GAUGE INVARIANT CONSTRUCTIONS IN YANG-MILLS THEORIES

by

POONAM SHARMA

A thesis submitted to Plymouth University
in partial fulfilment for the degree of

DOCTOR OF PHILOSOPHY

School of Computing and Mathematics
Faculty of Science and Technology

March 2012

Poonam Sharma

Gauge Invariant Constructions in Yang-Mills Theories.

Abstract

Understanding physical configurations and how these can emerge from the underlying gauge theory is a fundamental problem in modern particle physics. This thesis investigates the study of these configurations primarily focussing on the need for gauge invariance in constructing the gauge invariant fields for any physical theory. We consider Wu's approach to gauge invariance by identifying the gauge symmetry preserving conditions in quantum electrodynamics and demonstrate how Wu's conditions for one-loop order calculations (under various regularisation schemes) leads to the maintenance of gauge invariance. The need for gauge invariance is stressed and the consequences discussed in terms of the Ward identities for which various examples and proofs are presented in this thesis. We next consider Zwanziger's description of a mass term in Yang-Mills theory, where an expansion is introduced in terms of the quadratic and cubic powers of the field strength. Although Zwanziger introduced this expansion there is, however, no derivation or discussion about how it arises and how it may be extended to higher orders. We show how Zwanziger's expansion in terms of the inverse covariant Laplacian can be derived and extended to higher orders. An explicit derivation is presented, for the first time, for the next to next to leading order term. The role of dressings and their factorisation lies at the heart of this analysis.

Contents

List of Figures

Acknowledgements

At this point I want to take the opportunity to thank everyone who supported me in one way or the other in the completion of this thesis.

First of all I would like to thank my supervisor, Professor David McMullan, for his invaluable support, guidance and inspiration throughout my PhD. His dedicated supervision and unique knowledge has made my research in Plymouth a valuable learning experience. I would also like to thank my second supervisor, Dr. Martin Lavelle, for his constant support especially in the completion of this piece of work. I am also very thankful to Dr. Kurt Langfeld and Dr. Tom Heinzl for always being there to answer my questions and suggesting ways out of my problems.

I am grateful to Plymouth University for the financial support during the research work. A great thank you to everyone in the School of Computing and Mathematics for providing such a friendly and warm environment. Special thanks to my friends and fellow colleagues, in particular to Yanwei Lu, Merfat Raddadi, Golnaz Shahtahmassebi and Christopher Harvey, whose friendship has made my time in Plymouth so special.

Last but certainly not the least I have to mention the unparalleled support from my family for their love and support. I would like to thank my husband, Nitin, for believing in me and making me strong and positive in the most difficult circumstances, my parents, for having the determination and motivation to make the most of some very difficult times, my elder sister and younger brother for the support and encouragement- this thesis is dedicated to my family!!

Author's Declaration

At no time during the registration for the degree of Doctor of Philosophy has the author been registered for any other University award without prior agreement of the Graduate Committee.

This study was financed with the aid of an Overseas Research Scholarship (ORS) and Graduate School Scholarship (GSS-PGR) from the Plymouth University.

Relevant scientific seminars and conferences were attended at which work was presented. Various papers were prepared for publication.

Publications:
Scientific Papers
The Factorisation of glue and mass terms in SU(N) gauge theories.
Martin Lavelle, David McMullan and Poonam Sharma.
Phys. Rev. D **85**, 045013 (2012).

Covariant Laplacian and the Volkov fields.
Martin Lavelle, David McMullan and Poonam Sharma.
(In preparation.)

Presentations given:
Poster Presentation on Volkov States and the Volkov Propagator.
Rutherford Appleton Laboratory, UK.
15-17 December 2010.

Gauge Invariance and Regularisation.
Plymouth University, UK.
2 July 2009.

Conferences attended:
Christmas Meeting of the High-Power Laser Science Community.
Rutherford Appleton Laboratory, UK.
15-17 December 2010.

PROTTEC Summer School.
University of Plymouth and University of Exeter, UK.
21-25 June 2010.

Annual Theory Meeting.
University of Durham, UK.
18-20 December 2008.

Word count of main body of thesis: 23,800

Signed:...................................

Date:...................................

Chapter 1

Introduction and Motivation

1.1 The Standard Model

The standard model of particle physics must rank as one of the greatest achievement of the 20th century. It was formulated in the 1970's and tentatively established by experiments in the early 1980's. Despite the word model in its name, it is a comprehensive theory that identifies the basic particles and specifies how they interact and gain mass. Everything we know with confidence about particle physics is included in the standard model. It is a relativistic quantum field theory based on the principles of special relativity, quantum mechanics and local gauge invariance. In this chapter we give a brief introduction of the standard model focussing primarily on the role of symmetries and in particular, that of gauge invariance. For a more comprehensive study we refer the reader to textbooks and resources [1–9]. A detailed historical account is given in the book by Pais [10].

The standard model describes the three fundamental forces of nature, the electromagnetic, strong and weak forces and the interactions amongst the particles that experience these forces. Although all the matter in the universe is held together by gravity, it is not yet understood as a quantum theory and is thus not part of the standard model [11, 12]. Each of the forces in the standard model can be described in terms of matter fields, quarks and leptons, interacting via force-carrying fields.

Generally, the language of standard model is quantum field theory (QFT) [13–19] which is a many-body theory where the number of particles involved in an interaction (or collision) is not conserved. It is used to describe the particles not as discrete entities but in terms of mathematical entities, called fields. Particle interactions are in large part dictated by the symmetry principles which are described by the Lagrangian. A theory was developed in the early 1930's that was relativistically invariant and quantised which explained the electromagnetic interactions of electrons e^- and photons γ. This theory, the quantum version of Maxwell theory, is called Quantum Electrodynamics (QED). It is a gauge theory with symmetry group $U(1)$ which is specified by its set of field variables and its Lagrangian density $\mathcal{L}(\psi(x), A_\mu(x))$ where the particles, electron and photon participating in the interactions now become fields, $\psi(x)$ and $A_\mu(x)$. The photon which is the quantum of light is now described in terms of the vector field (potential) $A_\mu(x)$ where the label μ is a space-time vector with four values at each point. However photons have only two degrees of freedom corresponding to its polarisation states. This implies that there are unphysical components to A_μ that require some mechanism to remove them. In much the same way the spinor fields, $\psi(x)$ that enter the Lagrangian cannot be identified with the charged particles because of the gauge symmetry and hence require gauge invariance [20–23] to properly define them. The rest of the standard model is obtained by extending the gauge principles to produce theories of the other fundamental forces.

All physical quantities and electromagnetic fields must be gauge invariant. The distinctive nature of gauge invariance is based on the symmetry transformations being local. For a local symmetry, the element of the symmetry group $U = e^{i\theta(x)}$ is a function of space-time coordinate x^μ whereas for a global symmetry a fixed group element θ that is independent of x acts on fields at different space-time points. One of the important consequences of gauge invariance in these quantum field theories are the Ward identities [24, 25]. These are relations between, for example, vertex functions and propagators which play a key role in the proof of renormalisability of this theory. Such Ward identities[1] will be discussed in more detail in the next chapter of this thesis in the context of QED.

Throughout this thesis our aim will be to understand how to construct gauge invariant objects.

1.2 Overview of Thesis

The next chapter describes the tools needed to analyse gauge invariance in an abelian gauge theory and deals with the perturbative calculations to one-loop. This allows us to then present a new regularisation scheme originally introduced by Wu. We apply Wu's scheme to calculate one-loop integrals in QED which are summarised explicitly. Chapters 3, 4 and 5 deal with how the principle of gauge invariance can be applied to construct gauge invariant mass operators. Chapter 3 is divided into two parts. The first part briefly reviews non-abelian gauge theory and the Lie algebra which is needed for calculations in the following chapters. A brief introduction to Yang-Mills theory is presented along with the study of various representations of the Lie algebra. In the second part of that chapter the general idea of the dressing approach is introduced which is then used to construct gauge invariant configurations and the physical field strength. The relations between two different gauge invariant field strengths in non-abelian Yang-Mills theory are discussed. Chapter 4 deals with

[1]More generally in non-abelian theories, these type of identities are often known as Slavnov-Taylor identities [26].

the covariant Laplacian and some of its properties are discussed. This then allows us to introduce, understand and derive Zwanziger's expansion in Chapter 5. Chapter 6 summarises the conclusion and outlook.

Chapter 2

Gauge Invariant Calculations

2.1 Abelian Gauge Theory

In this chapter we will study the simplest gauge theory, which is the $U(1)$ gauge theory that describes quantum electrodynamics (QED). This chapter is divided into two main sections. In the first we will give a brief description of the QED Lagrangian and the perturbation theory with emphasis on the machinery required to produce one-loop integrals in QED. However as we will see the integrals involved are plagued by various divergences so to make them finite various regularisation schemes are adopted. We describe the renormalisation technique which is used to remove these divergences and discuss the consequences of gauge invariance, in terms of Ward identities. Along the same lines, in the second section we will discuss the impact of gauge invariance in field theory which has recently been proposed by Wu [27] in terms of various regularisation independent identities. Wu introduced these identities in a very general context and claimed that they are all that is needed in order to maintain gauge invariance. We will investigate these identities in QED and see their role in deriving the Ward identities. We will then investigate the origin of these identities.

Quantum electrodynamics is the theory of light interacting with charged matter *i.e.* photons and leptons where the photons are described by the quantised Maxwell field $A^\mu(x)$ via the Lagrangian density,

$$\mathcal{L}_M(x) = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) \quad \text{with the field strength} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{2.1}$$

and leptons (electrons, muons etc.) are identified with the quantised Dirac field $\psi(x)$ with the Lagrangian density,

$$\mathcal{L}_D(x) = \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x)\,, \tag{2.2}$$

where γ^μ are the usual γ matrices. The interaction between these two sectors of QED is determined by the Lagrangian

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - e\bar{\psi}\gamma^\mu A_\mu\psi \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi\,,\end{aligned} \tag{2.3}$$

with the requirement that the Dirac field in (2.2) possess local symmetry. Note that in (2.3) as compared to (2.2) we have replaced $\gamma^\mu\partial_\mu$ by $\gamma^\mu D_\mu$ where $D_\mu = \partial_\mu + ieA_\mu$ is called the covariant derivative. The necessity of introducing the covariant derivative is that it transforms covariantly

$$D_\mu\psi(x) \to e^{-ie\theta(x)}D_\mu\psi(x)\,, \tag{2.4}$$

under the local gauge transformations

$$\psi(x) \to e^{-ie\theta(x)}\psi(x)\,, \tag{2.5}$$

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\theta(x)\,, \tag{2.6}$$

where $\theta(x)$ now depends on each space-time point. So, the coupled Dirac equation is:

$$(i\gamma^\mu D_\mu - m)\psi = 0\,. \tag{2.7}$$

The new interaction term in (2.3) may be written as

$$\bar{\psi}\gamma^\mu A_\mu \psi = j^\mu A_\mu\,, \tag{2.8}$$

with $j^\mu = (\rho, \mathbf{j})$ where ρ and $\mathbf{j}$ are the charge and the current densities. The interaction term thus describes the coupling of the gauge field A^μ to the Dirac current.

The conservation of current

$$\partial_\mu\, j^\mu(x) = 0\,, \tag{2.9}$$

follows from the equations of motion. The existence of a conserved current is a direct consequence of Noether's theorem [28] which states that we get a conserved current under a global symmetry, see *e.g.* [29, 30]. This current is the Noether's current associated with the global gauge transformations *i.e.* those θ which do not depend on space-time position.

