- (pinball question) I used f to visualise the problem. Once the problem was sorted in my mind I then put the mathematics around the picture in my mind. (engineering graduate, pilot study)
- (Q1) I can understand (number crunch) e & c, but I prefer to visualise d & f. (engineering graduate)
- (Q1) b, f is how I think about the problem, visualise. (engineering graduate)



- graphical representation shows response and allows visual comparison against others. (engineering graduate)
- I always have a graph in my mind first, then I really think about the problem. (engineering graduate)
- I would rather have a drawing... (final year engineering student)
- Visible representation important (final year engineering student)

show the respondents thinking of themselves as visual people.

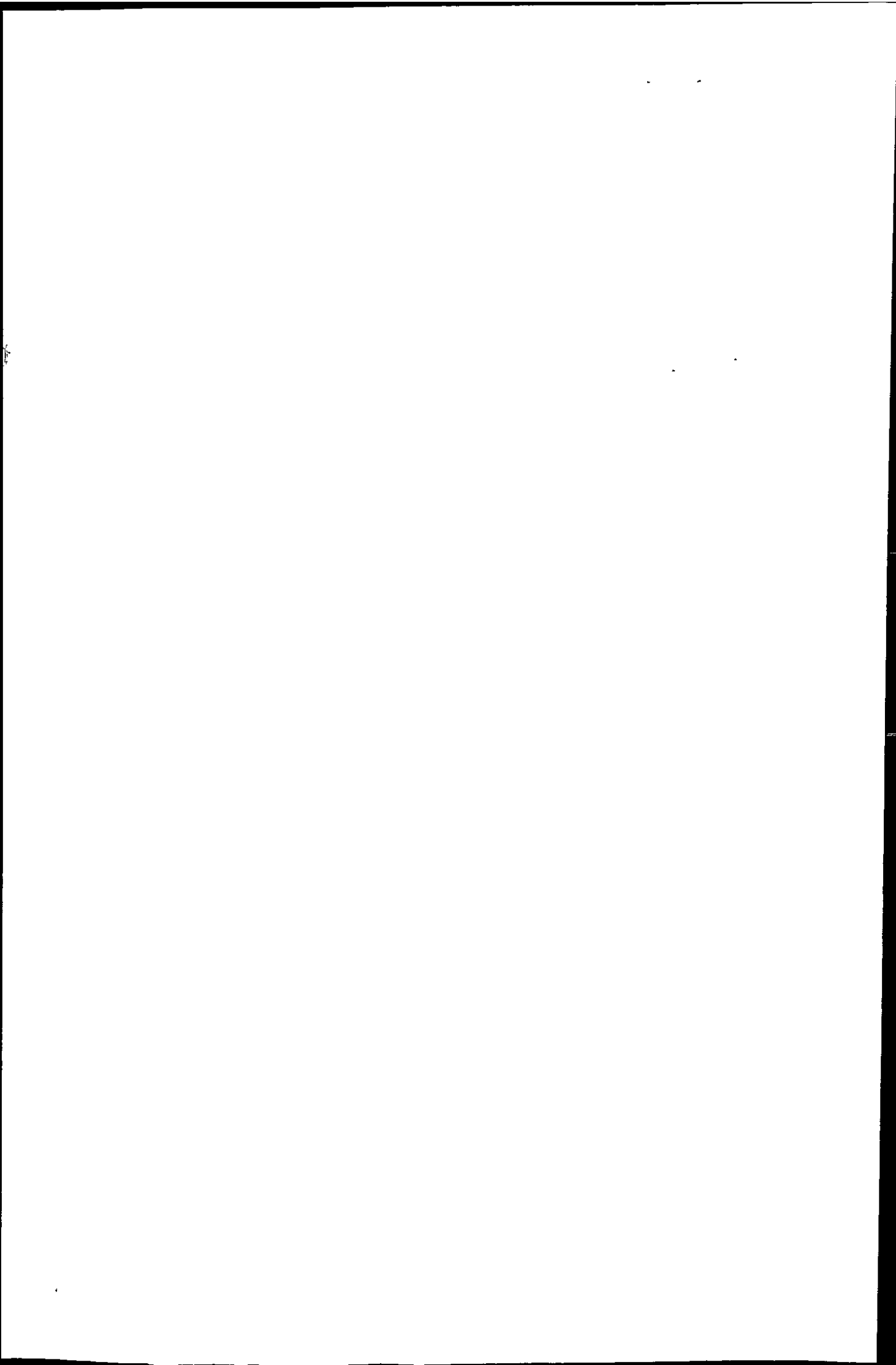
Some diagrammatic options, however, were not popular.

In question 2 option (a), the statement " $f'(x)$ is the slope of the tangent to a graph of y against x " was the most popular with all students and in question 4 the most popular response was (a), the statement " q is the area under the curve $y=x$ " although diagrams were available as options.

In question 3 the phase plane diagram was among the least popular, although a respondent on the pilot study commented "Never seen c before. It's good if that is a valid representation of the problem" (final year engineering student, pilot study). In question 4 options (c) and (e) were less popular than option (a) quoted above and in question 2 option (b) remained in the middle of the field.

In other words, although the respondents declare themselves to be visual people in general, options such as 2(a) and 4(a) which are sentences which are virtually drilled into learners of mathematics at an early stage are hard to dislodge as the correct and automatic response, and an unfamiliar graphical representation is not generally acceptable, for example:

- c shows nothing unless you specifically studied the subject. (final year engineering student)
- b, c and d do not convey the meaning at all well unless you have a detailed understanding. (final year engineering student)
- Unfamiliar with c (engineering graduate)
- I have never come across answer c. (engineering postgraduate, pilot study)
- c, d, b don't mean much to me. (engineering postgraduate, pilot study)
- c means nothing to me (engineering postgraduate, pilot study)



- I have never come across a representation of this type of question as shown in figure c. (final year mechanical engineering student, pilot study)

but on the other hand, the representation is valued by those to whom it is familiar:

- c defines/represents both velocity and displacement whereas f represents only y with time. (engineering postgraduate, pilot study)
- f perfectly describes for me a damped osc. c is what I would use next because it gives more better (*sic*) information. (first year computer systems engineering student, pilot study)

and also by some who, though not having seen the representation, can see it has possibilities:

- Never seen 'c' before. It's good if that is a valid representation of the question. (final year engineering student, pilot study)

We also find comments such as:

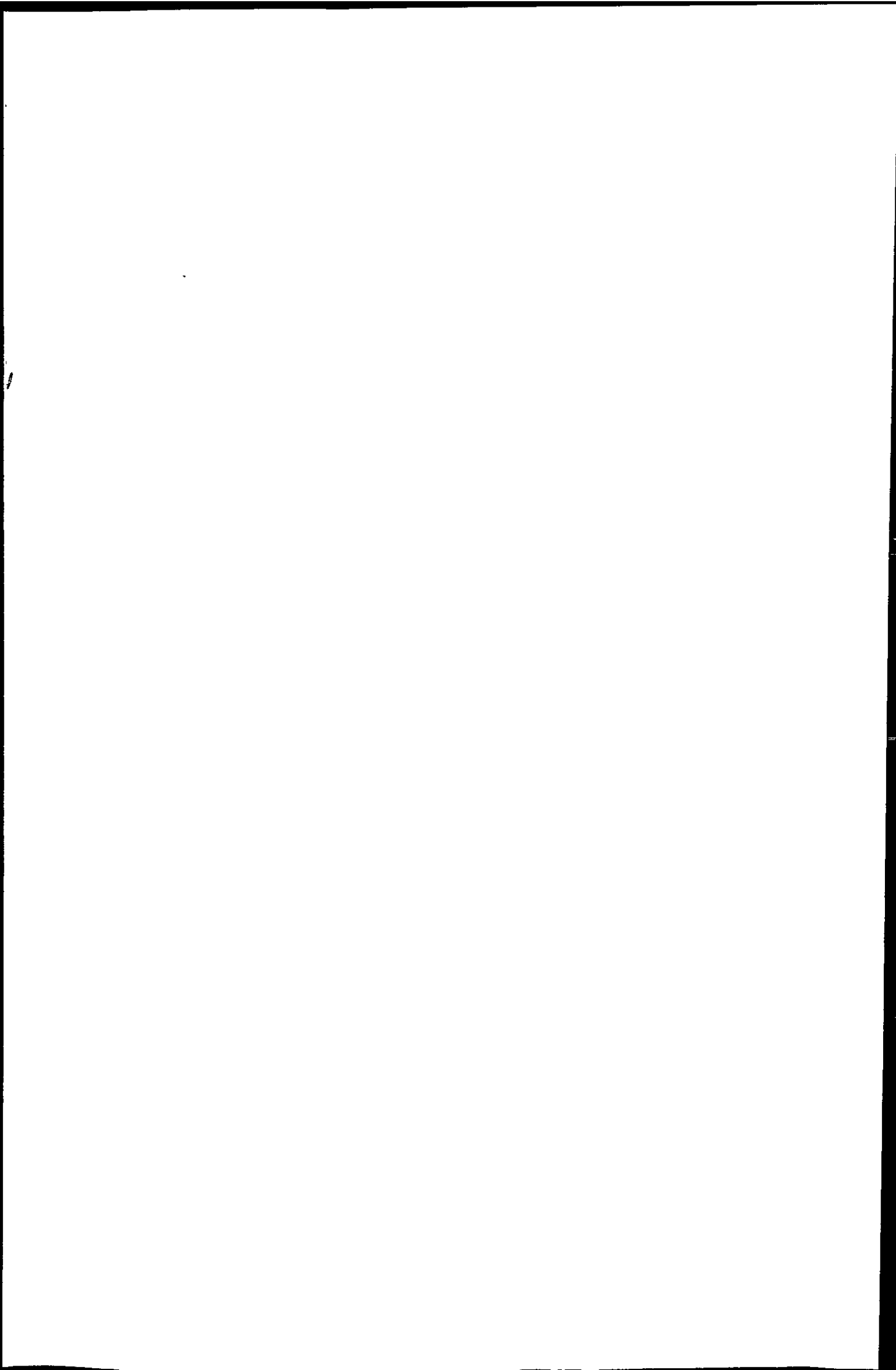
- Words first to get an idea of the problem. Then a 'diagram'. Then some maths = Greek! (engineering graduate, not a Greek student)
- as an engineer I tend to represent problems like this first verbally, then graphically, and as a last resort mathematically. (engineering graduate)
- Written and graphical solutions seem easier (final year engineering student)

which show the respondents representing themselves primarily as verbal, then as visual thinkers.

6.6 *Attitude to mathematics*

In addition to the above comments on their feelings about mathematics, some respondents made some more trenchant observations on the relation of mathematics to the engineer.

- Having not used calculus for 18 years, I'm guessing here (practising engineer)
- I have managed to forget differential equations for the last 40 years with great success! (practising engineer)
- I can't remember how differentiation works (so I've guessed) (engineering graduate)
- I've never really seen a link between the maths and the results of engineering and understanding. (engineering graduate)
- Not mathematically inclined (engineering graduate)
- (Q6) This makes me realise how long ago I did my engineering maths course and how little I've used it since!!! (engineering graduate)
- I was never very good at integration. Couldn't learn the tricks. (engineering lecturer, pilot study)



- Differential equations for engineers are on a need to know basis. For exams I needed to know, now I don't! (engineering graduate, pilot study)
- (differential equations question) I may be an engineer but to be quite honest, formulas like that are a complete waste of time and energy. (final year engineering student, pilot study)
- I am more practical than academic and I hate maths! (engineering graduate, pilot study)
- I don't know much about maths. All I know is about big bits of metal, Chevy V8's and Holley carburettors. The only maths I can do is what the magic calculator can do. (final year engineering student, pilot study)

It is worth bearing in mind that all the engineering graduates surveyed are enrolled on a course of postgraduate study, either MPhil, MSc or MBA, so they cannot be regarded as study-phobic.

Almost identical statements come from a student who was interviewed in the evaluation of the courseware as reported in chapter 15.

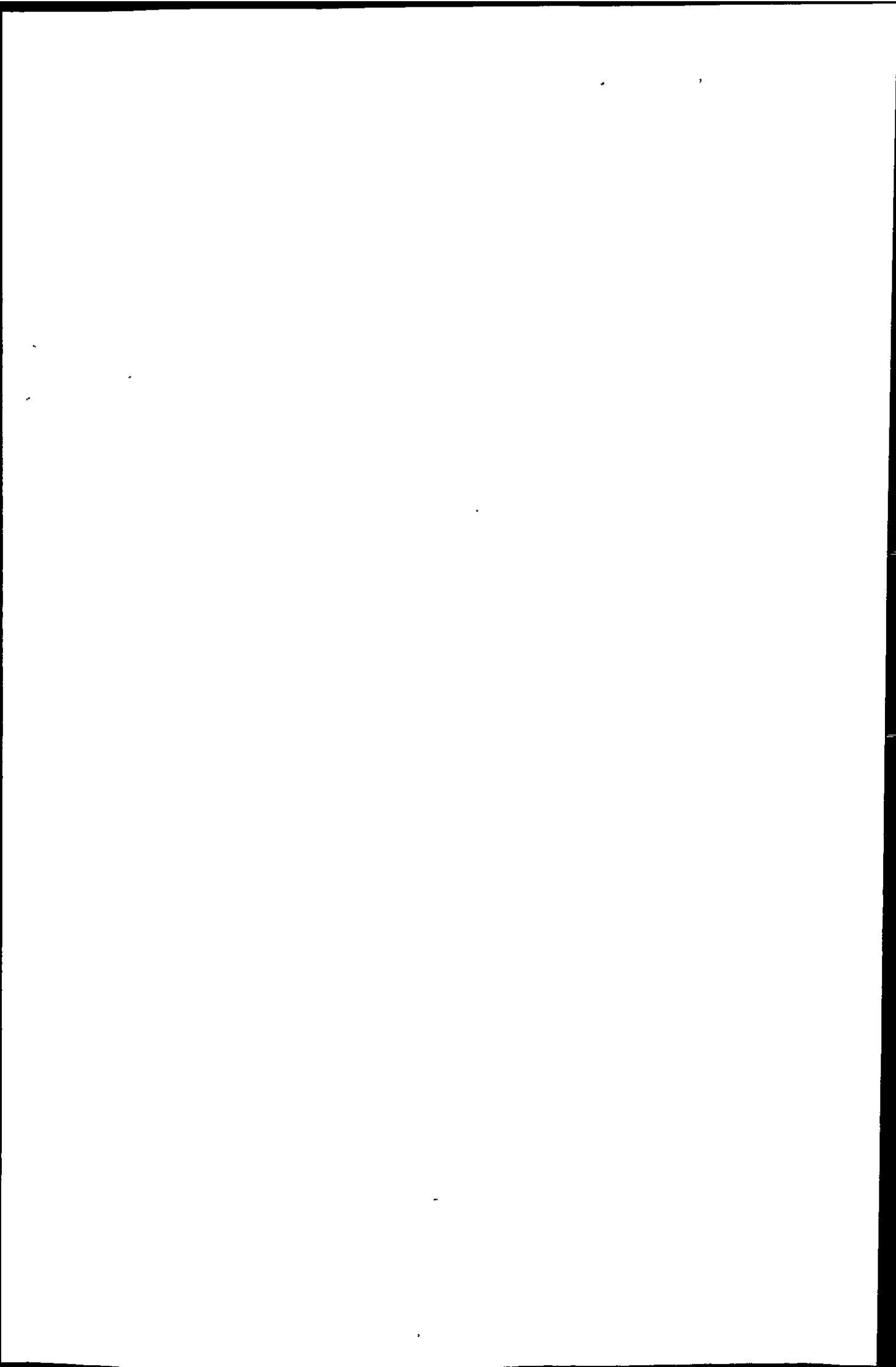
- Well, given a reference, I'm happy enough with understanding the calculus- I've forgotten all the transforms myself. When you use them a lot you know them, you just click them in, but I've forgotten all that.
- My maths is very rusty- I haven't been using it for a year and I haven't had to use it so far this year.
- I haven't been using any maths for the last year being on placement so my maths is very rusty.
- I'm trying to avoid mathematics this year.

These comments may be summarised as:

- * Mathematics is found in books,
- * Real engineers don't use mathematics in their jobs,
- * Mathematics is something you learn for exams and then forget.

These comments are reflected in the overall low popularity of algebraic forms of options: 1(c) and (e), 2(d), 3(b) and (d), 4(f), 5(c), and the short lists in the responses to Q6.

Another aspect of the attitude to mathematics is that the engineering respondents regard themselves quite strongly as practical, applied people. In addition to the last two



comments quoted from the questionnaires above, we have remarks such as the following, showing a feeling that engineering knowledge is applied knowledge.

- Only the old fashioned mechiees are likely to know what a dashpot is. And anyone who has worked on a Stromberg carburettor. (engineering lecturer, pilot study, response to mass-spring-damper question)
- When you hit a big bump at 80 mph in your 5.0 litre V8 Landrover, it goes 'bang' and it launches to the moon. When it lands it goes 'bang' again. (final year engineering student, pilot study, response to mass-spring-damper question)
- Derivative of y is what? Derivative of x is what? Are they voltages, decibels, time, frequency or APPLES? (final year engineering student, pilot study, in response to differentiation question, in which the word 'derivative' does not appear)

6.7 *Depth of representation*

Changes may occur as a direct response to teaching, or else as a result of use and familiarisation, resulting in the maturing of concepts. This would be characterised by a greater meaningfulness of concepts, a greater richness of associations between concepts and an encapsulation of the concepts.

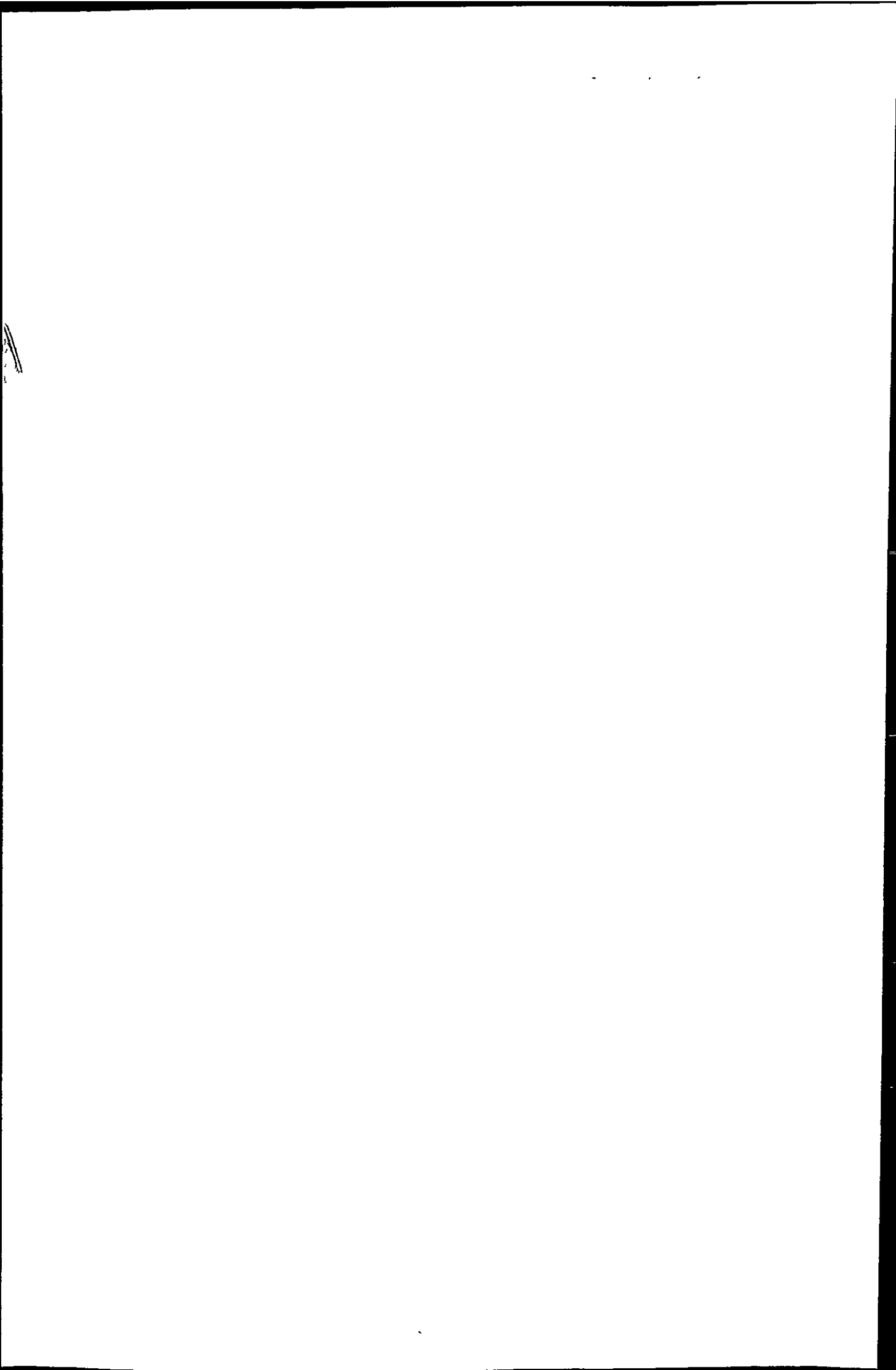
6.7.1 Growth in meaning

In Q2, option (c) states that dy/dx tells you how quickly something is changing. This option gains steadily in popularity after the first year in the engineering groups, but not in the mathematics groups. An engineering graduate commented, having completed the questionnaire, that he learned to understand the differential as a rate of change during his year of industrial experience, and that was when the idea gained meaning for him.

Other comments from the questionnaires on this option are:

- My extra learning has taught me that $dy/dx = \text{gradient}$ which I now understand as a rate of change in a business environment. (engineering graduate)
- c is the overriding image in my mind. (engineering graduate, pilot study)
- c is how I subvocalised the question. (engineering graduate, pilot study)

In the engineering graduates, (c) is the most popular option, but in the practising engineer group it has just been overtaken again by (a), the slope of the tangent.



The corresponding option in Q4, "The integral tells you how things build up" (option(d)) is the least popular option with both engineering and mathematics undergraduates, but whereas its popularity decreases from the first to final year mathematics students, among the engineering groups its popularity increases steadily. This could be interpreted as a steady increase in the meaningfulness of the concept in engineering applications, and its lag compared with the corresponding option 2(c) may be explained by the way integration depends as a concept on that of differentiation. Understanding of integration follows that of differentiation: individual making of meaning in integration also seems to follow that in differentiation.

6.7.2 Richness of association

Changes in popularity across the first four groups of engineers tested and across all three groups of mathematicians can more confidently be attributed to development, since their experiences of higher education have all been comparatively recent, and similar. Comparisons with the practising professional engineers are more problematic as these latter graduated in general in 1978, and from old universities, so their experience of higher education was different from the younger groups. Where we see a trend across the first groups not continuing into the last group, we cannot predict whether the present engineering graduates are likely to show the same characteristics in, say 15 years' time.

An example of this sort of trend is option 4(f), the integral interpreted as the inverse of differentiation. There is a steady increase in popularity across the first four groups which I would like to interpret as an increase in the depth of representation as the concept of integration becomes more related to that of differentiation.

6.7.3 Encapsulation

It seems that one of the ways in which knowledge changes as expertise is developed is that the concepts become more richly related, and that groups of related concepts are chunked together within a group. The richness of association has been discussed above; the chunking together or encapsulation follows from that stage. Several authors refer to this process in different ways, so for instance we have Tall's *procept* (Tall specifically refers to procepts as encapsulation in Tall, 1995), Anderson's *knowledge compilation* (1982, see Royer et al, 1993); Schoenfeld's *heuristics and algorithms* (1985), and Bandura's *schema construction* (1977). The interest in concept images in this case is to discover the *indexical* image: that is the image which is used by the individual as a label which evokes the schema as a whole. In semiological terms, we are looking for the most powerful signifier for the concept.

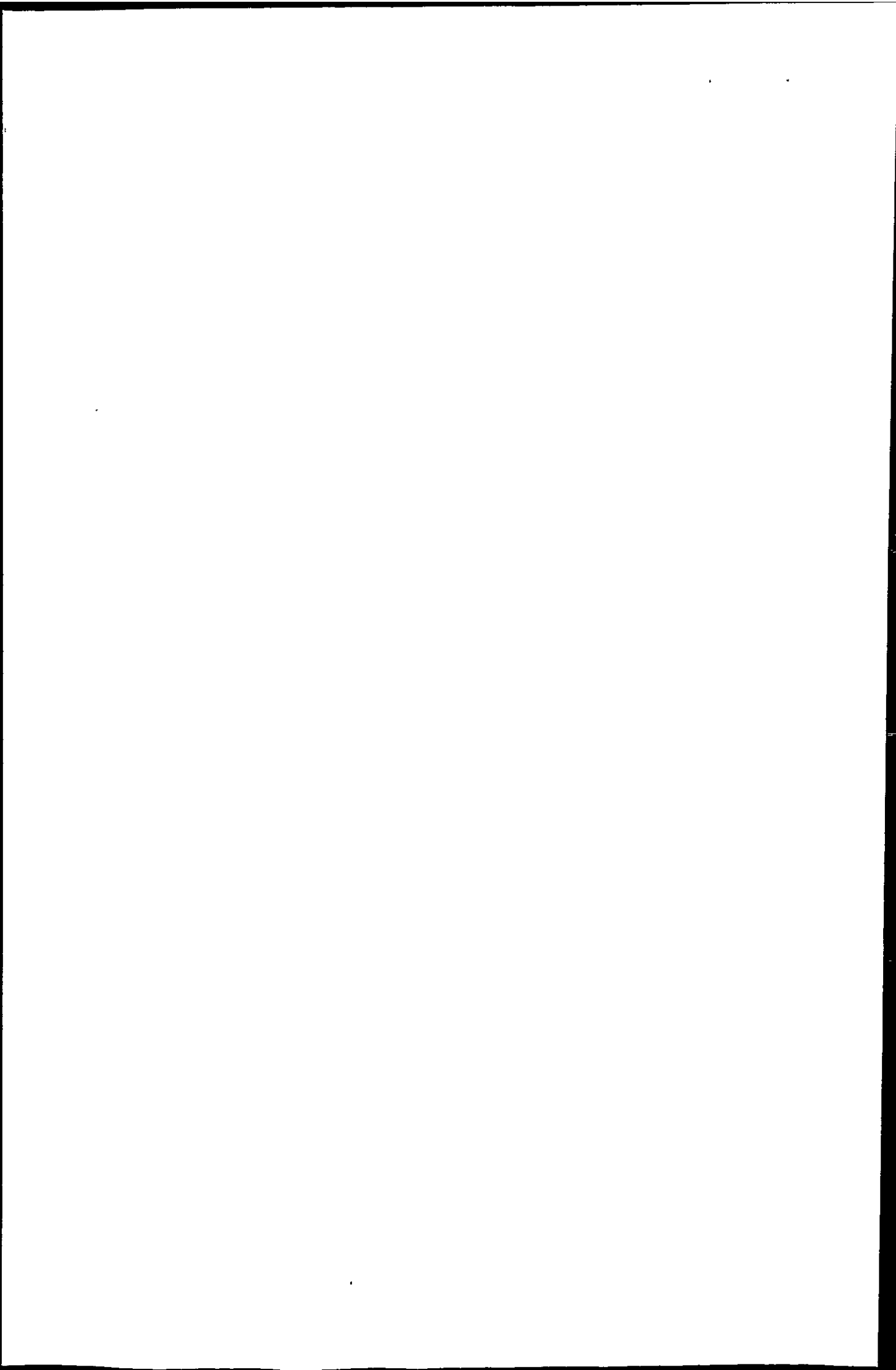
We also see that respondents may be aware of having more than one evocative image, depending upon context.

In the responses to the questionnaire, the options which appear to relate to encapsulation are 1d ("A simply supported beam...") and 3e ("Damped harmonic response"), where the given case is expressed as a particular instance of a class whose general solution is known.

6.8 Conclusions

6.8.1 On the analysis of the results.

The questionnaire yielded two types of empirical materials. The data on preferences was basically quantitative, although quantitative data about a qualitative subject (such as preferences) is a problematic entity. Some statistical analysis on this data was possible, and some more will be examined in the chapter on component analysis, but the small



size of some of the groups (particularly after the first year groups) meant that interesting results were not statistically significant enough to make strong statements about them.

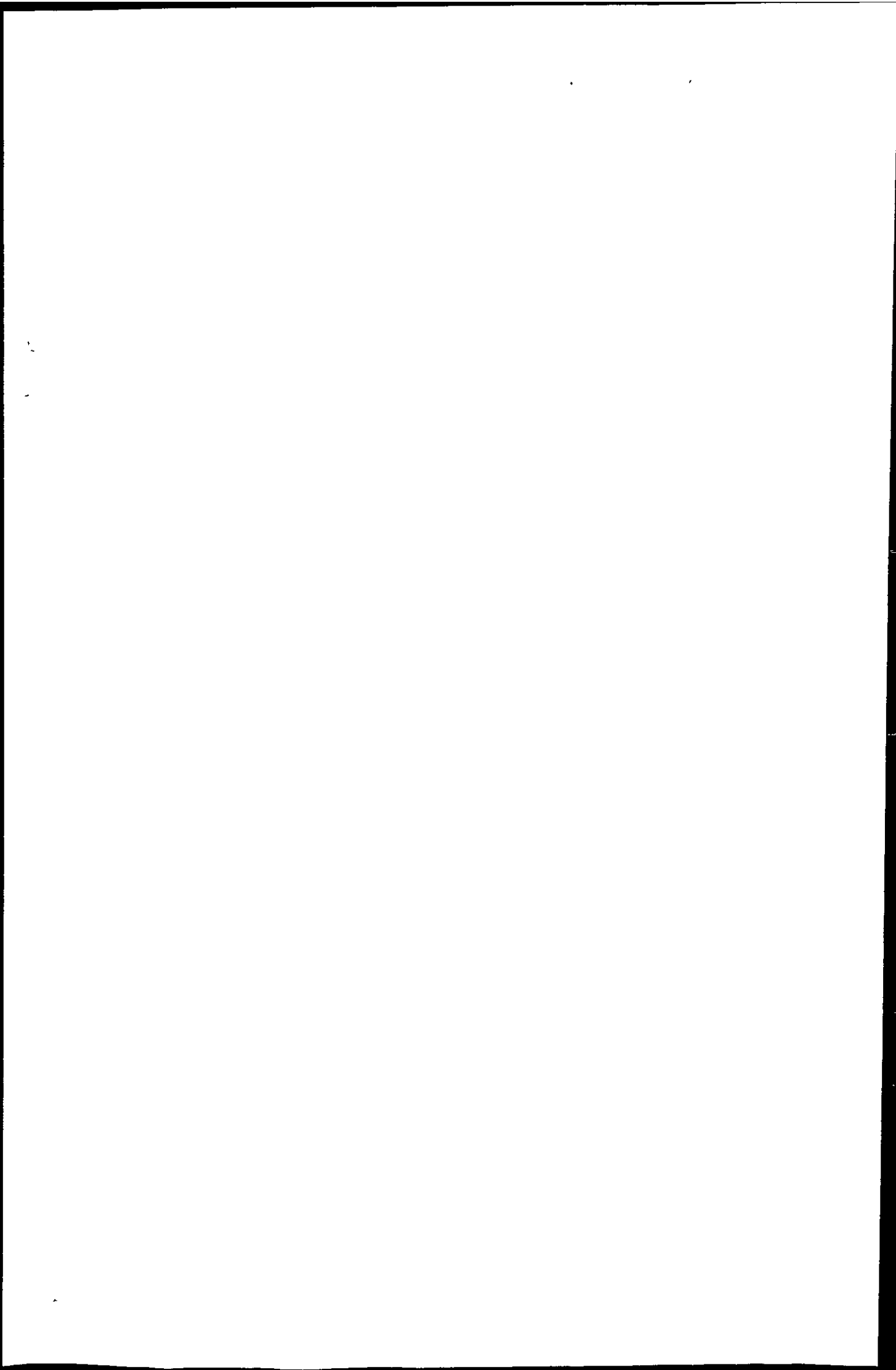
One reason why it was difficult to obtain large numbers of responses from the final year students is that it was difficult to locate groups being taught together, as they tended to have opted for different module choices in the final year. In the first year it was much easier to find large groups of students being taught together. The decline in numbers of responses from the first to the final year is thus not solely due to dropout from the degree courses.

The analysis of the themes brought out by the comments is basically qualitative, although it is possible to perform quantitative analysis such as comparison of the frequency with which themes are mentioned. Again there are not really enough responses here to justify such an undertaking.

6.8.2 On the mathematical representations of engineering students.

The engineering students, although in general they appear to regard themselves as visual people, seem to prefer verbal representations of mathematical concepts. This may be because they truly do prefer verbal representations, or may be because their individual visualisations are idiosyncratic and do not coincide with the diagrams presented to them. In "mechanics" questions the diagrammatic representations are the preferred option. As Presmeg (1986) and Tall (in Tall, ed., 1991) point out, visualisation is not generally encouraged in tackling mathematics problems, and so there are few standard mathematical diagrammatic representations. Mathematical pictorial representations are private.

In mechanics problems, in contrast, drawing a diagram is the first stage of the standard solution procedure. As well as being "approved by authority", mechanics diagrams have standard, public forms which are easily recognised by students and reproduced by the



designers of questionnaires.

In all the questions, there was a scatter of preferred options. Although we can generalise about a group having a preferred option, very rarely was that option preferred by the whole group. In some cases, an option which was preferred by some members of a group was strongly rejected by others.

6.8.3 On the questionnaire.

Different questions yielded different types of results.

Question 1 revealed two unexpected sets of misconceptions. The rigidity misconception is well known, but the wrongly estimated shape of the bent beam is not well documented. The practical importance of the misconception is probably slight, given that the people who hold it have presumably never had an experience which would cause them to change their minds. It is also a reminder that misconceptions are a part of the mental luggage of most of us.

Questions 1, 4 and 6 also showed changes in responses as a result of teaching. This is an encouraging result, in that such changes could be seen at all. New learning can be seen as a sharp peak in the popularity of an option, shortly after its acquisition, which falls off afterwards.

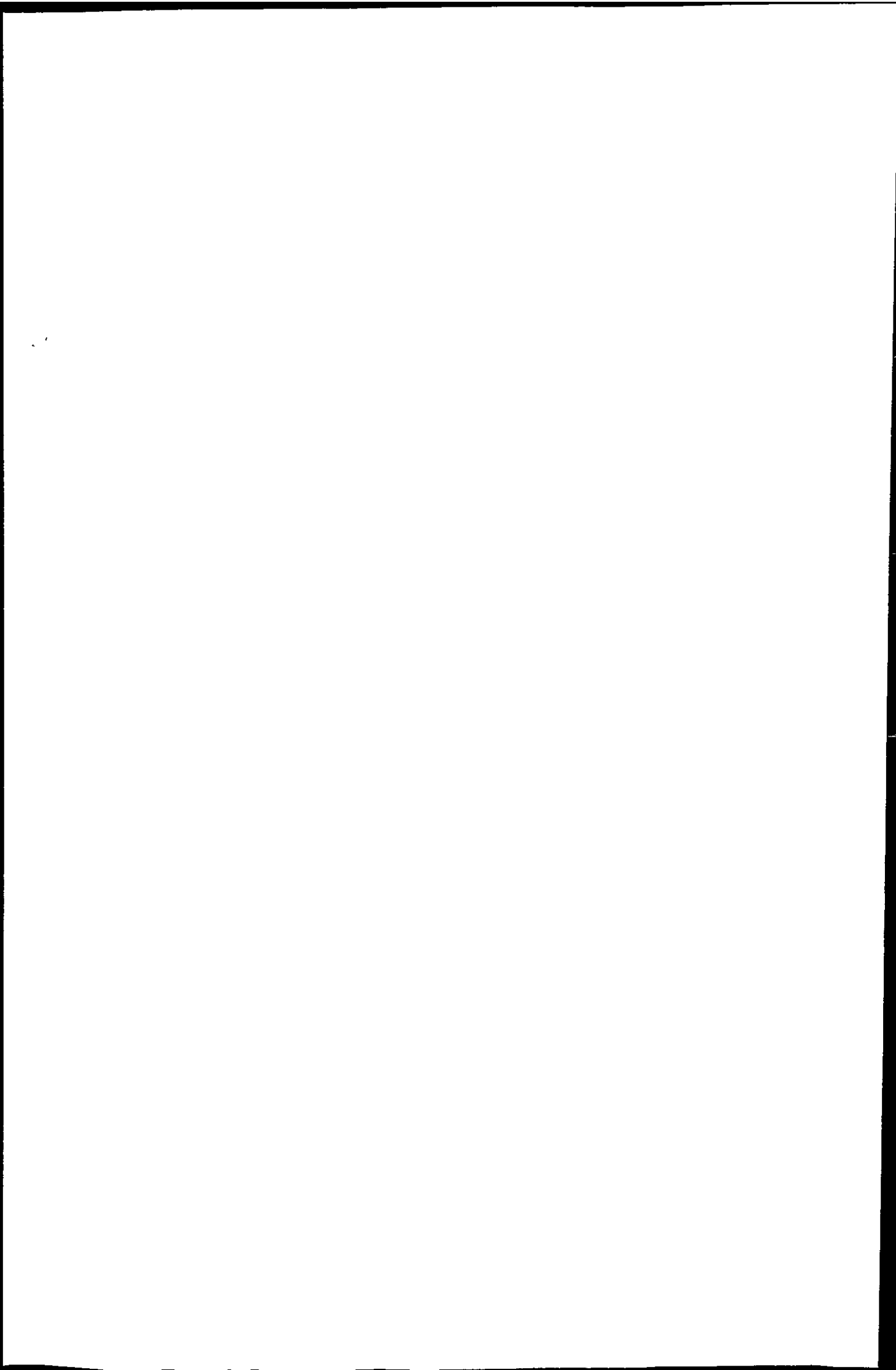
Question 2, and to some extent question 4, showed changes in the engineering students' responses as a result of experience. Learning through experience is shown as a gradual rise in the popularity of an option, without there necessarily being a drop afterwards.

These changes (in response to teaching and to experience) did not appear in the same way in mathematics students' responses, so it may be concluded that they were particular to the engineering students and not part of the general process of maturation through a university degree course. It is not suggested that there are no changes in

mathematics students' mathematical ideas: the target concepts used in this study were chosen to be particularly appropriate to engineering students rather than to mathematics students.

Question 3 showed strongly polarised reactions to a graphical representation, the phase plane diagram. Among those who were familiar with it, and perhaps some who were not, it was a popular option. Amongst most of those who had studied the topic of damped harmonic motion without using the representation it was strongly rejected. One of the themes emerging from the comments was the increased confidence with which options were rejected by those who had studied a subject.

Question 6 produced comments from students which revealed some attitudes to mathematics: that mathematics has correct and incorrect answers, that most mathematics is not really relevant to engineers and that mathematics can be found by looking in the appropriate texts.



7. Component and factor analysis

7.1 *Introduction*

The interpretation of the responses in the previous chapter depended on considering the set of responses to each question as a separate entity. In order to look for patterns in responses from question to question, we may turn to component analysis. This is an analysis of the correlations between variables: it does not claim to address causal relationships but rather seeks to identify patterns in the relationships between variables. In this chapter we will review the theory and process of component and factor analysis, and in the next chapter we will examine the results of component analysis on the responses to the questionnaire.

7.2 *Factors in statistics*

The idea of factors enters into statistics in two distinct ways. In the first, the factors are already known, and their interaction is the object of interest: for example in the treatment of crops, the factors may be irrigation and use of fertilisers, or in mathematics teaching, the use of an explorational approach and the employment of graphic calculators. Experiments are designed such that the factors are applied and controlled for separately and in combination, and then the results from the different treatments are compared. It would be possible to divide the students who responded to the questionnaire into (a) mathematicians and engineers, and (b) first years (early and end) and final years, and regard those as treatment factors, and devise a criterion for examining their within-group and between-group variances.

On the other hand, it may be impossible to carry out such controlled experiments, but nevertheless it may be felt that the patterns of correlation of experimental variables are showing that some underlying factors are operating. Under such conditions one may

11

12

13

14

try to induce the nature and relative importance of such factors through techniques known as factor or component analysis. It is in this second sense that we are interested in factors here.

7.3 Objectives of component and factor analysis.

As stated in the introduction, when it is suspected or hoped that underlying a number of variables there exists some simpler structure, component or factor analysis can be used to induce what those components may be.

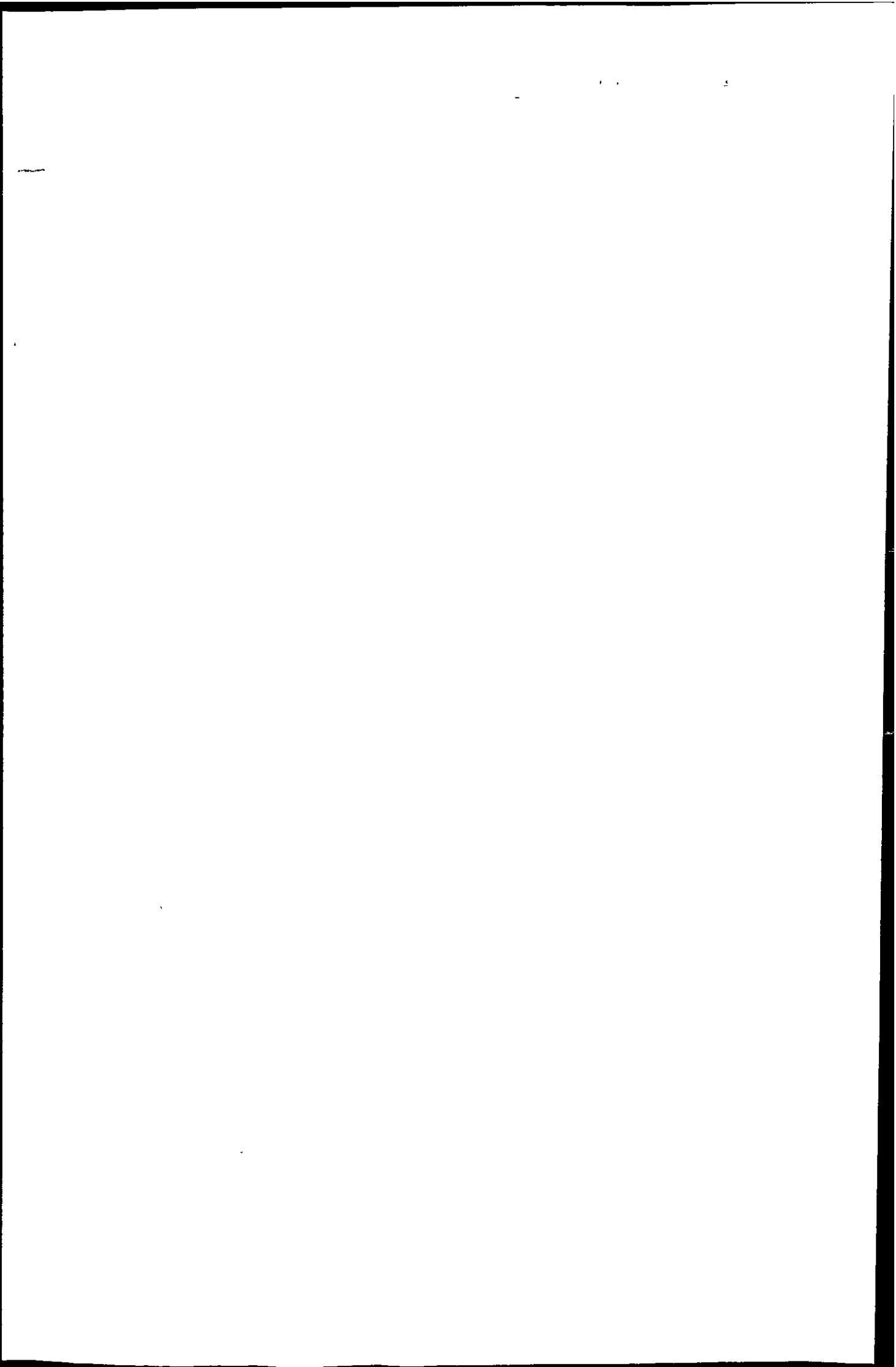
Component analysis may also be used to identify variables in a set of data which are essentially measuring the same thing, and which may give undue weight to that component when the variables are being used to group cases, as they may be in market research. For example, if out of a set of ten variables, two cases matched exactly on seven, but were different on three, it would be tempting to say that the two cases were very similar. If however component analysis revealed that these seven variables all loaded heavily on a single component, but that the other three were unrelated, then the two cases would in fact be identical on one variable out of four, and their perceived similarity would be much lower.

A central aim of factor analysis is the "orderly simplification", to use Burt's phrase, of a number of interrelated measures.

(Child, 1970, p1)

The main purpose of PCA [principal component analysis] (or FA [factor analysis]) is to reduce a system of correlated variables to a smaller number of new variables which, one way or another, will be of use in dealing with a multivariate problem.

Jackson, 1991, p424)



Factor analysis techniques can meet any of three objectives:

1. Identify the structure of relationships among either variables or respondents...
2. Identify representative variables from a much larger set of variables for use in subsequent multivariate analysis.
3. Create an entirely new set of variables, much smaller in number, to partially or completely replace the original set of variables for use in subsequent techniques.

(Hair et al, 1984, pp 368-371)

It is always tempting to seek to simplify to find structure and order in data and it is one of the criticisms of component analysis that it may show up spurious relationships which are a mere coincidence. On the other hand, if variables which have no apparent reason for loading on the same component turn out to do so, there may be a reason for it. For example, if shoe size, longevity and IQ load significantly on the same component, this may be a statistical vagary, or may reflect some other phenomenon such as early nutrition. The analysis itself will not distinguish between the possible explanations.

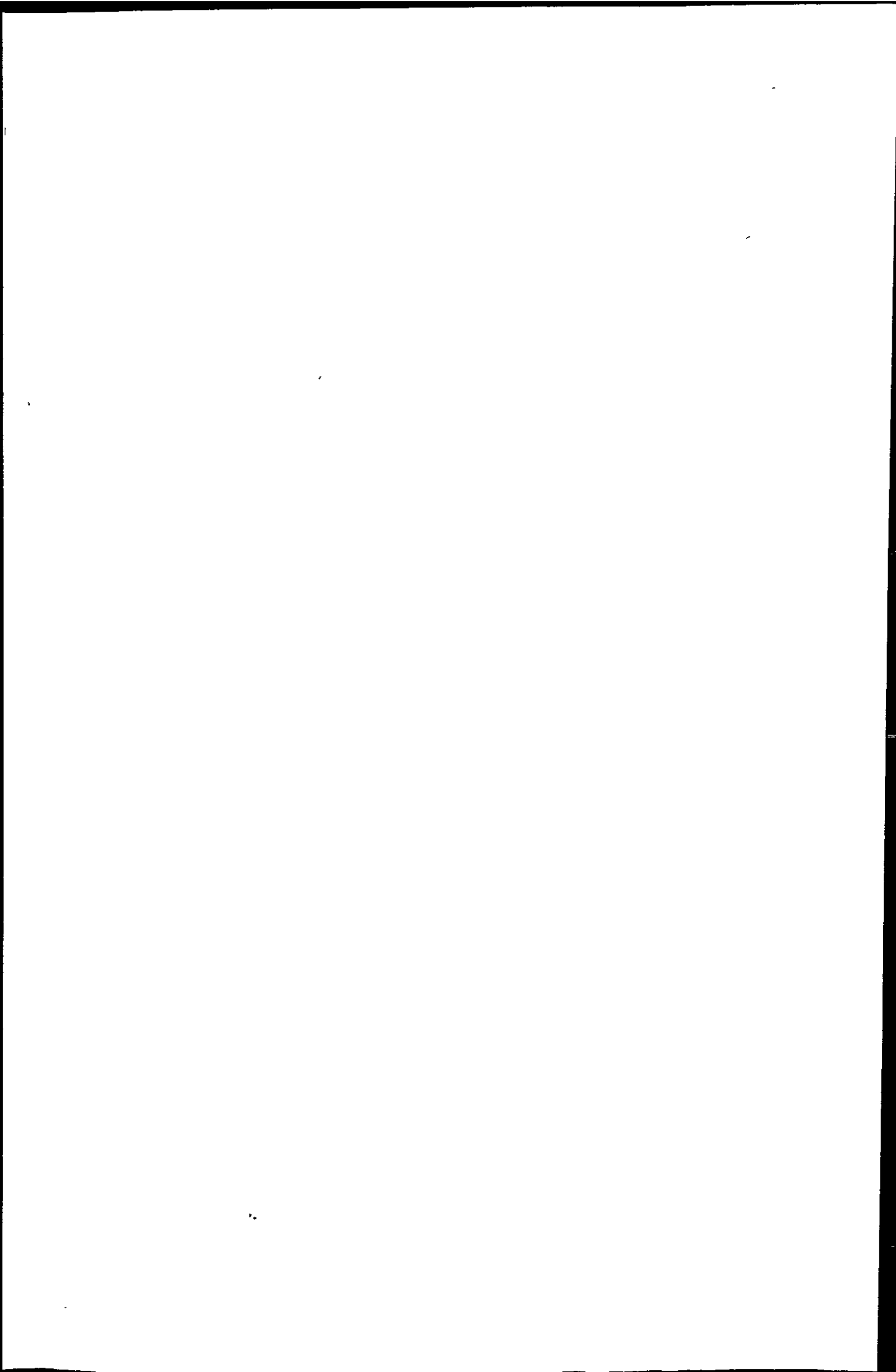
7.4 What is component analysis?

Component analysis is a method of treating data to reduce its complexity in an orderly and reproducible manner, in order to extract some meaning and pattern.

Factor analysis is a generic name given to a class of multivariate statistical methods whose primary purpose is to define the underlying structure in a data matrix. Broadly speaking, it addresses the problem of analyzing the structure of the interrelationships (correlations) among a large number of variables by defining a set of underlying dimensions, known as factors.

(Hair et al, p366)

The underlying idea is described graphically by Alt (1990) and by Child (1970) in similar terms. A set of cases is characterised by the scores of each measured on a set of variables. The correlations between the variables may be presented in a matrix. The terms in this matrix lie in the range $-1 < r < 1$. If these values are regarded as the cosines of angles,



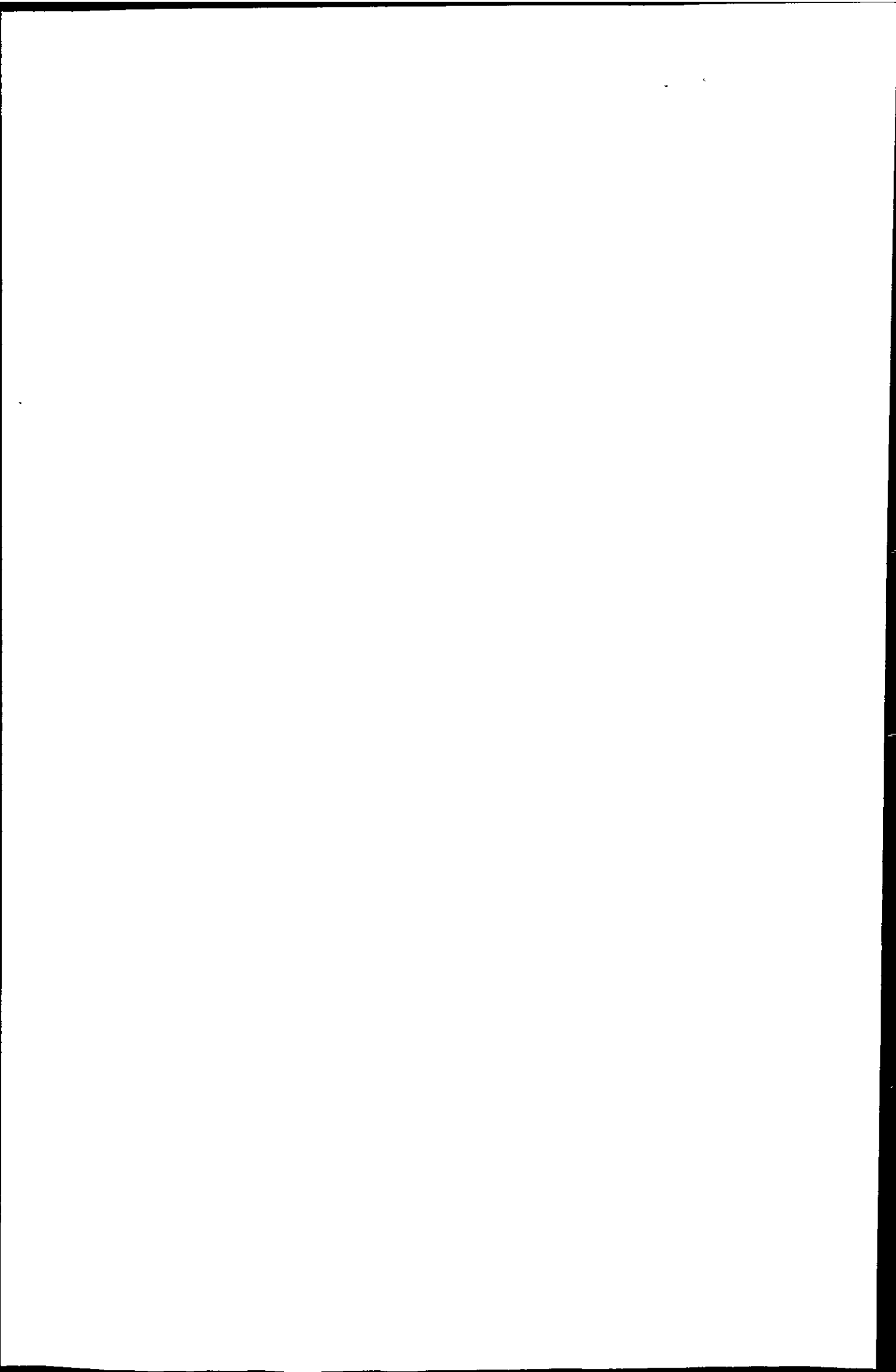
then the variables may be portrayed by vectors whose spatial relationship is defined by the cosines of the angles between them. These vectors lie in a space of up to p dimensions, where p is the number of variables. (If any three vectors happen to be coplanar, the number of dimensions needed is reduced.)

The "trick" in component analysis is to define a set of axes in this space which better describes the space than the set of vectors arranged within it, and then to interpret what the axes, that is the factors or components, represent.

The axes defined in a component analysis also correspond to the eigenvectors of the matrix under consideration. The largest eigenvalue corresponds to the first axis to be extracted, which is in the direction of the resultant of the vector variables. When all the (geometrical) components in this direction are subtracted from the variable vectors, the next axis is determined as the longest axis of the residuals, and so on.

The size of the eigenvalue indicates the amount of the total variance which is accounted for by the component. The sum of the values of the eigenvalues is the same as the number of variables, and the number of components. However, certain criteria may be used to disregard the smaller eigenvalues, and thus to reduce the effective number of variables. The correlation between a variable and a component (the cosine of the angle between the variable vector and the axis) is known as the loading of the variable on the factor or component.

The analysis thus far will extract what are known as the principal factors or components. The next stage is to "rotate" the axes to try to optimise the loadings of the variables on the factors or components. Various techniques exist which seek to optimise the rotation according to different criteria but the most commonly used appears to be VARIMAX, which attempts to align the axes so the sum of the squares of the loadings of the variables on the axes is maximised, subject to the axes remaining orthogonal. This



is then used to identify clusters of variables which appear to vary together, and use them to deduce the existence of the component which underlies and affects them all.

If instead of the self-correlations on the diagonal of the correlation matrix, an estimate of the "communality" of the variable is used, the axes are called factors. If the self-correlation is used, the axes are called components. The distinction between factor and component analysis is discussed in 7.6 below.

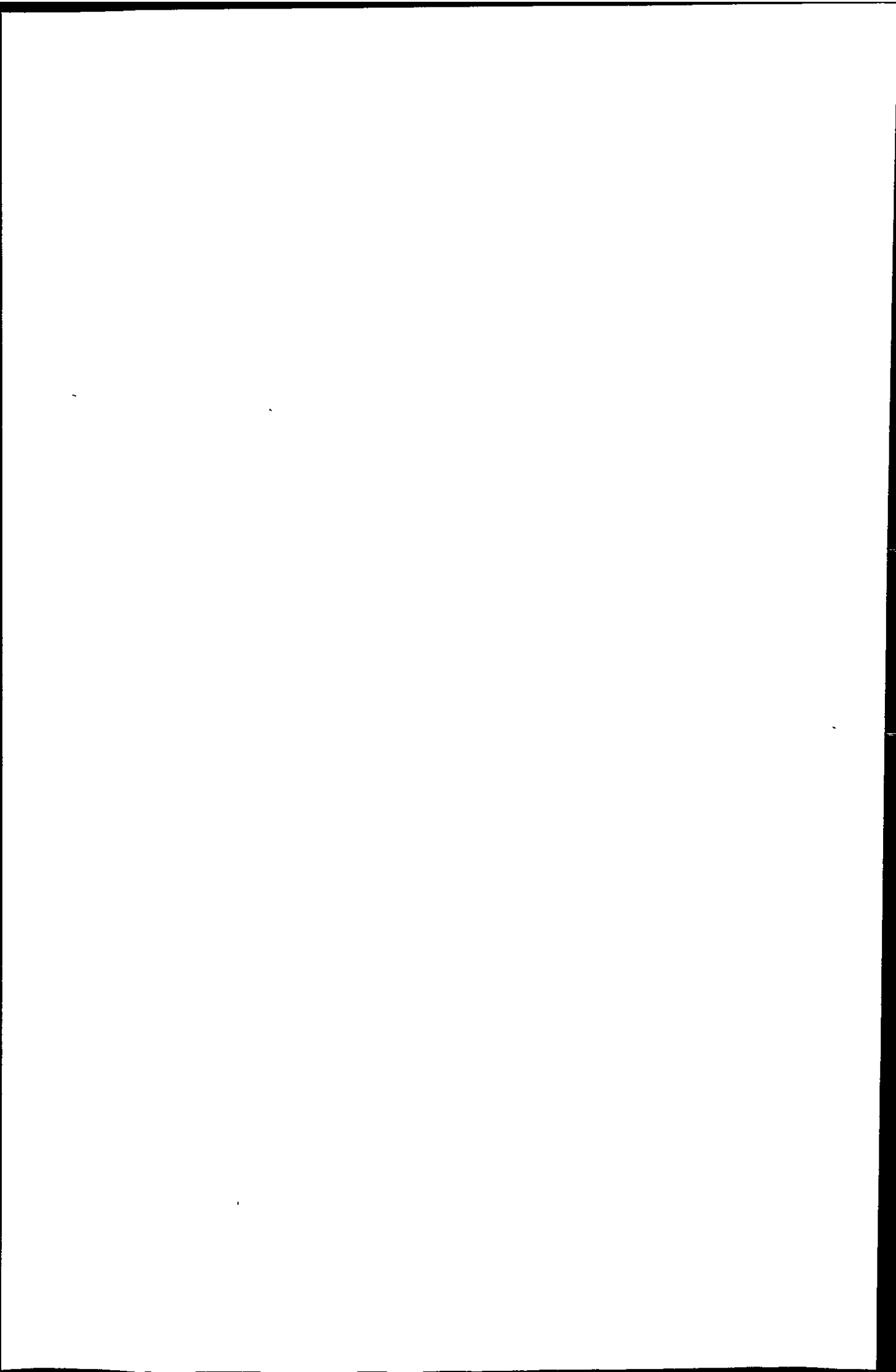
7.5 How it works: practical considerations

7.5.1 How many variables?

Hair et al (p373) recommend that the number of variables should be minimised while keeping five or more variables per proposed component if a given model is being tested. Component analysis is of most use in finding patterns of correlated variables, and a component consisting of a single variable will not show up strongly.

7.5.2 Sample size

A sample size of less than 50 is not recommended, and at least 100 is preferable. In general the sample should have at least five times as many observations as variables, and a ten-to-one ratio is more acceptable. Hair et al (p373) point out that with 30 variables there are 435 correlations in the component analysis. At a .05 significance level some 20 of these may be deemed significant and appear in the component analysis by chance. Increasing the cases-per-variable ratio should minimise the chances of over-fitting the data.



7.5.3 Suitability of data.

Given the above consideration, the data correlation matrix must have enough significant correlations to justify using component analysis. Hair et al (p374) recommend that component analysis should only be used if a "substantial number" of correlations greater than .30 can be found in the matrix. Alt (1990, p66) also recommends the use of .30 as a cut-off value. This represents a significance of better than .01 for a sample size of 100.

Other tests which the data should satisfy are the Bartlett test of sphericity and the Measure of Sampling Adequacy (MSA).

The Bartlett test calculates the statistical probability that the correlation matrix has significant correlations among at least some of the variables (see also Jackson p33 for the procedure of the test). The hypothesis is that the last $(p-k)$ eigenvalues are equal, where p is the number of variables and k is the number of components to be retained. As the sample size is increased, the test becomes more sensitive to detecting correlations among the variables.

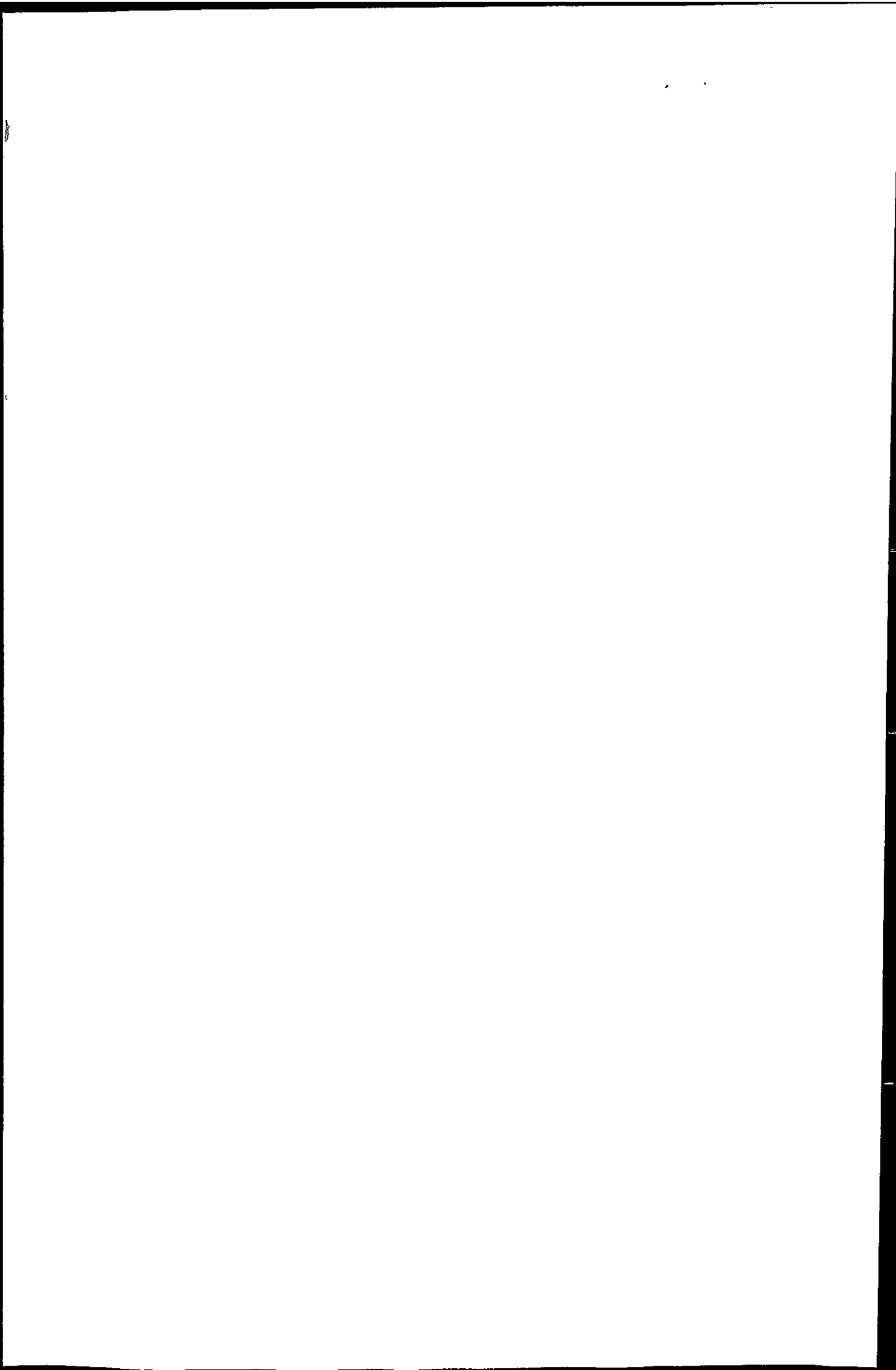
The MSA is another measure used to quantify the degree of intercorrelations amongst the variables and the appropriateness of component analysis. The index ranges from zero to one and may be interpreted as follows:

.90 or above, marvelous; .80 or above, meritorious; .70 or above, middling; .60 or above, mediocre; .50 or above, miserable; and below .50, unacceptable.

(Hair et al, p374)

Changes which cause the MSA to increase are: increased sample size, higher average correlations, increased number of variables and smaller number of components extracted.

Individual MSAs may be calculated for each variable, and any which fall in the unacceptable range should be discarded before component analysis is carried out. In



their illustrative example, however (p393), Hair et al discard only enough low-scoring variables to bring the overall MSA above .50, while retaining two others whose individual scores are below .50. They regard the data set as acceptable because the overall MSA is over .50 and over half the correlations off the diagonal are significant at the .01 level.

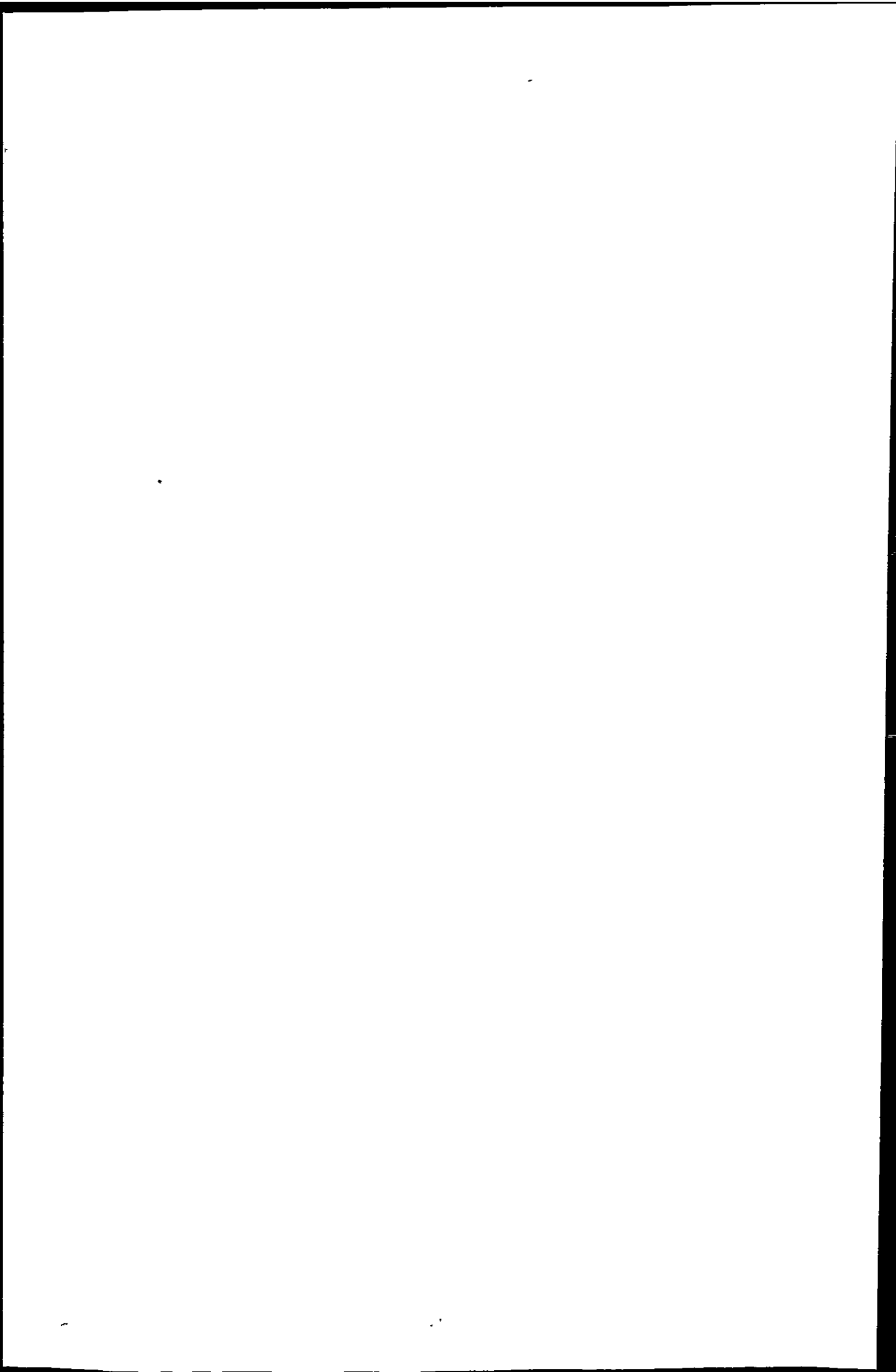
Jackson (p41) refers to Hotelling's "sand and cobblestone" theories of the mind, with regard to test batteries. A "cobblestone" situation is where a few components fairly well characterise the data. "Sand" means that there are many low correlations and the major resultant components are small with some possibly indistinguishable. A "sand" situation is probably not suitable for this type of analysis.

7.5.4 Correlations translated into angles between vectors.

The variables may be regarded as vectors and the correlations between them as the cosines of the angle made by any two vectors. Thus if the correlation is 1, the vectors are parallel and in the same direction, a correlation of -1 means the vectors are parallel but opposite, and zero means the vectors are at right angles. A correlation of 0.707 puts the vectors at 45° .

The next stage in component analysis is to create a set of vectors whose directions are determined by the angles between them. It would be unusual for the angles to allow three vectors to lie in one plane, and in general we have as many dimensions as variables.

In addition to the set of vectors, the analysis needs to define a set of axes by which to orient them.



7.5.5 How many components?

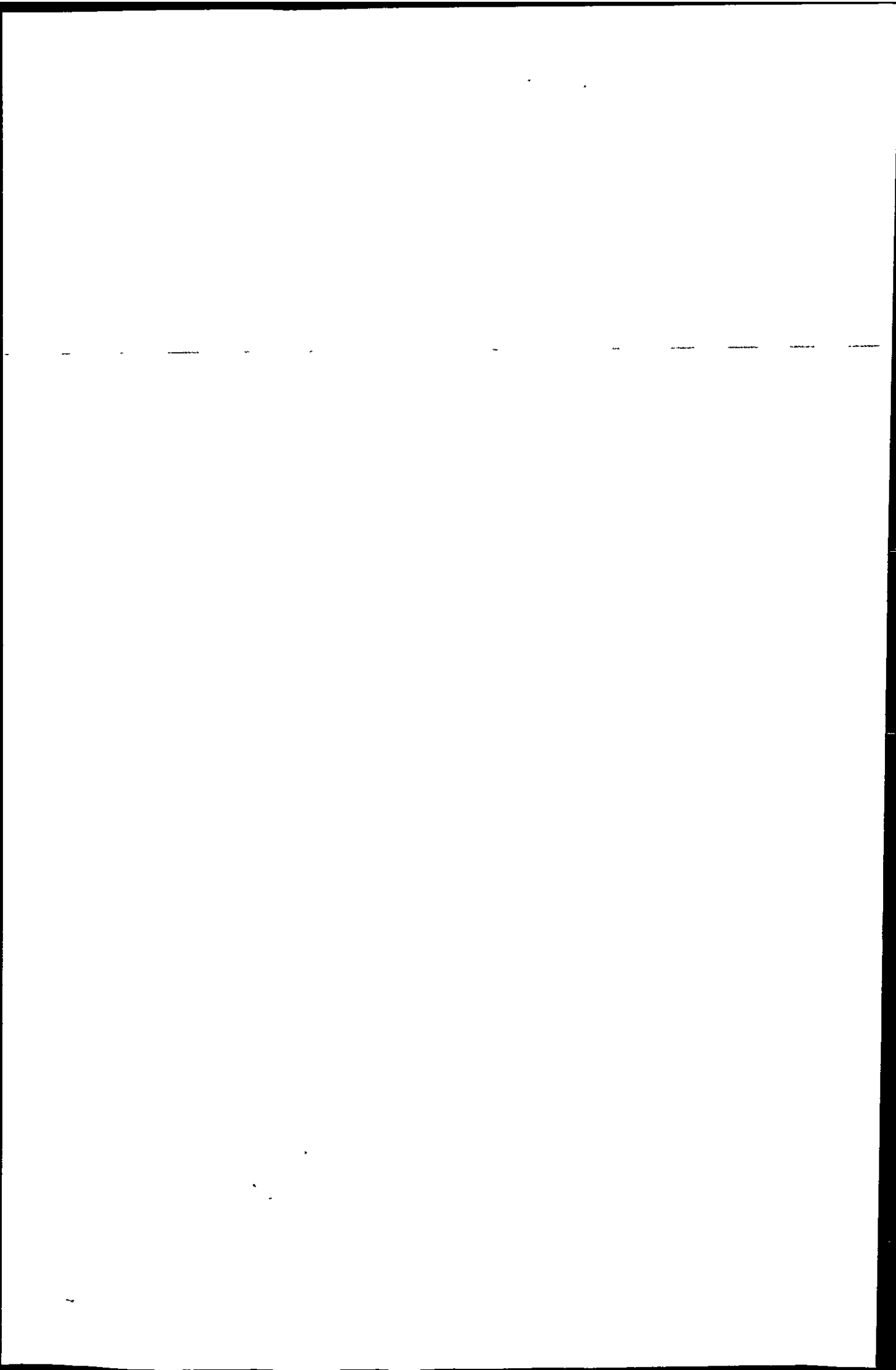
Once the eigenvalues of the variable correlation matrix have been extracted, the decision must be taken as to how many of them are significant. There are several tests which may be used to determine this. They include the Scree test, the eigenvalue size test and proportion of variance explained.

The Scree test is a graphical technique due to Cattell. The values of the eigenvalues (roots) are plotted against their corresponding root number. The term scree refers to the pile of rubble at the base of a cliff. The retained roots will correspond to the cliff and the rejected ones the rubble. The last few roots are much smaller than the first ones and are nearly equal in value. The scree test criterion is that the components up to and including the first of these should be accepted. The problem with the scree test is that the break between the cliff and the scree is not always well-defined, and in some applications there may be several breaks. This then means that personal judgement is needed to decide where to draw the line.

Another criterion is that eigenvalues less than unity should be rejected. This corresponds to the mean root size for component analysis, and the argument is that any component rejected by this criterion will have a smaller root than the contribution of the average variable. Jackson (p47) describes this criterion as being widely used in the fields of psychology and education.

Extracting components up to a given proportion of variance explained is not recommended by Jackson (p44), as "there is nothing sacred about any fixed proportion". Hair et al (p378) give the values for this criterion as varying between 95% in the natural sciences and up to 60% in the social sciences.

Other reasons for choosing a given number of roots include the suspicion for extrinsic reasons that it is appropriate, or that the smaller roots consist of uncontrollable inherent



variability. There is no single accepted criterion for selecting the number of roots to extract.

Hair et al (p378) point out that the first components will be those which are homogenous throughout the whole sample. Variables that are better discriminators between the subgroups in an inhomogenous sample will load on later components, which may not be selected according to the above criteria. If the analyst is interested in identifying the components which discriminate between the subgroups, then extra components should be extracted, which may be rejected in a later rerun of the solution if they do not prove useful.

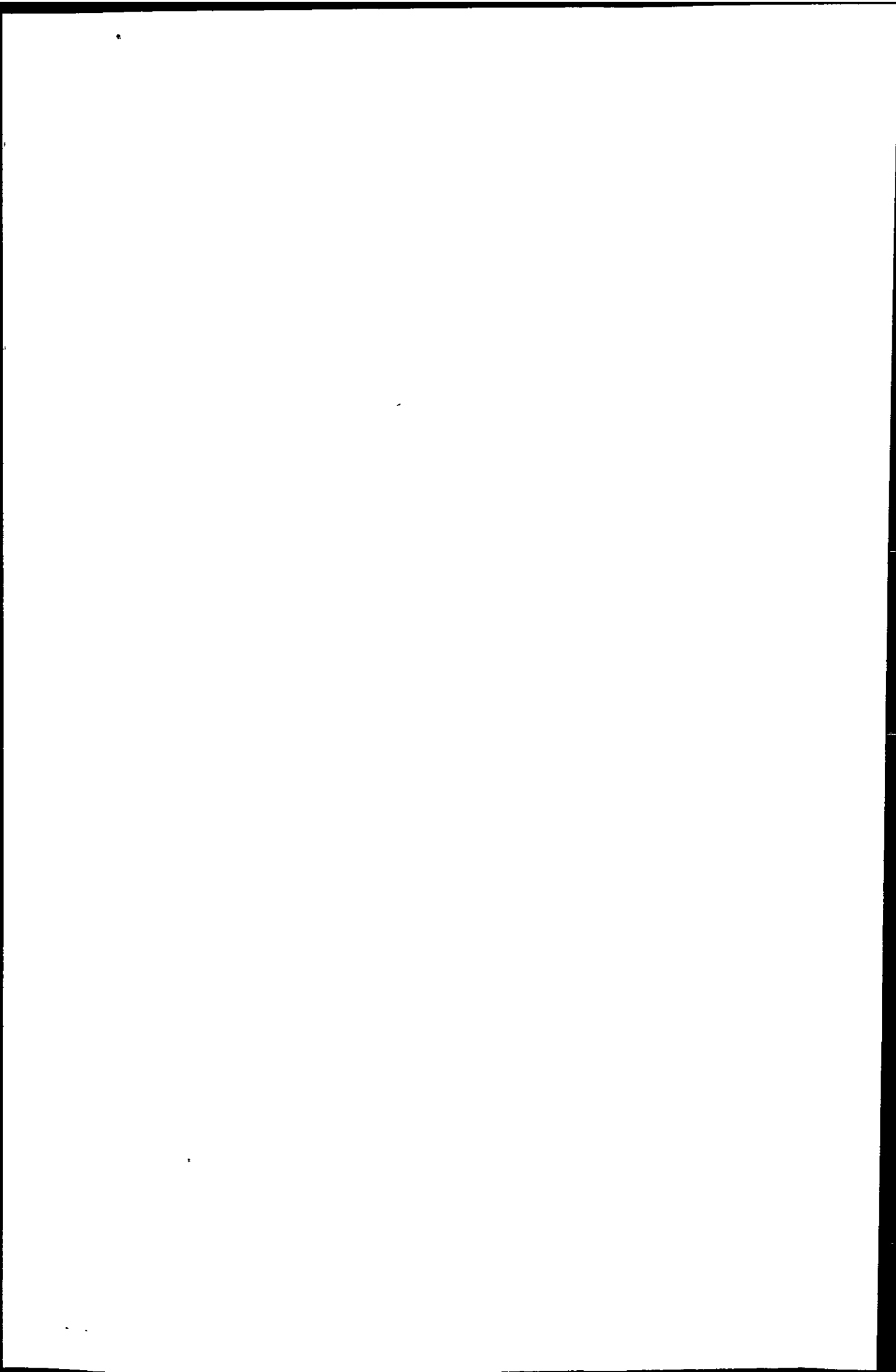
7.5.6 Interpreting the components

The first step in identifying what a component represents is to see which variables load significantly on it. Hair et al (p385) give a table (Table 7-1) by which to select the significant component loadings based on sample size.

Component loading	Sample size needed for significance at .05 level
.30	350
.35	250
.40	200
.45	150
.50	120
.55	100
.60	85
.65	70
.70	60
.75	50

Table 7-1: Dependence of significance on sample size (Hair et al, 1984).

However, they state that researchers often use a rule of thumb and work on practical rather than statistical significance. In this case component loadings less than ± 0.30 are ignored, and loadings of ± 0.50 are considered practically significant. As the loading is the



correlation between the variable and the component, the square of the loading is the amount of the variance of the variable accounted for by the component. Thus a loading of .5 means that the component accounts for a quarter of the variance of the variable.

Loadings of the order of .80 are not typical and of are great practical significance.

Some of the variables which load significantly on a component will be positive and some negative.

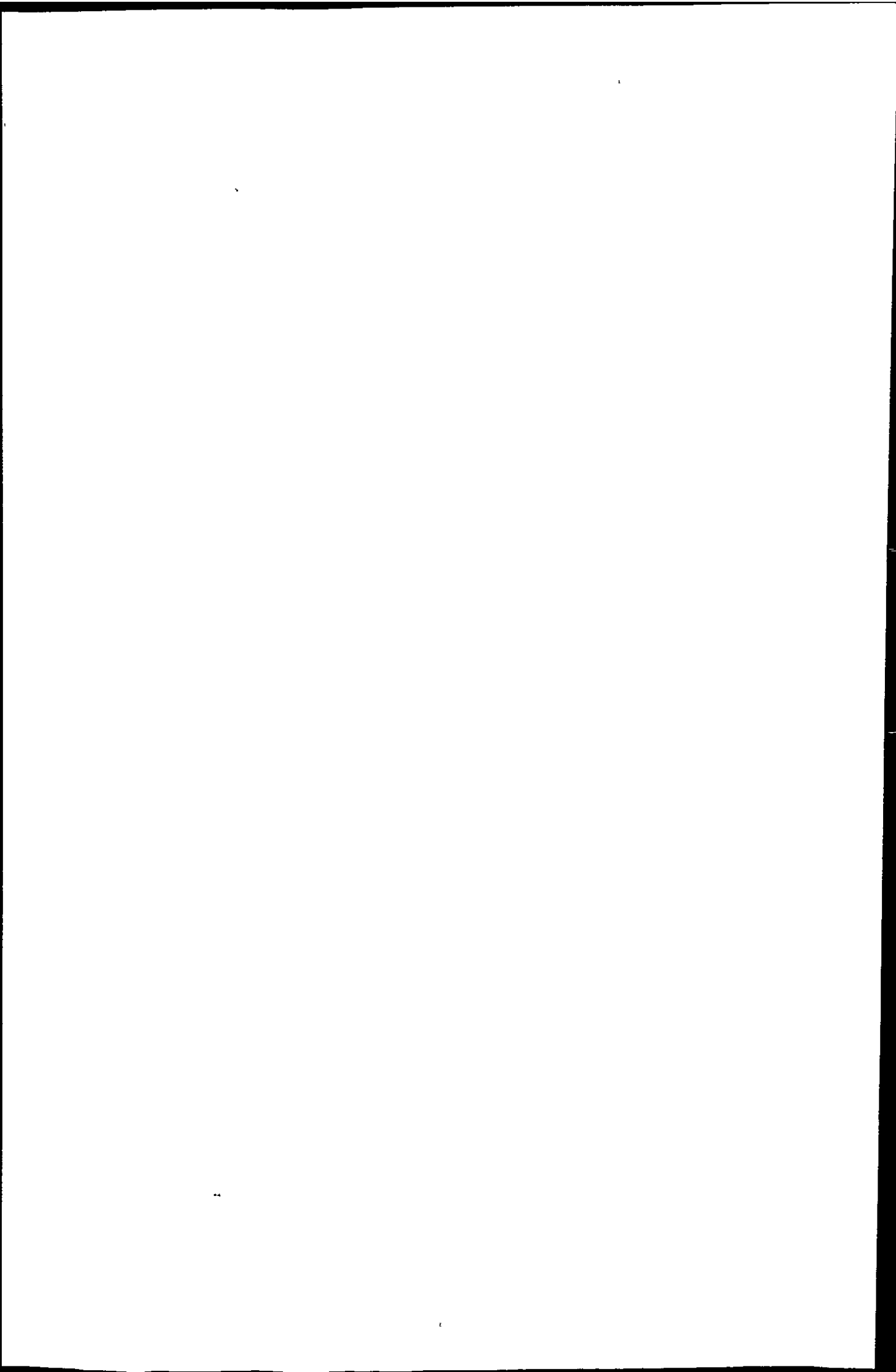
7.5.7 From principal components to factors.

The analysis has in fact been a principal components analysis. The axes we have found were defined by the eigenvectors of the correlation matrix, and were the principal axes of the space defined by the variables. These axes are unique, defined by the correlation matrix, and two researchers analysing the same data would arrive at the same set of axes. For this reason they are preferred by some researchers, as they are independent of the analyst, and so may be regarded as objective. Other researchers prefer to use factor analysis proper, which requires some subjective judgement, to define axes which may be more meaningful. For a discussion in depth of the differences and similarities between the methods, see Jackson Chapter 17 (pp 388-424) "What is factor analysis anyhow?".

7.6 *Relationship between factor analysis and principal components analysis.*

There are two basic models which can be adopted in factor solutions. They are known as the *factor analysis* and the *component analysis* models. Without being too technical, the distinction is that in factor analysis some account is taken of the presence of unique variance whereas in component analysis the intrusion of unique variance is ignored. In a component analysis the unique variance becomes merged with the common variance to give hybrid "common" factors containing small amounts of unique variance; but not enough in the first few important factors, according to some authorities, for us to be worried about the overall picture obtained from the analysis.

(Child, p36)



The 'names' principal components analysis and factor analysis are frequently used in a fairly loose fashion in survey research to the extent that when researchers have actually carried out a principal components analysis they often report that they have carried out a factor analysis. The reason for this is that the two procedures are quite similar, or, put another way, the differences between them are not immediately obvious.

(Alt, p48)

In most applications, both component analysis and common factor analysis arrive at essentially identical results if the number of variables exceeds 30 or the communalities exceed .60 for most variables.

(Hair et al, p367)

The method of principal components is primarily a data-analytic technique that obtains linear transformations of a group of correlated variables such that certain optimal conditions are achieved. The most important of these is that the transformed variables are uncorrelated.

(Jackson, p1)

PCA explains variability, FA explains structure or correlations. PCA is trying to reduce the diagonals of S [the sample covariance matrix], while FA is reducing the off-diagonals.

(Jackson, p391)

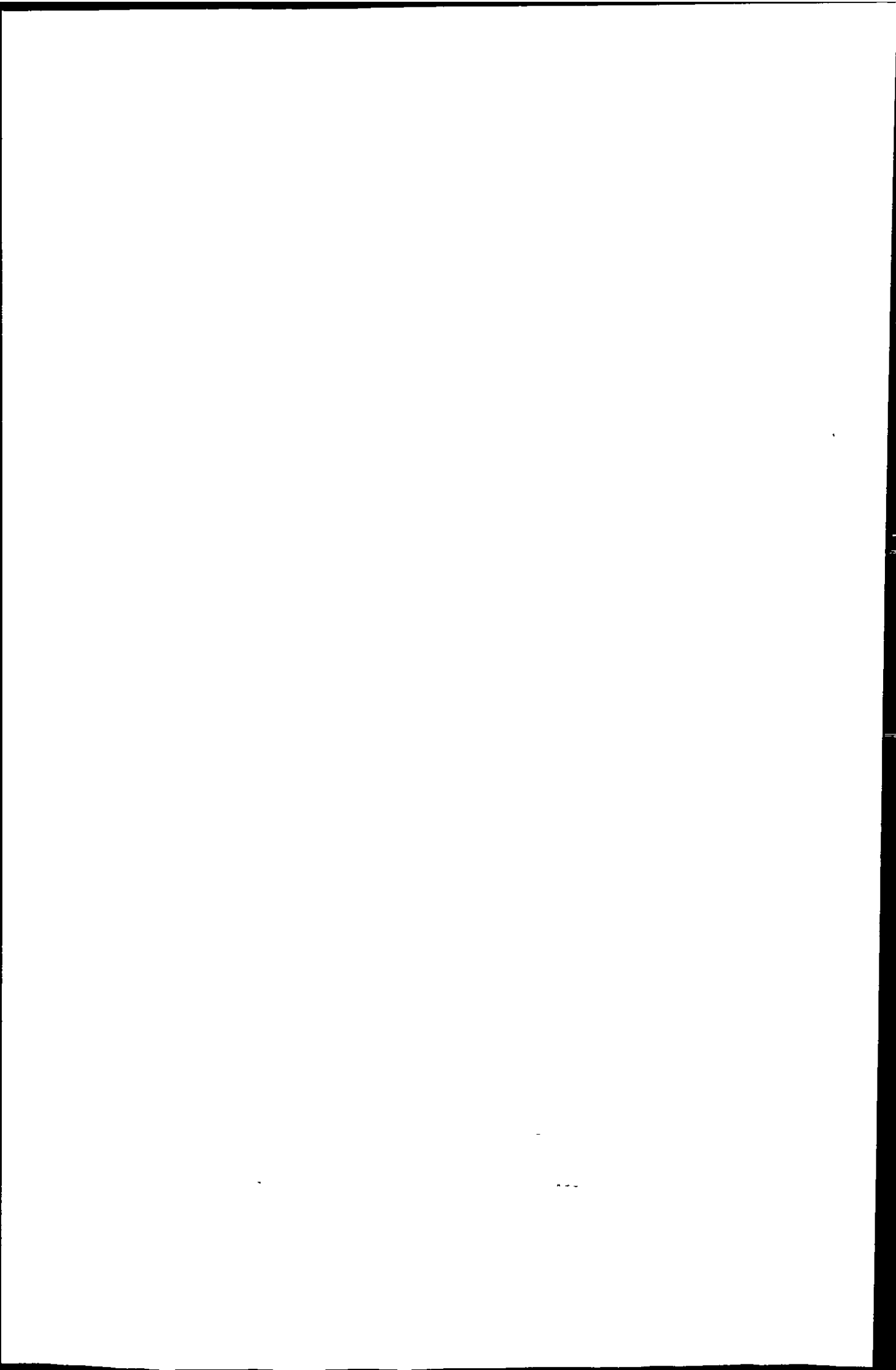
The term "factor analysis" is commonly used to describe any data analysis technique which seeks to distil a reduced number of variables to replace the experimental variables. Principal components analysis is however a computationally simple method which yields a unique solution. This makes it easier to use than factor analysis proper.

Factor analysis requires that the self-correlations be replaced by communalities, which have first to be estimated, and then a solution iterated towards. The communalities are estimates of the shared or common variance among the variables. The difference between the value of the communality and unity is an estimate of the error and the variance specific to that variable. The estimating of the communalities may lead to problems, as the iteration may lead to a value less than zero or greater than one, which is not allowed, and the iteration process itself may not converge after an acceptable



number of repeats. The first estimate of communality is often the largest value in each row of a correlation matrix (except of course the unity value on the diagonal).

Another distinction lies in that in factor analysis the number of factors extracted affects the value of the estimated communalities, which in turn alters the matrix whose roots are being extracted. Thus the first root of a two factor solution will be different from the first root of a three solution, and so on. In component analysis the matrix is not affected by the number of roots extracted, and so subsequent components simply supplement the existing ones. The sum of the squares of the communalities equals the total proportion of the variance accounted for by the factors extracted. Various factor analysis methods are available in SPSS. They are described in Jackson pp 398-405.



8. Component analysis of the questionnaire data

8.1 *Introduction*

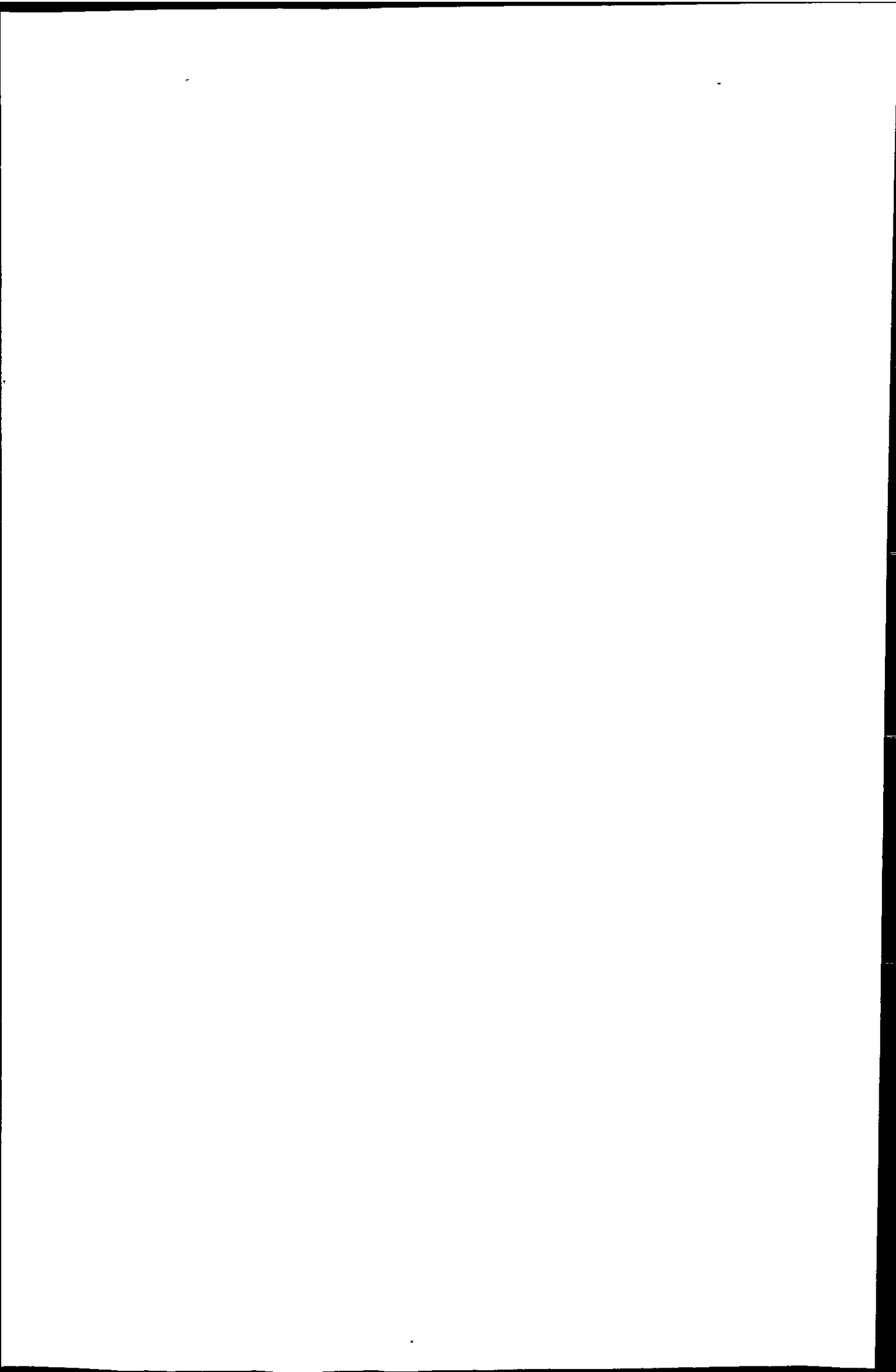
In chapter 7, the process of component analysis was described. The process was applied to the responses to the questionnaire to investigate the patterns of responses, particularly between responses to different questions. The responses to a question consisted of ordering the six options, which means that a degree of correlation between the options for any one question is forced. It would not be surprising if the analysis threw up components consisting of responses to one question. Components showing correlations between options from different questions will be more interesting and more practically significant.

The components derived from the analysis are then examined to find a possible interpretation of what each may represent, and the scores on each component for each group of respondents are found. Finally the component scores are used to determine canonical discriminant functions to see how far these scores can be used to sort the respondents back into their original groups.

It is recognised that the questionnaire was not designed with component analysis in mind and this leads to the components being difficult to interpret. This analysis leads to a redesign of the questionnaire which is proposed in the next chapter.

8.2 *Typology and taxonomy*

It has been suggested that classification systems can be divided into typologies and taxonomies (e.g. Meyer, Tsui and Hinings, 1993). Typologies start with theoretical considerations, predicting the groups into which individuals should be expected to fall, based on the known or expected available possibilities: for example, age groups, gender and previous education may be criteria according to which a theoretical



typology of students might be built up. A taxonomy is a more empirical system based on observation and experience: the individuals are grouped according to characteristics which appear upon observation. A particular group of students may be characterised, for example, by whether they ask many or few questions in class, or none at all. Either method of classification may yield similar groupings of individuals, while being methodologically very different.

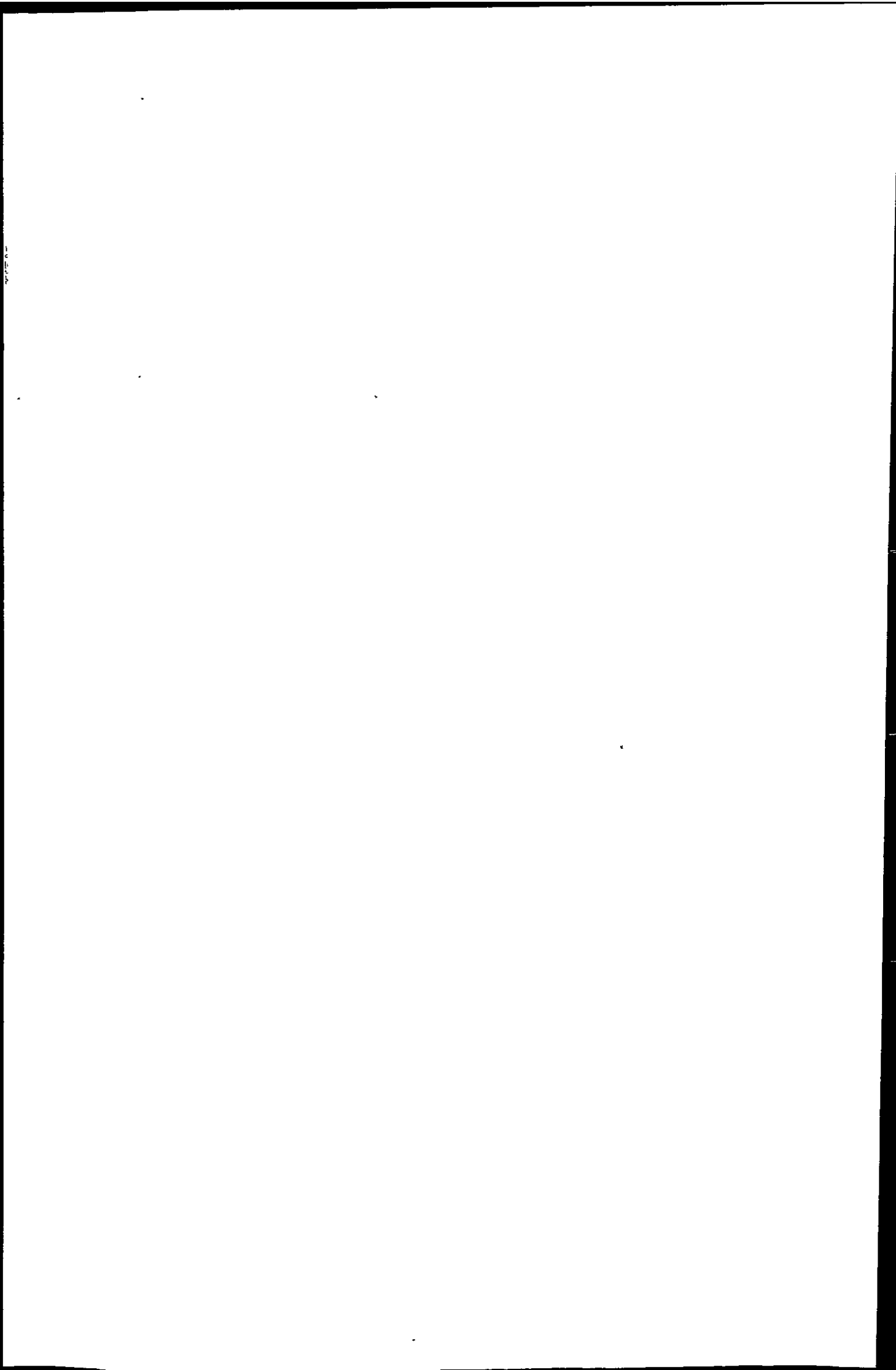
A typological classification of the respondents may be made by dividing respondents into engineers and mathematicians, and by experience. The component analysis of the questionnaire results may however be regarded as resembling a taxonomy, as it may be used to classify the respondents according to observed features. We may then compare the results of the two types of classification.

8.3 The analysis: in brief

The questionnaire was not designed for component analysis, and the number of variables is rather lower than the ideal five per component expected. These components were expected to correspond to the three modes of representation: verbal, visual and algebraic, to depth of representation, and to whether the question was a "mathematics" or "mechanics" question.

At the same time the options were not designed to fall neatly into groups to correspond with expected components, with at least one option per expected component per question. However there was felt to be enough resemblance between some of the options to attempt a component analysis to see if any sense could be made out of it, with the caveat that one should beware of over-interpreting the components.

Questions five and six were omitted from the analysis, on the grounds that they were felt to be unsatisfactory questions, and that they had no graphical options. This left

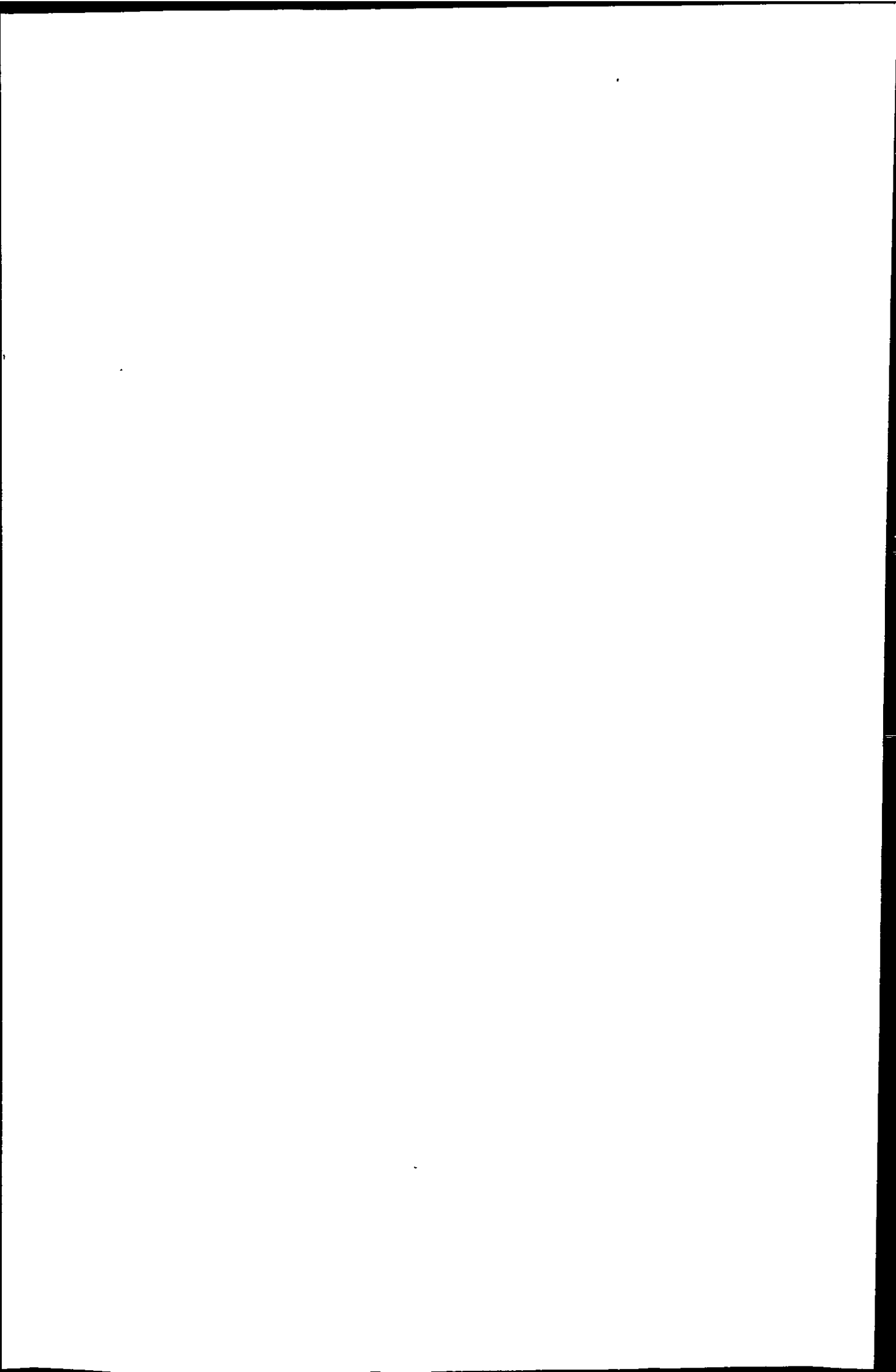


24 variables, and with 209 respondents the requirement of at least five respondents per variable was well fulfilled. Unlike many types of statistical analysis, component analysis copes well with a non-homogenous sample. The components which distinguish between clusters in the sample tend to have smaller eigenvalues than components which are common to all clusters.

The analysis was carried out using SPSS. The Bartlett Test of Sphericity yielded a value of 1301.2, significant at $P < .00001$, but the Measure of Sampling Adequacy (MSA) was only .23845 which falls short of the 0.5 value recommended by Hair et al (1984). This was an indication that the correlations between the options were not high in general, and that the data was not ideally suited to component analysis. However the solution recommended, that is to eliminate the variables with the lowest MSA values, would result in an unacceptably small number of variables considered, so the analysis was continued, but the results would be regarded with some caution.

The scree test criterion indicated that a break point came at the tenth eigenvalue, accounting for 67.3% of the variance. The components were rotated using VARIMAX to give ten components. At this level, none of the components was represented by less than two variables with a loading of 0.4 or greater, which was felt to be a satisfactory result. The "eigenvalues greater than 1" criterion also indicated that 10 factors should be extracted.

The "sign" of the components has been left as SPSS designated, although some of the components would be more easily described with their polarity reversed. This is mentioned at the appropriate points in the chapter.



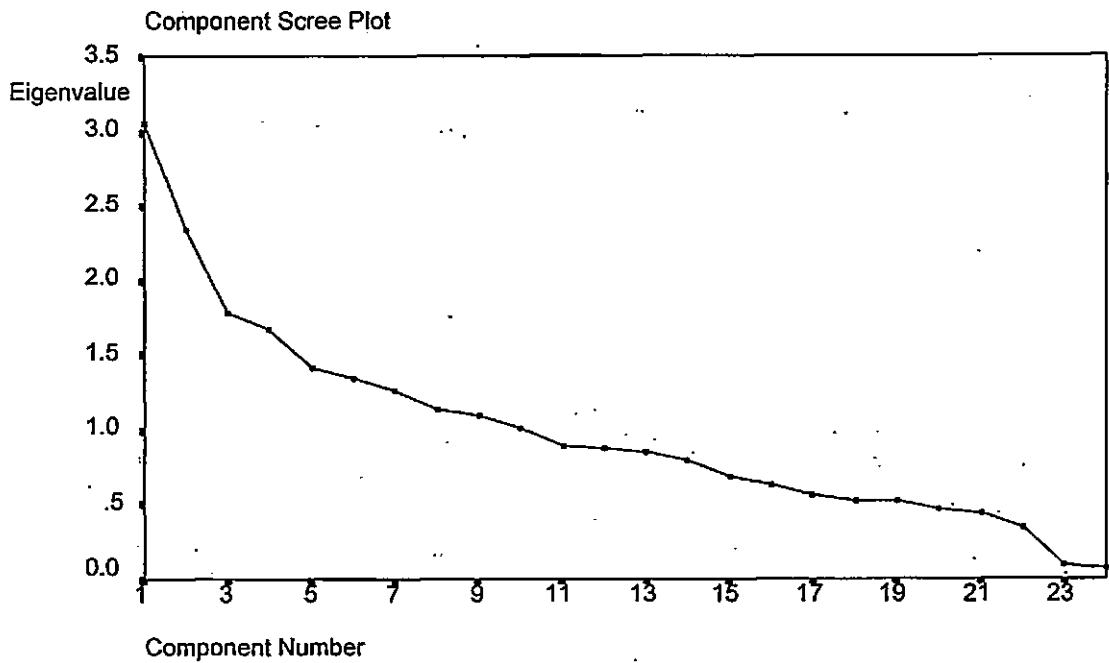
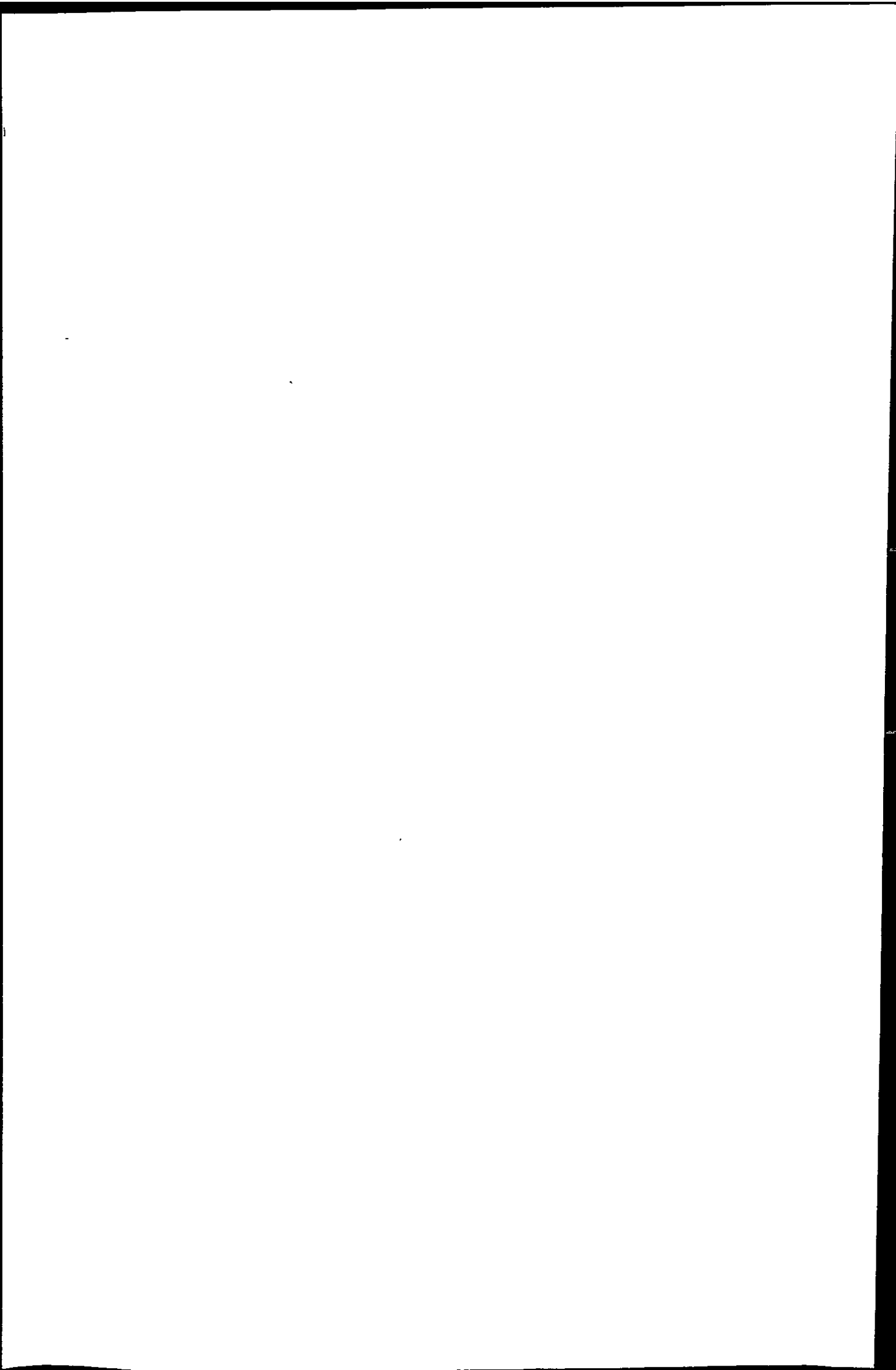


Figure 8-1: Component Scree Plot

Component → Option↓	1	2	3	4	5	6	7	8	9	10
3d	.78248									
3b	.63327									
3a	-.62050			.48014						
2d	.55223									
2e	-.45074									
1e		.81308								
1c		.79398								
1a		-.47404		.47143						
4b			-.72814							
4e			.69476							
3f				-.66201						
4a				.54664					-.44838	
2c					-.85996					
2f					.60372					
4d					-.52471					
3c						-.75997				
3e						.61661				
2a							.77162			
2b							-.63377			
4f								-.79630		
4c								.67916		
1f									-.77005	
1b										.79043
1d									.40077	-.59284

Table 18-1: Loadings of options on components



8.4 What do these components represent?

The components were investigated to see (a) what they might represent and (b) how well they separated the respondents into either engineers and mathematicians or more and less mature practitioners. Options which load negatively on a component are shown in brackets.

8.4.1 Component 1

Option		Loading
3d	$y = ae^{-\lambda} \cos \omega t$.78248
3b	$\ddot{y} + k\dot{y} + \omega^2 y = 0$.63327
(3a)	The mass bounces up and down, going less far each time, until it settles back to its original position.	-.62050
2d	$f'(x) = \lim_{(x_2 - x_1) \rightarrow 0} \frac{y_2 - y_1}{x_2 - x_1}$.55223
(2e)	As you zoom in more and more closely to a small section of the curve, it seems to straighten out. The slope of the tiny straight section is dy/dx at that point.	-.45074

Table 8-2: Loadings on component 1

This component would be better described with its polarity reversed. It loads positively on unpopular options and negatively on more popular options. It is one of the more significant components because it brings together options from more than one question. It is also interesting that the options are from a "mathematics" question (Q2) and a "mechanics" question (Q3).

The two most "wordy" options on the questionnaire are opposed to the most "algebraic" in the questions being analysed. A positive score on this component would indicate comfort with algebra and dislike of informal verbal description. The mathematicians score more highly on this option as their level of experience increases, and the engineers seem to become more comfortable with the algebraic

notation as they progress, having preferred the verbal descriptions throughout their first year. However the practising engineers are less happy with the algebra, and score similarly to the first year engineers. This phenomenon (the practising engineers resembling the first year students) recurs in this analysis, for example in components 2, 5, 6, 7 and 8.

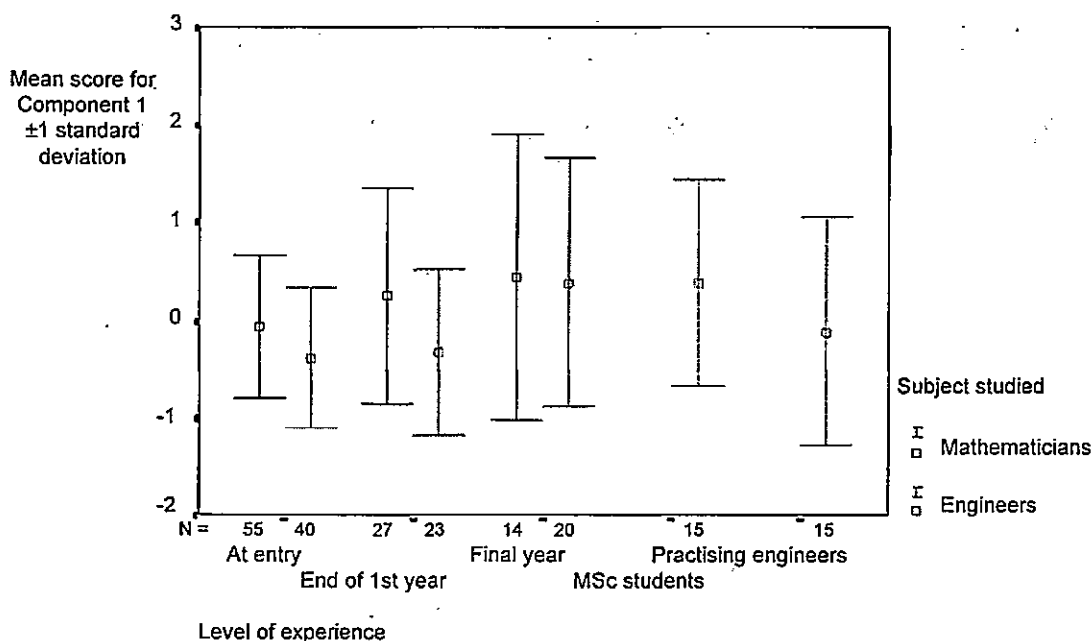


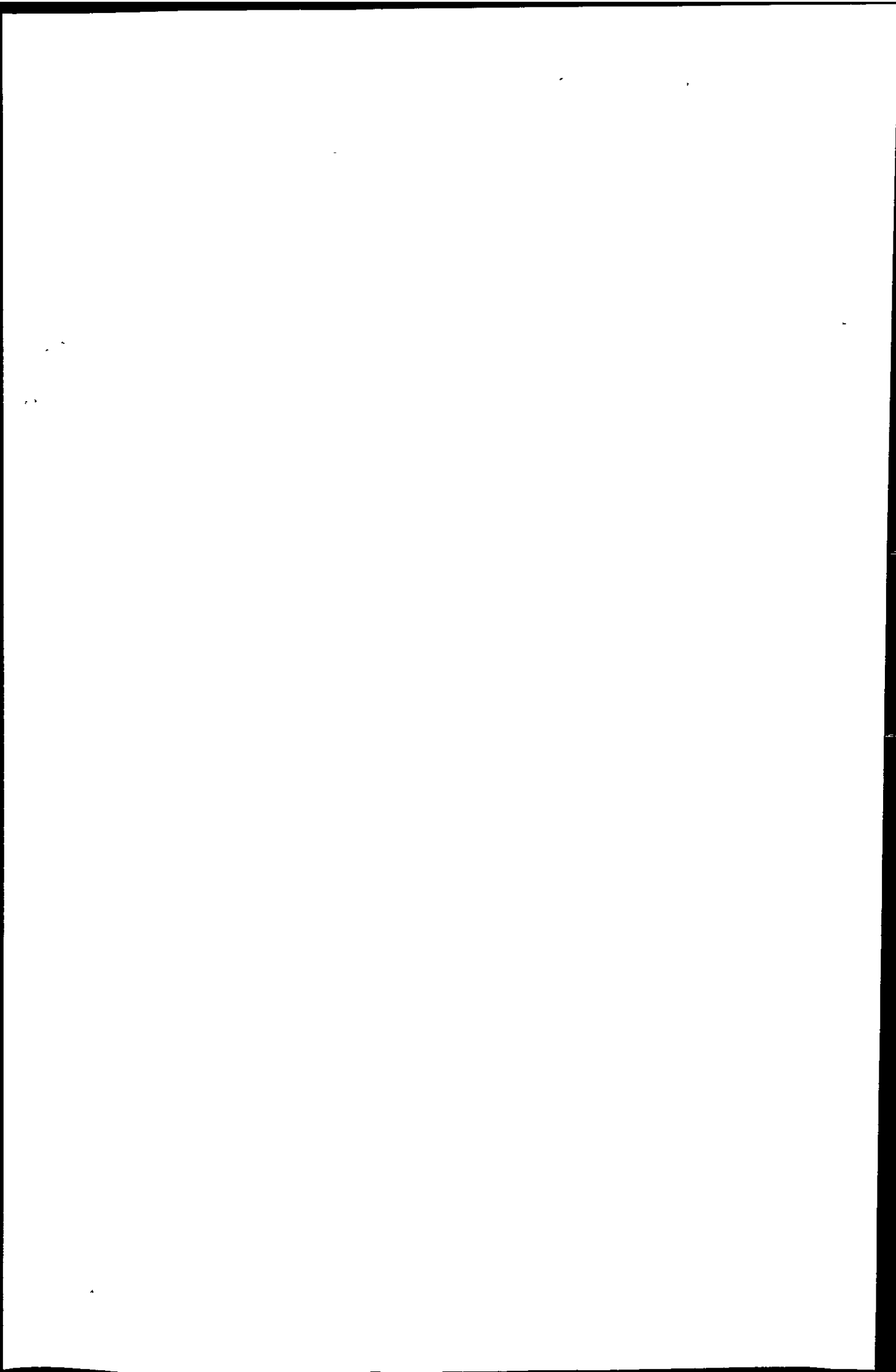
Figure 8-2: Scores on component 1

8.4.2 Component 2

Option		Loading
1e	Shear force $S = \int F dx$ Bending Moment $M = \int S dx$.81308
1c	Bending Moment $M = k \frac{d^2 y}{dx^2}$.79398
(1a)	The beam bends under the weight of the gold bar.	-.47404

Table 8-3: Loadings on component 2

This option contains the two expressions which the engineers would meet for the first time in the course of their first year of study, and which mathematicians would probably not meet. Thus the score for the engineers jumps up over the course of the first year and then subsides to what might be regarded as its natural level. The



“jump” in postgraduate engineers is less easy to explain, but again the practising engineers resemble the first year engineers. All the variations are fairly small compared with those for component 6, for example. The component contains options from one question only.

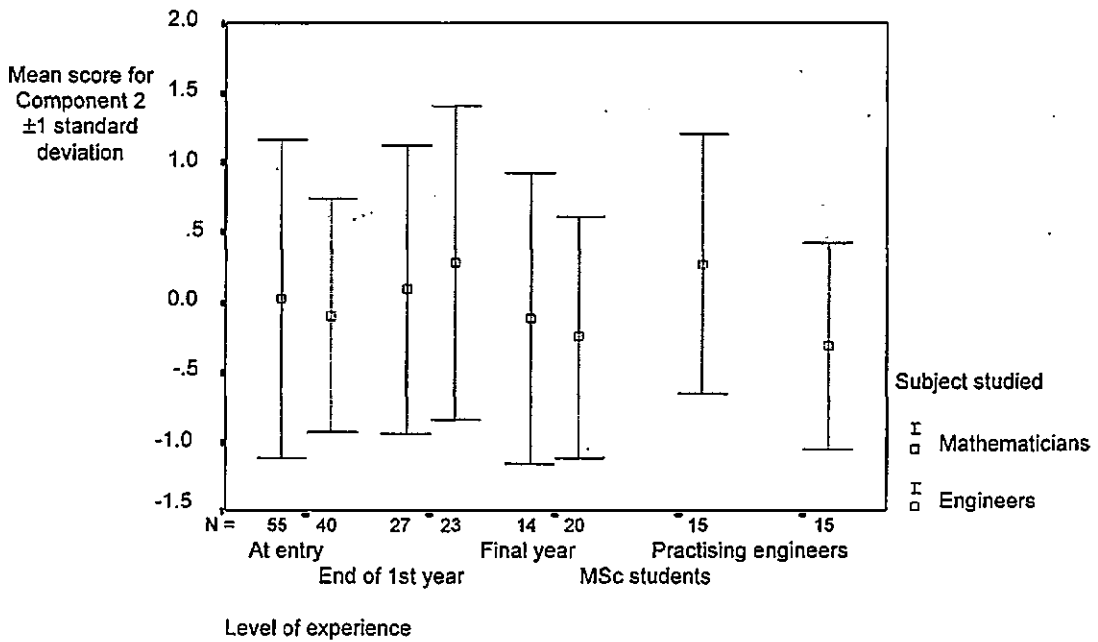


Figure 8-3: Scores on component 2

8.4.3 Component 3

Option		Loading
(4b)	$q = \frac{x^2}{2} + C$	-.72814
(integration carried out)		
4e	<p style="text-align: center;">(area under curve diagram)</p>	.69476

Table 8-4: Loadings on component 3

This component contains options from question 4 only. It loads oppositely on two options which are different both in mode and in underlying concept.

1. 2. 3. 4. 5. 6. 7. 8. 9. 10.

1.

2.

3.

4.

5.

6.

7.

8.

9.

10.

The “integration carried out” option was most popular with the mathematicians on entry, and is different both in mode and in embodied concept from the “area under curve” diagram. At all levels of experience the engineers seem to prefer the diagram, particularly the postgraduates and practising engineers. The tendency to the negative among the undergraduate engineers may be due to sustained practice in integration during the degree course, with a positive trend afterwards as less is done.

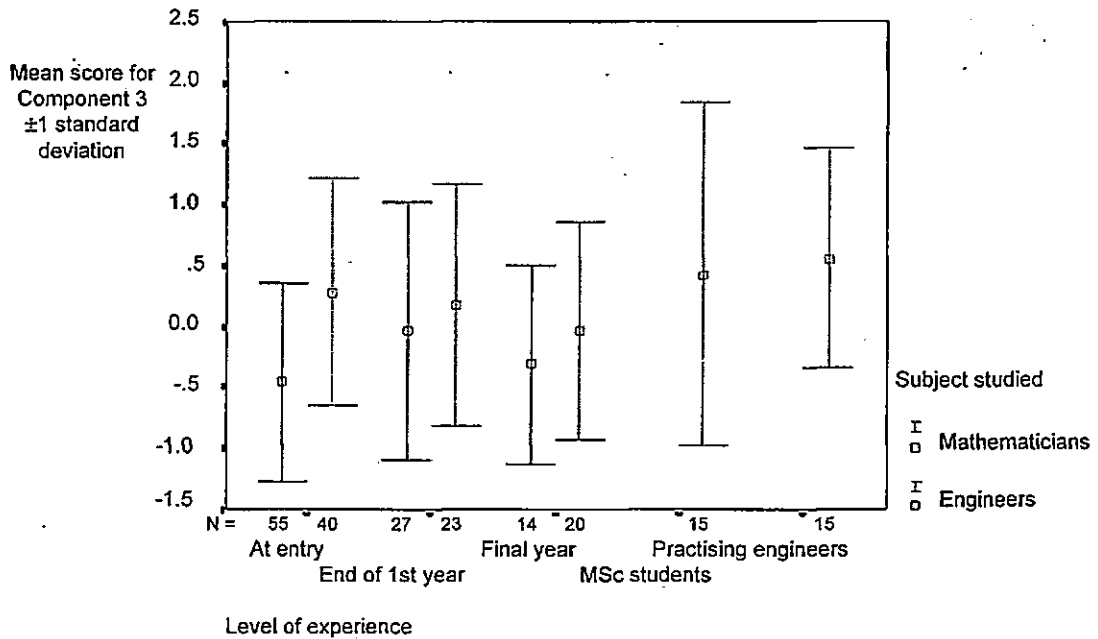
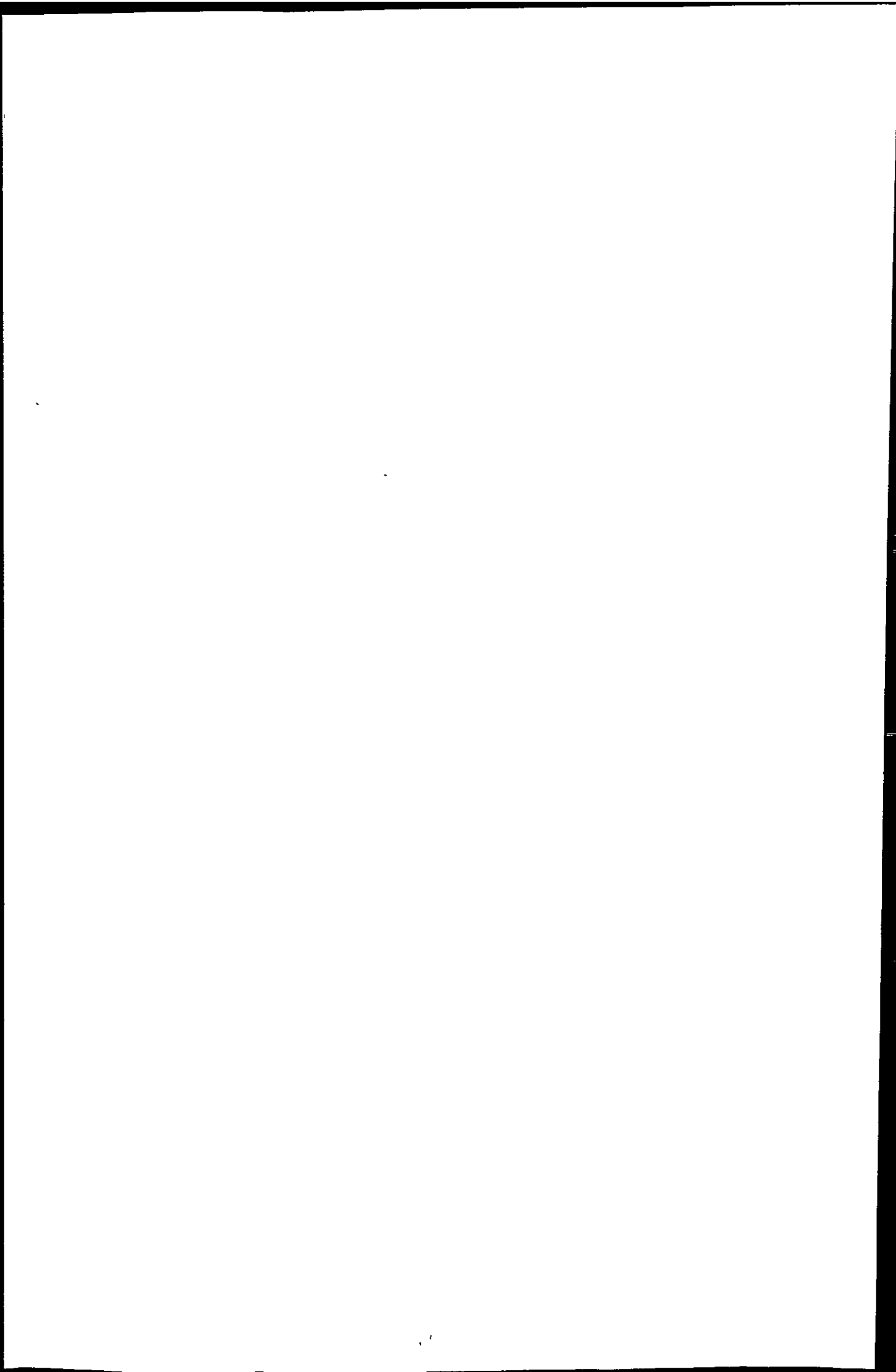


Figure 8-4: Scores on component 3



8.4.4 Component 4

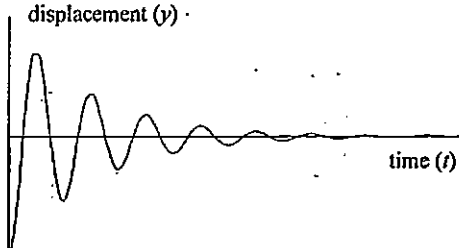
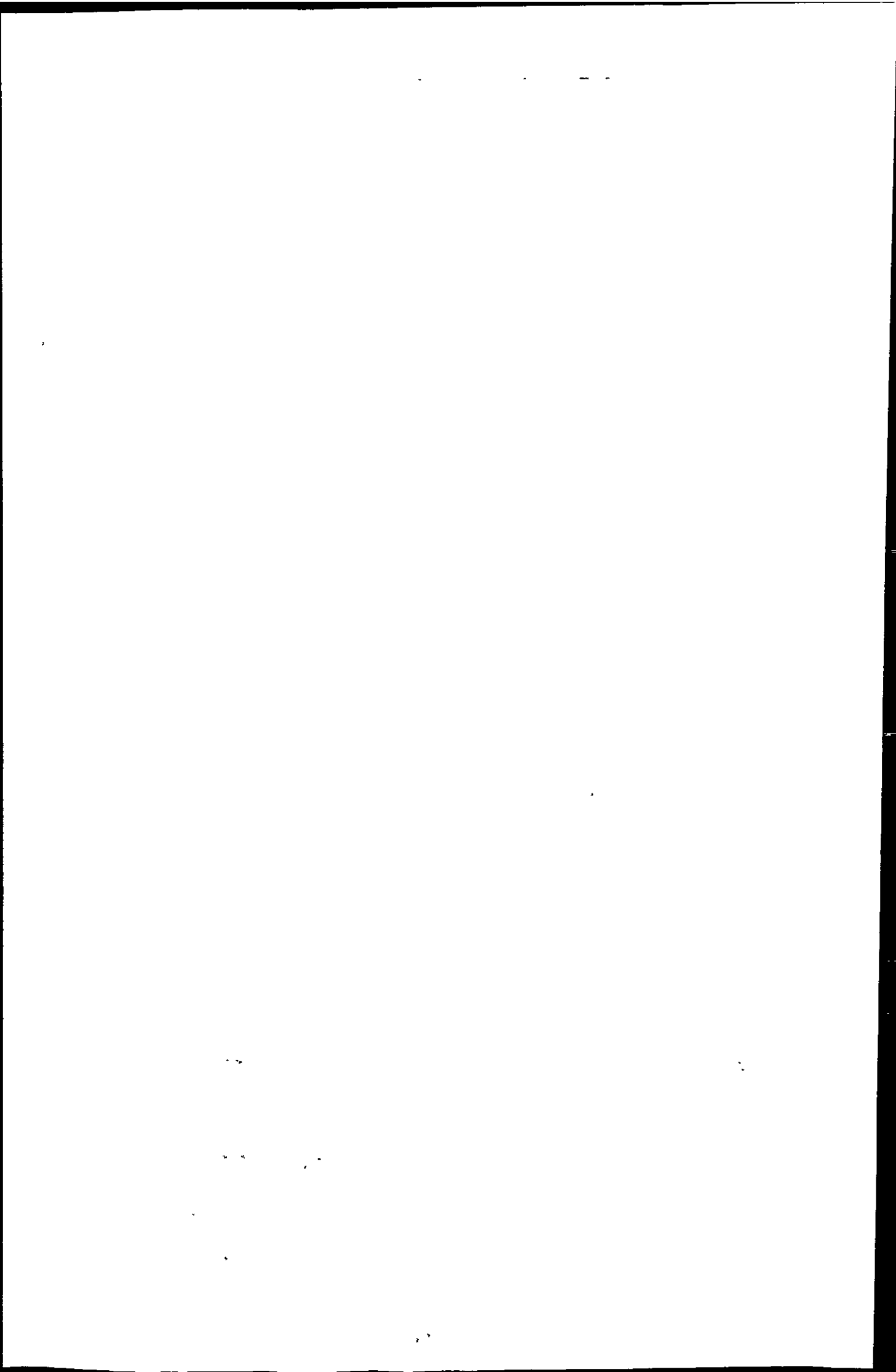
Option		Loading
3a	The mass bounces up and down, going less far each time, until it settles back to its original position.	.48014
1a	The beam bends under the weight of the gold bar.	.47143
(3f)	<div style="display: flex; align-items: center;"> <div style="margin-right: 10px;">displacement (y)</div>  </div> <p style="text-align: center;">(displacement-time diagram)</p>	-.66201
4a	q is the area under the curve $y = x$.	.54664

Table 8-5: Loadings on component 4

This component contains options from three questions: “mechanics” questions 1 and 3, and “mathematics” question 4.

It loads positively on verbal options and negatively on a diagrammatic option. It appears to be a verbal-versus-visual component for mechanics questions, and also a shallow-versus-depth component, as the options on which it loads positively all decrease with level of experience. For the mathematicians the component scores are almost constant with level of experience: for the engineers the scores rise almost steadily with a slight drop between the end of the first year and the final year of the degree. This drop may be due to a “new learning” peak (see conclusions to chapter 6) in that the damped harmonic motion graph would be relatively new material to engineering students in their final year. This option loads negatively on this component.



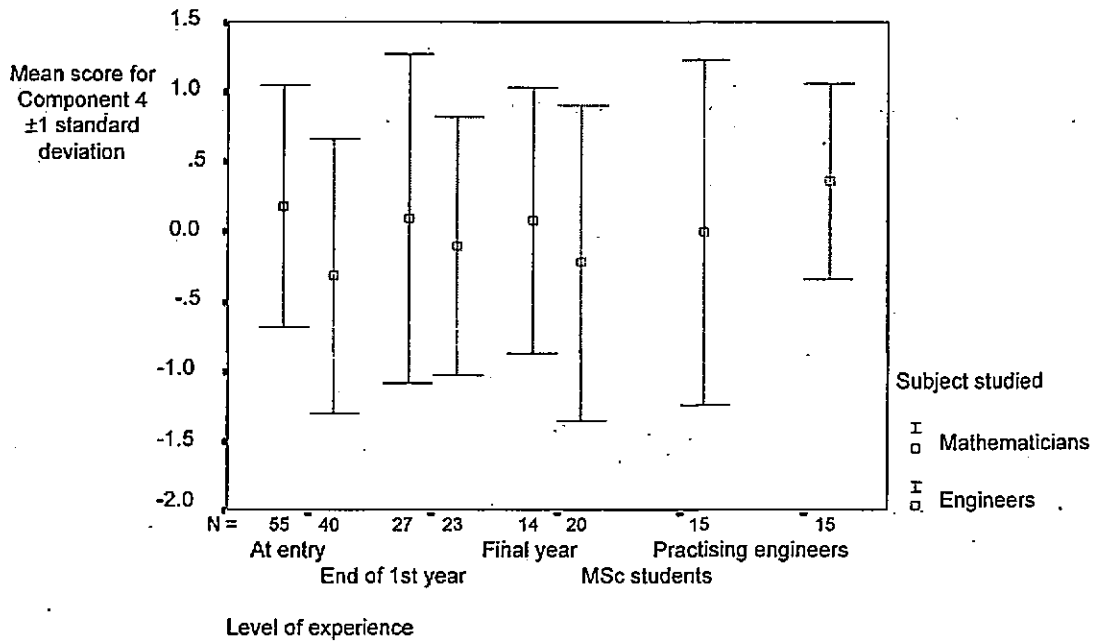


Figure 8-5: Scores on component 4

8.4.5 Component 5

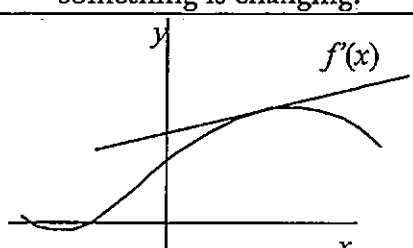
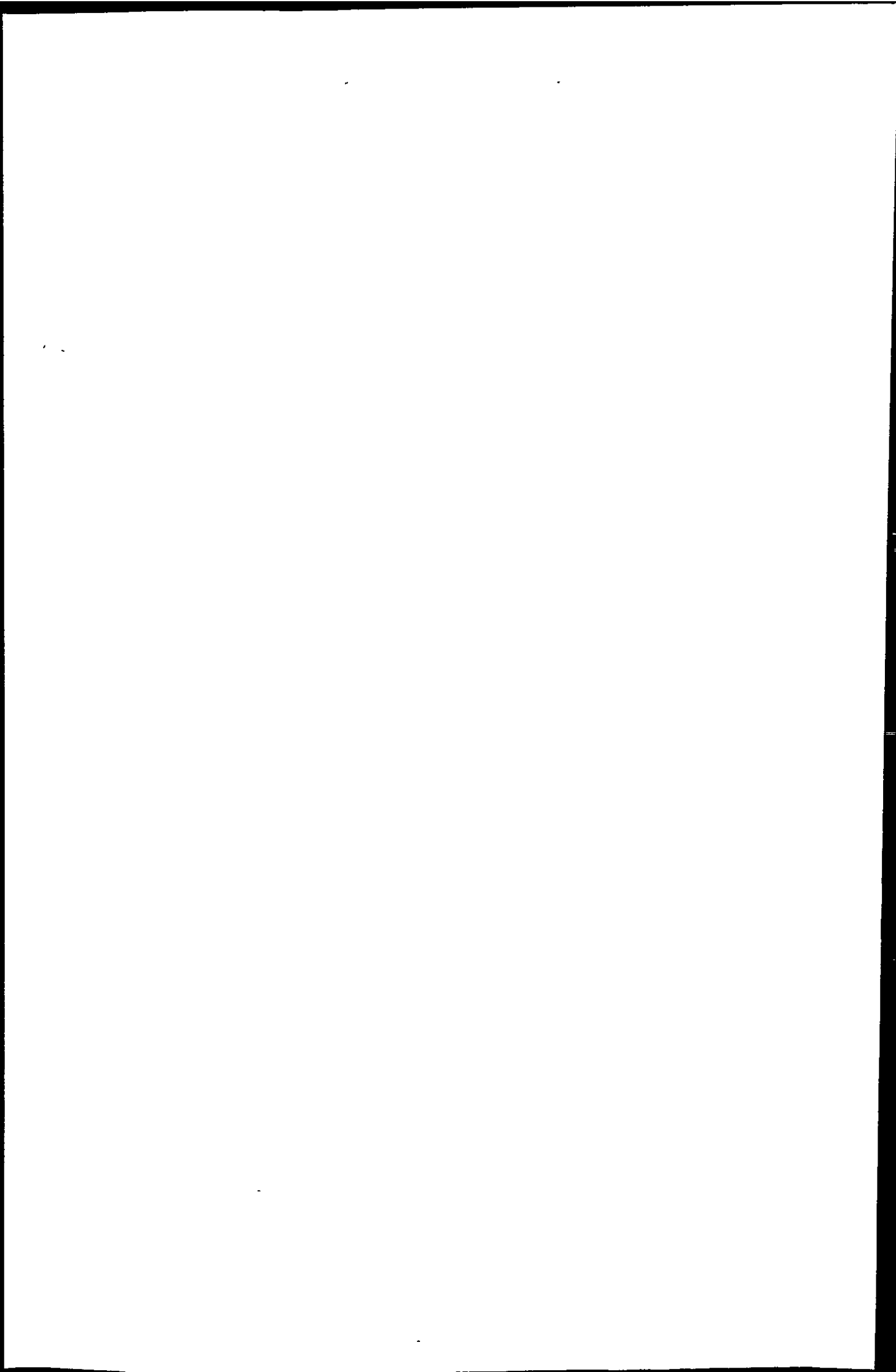
Option		Loading
(2c)	dy/dx tells you how quickly something is changing.	-.85996
2f		.60372
	(tangent diagram)	
(4d)	The integral tells you how things build up.	-.52471

Table 8-6: Loadings on component 5

This component brings together the two “sensemaking” options from the “mathematics” questions and opposes them to an option which is different both in mode (diagram as opposed to verbal) and content (graphical interpretation as opposed to sensemaking). Given the orientation of the component (sensemaking is negative) it is not surprising that the engineering respondents tend to show a decreasing score on this component. The peak for engineering students at the end of their first year looks



like a "new learning" peak, but unless the diagram is met for the first time by the students during their first year, which seems unlikely, it is difficult to say what has caused it.

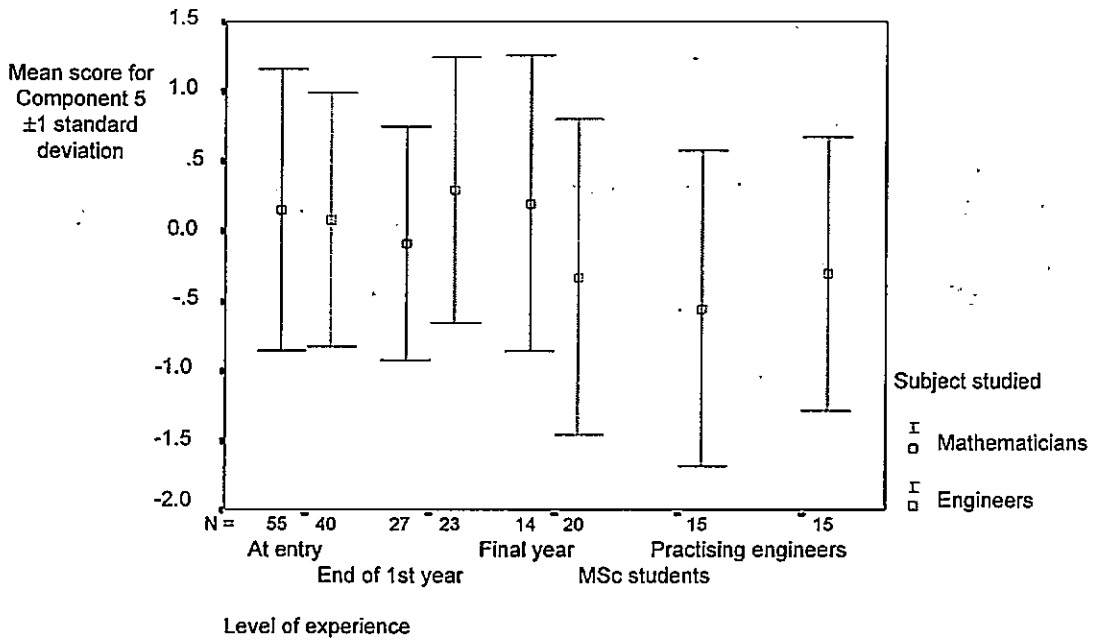


Figure 8-6: Scores on component 5

8.4.6 Component 6

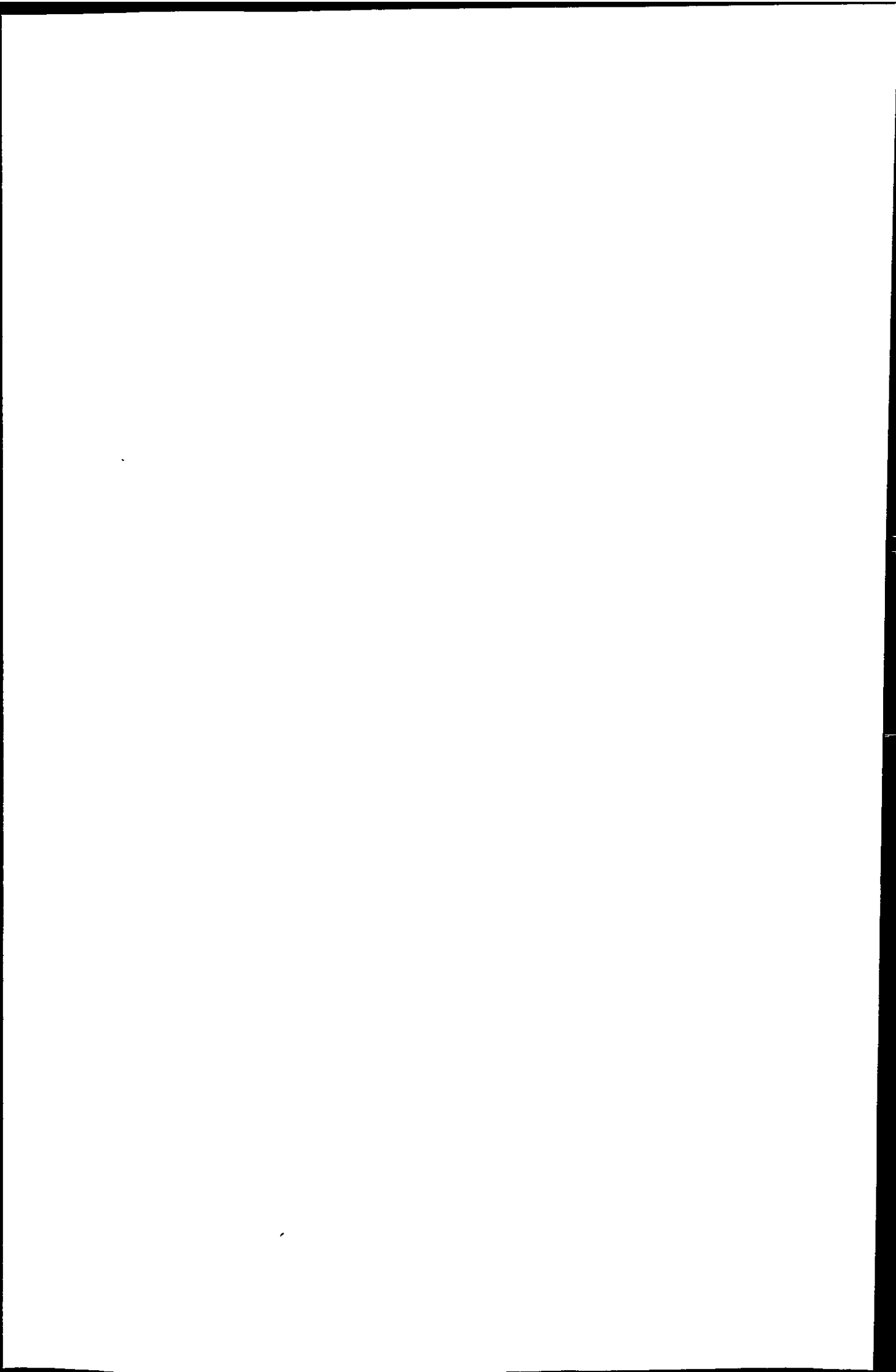
Option	Velocity (dy/dx)	Loading
(3c)		-.75997

(phase plane diagram)

3e	Damped harmonic response	.61661
----	--------------------------	--------

Table 8-7: Loadings on component 6

This is another option in which the practising engineers resemble the first year engineers more than they do the postgraduate engineers. There is a clear trend across the first four groups of engineers away from the phase plane diagram and towards the concise verbal option. Some of the practising engineers and the postgraduates have



met the phase plane diagram in their studies, which could account for their not rejecting it as strongly as the undergraduates.

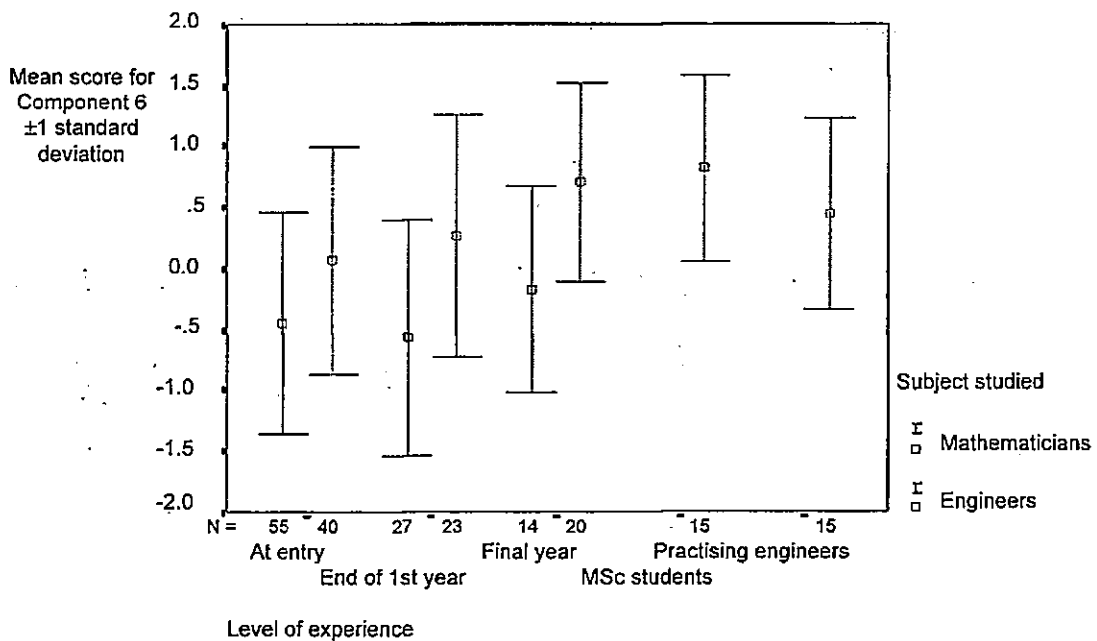


Figure 8-7: Scores on component 6

8.4.7 Component 7

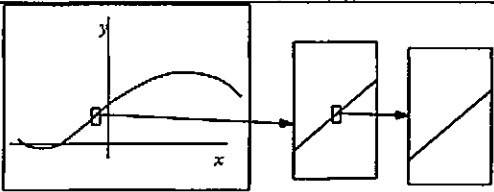
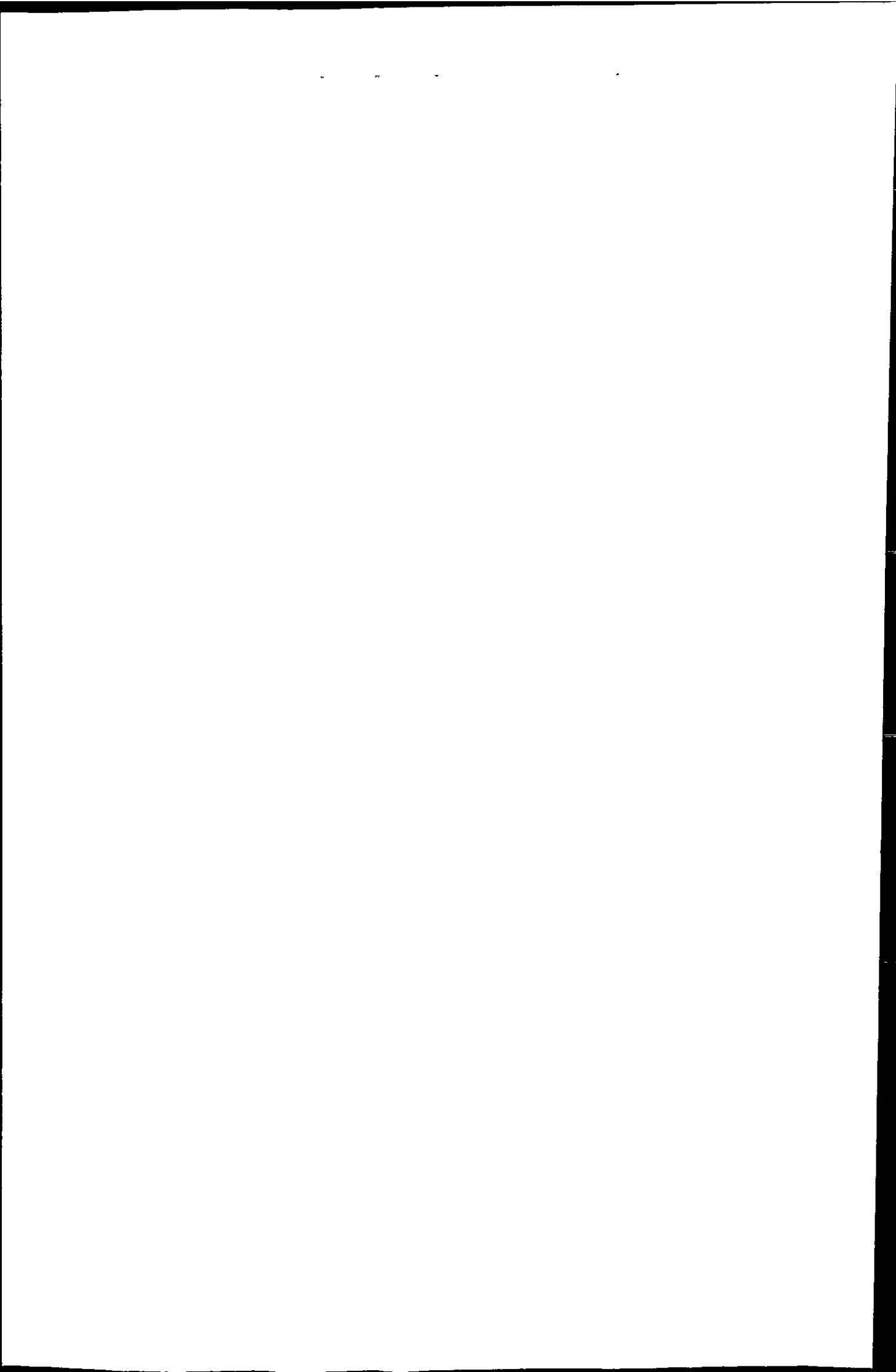
Option	Loading
2a $f'(x)$ is the slope of the tangent to a graph of y against x .	.77162
(2b)  (zoom in diagram)	-.63377

Table 8-8: Loadings on component 7

The two options which load on this component are radically different representations of the derivative, one of which (2a) is commonly taught and the other (2b) is taught in, for example, SMP A level. The two options are also in different modes of representation.

The greatest difference between the engineers and the mathematicians is at entry, where the mathematicians prefer the "slope of the tangent" option quite strongly.



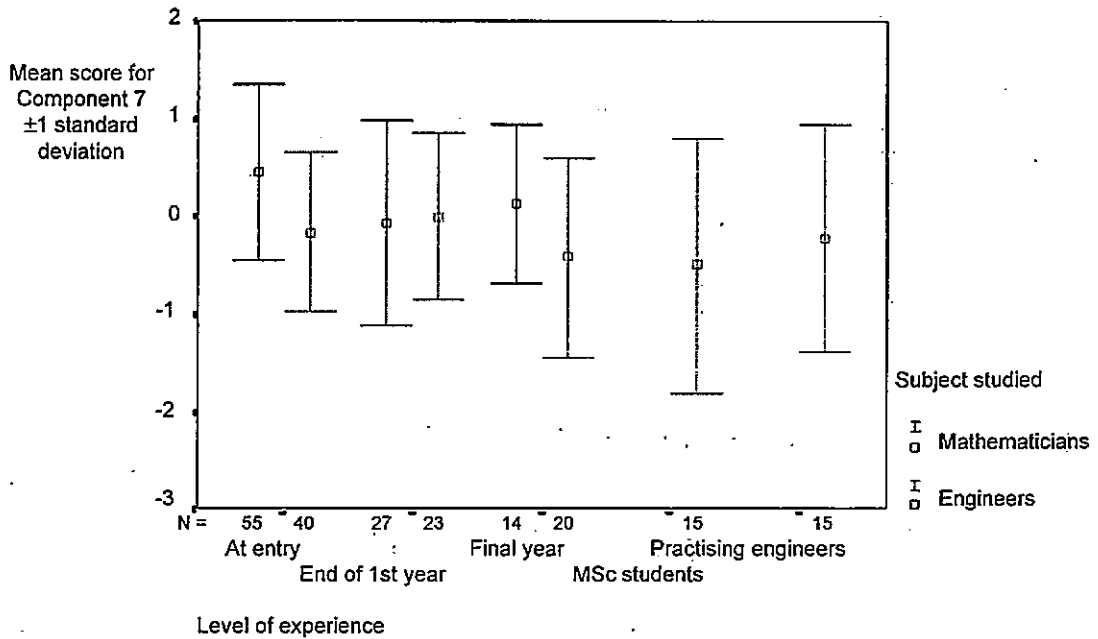


Figure 8-8: Scores on component 7

8.4.8 Component 8

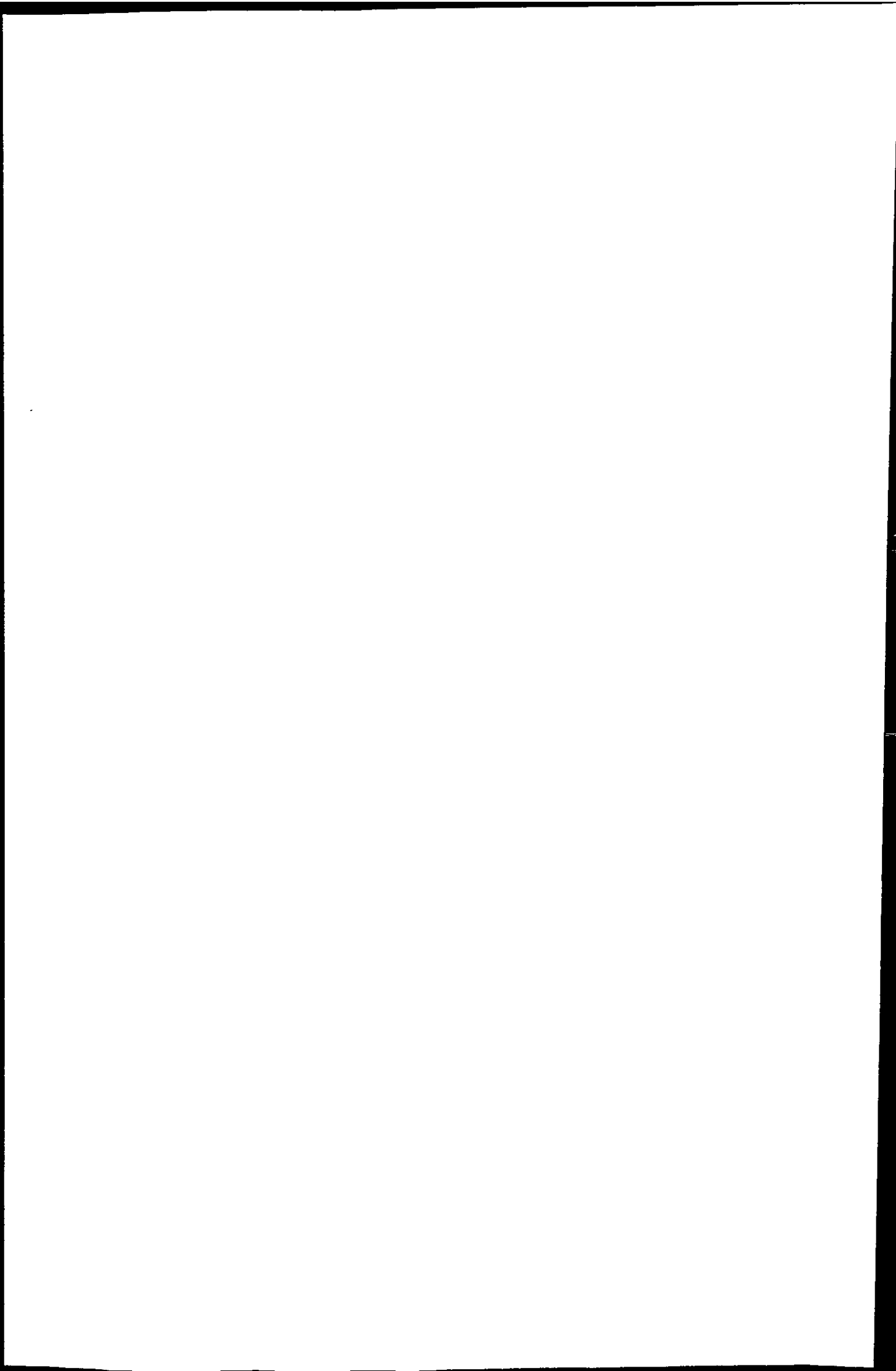
Option	Loading
(4f)	-0.79630
$\frac{dq}{dx} = x$ (integration is the reverse of differentiation)	
4c	0.67916

$q = \frac{x^2}{2} + C$

(integration carried out diagram)

Table 8-9: Loadings on component 8

Both these options come from question 4, and show both different underlying concepts and different modes of representation. It is not possible to say which is more significant. It is possible that the peak in this component in engineering students at the end of their first years is a “new learning” peak associated with teaching about integration over the course of the first year. The practising engineers are very like the end of first year engineers in this component.



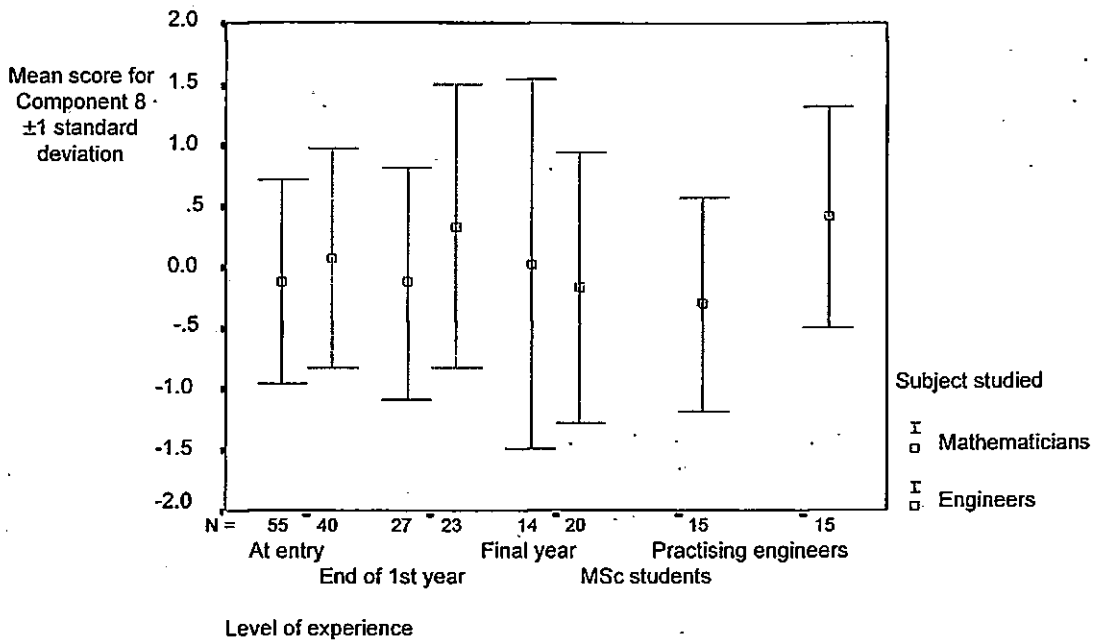


Figure 8-9: Scores on component 8

8.4.9 Component 9

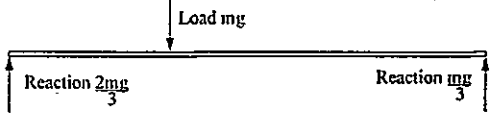
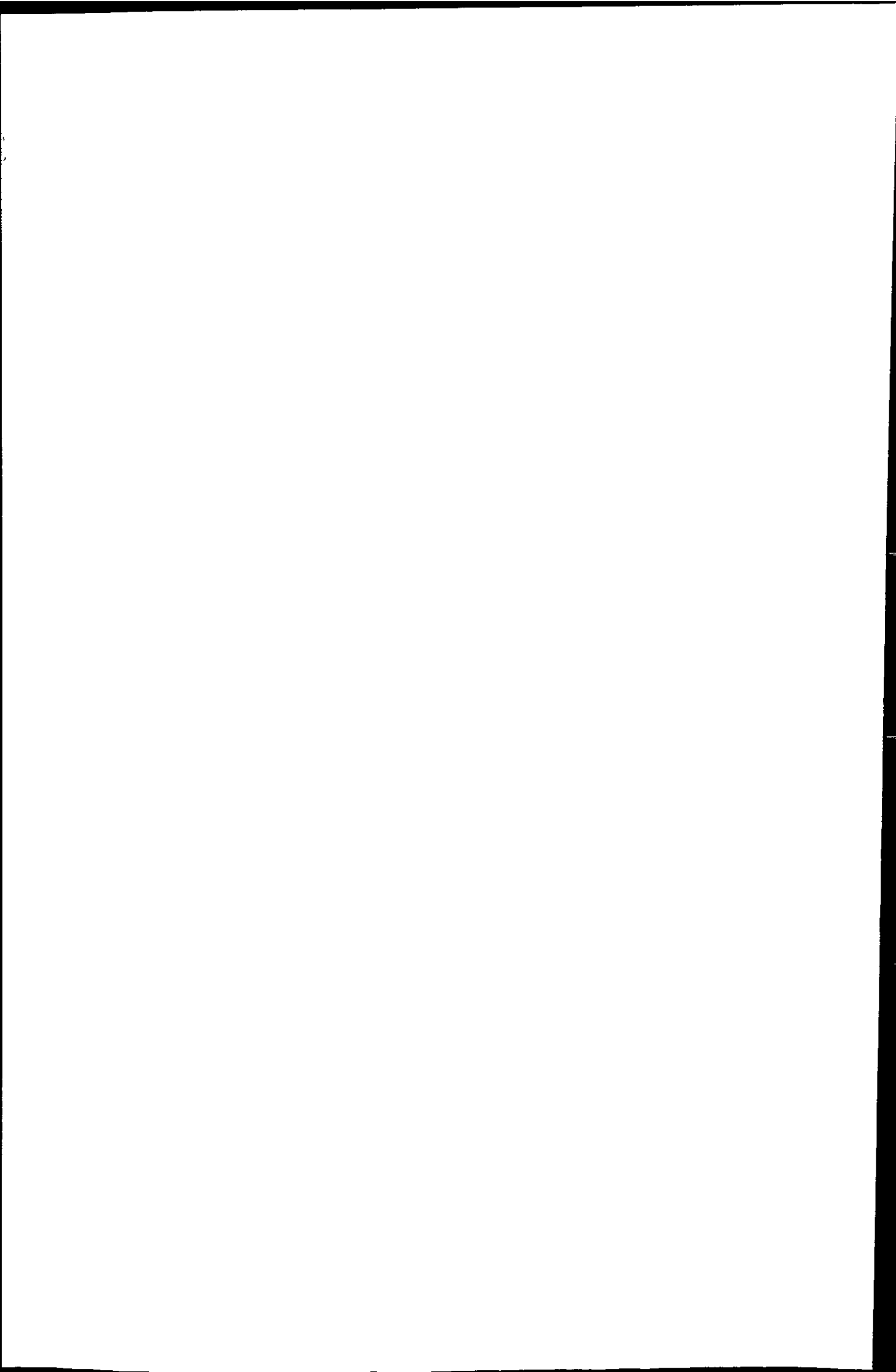
Option		Loading
(4a)	q is the area under the curve $y = x$.	-44838
(1f)	 <p>(loads diagram)</p>	-77005
1d	A simply supported beam with a point load at one-third span.	.40077

Table 8-10: Loadings on component 9

This component does not seem to represent a diagrammatic-verbal opposition, but it does contrast naïve and more mature concepts.

I suspect that the low scores in this component for the engineers in their first year are a “new learning” peak, representing covering integration (end of first year) and beam bending (beginning of first year). It is difficult to say why the postgraduate and practising engineers drift back towards the first year level.



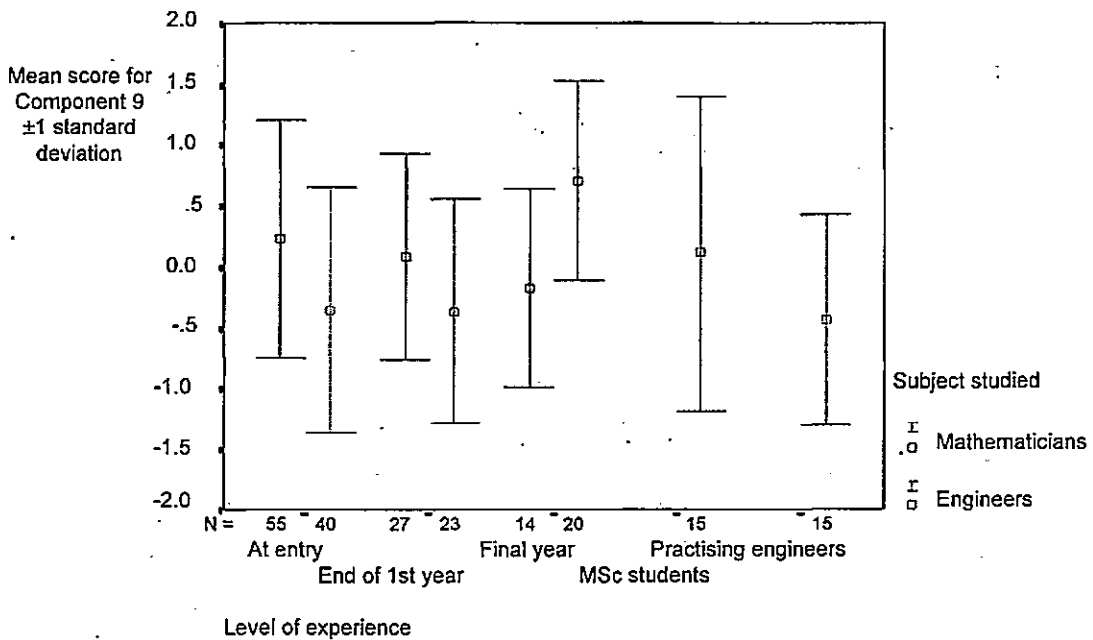


Figure 8-10: Scores on component 9

8.4.10 Component 10

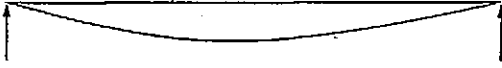
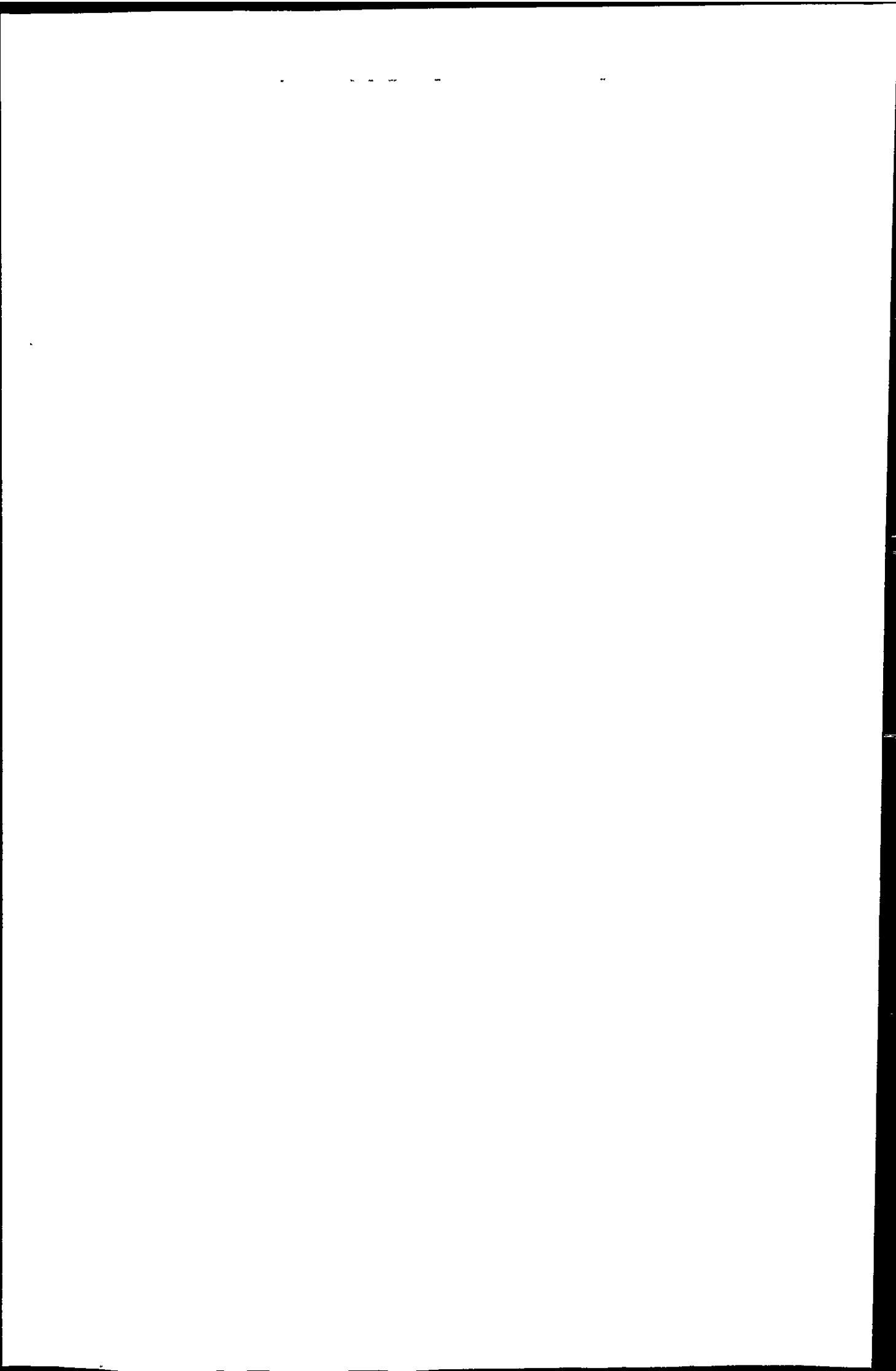
Option	Deflected shape	Loading
1b		.79043
(1d)	A simply supported beam with a point load at one-third span.	-.59284

Table 8-11: Loadings on component 10

This component opposes two options from question 1. The options are different in both mode and concept. Option 1d was intended to be a form of encapsulation: the summary of the target concept as a member of a class of standard solutions. The deflected shape is somewhat unexpected, and may be more acceptable to more experienced engineering students.



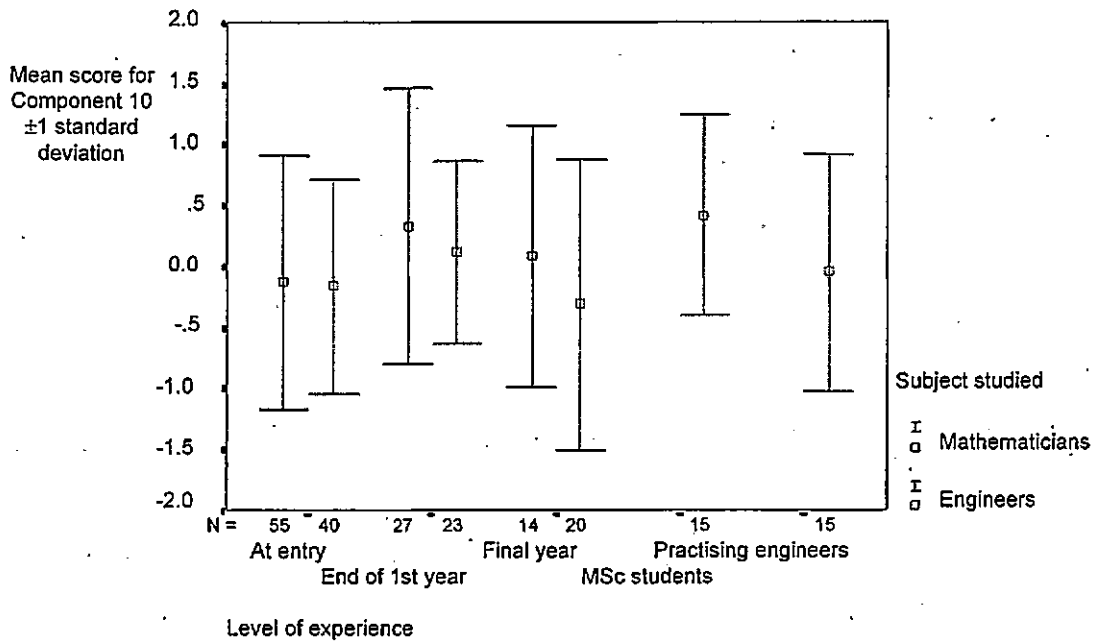


Figure 8-11: Scores on component 10

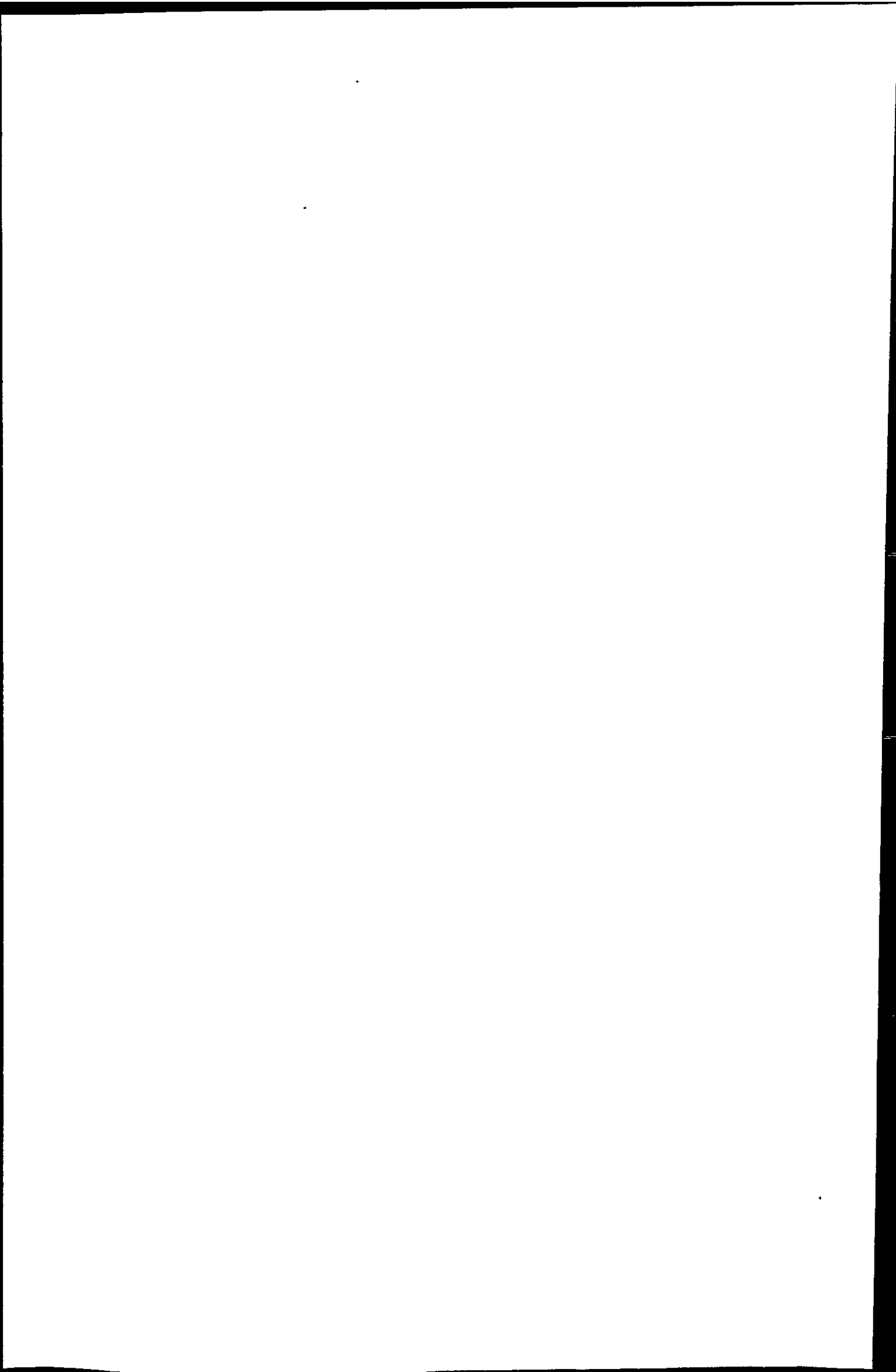
8.5 Discussion

8.5.1 Canonical discriminant functions

Given that the data is classified already according to the typology of subject studied and level of experience, the values obtained for the component scores for each respondent may be used to test how good a match the taxonomy of the component analysis is to the typology.

A form of regression analysis is used to obtain coefficients for linear equations in the component scores which best classify the cases back into their actual groups. Cases which are wrongly classified indicate overlap of the groups which cannot be resolved using the given variables.

When such an analysis of the respondents was performed with SPSS using the component scores as the variables, the reclassification shown in Table 8-12 was obtained.



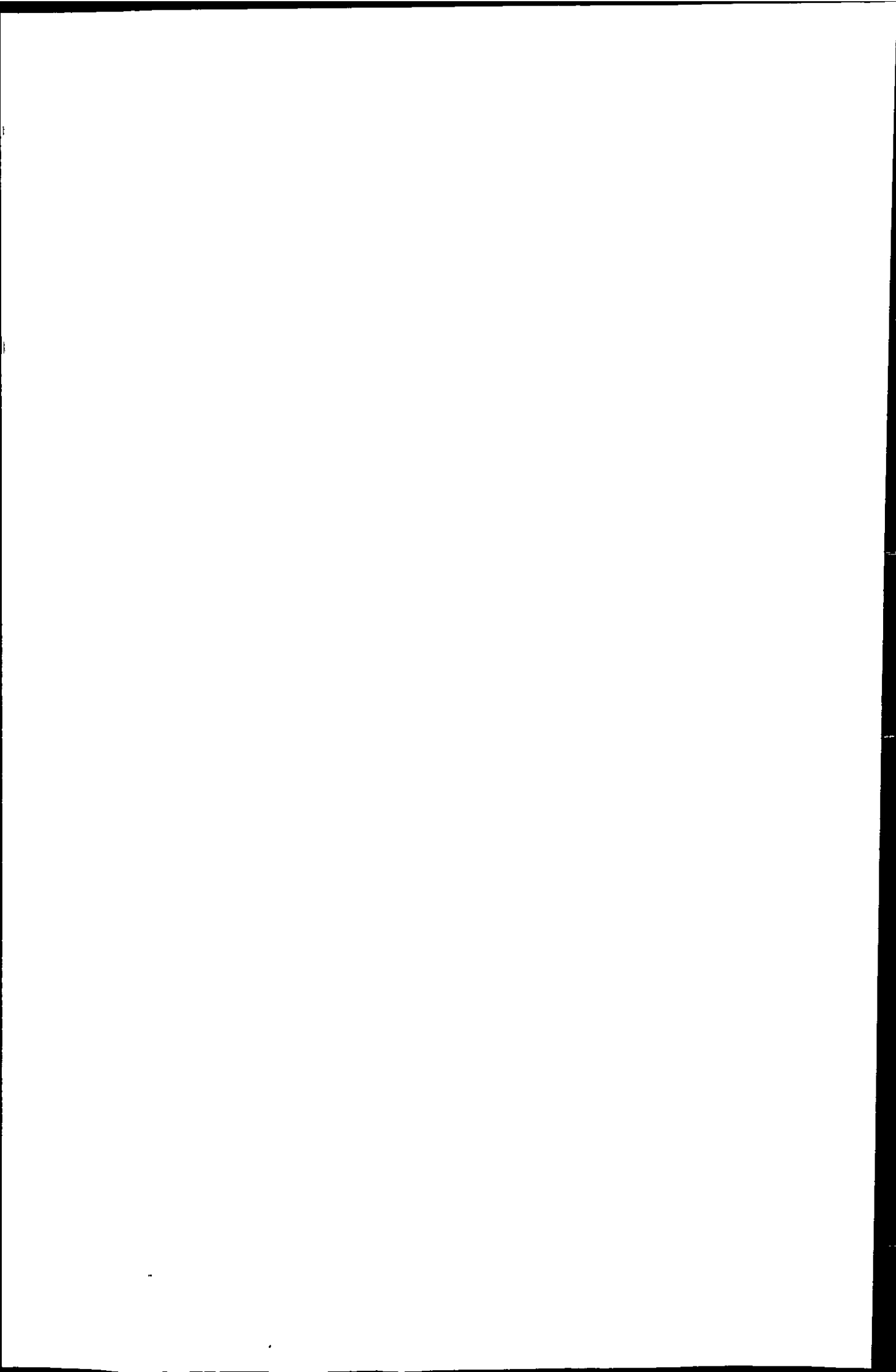
Actual Group	No. of Cases	Predicted Group Membership							
		maths entry	maths end 1st yr.	maths final	eng entry	eng end 1st yr.	eng final	post-grad eng	prac eng
maths entry	55	43 78.2%	5 9.1%	0	3 5.5%	3 5.5%	0	1 1.8%	0
maths end 1st	27	10 37.0%	10 37.0%	0	3 11.1%	1 3.7%	1 7.4%	2	0
maths final	14	3 21.4%	1 7.1%	8 47.1%	1 7.1%	0	0 7.1%	1	0
eng entry	40	5 12.5%	1 2.5%	0	24 60%	2 5%	2 5%	1 2.5%	5 12.5%
eng end 1st	23	1 4.3%	0	0	9 39.1%	12 52.2%	0	0	1 4.3%
eng final	20	1 5.0%	0	0	3 15.0%	3 15.0%	11 2.0%	1 7.0%	1 6.7%
postgrad eng	15	2 13.3%	0	1 6.7%	2 13.3%	0	2 13.3%	7 46.7%	1 6.7%
prac eng	15	2 13.3%	0	0	3 20.0%	2 13.3%	0	1 6.7%	7 46.7%

Table 8-12: Comparison of predicted with actual group membership

Each row shows the number and percentage of the members of an actual group classified in each predicted group. A perfect classification would be a diagonal array with 100% in each diagonal cell (shown in bold).

The percentage of "grouped" cases correctly classified is 58.37%, that is 122 cases; the number of cases where 1a mistaken for 1b, or vice versa, in the correct subject is 26. Thus 148 or 70.8% of cases were almost correctly classified. This indicates that the overlaps between the groups are small. Although the components do not individually distinguish well between types of respondent, it appears that in combination they do.

Of particular interest is that 5 of the engineering students at the beginning of their studies were classified with the practising engineers, and five of the practising engineers were classified with the first year engineers. This seems to indicate either that the practising engineers regress to mathematical ideas of the type they held at the start of their studies (See the comments in chapter 6 from practising engineers on



trying to avoid mathematics) or that this group was originally so different from the Plymouth students that any comparison is invalid.

Another interesting point is the overlap between the first year mathematics and engineering students. Of the first year mathematics responses, ten are mis-classified as first year engineering responses, and of the first year engineering responses, seven are mis-classified as first year mathematics responses. This overlap indicates that the two groups are not completely disparate, but that there is a degree of similarity between them. Either they have not been socialised into their subject groups, or those students who have chosen the "wrong" subject have not yet dropped out.

Clumping the groups together by subject studied gives Table 8-13.

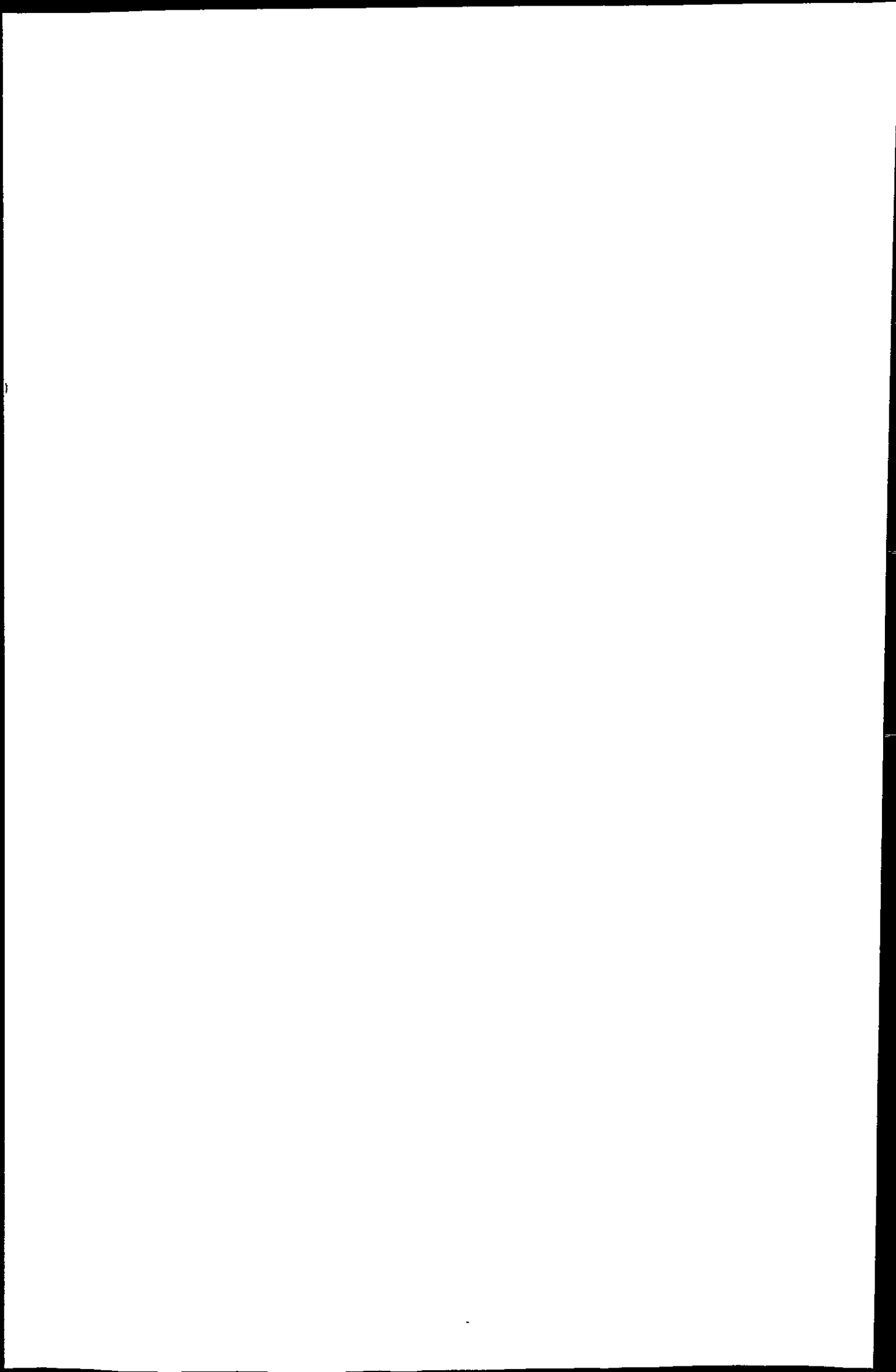
		Predicted Group Membership	
		Mathematics	Engineering
Actual groups	Mathematics	80 (44.1)	16 (51.9)
	Engineering	13 (51.9)	100 (61.1)

Table 8-13: Comparison of predicted with actual groups: (subject studied only)
(Figures in brackets show expected random redistribution)

The proportion of cases wrongly classified by subject studied was 13.9%. The value of χ^2 for this distribution is close to 108, and the value needed for significance at the 0.1% level for three degrees of freedom is only 16.27. Hence the reclassification could be regarded as successful. Testing the significance at a finer level is problematic as the expected class sizes drop below five, at which point the calculation of χ^2 is invalid.

8.5.2 The components

The component analysis produced components which were interesting in two different ways: those which could be interpreted easily and those which distinguished significantly between groups. Component 5 fulfils both these criteria: it seems to represent an assignment of meaning to calculus ideas and it distinguishes significantly



between final year engineers and mathematicians in particular. It was satisfying that the two verbal options which load significantly on this component were associated by the analysis, and gave some confidence in the analysis.

Six of the components loaded on options from one question only. These were components 2 and 10 which loaded on question 1, component 7 which loads on question 2, component 6 on question 3 and components 3 and 8 on question 4. These all oppose different modes of representation, either algebraic and verbal (component 2), algebraic and diagrammatic (Components 3 and 8), or diagrammatic and verbal (components 6, 7 and 10). These components were particularly difficult to interpret when they loaded on only two options (components 3, 6, 7, 8 and 10).

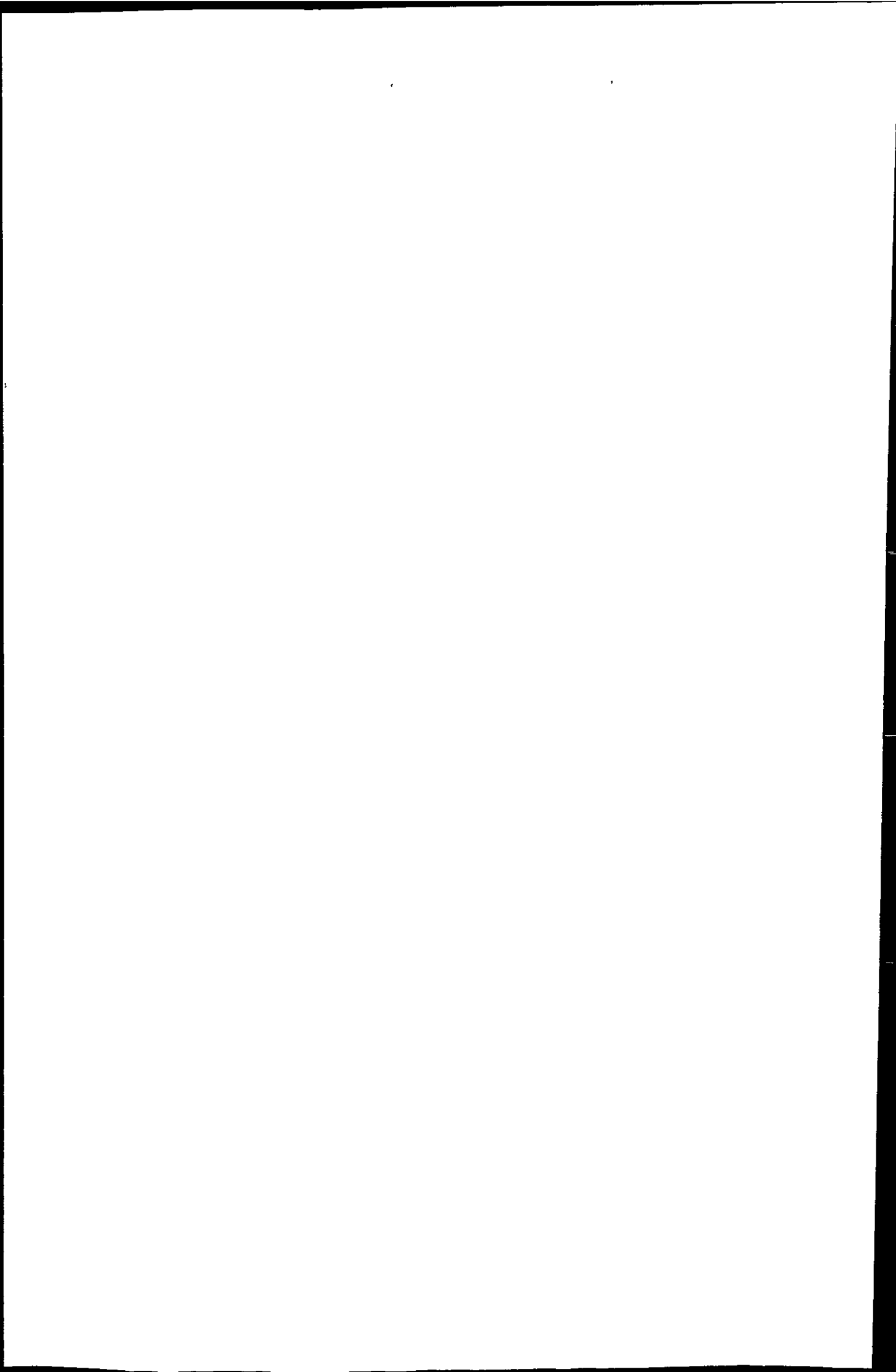
In components 1, 2, 5, 6, 7 and 8 the practising engineers' scores seemed to revert to being like the first year engineering students'. This phenomenon would explain the way that the canonical discriminant function analysis failed to distinguish between these groups. Why the groups are similar is a question which requires further investigation to answer sensibly.

8.6 Conclusions

8.6.1 On the analysis

Component analysis brought out some interesting features from the questionnaire results. In some ways the engineers and mathematicians appeared to be similar, in others they started different and became more similar, and others they started similar and diverged.

The canonical discriminant function separated the groups of respondents with a fair degree of success, apart from the practising engineers and the engineering students at the start of the course. This result indicates that it would be useful to survey some



University of Plymouth engineering graduates some 5-10 years into their careers to see if they also "regress" .

Some of the components were easy to identify as meaningful: others, particularly those which loaded significantly on only two options, were difficult to interpret. This shows up the importance of having enough variables per component in the analysis.

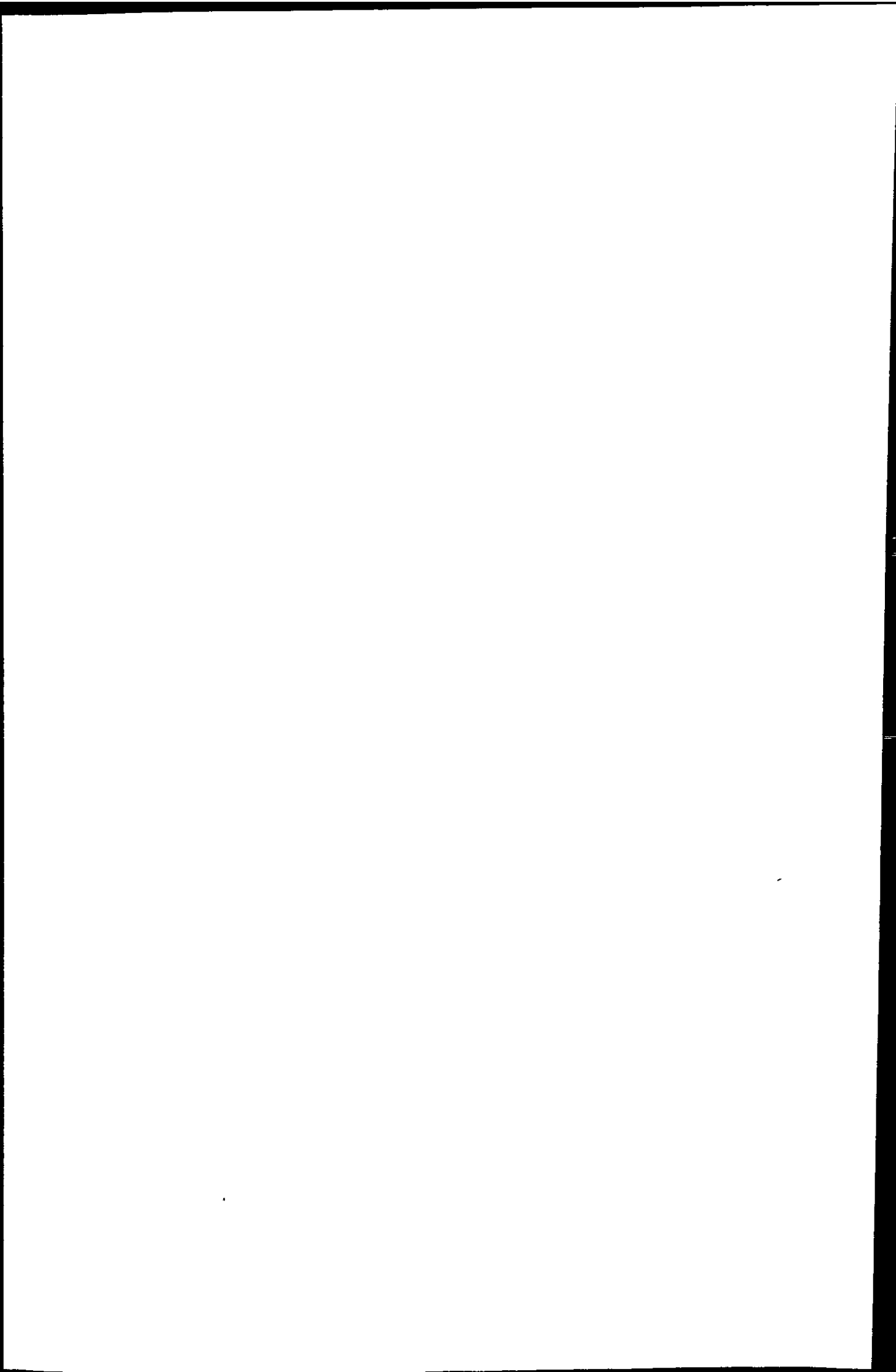
8.6.2 On the suitability of the data

The results were not strictly suitable for component analysis, so the features identified must be regarded with caution. In particular the Measure of Sampling Adequacy (MSA) for the sample as a whole and for all the options except those to Question 1 was below the recommended level of 0.5.

The number of components chosen was a compromise between accounting for enough of the variance of the sample (more components) and having enough options per component to identify the component (fewer components). As Hair et al (1984) point out, this is a matter of judgement, and ten components coincided with a break point on the scree plot, accounted for more than 50% of the variance and did not create any single-variable components. In addition, the rotation using SPSS converged within 25 iterations, and thus it was possible to perform the analysis with this number of factors.

8.6.3 On the questionnaire

The analysis has shown another way in which the questionnaire may be improved, namely by increasing the number of questions and by more consciously aiming the options towards particular components. This will be taken into account when redesigning Questions 5 and 6 in particular.



9. Revised questionnaire

9.1 Introduction

The component analysis described in the last chapter and the experience of administering the questionnaire described in chapter 5 showed that there were shortcomings in the questionnaire. On the basis of those results, and of the theory described in the previous chapter, the questionnaire was re-examined critically, and each question revised to follow a common format. In this chapter a set of six revised questions is proposed.

Questions 1-4 are largely as before, with minor adjustments, except that the invited response is in the form of a 5-point Likert scale for each option. This has two benefits: it should make data input easier and it makes the six scores for the options for a given question independent. The diagrams have also been redrawn to improve the quality of the lines.

It was felt that the use of a Likert-type scale would reduce the number of extreme ratings (as respondents tend to avoid the extreme ends of such a scale) and so with a smaller number of values for each variable, the internal correlations would be higher and the number of factors would be reduced.

There are now 36 variables which it is hoped will be suitable for component analysis. This would allow for up to 7 components at a rate of 5 variables per component. The components which are anticipated are: mechanics/mathematics type question, depth of concept, algebraic mode, verbal mode and diagrammatic mode: that is five probable components.

The space for comments has been retained as some of the comments returned by respondents have been so illuminating at various stages of the research.

The adjustments are as follows.

9.2 Question 1: beam bending (statics)

Option 1b, the deflected shape diagram: the load arrow now touches the line of the beam. A colleague suggested that he found this easier to understand. The consistently unpopular option 1e, two expressions giving the relationship between shear force, bending moment and applied loads in an integral format, is replaced by a pair of equations for the deflection of the beam, analogous to option 3d.

9.3 Question 2: differentiation

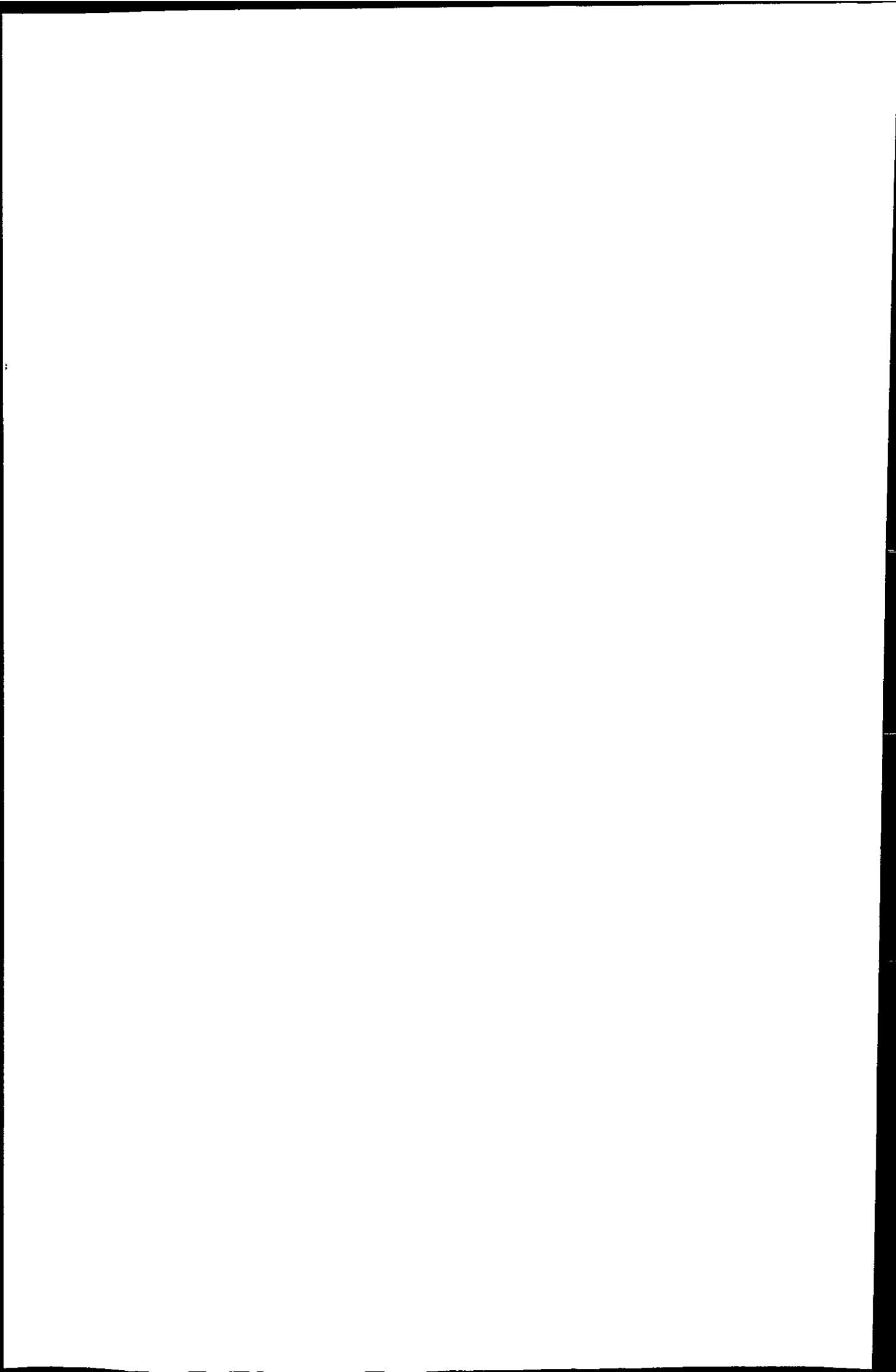
Option 2e, the "As you zoom in..." statement was an unpopular option which gave the question three verbal options with only one algebraic. It was replaced with a simple algebraic option expressing the idea that integration is the reverse of differentiation, analogous to 3f.

9.4 Question 3: mass/spring/damper system (dynamics)

Option 3c, the phase plane diagram was again an unpopular option. It was replaced with a diagram of the mass, spring and damper, which gave a simple diagrammatic representation, of the type which one might draw in beginning to solve the case, and analogous to option 1f.

9.5 Question 4: integration

Option 4b, the solved integral has been replaced with a statement of the integral as a limit, analogous with 2d.



9.6 Question 5: pinball (unfamiliar kinematics)

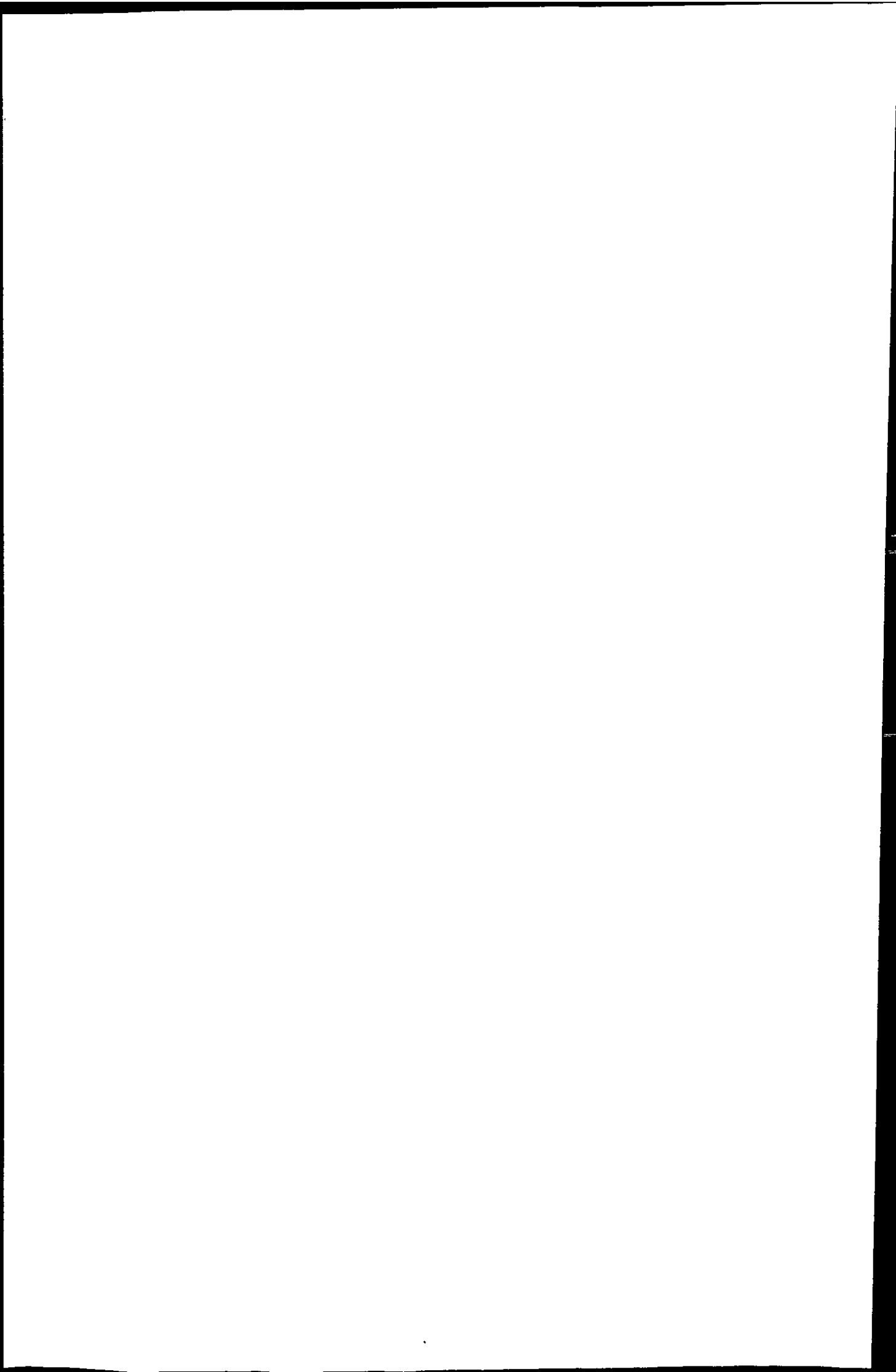
The options to question 5 were completely revised. Option 5b, a diagram of the forces acting on the ball while in contact with the spring, was given as analogous with options 1f and 3c. Option 5c is a statement about the motion of the ball using some technical language, analogous to options 1d and 3e. Option 5d is a differential equation describing the motion of the ball while in contact with the spring, analogous to 1c and 3b. Option 5e is a displacement-time diagram analogous with options 1b and 3f. Option 5f is a simple statement in non-technical language about the behaviour of the system analogous with 1a and 3a.

9.7 Question 6: differential equation: exponential growth

Question 6 was also completely revised. Instead of six algebraic options there are now two algebraic, two diagrammatic and two verbal options, to match all the other questions. They have also been designed to match the options in questions 2 and 4.

9.8 Revised questions

The complete set of revised questions appears on the following pages.



9.8.1 Question 1

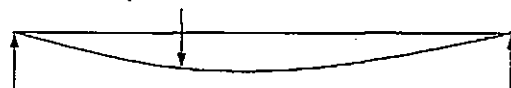
A plank 1.5m long is placed on two bricks very near its ends. A bar of gold is placed across it 0.5m from one end. Score the following according to how well they represent this to you.

(a) The beam bends under the weight of the gold bar.

Nothing like it

Exactly right

(b) Deflected shape



Nothing like it

Exactly right

(c) Bending Moment $M = k \frac{d^2x}{dy^2}$

Nothing like it

Exactly right

(d) A simply supported beam with a point load at one-third span

Nothing like it

Exactly right

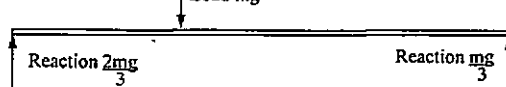
(e) $x < \frac{l}{3}; y = -EI \frac{mg}{9} (x^3 - \frac{2}{9} l^2 x)$

$$\frac{l}{3} < x < l; y = -EI \frac{mg}{6} (\frac{1}{27} l^3 - \frac{12}{27} l^2 x + lx^2 - \frac{1}{3} x^3)$$

Nothing like it

Exactly right

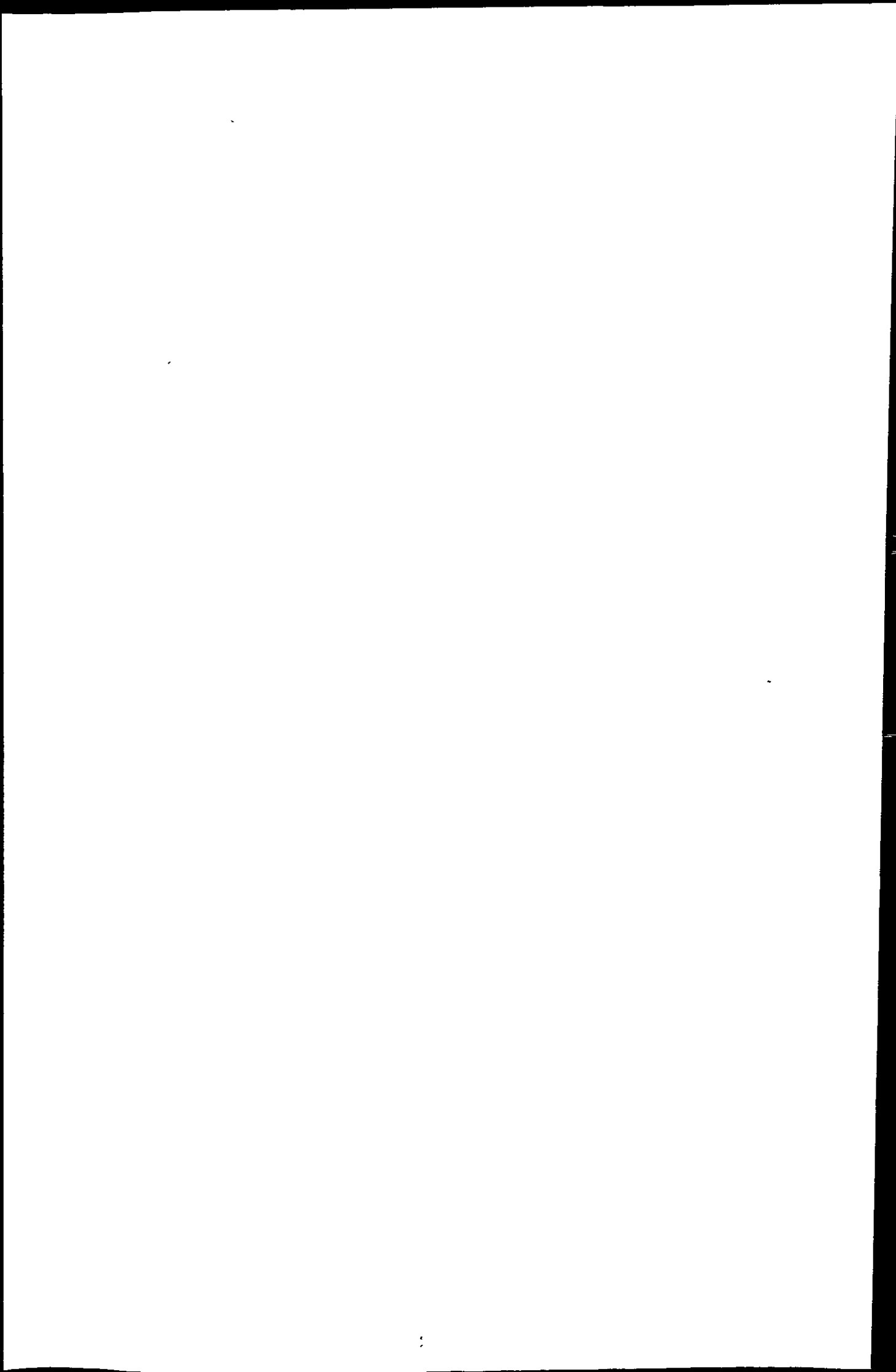
(f) Load mg



Nothing like it

Exactly right

Comments



9.8.2 Question 2

$$\frac{dy}{dx} = f'(x)$$

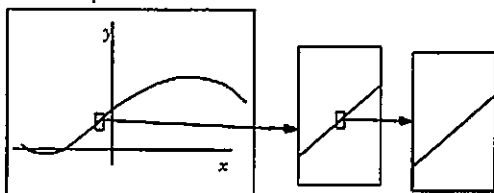
All the below may be associated with the statement above. Please score them according to how closely they are linked to it in your mind.

(a) $f'(x)$ is the slope of the tangent to a graph of y against x .

Nothing like it

Exactly right

(b)



Nothing like it

Exactly right

(c) dy/dx tells you how quickly something is changing.

Nothing like it

Exactly right

(d)

$$f'(x) = \lim_{(x_2-x_1) \rightarrow 0} \frac{y_2 - y_1}{x_2 - x_1}$$

Nothing like it

Exactly right

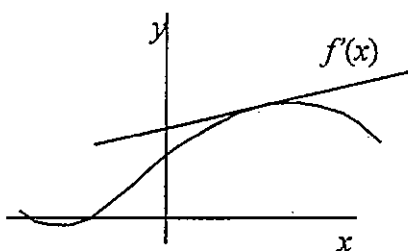
(e)

$$y = \int f(x) dx$$

Nothing like it

Exactly right

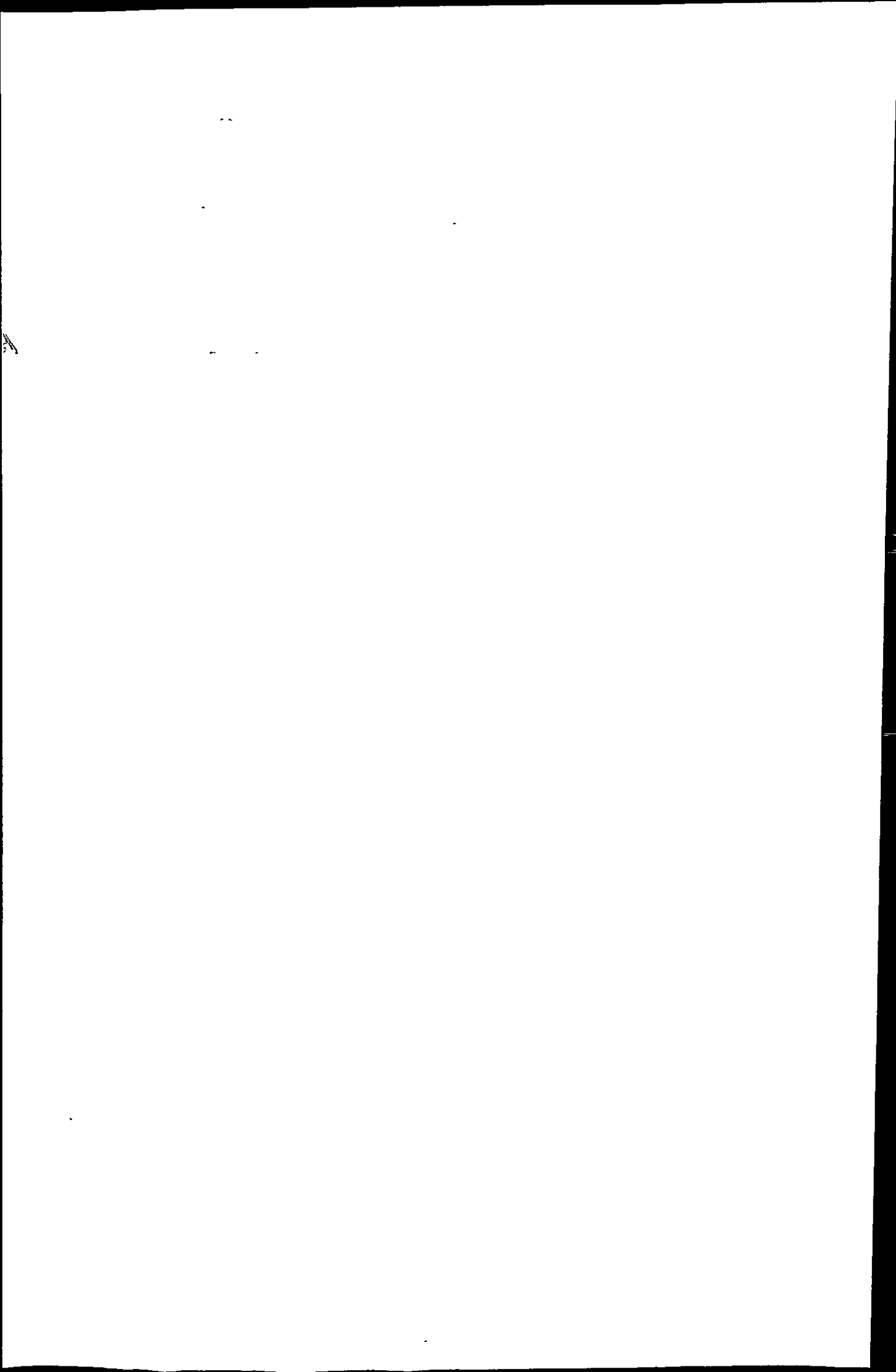
(f)



Nothing like it

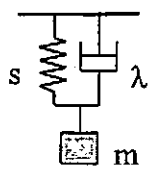
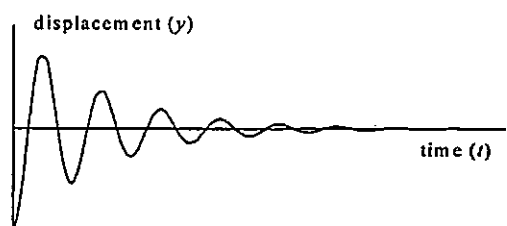
Exactly right

Comments

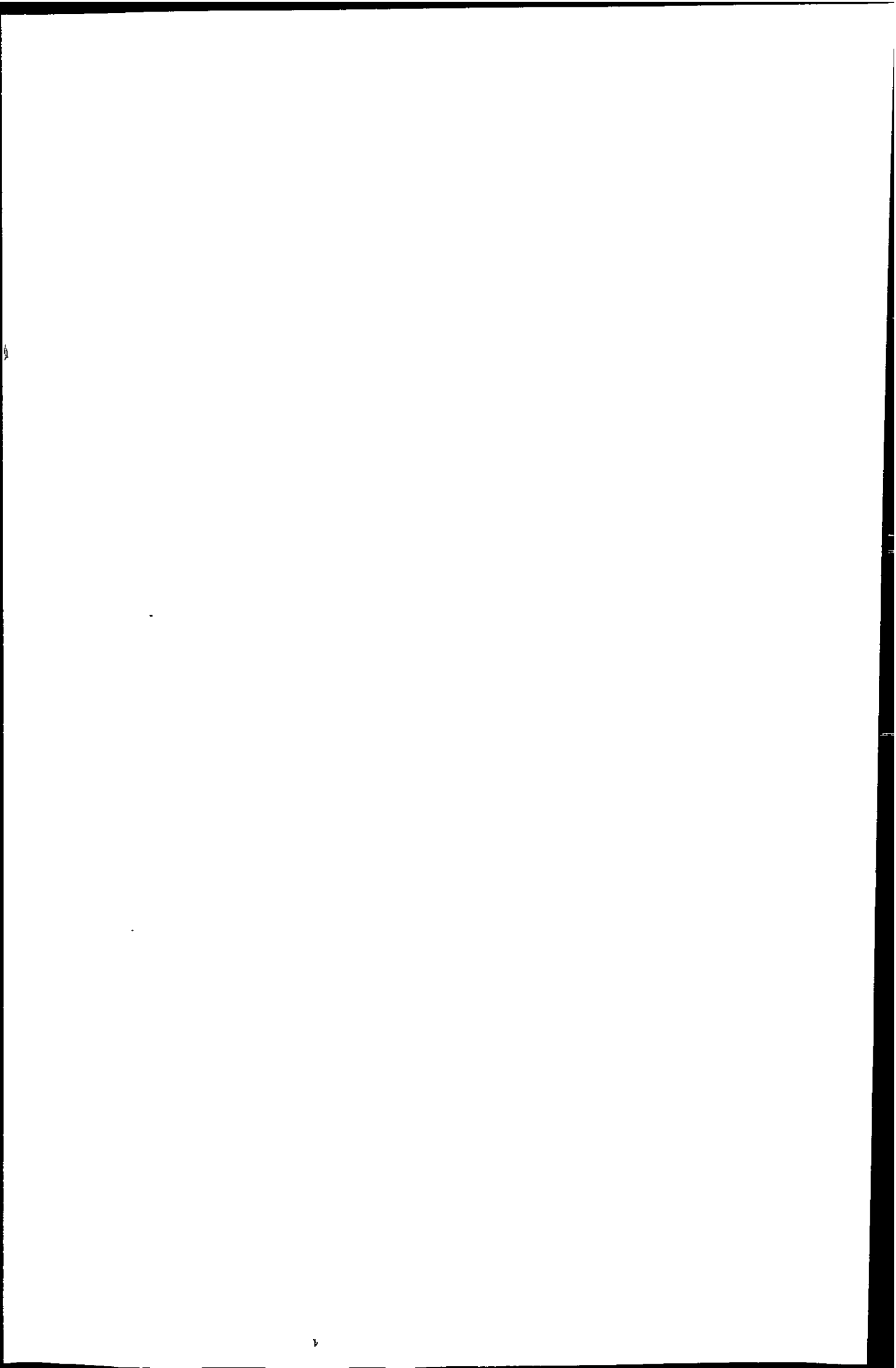


9.8.3 Question 3

A mass suspended from a spring and dashpot is pulled down from its equilibrium position and released. Please score the following according to how they best match this for you.

(a)	The mass bounces up and down, going less far each time, until it settles back to its original position.	Nothing like it	Exactly right
		<input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/>	
(b)	$\ddot{y} + k\dot{y} = \omega^2 y = 0$	Nothing like it	Exactly right
		<input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/>	
(c)		Nothing like it	Exactly right
		<input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/>	
(d)	$y = Ae^{-\lambda t} \cos \omega t$	Nothing like it	Exactly right
		<input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/>	
(e)	Damped harmonic response	Nothing like it	Exactly right
		<input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/>	
(f)		Nothing like it	Exactly right
		<input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/>	

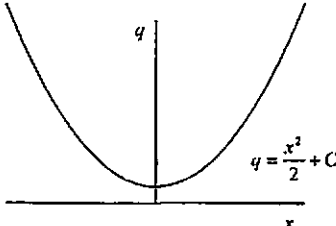
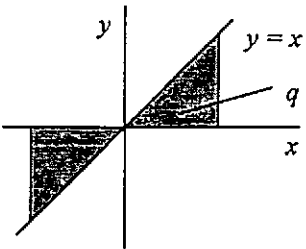
Comments



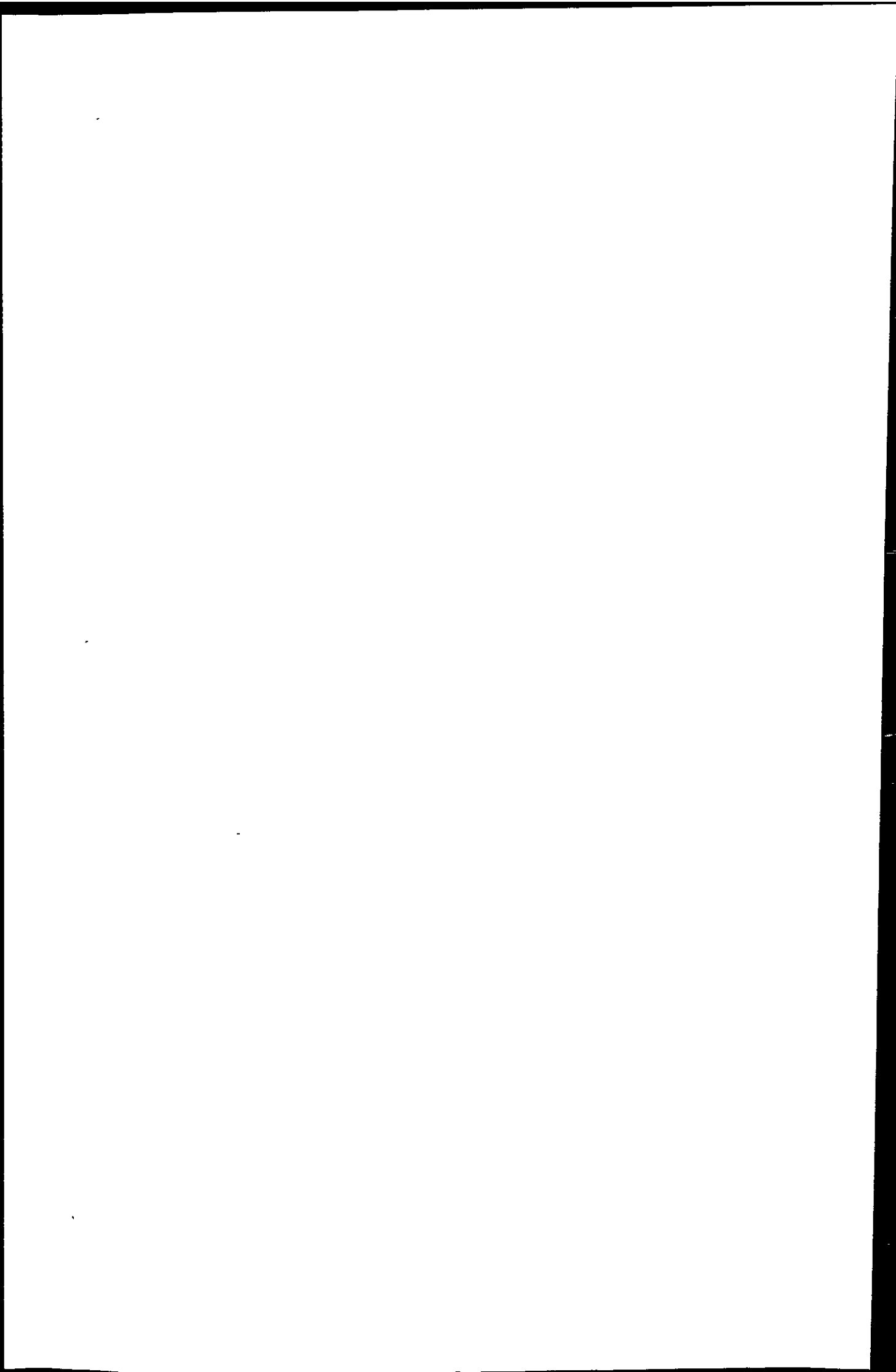
9.8.4 Question 4

$$q = \int x dx$$

(a) - (f) below may all be associated with this statement. Please score them according to how closely they fit the way you think of it.

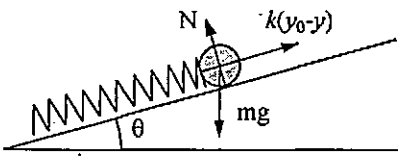
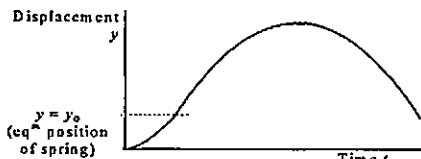
(a)	q is the area under the curve $y = x.$	Nothing like it	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	Exactly right	<input type="checkbox"/>	<input type="checkbox"/>
(b)	$q = \lim_{(\Delta x \rightarrow 0)} \sum x \Delta x$	Nothing like it	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	Exactly right	<input type="checkbox"/>	<input type="checkbox"/>
(c)		Nothing like it	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	Exactly right	<input type="checkbox"/>	<input type="checkbox"/>
(d)	The integral tells you how things build up.	Nothing like it	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	Exactly right	<input type="checkbox"/>	<input type="checkbox"/>
(e)		Nothing like it	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	Exactly right	<input type="checkbox"/>	<input type="checkbox"/>
(f)	$\frac{dq}{dx} = x$	Nothing like it	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	Exactly right	<input type="checkbox"/>	<input type="checkbox"/>

Comments

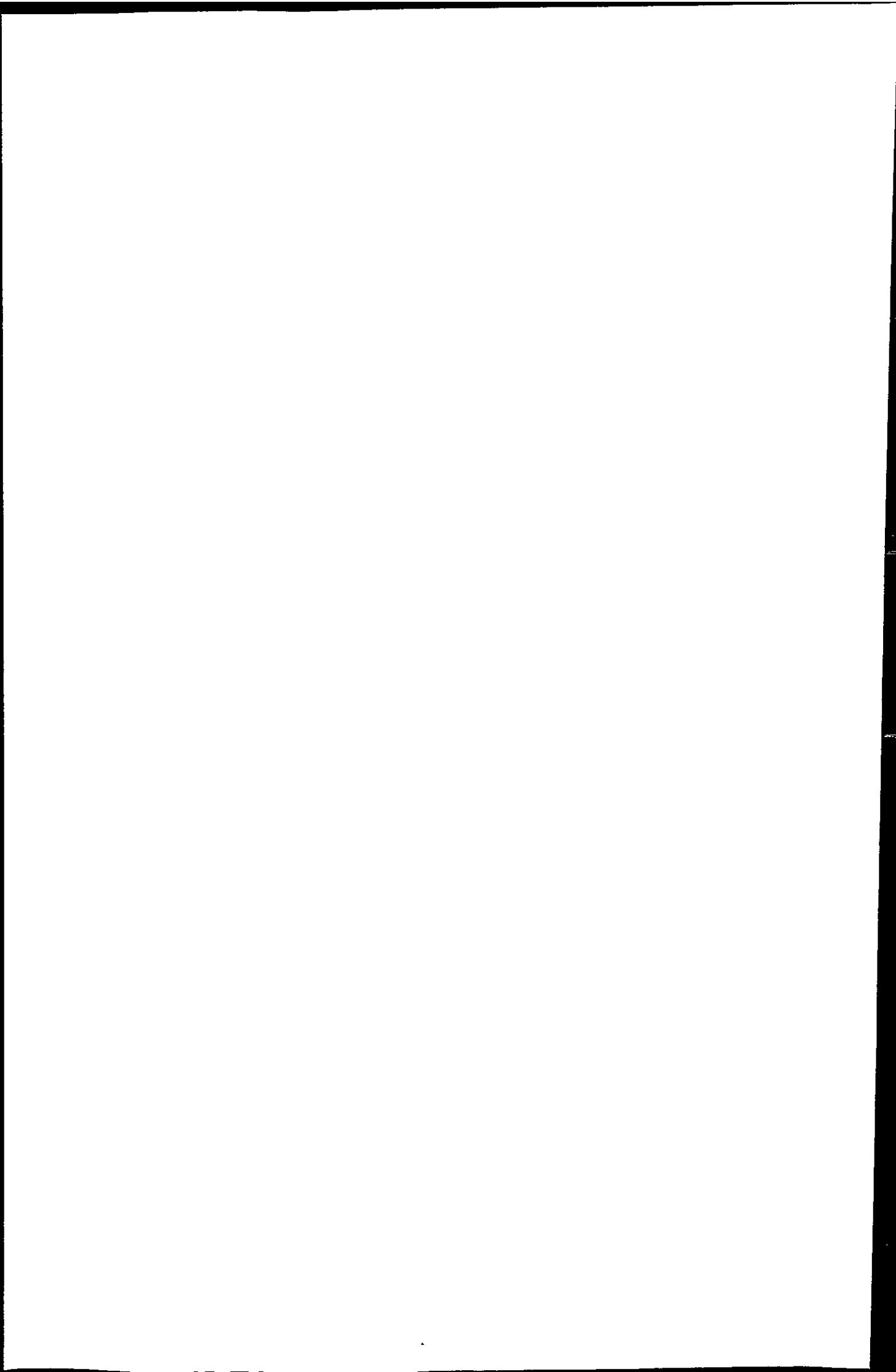


9.8.5 Question 5

In a pinball game, a ball is fired by releasing a taut spring behind it, propelling the ball out at speed. Score the following according to how well they represent this to you.

(a)	$y < y_0$; cosinusoidal motion $y > y_0$; parabolic motion.	Nothing like it	Exactly right
		<input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/>	<input type="checkbox"/> <input type="checkbox"/>
(b)		Nothing like it	Exactly right
		<input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/>	<input type="checkbox"/> <input type="checkbox"/>
(c)	The ball is accelerated against gravity by the spring until the spring reaches its equilibrium position, then rolls up and down the slope under gravity.	Nothing like it	Exactly right
		<input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/>	<input type="checkbox"/> <input type="checkbox"/>
(d)	$m\ddot{y} = k(y_0 - y) - mg\sin\theta$ while in contact with the spring.	Nothing like it	Exactly right
		<input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/>	<input type="checkbox"/> <input type="checkbox"/>
(e)		Nothing like it	Exactly right
		<input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/>	<input type="checkbox"/> <input type="checkbox"/>
(f)	The further you pull back the spring, the faster the ball will go.	Nothing like it	Exactly right
		<input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/>	<input type="checkbox"/> <input type="checkbox"/>

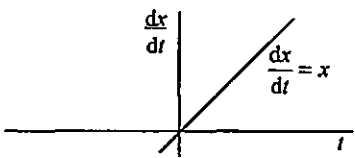

Comments



9.8.6 Question 6

$$\frac{dx}{dt} = x$$

Score the options below according to how closely they resemble the way you think about this equation.

(a)	$\frac{dx}{dt} = e^t$	Nothing like it	<input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/>	Exactly right	<input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/>
(b)	Exponential growth	Nothing like it	<input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/>	Exactly right	<input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/>
(c)		Nothing like it	<input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/>	Exactly right	<input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/>
(d)	$x = ke^t$	Nothing like it	<input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/>	Exactly right	<input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/>
(e)		Nothing like it	<input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/>	Exactly right	<input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/>
(f)	The quantity x is snowballing at an ever-growing rate.	Nothing like it	<input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/>	Exactly right	<input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/>

Comments

10. Engineering Students

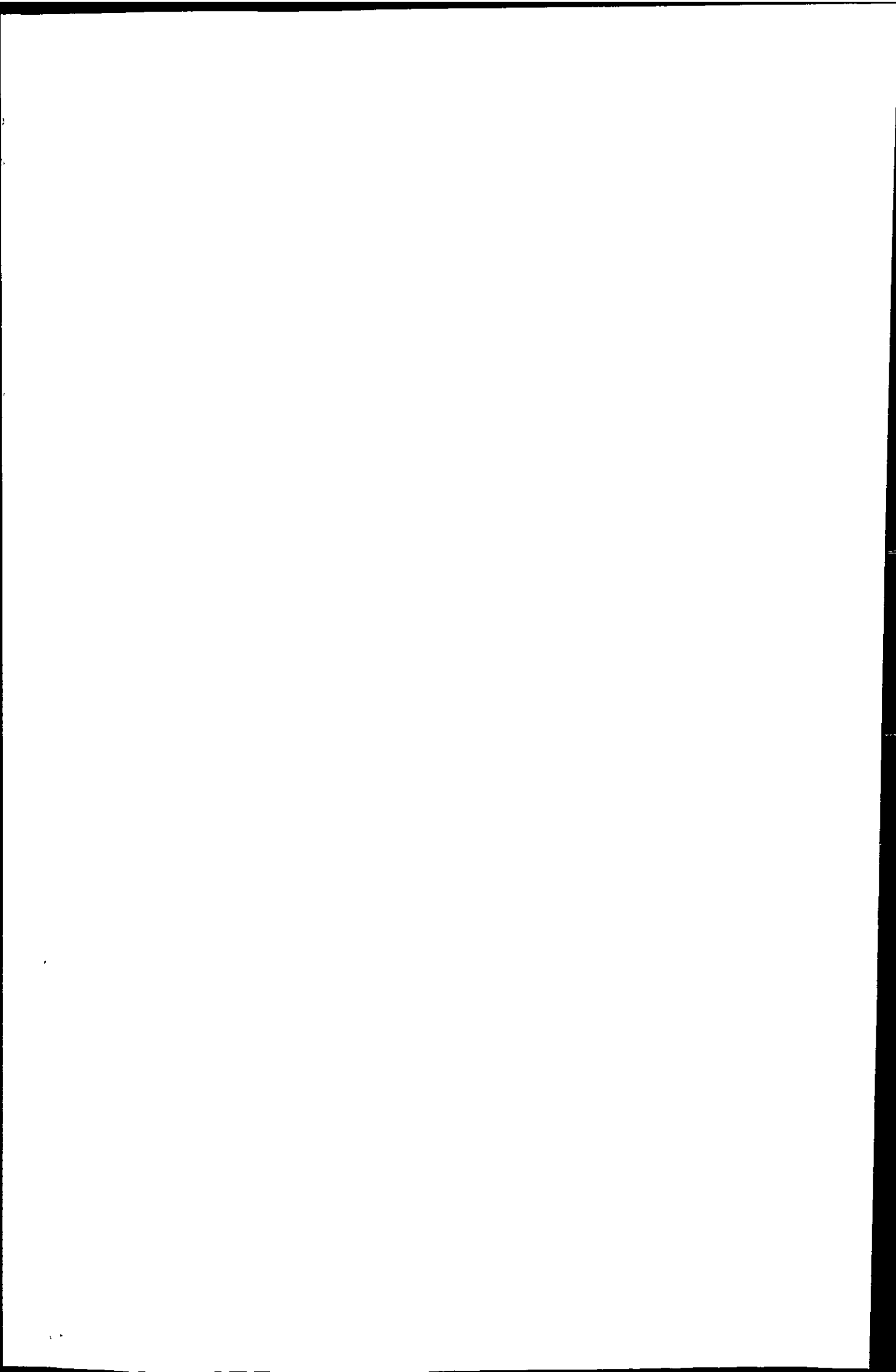
10.1 Introduction

In the previous chapters, it has been shown that there do appear to be differences between the engineering and mathematics students studied. In the past, other researchers have looked at different aspects of differences between engineering students and other groups to see what makes them special. This choice has been motivated by various reasons.

Engineering students are perceived as being "different". Engineering is an applied discipline, and engineering students want to "be engineers" rather than to "study engineering". It is also, despite Finniston (1980), predominantly seen as masculine. (See Galbraith, 1992; Wilson et al, 1993; Ethington, 1988; Hackett et al, 1992).

Most universities have engineering students, and in some places (Cambridge, for example) this is the largest single group of students in the university. Thus engineering students are accessible to many investigators, whereas students of say, Sanskrit, are harder to come by. These two reasons make engineering students a popular control group to compare with other disciplines. (e.g., Mikellides, 1989; Galbraith, 1992)

Engineering as a discipline is perceived by some as important for the social and economic welfare of the nation, in maintaining and improving the physical infrastructure and the health and innovation of manufacturing industry. The Finniston report pointed out a possible future shortfall in engineering graduates in the UK, and despite an expansion in the number of university places for engineering students, in autumn 1995, 11.3% of employers were having problems recruiting engineering graduates (Careers Research and Advisory Service, CRAC, 1996, p82).



10.2 The engineering student

Students of engineering have something of a reputation for being an extreme type, or a breed apart. Numerous studies seem to confirm this stereotype, for example by using engineering students as a comparison group (see for example, comparison with elementary teaching students (Galbraith, 1992), architectural students (Mikellides, 1989), commerce and social science students (Guimond and Palmer, 1990)). In her study of "the young worker at college" Venables (1967) found that engineering students even differed physically from the run of students.

The young engineers were predominantly muscular in body build: many more people who were either long and thin, or fat, turn up in the University population (p53)... Most students in mechanical engineering expressed a preference for a muscular build and those who were only marginally in this category tended to see themselves as more muscular than they actually were (p161).

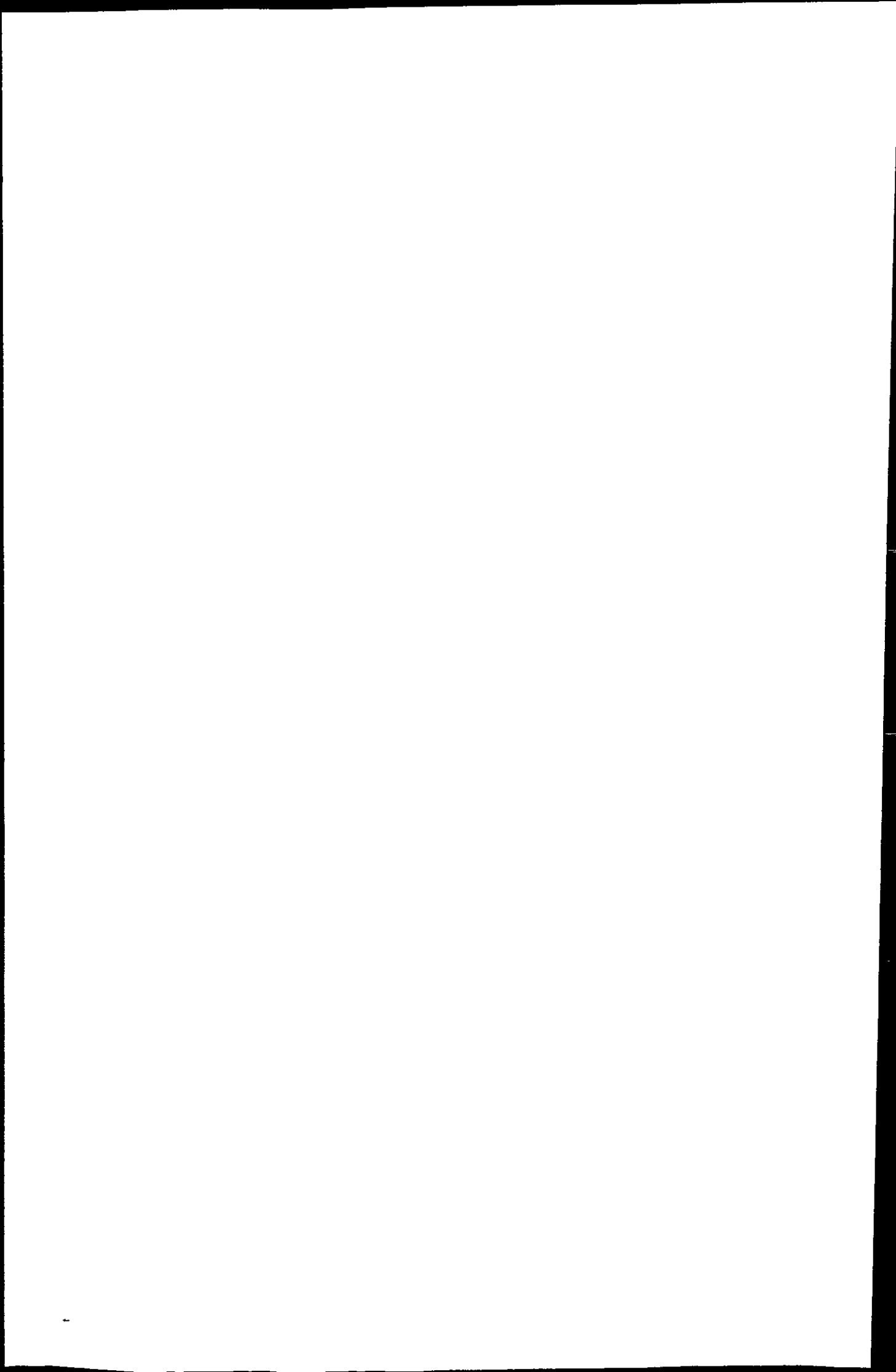
Venables (1967)

Where personality is concerned, Crowther (1997b) describes the majority of engineering students as "phlegmatic, practical people" who "consequently... show steady, reliable and extremely practical characteristics".

10.3 Learning style.

It is the general experience of teachers that students respond differently to the same lessons. Students are individuals, and apart from any differences in ability, they bring their own history, and their own preferences to learning. These preferences are broadly summarised as personality types or styles, and when they affect the way students learn or approach learning, they may be called learning styles.

In this section are described three different measures of personality types portraying engineers as an extreme type when compared to students of other subjects. The measures are independent and do not come from the same family tree.



10.3.1 Kolb's Experiential Learning Model

Kolb (1981) devised a model of experiential learning in which learning is conceived as a four-stage cycle. Concrete experience (concrete stage) is the basis for observation and reflection (reflective stage) from which an idea or theory is formed (abstract stage). The implications of the theory can be tested (active stage), leading to new concrete experiences (concrete stage again).

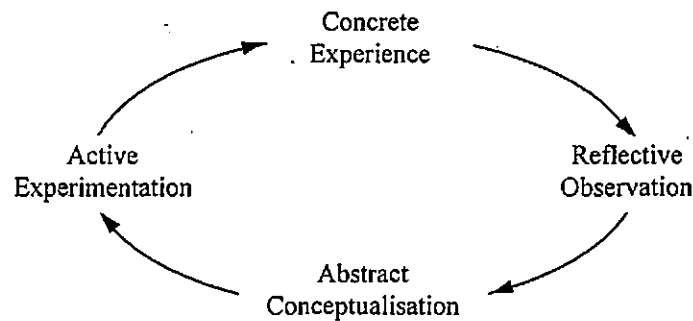
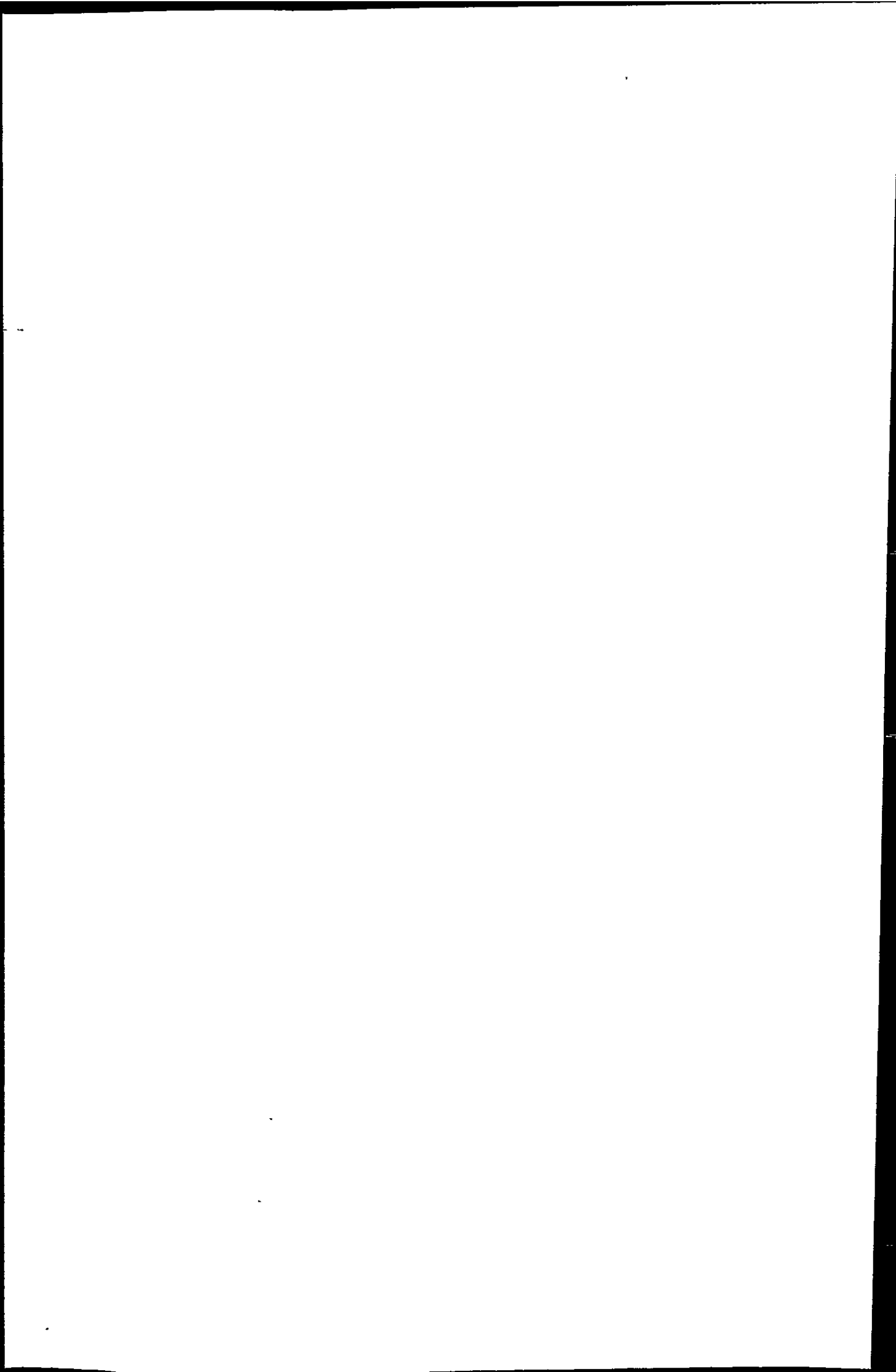


Figure 10-1: Kolb's Experiential Learning Model

Effective learners need to develop skills in all four areas, but as two pairs of polar opposites of capabilities are needed (concrete-abstract and active-reflective) learners will tend to be better at one part of each pair than the other. These characteristics can be regarded as lying on a pair of orthogonal axes, and individuals characterised according to the quadrant into which they fall.

The instrument devised by Kolb for measuring learning style differences along the two basic dimensions is a self-descriptive inventory called the Learning Styles Inventory (LSI). The earlier form of LSI consisted of groups of four adjectives. Respondents were asked to rank these in the group according to how accurately they felt the adjectives described them. A later revision consisted of sentences starting in such ways as "I learn best when..." with respondents ranking the four alternative endings for each sentence.

Thus he describes four types of learners. (Table 10-1)

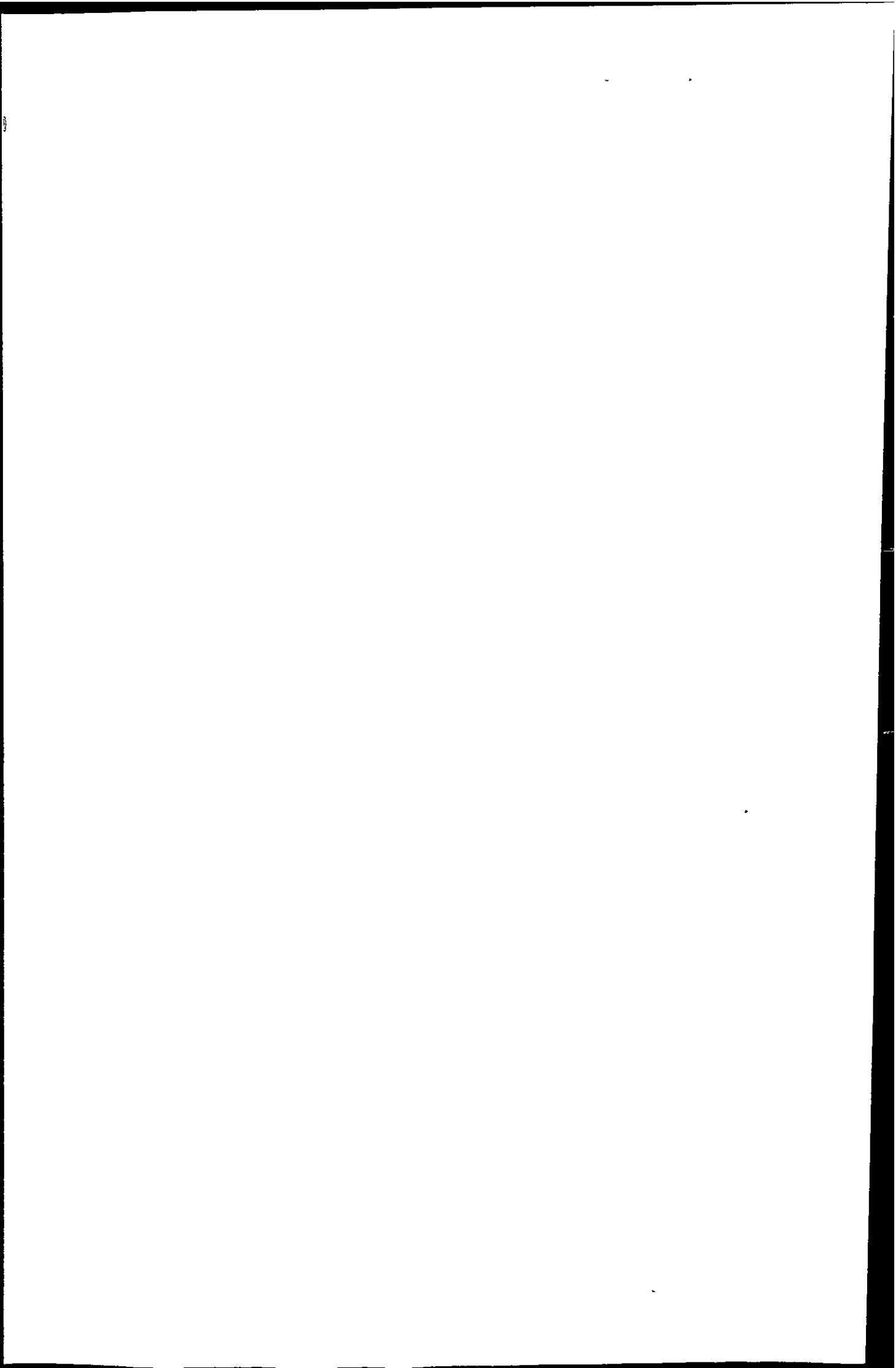


	Preferred styles	Key strengths	Preferences
<i>Convergers</i>	active experimentation and abstract conceptualisation	Concrete application of ideas.	Prefer to solve problems which have a single right answer and to deal with things rather than people
<i>Divergers</i>	concrete experience and reflective observation	Viewing a concrete situation from many perspectives, and generating ideas	Interested in people and tend to be imaginative
<i>Assimilators</i>	abstract conceptualisation and reflective observation	Producing theoretical models	Less interested in people than ideas, and less interested in practical applications than in the soundness of theories
<i>Accommodators</i>	concrete experience and active experimentation	Carrying out plans and experiments: more at home with taking risks than people with the other three learning styles	Doers: when the theory does not match the evidence, they will tend to discard the theory

Table 10-1: Kolb's classification of learning styles

Kolb administered his Learning Styles Inventory to a sample of 800 practising managers and undergraduate students in management, and related their learning styles to their undergraduate major subject. He found that although they shared a common occupation, their learning styles were strongly associated to their undergraduate educational experience. He found that only two groups of managers fell into the abstract/active convergers quadrant: nurses and engineers. Business majors appeared as concrete/active accommodators, human scientists as concrete/reflective divergers, and mathematicians, physical scientists and economists as abstract/reflective assimilators. He then turned to data produced in other studies.

Biglan (1973) had administered questionnaires to faculty members in the University of Illinois, asking them to group together subject areas on the basis of similarity, without any labelling of the groupings. Biglan identified dimensions to account for the similarity groupings. He found that the dimensions accounting for the greatest variance in the data could be described as hard-soft and pure-applied. Kolb identifies these with his abstract-concrete and reflective-active axes, and finds striking similarities between the distribution of Biglan's subject areas and his college majors of managers on these axes. For example, engineering falls into the hard/applied

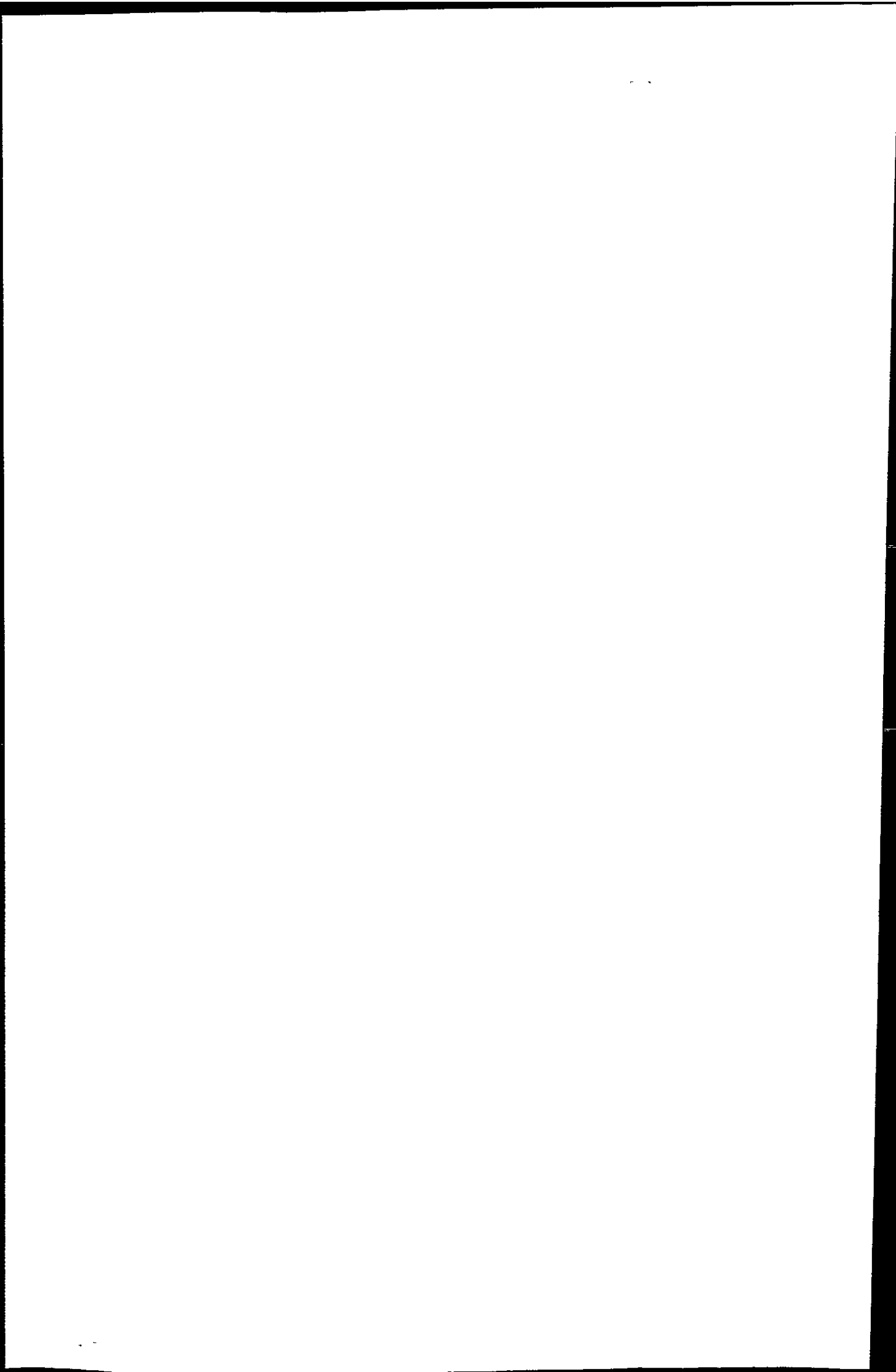


quadrant, accounting and finance (reckoned as equivalent to business) into soft/applied, human sciences in the soft/pure quadrant, and mathematics and physical sciences in the hard/pure area.

Finally Kolb analysed the results of the data collected by the Carnegie Commission on Higher Education in their 1969 study of representative American colleges and universities. Some 32,963 questionnaires from graduate students in 158 institutions and 60,028 from faculty members in 303 institutions had been tabulated, and he found proxies for his axes in the degree of consulting work carried out by the faculty (active/reflective axis) and the importance of mathematics or humanities as prerequisites for students in their faculty (abstract/concrete axis). When faculties were plotted against these axes, he found, once again, that engineering fell in the abstract/active quadrant, human sciences in the concrete/reflective area, and mathematics and natural sciences in the abstract/reflective quadrant. Subjects in the concrete/active area included law, medicine, education and architecture, not covered in his original survey.

He concluded that different faculties do indeed have different cultures, and that what constitutes knowledge in fact varies widely from one to another. From his analysis, engineering stands as an extreme group, removed from mathematics and natural sciences, in its dominant philosophy, theory of truth, inquiry strategy, typical inquiry method, how knowledge is portrayed, and basic units of knowledge. Correspondingly, it is studied by students with different priorities, skills and personalities. Above all, students with preferred learning styles which do not match those of their faculty become unhappy, alienated, and likely to drop out.

Brown and Hayden (1989), in a study on two different types of higher education institutions, administered Kolb's Learning Style Inventory to 222 students of arts (liberal arts, fine or applied arts), science (mostly computer science or mathematics),

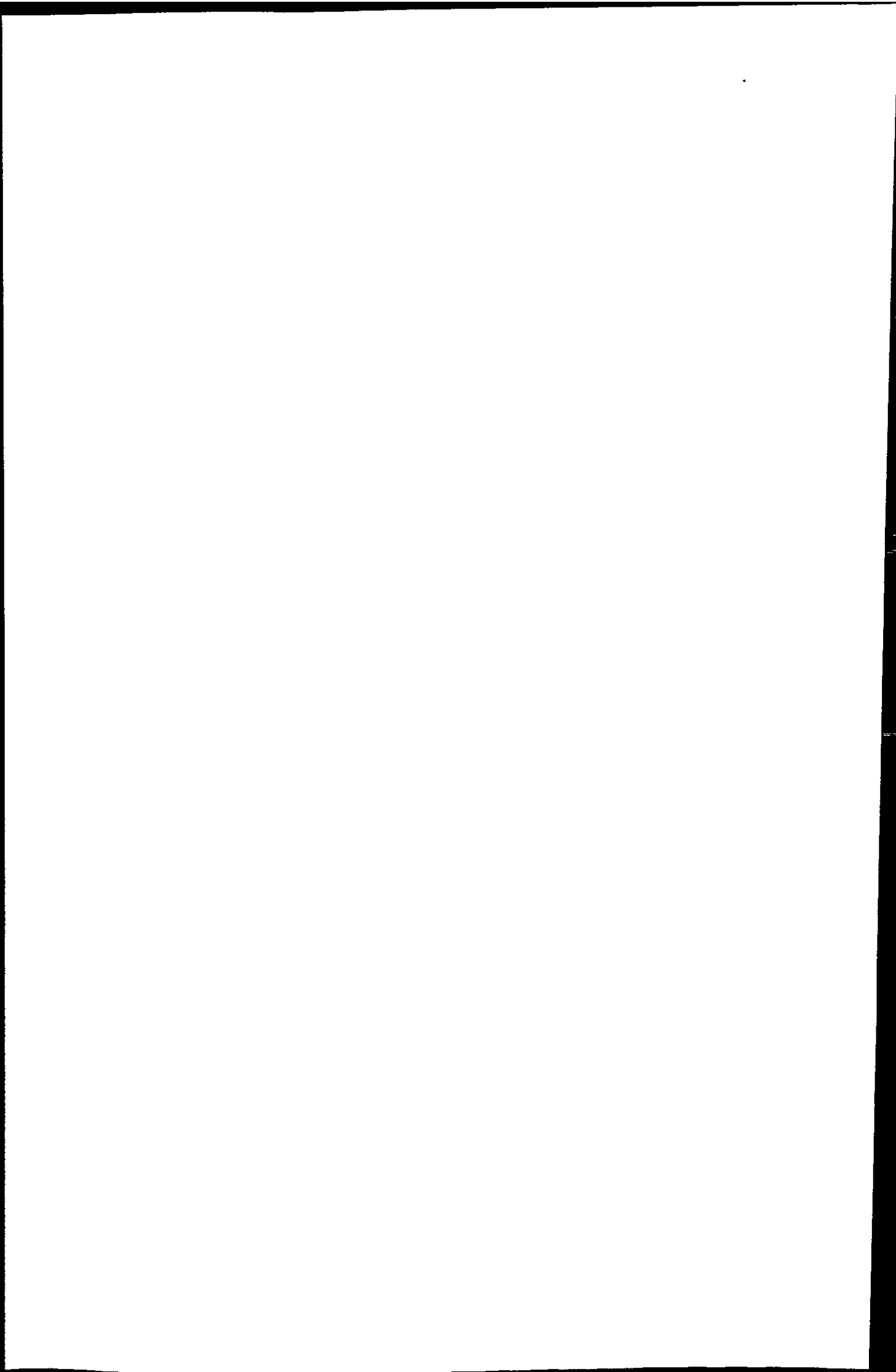


and business, plus 16 engineering students from one of the institutions. They found that, in Kolb's classification, 0% of the engineering students were accommodators (compared with 39% of science students), 31% divergers (39% in science), 25% assimilators (14% in science), and 44% convergers (7% in science). (The distribution of the engineering students is not significantly different from an even distribution at the 5% level, using a χ^2 test.) The other subject group most closely resembling the engineering students were business students at the same Institute of Technology, where the view of education was narrow, seeking to prepare students for careers. The authors comment that the group of engineering students particularly well supported Kolb's theory. Overall, convergers were the smallest group at either institution, so engineering students differed as a group not only from the science (computer science and mathematics) students but from the student body taken as a whole.

10.3.2 Myers-Briggs Type Indicator

McCaulley (1976) used the Myers-Briggs Type Indicator (MBTI), based on Jung's analysis of personality types, to ascribe classifications to 3362 students across 17 fields of study at the University of Florida. She found that on almost all the criteria within the indicator, engineering students occupied extreme positions, and that the results were statistically significant at the 1% level or better. For example, 63% of engineering students preferred introversion to extroversion, compared to 48% of the total student sample, and exceeded only by pharmacy, where 69% preferred introversion. Of physical sciences students, 51% preferred introversion.

Extroversion is that attitude where attention flows out to the objects and people of the environment, and introversion is where "energy flows from the object back to the subject". In the population in general 25% prefer introversion.

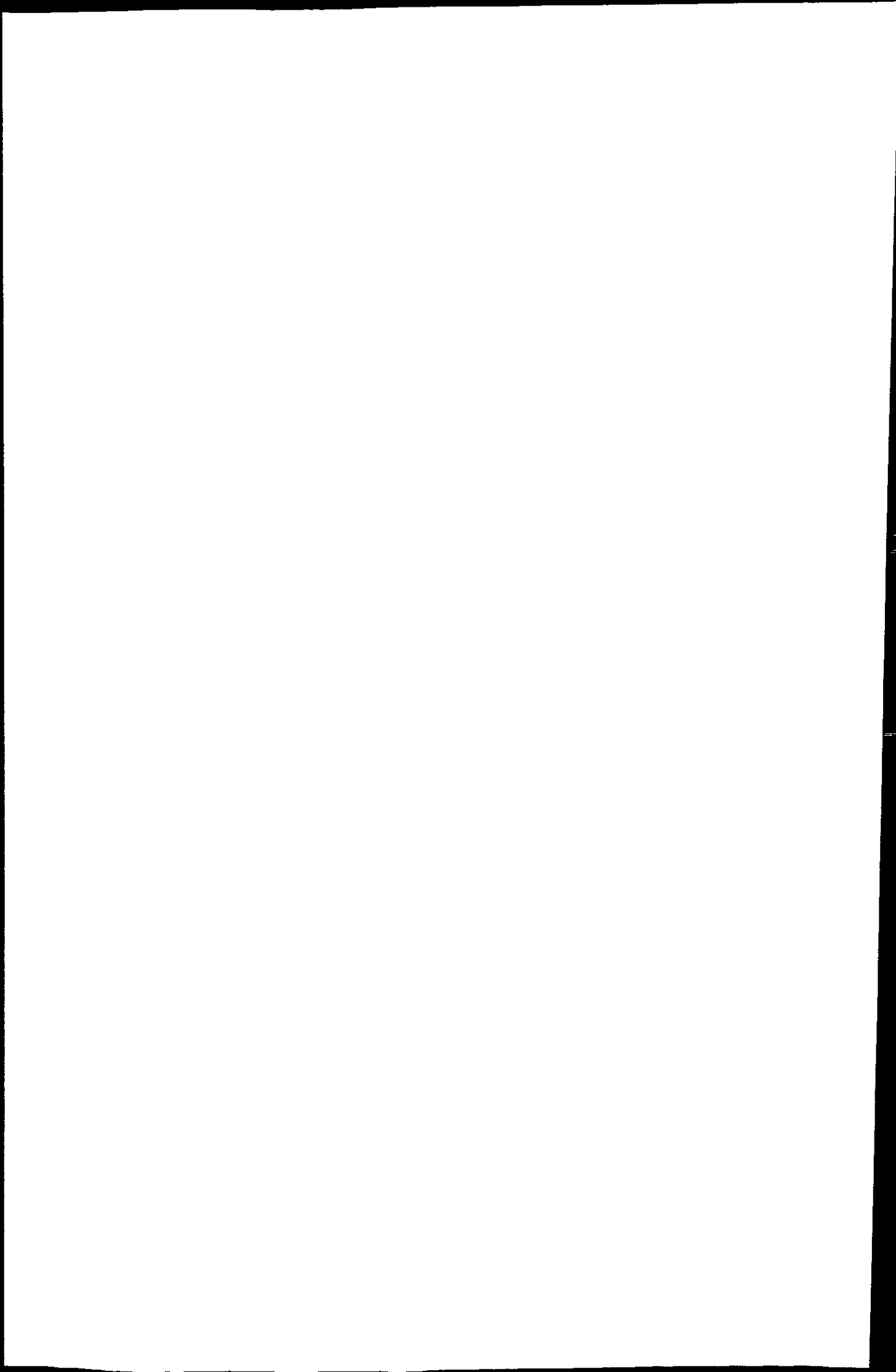


On the perceiving (sensing/intuition) axis, engineering students occupied a moderate position, 51% preferring intuition, compared to 53% of the total sample, and 66% of physical science students. This was the only axis on which engineering students did not differ widely from the whole sample. Sensing and intuition are two different ways of perceiving. Sensing is perceiving through observation, and grounded in what is observable and real. Intuition depends on the "mind's eye", through which one sees also relationships between events, and also imaginatively. The general population is evenly divided in its preferences on this axis.

The judgement axis in Jung's theory is divided into thinking and feeling. Thinking is the application of rules of cause-and-effect and objective analysis, and feeling is a way of prioritising which takes into account the human side of problems, and their solution. Of the engineering students, 53% preferred thinking, compared with 55% of physical science students who preferred feeling, and 63% of the whole sample who preferred feeling. No other group of students had a majority of thinkers.

The final preference in the typography is between perceiving and judging. In a judging attitude, we take in just enough information to make a decision. In a perceiving attitude, we are in no hurry to decide, but take in all there is to know about a situation. Engineering leads the field with 63% of students preferring judgement, while the whole sample is evenly balanced, and 58% of physical science students prefer judgement.

As a whole, engineering students can be characterised as thinking judging (TJ) types, described by McCaulley as the most tough-minded of the types, although, as she points out, there are people of every type studying engineering. They also differ from physical science students, who are markedly more intuitive, less introverted, more inclined to feeling, and less judging. Unfortunately, mathematics students, with whom engineering students are often compared, did not feature in this survey.



10.3.3 Approaches to Studying

Ramsden and Entwistle (1981) carried out a survey of 2208 students from 66 departments of Engineering, Physics, English, History, Psychology and Economics in British higher education institutions. They investigated student attitudes to learning and their perceptions of their departments. Their study is thus one of attitudes rather than an underlying style. The results for engineering students (as compared with students of the other five disciplines in the study) were as shown in Table 10-2.

APPROACHES TO STUDYING

Meaning Orientation

Deep approach	Active questioning in learning	low
Inter-relating ideas	Relating to other parts of the course	low
Use of Evidence	Relating evidence to conclusions	highest
Intrinsic Motivation	Interest in learning for learning's sake	low

Reproducing Orientation

Surface Approach	Preoccupation with memorisation	moderate
Syllabus-boundedness	Relying on staff to define learning tasks	highest
Fear of Failure	Pessimism and anxiety about academic outcomes	highest
Extrinsic Motivation	Interest in courses for the qualifications they offer	highest

Achieving Orientation

Strategic Approach	Awareness of implications of academic demands made by staff	high
Disorganised Study Methods	Unable to work regularly and effectively	high
Negative Attitudes to Studying	Lack of interest and application	moderate
Achievement Motivation	Competitive and confident	highest

Styles and Pathologies

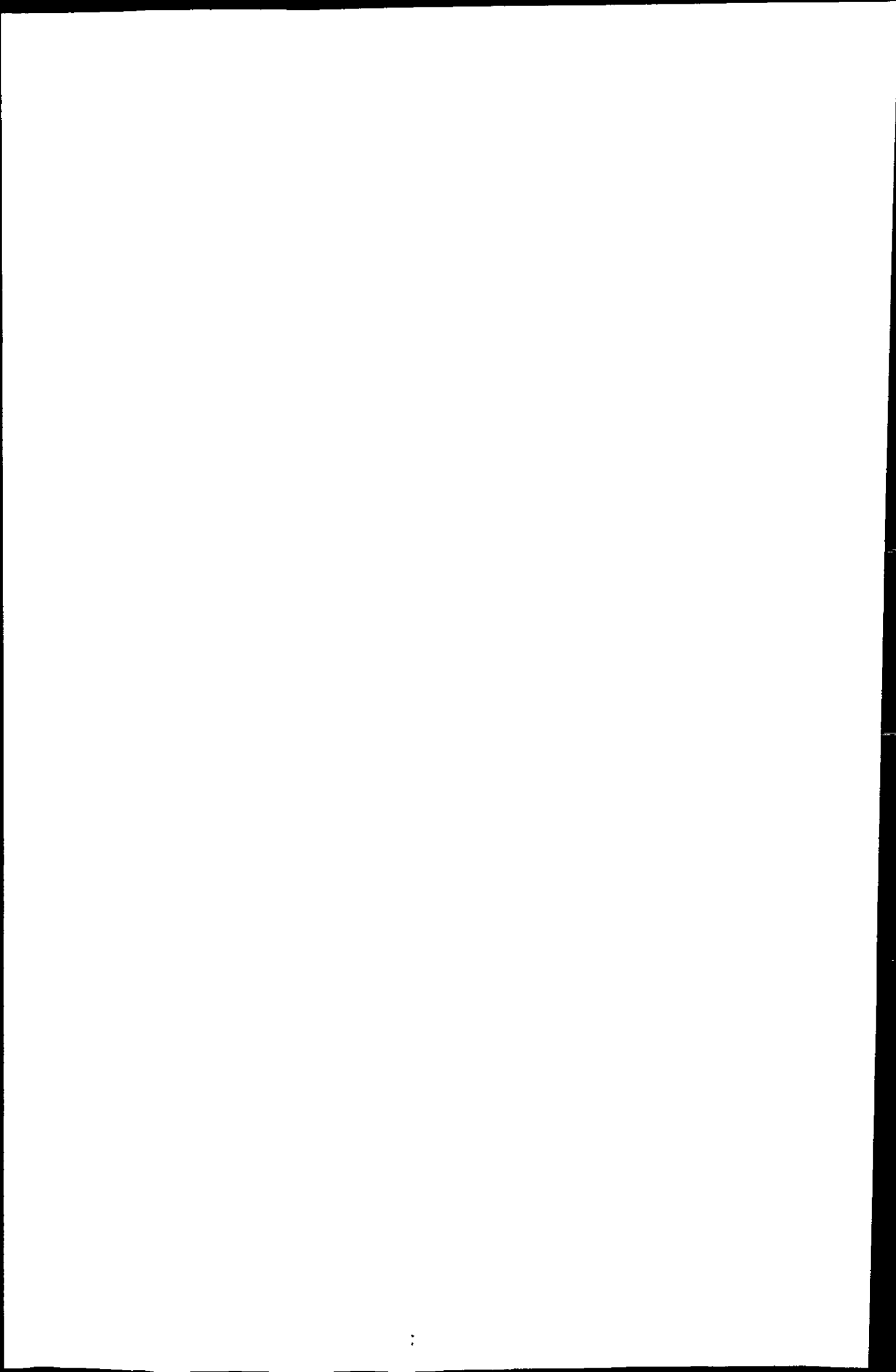
Comprehension Learning	Readiness to map out subject area and think divergently	low
Globetrotting	Over-ready to jump to conclusions	moderate
Operation Learning	Emphasis on facts and logical analysis	highest
Improvidence	Over-cautious reliance on details	high

PERCEPTIONS OF COURSE

Formal Teaching methods	Lectures and classes more important than individual study	highest
Clear Goals and Standards	Assessment standards and ends of studying clearly defined	highest
Workload	Heavy pressures to fulfil task requirements	highest
Vocational relevance	Perceived relevance of course to careers	highest
Good Teaching	Well-prepared helpful committed teachers	lowest
Freedom in Learning	Discretion of students to choose and organise own work	lowest
Openness to Students	Friendly staff attitudes and preparedness to adapt to students' needs	moderate
Good Social Climate	Quality of academic and social relationships between students	moderate

Table 10-2: Engineering students' attitudes to studying (Ramsden & Entwistle, 1981)

Engineering students, according to this analysis, were motivated to study by the desire to gain the qualification offered. (This agrees with Crowther's 1997b finding



that engineering students are motivated by short-term goals.) Since this was their aim, they followed the rules in a rational manner, depending on the staff to set targets and standards and to define the material to be learnt. Peers are more important as competition than as a social group, and staff are the taskmasters rather than sharers in a community of learners. Knowledge is important as a means to the end of becoming an engineer.

10.4 Socialisation

It may profitably be asked how much of the difference is due to the type of teaching and the taught material the students have encountered during their degree course, and how much is due to the personality differences between the types of students who choose mathematics or engineering degrees in the first place. McCaulley (1976) argues that the initial differences between students in different departments are emphasised as they continue through the course. She points out the importance of peer socialisation, and there is also the effect of adopting the tone and attitudes they associate with members of their chosen professional group, namely the staff, and in the case of engineering students, professionals they have met in their work experience year.

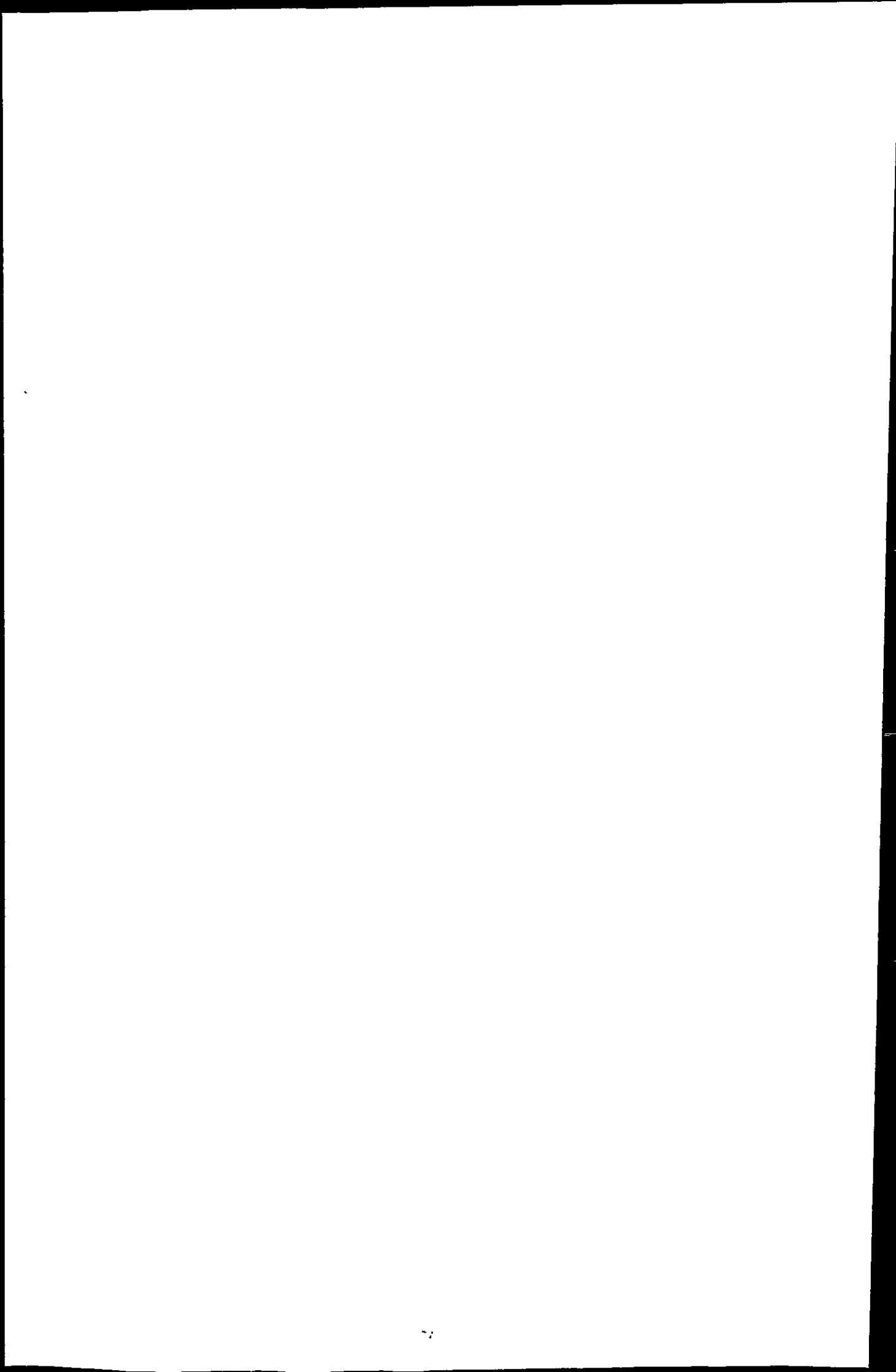
Guimond and Palmer (1990) found evidence of socialisation when they explored the ways that students in commerce, engineering and social science attributed the causes for poverty and unemployment. First year students in the three groups were indistinguishable in their beliefs: in the third and fourth years they had diverged strongly, with social science students blaming "the system" more, commerce students less, and engineering students the same as in the first year. They argue that the differences among upper year students are a result of differences in the socialisation of students in the subject groups.

White (1972) points out that Durkheim's insight (1956) was that education must be socialisation. The knowledge and culture handed on in education falls into two categories: that which all members of the given society should possess, and that which is specifically suited to the individual's place in the society, or their future occupation. Whereas White argues that this second category is of dubious morality in the teaching of children, in the context of the education of a young adult who may be deemed to have chosen a profession, by virtue of having chosen a vocationally titled degree course, this objection disappears. Indeed one may say that it is proper for an engineering student to be taught not only the subject matter of engineering, but also the manner of being an engineer.

Clark (1994) suggests that acculturation to the profession is an integral part of professional preparation, through appropriate mentorship.

Cooper and Millar (1991) investigated the personality types of faculty and students in a business school, and found the intuitive style to predominate among faculty and the sensing style among students. That is, the students preferred a concrete approach, and the faculty an abstract approach. McDermott (1991) suggests a similar mismatch occurs in physics: that lecturers in physics tend to teach in an abstract way, to save students the effort of making their own generalisations, forgetting that they had learnt from the particular to the general, the concrete to the abstract, and indeed gained intrinsic motivation therefrom.

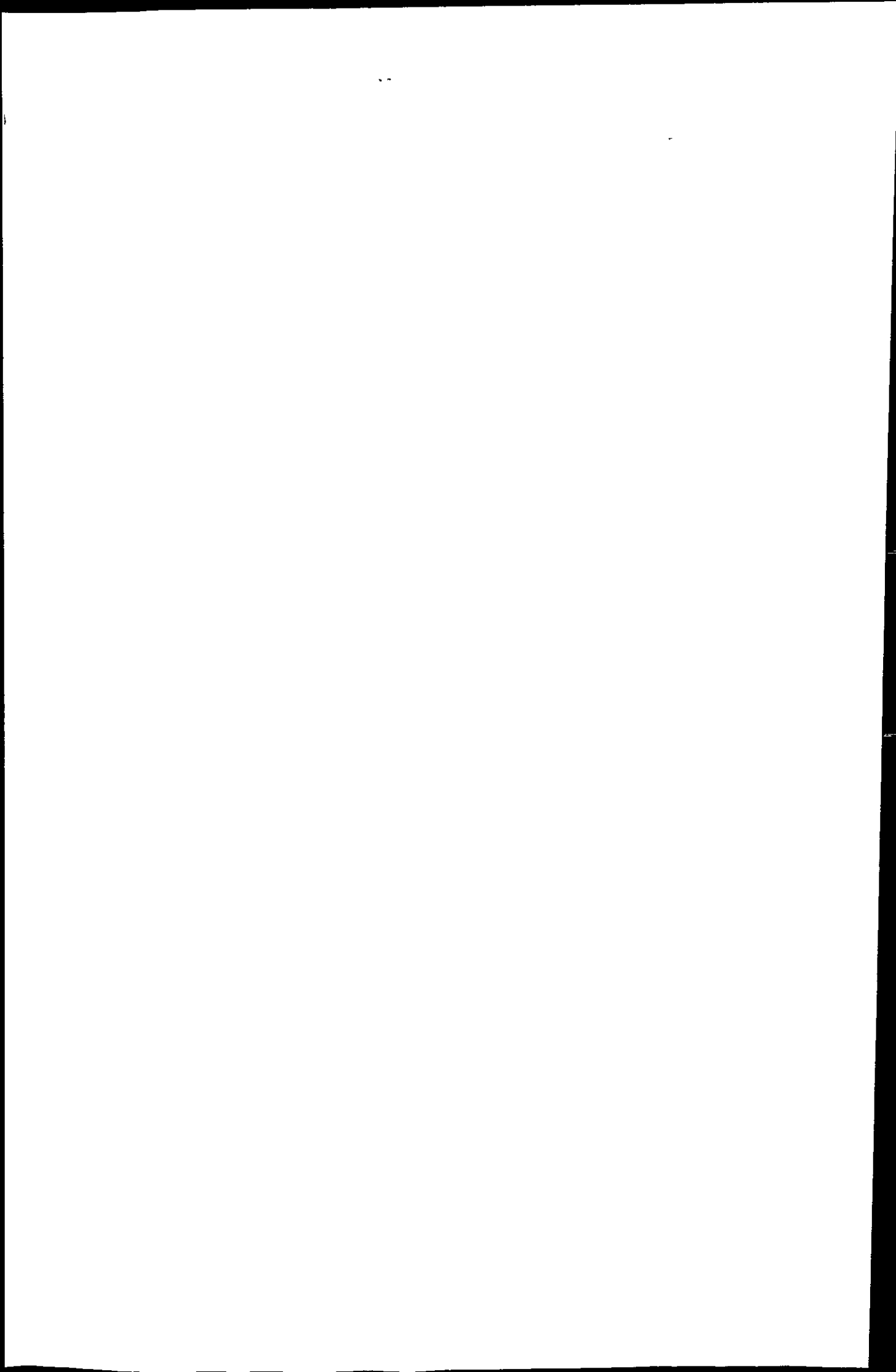
Brown and Cross (1992), using the Gough and Heilbrun (1980) adjective checklist (ACL) found differences between the engineering students they tested and the previously measured norm for engineers in the population. The students were more sociable and outgoing than their practising counterparts, and had a more global approach to problem-solving. They suggest that the population of engineering



students may have changed, but other explanations could be that not all engineering students enter the profession, those unsuited by personality moving elsewhere, or that there is an overall evolution of personality, perception and cognition as observed by Perry (e.g. 1988)

10.5 Conclusions

Overall, we can probably conclude that engineering students do have a different personality profile from mathematics or physical science students, and that this will be reflected in their preferred learning style. It would be wise to respect this difference when trying to teach the mathematics and physical science which form an important part of the engineering syllabus. In particular they differ from the lecturers who teach them in these subjects, particularly in their motivation, and lecturers should bear in mind Vinner's criteria (see Chapter 5): that a student's mental constructs should be confronted if in doing so a valid educational purpose is served, and otherwise that the teacher should teach in accordance with those constructs.



11. Epistemological orientations, paradigms of curriculum, and learning theories

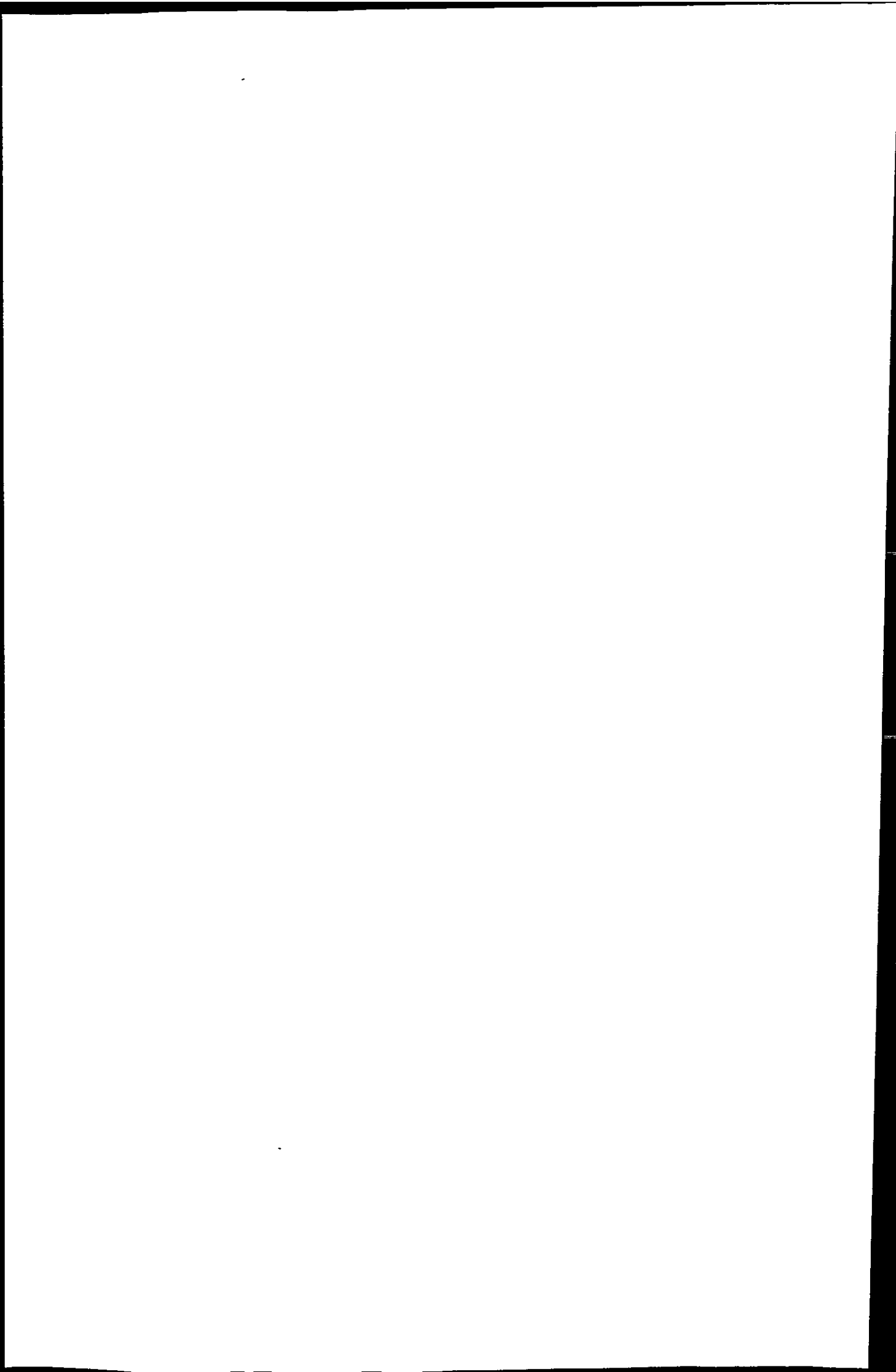
11.1 Introduction

The study described so far has concentrated on characterising engineering students and the ways in which they think of and do mathematics, particularly in comparison with mathematics students. As was stated in the Introduction (chapter 1), alongside the concern expressed in the various reports for the standard of mathematics skills of engineering students was a hope that the use of computer courseware would be helpful in improving the situation. It was proposed therefore that a piece of courseware should be written to help engineering students reinforce their mathematical concepts through the use of a mathematical modelling framework.

In chapters 12 and 13 the research relating to mathematical modelling and the design of computer courseware will be discussed, but at this point it is apposite to summarise some of the underlying theories which inform the design of any teaching materials.

11.2 Epistemological orientations, paradigms of curriculum, and learning theories

What is education? how do we learn? and what do we believe about learning? These are closely linked but not identical questions. The reasons for teaching mathematics and indeed the definition of what mathematics is will vary with the *curriculum paradigm* adopted. The way it is taught will depend on the *learning theory* held by the teacher, and the expectations held by the learner will depend on their theory of learning and their *epistemological orientation*.



In addition, these internal structures will affect our view of the nature of mathematics, and the reasons why anybody, and specifically engineering students, should need (or want) to learn mathematics.

We must remember though that the internal world of the learner is not directly knowable (the behaviourists' main argument), so that everything we say about the learning process is a description of a model of that process, and that no model can contain all truth. Sometimes one model will be more useful, sometimes another, and sometimes aspects of several may be combined to suit a particular purpose. In particular as the learner becomes more mature and graduates from novice to expert in a given field the learning process may change, and different models are needed.

11.3 *Epistemological orientations*

What is the nature of knowledge itself? How is it acquired, and who decides what is knowledge?