Because of gauge invariance, there are complications when we quantise the theory. A naive quantisation of the Maxwell theory fails for a simple reason: the photon propagator does not then exist. This is because the canonical momenta for the temporal component of the gauge fields vanish. If $A_\mu(x)$ is the vector potential then the momentum π^μ conjugate to $A_\mu(x)$ is

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu(x)} = -F^{0\mu}\,, \tag{2.10}$$

and the momentum π^0 conjugate to $A_0(x)$ is

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0(x)} = 0\,, \tag{2.11}$$

which vanishes implying that $A_0(x)$ is not a dynamical field. This means that this gauge theory is a constrained system. However, the momentum π^i conjugate to $A_i(x)$ is

$$\pi^i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i(x)} = -F^{0i} = E^i\,, \tag{2.12}$$

which imposes Gauss' law and is now an additional constraint on the system. The physical consequence of this is that the potential A_μ which is used to represent the photon with four degrees of freedom now really has two degrees of freedom that corresponds to the two photon transverse polarisation states. Gauge invariance is the redundancy in this covariant description of the physics. We undo this "over-counting" (at least in the abelian theory), by gauge fixing which corresponds to adding a term to the Lagrangian to break gauge invariance. Physical quantities are gauge invariant and, hence, should not depend on the choice of gauge.

Using this procedure we now have to modify the Lagrangian density (2.3) by adding an extra term such as $\frac{-1}{2\xi}(\partial_\mu A^\mu)^2$ where ξ represents a real arbitrary constant parameter known as the gauge fixing parameter. Therefore the Lagrangian density gets modified and we have:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{\partial} - m)\psi - e\bar{\psi}\not{A}\psi - \frac{1}{2\xi}(\partial_\mu A^\mu)^2\,, \tag{2.13}$$

where we define $\not{\partial} = \gamma^\mu\partial_\mu$ and $\not{A} = \gamma^\mu A_\mu$. This additional term, called the gauge fixing term, breaks gauge invariance and allows us to define the propagator for the gauge field. Other choices are allowed but this Lorentz class has the advantage of being covariant.

The photon propagator (of momentum q) for an arbitrary value of ξ is given by:

$$iD_{\mu\nu}(q) = \frac{-i}{q^2 + i\epsilon}\left(g_{\mu\nu} + (\xi - 1)\frac{q_\mu q_\nu}{q^2}\right). \tag{2.14}$$

Naively, physics should not depend on ξ. However for different values of ξ we have different gauges and some of these have distinct computational advantages *i.e.*

when $\xi = 0$, the gauge is called the *Landau* gauge,

when $\xi = 1$, the gauge is called the *Feynman* gauge and this is the gauge we will be working in. The propagator[1] in this gauge is:

$$iD_{\mu\nu}(q) = \frac{-ig_{\mu\nu}}{q^2 + i\epsilon}. \tag{2.15}$$

Given the Lagrangian density (2.13), how can we perform calculations in QED? The most widely used route to calculate the perturbative predictions of any process is to identify the Green's function of the theory with the corresponding Feynman diagrams [31]. For any process we can draw a diagram, in terms of Green's functions and extract mathematical expressions from them. For example, the two-point Green's function or propagator gives the amplitude for a particle to travel from one place to another in a given time. The probability amplitude for particle interactions are calculated using Feynman diagrams in momentum space that are illustrated in Appendix A.1. In order to relate general Green's functions to the physical results in field theory one makes use of the LSZ (Lehmann Symannzik Zimmermann) description of the S-Matrix formalism. The details of that method can be found in Appendix A.2.

[1] To avoid singularities and impose the correct causal structure we make use of $i\epsilon$ prescription. However, this will often be suppressed in our calculations.

As a typical example of a calculation in QED let us consider the full photon propagator represented by a series of graphs shown in Fig. 2.1. The diagram illustrates

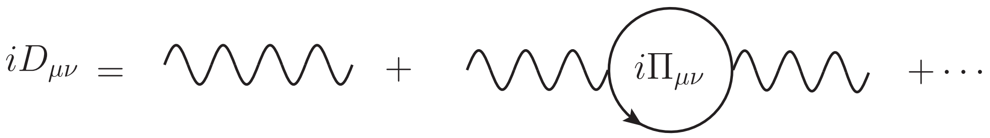


Figure 2.1: Full photon propagator.

that the full photon propagator can be described by the sum of a tree level diagram and the contribution from the radiative corrections written in the form:

$$iD_{\mu\nu}(q) = iD^{(0)}_{\mu\nu}(q) + iD^{(0)}_{\mu\rho}(q)(i\Pi^{\rho\sigma}(q))iD^{(0)}_{\sigma\nu}(q) + \cdots\,, \tag{2.16}$$

where the bracketed superscript refers to the power of the coupling. In Feynman gauge the full propagator (2.16) can be written as

$$\begin{aligned} iD_{\mu\nu}(q) &= -i\frac{g_{\mu\nu}}{q^2} + \Big(-i\frac{g_{\mu\rho}}{q^2}\Big)(i\Pi^{\rho\sigma}(q))\Big(-i\frac{g_{\sigma\nu}}{q^2}\Big) + \cdots\,, \\ &= -i\frac{g_{\mu\nu}}{q^2} + \Big(\frac{-i}{q^2}\Big)(i\Pi_{\mu\nu}(q))\Big(\frac{-i}{q^2}\Big) + \cdots\,, \end{aligned} \tag{2.17}$$

where $i\Pi_{\mu\nu}(q)$ denotes the photon polarisation tensor at one-loop also called the vacuum polarisation (with external photon lines omitted). The Feynman diagram for this is given by Fig. 2.2. Now applying Feynman rules to the one-loop graph of

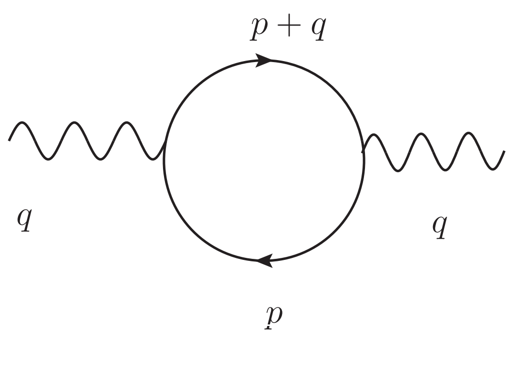


Figure 2.2: One-loop correction to the photon propagator.

Fig. 2.2 we obtain:

$$i\Pi_{\mu\nu}(q) = -e^2 \int \frac{d^4p}{(2\pi)^4}\ \mathrm{tr}\left(\gamma_\mu \frac{1}{\not{p} - m}\gamma_\nu \frac{1}{\not{p} + \not{q} - m}\right), \tag{2.18}$$

If we simplify the integrand (2.18) using the properties of gamma matrices as summarised in Appendix A.3, then we find that the integration over the loop momentum involves integrands such as:

$$\int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 - m^2)[(p+q)^2 - m^2]}\,. \tag{2.19}$$

At large momentum p, this integral clearly diverges as the integrand tends to $1/p^4$ and

$$\int \frac{d^4p}{(2\pi)^4} \frac{1}{p^4} \sim \infty\,. \tag{2.20}$$

We call this an ultra-violet (UV) divergence where the loop-momentum p takes arbitrary large values (that is, they come from region when $p \to \infty$). In the same way we can have infra-red (IR) divergences which are associated with the massless fields such as the photon, which come from the region where $p \to 0$ when external lines are on-shell. Similar stories hold for the one and higher loop contributions to the electron propagator and to the three point vertex as well. To make sense of these divergences we have to regularise our integral which is the next topic.

2.1.1 Regularisation

The most important feature required from a regularisation is that it should maintain the basic symmetry principles of the theory, such as gauge invariance, translational invariance, Lorentz invariance [32, 33] etc. Many regularisation methods have been introduced over the years. For example:

(1)Cut-off Regularisation: Here we set an upper bound 'Λ' to the loop momentum and then take the limit $\Lambda \to \infty$ at the end of the calculation. This is a very direct method that is easy to understand but it leads to the violation of gauge invariance because of the cut-off parameter. Therefore this method is not convenient in dealing with the theories that have local symmetries.

(2)Pauli-Villars (PV) Regularisation: In Pauli Villars regularisation [34], the propagator is modified by the addition of a fictitious field of mass $M > m$ *i.e.*

$$\frac{1}{p^2 - m^2} \to \frac{1}{p^2 - m^2} - \frac{1}{p^2 - M^2}\,, \tag{2.21}$$

where the limit, M tends to infinity, is taken at the end of a given calculation. Gauge invariance is preserved in abelian theories but this method becomes problematic when non-abelian theories are considered.

(3)Dimensional Regularisation: This is one of the most popular regularisation schemes as it maintains gauge invariance even in non-abelian gauge theories. Dimensional regularisation consists in modifying the dimensionality of integrals like (2.19) so that they become finite. The idea is to treat the loop integral as integrals over D-dimensional momenta with $D = 4 - 2\varepsilon$ where the limit $\varepsilon \to 0$ is taken at the end of a calculation. For example, the following integral

$$\int d^4p \frac{1}{(p^2 + m^2)^2} \approx \int^{\Lambda} \frac{dp}{p} \approx \ln \Lambda\,, \tag{2.22}$$

has a logarithmic (log) divergence when we put an upper cut-off Λ, in the momentum integral to regulate it. However, in D-dimensions the integral is modified as

$$\int \frac{d^{4-2\varepsilon}p}{p^4} \sim \int \frac{dp}{p^{1+2\varepsilon}}\,, \tag{2.23}$$

where we have effectively increased the power of p in the denominator making the integral finite. Space-time is then viewed as D-dimensional and UV singularities reveal themselves as pole terms in ε. We summarise the prescription for this scheme in Appendix A.4.

2.1.2 Renormalisation

We have seen above various ways to regularise our integrals. In order to remove the infinities appearing in loop integrals we must develop a way such that these infinite integrals make sense in perturbation theory. This method is called renormalisation[2] which was systematically developed in QED by Freeman Dyson [38] in 1948. The discussion of these infinities has been in terms of bare fields and bare parameters appearing in the Lagrangian (2.13). The main idea of renormalisation is to correct the original Lagrangian of QED by an infinite series of counterterms. The original Lagrangian which is a function of (m, e, A, ψ) is not the physical Lagrangian but a bare Lagrangian

$$\mathcal{L}_b = \bar{\psi}_b(i\partial\!\!\!/ - m_b)\psi_b - e_b\bar{\psi}_b A_{\mu b}\gamma^\mu\psi_b - \frac{1}{4}F_{\mu\nu b}F^{\mu\nu b} - \frac{1}{2}(\partial^\mu A_{\mu b})^2\,, \qquad (2.24)$$

consisting of the parameters (m_b, e_b, A_b, ψ_b) that are not measurable and hence are infinite. The UV divergences [39] in our calculations show up if we try to express our results in terms of the parameters of the bare Lagrangian. In order to save this situation we adopt the method of renormalisation so that the result of any physical quantity expressed in terms of new parameters in any perturbation theory is finite. This method enables us to pass from the bare theory to finite predictions. In order to achieve a well defined theory for QED and the non-abelian Yang-Mills theory, all

[2]The idea that gauge theories are renormalisable was first put forward by G.'tHooft [35–37] and Veltman [36] in 1973.

we have to do is, relate the bare parameters to the physical (renormalised) ones by multiplicative factors called renormalisation constants z_1, z_2, z_3, z_m as:

$$\psi_b = \sqrt{z_2}\psi_r \qquad z_2 = 1 + \delta_2\,, \tag{2.25}$$

$$A_b = \sqrt{z_3}A_r \qquad z_3 = 1 + \delta_3\,, \tag{2.26}$$

$$m_b = z_m m_r \qquad z_m = 1 + \frac{\delta m}{m}\,, \tag{2.27}$$

$$e_b = \frac{z_1}{z_2\sqrt{z_3}}e_r \qquad z_1 = 1 + \delta_1\,, \tag{2.28}$$

where $\delta_1, \delta_2, \delta_3, \delta_m$ are counterterms. These counterterms will compensate for the UV divergences (but there are still finite terms left). In some cases all the necessary counterterms can be obtained by modifying the parameters that appear in the original Lagrangian. Thus, when counterterms are added to the bare field the Lagrangian (2.24) becomes:

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}(F_r^{\mu\nu})^2 + \bar{\psi}_r(i\partial\!\!\!/ - m)\psi_r - e\bar{\psi}_r\gamma^\mu\psi_r A_{r\mu} - \frac{1}{2}(\partial^\mu A_{\mu r})^2 \\ &- \frac{1}{4}\delta_3(F_r^{\mu\nu})^2 + \bar{\psi}_r(i\delta_2\partial\!\!\!/ - \delta_m)\psi_r - e\delta_1\bar{\psi}_r\gamma^\mu\psi_r A_{r\mu}\,.\end{aligned} \tag{2.29}$$

To complete the specification of the renormalised theory we need to fix the four unknown constants. There are various renormalisation schemes such as the MS scheme [40] , $\overline{MS}$ scheme [41] or the on-shell scheme that one can use to specify these constants. For the calculations presented in this chapter we shall use the on-shell scheme as this scheme gives the route to S-Matrix calculations in QED.