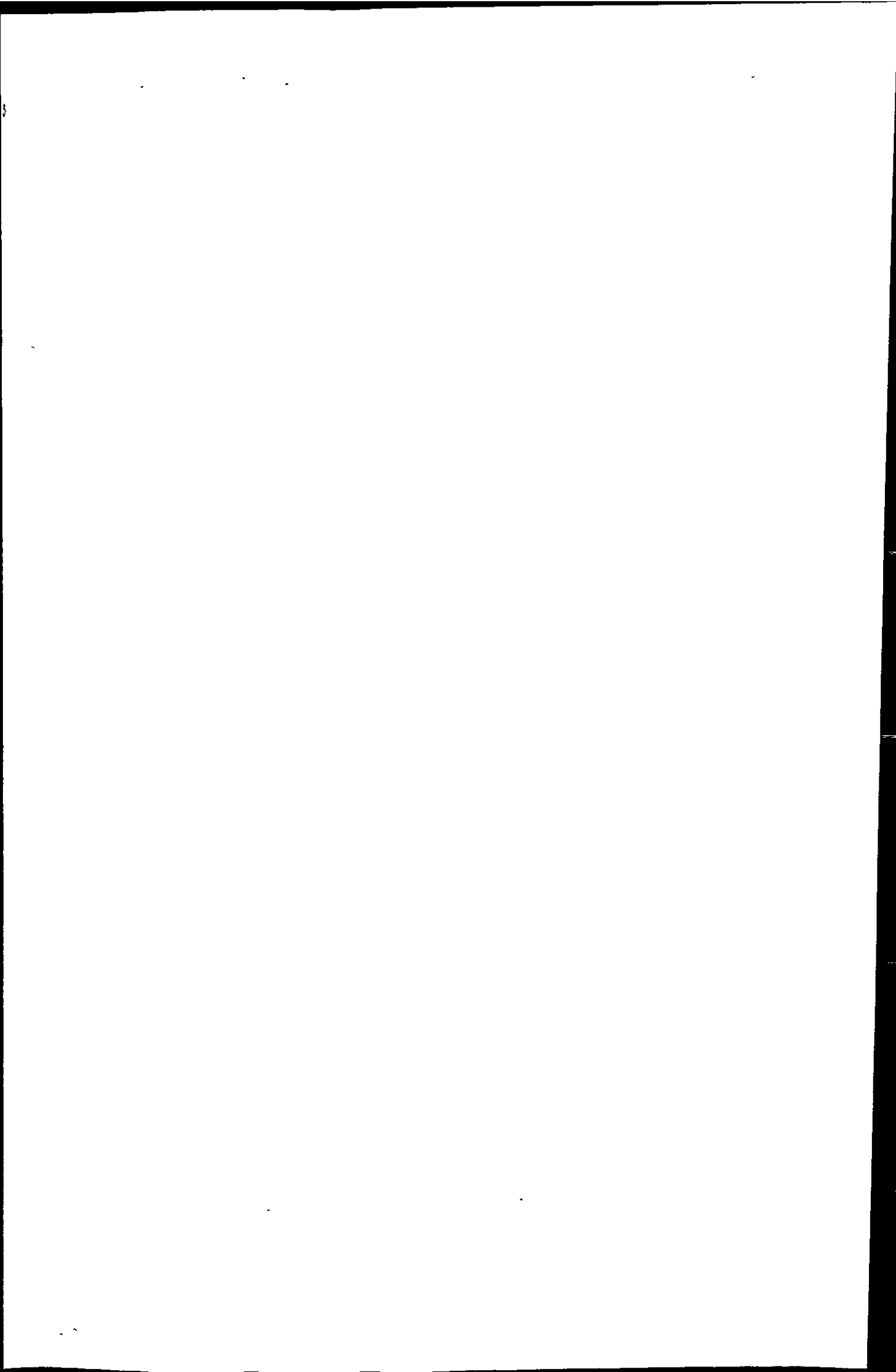
Three basic epistemological orientations are outlined in Table 11-1.

11.3.1 Transmission model

My name it is Benjamin Jowett:
I'm Master of Bailliol College.
If a thing is knowledge I know it,
And what I don't know isn't knowledge.

(anon, 19c)

This Clerihew shows a particular attitude to the possession of knowledge: that Authority has the right to define knowledge. Thus it is possible in this orientation to define a canon of works in literature, music, and the fine arts which are *Important*; a definitive history curriculum based on, for instance, battles and dates, which any child may be expected to quote; and a mathematics curriculum consisting of a list of topics which one should master at given ages. It is typified by questions of the type "what is the main export of Ceylon?" (answer, tea).

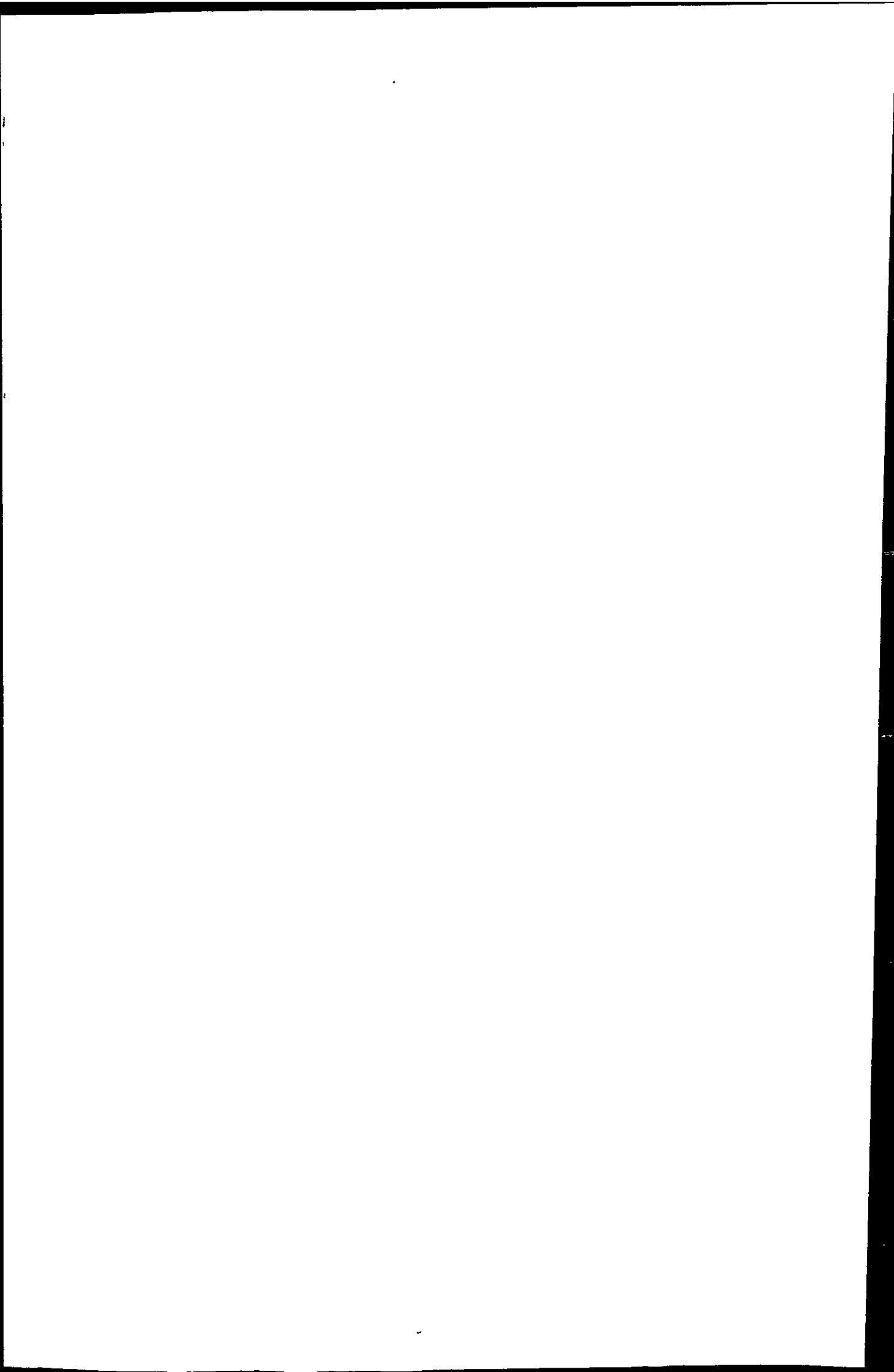


Model	Knowledge	Learning	Sense of authority	Corresponding learning theory, curriculum paradigms
Transmission	Static and objective. Knowing is closed, linear paradigms. Quantity and breadth and mastery of the content is emphasised	Transmitted from teacher or text to student. A linear and simple action. Can be tested through achievement tests	Teacher-centred. Teaching is emphasised and the learner dependent on the teacher. Teacher is responsible for deciding the learning outcomes and the design of the learning environment	Behaviourism, Information processing Curriculum as technology
Transaction	Dynamic and alive. Knowing is based on learning strategies. Quality of learning emphasised	Through the interaction between the learner and his/her environment. Co-operative activities, problem solving and higher order thinking.	Student-centred. Learners are responsible with the teacher for their own learning. Teacher has control of the situation but is not authoritarian.	Pask's conversational learning Situated learning Social learning Experiential
Transformation	Dynamic and changing, contextual: a construction of the community of learners	Construction of knowledge by the learner.	Community of learners. Visible authority does not exist. Teacher uses the power of the environment when new knowledge is created: transformed. Complexity, openness and creativity are emphasised.	Post-modernism Curriculum for personal relevance

Table 11-1: Epistemological orientations (from Brody, 1991, in Berry and Sahlberg)

Since knowledge is defined by the establishment, that which is known by groups outside the establishment (such as folklore, scientific theories which diverge from the accepted view, or alternative interpretations of history) is not knowledge.

Knowledge is monolithic and not open to discussion or interpretation. Knowledge is possessed by the learned, and may be passed from them to the unlearned. This view of knowledge, however, is static, and does not allow for changes such as the emergence of new branches of science or mathematics, or discoveries in history or biology. As Cox et al (1995) put it : "twenty years ago it was just possible to provide



a fair proportion of the class with both the fundamental concepts [of engineering mathematics] and the most useful methodology. It is now questionable whether one can do either with even those topics taught twenty years ago, not to mention the new topics now required.”

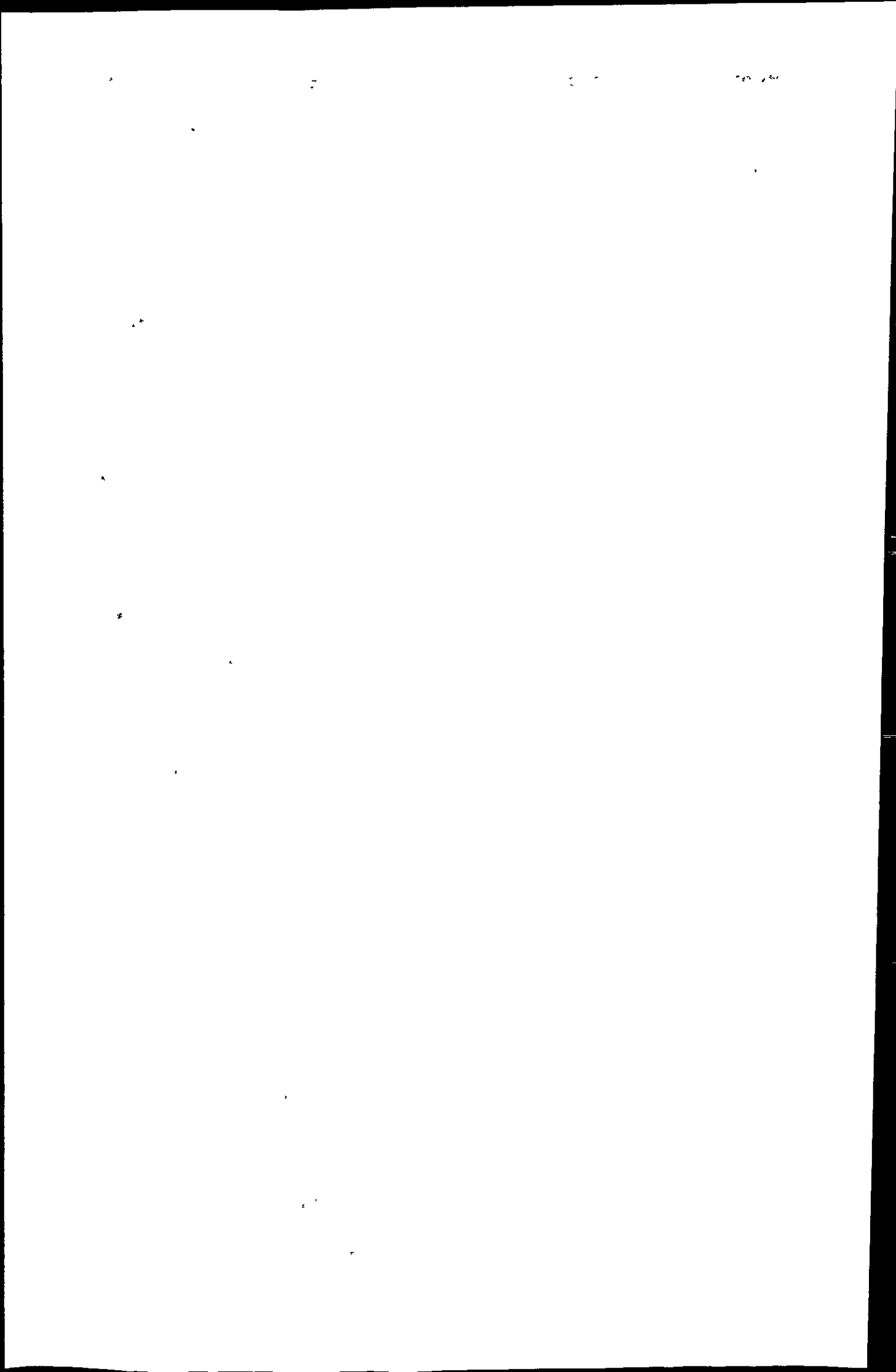
Where the matter to be learned is a skill which can be built up through drill and practise, which will be tested by performing the skill to a given degree of speed and precision and especially where the skill can be broken down into a set of subskills which can be learnt separately and then combined, then this model is highly appropriate.

According to Perry (1981), the naïve learner looks to authority to “tell me what I need to know”, and moving to a more mature model of learning is a painful process involving the loss of old certainties. To this extent the learning of skills through the transmission method may provide the cognitive equivalent of comfort eating: a temporary flight from complexity into certainty where working diligently and following the rules brings rewards. However in the large part of tertiary education we are seeking to make the student an independent learner and they should not rely on this form of learning.

11.3.2 Transaction model

The emphasis moves from teaching to learning, but the teacher is still in charge, setting the agenda. The teacher’s model of reality is the one which the student seeks to acquire. This is predicated on the greater experience and deeper understanding of the teacher. Experts teach novices to become like them.

Cox et al (1995, cited above) argue for the teaching of prototype methods within engineering mathematics, so that engineers will be able to understand the concept of transforming variables and functions by reference to Laplace transforms as a

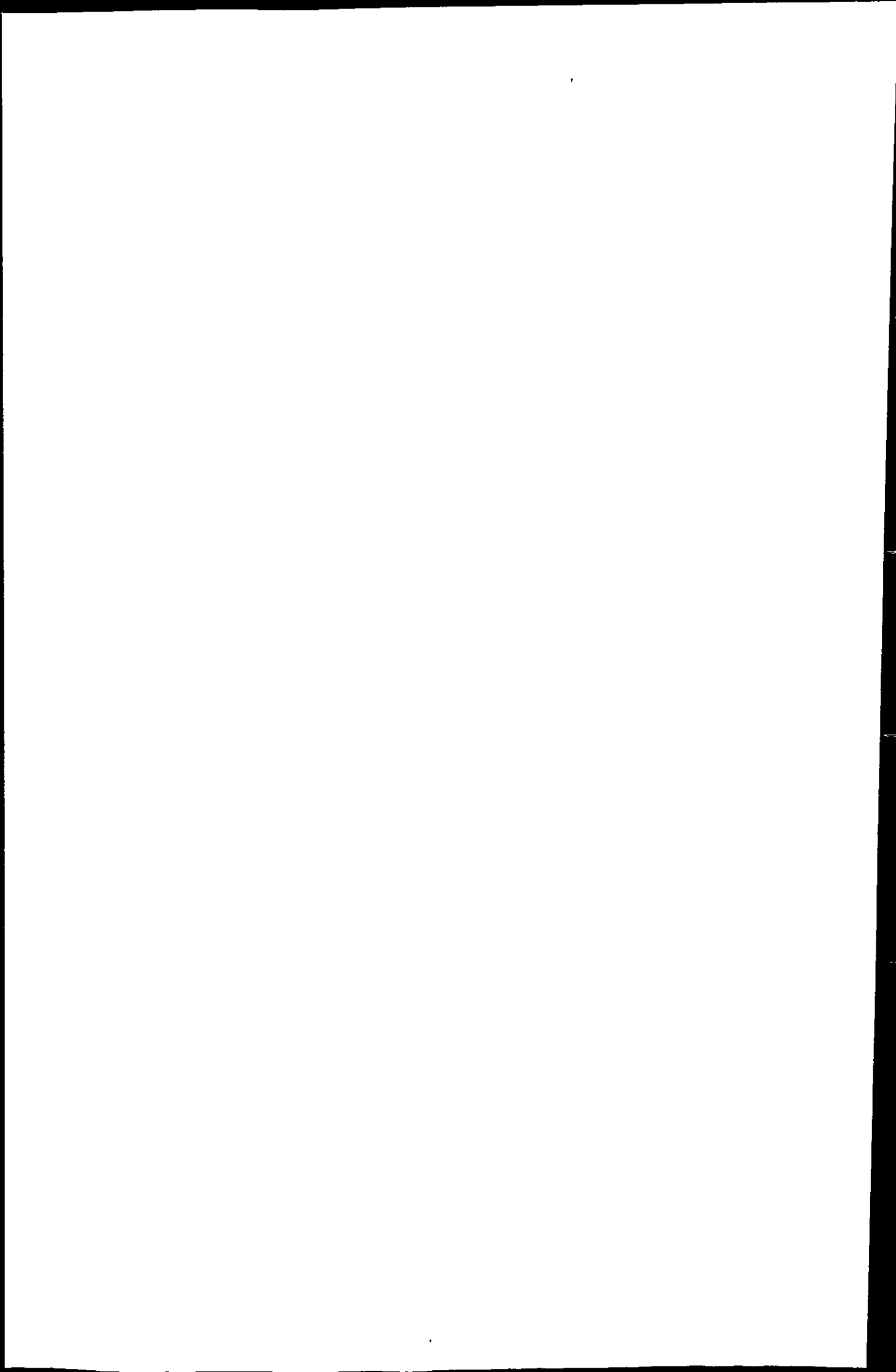


prototype, or operator methods by reference to the *D*-operator. They conclude "it is suggested that we move away from the contents-based methodology for curriculum design and instead base it around the interweaving of engineering objectives and the reasons for which mathematics is included." This is a strategic view of mathematics learning for the engineer, where the concepts of, for example, "transform" or "operator method" are built through learning particular skills, so that the student will recognise what is happening when analogous methods are performed by a computer.

Conceptual understanding is a construct of the transaction model, whereas the transmission model emphasises declarative and procedural knowledge. The student builds understanding of concepts through relating one concept to others. A well understood concept leads the student to make predictions which match those the teacher would make, and to be able to use and apply the concept as the teacher would define appropriately. The teacher can conclude that the student's mental model of the concept matches the model generally accepted as correct.

In the conversational model, (Pask, advocated by Laurillard, 1993) the student and the teacher exchange predictions about behaviour of a system according to their mental models of the system. When the student's predictions convince the teacher that the two models match well enough, the teacher concludes that the student has "understood" the system. This process depends on the teacher understanding the system and the possible variations of models well enough to predict where differences may occur, in order to explore the student's predictions in those sensitive areas. The system may be anything from a physical system to a mathematical process to a language to a set of social circumstances.

Vinner (1991) defines understanding as the possession of a concept image. When the concept is mentioned, some image is evoked in the student's mind, and the concept means something to the student. The image need not be the same as that which the



teacher has, and indeed if the student's image does not overlap with the teacher's then the student's understanding may be said to be faulty.

According to Perry's (1981) analysis, most university students should be able to cope with being responsible for their own learning under the direction of the teachers.

Questions and protests about the relevance of material (especially mathematics in engineering, see Cox et al, 1995; Coxhead, 1997) are characteristic of Perry's

Oppositional students.

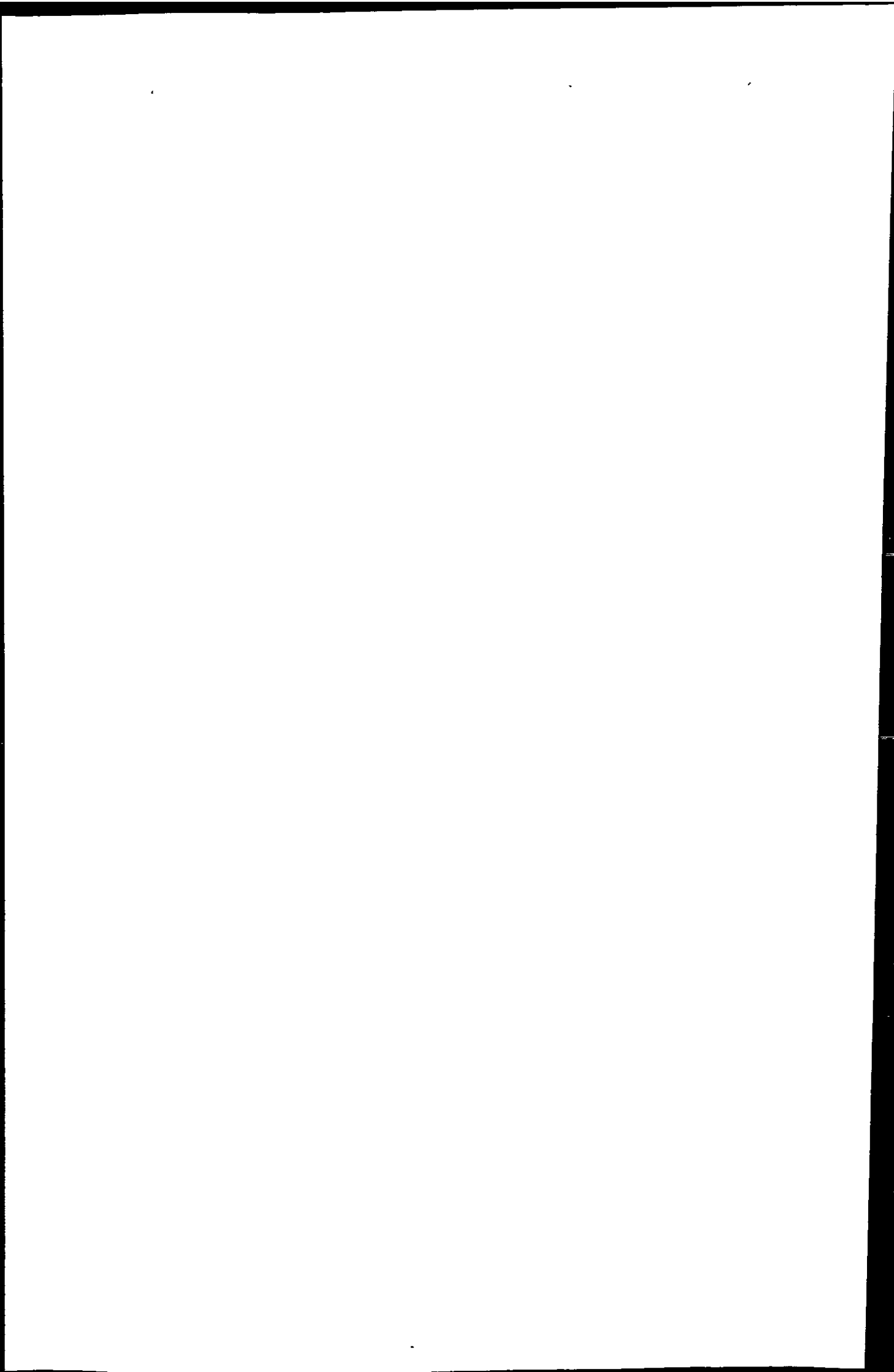
In this model, the teacher defines the space which the learner explores, and sets the rules which bound the student's freedom.

11.3.3 Transformation model

The transformation model is a more radically "democratic" view, in that learning is a process shared by the community of learners.

This model may be regarded as particularly appropriate in the context of research in Higher Education, where learning is a shared process. Perry (1988) describes a mature relationship of the learner to knowledge as when Authority becomes "resource, mentor and potentially colleague in the consensual estimation of the interpretation of reality".

The model may also apply well to situations of mutual tutoring where students work together to make sense of coursework, laboratory work (see Brown, 1994), lecture notes or reading. Under these conditions the students will normally be operating within each other's zones of proximal development, and in an ideal position to help one another. Clark (1994) also hypothesises that since the knowledge of the expert is different in kind from that of the novice, and not just in degree, someone who is closer developmentally to the learner "may serve as a better touchstone for the student's own reflection".



Authority is no longer a potential colleague, authority has become a colleague.

According to Perry (1981), if the learner is not mature enough in position then being confronted with a teacher who presents this model will bring pain, incomprehension and rage. The learner must have developed beyond an "anything goes" multiplicity in order to cope with the ambiguities of this model.

Perry points out that there may be a disjuncture between a learner's position in different areas of development, as well as a transfer ("d calage") of maturity. A mature student who has accepted commitment (in the face of ambiguity) in settling down, finding a spouse, choosing a home, may still seek for the old certainties in learning mathematics.

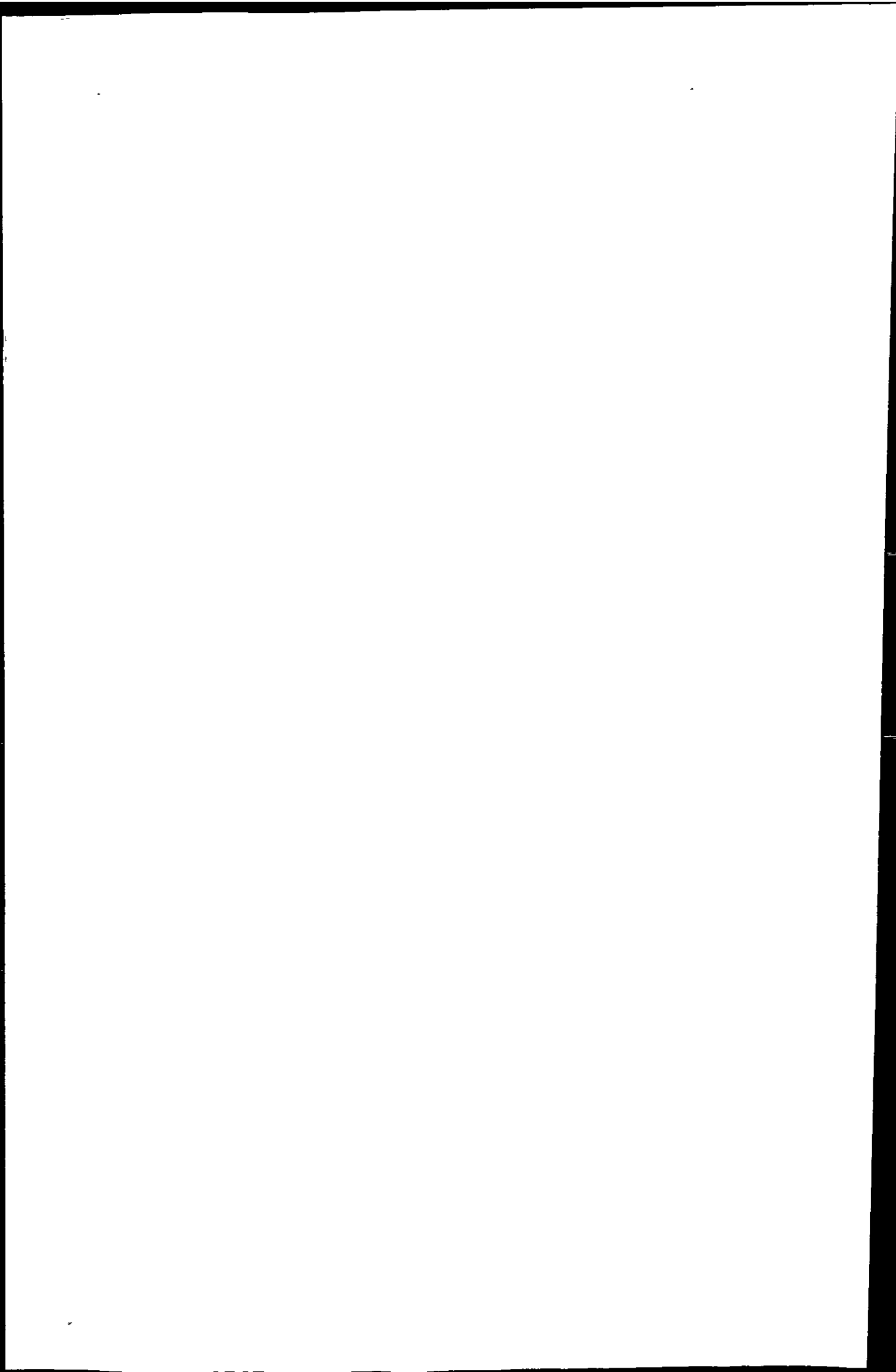
In this model the attitude of the learner to mathematics is not that mathematics is a reflection of absolute truth, but that it is an activity with rules by which one agrees to abide. This attitude is well beyond the grasp of most engineering students.

11.4 Learning theories

Some learning theories which have currency at present are summarised in Table 11-2.

As with epistemological orientations, many teachers and designers of learning materials are not consciously aware of the learning theory to which they subscribe.

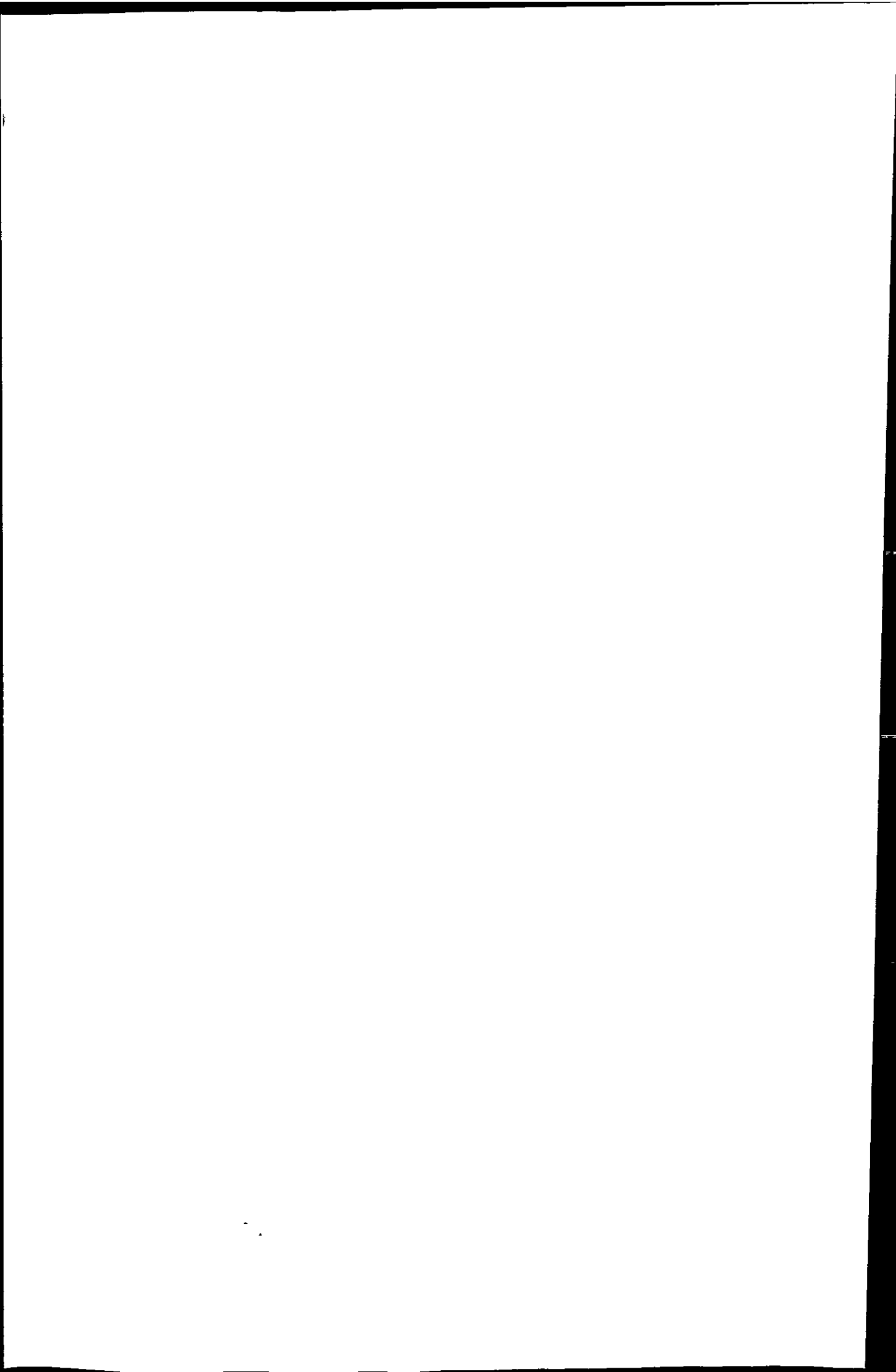
I have related these learning theories to the three epistemological orientations discussed above. Epistemological orientations are instrumentalised through learning theories: that is, given what we believe about knowledge, learning theories tell us how we might go about sharing the knowledge we have with others. Again, given that different epistemological orientations may be useful for different purposes, so we can see that different learning theories may also be useful in different applications.



	The mind	Knowledge	Learning	Teaching
Transfer models				
Behaviourism (Skinner)	black box	behaviour	training, reinforcement	control of the learning environment
Information processing (Gagné)	computer	object to be transferred	acquisition of rules, concepts and procedures	mapping expert's cognitive map onto learner
Transaction models				
Constructivism (Piaget, Skemp, Ausubel, et al)	inner representation of outer reality	residing in the individual mind	internal construction of meaning	negotiated construction of meaning
Experiential (Kolb)			a cycle of experience, observation, hypothesis formation, and hypothesis testing	guiding learner
Social learning (Bandura)		scripts and behaviour patterns	observation and rehearsal, apprenticeship. acquiring characteristics of admired others	modelling: the significant other
Situated learning (Collins, Brown & Newman...) (also known as cognitive apprenticeship)		develops from the complex interaction of students with technology, people and the other information available in a situation	enculturation, perceptual attunement: social construction of knowledge through discourse	guide students' attention to the invariant features which are meaningful across a class of situations: planning the assistance students will need
Connectionism (Papert)	brain (mind/body dualism eliminated): a material machine	pre-symbolic, pre-representational: socially and environmentally distributed	inseparable from performance: acquisition of meaningful patterns	providing examples and experiences from which patterns may be abstracted
Transformation models				
Post-modernism (Hlynka, Faulconer)	being-in-the-world: interactions and relationships are the starting point for understanding the human condition.	interpretation of the text which is life: not a one-to-one representation corresponding to an external reality.	the interpretive process	being another member of the social group

Table 11-2: Learning theories (elaborated from Wilson et al, 1993)

In the context of being an engineering student, several processes may be occurring at once. A lecturer may model to a student (social learning) how to behave while giving drill-and-practice exercises (behaviourist) while the student learns how to recognise different problem types (connectionist).



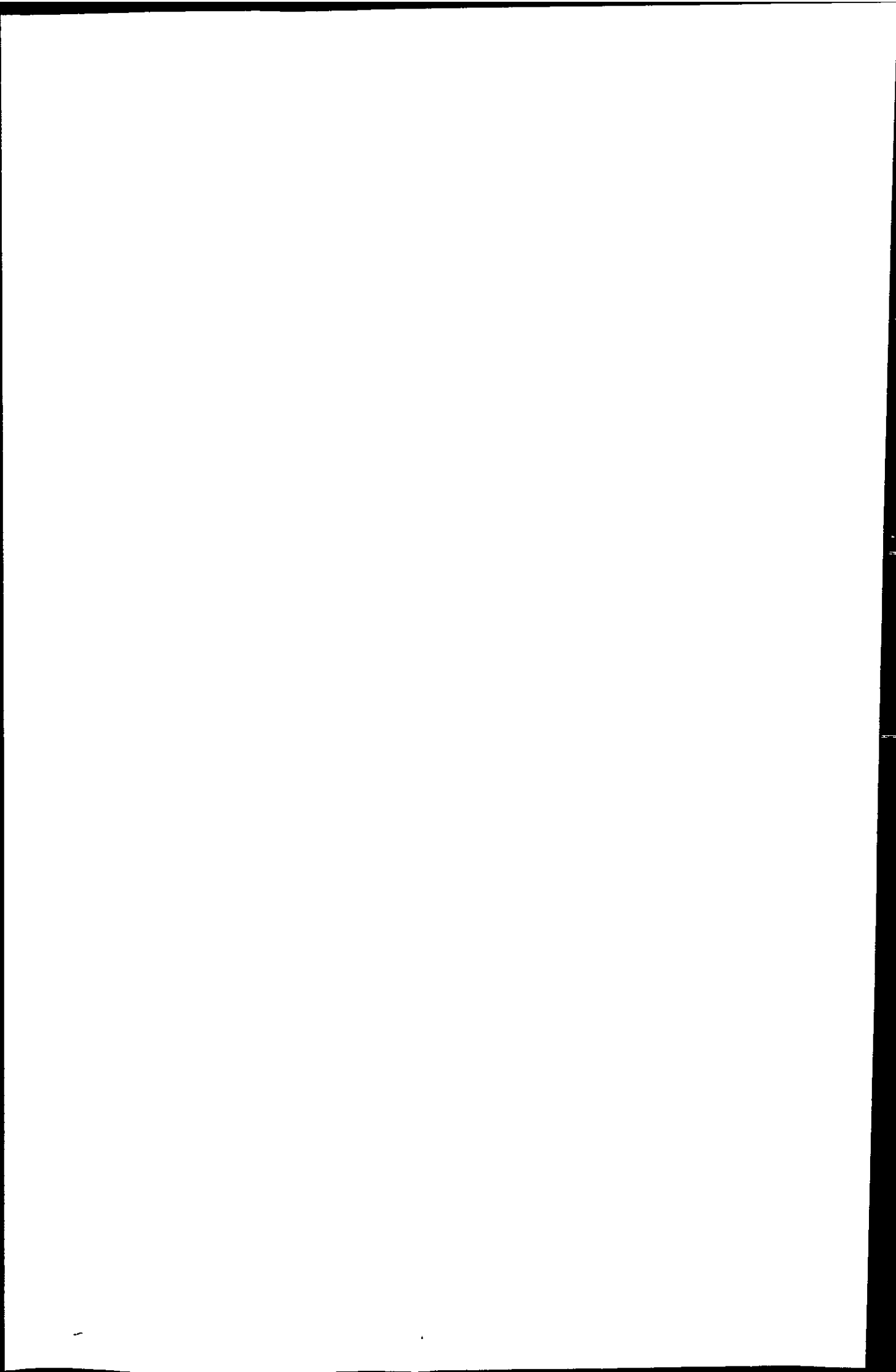
A naïve view of postmodernism relates it to Perry's (1988) *Multiplicity* where anything goes because we have no way of choosing between interpretations. A more sophisticated view would be related to his *Relativism and Commitment*, where it is recognised that different interpretations are possible, but some are more defensible (and useful) than others.

11.5 Paradigms of curriculum

What we teach and how we teach it is influenced strongly by our beliefs about what education is and what it is for. These questions affect the way we choose our intended learning outcomes and the way we assess what has been learnt. They affect how we allocate marks in assessing student work: giving credit for effort, originality, suitable method of working or correct answer.

These beliefs and values are rarely tested, and seem to be taken for granted, but individuals having different views on the meaning of education may run into conflict and fail to work together with catastrophic consequences. This applies particularly when the individuals are in a learner-teacher relationship. If the expectations on mark allocations differ, the teacher may be seen as an unfair marker, and then if negotiations do not explore *why* the marking scheme is as it is, the teacher is seen as imposing a decision in a power relationship. Another possible result of such a clash is that the learner decides he or she is learning nothing, or not what he or she wants to learn, or that he or she is very uncomfortable in a particular class. At least the teacher should be aware of this possible source of conflict, so as to decide whether to confront it, or avoid it.

One way of classifying the different systems of belief about the purposes of education (curriculum paradigms) has been proposed by Eisner (1974, cited in Helsel, 1987).

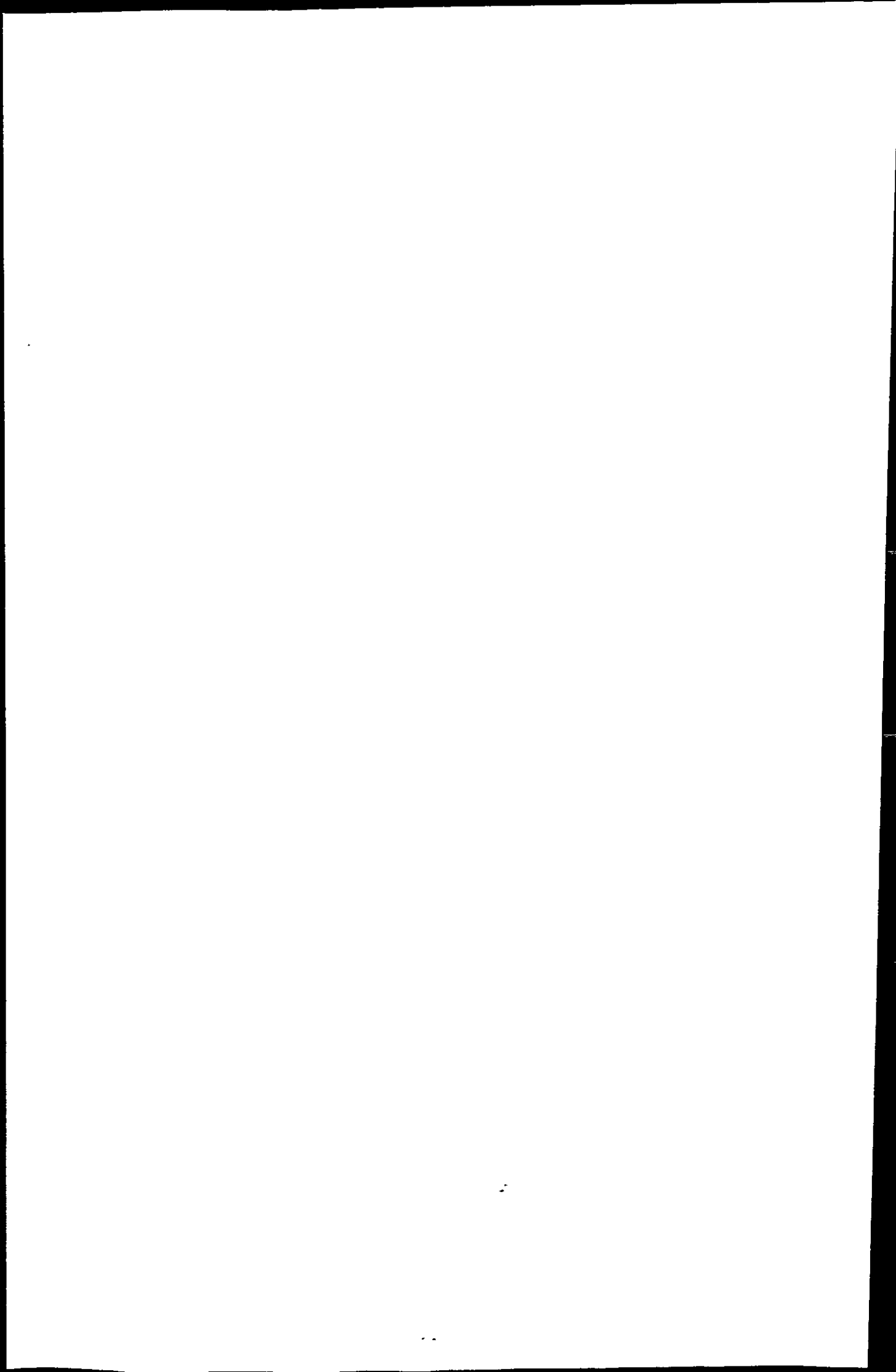


Helsel recommends this classification as comprehensive, succinct and dividing curricular orientations into distinct and mutually exclusive categories.

These can be characterised as in Table 11-3.

Curriculum paradigm	Education	Learning	The student	Paradigms of learning theory
development of cognitive processes	The human mind organises reality via cognitive structures. Education is the development of higher levels of structure	Learning is the raising of cognitive structures to higher levels of functioning by the development of metacognition.	An active, purposive individual in his or her own learning and development	Constructivist
curriculum as technology	Prescribed skills and knowledge are to be instilled into the student.	Learning is measured by the achieving of prescribed measurable behaviours on the part of the student.	A passive being to be conditioned to respond with desired behaviours to stimulus-response patterns	Behaviourist, Information processing, mastery learning
curriculum for personal relevance	To develop a human's higher needs, motives and capacities.	Learning is becoming a better, more fully developed person.	A unique individual with a singular perspective that should serve as the basis for his or her interpretation of the curriculum	Upper levels of Maslow's hierarchy
curriculum for social relevance-social adaptation	A mechanism for meeting the critical needs of society.	Learning is acculturation and socialisation.	A passive recipient who is schooled to take a place in the existing social order	Social learning
curriculum for social relevance-social reconstruction	To emancipate people from oppressive social conditions such as economic structures, linguistic biases or political inequalities	Learning is politicisation.	An individual committed to involvement in constructive social redirection and renewal	Social learning
academic rationalism.	The fullest possible evolution of the learner's mental capacities.	Learning is becoming a learned and cultured person.	A rational being who is expected to command essential facts and skills that undergird the intellectual disciplines of Western culture.	Liberal education
	(Eisner)	(Maull)	(Helsel)	(Maull)

Table 11-3: Paradigms of curriculum (elaborated from Helsel, 1987)



11.6 Applicability of the paradigms to engineering education

11.6.1 development of cognitive processes

This paradigm probably should form the dominant motivation in engineering education. We ask the students to be active, to take responsibility for learning, as they will have to take responsibility for their actions in future life. However the students, at least to begin with, tend to adopt an attitude more in line with skills acquisition (curriculum as technology) and socialisation (curriculum for social relevance- social adaptation). As Ramsden and Entwistle (1981) point out, engineering students tend to adopt a passive attitude to their learning.

11.6.2 curriculum as technology

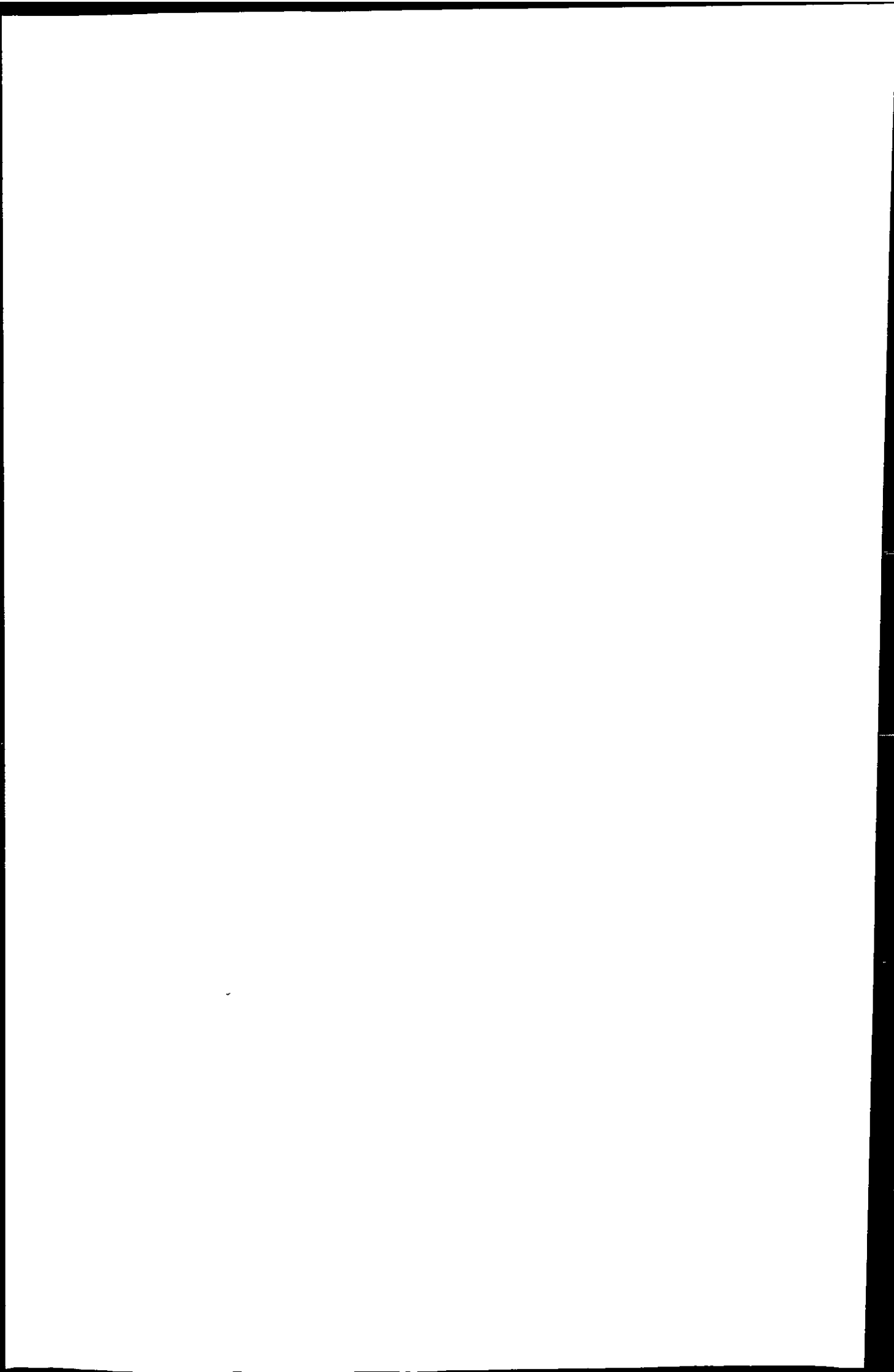
Where teaching takes the form of training with drill-and-practice exercises, tests which closely resemble exercises performed during the course and examinations which ask students to reproduce learnt material, we see evidence for this paradigm. It is held by many students in their early studies and if we define the syllabus through learning outcomes and competences this again reflects that paradigm.

11.6.3 curriculum for personal relevance

The development of the self is rarely emphasised in engineering education.

11.6.4 curriculum for social relevance- social adaptation

As engineering studies are regarded by many students as a preparation for a career in engineering, a social relevance curriculum paradigm is implied from the student point of view: the student expects to be equipped to play his or her part in society *as an engineer*. The student asks "what use is this?", meaning not "to me as a human adult"



but "to me as an engineer". The question of socialisation of engineering students was discussed in chapter 10.

11.6.5 curriculum for social relevance- social reconstruction

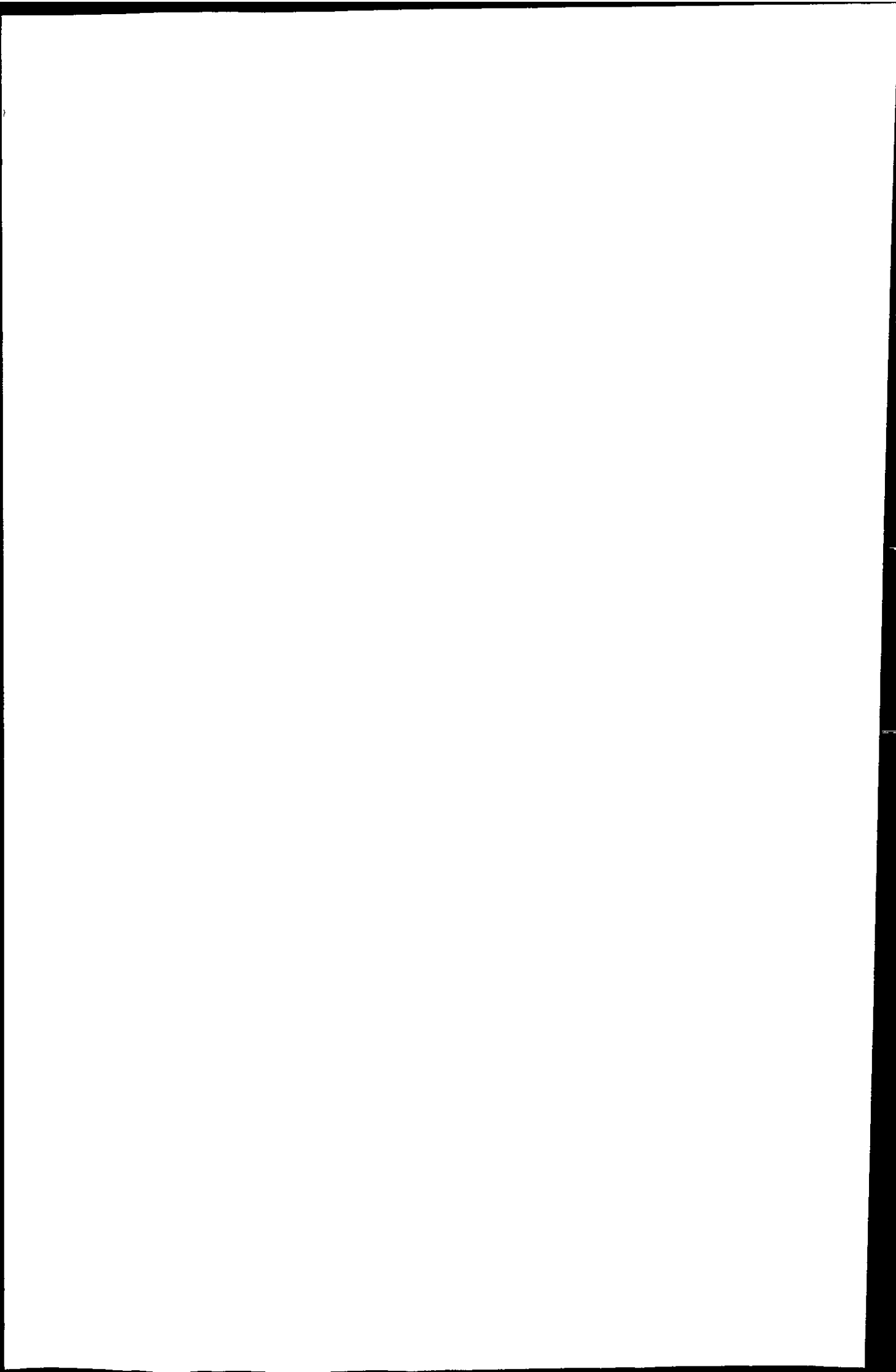
This is probably applicable in a context such as "green engineering", where the aim is to empower the students to take part in the enterprise of changing the face of engineering. In mainstream engineering education the aim is more to produce a socialised individual who will be a productive employee.

11.6.6 academic rationalism.

On the whole, this paradigm is probably foreign to engineering education. Although the engineer would accept the notion of being a rational being, the rest of the paradigm would be unacceptable on two counts. Firstly: that the engineer is pragmatic rather than academic and secondly: that the intellectual disciplines of Western culture, whereas they profit from the great engineering achievements of the past 200 years (for example, roads, railways, gas, electricity and water supplies) consistently undervalue these achievements in contrast with those of artists and "intellectuals".

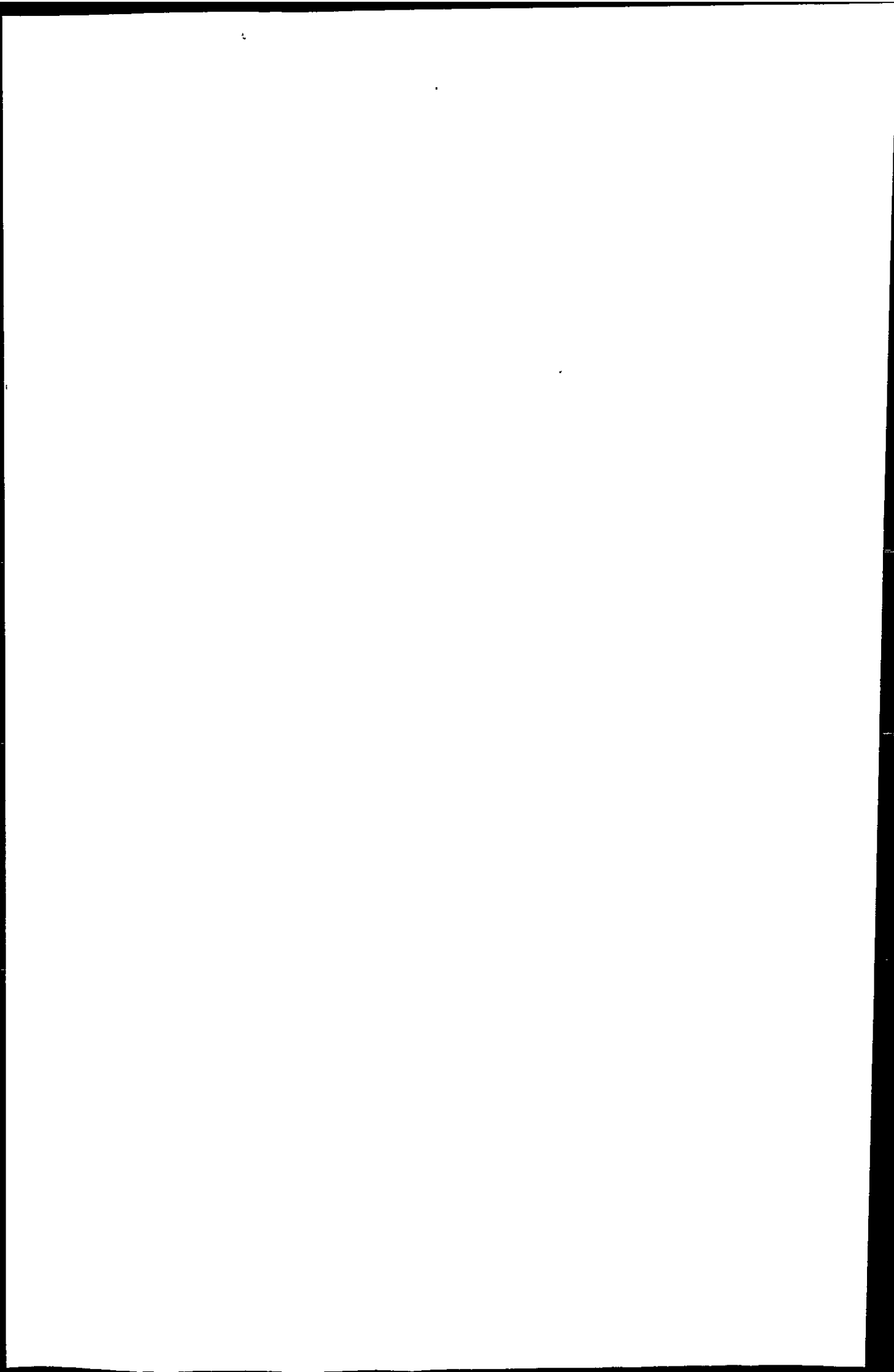
11.7 Endnote

A contrasting view is that taken by Perkins & Simmons (1988): their "first order theory of instruction" states that "people learn much of what they have direct opportunity and some motivation to learn, and little else". This is not quite as permissive as it may seem: although they proclaim that instructional style is subordinate to opportunity and motivation (which makes learning sound like a criminal activity), they insist that the instruction should address all their four frames or dimensions of knowledge (content, problem-solving, epistemic and inquiry)



11.8 Conclusions

Different learning theories and so on have useful ideas to offer, depending on the material to be taught, the prior knowledge of the students and their cognitive development, and the context in which they are learning. Whereas none of the theories mentioned above should be dismissed out of hand, none of them is complex enough to contain a model of an entire human being. Romiszowski's (1986) model (mentioned in chapter 12) on the other hand, which attempts to unify many of these theories, has become too complex to be immediately useful.



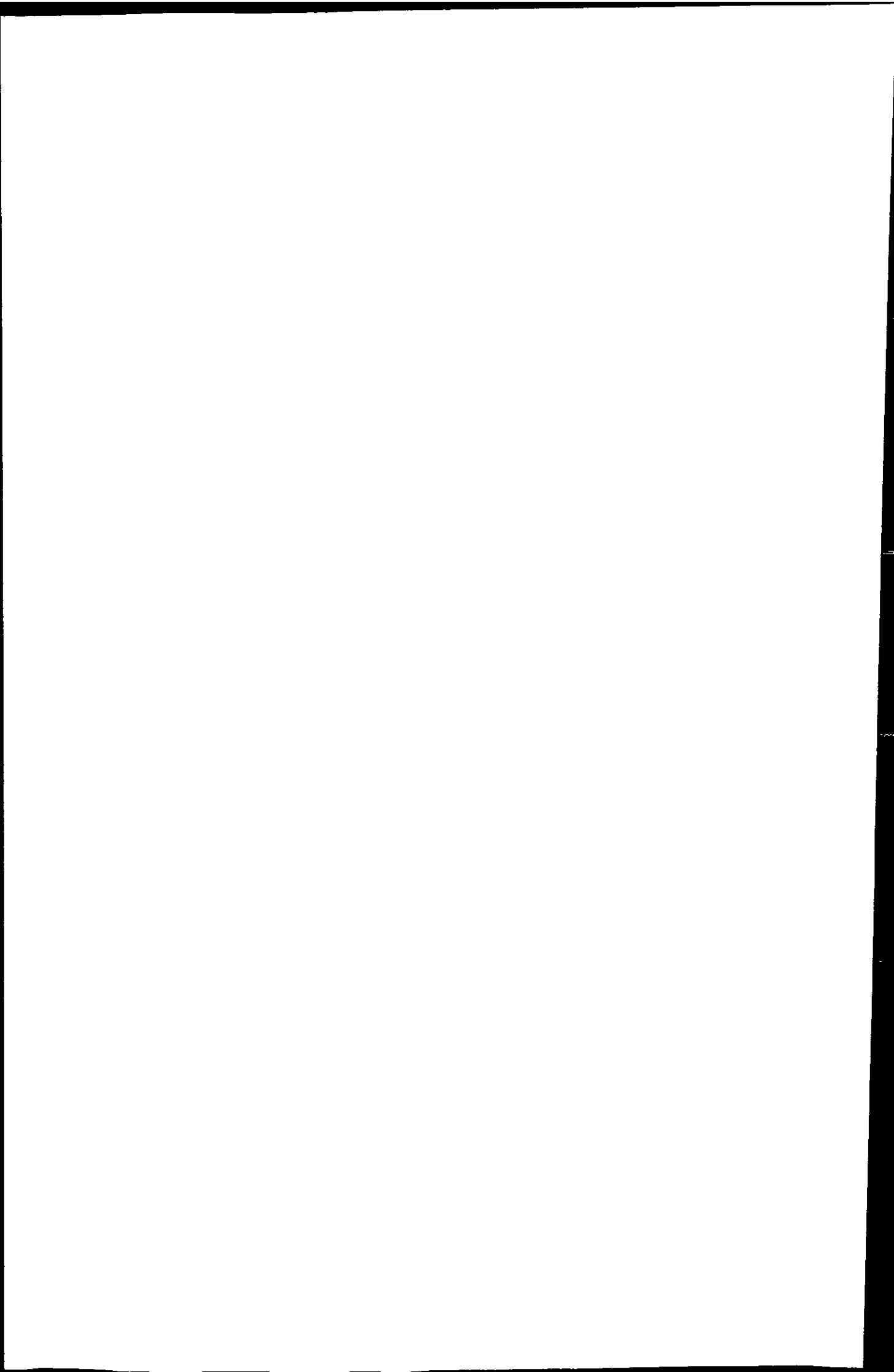
12. Computer aided learning (CAL) and computer aided instruction (CAI)

12.1 Introduction

In the last chapter it was pointed out that there are many ways of framing how students learn. In the field of designing courseware this is typified by the tension between the terms CAL and CAI. From the semantic point of view it is meaningless to speak of instruction without considering whether learning is taking place, and so from this point of view, as well as from a general philosophical bias towards considering the experience of the student as the most important part of the education process, I prefer the term CAL. This implies that the evaluation of the product should be carried out from the students' point of view rather than as an artefact in itself.

12.2 Medium and message

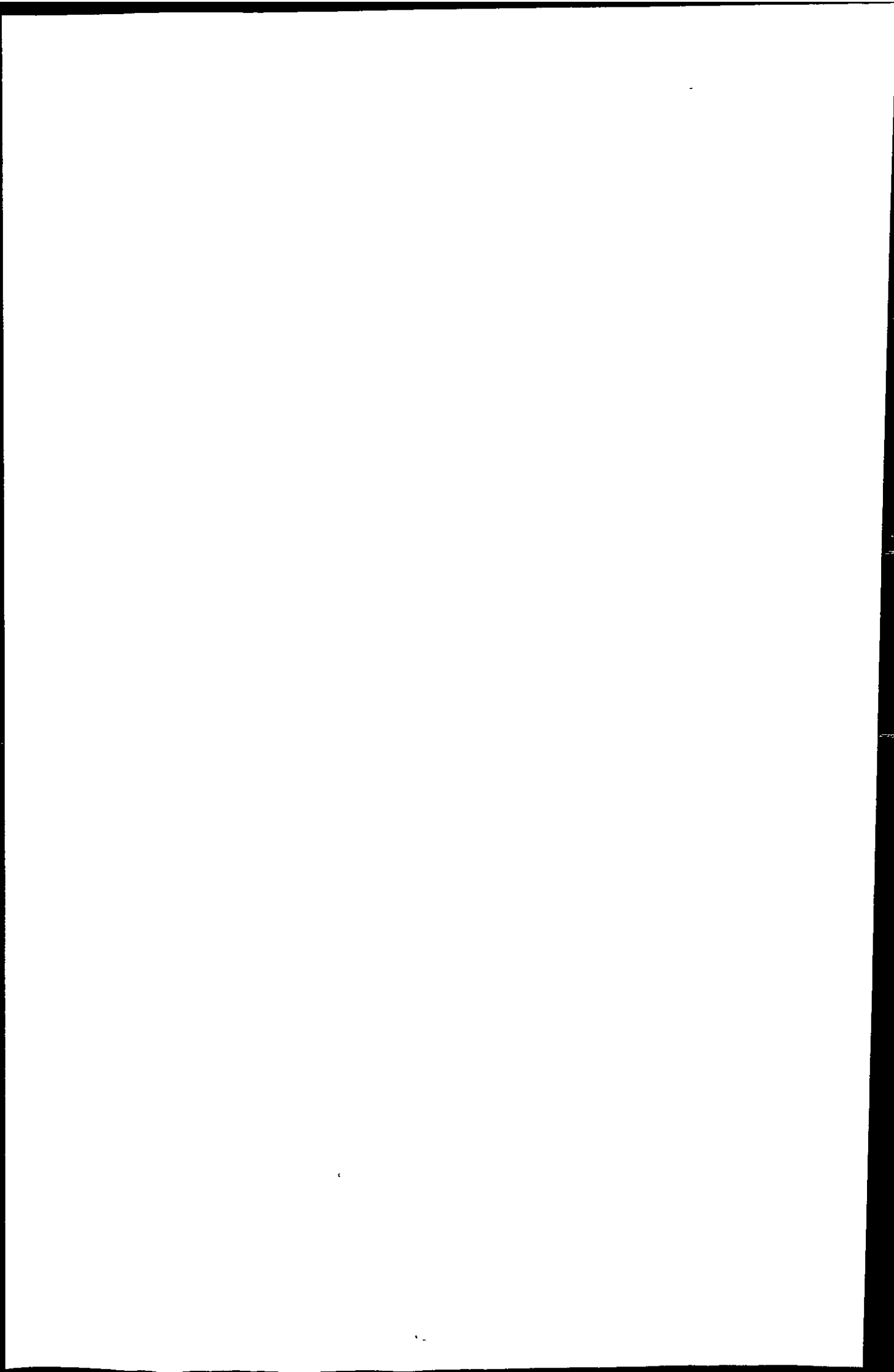
In 1983, Clark proposed that media should be regarded as mere "vehicles" in which knowledge was "delivered" to learners, and that discussion of the effects of media on learning should be suspended. Kozma (1991) replied that this analysis regarded the learner as the passive recipient of knowledge, rather than actively collaborating with the medium to construct knowledge. Different media of delivery meant that students learnt the same materials in different ways. The question of whether the computer delivers to a passive recipient or helps an active learner is the philosophical difference between the use of the terms computer aided instruction (CAI) and computer aided learning (CAL), although advocates of the CAL paradigm may use terms such as CAI and CBI (computer based instruction).



12.3 Competing models: CAL and CAI

Instructional design is a term used for the design of teaching materials and teaching strategies. Romiszowski (1986, p59) defines instruction as "a goal-directed teaching process which is pre-planned". Although Laurillard (e.g., 1987) argues for the adoption of a less didactic model of the teaching-learning model, in general the concern is mainly with the accurate transmission of accepted knowledge from the teacher to the learner: in other words the term "instructional design" embodies a transmission model of learning although not all practitioners accept the model. For example, Plowman (1989) states "Learning theories are embedded in the design of interactive video whether the designer has incorporated them intentionally or not... Most designs seem to rest on the belief that the student's mind is a *tabula rasa* and the knowledge in the program exists in a vacuum which is to be transferred straight from screen to mind without any other mediation than the occasional input via the keyboard... Most design manuals tend towards instruction based on drill and practice and simple branching designs." Reviewing users of computers in schools, Kurland and Kurland (1987) found there was an ideological struggle between Skinner-based behaviourist CAI advocates and Piagetian-developmental LOGO advocates.

Despite Wildman's (1981) contention that the 1970s had been the era of development of cognitive theory, which had supplanted the previous dominance of the behaviourists, Hannafin and Carney (1991) found that instructional practice was still dominated by behavioural strategies focusing on imposed methods which elicit the desired response. They suggested that cognitive psychology based strategies, presumed to increase the depth of processing, would improve the quality of encoded knowledge. They conclude that "it is the learner, not the designer, who mediates the possibilities of lesson strategies and activities". Hennessy and O'Shea (1993) also



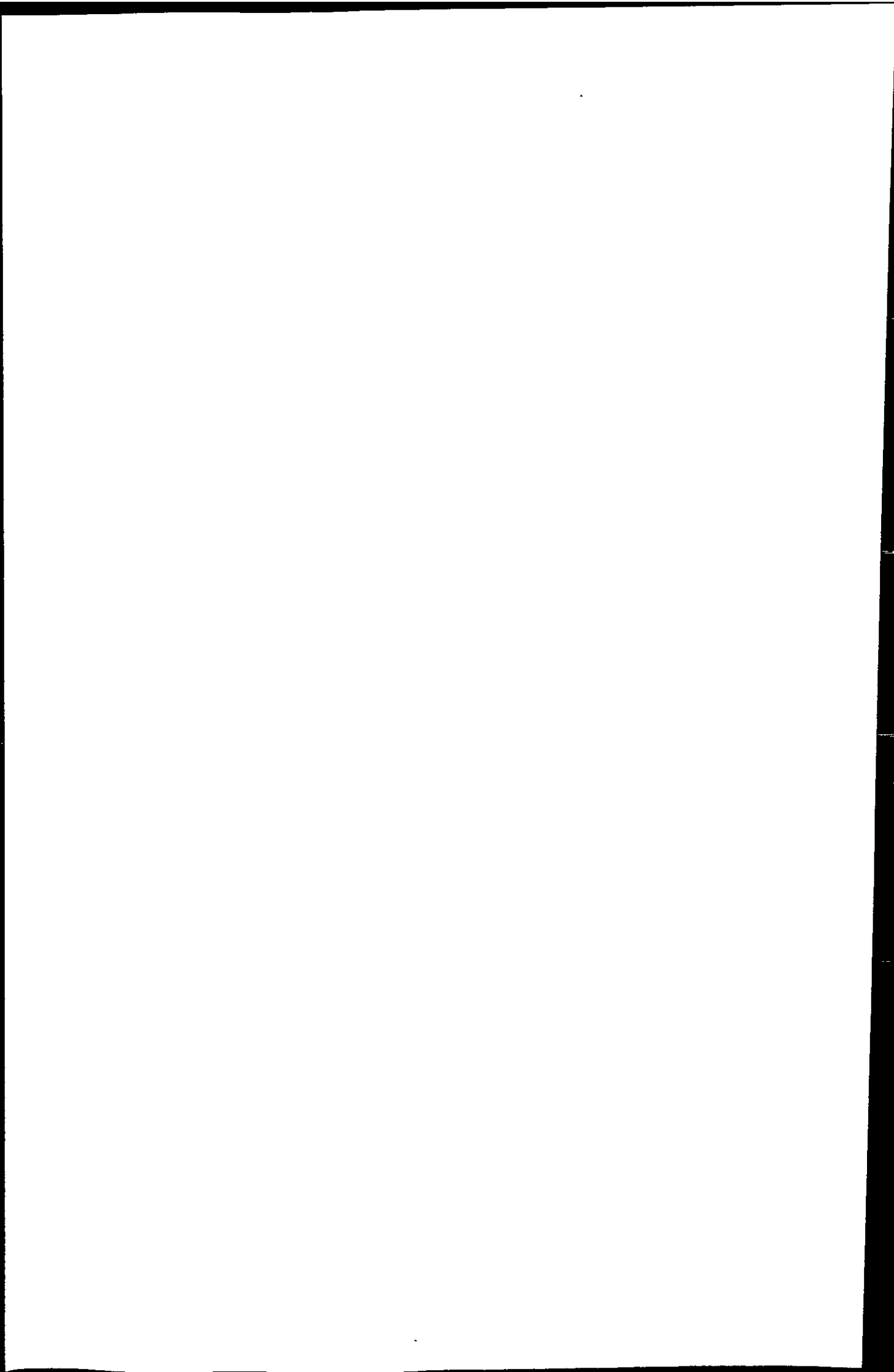
found that students attribute their own meanings to simulations and that they may refuse to accept the challenges thrown by the simulation. Smith (1988) points out that it is a commonly accepted view in semiotics that the interpretation of a text is handed from the author to the reader, and that any text is open to other readings than those intended by the author. For example, in a lesson where the teacher was using a program intended to teach the hidden complexities of simple conversations, the children thought they were being taught to use the computer. Burton (1997) describes learning as a "process of meaning making by learners not of being handed meaning by their teachers" which means that any "new piece of information is encountered and understood" heterogeneously by members of a class.

Julie (1991) also argues for the use of the computer in helping the student form semi-concrete concepts, at a level between concrete and abstract, for example by the use of images. The computer may thus be used as an introductory device rather than for drill on concepts already taught.

Clark (1994) considers instructional design in the context of professional education. He contrasts the instructional view of professional education as accumulation of competencies with the perception of a four-fold nature of that education:

"acculturation to the profession, development of associated competencies, thinking about the competencies and thinking about thinking about the competencies".

From the above, I conclude that many designers of CAL material tend towards a didactic and transfer model of learning, although there is a stream of criticism of this model. This may be because people who are interested in writing programs use a private metaphor of computer programming for human learning.



12.4 Lessons from Interactive Video

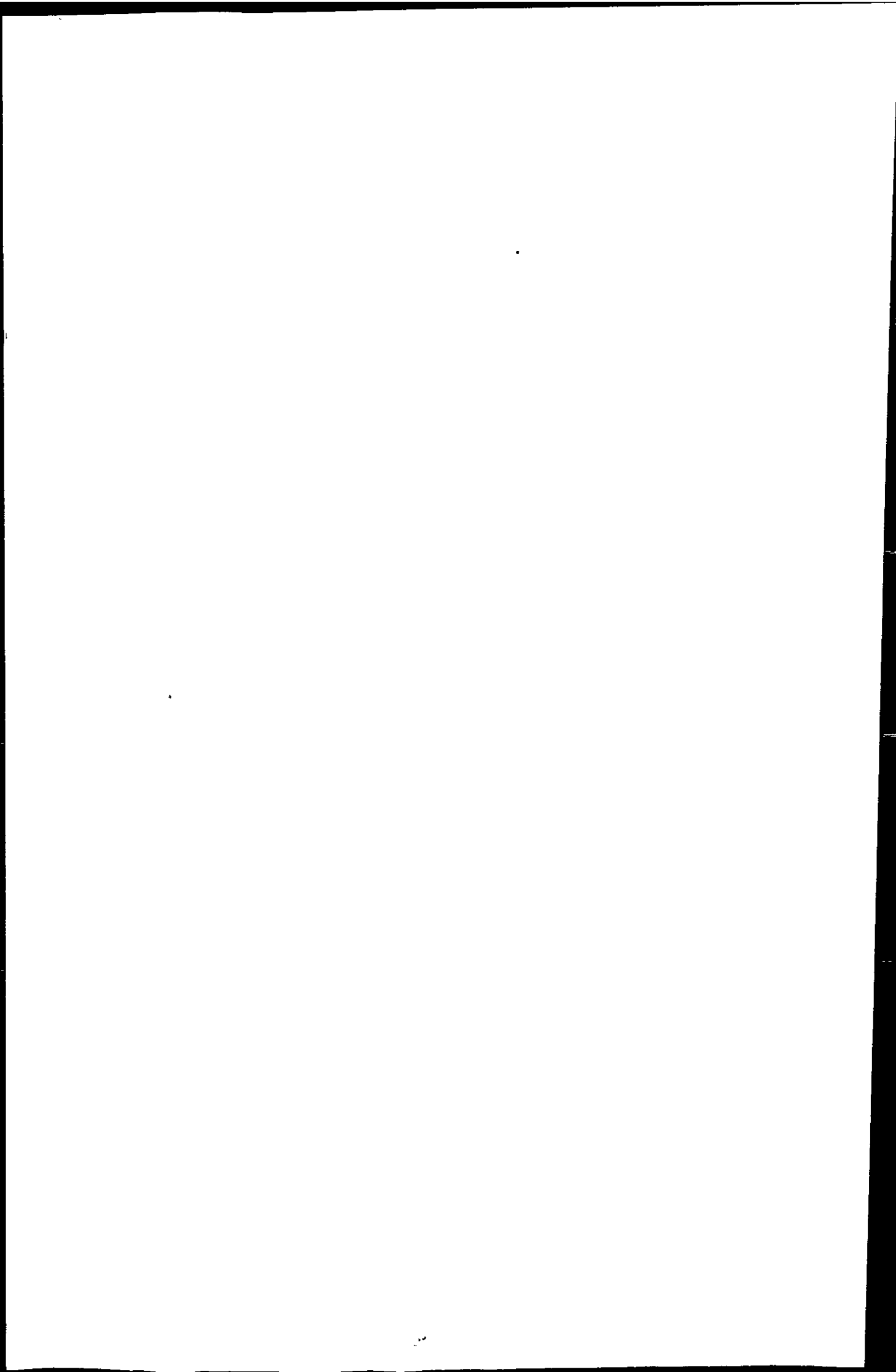
Much work on instructional design is based on interactive video (IV), which failed to have the impact that was expected in schools or universities. (see Norris et al, 1990)

Other work assumes much lower computer capabilities, such as displays and processing speeds, than are now available. However the conclusions which have been reached are sometimes applicable to the new generation of computer-aided learning (CAL) packages. For example, Plowman (1988) points out that IV has the authority of the TV (It must be real!), and the illusion of control over the disk, although users cannot really add their own contributions. She cautions against the use of multiple choice questions with the implication that the only questions worth asking are those with a correct answer.

Megarry (1988) speculated that the future of CD technology would be different from the "false dawn" of videodisk, given the commitment of Phillips and Sony to work from a common standard, the fallout benefits from CD audio technology, such as cheaper pressing and mass-production of CD readers in the audio context, the small physical size of CDs, and their all-digital format.

The advice in the literature tends to fall into two categories: that based on a developed learning theory, which is prescriptive (e.g. Romiszowski, Laurillard) and that which is based on avoiding hindrance, such as "do not make writing so small it is illegible", and more sophisticated equivalents. A smaller third stream is led by Malone (1981) who looked for intrinsic motivation factors in computer games and sought to apply the principles of challenge, fantasy and curiosity to CAI. (see for example Middleton, 1995)

Most research into the effectiveness of CAL/CAI takes the form of a classical experiment. Brown (1994), however, argues that it is impossible to control for



learning in this way, particularly since each piece of courseware has involved innumerable design decisions, some of which will be helpful and others of which will hinder use, and that learning results from a rich variety of interactions between individuals and their environment. He proposes that evaluation should be carried out by observing users in action and by debriefing and discussion.

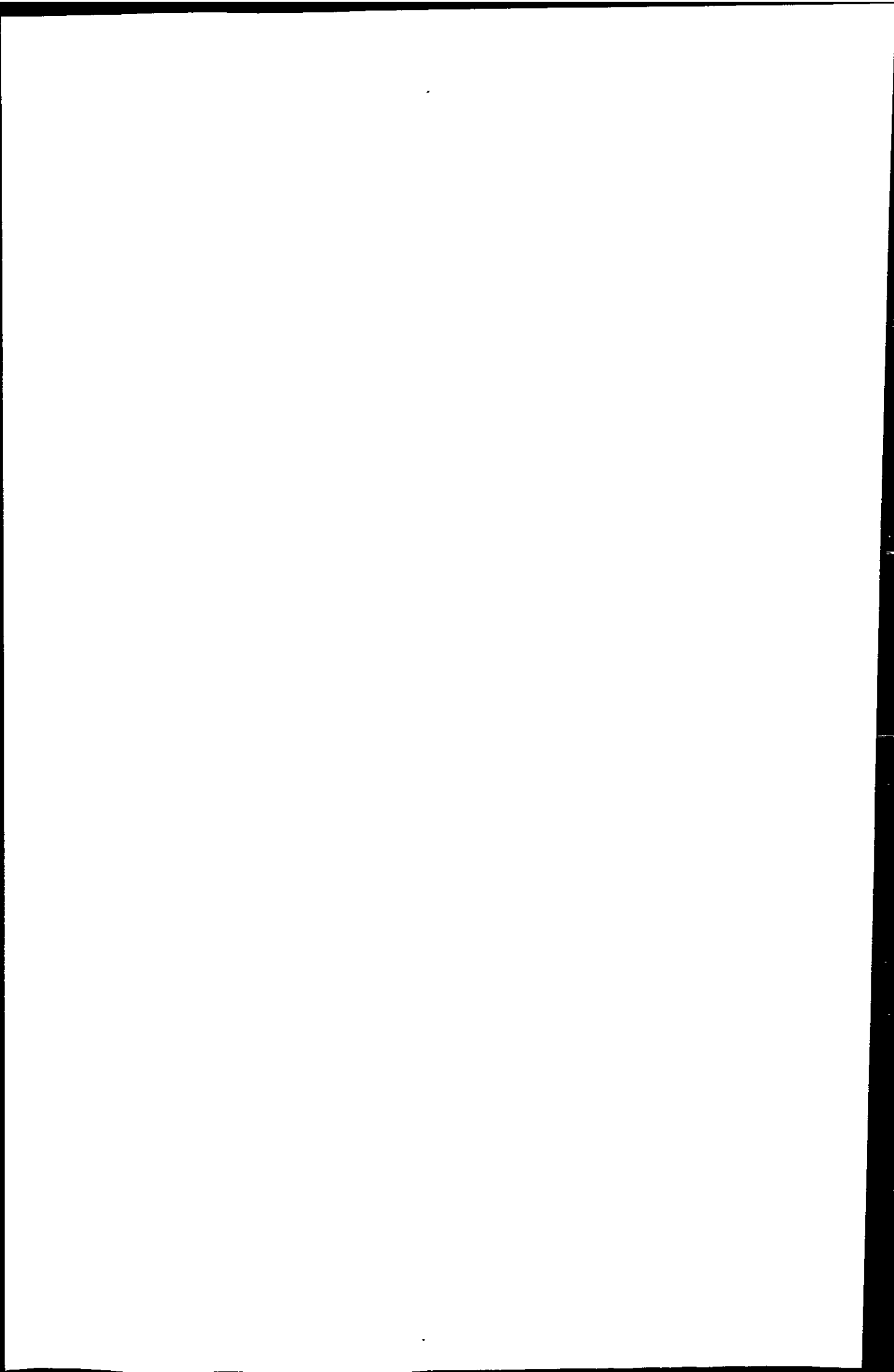
Dick (1981), with some foresight, warned that although the videodisk combined with the computer was being hailed by some as the "ultimate teaching machine", the experience of the 1970s should show that there was always another development coming down the road.

12.5 Freedom to roam

An important aspect in the design of courseware is the order in which material is presented, and who decides that order. The material in a film or television programme is accessed in strict sequential order: material on tape or disk may be skipped, fast-forwarded, paused or rewound. Material on a computer may be accessed in random order to be decided by the program or the user, or by a combination of the two.

Bartolomé (1992) proposed a scale of measuring the degree of interactivity of a system: (Table 12-1)

The level chosen depends on the instructional paradigm (levels 1 and zero are linked by the authors to behaviourism), the sophistication and the age of the students, and the nature of the material taught.



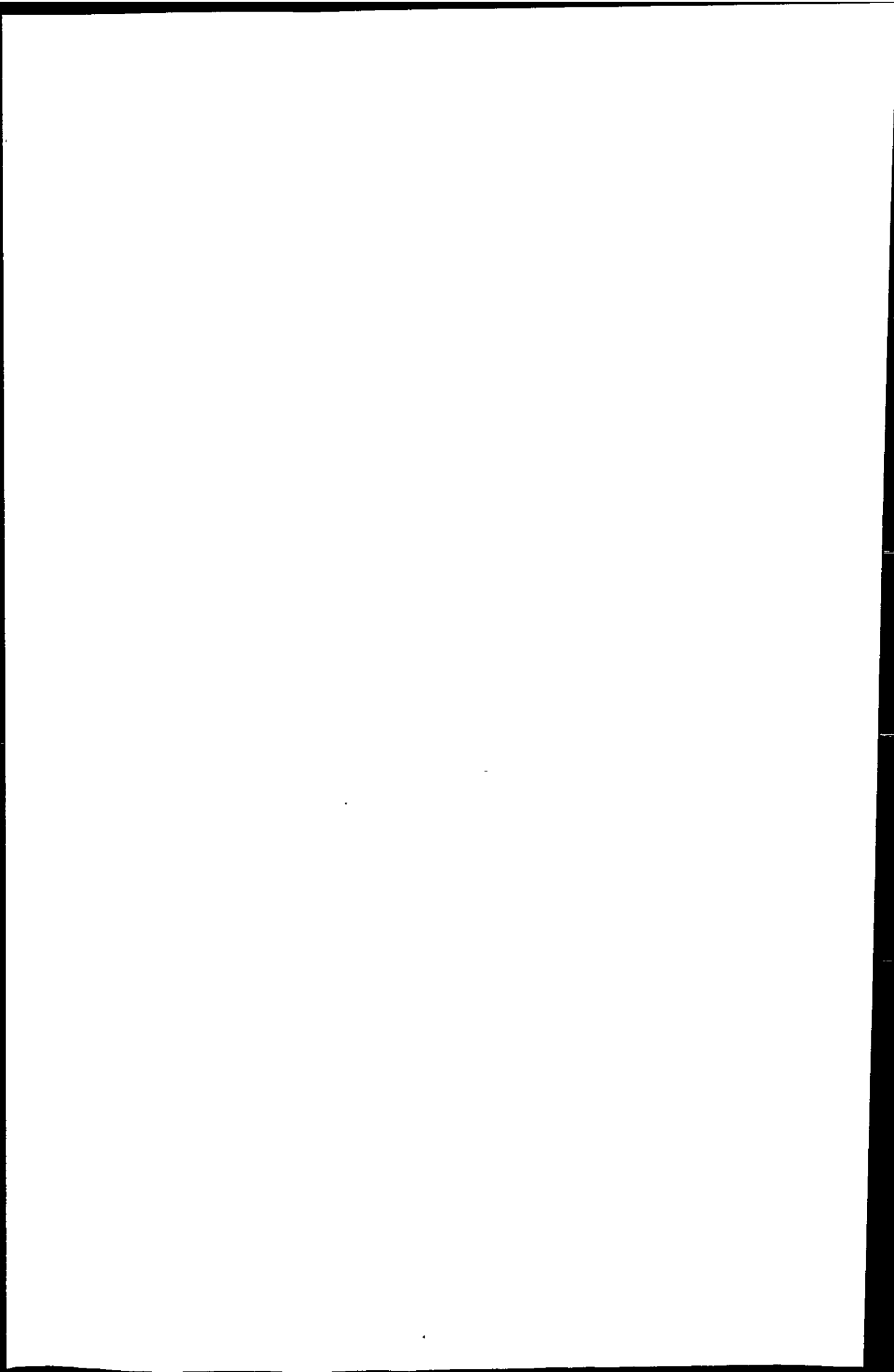
Level zero	Computer completely in control
	0.0 No motor activity called for
	0.1 Motor activity called for: e.g. press the return button
Level 1	System chooses the next information given as a result of the learner's previous responses
Level 2	Learner chooses next information given
	2.1 From a menu
	2.2 Some help given in choice
	2.3 Directed choice
Level 3	Learner chooses what is shown next and how it is shown
Level 4	Learner also chooses source of information

Table 12-1: Bartolomé's classification of degrees of interactivity (1992)

Gagné (e.g. 1981) specified that instruction should follow a prescribed order of events: orientation, presentation, sequence, encoding and retrieval, and he gives a comprehensive guide in tabular form of appropriate strategies for each stage according to the type of learning outcome. The order of presentation is to be strictly controlled by the author. Bartolomé (ibid.) warns that this can cause frustration. At the opposite pole, designers of hypertext systems impose a very loose ordering on the material. Users are free to follow or ignore links, and to find their own path through the material. This looser structure is felt to reflect better the relational structure of human knowledge, but the freedom of "hyperspace" can bring a feeling of disorientation (Frau et al, 1992).

12.6 Learning theory-based research.

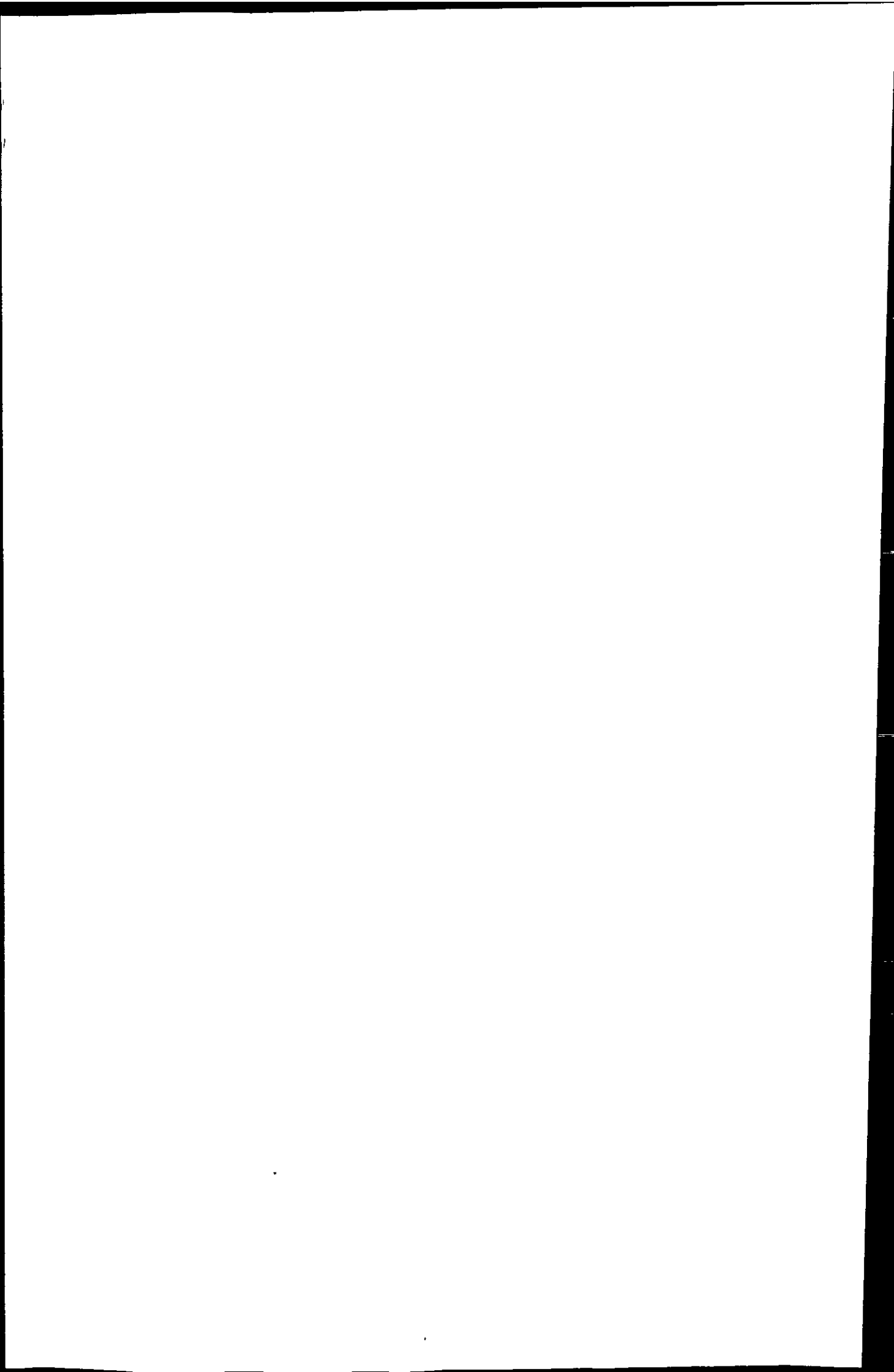
Laurillard (1987) has elaborated Pask's conversational model of learning as applied to courseware design. She has compared different methods of teaching with ideal computer-assisted learning environments and summarised the results in a table (Table 12-2).



	Print	Audio-vision	Television	Video	Self-assessed questions	Hypertext	Multi-media resources	Simulation	Microworld	Modelling	Tutorial program	Tutoring system	Tutorial simulation	Audio conferencing	Video conferencing	Computer conferencing	Computer supported collaborative work
1 T can describe conception	✓	✓	✓	✓	0	✓	✓	0	0	0	✓	✓	✓	✓	✓	✓	0
2 S can describe conception	0	✓	0	0	✓	✓	✓	0	✓	✓	✓	✓	✓	✓	✓	✓	✓
3 T can redescribe in light of S's conception or action	0	0	0	0	0	0	0	0	0	0	✓	✓	✓	✓	✓	✓	0
4 S can redescribe in light of T's redescription or S's action	0	✓	0	0	✓	✓	✓	0	0	0	✓	✓	✓	0	0	✓	0
5 T can adapt task goal in light of S's description or action	0	0	0	0	0	0	0	0	0	0	✓	✓	✓	0	0	0	0
6 T can set task goal	0	✓	✓	✓	✓	0	0	✓	✓	✓	✓	✓	✓	0	0	0	✓
7 S can act to achieve task goal	0	✓	0	0	✓	0	✓	✓	✓	✓	✓	✓	✓	0	0	0	✓
8 T can set up world to give intrinsic feedback on actions	0	✓	✓	✓	0	0	✓	✓	✓	✓	0	✓	✓	0	0	0	✓
9 S can modify action in light of intrinsic feedback on action	0	✓	0	0	0	0	✓	✓	✓	✓	0	✓	✓	0	0	0	✓
10 S can adapt actions in light of T's description or S's redescription	0	✓	0	0	✓	0	0	0	✓	✓	✓	✓	✓	0	0	0	✓
11 S can reflect on interaction to modify redescription	0	✓	0	✓	✓	0	✓	0	✓	✓	0	✓	✓	0	0	0	✓
12 T can reflect on S's action to modify redescription	0	0	0	0	0	0	0	0	0	0	✓	✓	✓	0	0	0	0

Table 12-2: Media comparison chart (Laurillard, 1993)

This may be contrasted with the view taken by, for example, Romiszowski (1986) in "Developing Auto-Instructional Materials". (Romiszowski attempts to integrate most theories of learning up to the present day in an overarching theory which looks rather like Kolb's, whom he does not reference, which takes 13 pages of dense tables to expound.) It may be seen from Romiszowski's diagram (Figure 12-2) that his model is not a simple one.



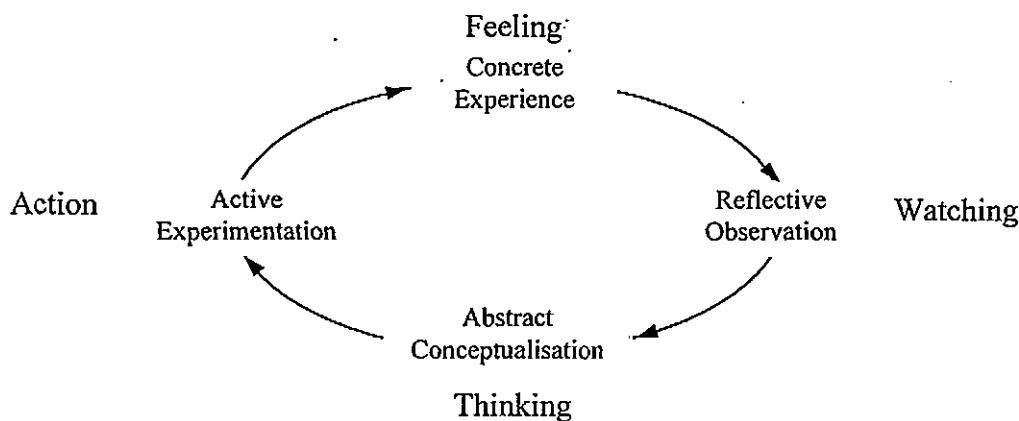


Figure 12-1: Kolb's cyclical learning model (see Kolb, 1981)

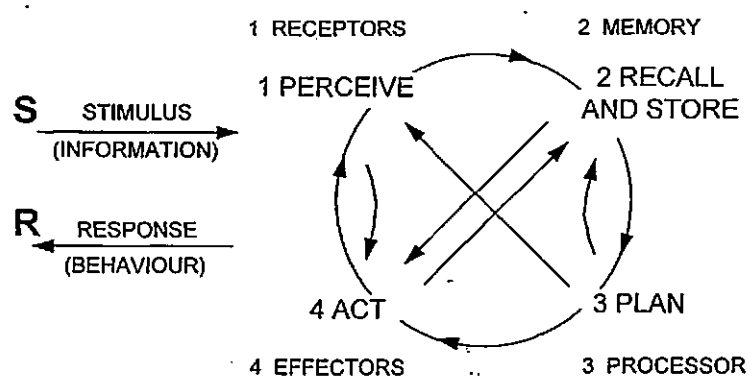


Figure 12-2: Romiszowski's learning model (see Romiszowski, 1986)

Both these models owe much to hermeneutical ideas, and in particular Ricoeur's (for example 1981) hermeneutic circle. Ricoeur (quoted by Brown, 1997) suggests that the individual switches between seeing the world as something of which he is a part and something which he can objectify and act on (that is between active experimentation and passive reflection, in Kolb's terms).

Instructional design tends to be linked with ideas such as *mastery learning*, where the student is expected to develop a high degree of mastery of the material in a unit of instruction (that is, to be able to answer, say 90% of questions on the material correctly in a post-test) before proceeding to the next stage. The terms "Stimulus" and "Response" also tend to appear in Instructional Design literature. Romiszowski states that in instructional design there should be three types of communication

channel between learner and instructor: these may be labelled stimulus, response and reinforcement or feedback. (p89)

Instructional design is driven by learning objectives which may be stated and verified, and tends to be reductionist, in that it is easier to test sub-units and low-level skills such as recall facts or reproduce a given performance than higher level ones such as synthesis or evaluation. Thus instructional design tends to be accompanied by quantitative research, with pre-tests to check that the subjects have the prerequisite knowledge and not the knowledge to be transmitted before exposure to the instruction, and post-tests to determine to what extent the requisite knowledge has been transmitted. This research then determines the effectiveness of the material in achieving the stated aims of the instruction. Again it is more difficult to quantify the acquisition of higher level skills, so the research tends to concentrate on lower-level skills.

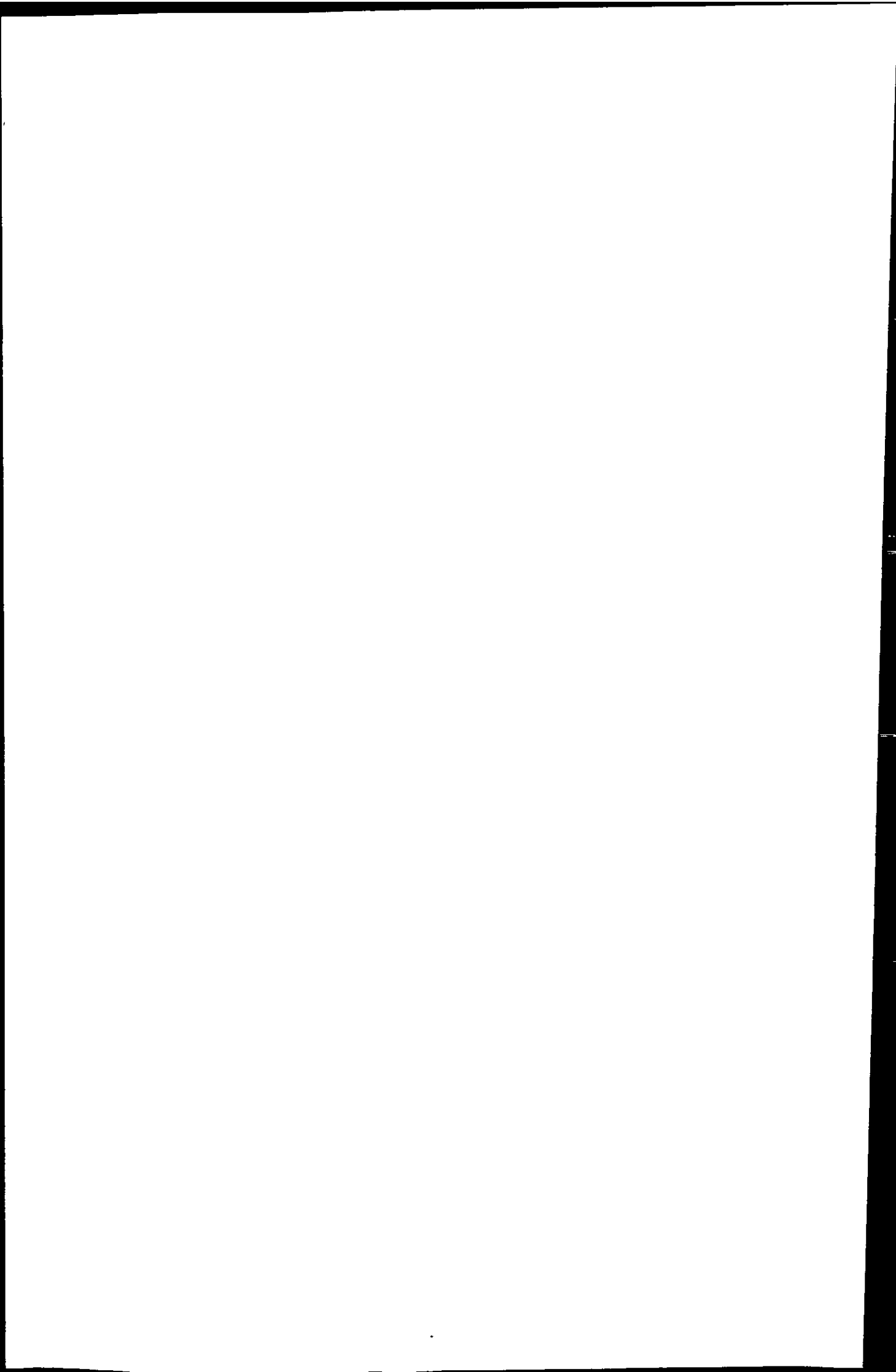
Gagné's theory of instruction states that instructional design should comprehend five phases, namely orientation, presentation, sequence, encoding and retrieval.

Romiszowski is heavily influenced by Gagné in his lesson planning in this respect.

Much of the research in the "avoiding impediments" falls into the presentation aspect of design.

Götz (1991) concludes that

"interactive learning programs often break down as a result of an inaccessible didactic construction whose media potential is only partially exhausted. Rigid user and limited interaction possibilities lead to the assumption that the foundation of learning programs lies more in informatics than it does in didactics. A stronger reliance on pedagogical aspects appears to us to be more desirable."



12.7 *Avoiding impediments to learning*

Merrill (1988) suggests guidelines for good instructional design under three headings: instructional design, screen design, and human factors.

12.7.1 instructional design

Avoid merely putting text on the screen: avoid mere page turning.

Avoid generality rich but example poor presentations. Each idea should be presented by a generality, examples and practice.

Avoid remember only practice.

Use attention focusing devices to relate examples to generalities and to point out critical characteristics of the illustrative material.

Promote active mental processing by asking rhetorical questions and engaging the student in a conversation which requires constructed (as oppose to multiple choice) responses to which we do not provide right-wrong feedback but rather an anticipation of a reasonable reaction.

Provide expository examples as well as practice

Merrill (1988)

12.7.2 screen design

No scrolling for educational programs.

The student should control text output. Never erase critical information until the student indicates readiness to proceed OR provide a way for the student to repeat dynamically presented information.

Use dynamic displays in which timing of text output, inverse text, flashing and animation are used for stress and emphasis.

Dark letters on a light screen will appear less confined and more natural to the student

Leave plenty of white space and erase information when it is no longer needed.

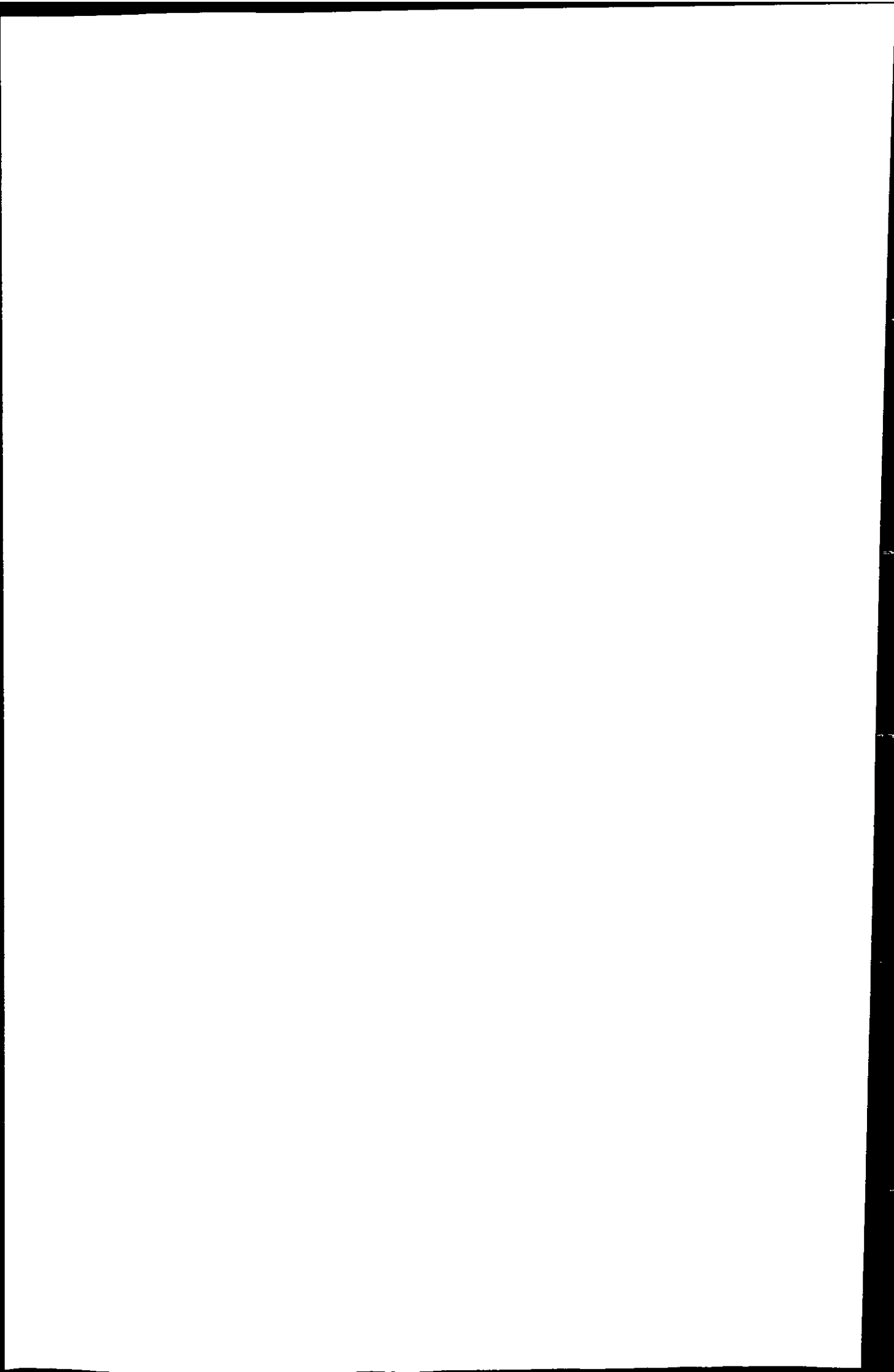
Use short lines and separate natural phrases or ideas on each line.

Do not full justify text on the screen.

Do not present information in all upper case except for emphasis.

Use a variety of text styles to indicate different kinds of messages.

Merrill (1988)



12.7.3 human factors

Provide a way for the student to skip to the major sections of the program in order to preview, review or repeat portions of the material.

Provide some kind of location indicator so that the student knows where he or she is in the total program.

Allow the student to "turn" the pages by going back to the last page, repeating the current page or going forward to the next page.

Minimise unnecessary typing by using a pointing device wherever possible.

Monitor the student's activity and provide advice when potentially decremental action is taken. In most cases, provide a mechanism for the student to override the advice.

Allow the student to select how many examples they need to study.

Provide optional help, do not force every student through the most detailed presentation.

Provide a means of escape from any lengthy activity but advise the student about the consequences of such an escape.

Provide adequate directions, including all the options available to the student. If possible, list the available options on the screen or make them accessible with the press of the [F] key. Use the most natural procedure.

Plan disk access to avoid long waits while the computer retrieves information.

Merrill (1988)

Dahl (1990) gives the following rules of thumb for writers of CAL packages..

Be consistent. The same user action should always lead to the same result.

The user should be in control. The user should choose his/her actions.

Give feedback.

Let the user re-enter erroneous input.

Cope with inputs like dividing by zero.

Allow experienced users to use shortcuts.

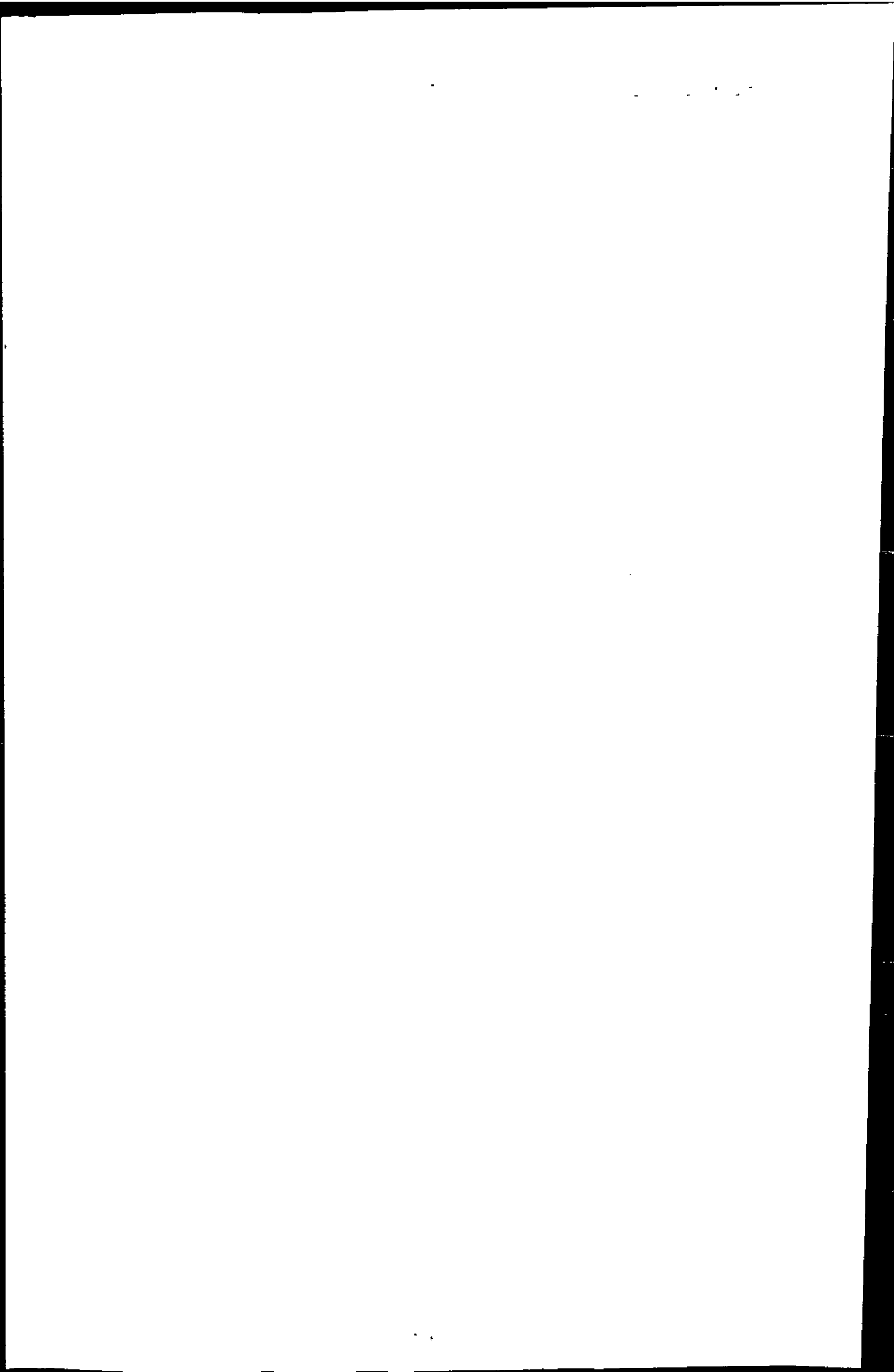
Keep displays simple. Do not put too much on screen at once.

Allow the user to cancel terminal actions (that is, to decide against quitting).

Dahl (1990)

12.7.4 Screen design

These guidelines are reinforced by Sweeters (1985), Madge et al (1986) and Sandals (1987) who also emphasise the importance of using space freely, consistent use of



screen areas, the use of overlays to add bits of information at a time, and the need to choose colours carefully.

12.7.5 Graphics

Perhaps one of the most significant points to come from the research is the overuse of graphics. Graphics which are not instructional or which seek to clarify an already clear process should be avoided... Unnecessary graphics can impede the course of learning by slowing the pace of the course and distracting the student.

Use graphics to clarify difficult concepts.

Use a combination of different kinds of graphics: flowcharts, graphs, maps, realistic pictures and analogical pictures.

(Madge et al, 1986)

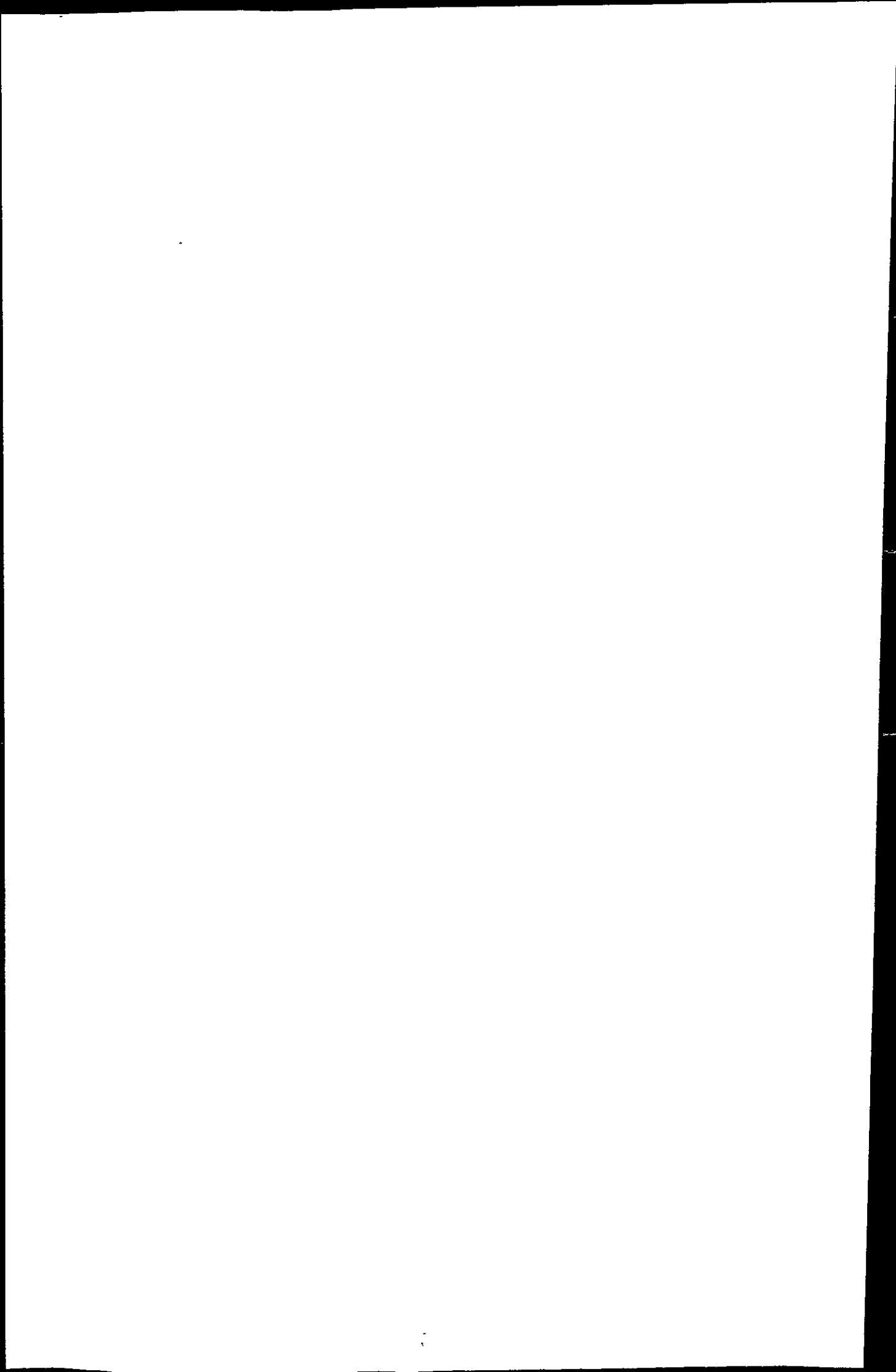
Present blocks in this order: graphics (so they don't distract from the text); text; directions...

Consider using a graphic on each display. Consider employing all the following types of graphics: realistic (portrays an instance or example of a concept. A realistic graphic); analogic (relates a concept to more familiar, similar things. A more far-fetched but personal graphic); logical (a highly schematised visual such as a flow-chart, graph or map. An abstract but logical representation).

(Sweeters, 1985)

The origin of the classification of graphics into representational, analogical and logical appears to be Knowlton (1966).

Clarke (1992) surveyed the use of graphics in CBL packages, and found the distribution shown in Table 12-3. He found these results to be in agreement with an American study (Alesandrini, 1985), and suggested that graphics were used by designers as optional extras and that most graphic screens consisted of a small image with supporting text.



	Industrial training	Commercial training	Basic education	Training teachers	Overall
Packages surveyed	10	13	5	6	34
Amount of text (%)	45	67	70	67.5	64
Amount of graphics (total) (%)	55	33	30	32.5	36
Representational graphics (%)	33	15	16	22.5	22
Analogical graphics (%)	2	3	1	2.5	2
Logical graphics (%)	20	15	13	7.5	12

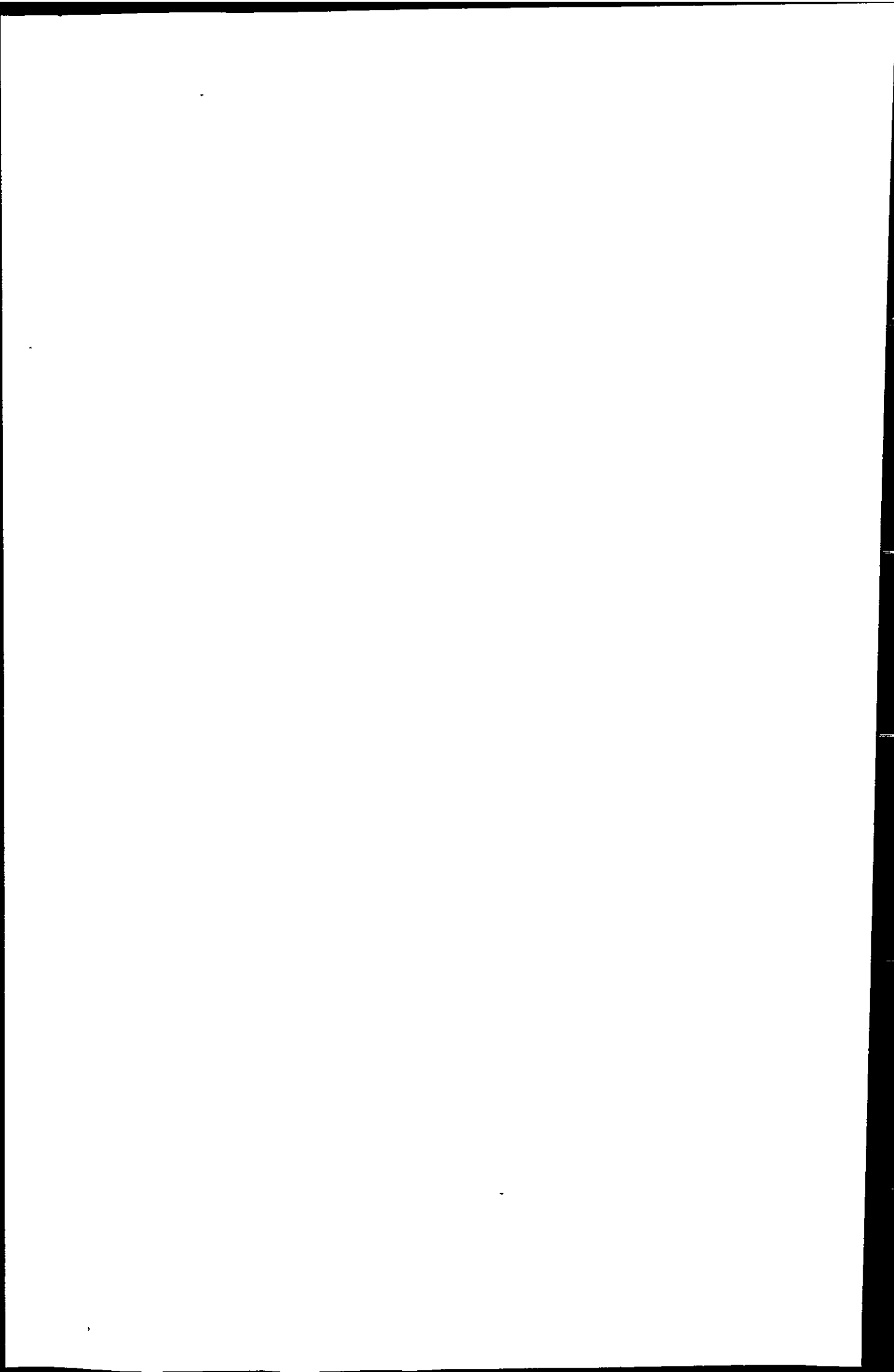
Table 12-3: Use of graphics in CBL packages (Clarke, 1992)

Kozma (1991) in a review of the learning effects of different media, finds a consensus that pictures have positive effects under certain circumstances.

The use of pictures with text increases recall, particularly for poor readers, if the pictures illustrate information central to the text, when they represent new content that is important to the overall message, or when they depict structural relationships mentioned within the text.

Kozma (1991)

He suggests that the combination of text with pictures presents the learner with two symbol systems (verbal and pictorial) and "facilitates the construction of the textbase [a collection of summary-like statements which represent the gist of the text] and the mapping of it onto the mental model of the situation." Likewise Spencer (1991) suggests that illustrations convert information from uni-modal to bi-modal form. Media which combine both modal forms and both storage systems will be most effective. Dwyer and Dwyer (1987) stress the importance of the time during which the learner interacts with the material and rehearses information. If the time is short, there will be no rehearsal and the information is not elaborated upon and transferred to long-term memory. If the time is adequate, and the rehearsal requires some form of action, such as taking notes or writing an answer, then the elaboration has time to take place, and the material will be remembered. Translation from a visual mode of presentation to a verbal mode for storage can also provide the time and rehearsal needed for transfer to long-term memory.



Sweller et al (1990) on the other hand suggest that "instructional material that requires learners to mentally integrate disparate sources of mutually referring material (e.g., text and diagrams)... interferes with learning by misdirecting attention and imposing a heavy cognitive load". They propose that text and diagrams should be presented as a united entity, and that text should be added to diagrams at the point at which new entities are added to the diagram, in order to reduce the load of switching between diagram and text.

Goldenberg (1988) suggested that students could find some visual representations confusing, for example the interpretation of whether a straight line had been moved up or to the right depended on the slope of the line and the shape of the window. Demonstrating the effects of changing the value of a parameter before having firmly established the difference between a parameter and a variable had the effect of making the notion of the variable very hazy.

12.7.6 Animations and sounds

It is tempting to exploit the possibilities of the medium by using animations and sounds to enliven the courseware. However various researchers warn against this, as students find inappropriate sounds and animations distracting. (see for example, Sandals, *op cit*, Malone, *op cit*)

Millheim (1993), for example, suggested the following guidelines for the use of animation in CAL material.

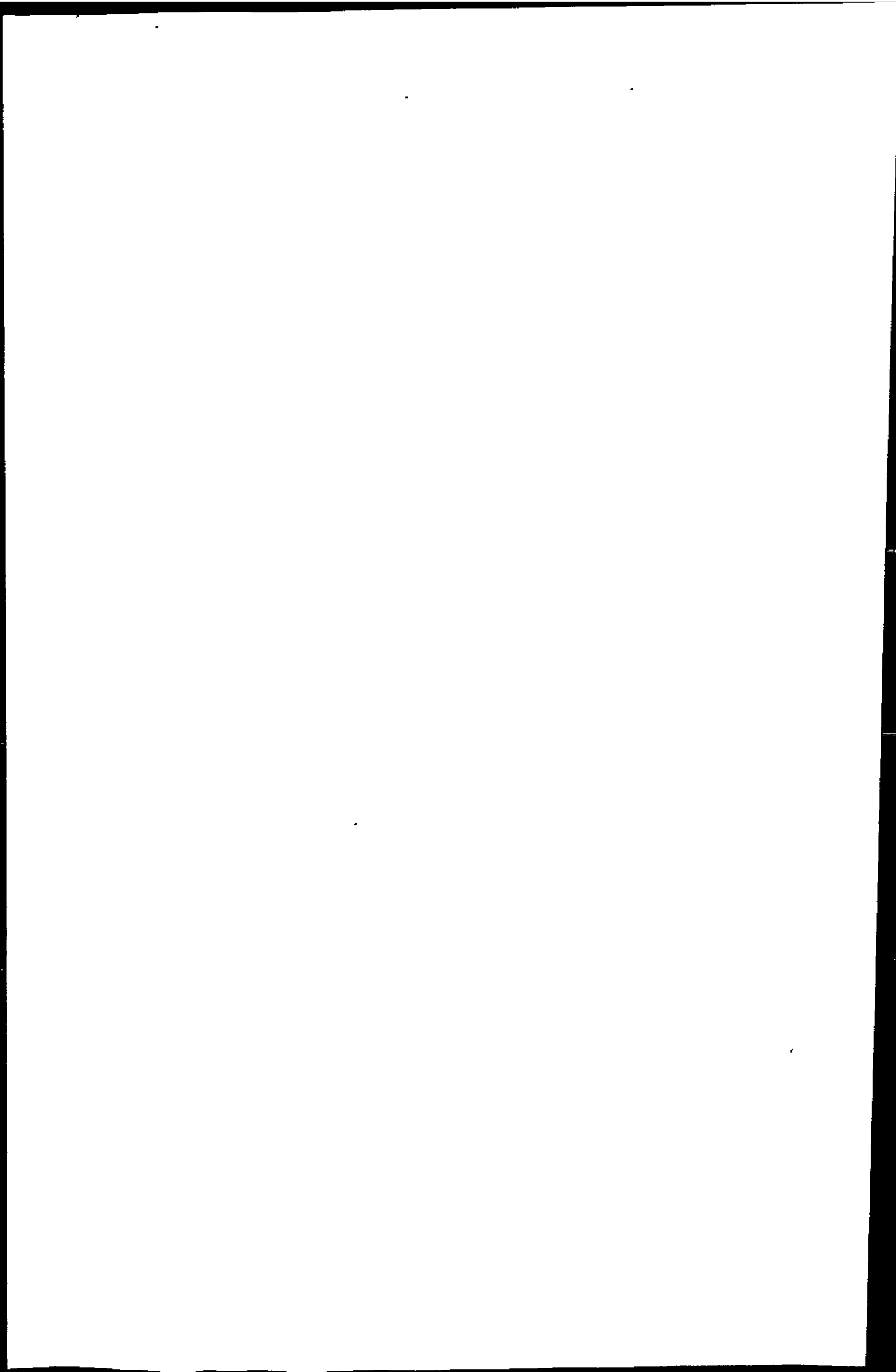
- Develop simpler animations rather than complicated ones.

- Design animation so that important information can easily be perceived.

- Include options for varying the speed of an animated presentation to provide emphasis at various points during the sequence.

- Use animation that relates directly to important objectives or features within an instructional lesson.

- Use animation when the instruction includes the use of motion or trajectory.



Use animation when the instruction requires visualisation, particularly with spatially-oriented information.

Use animation sequences to show otherwise invisible events.

Use coaching techniques that assist the learner in interpreting the simulation as well as free time before of after an animation sequence to allow for better understanding.

Use interactive, dynamic graphics.

Use animation to gain a learner's attention or increase motivation.

Avoid overuse of animation since it can be distracting to learners.

Avoid use of animation with novices who may be less able to attend to relevant details or cues within the sequence.

Millheim (1993)

12.7.7 Use of video

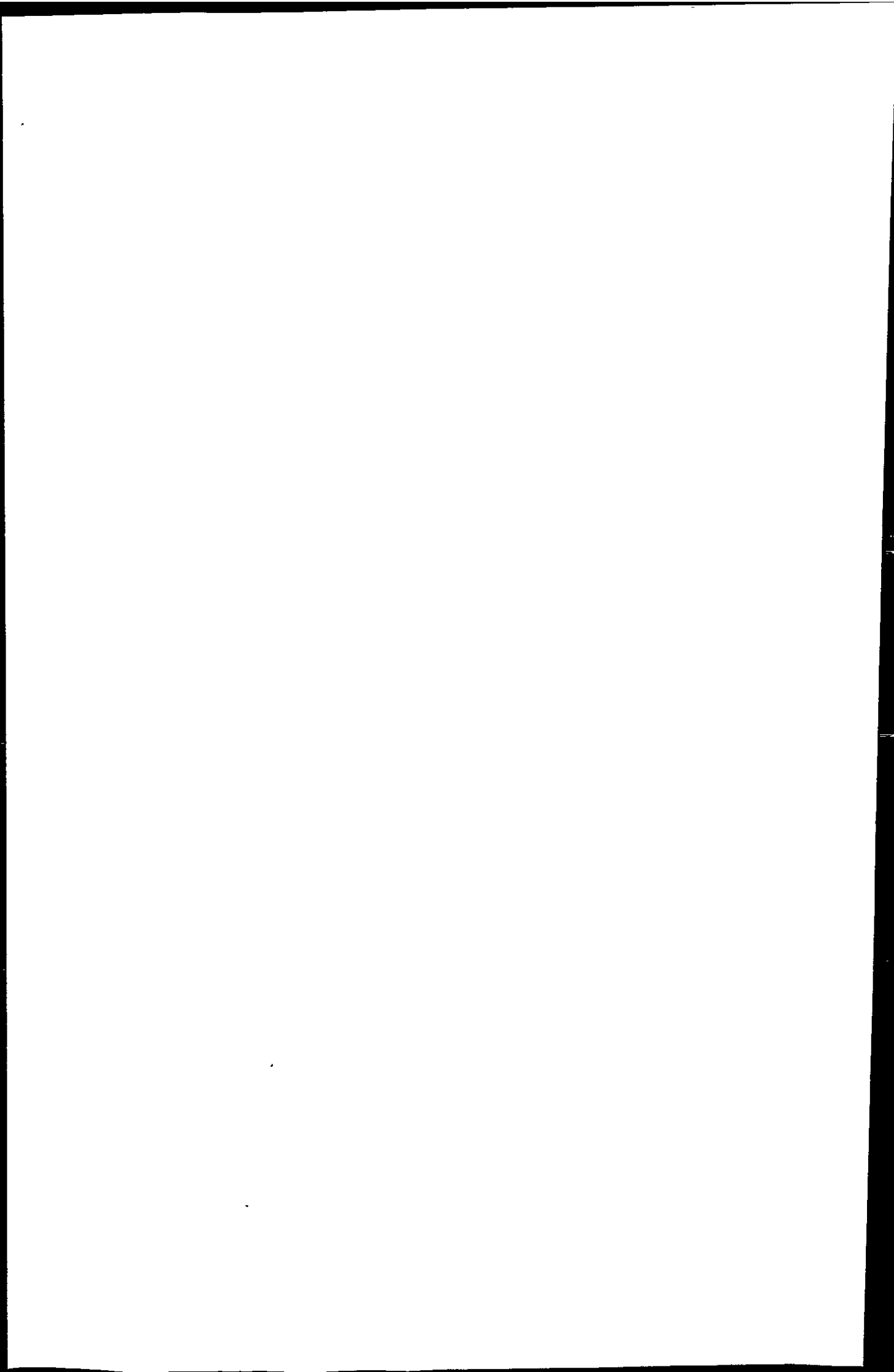
Graham (1991) suggests that video footage can overcome the difficulty of bringing "real" problems into the classroom and is more engaging than computer simulation.

The footage however should be shot under the direction of a scientist since panning, zooming and cutting make it difficult if not impossible to take meaningful measurements from the film. Applications such as weightlifting (as an example of sports science), motorcycle racing and cell division are suggested as appropriate.

Gautreau et al (1987) add that film of cars colliding with a wall and rebounding and two cars crashing (of the sort shot for vehicle safety tests) may be analysed and conclusions drawn about force and momentum. Measurements of the acceleration due to gravity on the moon can be made from a video sequence, their overall conclusion being that the use of video adds reality to a standard exercise.

12.8 *Matching instruction to learner*

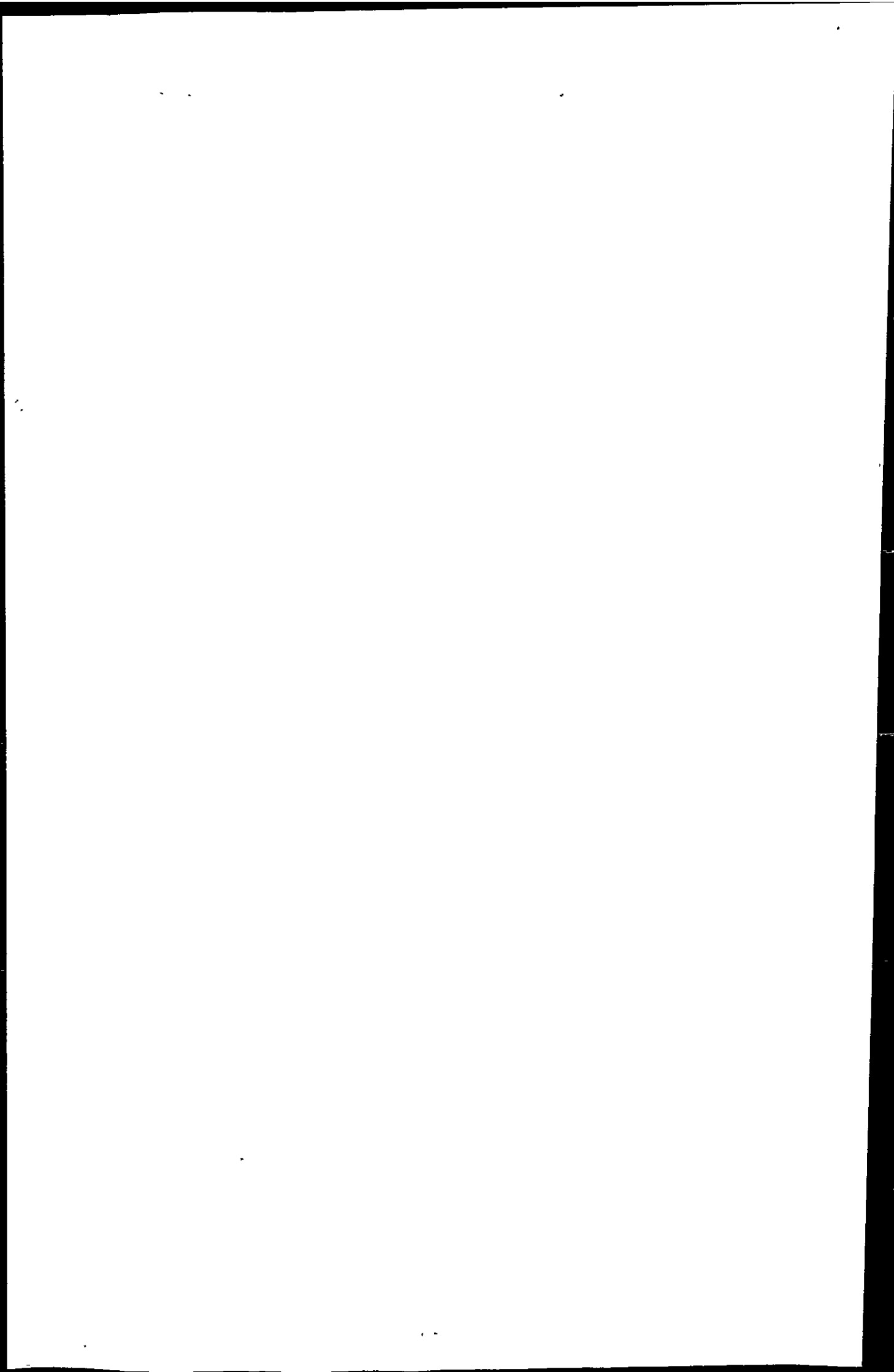
Dick (1981) predicted that if the increased complexity available in teaching technology were to be used effectively "it will require instructional design models which emphasise detailed alternative strategies of instruction for different types of



learner". Plowman (1989) describes such a system, and a possible means of picking one's strategy according to preferred learning style, but points out that even sorting learners according to two dimensions would require four different scripts. One difficulty with matching the instruction to the learner is that there are so many ways of categorising learners. The evidence that a prescriptive technique exists for successfully matching learner to material is at best sketchy.

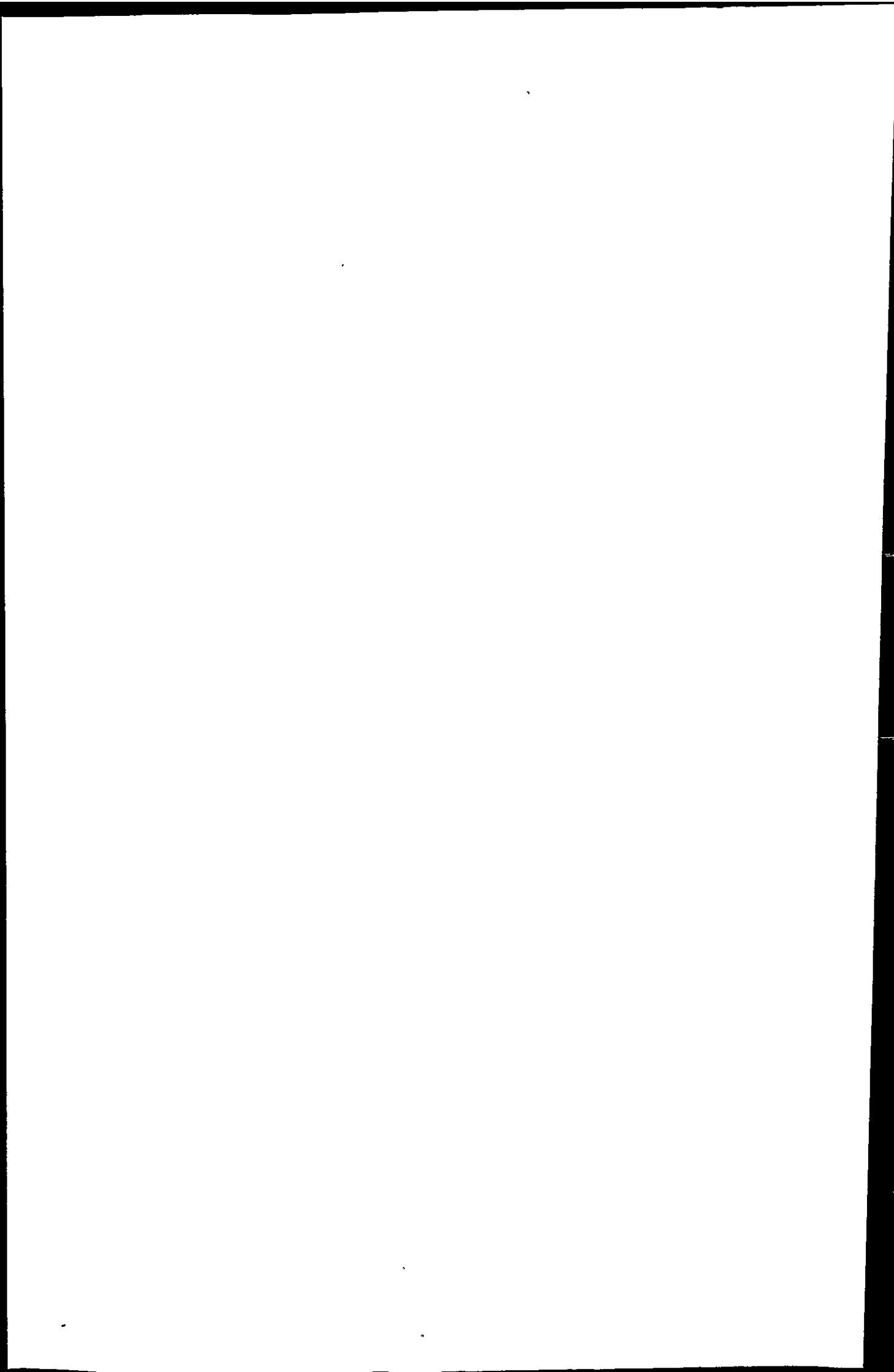
Several authors have explored the effects of matching or mismatching the instruction with the learner in various ways. Carlson (1991), for example, found that students with a deductive style had difficulties when clear instructions were not given: students with an inductive style were thought to prefer creating their own concepts after considering many examples but did not have as much difficulty as mismatched deductive students. No significant difference was found in the content learnt, but the sample size was relatively small (53 students).

Using the Myers-Briggs Type Indicator (MBTI, see Chapter 10) to categorise students, Matta and Kern (1991) found that sensing and introverted individuals tended to learn Lotus 123 better in an IV context than in a classroom, but their results were inconclusive. They eliminated individuals with significant prior experience of IV, programming and Lotus 123 which may have removed some personality types preferentially. Conwell et al (1987) postulated that Intuitive Thinking (NT) types would like a teaching style emphasising theory and logic, and Sensing Feeling (SF) learners would prefer factual knowledge and subjective experience. They found no significant differences in the change of scores on the pre- and post-tests between learners who were matched to the teaching style and those who were mismatched. Cooper and Millar (1991) found that in a college of business



the teaching staff tended to be Intuitive types and the students were predominantly Sensing types, and proposed that this clash of styles could be source of dissatisfaction. The field-dependent (FD) field-independent (FI) dimension of cognitive style is supposed to measure whether individuals rely primarily upon an external frame of reference (FD) or an internal frame (FI) in processing information. FI students are thought to use a more hypothesis-testing approach to problem solving, and FD students to prefer observation to gather information. Garlinger and Frank (1986) conducted a meta-analysis and review of studies on the effects of matching and mismatching teacher and student styles on this dimension, and found the overall effects were minimal. MacGregor et al (1988) studied 59 students in a remedial algebra class. Of these students, 44 were found to have an FD style, 14 intermediate and only one a field-independent style. They found CAI to be of more benefit to FD students than to those having an intermediate style: however the largest effect was that of the instructor. Abouserie et al (1992) found that FD university students had a more positive attitude to CAL in physiology than FI students, but that neither style nor attitude correlated significantly with achievement in the subject. The CAL type used was very structured, and the research begged the question whether a more hypertextual style would have appealed to FI types.

The Gregorc Style Delineator, which has two axes, concrete-abstract and sequential-random, was used by Lundstrom and Martin (1986) to determine the style of 132 psychology students. The effect of the teacher's preferred style was then investigated on the students' learning. They found no significant effects, and suggested that students were able to use their non-preferred style when the situation demanded it. Riding and Douglas (1993) suggest that whereas cognitive style is stable, and may be described on two dimensions, verbaliser/imager and wholist/analytic, cognitive



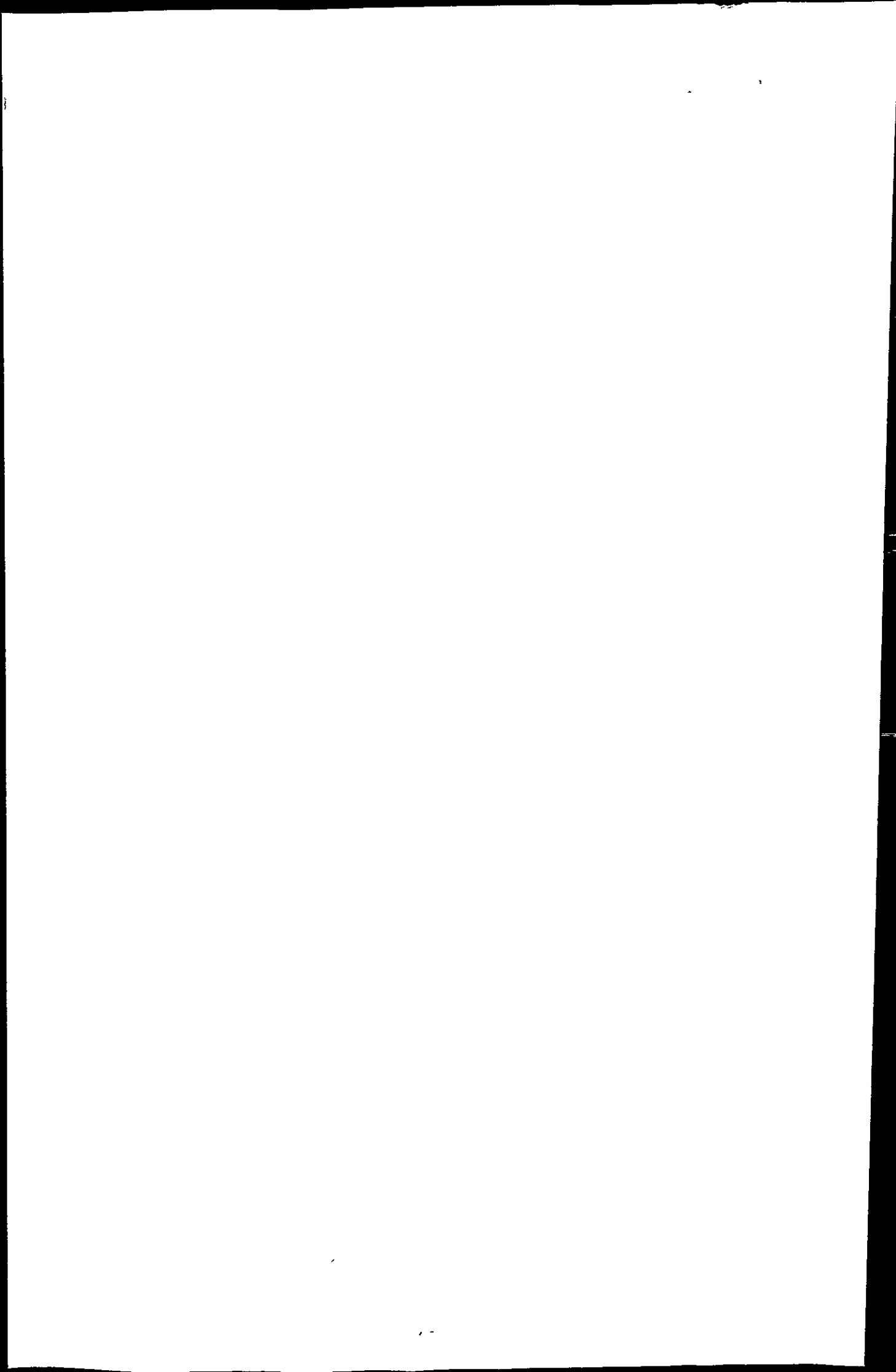
strategies are flexible and depend on circumstances. They found that even verbalisers recall material better when pictures are included in the text, and imagers recall much better when they have pictures to recall. Even when students are able to form mental visual images, Presmeg (1986) found that these images were discarded when the student was habituated to a procedure: that visualisation was used as a mnemonic aid.

Not all authors used cognitive or learning styles as a way of categorising learners. Malone (1981) found that girls did not like the fantasy of throwing darts at balloons to indicate their answers while boys did. He suggested that in such a context students should be allowed to choose their own fantasy.

Prosser (1987) suggested that prior subject knowledge was as important as study style in determining how students learnt. Students with prior knowledge in the general area of the new knowledge already had a framework into which to incorporate new learning. Students without prior knowledge had to depend on rote learning. Moran (1991) went further, suggesting that the student's prior knowledge in a domain and metacognitive skill were the predominant factors in learning and that the self-analysis provoked by learning styles research, leading to metacognition, was probably its most valuable result.

Jones (1993) suggests that full-time students find IV gave an interesting variation in delivery, whereas people in full-time employment tended to regard IV as television, and took a more relaxed attitude to material delivered through it.

Taking a developmental view of the young adult student, Perry (1988) describes the transformation of the relationship of the learner to knowledge. Authority, from having been the source of all knowledge becomes a resource, mentor and potential colleague in the consensual interpretation of reality. Students at different positions in



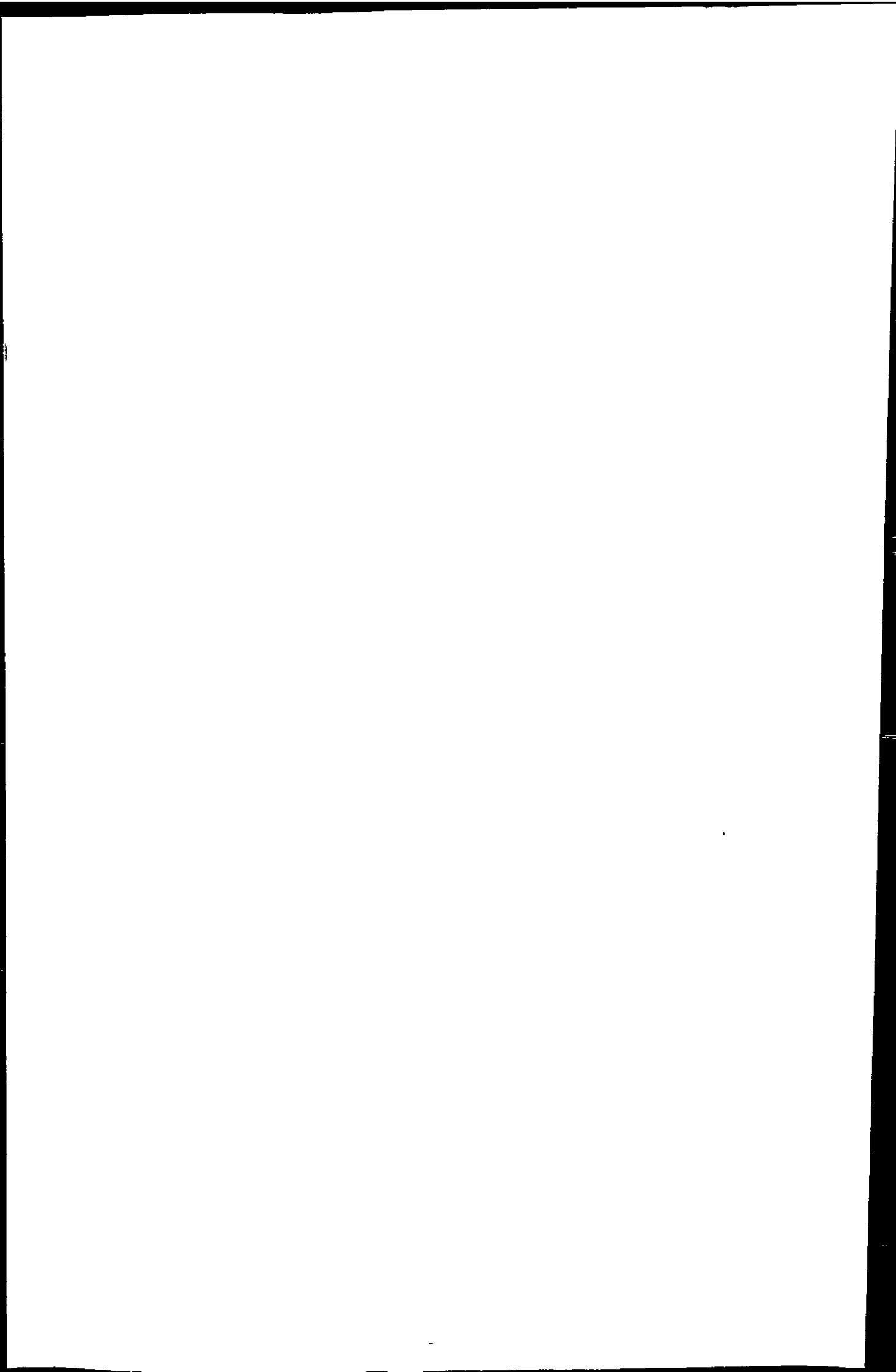
their journey will react differently to different styles of courseware, and the passage between positions in this development of meaning do not occur simultaneously in all the students in a classroom.

12.9 The role of the teacher/instructor/tutor

As mentioned above, MacGregor et al (1988) found the most important factor in determining the success of students was the instructor in whose group they found themselves. In other studies, for the sake of repeatability, the role of the tutor was deliberately minimised for the experiment. More recent authors have started to explore the importance of the teacher in learning with CAI/CAL materials, as the dream of the "perfect teaching machine" has faded.

Kurland and Kurland (1987) in the conclusion to their review paper, state that "The teacher remains the single most important instructional agent in the classroom, therefore students must respect and listen to their teacher." For Götz (1991), however, whose vision is that of a drop-in independent learning centre, the role of the instructor has disappeared entirely.

The general consensus is that the teacher's role is to assess the individual student's weaknesses and misunderstandings (Mackie 1992), to assess the knowledge and the needs of the individual and whether the system can meet them, to choose appropriate topics for the user, to decide the appropriate level of depth and channel for information, to provide connective tissue to join topics (Midoro et al, 1988) to guide the student's attention to features of the situation that are invariant and therefore meaningful across a class of situations, to plan the assistance students will need (Young 1995). In other words, the instructor is the best interface to match the individual student to appropriate instruction.



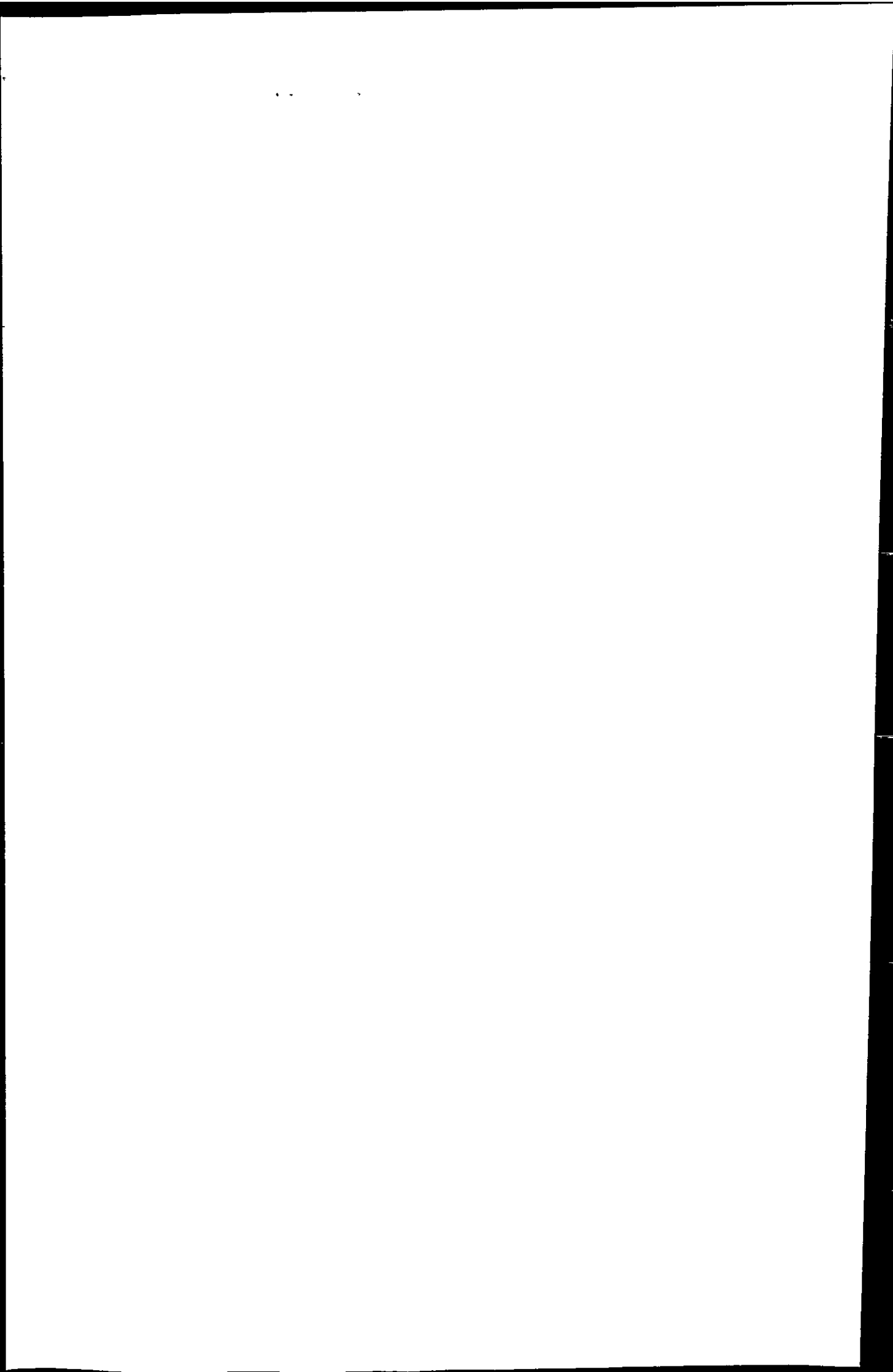
According to Jones (1993), the major key to the success of IV is the commitment of the tutor. This paragon should be familiar with contents and means of delivery, decide best format for delivery, decide what supporting structure is needed and decide how to evaluate effectiveness of learning.

For Laridon (1990 a, b), the instructor must have the same learning paradigm as the courseware, and that paradigm will determine the role as it has done historically.

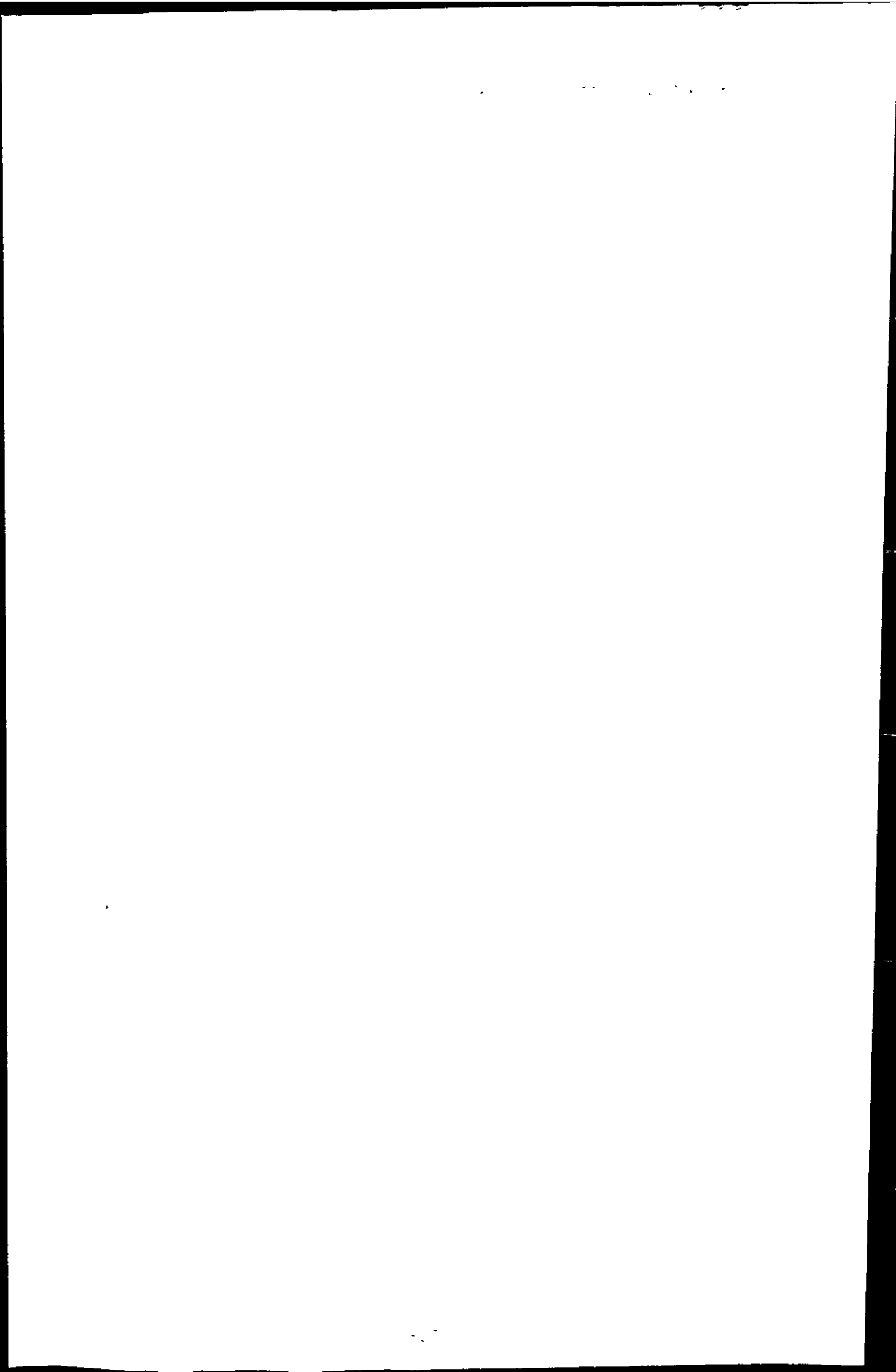
12.10 Computers in engineering education

Computers have been used as teaching aids for mathematics for engineers in a number of ways (see Maull et al, 1995). These include the following.

- a) the computer as a dumb tutor in a drill and practice session, providing a stream of questions and responding right or wrong as appropriate. The student is fed examples until he/she demonstrates an ability to perform which satisfies a pre-set criterion. This is often seen in the context of a mastery learning didactic paradigm. (e.g. Rae, 1993)
- b) intelligent tutoring where the program attempts to diagnose the particular misconceptions held by the student and difficulties held by the student according to his/her responses to mathematical questions. This is still in its infancy, but see, for example, Laurillard 1987, Khasawneh, 1994.
- c) programming, where the student designs and writes programs to solve particular classes of mathematical problem. The argument is that the student thereby develops a deeper understanding of the process involved by analysing it logically and reproducing it in terms of code. (e.g. Adams & Stephens, 1991)
- d) the application of in-house produced software which solve particular classes of mathematical problems, to enable students to check their solutions against the



- computer's, and facilitate self-marking of exercises. (e.g. Coull & Simmonds, 1988)
- e) the use of commercially available software to scaffold the student's exploration of mathematics by showing the solution to problems the student is not yet able to tackle by hand and demonstrate their relationship to maths already known. See, for example, Kutzler, 1994. Lawson (1995) points out that there are competing learning objectives in this context: using the package and understanding the mathematical contents of the worksheet.
 - f) the use of software to perform tedious calculation so that results can quickly be obtained and generalised. For example, the plotting of a family of curves to explore the effect of varying parameters. (e.g. Watkins, 1993)
 - g) the use of spreadsheets to perform iterative calculations and to find approximate numerical solutions in a manner transparent to the student. See for example, Lee et al 1987, Arganbright, 1993, or Fraser & Thorpe, 1994.
 - h) microworlds and simulations where hypotheses can be explored and tested by the student in a mutually safe (unthreatening and unbreakable) environment. See for example, DeCorte, 1994, Abel, 1990, Whitelock et al, 1993, Lindstrom et al, 1993.
 - i) use of a simulation language such as STELLA to allow students to create their own simulations. The assumptions made in the simulation are made explicit, and the students can check the behaviour of their models against their experience of reality.
 - j) use of the computer to drive recorded teaching material, for example on interactive video or CD.



k) hypertext or hypermedia environments to be explored by the student in pursuit of information.

m) live production of audio-visual material in the course of and in support of lectures.

(e.g. Marshman & Ponzio, 1987)

Smith (1992) surveyed the use of computers across 72 departments of engineering and classified the use according to 18 categories. 16000 students in the survey used the computers for an average of about 20 hours each per year. The overall use in student hours for some of the categories is shown below. The difference between "instructional" type software and software tools puts the arguments about this software firmly in context. In the study of engineering, computers were used as tools for doing, not tools for learning.

Category	Student-hours
Hypermedia	180
IV	300
CAD/CAM	36632
Spreadsheets	43405
Word processing	52806

Table 12-4: Use of computers by engineering students (Smith, 1992)

On the other hand, as Smith points out, "the prospect of being able to replace the drudgery of supervising tutorial examples classes must be highly attractive to most academics".

12.11 Choice of authoring package

Serious thought was given to the package which would be used to author the package, and of those available the best contenders were Authorware, Visual Basic and Director.

Authorware is specifically intended for the authoring of courseware and is recommended by Beevers et al (1992) as a package for authoring mathematical CAL materials. The design process consists of the building of a flowchart which imposes a sequence on the contents. The flowchart may be made to branch according to the student's responses to questions: either feedback to right/wrong answers, or according to the student's preferences. The courseware is intended to be interactive to the extent that the computer's responses at each stage depend on those programmed to anticipate the student's input. The student, however, has no control over the path through the material: the sequence is determined by the program. This corresponds to either an imposed sequence, or a limited adaptive design. (Hannafin & Phillips, 1987)

Visual Basic is a sophisticated general-purpose package for designing packages to run in a Windows environment. The screen design has a very Windows "feel" in that the objects created and manipulated by the package such as buttons, scroll bars and windows have a similar appearance to those encountered in familiar Windows packages such as Word, Excel, etc. Programs are event-based, in that the objects on-screen react to events such as clicking on buttons or scrollbars, or typing in a response in a box. The mathematical facilities of Basic mean that it can be used to generate drill and practice examples and to calculate correct answers, and it can also be programmed to take into account previous right/wrong answers when generating subsequent questions. The package may be used to write adaptive programs, sensitive to learner differences, or indeed programs with an imposed sequence or with total student freedom. (Hannafin & Phillips, 1987)

Macromind Director is an authoring package for animation, presentation and interactive multimedia. It can be used to build either linear, sequential or branching

packages or programs allowing random access at any point through an indexing mechanism, or indeed a mixture of the two. It can incorporate different media into presentations: sound, images, movies or animations, and can also be scripted to allow for interactivity. It has no inbuilt facility for such things as collecting a student's scores in tests, or drawing graphs, but these could be written. Buttons and other control features within a Director movie have to be designed by the author from scratch, giving great freedom, but more work than Visual Basic. Its particular strength from our point of view lies in designing "hot" areas of the screen which can be scripted in various ways to control the flow of the program.

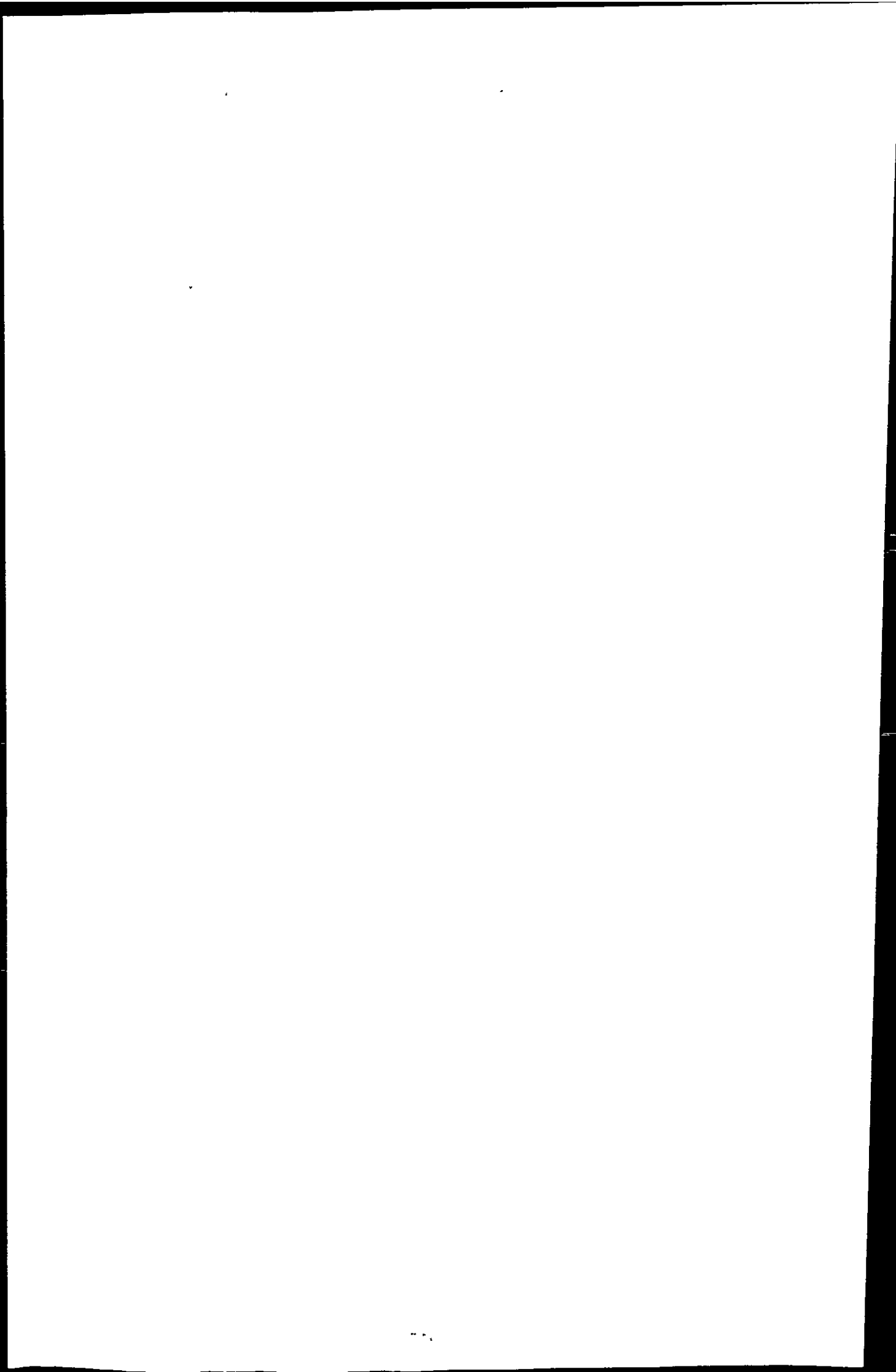
The final package considered was Hypercard, which runs on Macintosh computers only, and only in black and white. It is extremely flexible, and has been used to write successful hypertextual multimedia packages. In the end the availability of Director and a suitable PC on which to run it, together with the colour facility of Director and the author's greater familiarity with the PC led to the choice of Director as authoring package.

12.12 Conclusions.

From the wealth of literature, the following overall conclusions were drawn.

What is learnt depends largely on the individual student. Important factors are the student's prior knowledge and metacognitive skills, and the meaning attributed to the package by the student.

Whereas drill and practice exercises are regarded as an efficient means for the development of skills, particularly simple skills, it was felt that in the context of enriching concepts a more hypertextual format was appropriate.



There is little evidence that any measured factor is a good predictor of how well a given student will learn with a given package, and it is probably better to allow the student leeway in deciding how to proceed through the package, while ensuring that there is enough structure to ensure the student does not feel lost.

The working of the package should be transparent to the student, that is the student should not be expected to understand how it works except as a metaphor.

Appropriate graphics will include realistic illustrations to allow the student to orient the new learning within existing knowledge and symbolic illustrations to structure the new learning. Animation and sound will be used sparingly.

11

13. Mathematical modelling

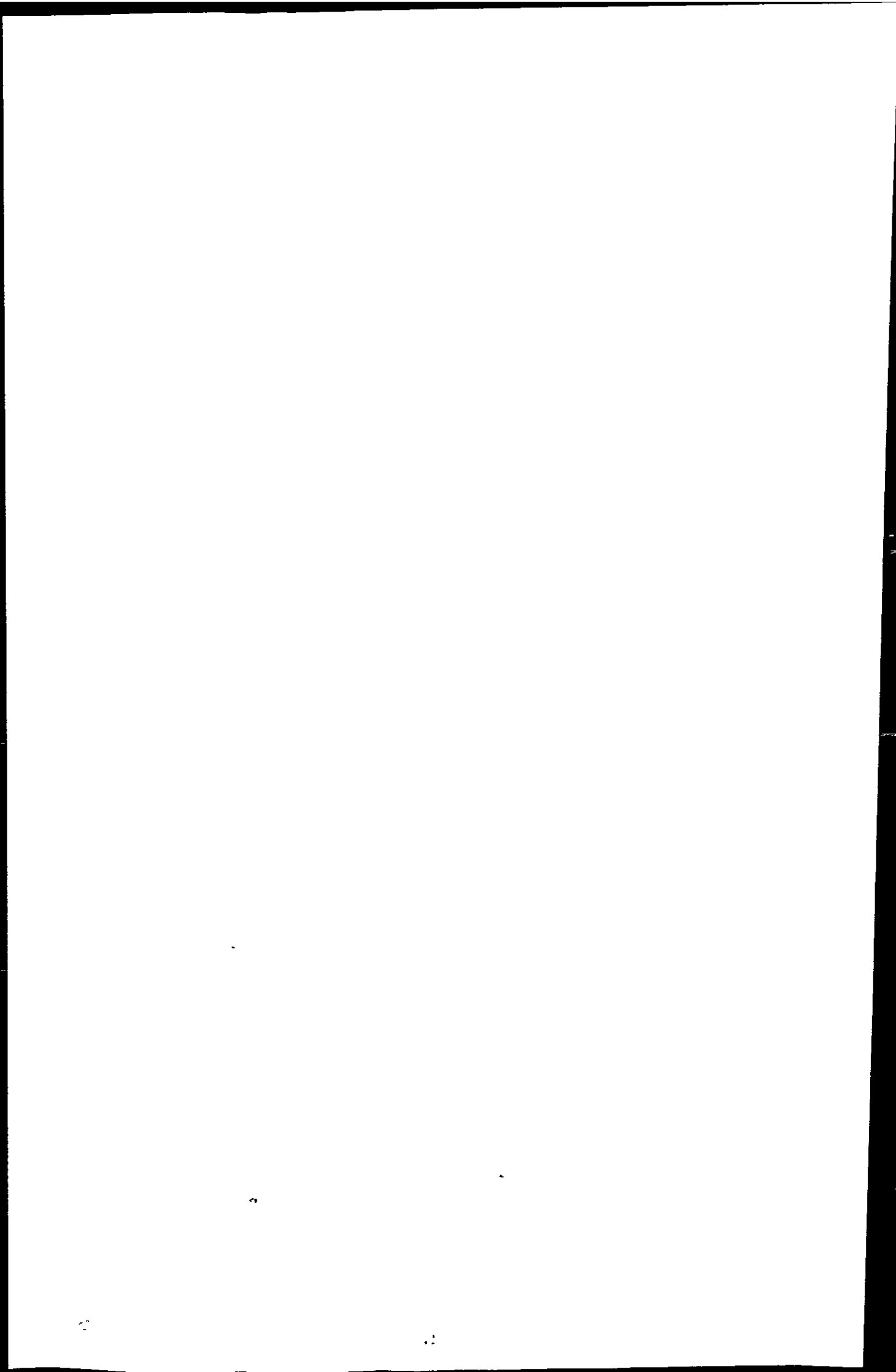
13.1 Introduction

We have seen in chapter 2 how mathematics has been construed and interpreted by different authors. In the present chapter, we concentrate on one aspect: the application of mathematics through mathematical modelling. Modelling is the context in which most engineers use mathematics in their working lives, and it was found from the results of the questionnaire (see chapter 6) that the greatest maturing of engineering students' mathematical concepts occurred when they were seen in the context of their applications. Since the aim of the courseware was to promote such a maturing of concepts, it was felt that presenting them in the context of mathematical modelling might achieve that aim. The engineering students who were observed carrying out a modelling exercise (reported in chapter 3) did not appear to be making a connection between setting up and solving mathematics and modelling a physical system. The way that the modelling structure was presented and used in the courseware will be described in the next chapter.

The particular relevance to most engineers is in the modelling of physical systems although statistical modelling is of increasing importance in production engineering.

13.2 What is mathematical modelling?

It may fairly be said that every mathematical formulation of a general law is a mathematical model, whether it is a physical law such as Hooke's law, an economic law or a population law. Each of these models has been obtained by a modelling process. Even when individual values for some property or quantity (number of fish in a tank, extension at a given load) may be measured, the process of interpolation to



give intermediate values is itself a modelling process, involving the making of assumptions about the situation.

Mathematical modelling is distinguished from pure mathematics in that the questions it addresses are based in some way in the real world (see Figure 13-1, McLone, 1984). It differs from classical applied mathematics in that the modeller has to build the connection between the real world and the mathematical world without necessarily the blueprint of an existing law as a guide. It is the activity of the authors of laws. (Whether the laws were invented or discovered is too big a question to argue here.)

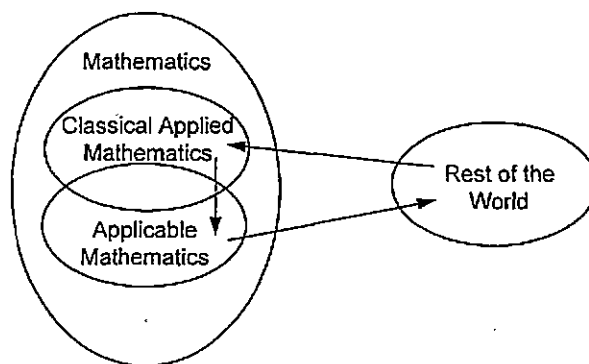


Figure 13-1: McLone, 1984

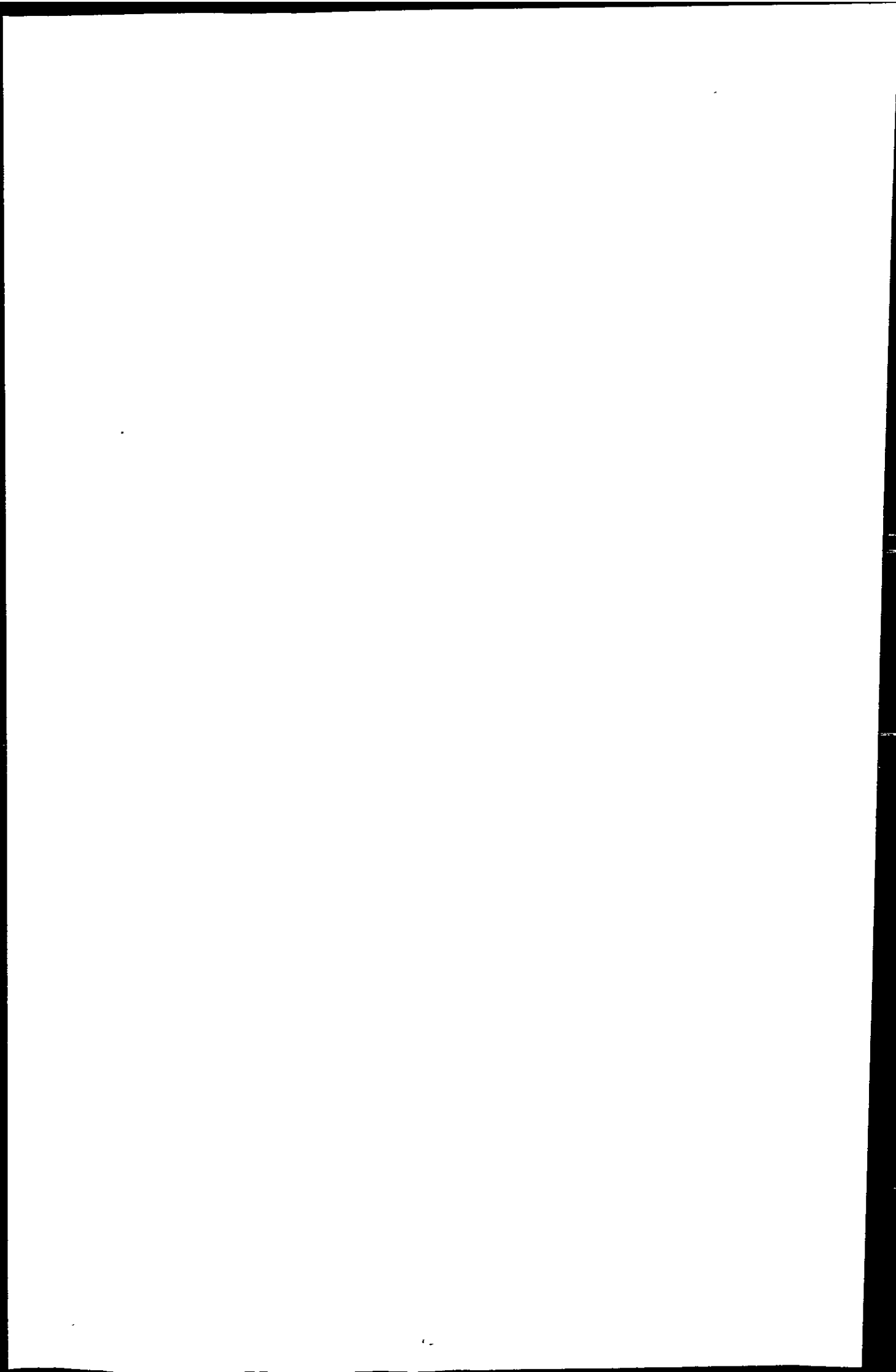
13.3 *Mathematical modelling and engineering*

It is generally agreed that mathematical modelling is important for the engineer. This reflects the idea that mathematics is a tool for the engineer, and a language for the description of real or potentially real entities (for example stresses in bridges, flow in pipes, traffic flows) in abstract terms.

Modelling and engineering applications

There is a cornerstone requirement for engineers to model and to be able to solve modelled problems. Most writers agree that mathematical modelling is perhaps the most constricted psychological bottleneck in the entire mathematical learning process and that the debate upon how to teach it is likely to continue and may never be resolved. Opinions among educators are not surprisingly divided between those who favour a top-down approach followed by skill learning to those who feel that skills are paramount and that the teaching of modelling is vague and wasteful. ... Many forms of modelling are essential to engineer formation.

SEFI, 1992, pp 20-21



To engineers, mathematics is a way of expressing physics and engineering precisely. It is rare that one needs mathematics pure and simple. One finds then that in looking for examples where mathematics is needed one is frequently encountering the simultaneous need for physical understanding.

IChemE/ICE/IEE/IMC/IMechE/LMS/IMA, 1995, p13

In the particular context of engineering education, mathematical modelling has at least four important aspects: modelling is the way that practising engineers use mathematics, so engineering students need to be able to do it; and seeing mathematics used in context enables students to see its relevance, which is a strong motivating factor to help them overcome reluctance to tackle a subject seen as difficult. A third aspect is that if the subject matter is seen as relevant, the mathematics lecturer is then seen as someone who teaches useful mathematics, an attitude which spills over into other parts of the mathematics syllabus. Finally, as Shaw (1989) points out, the building of a satisfactory mathematical model may be the source of great satisfaction, the creative "eureka" experience.

13.4 The process of mathematical modelling

The process of mathematical modelling is normally described in terms of stages.

Authors vary in the way they divide the stages, and in the extent to which they include formulation and verification stages.

Mathematical models

The application of mathematics to physical problems involves three stages:

- a) Idealization of a physical situation and formulation in mathematical terms
- b) Manipulation of the mathematical symbolism
- c) Interpretation of the results in physical terms.

In a particular pilot questionnaire which we sent to about 100 engineers we asked which of these three stages they found most difficult, and 70 per cent said the formulation of the problem. A contributory reason may be that much mathematics teaching stresses the manipulative aspect (b) above at the expense of the model building aspect (a) and in addition fails to relate different models to each other.

OECD, 1966, p33

Mathematical models...

There are three essential steps in the solution of a problem in applied mathematics. In the first step the problem is stated in mathematical terms. This

means that the relevant variables are identified and that mathematical relationships are identified between them, either by using physical laws or empirical evidence, or by hypothesis. The second step consists of the solution of the mathematical relationships, either by standard mathematical techniques, or, if these prove intractable, by numerical methods with the aid of a computer. Finally, in the third step, the solution is expressed in a form which enables one to interpret it and draw physical conclusions from it.

Jaeger & Starfield, 1974, p1

More recent publications tend to describe the process in diagrammatic form. Some follow flow chart conventions with activities in boxes joined by arrows, while others put nouns in boxes and label the arrows with verbs to describe the process by which one moves from one state to another. Whatever the convention, the diagrams have a strong family resemblance, as can be seen from the following sections.

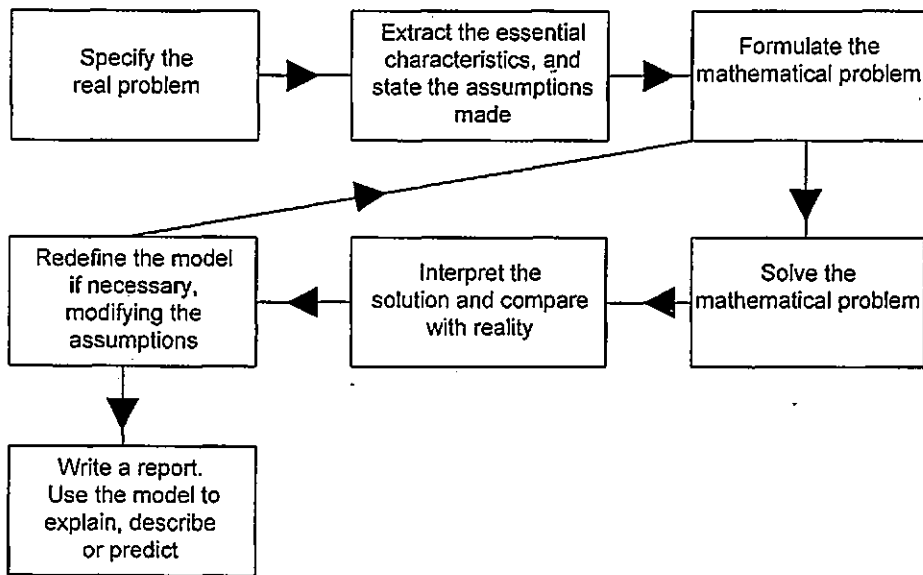
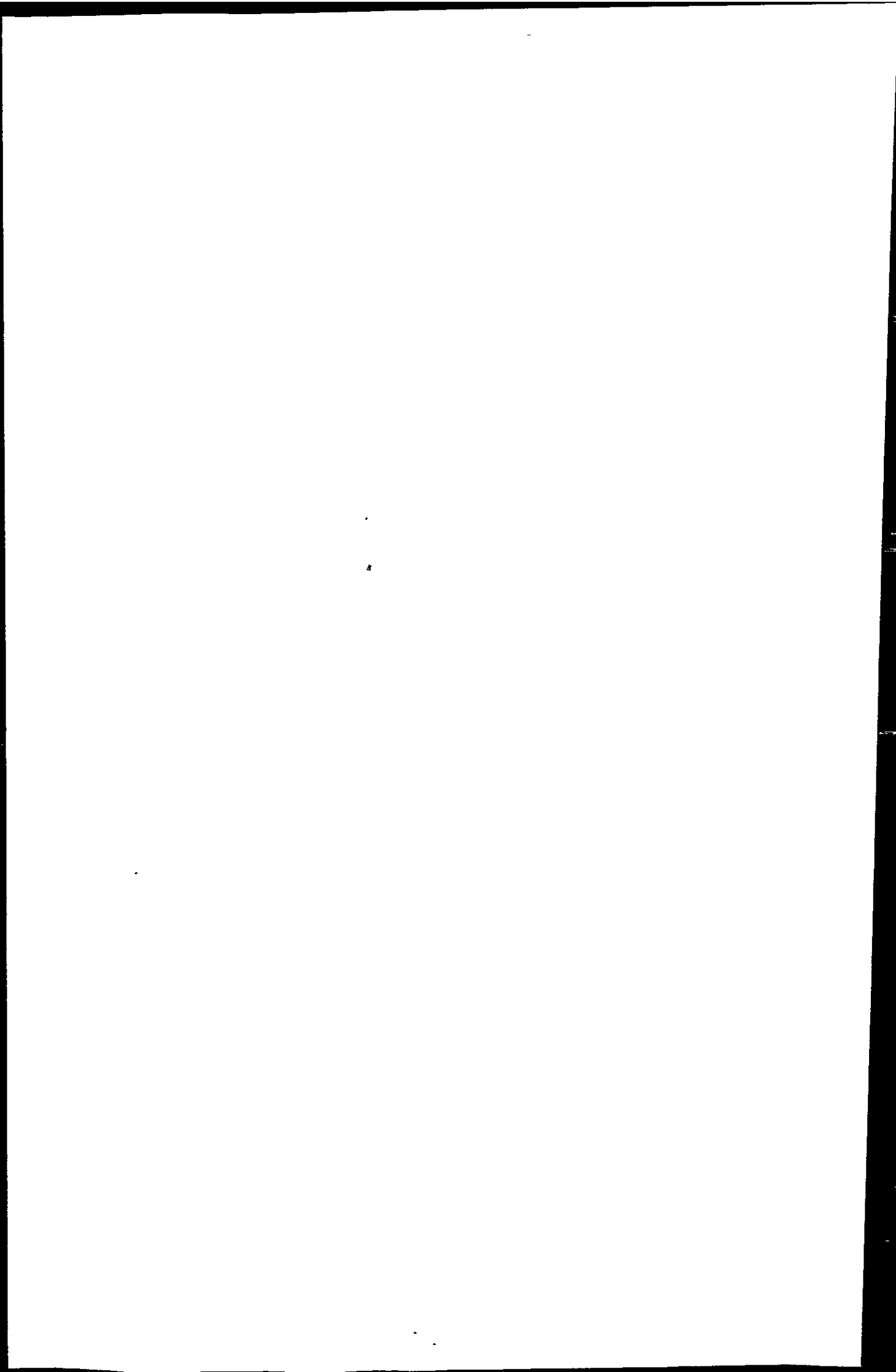


Figure 13-2: Hart & Croft, 1988

13.5 Modelling paradigms

There may be said to be two types of models: those intended to predict behaviour and those intended to lead to understanding behaviour. For the first type, an acceptable degree of precision over the field considered is more important than describing the general shape of the behaviour, whereas for the second, it is the general shape which allows one to describe the phenomenon and the processes underlying it,



although the model may need parameters to be determined before it is useful for prediction.

To some extent corresponding to these two motivations for modelling, there are two dominant paradigms of mathematical modelling, shown by the two sides of the MEI mathematical modelling flowchart. (Figure 13-3) (MEI, 1994).

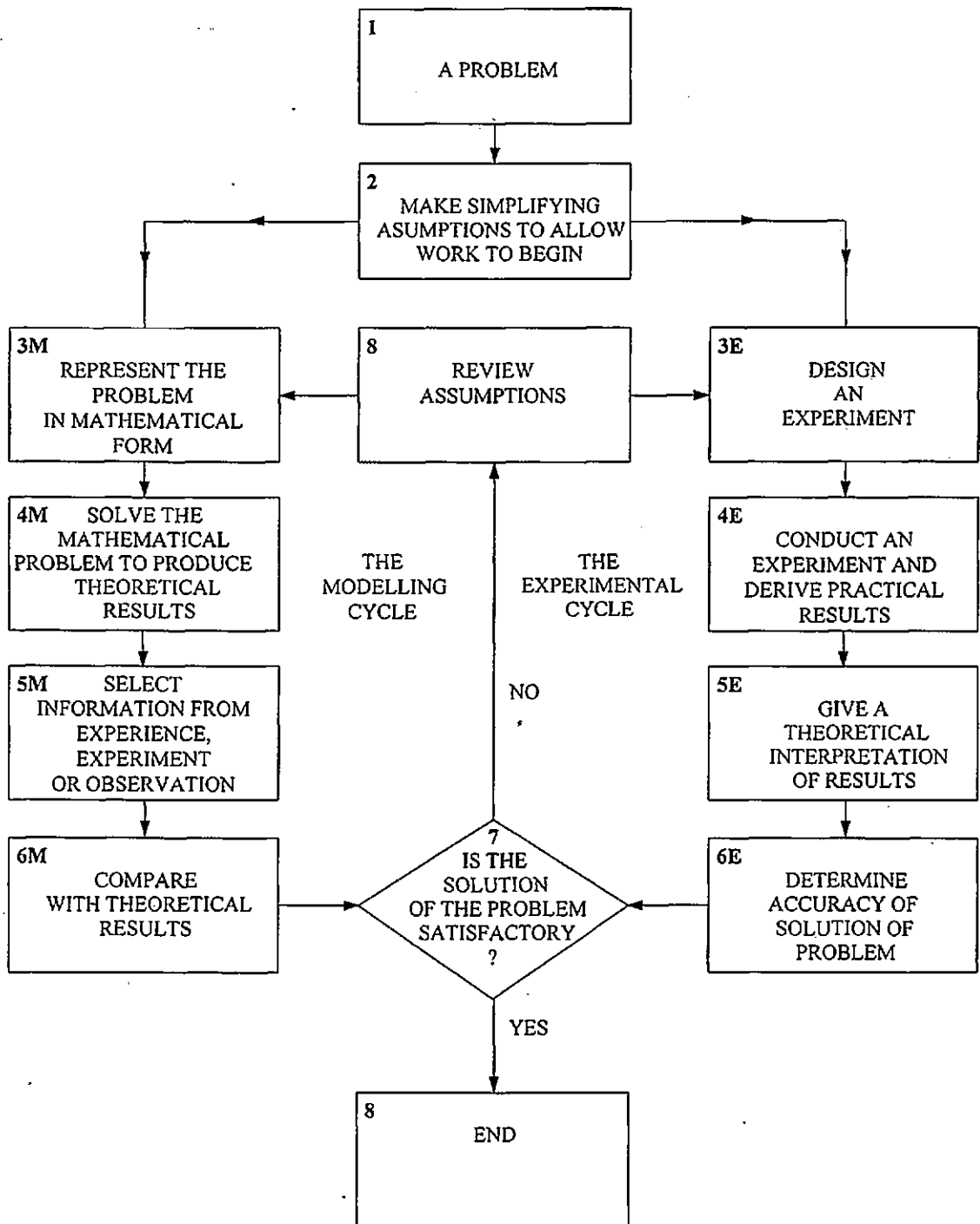
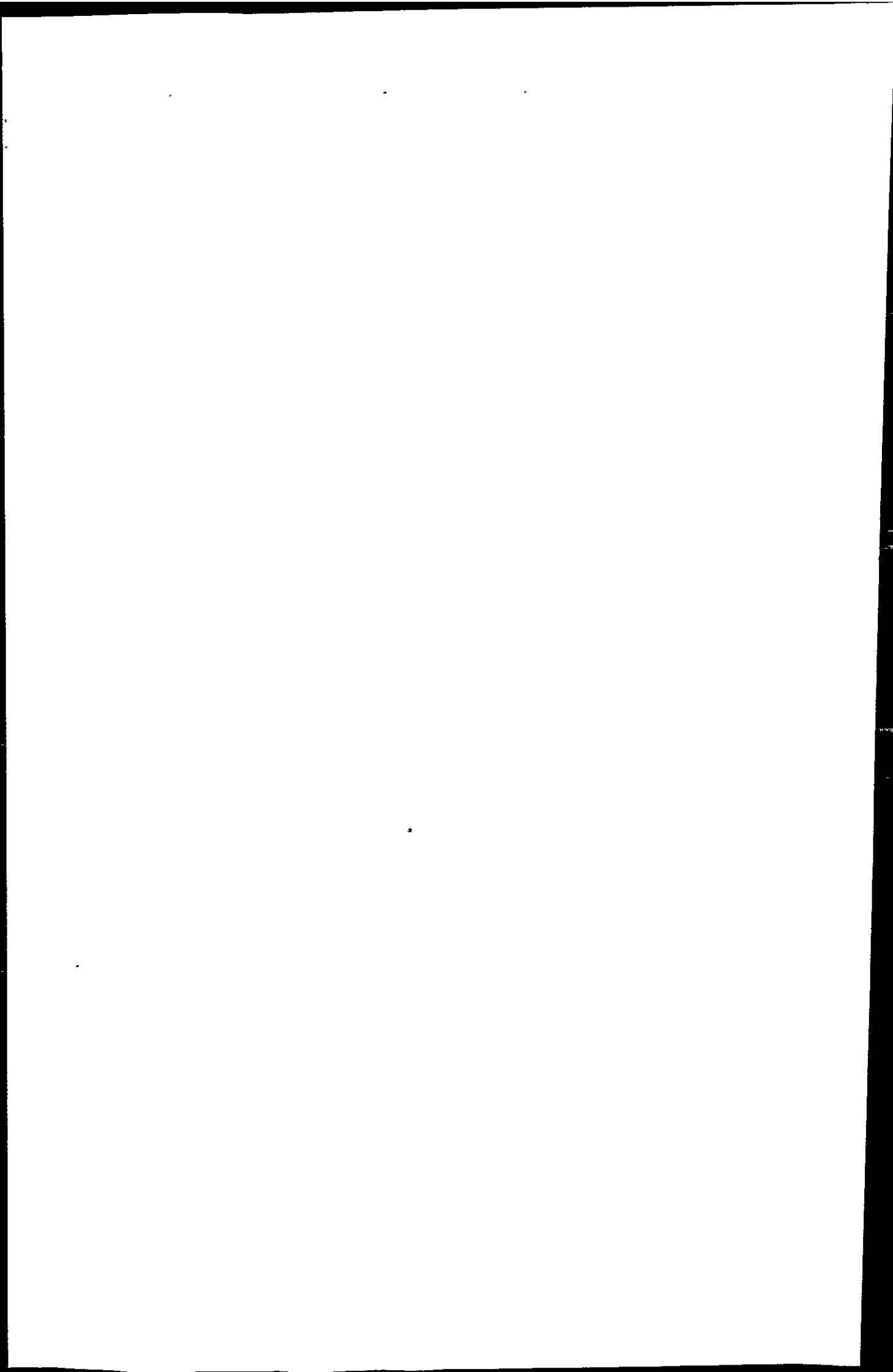


Figure 13-3: MEI mathematical modelling flowchart (MEI, 1994)

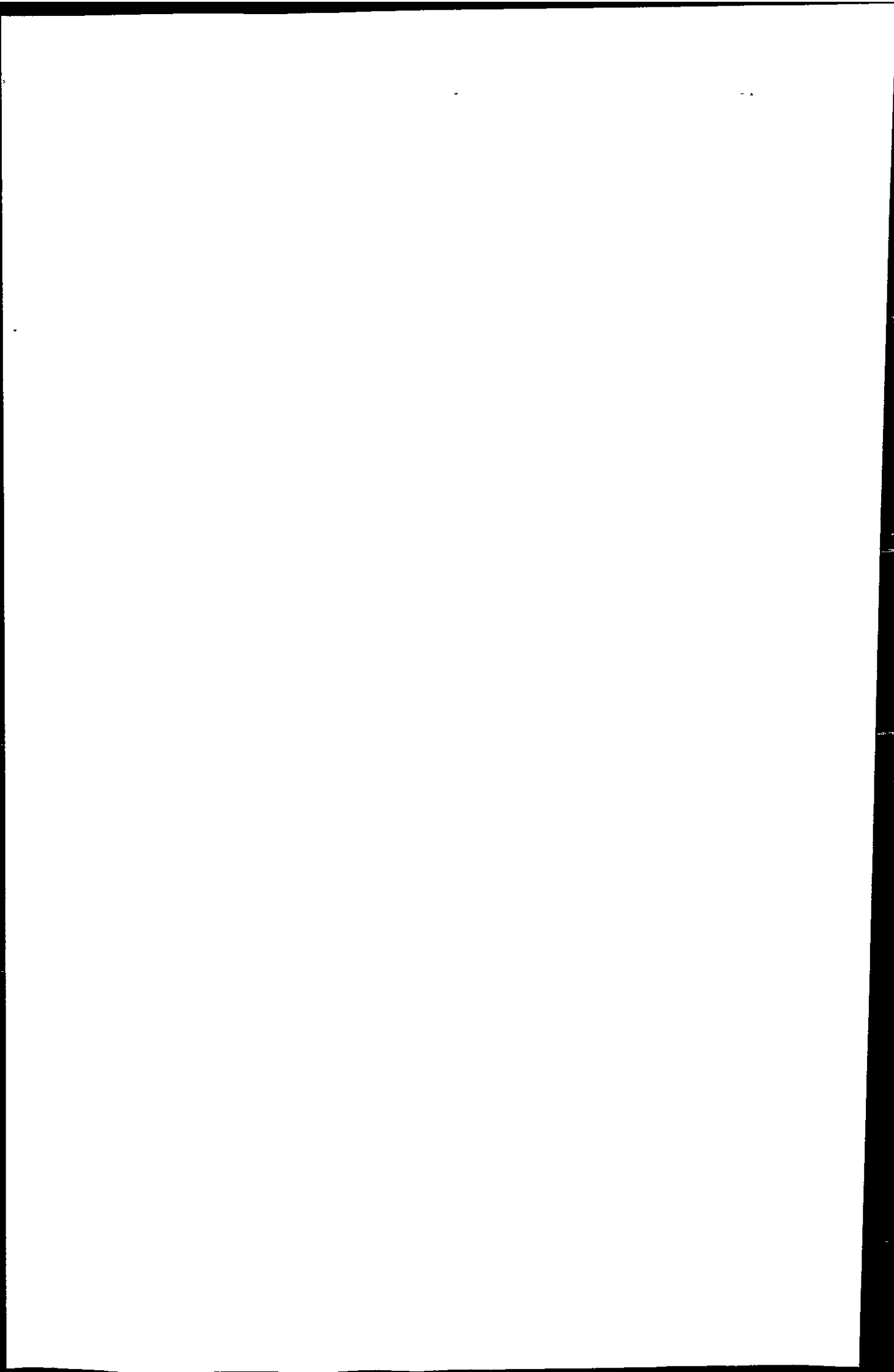


The right-hand path (3E-6E) represents the *empirical* paradigm, which suggests that data is collected, and its form is studied. A curve fitting exercise allows the modeller to suggest the law obeyed by the data, and the thoughtful modeller may then suggest the origins of the parameters. The weakness of this paradigm is that a real phenomenon has to exist and to be observed in order for data to be collected before it is fitted. Its strength lies in the motivation of having observed the phenomenon and seen what it really does, allowing the mathematics to be seen as relevant and meaningful. Empirical models are often employed for prediction rather than understanding since they do not probe the underlying relationships of the phenomenon observed.

The other (represented by the left-hand path, boxes 3M-6M) is the *theoretical* paradigm, which suggests that the processes underlying the phenomenon be studied, appropriate laws suggested, and the laws compared with the data obtained. This paradigm is also analogous to classic scientific method, where a hypothesis is stated, and its implications considered, then a crucial experiment is devised and carried out to test whether the implied consequences come to pass.

It is suggested (e.g., Tunnicliffe, 1981, p11) that the latter paradigm is appropriate where the model is intended to enhance understanding of a physical situation, and this is the paradigm that is used in the case studies described in the package and in this thesis.

However it is important to remember that the bases of physical science were descriptive, that for instance our understanding of, for example, gravity is empirical, and all our models of gravitational attraction derive from observation rather than an understanding of the physical processes at work. The same has been true of many



other areas where the technology (this is how the thing behaves, so we can use it) has led the science (this is why the thing behaves this way).

The two paradigms may also be compared with Kolb's (1981) learning cycle. The empirical paradigm may be said to begin with concrete experience, and the theoretical with abstract conceptualisation. Ideally, as in the MEI diagram, the whole circuit should be completed, but often the students are kept on one side or other of the laboratory/classroom boundary (my addition to diagram).

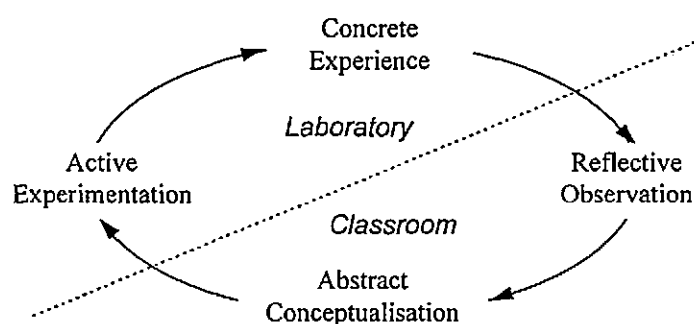
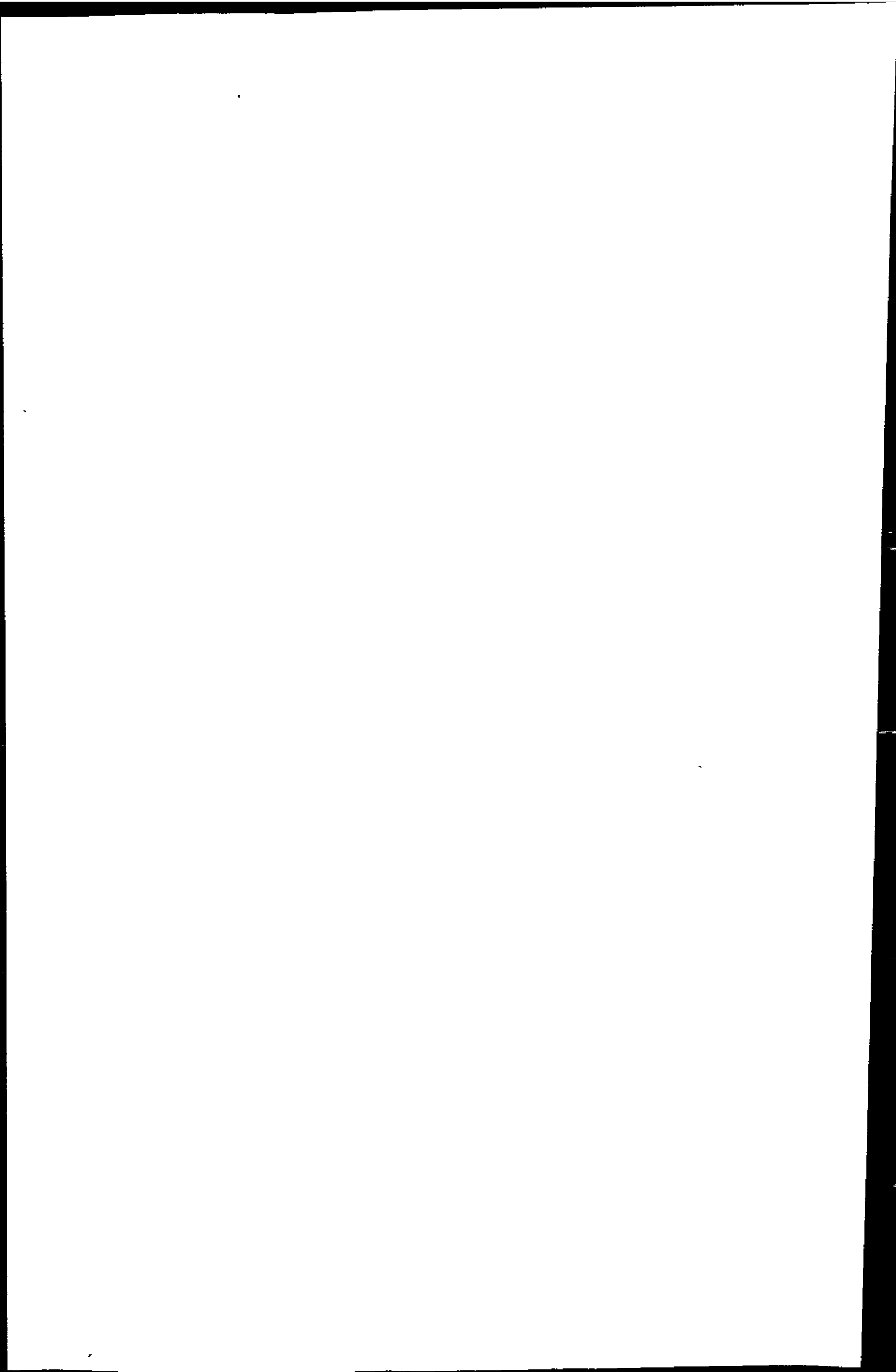


Figure 13-4: Kolb's experiential learning diagram (Kolb, 1981)

13.6 *The theoretical paradigm*

The whole structure of the courseware in this project is based on the Open University (OU) mathematical modelling flowchart. (Berry & Houston, 1995, Tunnicliffe, 1981, possibly derived from Penrose, 1978) This flowchart employs the theoretical paradigm, but as we shall see, it is not sufficient to look at the flowchart to know what paradigm is being used.



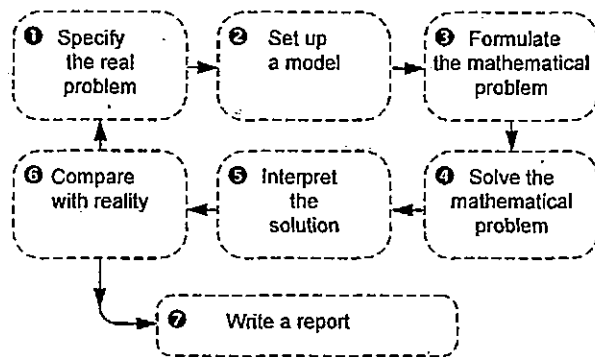
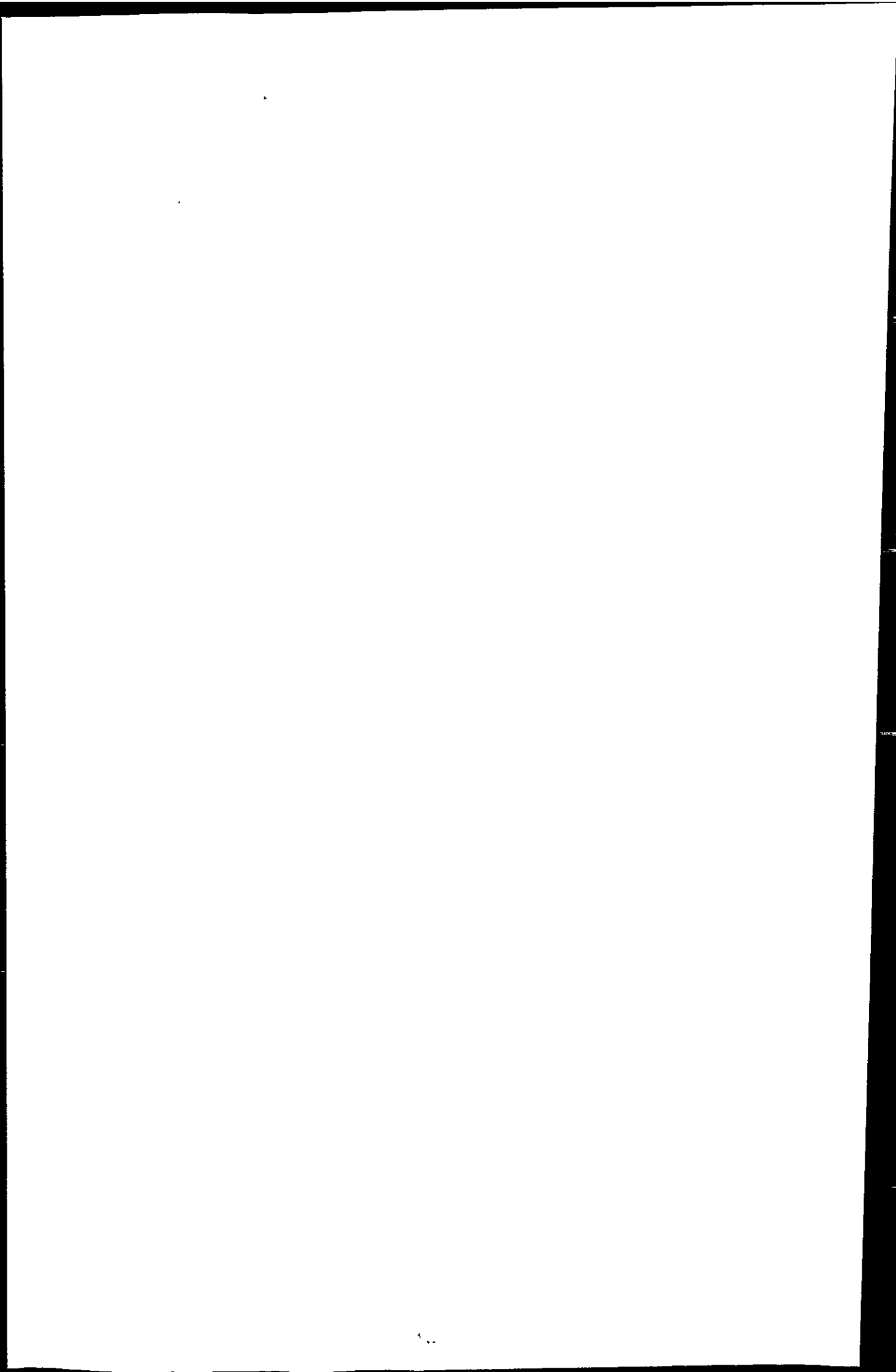


Figure 13-5: OU flowchart analysing the process of mathematical modelling, (Tunnicliffe, 1981, p5)

In the OU unit on animal populations, Tunnicliffe introduces data at the start to demonstrate how animal populations may change over time, as many students may not be familiar with the phenomenon. This introduction of data then leads to a discussion of an empirical, curve-fitting model, before theoretical models are introduced. This can lead to confusion in the students' minds as to what is being described in the flowchart.

13.6.1 Two worlds

Galbraith and Haines (1997) modify the OU flowchart, promoting "Refining model" to a box of its own, and putting all the mathematics in the same box. They divide the universe into the real world and the mathematical world, where "Formulating model" and "Evaluating solution" lie on the interface between the worlds. This notion of the modelling cycle moving between the real world and the world of theory, then back again, is a useful one, but their interest is in the mathematical part of the cycle, which means that they consider neither the way that mathematics is extracted from the real world, nor how the model is evaluated.



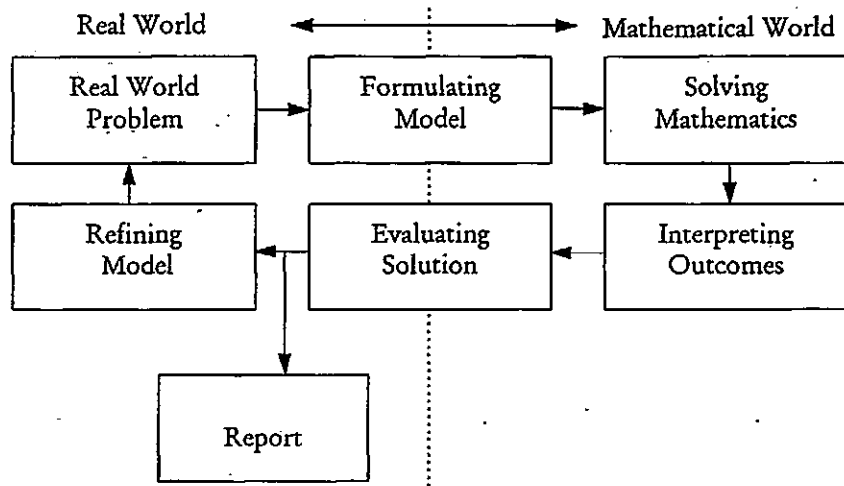


Figure 13-6: Galbraith and Haines, 1997, after OU flowchart

13.6.2 States and stages

Ikeda (1997), after Burghes and others, proposes a different diagram, showing five states (in the boxes) and four stages in the process. The real problem is recognised as outside the modelling loop, but affecting it at the level of the classroom model and the “Real solution”. The “states” in this model correspond to the arrows in the OU flowchart, and the “stages” to the boxes. There may be a slight difficulty with language in this diagram: “real solution” implies that the solution is unique, and “real-world solution” would probably be a better term.

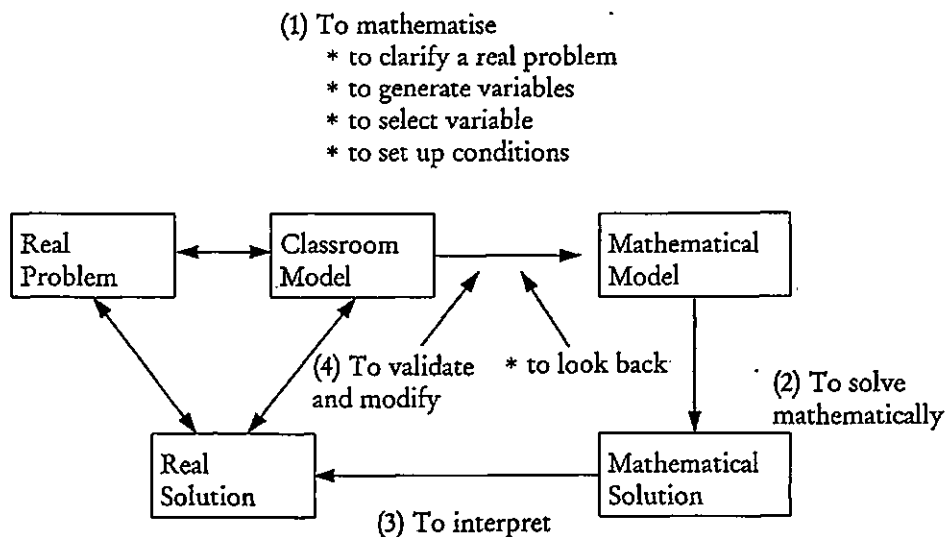
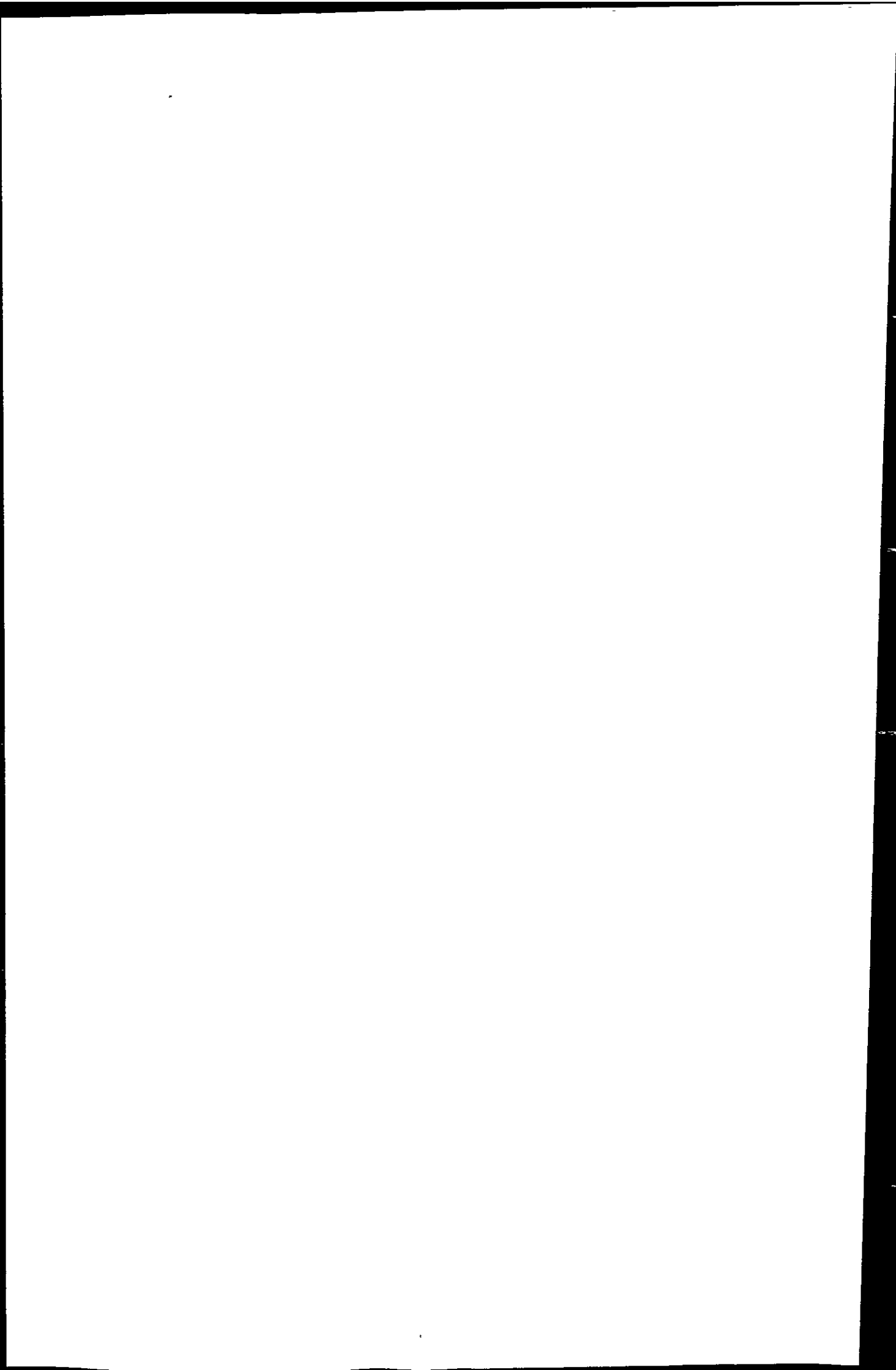


Figure 13-7: Ikeda, 1997, after Burghes et al



When the diagram is transformed, by turning the boxes into arrows and vice versa, it looks more familiar: The arrow for “real problem” is awkward, but Ikeda does not label the arrows on his diagram showing the relationship between the “real problem”, the “classroom model” and the “real solution”. On the other hand the activities in the boxes clearly correspond to the boxes in the corresponding positions in the OU flowchart.

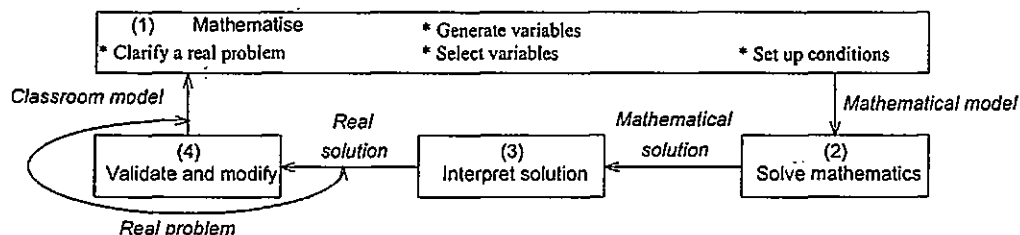
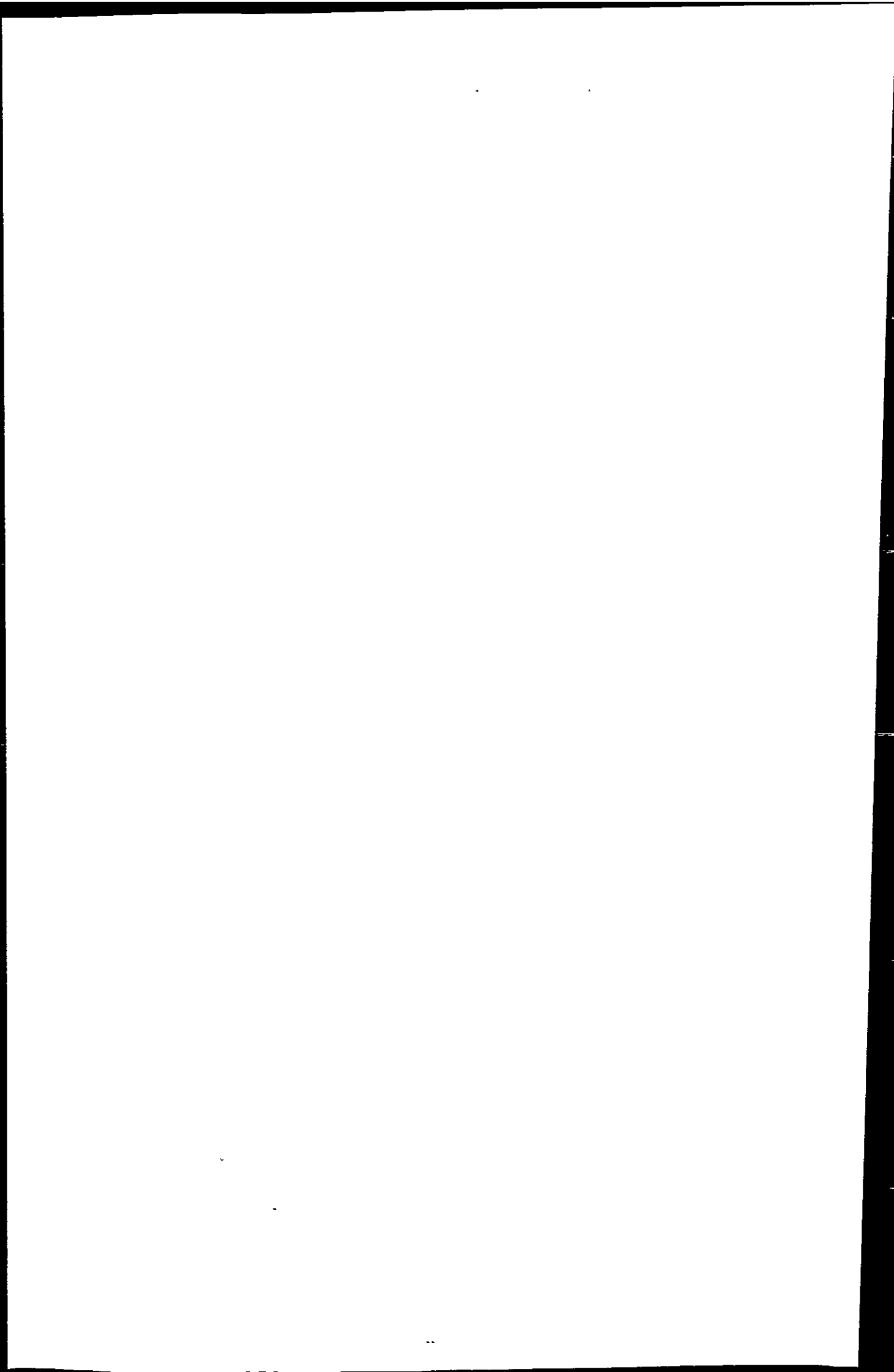


Figure 13-8: Ikeda diagram transformed for comparison with OU diagram

13.6.3 Iteration

Moscardini et al (1984) emphasise the iterative nature of the process, but make the process appear more linear than the OU flowchart. The “advancement of understanding” arrow implies that if an iteration loop means that the modeller moves back to the left to adjust the model, understanding is thereby retarded. They also de-emphasise the validation stage by implication, since its box is both smaller and less full of detail than the other stages.



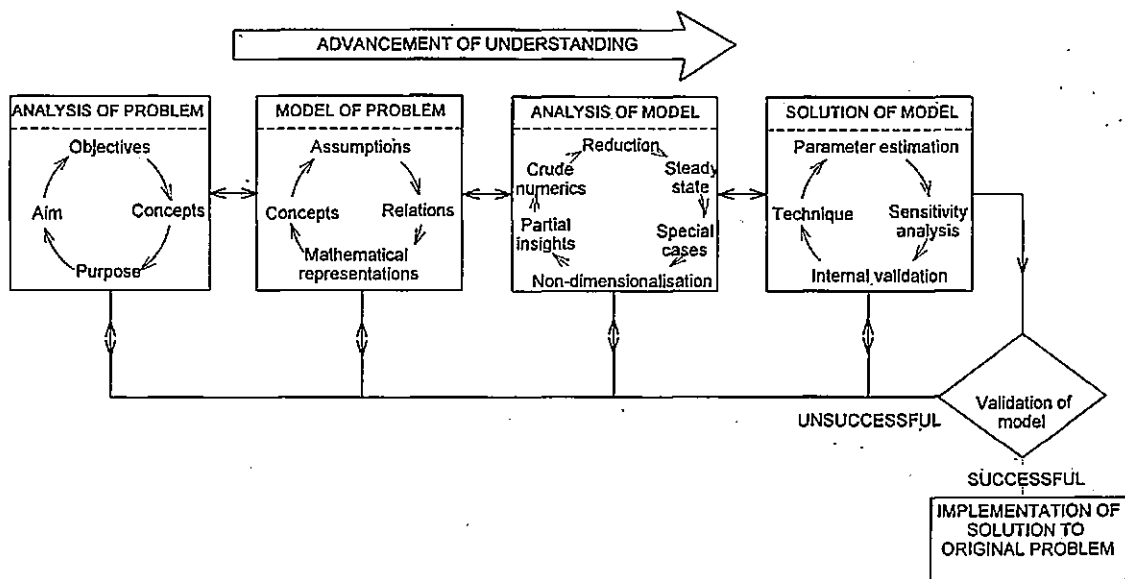


Figure 13-9: Moscardini et al, 1984

13.6.4 Eight-box diagram

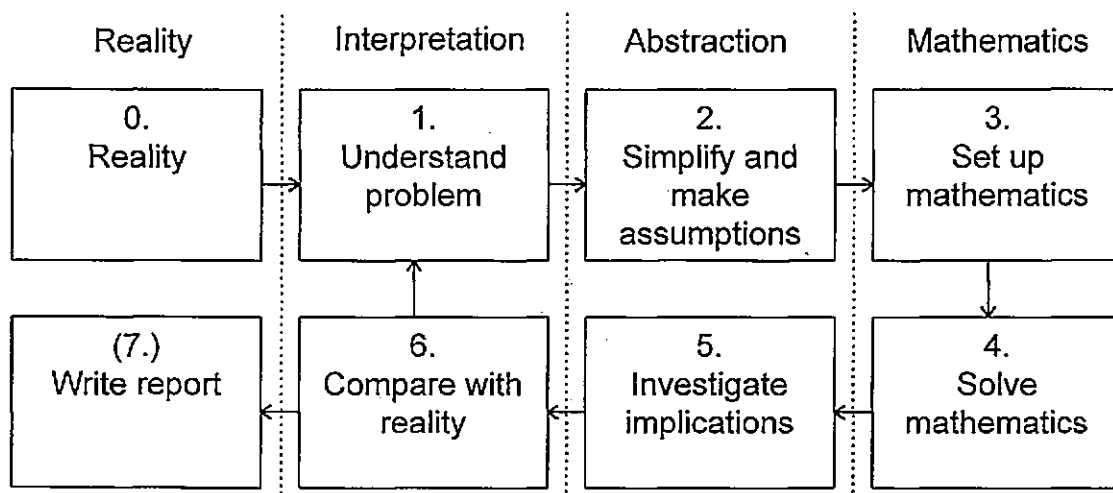
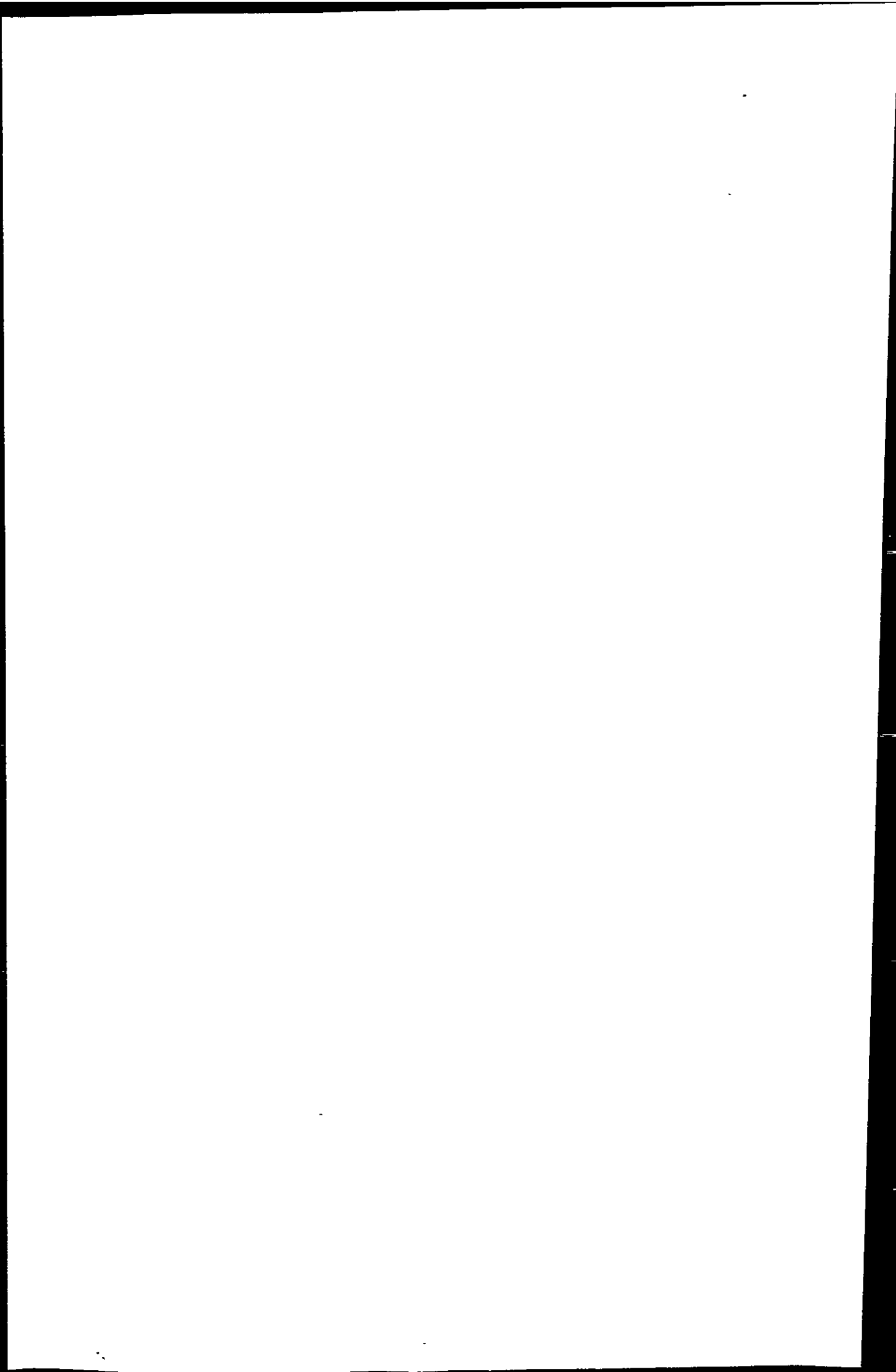


Figure 13-10: Modified mathematical modelling flowchart

This is the diagram on which the structure of the programme is based. It is virtually identical to the classic OU diagram, except that box 0 acknowledges that reality is outside the mind of the modeller, and that the modelling starts with something slightly different, that is the modeller's understanding of the problem. This box may validly be criticised as its contents are a noun, rather than a verb. No suitable verb presented itself during the writing of the programme.



13.7 Empirical modelling

In studying Figure 13-11 we see the familiar-sounding stages of Simplification (simplify and make assumptions), Mathematization (set up mathematics), Transformations (solve mathematics), Interpretation (investigate implications) and Validation (compare with reality). (Oddly, both the verbs which label their arrows and most of the nouns in the boxes take “-ation” forms which leads to some ambiguity.)

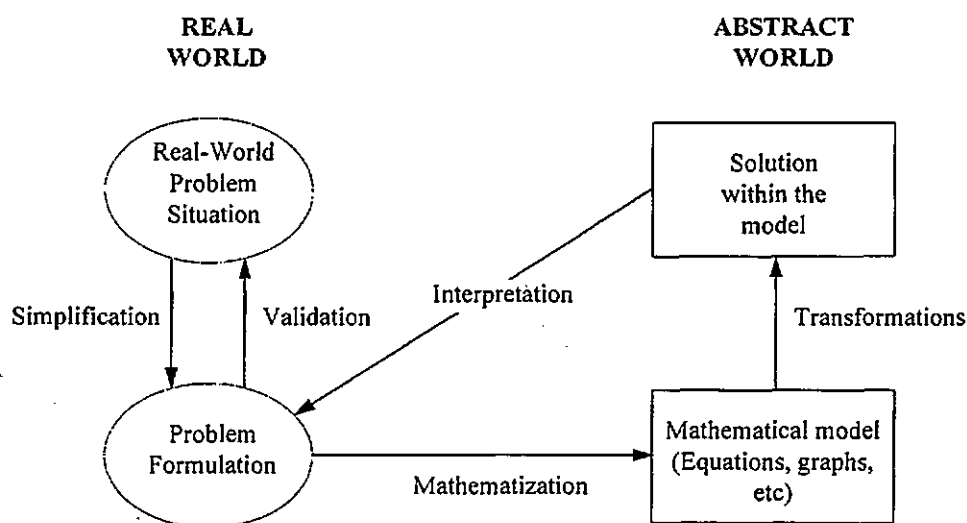
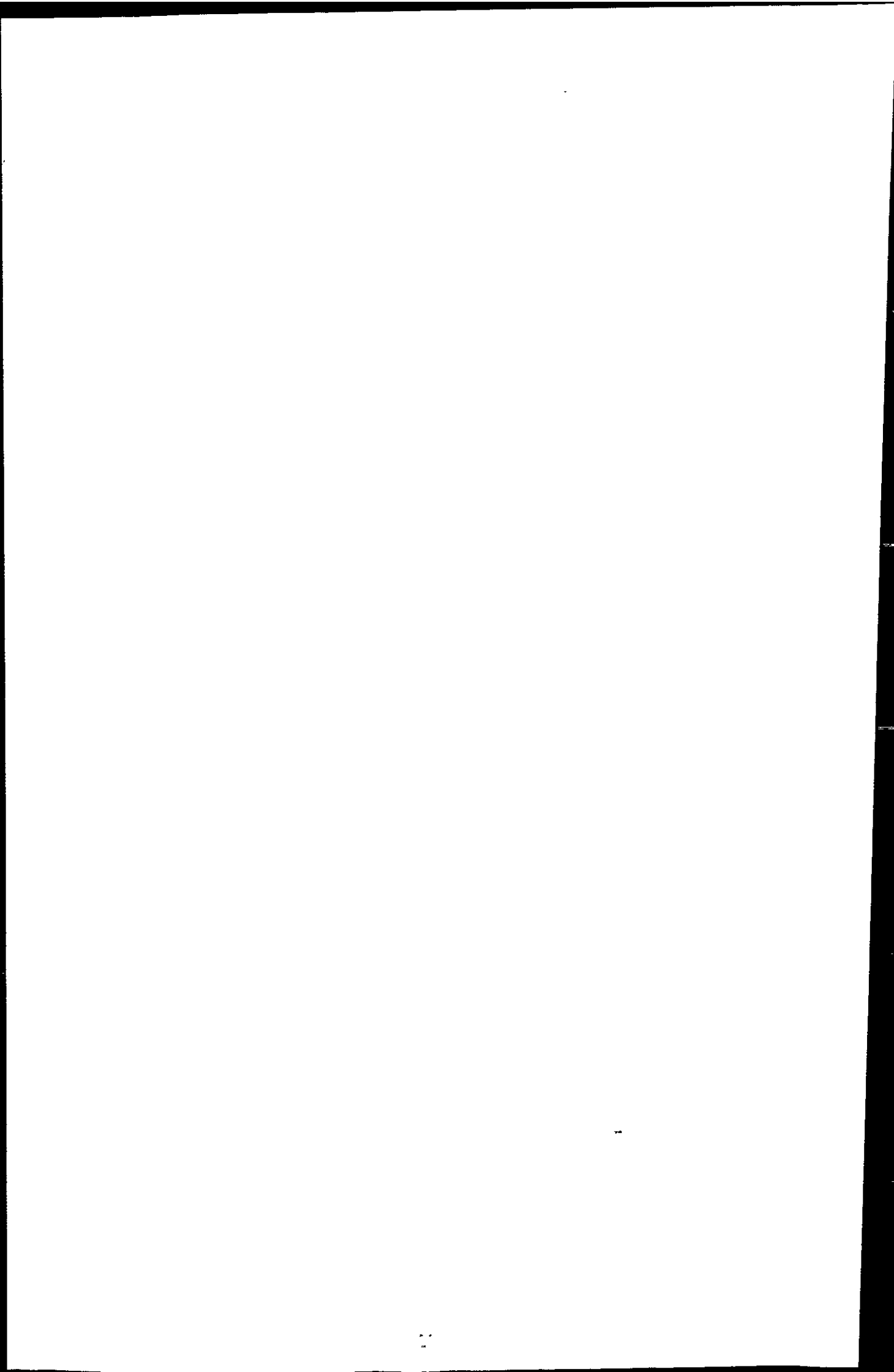


Figure 13-11: The NCTM Standards' (1989) characterisation of mathematical modelling (after Hodgson and Harpster, 1997)

However, reading of Hodgson and Harpster's article makes it clear that the Mathematization stage consists for them of collecting data and finding a graph of an equation which fits the data. Thus although the diagram appears to describe theoretical mathematical modelling, just as the Open University diagram does, in fact the empirical paradigm is being expressed.

The difference between the two paradigms is blurred by the need for students who are not familiar with a certain application to watch it happening in order to work out



what are the underlying mechanisms or processes before attempting to make a theoretical model. The major difference is that this is a watching without measuring: no data should be collected at this stage and the observation should be qualitative rather than quantitative. Unfortunately resources may determine that only one visit to the phenomenon can be made, and data collection and observation must be combined. In that case it should be made clear to students that the data should not be used until the stage of "compare with reality".

The empirical modelling paradigm is rarely explicitly expressed, except in the MEI flowchart in Figure 13-5. However, observation of engineering students has shown that the way they try to deal with mathematical modelling is to believe in the empirical paradigm.

13.8 Modelling behaviour of engineering students

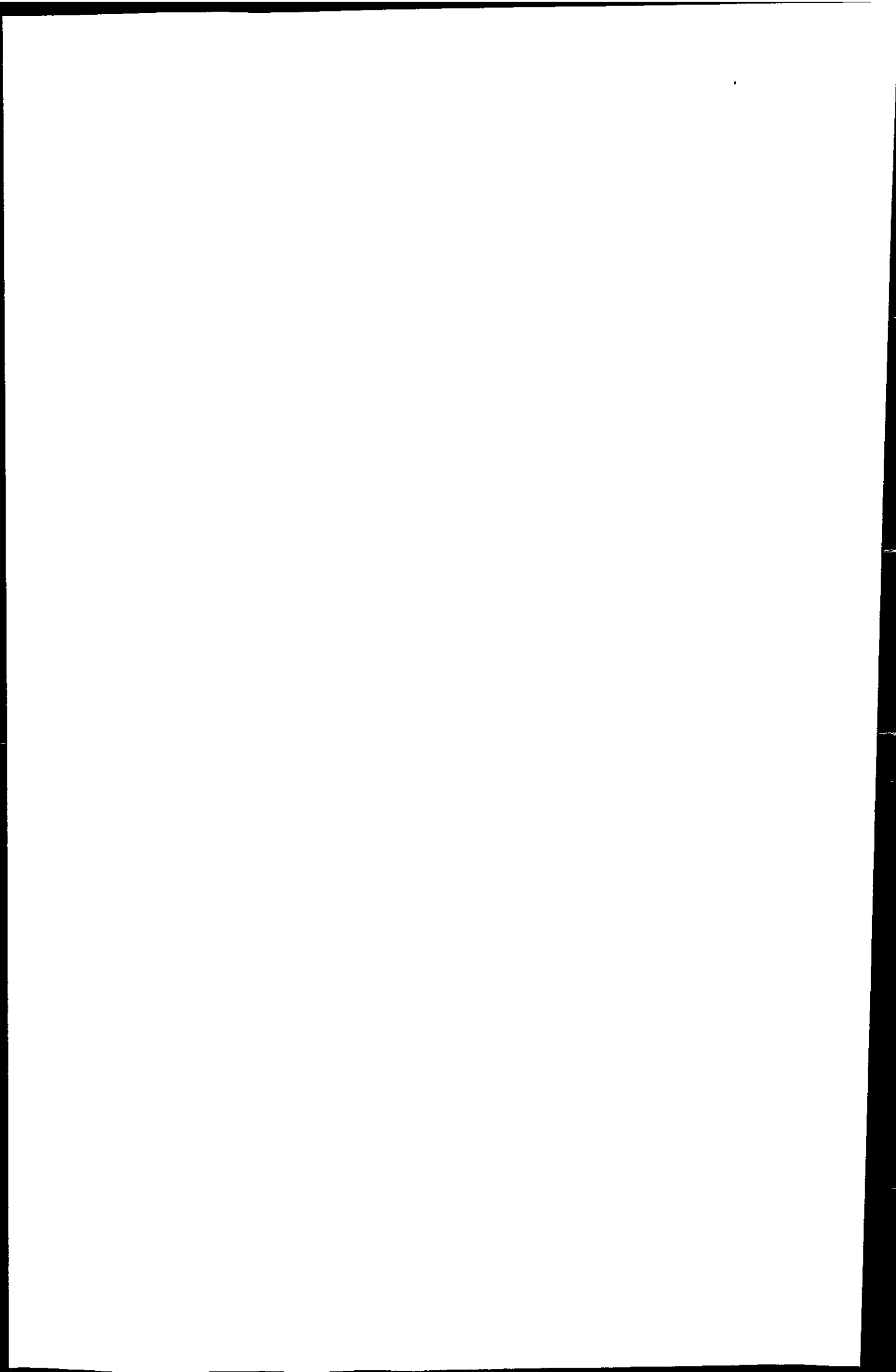
When asked how they would find the flow rate of the water in the cascade exercise, the engineering students said:

Adrian: What you'd do is to set it up with a variable input into it and we'd have to maintain the water in litres for the flow rate. Maintain it at say five litres and measure the water coming out the bottom in a given time.

When it was suggested that marking the height of the water on the side of the tank at regular time intervals might be a reasonable approach the reply from the students was not encouraging.

Jolyon: Very hard to achieve I suppose.

The way in which they seek to mathematise is to find a ready-made model. The data can be compared with the results recorded by a previous engineer in the form of an empirical relationship. This can be found by looking in the appropriate place, either a textbook or notes.



Jolyon: Well I expected perhaps to go into the same equations for steady flow theory, that sort of thing which I know we've done - in previous years - without our notes and that sort of thing. Engineers don't remember equations. We go and look them up in books. We don't derive things from first principles and - we tend to anyway - just to take it from the vantage of theory, and then applying it.

WMM: So did you find it hard that you were actually being asked to create the equations?

Adrian: Well, yes, if you like

Jolyon: Though we could probably do it with a book in front of us.

WMM: Is it something you were asked to do ever in the course?

Adrian: Well I'd say if we ever did have this we'd have more of a formula to start with.

Jolyon: Well we generally work through the theory which they tend to like, make us - the teachers - try to understand it, and then like - apply the results. It's very rare that we do anything from first principles like this.

I remember that we did in HITECC.. - we did a mathematical model and the particular one I did was the optimal speed of rotation of a tumble drier and that worked well and we actually took that from equations and then Tony sort of encouraged us to do it and we sort of had it - centrifugal force against centripetal force and acting against gravity - sort of worked out from there, rather than doing it practically. So I would perhaps have expected to go on to some equations - but just doing it practically shows what happens but you can't always do things practically like building a bridge..

Adrian: You can build models though.

And

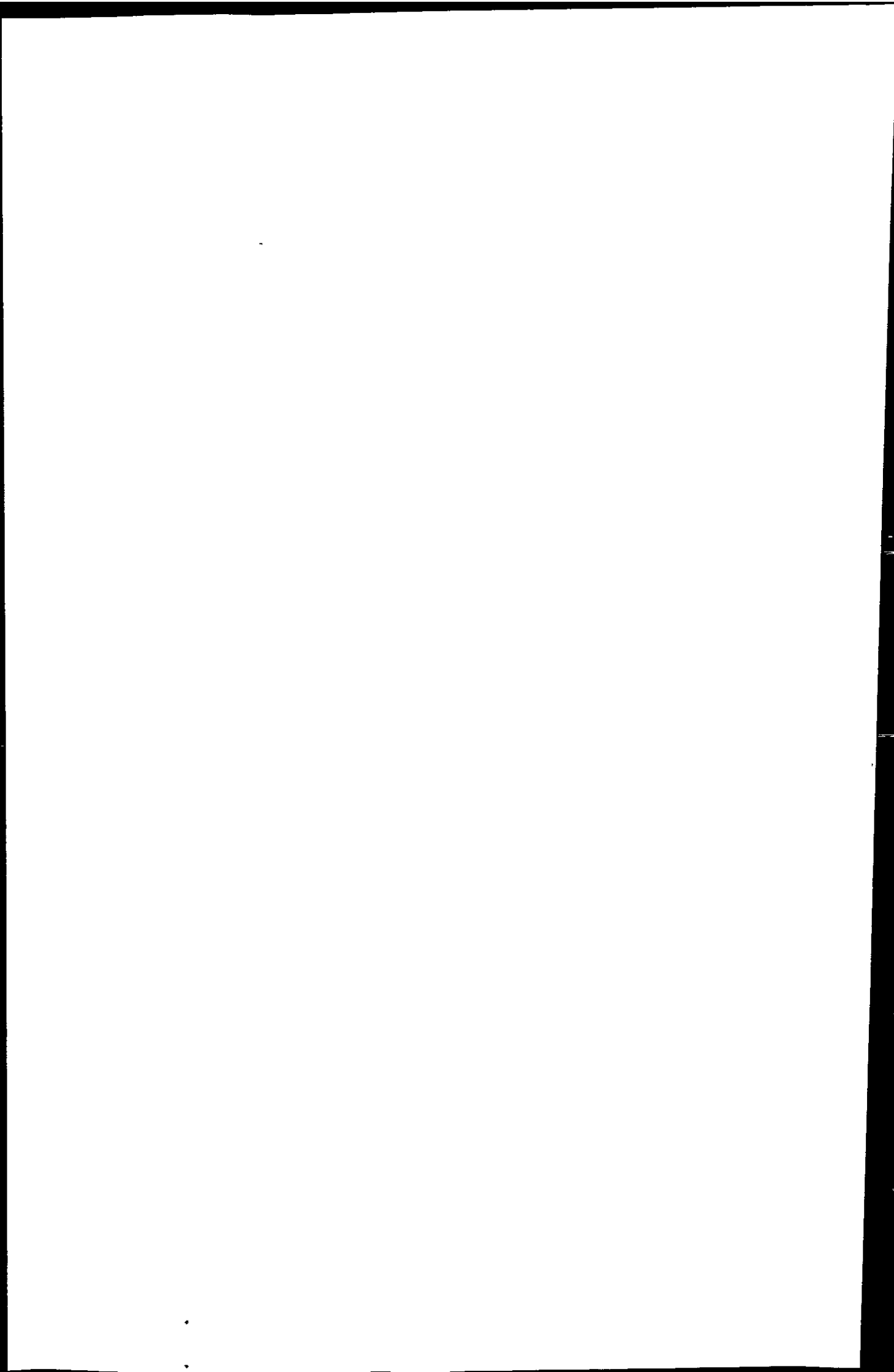
Jolyon: I felt that the maths side of it lets us down a bit on what we've done in the course.

Adrian: Unless it's like what a lecturer said if you can sort of feel you can remember what he did in the second year it's quite easy but everyone just sort of forgets it. You know you can do it and you know you can look it up how to do it.

For engineering respondents to the questionnaire a similar message came through:

- My mathematics skills are not brilliant and I need reference back to old notes. (Engineering graduate)
- If I saw these in real life, I have a good book I can look things up in. (Engineering lecturer, pilot study)
- I would probably only get as far as (f) before I looked in a book. (Engineering graduate, pilot study, beam bending question)

For these students the process appeared to be as shown in Figure 13-12.



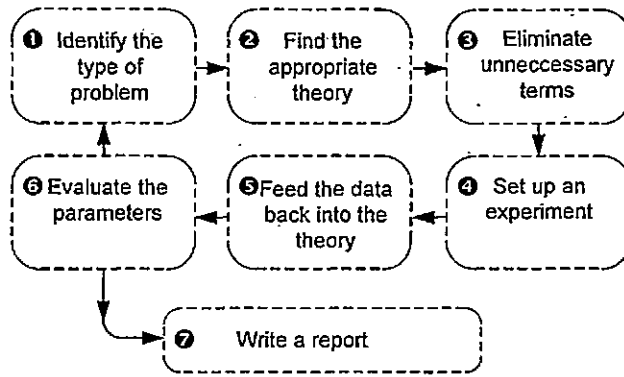


Figure 13-12: The engineers' modelling cycle

(1) Identify the type of problem

Jolyon: We've done a similar thing as an exercise.. Practical.. Quasi-static flow which was based totally around this. How long it takes a container of water to empty into another one- And if we knew that for each container- over a period of time- and compared the times- then (sketching) the whole thing is we would probably get a graph where the two coincide - where the two would be a maximum.

Jolyon:(Pointing to taps) We have to assume that the flow rate through here is going to be the same as the flow rate through here. There's roughly the same difference in head, diameter of tube.

Adrian: As the pressure varies, the flow is going to be changing.

Jolyon: Inside the tubes a steady flow job is developed..... It's a maximum

Adrian:(Drawing) What we're interested in is in.. maximum volume in B is this one here.

(2) Find the appropriate theory

Adrian: Trying to go back to basic principles here

Jolyon:(Studying tanks) What if we apply Bernoulli to the area between each tank? Both $p_1 v_1$ is going to be equal.

Jolyon: (Takes out calculator and from it a formula card) Cause I've got Bernoulli's law on here.

(3) Eliminate unnecessary terms

Adrian:(Taking calculator card again) Looking to see what I can get rid of

Adrian: This thing actually cancels out. doesn't it, because you take that away from each side and divide through

Jolyon: You want this squared and then

Adrian: Yes it's just rearranged, isn't it.

(4) Set up an experiment

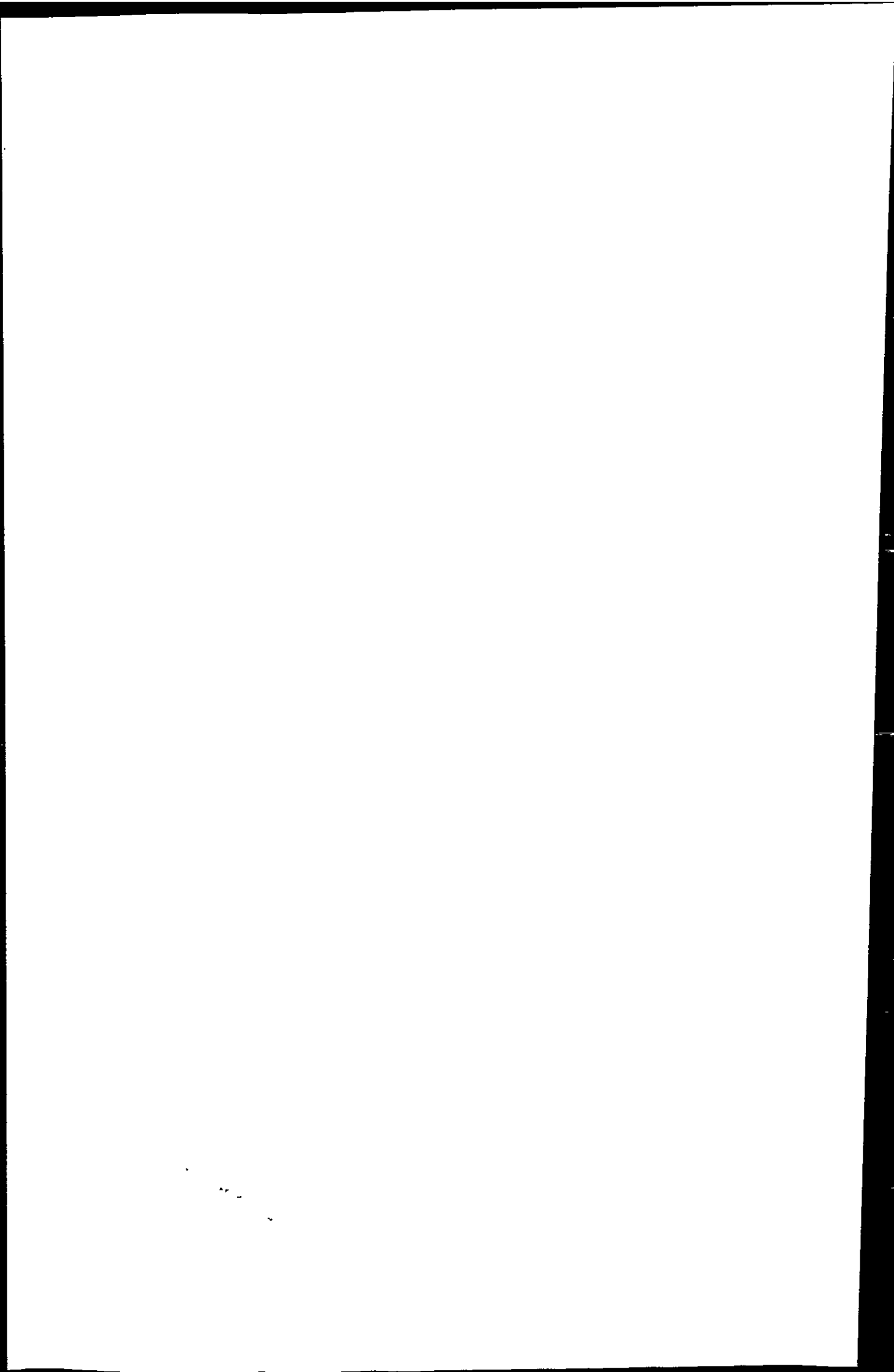
WMM: How could we test your intuitions?

Jolyon: Fill it up: we'd just do it.

WMM: Well why not?

Jolyon: I thought the idea was to get a mathematical model of it rather than just sort of measure it.

WMM: It might be helpful to test your intuitions.



Jolyon: Yeh

Adrian: Come on then, let's fill it up.

(5) Feed the data back into the theory

Adrian reading data and plotting. Sketches in line

Jolyon: Should it be linear? Because it's proportional to v^2 isn't it? Dunno if that's. Ordinate scale is volume.

Adrian: The volume's the same. It's the time that's changing.

Jolyon: You change the volume as well. You lose it out the tap. As time goes on, you're losing volume and pressure.

Adrian: Well?

Jolyon: The top doesn't vary but it does vary here (indicates tap) because of the head.

Jolyon: If that's the case, what comes out of the bottom has gained by the top one. You'd also lose the bottom one at a linear rate.

Adrian: Any tips?

WMM: What sort of relationship does it look like?

Adrian: There isn't any shape showing clearly there. I thought it tends to be linear.

WMM: What does the graph represent?

Jolyon: It relates the amount of volume to how long it's been going.

WMM: So what does the slope of the graph represent?

Adrian: The rate of flow

WMM: So..

Adrian: So when the slope is nought there's no volume in...

WMM: Cause you've still got about 20 mm of drop there: when it's empty - in inverted commas - you've still got about 20 mm of drop there. So - and so can you deduce some sort of relationship between the flow rate and the height of the water?

Adrian: The higher the water the greater the flow rate

WMM: That sounds reasonable. So you've got that there is a relationship between flow rate and the height of water. What do you think that relationship might be.

Adrian: Well, the slope of the line.

WMM: Well, do you think it's- It's clearly not independent of the height, so you could write down an equation that says the flow rate is a function of the height. And what sort of function do you think that is? Do you think it might be?

Jolyon: Well, it's obviously not linear, from those results.

WMM: No- yes- if it were a straight line it would be independent of height. So you know it's some sort of function of the height.

Adrian: We thought it might be some sort of square.

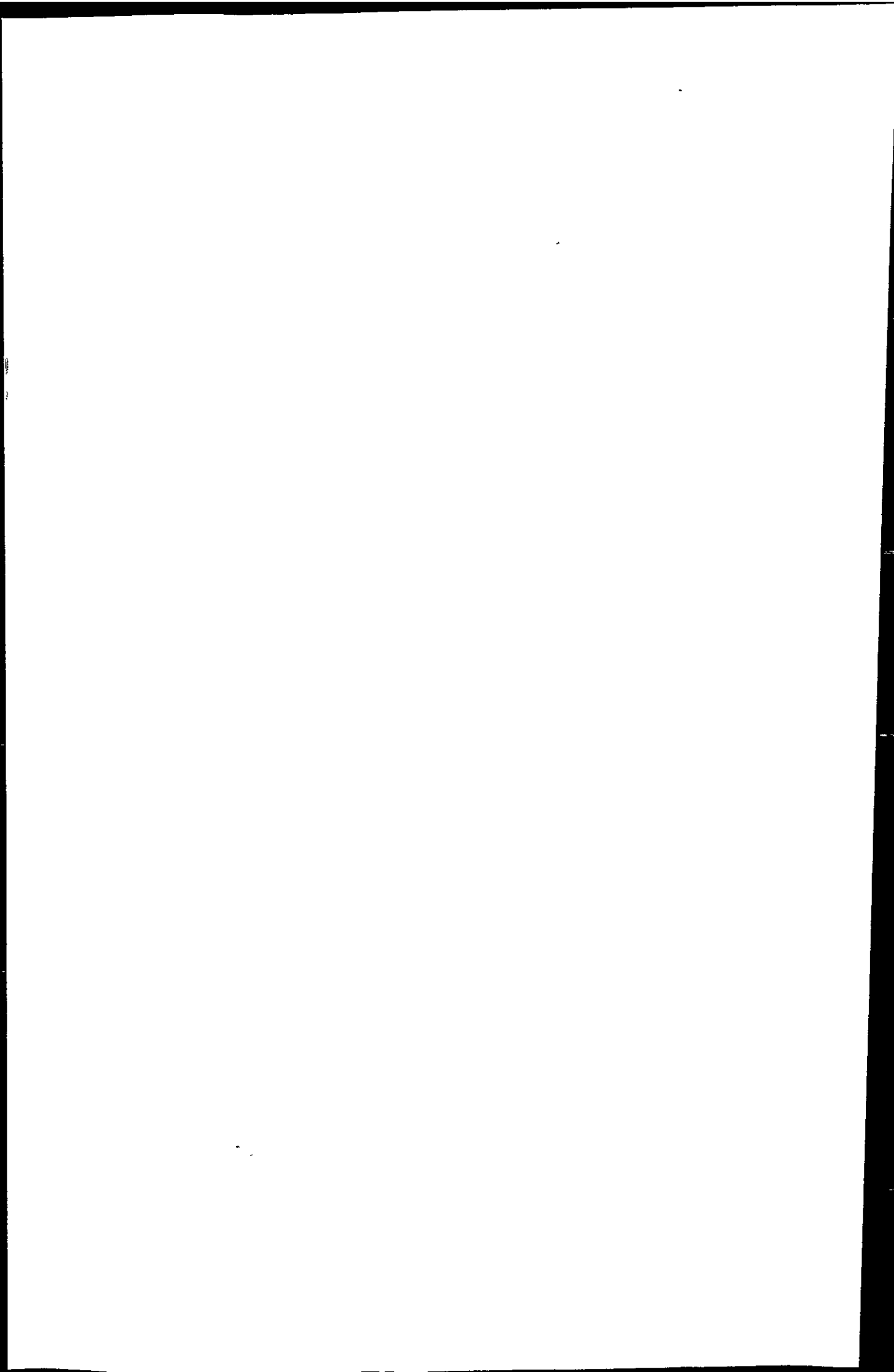
WMM: How would you test what the relationship between flow rate and height is? If you're suggesting it's a quadratic, how would you test if it's a quadratic?

Jolyon: Surely you'd have that by seeing the results.

But we don't really know what's going on- we're not really sure what's happening between.. each container. So up to now we've only done experimental - and what we've got there - doesn't really show enough - doesn't really tell us enough about the flow rate against the height of the water.

(6) Evaluate the parameters

Adrian using calculator.



Adrian: This could be some big pig.. Which kids us. The maths bother me. I make it point three three.. nought point three three. Ah here it is.. The max..

Adrian: That's the maximum innit.. The maximum level.. So when point nought three mass flow rate occurs.. what time that occurs.. is the time when it hits the maximum.

Jolyon: So we could... if we had an equation for that line, we could differentiate that and find a maximum, couldn't we?

Adrian: Yes, but it's..

Jolyon: Which is what we're after.

Adrian: Mmm.

Jolyon: But it still doesn't explain why it stays at that level.

Adrian: How long does it take to get to that point there, then it stays there?

Jolyon: Not particularly.. Take the.. Determine half empty..

Adrian: It's not so bad. (Writing on a paper he doesn't seem to have left with me) Four and a half, that occurs.. (Using calculator) Two and a half, will occur..

Adrian: one and a half..

Jolyon: So?

Adrian: The time that occurs. The time that occurs... So we're saying that our maximum height be reached.. How high was it do you reckon?

Jolyon: (looking at carboy) Two - twoish

Adrian: Two point two. (using calculator) Point three o five.. so..

Jolyon: Between those two values

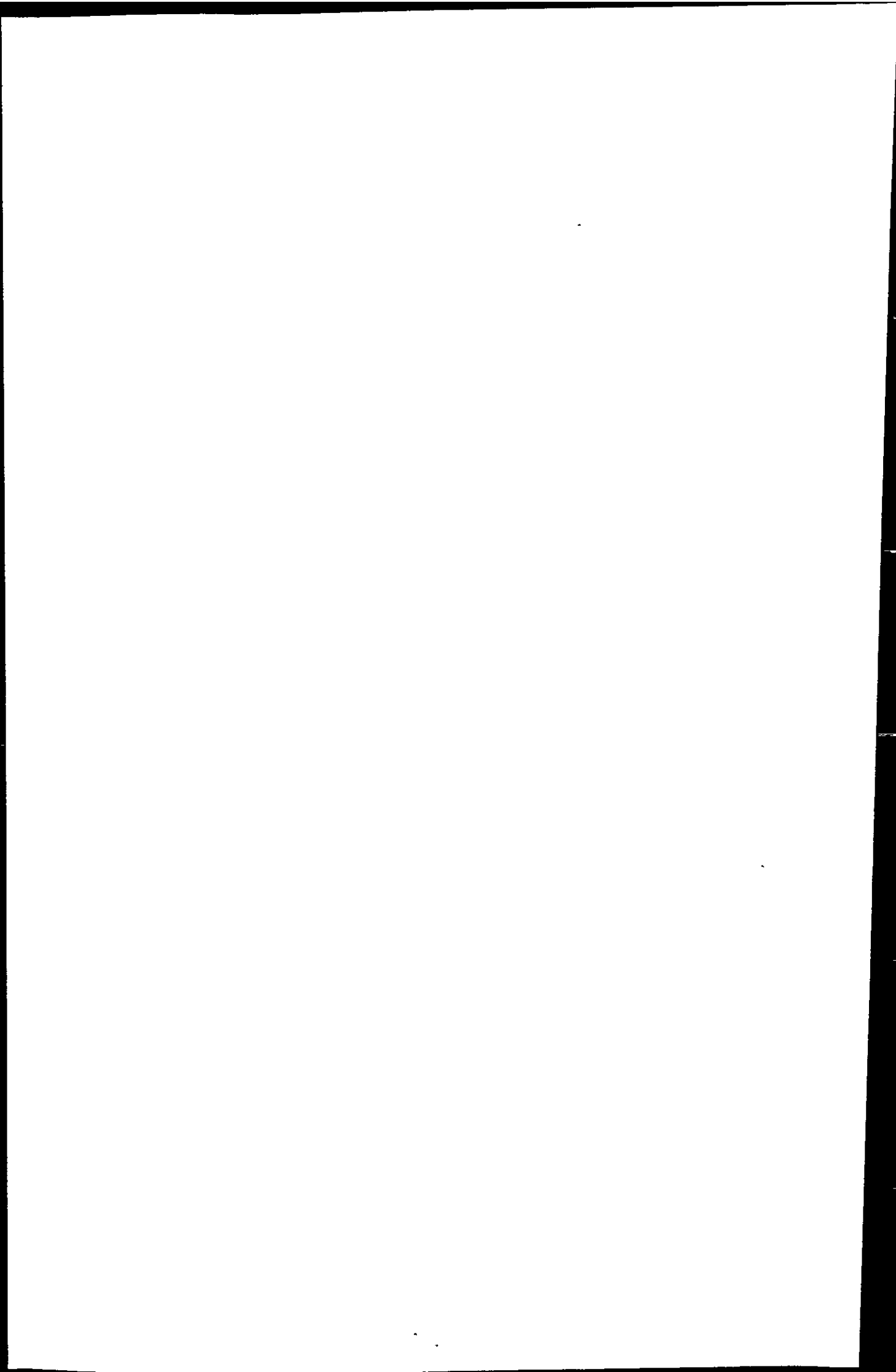
Jolyon: You're saying after 69 seconds that makes that a maximum.

Adrian : Yep that is my prediction.

Jolyon: It's incredibly dodgy I reckon.

Although in the early stages one student suggested they should consider the tanks separately, as the session progressed, they remained fixed on the question asked: that is, when the water level would be maximum in the second tank. In contrast the mathematics students had realised that in order to understand the flow with input and output in the second tank they had to understand the simpler case of the top tank first, and spent a considerable time working on that.

Crowther's (1997b) finding that engineering students like to be taught mathematics using familiar, understood applications reflects an empirical modelling attitude: the data has been found, now we can fit some mathematics to it.



A different group of final year Manufacturing Systems Engineers, discussing mathematical modelling, reflect the same views: that you choose the appropriate model from a relatively narrow selection, and plug in the parameters. (Martin is a German student, and his English is good but not perfect.)

Martin: Usually finding the formula is never really a big problem. You choose out of the seven, tick one that must be right, and then you have the problem that ..

John: Good old engineering guess (laughter)

Martin: It is the only one which has all the variables I know or any of those must be the one. But where we get all the factors from if there is no letter

John: You can look them up

Martin: Then we can find them or how to guess them.

I found that sometimes that is a big problem.

And so usually you get told by a lecturer the wall has a T of 20 or something, and then you get your..

John: Yeh, the data that's stored you can get that out of reference books .

You get all different k values and C values and you can..

Martin: When you are starting to find those it can be quite a long wait. Might be even longer than finding the right equations and solutions. You get from simple mathematics into a high mathematics problem in finding them.

John: No, you look them up.

Martin: But the conductivity of this wall [gestures] is not written down in a book. You have to..

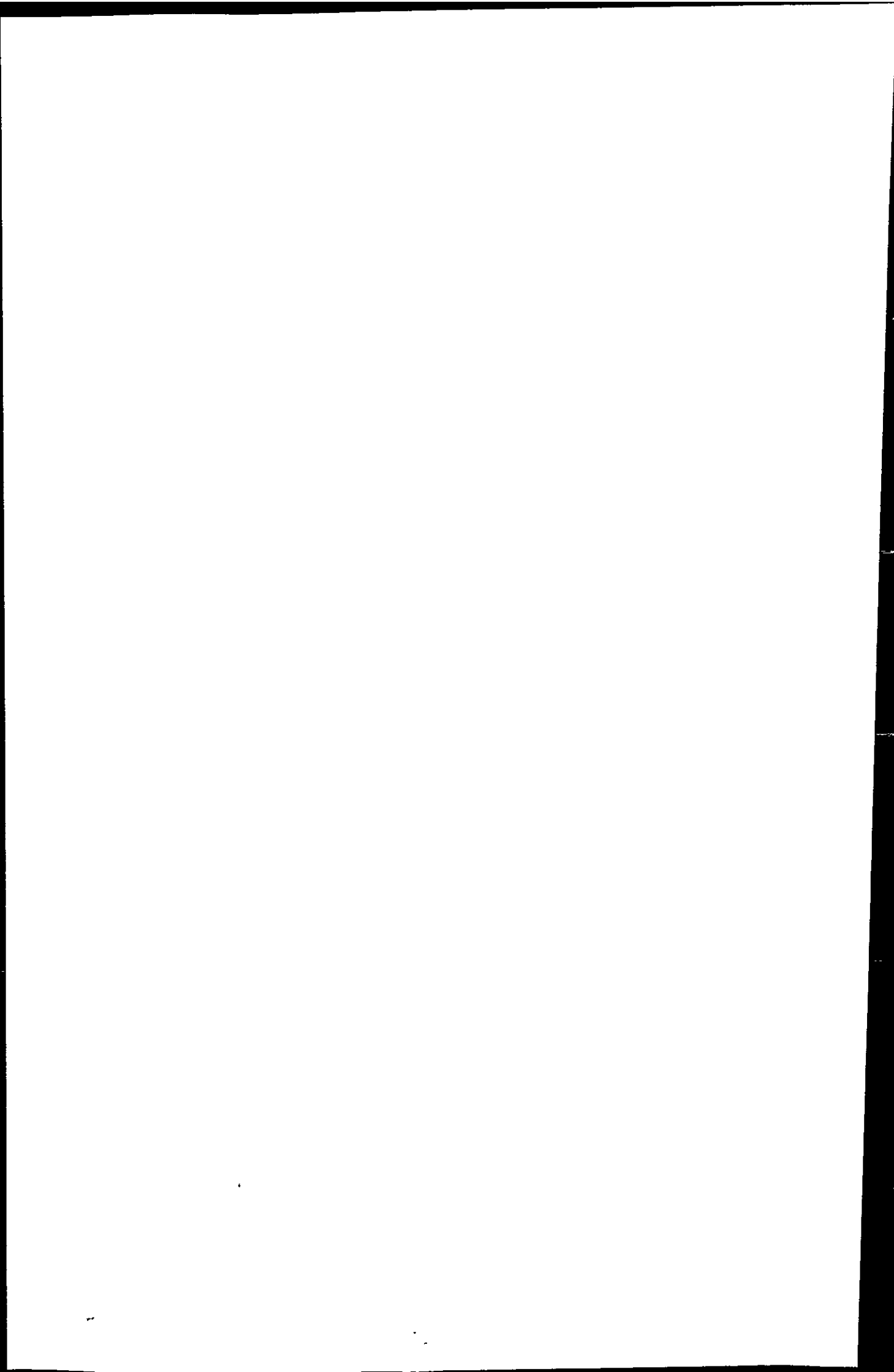
John: But you know what the wall is made of, you know how thick it is, so you can go and look that information up in tables.

Martin: It goes quite a few steps back.

One student does wonder about where the models come from before they appear in the reference books, but does not seem to have a clear grasp of the mathematical modelling cycle. A model reflects the assumptions on which it is founded, and a static strength model will not predict dynamic behaviour such as resonance. "Compare with Reality" does not imply that we have to build the real bridge, but that we can check the implications of the model against the behaviour of other real structures.

Gareth: What if the model is such a situation that we can't actually get and physical data from it? Like it hasn't been created, building a bridge?

John: Then you have to measure it.



Gareth: How?

John: Seems obvious. From what Martin was talking about, conductivity values, you can measure those.

Gareth: If it's for an unknown.

John: I mean if it's for an unknown thing, if the material hasn't been invented yet then fair enough you can't look it up but you can't measure it either, but what are you modelling on something that's not known?

Gareth: You might be doing a feasibility study or something.

John: But then you'd know the properties of the material you're looking at or you'd be looking at specific properties. You'd be working from the back end to try to identify what specific properties you're looking for from the material, wouldn't you?

Gareth: It's just that this step here where it says "compare with reality", you may not be able to do that.

I mean I'm sure that when they built that bridge in America which destroyed itself when it reached resonance. (The Tacoma Narrows Bridge)

John: It begins with a T doesn't it?

Martin: The swinging bridge.

Gareth: They couldn't compare with reality until they'd built it. You could do all your modelling- that's where I'm saying the assumptions are very..

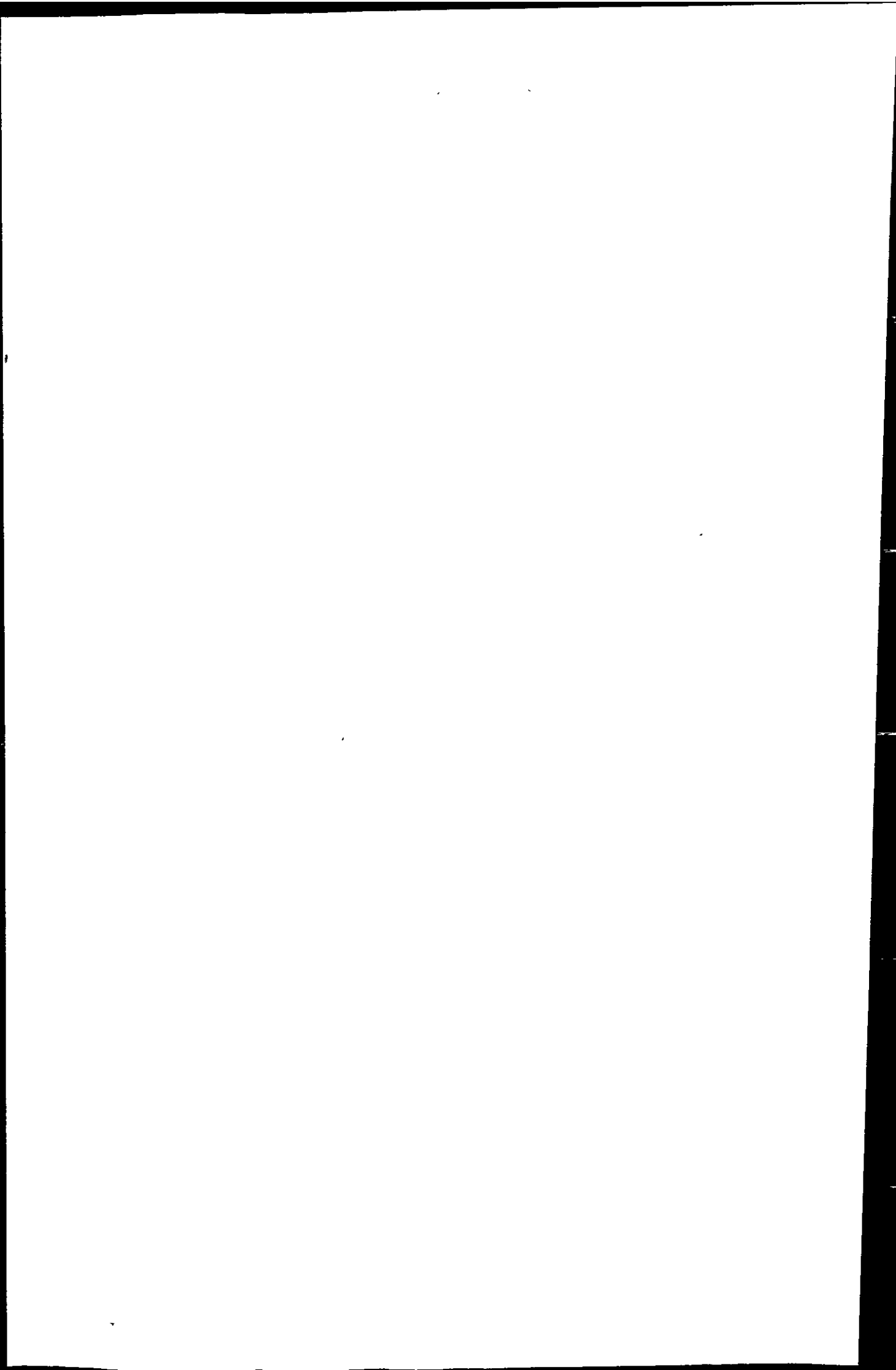
John: The assumptions are there, aren't they. They are assumptions.

These things don't actually happen just the same as you think. The only way is you can simplify it so that at this level you can solve the mathematics.

13.9 Conclusions

Various authors, for example Mustoe (1992), SEFI (1992), IChemE et al (1995), have argued for the importance of teaching mathematical modelling explicitly to engineering students, although they differ in how and when it should be done.

However, in most engineering courses most of the elements of the modelling cycle are taught, but in isolation. In structures, mechanics, fluids and thermodynamics, for example, general physical laws are taught, assuming that the phenomena to which they apply are familiar to students. In mathematics, techniques which are useful for manipulating the algebraic expressions are taught, without generally referring to the laws themselves. Mathematics appears to be external to the student: it comes from a formula sheet. In chapter 6, particularly in the comments made by engineering respondents, we see this idea manifested strongly. In the practical side of the course,



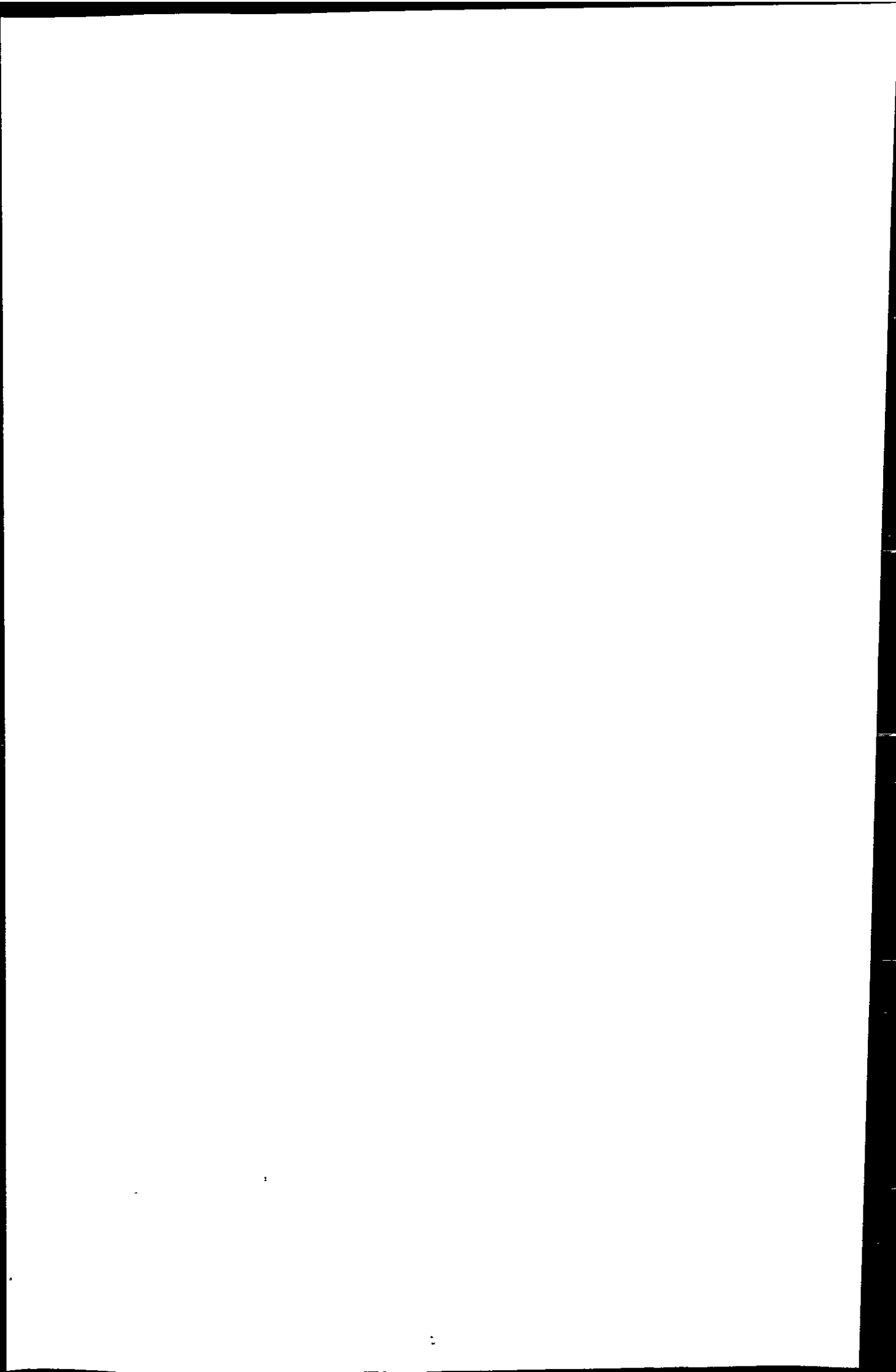
the laws are again quoted, experiments are carried out and their results compared with the theoretical predictions, but it is often a case of calibrating apparatus or trying to make one's results agree as closely as possible with theory rather than checking the validity of the theory.

In other words, if in an experiment (or a demonstration, because it is intended that the results will be what the designers expect) the results disagree with theory this is a matter of sloppy technique, not faulty theory, and the real world is less valid than the theoretical world. Engineers, though, believe in the real world, which means that experiments join mathematics in the fantasy world of academic study.

What is not often taught is how to recognise when theory is not valid, and how to refine or reformulate the model to cope with that, nor how to set up mathematics for a variant on a familiar situation.

A case in point is that there are two expressions generally in use to calculate the effects of gravity on a body. $F = mg$, the "linear law" is valid close to the earth's surface where the acceleration due to gravity may be regarded as a constant. $F = GMm/r^2$, the "inverse square law" applies when variations in the distances between the bodies vary significantly. The linear law is a special case of the inverse square law where M and r are sensibly constant. However it is normally taught as a fact that the linear law is true, and not that it is a model.

Finally, we should point out that since any given situation may be modelled in different ways (such as the linear law and the inverse square law of gravitational attraction), students may take different approaches to a problem and arrive at similar conclusions (Graham E, 1997). A model which is simple to use and gives accurate predictions is a useful model, whatever the modelling approach taken.



14. Design of the courseware

14.1 Introduction

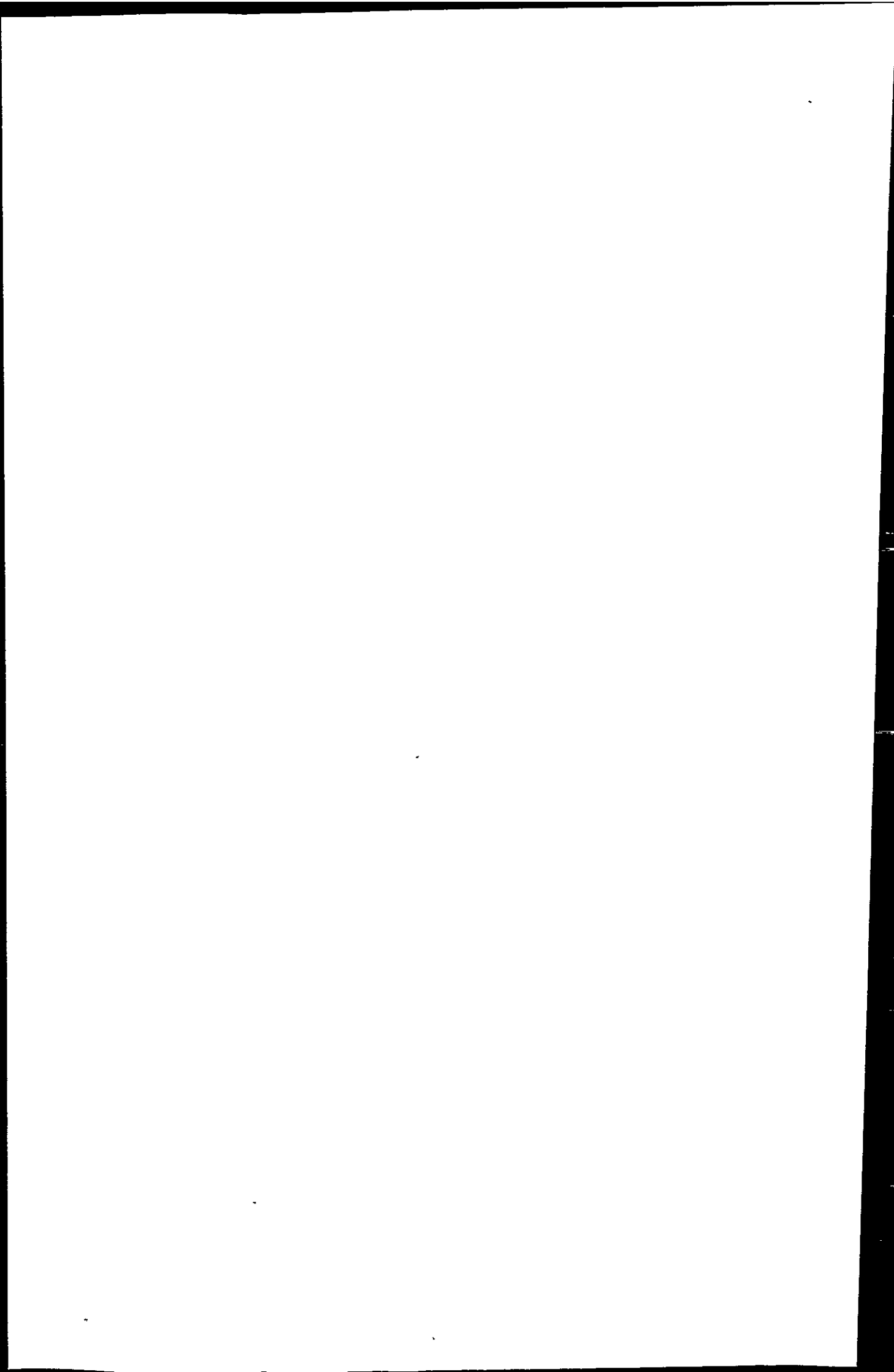
The specific design of this software involved decisions about *style* and about *content*. The literature concerning principles of style has been reviewed in chapter 12, and chapter 13 explores the mathematical modelling principles which have informed the structure of the package. In this chapter the details of the design of the package are discussed.

14.2 Overall principles

Given that the engineering students responding to the questionnaire showed a preference for verbal explanation when tackling mathematical subjects, it was decided to include plenty of verbal explanation. Despite the students' dislike of algebra, mathematical ideas would have to be expressed in algebraic form, but diagrams and graphs would be used to reinforce or support the mathematical ideas wherever possible.

Some of the conclusions from Chapter 12 concerning package design are reiterated below.

- Whereas drill and practice exercises are regarded as an efficient means for the development of skills, particularly simple skills, it was felt that in the context of enriching concepts a more hypertextual format was appropriate.
- There is little evidence that any measured factor is a good predictor of how well a given student will learn with a given package, and it is probably better to allow the student leeway in deciding how to proceed through the package, while ensuring that there is enough structure to ensure the student does not feel lost.



- The working of the package should be transparent to the student, that is the student should not be expected to understand how it works except as a metaphor.
- Appropriate graphics will include realistic illustrations to allow the student to orient the new learning within existing knowledge and symbolic illustrations to structure the new learning.
- Animation and sound will be used sparingly.

This courseware is still a prototype, and aspects of the design which did not work are naturally open for negotiation.

14.3 Structure and embedded metaphor

The whole structure of the courseware is based on the Open University mathematical modelling diagram. (see, e.g., Tunnicliffe B, 1981)

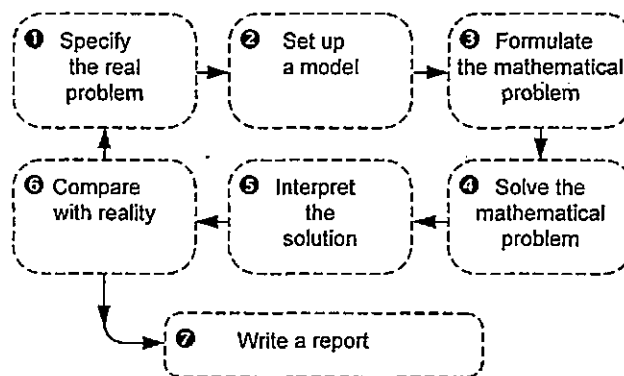
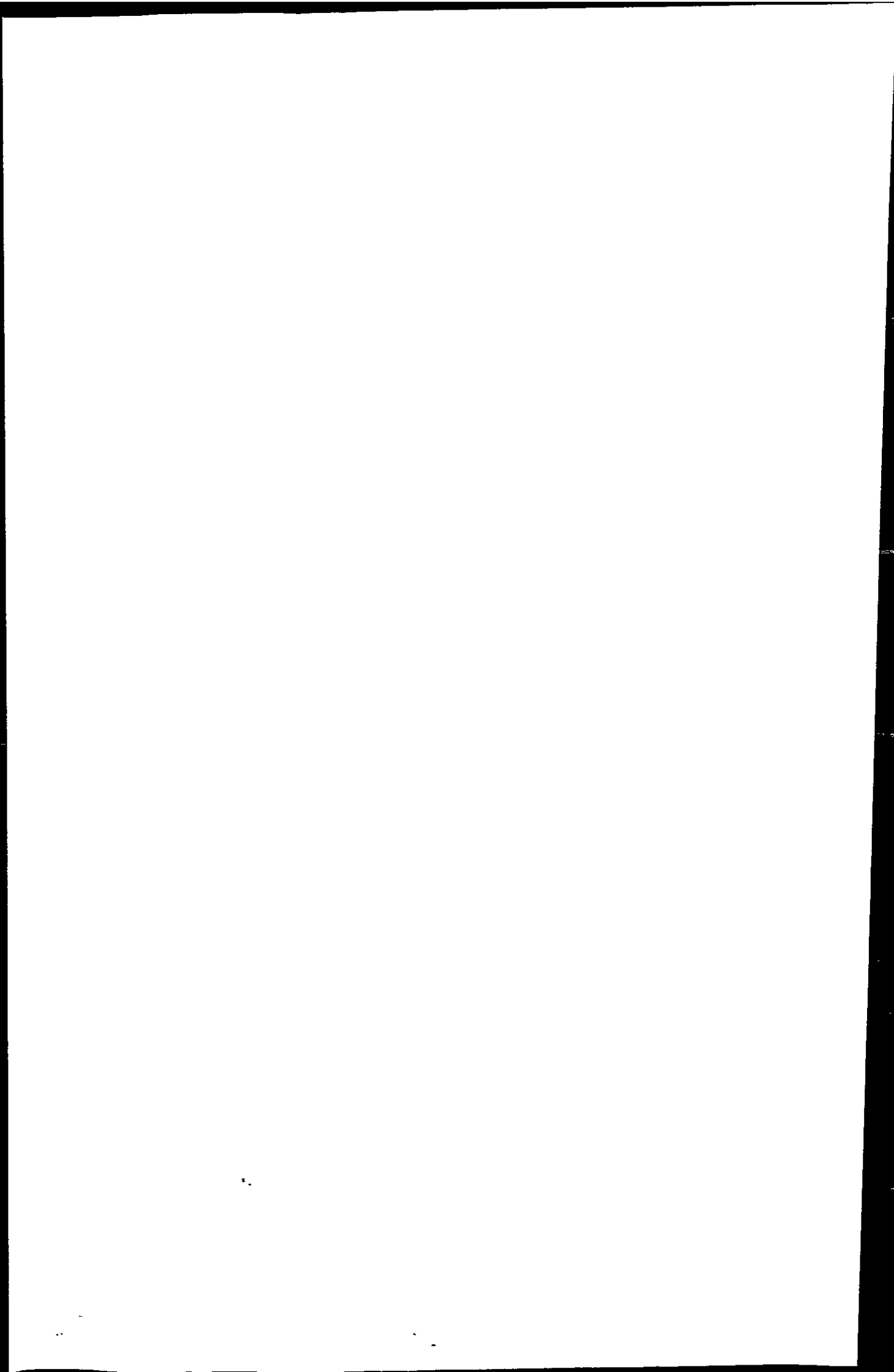


Figure 14-1: A flowchart analysing the process of mathematical modelling, (Tunnicliffe, p5)

The diagram is slightly modified as shown in Figure 14-2 to an eight-box diagram. As the modeller moves from left to right in the first part of the cycle, the model becomes progressively more abstracted from reality. In the second part, the modeller moves back from mathematics through interpretation to the real world where the report exists.



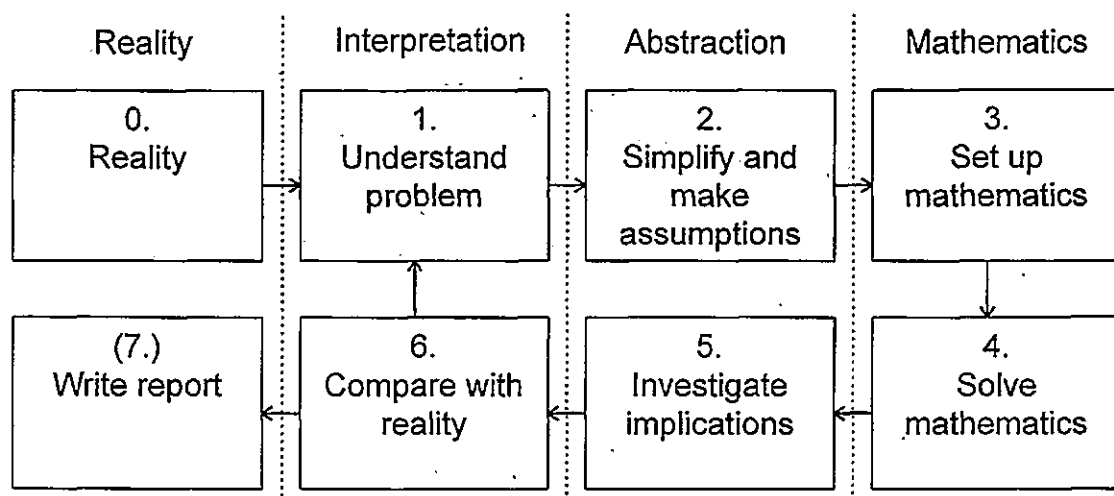


Figure 14-2: Modified mathematical modelling flowchart

These boxes are treated as rooms in a multi-storey building, where the ground floor is an introduction to modelling, and the successive floors are mathematical models of increasing mathematical and modelling subtlety. The user moves from room to room around the building. Any room can be accessed from the home page, which shows the diagram and a panel analogous to a lift panel in which a modelling example (or floor number) may be chosen. Rooms are also accessed from the previous room by proceeding through the model (around the floor).

14.4 Navigation

The home page is accessible from any point in the package via a single mouse click on the appropriate button. This was felt to be an important orientation feature, so the user would always be able to “get home” easily. The exit button is always available, in the same way, but in that case there is a check screen to ensure the user really intends to quit, rather than terminating the program as a result of an accidental mouse click in the wrong place. The navigation buttons are arranged along the bottom of the screen. Buttons for functions which are not available are greyed out.