2.1.3 Consequences and Examples of Gauge Invariance

As seen earlier the principle of gauge invariance under local phase transformations completely specifies the interaction between a fermion and the gauge boson. This symmetry leads to Ward identities in QED which are a reflection of the continuous symmetries present in the system. As we have seen earlier that classically, charge is conserved, so the Ward identity is this claim at the quantum level. For a comprehensive study on Ward identities we refer the reader to go through the following references [30, 33, 42–46].

The Ward identity is an identity which describes physically possible scattering processes with S-matrix elements and thus have all their external particles on-shell. To understand this we assume that there is an arbitrary QED process with the amplitude $\mathcal{M}(q)$ that involves an external outgoing photon with momentum q and the polarisation vector $\epsilon^*_\mu(q)$. This process can be written as

$$\mathcal{M}(q) = \epsilon^*_\mu(q)\mathcal{M}^\mu(q)\,, \tag{2.30}$$

where we have factorised the $\epsilon^*_\mu(q)$ dependence and $\mathcal{M}^\mu(q)$ is the rest of the amplitude. However we have seen earlier that the interaction term $j^\mu A_\mu$ in QED is responsible for the creation of external photons. Therefore the amplitude $\mathcal{M}^\mu(q)$ (in momentum space) can be defined by including the fourier transform of the matrix element (of the Heisenberg field) of the Dirac vector current as:

$$\mathcal{M}^\mu(q) \sim \int \mathrm{d}^4x\; e^{iq\cdot x}\langle f \mid j^\mu(x) \mid i\rangle\,, \tag{2.31}$$

where i, f are the initial and final states which include all the particles (except the photon in question). Now if we do the scalar product of q_μ into (2.31), that is,

$$q_\mu\mathcal{M}^\mu(q) \sim \int \mathrm{d}^4x\; e^{iq\cdot x}\partial_\mu\langle f \mid j^\mu(x) \mid i\rangle\,, \tag{2.32}$$

where we have used the equivalence of $q_\mu e^{iq\cdot x}$ with $-i\partial_\mu e^{iq\cdot x}$ in momentum (Fourier) space and also integrated by parts.

Using the current conservation equation (2.9) we have for (2.32)

$$q_\mu \mathcal{M}^\mu(q) = 0\,, \tag{2.33}$$

which implies that the classical conserved vector current introduced in (2.9) is in fact conserved at the quantum level.

The first example of the use of the Ward identity [47] is seen when we regulate a divergent loop integral in high order processes. The Ward identity allows us to properly determine the exact form of the photon propagator. If $\Pi_{\mu\nu}(q)$ is the sum of all 1-particle-irreducible insertions into the photon propagator with momentum q then according to the Ward identity

$$q^\mu \Pi_{\mu\nu}(q) = 0\,. \tag{2.34}$$

By Lorentz invariance the only tensors that can appear in $\Pi_{\mu\nu}(q)$ are $g^{\mu\nu}$ and $q^\mu q^\nu$. This constraint allows us to make the following decomposition

$$\Pi_{\mu\nu}(q) = (q^2 g_{\mu\nu} - q_\mu q_\nu)\Pi(q^2)\,, \tag{2.35}$$

where $\Pi(q^2)$ is regular at $q^2 = 0$. To see this we write the exact photon two point function (2.17) as

$$\begin{aligned}
iD_{\mu\nu}(q) &= -i\frac{g_{\mu\nu}}{q^2} + \left(\frac{-i}{q^2}\right)\Big\{i(q^2 g_{\mu\nu} - q_\mu q_\nu)\Pi(q^2)\Big\}\left(\frac{-i}{q^2}\right) + \cdots \\
&= \frac{-i}{q^2}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \frac{-i}{q^2}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\Pi(q^2) + \frac{-iq_\mu q_\nu}{q^4} + \cdots \\
&= \frac{-i}{q^2(1-\Pi(q^2))}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \frac{-iq_\mu q_\nu}{q^4} + \cdots\,.
\end{aligned} \tag{2.36}$$

For any S-matrix calculation the terms proportional to q_μ or q_ν vanish when the propagator is coupled to external charges j^ν or currents owing to current conservation. Hence we can summarise (2.36) as,

$$ij^\nu D_{\mu\nu}(q) = \frac{-ig_{\mu\nu}}{q^2(1-\Pi(q^2))} j^\nu \,. \tag{2.37}$$

The exact propagator has a pole at $q^2 = 0$ which implies that the photon has zero mass and receives no correction from higher orders.

Suppose we consider a theory with massive vector bosons [48] then the propagator in momentum space is defined as:

$$iD_{\mu\nu}(q) = -i\frac{(g_{\mu\nu} - q_\mu q_\nu/m^2)}{q^2 - m^2} \,. \tag{2.38}$$

If we wished to calculate the photon propagator from (2.38) then we must set $m = 0$, which is problematic for the numerator part. This is where the Ward identity comes into use according to which any term in the photon propagator that is proportional to q_μ or q_ν does not contribute to S-matrix element and can be ignored thus contributing to the photon propagator (2.15). Physically, what this identity means is the longitudinal polarisation of the photon which arises in an arbitrary ξ gauge is unphysical and disappears from the S-matrix.

Using current conservation also leads to a Ward identity in the vertex function $\Gamma^\mu(p_1, p_2; q)$ represented by Fig. 2.3. The effective vertex captures the full electromagnetic properties of a spinor with one incoming (with momentum p_1) and one outgoing electron (with momentum p_2) interacting with an external photon with momentum $q = p_2 - p_1 \neq 0$. Thus for the fermion-boson vertex of quantum electro-

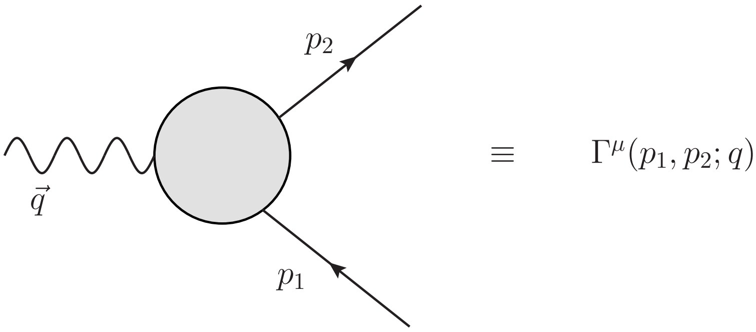


Figure 2.3: Vertex function.

dynamics at tree level we have

$$\begin{aligned} ieq_\mu \Gamma^\mu(p_1, p_2; q) = ieq_\mu \gamma^\mu &= ie(p_2 - p_1)_\mu \gamma^\mu \\ &= ie(\not{p}_2 - m - \not{p}_1 + m) \\ &= ie\big[S_0^{-1}(p_2) - S_0^{-1}(p_1)\big] \, , \end{aligned} \tag{2.39}$$

where $S_0^{-1}(p) = (\not{p} - m)$ is the electron self energy at tree level. However in the limit when $q \to 0$ we obtain

$$\lim_{q \to 0} \Gamma^\mu(p_1, p_1) = ie\gamma^\mu \, , \tag{2.40}$$

which describes an interaction of the spinor with a static potential, measuring electric charge.

The result (2.39) can be generalised to all orders of the perturbation theory for which we consider Fig. 2.4 that contains on the left hand side, the three-point vertex function and on the right hand side the exact electron propagators evaluated at p_1 and p_2. We can now write the amplitudes for both the sides of Fig. 2.4 in which case the Ward identity is:

$$S(p_2)[ieq_\mu \Gamma^\mu(p_1, p_2)]S(p_1) = e[S(p_1) - S(p_2)] \, . \tag{2.41}$$

To further simplify this equation we multiply the left and right hand side by the

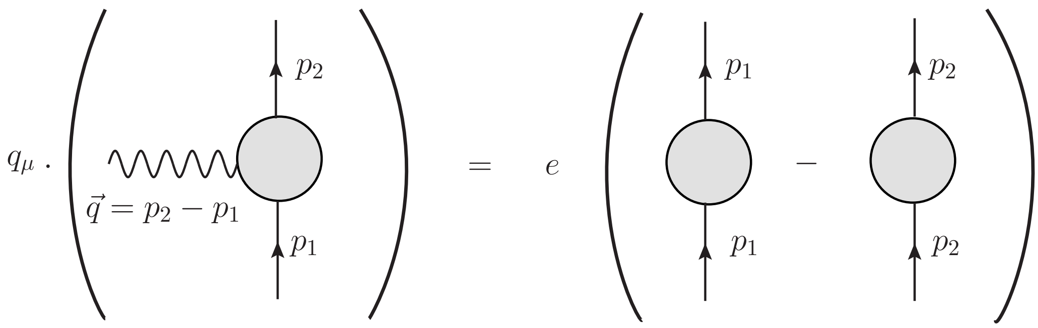


Figure 2.4: Example of Ward identity.

Dirac matrices $S^{-1}(p_2)$ and $S^{-1}(p_1)$ which gives:

$$iq_\mu \Gamma^\mu(p_1, p_2) = [S^{-1}(p_2) - S^{-1}(p_1)] \, . \tag{2.42}$$

The Ward identity can be used to obtain the general relation between the fermion two point function (self-energy) and the vertex function in QED which states the equivalence of the renormalisation scale factors, z_1 for vertex and renormalisation scale factor, z_2 of self energy to all orders *i.e.*

$$z_1 = z_2 \, . \tag{2.43}$$

This implies that if there are divergent parts in the fermion self-energy graphs at higher order, then they must equal those present in the vertex correction graphs at the same order. This relation guarantees the exact cancellation of infinite rescaling factors. As a consequence of (2.43), equation (2.28) reduces to

$$e_r = \sqrt{z}_3 e_b \, , \tag{2.44}$$

which clearly illustrates that the charge renormalisation is not dependent on fermion self energy or vertex modifications but originates from the photon self energy effects.