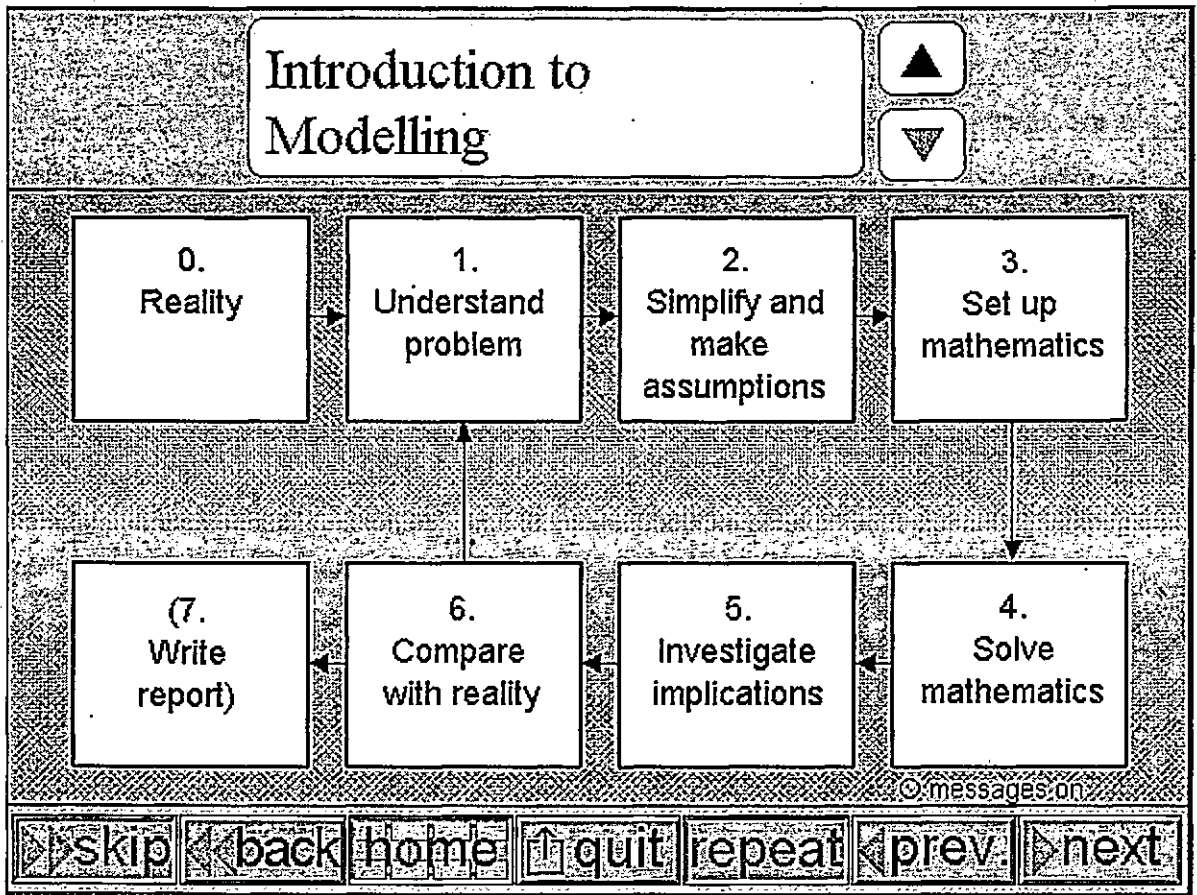


Figure 14-3: Home page

14.5 Contents

The contents of the package are an introduction to modelling, explaining briefly what happens in each stage, and a series of six case studies in which a physical situation is subjected to the mathematical modelling process.

For simplicity a single feedback loop has been shown on the diagram, but in real modelling there may be many, as mentioned below.

14.5.1 Box 0. Reality

Reality cannot be put onto a monitor screen, so any attempt to put it there is somewhat artificial. In our case it was felt that the best compromise was to use a photograph of the object, and then move on swiftly to the next section.

//

14.5.2 Box 1. Understand problem

This is the point at which the modeller makes contact with the real problem. It is stated what the problem is and what physical processes are understood to be involved. The real problem becomes conceptualised by the modeller, so passing from the real world into the world of ideas.

14.5.3 Box 2. Simplify and make assumptions

The physical processes which are being considered generally obey laws which require some aspects of reality to be ignored. The physical conditions cannot be completely modelled, or even known, so assumptions will have to be made. The conceptualised problem is idealised.

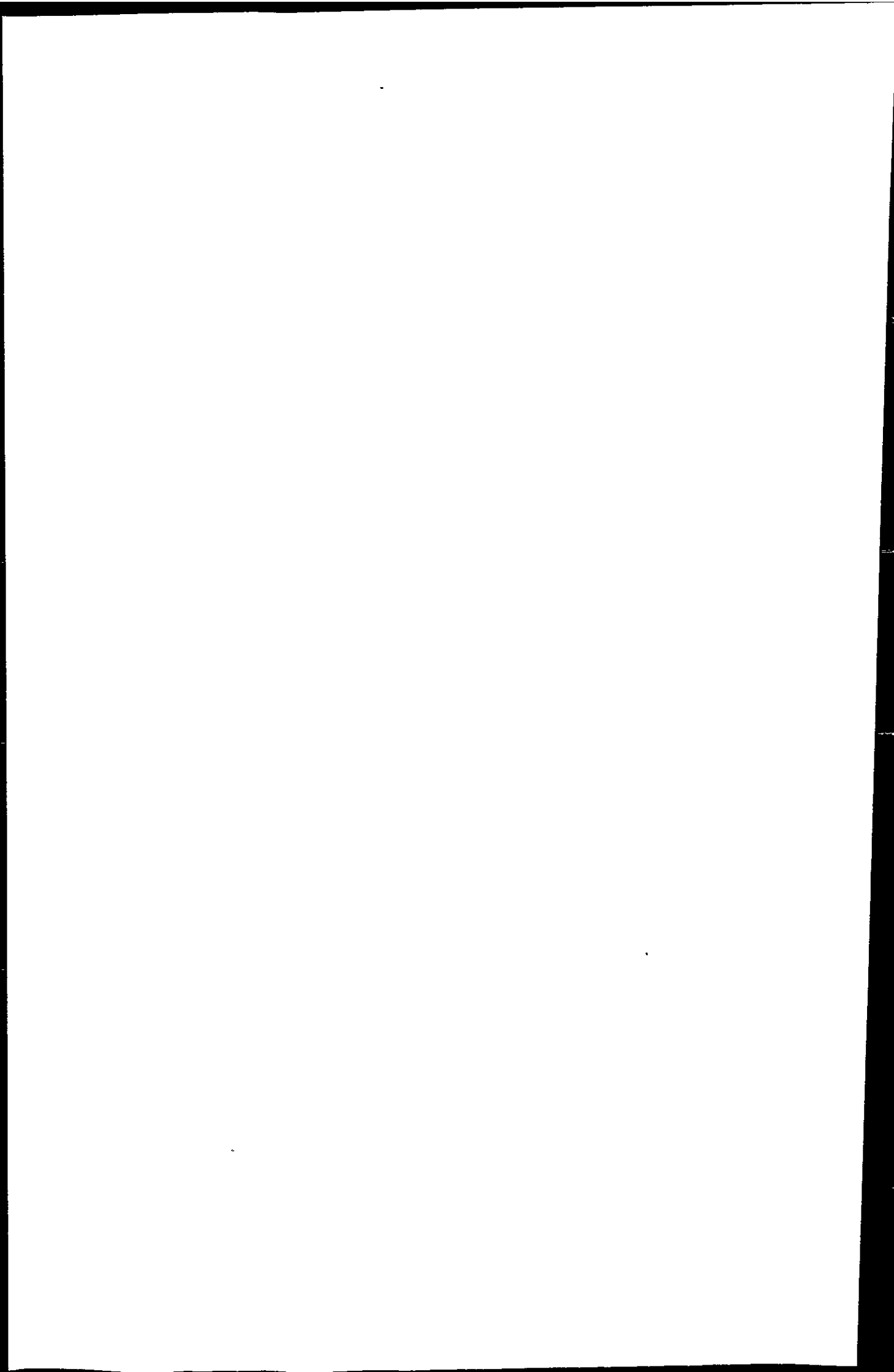
The assumptions are simply presented as a list, because the order in which they are made is not strictly relevant.

14.5.4 Box 3. Set up mathematics

The physical laws which are assumed to pertain to the problem are expressed in algebraic form, and the quantities involved are classified as known or unknown; constants, variables or parameters. From the idealisation is abstracted an expression or set of expressions which is purely algebraic in form, and which is amenable to mathematical manipulation. At this stage it may be found that more assumptions have to be made, for example about the relationships between the quantities considered.

14.5.5 Box 4. Solve mathematics

The mathematical system obtained in the previous phase (in these case studies, a first order differential equation) is solved. There is no reference needed to the physical or



other significance of what is going on: now we are in the purely mathematical world. The end result of the phase is another mathematical expression or set of expressions, but this time they are ready to be interpreted in physical terms. It may be found that the physical situation leads to a mathematically indeterminate or insoluble situation. Then either more assumptions may need to be made, or a different physical model (understanding the problem) adopted, or the mathematics framed differently (set up mathematics).

14.5.6 Box 5. Investigate implications

The next phase consists of looking at the mathematical model produced in the previous phase and seeing what it predicts. What sort of relationship between the variables does it imply? What are the effects of varying the parameters? What are the relative sizes of the effects? Do these seem reasonable? At this stage we are moving from the abstract mathematics back to our mental model of reality. If the implications are not reasonable, then we need to look back through the previous stages to determine at what point things need to be altered. It may be that the model is valid only over a limited range, if a constant value was assumed for a parameter.

14.5.7 Box 6. Compare with reality

Here the model is compared with real data. The program contains real data for each model to be measured against, either in the form of photographs or measurements taken in a real experiment. How well do the two match? Is the overall shape of the model consistent with real data? How confident do we feel in extrapolating from the known into the untested? In this phase the conceptual model moves back into contact with the real world.

100

100

100

100

14.5.8 Box 7. Write report

It is suggested that the report should be written for a variety of reasons. These include recording the process for oneself and others to follow in future, clarifying for oneself what has been done, and helping oneself to remember by the act of verbalising the process. The report exists now as an entity independent of the modeller, and the circle back into the real world has been completed. Because it is real it cannot exist in the software any more than the reality which is being modelled, so the title is in brackets.

14.6 *Case studies*

The contents of the different levels are summarised below. All the cases are applications of first order differential equations. This decision led to the exclusion of some visually appealing material, such as resonance, but it was felt that for reasons of consistency and simplicity it was better to limit the scope of the program. As will be seen, even first order equations required some algebraic subtlety in the higher level examples. (e.g. numbers 4 and 6)

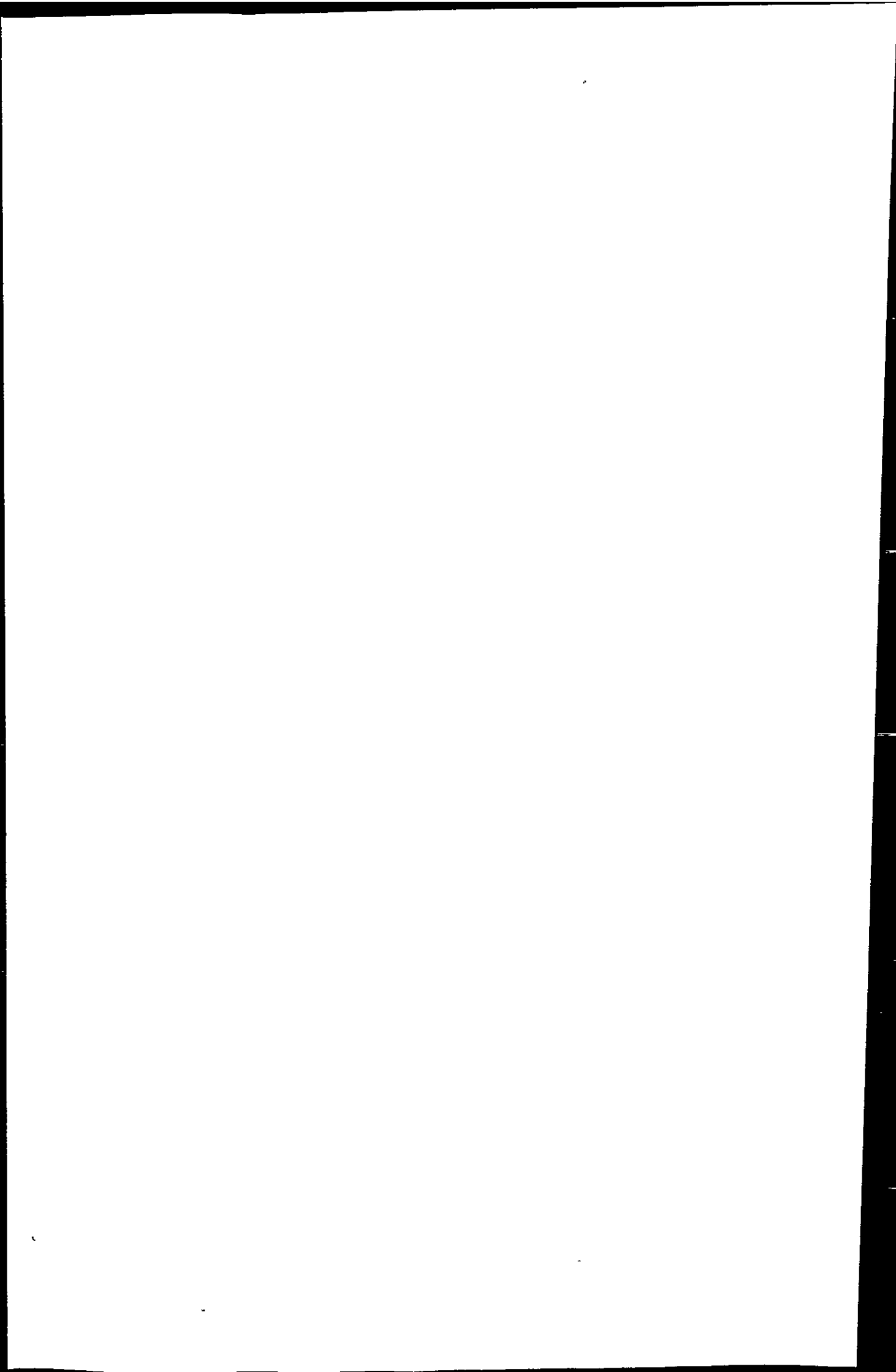
14.6.1 Level zero: introduction to modelling

The stages above are described in the context of a hypothetical model of a roller-coaster ride.

14.6.2 Level 1: the suspension bridge

Assuming that the weight per unit length of the decking is uniform, the analysis predicts that the shape of the chain will be a parabola.

A selection of photographs, both suspension and arch, are included so the model can be checked against reality. The user is asked why the analysis should be applicable to



a lightweight compression arch bridge. An over-the-deck arch is included so that the transition between the two types is smoothed.

This model does quite accurately match reality, so there is no need to adjust it.

14.6.3 Level 2: the cup of coffee

Rather than simply stating Newton's exponential law, the problem is treated as a conduction problem, a special case of diffusion through a membrane, with the thermal energy of the coffee as a reservoir. This leads to assumptions such as the liquid being stirred to keep it at constant temperature throughout.

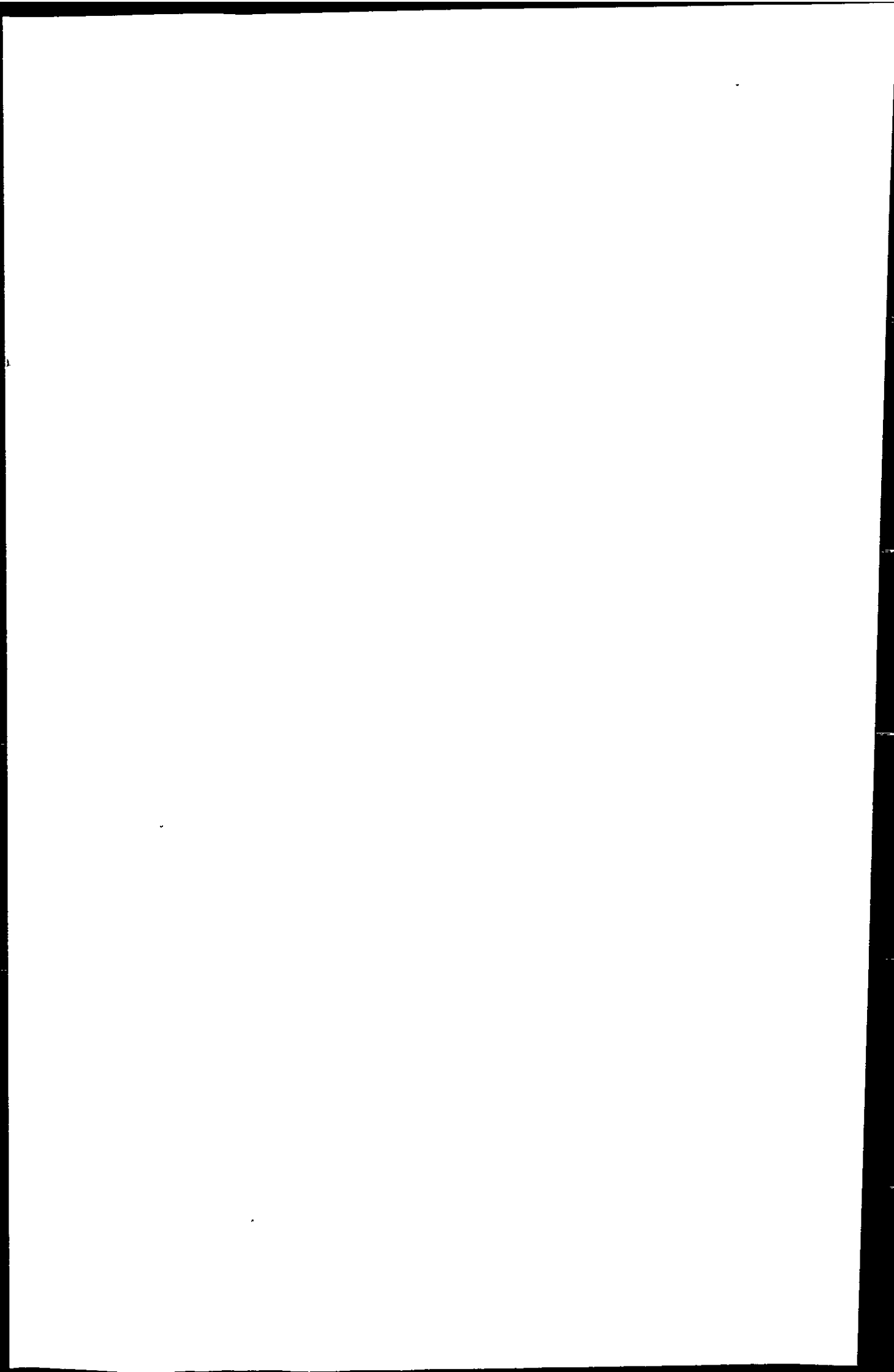
The classic Newtonian expression $T = T_A (1 + Ce^{-Kt})$ is obtained, and compared with the results of an experiment carried out with a real cup of hot water. The results are found to agree fairly well, but a better agreement emerges when the value of room temperature is adjusted upwards. An explanation, that there is a boundary layer effect, is suggested.

14.6.4 Level 3: the water tank

This is the first of a set of three circuits around the cycle. The way in which water flows out of a tank with a hole in the bottom is found to follow a parabolic law, and this agrees well with results from an experiment. However it is suggested that this does not match the normal way in which water is drawn from a tank, which is through a pipe. The program then leads the user onto the second circuit in modelling example 5, the tank with a pipe.

14.6.5 Level 4: the freely hanging chain

The modelling of the catenary is less straightforward than that of the suspension bridge, as the weight per unit span of the chain varies along its length. Solving the



differential equation is algebraically subtle, though not mathematically difficult. The expansion of $\sinh(x)$ is compared with the equation for a parabola, catenary curves with different values of the parameter may be compared with parabolae, and photographs of freely hanging chains may be measured to compare with the predicted shape.

14.6.6 Level 5: the tank with a pipe

As a first step, the pipe is treated as an extra depth of water, which leads to another parabolic model. However, this does not compare well with experimental results, and another circuit is proposed. (Level 6)

14.6.7 Level 6: the tank with pipe losses

The losses in the pipe are assumed to be proportional to the speed of the water in the pipe. This leads to an expression which is a quadratic in (dy/dt) . This type of differential equation is not often covered in engineering mathematics courses. A solution in terms of a parameter which happens to be the same as dy/dt at any given moment is proposed. The predictions for the flow out of the tank depend on which term in the expression for y predominates. If there is negligible pipe loss, the same parabolic solution as in the previous loop is found. If the pipe loss term predominates then the solution is an exponential decay. This latter is found to match the experimental results, and so it is accepted as a valid solution.

14.7 Form: how the advantages of the medium were employed

Transitions between sections are shown as passing from one "room" to another through a screen suggesting a pair of doors. This makes more explicit the progress through the modelling cycle, and also confirms to the user who has arrived at the section from the "home" screen that they have arrived at the expected point.

100

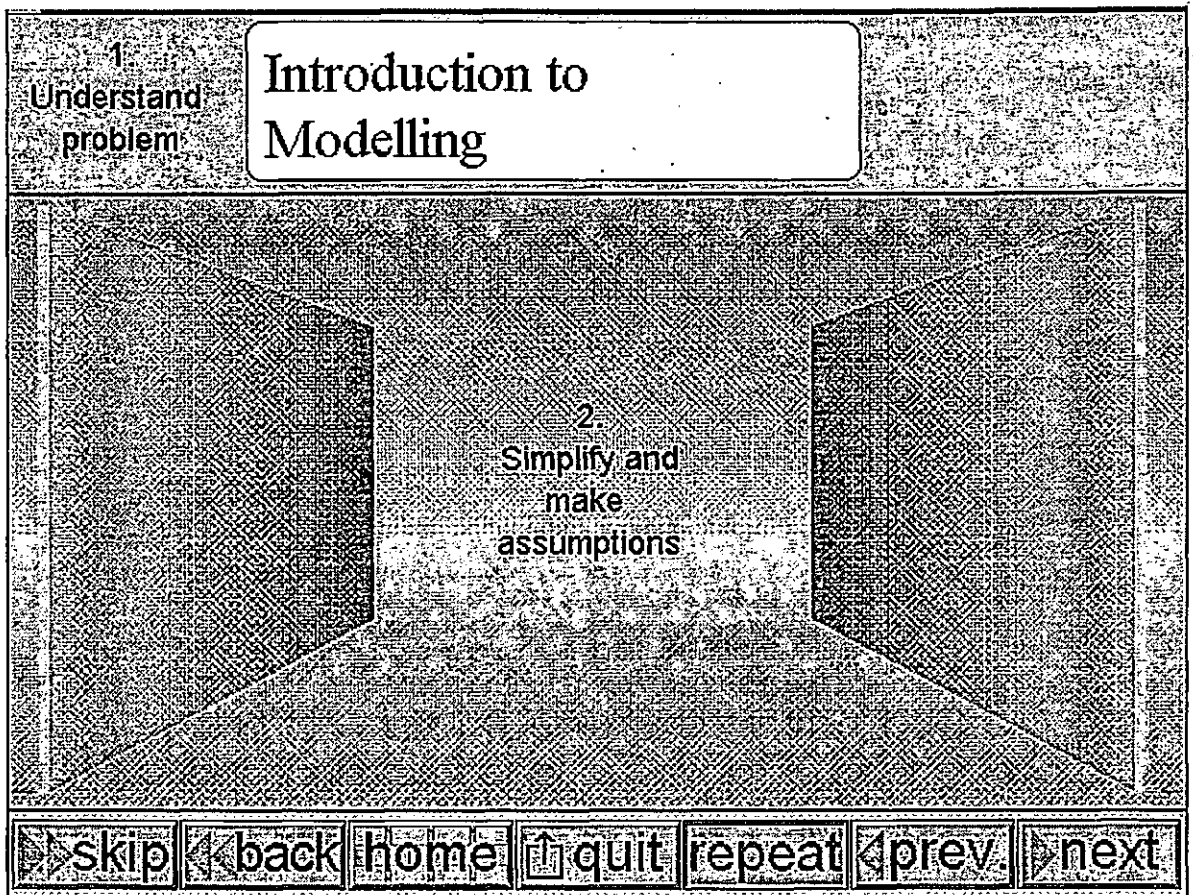
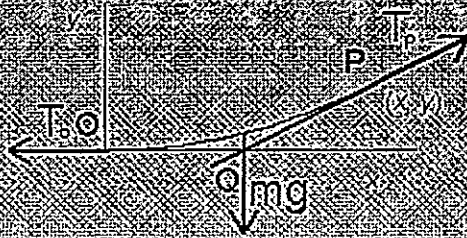


Figure 14-4: Transition page

In all the levels, text is built up in sections, allowing the user to progress at an appropriate pace. The new text is shown in black, and previous text in dark grey, so it is clearly visible, but it is obvious what is new and what old. All the text on the screen at one time is related. When a page is full, important equations are kept visible by scrolling them from the position in which they originally appear on the screen up to the top. This emphasises the continuity of the algebra.

100

The suspension bridge Modelling example 1



Let us call the co-ordinates of P (x, y) .

The coordinates of Q must then be $(x/2, 0)$ and the gradient of line QP is

$$\frac{y - 0}{x - (x/2)} = \frac{2y}{x}$$

$$x - (x/2) = x/2$$

This is also the gradient of the cable at (x, y) .

so we can say

$$\frac{dy}{dx} = \frac{2y}{x}$$

$$\frac{dy}{y} = \frac{2}{x} dx$$



Figure 14-5: Page with text and diagram

At many stages in the design process, animated sequences were discarded as being possibly distracting and adding nothing in terms of helping understanding. A few animated features were included where it was felt that they enhanced understanding. One example is the black line PQ in the figure above which is drawn from P through Q as an animation.

14.7.1 Level zero

No "special effects" were used in this level.

14.7.2 Level 1: the suspension bridge

One of the major advantages of software over plain text is that it makes it possible to build diagrams progressively, showing the sequence of drawing them, which may be done on a board or overhead projector, but not in a traditional text. This technique

is employed to portray the successive abstraction from a profile of a suspension bridge to a diagram from which it may be deduced that the gradient at a point is proportional to half the distance from the centreline. See Figure 14.5 above.

A collection of photographs of bridges is included and each photograph may be overlaid with a grid so the user can take measurements of the shape of the bridge.

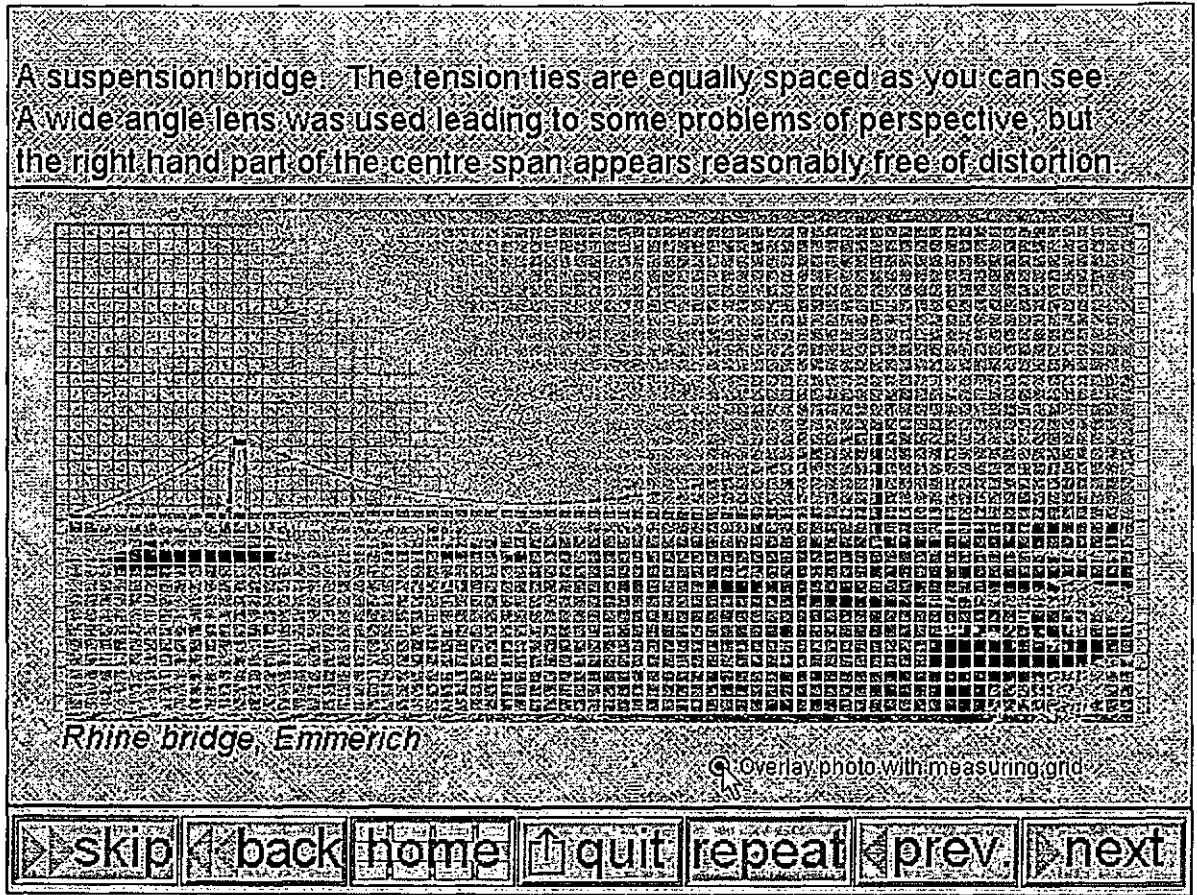
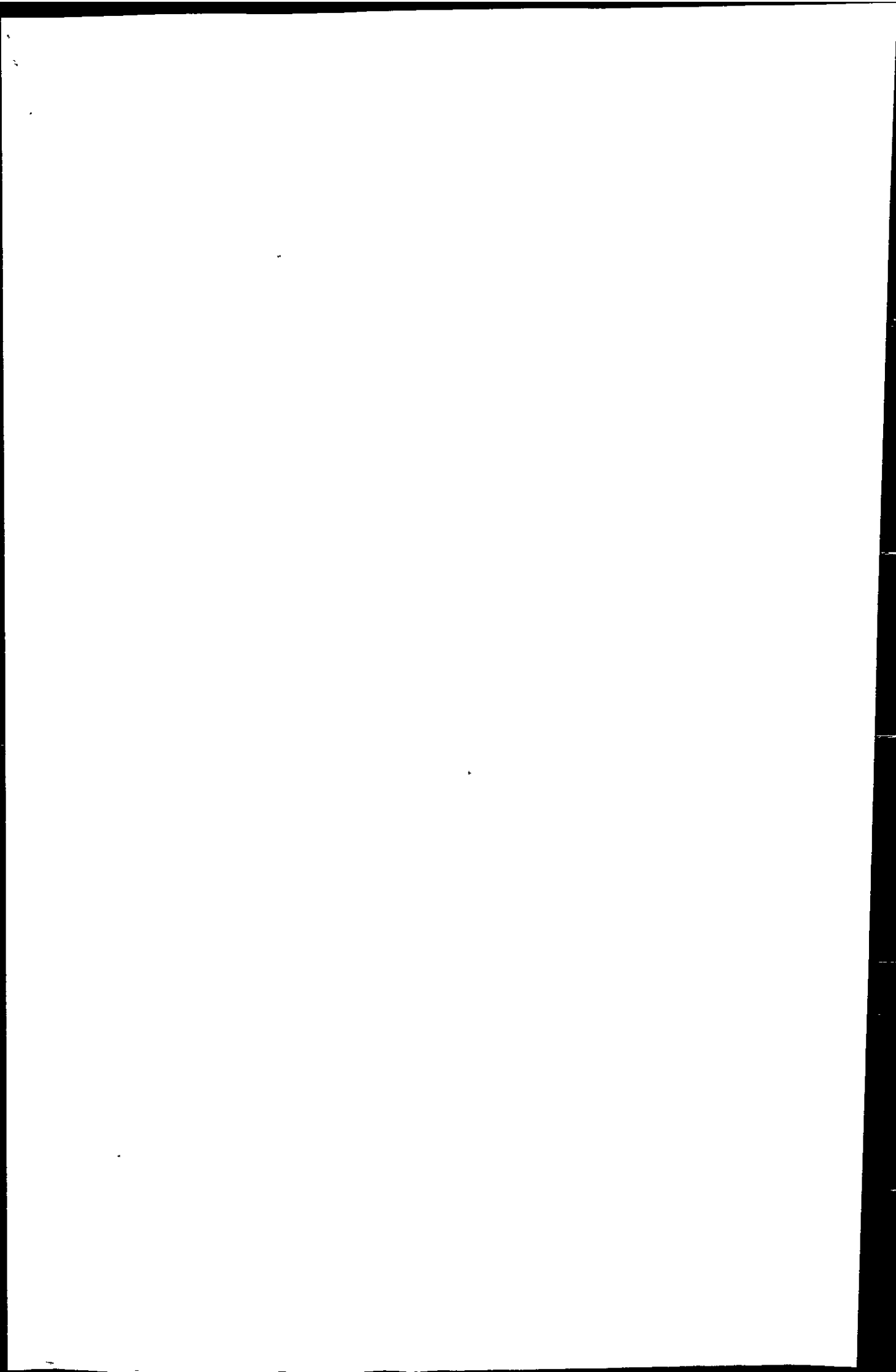


Figure 14-6: Photograph with measuring grid overlaid

14.7.3 Level 2: the cup of coffee

Animation is used to depict the different ways in which heat is lost from the cup, and that the coffee is stirred.



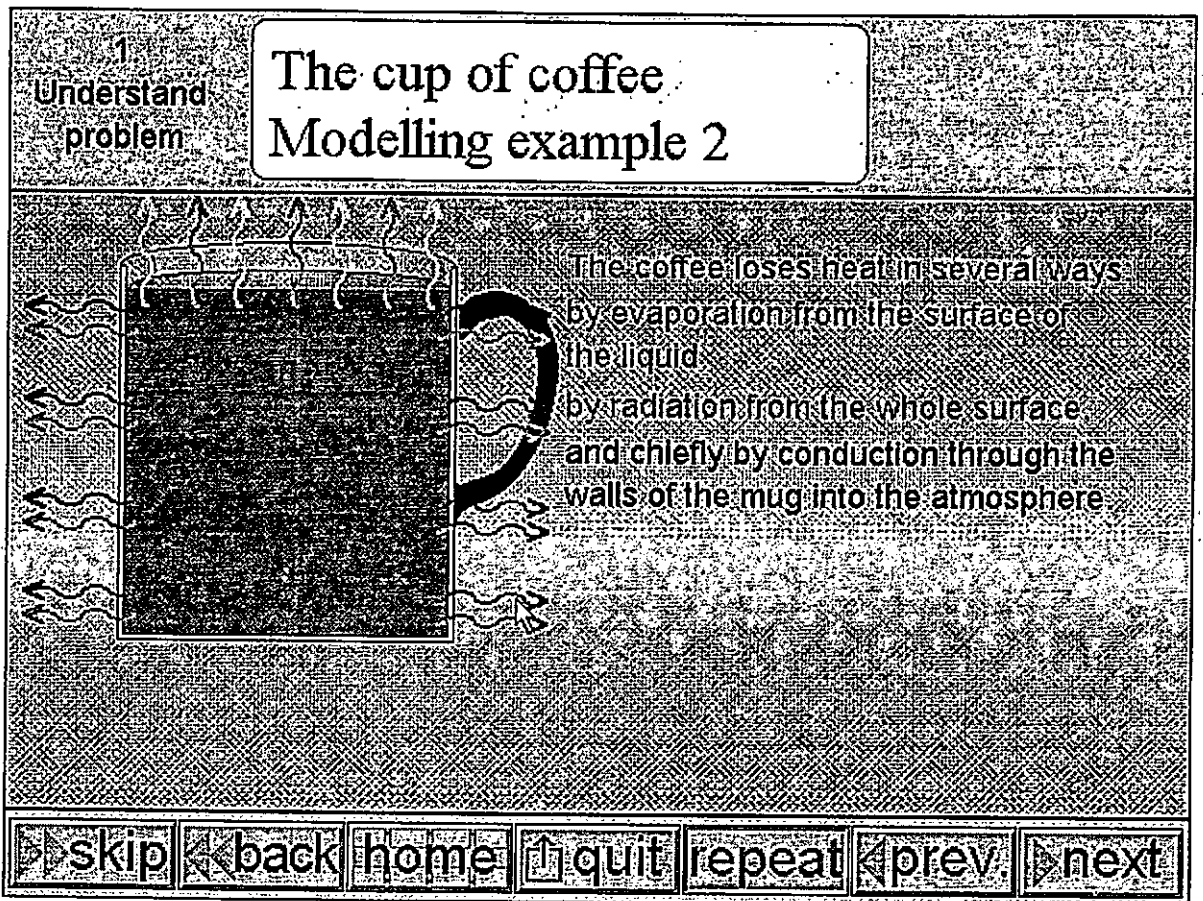


Figure 14-7: Animated screen

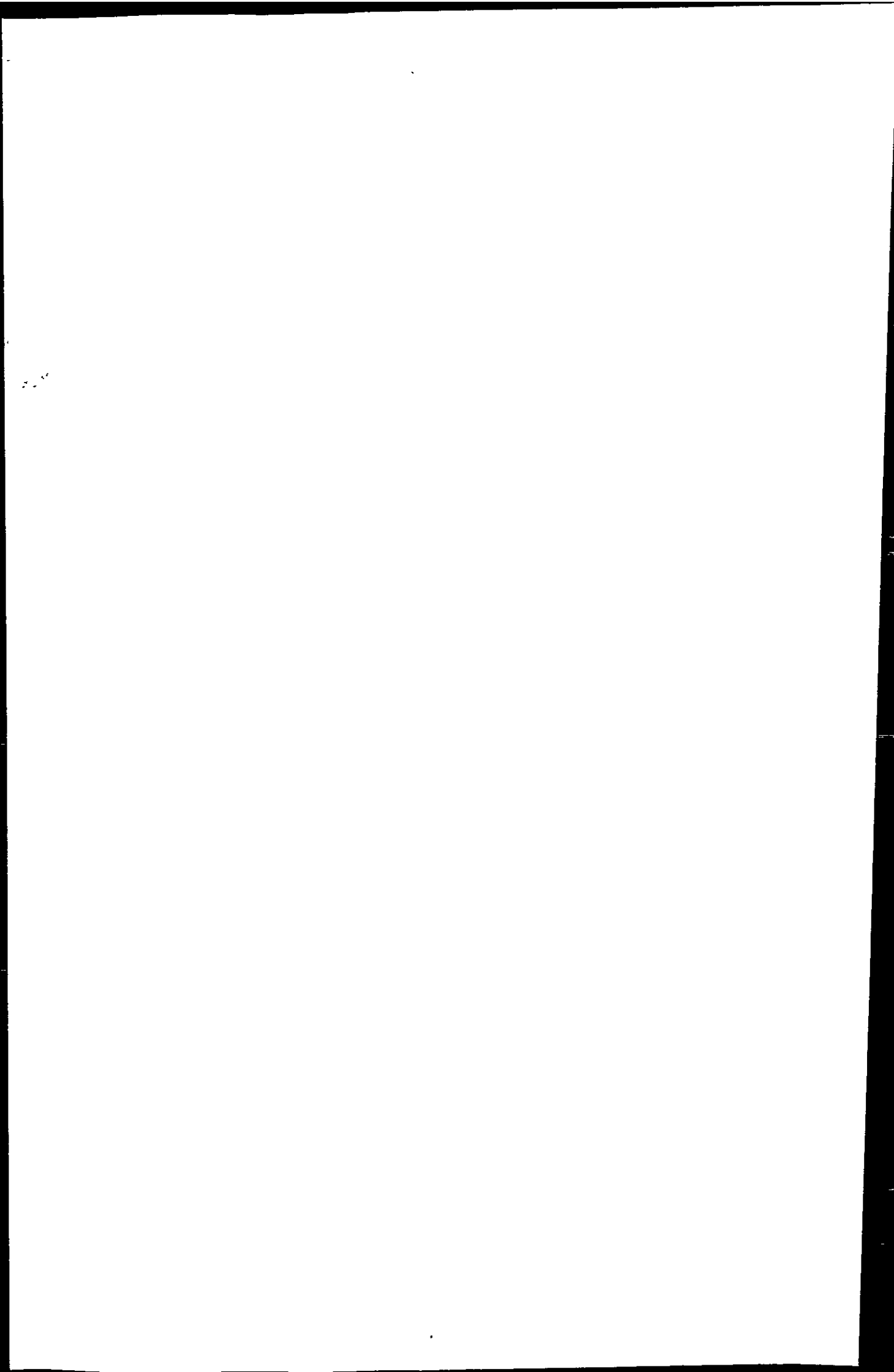
The graph showing the effects of changing the initial temperature is animated to show that the effect of starting at a lower excess temperature is to move the curve to the left.

14.7.4 Level 3: the water tank

The diagram of the water emptying from the tank is animated.

In the “set up mathematics” section, some of the terms are coloured red. Running the cursor over these terms causes a window to open with an explanation of the terms.

This allows a faster user to run through without interruption, but a slower user will be able to stop and see the explanations.



3
Set up
mathematics

The water tank Modelling example 3

Run the cursor over text in red for an explanatory note

at time t at time $t + dt$

In the control volume shown between t and $t + dt$

loss in PE = $(\rho A \cdot dh) gh$

gain in KE = $\frac{1}{2}(\rho A \cdot dh) v^2$

The volume of water that flows in time dt is $A \cdot dh$. (that is cross-sectional area \times height)
Mass = density \times volume, hence the mass of water flowing in unit time is $(\rho A \cdot dh)$

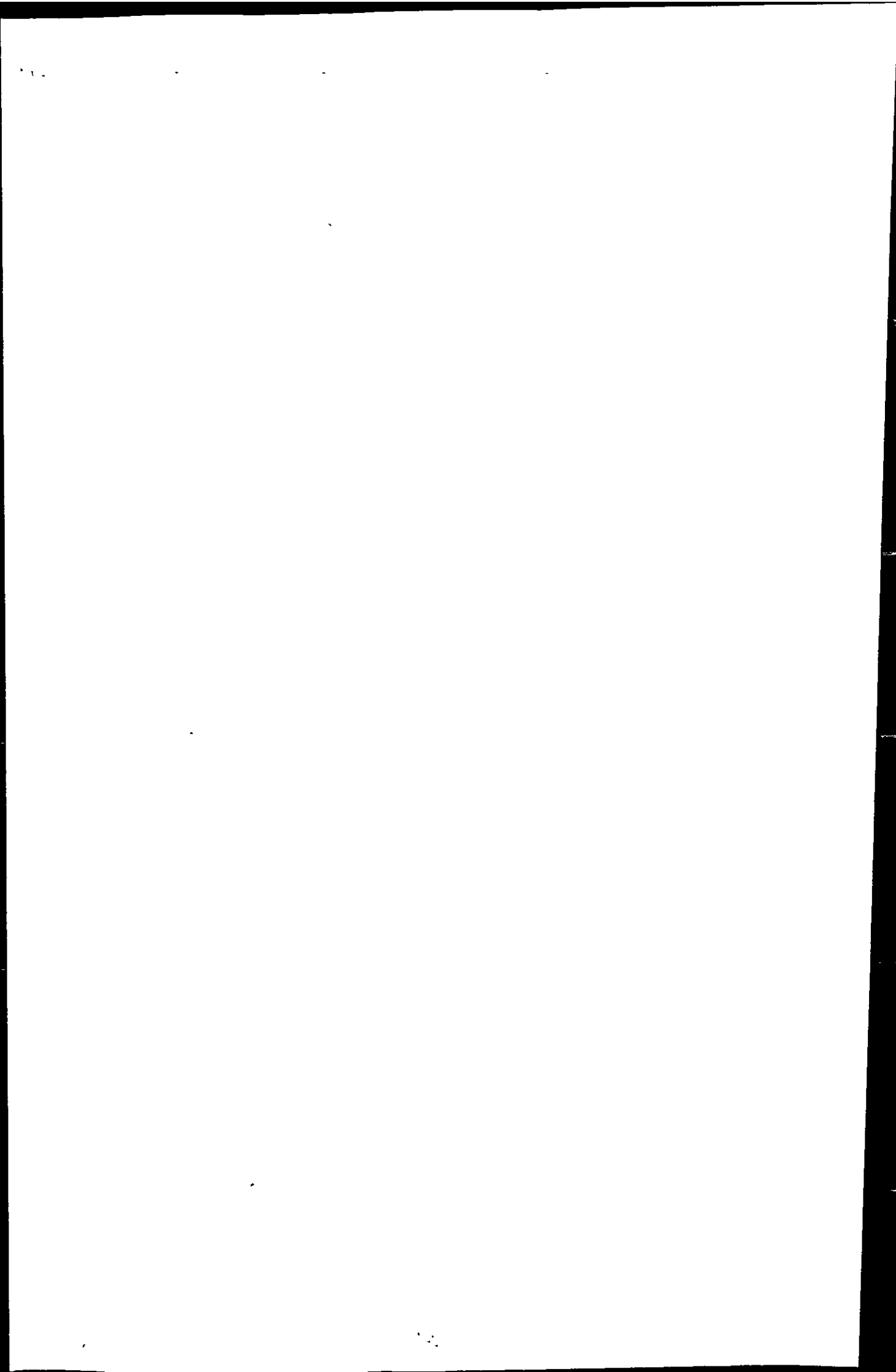
$(\rho A \cdot dh)$ is the mass of water which passes through the nozzle in time dt , where ρ is the density of the water

skip back home quit repeat prev. next

Figure 14-8: "Hot words" help facility

14.7.5 Level 4: the freely hanging chain

A graph showing the shape of the freely hanging chain compared with a parabola has a set of radar buttons which allow the catenary parameter to be set at different values and the differences between the curves to be compared. At small values of the parameter, the two curves the difference is minimal.



5.
Investigate
implications

The hanging chain Modelling example 4

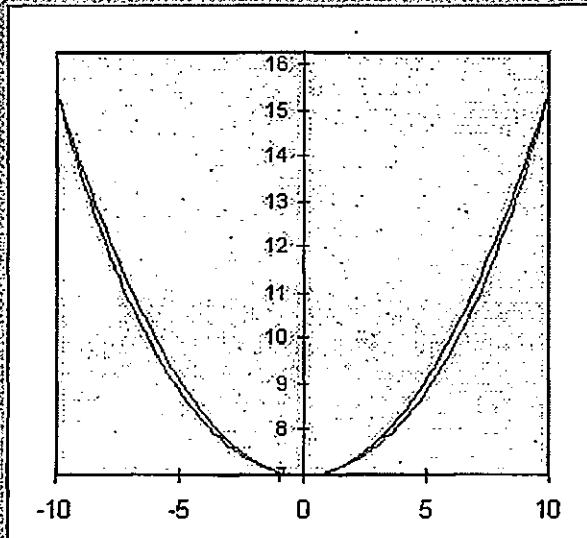
Compare the shape
of a parabola with a
catenary for various
values of w .

They become
indistinguishable at
large values of w .

Why is this?

Value of w

- 1
- 2
- 3
- 5
- 7
- 10
- 100
- 1000
- 10000



skip back home quit repeat prev next

Figure 14-9: Comparison of parabola and catenary

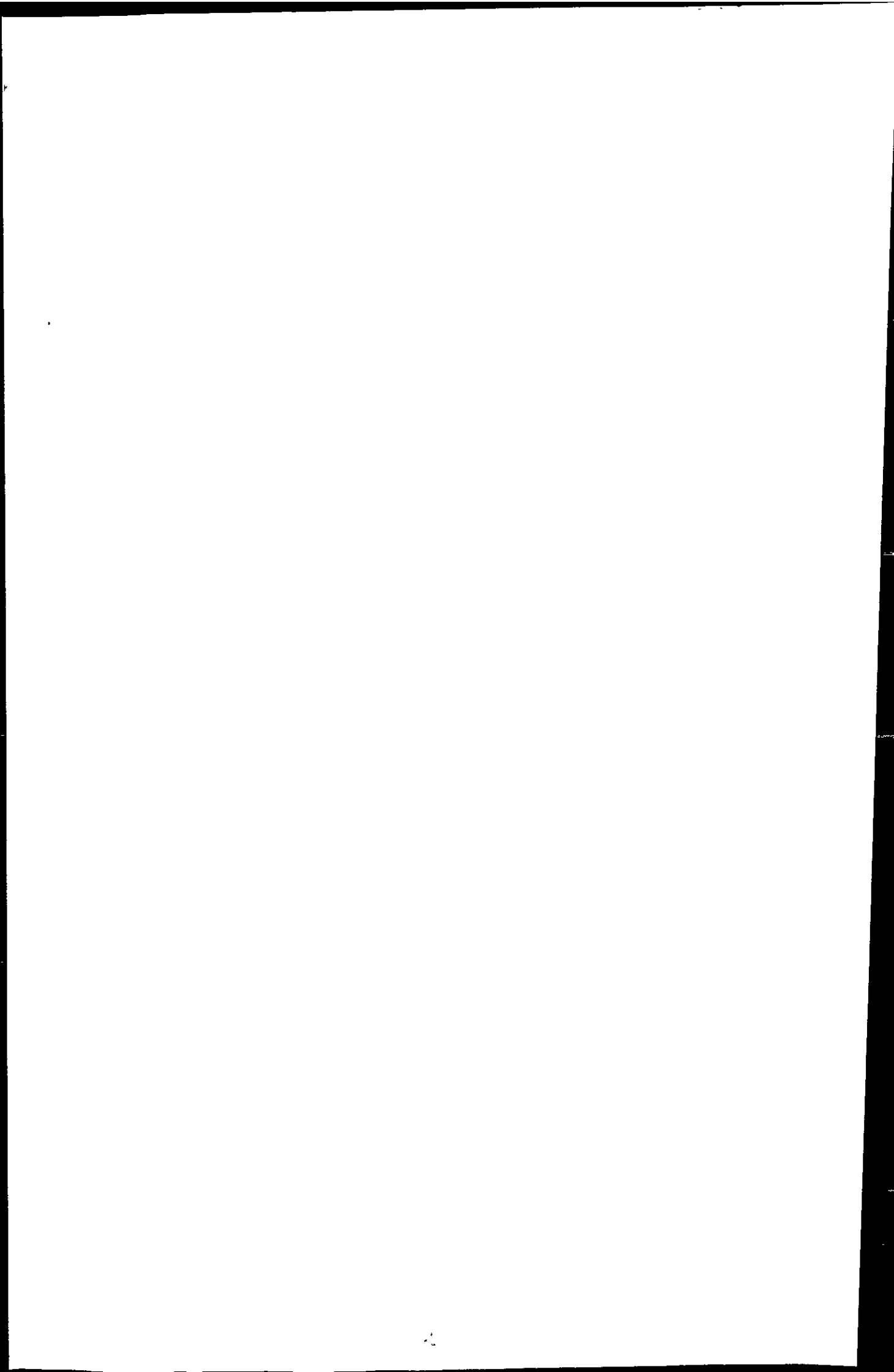
Photographs of freely hanging chains with the suspension points at a range of distances apart can be overlaid with a measuring grid to obtain data to compare with the model's predictions.

14.7.6 Level 5: the tank with a pipe

No "special effects" were used at this level.

14.7.7 Level 6: the tank with pipe losses

As the equations were built up in the "set up mathematics" section, a box on the right-hand side of the screen was used as a window to explain and comment on the progress in the left-hand column. The idea was to allow slower users to read the right-hand column, to help them understand what was going on, while faster users could skip through if they were able to follow the left-hand column.



3
Set up
mathematics

The tank with pipe losses Modelling example 6

Our energy balance equation becomes

Loss in PE = gain in KE + pipe losses

Putting the total height from the exit of the pipe to the water level ($h+H$) as h , for simplicity, we have now

$$\text{Loss in PE} = mgh = (\rho A \cdot dl) \cdot g \cdot h$$

$$\text{Gain in KE} = \frac{1}{2} mv^2 = \frac{1}{2} (\rho A \cdot dl) \cdot \left(\frac{A \cdot dl}{a \cdot dt} \right)^2$$

$$\text{pipe losses} = F_s = K \left(\frac{A \cdot dl}{a \cdot dt} \right) \left(\frac{A}{a} \right) dl$$

Remember that

work done = force \times distance moved in the direction of the force

Viscous resistance is proportional to the speed of the fluid flow:

$F = Kv$ where K is a constant which depends on the dimensions of the pipe and the viscosity.

Distance moved = speed \times time

Hence work done = $K \cdot v^2 / dt$,
where $v = (A/a) \times (dl/dt)$,
as shown above.

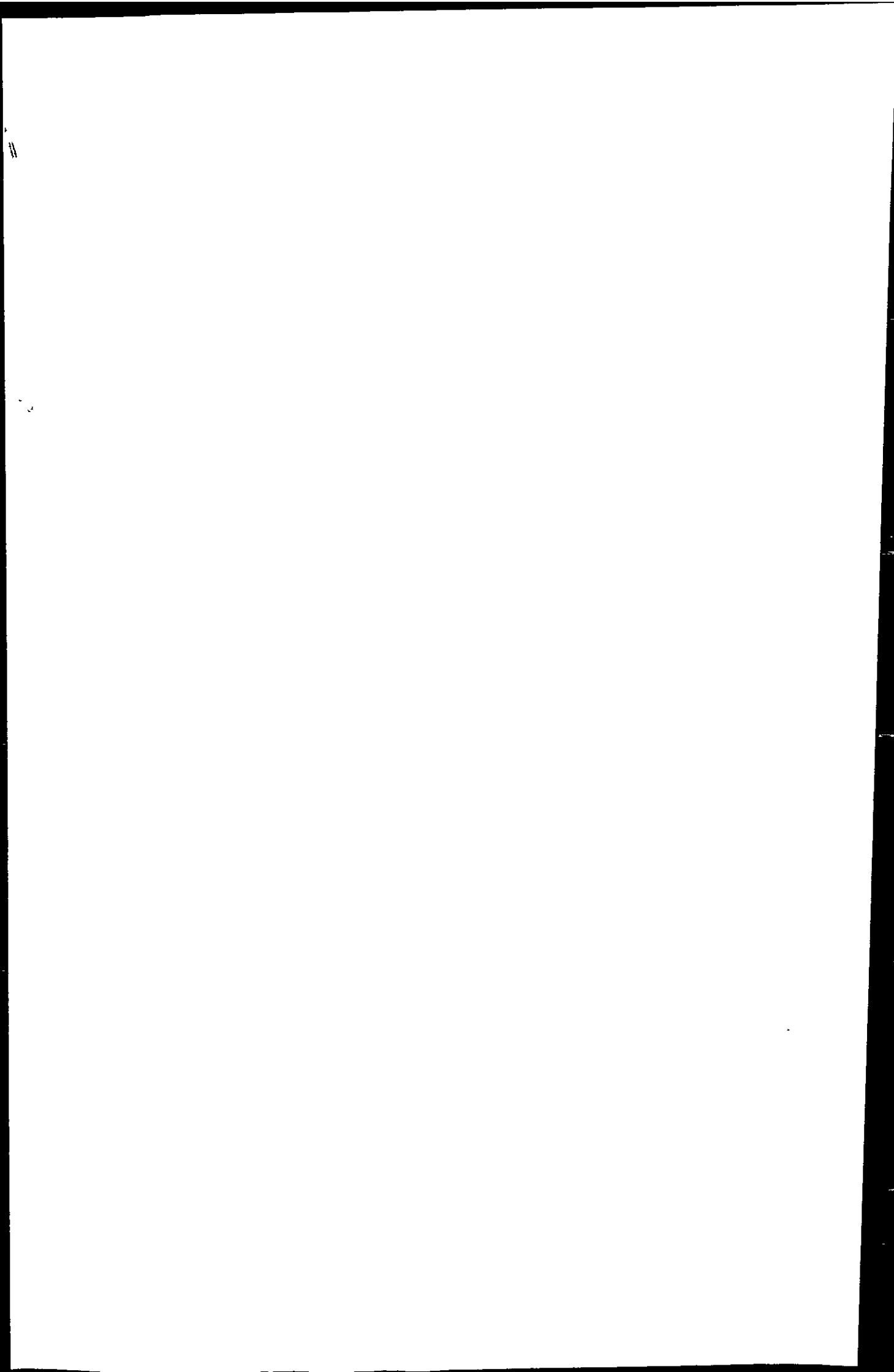


Figure 14-10: Commentary in right-hand column

14.8 Conclusion

As part of the research conducted Intermedia was used to provide access to information for two undergraduate courses. The results were inconclusive for the students but there was clear evidence that those responsible for authoring underwent significant changes in thinking style. (Begoray, 1990)

One of the most powerful parts of the experience of authoring the package in this project was the manipulation of equations as bitmapped objects, and the metaphorical cutting and pasting operations involved. Rearranging the mathematical objects on screen and substituting terms became very meaningful. It would have been good to have found a way to allow students to share that experience. Again, building the mathematical models included in the package was an opportunity to understand the processes involved and make them explicit.



15. Evaluation of the courseware

15.1 Introduction

Rigorous evaluation procedures are available and have been described by, for example Cronbach, 1988; Alkin, 1988 and Laurillard, 1988. By this stage in the project, however, we had come to feel that mathematical modelling skills were best learnt by carrying out mathematical modelling, and that the concepts of mathematical modelling were also probably best developed in the context of reflection on those skills, and in discussion with peers and teachers. The best that could be hoped for is that the courseware might provide a useful introduction to the subject, especially as it tries to establish a mathematical rather than an empirical pattern for modelling.

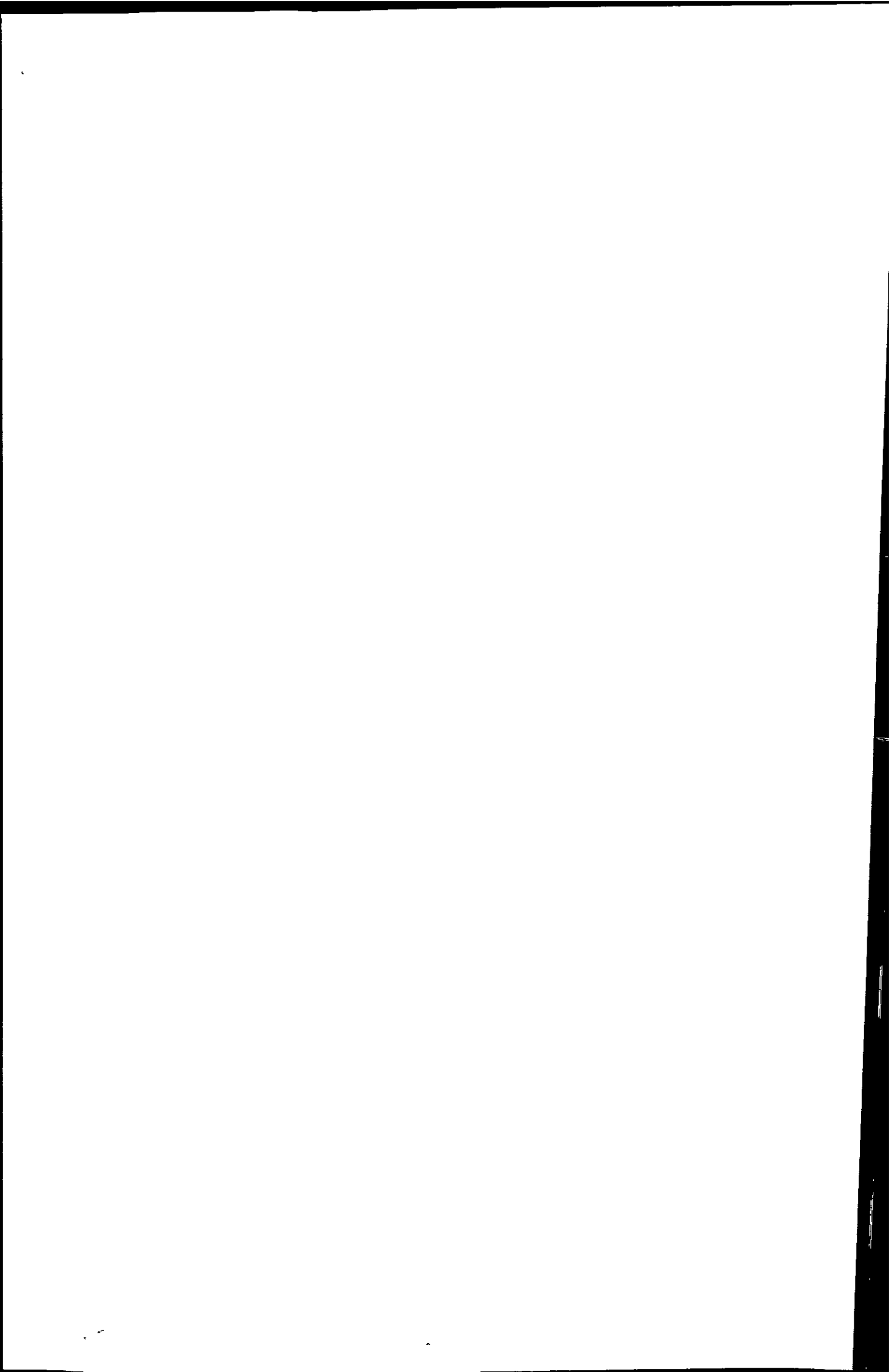
The software described in Chapter 14 was therefore evaluated by asking some final year engineering students to come and use it, and to comment upon it afterwards.

They were video recorded, in the case of one group of students, operating the courseware, and in both cases being interviewed.

Jed, a final year mechanical engineering student, used the package by himself, and went conscientiously through all the models. He was interviewed by himself immediately after using the package.

John and Gareth, mature students in the final year of the Manufacturing Systems Engineering (MSE) degree, looked at three models together with Martin, who is a German student also following this final year course. The UK students had taken a common first and second year course with the mechanical engineering students. The group of three was then interviewed together over a buffet lunch.

The video recordings were transcribed and the complete transcription may be found in appendix C



15.2 Style

Some aspects of the package were viewed positively by the students.

15.2.1 Overall impressions

The students were generally positive about the program overall. Gareth appreciated the sense of reality given by the inclusion of photographs. Jed compared the package favourably to a lecture.

WMM: What did you feel about it aesthetically, about the look of the program?

Gareth: I thought it was quite good, yeh, with the imported graphics, like the picture of the bridge. It was nice to see something in reality, that you're actually modelling from, so that picture just sets the scene, doesn't it? You can see the bridge, see the cable, everything.

Jed: It's a lot more interesting -, it's easier to grasp than standing in front of someone who's telling you about it.

15.2.2 Commentary

In general the students felt that the level of commentary was terse but adequate.

WMM: What about the level of the commentary- of the explanation that was going on?

Jed: I thought that was quite good.

WMM: Did you feel it was too high a level, too low a level?

Jed: I felt it was about right.

John: Well it was the beauty of that was the explanations were very short and simple. In a lot of books they're so verbose about what they're trying to talk about, when you analyse it, figure out what they're saying they could have said it in about four words: "This does not work".

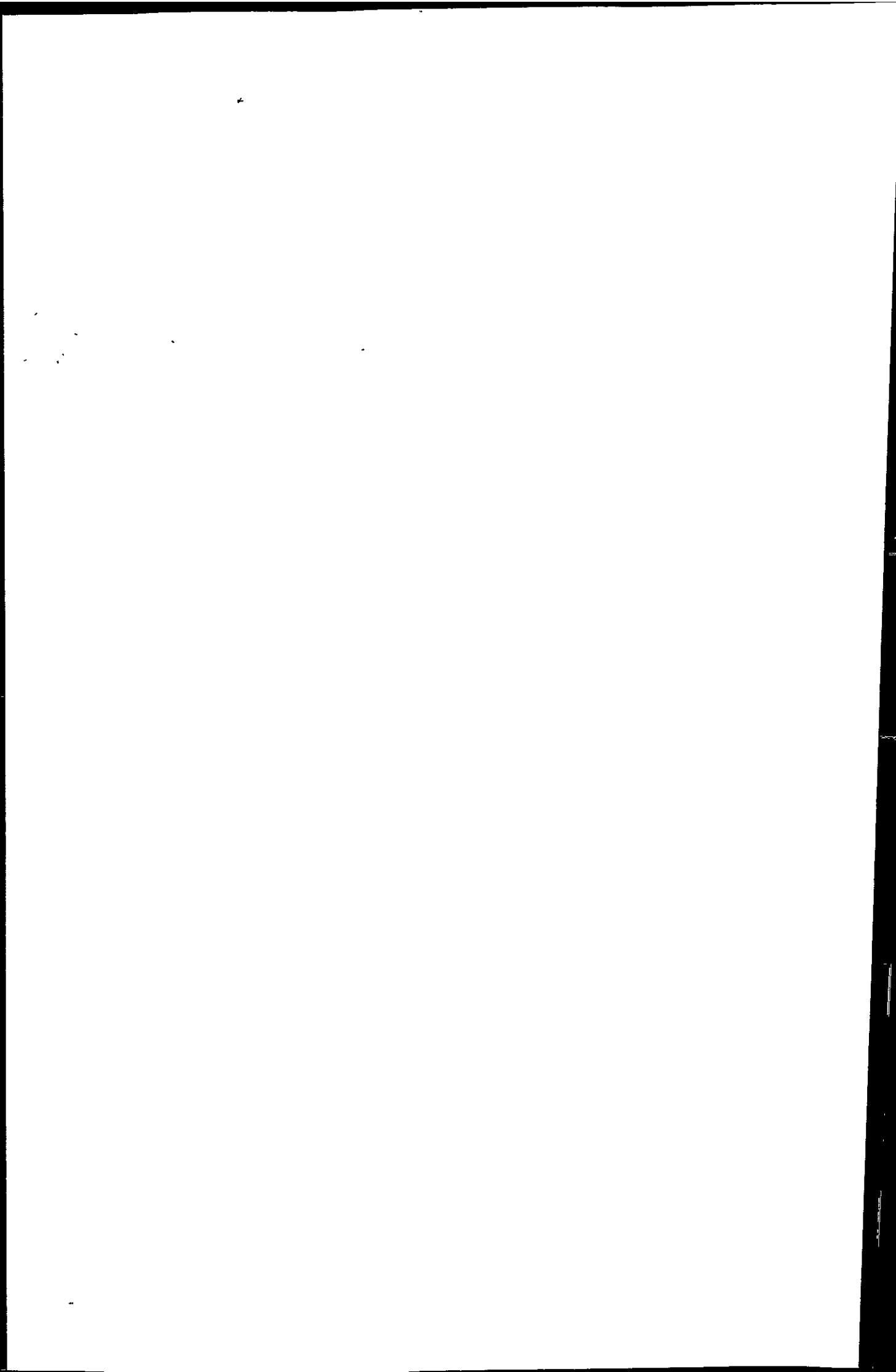
15.2.3 Navigation: the home page structure

The students understood the underlying structure well, although they had some criticisms.

WMM: The home page worked as a way of navigating?

Jed: Yep. You can base yourself from there. That worked quite well.

Jed: Perhaps on some of the longer modules, for instance investigate implications or compare with reality I thought it could perhaps do with a page numbering system or some sort of scrolling system. For instance I got about halfway through, I wanted to pop back and look at this page but you have to go right back and go



through it again.

I felt some sort of scrolling system perhaps would be useful.

WMM: Did you find using the home page easy to operate?

John: It was the same as doing it on the Internet. Might be nice if you put a bookmark in it, so you could bookmark where you are in it so if you if were disturbed in the flow of concentration, you could bookmark. You could then go back exactly to that point.

The criticisms seem to centre around the difficulty of going directly to a page in the middle of a section. The suggestion made to Jed, of “tabs” along the sides of the pages would seem to answer the problem.

Jed also had some detailed suggestions to improve the navigation: that the arrow keys on the keyboard could also be used for moving through the program, and that the “skip” and “back” buttons could be switched to match the “next” and “prev.” buttons.

15.2.4 Pace

The two groups had different interpretations of “pace”. Jed understood it to mean the rate at which the mathematics was explained; John, the rate at which the page was turned.

Jed: It went through too quickly for me because I haven't been using it but I think if I'd just come out of the maths modules that would be fine.

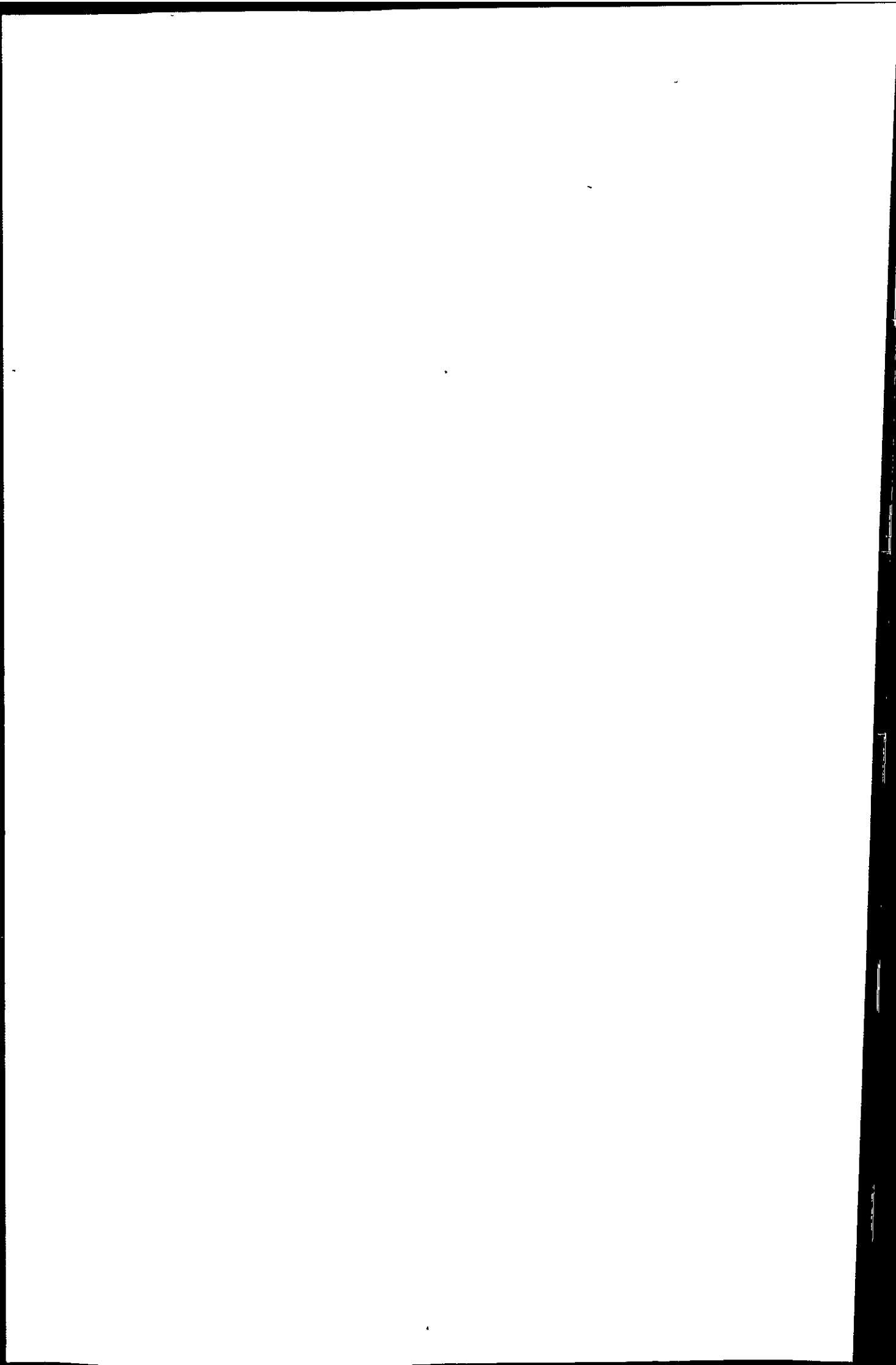
John: It went as fast as you wanted to click the button.

15.2.5 Help

Red “hot” words were the preferred help style.

Martin: Click on the red word and then it comes up as the easiest way, like Internet. If you have to go to a help menu on the right hand side, click down several points, keep going, it would..

Jed: I quite liked the red word although the danger of that is people might be lazy and just not do it, seeing they ought to be able to figure it out and skip past it.



15.2.6 Specific points

Jed appreciated the way the new text on each screen was shown in black while the existing text faded to dark grey.

Jed: I like the way the text changes between the active and the inactive. It keeps in focus else it ends up being just a big screen of words.

15.3 Content

15.3.1 The case studies

Were the case studies appropriate? Gareth and John felt that if the package was given to second year students they might find the applications unsophisticated.

Gareth: In the second year is where you do quite a lot of thermodynamics work. although the coffee cup was there it's not really at the level of a second year degree student.

John: Yeh, I thought something about most of that.

Gareth: You're going into gas turbines and steam plants and things like that so that kind of work was done a long time ago, i.e. foundation year, first year, so if you're pitching it at second years I think that you should be looking at an example of say, a gas turbine engine would be more appropriate.

Gareth: Just trying to think what would be more apt, really. Trying to think of examples we did last year. Can I think of one?

John: Water flowing through a tank? There was a very basic case study- the tank was straight, constant cross sectional area equal all through, then you could have another one with changing volume, a changing cross-sectional area rather.

Jed, however, felt that the case studies were suitable.

WMM: What did you feel about the context? Did you feel the case studies were relevant?

Jed: Yes I thought they were quite useful : the case studies. They were suitably practical.

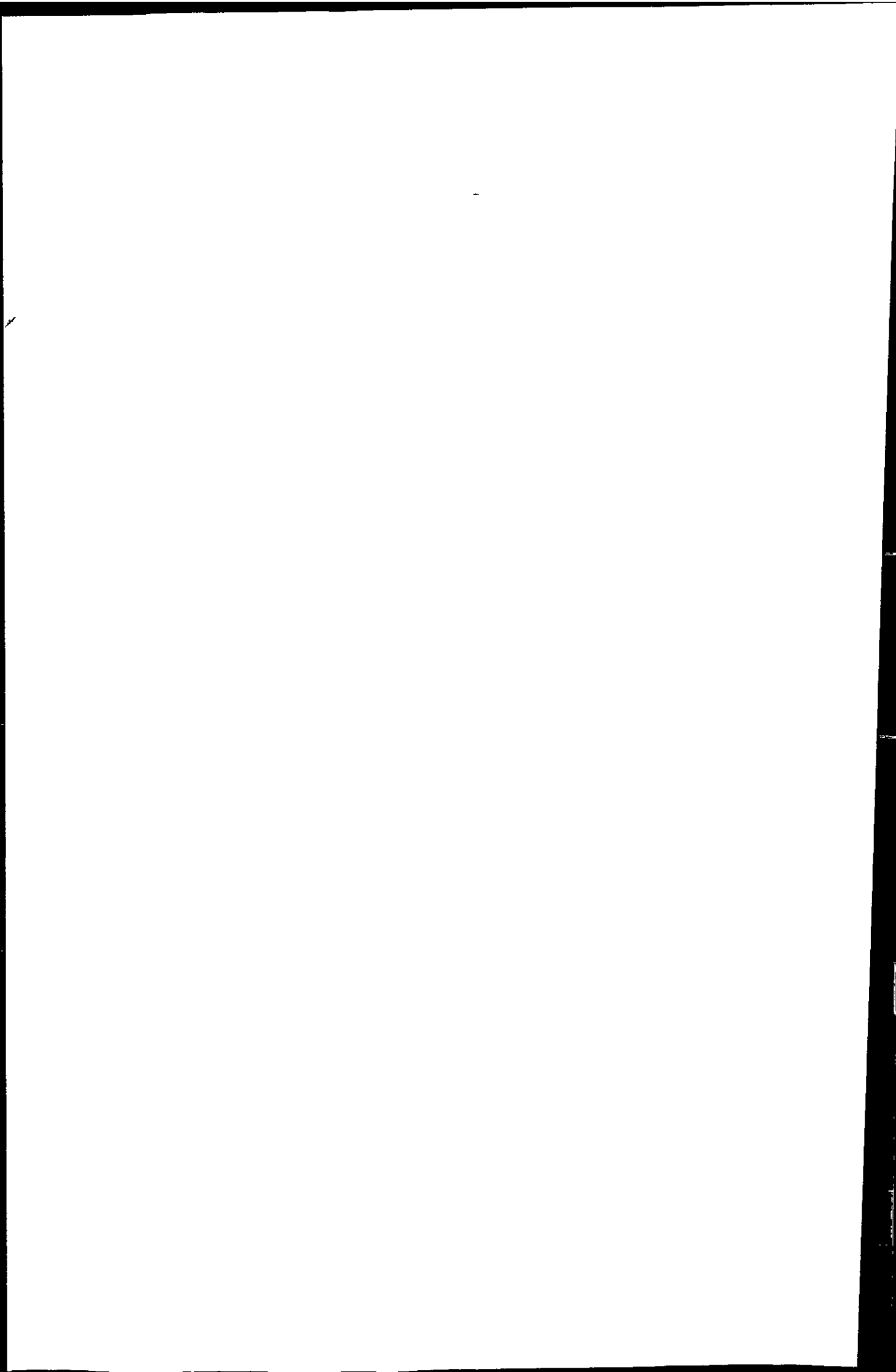
WMM: You didn't think they were.. Did you feel your intelligence was being insulted by any of them?

Jed: Not really, no. I haven't been using any maths for the last year being on placement so my maths is very rusty.

WMM: Is there anything you've come across in your course that you feel would make a good case study?

Jed: A good model, mmm.

Nothing that springs to mind , but I think those are good choices, the way they cross over to the water tank with the hose and the hose with losses.



15.3.2 Level of mathematics.

John and Gareth felt that the level of mathematics was appropriate for students early in their engineering courses. They had not looked at the later models, however, in which such unappetising items as a quadratic in (dy/dt) appear.

John: I think it should be aimed at first year because they've already... That's more of an introduction.. They've already done a lot of the stuff that's in that book before they get into the second year. It would be like, regressive, if you like. It would be better if that were introduced earlier rather than later on.

Gareth: I mean even to the point where on the foundation year, you're doing differentiation at that stage and I think it's a key point to get across that this is a tool used by engineers to model situations which they are trying to overcome. Cause you can get lost in the maths without seeing the relevance to the real world in which we're living. Whereas that's quite good with the explanations like the coffee cup and the bridge and things like that, why we actually use differentiation.

Jed did not seem to have engaged with the mathematics content. Dwyer and Dwyer's (1987) research, as described below, appears to be relevant here as well. Because the interactivity is low, Jed has not spent enough time in contact with the information, either in transforming it or in some psycho-motor activity, to elaborate it, and it has just washed over the surface.

WMM: The level of the maths varies quite a lot from the first one through it. Whereabouts do you feel happiest?

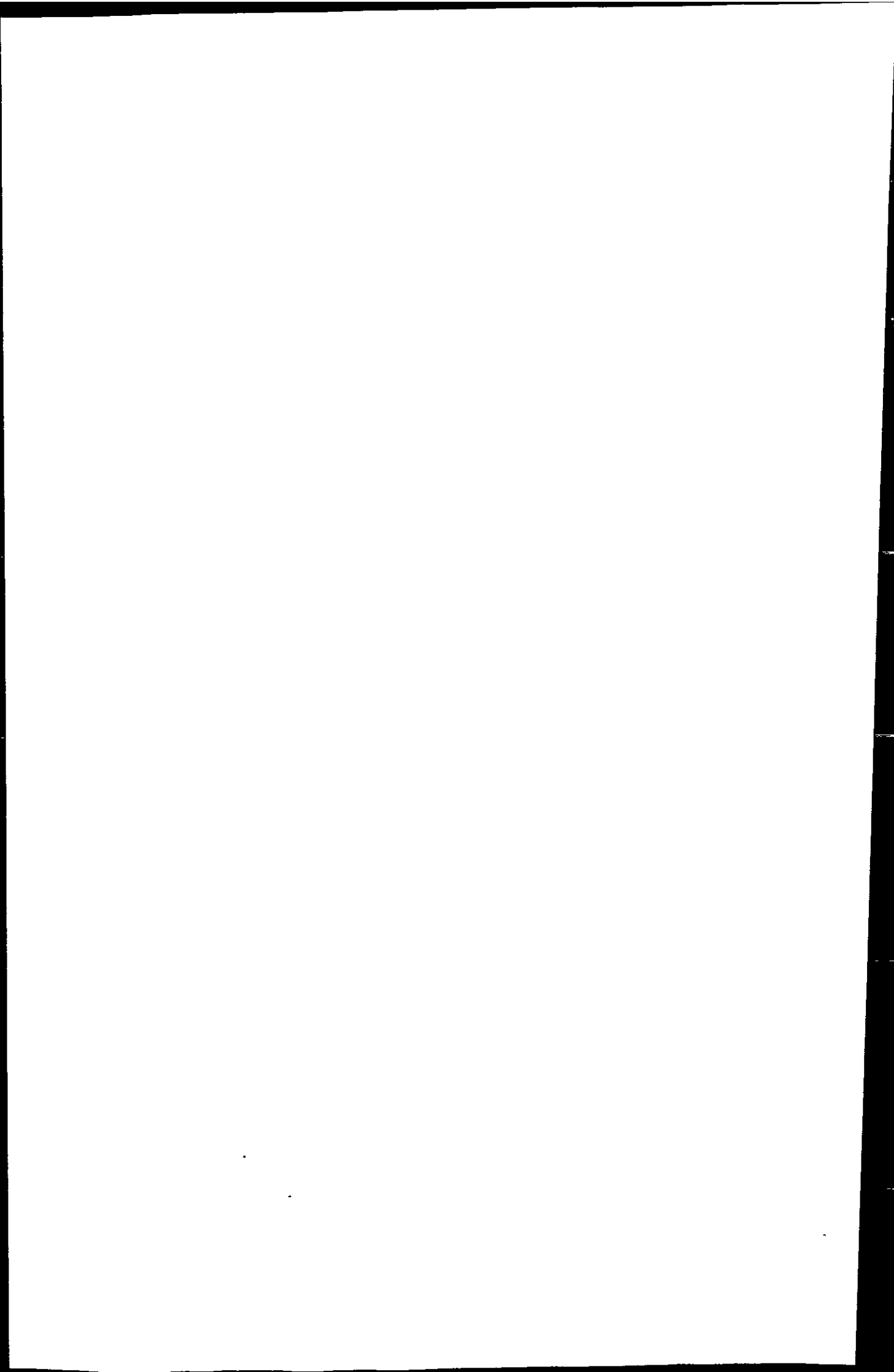
Jed: Well, given a reference, I'm happy enough with understanding the calculus- I've forgotten all the transforms myself. When you use them a lot you know them, you just click them in, but I've forgotten all that. My maths is very rusty- I haven't been using it for a year and I haven't had to use it so far this year.

Jed: I sort of tended to follow what was going on without actually examining the maths. I understood what was happening without doing the sums as it were in my head.

15.4 Discussion

15.4.1 Learning processes

Although the details are different, both John's and Gareth's strategies for learning incorporate a transformative element: translation into John's own words, transformation into Gareth's pictorial form. This elaboration reflects Dwyer and



Dwyer's (1987) findings on the depth of information processing and the effect on students' ability to acquire and retrieve information. They found that when students converted information from the visual to the verbal mode, and performed such physical activity as taking notes or writing an answer down, the interaction between the learner and the information was extended long enough for in-depth information processing to take place, which would make the student more likely to be able to recall the information in the long term.

John: I learn by writing things down. I can write everything down in my own words. In my notes you have to translate.. But then you know that because you had all my notes. I can write them up in my own words.

Gareth: I know if I'm studying for an exam I'll just get one big sheet of paper and put like what it is in the middle and then just draw it an all around do various shapes, like somebody's goals then I'll draw a set of goals and then when I try to remember it...

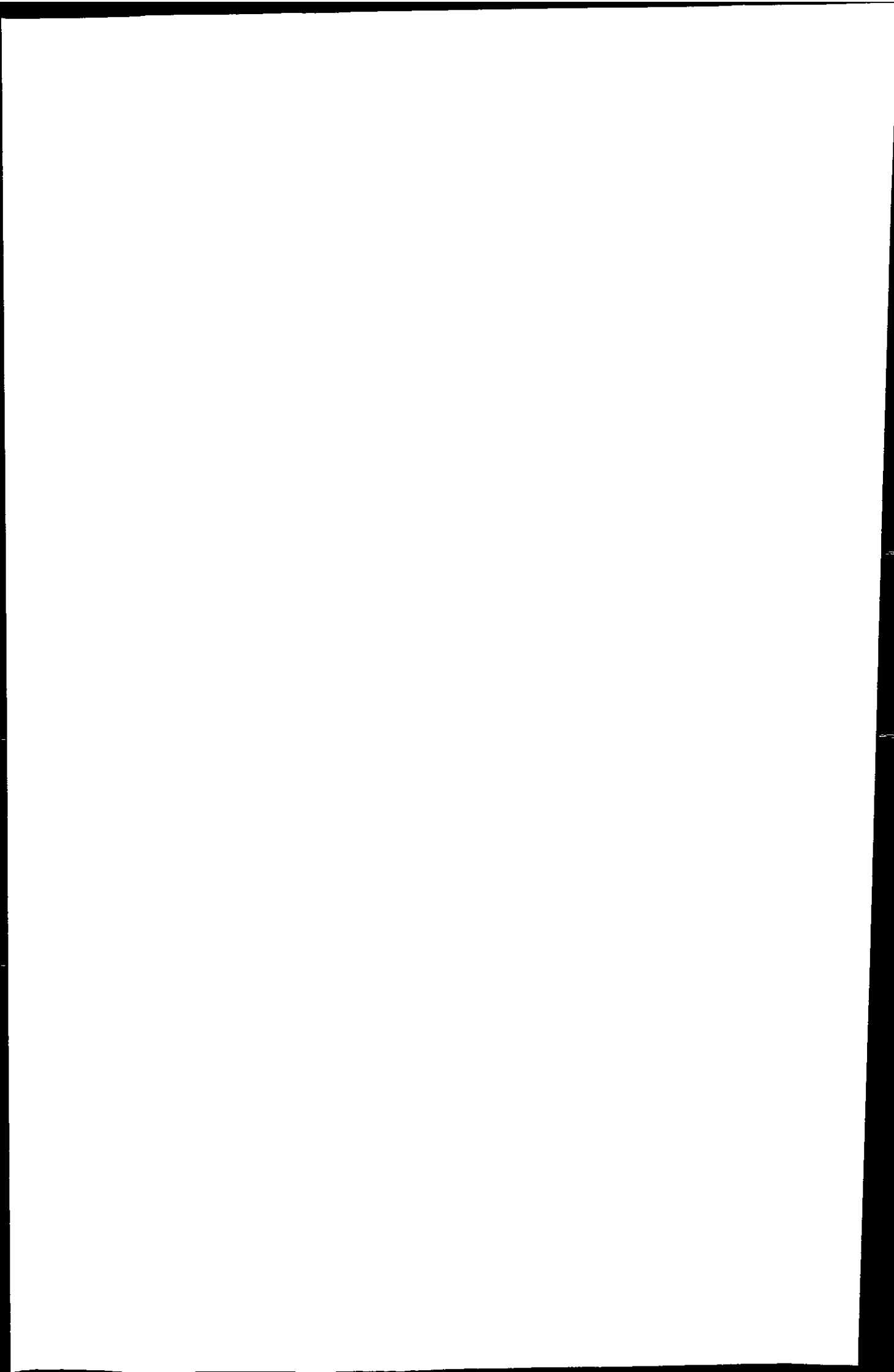
Jed also liked the idea of a suggestion for spending time on a model, because he implies that in general lectures are too rushed to have time to think.

Jed: I liked particularly - There was particularly in that section, the bridge I think it was, there were suggestions for something to think about that sort of came over-further investigations you can do. I particularly liked that because often when you're being taught the lecturer's desperate to get through the subject so they don't have time to stop and talk about that. That's quite useful - keep people thinking hopefully.

15.4.2 On the effects of using the package

What happened to the students as they were using the package?

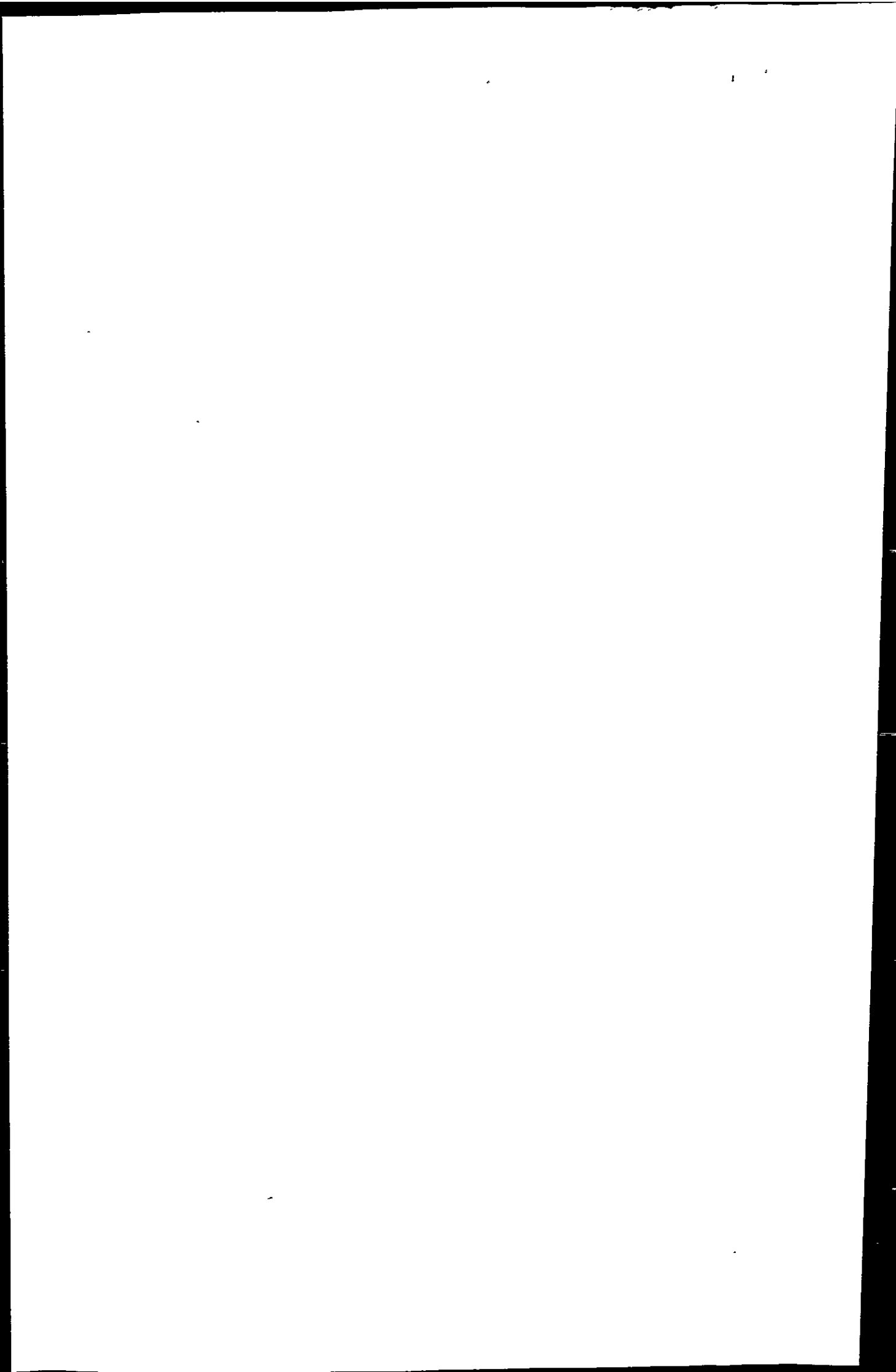
The most encouraging aspect was that the MSE students were provoked into a spontaneous discussion of the nature of modelling. This discussion seemed to start with a semi-audible comment by Gareth that the model of the coffee cup predicted that the coffee would never reach room temperature, but that of course it would in reality. At that level he had engaged enough with the program to test it against his internal model and to begin a conversation with it.



Jed was conversing with the program in that he wanted to go back and see what had gone on earlier, because he wanted an easier way of going back than provided by the buttons. On the other hand, he did not engage with the mathematics, as he says: "I sort of tended to follow what was going on without actually examining the maths. I understood what was happening without doing the sums as it were in my head." I suspect that the MSE students did not engage with the mathematics either, since they dismissed that aspect as simple, and suggested ways of forcing users to work through by including multiple choice questions at that point.

The idea of using such a stratagem raises other points. The part of the modelling process which is the most challenging for engineers is the transition between the real world and mathematics, that is simplifying and abstracting the mathematical problem (OECD, 1966, as quoted in chapter 12). This is the part of the cycle in which it is most important for users to engage. It is also the part in which correct answers are most difficult to define (see Graham E, 1997). If some form of what appears to be assessment is included, then students will concentrate on the part that is assessed (Hargreaves, 1997).

In the package as it stands, there was some engagement with the "investigate implications" shown by Gareth's comment. This was an intrinsic engagement, as no extrinsic provocation such as self-testing was present. If such a provocation is included elsewhere in the package, in order to force engagement in parts with less intrinsic interest, it will down-play the perceived importance of the parts which do have intrinsic interest. It would be difficult to include self-testing of a simple multiple choice type in the most important and difficult part of the cycle, that is the simplification and abstraction stages. We have seen in chapters 3 and 12 that these are the stages that engineering students find the most difficult.



In summary, as Connell (1997) observes:

Typically what is done is to model expert performance, which does not include all the muck and mess which the expert went through to develop it. The learner in such a system is presented with what represents the end-level trace of an incredibly complex act of meaning construction. It is little wonder that so many students tune out of [integrated learning system] programs.

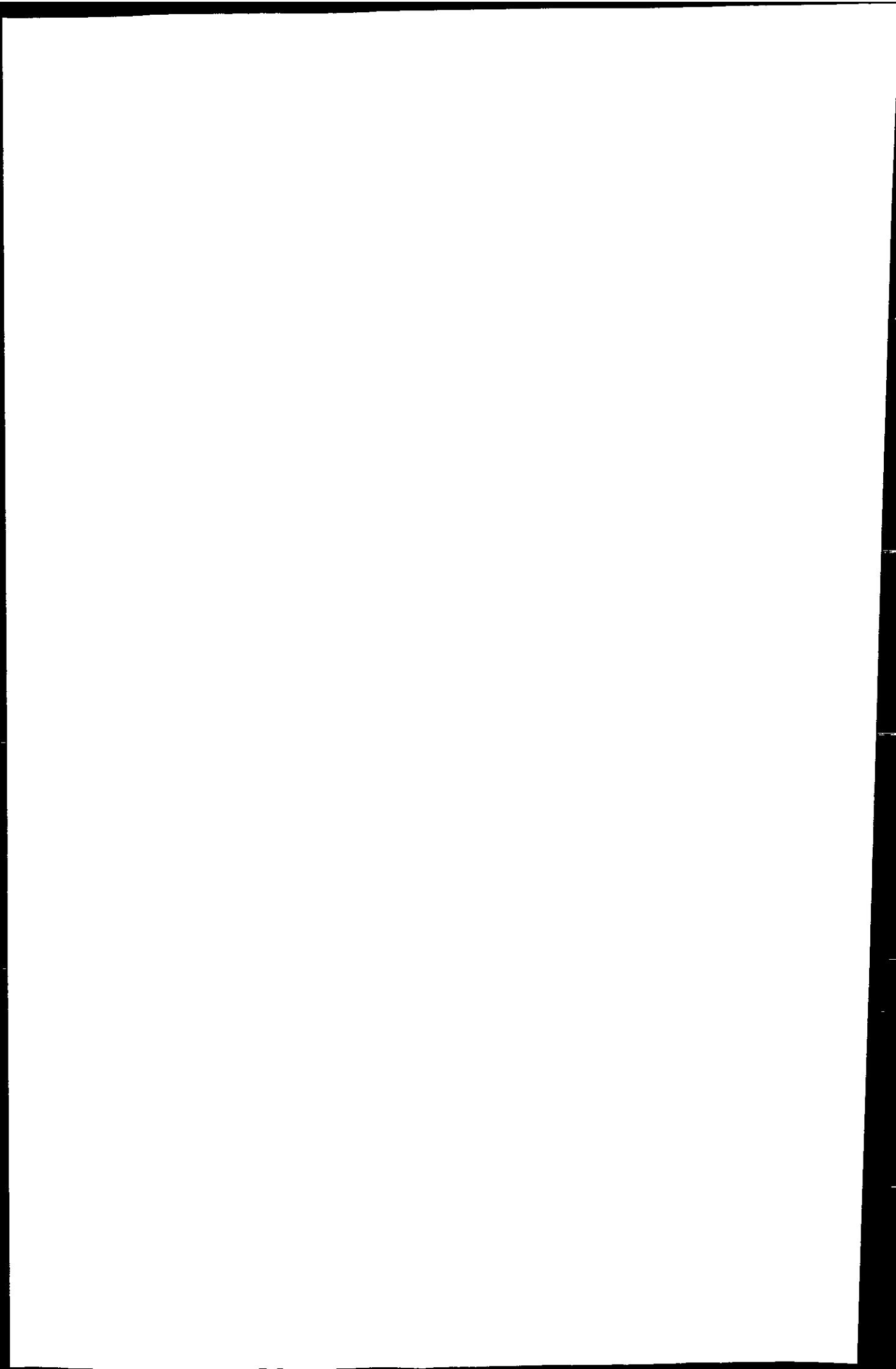
15.5 Conclusions

15.5.1 On the package

One of the rules of instructional design is that the author should know who the package is aimed at. In this case the target was "engineering students". This proved to be far too vague: the students did not feel that it was addressing their needs, although we saw in chapter 3 that their analogues had been unable to produce a model of the cascade problem.

The interactivity suggested by John reflects a behaviourist paradigm, verging on mastery learning, where the aim is to find the correct answer. The stage at which he suggests it should be used: "can you perform this integration?" is probably not the most critical point of the cycle, and the object of the program is to build up a concept of the modelling cycle through repetition, showing how the same stages apply although the model may change and refinement may be necessary at different stages. This is what Jed identifies as "*the homogeny of the principles behind modelling. The principles behind it.*"

It is recognised that the package needs more interactivity built into it, and that this should be made more obviously available. The group of students did not look at the bridge pictures, because the button leading to those pictures was not in the same place as the "next" button. A better approach to interactivity may be to interleave a page asking the student to score the package on how well the student feels they have understood it so far, which they cannot pass until they have completed it. The

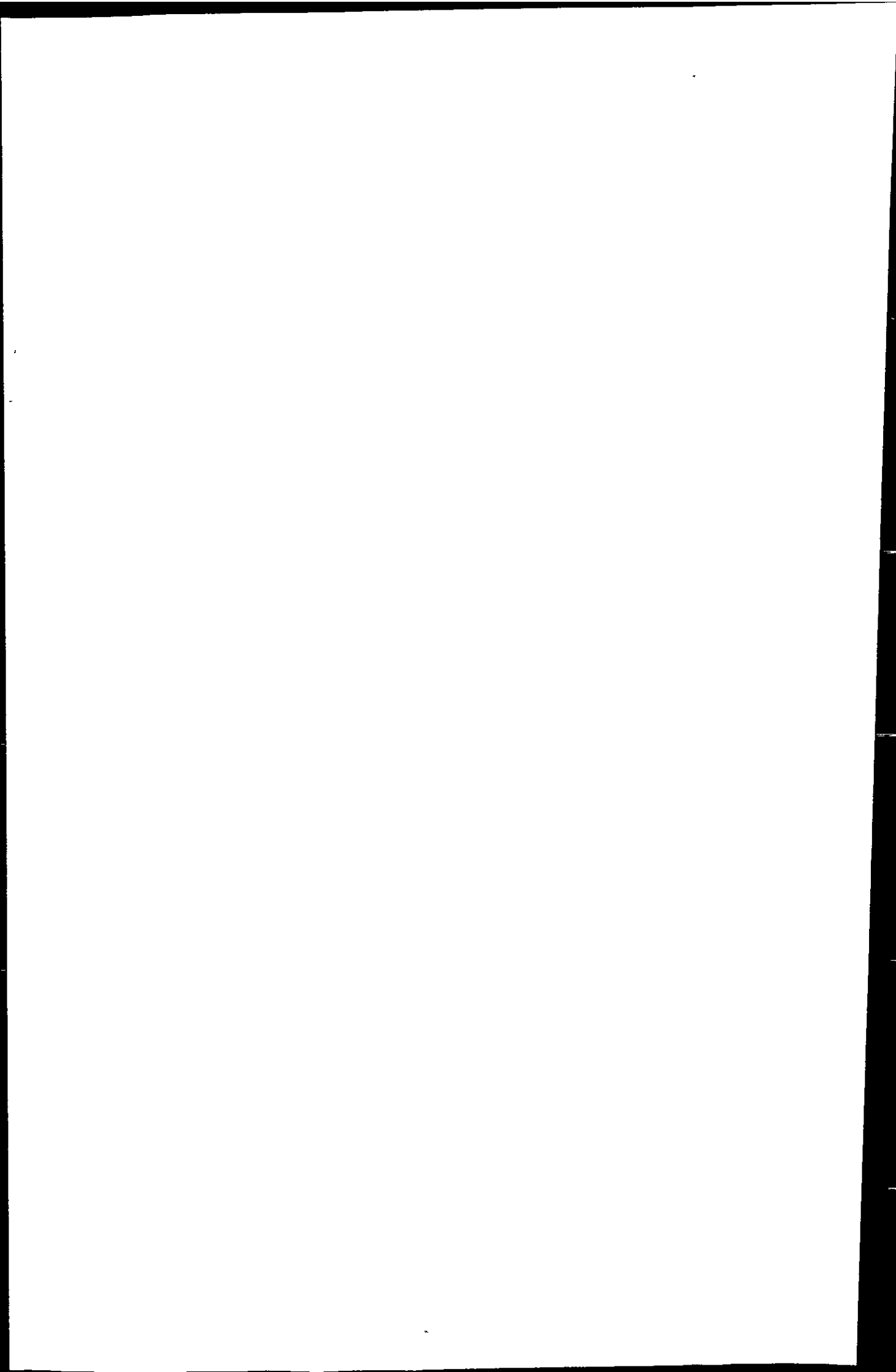


student would be asked to reflect on their understanding but the context would be that it was the package and not the student under test.

On reflection, the place for this sort of package is probably in the context of a teacher-led course, and not in isolation. This reflects the findings of various researchers. Heard's (1978) engineering student respondents wanted more teacher contact, Brown (1994) found that the role of the teacher was very important to engineering students in the context of a computer supported mechanical engineering course, Ramsden and Entwistle (1981) found that engineering students were dependent on teachers to direct them what to do and Crowther (1997b) suggests that engineering students gain motivation from teacher-centred instruction rather than working independently.

15.5.2 On engineering students

The subject of mathematical modelling appears not to have been taught explicitly in the engineering degree course. The students seem to have incompatible ideas on what mathematical modelling is. Martin, although he has followed the German rather than the English education system up until now, reflects the pattern suggested in chapter 13 quite strongly. He suggests that there are about seven basic formulae, from which you choose the appropriate one by deciding the variables you have and the ones you need, and then the most difficult part for him is determining the parameters. John has more of a grasp that setting up the equations is involved, but when he describes how a report should be written he makes reference to using ready-made models (Bernoulli, specific gravity). Jed also felt that he would have problems locating the variables, though he would know where to go to for modelling. This suggests a similar view to Martin's except that Martin may keep the formulae he needs in his memory. Jed's attitudes to mathematics shown by this interview are that



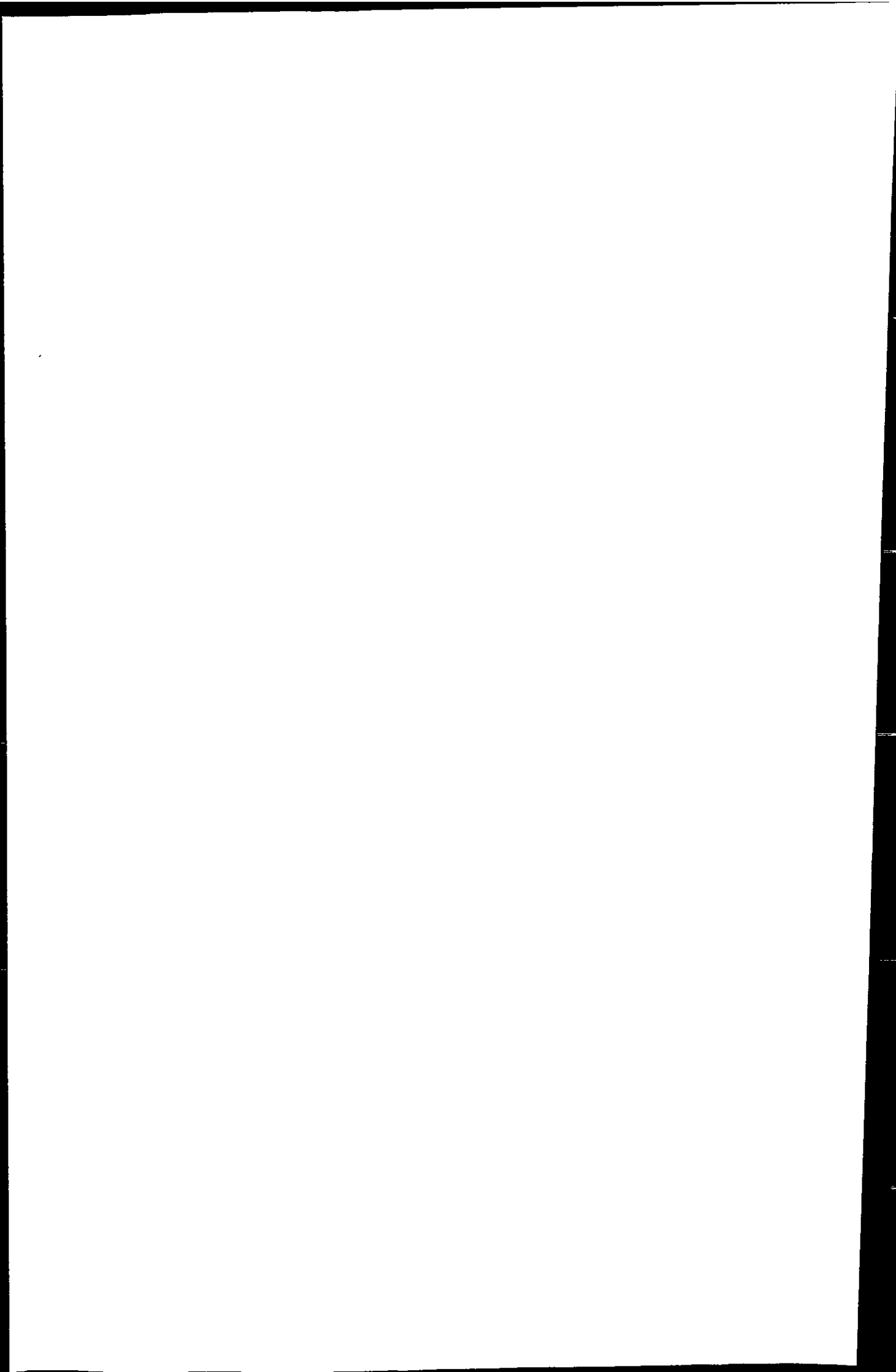
he would like to avoid mathematics, he has not used mathematics during his year in industry, and that he would be able to look up what he needs. These reflect the attitudes found in the comments on the questionnaire in chapter 6.

Neither group were able to suggest applications which would make good case studies (the water tank was effectively dealt with in model 3), although Gareth felt that second year students would be looking at gas turbines. This also suggests that none of the students had been in contact with mathematical modelling during their degree course (since the foundation year).

Both groups pointed out that students (particularly engineering students) would not go out of their way to use such a package, unless (John felt) they felt that it specifically addressed a model they might need for an assignment. This reflects Ramsden and Entwistle's (1981) view of engineering students as syllabus-bound.

15.5.3 On a possible redesign

It would be possible to redesign the package on a more open plan, showing a representation of the thing to be modelled and then making suggestions for possible modelling strategies. However, as Graham (E, 1997) points out, students will approach the same problem in different ways and the diagnostics required would be highly sophisticated. We conclude that the best use of the package would be as an introduction to modelling, but that the level of the mathematics, if not of the algebra involved, is probably too high for many first year engineering students.



16. Conclusions and suggestions for further study

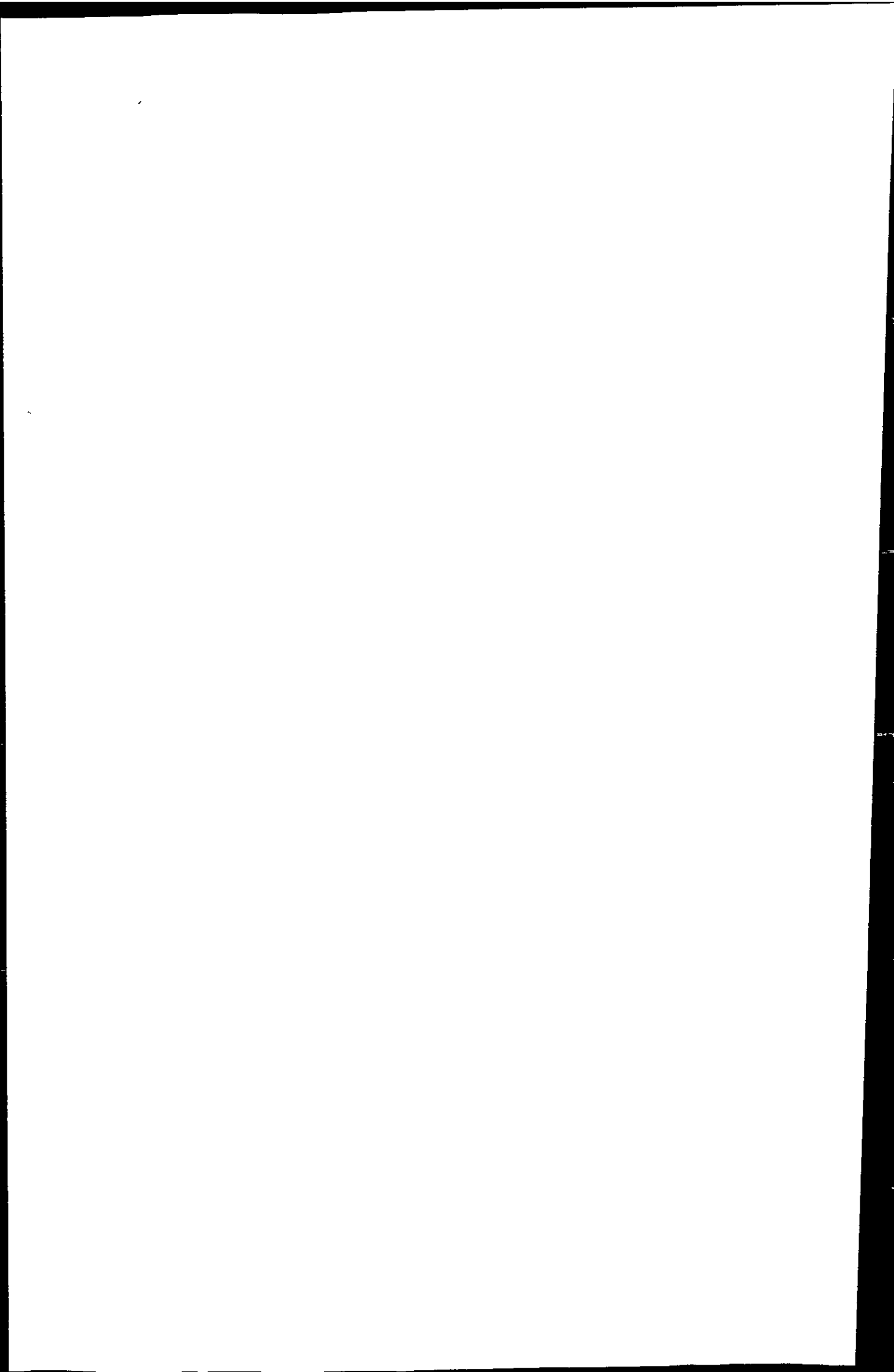
16.1 Introduction

The important outcomes of this study include the design and use of an instrument to investigate the concept images of engineering students; the subsequent mapping of these concept images and the description of the engineers' mathematical modelling cycle. Previous research on concept images has not addressed the mode in which the concept images are held, (see for instance Vinner, 1991). Research on the mathematics of engineering students has concentrated either on their mathematical skills and declarative knowledge (see for example Sutherland & Pozzi, 1995; Crowther, 1997a) or on their attitude to mathematics (see for example Shaw & Shaw, 1995, 1997; Crowther, 1997b).

16.2 Concept images: mode and depth

The responses to our questionnaire provide some evidence that the engineering students did hold different concept images from the mathematics students and that there was a development in their concept images which was not detected in the mathematics students. These differences are described in the sections below.

It is not implied that mathematics students' concept images do not develop, but rather that the research was designed to concentrate on concepts more central to engineers than to mathematicians. These results are an original contribution to our understanding of the development of engineers' concept images in that published research has concentrated on the mathematical skills and declarative knowledge of engineering students.



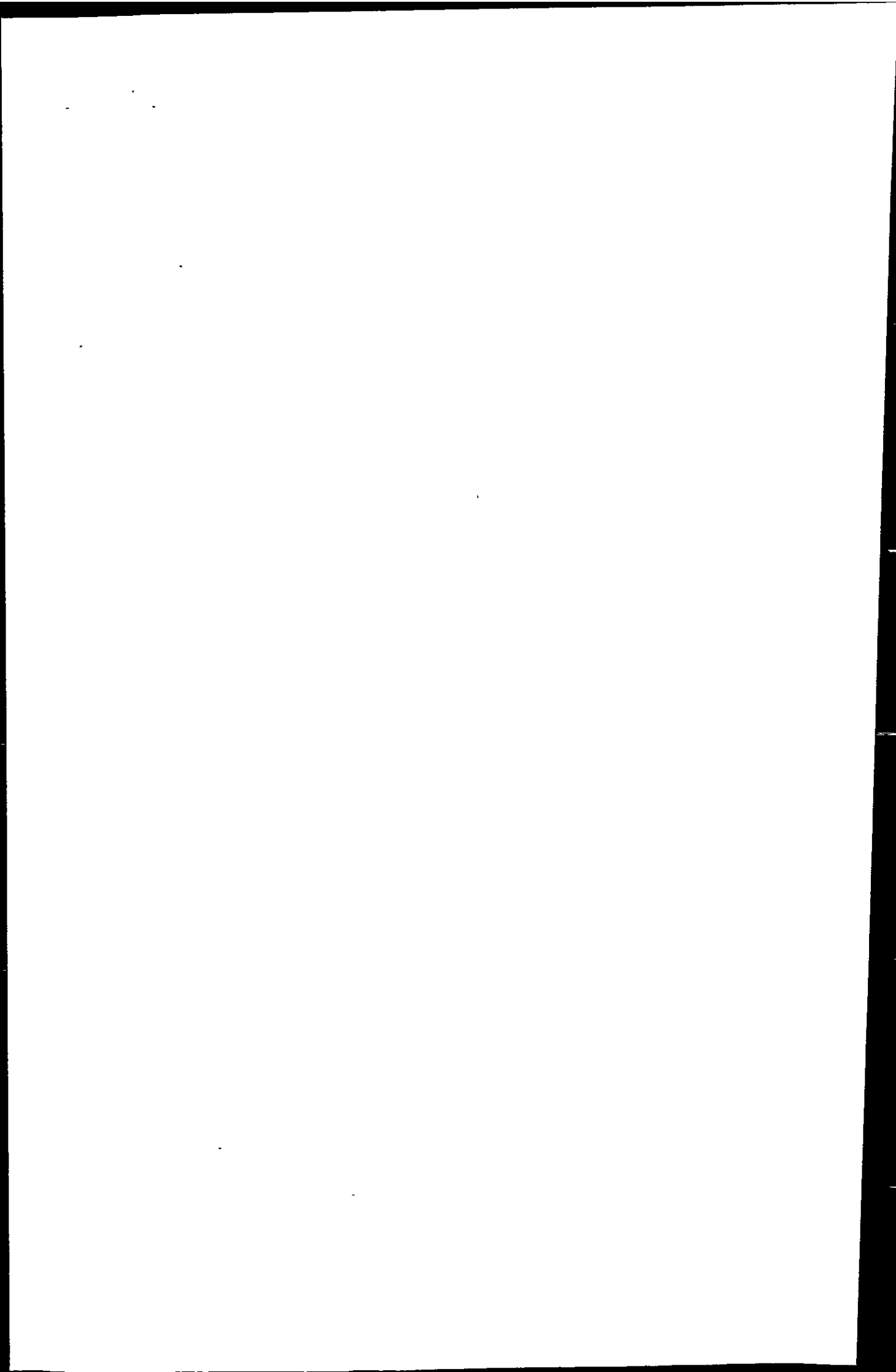
16.2.1 Mode of image

Although many engineering students would describe themselves as visual people (Crowther, 1997b), the diagrammatic options on the questionnaire were only popular with them in a "mechanics" context. In a "mathematics" context, respondents preferred verbal options where these were available. This could have one of several causes. It may be because their visual representations were private and did not coincide with the given options, since drawing a diagram is not a standard step in tackling mathematics questions. It may be that they have not engaged enough with the mathematical concepts to form visual images, or it may be that the verbal mode is their preferred mode in conceptualising mathematics. The distance in iconicity between an abstract mathematical idea and a slightly less abstract representation in words is less than that between the mathematical idea and a visual representation.

16.2.2 Depth of image

The change between a novice's and an expert's concept system can be characterised on three dimensions by (a) an increased perception of the concepts as meaningful, (b) an increased linkage and richness of relationships between concepts and (c) an encapsulation or clumping of concepts (Royer et al, 1993). These aspects are discussed below.

We found evidence of two different ways in which engineering students' concept images appeared to mature and deepen. The first was a response to teaching, where the pattern of preferences changed after a topic had been taught, but reverted to the "baseline" pattern after a time. The second type of change was a response to experience, where the change was more gradual, but the pattern did not seem to revert.



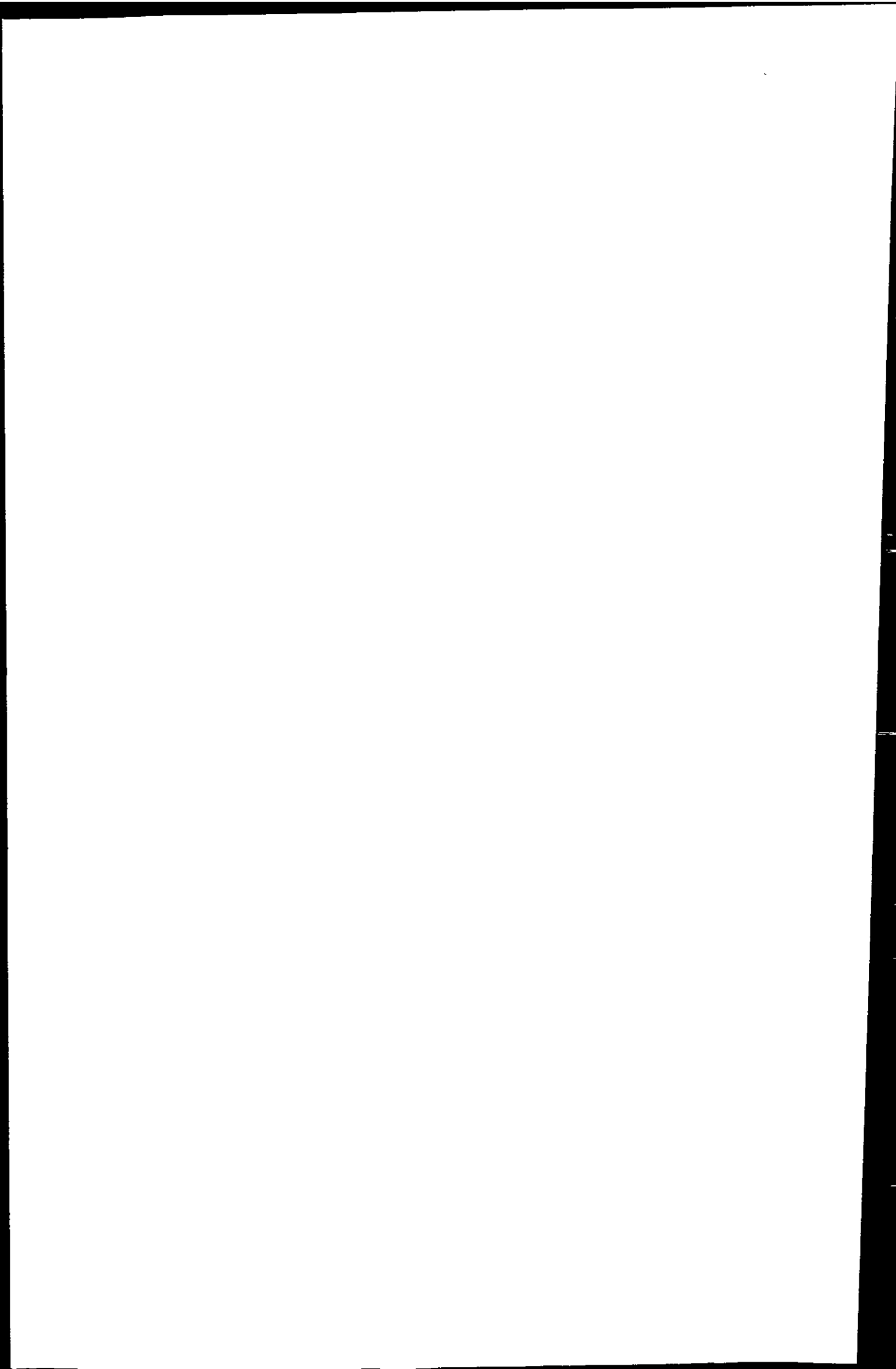
The practising engineers proved a difficult group to account for, because in many ways their concept images were more similar to those of first year engineering students than those of postgraduate engineers. Possible explanations might be that because of their age they had undergone a different school and university experience from current students, being older than the current students; or that they were intrinsically dissimilar people, having been a group of Ministry of Defence sponsored students who had all attended "old" universities and who were now working largely as managers.

16.2.3 Growth in meaning

One aspect of maturing of a concept was shown by the growth in popularity among engineering students of the options where a mathematical concept was expressed as a sentence describing what "it tells you". Growth in meaning may be seen as proceeding in two directions: making a relationship between the mathematical concept and the outside world, so the world is interpreted through mathematics (see Wilson et al, 1993), and seeing how that concept had meaning particularly in a work situation (see for example, Lave, 1996), so mathematics is interpreted in terms of the everyday world.

16.2.4 Richness of association

Some small evidence of an increasing richness of association between mathematics concepts was found in the engineering students, in that an option which portrays integration as the reverse of differentiation gained in popularity. This was not an aspect which the questionnaire was designed to explore so it was interesting to find it. Although the process of integration is taught as being the reverse of the process of differentiation, the students in the early part of their studies did not identify the



relationship in the same way that they readily chose the description of integration as "the area under a curve". This development may be related to the application of integration in areas such as hydraulics and stress analysis where a total force may be equated to the sum of the stresses across an area.

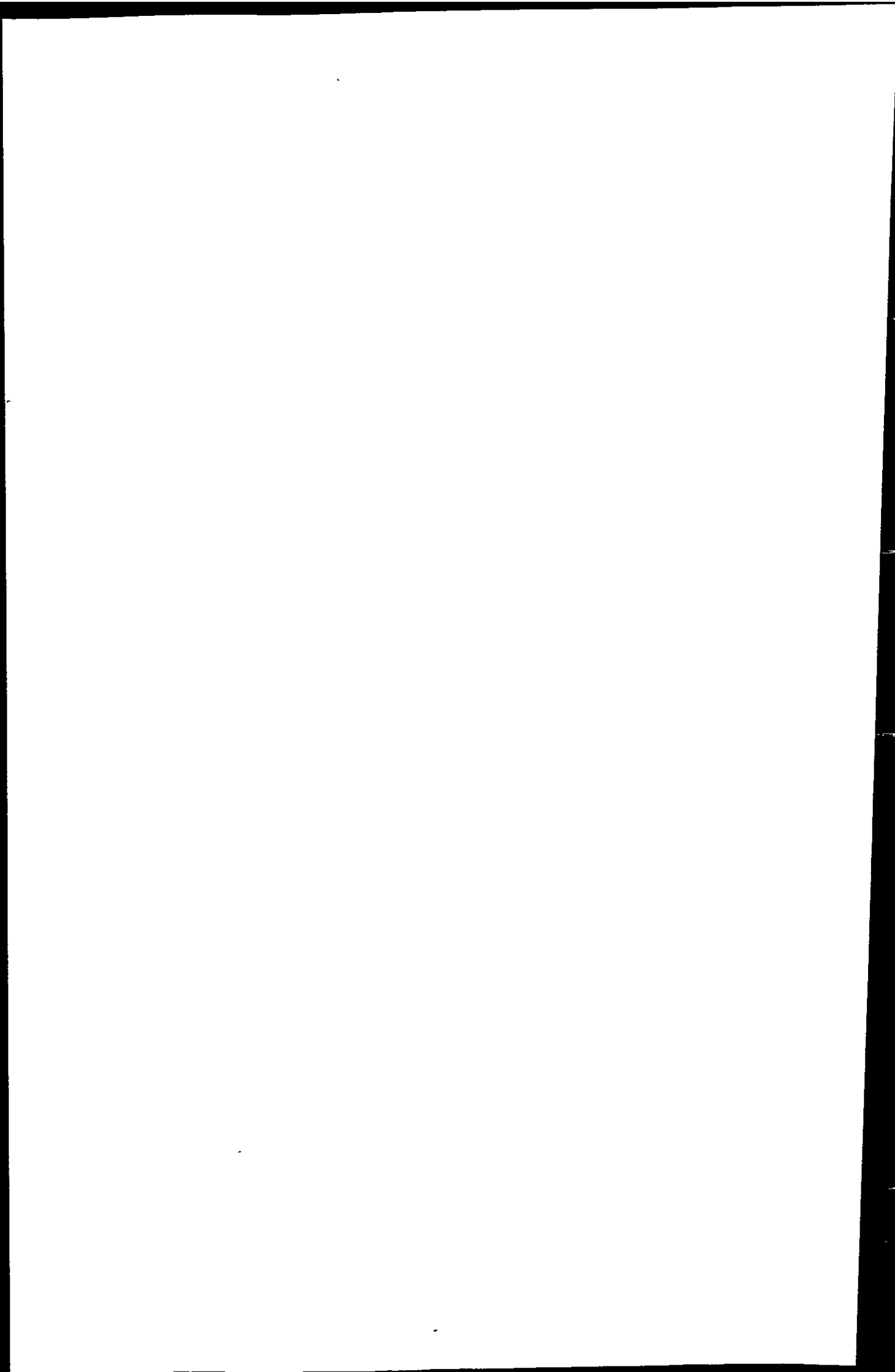
16.2.5 Encapsulation

There was some small evidence that the engineering students' mechanics concepts were becoming encapsulated across the course. This was shown when options which used a general heading under which the particular case might be classified increased in popularity. The statement describing the beam as simply supported with a load at one-third span, and the description of the mass-spring-damper system as performing damped harmonic motion both increased in popularity over time among the engineering students.

16.2.6 Comparison with mathematics students

We believe from the results of this study that the mathematical development of engineering students is different from that of mathematics students, particularly in the way in which they give engineering meaning to certain mathematical concepts. At entry, there were strong similarities between the mathematics and engineering students' patterns of responses but by the final year the groups had diverged. This confirmed the differences observed between the mathematics and engineering students carrying out the cascade modelling exercise.

There is evidence in the literature that engineering students are socialised into ways of thinking and behaving, and we may ask whether the difference found stems from socialisation, from the interactions between students and their peers, lecturers and other professional contacts, or whether there is also a second acculturation process



through their discovery of what is useful in the context of their study and work.

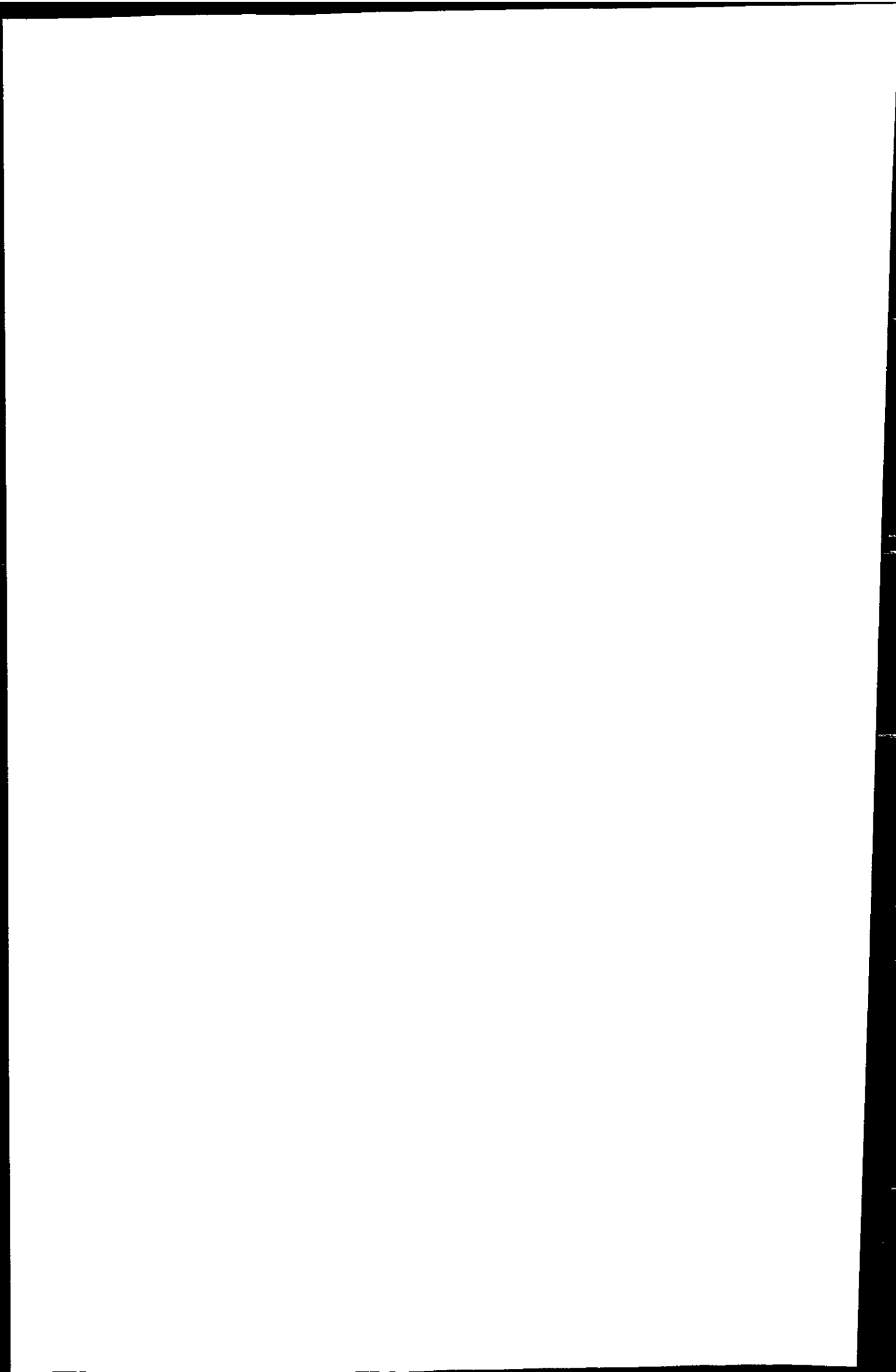
16.2.7 Summary

We believe that the study has for the first time shown changes in engineering students' mathematical concepts as they progress through their studies. These changes are not shown in mathematics students, the virtual control group, so they may be thought of as specifically engineering changes. It was unexpected that the engineering students seemed to have verbal concept images in mathematics, but pictorial ones in a mechanics context.

16.3 *Implications for teaching*

An important element of engineering mathematics which we have demonstrated is the growth of meaning of mathematical concepts in external terms. This creates gains in three fields: the students have more attachment points for their concepts, the students have intrinsic motivation to develop their understanding since it relates to their chosen studies and the students are able to apply their mathematical knowledge in the engineering fields to which they are appropriate.

If this growth in meaning is seen as both a valuable and intrinsic part of engineering mathematics, then its development should be encouraged as early as possible in the engineering mathematics curriculum. As present it appears to occur mainly during the students' "year out" while they are in contact with mathematics as it is embedded in engineering experience. In order to accelerate its development then mathematics should be experienced as embedded within engineering at an early stage of the degree course, and this implies that mathematical modelling should be used as a context for the use of mathematics at an early stage in the course.



16.4 Engineering students' attitudes to mathematics

Attitudes to mathematics form another aspect of the internal mathematical world of engineering students and some were expressed by engineering students in the questionnaire responses, the modelling exercise and the courseware evaluation interviews. They may be summarised as follows.

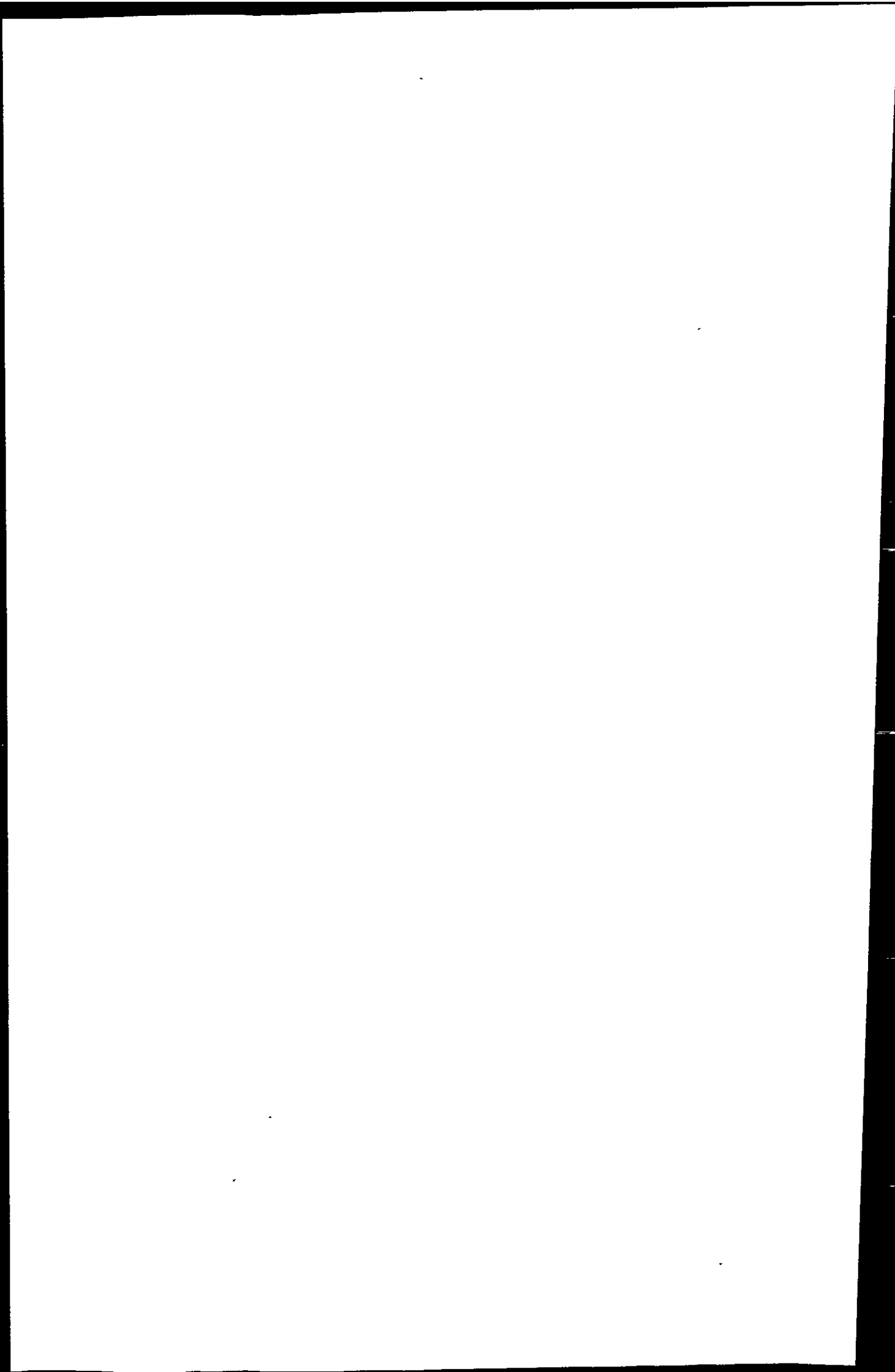
1. Mathematics is found in books: it is external to an engineer,
2. Mathematics is not really relevant to engineers,
3. Mathematics is something you learn for exams and then forget,
4. Mathematics has right and wrong answers.

Of these attitudes, the first two seem to refer to mathematics in general, and the second two to mathematics as a part of the university syllabus.

16.4.1 Mathematics as external

Engineering students appear to regard mathematics as something external to them, which they would find in books or notes. When mathematical skills and concepts become internalised as part of other processes and conceptual structures, we suspect that they are no longer regarded as being mathematics. O'Kane (1995) makes the claim that "all fundamental concepts in engineering science have been given their most precise expression by mathematicians working in Rational Mechanics", and that engineers do not recognise the mathematical content of their everyday concepts.

Engineers find meaning in mathematics when it meets their experience: for example, the derivative as a rate of change, which has more meaning to an engineer than the more abstract notion of the slope of the tangent. There is a development in the extent to which such concepts become meaningful to engineers, but this appears to relapse while they are in practice. This aspect would be worth further research.



16.4.2 Mathematics as not relevant to engineers

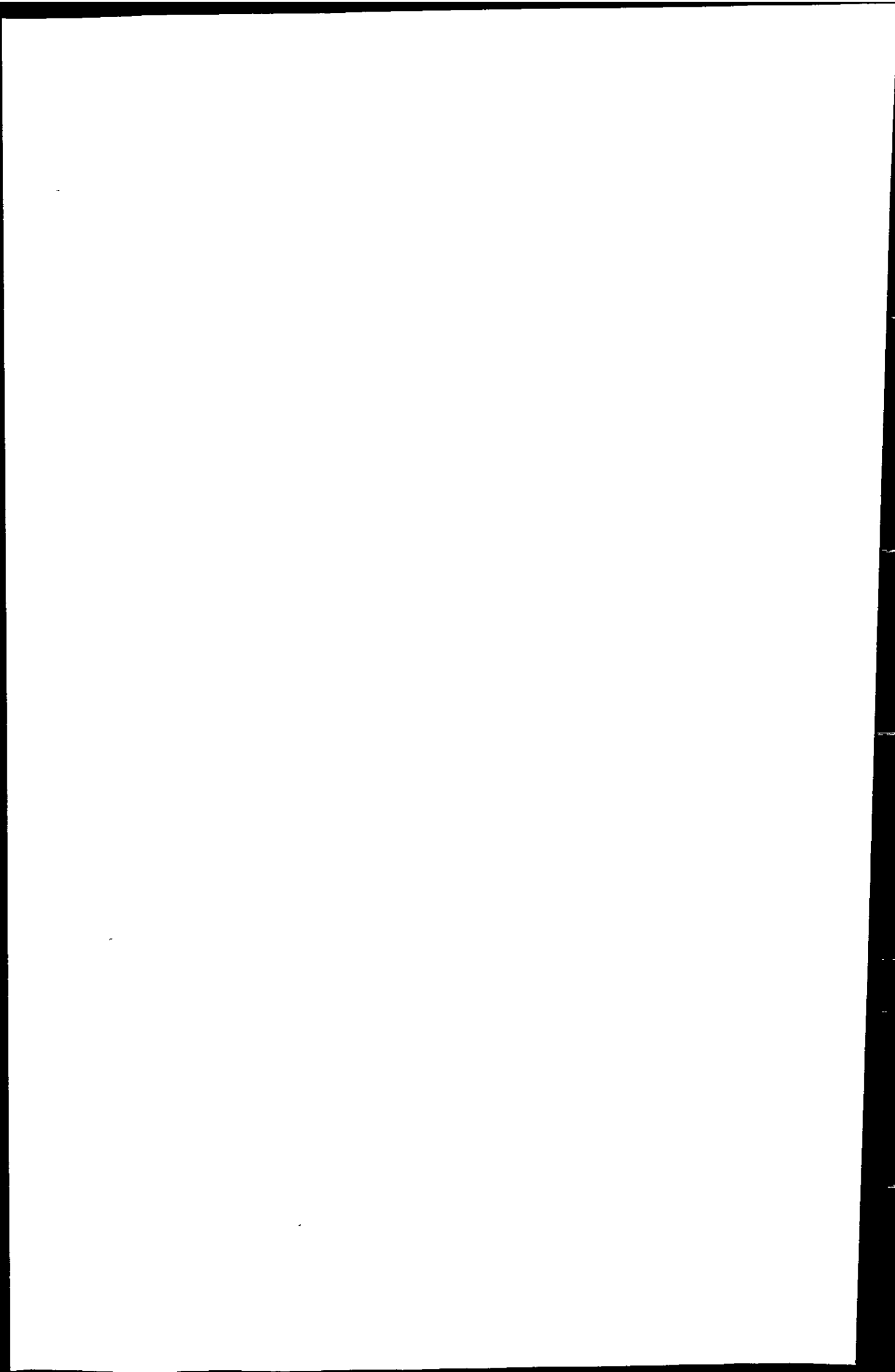
The engineering students observed were not using mathematics as a way of describing or interpreting experience (cf. Hirst, 1972). They say they do not use mathematics when they are working in industry. They even claim to try to avoid mathematics. These claims should be seen in the light of the proposal that once something is seen as useful to the engineer it ceases to be regarded as mathematics.

Separation from mathematics is probably a disadvantage in the practice of engineering and may also rob the engineer of a source of satisfaction: successful mathematical modelling (Shaw, 1989). Although mathematical modelling is now a recommended part of the engineering core (IChemE et al, 1995) it is as a short course which we feel does not capture the essence of the subject: I recommend that it should be integrated into the mathematics of the engineering course from early on, as Cross (1983) does for most mathematics courses.

16.4.3 Mathematics as something you learn for exams and then forget

Engineering mathematics has been variously described as a tool, a language and a competence. We feel that to regard mathematics as a competence implies that it is a skill which may be acquired in order to pass a test of competence, and then ignored. A language is a better metaphor for engineering mathematics, as it implies a living entity which is used to communicate content about something else, and which is subject to change as usefulness dictates. A tool is a metaphor of mixed usefulness. On the one hand a tool is external to the user rather than internal like a language. On the other it implies usefulness, and being a means to an end.

Even some authors who insist on rigour in engineering mathematics defend this as a way of teaching intellectual rigour to engineering students (implying that intellectual rigour is not present in their other subjects, which implies that without mathematics



engineering would have no intellectual rigour) and not as an intrinsic part of the mathematics needed.

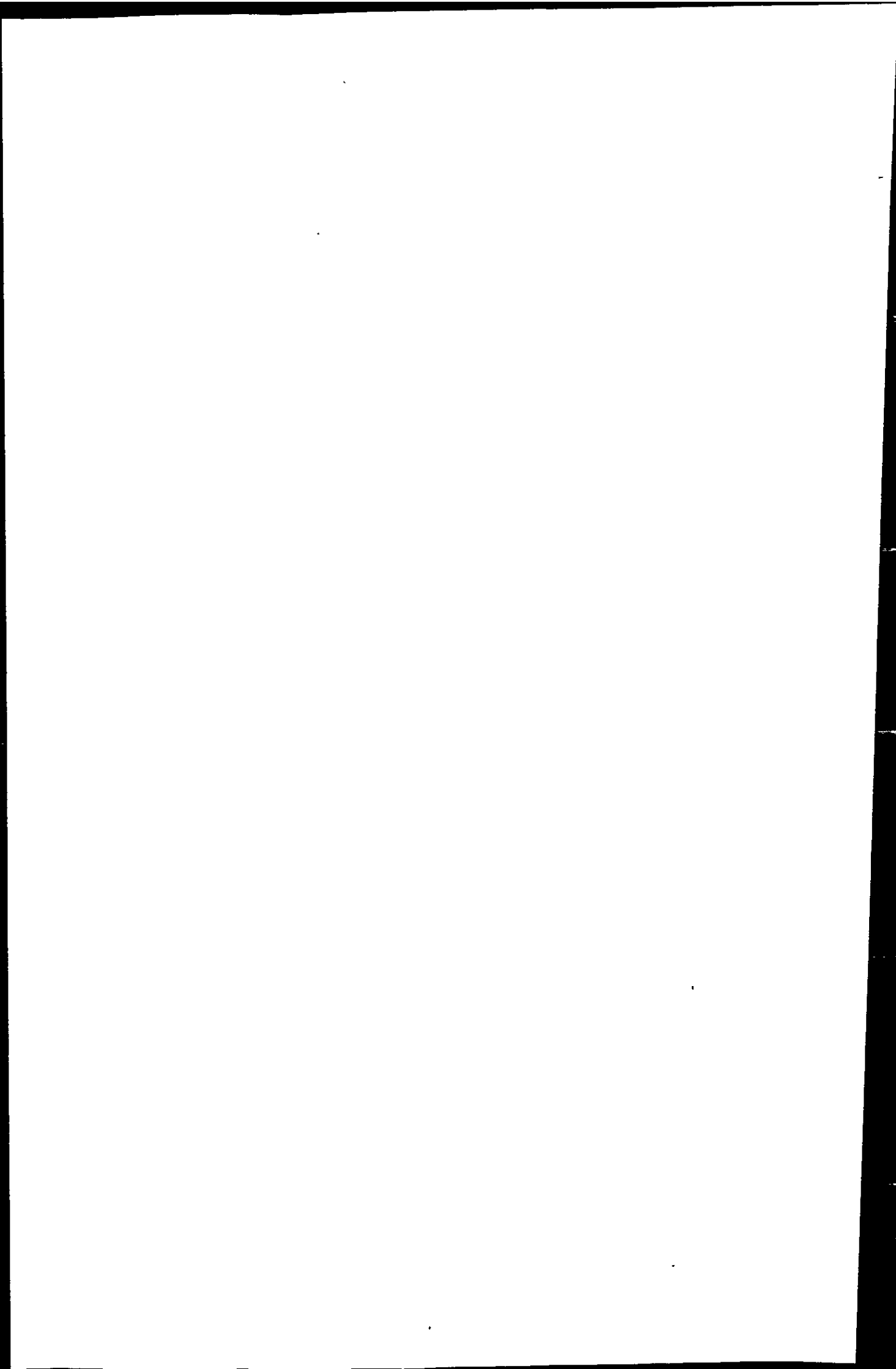
We have sympathy with the authors (Cox et al, 1995) who recommend that sample transformation and operator methods, for example, be taught to engineering students, and that students should become proficient with symbolic algebra systems since there is simply not time in the engineering mathematics course to teach all the mathematics an engineer may require.

On the thorny question of relevance, there is an argument which says "if you can't find an engineering application for this piece of mathematics, so you can teach it in the context of that application, why are you teaching it to engineering students?".

The counter-argument is that learning a new application and new mathematics simultaneously is too hard: in general it is easier to teach the mathematics first and then to teach an application which will use the mathematics later. Crowther (1997b) suggests that any three-dimensional, practical or easily visualised application would help engineering students to give meaning to mathematical concepts.

16.4.4 Mathematics has right and wrong answers.

This opinion is dangerous in engineering students as it causes a false expectation of mathematical modelling. The ability to construct and use mathematical models is a vital piece of engineering knowledge. It is the way that engineers use mathematics in their professional lives, and it allows engineering students to give content and meaning to mathematical concepts. However mathematical modelling is an intrinsically messy process, and the critical property of a model is not its correctness but its usefulness (or "fruitfulness"; see Gilbert et al, 1998). Again I suspect that a process which produces messy answers is regarded by engineers as not mathematics.



The model used in this study for mathematical modelling has been a modification of the Open University flow chart (Figure 16-1).

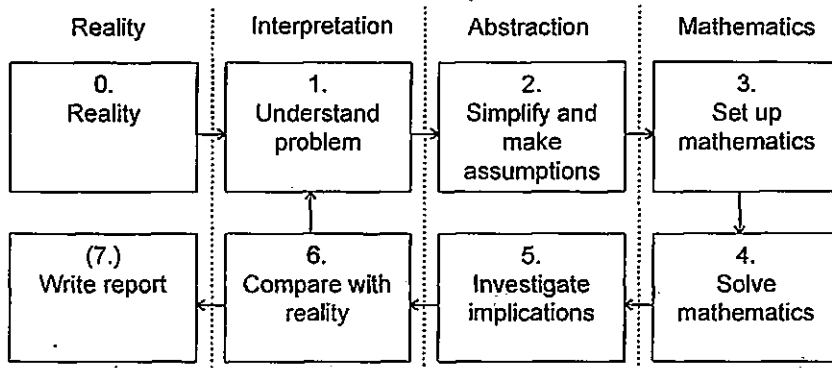


Figure 16-1: Modification of OU modelling flowchart

The mathematics taught to the engineering students in this study does not appear to be directed towards making them proactive mathematical modellers, but more to allowing them to follow the mathematical arguments used by others in constructing well-known physical laws (or “formulae”), and to manipulating the formulae once given. These formulae then become the basis for the engineer’s modelling process, which appears to be different from both classical mathematical modelling and classical empirical modelling. The process is described in Figure 16-1, a new model, reproduced from chapter 13.

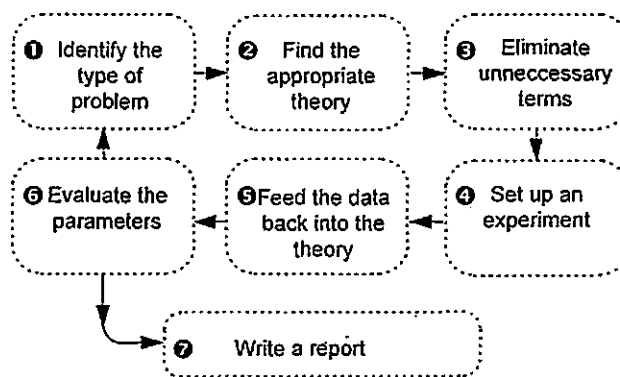
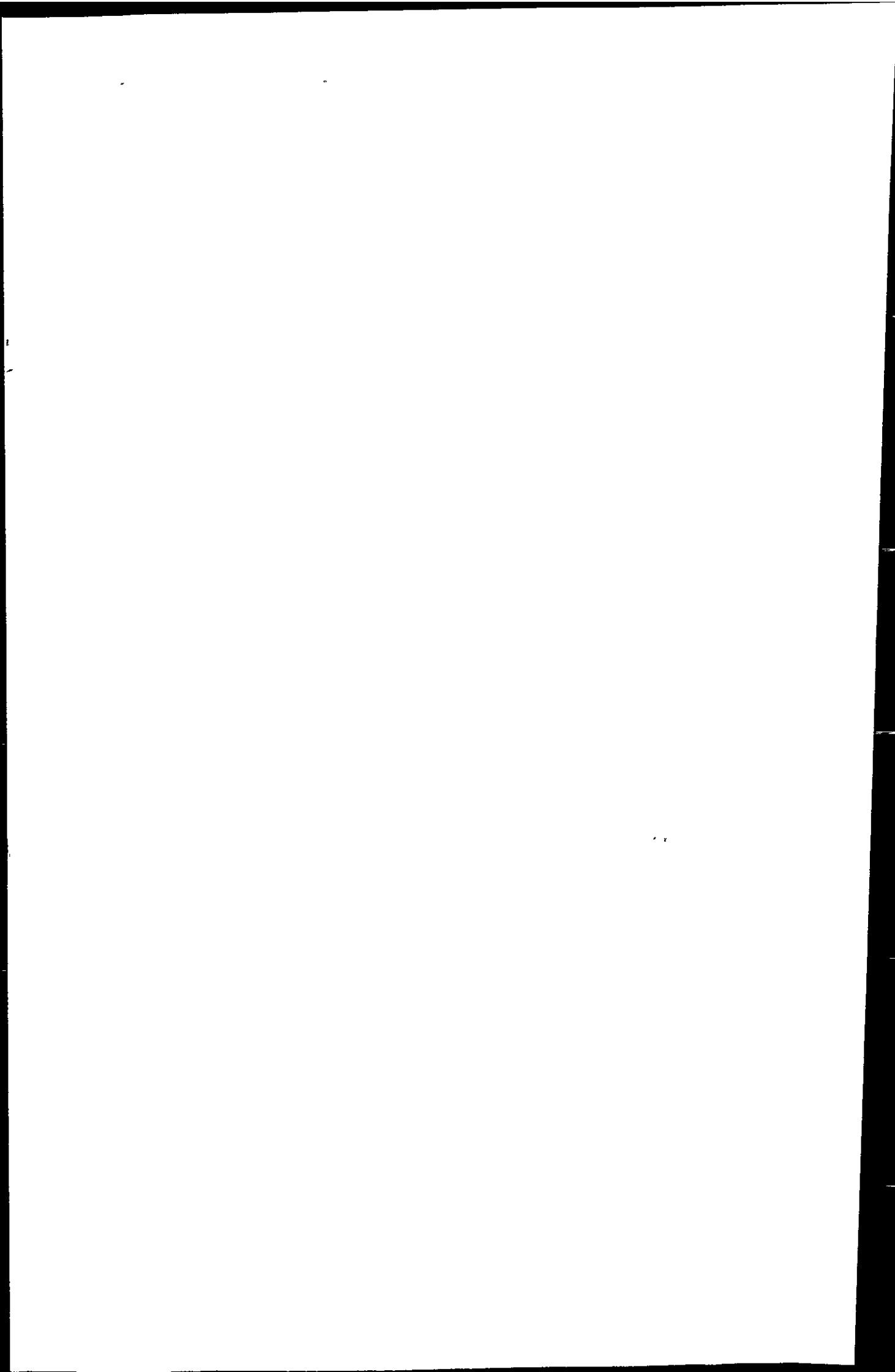


Figure 16-1: The engineering modelling process

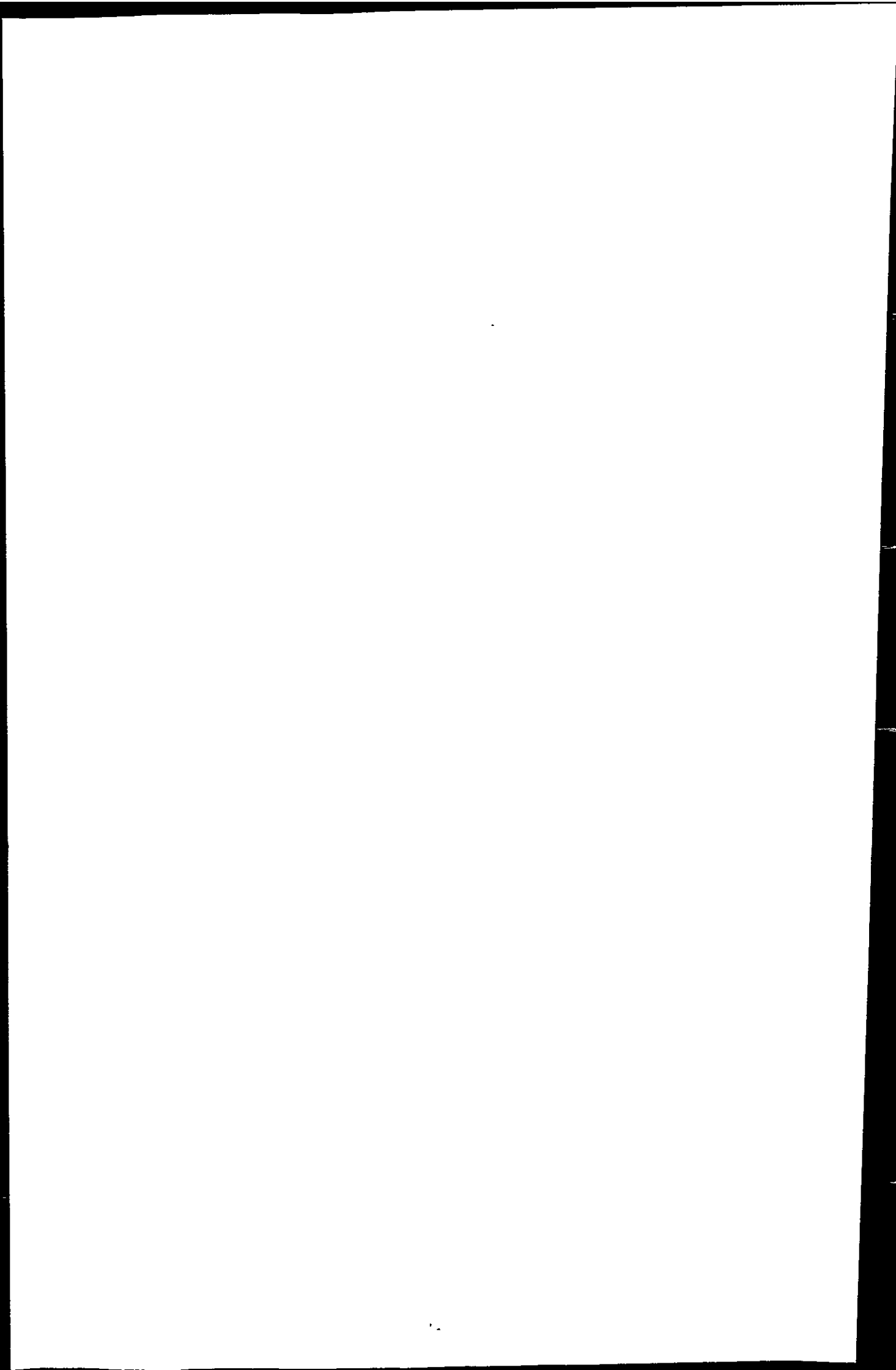
I would argue that the critical part of mathematical modelling lies in the central part of the OU diagram: in the “interpretation” and “abstraction” streams, where the possibility for individual variation is the greatest (Graham E, 1997). This was



intended to be the focus of the courseware: extracting the mathematics from the real world situation and interpreting the model in terms of how well it matched reality. It is all too easy to show students a polished model of reality, but the engineering student needs to be aware of the process of building the model and then of interpreting it in terms of the real world. Part of the problem is the desire of the teacher or the professional to present the learner or other audience with the final product of their reasoning, without showing any of the scaffolding with which it was built (see, for example, McDermott, 1991; Gauss's habit of destroying his working notes), which is related to the absolutist view of mathematics, in which the struggle of the mathematician to produce the final knowledge is of no interest. This is analogous to mathematics being regarded as a theatrical performance in which only the final event is of interest, or a building which stands after the scaffolding has been removed. However if mathematics is regarded as an endeavour, the process of building is important, so that it may be replicated with variations to arrive at an extension of the edifice. On the other hand, slavishly following the same process will not lead to any growth in knowledge: the message is not "this is the way mathematics is done" but "this is the way we did this bit of mathematics".

16.5 The courseware

The courseware failed to engage the students, by presenting a picture of how the model had been built rather than allowing them to take part in the building process. The interactivity proposed by the students suggested that they were expecting a right-wrong type of answer, because to them mathematics has right and wrong answers. Writing the courseware was an experience which produced cognitive change in the author. When it was tested it proved to have been too loosely aimed. Although the case studies are the practical or easily visualised type of applications recommended by



Crowther (see above), and the mathematics is deceptively simple, some of the manipulation, for example in the catenary and in the tank with pipe losses, is probably more subtle than many engineering students would appreciate.

The main shortcoming with the courseware is that it did not engage the students in the modelling process, and this is probably due to its book-like appearance, that it presented the process in too polished a guise, and that did not allow students to participate in the decisions taken. It is probably an indication of the lack of engagement of the students with the courseware that they did not recognise the subtlety of the manipulation.

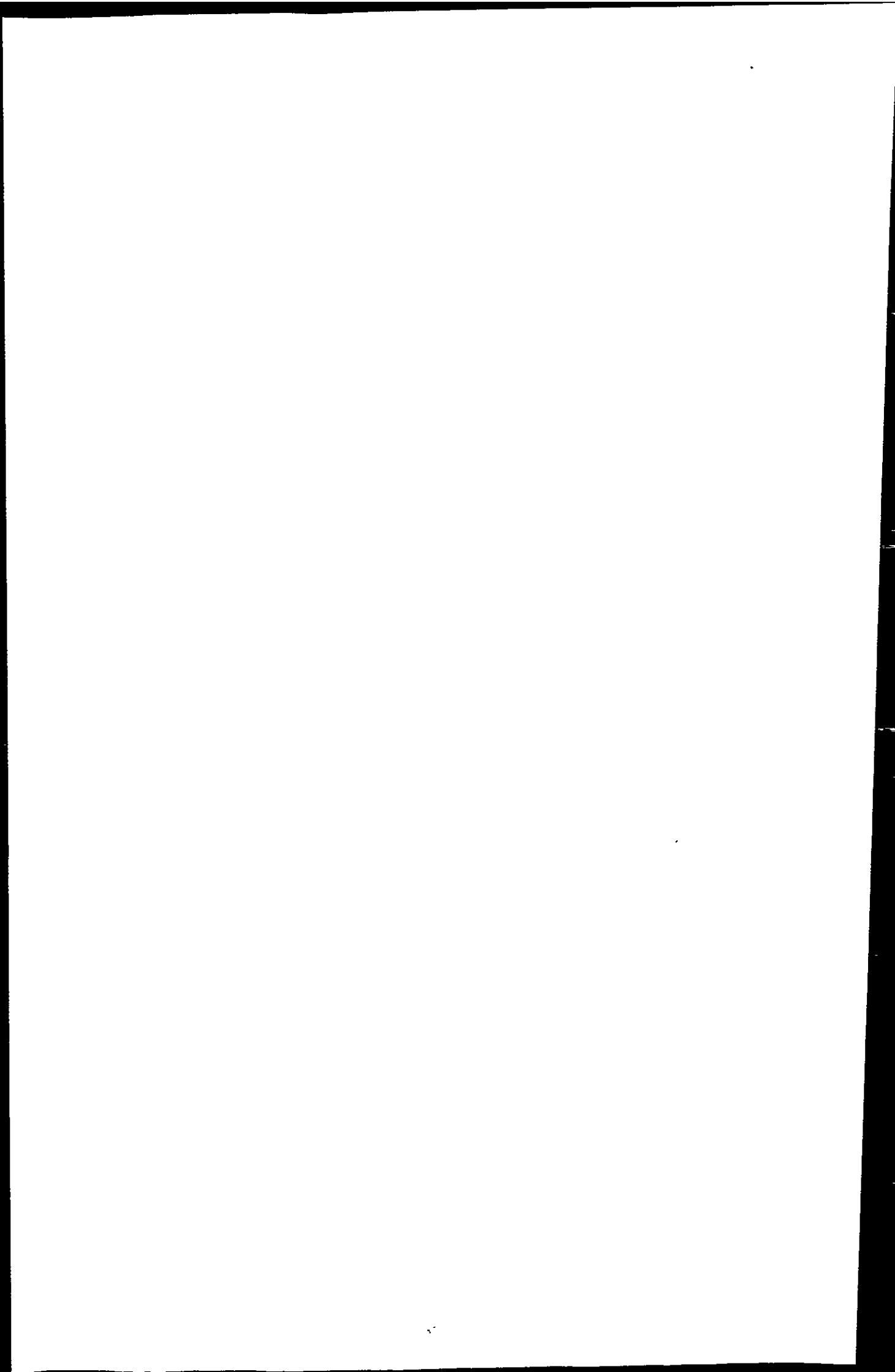
I see such courseware as being useful at two points in the engineering mathematics course: as an introduction using the simpler cases in the first year and as a recap to refresh the memories of final year students coming back from a year away from study. The courseware will need some modification before it is practically useful, to invite more engagement in the simplification and abstraction sections.

16.6 Research methods

Several research methods were used in this study: observation, survey, content analysis, factor analysis and interview. This spread of methods allowed a degree of triangulation in the study, although all were approached qualitatively, rather than quantitatively. The study was overall narrow and in depth rather than broad and shallow in scope, which is an approach which gives a rich qualitative picture in general.

16.7 Misconceptions we all may have

One of the questions in the questionnaire provoked strong comments from respondents of all levels of experience. These responses gave evidence of two



widespread misconceptions, one of which is the well-known case of rigidity given priority over stiffness, and the other of which, described for the first time, appears to be the priority of the behaviour of a string over the bending behaviour of a beam.

The conclusion I draw from these misconceptions and their widespread distribution is that they probably represent a widespread distribution of other misconceptions which most of us are carrying about with us, which are presently causing very little harm, and which would be dispelled if we were to run into the phenomenon in practice. However it is a useful *memento mori*, as it were, to remember that we may all be fallible, as well as mortal.

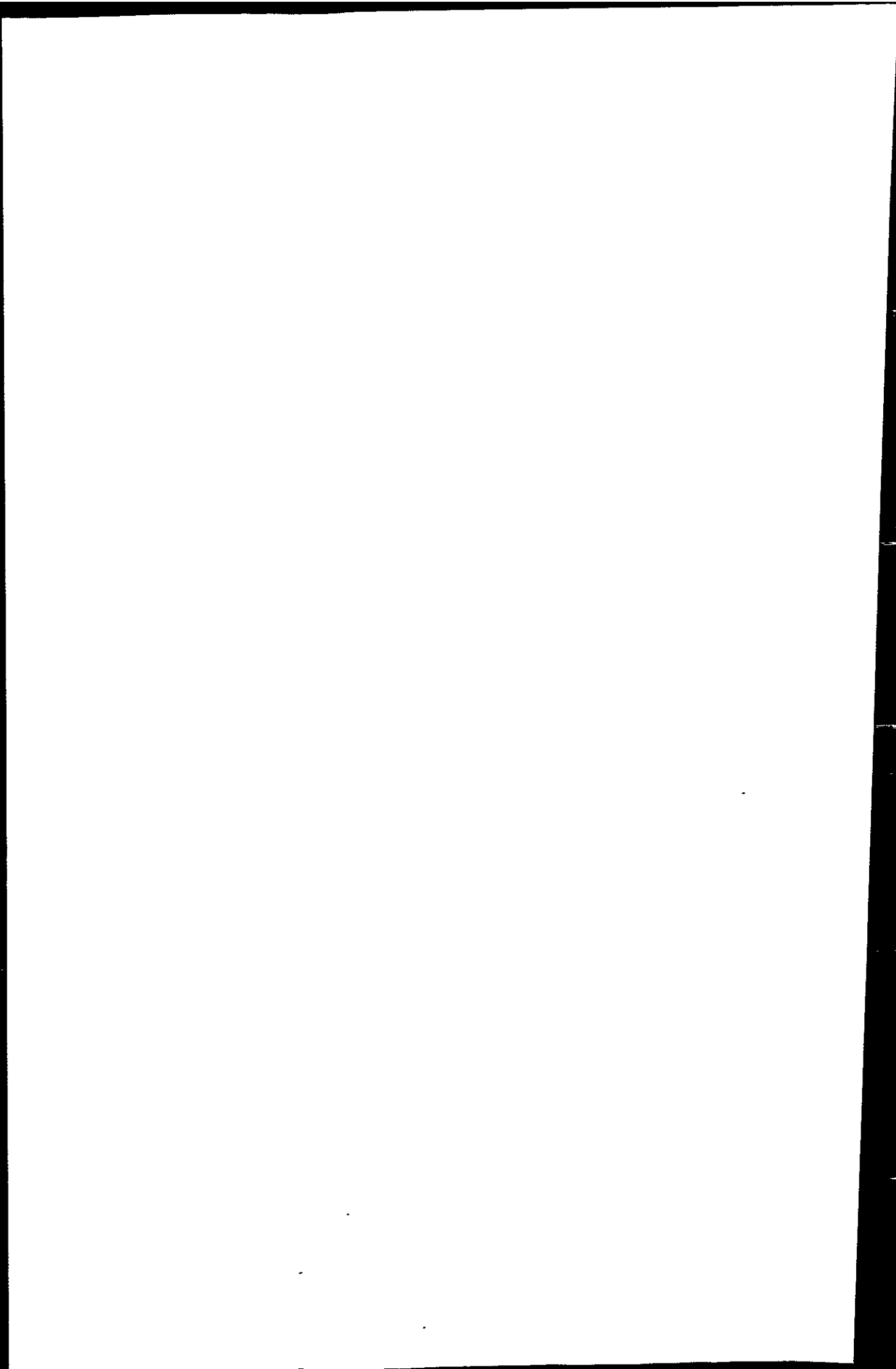
16.8 Questions arising from the research

Three new questions are now begging to be answered.

The first is whether the effects I have observed in this study, of the changing concept images of engineering students and the increasing difference between engineering and mathematics students, may be found in other universities and whether they may be found again at Plymouth. This is a question of confirming the findings for other groups of students, and broadening the scope of the study.

Given that the effects are confirmed, the question arises why the practising engineers appear to have reverted to a similar state to students at entry. This would be a matter of tracing engineers at different stages of their professional lives and trying to document the process of regression if it indeed happens.

Finally, I mentioned in chapter 6 the suspicion that engineers cease to consider concepts mathematical as they find them useful or comprehensible. This process is a topic which would be appropriate to investigate through a questionnaire which could be designed while applying the lessons learnt during this study.



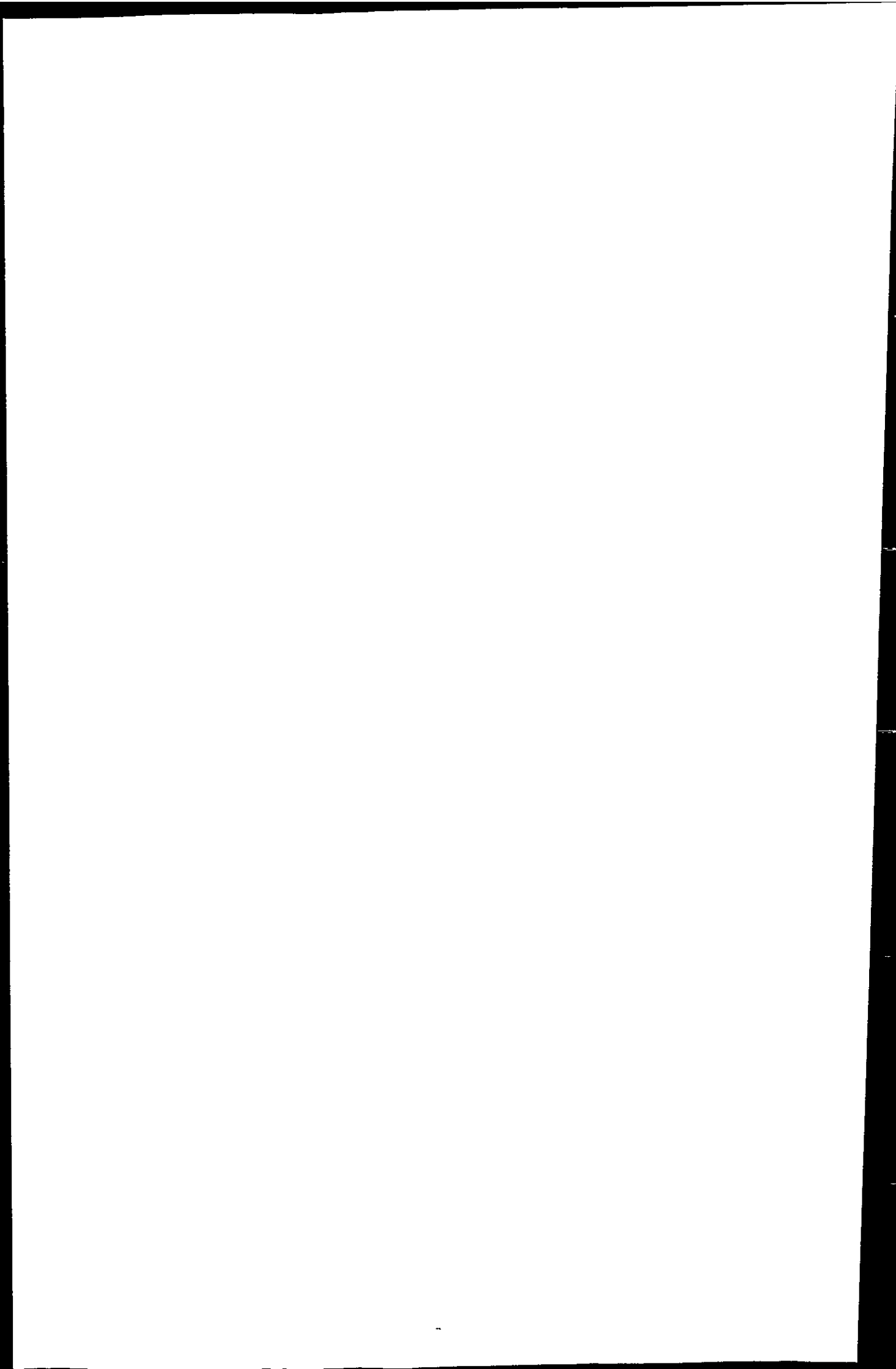
APPENDIX A: TRANSCRIPTIONS OF RECORDINGS OF STUDENTS PERFORMING CASCADE MODELLING PROBLEM

30 June 1994. Two final year mechanical engineering students. 3 vessel problem
Students have the apparatus set up with tubes but no water on a table in the window of 1
Rowe St. The camera is operated by a student from Media lab arts degree.

Tape 1

Note: there are inaudible passages on the tape where the students are muttering to themselves or to one another, but there are also long periods of silence as they study pieces of paper or the apparatus.

Time on tape	Speaker	
00:12	WMM	The situation today is that I've asked you to come and do a mathematical modelling problem to help me in my research into teaching maths to engineering students. The problem is this: you have three vessels as described on the sheet. Tank 1(sic) emptying into tank B and tank B into tank C. Clearly the level of fluid in each tank will vary. So the question is to determine.. To predict when the level of fluid in the middle tank is greatest and what that greatest volume is. So... Are you happy with that?
01:00	Adrian	I think so, yes.
	WMM	So what I intend to be doing is to be taking notes, really, of the things you are doing, to watch you doing this and intervene if necessary. I hope not to intervene at all. The pink little bottle has got some pink fluid in it which you can dye the water to make it a bit more visible. And it's got phenolphthaleine in it which is laxative so please don't drink it. (laughter) OK, thank you
Early stages		
02:07	Adrian	Right. (Students settle down to look at sheet) OK So how does the water go to exit
03:11	Jolyon	There's a certain amount before it actually goes in there, so we've only got so much head.
03:54		Adrian sketching diagram.
04:22	Jolyon	(Pointing to taps) We have to assume that the flow rate through here is going to be the same as the flow rate through here. There's roughly the same difference in head, diameter of tube.
	Adrian	As the pressure varies, the flow is going to be changing.
	Jolyon	Inside the tubes a steady flow job is developed..... It's a maximum
	Adrian	(Drawing) What we're interested in is in.. maximum volume in B is this one here.
05:37	Adrian	Shall we go home?(Laughter)
05:41	Adrian	A wild guess
	Jolyon	(Sketching) Well I'd have thought the answer is something like- do that Do it experimentally. start with that
06:21	Adrian	(Drawing vertical arrow on sketch) See that distance
	Jolyon	The pressure on there is mgh
06:55	Adrian	We could do with the air really
	Jolyon	That could give us some help because we obviously need ...
08:10	Jolyon	What if we split it up so we've got the initial tank of water. We need to know how long it's going to take for each litre of period of water.. So you can take..
The formula card appears		
08:54	Adrian	Trying to go back to basic principles here
	Jolyon	(Studying tanks) What if we apply Bernoulli to the area between each tank? Both $p_1 v_1$ is going to be equal.
10:20	Jolyon	(Takes out calculator and from it a formula card) Cause I've got Bernoulli's law on here
	Adrian	Ha Smart!



10:50 Jolyon Use this equation just to see
 11:10 (Adrian picks up card and studies it)
 11:34 Adrian Quite a smart formula
 Jolyon Yeh

Jolyon reaches a physical understanding and makes a prediction in general terms

11:58 Jolyon See we just don't want mass flow rate, we want to (believe) the formula as well
 Jolyon (Sketching) cause somewhere on the line we're going to have to plot this ..
 Time
 Adrian It's a matter of differentiation then I suppose.
 12:28 Jolyon We've got - say volume over here. Initially very small and it's going to rise
 quite quickly initially because nothing's coming out the bottom.
 Adrian Yeh but if you look at this diagram you're assuming that the exits are at the
 bottom.
 13:02 Adrian So it appears for me you're going to have a high flow rate which soon moves
 off dunnit?
 13:22 Jolyon There will come a point when the level in the top.. The two flow rates are going
 * to be equal when we've got the same head in it after which it will start going
 down
 Adrian Yeh, yes right
 13:50 Adrian The time it takes for this to half empty, these two will have the same volume
 Jolyon No because some will be coming out at the same time. When that's half empty
 you'll still have..
 14:11 Adrian (Point to tap in tank B) Yeh. I'd say we were assuming this was level. Not the
 initial volume at the bottom.
 Jolyon Yeh.. You'll see what's there. When that's half empty that isn't going to be
 half full.
 Adrian No they're the same size.
 Jolyon Yeh, but it'll be flowing out of them as well.
 Adrian Yeh
 * Jolyon You'll get a point when they are at the same level but not when that's half
 empty. Probably some time after that.

Some attempts to manipulate the formula

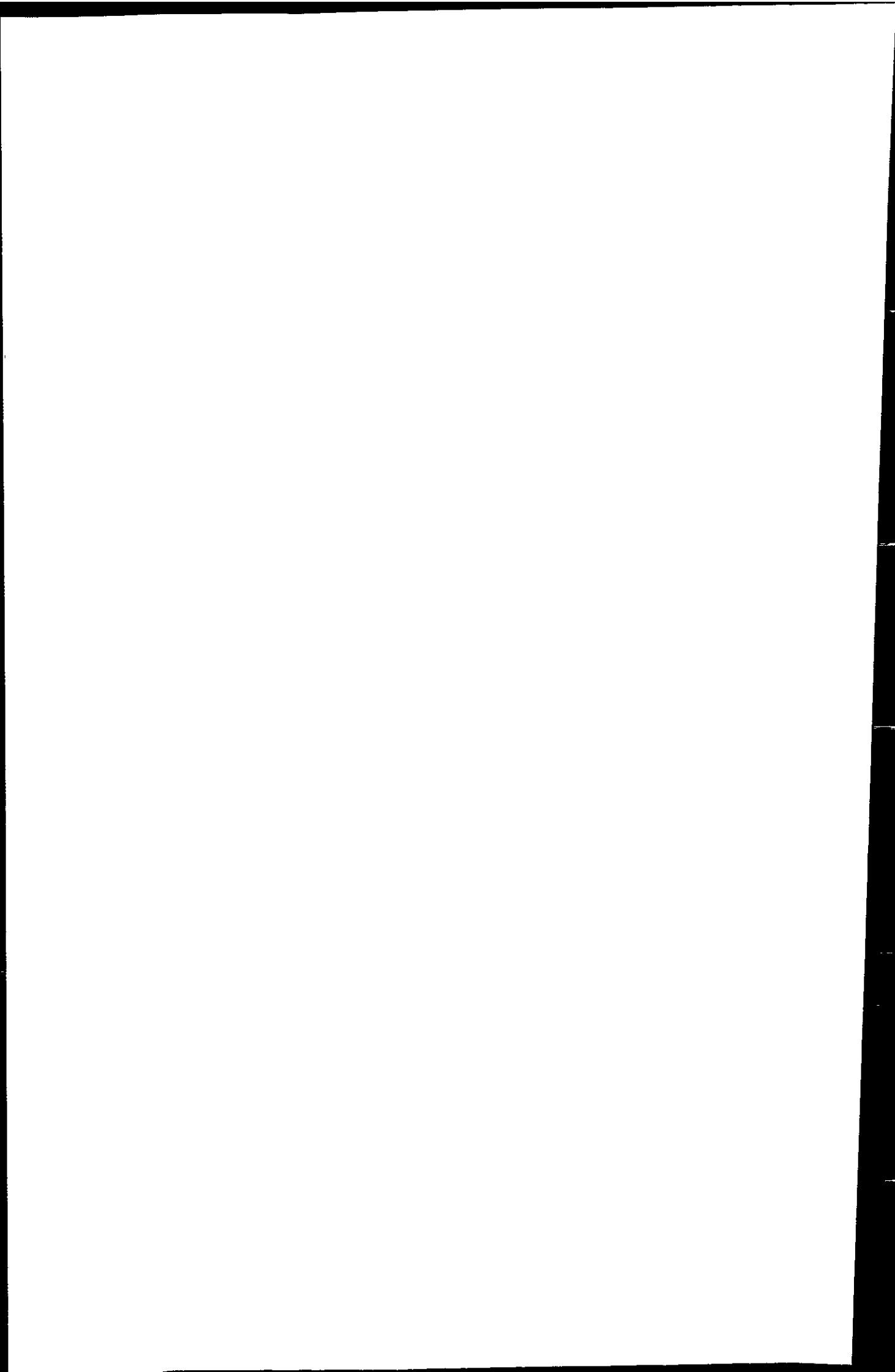
15:16 Adrian Got some stuff over here
 15:36 Jolyon You've got the other here which is () you don't have to () the maths while
 we do this
 Adrian No that's going to vary as time goes on, isn't it. () wit respect to time but
 actually..
 Jolyon Yes
 16:10 Jolyon I'm just trying to get the flow rate coming out of the bottom then we can use it.
 Adrian Yep
 17:11 Adrian We're going to have to assume the losses. There will be because of the
 coefficient of viscous.. Velocity is going to be zero. p_1 is going to be zero
 Jolyon h_2 is zero. h_1 will be..
 18:37 Adrian We need pressure to...
 Jolyon (Pointing) That's zero
 19:30 Adrian (Taking calculator card again) Looking to see what I can get rid of

Refocusing

21:50 Adrian Where is this taking us? What are we trying to get out of this?
 Jolyon Flow rates for varying heights of water
 Adrian Then that will be the same at the bottom as well
 Jolyon Like I said, what's coming out of the bottom is the same as the top one.
 Adrian (Writing) We know g
 Adrian Did you do that plot with steady flow?
 23:30 Jolyon (Laughing) Two years ago

More manipulation: capitulation

24:15 Adrian This thing actually cancels out. doesn't it, because you take that away from



each side and divide through
 Jolyon You want this squared and then
 Adrian Yes it's just rearranged, isn't it.
 26:00 Jolyon Well do you fancy giving us any pointers then, because we're not getting anywhere.

Intervention 1: permission to play

26:23 WMM Have you got any intuitions about what you think might happen?
 Jolyon We've done a similar thing as an exercise.. Practical.. Quasi-static flow which was based totally around this. How long it takes a container of water to empty into another one- And if we knew that for each container- over a period of time- and compared the times- then (sketching) the whole thing is we would probably get a graph where the two coincide - where the two would be a maximum.
 27:10 WMM How could we test your intuitions?
 Jolyon Fill it up: we'd just do it.
 WMM Well why not?
 Jolyon I thought the idea was to get a mathematical model of it rather than just sort of measure it.
 WMM It might be helpful to test your intuitions.
 Jolyon Yeh
 Adrian Come on then, let's fill it up.

Filling the tanks

27:52 Adrian (Pointing at tap level of tank B) We want to fill this one up so it..
 28:50 Adrian If you just fill it up to the five litre mark.
 Jolyon That's spot on, mate.
 Adrian When it's empty it's still got all that
 Jolyon Shall we fill this one (tank B) to there? To this line first?

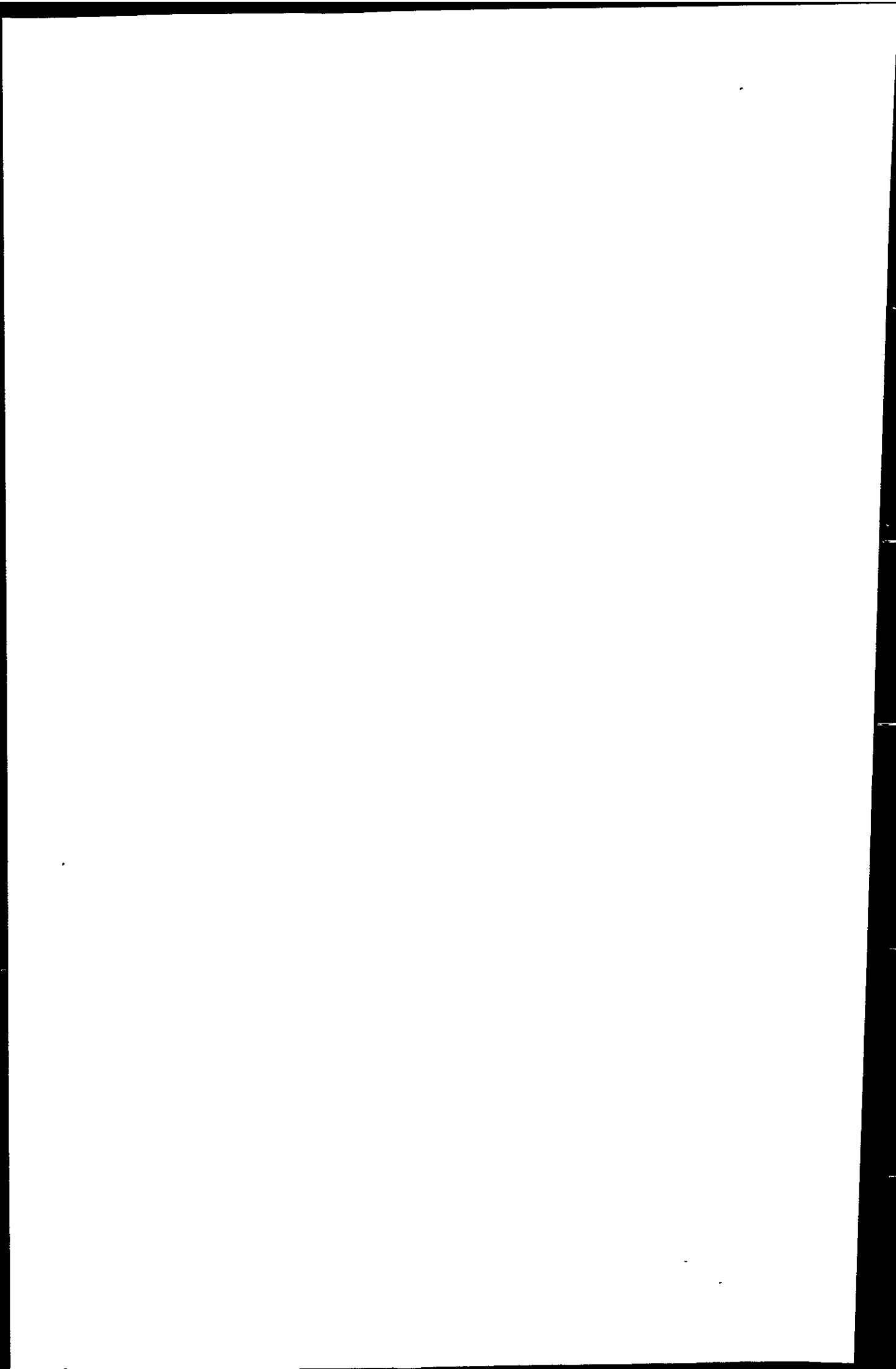
A gap in recording. The students are now sitting looking puzzled and unhappy

30:00 Adrian The liquid is leaving the top container at a uniform mass flow rate. Of what volume.
 Jolyon So?
 Adrian So
 Jolyon No different pressure either. I'm not so sure about this formula anyway
 Adrian That must be it most likely. The time that..
 Jolyon All I can say is it seemed to reach a maximum near the first taking water, albeit very slowly.

Intervention 2: How did the apparatus differ from the thing you are trying to model?

WMM attempts a Socratic dialogue.

32:28 WMM Can I come in here?
 You found that it didn't do what you expected it to do. And you found that it emptied at virtually a uniform rate.
 Jolyon Well, yeh it did
 WMM which was not what you were expecting. Why do you think it did that?
 Adrian The difference in height was causing pressure difference
 WMM Yep
 Adrian But it isn't enough to vary- say if the thing was 10 metres high then you would've got enough pressure difference to matter.
 Jolyon It would have been more visible, wouldn't it?
 WMM What I'm questioning really is: what the height is. What are you measuring the height from?
 Adrian From the top. From the level it is at now to the level it was beforehand.
 WMM Well, that's the change in height- but what's the significance of zero height in the physical significance of height?
 Jolyon Well we were taking the height from the floor to each one.
 WMM What's the sig, um, what is it about h_0 that's, I mean I'm asking really what's the physical significance of h .



Another tack

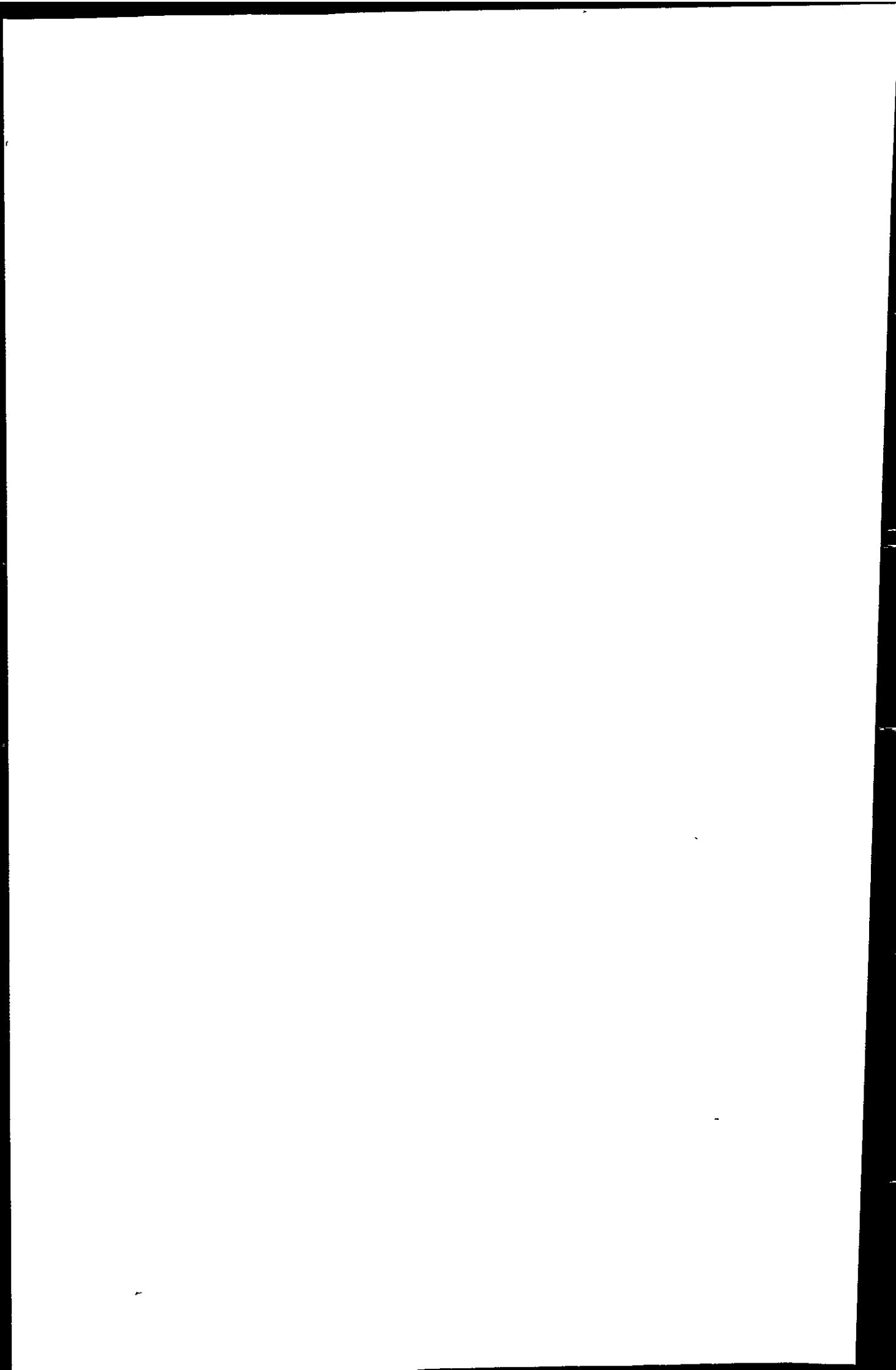
- 33:12 WMM Does the apparatus match the diagram?
Adrian (Looking at diagram) Um- no apparatus will exactly match the diagram but you can assume it's being just by putting the bottom amount of water in the vessels.
WMM So you've got a little lip, but what else is the difference between the apparatus and the diagram.
Jolyon Well nothing apart from what's left in the bottom of the container.
Adrian Water is forced down into here instead of horizontally
WMM So it's got no horizontal velocity. What about.. What is it that's affected by h apart from the flow rate?
Adrian The pressure
WMM The pressure, yup. h_0 is the place where the pressure is equal to atmospheric pressure. So here is it apart from at the surface of the liquid that water's at atmospheric pressure?
Jolyon It's at atmospheric pressure when it leaves this tube.
WMM Exactly, yes.
Adrian So pressure at entrance and exits to the tube is the same then.
WMM Is it?
Jolyon At the water surface it is.
WMM Well it's the same at the water surface and at the point where it leaves, cause that's what's driving the flow.
Jolyon Yes but where the water's higher the pressure's getting to be different there than
WMM Yes. So could you adjust the apparatus to make it more similar to the diagram.. so it matches better?
36:14 Adrian By taking the rubber..
Jolyon We could take the rubber off but I mean then I don't see any difference. Cause you still.. whether atmospheric pressure's going- to be here or there (indicating top and bottom of tube) but the restriction..
Adrian I'm not sure it's going to make any difference
WMM Do you think that taking the pipes off will make a significant difference?
Adrian Some difference, yes, but not...
Jolyon You haven't got any losses in the tube then, have you.
37:06 Adrian So you increase the flow rate leaving the tap
Jolyon I suppose the pressure's less because you've got the head of water from the tap to the bottom of the tube.
37:41 Jolyon (Removing rubber tubes) One way to find out.

We set up the apparatus again

- Students empty tank C into tank A.
Jolyon We don't need to do that (As Adrian drains tank B)
WMM May I suggest you sit it on something- the middle carboy.. I think you'll also need to sit the bottom carboy on something.
Jolyon That won't affect the top one?
Adrian No
(Middle carboy sitting on spool)
Adrian You hold it (the bottom carboy)
(Jolyon trying to spot when the level in B hits a maximum, holding bottom carboy. Level in top carboy starts at 5 litres))
Adrian Ready
Jolyon Yes
Adrian Keep your eye on that. Three two one go
Jolyon It's taking longer. A lot longer
43:45 Jolyon It's constant. It's staying that way
44:00 Adrian Right then that's a lot better
Jolyon Doesn't significantly change the result of this but it's changed the top one. This one's still at a maximum- well from about then on

A dialogue: how much have things changed or improved?

- 45:45 Adrian Right. That performed more as we expected: The first litre at 21, the second 24,



the third 31 and the fourth 41. That was more how we expected it to go, wasn't it?

Jolyon So what did you say initially it was?

Adrian Initially it was virtually constant for a time: first litre too 22, the second 24, the third 31 and the fourth 41. So that was more how we expected it to go, wasn't it? From the top.

Jolyon I suppose so.

Adrian Initially it was virtually constant time really. First litre took 22, second 24, third 31 and the fourth 41, so that was more what we expected, yes?

Jolyon (Pointing at tank B) Cause the difference with the tube was less.

46:30 Jolyon I seem to remember that somewhere: the thing we did was vary the length of that tube.

Adrian It has changed it a lot, hasn't it?

Jolyon The thing is still reaching a maximum with the amount of water at the top.

47:44 Adrian So when you said it was a maximum, yep, was it a noticeable maximum or did it sit at that level for a long time?

48:04 Jolyon It went up to just over 2 then hovered around there- couldn't really notice anything until the last little bit when it started dropping again.

48:48 Jolyon That's assuming the sizes of these are the same.

Adrian Yes. They are the same, aren't they

49:20 Jolyon The difference between this and the previous to compare- the pressure differences between there and there- and this one between there and there.

50:26 Adrian I'd say the maximum is an interesting point where it starts losing more than it's gaining.

Jolyon Well I would have thought it's obviously (μ) that's wrong and it's starting losing more than gaining at this level (B) is the level in there (A) i.e. over halfway down.

Adrian I expect the level in there (B) got up to there too soon and only the last bit (A) will start dropping.

Jolyon If it was the same out of here and out of here this level (B) wouldn't change at all.

Adrian looks at papers. Jolyon looks at apparatus.

53:11 Jolyon takes a sheet of paper

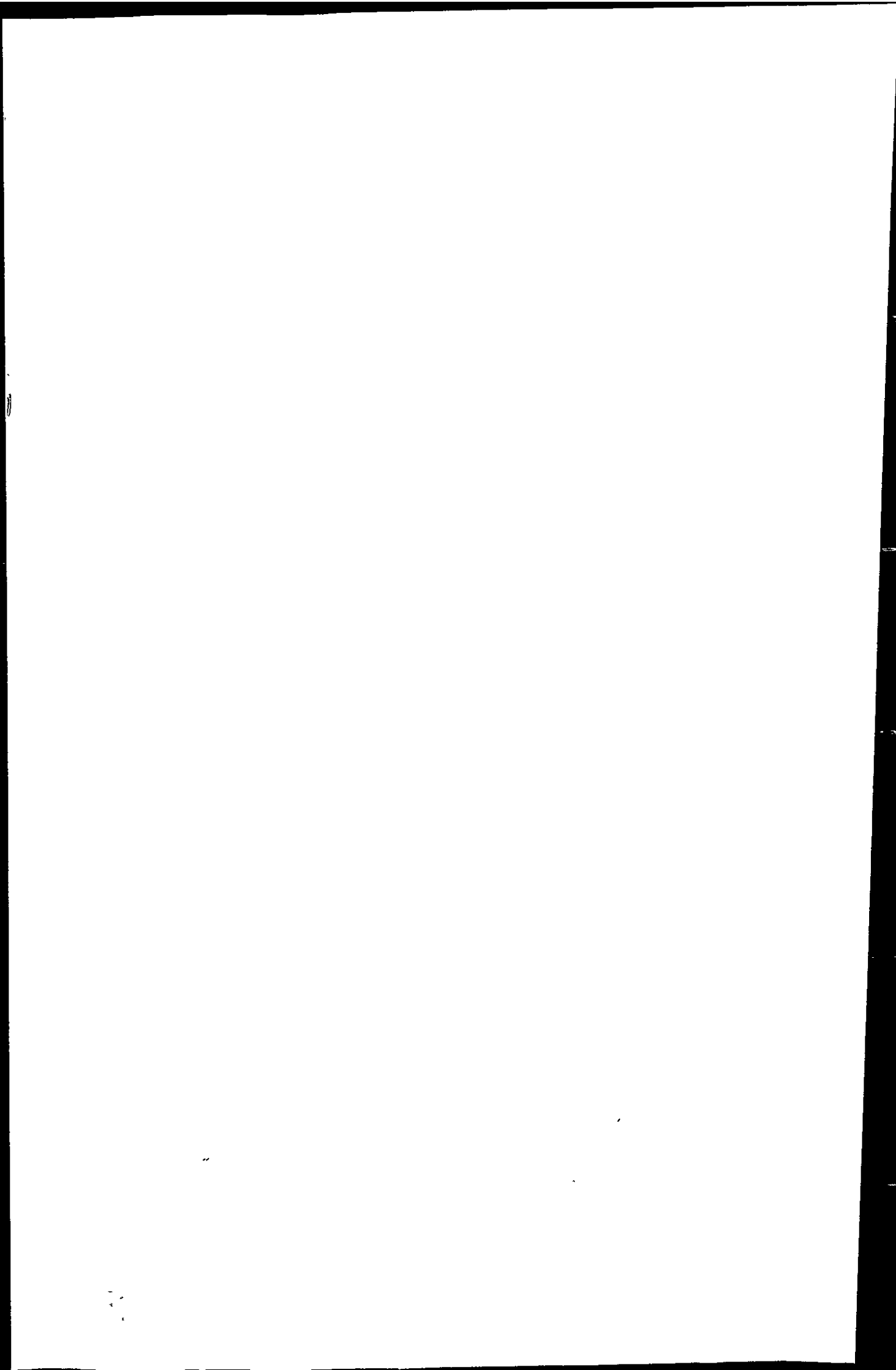
53:35 End of tape.

Tape 2

Time on tape	Speaker	
00:01	Adrian	So the maximum is going to occur on that when it starts emptying more than it's gaining - yeh- first of all its gaining water more than it's emptying, and it rises.
00:29	Jolyon	I mean if that's the case - if you're saying that - then why is it still at a maximum when that's nearly empty?
00:45	Adrian	There's not much in it. It's not really visible. I would agree that should be the case

Intervention 3: Plot a graph?

01:35	WMM	Can I intervene?
	Adrian	You certainly can
	WMM	May I make a suggestion? Have you used all the information that you've actually got?
01:47	Jolyon	I think so
	WMM	Well it appears to me you could try plotting the er the figures you got from the top one emptying and that might lead you down an interesting road.
	Jolyon	Shall we make a note of each..
	WMM	You've got the times it took to go past each little mark in the top one. That may lead you down an interesting road.
02:50	Adrian	Shall we plot the time?
		Adrian draws
		Jolyon takes paper and sketches



Break in film

Is there a linear relationship?

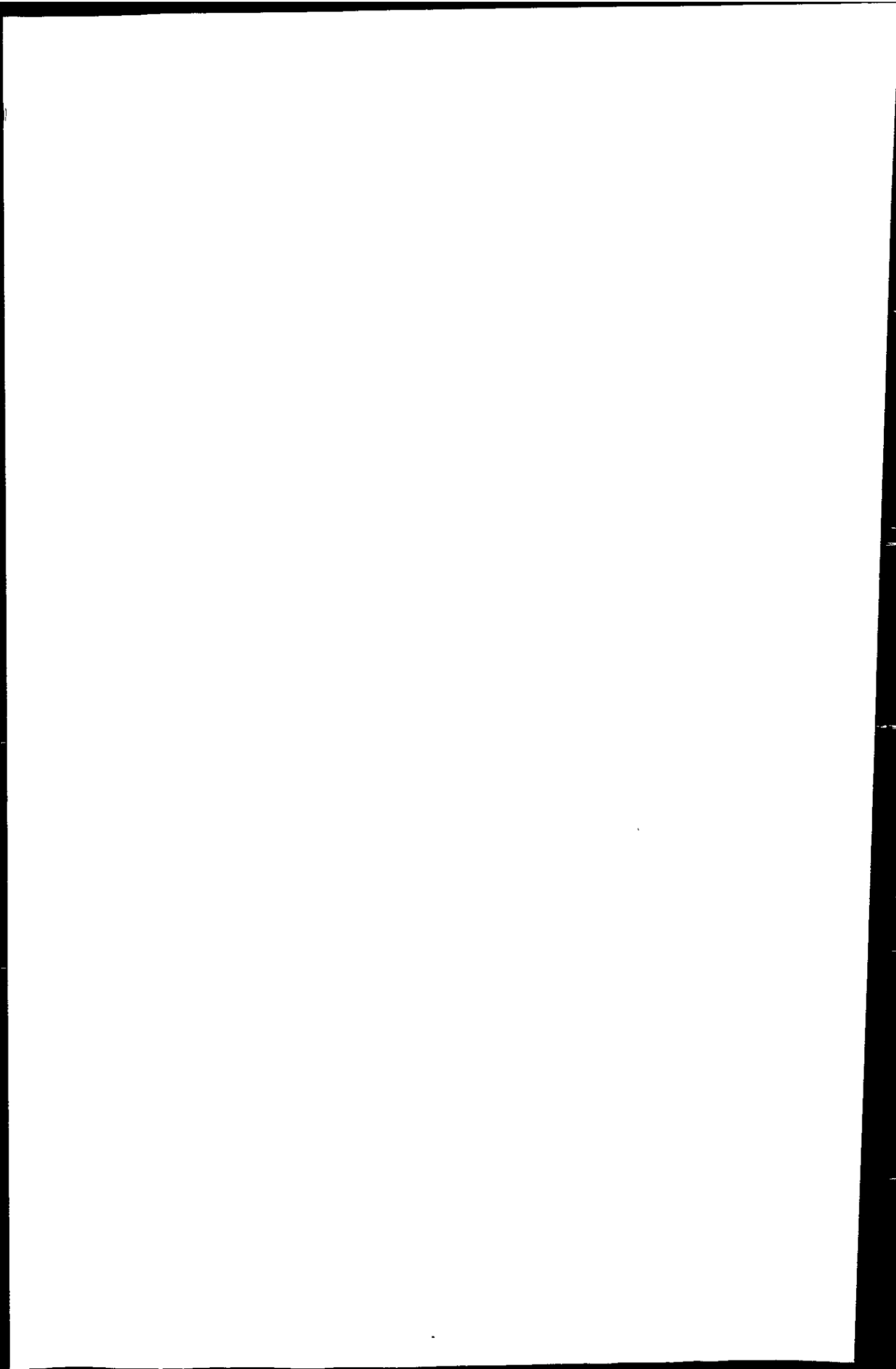
- 04:41 Adrian reading data and plotting. Sketches in line
Jolyon Should it be linear? Because it's proportional to v^2 isn't it? Dunno if that's.. Ordinate scale is volume.
Adrian The volume's the same. It's the time that's changing.
Jolyon You change the volume as well. You lose it out the tap. As time goes on, you're losing volume and pressure.
Adrian Well?
Jolyon The top doesn't vary but it does vary here (indicates tap) because of the head.
06:20 Jolyon If that's the case, what comes out of the bottom has gained by the top one. You'd also lose the bottom one at a linear rate.

Intervention 4: What's the relationship between height and flow rate?

- 08:00 Adrian Any tips?
WMM What sort of relationship does it look like?
Adrian There isn't any shape showing clearly there. I thought it tends to be linear.
WMM What does the graph represent?
Jolyon It relates the amount of volume to how long it's been going.
WMM So what does the slope of the graph represent?
Adrian The rate of flow
WMM So..
Adrian so when the slope is nought there's no volume in
WMM Cause you've still got about 20 mm of drop there: when it's empty - in inverted commas- you've still got about 20 mm of drop there. So - and so can you deduce some sort of relationship between the flow rate and the height of the water?
Adrian The higher the water the greater the flow rate
WMM That sounds reasonable. So you've got that there is a relationship between flow rate and the height of water. What do you think that relationship might be.
10:15 Adrian Well, the slope of the line.
WMM Well, do you think it's- It's clearly not independent of the height, so you could write down an equation that says the flow rate is a function of the height. And what sort of function do you think that is? Do you think it might be?
10:47 Jolyon Well, it's obviously not linear, from those results.
WMM No- yes- if it were a straight line it would be independent of height. So you know it's some sort of function of the height.
Adrian We thought it might be some sort of square.
WMM How would you test what the relationship between flow rate and height is? If you're suggesting it's a quadratic, how would you test if it's a quadratic?
12:23 Jolyon Surely you'd have that by seeing the results.
But we don't really know what's going on- we're not really sure what's happening between.. each container. So up to now we've only done experimental - and what we've got there - doesn't really show enough - doesn't really tell us enough about the flow rate against the height of the water.

You can only measure flow rate under steady flow conditions.

- 13:07 WMM So you carried out an experiment to.. How would you try an experiment that would give you a sort of .. the relationship between height and flow rate?
13:40 Adrian What you'd do is to set it up with a variable input into it and we'd have to maintain the height in litres for the flow rate. Maintain it at say five litres and measure the water coming out the bottom in a given time.
*
14:18 Jolyon Do the same for four, three, two and one litre which would probably give us a more accurate result.
WMM Could you not do something simpler than that? (Laughter)
14:40 WMM Well, as the water's running out, it's actually, successively, at every height



that you've got there. So, um, if you can work out a way of er estimating or um..

15:15 Adrian You want to get the flow rate for each thing of water.. each..
 Jolyon Yeh - that's what the volume is .. Water in .. how much it is.

16:08 WMM So what would help you then? rather than just measuring each litre..
 Jolyon Well, just maintain the height of the er water in the top.

16:08 WMM What if you were, for example, to er time the time it took to go from 5.2 to 4.8 litres?
 Adrian We haven't got enough calibration on the..
 Jolyon It'd give you a rough guide I suppose
 Students study papers.

17:44 WMM Another approach might be to mark - to put a scale on the side of the carboy and mark at regular time intervals. So you get slightly more points in there.
 Jolyon Very hard to achieve I suppose.

18:23 Adrian I thought we could keep the head the same.

A relationship between flow rate and height in litres.

19:00 Adrian sketching and jotting. Jolyon looking at the tanks
 Adrian uses calculator

19:55 Jolyon That's not constant is it?
 * Adrian No, it's not constant is it.
 The first could be the height for ranges.
 So if we say that was between four and five, then we assume that that occurs at four and a half.
 Jolyon What are those for?
 Adrian (pointing) between the scales - the graduation - yep
 Jolyon Oh I see, OK.

20:43 Adrian If we say it takes () seconds to empty that, then the average between the two is going to be what comes up on our screen. So now we know the mass flow rates at various levels
 Jolyon So what will that give us? So..
 Adrian doing some more writing and I cannot find a sheet looking like the one he was working on.

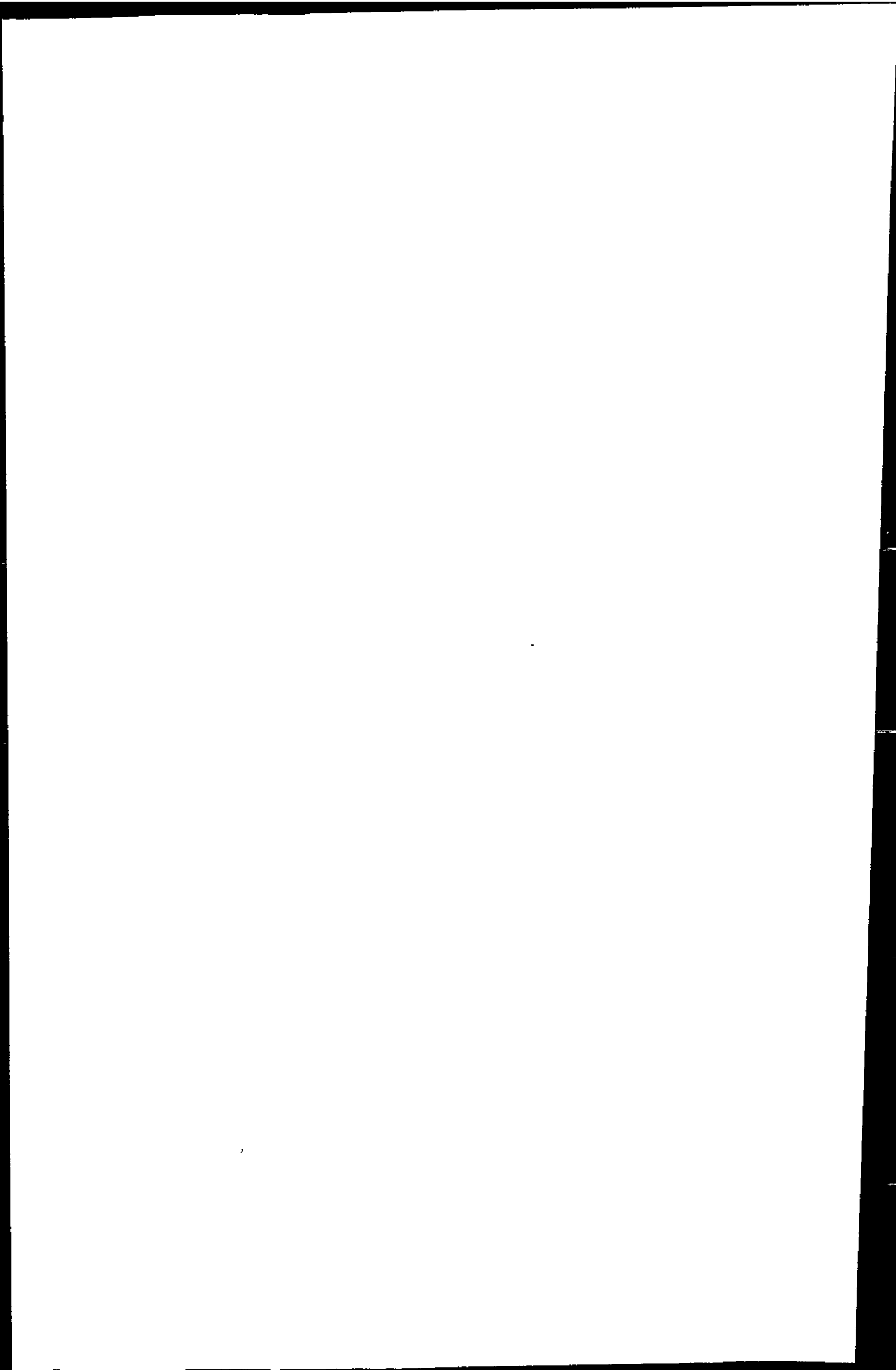
Trying to come to terms with tank B

23:30 Adrian So we know that this has got to be less than that. What we want is the volume in terms of the height - yeh?
 Jolyon Then the volume is turning up - yup. So is the height.
 Adrian Just writing the two to make a table.
 Jolyon Although effectively you've got to take a line..
 Adrian If we see this reasonably.. It starts gaining
 Jolyon It's also losing.
 It's losing..
 It's not losing at the rate it comes out of here otherwise it wouldn't gain..
 Adrian So the rate it's gaining is overtaken by the..
 Jolyon You'd think they were equal here
 Adrian Yeh..
 Adrian drawing the graph of mass flow rate against height.

27:38 Jolyon Turning there.. flowing in at that rate. That's what's coming in the water.. if it takes that form.. flowing less volume as time goes on.. right?
 Coming out.. at the bottom one.. but it's happening slower because we assume there's already water in here in fact that might even vary sometimes.
 So if you can those two that's why it seems to reach that height and stay there till it's almost empty.

28:52 Jolyon Perhaps we're going to get a simple output.
 Jolyon sketching graph.

30:00 Adrian Outstanding - guesstimation.
 Jolyon Does it prove anything?



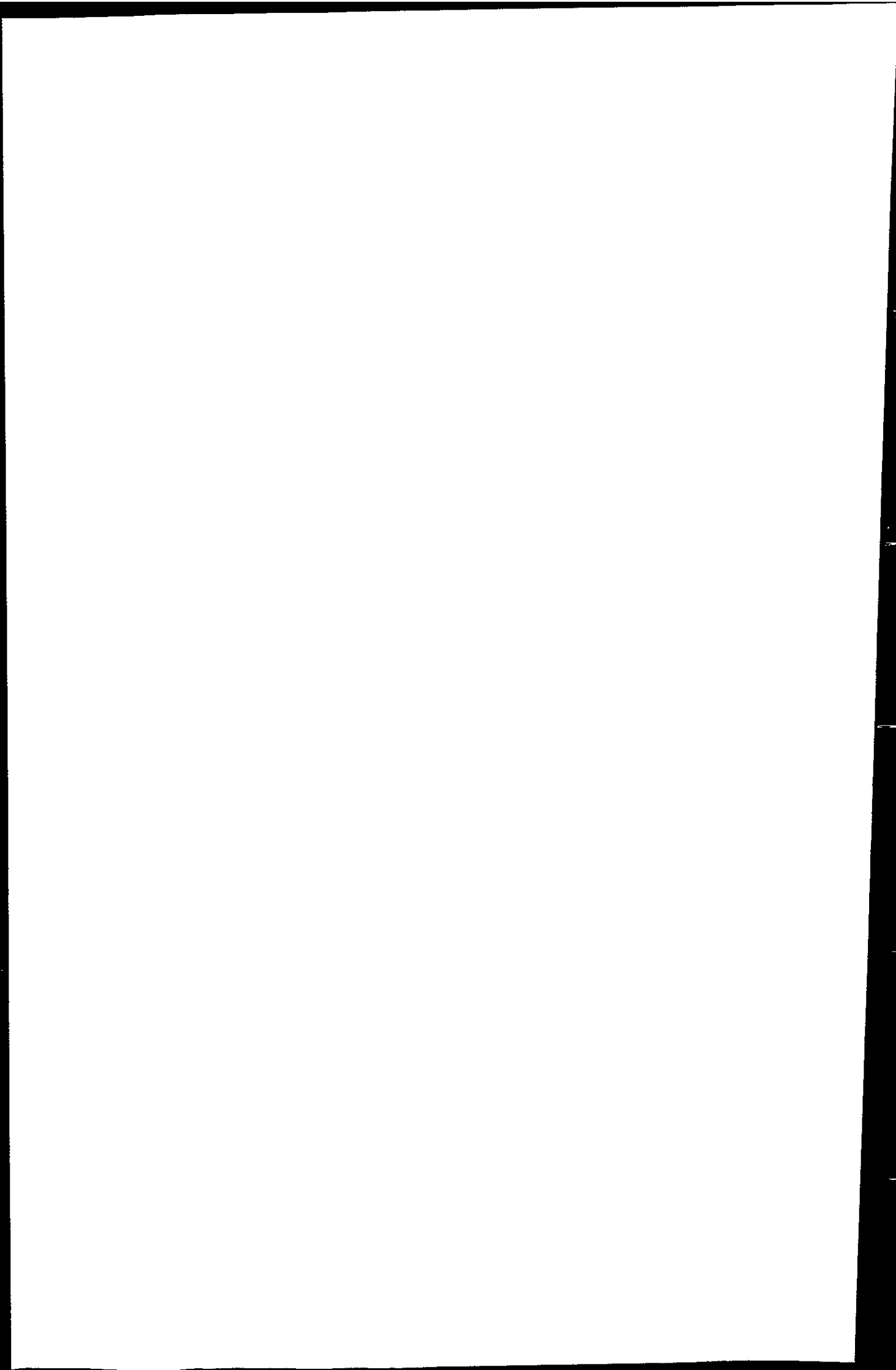
Adrian We can specify the mass flow rate at any height
 Jolyon We need to know that in terms of h though. Cause what's going to happen here (points at tap level) if h is going to go up.
 Adrian It's going to - some height and mass flow rate. So in the lower container it's gaining at this rate but it's losing at that rate until we get to a point..
 Jolyon But as it carries on it doesn't seem to reverse.
 Adrian No
 32:12 Adrian It gets me up that these. That the two nozzles that aren't...
 Jolyon I'm sure you'd notice it because it limits and just stays there until it's going to be empty.

Struggling to come to terms with the implications of water flowing in and out at different rates.

32:30 Adrian It comes in at this flow rate. It goes out at that flow rate.
 Adrian using calculator.
 33:20 Adrian This could be some big pig.. Which kids us. The maths bother me.
 I make it point three three.. nought point three three..
 Ah here it is.. The max..
 33:28 Adrian That's the maximum innit.. The maximum level.. So when point nought three mass flow rate occurs .. what time that occurs.. is the time when it hits the maximum.
 33:50 Jolyon So we could... if we had an equation for that line, we could differentiate that and find a maximum, couldn't we?
 Adrian Yes, but it's..
 Jolyon Which is what we're after.
 Adrian Mmm.
 Jolyon But it still doesn't explain why it stays at that level.
 34:45 Adrian How long does it take to get to that point there, then it stays there?
 Jolyon Not particularly.. Take the.. Determine half empty..
 Adrian It's not so bad. (Writing on a paper he doesn't seem to have left with me)
 Four and a half, that occurs.. (Using calculator) Two and a half, will occur..
 36:20 Adrian one and a half..
 36:30 Jolyon So?
 Adrian The time that occurs. The time that occurs... So we're saying that our maximum height be reached.. How high was it do you reckon?
 Jolyon (looking at carboy) Two - twoish
 Adrian Two point two. (using calculator) Point three o five.. so..
 38:20 Jolyon Between those two values
 39:30 Jolyon You're saying after 69 seconds that makes that a maximum.
 Adrian Yep that is my prediction.
 Jolyon It's incredibly dodgy I reckon.
 Adrian Well, let's do it again then, try it.
 Jolyon But going by that, that doesn't say it's going to stay a maximum height, does it?
 Adrian No.
 40:15 Adrian Unless we calculate what the volume is
 40:30 Adrian You trying it?
 Jolyon I suppose we are - can at least

A frustrating experimental run. Adrian sees the behaviour of tank B for the first time, and is not impressed. Adrian interprets in terms of improving the apparatus.

41:00 Students set up apparatus again. Jolyon pours fluid from tank C into A and holds tank C in his lap. They adjust the starting levels of fluid. It seems to be less pink than it had previously been.
 41:30 Adrian It's on five litres.
 Adrian Do you want to mark it every so often?
 Jolyon I'd say we're not fussed about it this morning.
 WMM If you want to mark it every so often I can give you some stuff to do that.
 Jolyon Yeh, we could
 WMM Do you want to mark it every so often?



42:30 Adrian Yeh
 Adrian puts transparent tape on middle carboy
 WMM You might want to mark some sort of register on there, then take it off and er..
 Adrian puts marks at tap level, and sits next to Jolyon.
 Jolyon So you're saying sod the top one then.
 Adrian Hm?
 Jolyon Just let the top one go.
 Adrian Well, yeh. That's going to be the same as last time, innit?
 Jolyon Yeh
 Adrian OK
 Jolyon Ready, steady, go.
 Adrian marks level on middle carboy at intervals.
 43:45 Adrian We'll mark at one and a half and then..
 44:14 Jolyon Thirty
 44:30 Jolyon About two and a half now.. That's stopped dead at the maximum innit Still
 going up though..
 44:38 Jolyon Three and a quarter
 44:47 Jolyon The max.
 Adrian Constant at.. No..
 45:08 Jolyon See, it's still got something..
 45:20 Adrian Three and a quarter
 Jolyon It's going a bit now thank goodness
 Adrian What we could do with is something up front here that stands out and shows
 that .. I don't know how we can do that..
 Jolyon Take it off the kettle through there.
 45:37 Adrian Yeh

Adrian insists on a change in experimental procedure.

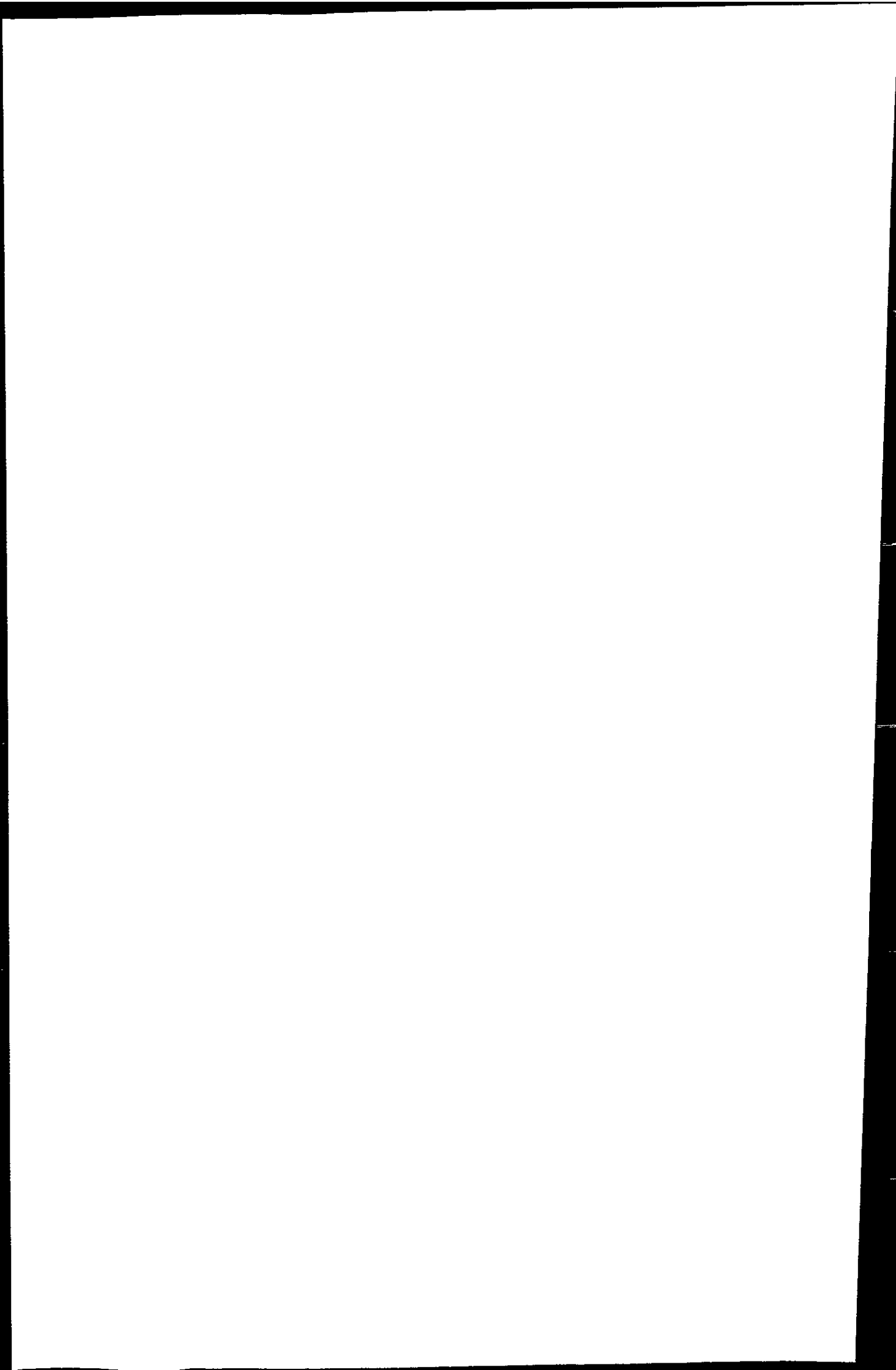
46:15 Adrian How about we try to mark it, say every ten seconds or something - and see
 how long it takes to reach that maximum, yeh?
 Jolyon But you can't tell when it gets to the maximum, because..
 Adrian Well, yeh, I think that's the best thing, innit?
 Jolyon Shall I fill it up again then?
 46:40 Adrian Yeh

Adrian feels his estimate is vindicated.

Students fill up the apparatus as before
 47:36 Adrian Yeh so if you just call out every ten seconds, then I'll put a mark on it.
 47:51 WMM Do you want another piece of tape?
 Adrian No
 Jolyon It'll be all right.
 Adrian Every ten seconds
 48:02 Jolyon That's all right
 Adrian You're going to go by your watch, right?
 Jolyon If you like
 Adrian You say when
 48:37 Adrian five, four, three, two, one - here we go..
 Can't get it (the tap on the top tank will not open)
 48:50 Adrian five, four, three, two, one - go.. Here we are
 Jolyon call out the time at ten second intervals: Adrian marks on the tape on
 tank B. After "sixty" Adrian does not mark the tape
 50:25 Adrian (Pointing at the marks) Ten, twenty, thirty, forty, fifty, sixty seconds.
 Jolyon So 69 is.. just a lucky guess only.
 Adrian Ah! (Stands up) Proof!

Adrian attempts to improve technique further.

50:55 Adrian (Putting metre ruler into tank B) Get this thing wet.
 Run the water down that we'll have a far better level: not going to be
 splashing as much is it? Try it-one more time?



Jolyon So we'll get a maximum height, you reckon?
 Adrian Well yes cause if we do it every ten seconds
 Jolyon. you can see the end where the water starts dropping back from where it's
 went.
 Adrian No, why I want to put that in there is to stop the water splashing so much so
 we've got a more.. Yeh?
 51:55 Adrian Pouring onto the ruler, should run down a bit smoother..
 52:40 Jolyon Is this the fifth time?
 Adrian Something like that
 53:20 Jolyon (Holding tape against tank B) Stick it by the side, d'you reckon?
 53:37 Tape ends

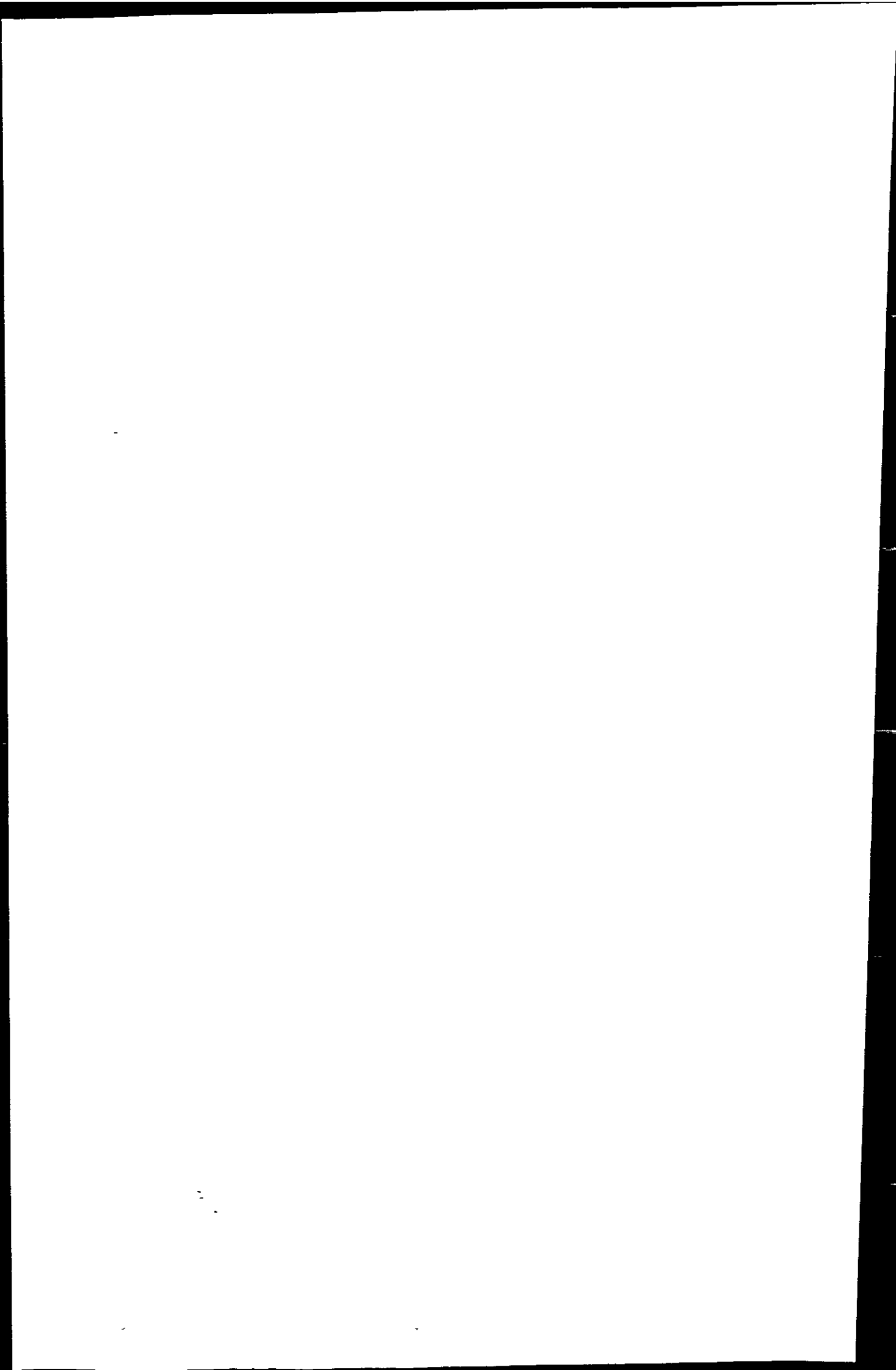
Tape 3

Another experimental run.

00:08 Adrian Five, four, three, two, one, go.
 Jolyon calling out at 10 second intervals: Adrian marking level on tank B.
 01:40 Jolyon So all we've got to do is draw a line up to your curve.
 Adrian Yeh, when it dips - it's max over two now.
 Jolyon We do the line up about two point two.
 Adrian It's about two really is the max you've got there.
 Jolyon Right - your answer's 69 seconds.
 Adrian Well more or less - we'll rework it a bit.
 03:00 Adrian uses calculator again.
 04:08 Adrian About 76 - 76 seconds.
 Jolyon Same time.
 Adrian Well, there we go. So that's how we estimated the time.
 What will the maximum volume be?
 04:45 Jolyon We don't need to know that, do we?
 Adrian (Reading from sheet) What will the maximum volume be?
 Jolyon The thing is, doing it practically, we know what the maximum volume is..
 05:17 Adrian Two point two litres
 Jolyon It was a bit over that...
 Adrian What else do we know?

We debrief.

05:58 Adrian How's that?
 WMM Fine
 Do you mind if we have a bit of a debrief?
 Adrian Sure
 Jolyon We can't prove anything though
 06:14 WMM The.. you seem to have gone about it in a very pragmatic way.
 Adrian Yep
 WMM There's nothing wrong with that
 Adrian That's what I thought because you said like to mathematically model it but
 then to use the experimental results
 Jolyon It's a bit dodgy
 Adrian what was getting back to the point of mathematically modelling things..
 WMM Right...
 The idea was that we gave the identical question to some maths students to -
 some maths, just graduated students so it was fair to give them an identical
 question.
 What I've been really interested to see is the way - the different way you've
 tackled it. Completely different way you've tackled it
 They didn't touch the apparatus until an hour in - so it was completely
 different - and they went in and they set up equations - making assumptions
 about the function that the flow rate was a function of height - making
 assumptions about that- working it through - going about the whole thing
 mathematically - in fact they made the assumption that dh/dt is proportional to
 h and if you go into it and carry out the experiment you find out it's



proportional to the square root of h so you can then set up a differential equation which will -er- which will predict the shapes of the curves . You get that dh/dt is a constant times the square root of h : you shuffle it around and integrate both sides and you get functions out which you can check by doing log plots - and that was one of the things I was inviting you to do when I was asking you - could you check if this was a quadratic - did you suggest it was a quadratic?

Adrian Yep

WMM So that was what I was inviting you to do - so I found it very interesting - I found it extremely interesting that you tackled this in a completely different way

Jolyon Didn't expect to tackle it this way, though.

Adrian I mean like obviously there was a minimum and a maximum in the realms of differentiation or like, but its so long since we've done it in maths, a year ago.

09:09 WMM Really? What was Jolyon going to say about not expecting to tackle it this way?

Jolyon Well I expected perhaps to go into the same equations for steady flow theory, that sort of thing which I know we've done - in previous years - without our notes and that sort of thing. Engineers don't remember equations. We go and look them up in books. We don't derive things from first principles and - we tend to anyway - just to take it from the vantage of theory, and then applying it.

09:50 WMM So did you find it hard that you were actually being asked to create the equations?

Adrian Well, yes, if you like

Jolyon Though we could probably do it with a book in front of us.

WMM Is it something you were asked to do ever in the course?

Adrian Well I'd say if we ever did have this we'd have more of a formula to start with.

Jolyon Well we generally work through the theory which they tend to like, make us - the teachers - try to understand it, and then like - apply the results. It's very rare that we do anything from first principles like this.

10:40 Jolyon I remember that we did in HITECC - we did a mathematical model and the particular one I did was the optimal speed of rotation of a tumble drier and that worked well and we actually took that from equations and then Tony sort of encouraged us to do it and we sort of had it - centrifugal force against centripetal force and acting against gravity - sort of worked out from there, rather than doing it practically. So I would perhaps have expected to go on to some equations - but just doing it practically shows what happens but you can't always do things practically like building a bridge..

Adrian You can build models though.

11:42 WMM Was it a surprise to you when the first time you ran it you got a practically constant flow rate out of there?

Adrian Yeh

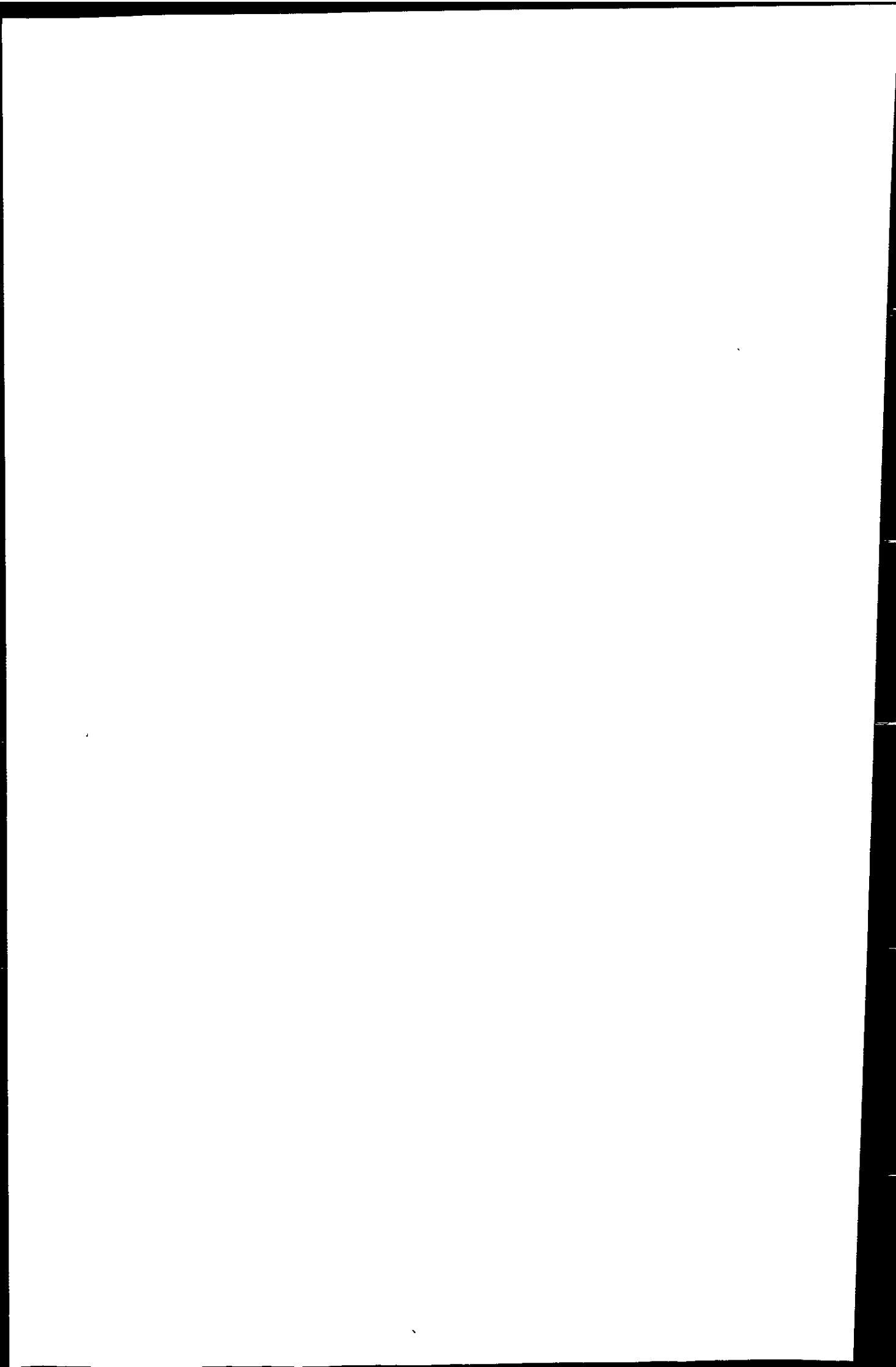
Jolyon Yeh I didn't realise about that - about coming out of the tube which I could relate to because the steady flow stuff we did we had a tube which ran down and we had to find the time it takes to empty a basin full of water and that was dependent on the hill.. But to derive tat from the equations I probably couldn't do

12:27 WMM Did it give you any insight into questioning the apparatus?

Jolyon I think it's all right for what it is. Basic shape is the same. You have to make some assumptions that these are the same. Probably no loss..

12:59 Adrian Did you put those rubber tubes on there on purpose?

WMM I'll explain the history of those rubber tubes. I put them on there for the maths students without thinking about it. to give you a nice steady flow and stop them splashing about all over the place and also to try to reduce the splash in the carboys and the maths students did an experiment with just the top one marking off the levels at intervals and the levels at intervals were - practically constant intervals and it was at that point I realised that there was a problem with the apparatus and I thought - I thought about giving you the apparatus. without the tubes but I thought it was probably better to give it to you with the tubes because it was the same starting position



13:55 Adrian The same
WMM Also because I thought it was a really interesting - two really interesting things. One is that you can't always rely on the apparatus reflecting the same reality as the picture.

14:12 Jolyon We should have been able to work that out if we went through a series of equations because that would have been a difference of head there. We'd have picked it up.

14:25 WMM and the physical significance of h: that it's from free surface to free surface, and also that the pipe losses are more than compensated by the change in h. I'd guess if anything that the pipe would slow the flow down.

15:20 Adrian It would do - yes theoretically there'd be a loss but with the additional height
Jolyon It would be the same head.
WMM So my intuition was the pipe would slow it down but in fact the extra head provided was a bigger influence.

Adrian Yep

15:30 WMM Than the pipe losses. I found that quite exciting. Have you any got other comments about it?
Jolyon I felt that the maths side of it lets us down a bit on what we've done in the course.
Adrian Unless it's like what a lecturer said if you can sort of feel you can remember what he did in the second year it's quite easy but everyone just sort of forgets it. You know you can do it and you know you can look it up how to do it.
WMM Were there any other resources you'd have liked? Apart from your text books or your notes with it in
Jolyon Well I would have done it mathematically but I mean you could get more accurate results with that - put flow meters on, that sort of thing.

16:30 WMM So there wasn't anything like graphics packages - cause I left DERIVE on there in case you wanted it.
Jolyon Right well we needed to set up the equations to put something into it.

16:49 WMM Did you use the graphics facility on your calculator?
Adrian No I didn't. It's a bit of a mystery.. It's all right when you've got the equation to plot and you can pick off the minimums and maximums, but when you've got a set of results to put in I'm not too sure what to use.
Jolyon Silly really - we did better than this on HITECC and now it's four years later.
WMM Can I just say you absolutely confirm my prejudice about what engineers are: stereotype engineers: that you're happy measuring things and carrying out tests and..

17:40 Adrian Yeh but the thing about doing it that way is like you say is if we'd done it purely theoretically and then used that equipment as it was first set up we'd have thought it was miles out and we've have thought it was the theory that was wrong.
Whereas we know
We had to assume so much.
Jolyon Doing this practically was only applicable to this. We do it in a reservoir: totally different. So that's where the theory would probably hold.

18:18 Adrian If you can get your hands on a model you can test it.

We say goodbye and thank you.

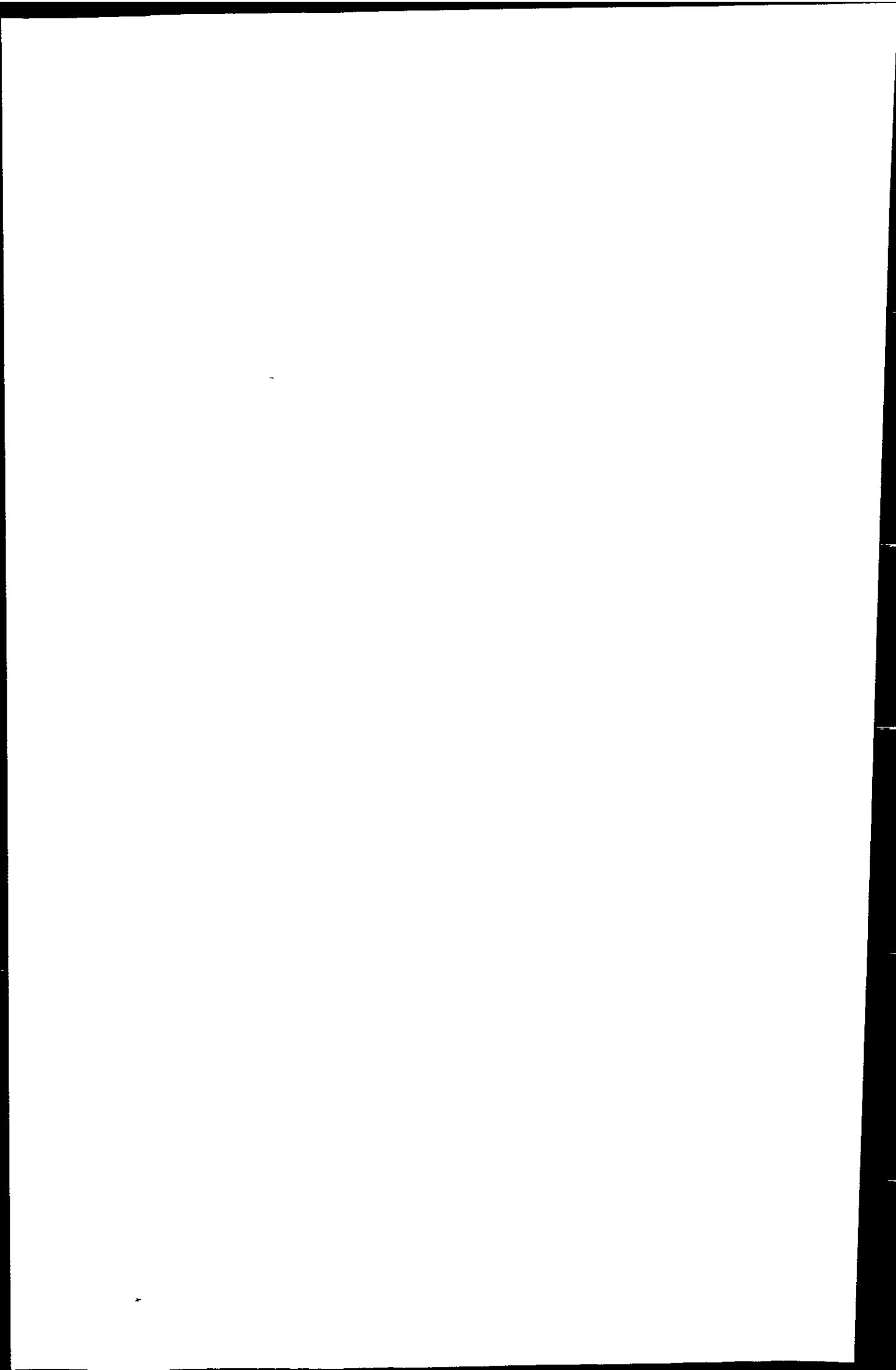
18:29 WMM It's very kind of you
Can you leave all your bits - all your paper here because they are very interesting.
Adrian If you can read them, yes.
WMM I think so, that's great. Thank you very much indeed

18:58 WMM Your watch has got zeros on it
Adrian It must be midnight.
WMM A bit worrying..

19:10 WMM That's been very helpful, thank you

19:10 Adrian Some of the others in our year would've sorted it..

Tape ends



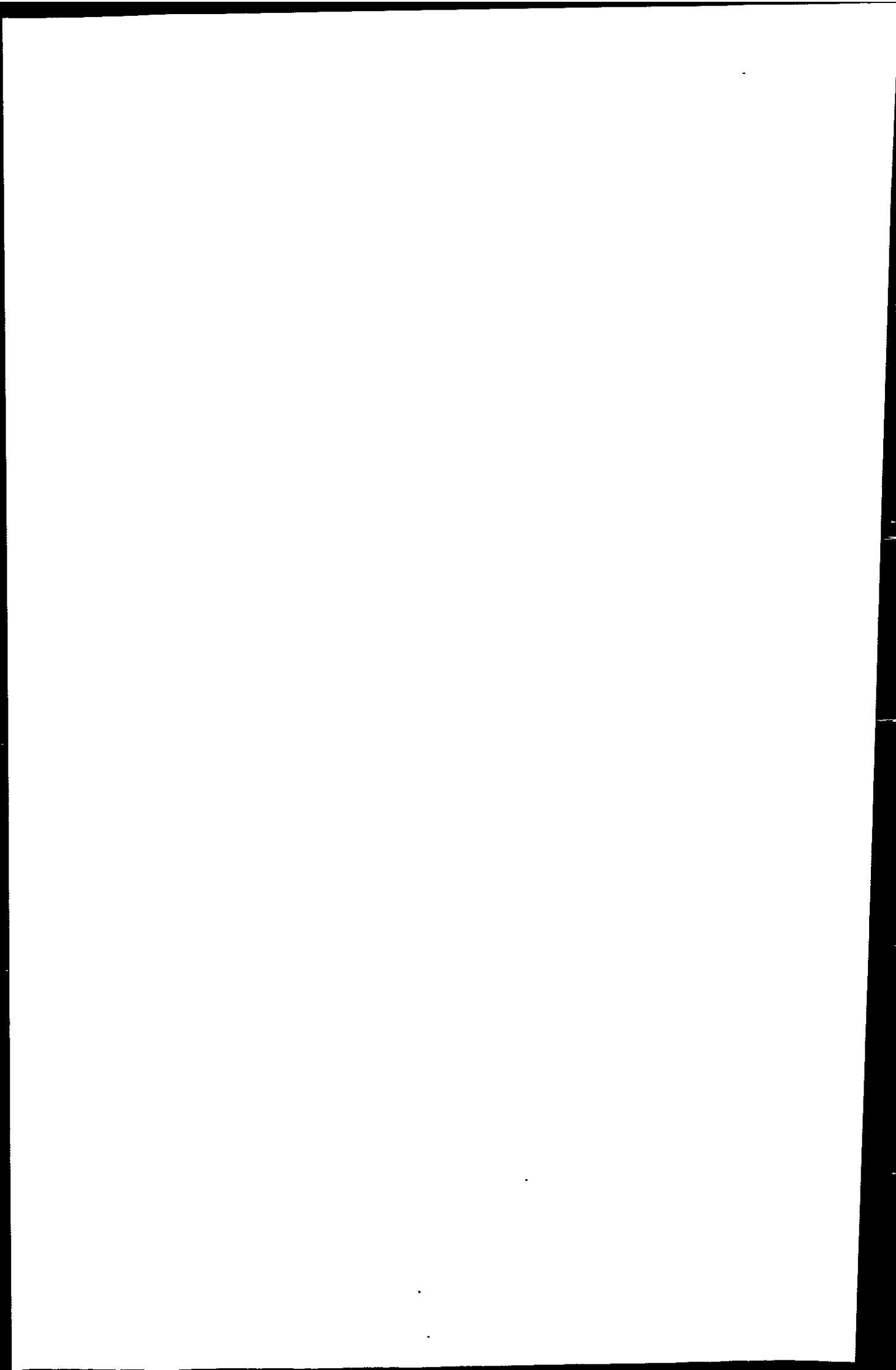
Mathematical modelling with five final year maths students

Single long tape: recorded in 1 Rowe St by cameraman from Hoe Centre studio.

Note. There are unintelligible passages here students are all talking together, or are talking very quietly, but less silence than on the engineering students tape.

WMM is the author, RKP is a colleague.

Time on tape	Speaker	Notes
00:30	WMM	This solution, by the way, is slightly laxative (laughter) So please don't get it on your fingers and lick it. It's just to colour the water.
	Ann	Oh, right
	RKP	Well, good afternoon, everybody and thank you very much for coming to take part in this. The problem which you've got in front of you is you've got three tanks of water, one draining into another and that's draining into a third one. You start with the top one full and both of the other two empty and you set it draining. At some point - um - the water - um - there's water in the middle tank - and then ultimately all the water will be in the bottom tank and the top two will be empty. The question is really when is the top tank at maximum volume and what will that maximum volume be. Now we've got some - there's some equipment there to let you do experiments. (laughter)
	Ann	You mean the bottom one.
	Others	The middle tank (laughter)
01:34	RKP	Let me say that again.. In the initial state the top tank is full and the other two are empty. In the final state all the water has gone through into the bottom tank and the middle tank and the top one are empty. Sometime in the middle, the middle tank will have a maximum volume. the question is to produce a mathematical model which will describe when it is and how much of the volume is actually at the greatest in the middle tank. We've set up - put- arranged some equipment so you can do some experiments on this. The little medicine bottle there has got some pink liquid in it which can be diluted up to the requisite volume of water - with water and it will then give you a nice pink solution which we would prefer not to be spread over our nice green carpet. (laughter). There are two purposes for this thing this afternoon: and for both of which it's being videoed. We - from my point of view I'm hoping that we shall get out of it a video on how people attack a problem like this which can be used for teacher training - for inservice training courses for teachers. And we would hope that your afternoon could be condensed into fifteen - twenty minutes of the key points of when you have moments of inspiration when you're stuck and whatever.
03:16	Ann	So you'll just video out some bits.
	RKP	You shouldn't - don't feel nervous about it. If you make a whatsit just like I did then we'll just edit out.
	Jason	Not so fun to watch at the end then (Laughter).
	RKP	The other purpose which is closely related is that Wendy's PhD research is in how students learn differential equations and she will be sitting also taking notes and seeing actually the processes you go through in solving this problem. You can talk as much as you like among yourselves of course. If you are stuck I shall be working over there and you are welcome to come and ask me to give you help. Wendy will be there taking notes and you are welcome to ask her for help. But initially give yourselves a reasonable amount of time to get into the problem before asking for help. But then if you find you're getting nowhere, for goodness sake do ask, and we will of course do what we can to direct you. Well, good luck!



We begin our deliberations.

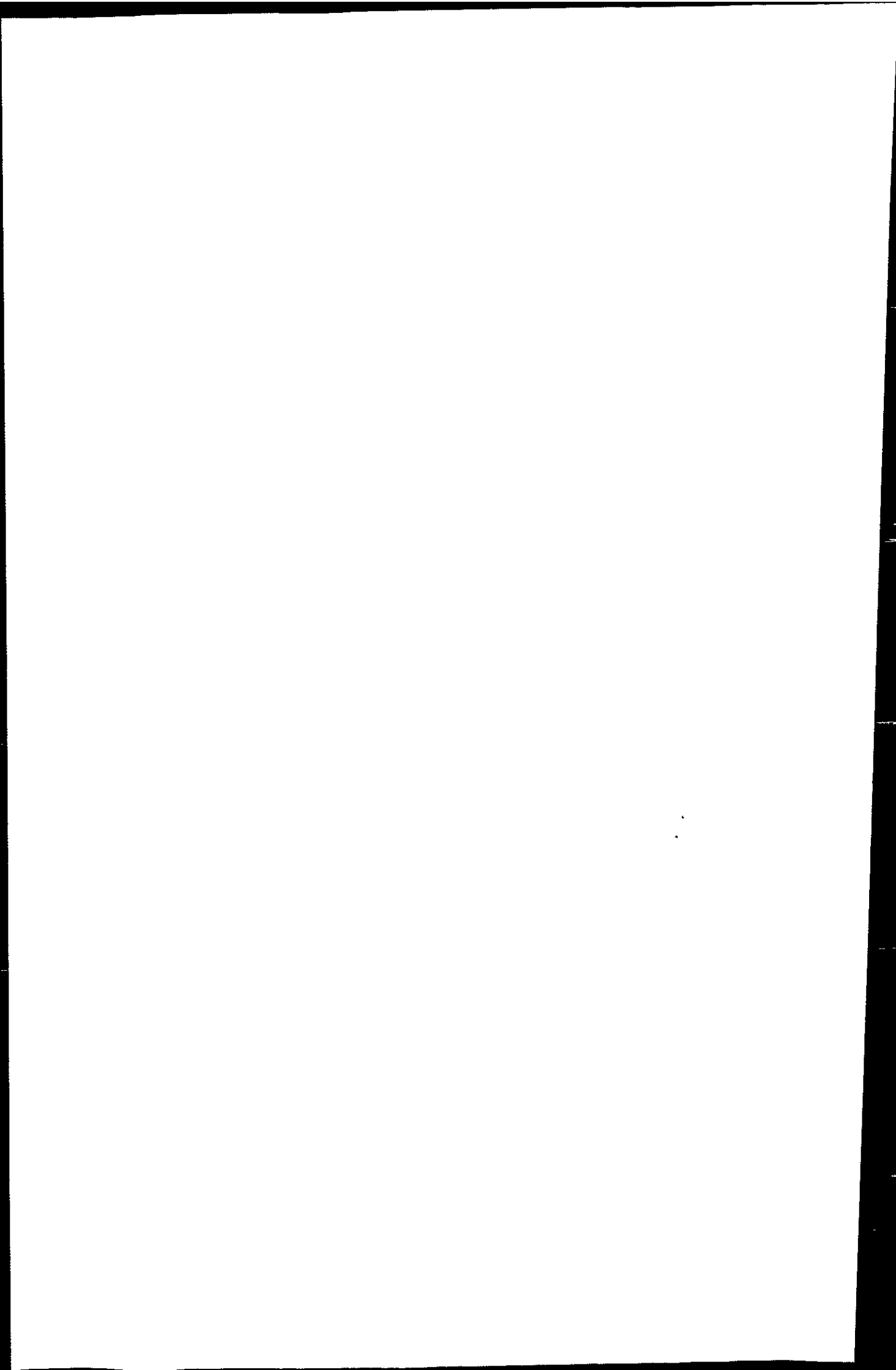
- 04:28 Ann Where do we start?
Jason It doesn't matter. I've got the formula. The flow rate is a function of the volume in that.
Ann The only thing that can affect that is the diameter of - I presume it's linear..
Jason No, the height - I mean the height - the volume of - cause that affects the pressure at the tap.
Derek Since I think you've got the taps the same in both I mean that will not really come into it so I think it's just some function of volume in both bottles like, shouldn't it.
Ann You don't think the size of that the water's flowing out..
Derek Yes but all the other things being the same, that won't change through the run of it.
Ann No - it's just a constant.
Derek No - so its the rate throughout is going to change as the volume in both bottles change.
Ann Due to the change in pressure or gravity or?
Derek Yes I think it's got to be the volume cause obviously when you start off the rate of flow in has got to be bigger than the flow out and that's
Jason What even with that's with the volume but like with the height
Derek Not unconnected - Yeh. Well no the height's going to come in
Anthony So it's the point of gravity then?
Ann Yes it's gravity what's..
Anthony So we can think of a differential equation for each one or something?
Vicky Doesn't seem enough liquid for it to..

Ann makes her first bid to try it out, while an expression relating flow rate and volume is proposed.

- 06:12 Ann We could always put the liquid in and do it and see.
Anthony No
Ann You don't think you need to do it? Just sit there and look at it and..
Vicky You could actually put the pink liquid in and actually test it to see what the fits will be.
Anthony If $dv_1/dt = kv_1$ then that gets smaller..
Derek But if that is going to increase then it will increase and
Vicky because it's otherwise then you'll need to
Ann No you fill it up with water and put a bit of that in to turn it pink.
07:10 Derek In the first instance then, for example, $dv_1/dt = -kv_1$
Vicky oh, couldn't you do it for each tank? and then..
Derek dv_2/dt
Vicky yes

Complications emerge in the proposed model. The volume in the second tank is affected by a flow in and a flow out.

- Ann Surely once it's going it's what's going in.
Anthony You've got two lots flowing.
Derek That's the point
Ann You've got so much flowing into here until it gets up to this level and anything comes out.
07:26 Anthony So what are you going to write for the second one because you've got stuff coming in and still going out?
Derek Well I've written down to start off $dv_2/dt = kv_1$ which is the amount flowing in minus kv_2 which is what's going out.
Ann Yup
Derek And that's
Ann But initially nothing's going to come out there is it?
Derek Well that's.. that'.. that's.. not part of the model. I mean, look at



those diagrams there (indicates sheet) You assume that to simplify it anything that any water that goes in there will..

- Ann Any going in's going out.
Derek In other words the opening's right at the bottom. I mean I think we could refine that model..
Ann Okay, yep, mm

The flow rates vary both with time and with the volume in the tank they are leaving.

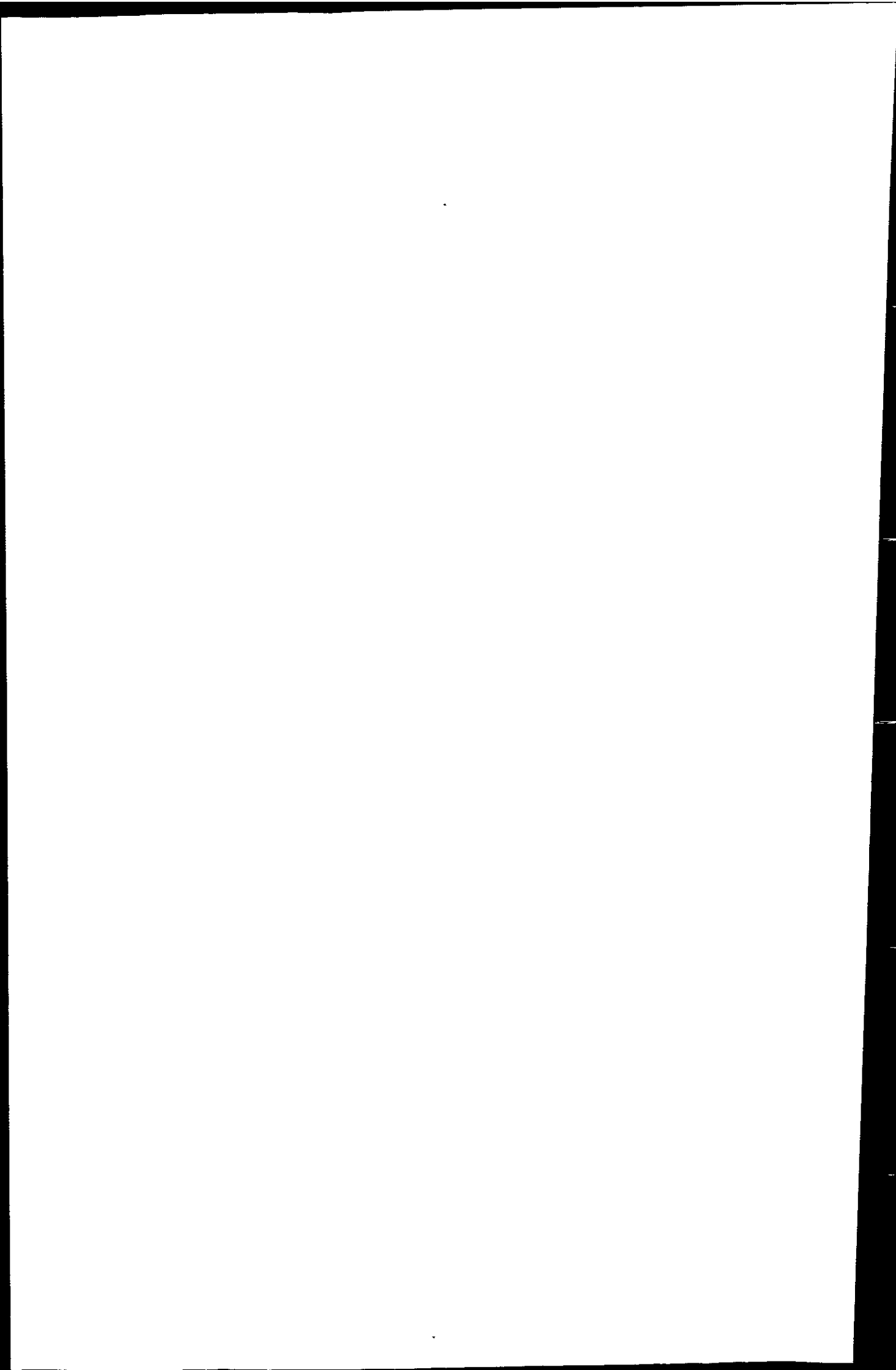
- Derek But if we're going to start thinking about it to start with.. You may as well start with the diagram you've got there.. Assume that any water's going to affect the outflow.
Ann So if you're going to make the assumption that what's going in there's coming out, then you're never going to have anything more in there, are you?
08:36 Derek No, only if the rate of flow out is always going to be the same as what's flowing in. And if it's, say, proportional to the volume, which is my first suggestion might be, then when you start off you're going to have a smaller volume in the second bottle, so your rate of outflow is going to be less, and that's why the volume increases up to some point, and then when there's more in that than in that, then the rate of flow out will be greater than the rate of flow in, so you..
Ann Yeh
Derek But um.. If we assume that this is going to be a decent model and solve it, then how are we going to test it?
09:08 Vicky Fill it up
Jason But it's still going to be more than just those two equations there (takes paper from Derek)
Derek Well obviously

The volume in the bottom tank is immaterial.

- Vicky Well, haven't you got one for the bottom one?
Jason Well, what you want is the maximum volume in there which is totally independent of the volume in the bottom one.
09:26 Anthony Do you want to start with the bottom one. What happens in the bottom one?
Jason We don't need the bottom one
Derek Not much
Ann The bottom one's just going to collect the water
Jason So it won't go on the carpet (laughter)
Anthony The volume in that's just related to the volume above.
Jason Yeh but that is also a function of that (laughter)
10:01 Anthony (Passes notes to Ann) Well have a look at my notes and see what you think. (There are about 3 line on the page)

Anthony is unhappy that the model does not include any physical reasoning.

- 10:10 Anthony I don't know how I'm going to tie in v_1 and v_2 to gravity
Derek Well, that's probably a function of g , a constant.
Ann Are v_1 and v_2 your rates of flow in and out?
Anthony We want to work out the force. It'll be from here.. from the outflow
Derek So that will be related to the max.
Jason Under very high pressure it'll come out all the way out and go psst (describes parabola with his hand) (laughter)
Vicky But it couldn't do on this (pointing at tube)
Anthony If we write down the force
Ann That's why there's the tubing.
Anthony listen, listen. If we write down the force for each amount of liquid for each time won't that be the same as the flow rate or differential?
Derek You might be going too far back on that one.



11:04 Anthony Mmm
Ann It's mgh

Trying to solve the equations.

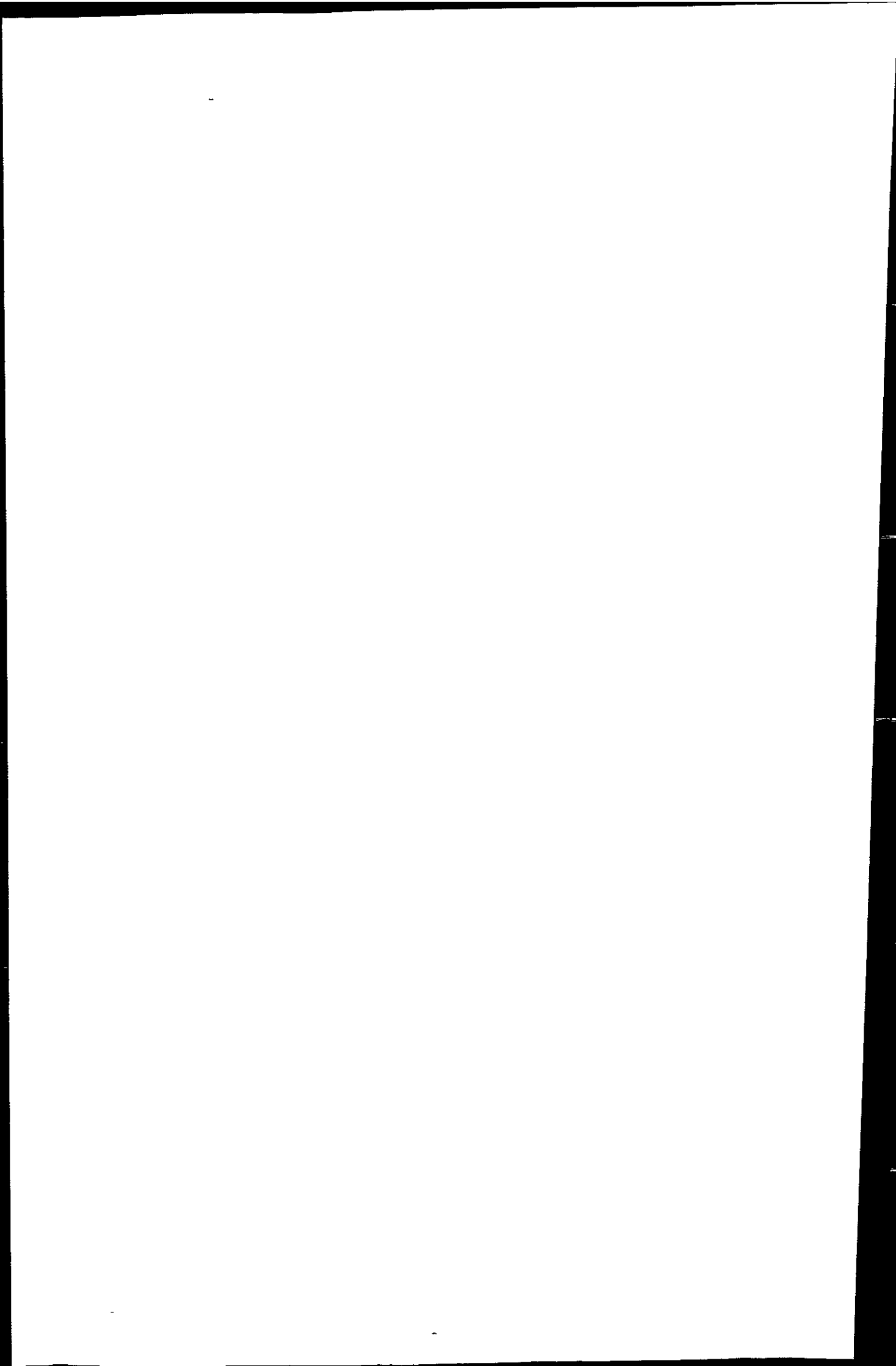
11:08 Jason So v_2 is irrelevant for that one.
Derek V_2 won't matter
Jason So you just want to solve these two?
Derek Yeh we could I suppose.
Ann (Indicating top tank) That's just the change in..
Derek I mean yeh like do we want to find the slope in one
so that
Let's assume $v = e^{-t}$
Anthony You're stating what? You're starting with the second one.
Derek That's the rate of change of volume of the second one equals um
the rate of flow in which is dv/dt the rate there multiplied by the
flow.
Anthony So what's v_2 ?
Derek That's the volume in the second bottle.
Anthony No it's not. You can't say it's v_2 is this, like that, with a minus
sign, can't be right, can it, cause you've got two flows. You've
got the one going out and the one coming in.
12:05 Jason That is minus dv_1/dt .
Anthony There's got to be two of these.
Derek Yeh. That's the other one.
Jason That's this one and dv_2 is here.
Anthony I hadn't thought of that.
12:21 Anthony Um.
12:29 Vicky So you've got something you can sub back in there. Is that what
you want?
Derek I can use v_1 as a .. no.. A maximum there just as..

Investigating the implications.

Ann So your flow from the top one is a decreasing function. It's a
maximum to start off with.
Derek I'm staying when you start off the flow is a maximum out of there
and it's just going to decay.
Ann Yeh,
Vicky It'll be an exponential.
Derek Basically my assumption is that it's minus kv_1 . That'll give you
an exponential.
Vicky Yes, right
13:24 Ann Presumably this one (tank B) is going to be the opposite what-do-
you-call-it.
Vicky I don't know cause it's losing at the same time. But then it won't
until it hits the stage where it's (indicates lip). Hmm, it's not easy.
Ann I imagine that the assumption is that it starts at the bottom.
Vicky In which case it won't actually do it. It's not going to be.. Oh, no
all right.

A prediction of when the maximum in the middle tank will occur, in terms of flow rates in and out.

Ann See the rate of flow out of that's (top tank) going to be a maximum
initially and the rate of flow out of this one (middle) is going to be
a minimum.
Derek Yeh
Ann And there's going to be some point where they're equal.
Derek Yeh but
Ann And that will presumably be when it's at its maximum value.
14:03 Derek That's when they're the same



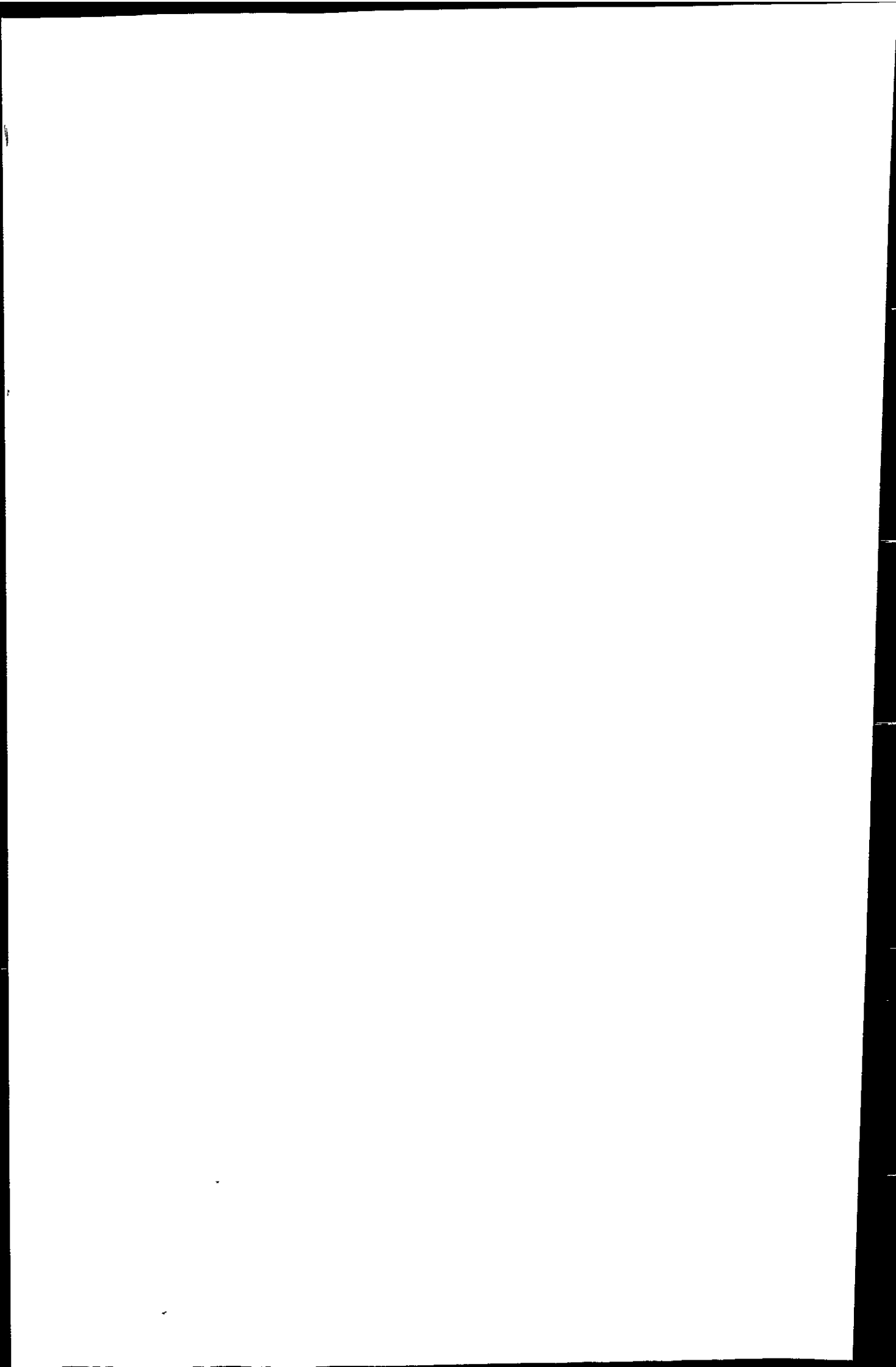
Ann That's when you have your maximum
 Derek You get the maximum
 Derek Yeh, I guess that makes sense, because, when the rate of flow in that (top tank) than the rate of flow out, it's increasing, and when the rate of flow out becomes greater than the rate of flow in, that's when it's going to start decreasing, so that will be the maximum.
 14:24 Derek Meaning what?
 14:40 Jason Yeh, like, if you put that in there, it's going to be $vt = e^{-t}$.
 Ann You've got your time.
 Vicky What you want is your time at t , and then when you integrate out you'll want it.
 Derek I suppose we have to, like, practically solve this, yeh? So it's going to be when..

An exponential will not allow the tank to empty in finite time.

15:10 Vicky Haven't we done something like this before?
 Derek Thing is..
 Jason Well, with that function, we won't be going to reach zero until infinity. That won't be empty in finite time.
 Derek Can you construct a simple mathematical function which will actually do that? That's going to have something in to start with and is going to fall to zero? And actually be zero there?
 Ann Cause it's got to be a
 Derek Yeh, cause it's got to be a function of..
 Ann Well I don't know if it's

Is it a function of the volume or the depth.. or what?

15:55 Derek It's got to be some function of the volume itself, hasn't it?
 Ann Is it a function of the volume or a function of the height of the water?
 Jason The height of the water is directly linked to the volume anyway, isn't it?
 16:10 Derek Yeh, you mean the level. The height would mean..
 Jason mm. If you had the bottles a bit higher than the tube it's not going to make much difference.
 Ann Well, I wasn't thinking of changing it.
 Derek Yeh, well, if we rerun this a few times, I think what you're saying is the only thing we'll change is the volume in the two bottles, isn't it?
 Ann Instead of considering it as a volume, can't we consider it as a height?
 Vicky As in the depth of water.
 Ann I mean..
 Jason You mean multiply by diameter?
 Derek Because diameter is going to remain constant anyway, to a decent approximation, so the height of the level is going to be equivalent to the volume.
 Vicky So surely it's even simpler to work in terms of height, rather than volume
 Derek Well yeh, that makes sense, yeh.
 17:03 Jason So instead of v we have h then.
 Vicky Well, yeh (laughter)
 Jason But v 's easier to write than h .
 Derek Yes cause you've got two lines and..
 Vicky But no, measuring-wise height is easier to work with.
 Ann Yeh, brilliant
 Derek Yeh, fair enough
 Vicky Anthony's brainstorming over there.
 Derek What's the force then?
 Jason $\rho \cdot g \cdot h$ - something like that then probably
 Ann It's ρgh , isn't it, the pressure of the water?
 Jason Foul but



17:42 Anthony Thank you
 Vicky You must have to take into account gravity somewhere
 Ann Yes, that's what
 Vicky But rather than this (paper with exponentials on)

18:00 Derek I think the volume would give you the pressure which with the gravity would give you the rate of outflow and so this is all probably.. I think it will be.. I think it's going to.. It'll come across..
 Jason What, the pressure?
 Derek I think your concerns about g on that side..
 Ann Yeh but if you're assuming that everything is constant you're not going to get any change, are you?
 Derek No I'm assuming the outflow is not constant and the main thing that's not constant is the volume in there. The rate of flow's proportional to the volume isn't it? The rate of outflow.

18:46 Vicky As in the volume of the water.
 Derek The volume of the water is directly linked to the mass of the water which is directly linked to the pressure.
 Ann Yeh, that's what I'm saying. I know we keep talking about volume but the rate at which height is changing is the rate at which pressure's changing. Pressure is ρgh . ρ 's a constant, gravity's constant so the only thing that's changing in that is the height, if you see what I mean.

19:16 Jason And as the height changes, the volume changes because the diameter and the pipe are constants so

Discussion about dimensions.

Anthony Well the diameter is constant, and we've got time over here so we must have t somewhere.
 Jason Well no because no because K will have dimensions of one over t.
 Derek Who brought him anyway?
 Ann Say that again?
 Derek We've written down dv/dt .
 Jason Yeh so the constant has units of one over time. What about those other things we did? The wave equation where we got c^2 . That's got dimensions of length over time squared.
 Derek $mdv/dt = kv^2$

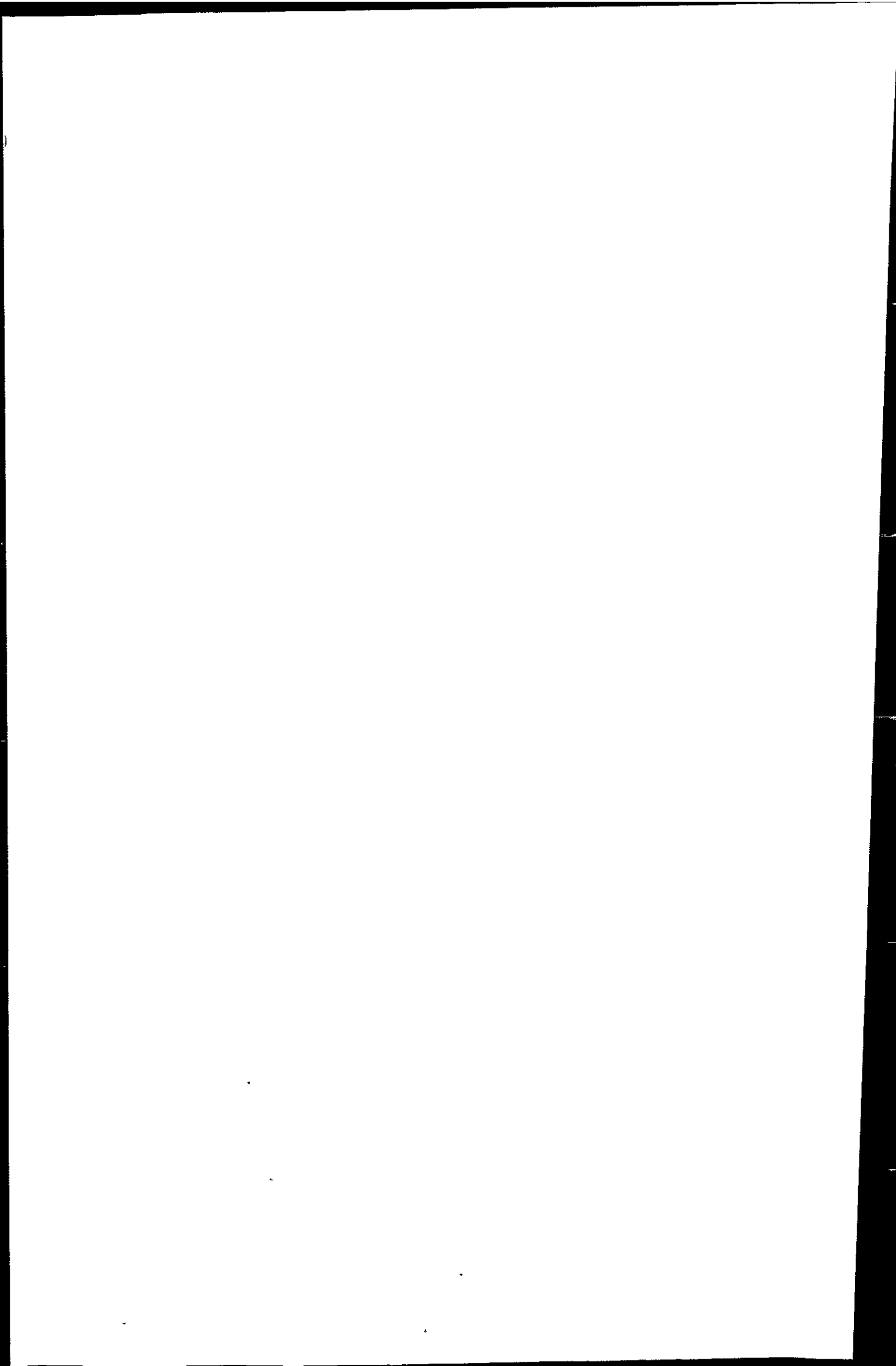
An interesting idea which is not followed up.

Vicky Could we do it in terms of energy? Potential up there and then kinetic energy. Cause that's ρgh , isn't it? Potential energy, mgh , that's what I mean. And then kinetic will be the rate it's going through it, anyway.
 Well, no, then you're going to know it's the same down here as it was up there. You're going to lose the same amount as was up there initially.

An interesting fallacy, and an interpretation of its physical significance. A hint that it may not be correct.

20:33 Jason Thing is you said it was going to be at its maximum when that equals that. That's not true basically.
 Derek That's interesting, go on
 Jason Which would mean $2kv_1 = kv_2$
 Derek That's an interesting point, isn't it.

21:00 Jason That would give me $v_2 = 2v_1$.
 Ann That flowing into there is going to be greater than that flowing out with the pressure difference so it flows in faster than it's flowing out.
 Derek That's something to check, isn't it? At some point the rate of flow out of this one is going to be equal to the rate of flow into this one and that's going to be ..



Vicky When they're the same.
 Ann When the height in that is the same as that on
 Jason But it can't be, can it?
 Derek Yes you can.
 Jason But that would lead to us -oh- at that time. So it isn't.
 Derek So what you get is .. the maximum you've got in that is..
 21:29 Ann At the start of it that is lower than that one anyway. So you've got
 to have more in there (middle tank) than in that one (top tank) for
 it to be balanced anyway.
 Derek That's an interesting result, isn't it?
 Jason So that's equal then.
 Ann You get to a point where the rate flowing into this one is the same
 as the rate of flow out in which case it's going to get to a constant.
 21:55 Jason Well, let's try something out.
 That that $dv_2.. V_2... + Av$ is $kv_1...$
 Ann The rate of flow in is equal to the rate of flow out until there's no
 more flow in and then it'll start decreasing.
 Vicky Except the very last bit.
 Ann The rate of flow in - to start with - is more than it is out. At some
 point it's going to be equal. The rate of flow in is going to equal
 the rate of flow out but it's going to be when there's more in this
 one because that one's higher anyway.
 Vicky So if you're increasing height, are you saying it's in terms of, well
 with respect to the ground or..?
 Ann Yeh, if you can take zero h as the floor that one's already got more
 height to start off with, haven't you?
 22:41 Vicky Yeh, but the..
 Ann So any
 Vicky So you've got to measure the water then?
 Ann If they're both half full, that one's still got more pressure by virtue
 of where it is, yeh?
 Vicky Yeh
 Ann Yeh So let's for the sake of argument say it's two-thirds or
 something there's got to be a point where more water in here than
 you have in there but the rates of flow are the same and it's going
 to continue the same until that one's empty, and once that one's
 empty then this one will go down.
 22:56 Derek Have you got some rough paper so I can trace this out?
 Jason So v_2 is that.
 Derek So that means that that can't be right, can it? That's just a decay
 which means that it starts at a maximum.

At last we agree to fill it up and see.

Vicky I'm just wondering whether it would be easier to fill it up and have
 a look.
 23:24 Ann Well, that's what I said.
 Can we run this thing and see what it does?
 Derek Yeh, why not
 Ann Cause I predict that it's going to do.. It's going to get to some
 point, stop, remain level until that one's empty and then start
 draining into the bottom.
 23:38 Anthony (Looking at lip) Actually this isn't the same as what we've got
 because..
 Derek So it's not at a maximum.. It's going to have a level maximum
 rather than a..
 Ann And it's going to remain at that point for.. until that one's empty.
 It's going to come into here and go out of there at different rates
 until you get to a point when the flow in is equal to the flow out,
 and it will remain at that point until that one's empty, and then it
 will go on draining into the bottom.
 24:01 Anthony The hole's here.
 Ann Can we see if that's what it does?

Effect of a Self-Management Program on the Performance of a Complex Task

John M. Johnston, Robert M. Gable, and Robert M. Gable

Department of Psychology, University of North Carolina at Greensboro

Abstract: A self-management program was developed to improve the performance of a complex task.

Keywords: self-management, complex task, performance

The purpose of this study was to evaluate the effectiveness of a self-management program designed to improve the performance of a complex task.

The study was conducted in a laboratory setting and involved 20 participants who were randomly assigned to either a self-management group or a control group.

The self-management program consisted of a series of self-monitoring and self-reinforcement procedures that were designed to help participants identify and correct their own performance errors.

Results of the study showed that the self-management group performed significantly better than the control group on the complex task.

These findings suggest that a self-management program can be an effective way to improve the performance of a complex task.

The present study was supported by a grant from the National Institute of Mental Health.

Correspondence should be addressed to John M. Johnston, Department of Psychology, University of North Carolina at Greensboro, Greensboro, NC 27402.

© 1998 by John Wiley & Sons, Inc. All rights reserved. This article is a U.S. Government work and, as such, is in the public domain in the United States of America.

0893-3200/98/2501-0001\$05.00

DOI: 10.1002/9781118530461.ch01

Published online 10 January 2008 in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/9781118530461.ch01

Jason I'll get the water.
 Vicky No but if you fill it up anyway it isn't going to matter. It's not going to affect it that much. It'll only affect it when it's really low.

Derek expounds the fallacious argument.

24:20 Derek Jason suggested this. You know I said that $dv_1/dt = -kv_1$, yes? $dv_2/dt = kv_1 - kv_2$, yes? Now Jason says that at - um - the time that's a maximum, those two are the same. So if we equal those we get $-kv_1 = kv_1 - kv_2$, and otherwise, $v_2 = 2v_1$. Now this is only for one t so it won't be for any other point but what you're going to find is when two thirds of that are in there, that's when there's twice the amount in there as in there, that's when the maximum will be.

25:10 Ann Well, as for two thirds, I couldn't have predicted what it would be. I just guessed the two thirds like for argument. (laughter)

Derek But I believe it's going to.. it's going to be a smooth curve.

25:29 Jason We want a volume in here that's.. shall we do something decent in here like a three?

Derek Yes you do but..

Jason Shall I drain it down to three then?

Anthony Well, the tap's at one so really want about four, don't we?

Ann Do you want to get this (middle tank) so it's level.. So it's at the point where it's ready to flow out so we've actually physically got what we've got in the drawing.

Derek Yeh, fill it up so it's.. This tap is open is it all the time? (middle tank)

Anthony Dunno- don't know where open is.

Derek Just make sure that this one's in first (tucks tube into bottom tank)

Ann As soon as it starts to..

Derek No, that's closed..

Ann Unscrew the top.. Take the lid off. (applause as the water starts to flow)

Vicky Here it comes out

Derek As soon as that starts coming out close that tap.

Jason Right, are we going to start, are we?

Ann Yeh.

Having filled the apparatus, we discuss some more.

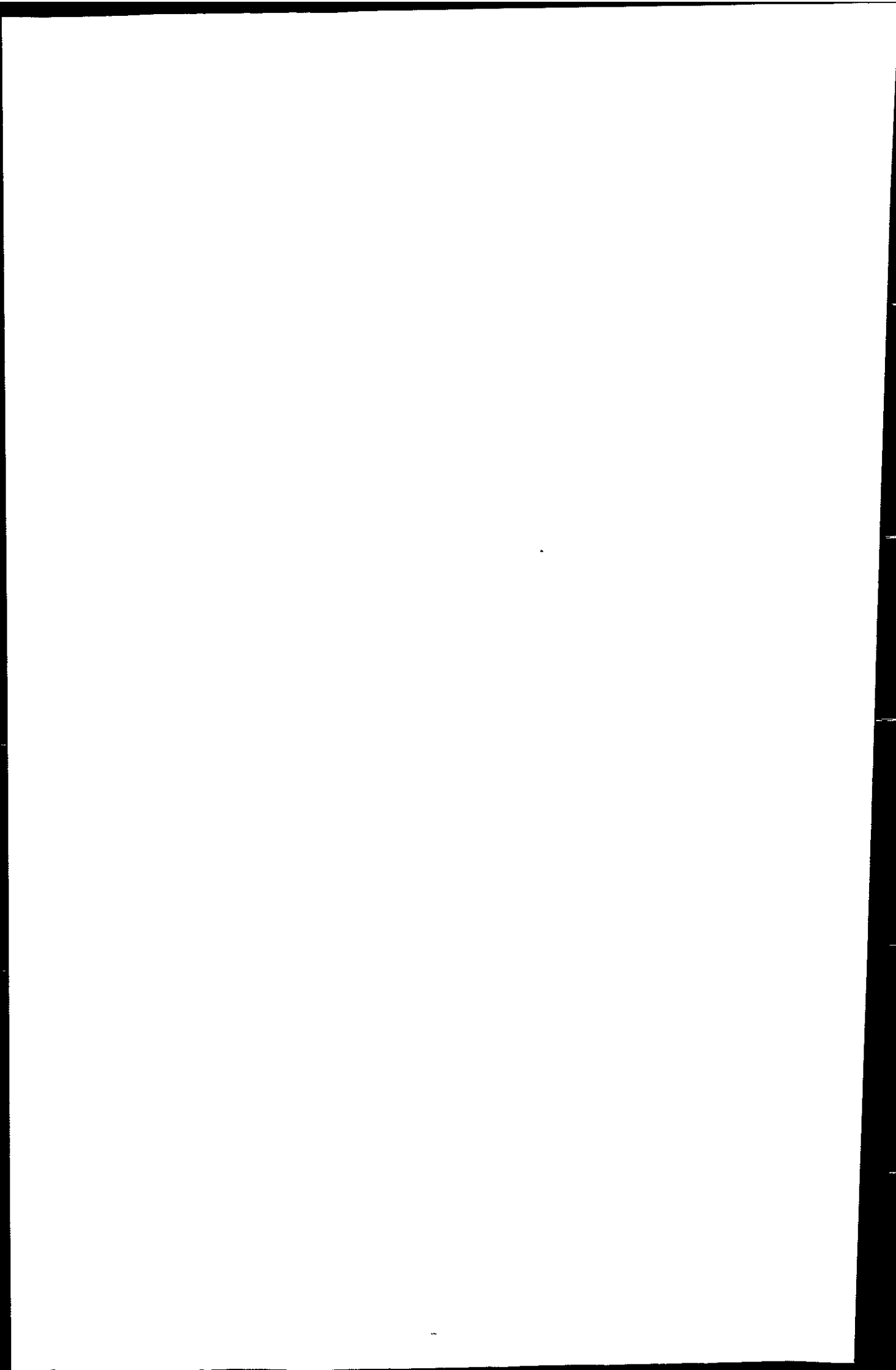
26:55 Jason Right, so what are we going to say is happening?
 Derek So that's on one then, (middle tank) so we say that on one is equal to zero.
 Jason Right, so one is equal to zero
 Ann This is gradually going to fill up but the rate going out is going to be less than the rate going in. Then it will stay constant until that one's empty, and then it will start draining into the bottom one.

Mathematical versus descriptive solutions, while the apparatus is running.

27:17 Derek The problem I've got with that, Ann, is it's got to be modelled by some differential equation and so we've got to have.. It's got to be some functions..
 Ann I mean I might be wrong. I'm just saying..
 Jason We'll just do like.. a Heaviside.. (laughter)

27:33 Ann Yeh, well.. just see what happens.
 Jason opens tap.
 Ann Unless at that particular point the rate of flow out becomes greater than the rate of flow in.

27:55 Anthony Do you want to open that (middle tap) cause it's
 Jason It's going to be maximum where these two volumes are the same..
 Ann But is it going to remain at that or just going to be an instantaneous maximum?
 Jason It's going to be when the two are the same then that's going to go



up and that's going to go down, and then that's going to (scratches up and down on middle tank). As soon as they're the same, that one's on two point two.

Ann Are they going to be the same for an instant.. Aren't the..

So what does happen?

Jason Yeh - only for an instant. That one's at one point nine - that one's at one point eight. Right, They're both the same now. They won't stop like because..

Ann That one's slightly higher anyway.

28:30 Jason Now that one's lower and that one's still about the same.

Ann Is it - no - it's still increasing?

Jason It's not increasing.

Ann Yes it is.

Jason You look at the scale. It's not increasing.

Vicky I can't see the scale.

Ann So it is staying the same.

28:42 Jason It's staying the same. No I think it's coming down now.

Ann It's coming down again now.

Derek If we were to run this a couple of times, trying some characteristics, we're going to have to start with the same initial conditions.

Ann Yeh

Derek So what was it like? You started off, Jason. (laughter)

Ann Yeh

28:58 Jason Oooh, about three point two, three point three..

Derek Threeish, right then, bout that. Why don't you fill it up to four to start with then and keep it?

Jason Well I did fill it up to four but we drained half of it.

Derek Well you should have filled it up again, shouldn't you?

Jason Well, I know that now.

And how shall we interpret it?

Ann I think we're agreed that it's at its maximum at the point when the flow in equals the flow out.

Jason It's at its maximum where they're both the same.

Derek Cause what's wrong with this in that case..

Ann What's the same?

Jason The volumes.

Ann The volumes are the same?

29:28 Jason But then that would be the same as the flow rate because we've done them both.

Derek If the constants are different for each bottle..

Jason Yeh, We've assumed both constants. If that constant - like- all it depends on are that hole.

Anthony Is that (top) running now?

Jason Yeh, they're both running now?

30:01 Derek The constants.. The two taps.

While we refill the apparatus and put the pink dye in. Getting the same initial conditions is important.

Jason Right. I'll pour this up to the top bottle.

30:27 Anthony Right, so that's about one litre. Don't drop it.

Jason I don't reckon this is a good idea, actually.

Ann Let's take it over the sink.

Derek Cowards. (laughter)

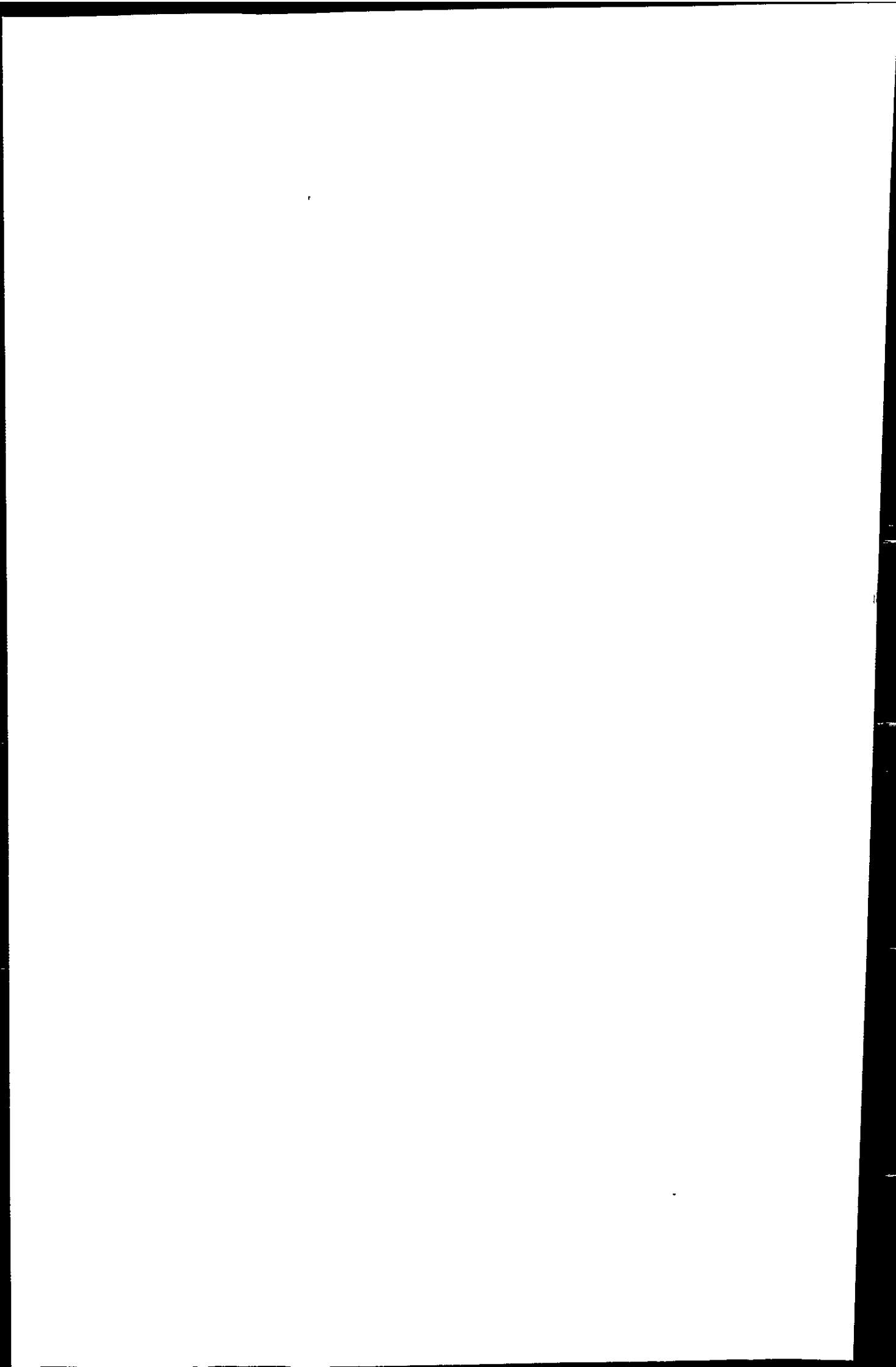
Ann Shall we put the pink dye in?

31:12 Jason Will it help?

Ann Yes, please.

Derek So, Jason, will you dye the water?

Jason So what did you want it on, Derek?



Derek Four.
 Jason adds more water.

31:50 Ann Although it's slightly higher, that's not a factor.
 Vicky Oh no, not with respect with the bottom one.
 Ann But..
 Vicky That is a maximum when that is the same as that you see.
 Ann So when the height is equal to that height it's going to be...

32:02 Jason If you want the pink dye in it's going to be more. (laughter)
 Ann Well put it in and drain it off.
 Derek It's a bit low anyway.
 Vicky If you put it in when you're at the sink then..
 Ann I don't know how much of that you'll need. You might only need a little bit. you don't know how pink it is.