More discussion and examples of the use of Ward identities will be in the later sections of this chapter.

2.2 Wu's Regularisation Scheme

In order to evaluate loop integrals in Feynman diagrams one makes use of the Feynman parameterisation method (details of which are included in Appendix B.1). This method enables us to reduce loop integrals in Feynman diagrams into irreducible loop integrals, which are generally of the form

$$\int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 - M^2)^n}\,, \tag{2.45}$$

where p is the loop momenta, M^2 are mass-factors which depend on Feynman parameters and n is a parameter which takes into account various processes. By power counting in four dimensions, when $n = 1$ we have quadratic divergences, when $n = 2$ we have logarithmic divergences and for $n > 2$ the integral converges.

There are various techniques available to make the quantum field theory finite and physically meaningful. However, not all of regularisation schemes preserve all symmetries of the original theory. In particular, the construction of a regularisation which respects the non-abelian gauge symmetry has turned out to be a difficult task. As seen earlier, although dimensional regularisation is the most popular gauge symmetry preserving regularisation however it is nice to have alternatives. This has been considered by Wu [49, 50] where he introduced a set of regularisation independent consistency conditions which confirms the gauge invariance of one-loop graphs in QCD.

In this section we will introduce Wu's identities and describe the set of consistency conditions which he claims ensures the gauge invariance for all one-loop graphs in the non-abelian theory. We will consider the abelian theory and try to understand Wu's identities. Using Wu's consistency conditions our first approach will be to verify the Ward identity in an arbitrary regularisation scheme. Once this is achieved we will then verify that Wu's identities hold in various regularisation schemes such as dimensional regularisation and Pauli Villars both of which respect gauge invariance in QED. To study Wu's regularisation method we will first verify that the Ward identity (2.34) holds in QED. It is also seen that with the use of Wu's conditions, the one-loop calculations can be performed rapidly.

To investigate the gauge symmetry preserving conditions in QCD at one-loop, Wu considered a set of irreducible loop integrals that are evaluated from the one-loop Feynman diagrams. These integrals (in his notation) are given by:

$$I_{-2\alpha} = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - M^2)^{2+\alpha}} , \tag{2.46}$$

$$I_{-2\alpha\,\mu\nu} = \int \frac{d^4k}{(2\pi)^4} \frac{k_\mu k_\nu}{(k^2 - M^2)^{3+\alpha}} , \tag{2.47}$$

$$I_{-2\alpha\,\mu\nu\rho\sigma} = \int \frac{d^4k}{(2\pi)^4} \frac{k_\mu k_\nu k_\rho k_\sigma}{(k^2 - M^2)^{4+\alpha}} , \tag{2.48}$$

where the number -2α in the subscript labels the power counting dimension of energy momentum with $\alpha = -1, 0, 1, \cdots$. In the above equations M^2 is a mass factor that depends on the Feynman parameters and the external momenta. For different values of α these integrals take different values.

For $\alpha = -1$, we have quadratically divergent integrals:

$$I_2 = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - M^2)}; \quad I_{2\mu\nu} = \int \frac{d^4k}{(2\pi)^4} \frac{k_\mu k_\nu}{(k^2 - M^2)^2}; \tag{2.49}$$

$$I_{2\mu\nu\rho\sigma} = \int \frac{d^4k}{(2\pi)^4} \frac{k_\mu k_\nu k_\rho k_\sigma}{(k^2 - M^2)^3} . \tag{2.50}$$

If $\alpha = 0$, we get logarithmically divergent integrals:

$$I_0 = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - M^2)^2}; \quad I_{0\mu\nu} = \int \frac{d^4k}{(2\pi)^4} \frac{k_\mu k_\nu}{(k^2 - M^2)^3}; \tag{2.51}$$

$$I_{0\mu\nu\rho\sigma} = \int \frac{d^4k}{(2\pi)^4} \frac{k_\mu k_\nu k_\rho k_\sigma}{(k^2 - M^2)^4} . \tag{2.52}$$

Finally, when $\alpha = 1$, the integrals converge:

$$I_{-2} = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - M^2)^3}; \quad I_{-2\mu\nu} = \int \frac{d^4k}{(2\pi)^4} \frac{k_\mu k_\nu}{(k^2 - M^2)^4};$$

$$I_{-2\mu\nu\rho\sigma} = \int \frac{d^4k}{(2\pi)^4} \frac{k_\mu k_\nu k_\rho k_\sigma}{(k^2 - M^2)^5} . \tag{2.53}$$

Wu introduced a set of consistency conditions which he claims ensures the gauge invariance of the above mentioned one-loop graphs in QCD and he further claims that these conditions are regularisation independent. These expressions require a regularisation prescription, $I \to I^R$ which must preserve gauge invariance. In general there are six regularisation-independent consistency conditions which maintain gauge invariance and satisfy Ward identities.

- For quadratically divergent one-loop integrals

$$I^R_{2\mu\nu} = \frac{1}{2} g_{\mu\nu} I^R_2 , \tag{2.54}$$

$$I^R_{2\mu\nu\rho\sigma} = \frac{1}{8} g_{\{\mu\nu\rho\sigma\}} I^R_2 . \tag{2.55}$$

- For logarithmically divergent one-loop integrals

$$I^R_{0\mu\nu} = \frac{1}{4} g_{\mu\nu} I^R_0 , \tag{2.56}$$

$$I^R_{0\mu\nu\rho\sigma} = \frac{1}{24} g_{\{\mu\nu\rho\sigma\}} I^R_0 . \tag{2.57}$$

- For convergent one-loop integrals

$$I^R_{-2\mu\nu} = \frac{1}{6} g_{\mu\nu} I^R_{-2}\,, \tag{2.58}$$

$$I^R_{-2\mu\nu\rho\sigma} = \frac{1}{48} g_{\{\mu\nu\rho\sigma\}} I^R_{-2}\,. \tag{2.59}$$

where we have used the notation

$$g_{\{\mu\nu\rho\sigma\}} \equiv g_{\mu\nu} g_{\rho\sigma} + g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\rho\nu}\,. \tag{2.60}$$

The above conditions are the regularisation-independent consistency conditions for maintaining the gauge invariance of theories. In our calculations we will use the two conditions (2.54) and (2.56).

2.2.1 Verification of the Ward Identity in Wu's Scheme

In this subsection we will show that the first Ward identity, $q^\mu \Pi_{\mu\nu}(q) = 0$, holds when Wu's conditions are used. We start with the photon self energy (2.18) to obtain

$$\begin{aligned} i\Pi_{\mu\nu}(q) &= -e^2 \int \frac{[d^4p]}{(2\pi)^4} \mathrm{tr}\left(\gamma_\mu \frac{1}{\not p - m} \gamma_\nu \frac{1}{\not p + \not q - m}\right) \\ &= -e^2 \int \frac{[d^4p]}{(2\pi)^4} \frac{1}{(p^2 - m^2)} \frac{1}{\{(p+q)^2 - m^2\}} \\ &\qquad \times \mathrm{tr}\left(\gamma_\mu (\not p + m) \gamma_\nu (\not p + \not q + m)\right)\,. \end{aligned} \tag{2.61}$$

Note that in the above equation square brackets indicate any regularisation scheme that respects Wu's identities. Using properties of traces we get:

$$\mathrm{tr}\Big[\gamma_\mu (\not p + m) \gamma_\nu (\not p + \not q + m)\Big] = 4\Big[2p_\mu p_\nu + p_\mu q_\nu + p_\nu q_\mu - g_{\mu\nu}(p^2 + p\cdot q - m^2)\Big]\,. \tag{2.62}$$

For the denominator, making use of the Feynman parameterisation introduced in Appendix B.1 we have,

$$\frac{1}{(p^2-m^2)[(p+q)^2-m^2]} = \int_0^1 dx \, \frac{1}{\Big((p^2-m^2)x+(1-x)[(p+q)^2-m^2]\Big)^2} \, . \tag{2.63}$$

Consider the denominator:

$$\begin{aligned}(p^2-m^2)x+(1-x)[(p+q)^2-m^2] &= p^2+q^2+2p\,q-m^2-2p\,qx-q^2x \\ &= (p+\bar{x}q)^2+q^2\bar{x}x-m^2\,,\end{aligned} \tag{2.64}$$

where $\bar{x}=1-x$.

On substituting (2.64) into (2.63) we find:

$$\frac{1}{(p^2-m^2)[(p+q)^2-m^2]} = \int_0^1 dx \, \frac{1}{\big((p+\bar{x}q)^2+q^2\bar{x}x-m^2\big)^2} \, . \tag{2.65}$$

After inserting (2.65) into (2.61), the expression for $i\Pi_{\mu\nu}(q)$ becomes:

$$i\Pi_{\mu\nu}(q) = -4e^2 \int \frac{[d^4p]}{(2\pi)^4} \int_0^1 dx \frac{\Big[2p_\mu p_\nu + p_\mu q_\nu + p_\nu q_\mu - g_{\mu\nu}(p^2+p\cdot q-m^2)\Big]}{\big((p+\bar{x}q)^2+q^2\bar{x}x-m^2\big)^2} \, . \tag{2.66}$$

The above expression contains two terms in the integrand which can be evaluated separately, that is, for the first part we define

$$\mathcal{I}_{\mu\nu} = \int \frac{[d^4p]}{(2\pi)^4} \int_0^1 dx \frac{2p_\mu p_\nu + p_\mu q_\nu + p_\nu q_\mu}{\big((p+\bar{x}q)^2+q^2\bar{x}x-m^2\big)^2} \, . \tag{2.67}$$

Making use of the shift, $p + \bar{x}q \to p$, we have for this part

$$\begin{aligned}\mathcal{I}_{\mu\nu} &= \int \frac{[d^4p]}{(2\pi)^4} \int_0^1 dx \frac{2(p_\mu - \bar{x}q_\mu)(p_\nu - \bar{x}q_\nu) + (p_\mu - \bar{x}q_\mu)q_\nu + (p_\nu - \bar{x}q_\nu)q_\mu}{\big(p^2 - (m^2 - q^2\bar{x}x)\big)^2} \\ &= \int \frac{[d^4p]}{(2\pi)^4} \int_0^1 dx \frac{2p_\mu p_\nu - 2\bar{x}(p_\mu q_\nu - q_\mu p_\nu) + 2\bar{x}\bar{x}q_\mu q_\nu + p_\mu q_\nu - 2\bar{x}q_\mu q_\nu + p_\nu q_\mu}{\big(p^2 - (m^2 - q^2\bar{x}x)\big)^2} .\end{aligned} \tag{2.68}$$

The terms odd in p have been dropped since they give zero, so we find

$$\begin{aligned}\mathcal{I}_{\mu\nu} &= \int \frac{[d^4p]}{(2\pi)^4} \int_0^1 dx \frac{2p_\mu p_\nu + 2\bar{x}\bar{x}q_\mu q_\nu - 2\bar{x}q_\mu q_\nu}{\big(p^2 - (m^2 - q^2\bar{x}x)\big)^2} \\ &= \int \frac{[d^4p]}{(2\pi)^4} \int_0^1 dx \frac{2p_\mu p_\nu - 2x\bar{x}q_\mu q_\nu}{(p^2 - M^2)^2} ,\end{aligned} \tag{2.69}$$

where $M^2 = m^2 - q^2\bar{x}x$. Identifying the integral above with the identities (2.49) and (2.51) we can rewrite equation (2.69) as,

$$\mathcal{I}_{\mu\nu} = \int_0^1 dx[2I_{2\mu\nu} - 2x\bar{x}q_\mu q_\nu I_0] . \tag{2.70}$$

Now we evaluate the second part of (2.66) defining

$$\mathcal{J}_{\mu\nu} = \int \frac{[d^4p]}{(2\pi)^4} \int_0^1 dx \frac{g_{\mu\nu}(p^2 + p.q - m^2)}{\big((p + \bar{x}q)^2 + q^2\bar{x}x - m^2\big)^2} , \tag{2.71}$$

and use the shift, $p + \bar{x}q \to p$. This yields:

$$\mathcal{J}_{\mu\nu} = \int \frac{[d^4p]}{(2\pi)^4} \int_0^1 dx \frac{g_{\mu\nu}((p - \bar{x}q)^2 + (p - \bar{x}q)q - m^2)}{\big(p^2 - M^2\big)^2} . \tag{2.72}$$

where $M^2 = (m^2 - q^2\bar{x}x)$. Expanding (2.72) we find linear terms in p vanish and we are left with,

$$\begin{aligned}\mathcal{J}_{\mu\nu} &= \int \frac{[d^4p]}{(2\pi)^4} \int_0^1 dx \frac{g_{\mu\nu}(p^2 + \bar{x}\bar{x}q^2 - m^2 - \bar{x}q^2)}{\left(p^2 - M^2\right)^2} \\ &= \int \frac{[d^4p]}{(2\pi)^4} \int_0^1 dx \frac{g_{\mu\nu}(p^2 + \bar{x}\bar{x}q^2 + x\bar{x}q^2 - x\bar{x}q^2 - m^2 - \bar{x}q^2)}{\left(p^2 - M^2\right)^2} ,\end{aligned} \tag{2.73}$$

where we have added and subtracted the $x\bar{x}q^2$ term.

Hence,

$$\begin{aligned}\mathcal{J}_{\mu\nu} &= g_{\mu\nu} \int \frac{[d^4p]}{(2\pi)^4} \int_0^1 dx \frac{1}{(p^2 - M^2)} + \frac{q^2\bar{x}(-x + \bar{x} - 1)}{(p^2 - M^2)^2} \\ &= g_{\mu\nu} \int \frac{[d^4p]}{(2\pi)^4} \int_0^1 dx \Big[\frac{1}{(p^2 - M^2)} - \frac{2x\bar{x}q^2}{(p^2 - M^2)^2}\Big] \\ &= g_{\mu\nu} \int_0^1 dx [I_2 - 2x\bar{x}q^2 I_0] .\end{aligned} \tag{2.74}$$

In (2.74) we have written the integrals I_2 and I_0 using Wu's notation (2.49) and (2.51). Combining (2.70) and (2.74) and substituting in (2.66) we get:

$$i\Pi_{\mu\nu}(q) = -4e^2 \int_0^1 dx [2I_{2\mu\nu} - 2\bar{x}x q_\mu q_\nu I_0 - g_{\mu\nu} I_2 + 2x\bar{x} g_{\mu\nu} q^2 I_0] . \tag{2.75}$$

Substituting Wu's first identity (2.54) into (2.75) we find

$$\begin{aligned}i\Pi_{\mu\nu}(q) &= -4e^2 \int_0^1 dx [2\tfrac{1}{2} I_2 g_{\mu\nu} - 2\bar{x}x q_\mu q_\nu I_0 - g_{\mu\nu} I_2 + 2x\bar{x} g_{\mu\nu} q^2 I_0] \\ &= -8e^2 q^2 \int_0^1 dx\ \bar{x}x I_0 q^2 \Big(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\Big) .\end{aligned} \tag{2.76}$$

Extracting the transverse projector, $P_{\mu\nu} = g_{\mu\nu} - \dfrac{q_\mu q_\nu}{q^2}$ in (2.76) we obtain

$$i\Pi_{\mu\nu}(q) = -8e^2 q^2 P_{\mu\nu} \int_0^1 dx\ \bar{x}\ x I_0 q^2 , \tag{2.77}$$

from which it is immediately obvious that $q^\mu \Pi_{\mu\nu} = 0$. Therefore, we have seen that the first Ward identity holds in an arbitrary regularisation scheme which satisfies Wu's identity.

In the next section we will verify Wu's identity using various regularisation schemes. To start with we will first check that Wu's consistency conditions hold in dimensional regularisation and the PV scheme. A cut-off scheme does not maintain gauge invariance and hence Wu's identities should not hold in that scheme.

2.2.2 Wu's Identities in Dimensional Regularisation

As described earlier in Section 2.1.1 the method of dimensional regularisation [9] is the most effective method that preserves the symmetries of QED and also of a wide class of more general theories. In this subsection we will verify some of Wu's identity using the method of dimensional regularisation. The momentum integrals will be solved in Euclidian space using (B.8) from Appendix B.1. We want to show that (2.54) is obeyed *i.e.*

$$\int \frac{d^D k}{(2\pi)^D} \frac{k_\mu k_\nu}{(k^2 - M^2)^2} = \frac{1}{2} g_{\mu\nu} \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 - M^2)} \, . \tag{2.78}$$

To prove this we start with the left hand side where the integral $I_{2\mu\nu}$ is a second-order rank tensor that is just proportional to $g_{\mu\nu}$ *i.e.*

$$\begin{aligned} I_{2\mu\nu} &\propto g_{\mu\nu} \\ &= \mathcal{A} \; g_{\mu\nu} \, . \end{aligned} \tag{2.79}$$

In the above equation $\mathcal{A}$ is a constant that can be determined by contracting the above expression by $g_{\mu\nu}$ on both sides such that:

$$\begin{aligned}\mathcal{A} &= \frac{1}{D} g^{\mu\nu} I_{2\mu\nu} \\ &= \frac{1}{D}\int \frac{d^Dk}{(2\pi)^D}\,\frac{g^{\mu\nu}k_\mu k_\nu}{(k^2-M^2)^2} \\ &= \frac{1}{D}\Big[\int \frac{d^Dk}{(2\pi)^D}\,\frac{1}{(k^2-M^2)} + M^2\int \frac{d^Dk}{(2\pi)^D}\,\frac{1}{(k^2-M^2)^2}\Big]\,,\end{aligned} \tag{2.80}$$

where we have added and subtracted M^2. To evaluate the above integrals we make use of (B.8) to obtain

$$\int \frac{d^Dk}{(2\pi)^D}\,\frac{1}{(k^2-M^2)} =: I_2 = -\frac{i}{(4\pi)^{\frac{D}{2}}}\Gamma\Big(1-\frac{D}{2}\Big)(M^2)^{\frac{D}{2}-1}\,, \tag{2.81}$$

and

$$\int \frac{d^Dk}{(2\pi)^D}\,\frac{1}{(k^2-M^2)^2} = \frac{i}{(4\pi)^{\frac{D}{2}}}\Big(1-\frac{D}{2}\Big)\Gamma\Big(1-\frac{D}{2}\Big)(M^2)^{\frac{D}{2}-2}\,. \tag{2.82}$$

Inserting these into (2.80) we find:

$$\begin{aligned}\mathcal{A} &= -\frac{i}{(4\pi)^{\frac{D}{2}}}\frac{1}{2}\Gamma\Big(1-\frac{D}{2}\Big)(M^2)^{\frac{D}{2}-1} \\ &= \frac{1}{2}I_2\,.\end{aligned} \tag{2.83}$$

Therefore,

$$I_{2\mu\nu} = \frac{1}{2}g_{\mu\nu}I_2\,. \tag{2.84}$$

We have thus proved that Wu's first identity hold in dimensional regularisation at one-loop. The other identity (2.56) follows in much the same way.

More generally it is found that in D-dimensions Wu's conditions (2.54)-(2.59) follow some pattern (as shown in Appendix B.2) which corresponds to the formula

$$I^n_{\mu\nu} = \frac{g_{\mu\nu}}{2(n-1)}I^{n-1}\,, \tag{2.85}$$

for $n \geq 2$ where

$$I^n_{\mu\nu} = \int \frac{d^D k}{(2\pi)^D} \frac{k_\mu k_\nu}{(k^2 - M^2)^n} \quad \text{and} \quad I^n = \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 - M^2)^n} \,. \tag{2.86}$$

In the next section we will verify that Wu's identities hold in the PV scheme.

2.2.3 Wu's Identities in the Pauli Villars Scheme

In the PV scheme we modify, say a fermionic propagator of mass M by assuming that there are fictitious fermions of mass Λ_i. These fictitious particles are the contribution from some other generalised Pauli Villars fields which have large masses compared to the original mass M of the propagator. In this method we subtract off the same loop integral but with a much larger mass. To see how Wu's identities work in the PV scheme we will start with a simple example. We want to show that Wu's identity (2.56) holds in the PV scheme *i.e.*

$$\int \frac{[d^4k]}{(2\pi)^4} \frac{k_\mu k_\nu}{(k^2 - M^2)^3} = \frac{1}{4} g_{\mu\nu} \int \frac{[d^4k]}{(2\pi)^4} \frac{1}{(k^2 - M^2)^2} \,, \tag{2.87}$$

where square brackets indicate any regularisation scheme that respects gauge invariance which in our case is the PV scheme. It is clear that the above integrals contain a logarithmic divergence. Starting with the right hand side of (2.87) we expand the integrand in inverse power of k^2:

$$\begin{aligned} \int \frac{[d^4k]}{(2\pi)^4} \frac{1}{(k^2 - M^2)^2} &= \int \frac{d^4k}{(2\pi)^4} \Big[\frac{1}{(k^2 - M^2)^2} + \sum_i \frac{C_i}{(k^2 - \Lambda_i^2)^2} \Big] \\ &= \int \frac{d^4k}{(2\pi)^4} \Big[\frac{1}{k^4} \big(1 + \frac{2M^2}{k^2} + \frac{3M^4}{k^4} + \dots \big) \\ &\qquad + \sum_i \frac{C_i}{k^4} \big(1 + \frac{2\Lambda_i^2}{k^2} + \frac{3\Lambda_i^4}{k^4} + \dots \big) \Big] \,, \end{aligned} \tag{2.88}$$

where C_i are the coefficients. In (2.88) we find that both the integrals contain logarithmic divergence and the higher terms are UV finite. The only condition for the integral to be finite is:

$$1 + \sum_i C_i = 0\,. \tag{2.89}$$

We now consider the left hand side of (2.87) where we rewrite $k_\mu k_\nu$ in terms of k^2:

$$\begin{aligned}
\int \frac{[d^4k]}{(2\pi)^4} \frac{k_\mu k_\nu}{(k^2 - M^2)^3} &= \frac{1}{4} g_{\mu\nu} \int \frac{[d^4k]}{(2\pi)^4} \frac{k^2}{(k^2 - M^2)^3}\,, \\
&= \frac{1}{4} g_{\mu\nu} \int \frac{d^4k}{(2\pi)^4} \Big[\frac{k^2}{(k^2 - M^2)^3} + \sum_i \frac{C_i k^2}{(k^2 - \Lambda_i^2)^3} \Big] \\
&= \frac{1}{4} g_{\mu\nu} \int \frac{d^4k}{(2\pi)^4} \Big[\frac{1}{(k^2 - M^2)^2} + \frac{M^2}{(k^2 - M^2)^3} \\
&\qquad + \sum_i \frac{C_i}{(k^2 - \Lambda_i^2)^2} + \sum_i \frac{C_i \Lambda_i^2}{(k^2 - \Lambda_i^2)^3} \Big]\,,
\end{aligned} \tag{2.90}$$

where we have added and subtracted M^2 and $C_i \Lambda_i^2$. Combining the first and third terms of (2.90) using the first line of (2.88) we have

$$\begin{aligned}
\int \frac{[d^4k]}{(2\pi)^4} \frac{k_\mu k_\nu}{(k^2 - M^2)^3} &= \frac{1}{4} g_{\mu\nu} \int \frac{[d^4k]}{(2\pi)^4} \frac{1}{(k^2 - M^2)^2} \\
&\quad + \frac{1}{4} g_{\mu\nu} \int \frac{d^4k}{(2\pi)^4} \Big[\frac{M^2}{(k^2 - M^2)^3} + \sum_i \frac{C_i \Lambda_i^2}{(k^2 - \Lambda_i^2)^3} \Big] \\
&= \frac{1}{4} g_{\mu\nu} \int \frac{[d^4k]}{(2\pi)^4} \frac{1}{(k^2 - M^2)^2} + \frac{1}{4} g_{\mu\nu} \Big(\frac{M^2}{M^2} - \frac{\Lambda_i^2}{\Lambda_i^2} \Big)\,.
\end{aligned} \tag{2.91}$$

The last two terms cancel to yield:

$$\int \frac{[d^4k]}{(2\pi)^4} \frac{k_\mu k_\nu}{(k^2 - M^2)^3} = \frac{1}{4} g_{\mu\nu} \int \frac{[d^4k]}{(2\pi)^4} \frac{1}{(k^2 - M^2)^2}\,, \tag{2.92}$$

i.e.

$$I_{0\mu\nu} = \frac{1}{4} I_0 g_{\mu\nu}\,, \tag{2.93}$$

Hence Wu's identity is shown to hold in the PV scheme. All of the Wu's identity have been checked in this way.