Derek Is it poisonous, Jason?
 Jason Drink it and I'll find out tomorrow. (laughter)
 Have a curry tonight
 Ann I don't think you'll need to with that.
 Derek Drink it and then check it.
 Vicky Oh look, it's going pink - - ish.

32:50 Jason It's not very good, is it?
 Derek Oh, chuck the lot in
 Jason So what do we think is going to happen in this tank?
 Vicky I think the pink's going to get diluted.
 Jason Shall we start?
 Vicky What did you do that again for?
 Jason Yeh, well, I know we're going to do it again.
 Vicky Shall we test our theory while we've actually got a theory?
 Jason But do we know what our theory is?

33:30 Vicky When we've got a theory.
 Derek Who's got a theory - Ann?
 Ann We know it's a maximum when they're the same.
 Vicky Yeh, the-top and the middle on
 Derek And that makes sense, doesn't it?
 Ann Yeh.

33:49 Vicky It's dripping. (Looks at top tap) Well, we've got one drip.

Is there a problem with the model?

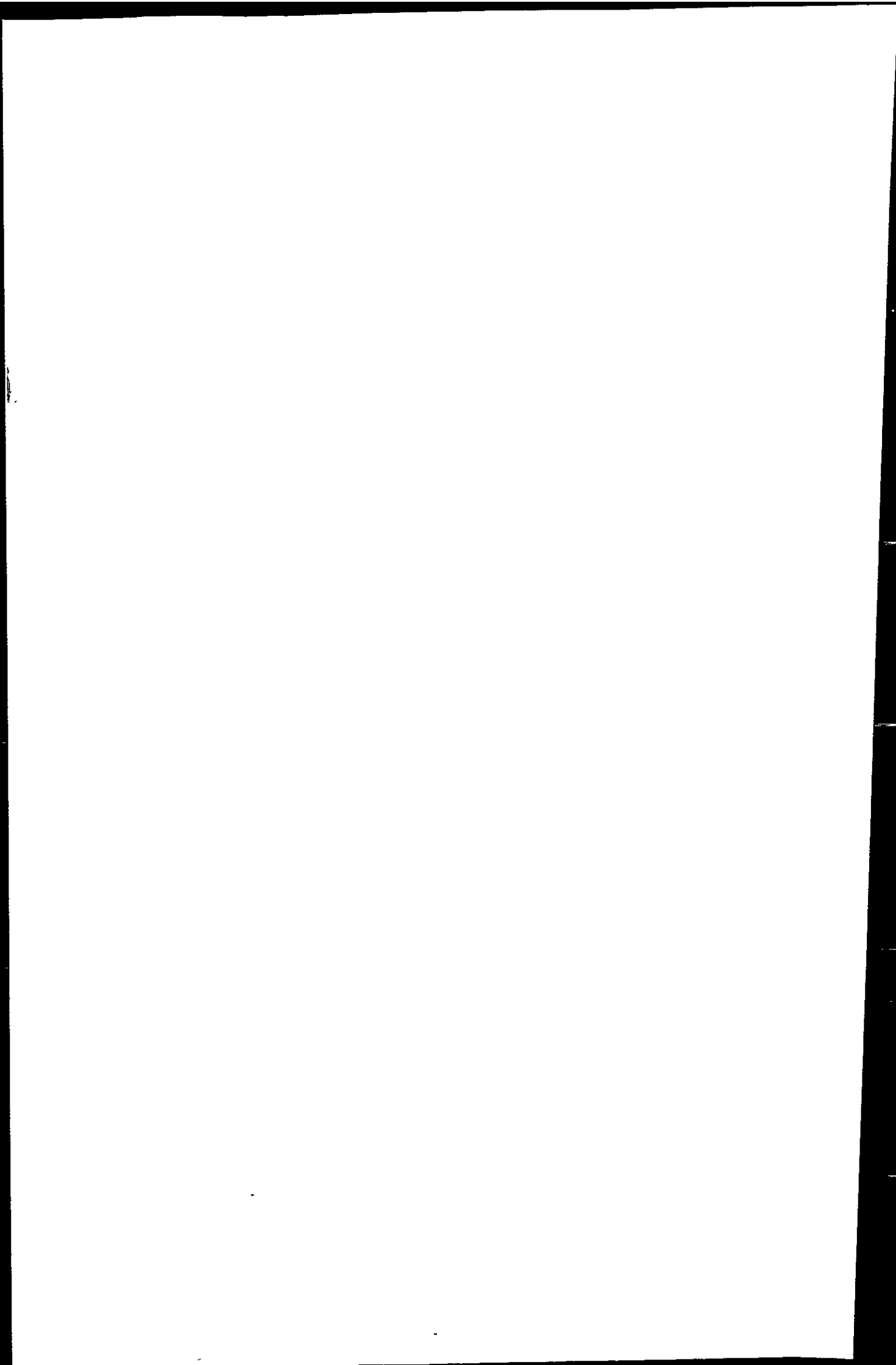
34:10 Anthony I don't like these constants.
 Jason No? I reckon they're great but..
 Anthony It's different constants.
 Ann Why is it different constants?
 Derek if k is proportional to g..
 Anthony It's going to be density over..
 Jason (Smiling) Cause the further you get from the centre of the earth, the less gravity is, and that one's (indicates difference in height of top two bottles, and laughs).
 Ann Yeh, that's what I was saying but I think you've got to take that as being negligible.

34:28 Tape stops

We list our assumptions.

34:40 Anthony One: hole in the bottom.
 Jason Viscous, incompressible and irrotational. (laughter)
 Ann We've got a Newtonian flow, have we?
 Jason Yeh, I'm sure it doesn't matter.

34:56 Ann Of course it matters. We've got to have those things written down.
 Anthony This is where we've got the Navier-Stokes equations.
 Derek Irrotational is it?
 Anthony Bernoulli's equation
 Derek Newtonian flow
 Jason Assume negligible viscosity.



And revise the original basis of our analysis.

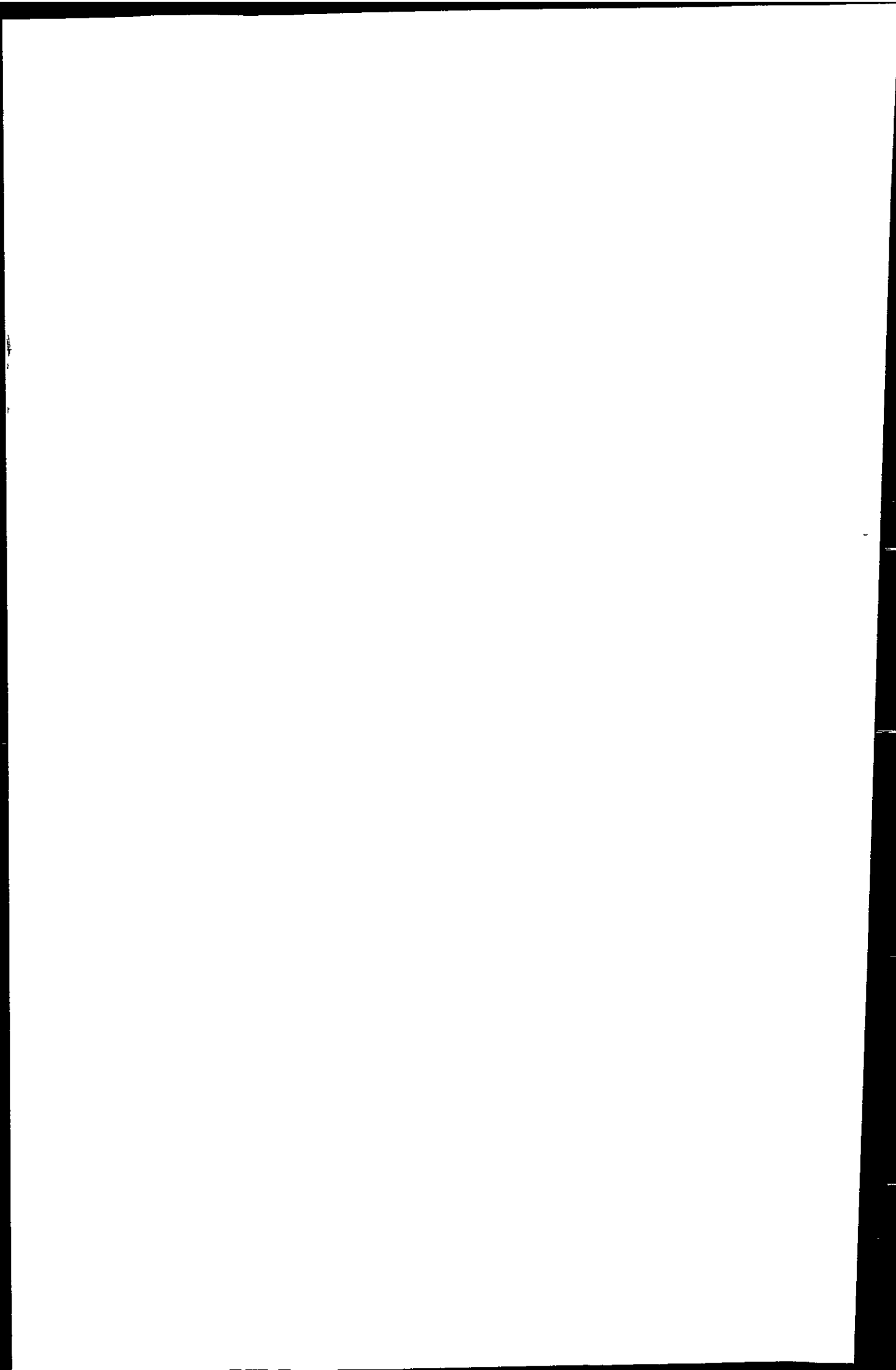
- 35:18 Ann We've come to the conclusion that it's a maximum at the point where the flow in equals the flow out:
Jason Mmm. No
Ann No? Well, it's maximum when the volume of water in here is equal to the volume of water in there.
Derek Yep
Jason Yep
Derek As soon as there's more in there than there is in there then the flow out will be more than the flow in.
Jason So it's got to have the maximum.. the maximum has got to be when the two volumes..
Ann When the two volumes are equal.
Jason (writing) $v_1=v_2$.
36:08 Derek Yep. So what we want to do then is to watch at that one closely and see when you reckon it starts coming down and then if he calls out see what the two volumes are then and if they're the same we're satisfied it makes sense.

We discuss the apparatus again, and make a tentative prediction of the maximum volume in the middle tank.

- Ann I think they were but we can run it again and see. Does that sound sensible to you, Anthony? You're looking so..
Anthony It sounds very sensible.
Derek We want a quantum fluid. (laughter)
Jason So what are we looking for then?
36:30 Ann The point at which it's steady.
Derek The second bottle.
Jason Right. The top one's at four point one.
Anthony I don't know if it's useful we write down four point one.
Derek Hang on - is that tap open then?
Anthony It isn't right on one, is it? (the middle tank)
Jason I mean those taps could easily affect the flow rates. We should change these around in a minute.
Anthony That's on point seven.
Jason No, that's not on one.
Derek Those bottles look pretty much the same to me.
Jason That one's on nought point seven. That one's on four point one. So when they're the same what are they both going to be on?
Midway between the two.
Derek Two eight won't they?
37:25 Jason They should be on two point four, right? (turns on tap)
Derek So Jason, watch the top bottle and see when you think is the maximum and put your finger on that point. If you could call out as well.
Ann We could turn it off when it gets to that.
Derek No, you can't check it's right then can you, cause it will be at a level.. Cheat!
37:40 Ann I thought you could turn both taps off.
Derek What I suppose it's best to do is to follow it up there and when it gets below that to register where it is.

We realise the prediction was based on a false assumption.

- Jason No, it's not going to get to two point four anyway.
Ann What did you think?
Vicky What numbers did you have?
Jason Two point four's rubbish. Cause it's going out, isn't it.
Derek Course it is.
Jason Complete rubbish, that.
38:11 Derek So keep your eye on that. When you think it drops, mark that and shout out and watch that and..



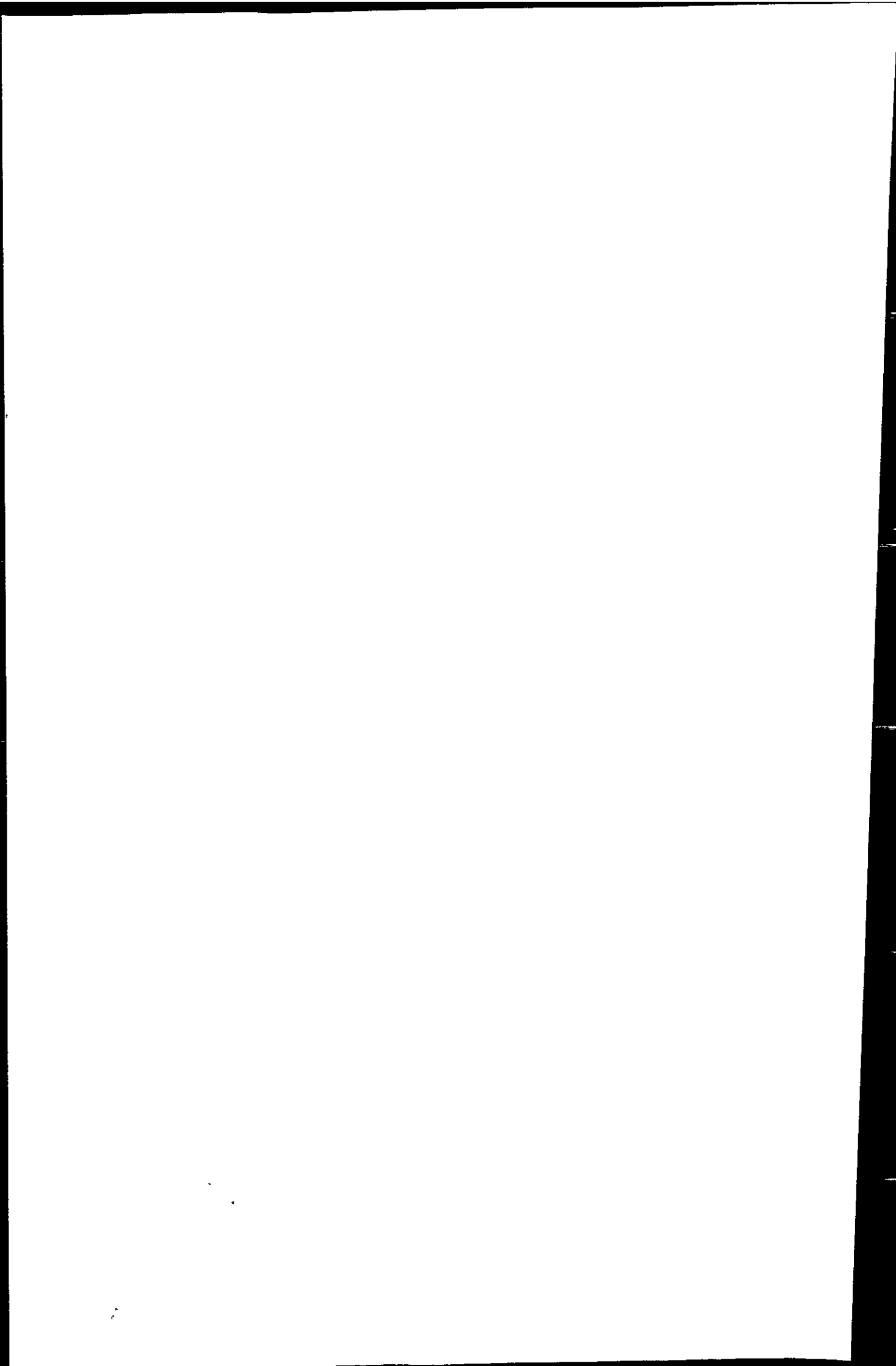
Jason What, and see when that's a maximum?
 Derek And keep an eye on that.
 Jason I think I might call it out actually.
 Anthony I think that's about nine
 Jason It's falling now
 Derek No

And that something else is wrong.

38:42 Vicky I'm sure it's not gone as high as it was before.
 Jason Yes, I reckon it was there.
 Vicky Oh Crunchy!
 Ann It has, hasn't it?
 Vicky It hasn't hit
 Ann It went higher than that last time.
 Jason That's the peak diameter: it's less viscous. (laughter)
 Derek Did you have the same level of water in the top. To start..
 39:03 Jason Yeh, about.
 Derek No, we had a lot more, we had a lot more.
 Derek That's interesting because you'd expect if there was more in that
 you'd expect that level to be higher at its maximum.
 Jason Yeh
 Derek Interesting, isn't it.
 Jason I mean it didn't get half as high that time.
 Vicky Yeh
 Jason It got to one point seven last time.
 Vicky Why was that?
 Jason And only got to about one point two
 39:31 Ann They wouldn't be.. The taps were both open the same..
 Jason Yeh, but they won't be different.
 Derek You have to assume that makes no difference otherwise you're
 never going to work out..
 Ann Yeh, but we need some consistency.
 Vicky Why did it go higher last time?
 Ann In our experiments?
 Derek So we have to assume that much (twisting gesture as if turning tap)
 doesn't make a lot of difference.
 Ann Why did it reach a higher level the last time we did it?
 Jason Cause no two things are ever exactly the same. (laughter)
 Ann Significantly higher level last time.
 Derek Heisenberg's principle. Cause when we were watching it
 something different happened. (laughter)
 40:11 Ann Yeh, oh
 Derek Forget the experiments.
 Jason Try something different.
 Vicky But did we conclude from that one that it was when that one as the
 same height as that one. No, we weren't watching, were we?
 (laughter).

So what was wrong with the model? Was it the mathematical assumption?

41:21 Jason Well, I don't see there's anything wrong with that.
 Derek I think we should run this again.
 Vicky If anything it's just..
 Jason Whatever we're saying is it doesn't get there in a finite time which
 is true, eh? because that will never get empty in a finite time.
 41:32 Derek (Points to volume below tap in top tank) That's a Heaviside
 function, isn't it though?
 Vicky That will never get empty with the thing at that height anyway, so..
 Ann But we're modelling it as though it would.
 Vicky I'm just arguing their exponential thingummy.
 Jason I mean what we want is a linear function really, or a quadratic.
 41:57 Vicky The other thing's if you have two differently shaped things. That's
 why it's all working:



Ann Yeh, I know but
Vicky Cause we're saying if we just had two rectangular, it'd be easier to measure.

42:40 Jason can we use that model...
Jason Now we've got to think about it.
Derek What happened to the piece of paper I started writing on and other people have taken and sort of worked on? (Jason passes paper)

Derek Thank you.
Jason It's got to be something more than $dv/dt = -kv_1$.
Derek Yeh, I mean you can solve it quite simply and get an answer.
Mmm
Jason Well it's probably an exponential because it's got to be something decreasing, isn't it?

43:10 Derek Yeh.
Derek Yeh, but I mean that doesn't have to be as such.
Jason You see there's something wrong in that, I mean that's right.
Derek You reckon v is that simple.

43:50 Jason I reckon the top one is that simple, but when you say that, that's obviously not true, because it's got to go up and then sort of down.
Derek If you start with these, if you'll pardon me questioning your sort of.. Why should we assume that k in these two is both the same?
Anthony Cause they are the same.
Derek Same bottle, same taps, yes?
Anthony That's a good point actually because that's bigger than that.
Derek This is the third one. Why are we bothering with that?

45:00 Ann The things that affect the rate of flow out of that are exactly the same as the things that affect the rate of flow out of that.
Vicky Yeh, unless..

45:21 Jason They're solved all right, aren't they, Derek?
Derek Yeh.
Jason And the rate of change of the volume of this one (middle tank) is what's going out, which is what's coming in.. what's coming in is dv_1 , so I can't see how it can be any different.

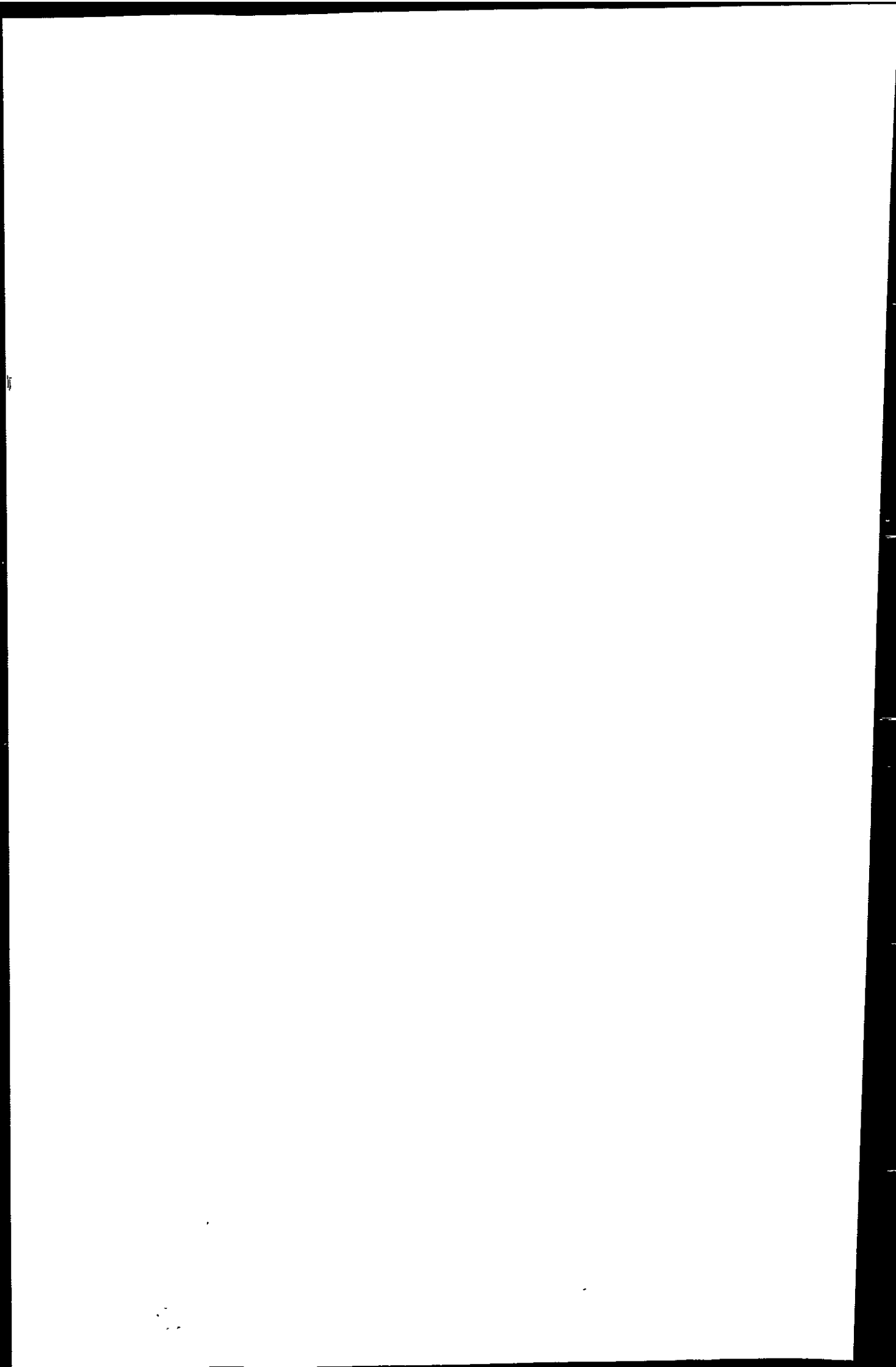
Derek Yeh
Jason But the.. But.. minus k_2v_2 .
45:50 Ann But the maximum height is the instant the rate of change equals the others.
Vicky That's the middle one.
Ann Mmm.

Was it the assumption about the conditions for a maximum?

46:24 Ann The rate of change of that one's the same as the rate of change of that one.
Vicky Yeh, the rate of change. But you've got to remember that one's also coming out as well.
Ann Yeh.
Vicky So you've got..
Ann So that's dv_1 and that's dv_2 .
Vicky No if you - no, because you've got the volume, that's dv_1 , as that's changing, and that's dv_2 .
Ann But that doesn't really matter, because what you're interested in is what's in here, aren't you.
Vicky But if you say dv_2 equals something, you've got to say something's coming out, and you've got to take into account that something.

47:26 Vicky But the problem is, we don't really.. You should be able to rewrite that in terms of v_2 say, because you shouldn't need the volume in the third one. You shouldn't bother to look at that, should you.- I don't think you do - all you need to know is what it's losing there. So in effect we're looking at the change.

47:54 Jason Do you reckon they're different constants or not?
Ann I can't see how they can be.



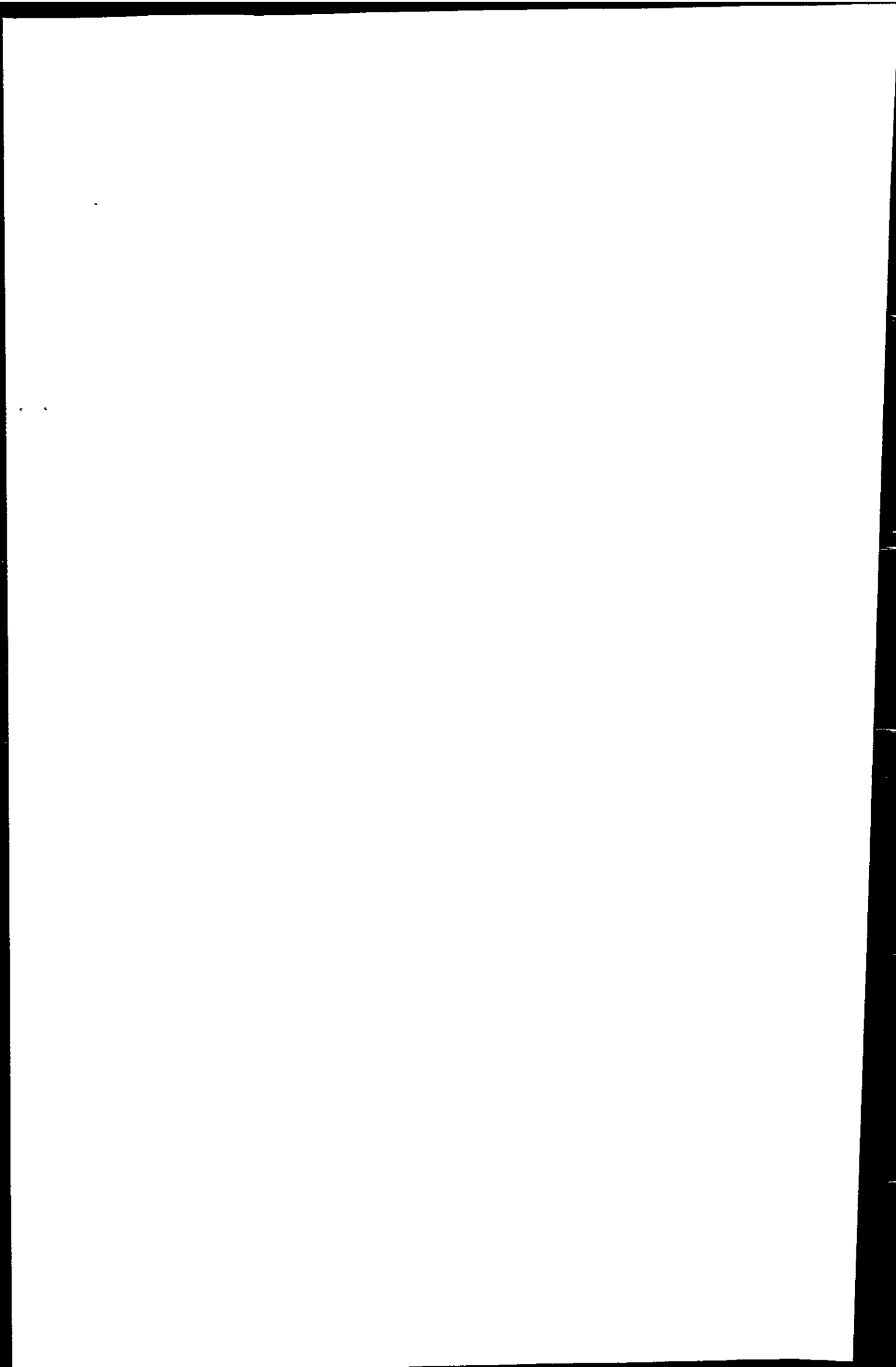
48:00 Jason It doesn't matter then.
 Ann Because the only things that are affecting them
 Derek The only thing that might make a difference is gravity and I think it's negligible.
 Ann It's not sufficiently different height to make any difference.
 Derek The two bottles are the same. I think we should assume the two taps are the same as we would..
 Ann Yes we're assuming all those things are constant.
 Derek So the rate of outflow is only dependent on the volume in there.
 Ann Yeh. Mmm

Jason has found a solution but does not find it convincing.

48:42 Jason (showing paper) I mean it's that but that is still going to decrease at all times.
 Vicky Let's have a look. You've got an exponential again, haven't you?
 Jason Unless k is a .. No. Unless k changes between them two. But k can't affect it that much.
 49:34 Derek Jason, have you done it?
 Jason Yes, I know.. I've done it right though.
 Derek You agree with me, yeh?
 Jason But still unless k is at some time negative.
 Derek But no, k is constant, isn't it? But this is still.. You've got two decaying exponentials. That's no good.
 49:55 Jason The only thing is, if k_1 is different.
 Derek If B is negative or something, then you'll get something like that.
 Jason Mmm, yeh.
 50:12 Derek Which is what you want, I think. I'm expecting that somehow.
 Jason e^{-B} is always positive though, isn't it? So if you've got a plus and a minus it's always going to be decaying, isn't it? Do you agree?
 Derek Yeh but what if B or A , one of the constants is negative?
 Jason Yeh, but that's going to be positive, the e^{-2k} , that's going to be positive so whatever there's plus or minus it's going to get smaller, isn't it?

More discussion about constants

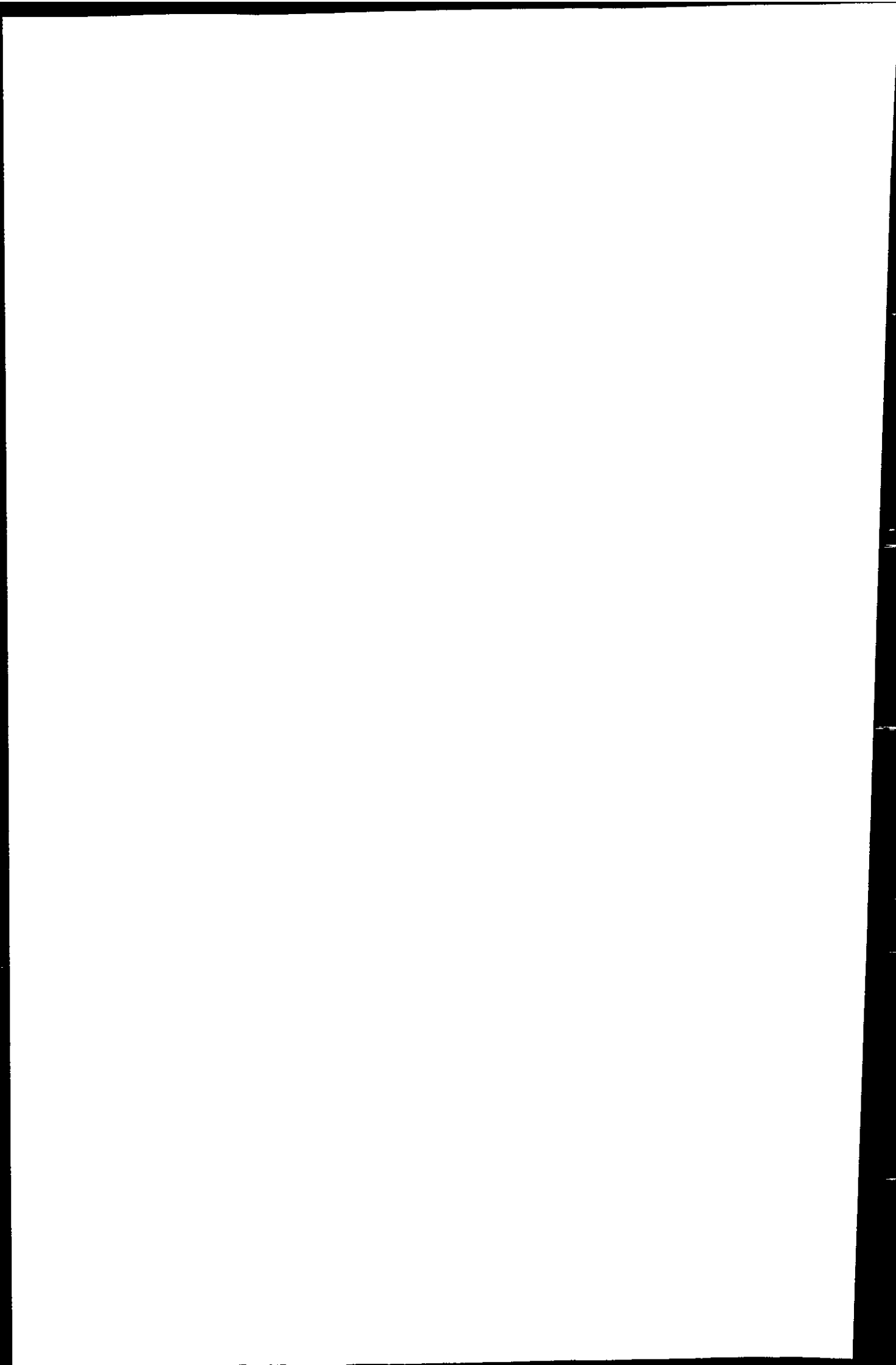
50:58 Derek What are you suggesting? Is it anything in particular?
 Ann You see what you're saying your constant is..
 Jason k
 Ann Yeh, but it's
 Derek Don't push it, whatever you do.
 Ann Yeh, but what is it? A box of eggs? What?
 Anthony It's a constant, like in maths.
 Jason It's a function of mark, time passed, protons, neutrons, electrons, viscosity, incompressibility, inviscid,
 Ann Gravity.. (laughter)
 51:30 Jason I thought you two (Ann and Vicky) did a maths modelling project anyway.
 Ann Yeh, but..
 Jason Didn't you do this one?
 52:12 Derek Yeh because this is the total volume which is v_1 at t equals nought.
 The initial volume.
 Jason The initial volume v_1 at nought.
 Derek So that's what A is therefore. v_1 at nought
 Jason Yeh
 Anthony Yeh
 Vicky So that's your model.
 Derek v_2 at t equals nought is going to be zero. Which is therefore going to be.. Therefore this is going to be the initial volume and B plus this is zero which gives you the initial point.
 Ann In which one?
 Jason What? In what way?
 52:50 Derek v_2 of t is v_2 minus kt . So A equals minus $2kt$. If B is negative and



equal in size to A then..
 Ann How can B be negative?
 Derek It is
 Jason Cause v2 at nought is nought and therefore you've got A plus B is nought, and A is v1 at nought so B is v1 at minus nought.
 53:20 Vicky Well no because it's minus 2kt
 Derek e^{-2kt} is smaller in mod than e^{-kt} , isn't it?
 Jason e to the.. Yes it is.
 Derek Which means therefore that v2 will go negative.
 Jason Mmm, well yeh.. perhaps we got all our things the wrong way.
 Derek Which is absolute rubbish.
 Jason Unless you do the.. Because you've got..
 53:46 Vicky Yeh, it sounds surprising, doesn't it?

Another interesting suggestion is ignored.

54:30 Jason Make it a quadratic function.
 Vicky That's.. I'm not convinced..
 Derek No, this has got to be wrong. I'm just writing down why this is wrong.
 Vicky Why
 Jason Cause Derek did it.
 55:04 Vicky Well the other thing is that we're saying at some point v2 equals v1.. v2 equals nought. Just by looking at this (apparatus) At the maximum.. Which doesn't make sense.
 Derek What are you saying?
 Vicky Well from what you've got here, you're saying that at some point v2 is v1 when v1 is at the maximum..
 Derek Yeh, so..
 Vicky So for v2 to equal v1, you've got v1 equals this (points to paper).
 Jason No, well I didn't think that was true anyway at the same time.
 Vicky That equals that
 Derek Yeh, I see what you're saying
 Vicky You're going to have B equals nought, which is a different solution from B equals..
 Derek Yeh, you know what I'm saying is something's wrong here.
 Jason Yeh, there must be something wrong in your equations.
 Derek What if the k's were different then?
 Jason Well if the k's were different you'd get v2 is $Be^{-(k_1+k_2)t} + Ce^{-k_2}$
 56:17 Jason Well, yes mate. So if k1 and k2 were opposite sign, which they're not..
 Derek No it can't be. It's a simple physical relationship. (laughter)
 Ann You didn't say that
 Ann Pass me that.
 Derek Get your own for God's sake!
 Vicky No - you've scribbled on this one.
 Ann. Don't look at mine! Your writing is very tidy today, Derek.
 Vicky It's very large today, Derek.
 Ann What are you doing?
 Jason It's not his paper. He writes small to save money. (laughter)
 57:07 Vicky They've done something wrong here - or it's wrong, or it's total rubbish.
 Jason The equation must be wrong.
 Derek Can I have a look at those equations.
 Jason I think they're right.
 57:22 Jason The solution is right as well.
 Derek Right, well, I don't know. I'm just writing it down.
 57:50 WMM Roger's the man to ask if you're floundering.
 Jason Shall we ask for help?
 Anthony No way. No, no.
 Jason Ask his opinion
 Anthony No, we don't want help. We've got two hours to do it in.
 Jason Two hours!



58:20 Vicky Yeh, it's seven o'clock now.
 Derek This is just me gobsmacked to see how we've all approached it differently. If we'd had to think we'd have all gone to the library for similar questions.
 Anthony I think we'd have photocopied them. (laughter)

Yet another interesting suggestion not followed up.

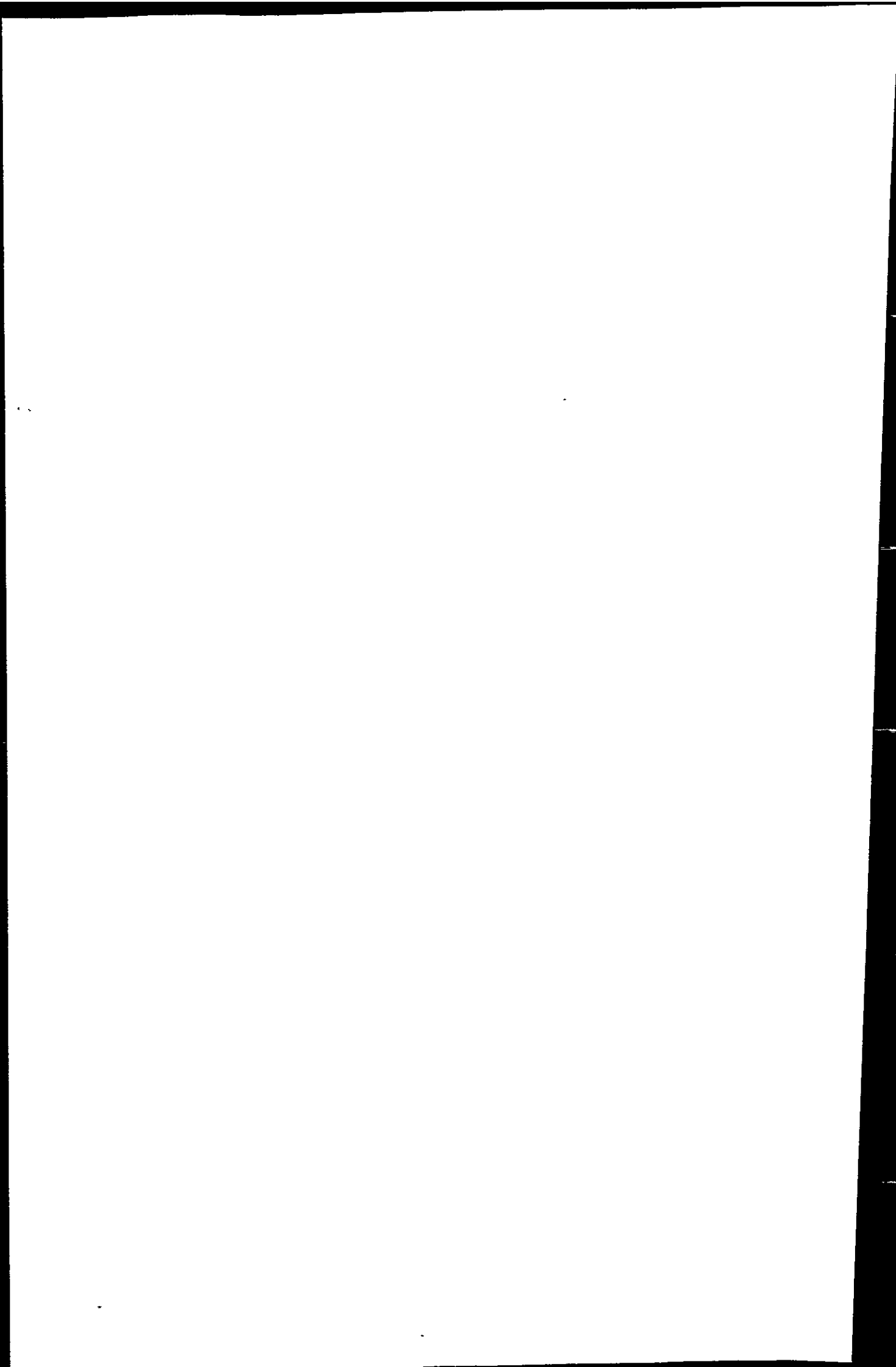
59:20 Jason Why don't we look at pressure?
 Anthony No, you don't need to worry about pressure?
 Jason Cause then you've got pressure is what pgh?
 Derek Oh, that's far to technical.
 Jason And pressure is force over area..
 Derek Oh dear, area..
 Jason And force is mass times acceleration.
 Derek Force is mgh, yeh? Area is the same and h. is what changes and so therefore you can..
 Anthony But then you can measure that directly.
 Jason. Acceleration.. acceleration.. acceleration is what? Acceleration.. acceleration. and then you get a second degree equation, and you get two integrations..
 Ann mgh is potential energy.
 Jason The boundary conditions are exciting..

Time for a break, but we are unwilling to stop. We discuss textbooks and TV.

1:00:22 Ann I'll go and find you a book, Anthony.
 RKP At this point you should have a cup of coffee.
 Derek Hope there's some nice biscuits as well.
 RKP Are you happy to take a break for a few minutes and have a cup of coffee? You've worked hard and I think we need a bit of time to let things mull over in your minds.
 1:00:45 Ann (to Anthony) Have a book.
 Derek Cheat
 Ann It's a differential equations book. (laughter)
 Derek How to solve differential equations.
 Vicky Page 63.. Page 63. It's got some interesting things on it.
 Jason Bostock and Charles. It's a useless book that is
 Vicky Oh no, that's a good book that is.
 Anthony A useless twelve quid's worth
 Vicky That's what got me here.
 Ann I know a lot of people say the only thing that's good for is propping up your bedroom table.
 Jason At school they refused to use it.
 Ann Really?
 Derek It's brilliant
 Anthony We used it.
 Derek You know, Jason, (unintelligible) Did you ever see that on television? It's obscene now.
 Jason That's a personal opinion. It's brilliant.

We return to the matter in hand. Derek checks his maths.

1:01:46 Derek Show us your maths again.
 Jason Which one? The separate ones?
 Derek Yep, integral that, yeh
 Jason Yep, I did two k's on that. kv1dt. No, I did it.. Yeh, like you've done. No, hang on, what have you got there? Yeh, but v1, you know what v1 is. It's Be^{-kt}
 Derek Yeh
 Jason And so you get your.. the Be^{-kt}, yeh.
 1:02:22 Derek We've got a constant A though. We can work it out in terms of A because we know what it is.
 Jason It's not a B there, it's.. Yeh, and..



Derek And you've got A there and you take it across.. And you get v2dt.
 Jason What, is that with a separate k1 and k2 or not?
 Derek Um.. It's not actually.
 Ann I don't see how they can be different k.
 Jason All right then, it makes no difference.
 1:02:50 Ann Is that what you've just done, worked it out?
 Jason I mean if they're the same..
 Derek That isn't the point.
 Jason If they're the same you still get two exponential functions, you just get smaller and smaller.
 Derek I can't see any deductive step in there that's wrong, but this answer clearly doesn't work.

Jason suggests that if the maths are correct and still make a wrong prediction, then the assumptions must be wrong. Derek suggests and discards an alternative.

1:03:06 Jason So our assumptions are wrong to begin with?
 Derek So I wanted to see if I made a mistake in the maths. It seems that I was all right.
 Ann We're saying we aren't changing anything but the volume flowing in and the volume flowing out.
 Derek Yes, obviously.
 Jason But apart from that..
 Derek Some function of that and if it's just kv, I mean if it's kv² or something.
 Ann Yeh
 Derek That makes it more complicated.

An interruption about coffee follows. Then we return to the subject.

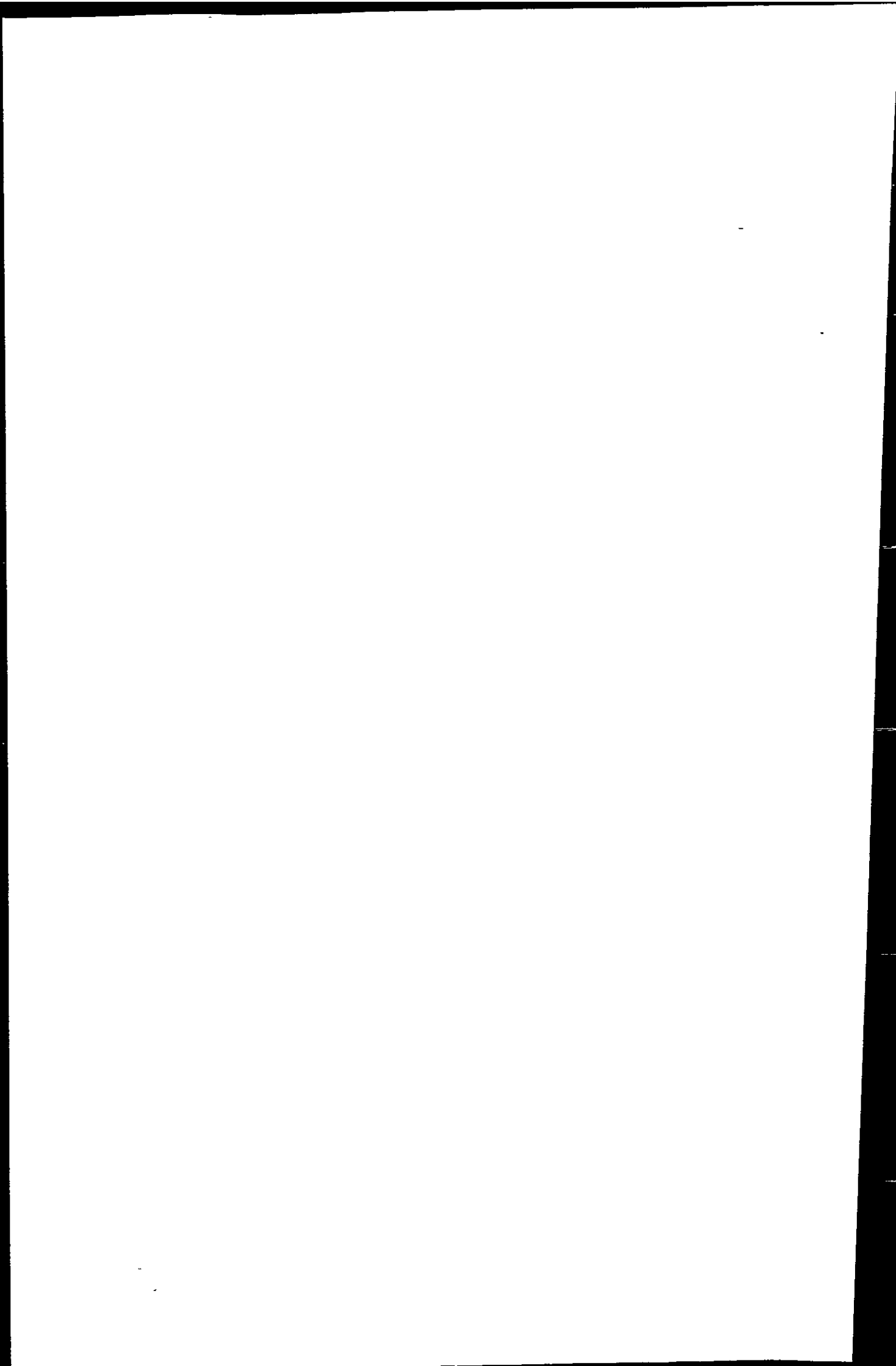
1:05:25 Derek Ann, can I have my piece of paper?
 Ann (returning paper) Yeh
 Derek I think I'll mark that with a big red pen. (draws ring on paper)
 Vicky What was that?
 Derek What do you make of that?
 Ann That is one of my favourite tricks.
 Derek It probably happened when I gave it to you. Just materialised when you handled it.
 Ann Yeh, I've just done that for you. (laughter)
 Vicky You've got nought equals b minus a?
 Anthony That means A equals B.
 Jason A equals B, which means?
 Vicky We've done something wrong somewhere.
 Ann How can a equal B?
 Jason Well, let's have a look at your Navier-Stokes equation. That's z, so that's like vz

Anthony has been writing on the folded paper. He has a solution but does not like it.

1:06:50 Anthony well, I've got a solution but I think it's wrong.
 Jason Oh, yeh?
 Ann Come on then, show it to us, Anthony.
 Anthony No, cause it's..
 Jason (reading) alpha e to the quarter t equals alpha over two..
 Derek That doesn't go negative, does it?
 Anthony No, of course it doesn't
 Ann No, that doesn't

Derek has a solution which looks promising.

1:07:25 Derek (reading from a different piece of paper) When t is nearly zero, that's about one, and that's one so that means it starts off at A, which is rubbish, cause that's zero, isn't it, and that decreases. That's zero, this bracket starts at zero and goes up to one, and



this starts at one and goes down to zero, so that product is going to be something at a max, so this looks quite good.

Ann Yes, that does, doesn't it.
RKP brings coffee.

1:07:43 Vicky No, yeh, that looks good.
Ann So what was your brainstorm there, Derek?
Vicky Something we said?
Ann Did it in a different format?
Derek No, I did it again and just sort of kept my signs consistent.
An Not introduce a negative when you didn't want one.
Vicky So if you diff it, and take it equal to zero, that should be where the maximum is, at that time.

Derek Yeh
Ann Go on then, try it.
Derek dv_2/dt equals minus kBe^{-kt} plus two kAe^{-2kt}
Vicky Hey, where have you got your B from?
Ann The B equals A anyway.
Anthony B is minus A
Derek B is the constant you get when you integrate the second..
Jason So it's just minus A, isn't it, so you might as well just put minus A.
Vicky Why don't you just put one.. Cause..
Derek So why don't I just write one? Okay.
Vicky Oh, I see what you're doing. You're working on the premise this is easier to differentiate than that. (There are two different forms of the expression for v on the paper.)

Derek I was just starting by differentiating that because I imagine I could have written that with one of those things with a bar across it but.. (laughter)
Shall I carry on or is it too exciting?

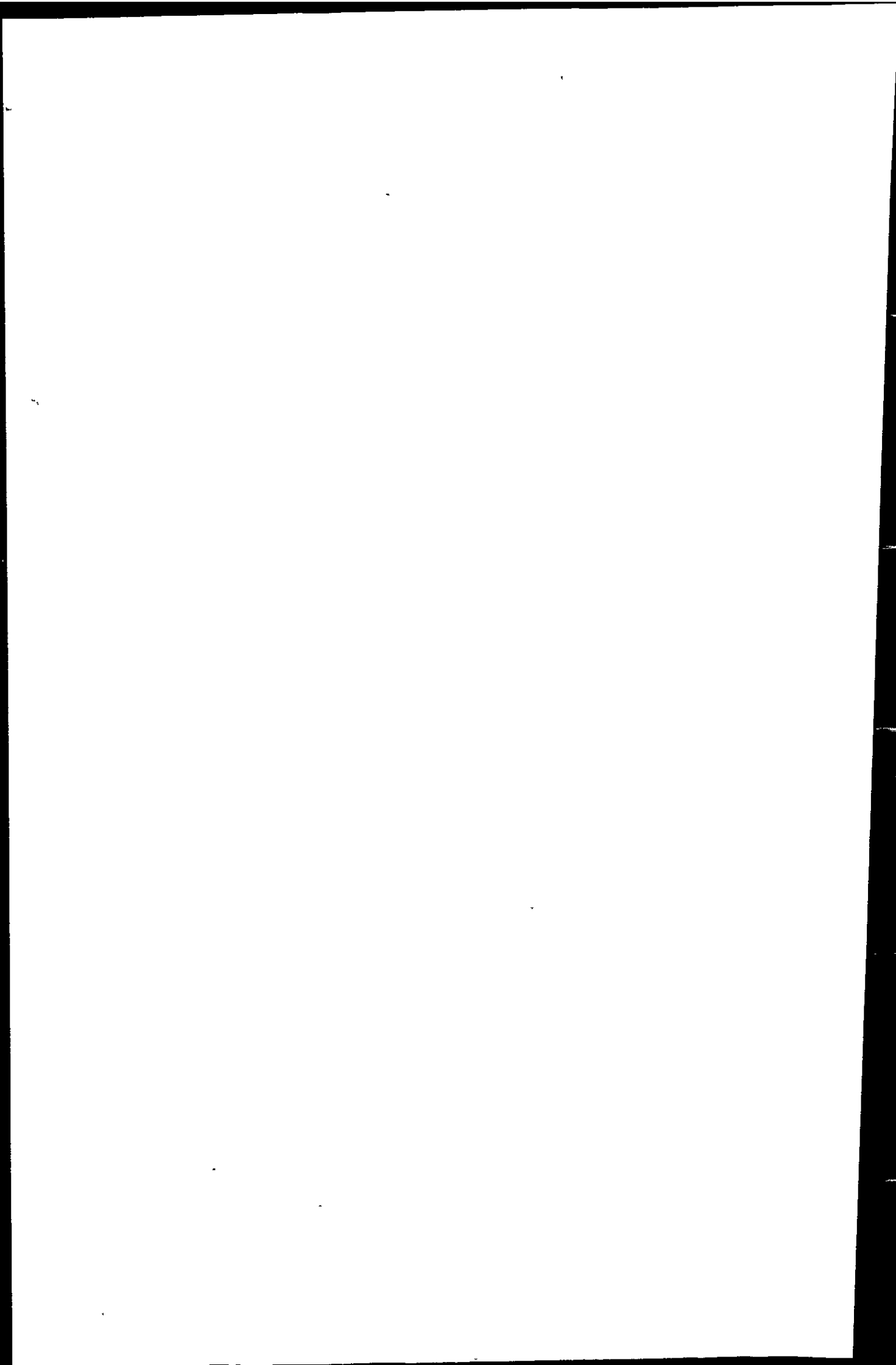
Vicky Differentiating

1:09:36 Break in film.

Derek explains the thinking.

1:09:40 RKP Okay, so what is your initial assumption about how your rate of flow.. how your water, your liquid, flows out?
Derek Well, we started working with the theory that the rate of flow is some constant times the volume in there which would lead us to, you know, work via some points that, as the water decreases so the flow out's going to decrease-so, so that in the second bottle the rate of flow out's going to be quite small because there isn't much water.
It's going to be coming through at a greater rate than it's going out, so that the level will rise.
Um, at some point there will be as much water in the second bottle as there is in the first, so that the rates of flow in and out will be the same, as there is then, is then less water in the first, that rate of flow will decrease and this will be higher as there increases the level of water, and of course that will then start dropping.
So we worked on a differential equation that dv_1/dt equals minus kv_1 , and it doesn't seem like that gives us a workable answer though the equations can be solved, so we don't think it like does the job

1:11:11 RKP Why do you think it doesn't do the job?
Derek Well, actually we worked out we got for the first part as I just mentioned dv_1/dt equals minus kv_1 . In the second, dv_2/dt equals kv_1 minus kv_2 . We assume the same constant in both, though that may not be correct. Working that out, we've actually got an answer for t, but
Anthony For the maximum
Derek For the maximum, but that gives an answer which doesn't depend



on the initial volume, so it can't be right.

RKP Yes, I suspect that your problem there is that if you work out. if you write out your algebra carefully you'll find there's that you've got a fault there probably more than in the modelling.

Derek Of course there is.. Just.. um (writing)

RKP Perhaps, because that should certainly give you an answer. Whether it's an answer that's actually physically right or not, you should certainly.. Those assumptions should lead you to a mathematical answer.

The question I would ask you is, is it really true that um the rate of flow is proportional to the volume that's already there?

Derek That's a good point, um cause we just made our assumptions about what we could see and started trying to work it out and we didn't really list our assumptions in any way.

How can you test your assumptions? We struggle towards describing a simple experiment

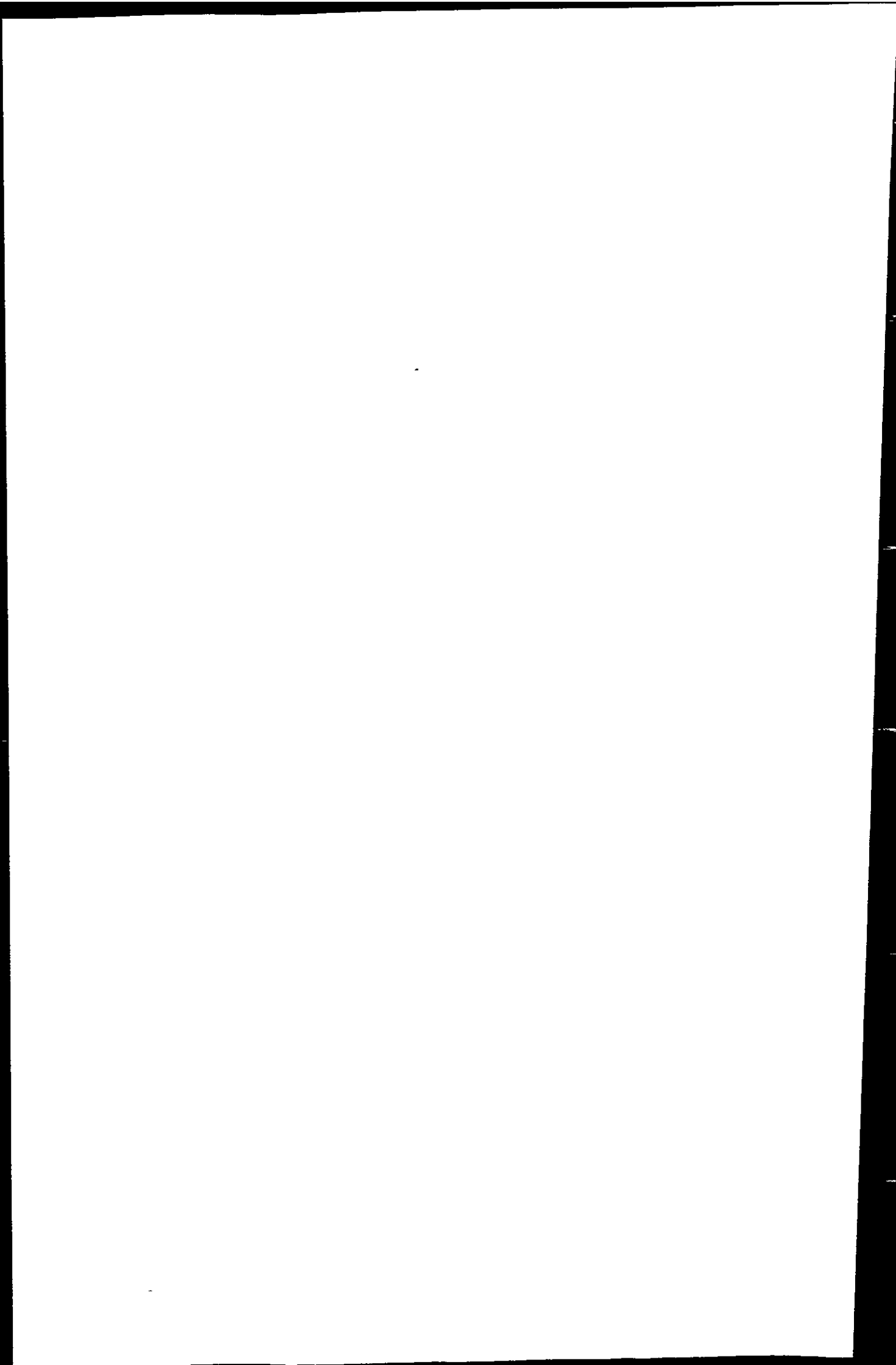
1:12:42 RKP How can you test your assumptions?
 Derek Umm
 Ann We need to time it.
 Derek Well, we didn't even think at all about measuring rate of flow depending on how much volume we've got, and that's something we could and probably should have done, yep.

1:13:03 RKP What experiment actually would you do to determine that?
 Ann Put half the amount of water in there, and see how long it took to flow out, and then put once in there and see how long it took to flow out.
 Jason You could time how long it took for like a litre to flow from there to there or something.
 Derek What we should do is simply empty one bottle into another and forget about the middle one. Just that'll make it much simpler and we can measure our assumptions to start with.
 Ann Mmm

1:13:27 Jason Yeh, you could see how it depends on.. I mean you could do it both from four to three and from three to two and see if one's quicker than the other.
 Derek Yeh, and that would work..
 RKP You can do it better than that, actually, can't you? You could.. You can give yourselves a bit of a scale there, you'll have to put - make a scale on it and just read the volume that's in it against the time and graph it, and see if that's consistent with the model you've..
 Jason Well, we could mark on there nought seconds, thirty seconds, a minute, and mark on it the time it's gone..

1:14:02 Ann Yeh, just see what's
 RKP And you're assuming it depends on the pipe, which it may or may not do
 Derek The height of water in there and the volume, they're pretty closely linked?
 RKP Oh yes, that's supposed to be straight-sided.
 Derek So it's a constant, sort of, area.
 RKP Yes, cause it's the height above the tap, isn't it?
 Ann We made that assumption.
 Derek We noticed the drawings have the water flowing from the very bottom, which is a simplification of the true problem.
 RKP I think that if you work through what you've done, carefully, you'll get, you know, just make sure that the maths is right. You'll get an answer. The answer will depend on that initial assumption. You ought to be challenging that now.
 Ann Okay, well..
 Derek Fine

1:18:03 Ann I've got a second hand on mine.



Vicky Yeh, so have I
Ann I don't actually have a stop watch or anything.
Jason Well, we don't want to stop it. We'll just write on the container and..

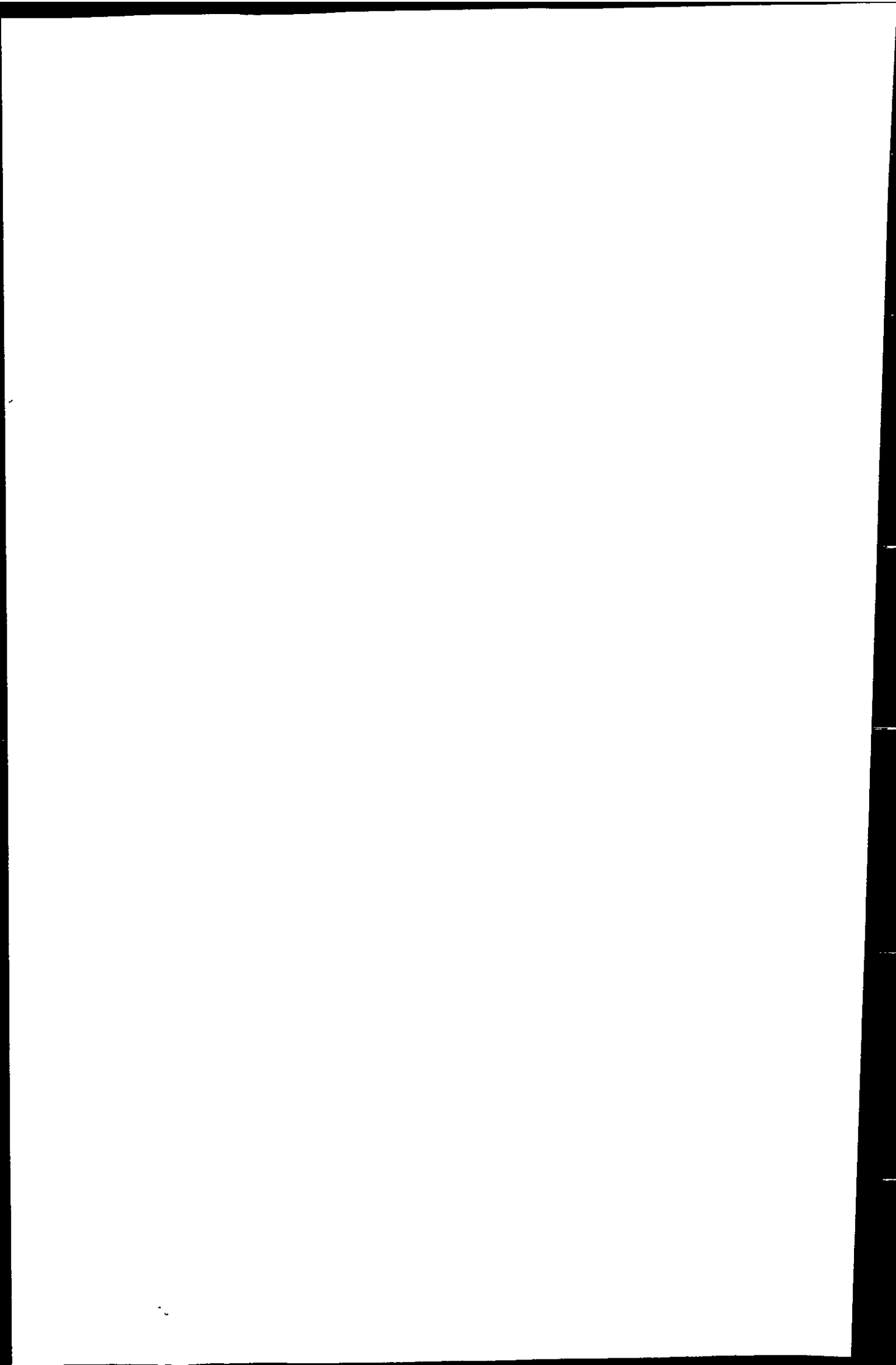
1:15:33 RKP repeats question for video.

The first test to test the assumptions.

1:16:02 Jason sticks vertical strip of tape on top tank.
1:16:19 Jason Shall I fill it up again?
Ann Yep
Jason Come on Ann (they carry jars to sink for refilling)
Break in tape
1:17:04 Anthony When it's stopped moving around.
Jason Just at the end.
Anthony Is it ready to go then?
Derek That's not a maximum, is it. Just the rate of flow.
Jason The top one isn't.
Derek So if you...
Jason What about the meniscus?
Anthony Why don't you lower it down to four? Why don't you lower it down to four and measure it from there?
Ann Yes, measure it from..
Anthony Let it go to four. Let it go to four.
Derek It doesn't matter.
Ann We're just looking at the difference in heights.
Derek It doesn't matter
Anthony we don't want to fiddle with funny numbers.
Ann The numbers don't matter. You're just looking at a physical..
Derek Just mark it every ten.
Jason Who's looking at their watch?
Anthony Is this one open? (middle tap)
Jason Tell me to mark it on the minute.
Vicky When it's convenient.
Ann You want it every ten seconds.
Jason Well call it and if it's too close I can't mark it.
Derek Well, if it's too close we'll run it again, but try it for the moment, shall we?
1:17:45 Jason Well, I'll do it every other ten if it's too quick.
Vicky We'll start with ten seconds.
Ann Okay it's coming up to the minute. Right..
Jason Now?
Ann Mmm
Jason Right, call out ten.
Ann Ten, ten, ten, ten

A glitch in timing.

1:18:38 Vicky Do you want to stop because I'm sure you went ten too close.
Jason Yeh, something's gone wrong here. We've got two tens far too close.
Vicky You did two tens too close together.
Jason They were about an inch apart and then we've got two about that far apart.
Vicky I'll do it on mine.
Anthony Why don't you use the stopwatch?
Derek I haven't got a stopwatch.
1:19:00 Jason Something went wrong even if it's not a stopwatch. They sort of went evenly and then we got.. (removes tape from tank)
Vicky It sounded like it was sort of five seconds, and then like..
Jason It did, yeh.
Vicky Its cause you've got no numbers on yours, that's why.
(Jason puts new strip of tape on bottle)



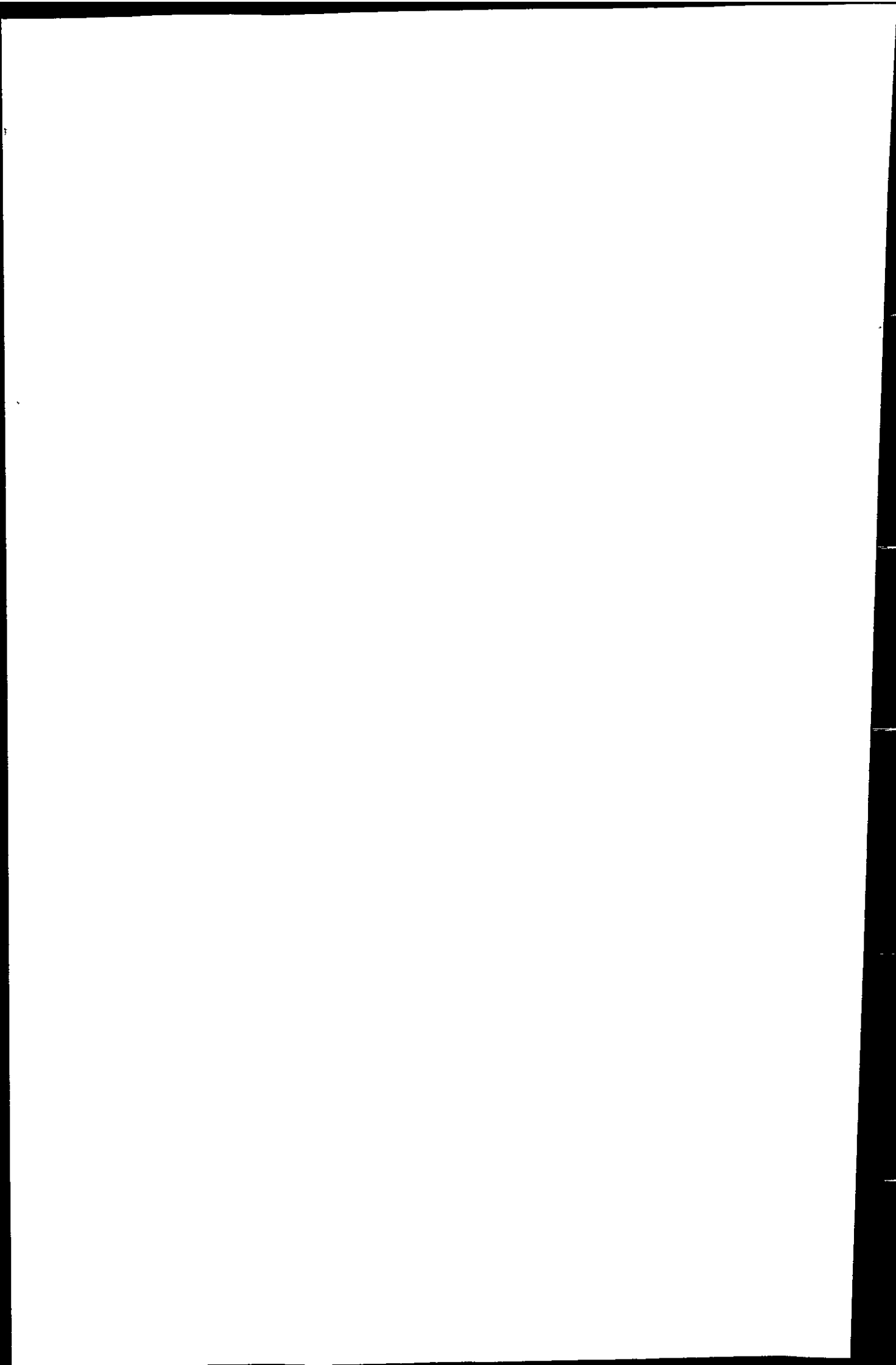
Derek Excuses made up.
Anthony Are you going to fill it to the top?
Jason What?
Vicky You'll have to fill it.
Jason Better than emptying it.

Filling the apparatus again to rerun the test.

1:19:34 Derek closes middle tap
Derek Bottle filling procedure again.
Vicky It's open, isn't it?
Derek No, I just closed it.
Jason I shall do it here. (fill the bottle, as opposed to over the sink)
Vicky Yeh
Derek Yeh, be daring.
1:20:10 Anthony (points at computer) We could just look at the computer and..
Vicky I can't see it from here.
Jason What clock?
Anthony There's a clock on the computer.
Derek Yeh, I can call that.
Anthony We'll just count the seconds down and you can mark it off.
Derek Or what we can do, Jason, is open it when you feel like it and
when it starts running call out and I'll call out the tens from there.
Vicky Oh no, do it from a reasonable sequence.
Jason Yeh
Anthony Yeh, when you hear a little bit of a trickle.
Vicky Well he can do it and look at it.
Derek Yeh, I will.
Vicky No, he can.
1:20:38 Jason I can't see the clock from here.
Derek When you feel like it, Jason. Just call and I'll tell you when.
Jason Yeh, hang on.
Vicky He can't even see it. He's too close to it.
Derek Zero, is it?
Jason Yeh
Derek Ten
Anthony No, forget it. This one's not open. (middle tap)
Jason It doesn't matter, we're not marking that one.
Derek Mark.
Jason Exactly even at the moment.
Derek Mark.. mark..
Jason It's slowing down
Derek Mark.. mark..
1:22:08 Jason We missed one there, didn't we?
Ann No, he didn't.
Jason There's a bit of a gap.
Derek Mark

A surprising result: the marks seem to indicate a steady flow rate.

Anthony I don't think the apparatus is very useful.
Jason Well I think what we can say from that is it's a constant.
Derek You should have ten marks there.
Jason (counting marks) One, two, three, four, five, six, seven, eight,
nine, ten.
Derek So..
Ann So if you break the flow it's not proportional.
Vicky It does look constant.
Jason It does look.. I mean these ones are a bit.. For the first it does
definitely. Up to probably where it's going to be a maximum.
Ann It will come out totally different.
Jason Well it's the slope we need.
1:23:05 Break in tape.
Close-up. Film of Jason marking tank as water runs out.



1:24:22

Break in tape. Back to assembled group.
Derek Can you help us please.
Ann It doesn't make that difference.

Intervention: Where do you measure the depth of water from, if the flow rate is proportional to height?

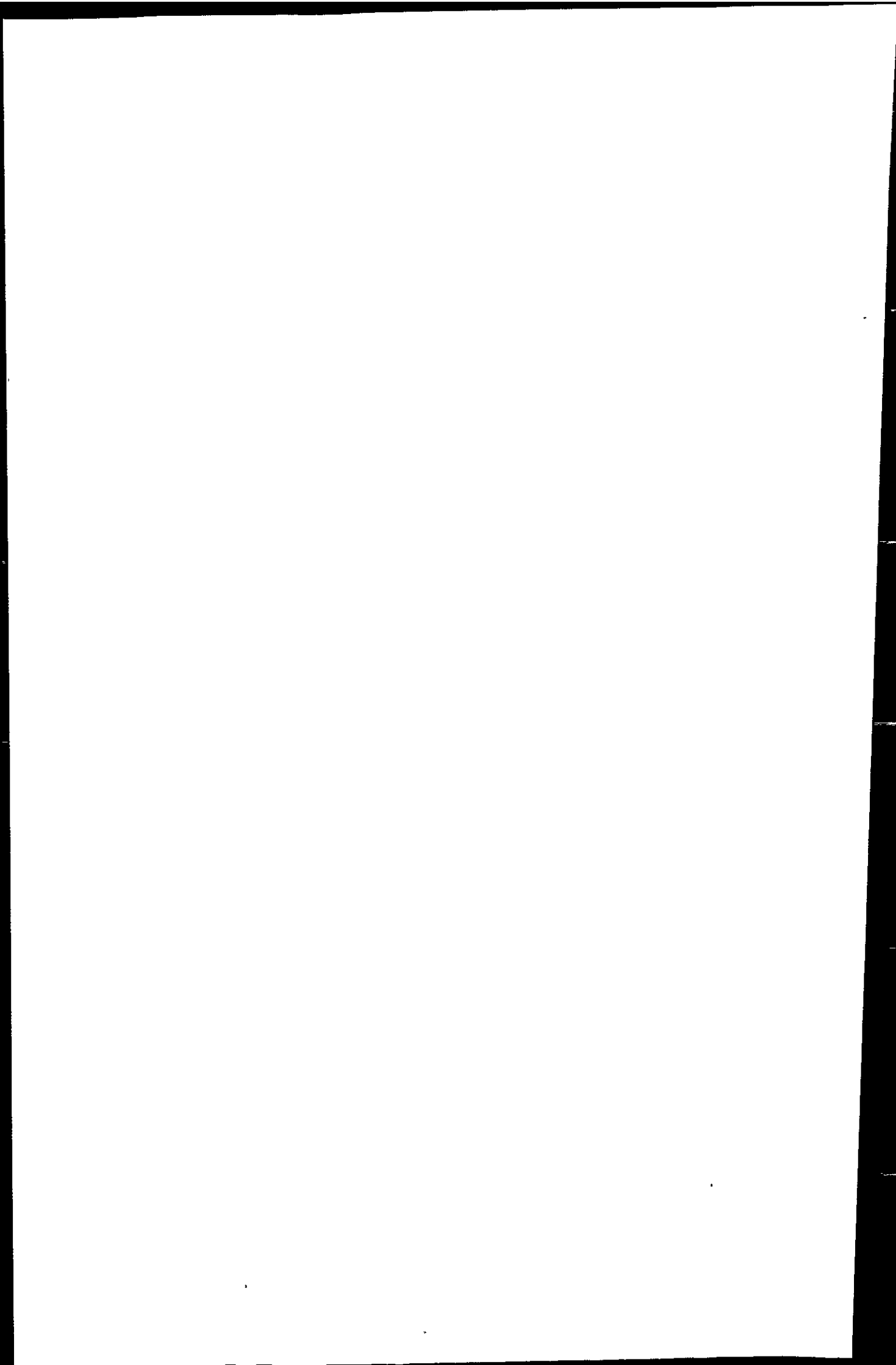
WMM Can I ask you a question about one of your assumptions?
Derek Not another one. (laughter)
Ann The assumption
WMM You've assumed the pressure is equal to ρgh , yeh?
Ann Yeh
WMM And what I'd like to ask you is where you're measuring h from.
Ann Well, we're taking zero as the floor level, if that's what you mean.
Derek (off camera) No, zero is here.
Anthony Centre of the earth.
WMM Why is zero there?
Anthony No it's the centre of the earth we're taking as zero h .
Jason No.
Anthony No, it is
Derek No, He's right, because..
Jason No, like anything there (top bottle) is like just a waste of space.
Ann The diagram is that it's flowing from the bottom, isn't it? So you've taken the level where it starts flowing out the tap. You measure from there to the top where the water would be the height in that can, won't it?
WMM What I'm trying to get at is where a better place to measure zero from might be. What in fact in your ρgh does h stand for?
Ann The height of the water in the can.
WMM Does it equal.. Is it equal to the height of water in the can?
Derek Yeh
WMM What about the height of the water is it?
Jason The height of where it's flowing from.
WMM The height of where it's flowing from. Where is it flowing from?
Derek There
WMM Is it? Is that actually where it's flowing from?
Derek Umm.. Ummm..
Jason Well, it's flowing out from the hole in the bottom.
WMM It's flowing through the hole, and then what happens to it?
Jason Goes down the pipe (points along tube with pen)

Is there anything special about the place you measure the depth of water from? What is a point, time or space?

WMM What's the magic about h equals zero? What is it that you're assuming happens at h equals zero?
Ann There's no flow..
Jason No flow.
WMM Well, no, sorry. At the position where the level is zero. I'm expressing myself very badly.
Ann There's no flow.
WMM Umm.. What is it about umm
Ann There's no change in the amount of water that's in the can.

Derek has an explanation which does not seem to involve pressure, potential energy, or anything.

Derek The reason we're measuring the height is because the volume's directly proportional to the height, and we think the volume is what forces the water out. So we're..
Ann We assumed the flow would change,
Derek So we're just measuring the height of the liquid as one of the dimensions of the volume. We aren't really trying to equate it to



its pressure at a particular area, or potential energy, or anything.
WMM I thought you.. A shame.
Derek Because if it was pgh.
WMM Yes.
Derek Relative to the centre of the earth, we discounted that because they are virtually the same relative to the centre of the earth. I think that's what you're trying to ask..

We establish that we have a constant flow rate, which is not what we expected. However this may be because we are not measuring accurately enough..

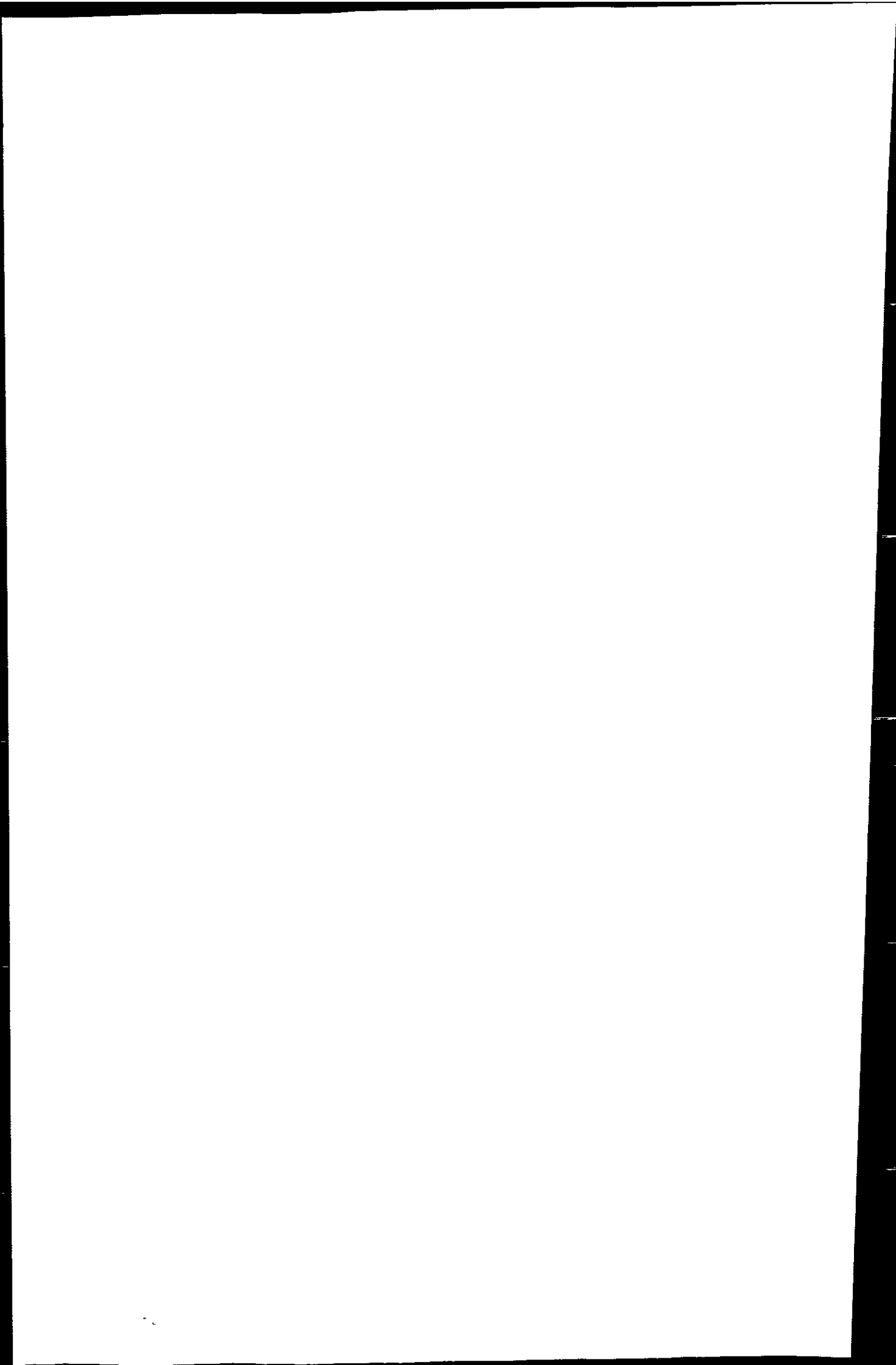
WMM No, it's not. That's not what I'm trying to ask. I'm trying to challenge basically why you think you've got very little difference in the size of your steps.
Derek Er.. Jason's got a steady hand.
WMM I'd
Jason You mean they're almost all the same.
WMM They're almost all the same, yes.
Jason That means it must be a constant flow rate, then.
WMM And why.. What is it about your measurement of h that might lead you to have a more steady flow rate than you might think you'd have, and what can you change to have a flow rate that varies more.
Derek Ah! The centre of mass of it, because that doesn't rise that.. No, that wouldn't make any difference, would it? That would just be half all the time.
Ann What I think you're trying to get at is if you're trying to measure it to the nearest tenth of a centimetre then they're identical but if you're trying to measure to something more accurate then they're not all the same.
WMM Clearly I am asking the wrong questions here.
Derek No, don't say that, just.. (laughter)

Let's talk about pressure..

WMM I must see if I can find a more constructive question.
Would you agree that at the position from which you measure the height, the pressure in the water is equal to atmospheric pressure?
yeh?
Jason Yeh
WMM Okay, you like that one?
Okay, so at what point in your top bottle and its bits and bobs attached is the er pressure in the water equal to atmospheric pressure?
Derek At the top
Jason On the surface
WMM It is at the surface and where else?
Jason Here at the tap.
WMM Why at the tap?
Derek Cause it's got..
Jason Well it's
Derek Actually it's here
1:29:50 WMM (claps) Yes, yes, yes, it is. At that point where it leaves the hole it should be equal to..
Ann Oh.
WMM So
Vicky So we should be looking at the height from the bottom of the tube.
WMM Yes

And we are not at all convinced we should.

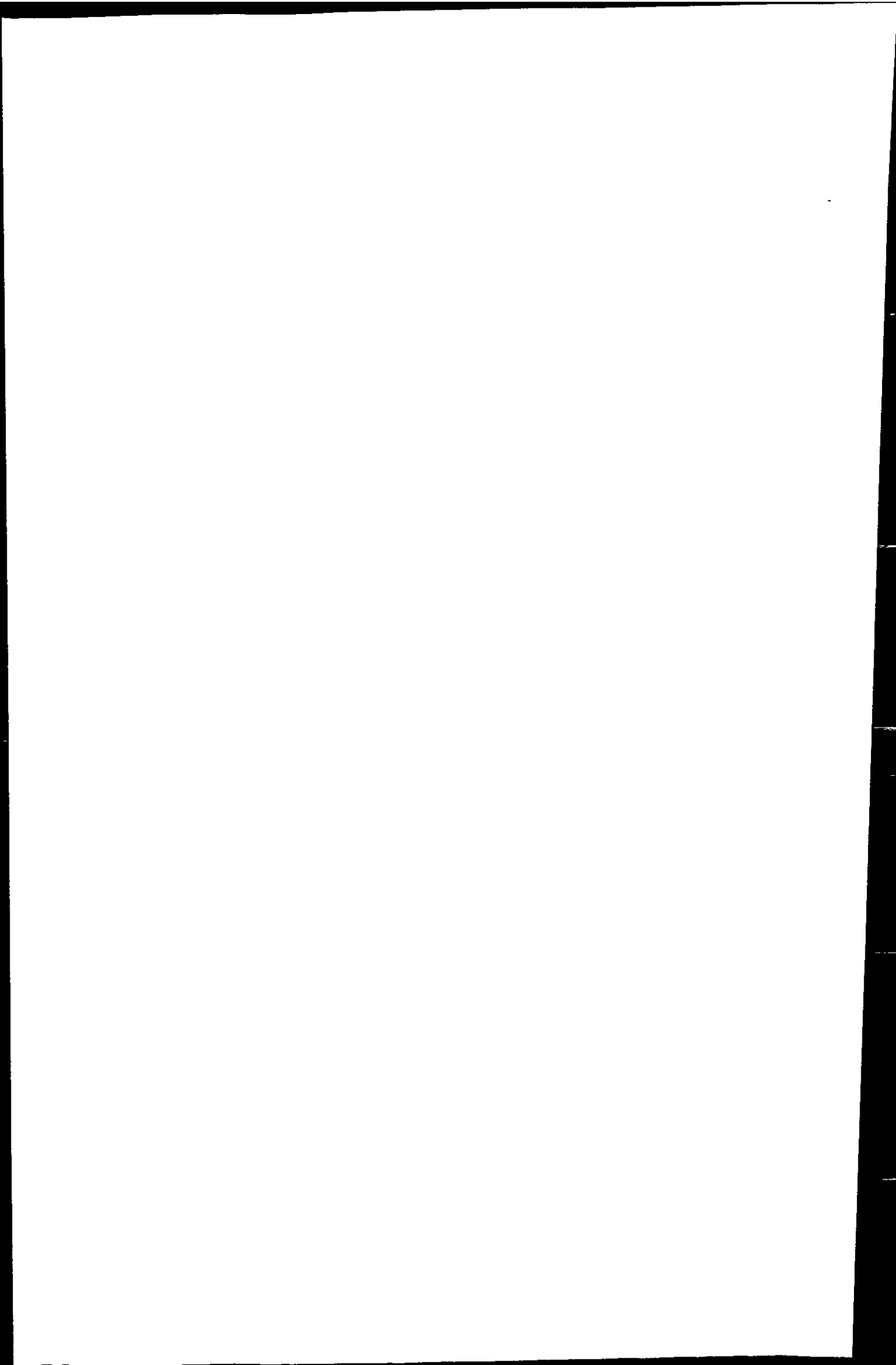
Derek It's the same amount of fluid.
WMM You could try that.



Ann I can't see what difference that would make.
 Jason Yes, but if it's the height from there, from the bottom of the tube, it's still coming down equally, isn't it?
 Ann I was going to say, it's still going to drop at those intervals.
 WMM Well, how about making.. doing an experiment to see whether the length of the tubes makes any difference?
 Jason Like take the tube off
 WMM Could you?
 Derek We've only got two tubes the same length.
 WMM Yep, that sounds reasonable.
 1:30:32 Derek How do we adapt the model, then? How will that help us out with the maths?
 Vicky Is that tube actually longer than that one?
 Jason Mmm, yep. Oh, we could change then over. No, try it without the tube first, see what happens then.
 1:31:25 Jason You can't get the tubes back on
 WMM If you get them wet they go..
 Jason That's closed now, isn't it? (top tap)
 Derek Hope so.

We try, although we get a bit wet and have to modify the apparatus a little.

1:32:10 Jason So it's like.. I'll hold it. (fills top tank from bottom one in situ)
 Jason So what's this written on the side of this? Right, leave this one (middle tap) shut.
 Ann Leave this one shut. And.. I'll hold it (middle carboy) until I see where the flow's going.
 Jason No, no, no. As soon as it starts to come out properly we do this. Because we're going to mark it again. (removes previous tape)
 Ann We're going to mark it again.
 Jason Ten seconds
 1:32:37 Derek Um, I'm just wondering if here's a funnel, though.
 Ann I'm just wondering if it's actually going to go in there without..
 Anthony I don't think it'll make a difference.
 Jason No, you don't want the tube on.
 Ann No, I know you don't, but I just want to make sure the water's gonna..
 RKP (offers phone directory) Here, that may help it stop splashing a bit.
 (Anthony puts directory under middle bottle.)
 Derek You don't have a funnel, do you?
 1:33:21 Anthony I was just wondering- That's all I'm making sure of
 Jason Right, who's going to do the timing then? Who's watching the clock then?
 Derek I'll do it with my watch then?
 WMM revives clock on windows.
 Jason I'll start it on ten seconds
 Derek As you wish
 Ann Oh crickey (water splashes from tap)
 Anthony Forget it.
 Jason Right. I'll need a few more books.
 1:34:05 RKP (Bottle now sitting on empty wire spool, putting a book under spool) We'll get it right up there.
 Jason Clock. It's coming up to a minute, isn't it?
 Derek Seven, nine, zero.
 Jason Shout it out
 Derek Ten, mark, mark.
 Anthony Is it getting smaller at all.
 Derek Mark, mark, mark, mark, mark..
 Jason Mmm, I think they're getting smaller, you know.
 Derek Mark
 Jason But they did last time we got towards the bottom.
 Vicky But is that significantly smaller, or..



Derek Mark
Jason We'll make this one the last one. Ah, no you want to carry on.
Derek Mark, mark
Anthony Wasn't so many marks last time.
Jason No, it's taking a longer time for some reason.
Derek Mark
Ann If it's flowing slower, you've got a shorter height. You notice the difference more, don't you.

Derek Mark
Anthony What did that prove?
Jason Proves that it's
Ann It is proportional, isn't it? Flowing through that pipe,
Jason seemed to change it
Ann Is speeding up the flow, didn't it, because that took a lot longer.
Derek Why is that?
Ann So it's slowing flower and you notice the difference a lot more.

APPENDIX B: MAIN QUESTIONNAIRE

Rubric

Please read this page first:

Read the following question:

Which is the odd one out?

house office

igloo flat

Some people will choose *house*, because it has more than one storey.

Some will choose *office*, because people don't live there.

Some will choose *igloo*, because it is made of ice.

Some will choose *flat*, because it doesn't have an 'o' in it.

There is no wrong or right choice here, but the way you answer gives a clue to the way you think.

The six examples that follow are rather like this question. They are not intended to trick you into giving a wrong answer, but rather to see which of a choice of answers you prefer.

For each example there is a list of six options. Please choose the one you think is the best fit, and then rank the rest from best to worst. If none of the answers seems to you to fit, then please put a better one in the comments box. Otherwise please use the comments box to explain your order of choice, or for any other comments on the question.

Please fill in the following details. I would like to know who you are for possible follow-up research, and nothing you write in this questionnaire will end up in your student records!

Name:

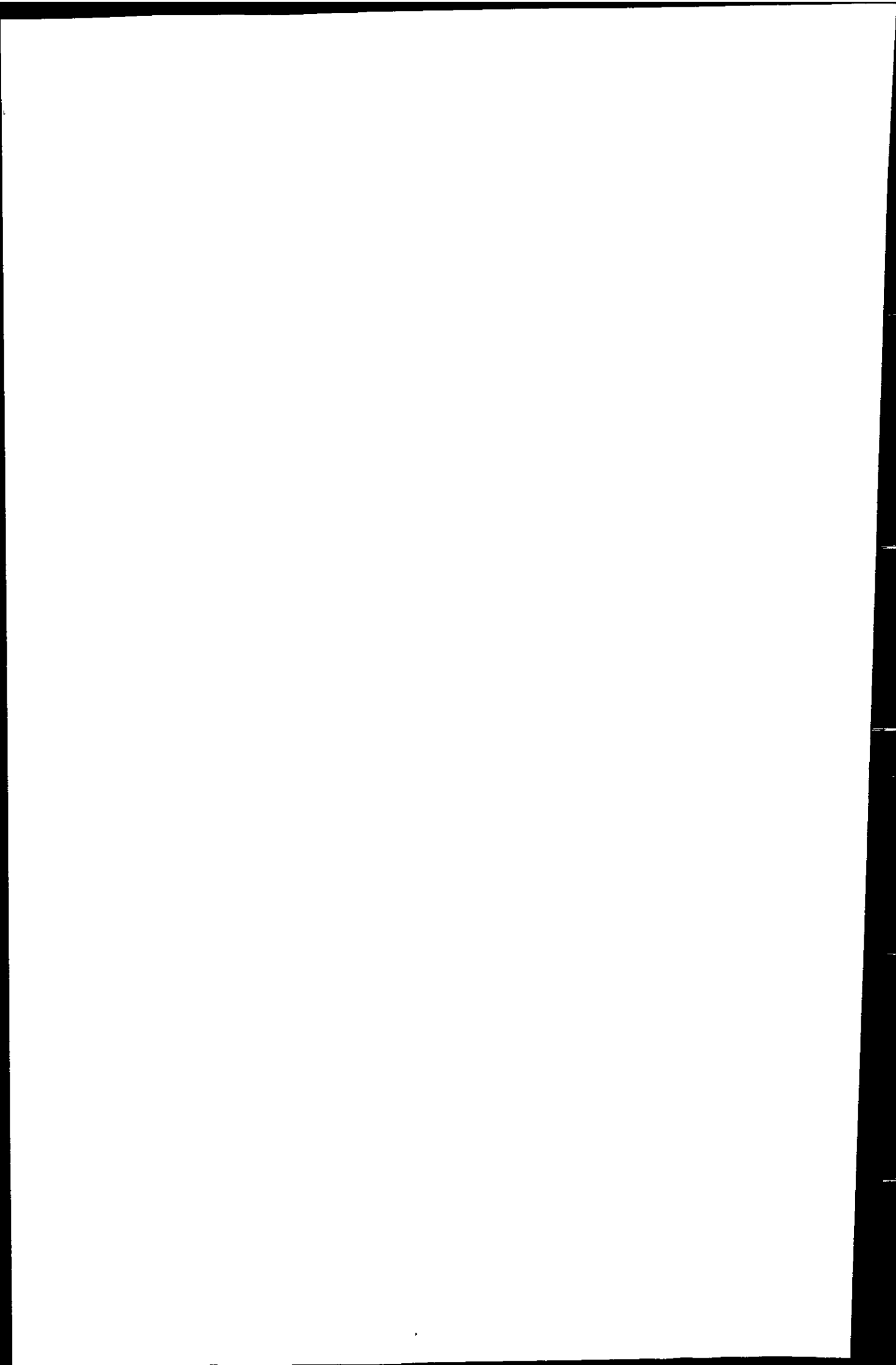
Course:

Year:

Date:

Thank you for helping me with my research into
the mathematical ideas of engineering students.

Wendy Maull
Centre for Teaching Mathematics



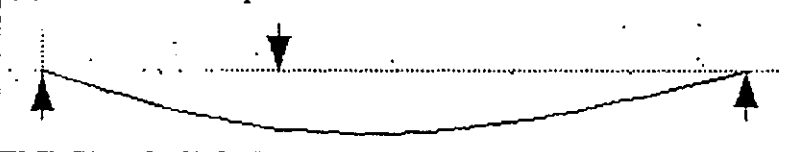
Question 1

A plank 1.5 m long is placed on two bricks very near its ends. A bar of gold is placed across it 0.5m from one end. Rank the following according to how well they represent this to you.

(a)

The beam bends under the weight of the gold bar.

(b) Deflected shape



(c)

Bending Moment $M = k \frac{d^2y}{dx^2}$

(d)

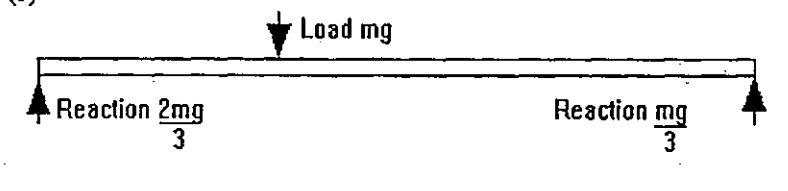
A simply supported beam with a point load at one-third span.

(e)

Shear Force $S = \int F dx$

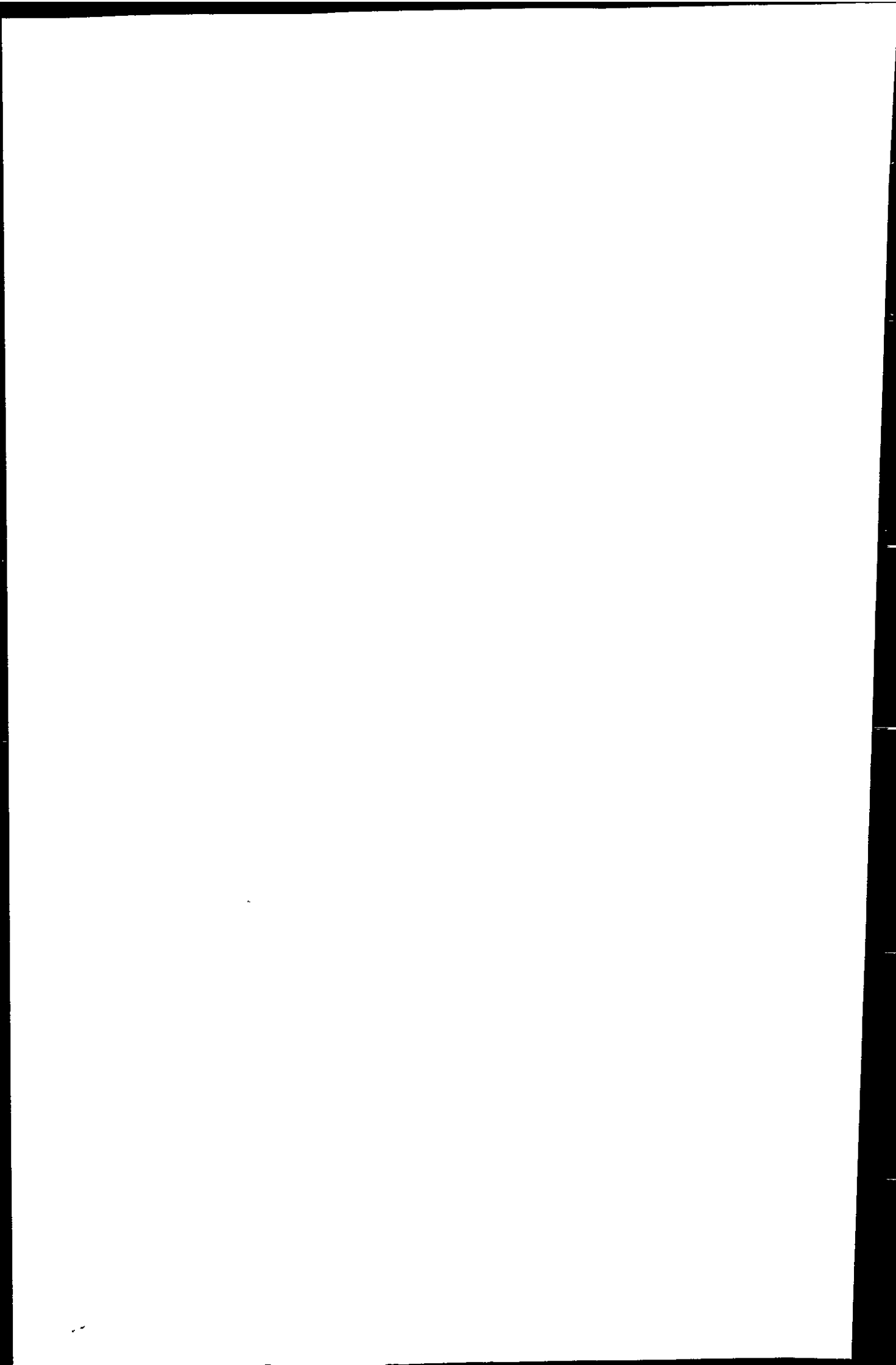
Bending moment $M = \int S dx$

(f)



Answer

Comments:



Question 2

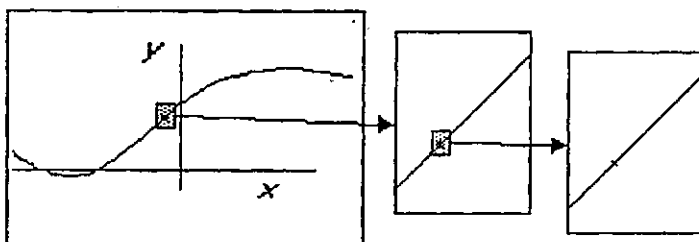
$$\frac{dy}{dx} = f'(x)$$

All of (a)-(f) can be associated with the statement above. Please arrange them in order of how closely they are linked to it in your mind.

(a)

$f'(x)$ is the slope of the tangent to a graph of y against x .

(b)



(c)

dy/dx tells you how quickly something is changing.

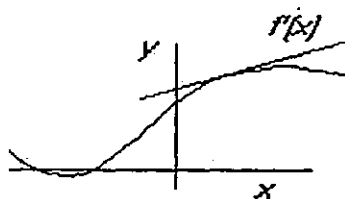
(d)

$$f'(x) = \lim_{(x_2 - x_1) \rightarrow 0} \frac{y_2 - y_1}{x_2 - x_1}$$

(e)

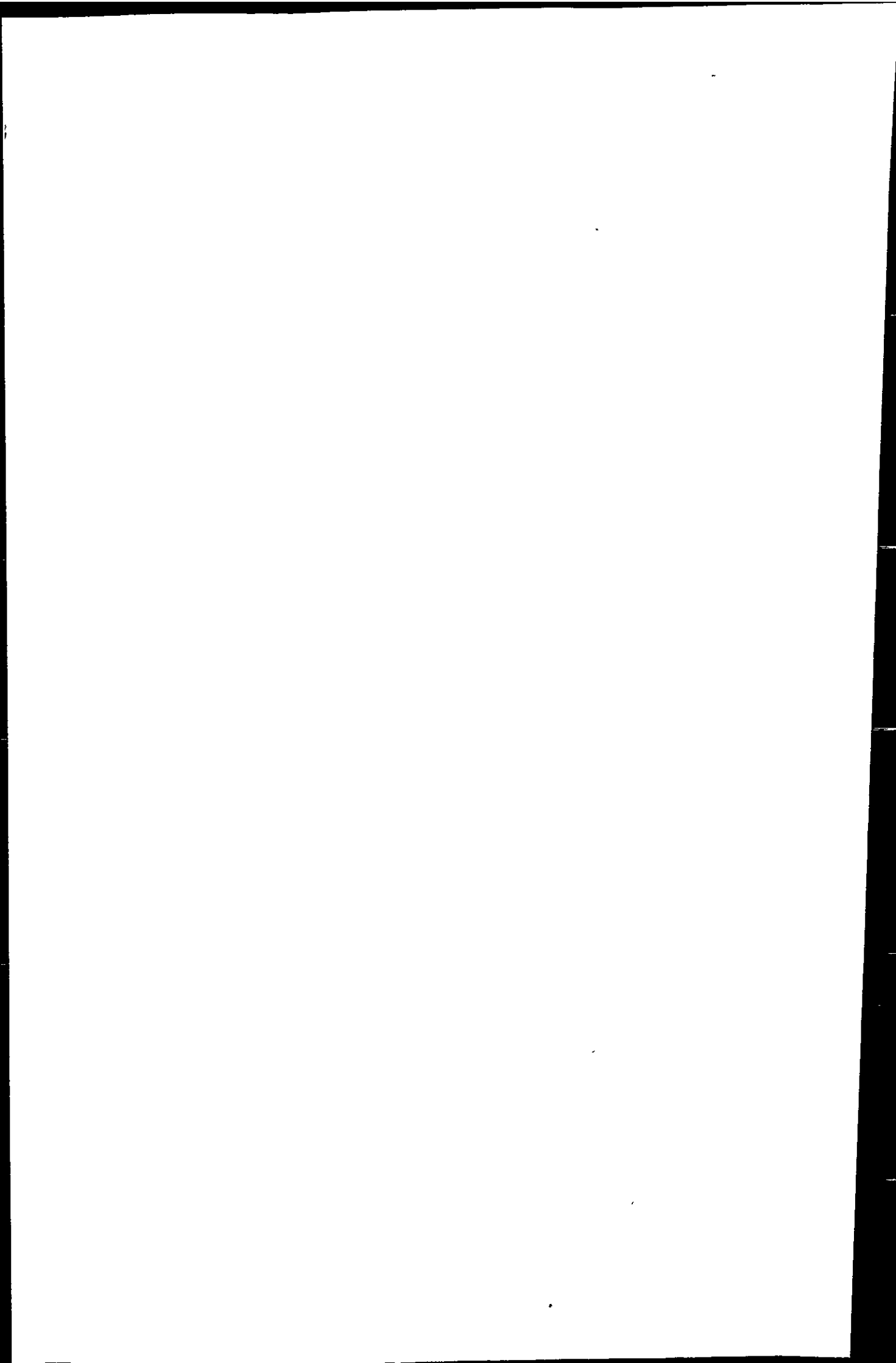
As you zoom in more and more closely to a small section of the curve, it seems to straighten out. The slope of the tiny straight section is dy/dx at that point.

(f)



Answer:

Comments:



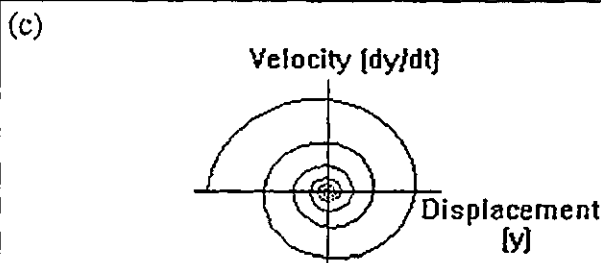
Question 3

A mass suspended from a spring and dashpot is pulled down from its equilibrium position and released. Which of the following do you think best describes what happens next?

Please arrange the answers in order of how well you think they describe the movement of the mass (best first, worst last).

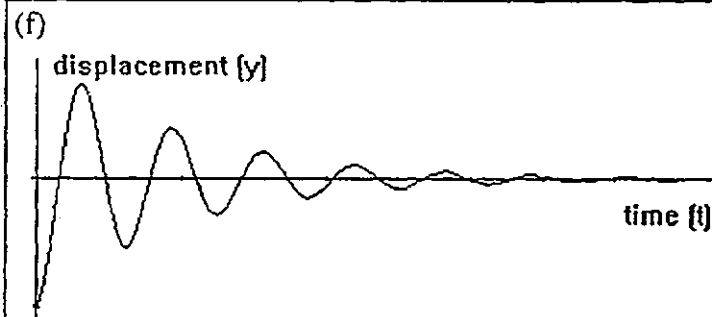
(a) The mass bounces up and down, going less far each time, until it settles back to its original position.

(b) $\ddot{y} + k\dot{y} + \omega^2 y = 0$



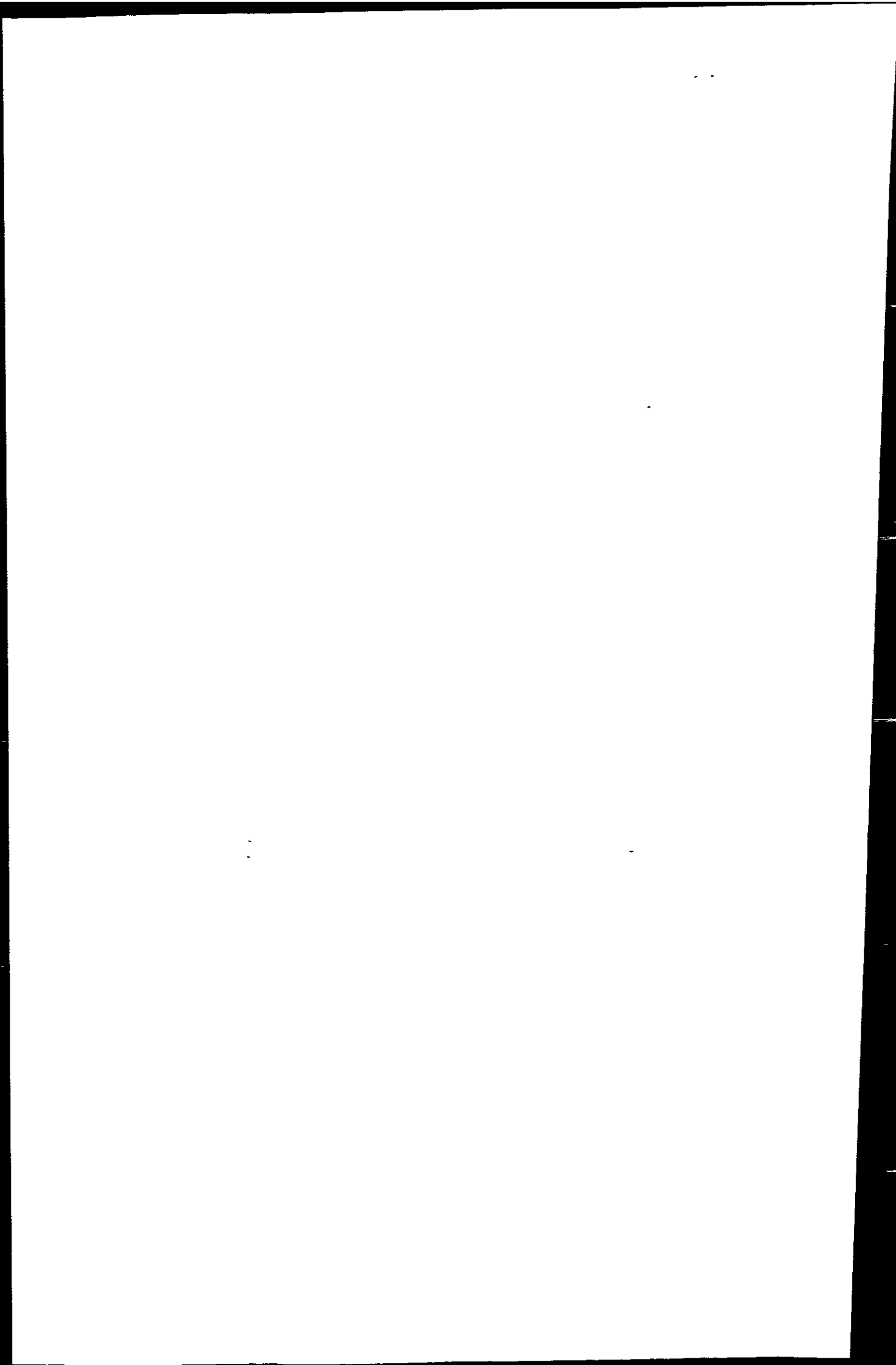
(d) $y = Ae^{-\lambda t} \cos \omega t$

(e) Damped harmonic response



Answer:

Comments:



Question 4

$$q = \int x \, dx$$

(a)-(f) may all be associated with this statement. Please arrange them in order of how closely they fit the way you think of it.

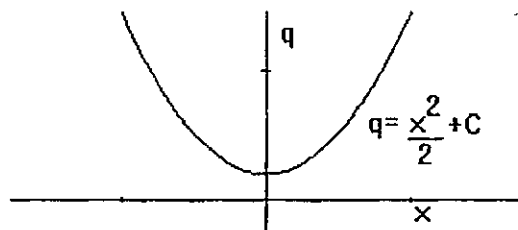
(a)

q is the area under the curve $y=x$.

(b)

$$q = \frac{x^2}{2} + C$$

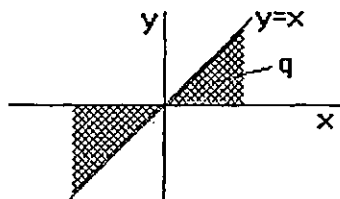
(c)



(d)

The integral tells you how things build up.

(e)

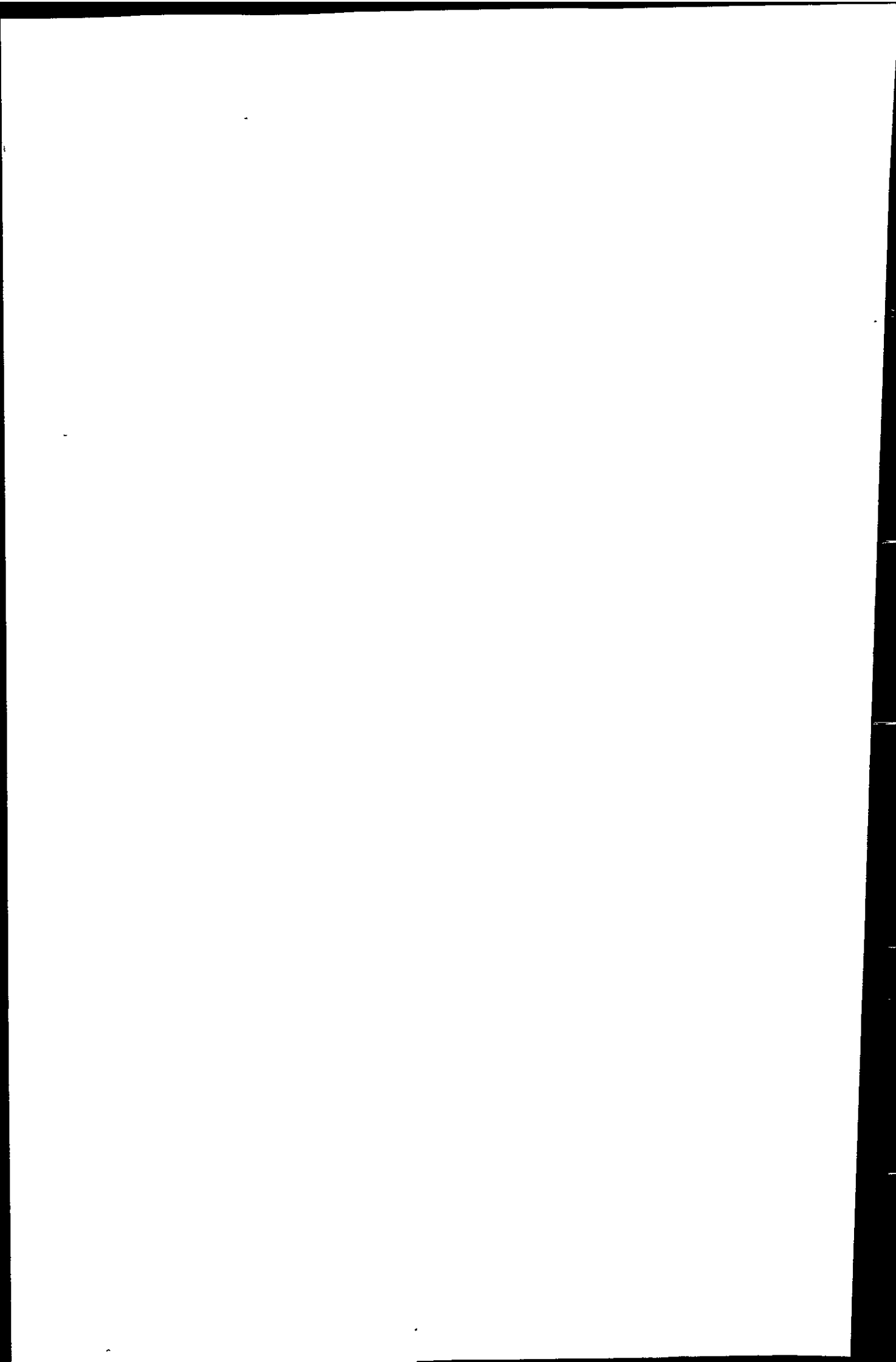


(f)

$$\frac{dq}{dx} = x$$

Answer:

Comments:



Question 5

In a pinball game, a ball is fired by releasing a taut spring behind it, propelling the ball out at speed. Arrange the following in order of how well they describe this to you.

(a)

Energy stored in spring = $\frac{1}{2}$ Force \times Extension

(b)

Energy imparted to ball = $\int F dx$

(c)

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = \frac{1}{2}kx^2$$

(d)

Change in momentum = $\int F dt$

(e)

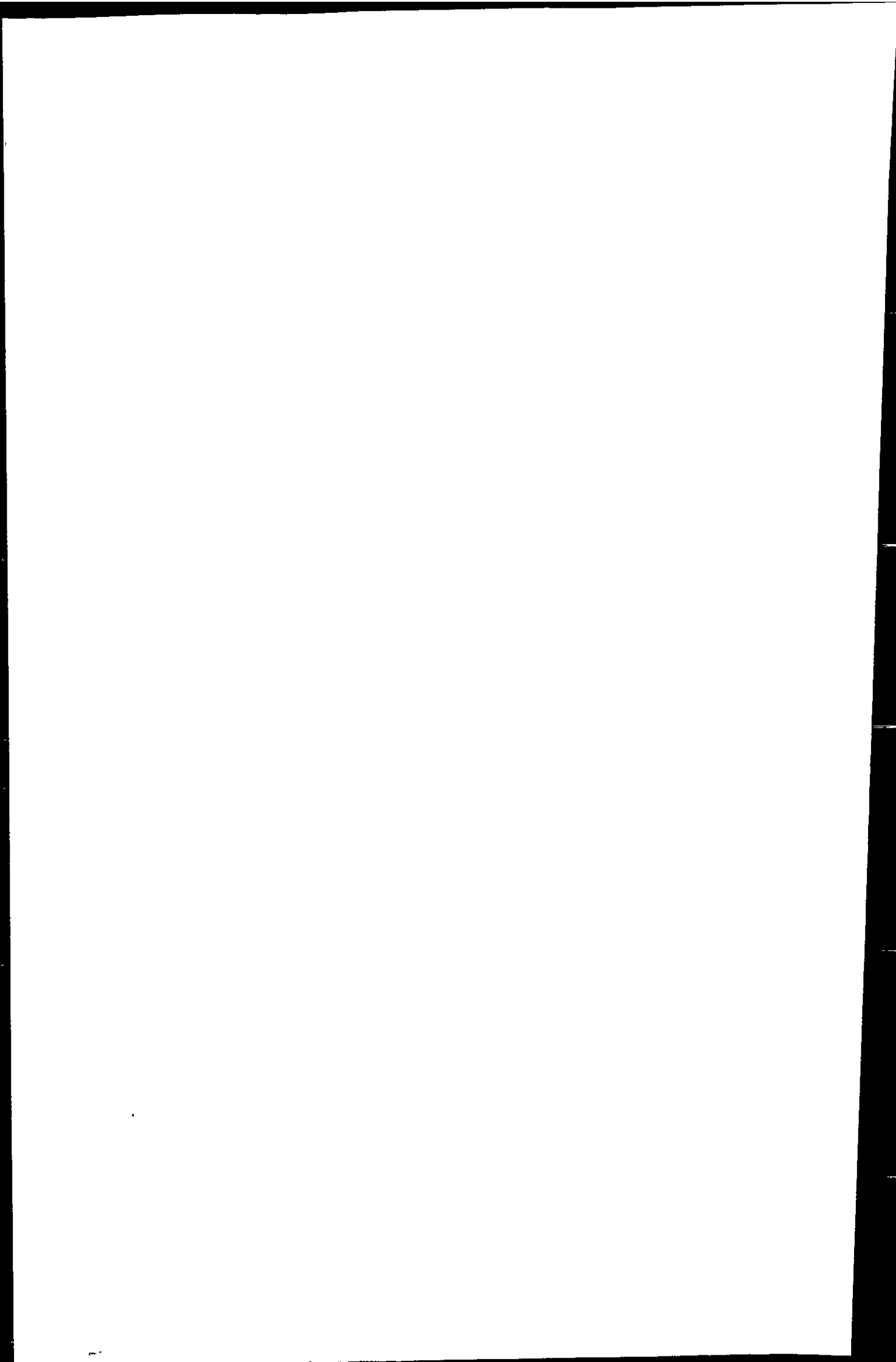
$$F = ma$$

(f)

The further you pull back the spring, the faster the ball will go

Answer:

Comments:



Question 6

$$\frac{dy}{dx} = e^x$$

Arrange the differential equations below according to how similar you think they are to the one above.

(a)

$$y^2 \frac{dy}{dx} = e^x$$

(b)

$$\int dy = \int e^x dx$$

(c)

$$\frac{dy}{dx} = e^y$$

(d)

$$\frac{dy}{dx} = mx + c$$

(e)

$$\frac{d^2y}{dx^2} = e^x$$

(f)

$$\frac{dy}{dx} = my$$

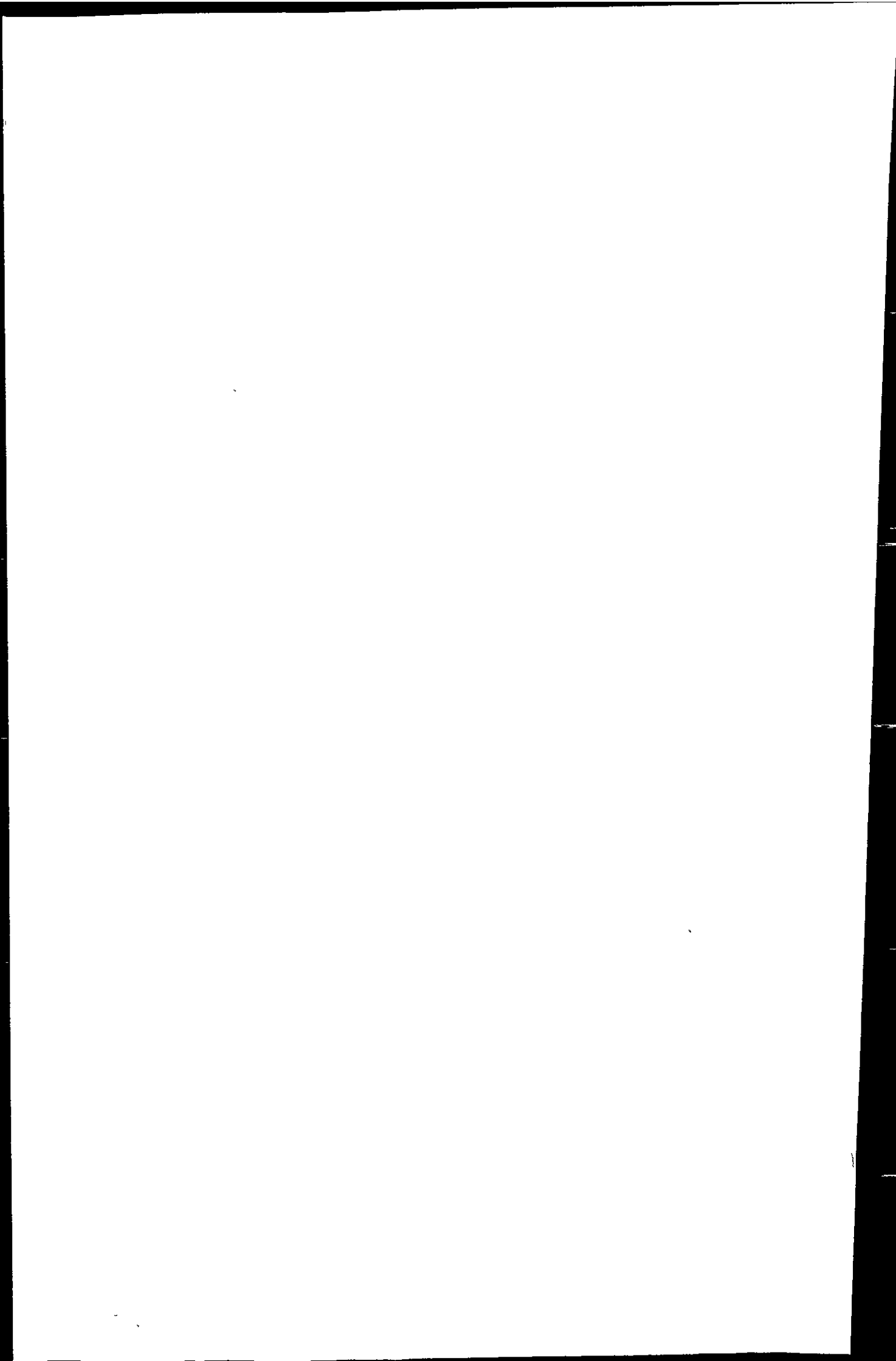
Answer:

Comments:

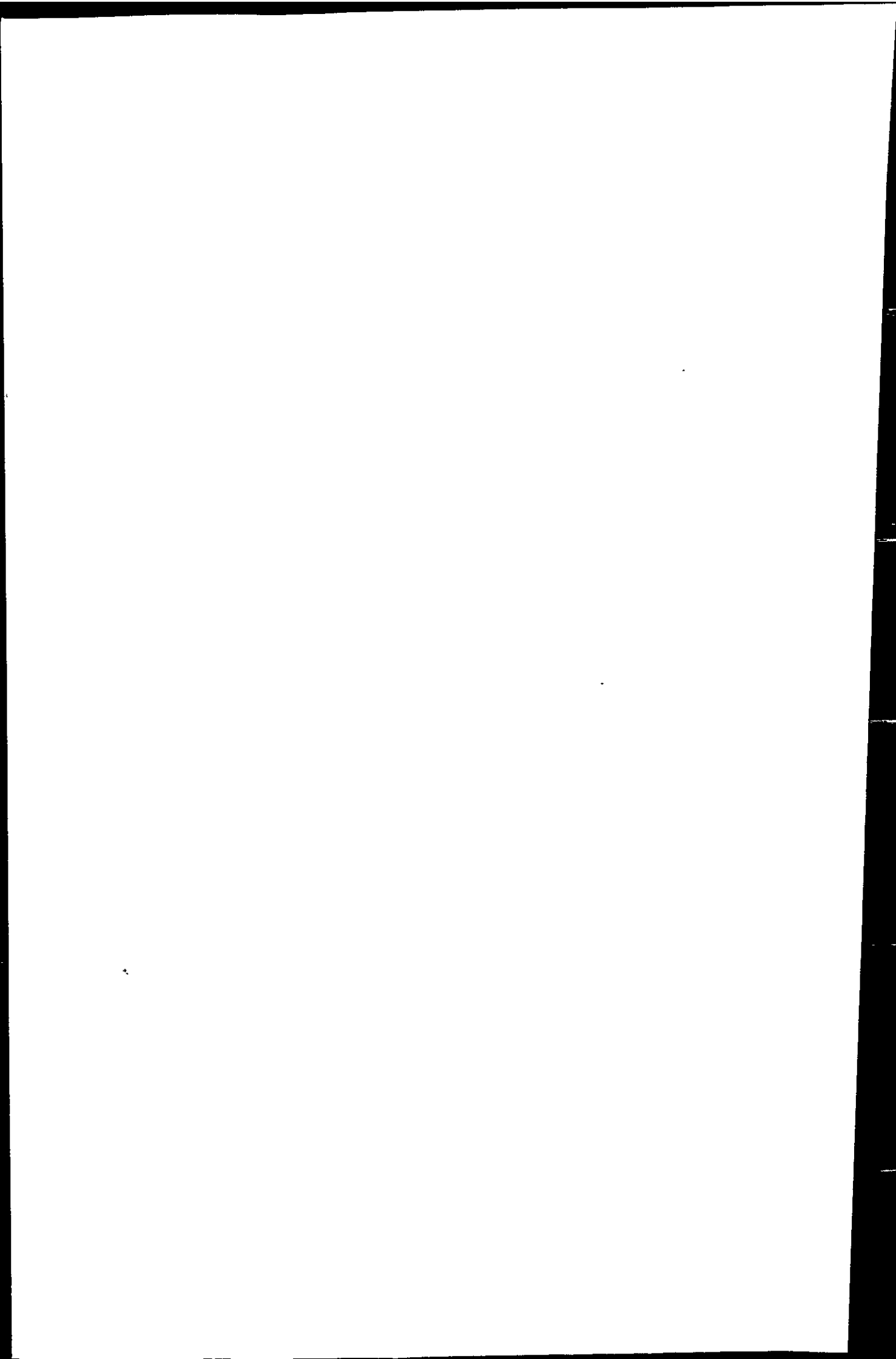
APPENDIX C: TRANSCRIPTION OF INTERVIEWS WITH STUDENTS TO EVALUATE THE COURSEWARE PACKAGE.

First evaluation interview with final year mechanical engineering student, recorded by video in CTM, 2 Kirkby Place R101, directly after using the package by himself, November 1997

- Jed It's nice to use
WMM So you like...
Jed It's interesting enough to stand examination, which is nice
I think perhaps keyboard alternatives, such as the two arrow keys for previous and next
Indistinct
Microphone and camera adjusted.
WMM What did you feel about the navigation? Did you find it easy to navigate round
Jed I thought it was pretty good.
Perhaps on some of the longer modules, for instance investigate implications or compare with reality I thought it could perhaps do with a page numbering system or some sort of scrolling system. For instance I got about halfway through, I wanted to pop back and look at this page but you have to go right back and go through it again.
I felt some sort of scrolling system perhaps would be useful.
WMM That's something to think about.
The home page worked as a way of navigating?
Jed Yep. You can base yourself from there. That worked quite well.
WMM Do you have any comments: did you have any glitches: did anything go wrong?
Jed Nothing went wrong. Well I think one of the pictures is inverse, is in negative
WMM Is it? do you remember which one?
Jed Intro to modelling: reality
WMM Ah! It's just bad colour. Thanks
Jed One thing I thought about the navigation was there are different numbers here. They don't really co-ordinate, moving from reality to understanding. I thought perhaps rather than have a whole page dedicated to just that if it just flashed up and fades into the next page perhaps.
WMM Right
Jed Rather than have to click again that just fades into the next.
WMM What did you think about the introduction. The bit before you get to the home page?
Jed Right, telling you how to navigate. Yes, that's clear. It just gives you everything there. Perhaps if these two [buttons] were swapped around to match those two..
WMM What did you feel about the context? Did you feel the case studies were relevant?
Jed Yes I thought they were quite useful : the case studies. They were suitably practical.
WMM You didn't think they were.. Did you feel your intelligence was being insulted by any of them?
Jed Not really, no. I haven't been using any maths for the last year being on placement so my maths is very rusty.
WMM What about the level of the maths?
Jed What is the package aimed for?
WMM It's aimed for giving concepts of differential equations in the context of mathematical modelling. So either you have looked at.. So you can either use it for introducing mathematical modelling to people who have done differential equations, yes, calculus, or for introducing calculus, differential equations to people who have done mathematical modelling.
Jed Right- it seemed quite useful for that. It's a lot more interesting -, it's easier to grasp than standing in front of someone who's telling you about it.
WMM The level of the maths varies quite a lot from the first one through it. Whereabouts do you feel happiest?
Jed Well, given a reference, I'm happy enough with understanding the calculus- I've forgotten all the transforms myself.
When you use them a lot you know them, you just click them in, but I've forgotten all that.
My maths is very rusty- I haven't been using it for a year and I haven't had to use it so far this year.
WMM What about the level of the commentary- of the explanation that was going on?
Jed I thought that was quite good



WMM Did you feel it was too high a level, too low a level?
 Jed I felt it was about right,
 WMM What about- did you feel that the mathematics was put in context or that there was a sort of break?
 Jed No- it seemed to go- to flow quite well really.
 WMM What about the pace- that it went too quickly or that it took you through at a snail's pace?
 Jed It went through too quickly for me because I haven't been using it but I think if I had just come out of the maths modulus that would be fine.
 WMM I had at one point thought of building in another layer in which the maths went a lot more slowly so you could switch from one layer to another
 Jed Yes you could do that or even clickons for each page - just sort of little maths subpages come up- it could be- I think for people from maths modules that's probably enough.
 WMM I think it's more aimed at somebody like you who's been out to try to encourage you to think mathematically.
 Jed Yes, mm, I see what you mean. I sort of tended to follow what was going on without actually examining the maths.
 I understood what was happening without doing the sums as it were in my head.
 WMM So were there any particular spots where you thought it was going really too fast.
 Jed No
 WMM Or conversely really too slow
 Jed No in general I liked the pace very much throughout.
 WMM There were different help styles used in one or two places where I was testing them out. One of them was where you had a red word..
 Jed Yes, I noticed that, I quite liked that
 WMM Another was where I put - used a split screen - put a lot of commentary down the right hand side.
 Did you prefer either of those?
 Jed I quite liked the red word although the danger of that is people might be lazy and just not do it, seeing they ought to be able to figure it out and skip past it.
 WMM So you haven't done any maths... Have you done any mathematical modelling in your course, ever?
 Jed Well, we did some basic stuff..
 WMM Specifically as mathematical modelling?
 Jed Yes, it tended to be in the mechanics modules. Did we do some in the maths modules...
 I don't really remember. It was such a long time ago
 WMM Do you feel that you've learnt anything in using it?
 Jed Perhaps in the actual.. In each particular application. Learnt what you would use to do a coffee mug. But on the other hand I felt I was aware of the homogeneity of the principles behind modelling.. The principles behind it.
 I would know where to go for modelling and what I would want to be looking at, just be inadequate in locating the actual variables.
 WMM Do you think you would find it useful to have something like that available?
 Jed I think it would be, really, yep. Just- I don't know, you mean like on the network or something?
 I think it would, certainly to go and enlighten oneself as to what is possible
 WMM Do you think anybody else would use it?
 Jed Well, it's the same old thing, being a student, nobody's going to use it, but, I mean it might be if there's a tutorial session as part of a module, you know, go through this. It's relatively... it's not... Nobody's going to be turned off by it so it would be all right.
 WMM Would you like a copy?
 Jed I won't, Thank you, no: Thanks for the offer. I'm trying to avoid mathematics this year.
 WMM It has been suggested that there are two ways we might encourage people to try it out and test it. One way would be to invite people to have a go at it and then have some lunch, or another way would be to put it on disks so that people could take it away and have a play with it at home, and just give a quick fill-in questionnaire. What do you think of either of those two?
 Jed I think either would really be useful. I mean it's cheaper for you to send it away on disks. I guess everyone could go through it at their own pace.
 WMM What is nice, sometimes when you get people using something together is that you get them talking to one another.
 Any other comments you would like to make?



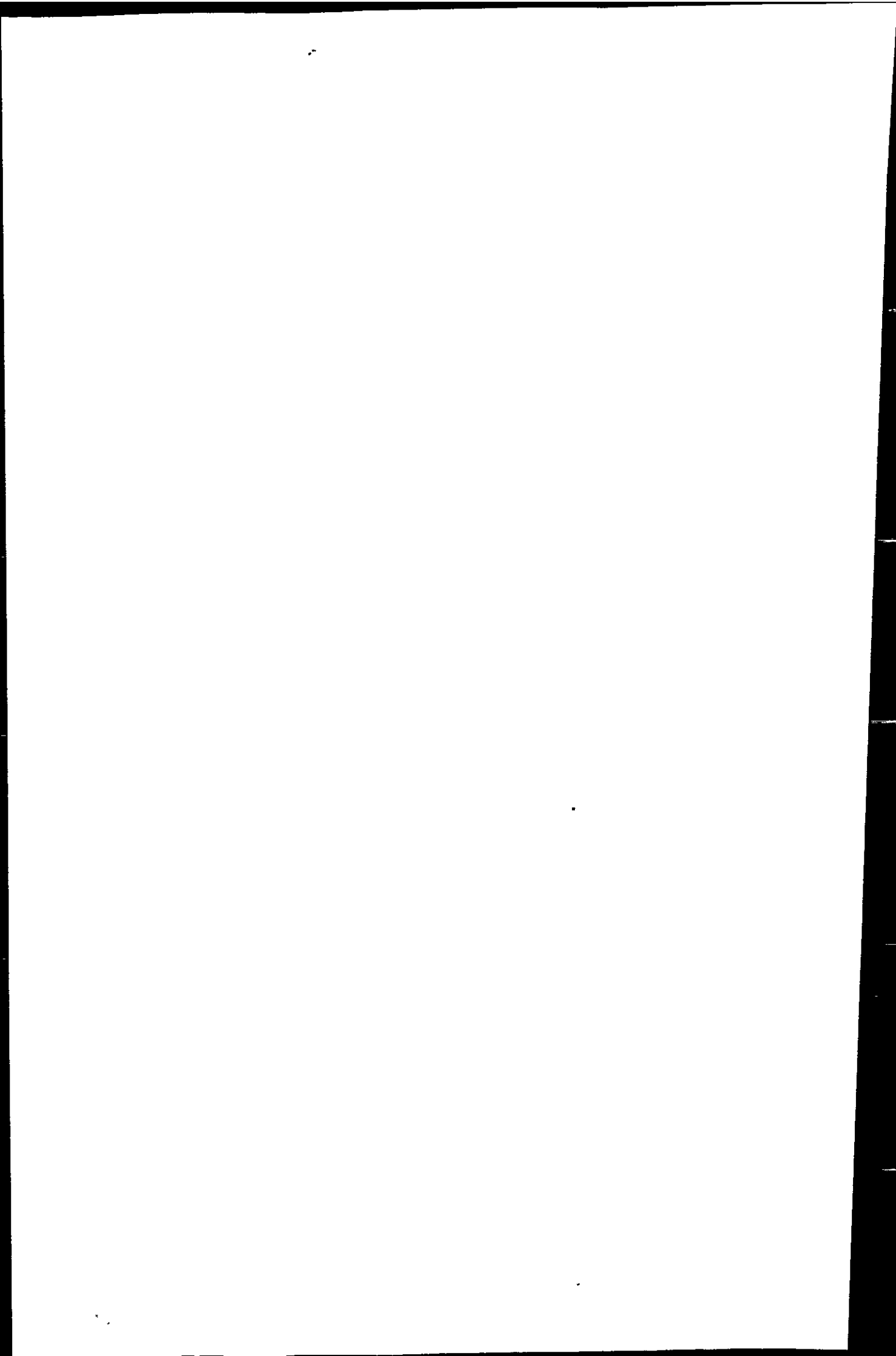
- Jed I liked particularly.. There was particularly in that section the bridge I think it was, there were suggestions for something to think about that sort of came over- further investigations you can do. I particularly liked that because often when you're being taught the lecturer's desperate to get through the subject so they don't have time to stop and talk about that. That's quite useful ... keep people thinking hopefully .
On the one about the water tank when you're comparing it with reality it suggests you compare it with example five and it goes straight into it when you click "next". I don't know if that was intended but it doesn't seem to finish it.
- WMM Yes the implication was that we'd modelled something but you can go round the cycle again in fact you can go round the cycle twice more.
- Jed Mmm, right. I like the way the text change between the active and the inactive. It keeps in focus else it ends up being just a big screen of words.
- WMM Do you feel the colours are easy or tiring or anything?
- Jed Generally I've heard yellow is supposed to be a tiring colour but I didn't find it to be so.
- WMM Anything you particularly liked?
- Jed The presentation- a nice package generally- I like it all really.
- WMM Or anything you particularly didn't like?
- Jed No not really. I didn't like having to click through so many pages on some of them when you went back a few .
It's just that some of the modules, I don't know how many pages, perhaps 10 pages, if you're in the middle it takes quite a while just to work your way back, but that's because I was skipping about a lot.
- WMM Well, that's fair enough because part of the idea is that you should be able to skip about. I'm not excessively keen on scroll bars.
- Jed No, they're difficult to apply.
I mean certainly the type of scroll bar you get on a lot of Microsoft stuff are horrible. Perhaps you see better scroll bars on the Apple, like sort of sound editing. It's got that sort of scroll bar where you move it.. (draws horizontal line).
- WMM I wonder if I could.. because it's pages rather than scrolling up an down, put tabs up each side..
- Jed You mean like a Filofax?
- WMM Yes, exactly, if that would be helpful?
- Jed Yes, good idea. Yep it would just allow you to pick out pages and turn back. Also as a reference:
- WMM Which tell you how far you are through a section, whether it's a big one or a small one, yes that's a good idea.
Is there anything you've come across in your course that you feel would make a good case study?
- Jed A good model, mmm.
Nothing that springs to mind , but I think those are good choices, the way they cross over to the water tank with the hose and the hose with losses.
- WMM Because I was a bit concerned that there were only three basic applications in there. There's cooling, the water flow and the chain and the fact that there are three modelling cycles doesn't cover the fact that there are only three applications.
- Jed Perhaps but I mean it's a broad field and they are pretty typical examples, really... applications.
- WMM Do you think it would be helpful to put in something that said something more about the particular mathematics up front in the title?
- Jed In the title, you mean like thermal consideration?
- WMM Like exponential decay, non-exponential decay.
- Jed Yep, something like that you could grab if you're looking for something in particular. Right..
- WMM Well, thank you very much.

Three final year Manufacturing Systems Engineering students (first two years in common with mechanical engineering students), at CTM in Kirkby Place. Martin is a German student with good if not perfect English. Gareth and John are mature students

After the students have used the package together we discuss it over lunch.

While using the package the students' only conversation is a spontaneous discussion of the nature of mathematical modelling.

- Gareth (inaudible) according to this the temperature will never be zero, that is room temperature. In actual real life it would at some time.
- Martin In a theoretical model we have to have some factors for the heat transfer and so on. Usually finding the formula is never really a big problem. You choose out of the seven, tick one that must be right, and then you have the problem that ..
- John Good old engineering guess (laughter)
- Martin It is the only one which has all the variables I know or any of those must be the one. But where we get all the factors from if there is no letter
- John You can look them up
- Martin Then we can find them or how to guess them.
I found that sometimes that is a big problem.
And so usually you get told by a lecturer the wall has a T of 20 or something, and then you get your..
- John Yeh, the data that's stored you can get that out of reference books .
You get all different k values and C values and you can..
- Martin When you are starting to find those it can be quite a long wait. Might be even longer than finding the right equations and solutions. You get from simple mathematics into a high mathematics problem in finding them.
- John No, you look them up.
- Martin But the conductivity of this wall [gestures] is not written down in a book. You have to..
- John But you know what the wall is made of, you know how thick it is, so you can go and look that information up in tables.
- Martin It goes quite a few steps back.
- Gareth What if the model is such a situation that we can't actually get and physical data from it? Like it hasn't been created, building a bridge?
- John Then you have to measure it.
- Gareth How?
- John Seems obvious. From what Martin was talking about, conductivity values, you can measure those.
- Gareth If it's for an unknown.
- John I mean if it's for an unknown thing, if the material hasn't been invented yet then fair enough you can't look it up but you can't measure it either, but what are you modelling on something that's not known?
- Gareth You might be doing a feasibility study or something.
- John But then you'd know the properties of the material you're looking at or you'd be looking at specific properties. You'd be working from the back end to try to identify what specific properties you're looking for from the material, wouldn't you?
- Gareth It's just that this step here where it says "compare with reality", you may not be able to do that.
I mean I'm sure that when they built that bridge in America which destroyed itself when it reached resonance.
- John It begins with a T doesn't it?
- Martin The swinging bridge.
- Gareth They couldn't compare with reality until they'd built it. You could do all your modelling- that's where I'm saying the assumptions are very..
- John The assumptions are there, aren't they. They are assumptions.
These things don't actually happen just the same as you think. The only way you can simplify it so that at this level you can solve the mathematics.



The students returned to silence and to perusing the package.

We discuss the package over lunch.

WMM Can I ask you please.. Do you have any general comments about using the package?

John What level is it aimed at?

WMM Probably second year engineering students.

John I think it should be aimed at first year because they've already... That's more of an introduction.. They've already done a lot of the stuff that's in that book before they get into the second year. It would be like, regressive, if you like. It would be better if that were introduced earlier rather than later on.

Gareth I mean even to the point where on the foundation year, you're doing differentiation at that stage and I think it's a key point to get across that this is a tool used by engineers to model situations which they are trying to overcome.

Cause you can get lost in the maths without seeing the relevance to the real world in which we're living. Whereas that's quite good with the explanations like the coffee cup and the bridge and things like that, why we actually use differentiation.

WMM Would it have helped you at any point to have had access to something like that?

John Not as a solely teaching aid. Certainly as a backup to go there when you're not sure of a point to go and go over something again because you can see it clearly on a point. You still need the lecturer interface but as a backup to that to give the student extra individual tuition, that's an excellent aid.

WMM What about the fact that it's done very much in the context of mathematical modelling? You did HITECC¹, didn't you So you have been explicitly taught mathematical modelling. How about either of the other two?

John I did HITECC too.

WMM [to Martin] So have you seen specific mathematical modelling during your engineering course?

Martin I have done the same with problems like in that program but whenever I got a bit confused, a bit lost after the introduction. There was mathematics all over the place and in the end I could not see what I have really done.

I knew about perhaps the bridge and that is how it must be but when I have done it I was lost, and this gives a bit of an overview of the mathematics. It was not the first time I saw the modelling stages and the comparison with reality.

It was a good thing. I really liked it.

WMM What did you feel about how easy it was to use and the structure of it? Was that helpful?

John Very simple to use, very clear instructions.

Gareth Is there a possibility that people are going to be looking for certain areas? Say they were given an assignment. Are they going to use that to find out how to do, so are they going to narrow the search into one area? Is that's why it's there, as an aid to helping people to understand certain problems or just overall?

WMM It was a set of examples for setting mathematics in the context of mathematical modelling.

Gareth Yeh, what I mean is what I was trying to say is, say someone was set an assignment and it was a shell being fired from a gun and its motion through the air. Are they going to look on there and not find it and go on and look somewhere else, rather than getting the feeling of why the program is there, not just to help people in specific problems but to understand a greater range of instances? (laughter) I know what I'm trying to say.