2.3 Derivation of Wu's Identity

So far we have verified Wu's consistency conditions and seen how they preserve gauge invariance for the regularised irreducible loop integrals in the abelian theory. In this section we will show the origin of these identities. We will see how these identities can be derived within the context of any regularisation that respects translational invariance. We will start with the derivation of Wu's first identity (2.54) written in the form:

$$\int \frac{[d^4k]}{(2\pi)^4} \frac{k_\mu k_\nu}{(k^2 - M^2)^2} = \frac{1}{2} g_{\mu\nu} \int \frac{[d^4k]}{(2\pi)^4} \frac{1}{(k^2 - M^2)} . \tag{2.94}$$

To derive the above identity we consider the following integral,

$$\mathcal{K} = \int \frac{[d^4k]}{(2\pi)^4} \log(k^2 - M^2) , \tag{2.95}$$

where square brackets indicate that we are working in the dimensional regularisation. We extend the dimension from $D = 4 \to 4 - 2\varepsilon$ and obtain

$$\mathcal{K} = \int \frac{d^D k}{(2\pi)^D} \log(k_\lambda k^\lambda - M^2) . \tag{2.96}$$

Under momentum translation $k_\lambda \to k'_\lambda = k_\lambda + \alpha_\lambda$, the above equation becomes:

$$\mathcal{K} = \mathcal{K}(\alpha) = \int \frac{d^D k}{(2\pi)^D} \log\left[(k_\lambda + \alpha_\lambda)(k^\lambda + \alpha^\lambda) - M^2\right] . \tag{2.97}$$

Differentiating with respect to α_μ

$$\frac{\partial \mathcal{K}(\alpha)}{\partial \alpha_\mu} = \int \frac{d^D k}{(2\pi)^D} \frac{2(k_\mu + \alpha_\mu)}{\left[(k_\lambda + \alpha_\lambda)(k^\lambda + \alpha^\lambda) - M^2\right]}, \tag{2.98}$$

and again with respect to α_ν we obtain:

$$\begin{aligned} \frac{\partial^2 \mathcal{K}(\alpha)}{\partial \alpha_\mu \partial \alpha_\nu} = \int \frac{d^D k}{(2\pi)^D} \Bigg\{ & \frac{-4(k_\mu + \alpha_\mu)(k_\nu + \alpha_\nu)}{\left[(k_\mu + \alpha_\mu)(k^\mu + \alpha^\mu) - M^2\right]^2} \\ & + \frac{2g_{\mu\nu}}{\left[(k_\mu + \alpha_\mu)(k^\mu + \alpha^\mu) - M^2\right]} \Bigg\}. \end{aligned} \tag{2.99}$$

Because of translational invariance $\mathcal{K}$ is independent of α, so setting $\alpha = 0$ in the above equation we end up with,

$$\int \frac{d^D k}{(2\pi)^D} \frac{k_\mu k_\nu}{(k^2 - M^2)^2} = \frac{1}{2} g_{\mu\nu} \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 - M^2)}, \tag{2.100}$$

or

$$I_{2\mu\nu} = \frac{1}{2} g_{\mu\nu} I_2 \,. \tag{2.101}$$

Thus Wu's first identity is derived as being a consequence of translational invariance in dimensional regularisation.

To derive Wu's second identity (2.56) we consider the following integral

$$\mathcal{J} = \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 - M^2)} = \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k_\lambda k^\lambda - M^2)}, \tag{2.102}$$

which under momentum translation gives:

$$\mathcal{J} = \mathcal{J}(\alpha) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{\left[(k_\lambda + \alpha_\lambda)(k^\lambda + \alpha^\lambda) - M^2\right]}. \tag{2.103}$$

Again differentiating twice with respect to α we obtain:

$$\frac{\partial^2 \mathcal{J}(\alpha)}{\partial\alpha_\mu\partial\alpha_\nu} = \int \frac{d^Dk}{(2\pi)^D} \left\{ \frac{8(k_\mu+\alpha_\mu)(k_\nu+\alpha_\nu)}{[(k_\mu+\alpha_\mu)(k^\mu+\alpha^\mu)-M^2]^3} - \frac{2g_{\mu\nu}}{[(k_\mu+\alpha_\mu)(k^\mu+\alpha^\mu)-M^2]^2} \right\}. \tag{2.104}$$

Setting $\alpha = 0$ (because of momentum translation) in (2.104) we find,

$$8\int \frac{d^Dk}{(2\pi)^D} \frac{k_\mu k_\nu}{(k^2-M^2)^3} = 2\int \frac{d^Dk}{(2\pi)^D} \frac{g_{\mu\nu}}{(k^2-M^2)^2}\,, \tag{2.105}$$

or

$$I_{0\mu\nu} = \frac{1}{4} g_{\mu\nu} I_0\,. \tag{2.106}$$

Thus, Wu's second identity can also be seen in dimensional regularisation to be a consequence of translational invariance.

2.4 Concluding Remarks

In this chapter we have shown that Wu's conditions must be satisfied for regularised loop integrals to preserve gauge invariance. Since Ward identities are a consequence of gauge invariance, Wu's conditions must also satisfy Ward identities that we have seen above. It is easily shown that dimensional regularisation and PV scheme satisfy these consistency conditions. But these identities do not hold in the cut-off regularisation as it breaks gauge invariance. We have focussed on the role of gauge invariance and have seen the connection between translational invariance and Wu's conditions in dimensional regularisation. We prefer dimensional regularisation over other methods as it ensures gauge invariance and the validity of the Ward identity to all orders of perturbation theory. Since gauge invariance is independent of the number of space-time dimensions, it is by construction a gauge invariant regularisation. Although Wu's approach provides a set of identities needed to maintain gauge invariance, we have seen it is simply a consequence of translational invariance in the momentum integrals.

Chapter 3

Non-Abelian Gauge Theory

3.1 Introduction

The extension from the abelian to the non-abelian theory introduces many complications into the identification of physical variables. The photon fields A_μ in the abelian theory now become the gluon fields A_μ^a in the non-abelian gauge theory with colour index a. Unlike the abelian theory, the field strength for the non-abelian fields is not gauge invariant which raises the question of how to identify physical field strengths. As we will see in this chapter, there are two possible routes that one can consider to construct gauge invariant expressions for the field strength. The interplay between these field strengths will play an important role in the construction of gauge invariant configurations in the later chapters.

To understand the nature of non-abelian theories we need to study some of the properties of the groups on which they are based and that will be outlined in this chapter. Some new identities will be explored which will provide the necessary background material for the rest of the thesis. Following this we will review the role of dressings in constructing gauge invariant configurations. This will allow us to construct two gauge invariant field strengths. We will show a factorisation which relates the two field strengths. This will be main equation needed in the construction of a non-abelian mass like term.

3.2 Lie Groups and Representations

Informally a Lie group is a group whose elements depend in a continuous and differentiable way on a set of real parameters. It is described as a smooth space upon which there is a continuous product which satisfies the normal group properties [51–56]. Therefore a Lie group is at the same time a group and a smooth finite-dimensional manifold.

More concretely we can view a given group through its representation. A *representation* of a group G is a mapping, U of the elements of G onto a set of linear operators. That is, to each element g of the group G we associate a matrix $U(g)$ acting on a vector space such that the group product g_1g_2 is fully captured by the matrix product:

$$U(g_1g_2) = U(g_1)U(g_2)\,. \tag{3.1}$$

Many of the key structures for a given group G are revealed by the collection of infinitesimal transformations close to the identity, that is the tangent space at the identity. This is the Lie algebra of the Lie group which can also be studied through its representation. Given the Lie algebra of such vector fields, the group structure is summarised in the commutator properties of the elements of the Lie algebra, that is,

$$[t^a, t^b] = f_{abc}t^c\,, \tag{3.2}$$

which is the abstract commutator of the vector fields t and a, b, c ranges over the dimension of G. In (3.2) the factors f_{abc} are the *structure constants* of the group

G that are anti-symmetric and satisfy the following Jacobi identity

$$f^{abp}f^{pcq} + f^{bcp}f^{paq} + f^{cap}f^{pbq} = 0\,, \tag{3.3}$$

which follows from the cyclic permutation of the commutators. The Jacobi identity must be satisfied in order for a given set of commutation rules to define a Lie algebra. For the group $SU(2)$ the structure constants f_{abc} are simply given by the antisymmetric Levi-Civita symbol ϵ_{abc}.

Just as we can talk about a representation of the group we can similarly talk about a representation of the Lie algebra, $t^a \to T^a$, where the vector fields t^a are now represented by the matrix fields T^a and thus is a representation if and only if we have the matrix commutator

$$[T^a, T^b] = f_{abc}T^c\,. \tag{3.4}$$

There is a close connection between the representation of the group (3.2) and the algebra (3.4) and this is captured by the exponential mapping from the Lie algebra to the group, which, in the context of a specific representation, is just the matrix exponential. Every Lie group G has an associated Lie algebra $\mathfrak{g}$ that is related to it via the exponential map. For example, suppose θ is an element of the Lie algebra $\mathfrak{g}$,

$$\theta = \theta^a t^a\,, \tag{3.5}$$

then this is represented by $\theta = \theta^a T^a \in \mathfrak{g}$. The exponential map relates the algebra element to the group element by, $g = e^\theta$ which, in terms of our representation is

$$U(g) = e^{(\theta^a T^a)}\,, \tag{3.6}$$

where the exponential here is just the usual matrix exponential.

There is a rich theory of such groups and their representations, but for this thesis we shall focus on the special unitary groups $SU(\mathrm{N})$ that is defined to be the subgroup of the unitary group $U(\mathrm{N})$ which in the fundamental representation consists of $N \times N$ unitary matrices with determinant $+1$. The Lie algebra associated with the group $SU(\mathrm{N})$ is denoted by $\mathfrak{su}(N)$ and is the vector space of complex, anti-hermitian $n \times n$ matrices with null trace. With this in mind, the indices a, b, c etc will range over the dimension of the group which for us will be $1, \cdots, N^2 - 1$. The representations are well understood for these groups.

The smallest (irreducible) representation is called the *fundamental* representation and it is described in terms of N-dimensional traceless anti-hermitian matrices

$$(T^a)_{ij}\,, \tag{3.7}$$

where the indices i, j etc. range from $1, \cdots, N$. There are $(N^2 - 1)$ linearly independent $N \times N$ anti-hermitian traceless matrices T^a.

Another important representation that can be constructed for any Lie algebra is called the *adjoint* representation [57] which consists of generators of the algebra written in the form:

$$(T^a)_{bc} = -f_{abc}\,, \tag{3.8}$$

where the structure constants f_{abc} correspond to the representation matrices. That this is a representation, follows from the fact that it satisfies the Lie algebra (3.4).

Conventionally, the generators are normalised according to

$$\mathrm{tr}\,(T^a T^b) = C_r \delta^{ab}\,, \tag{3.9}$$

with C_r the normalisation constant in any arbitrary representation r to which T^a belong. For the non-abelian theory the structure constants are not equal to zero *i.e.*, there are non-zero commutation relations between the generators of the gauge group.

3.2.1 $SU(2)$ Gauge Theory

The simplest example of a Lie group is the $SU(2)$ group that is a 3-dimension manifold, S^3, and plays an important role in the standard model. For the group $SU(2)$, the representation matrices:

$$U(g) := e^{\theta} = e^{\theta^a \tau^a}\,, \tag{3.10}$$

are unitary, where, for the fundamental representation, the matrix element, $(T^a)_{ij} = (\tau^a)_{ij}$ and i, j range over $(1, 2)$. Note that each τ_a is a traceless unitary 2×2 matrix with $U^\dagger = e^{-\theta}$, $UU^\dagger = U^\dagger U = 1$ and $\det U = e^{\mathrm{tr}\,\theta} = 1$ as required. This representation satisfies the following properties:

$$[\tau_a, \tau_b] = \epsilon_{abc}\tau_c\,, \tag{3.11}$$

$$\mathrm{tr}\,(\tau_a\tau_b) = -\frac{1}{2}\delta_{ab}\,, \quad \mathrm{tr}\,(\tau_a) = 0\,, \tag{3.12}$$

and

$$\tau_a^\dagger = -\tau_a\,. \tag{3.13}$$

The infinitesimal generators, τ_a, of the Lie algebra of $SU(2)$ are related to the Pauli matrices, σ_a by

$$\tau_a = -\frac{1}{2}i\sigma_a \quad \text{with} \quad a = 1, 2, 3\,, \tag{3.14}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.15}$$

At this stage we introduce a very useful identity called the Fierz identity for the fundamental representation of the group $SU(\mathrm{N})$ that is found in [58, 59]:

$$[\tau^a]_{ij}[\tau^a]_{kl} = -\frac{1}{2}\Big(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\Big), \tag{3.16}$$

where we are summing over the Lie algebra index $a = 1, \cdots, N^2 - 1$ and $i, j, k, l = 1, \cdots, N$.

For the group $SU(2)$ the above identity leads to

$$[\tau^a]_{ij}[\tau^a]_{kl} = -\frac{1}{2}\Big(\delta_{il}\delta_{jk} - \frac{1}{2}\delta_{ij}\delta_{kl}\Big), \tag{3.17}$$

where now the Lie algebra index a takes the values from $1, \cdots, 3$ and $i, j, k, l = 1, 2$. The derivation of these identities is relatively straightforward and follows from the fact that the right hand side can be constructed only out of the δ_{ij} tensors. With this in mind, equation (3.17) can be written as

$$[\tau^a]_{ij}[\tau^a]_{kl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}, \tag{3.18}$$

where fixing the constants α, β, γ and using the trace properties of matrix τ^a we can easily verify the result (3.17).

3.2.2 Relation between Representations

After the introduction of the Lie algebra for the fundamental and the adjoint representations we shall now try to understand how they are related. Suppose we start with the fundamental representation, $SU(2)$, that is, $U(g) = e^{\theta^a \tau^a}$ is a 2×2 special unitary matrix and $(\tau^a)_{ij}$ are given by (3.14). As is well known from quantum mechanics, the adjoint (3-dimensional) representation in terms of orthogonal 3×3 matrix is denoted by

$$R(g) = e^{\theta^a T^a} \, , \tag{3.19}$$

with $\det R = 1$ where T^a are the appropriate 3×3 matrices for this representation.

Given these representations, however, our claim is that we can also construct $R(g)$ directly from the fundamental representation via the representation matrix

$$R_{ab}(g) = -2 \ \mathrm{tr}\left(\tau_a U(g) \tau_b U^{-1}(g)\right) , \tag{3.20}$$

where R_{ab} has the properties that the elements are

- Real, $R^\dagger_{ab} = R_{ab}$,
- Orthogonal, $RR^t = R^t R = 1$ or $R_{ab} R_{cb} = \delta_{ac}; R_{ba} R_{bc} = \delta_{ac}$.

The superscript $\dagger$ and t in above represents the complex conjugate and transpose of R_{ab} matrix. These equations are straightforward to prove. To verify that (3.20) is a representation of the group we need to verify that (3.1) holds, that is,

$$\begin{aligned}
R_{ab}(g_1 g_2) &= -2 \ \mathrm{tr}\left\{\tau_a U(g_1 g_2) \tau_b U^{-1}(g_1 g_2)\right\} \\
&= -2 \ \mathrm{tr}\left\{\tau_a U(g_1) U(g_2) \tau_b U^{-1}(g_2) U^{-1}(g_1)\right\} \\
&= -2 \ \mathrm{tr}\left\{U^{-1}(g_1) \tau_a U(g_1) U(g_2) \tau_b U^{-1}(g_2)\right\} \\
&= -2 \{U^{-1}(g_1) \tau_a U(g_1)\}_{ij} \{U(g_2) \tau_b U^{-1}(g_2)\}_{ji} \\
&= -2 \{U^{-1}(g_1) \tau_a U(g_1)\}_{ij} \{U(g_2) \tau_b U^{-1}(g_2)\}_{kl} \delta_{il} \delta_{jk} \\
&= 4 \{U^{-1}(g_1) \tau_a U(g_1)\}_{ij} \{U(g_2) \tau_b U^{-1}(g_2)\}_{kl} [\tau_c]_{ij} [\tau_c]_{kl}
\end{aligned} \tag{3.21}$$

where in the last line we have used the result (3.17). Using the trace property

$$
\begin{aligned}
R_{ab}(g_1 g_2) &= 4\mathrm{tr}\,\{U^{-1}(g_1)\tau_a U(g_1)\tau_c\}\mathrm{tr}\,\{U(g_2)\tau_b U^{-1}(g_2)\tau_c\} \\
&= 4\mathrm{tr}\,\{\tau_a U(g_1)\tau_c U^{-1}(g_1)\}\mathrm{tr}\,\{\tau_c U(g_2)\tau_b U^{-1}(g_2)\} \\
&= R_{ac}(g_1)R_{cb}(g_2)\,,
\end{aligned}
\tag{3.22}
$$

and hence we see that (3.20) is a representation. Here we have constructed the adjoint representation of the group by using the fundamental representation results.

However earlier we talked about the adjoint representation of the Lie algebra given in terms of structure constants (3.8) so it is useful to verify that this also follows from the adjoint representation construction of the group. In other words the question that needs to be addressed now is, 'What does the adjoint representation look like at the algebra level'? To answer this we expand (3.19) infinitesimally for the bc component

$$
R_{bc} = (e^{\theta})_{bc} = \delta_{bc} + \theta^a [T^a]_{bc}\,, \tag{3.23}
$$

where $[T^a]_{bc}$ are the generators of the symmetry (rotation) group that we need to evaluate. In (3.20) we now make an expansion for U infinitesimally to yield:

$$
\begin{aligned}
R_{bc} := -2\ \mathrm{tr}\,\{\tau_b(1+\theta)\tau_c(1-\theta)\} &= -2\ \mathrm{tr}\,(\tau_b\tau_c) - 2\ \mathrm{tr}\,(\tau_b[\theta,\tau_c]) + O(\theta^2) \\
&= \delta_{bc} - 2\ \theta^a \mathrm{tr}\,(\tau_b[\tau_a,\tau_c]) + O(\theta^2)\,.
\end{aligned}
\tag{3.24}
$$

Using the identification, $\epsilon_{abc} = -2\ \mathrm{tr}\,\{\tau_a[\tau_b,\tau_c]\}$ and hence, comparing with (3.23) we find

$$
[T^a]_{bc} = -\epsilon_{abc}\,, \tag{3.25}
$$

where ϵ_{abc} is the structure constant for $SU(2)$. This thus confirms the general result (3.8). So in the adjoint representation the structure constants make up the basis of the Lie algebra. The relation (3.25) is obeyed by the structure constant of any Lie group SU(N), where these structure constants obey the Jacobi identity (3.3).

Now that we have studied the properties of the groups and their algebras we can move on to construct the Lagrangian for the non-abelian Yang-Mills theory.

3.2.3 Yang-Mills Lagrangian

We now want to generalise the transformations (2.5) and (2.6) to the case when $U(x)$ belongs to a non-abelian group G, and so construct a Lagrangian invariant under local gauge transformations. We will limit ourselves to the case when $G =SU$(N). We start by generalising equation (2.5) to a set of fermionic fields ψ_i that transform in a fundamental representation of SU(N) as

$$\psi_i(x) \to \psi_i^U(x) = U_{ij}(x)\psi_j(x)\,. \tag{3.26}$$

If we suppress these internal indices for the fermionic fields we have for (3.26):

$$\psi(x) \to \psi^U(x) = U^{-1}(x)\psi(x)\,, \tag{3.27}$$

where $U(x) = e^{g\theta}$ is the group element with $\theta = \theta^a(x)\tau^a$. Note that we are denoting the group element by $U(x)$ instead of $U(g)$ where we are suppressing g and writing x locally *i.e.* $U(g) \to U(g(x)) = U(x)$. Going through the same procedure as in QED local gauge invariance is preserved when the replacement $\partial_\mu \to D_\mu = \partial_\mu + gA_\mu$ (with g the coupling constant, the analogue of ie in QED) is made in the Lagrangian of the fermionic fields. The matrix covariant derivative $D_\mu(x)$ acting on the fermionic fields then transforms as

$$D_\mu\psi(x) \to (D_\mu\psi(x))^U = U^{-1}(x)D_\mu\psi(x)\,, \tag{3.28}$$

where $A_\mu(x)$ are the vector fields that can be written using the matrix notation in the fundamental representation as

$$A_\mu(x) := A_\mu^a(x)\tau^a \,. \tag{3.29}$$

In the above equation $A_\mu^a = -2\,\mathrm{tr}\,(A_\mu \tau^a)$ are the components of the vector potential. In this way the set of gauge fields A_μ^a, enters the Lagrangian with the transformation properties (in matrix form as)

$$A_\mu(x) \to A_\mu^U(x) = U^{-1}(x)A_\mu(x)U(x) + \frac{1}{g}U^{-1}(x)\partial_\mu U(x)\,, \tag{3.30}$$

that ensures the correct transformation for the covariant derivative. In deriving (3.30) from (3.28) we have made use of the result, $(\partial U)U^{-1} = -U(\partial U^{-1})$ where $U(x)$ is an element of group $SU(\mathrm{N})$ for each point x in space-time. The result (3.30) is the non-abelian generalisation of the result (2.6). If we expand (3.30) as a power series in the coupling

$$U(x) = e^{g\theta} = 1 + g\theta + \frac{1}{2}g^2\theta^2 + \cdots\,, \tag{3.31}$$

and

$$U^{-1}(x) = e^{-g\theta} = 1 - g\theta + \frac{1}{2}g^2\theta^2 + \cdots\,, \tag{3.32}$$

then A_μ is seen to transform as

$$A_\mu(x) \to A_\mu^U(x) = A_\mu(x) + \partial_\mu\theta(x) + g\big([A_\mu, \theta] + \frac{1}{2}[\partial_\mu\theta, \theta]\big)(x) + \cdots\,. \tag{3.33}$$