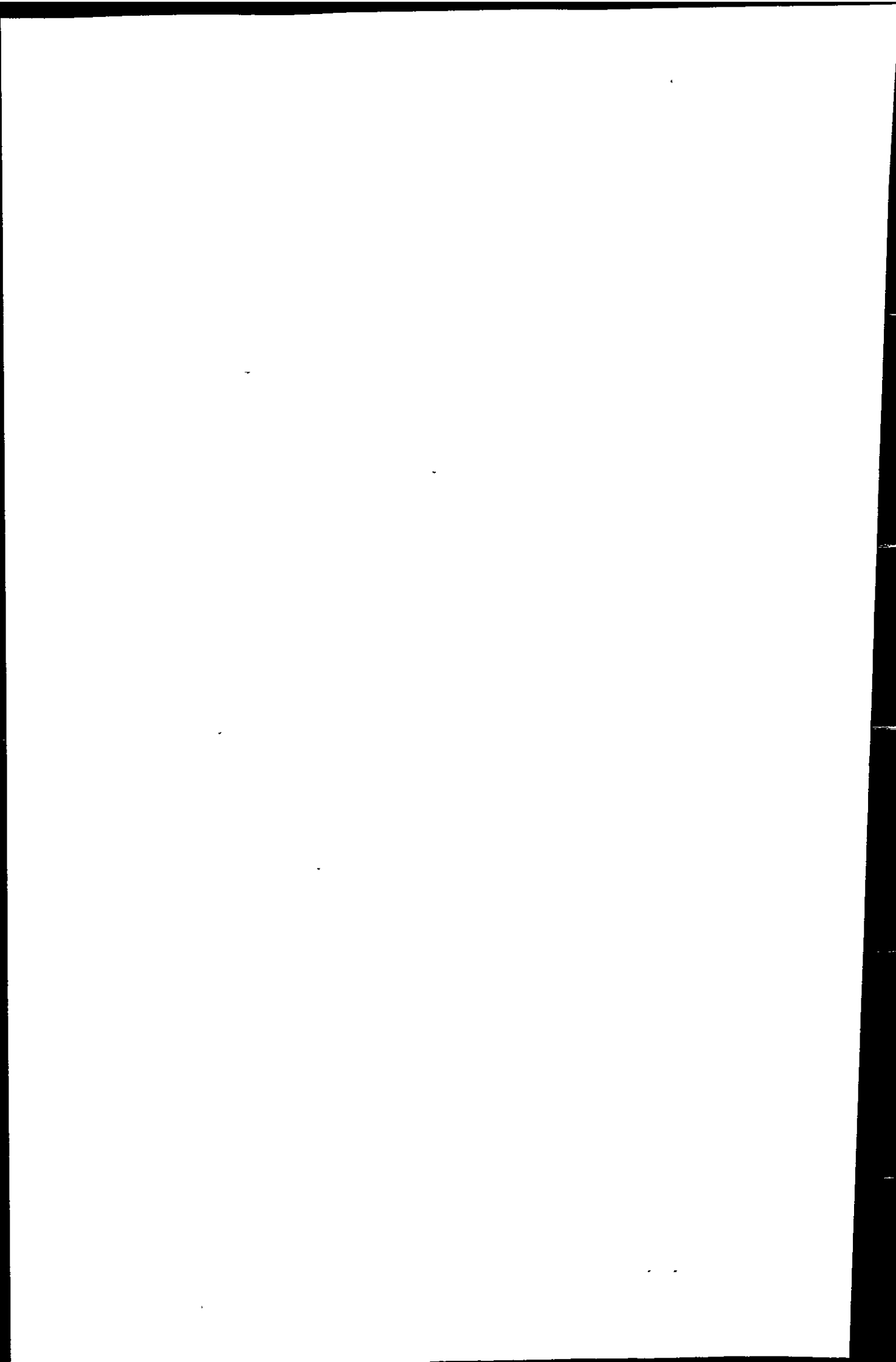
John I think what you're saying is there could be a tendency for students to leap to that to see if there is an example where the maths has already been done for them to save them work rather than them to sit down and actually work through the maths. A lot of students will say Ooh, it's on there.

Martin If you are doing usually you will not get the same example like a computer program, just for an example what could be the [inaudible] of the same problem, you are going through the examples to see what you have to do then you are following the steps with your specific...

John But you know as well as I do, Martin, that there are enough students out there who look to find exactly the example they are looking for.. Hey presto, that's my work done, solved.

Martin But it doesn't say the equation steps are explained a bit, but in an assignment you have

¹ HITECC: Higher Introductory Technology and Engineering Conversion Course, now known as the Engineering Foundation Year



- to do more I would say. You have to do it with your specific numbers, at least. It's like the statistics at the moment, you have to do the whole assignment with your numbers, but at least you have done it once even if you just copied your numbers.
- John I'm not saying you shouldn't. What I'm saying is there are students out there who will go, that's it and all I have to do is bung my numbers in and there's an answer, which isn't what it should be about.
- Martin But at least you have to follow the steps to make it your assignment with your numbers.
- John Yeh, but all they'd be doing is putting their numbers in.
- Gareth They wouldn't be getting an understanding of the maths they were doing they would just say I put my numbers in.
- John It's learning by rote.
- Gareth Rather than actually learning how to differentiate the equations.
- Martin You can do it in every subject this way.
- John What I think you could do with it is possibly is when you're doing some of the mathematics, give options for them to choose, say is it this, or this, or this, a, b or c, and you press a, wrong, press b, wrong. At least it gives them an opportunity to see if they can do the differentiation themselves².
- Gareth Make it a bit more interactive, rather than just following the steps. You may lapse into next, next, next, very interesting, what's next.
- John We were beginning to get into that mode after doing three. If you've got the interaction in there it would go somewhere to solving that.
- WMM What did you feel about it aesthetically, about the look of the program?
- Gareth I thought it was quite good, yeh, with the imported graphics, like the picture of the bridge.
- John It was nice to see something in reality, that you're actually modelling from, so that picture just sets the scene, doesn't it? You can see the bridge, see the cable, everything.
- John I don't know if you've been to the Business School and looked at any of the CAL things over there?
- WMM I haven't, no
- John Well they do.. Not only have you a similar sort of thing, it's not really any more interactive, you have the same options- there's also a couple of little tests at the end of each module to see if you've learnt what's going on. Not only that it's sound as well, so you put on a pair of headphones so you've got someone talking through the course so although you have it written down in black and white if you like on the computer screen you also have an overview sound so it's a discussion of what's going on as well. So that would be something you might consider introducing as well.
- WMM Did you find the sound helpful?
- John Sometimes, yes. It's not that it reads off all the words that are written on the screen. It's related., if you like, talks around the words that are written on the screen, rather than just saying the words that are on the screen. You can sit and listen and take notes from the words on the screen and listen to the general description of the tool as well. It takes a bit longer to go through because you tend to listen to the description then take the notes, because it's based on the same sort of thing.
- Gareth Well it's a well-known fact, isn't it that people learn in different ways. Some people are visual learners and so maybe that just having a visual learning aid may appeal to some people whereas to others who don't learn visually it won't have the same impact on them. So if you had an audio as well, you know, that way it won't narrow the focus of people which are going to learn from it.
- John You are also giving them the option of taking the earphones off. You don't have to listen to it.
- Gareth There is quite an extensive piece of work into how people learn and other things and visual is only one part of them.
- WMM Did you find using the home page easy to operate?
- John It was the same as doing it on the Internet. Might be nice if you put a bookmark in it, so you could bookmark where you are in it so if you if were disturbed in the flow of concentration, you could bookmark. You could then go back exactly to that point.
- WMM Actually if you go "home" and then you click next you should normally go back to that place where you were.
- John Right.
- WMM Which should normally act as a bookmark.

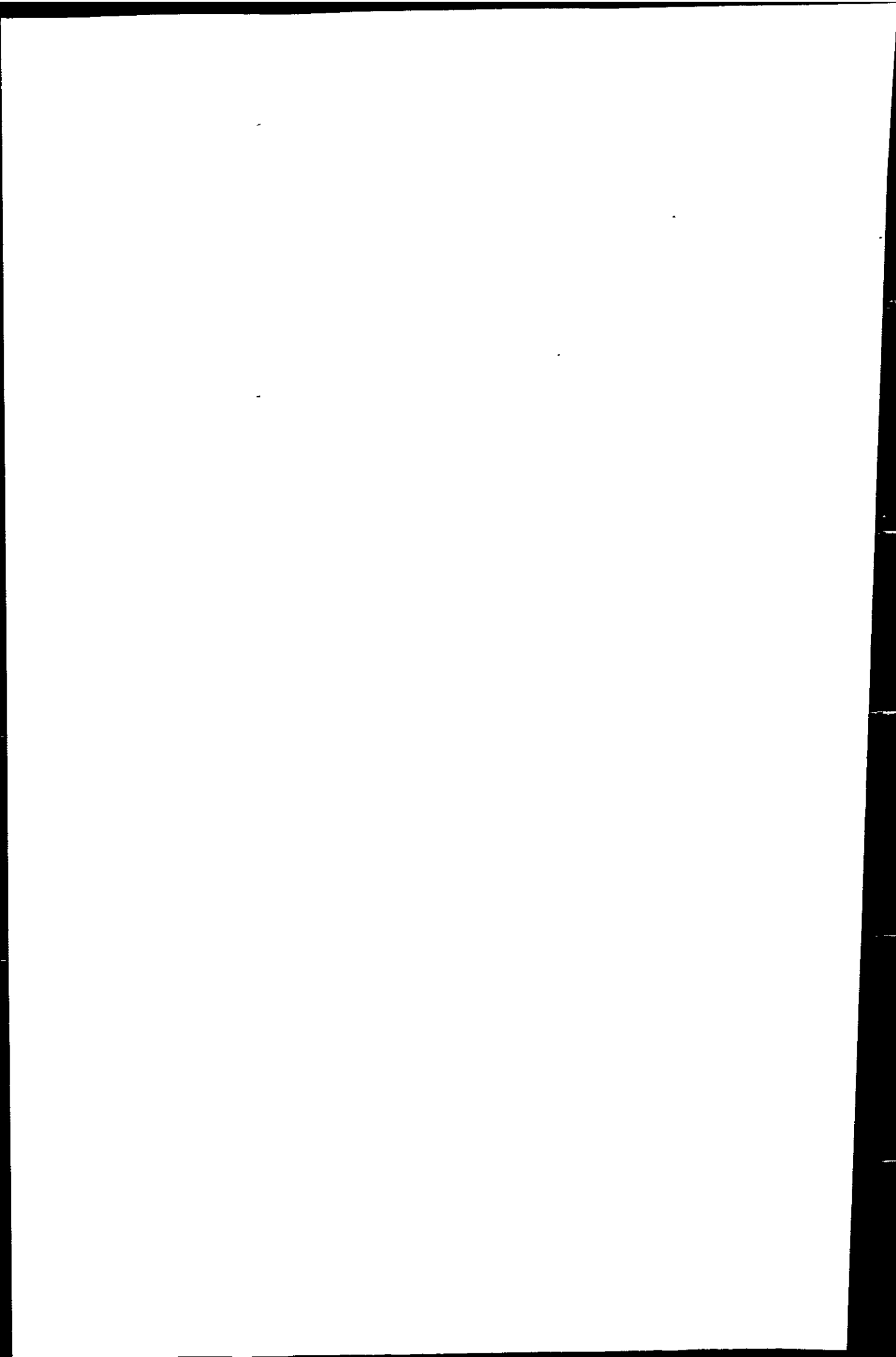
² The manipulation in the program was algebra and integration, no differentiation

- You did find a glitch.. I thought I'd got rid of all of them, but there we go..
Did you have any comments about the content? We've talked about the level of the mathematics.
- John On your report writing, you haven't said you should write anything about the theory of the mathematics behind it, which when I was taught reports we were told we should do as well. So if you were working on something like specific gravity you should do some specimen calculations on the calculation of specific gravity and then do your calculations beside it as well so you say for.... I can't think of an example.
Well, normally what you have to do is to do the theory behind the mathematics, usually, rather than just doing mathematics you have to put the theory behind that level of mathematics- where has this mathematics come from. Is it something you've invented yourself or is it something which someone else has come across, say Bernoulli's. Like the theory of Bernoulli's equation, and then you can relate in your discussion or in your mathematics where you've obtained these things from, these specimen calculations, but you haven't mentioned the need to put that theory stage in.
- Gareth He lost me there.
- WMM What did you think about the particular case studies. Were they appropriate?
- Gareth You're pitching this at mechanical engineers?
- WMM Yes
- Gareth Just trying to think what would be more apt, really. Trying to think of examples we did last year. Can I think of one?
- John Water flowing through a tank? There was a very basic case study- the tank was straight, constant cross sectional area equal all through, then you could have another one with changing volume, a changing cross-sectional area rather.
- WMM If you go on up to number.. Numbers 3, 5 and 6 are linked. 3 is just a straight tank and 5 is a first circle through putting on a tube
- John Yes, we did that one.
- WMM And 6 takes you into pipe loses, so that was how that one was developed.
- Gareth Yeh. In the second year is where you do quite a lot of thermodynamics work. although the coffee cup was there it's not really at the level of a second year degree student.
- John Yeh, I thought something about most of that.
- Gareth You're going into gas turbines and steam plants and things like that so that kind of work was done a long time ago, i.e. foundation year, first year, so if you're pitching it at second years I think that you should be looking at an example of say a gas turbine engine would be more appropriate.
- John What it would be nice to have is say a module on that early on that early on in the first year where you do your levelling up mathematics course. Stick that on at the same time as that..
Would you run that in one session, two sessions, three sessions?
- Gareth There's a lot of scope in there, isn't there, to either go bigger or smaller, if you see what I mean. To go down to the foundation level and then that's the first introduction to it and then in the first year they have a second go and in the third year - it gets progressively more in depth.
- WMM I was restricting myself to first order differential equations. In fact when you get up to the last one it actually gets pretty hairy, it starts out looking simple but..
- Gareth Is this supposed to be used as a training- as an aid, isn't it? An educational aid.
- WMM Yes
- John I think it should be introduced earlier on. The mathematics you are showing in that is covered in the first or second semester of the first year.
- Gareth Certainly if it's an educational aid I think you should have more interaction from the actual user, so rather than just reading through cause you can do that from a book.
Why have a computer program when you can just read from a book?
- WMM I agree about the amount of interactivity. The things that were there that can't be put on are the animations, the built up diagrams, the overlays..
- John Why can't you do that?
- WMM You mean in a book
- John No
- WMM That's what I've done that you can't put in a book.
What about the pace. Did you feel it went too fast, too slow, about right?
- John It went as fast as you wanted to click the button.
- Gareth Yeh, again, a little more interaction through the stages. Instead of just giving the answer, give a possible two or three answers would slow it down and then maybe that you would take a bit more of that in, would make you instead as John says you get to a

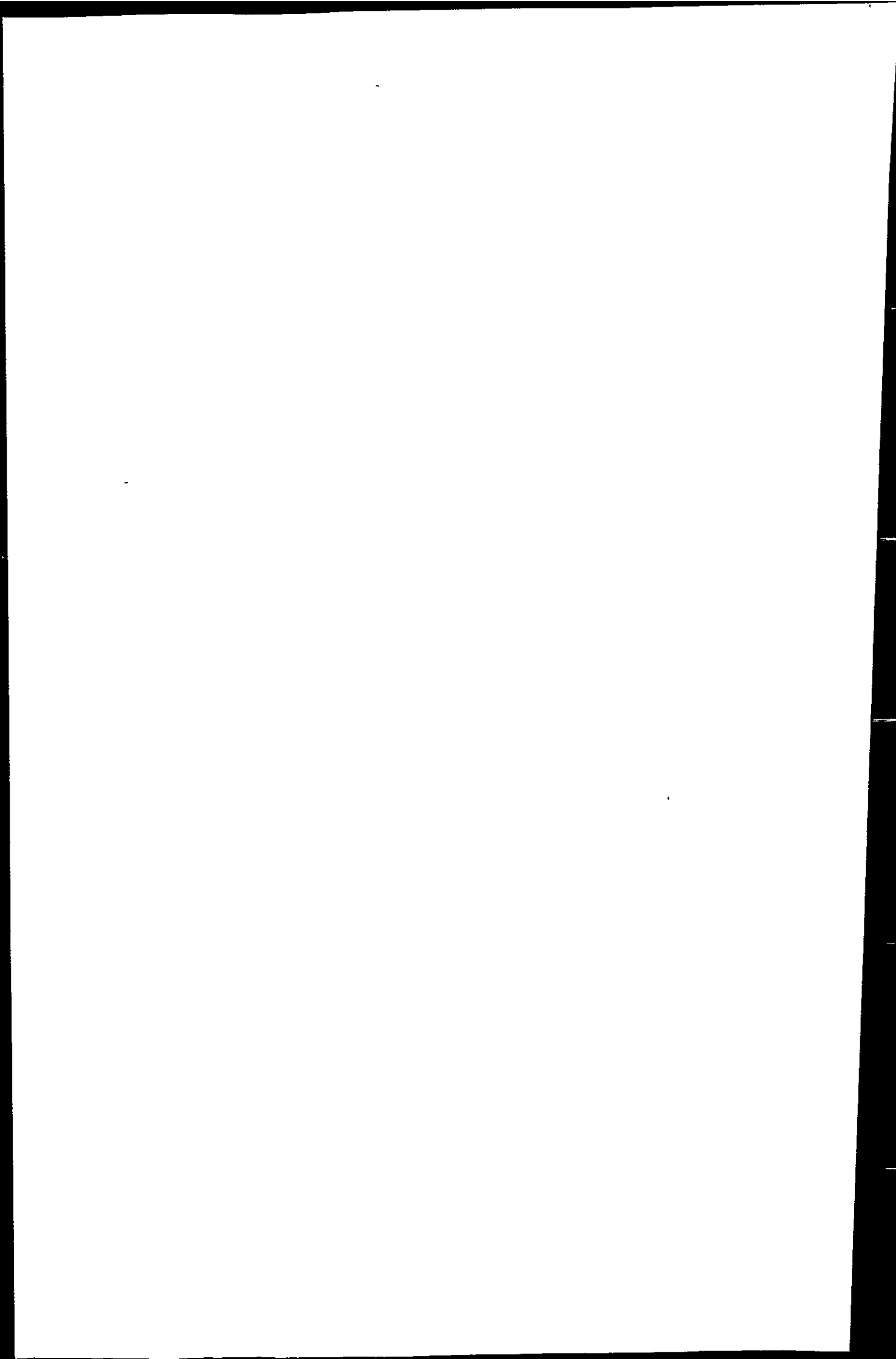
- stage where you are just clicking the button, seen that, seen that...
- John And have a little test at the end of each module to see if you've got the main concept of that module. Why not have a test, going back to this thing, it's only because I've been working on this CAL which is very similar to what you've been setting. Those test are held centrally and you can see how you've progressed through. If you do particularly badly on a test you can do it again: you can take it up to four times. I think it's based on there are 10 questions in an end of section test, an end of section, end of paragraph test if you like. It will randomly select five of those but when you finish the whole chapter you then have another test that records how you've done on fifteen out of thirty questions or ten out of thirty possible questions and so you're being tested all the time how you've done and that can identify to you "I didn't understand, I'd better go back and do that module again" which might be something you were well into.
- Gareth Yeh, and highlight to the actual users the areas they didn't really grasp the meaning of. Cause you might think "oh yeh, I got one or two questions wrong" but where you see a tally at the end and you think "oh, I didn't understand that as well as I thought I did".
- WMM Going back to the actual mathematical modelling itself which is something you did in HITECC. Is it actually explicitly covered anywhere in your degree- apart from HITECC?
- John Not explicitly, no
- Gareth No, but then at first year degree level you're supposed to have covered the basics and have a concept of what mathematical modelling is about, surely, so do you need to?
- WMM Not in most A levels
- John Depends if you do pure or applied. If you do applied you do maths modelling.
- Gareth Shows how long since I was at school, and I never did A level maths either.
- WMM Would something like that be useful in the introduction of mathematical modelling?
- John Most definitely
- Gareth Yes, to give a feeling of what mathematical modelling is, rather than.. Why there is a need to do mathematical modelling.
- WMM You were talking about how you could go and look the formula up in a book.
- John Yeh! Well we weren't actually talking about looking the formula up, we were talking about..
- Gareth Assumptions made
- John Looking up variables, so you could determine the specific capacity of whatever substance it is you were working with. Now that's something you could look up in a textbook, which is what we were getting into discussion with Martin.
- Martin I was thinking about for example the heat capacity of a wall is something I can read out of a book but how much capacity or what is the energy of the fire to the wall, how can I calculate this? So finding the data for the example to make it like reality I had a candle and a small fire, how long will it take? How much power has the candle light? The wall and everything is described somewhere in a book. Finding altogether so you can apply your formula, that's what I found always quite difficult, not setting up the equation.
- John Perhaps if you had at the end of each section, right, a reference section to show people where different information is available, where they could find some of the information for it to be useable, so they could if they had an assignment go and look to that to give them a guide. It's not going to tell them look on page 13 of book 33a in the library on the second floor, but it will say these are the terms you should be considering. That might be useful perhaps.
- WMM There were at different point different sorts of "help"- I don't know if you noticed... At one point there were things in red and if you ran the pointer over the thing in red it ... At another point there was a column on the left hand side..
- John Don't think we saw that one... (laughter)
- WMM Okay, What sort of help styles did you find useful or most useful, or...
- Martin Click on the red word and then it comes up as the easiest way, like Internet. If you have to go to a help menu on the right hand side, click down several points, keep going, it would..
- Gareth Again it may be how you learn things. If you are a visual learner if you have a show me key and you press that and then you have an option "how do I model a bullet", or something like that and then you press it and then it goes it shows you rather than having to read text and text and text sometimes.
- John Well it was the beauty of that was the explanations were very short and simple. In a lot of books. They're so verbose about what they're trying to talk about, when you analyse it, figure out what they're saying they could have said it in about four words:

- "This does not work".
- Gareth If you can't work it out, buy a bigger calculator.
- WMM So you think you would have found that of use in your first year.
- John Yep
- Gareth Yep- round at the levelling up period as well, so that everybody then had an understanding of what mathematics modelling- so far as the engineering world was about.
- WMM So again before you started tackling things like turbine engines.
- Gareth Yes, definitely
- John Should be done in the first year, beginning of the first year, or as soon as the mechanical side of it and the mathematical side of it is .. No we did integration and differentiation first level at foundation, didn't we? So you should be au fait with that.
- Yeh, right at the beginning.
- WMM You did some differentiation.. some differential equations in HITECC, didn't you?
- John Yeh, we had to do some mathematical modelling. I had to do one with a chain that was being dragged along the ground for a project in maths. Oh look- hanging chain- how useful that would have been.
- Gareth I seem to remember somebody not far away actually, who took me for a few lessons.
- MMM There's a theory in mathematical modelling that either the mathematics or the application should be at least a year old.
- Gareth Run that one over again.
- WMM There is a theory that either the mathematics or the application should be at least a year old . So you shouldn't have new mathematics and a new application.
- Martin This is useful
- WMM So either you are familiar with the mathematics or with the application.
- John That's all very well and good but in the first year you're having to do some mathematical modelling in mechanics. That would be ideal if you were an A level student and you'd done calculus at a level , if you were a foundation student, you'd done calculus on HITECC so you'd done calculus that is a year old.
- So your mathematics is a year old so that could be brought forward to the first year and done at the beginning at the first year to give people a chance of going through, rather than getting hung up trying to do...
- And that's all you... The only mathematics involved there is calculus.
- I don't know about you guys but when I was trying to set up and solve my first differential equations I was getting some mammoth things like I mean totally out of proportion because you weren't sure what it was you were actually looking for. Something like that would have identified the key points to point the way to what you were looking for and it would have been a help but the mathematics would have been a year old.
- Definitely, definitely.
- Gareth Have a word with Mansel³. (laughter)
- Have you had much feedback from lecturers and things?
- WMM I haven't. You would be the second group of people who've tried it out.
- John Are you going to try it out on lecturers?
- Gareth That would be interesting I think.
- John You may get a totally different perspective from the other side of the fence so to speak.
- WMM The lecturer side is very much more guarded.
- John Students tend to want to find the easiest way round things, where lecturers want to make it look as complicated as possible. (laughter) My perception anyway.
- Gareth Well, we're not going to get into that one, I don't think.
- WMM I did a questionnaire which I gave out to students and to lecturers at various stages, and I got back the questionnaires from students with a reasonable rate of return but with the lecturers it was.. They didn't want to... I don't know.. The lecturers were very much more guarded about returning the questionnaires.
- John I've been (inaudible) for my project. The company are also doing questionnaires for Investors in People. talking to the personnel manager up there, she said she only got 20% reply rate from the questionnaire she sent out, and she said this was standard. When I was doing my sample interviews, I was asking all the operatives, had they filled in these questionnaires, purely for my own benefit. About 95% said they had. However about 92% said they still had them in their bags and they couldn't be bothered

³ The mathematics lecturer who teaches the first year mechanical engineering students.



- to hand them in.
- WMM The way I got the best return was going to a class and saying "Please fill these in now", but what I didn't get back then was some of the very percipient comments I got back when people held on to them a while.
- But one of the things that seemed to come out was that I gave various options of "what do you think differentiation is- what do you think differentiation tells you?" and maths and engineering students when they came in at the beginning of their first year said "it's the slope of the tangent".
- John I think we had this. I think we were one of your people
- Gareth Yeh, absolutely
- WMM At the end of the first year it was still the slope of the tangent. Mathematicians in their final year it was still the slope of the tangent.
- Engineers in their final year tend much more and then onto TCAs, Teaching Company Associates it was very much more "it tells you how something is changing". So there had been a sort of changeover in understanding. But I put in plenty of diagrams, and in some cases the diagrams were well chosen, people chose them a lot, and in other cases they went for verbal explanations which is very odd, because a researcher called Kim Crowther went and interviewed about 85 prospective engineering students and they said they predominantly saw themselves as visual people. So there is a difference between how people see themselves and perhaps the way they answer things when they are not asked explicitly.
- Gareth I know I have very much a visual brain and only recently have I found that out but just by changing the way that you actually learn about something can increase how much you understand about it.
- John I learn by writing things down. I can write everything down in my own words. In my notes you have to translate.. But then you know that because you had all my notes. I can write them up in my own words.
- Gareth I know if I'm studying for an exam I'll just get one big sheet of paper and put like what it is in the middle and then just draw it an all around do various shapes, like somebody's goals then I'll draw a set of goals and then when I try to remember it...
- It was actually my girlfriend who's a teacher pointed this out and she showed me how to learn in this kind of way. She said she can see my eyes wandering around the sheet of paper as she asks me questions. I mean the piece of paper is not there, I'm just seeing it in my mind. she says it's quite amazing how people learn differently and then once you learn how you learn you can go on and learn even more, build on that.
- WMM There's a slight danger in locking too firmly into one way of thinking in that you get growth by challenging the way you tend to think and things where it makes your head hurt, are really causing you to grow and to come onto a different level, rather than just.. So there is a sort of dilemma whether you concentrate on teaching people the way which they find it easiest to learn or whether you sometimes confront people and move them out of that particular pattern, in order to force them onto another level.
- Gareth You may also force them away though
- WMM You have to do it deliberately.
- John We had a lecturer at college and he walked in one day, said "Hmm, assignment time". He threw a con rod down on the table, said "analyse that" and walked out.
- What could we do with that?
- This had thrown us because we hadn't been told what we had to do, we were just told "analyse that". A lot of people really took offence about it. I was student rep and it was the end of the course. They bitterly complained about this. It was absolutely... I'm not making any complaints about that because it wasn't about that...
- But he was effective in what he did because he mad you think. He wasn't... He didn't tell you how to do something. You had to figure it out what you were after. Then he came back the next day and started determining the centre of gravity, things like that.
- But you start.. "oh, what do I do here?"
- We have to shoot now because we have a lecture.



TEACHING ENGINEERING MATHEMATICS THROUGH COMPUTER TECHNOLOGY.

Wendy Maull, John Berry and Colin Stockel

University of Plymouth, Plymouth PL4 8AA, UK: Fax (0)1752 232772

ABSTRACT: In a recent report to the Engineering Council of the United Kingdom, "The Mathematical Background of Engineering Undergraduate Students" the authors point out that engineering students perceive that mathematics is difficult and irrelevant. This is a major barrier to any mathematical learning they may need, and it is suggested that it is a cause of reducing the mathematical content of some engineering degrees.

INTRODUCTION

The perception that engineering students have of mathematics as a difficult and irrelevant subject is not new and much has been done over recent years to alleviate the problem. The approach we propose in this paper is two-pronged;

- the relevance by setting mathematics within a modelling context, which situates it firmly in reality, and
- the difficulty issue is addressed by the use of computer algebra in a supportive environment.

The importance of showing the relevance of mathematics within realistic engineering applications may seem obvious since the engineer does not use mathematics in isolation of applications. However it is surprising that in some courses mathematics is still taught without showing its relevance¹. Mustoe² points out that useful and relevant applications can be shown, which is motivating and gives a context for situated cognition. Two kinds of scaffolding are thus available to the students: the mathematical scaffolding provided by the algebra package, and the familiarity of the setting.

In addition, we need to be aware that some engineering students view and do mathematics in a slightly different way from students of mathematics. They have

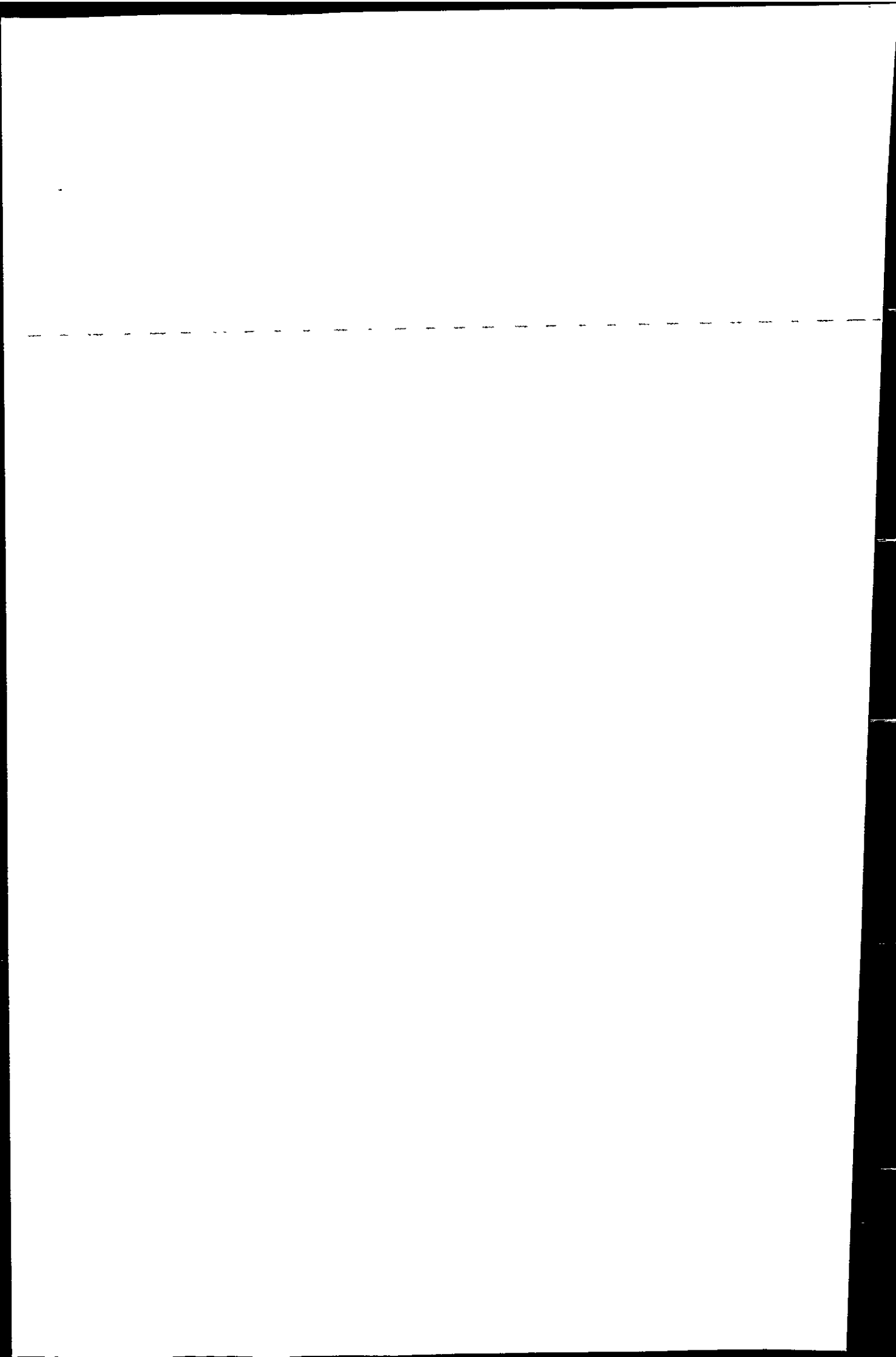
- possibly different concept images,
- different symbol sets,
- different attitudes to graphs.

Furthermore there is a dichotomy between the 'ball-park/back of an envelope type' calculations and precision calculations, both needed and used by engineers.

THE CURRICULUM

An important question when designing the engineering curriculum is 'What mathematics do engineering students really need?' Most of the work that has been done in this field³ covers the scope of the mathematical skills and techniques which may or may not be exercised in the pursuit of engineering studies and career. Mathematics, however, is more than a collection of named skills and techniques.

Dreyfus⁴ argues that advanced mathematical thinking includes the ability to switch fluently between modes of representation. This means switching between graphical, parametric, standard form, etc. This is what we expect of mathematicians. It may be that teachers who are mathematicians have had so much practice in this switching that they forget that it was



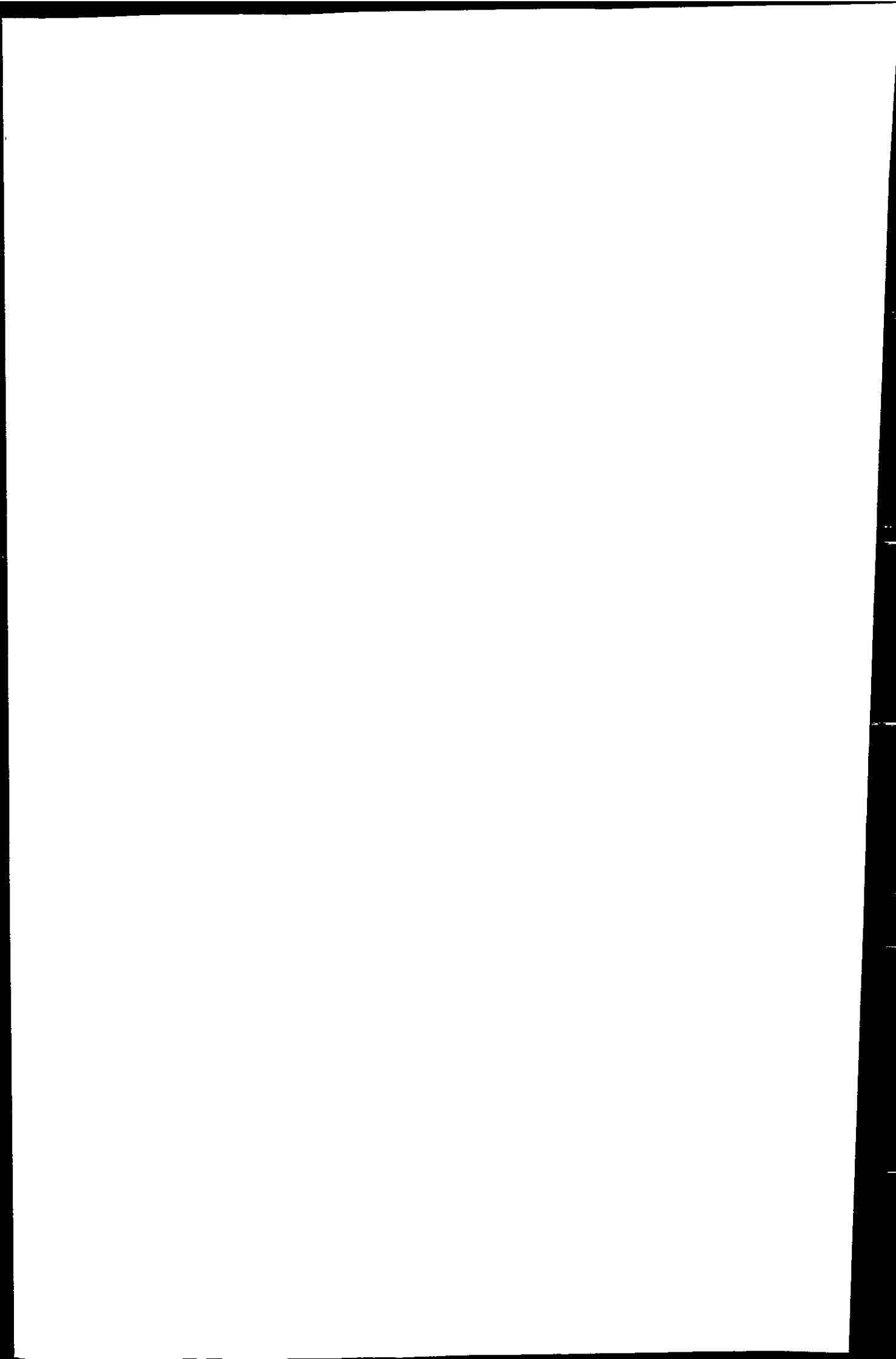
an acquired skill. Do we really need to train non-mathematicians in this sort of skill? Or do we need to recognise they do not yet have it and do not need to be taught it? The implication is that it would be useful to determine the form in which these mathematical concepts (concept images) are held by engineering students in order to access them most effectively. As part of our research a series of questionnaires is being designed in an attempt to induce students and staff to tell us how they best represent physical situations. It sometimes surprises people to discover that the way they think of situations is not the only way but sometimes not even the most common way. This relates to notions of cognitive learning theory such as connecting new knowledge to existing knowledge. We do not always know what we do know, if it is not accessed properly through our own particular index system.

For instance, a particular example is that a graphical representation of a physical situation is not thought of as being mathematically rigorous, and so may be scorned by pure mathematicians, but may nevertheless prove to be a highly effective way of communicating mathematics to engineering students. This is shown by the way engineering students sketch graphs to show their expected solutions, and use these sketches to communicate with one another. Visualisation is not a skill which is much encouraged in mathematics students⁵.

TECHNOLOGY IN LEARNING

Computers have been used as teaching aids for mathematics for engineers in a number of ways. These include

- the computer as a dumb tutor in a drill and practice session, providing a stream of questions and responding right or wrong as appropriate. The student is fed examples until he/she demonstrates an ability to perform which satisfies a pre-set criterion. This is often seen in the context of a mastery learning didactic paradigm⁶.
- intelligent tutoring where the program attempts to diagnose the particular misconceptions held by the student and difficulties held by the student according to his/her responses to mathematical questions. This is still in its infancy⁷.
- programming, where the student designs and writes programs to solve particular classes of mathematical problem. The argument is that the student thereby develops a deeper understanding of the process involved by analysing it logically and reproducing it in terms of code⁸.
- the application of in-house produced software which solve particular classes of mathematical problems, to enable students to check their solutions against the computer's, and facilitate self-marking of exercises⁹.
- the use of commercially available software to scaffold the student's exploration of mathematics by showing the solution to problems the student is not yet able to tackle by hand and demonstrate their relationship to mathematics already known¹⁰.
- the use of software to perform tedious calculations so that results can quickly be obtained and generalised. For example, the plotting of a family of curves to explore the effect of varying parameters¹¹.
- the use of spreadsheets to perform iterative calculations and to find approximate numerical solutions in a manner transparent to the student¹².



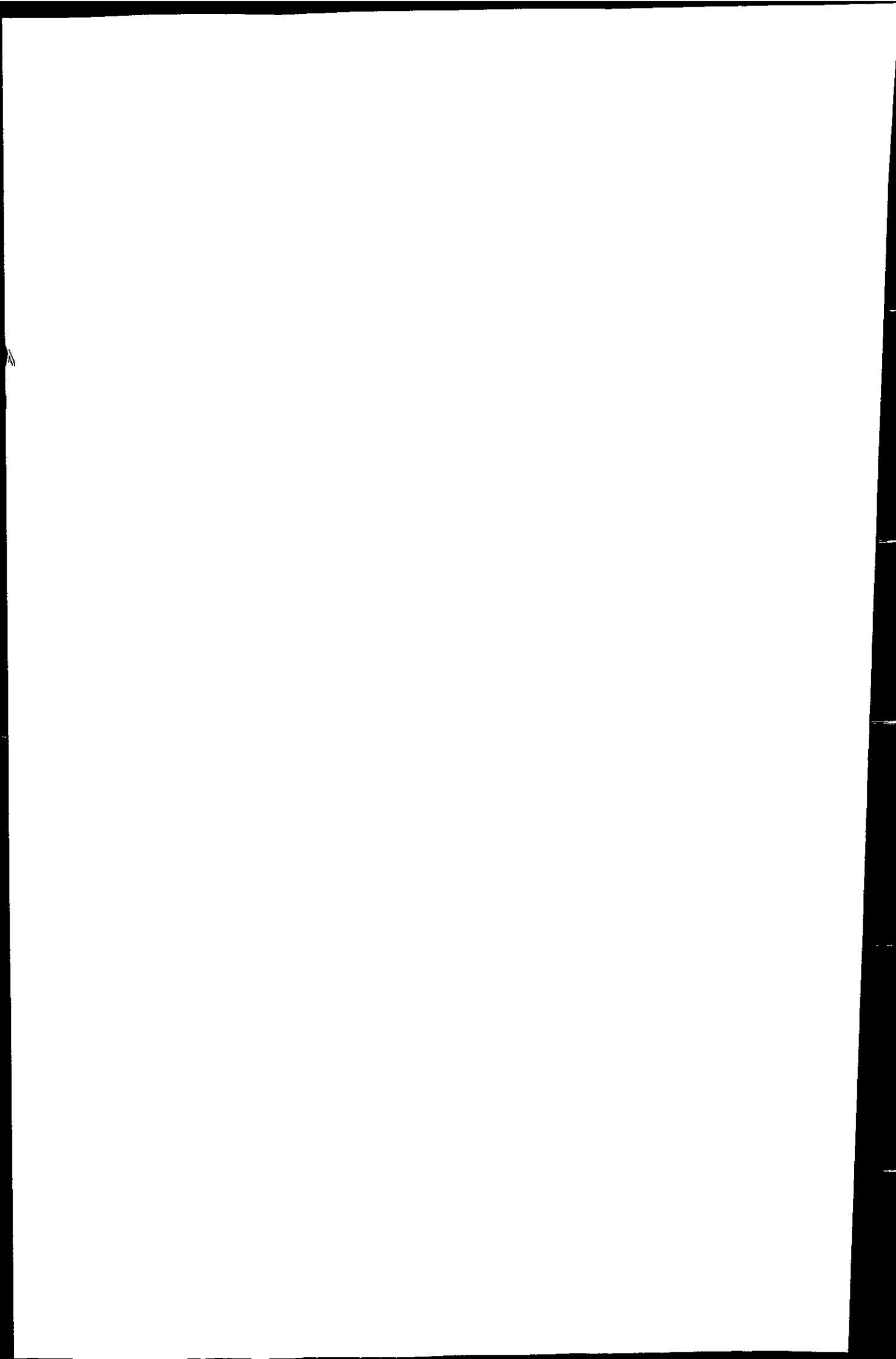
- microworlds and simulations where hypotheses can be explored and tested by the student in a mutually safe (unthreatening and unbreakable) environment¹³;
- use of a simulation language such as STELLA to allow students to create their own simulations. The assumptions made in the simulation are made explicit, and the students can check the behaviour of their models against their experience of reality.
- use of the computer to drive recorded teaching material, for example on interactive video or CD.
- hypertext or hypermedia environments to be explored by the student in pursuit of information.
- live production of audio-visual material in the course of and in support of lectures¹⁴.

The philosophies and models of learning employed in these different techniques are disparate, and the simple existence of computer use says nothing about the way the subject or the material are assumed to be seen by the student.

The authors will discuss appropriate applications of some of these learning aids in teaching mathematics to engineering students.

References.

- ¹ Sutherland R & Pozzi S. The changing mathematical background of undergraduate engineers. London: The Engineering Council, 1995
- ² Mustoe L R. Modelling or muddling through? Mathematics for engineering undergraduates. *IntJMathEducSciTechnol* 1992; 23: 575-582
- ³ Barry M D J & Steel N C. A core curriculum in mathematics for the European engineer: an overview. *IntJMathEducSciTechnol* 1993; 24: 223-229
- ⁴ Dreyfus T. Advanced mathematical thinking processes. In: Tall D, ed, *Advanced Mathematical Thinking*. Dordrecht: Kluwer Academic Publishers, 1991; 25-41.
- ⁵ Presmeg N C. Visualisation and mathematical giftedness., *Educational Studies in Mathematics* 1986; 17: 297-311
- ⁶ Rae A. Self-paced learning with video for undergraduates: a multimedia Keller plan. *British Journal of Educational Technology* 1993; 24: 43-51
- ⁷ Khasawneh A A. An intelligent CAI project for teaching mathematical concepts. *IntJMathEducSciTechnol* 1994; 25; 3, 325-332
- ⁸ Adams J L & Stephens L J. A comparison of computer-assisted instructional methods. *IntJMathEducSciTechnol* 1991; 22: 889-893
- ⁹ Coull J & Simmonds D G. Computer-assisted algebraic differential equation solution. *IntJMathEducSciTechnol* 1988; 19: 551-559
- ¹⁰ Kutzler B. DERIVE - the future of teaching mathematics. *International DERIVE Journal* 1994; 1: 37-48
- ¹¹ Watkins A J P. A new approach to mathematics for engineers. *IntJMathEducSciTechnol* 1993; 24: 689-702
- ¹² Fraser C J & Thorpe J R. The use of spreadsheets for modelling and analysis applications in mechanical engineering. Sheffield: Proc Conf on CAL in Engineering, 1994; 167-182
- ¹³ DeCorte E. Technology-supported learning in a stage of transition: the case of mathematics. *IFIP Transactions A-Computer science and technology* 1994; 46: 113-123
- ¹⁴ Marshman B, Ponzio P. Innovations with interactive graphics in teaching and learning about applied differential equations. Calgary: ProcConf on Computer Aided Learning in Post-Secondary Education, 1987; 311-313



Modelling the Mathematical Ideas of Engineering Students

W.M. Maull and J.S. Berry

Centre for Teaching Mathematics, University of Plymouth, Devon

Abstract: A questionnaire designed to elicit from undergraduate engineers the way they thought about selected concepts in mechanics and calculus (their concept images) revealed some strongly held misconceptions about simple bending. Possible origins are described and some implications are discussed.

1. Introduction

Observation of final year mathematics and engineering students working on a mathematical modelling problem revealed differences in the way the problem was approached by the two groups. It appeared that not only did they have different levels of mathematical modelling skills, but that they understood and represented key mathematical concepts, particularly in the area of calculus, to themselves in different ways.

A questionnaire was developed to elicit from students their mathematical concept images, in calculus and in some mechanics applications of calculus. After a pilot study, the questionnaire was applied to mathematics and mechanical engineering students at the beginning and end of their first year of study, to mathematics and mechanical engineering students in their final year of study, to students preparing for an MSc in Management of Technology, and to practising engineers.

The results of the project suggest that the ways that students hold such concepts are related to their mathematical skills, and that they do change and develop over the course of a degree programme. It is also suggested that engineering students should explicitly be taught mathematical modelling skills using applications where the mathematical content is familiar from an early stage in their degree course so that:

- the skills which they will need are well established by the time the students enter engineering practice;
- mathematics applications are seen and understood by the students as relevant to engineering studies;
- mathematics concepts are developed by engineering students in ways that will be useful to them.

In this paper we propose to explore one of the issues which has emerged from the responses to the questionnaire.

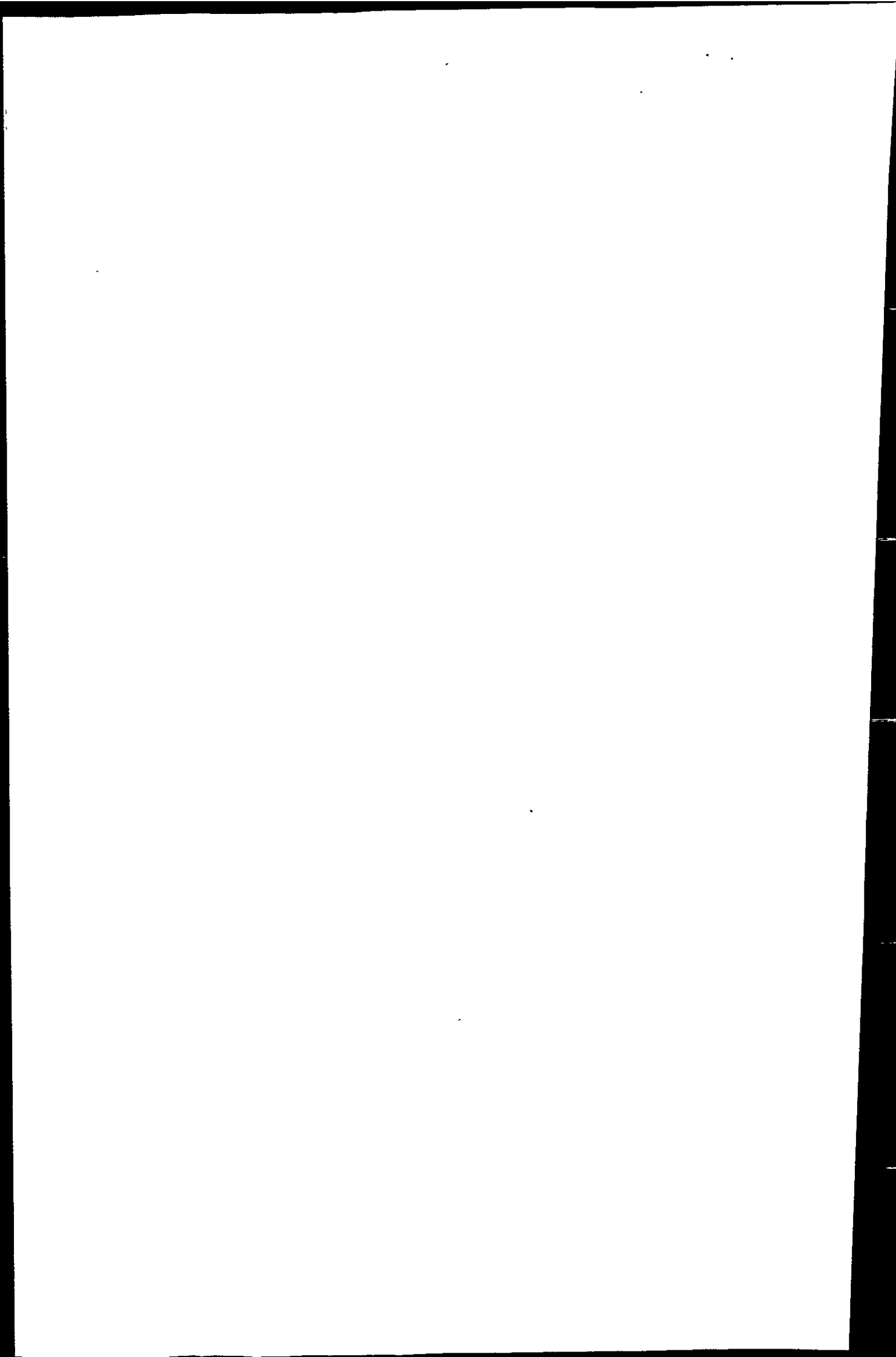
2. Concept images

The major premise of this research is that people hold concepts in different ways: they have different *concept images* [1] attached to the same concepts. These concept images may differ in *mode*, in *sophistication*, and in *content*.

These concept images may be revealed when people make statements which do not coincide with the predictions of the accepted or institutional meaning of the concept.

2.1 Mode of representation

It was suggested that engineering students may tend to think more pictorially than mathematics students, and so in the questionnaire the options proposed included diagrams, verbal descriptions and algebraic expressions, some of which expressed equivalent representations, to test whether this was indeed so. The implications for



teaching would be that:

- for easy teaching and learning one should appeal to the students' existing representations of concepts,
- when it is necessary to make students change or extend their concept images, time and effort must be allowed for the process to take place.

The idea of cognitive style is a construct of learning theory. It was first proposed by Galton [2] that individuals vary in their tendency to use visual or verbal representations of reality and of problems. This has led to such instruments as the Verbalizer-Visualizer Questionnaire (VVQ)[3] which seek to analyse an individual's preferences and to generalise from those preferences to make statements about the way that person relates to the world around. The whole notion of cognitive style has been criticised by various authors [4,5] but it seems that from earliest times philosophers have held different views as to whether one can think with or without mental pictures [6]. Mathematicians, too, differ in the way they construe problems before the final solution is presented to the world. (Poincaré, cited in [7])

2.2 Depth of representation

The second aspect we wished to explore was whether students' concept images became more sophisticated as they proceeded through their courses. It is suggested by Royer et al [8] that as a cognitive skill is acquired by a learner, the depth of problem representation increases. Novices attend to surface features whereas experts tend to identify inferences and principles that subsume the surface features. The implication is that experts represent information as chunks by labelling (for example) a chess position in terms of familiar games or positions. This takes place after the cognitive skill is acquired at the level where students can explain what they are doing or answer examination questions on a topic.

A partially understood concept cannot be used as fully as a deeply understood one. At a naïve level, concepts tend to be simple, isolated, and fragmented. As mastery of a subject is developed, the concepts become richer, and more highly linked. For example, the concept image of the derivative as the gradient of the tangent does not easily adapt to the derivative as a term in a differential equation, the derivative of a function as a function in its own right, the derivative telling one how something is changing. A rich image is full of possible meanings.

In particular, advanced mathematical thinking requires the ability to switch between representations [9,10] in order to work both intuitively and deductively on a problem. The implication for teaching is that students at an early stage will not be able to recognise the underlying structure of problems, but through practice their expertise will develop, and this continues to happen after they have acquired the basis of the skill.

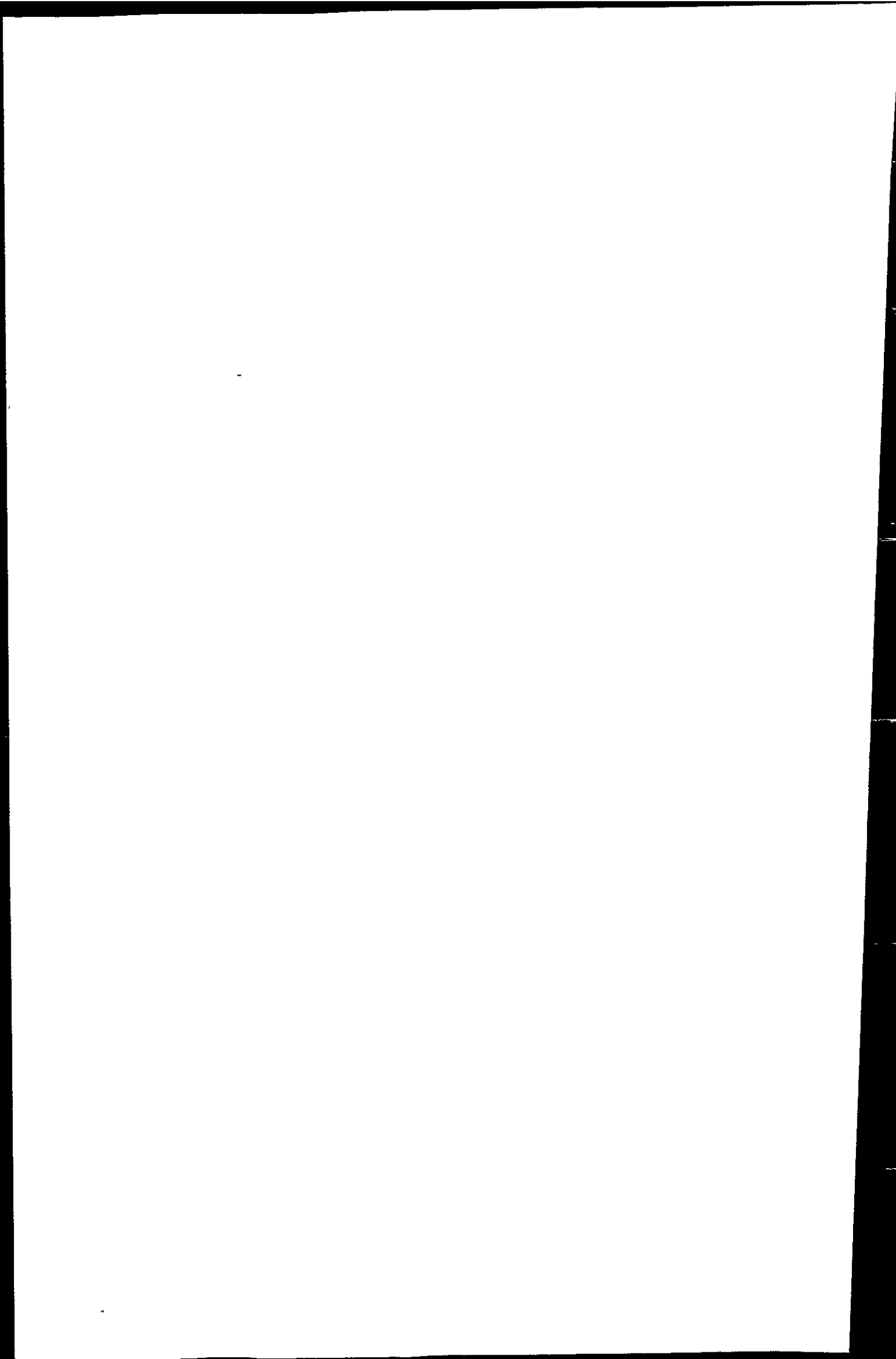
2.3 Content of representation

The content of a concept may vary both in quality and quantity. A concept may be susceptible to improvement if it is

- correct as far as it goes but incomplete;
- partly correct and partly in error;
- framed in non-technical terms.

According to Vinner, a concept is acquired when a concept image is formed.

Engineering and mathematics are not democratic subjects, where each person's concept image is as valid as any other. Using a slightly different framework[11], a concept is understood when an individual's private meaning matches the institutional meaning.



Students need to develop the correct contents to their concepts, and we need to know their current position in order to move them to where we would like them to be. The questionnaire was not, however, designed with revealing incorrect images in mind.

3. The bending of beams.

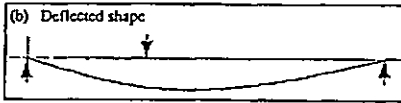
Simple bending is part of the basic tool kit of the engineer. The equation $\frac{f}{y} = \frac{M}{I} = \frac{E}{r}$ has been engraved on many hearts (with its more or less rude mnemonics), and the topic is part of the first year engineering syllabus, so a question on simple bending was included in the questionnaire to detect the effects of teaching on responses of engineering students. (Figure 1)

For the mathematics student, the bending of beams is sometimes used as an application of end conditions in the solution of differential equations, and the analysis underlies the whole concept of the spline curve. This made the question relevant to mathematics students in a slightly different way. In any case, it should not have come as a shock to any but those coming in at the start of their first year.

A plank 1.5 m long is placed on two bricks very near its ends. A bar of gold is placed across it 0.5m from one end. Rank the following according to how well they represent this to you.

(a) The beam bends under the weight of the gold bar.

(b) Deflected shape



(c) Bending Moment $M = k \frac{d^2y}{dx^2}$

(d) A simply supported beam with a point load at one-third span.

(e) Shear Force $S = \int F dx$
Bending moment $M = \int S dx$

(f)

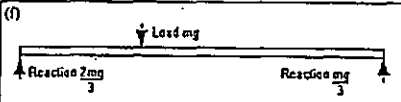


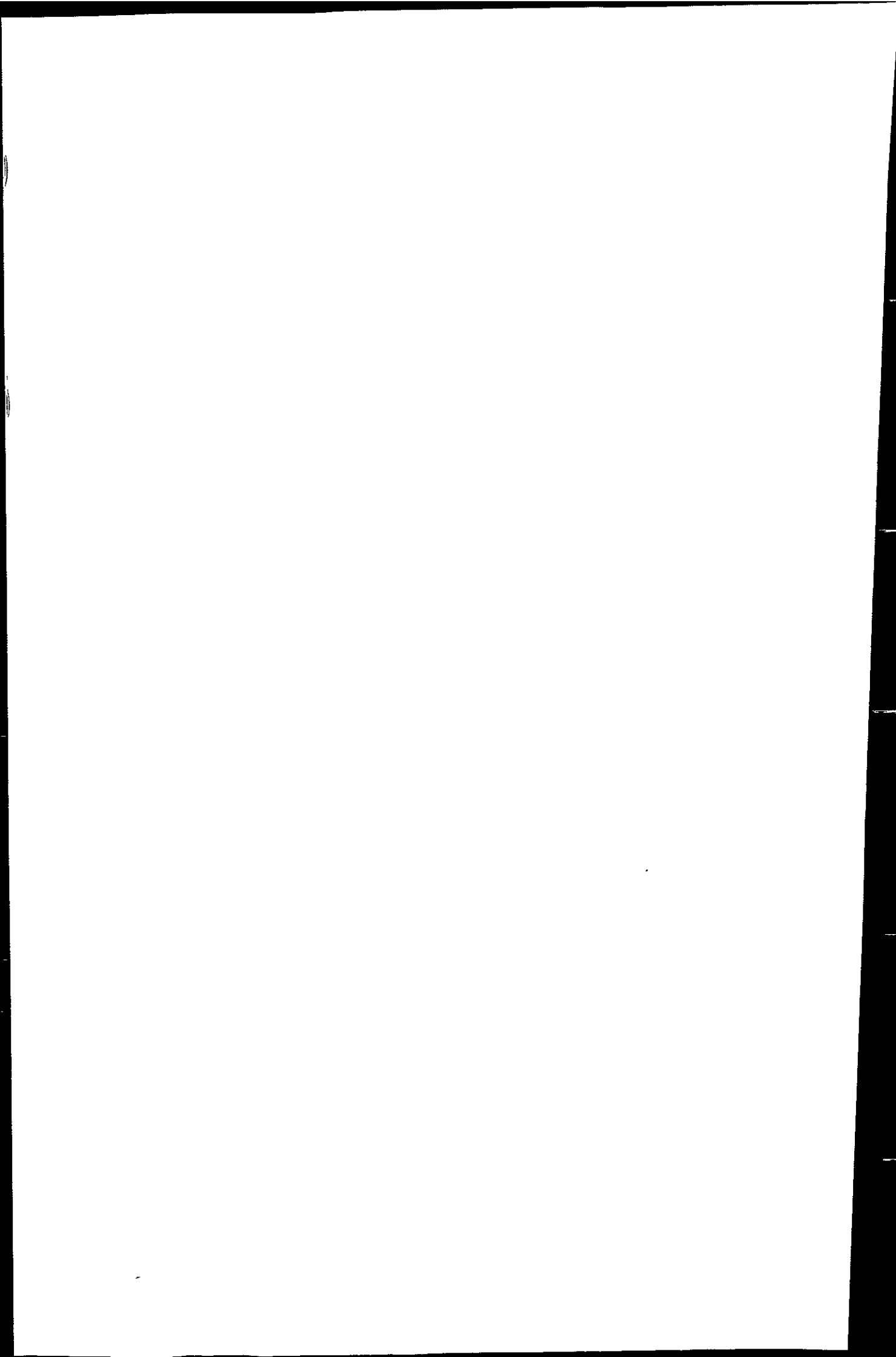
Figure 1. Beam bending question

4. Strong responses

It was not expected that any of the questions would arouse particularly strong feelings in respondents. The rubric to the questionnaire explicitly stated that there were no trick questions, but some respondents still objected so strongly to two of the given choices in the question on bending that they wrote comments about them.

4.1 The beam will not bend at all, or whether it bends depends on its thickness

Option (a) stated "The beam bends under the weight of the gold bar". This was included particularly so that respondents who had never seen an analysis of the case would not feel that the questionnaire was dealing with matters above their heads, and it was thought it would be popular with first year students at the start of their first year. Some respondents made comments such as:



- 1) "Nobody says it actually bends, so automatically assume rigidity." (Final year maths student)
- 2) "Not(a)- cos depends on thickness of plank" (First year mathematics student, start of first year)
- 3) "It depends on how thick the plank is (a)." (First year mechanical engineering student, start of first year)
- 4) "How thick is the plank? How heavy is the bar of gold?" (Second year computer systems engineer, pilot study)
- 5) "I feel a bit uncomfortable not knowing the weight of the gold or the thickness & width of the plank." (Manufacturing engineering lecturer)
- 6) "'a' may not be very valid- The deflection may be so small as to be negligible." (Practising professional engineer)

There is a graduation from assuming absolute rigidity to wondering whether the assumption is valid under the circumstances.

4.2 The point of greatest deflection must be under the load

Option (b) was a diagram of the deflected shape,

- 1) "(b) looks like the bar would be in the middle." (engineering student, start of first year)
- 2) "(b) is wrong" (final year mechanical engineering student)
- 3) "I would rather have a drawing but (b) looks wrong" (final year mechanical engineering student)
- 4) "not keen on (a) (too simplistic) and (b) (wrong?)" (maths student, start of first year)
- 5) "b is useless!" (maths student, end of first year)
- 6) "b isn't quite right, but I've assumed poetic license with the artist!" (practising engineer, graduated 1979)
- 7) "b (slightly changed)" (see Figure 2) (experienced maths and mechanics teacher)
- 8) "I don't recognise any of the equations and (b) doesn't look quite like what I'd expect!" (postgraduate, degree in Business Administration, A level maths)

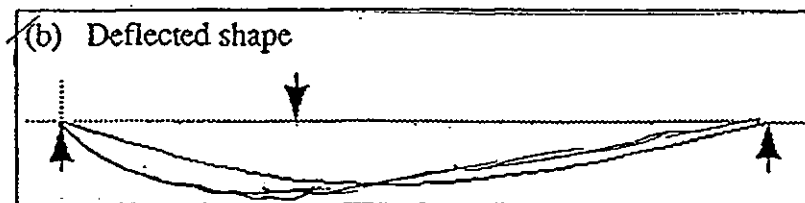
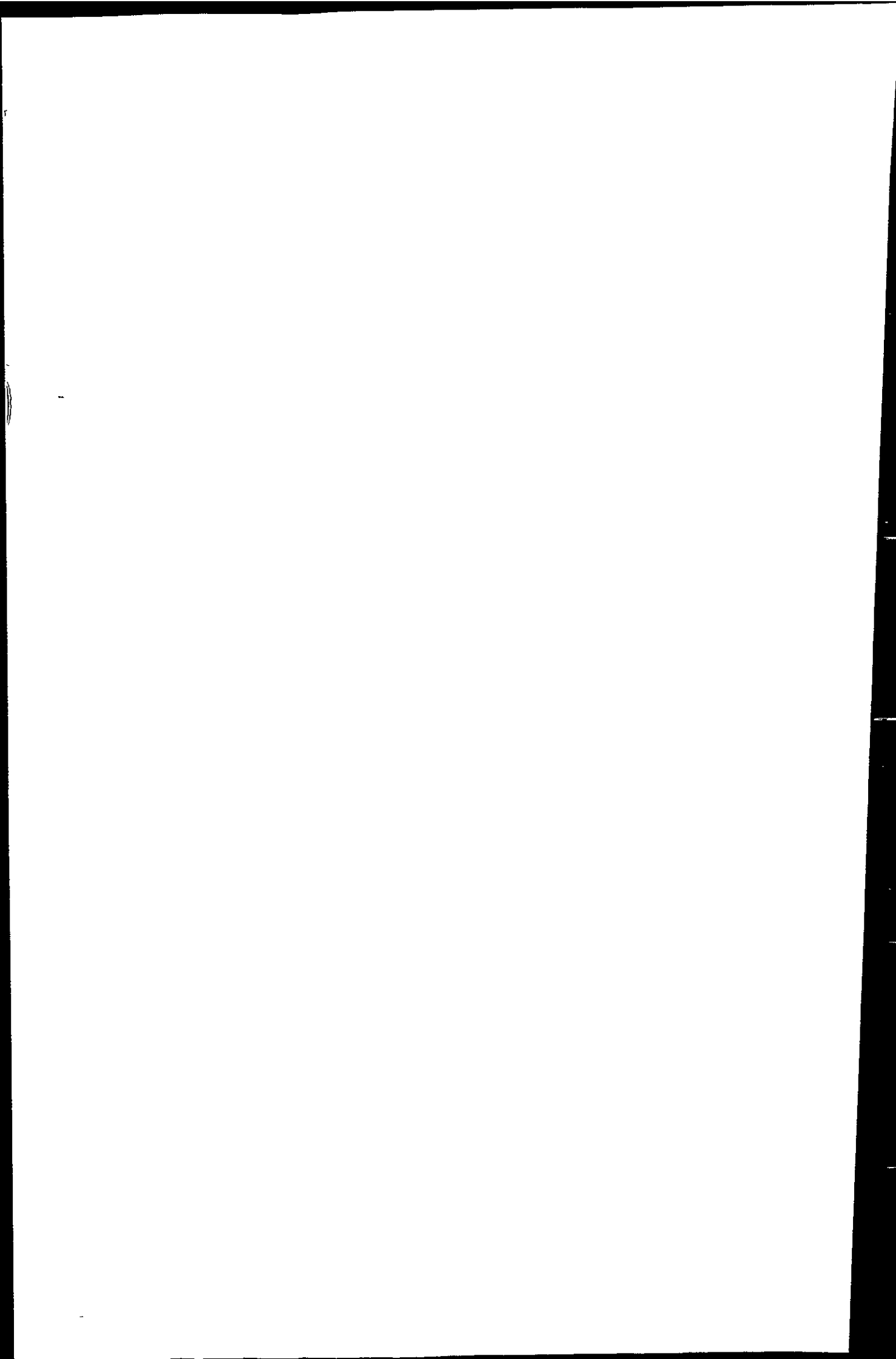


Figure 2. Modified diagram of deformed shape of beam.

5. Why do people think these things? Mental models of physical problems

These ideas do not come out of thin air, but are based on the mental models that the respondents hold. These models are not directly accessible to investigators, but the comments that have been given are predictions these respondents have made of the behaviour of the system according to their mental models. Given the predictions, it is possible to deduce the nature of the models. Anzai and Yokohama (cited in [9]) classify models as *experiential*, *correct scientific* or *false scientific*. Experiential models, which are derived directly from experience, do not have any technical or scientific content. The statement "The beam bends..." was intended to appeal to this type of model. A correct scientific model is a set of scientific concepts and relations that are correct and sufficient to capture problem information. Such a model would characterise the bending in terms of bending moment and shear force, loads and reactions, displacements,



stresses and strains. Incorrect scientific models are those which contain scientific concepts and relations, but incorrectly characterise the problem. It is this type of model which is shown in the comments quoted above.

5.1 Planks are, or may be, rigid.

The first set of comments represent the view of rigidity as the natural state of a beam, given that this is a frequently made assumption in statics problems. This is sometimes held at the same time as the concept that the deflection of a beam does depend on its dimensions, its loading, and, not specifically mentioned by our respondents, the material stiffness (Young's modulus) of the beam, which we see in 5.1 (b)-(e) above. It is perfectly possible to hold two opposite views on a physical phenomenon as long as they are not brought into direct conflict. The point is that these quantities do not affect *whether* a beam will bend, but *how much* it will bend: as comment 5.1 (f) points out, the bending may be negligible, but negligible is still not the same as non-existent.

5.2 The deflection must be greatest under the load

This idea may come from one of several sources:

- 1) Weightless strings and point masses
- 2) The lowest point is the lowest (potential) energy position
- 3) Shear dominated deflection

5.2.1 Point masses versus solid bodies

The first stage of modelling that students encounter in mechanics is of the idealised world of point masses, weightless strings and infinite bodies of infinite stiffness. In such a world, the nearest approximation to our weight on a beam is a weight hung on a loose horizontal string, one-third of the way between its points of suspension. For horizontal equilibrium, the weight would have to fall so that both parts of the string are under tension, pulling the string into an asymmetrical V-shape.

5.2.2 Potential energy

The powerful idea of potential energy being minimised would seem to mean that the weight must be at the lowest possible point, which must be the lowest part of the beam. The lowest part of the beam must thus be under the weight.

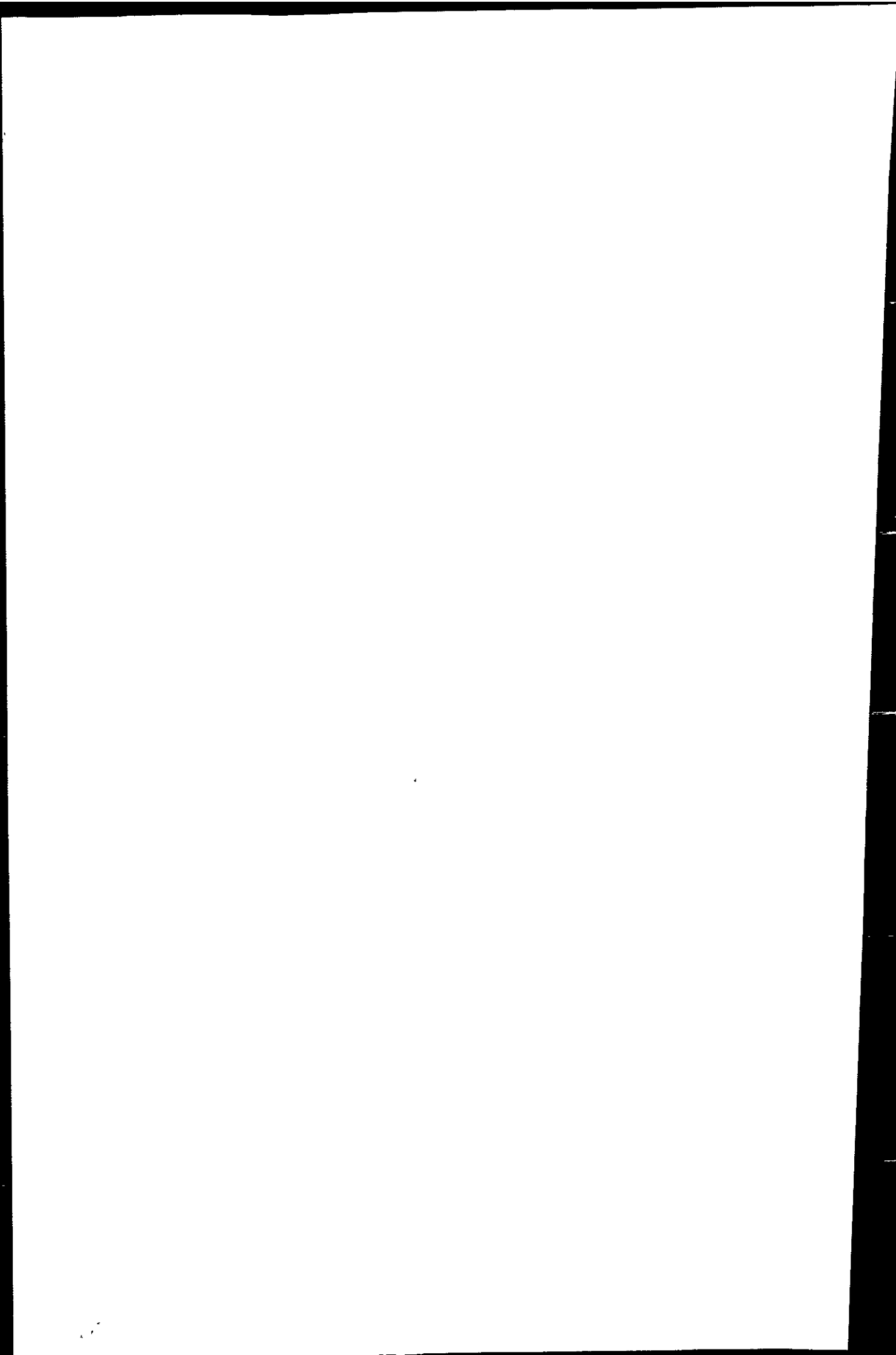
5.2.3 Shear dominated deflection

When beams are designed to use material to perform as efficiently as possible in bending, the notion of putting as much as possible into top and bottom flanges connected by a thin web emerges, and we have an I-beam. The stiffness of the I-beam in bending is greatly enhanced, but its stiffness in shear is related simply to the cross-sectional area. In extreme cases, the deflection due to shear, normally negligible, can dominate, so that the load is close to the lowest point of the beam. This would not happen in the case of a plank lying between two bricks.

It would be speculative to suggest which of these is the principal source of error, but it is suspected that for the students at least 5.2.1 above is the most important.

6. Discussion

The questionnaire was not designed to pick up incorrect mental models, but rather to tease out how people were holding mental representations of some engineering and mathematical concepts. Nevertheless it appears to have brought out into the open some alternative representations which we may not have discovered in teaching or discussion. We should ask ourselves how important these misconceptions are in the scheme of things. To most people they are probably never going to matter. To those to whom



they will make a difference, they will probably discover in time that they have been mistaken. However, particularly for those people who objected to the shape drawn in (b), the revelation comes as a shock. A comfortable assumption has been shaken, and it is unpleasant.

Given that it has been shown that people do have different concept images attached to the same concept, it is germane to ask the following question.

7. What is engineering mathematics?

It appears that one of the principal characteristics of mathematics is its intellectual rigour, and that in teaching mathematics to engineering students one of our aims is to teach them to think clearly [12,13]. On the other hand, one of the complaints about engineering students is that they lack feeling for mathematics and for whether the answers they produce are correct or not [14]. We have to decide whether we regard mathematics for the engineer as a mental discipline or as a tool. Given that engineering students suffer from crowded timetables, we have to be clear about the aspects of mathematics we want to develop in our students. We may have to sacrifice rigour and develop clear thinking in alternative ways, while attempting to enhance students' feeling for mathematics through the use of prostheses such as computer algebra and graphics calculators, and the use of modelling from an early stage.

References

1. Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics, in Tall, D (editor), *Advanced Mathematical Thinking*, Kluwer, Dordrecht, The Netherlands, 65-81
2. Galton, F. (1883). *Inquiries into human faculty and its development*, Macmillan, London
3. Richardson, A. (1978). Verbalizer-Visualizer: a cognitive style dimension, *Journal of Mental Imagery*, 2, 85-100
4. Moran, A. (1991). What can learning styles research learn from cognitive psychology? *Educational Psychology*, 11, 239-245
5. Tiedemann, J. Measures of cognitive styles: a critical review, *Educational Psychologist*, 24, 261-275
6. Thompson, S V. (1990). Visual imagery: a discussion, *Educational Psychology*, 10, 141-167
7. Tall, D. (1991). The psychology of advanced mathematical thinking, in Tall, D (editor), *Advanced Mathematical Thinking*, Kluwer, Dordrecht, The Netherlands, 3-24
8. Royer, J M, Cisero, C A and Carlo, M A. (1993). Techniques and procedures for assessing cognitive skills, *Review of Educational Research*, 63, 201-243
9. Dreyfus, T, Advanced mathematical thinking processes, in Tall, D (editor), *Advanced Mathematical Thinking*, Kluwer, Dordrecht, The Netherlands, 25-41
10. Presmeg, N G, (1986), Visualization and mathematical giftedness, *Educational Studies in Mathematics*, 17, 297-311
11. Godino, J D and Batanero, C, (1996), Institutional and personal meaning of mathematical objects, *Journal für Mathematik-Didaktik*, 17, 99-121
12. OECD, (1966), *Mathematical education of engineers*, OECD, Paris, France, p11
13. Barry, M D J, and Steele, N C (editors), (1992), A core curriculum in mathematics for the European engineer, *SEFI document 92.1*, , p15
14. Sutherland, R, and Pozzi, S, (1995), *The changing mathematical background of undergraduate engineers*, Engineering Council, London, p14

The first part of the document discusses the importance of maintaining accurate records of all transactions. It emphasizes that every entry, no matter how small, should be recorded to ensure the integrity of the financial statements. This includes not only sales and purchases but also expenses and income. The document also highlights the need for regular reconciliation of accounts to identify any discrepancies early on.

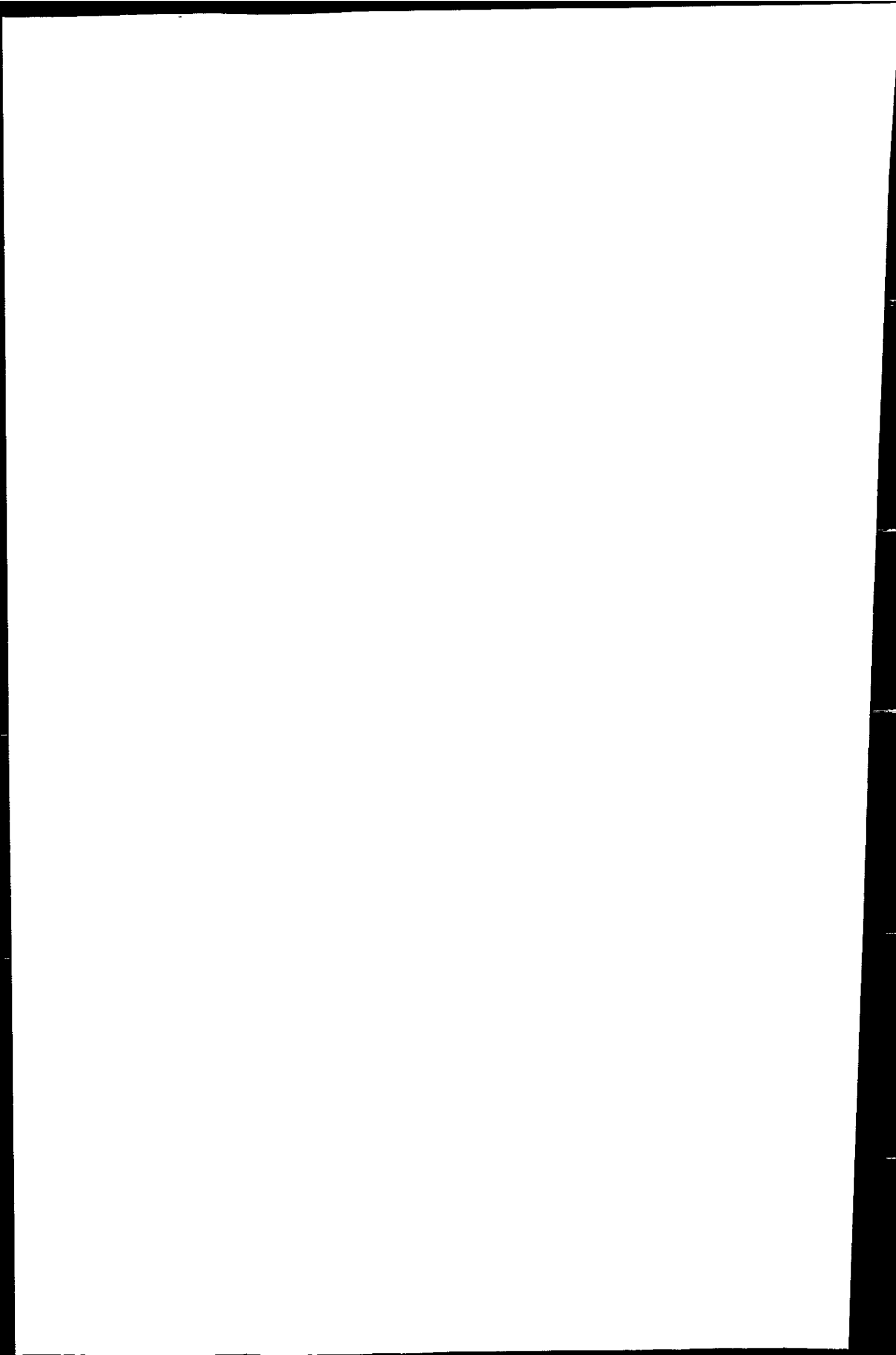
In the second part, the author provides a detailed breakdown of the accounting cycle. It starts with identifying the accounting period and ends with the preparation of financial statements. Each step is explained in detail, including the necessary journal entries and the use of T-accounts to organize the data. The document stresses that following these steps carefully is crucial for producing accurate and reliable financial information.

The third section focuses on the classification of accounts. It explains how to distinguish between assets, liabilities, and equity accounts, as well as how to categorize expenses and revenues. This classification is essential for the proper presentation of the balance sheet and the income statement. The document provides examples to illustrate how different types of transactions affect these accounts.

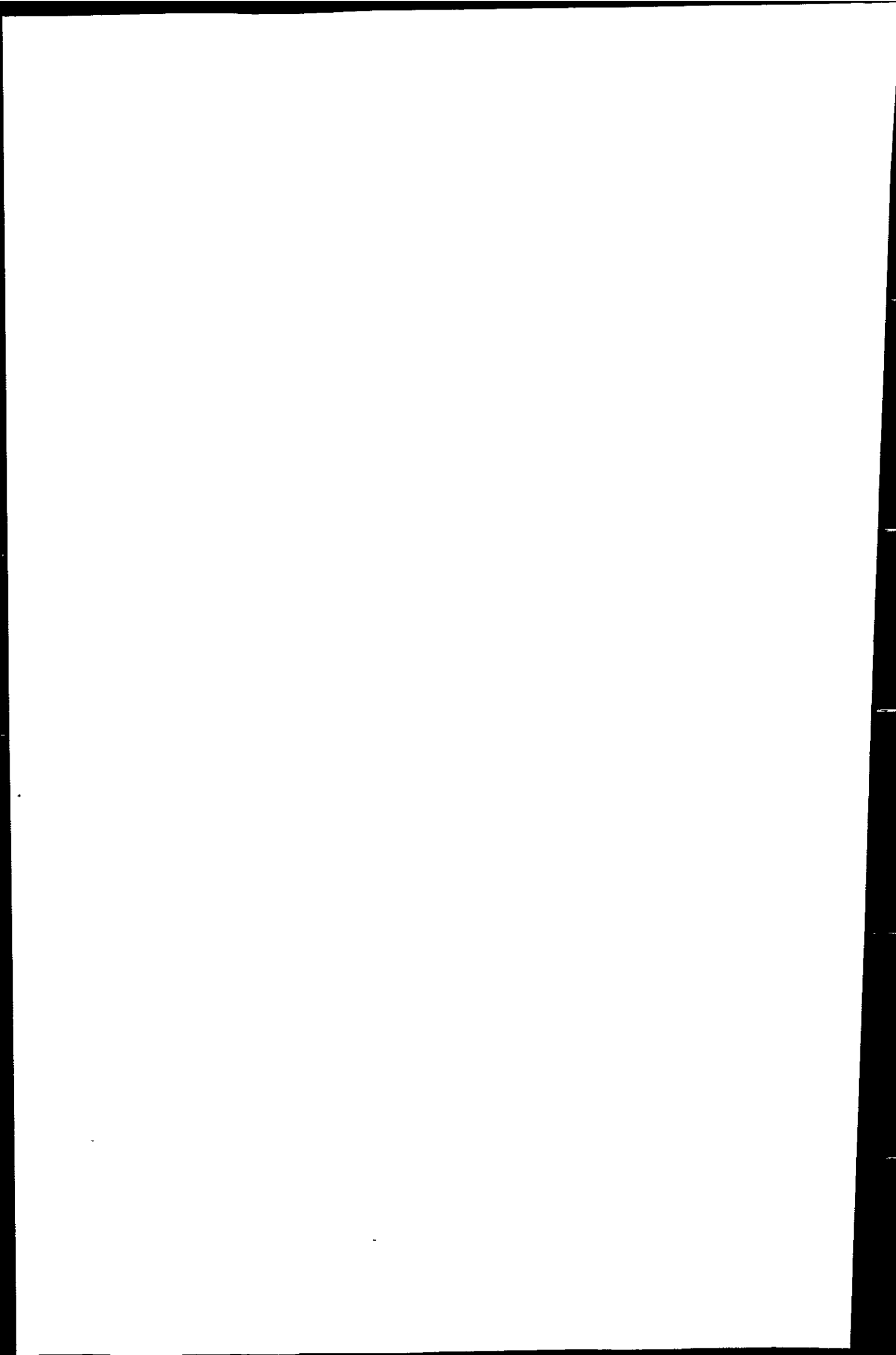
Finally, the document concludes by discussing the importance of internal controls. It suggests several ways to minimize the risk of errors and fraud, such as separating duties, requiring approvals, and conducting regular audits. The author notes that strong internal controls are not only a good practice for any business but also a requirement for many regulatory bodies.

REFERENCES

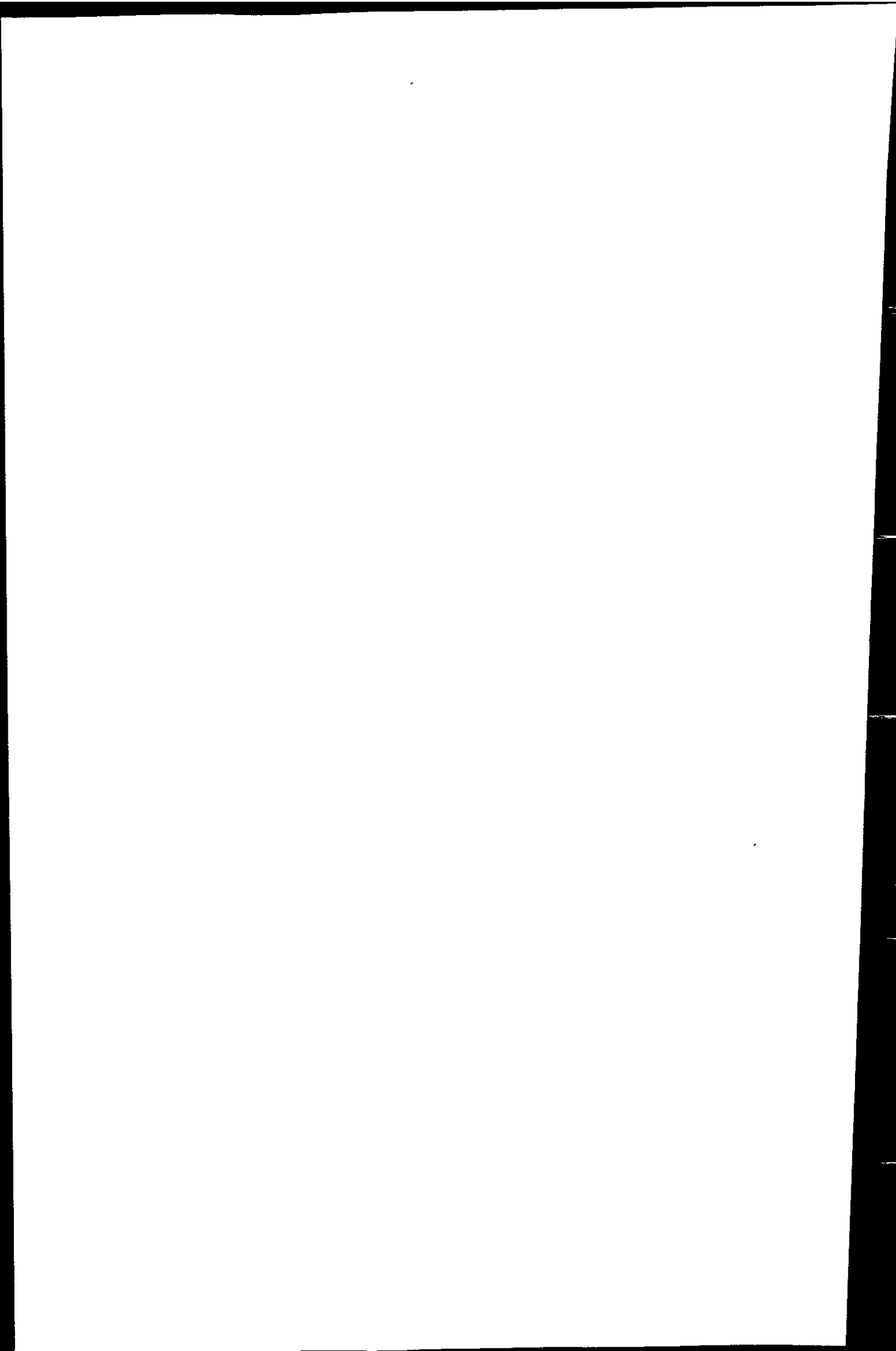
- Abel J F, 1990, Interactive graphics as a stimulus to creative play in engineering students, *CATS'90*, E Oñate et al (eds), Pineridge Press, Barcelona, 66-72
- Abouserie R, Moss D & Barasi S, 1992, Cognitive style, gender, attitude toward computer-assisted learning and academic achievement, *Educational Studies*, 18, 2, 151-
- Adams JL & Stephens LJ, 1991, A comparison of computer-assisted instructional methods, *International Journal for Mathematics Education for Science and Technology*, 22, 6, 889-893
- Alesandrini KL, 1985, Instructional graphics in commercial courseware, *26th ADCIS Proceedings*, Association of the Development of Computer Based Instructional Systems, Philadelphia, US, 188-195
- Alkin MC, 1988, Evaluation: who needs it? Who cares?, in Murphy & Torrance (eds), *Evaluating Education: Issues and Methods*, Paul Chapman Publishing, London, 314-329
- Alt M, 1990, *Exploring Hyperspace: a non-mathematical explanation of multivariate analysis*, McGraw-Hill, London
- Anderson JR, 1982, Acquisition of cognitive skill, *Psychological Review*, 89, 369-406
- Anghileri J, 1989, An investigation of young children's understanding of multiplication, *Educational Studies in Mathematics*, 20, , 367-385
- Anzai Y & Yokoyama T, 1984, Internal models in physics problem solving, *Cognition and Instruction*, 1, 397-450
- Appleby JC, 1995, "Making do" in mathematical modelling- what can be done with limited resources, in Mustoe LR & Hibberd S (eds), *Mathematical Education of Engineers*, Clarendon Press, Oxford
- Arganbright DE, 1993, Innovations in mathematics education through spreadsheets, *IFIP Transactions A - Computer science and technology*, Vol 34, 113-118
- Arzarello F, Bazzini L & Chiappini, G, 1995, The construction of algebraic knowledge: towards a socio-cultural theory and practice, *Proc PME19*, Recife, Brazil, 1, 119-134
- Bajpai AC, Mustoe LR & Walker D, 1989, *Engineering Mathematics*, Wiley, Chichester
- Bandura A, 1977, *Social Learning Theory*, Prentice-Hall, Englewood Cliffs NJ, USA
- Barry MDJ & Steele NC (eds), 1992, *A Core Curriculum in Mathematics for the European Engineer*, SEFi, Brussels, Belgium
- Bartolomé AR, 1992, Interactive learning and cognitive styles: the problem of design, *Learning Resources Journal*, 8, 3, 63-69
- Beevers CE, Foster MG, McGuire GR & Renshaw JH, 1992, Some problems of mathematical CAL, *Computers and Education*, 18, 1-3, 119-125
- Begoray JA, 1990, An introduction to hypermedia issues, systems and application areas, *International Journal of Man-Machine Studies*, 33, , 121-147
- Berry J & Houston K, 1995, *Mathematical Modelling*, Edward Arnold, London
- Berry J & Sahlberg P, 1996, Investigating students' ideas of learning, *Learning and Instruction*, 6, 1, 19-36
- Berry JS & Wainwright P, 1991, *Foundation Mathematics for Engineers*, Macmillan, London



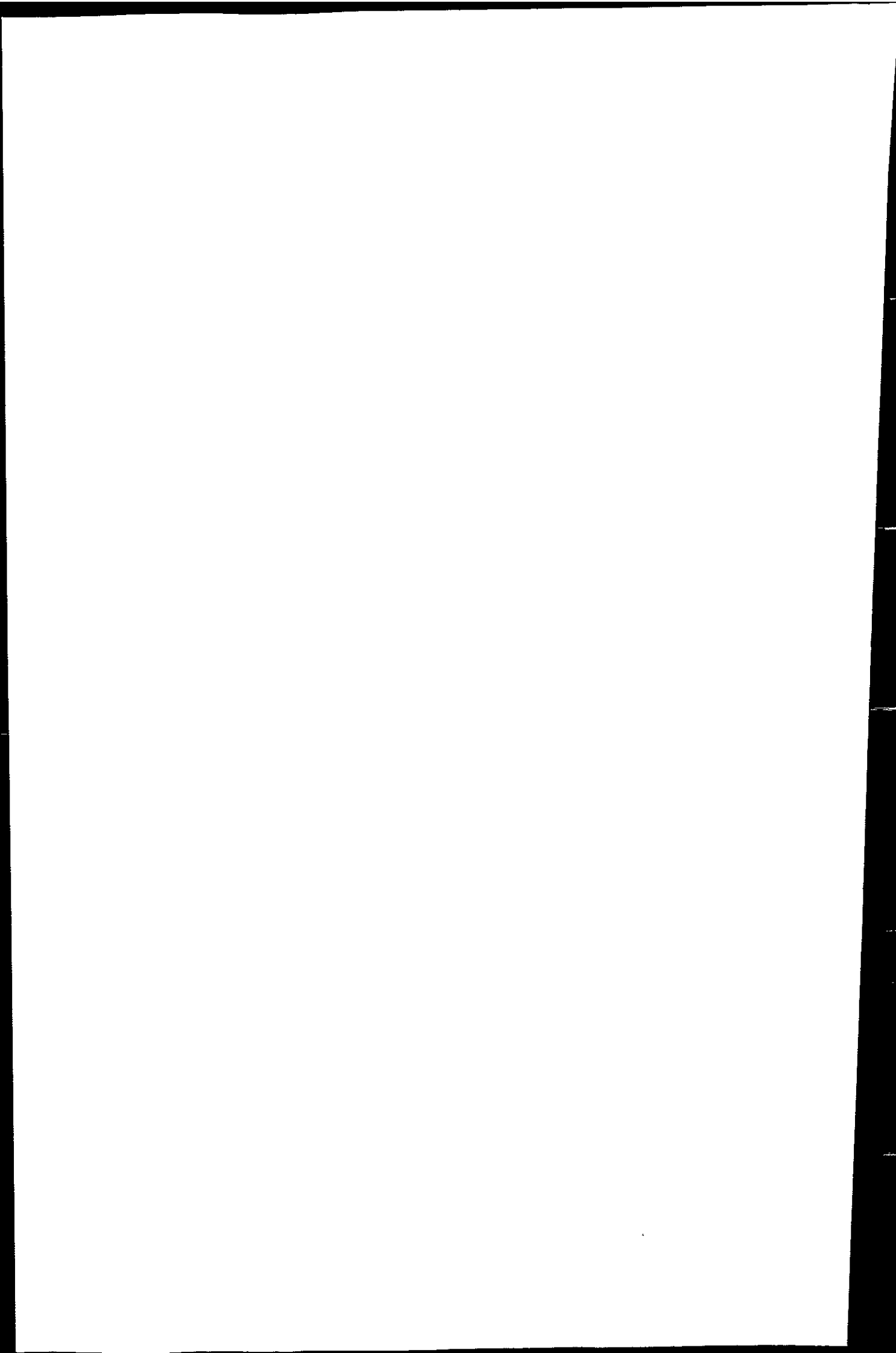
- Berry JS, Graham E & Watkins AJP, 1996, *Learning Mathematics through DERIVE*, Chartwell-Bratt, Bromley
- Berry JS, Northcliffe A & Humble S, 1989, *Introductory mathematics through Science Applications*, CUP, Cambridge
- Biglan A, 1973, The characteristics of subject matter in different academic areas, *Journal of Applied Psychology*, 57, , 195-203
- Booth, DJ, 1991, *Foundation Mathematics*, Addison-Wesley, Wokingham
- Brody CM, 1991, *Cooperative learning and teacher beliefs: a constructivist view*, paper presented at the annual meeting of the American Educational Association in Chicago, April 1991
- Brown AJ, 1994a, Evaluation of teaching and learning processes in a computer supported mechanical engineering course, in Eames IW & Johnson AR (eds), *Proc Conf on Computer Aided Learning in Engineering*, University of Sheffield, Sheffield
- Brown MS & Hayden RR, 1989, Learning styles- liberal arts and technical training: what's the difference?, *Psychological Reports*, 64, 2, 507-518,
- Brown NW & Cross EJ, 1992, A comparison of personality characteristics for entering freshmen, persistors and norm groups in engineering, *Educational and Psychological Measurement*, 52, 4, 939-944
- Brown T, 1997, *Mathematics Education and Language: Interpreting Hermeneutics and Post-Structuralism*, Kluwer Academic Publishers, Dordrecht, Netherlands
- Bruckheimer M, Gowar NW & Scraton RE, 1971, *Mathematics for Technology: a New Approach*, Chatto and Windus
- Burrell G & Morgan G, 1979, *Sociological Paradigms and Organisational Analysis*, Heinemann Educational Books, London.
- Burton L, 1997, The assessment factor- by whom, for whom, when and why, in Houston SK, Blum W, Huntley ID & Neill NT (eds), *Teaching and Learning Mathematical Modelling: Innovation, Investigation and Applications*, Albion, Chichester, 95-107
- Carlson HL, 1991, Learning style and program design in interactive multimedia, *Educational Technology Research and Development*, 39, 3, 41-44
- Challis N & Gretton H, 1997, Technology, core skills and the engineering mathematics curriculum, *The Mathematical Education of Engineers II*, IMA, London, 145-150
- Checkland P, Warmington A & Wilson B, 1983, Research on improving the management of production systems: an approach through systems methodology, Working paper abstracted from Wilson B, Berg CC & French D (eds), *The Efficiency of Manufacturing Systems*, NATO ARI Series, Plenum Publishing Corporation, New York
- Chi MTH, Feltovich PJ & Glaser R, 1981, Categorization and representation of physics problems by experts and novices, *Cognitive Science*, 5, 121-125
- Child D, 1970, *The Essentials of Factor Analysis*, Holt, Reinhart and Winston, London
- Chirgwin BH & Plumpton C, 1970, *A Course of Mathematics for Engineers and Scientists*, 2nd edition
- Clark R, 1983, Reconsidering research on learning from media, *Review of Educational Research*, 53, , 445-459
- Clark WB, 1994, Constructing an integrated model for professional education, *Educational Technology*, 34, 7, 38-43



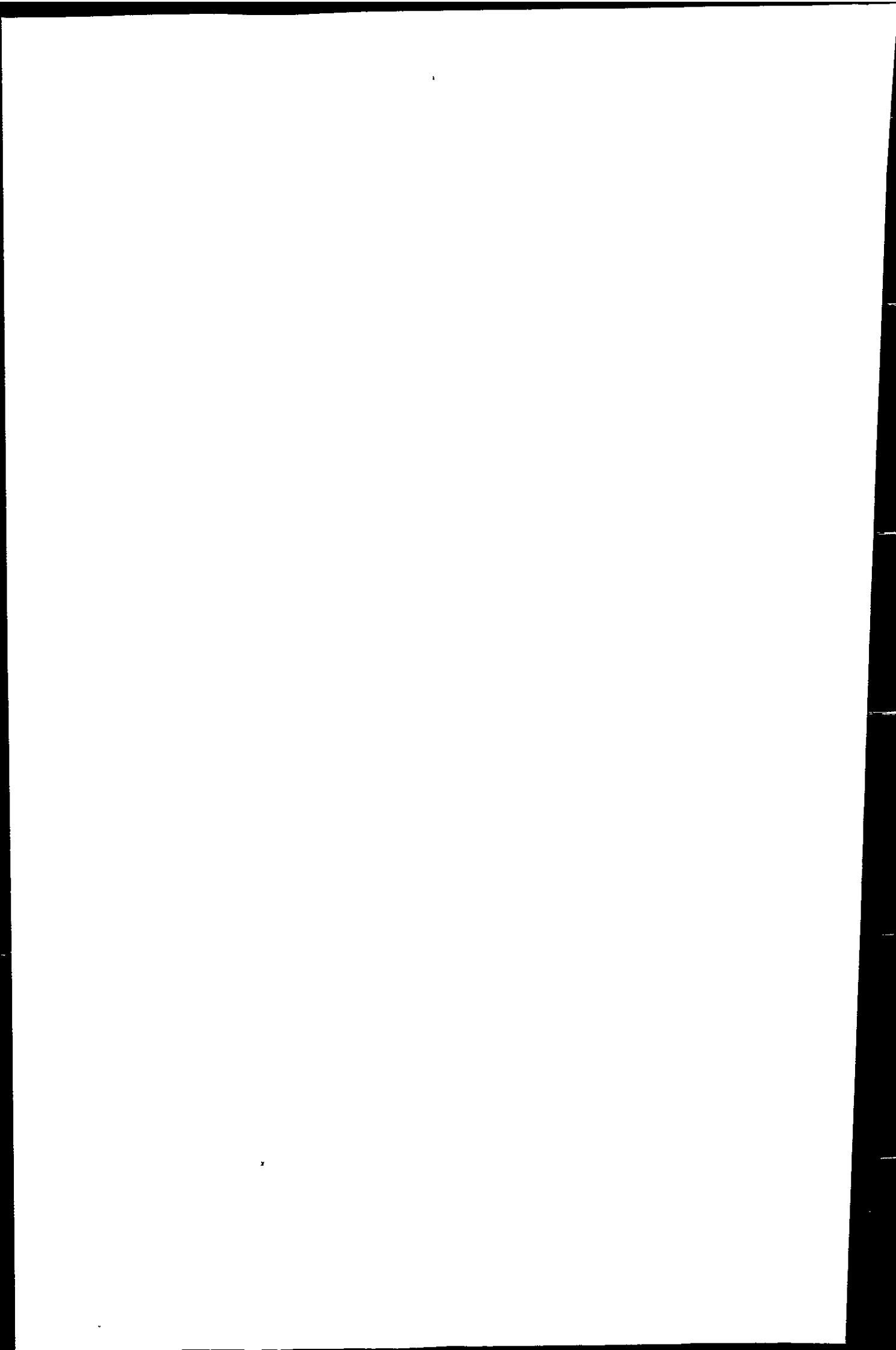
- Clarke A, 1992, How are graphics used in computer based learning?, *British Journal of Educational Technology*, 23, 3, 228-233
- Cohen L & Mannion L, 1994, *Research Methods in Education*, 4th edition, Routledge, London
- Connell ML, 1997, AI or IA: the choice is yours!, *Educational Technology Review*, 7, 27-29
- Conwell CR, Helgeson SL & Wachowiak DG, 1987, The effect of matching and mismatching cognitive style and science instruction, *Journal of Research in Science Teaching*, 24, 8, 713-722
- Cooper SE & Millar JA, 1991, MBTI learning style-teaching style incongruities, *Educational and Psychological Measurement*, 51, 3, 699-706
- Coull J & Simmonds DG, 1988, Computer-assisted algebraic differential equation solution, *International Journal for Mathematics Education for Science and Technology*, 19, 4, 551-559
- Cox W, Norris WT & Penny JET, 1995, Mathematics in engineering teaching at university- a practitioner's dilemma, in Mustoe LR & Hibberd S (eds), *Mathematical Education of Engineers*, Clarendon Press, Oxford, 251-266
- Coxhead C, 1997, Curriculum development and assessment in Northern Ireland, in Houston SK, Blum W, Huntley ID & Neill NT (eds), *Teaching and Learning Mathematical Modelling: Innovation, Investigation and Applications*, Albion, Chichester, 3-22
- CRAC, 1996, *Graduate Employment and Training (GET 97)*, Hobson, Cambridge
- Croft A, Davison R & Hargreaves M, 1995, *Introduction to Engineering Mathematics*, Addison-Wesley, Wokingham
- Cronbach L, 1988, Issues in planning evaluations, in Murphy & Torrance (eds), *Evaluating Education: Issues and Methods*, Paul Chapman Publishing, London, 4-35
- Cross M, 1983, How should mathematical modelling and computer technology influence mathematics degree courses?, *Teaching Mathematics and its Applications*, 2, 3, 120-125
- Crowther K, 1997a, Engineering degree students are deficient in mathematical expertise - why?, *International Journal of Mathematical Education in Science and Technology*, 28, 6, 785-792
- Crowther K, 1997b, Engineering students' approach to their mathematics education, in Hibberd S & Mustoe L (eds), *The Mathematical Education of Engineers II*, IMA, London, 79-84
- Culler J, 1982, *On Deconstruction*, Cornell, Ithaca, USA, p175
- Culler J, 1975, Reading and Misreading, *Yale Review*, 65, 1, 90-91
- Dahl OG, 1990, GIGS: graphical interface generating system, in *CATS'90*, Oñate E et al (Eds), Pineridge Press, Barcelona, 211-218
- DeCorte E, 1994, Technology-supported learning in a stage of transition: the case of mathematics, *IFIP Transactions A - Computer science and technology*, Vol 46, 113-123
- Dick W, 1981, Instructional design models: further trends and issues, *Educational Technology*, 21, 9, 29-32
- Dissa AA, 1987, The third revolution in computers and education, *Journal of Research in Science Teaching*, 24, 4, 343-363



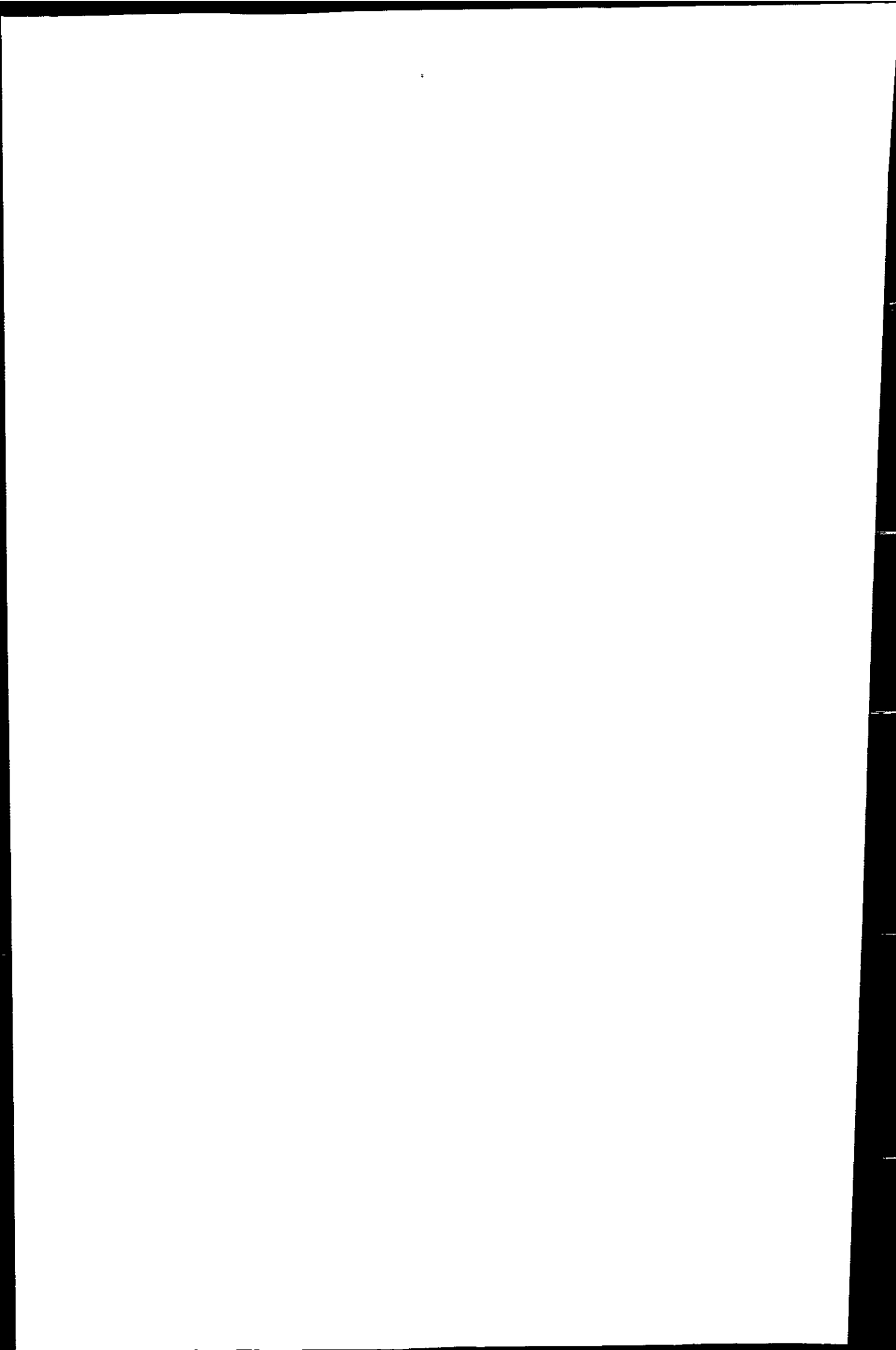
- Durkheim E, 1956, *Education and Sociology*, New York, p71
- Dwyer CA & Dwyer FM, 1987, Effect of depth of information processing on students' ability to acquire and retrieve information related to different instructional objectives, *Programmed Learning and Educational Technology*, 24, 4, 264-279
- Eames IW & Johnson AR (eds), 1994, *Proc Conf on Computer Aided Learning in Engineering*, University of Sheffield, Sheffield
- Engineering Council, 1990, Standards and Routes to Registration (SARTOR), 2nd Edition, Engineering Council, London
- Engineering Council, 1995, Competence and Commitment, Engineering Council, London
- Entwistle & Ramsden, 1983, *Understanding Student Learning*, Nichols Pub.Co, New York, 179-193
- Ernest P, 1991, *The Philosophy of Mathematics Education*, Falmer, London
- Ernest P, 1983, Thinking of a funny line, *Mathematical Education for Teaching*, 4, 2
- Ethington, CA, 1988, Differences among women intending to major in quantitative fields of study, *Journal of Educational Research*, 81, 6, 354-359
- Evans CW, 1992, *Engineering Mathematics*, 2nd edition, Chapman and Hall, Andover
- Eysenck HJ, 1962, *Know your own IQ*, Penguin, Harmondsworth
- Finniston M, 1980, *Engineering our Future: a report of the Committee of Inquiry into the Engineering Profession*, Cmnd 7794, HMSO, London
- Fraser CJ & Thorpe JR, 1994, The use of spreadsheets for modelling and analysis applications in mechanical engineering, *Proc Conf on CAL in Engineering*, Sheffield, UK, 167-182.
- Freudenthal H, 1991, *Revisiting Mathematics Education*, Kluwer, Dordrecht, Netherlands
- Frau E, Midoro V & Pedemonte GM, 1992, Do hypermedia systems really enhance learning? A case study on earthquake education, *Educational Technology and Teaching International*, 29, 1, 42-51
- Gagné RM, Wagner W & Rojas A, 1981, *Planning and authoring computer-assisted instruction lessons*, *Educational Technology*, 21, 9, 17-21
- Galbraith M, 1992, Understanding career choices of men in elementary education, *Journal of Educational Research*, 85, 4, 246-253
- Galbraith PL & Haines CR, 1997, Some mathematical characteristics of students entering applied mathematics degree courses, in Houston SK, Blum W, Huntley ID & Neill NT (eds), *Teaching and Learning Mathematical Modelling: Innovation, Investigation and Applications*, Albion Publishing, Chichester, 77-94
- Galton F, 1883, *Inquiries into Human Faculty and its Development*, Macmillan, London
- Garlinger DK and Frank BM, 1986, Teacher-student cognitive style and academic achievement: a mini-meta-analysis, *Journal of Classroom Interaction*, 21, 2, 2-8
- Gautreau R, Chawla H, Chang J, Chan Y, Lee ST & Savin W, 1987, Video discs in Physics education, *Proc. Int. Conf. on Computer Assisted Learning in Post-Secondary Education*, Calgary, Canada, 107-111
- Godino JD & Batanero C, 1996, Institutional and personal meanings of mathematical objects, *Journal für Mathematik-Didaktik*, 17, 2, 99-121



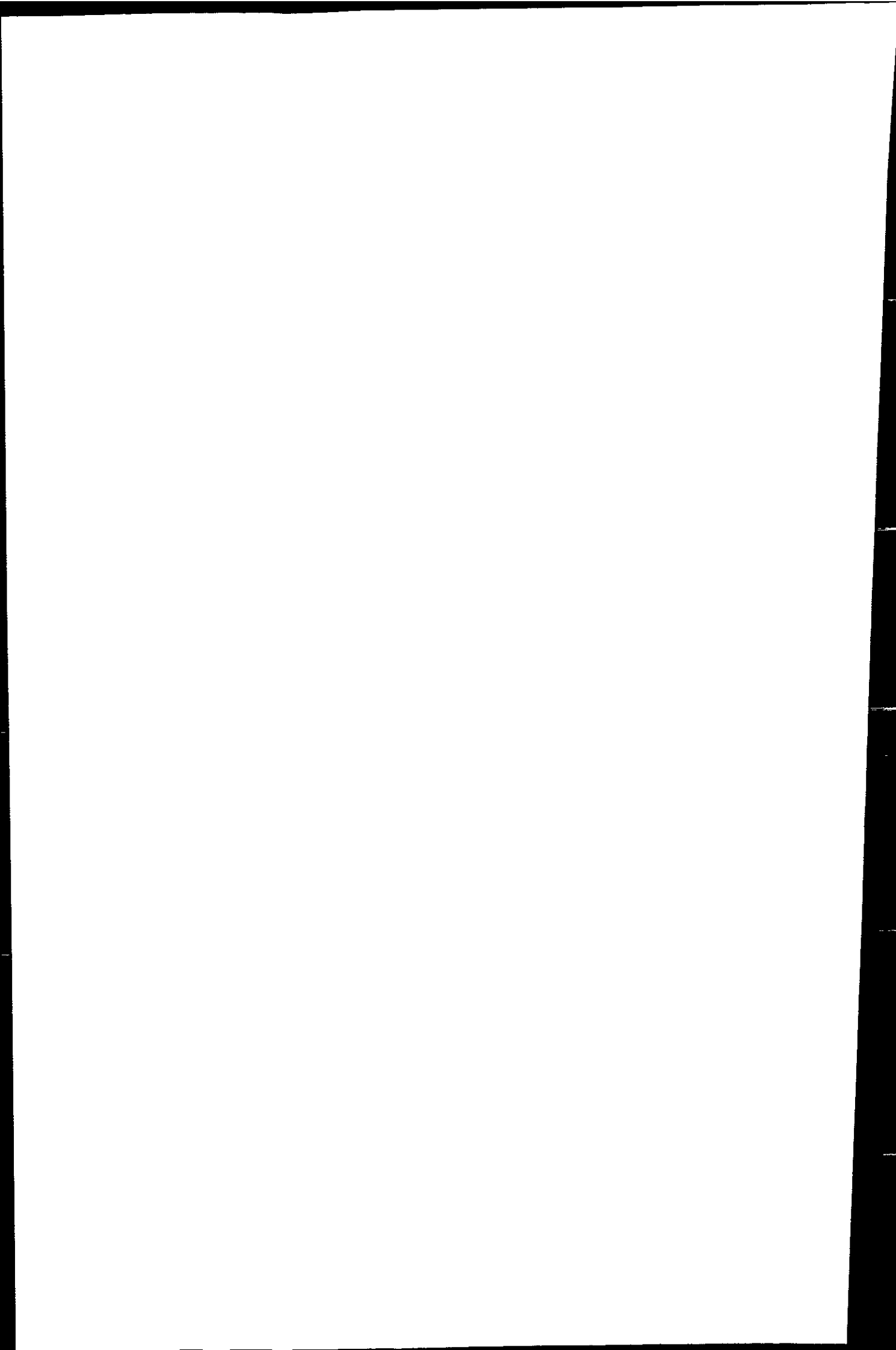
- Goldenberg EP 1988, Mathematics, metaphors and human factors: mathematical, technical and pedagogical challenges in the educational use of graphical representation of functions, *Journal of Mathematical Behaviour*, 7, 2, 135-173
- Götz K, 1991, Interactive learning with the computer, *Educational Technology and Teaching International*, 28, 1, 7-14
- Gough H & Heilbrun, 1980, *Adjective Check List*, Consulting Psychologist Press, Palo Alto, Ca, USA
- Goult RJ, Hoskins RF, Milner JA & Pratt MJ, 1973, *Applicable Mathematics: a Course for Engineers and Scientists*, Macmillan, London
- Graham D, 1997, Presentation at internal seminar, Seale Hayne,
- Graham E, 1997, The ways in which different students respond to the same mathematical modelling problem, *Teaching Mathematics and its Applications*, 16, 1, 19-22
- Graham GR, 1991, 'Let's see it for real' A new medium for an old message, *Physics Education*, 26, 6, 355-358
- Gilbert JK, Boulter C & Rutherford M 1998, Models in explanations, part 1: horses for courses?, *International Journal of Science Education*, 20, 1, 83-97
- Greer A & Taylor GW, 1989, *Mathematics for Technicians*, Thornes, Cheltenham
- Guimond S & Palmer DL, 1990, Type of academic training and causal attribution for social problems, *European Journal of Social Psychology*, 20, 1, 61-75
- Hackett G, Betz NE, Casas JM & Rocha-Singh IA, 1992, Gender, ethnicity and social cognitive factors predicting the academic achievement of students in engineering, *Journal of Counseling Psychology*, 39, 4, 527-538
- Hair JF, Anderson RE, Tatham RL & Black WC, 1984, Factor analysis, Chapter 7 of *Multivariate Data Analysis with Readings*, 4th edition, Prentice Hall International, London, 364-420
- Haney W, 1984, Testing reasoning and reasoning about testing, *Review of Educational Research*, 54, 4, 597-654
- Hanna G, 1991, Mathematical Proof, in Tall D (ed), *Advanced Mathematical Thinking*, Kluwer Academic Publishers, Dordrecht, Netherlands, 54-64
- Hannafin MJ & Carney BW, 1991, Effects of elaboration strategies on learning and depth of processing during computer-based instruction, *Journal of Computer-Based Instruction*, 18, 3, 77-82
- Hannafin MJ & Phillips TL, 1987, Perspectives in the design of interactive video: beyond tape versus disc, *Journal of Research and Development in Education*, 21, 1, 44-60
- Hargreaves DJ, 1997, Student Learning and assessment are inextricably linked, *European Journal of Engineering Education*, 22, 4, 401-410
- Hart D & Croft T, 1988, *Modelling with Projectiles*, Ellis Horwood, Chichester
- Heading J, 1970, *Mathematical Methods in Science and Engineering*, 2nd edition, Edward Arnold, London
- Heard TJ, 1978, The Mathematical Education of Engineers at School and University, BP Educational Service
- Helsel SK, 1987, The curricular domain of educational interactive video, *Optical Information Systems*, 7, 2, 107-112
- Hennessy S & O'Shea T, 1993, Learner perceptions of realism and magic in computer simulation, *British Journal of Educational Technology*, 24, 2, 125-138



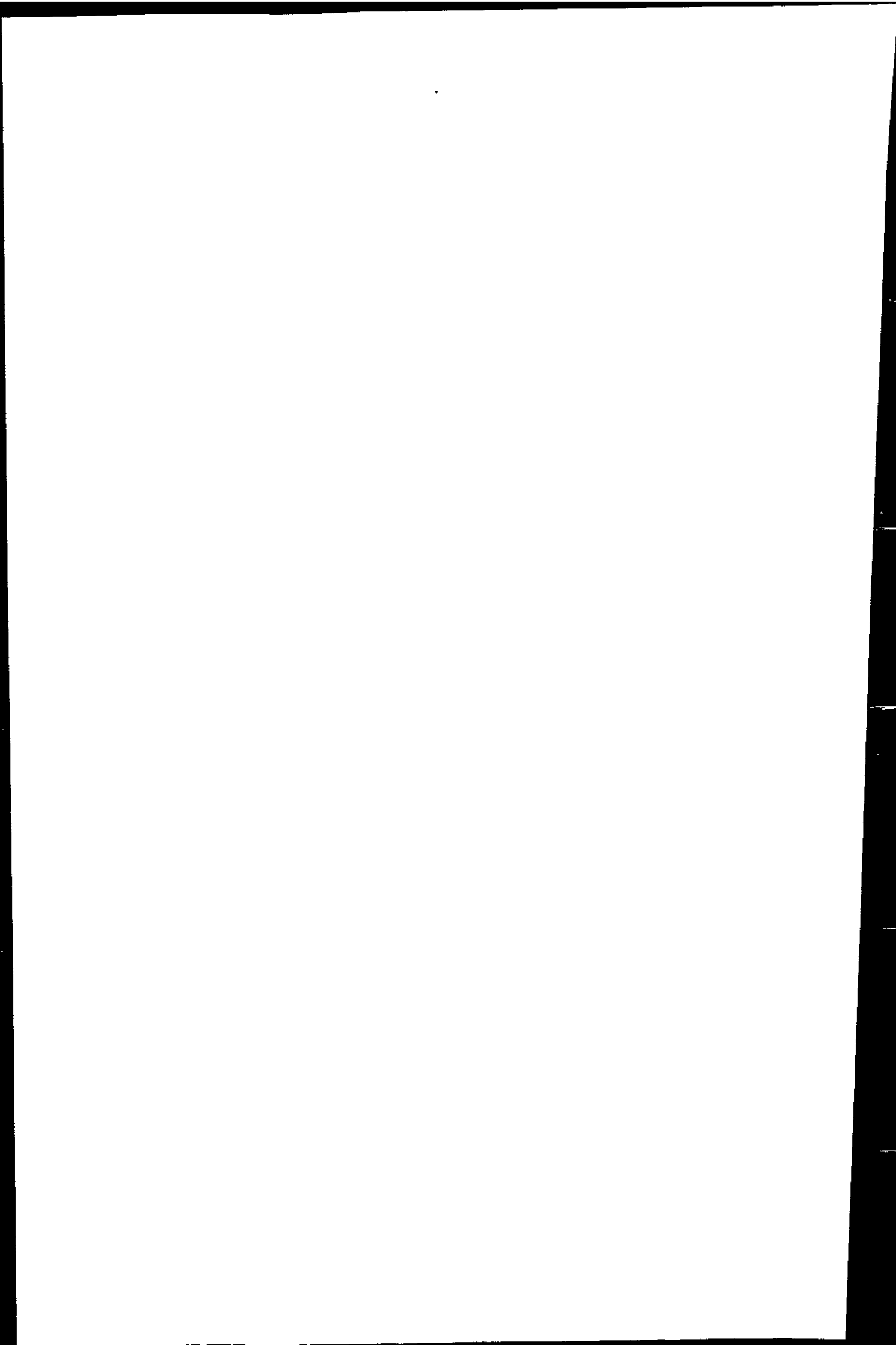
- Hirst KE, 1996, *Changes in Mathematics A level from 1996*, London Mathematical Society, London
- Hirst PH, 1972, Liberal education and the nature of knowledge, in Dearden RF, Hirst PH & Peters RS (eds), *Education and the Development of Reason*, Routledge and Kegan Paul, London, 391-414
- Hodgson T & Harpster D, 1997, Looking at mathematical modelling, classroom observations and instructional strategies, *School Science and Mathematics*, 97, 5, 260-267
- IChemE/ICE/IEE/IMC/IMechE/LMS/IMA, 1995, *Mathematics Matters in Engineering*, Institute of Mathematics and its Applications, Southend-on-Sea
- Ikedda T, 1997, A case study of instruction and assessment in mathematical modelling- 'the delivering problem', in Houston SK, Blum W, Huntley ID & Neill NT (eds) *Teaching and Learning Mathematical Modelling: Innovation, Investigation and Applications*, Albion Publishing, Chichester, 51-62
- IMarE, 1995, *Working group report on degree entry qualifications*, IMarE, London.
- Jackson JE, 1991, *A User's Guide to Principal Components*, John Wiley, Chichester
- Jaeger JC & Starfield AM, 1974, *An Introduction to Applied Mathematics*, Second edition, Oxford University Press, Oxford
- James G, Burley D et al, 1992, *Modern Engineering Mathematics*, Addison-Wesley, Wokingham
- James G, Burley D, Dyke P, Searl J, Wright J & Steele N, 1993, *Advanced Modern Engineering Mathematics*, Addison-Wesley, Wokingham
- Jones B, 1993, Effective use of interactive video with large groups, *British Journal of Educational Technology*, 24, 3, 113-121
- Jordan DW & Smith P, 1994, *Mathematical Techniques*, OUP, Oxford
- Julie C, 1991, Learning novel concepts in a computer-enriched environment, *British Journal of Educational Technology*, 22, 2, 99-109
- Khasawneh AA, 1994, An intelligent CAI project for teaching mathematical concepts, *International Journal for Mathematics Education for Science and Technology*, Vol 25, No 3, 325-332
- Kirby JR, Moore PJ & Schofield NJ, 1988, Verbal and visual learning styles, *Contemporary Educational Psychology*, 13, , 169-184
- Knowlton JQ, 1966, On the definition of a picture, *Audio Visual Communication Review*, 14, , 157-183
- Kolb D, 1981, Learning styles and disciplinary differences, in Chickering AW (ed), *The Modern American College*, Jossey-Bass, London, 232-255
- Kozma RB, 1991, Learning with media, *Review of Educational Research*, 61, 2, 179-211
- Kreyszig E, 1993, *Advanced Engineering Mathematics*, 8th edition, Wiley, Chichester
- Kurland DM & Kurland LC, 1987, Computer applications in education, an overview, *Annual Review of Computer Science*, 2, , 317-58
- Kutzler B, 1994, DERIVE - the future of teaching mathematics, *International DERIVE Journal*, 1, 1, 37-48
- Laridon PE, 1990a, The role of the instructor in a computer-based interactive videodisc education environment, *Educational Technology and Teaching International*, 27, 4, 365-374



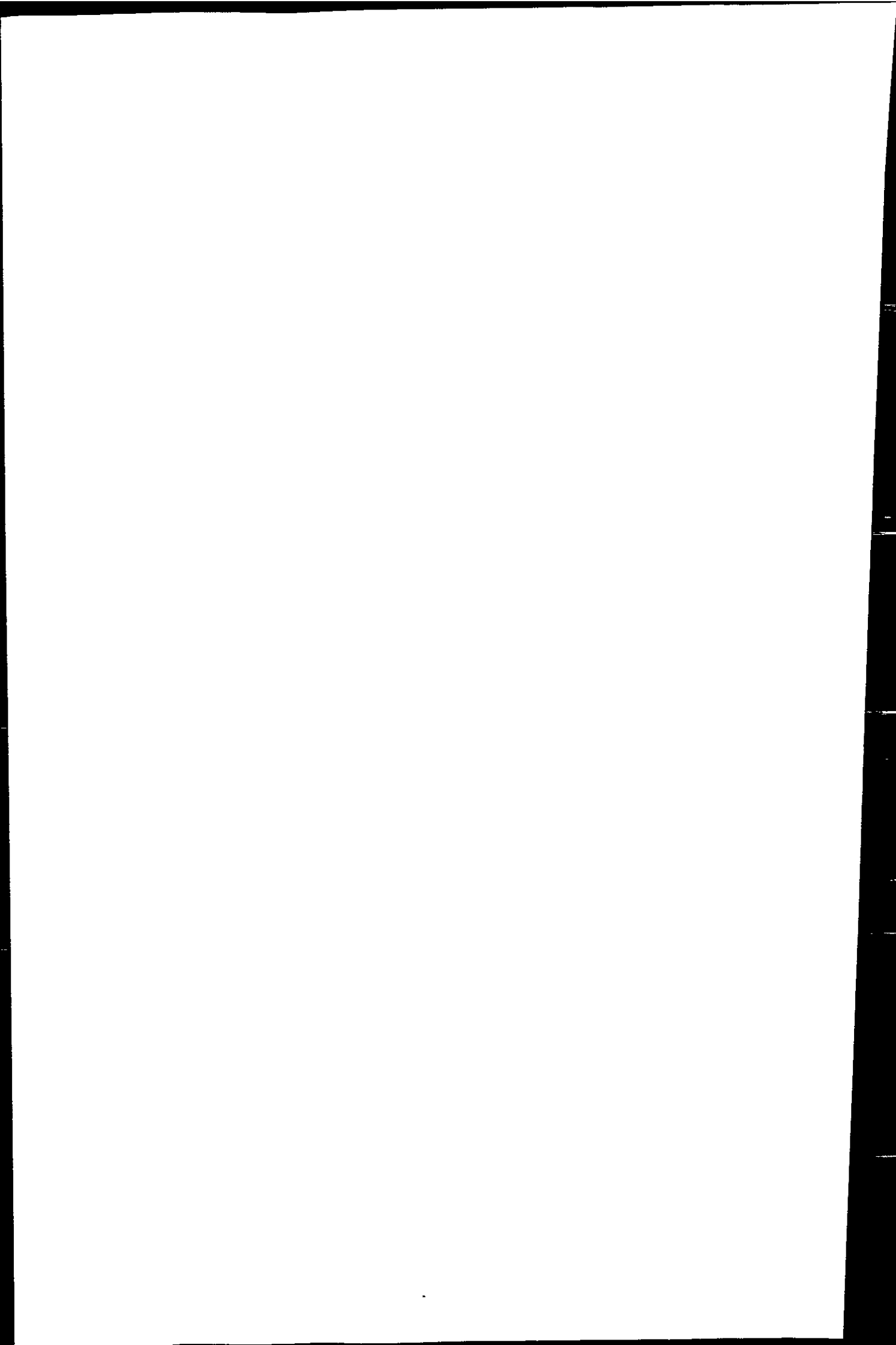
- Laridon PE, 1990b, The development of an instructor role model for a computer-based interactive videodisc education environment for learning mathematics, *Educational Technology and Teaching International*, 27, 4, 375-385
- Laurillard D, 1979, The processes of student learning, *Higher Education*, 8, , 395-409
- Laurillard D, 1987, Computers and the emancipation of students: giving control to the learner, *Instructional Science*, 16, 1, 3-18
- Laurillard D, 1988, Evaluating computer-based learning, *Proc Conf on Computers in Engineering Education*, Imperial College, London, 70-76
- Laurillard D, 1993, *Rethinking University Teaching: a framework for the effective use of educational technology*, Routledge, London
- Lave J, 1996, Teaching, as learning, in practice, *Mind, Culture and Activity*, 3, 3, 149-164
- Lawson DA, 1995, The effectiveness of a computer-assisted learning programme in engineering mathematics, *International Journal for Mathematics Education for Science and Technology*, 26, 4, 567-574
- Lee T, Graham & Wills B, 1987, Engineering problem solving using spreadsheets, *Proc. Int. Conf. on Computer Assisted Learning in Post-Secondary Education*, Calgary, Canada, 303-305
- Lindstrom B, Marton F, Ottoson T & Laurillard D, 1993, Computer simulation as a tool for developing intuitive and conceptual understanding in mechanics, *Computers in Human Behaviour*, 9, 2&3, 263-281
- LMS/IMA/RSS, 1995, *Tackling the Mathematics Problem*, London Mathematical Society, London
- Lundstrom KV & Martin R, 1986, Matching college instruction to student learning style, *College Student Journal*, 20, 3, 270-274
- MacGregor SK, Shapiro JZ & Niemec J, 1988, Effects of a computer-augmented learning environment on math achievement for students with differing cognitive style, *Journal of Educational Computing Research*, 4, 4, 453-465
- Mackie DM, 1992, An evaluation of computer-assisted learning in mathematics, *International Journal of Mathematics Education in Science and Technology*, 23, 5, 731-737
- Madge N, Meyer S & Sweezie J, 1986, Screen design for computer assisted learning courseware, *fifth Canadian Symposium on Instructional Technology*, Ottawa, Canada, 371-374
- Mahoney MS, 1972, in Gillespie CC (ed), *Dictionary of Scientific Biography*, Charles Scribner's Sons, New York, US, Vol 5 (Fischer to Haberlandt), p450
- Malone TW, 1981, Towards a theory of intrinsically motivating instruction, *Cognitive Science*, 4, , 333-369
- Marshman B & Ponzo P, 1987, Innovations with interactive graphics in teaching and learning about applied differential equations, *Proc. Int. Conf. on Computer Assisted Learning in Post-Secondary Education*, Calgary, Canada, 311-313
- Matos JF & Careira S, 1997, The quest for meaning in students' mathematical modelling activity, in Houston SK, Blum W, Huntley ID & Neill NT (eds), *Teaching and Learning Mathematical Modelling: Innovation, Investigation and Applications*, Albion, Chichester, 63-75
- Matta KF & Kern GM, 1991, Interactive videodisc instruction: the influence of personality on learning, *International Journal Of Man-Machine Studies*, 35, , 341-352



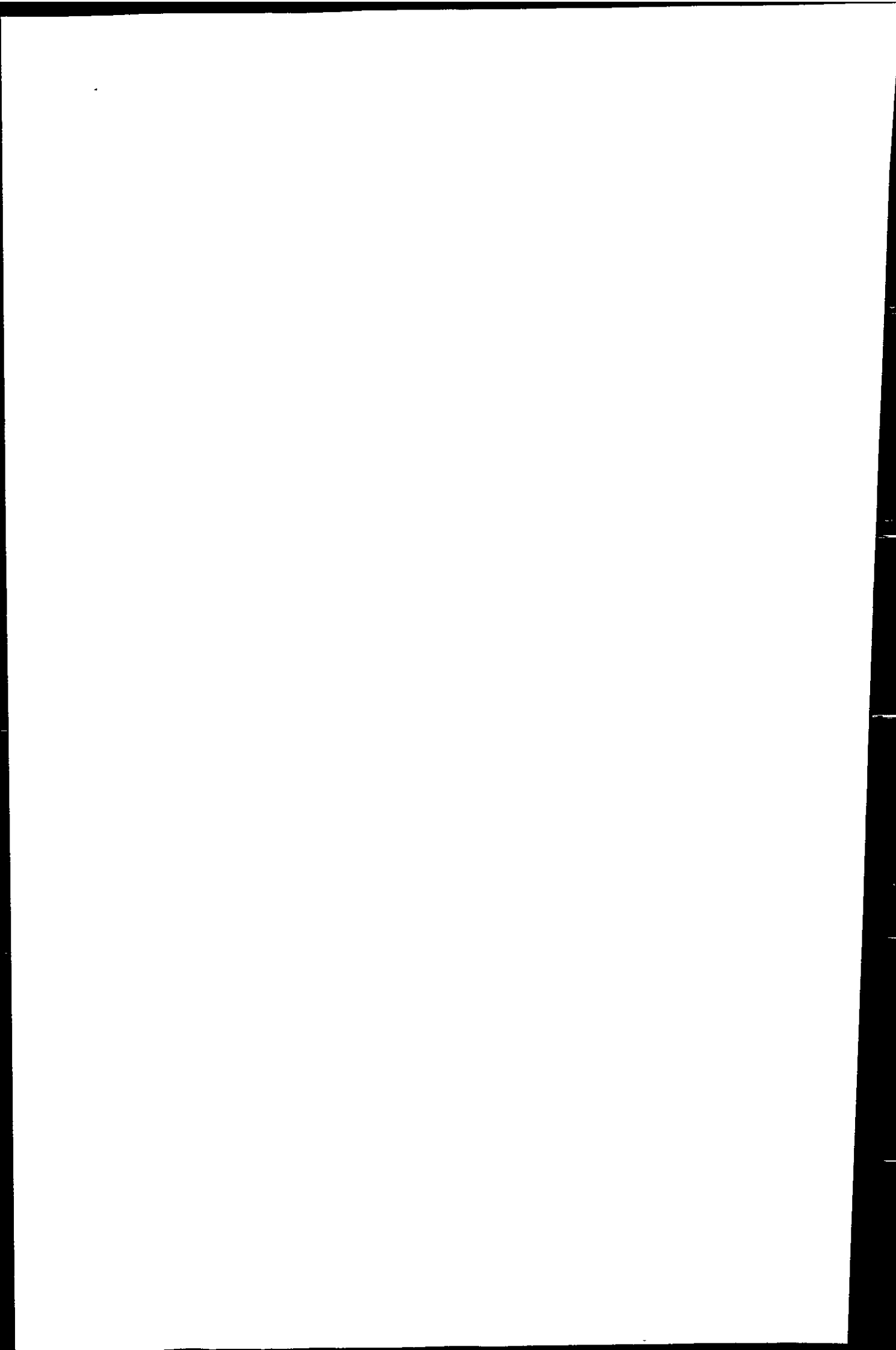
- Mauil WM, Berry JS & Stockel C, 1995, Teaching engineering mathematics through computer technology, in Dar-El EM, Karni R & Herer YT (eds), *Proc 13th Int Conf on Production Research*, Freund Publishing House, London
- Mauil WM & Berry JS, 1997, Modelling the mathematical ideas of engineering students, in Mustoe LR & Hibberd S (eds), 1995, *The Mathematical Education of Engineers*, Clarendon Press, Oxford,
- McCaulley MH, 1976, Psychological types in engineering: implications for teaching, *Engineering Education*, Vol 66, April, 729-736
- McDermott LC, 1991, Millikan Lecture 1990: What we teach and what is learned-closing the gap, *American Journal of Physics*, 59, 4, 301-315
- McLone RR, 1984, Can mathematical modelling be taught?, in Berry JS, Burghes DN, Huntley ID, James DJG & Moscardini AO (eds), *Teaching and Applying Mathematical Modelling*, Ellis Horwood, Chichester, 476-483
- Megarry J, 1988, Hypertext and compact discs: the challenge of multi-media learning, *British Journal of Educational Technology*, 19, 3, 172-183
- MEI, 1994, *MEI Structured Mathematics Students' Handbook*, MEI,
- Merrill MD, 1988, Don't bother me with instructional design, I'm busy programming: suggestions for more effective educational software, *Computers in Human Behavior*, 4, 1, 37-52
- Mestre JP, 1994, Cognitive aspects of learning and teaching Science, in Fitzsimmonds SJ & Kerpelman LC (eds), *Teacher Enhancement for Elementary and Secondary Science and Mathematics*, National Science Foundation (NSF 94-80), Washington DC, USA
- Meyer AD, Tsui AS & Hinings CR, 1993, Configurational approaches to organizational analysis, *Academy of Management Journal*, 36, 6, 1175-1195
- Midoro V, Chiocciariello A, Olimpio G, Persico D, Sarti L & Tavella M, 1988, Interactive video and artificial intelligence: a convenient marriage, *Programmed Learning and Educational Technology*, 25, 4, 299-309
- Middleton JA, 1995, A study of intrinsic motivation in the mathematics classroom: a personal constructs approach, *Journal for Research in Mathematics Education*, 26, 3, 254-279
- Mikellides B, Some questions concerning the interest profiles of prospective architects, *Journal of Architectural and Planning Research*, 6, 3, 240-250
- Millheim WD, 1993, How to use animation in computer assisted learning, *British Journal of Educational Technology*, 24, 3, 171-178
- Moran A, 1991, What can learning styles research learn from cognitive psychology?, *Educational Psychology*, 11, 3&4, 239-245
- Morgan AT, 1990, A study of the difficulties experienced with mathematics by engineering students in higher education, *International Journal of Mathematical Education in Science and Technology*, 21, 6, 975-988
- Moscardini AO, Curran DAS, Saunders R, Lewis BA & Prior DE, 1984, Issues involved in the design of a modelling course, in Berry JS, Burghes DN, Huntley ID, James DJG, Moscardini AO (eds), *Teaching and Applying Mathematical Modelling*, Ellis Horwood, Chichester, 96-106
- Mustoe LR, 1997, *Engineering Maths (sic)*, Addison-Wesley, Harlow
- Mustoe LR & Hibberd S (eds), 1995, *The Mathematical Education of Engineers*, Clarendon Press, Oxford.



- Mustoe LR, 1992, Modelling or muddling through? Mathematics for engineering undergraduates, *International Journal of Mathematical Education in Science and Technology*, 23,4, 575-582
- Noble B, 1969, *Applications of Undergraduate Mathematics in Engineering*, Macmillan, New York, USA
- Norris N, Davies R. & Beattie C, 1990, Evaluating new technology: the case of the Interactive Video in Schools (IVIS) programme, *British Journal of Educational Technology*, 21, 2, 84-94
- O'Carroll MJ, 1995, *Conclusions after a decade of decline*, in Mustoe LR & Hibberd S (eds), *The Mathematical Education of Engineers*, Clarendon Press, Oxford, 379-383
- O'Kane JP, 1995, Engineers and their mathematicians- foundations for a debate on mathematics in the engineering curriculum, in Mustoe LR & Hibberd S (eds), *Mathematical Education of Engineers*, Clarendon Press, Oxford, 361-378
- OECD, 1966, *Mathematical Education of Engineers*, OECD, Paris, France
- Ogden CK & Richards IA, 1923, *The Meaning of Meaning*, Routledge and Kegan Paul, London
- Osborne H, 1983, An experimental approach to the assessment of mathematical abilities, *Teaching Mathematics and its Applications*, Vol 2, No 1, 36-42
- Penrose O, 1978, How can we teach mathematical modelling?, *J. Math. Mod. for Teachers*, 1, 31
- Perkins DN & Simmons R, 1988, Patterns of misunderstanding: an integrative model for science, math and programming, *Review of Science Education*, 58, 3, 303-326
- Perry WG, 1981, Cognitive and ethical growth: the making of meaning, in Chickering AW (ed), *The Modern American College*, Jossey-Bass, London, 76-116
- Perry WG, 1988, Different worlds in the same classroom, in Ramsden P (ed), *Improving Learning: New Perspectives*, Kogan Page, London, 145-161
- Pipes LA & Harvill LR, 1970, *Applied Mathematics for Engineers and Physicists*, McGraw Hill, New York
- Plowman L, 1989, Learning from learning theories: an overview for designers of interactive video, *Interactive Learning International*, 5, , 165-174
- Poincaré H, 1924, *The Foundations of Science* (tr Halstead GB), Science Press, London, 383-394
- Porter RI, 1986, *Further Elementary Analysis*, Bell & Hyman, London
- Presmeg NC, 1986, Visualisation and mathematical giftedness, *Educational Studies in Mathematics*, 17, , 297-311
- Prosser M, 1987, Is prior knowledge of subject matter more important to the development and use of meaningful learning skills?, *Programmed Learning and Educational Technology*, 24, 4, 280-285
- Rae A, 1993, Self-paced learning with video for undergraduates: a multimedia Keller Plan, *British Journal of Educational Technology*, 24, 1, 45-51
- Ramsden P & Entwistle N J, 1981, Effects of academic departments on students' approaches to studying, *Br J Educational Psychology*, Vol 51, , 368-383
- Ricoeur P, 1981, *Hermeneutics and the Human Sciences*, Cambridge University Press, Cambridge, (Ed and Tr J Thompson)



- Riding R & Douglas, G, 1993, The effect of cognitive style and mode of presentation on learning performance, *British Journal of Educational Psychology*, 63, , 297-307
- Riley KF, 1974, *Mathematical Methods for the Physical Sciences*, Cambridge University Press, London
- Romiszowski AJ, 1986, *Developing Auto-Instructional Materials*, Kogan Page, London
- Rose N, 1988, *Mathematical Maxims and Minims*, Rome Press, Raleigh NC, USA
- Royer JM, Cisero CA & Carlo MS, 1993, Techniques and procedures for assessing cognitive skills, *Review of Educational Research*, 63, 2, 201-243
- Sandals LH, 1987, The role of screen design, graphics, colour and sound in computer based learning: how much is too much or too little?, *Proc. Int. Conf. on Computer Assisted Learning in Post-Secondary Education*, Calgary, Canada, 189-196
- Schliemann AD, 1995, Some concerns about bringing everyday mathematics to mathematics education, *Proc 19th PME Conference*, Recife, Brazil
- Schoenfeld AH, 1985, *Mathematical Problem Solving*, Academic Press, New York, USA
- Schoenfeld AH, 1991, An informal attack on the unfortunate divorce of formal and informal mathematics, in Voss JF, Perskins DN & Segal JW (eds), *Informal Reasoning and Education*, Erlbaum, Hillsdale NJ, USA, 311-344
- SEFI, 1992, *A core curriculum in mathematics for the European engineer*, SEFI document 92.2. Paris, France
- SEO/EC/SCSST, 1996, *A Mathematical Foundation*, Society of Education Officers, Manchester, Engineering Council, London, Standing Conference for Schools' Science and Technology, London
- Shaw MP, 1989, The eureka process: a structure for the creative experience in science and engineering, *Creativity Research Journal*, 2, 4, 286-298
- Smith D, 1988, Institutional contexts for interactive learning media (or semiology meets the 'Beeb'), *Programmed Learning and Educational Technology*, 25, 4, 340-344
- Smith PR, 1992, Computer mediated learning in engineering education, *Computers and Education* 19, 1/2, 37-47
- Sokal AD, 1997, What the Social Text affair does and does not prove, in Koertge N (ed), *A House Built on Sand: Exposing Postmodern Myths about Science*, Oxford University Press, Oxford
- Spencer AJM et al, 1997, *Engineering Mathematics*, Van Nostrand Reinhold, Wokingham, UK
- Spencer K, 1991, Modes, media and methods: the search for educational effectiveness, *British Journal of Educational Technology*, 22, 1, 12-22
- Spiegel MR, 1974, *Schaum's Outline of Theory and Problems of Advanced Calculus*, McGraw-Hill, New York
- Stephenson GM, 1973, *Mathematical Methods for Science Students*, Longmans, London
- Stewart J, 1988, *Calculus*, Brooks/Cole, Monterey, CA, USA
- Stroud KA, 1990, *Further Engineering Mathematics*, 2nd edition, Macmillan, Basingstoke
- Stroud KA, 1995, *Engineering Mathematics*, 4th edition, Macmillan, Basingstoke
- Sutherland R & Pozzi S, 1995, *The Changing Mathematical Background of Undergraduate Engineers*, Engineering Council, London



- Sweeters WG, 1985, Screen design guidelines, *26th ADCIS Proceedings*, Association of the Development of Computer Based Instructional Systems, Philadelphia, US
- Sweller J, Chandler P, Tierney P & Cooper M, 1990, Cognitive load as a factor in the structuring of technical material, *Journal of Experimental Psychology: General*, 119, 2, 176-192
- Tall DO (ed), 1991, *Advanced Mathematical Thinking*, Kluwer Academic Publishers, Dordrecht, Netherlands
- Tall DO, 1995, Cognitive growth in elementary and advanced mathematical thinking, *Proc 19th PME Conference*, Recife, Brazil
- Thompson SV, 1990, Visual imagery: a discussion, *Educational Psychology*, 10, 2, 141-167
- Tunncliffe B, 1981, *MST204.3: Animal populations: their growth and exploitation*, Open University, Milton Keynes
- Venables E, 1967, *The Young Worker at College*, Faber, London
- Vinner S, 1991, The role of definitions in the teaching and learning of mathematics, in Tall D (ed), *Advanced Mathematical Thinking*, Kluwer Academic Publishers, Dordrecht, Netherlands, 65-81
- Watkins AJP, 1993, A new approach to mathematics for engineers, *International Journal for Mathematics Education in Science and Technology*, 24, 5, 689-702
- Weltner K, Grosjean J, Schuster P & Weber WJ, 1986, *Mathematics for Engineers and Scientists*, Thornes, Cheltenham
- White PA, 1972, Socialization and education, in Dearden RF, Hirst PH & Peters RS (eds), *Education and the Development of Reason*, 113-131, Routledge and Kegan Paul, London
- Whitelock D, Taylor J, O'Shea T, Scanlon E, Sellman N, Clark P & O'Malley C, 1993, Challenging models of elastic collisions with a computer simulation, *Computers and Education*, 20, 1, 1-9
- Wildman T, 1981, Cognitive theory and the design of instruction, *Educational Technology*, 21, 7, 14-21
- Wilson BG, Teslow JL & Taylor L, 1993, Instructional design perspectives on mathematics education with reference to Vygotsky's theory of social cognition, *Focus on Learning Problems in Mathematics*, 15, 2&3, 65-86
- Young M, 1995, Assessment of situated learning using computer environments, *Journal of Science Education and Technology*, 4, 1,