To introduce the dynamics in a gauge invariant manner we introduce the antisymmetric field strength tensor $F_{\mu\nu}(x)$ written in the form

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + g[A_\mu, A_\nu](x)\,, \tag{3.34}$$

with components

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\mu\,. \tag{3.35}$$

This is not gauge invariant but transforms as

$$F_{\mu\nu}(x) \to U^{-1}(x) F_{\mu\nu}(x) U(x)\,, \tag{3.36}$$

in contrast to QED where $F_{\mu\nu}$ is gauge invariant. As a consequence of (3.36) we find that the trace, $\mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu})$ is gauge invariant and provides the non-abelian analogue of (2.1) for the kinetic term of the gauge fields. Using this we arrive at the gauge invariant Yang-Mills Lagrangian

$$\mathcal{L}_{YM} = -\frac{1}{2}\,\mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu}) = \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a}\,, \tag{3.37}$$

where we have used the normalisation condition (3.9). The quadratic part in the kinetic term describes the free propagation of the gauge fields, however there are also cubic and quartic terms that describe self interactions of the gauge fields yielding three and four point vertices. This means that gluons interact themselves via the colour force. When the Lagrangian of the matter field is added to the Yang-Mills Lagrangian, the resulting Lagrangian is given by

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi + \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a}\,, \tag{3.38}$$

which is by construction invariant under the gauge transformations (3.27) and (3.30).

3.2.4 Higher Representations

In the Standard Model one usually considers the matter fields to be in the fundamental representation and the gauge fields in particular, the field strength transform according to the adjoint representation of the group. What we want to see now is how this formulation works in higher representations. That is, we want to clarify the relation between the fundamental and the adjoint representation in terms of the tensor products and then understand how to build up higher tensor product representation of the adjoint representation fields. It must be stressed that in our discussion we will frequently switch between the matrix notation and component notation. We saw earlier from (3.26) how the N-component vector $\vec{\psi} = \{\psi^j, j = 1, \ldots, N\}$, defined both in the component and matrix notation transformed according to the fundamental representation of the group.

The word "tensor product" refers to one of the ways of constructing a bigger vector space out of two or more smaller vector spaces. It is also called Kronecker product or direct product. For example, if we start with two vector spaces, U that is m dimensional with the basis $\{\vec{e}_1, \vec{e}_1, \ldots, \vec{e}_m\}$ and W that is n dimensional with the basis $\{\vec{f}_1, \vec{f}_1, \ldots, \vec{f}_n\}$ then the tensor product of these two vector spaces is mn dimensional denoted by $\vec{e}_i \otimes \vec{f}_j$. Let us recall from quantum mechanics the fact that a combined system V, consisting of the sub-systems represented by Hilbert spaces V_1 and V_2, is represented by the tensor product $V = V_1 \otimes V_2$. The operators M_1 on V_1 and M_2 on V_2 can be combined to give the operator

$$M = M_1 \otimes \mathbb{I} + \mathbb{I} \otimes M_2 \,, \tag{3.39}$$

which acts on the combined system $V_1 \otimes V_2$ where $\mathbb{I}$ is the identity element. We can now extend this to study the tensor product of Lie-algebra representation. In our case, $V_1 = V_2 = \mathfrak{g}$, that is, the system corresponds to the Lie algebra and $M_1 = M_2 = (T^c)^{ab}$, that is, the operators correspond to the adjoint representation matrices. If we consider a representation of the Lie algebra $\mathfrak{g}$ then we can construct more representations using the machinery from the Lie algebra. Because tensors furnish representations of the group, we can therefore derive higher representations by taking tensor products of the fundamental representation.

In our theory the gluonic fields $F^a_{\mu\nu}$ are the basic fields and we want to view them as being embedded in higher representations. For example, if we define a Lie algebra valued field strength tensor $F_{\mu\nu} = F^a_{\mu\nu}\tau^a$ in the fundamental representation then the components of the field strength, $F^a_{\mu\nu} = -2 \text{ tr } (\tau^a F_{\mu\nu})$ transform in the adjoint representation of the group as

$$\begin{aligned} F^a_{\mu\nu} &\to -2 \text{ tr } (\tau^a U^{-1} F_{\mu\nu} U) \\ &= -2 \text{ tr } (\tau^a U^{-1} \tau^b U) F^b_{\mu\nu} \\ &= (U^{-1})^a{}_b F^b_{\mu\nu} \,, \end{aligned} \tag{3.40}$$

where $(U^{-1})^a{}_b = -2 \,\text{tr}\,(\tau^a U^{-1}\tau^b U)$ is an adjoint representation matrix.

Suppose we now start with our field strength in the adjoint representation

$$F^{ab}_{\mu\nu} := F^c_{\mu\nu}(T^c)_{ab} = 2F^c_{\mu\nu}\text{tr}\,([\tau^c, \tau^a]\tau^b)\,. \tag{3.41}$$

We want to identify how this field strength transforms. Applying the cyclic property of the trace, $\text{tr}\,([A, B]C) = \text{tr}\,([B, C]A)$, to (3.41) we find $F^{ab}_{\mu\nu} = 2 \text{ tr } ([\tau^a, \tau^b]F_{\mu\nu})$.

This then transforms as

$$
\begin{aligned}
F^{ab}_{\mu\nu} &\to 2\,\mathrm{tr}\left([\tau^a,\tau^b]U^{-1}F_{\mu\nu}U\right) \\
&= 2\,\mathrm{tr}\left([\tau^a,\tau^b]U^{-1}F^c_{\mu\nu}\tau^c U\right) \\
&= 2\,F^c_{\mu\nu}\mathrm{tr}\left([U\tau^a U^{-1},U\tau^b U^{-1}]\tau^c\right) \\
&= 2\,F^c_{\mu\nu}\mathrm{tr}\left([U\tau^b U^{-1},\tau^c]U\tau^a U^{-1}\right) \\
&= 2\,F^c_{\mu\nu}[U\tau^b U^{-1},\tau^c]_{ij}(U\tau^a U^{-1})_{kl}\,\delta_{il}\,\delta_{jk}\,,
\end{aligned}
\tag{3.42}
$$

where in the fourth line we have again used the cyclic property of the trace. Using (3.16) we can replace the Kronecker delta functions by fundamental representation matrices

$$
\begin{aligned}
F^{ab}_{\mu\nu} &\to -4F^c_{\mu\nu}[U\tau^b U^{-1},\tau^c]_{ij}(U\tau^a U^{-1})_{kl}([\tau^d]_{ij}[\tau^d]_{kl}) \\
&= -4F^c_{\mu\nu}\,\mathrm{tr}\left([U\tau^b U^{-1},\tau^c]\tau^d\right)\mathrm{tr}\left(U\tau^a U^{-1}\tau^d\right) \\
&= 2F^c_{\mu\nu}\,\mathrm{tr}\left([\tau^c,\tau^d]U\tau^b U^{-1}\right)(U^{-1})^a{}_d \\
&= 2F^c_{\mu\nu}f^{cde}\,\mathrm{tr}\left(\tau^e U\tau^b U^{-1}\right)(U^{-1})^a{}_d \\
&= -2F^{de}_{\mu\nu}\,\mathrm{tr}\left(\tau^e U\tau^b U^{-1}\right)(U^{-1})^a{}_d \\
&= F^{de}_{\mu\nu}(U^{-1})^b{}_e(U^{-1})^a{}_d \\
&= (U^{-1})^a{}_d\,(U^{-1})^b{}_e(F_{\mu\nu})^{de}\,.
\end{aligned}
\tag{3.43}
$$

Therefore we find that the field strength can be viewed as a matrix adjoint representation and in this representation its transformation is,

$$
F^{ab}_{\mu\nu} \to (U^{-1})^{ab}{}_{de}\,F^{de}_{\mu\nu}\,,
\tag{3.44}
$$

where

$$
(U^{-1})^{ab}{}_{de} = (U^{-1})^a{}_d(U^{-1})^b{}_e\,.
\tag{3.45}
$$

Up to now we have given the transformation rules but have not yet identified what representation we are dealing with. In order to find the representation matrices for (3.44) we expand the left hand side of (3.45) infinitesimally to obtain

$$(U^{-1})^{ab}_{\ de} = (1-\theta)^{ab}_{\ de} = \delta^a_{\ d}\delta^b_{\ e} - \theta^k (T^k)^{ab}_{\ de} + O(\theta^2)\,. \tag{3.46}$$

In the same way we apply an infinitesimal transformation to the right side of (3.45) and obtain

$$\begin{aligned}(U^{-1})^a_{\ d}(U^{-1})^b_{\ e} = (1-\theta)^a_{\ d}(1-\theta)^b_{\ e} &= (\delta^a_{\ d} - \theta^a_{\ d})(\delta^b_{\ e} - \theta^b_{\ e}) \\ &= \{\delta^a_{\ d} - \theta^k (T^k)^{ad}\}\{\delta^b_{\ e} - \theta^k (T^k)^{be}\} \\ &= \delta^a_{\ d}\delta^b_{\ e} - \theta^k \delta^a_{\ d}(T^k)^{be} - \theta^k (T^k)^{ad}\delta^b_{\ e} + O(\theta^2)\,.\end{aligned} \tag{3.47}$$

Now comparing (3.46) and (3.47) to order θ we find

$$(T^k)^{ab}_{\ de} = (T^k)^{ad}\delta^b_{\ e} + \delta^a_{\ d}(T^k)^{be}\,, \tag{3.48}$$

which are the representation matrices for the tensor product of the adjoint representation with itself. This then exposes what is the underlying mathematics of these constructions and thus allows us to generate higher order representations.

We can now generalise the above procedure to higher order and build up higher representation matrices by taking the tensor product of the tensor product. In order to find the representation matrices for this field we first derive a very important result that will be used in our calculations. From equation (3.40) we know that the matrix valued field $F^{ab}_{\mu\nu} := F^k_{\mu\nu}(T^k)^{ab}$ transforms as

$$F^{ab}_{\mu\nu} \to (U^{-1})^k_{\ l} F^l_{\mu\nu}(T^k)^{ab}\,, \tag{3.49}$$

and similarly the transformation for this field from (3.43) is

$$F^{ab}_{\mu\nu} \to (U^{-1})^a_{\ d}\,(U^{-1})^b_{\ e} F^l_{\mu\nu}(T^l)^{de}\,. \tag{3.50}$$

Comparing (3.49) and (3.50) we obtain a very important result

$$(U^{-1})^k_{\ l}(T^k)^{ab} = (U^{-1})^a_{\ d}\,(U^{-1})^b_{\ e}(T^l)^{de}\,, \tag{3.51}$$

which is used in the construction of higher representations.

Suppose

$$F^{abcd}_{\mu\nu} := F^k_{\mu\nu}(T^k)^{ab}_{\ \ cd} = F^k_{\mu\nu}\{(T^k)^{ac}\delta^b_{\ d} + \delta^a_{\ c}(T^k)^{bd}\}\,, \tag{3.52}$$

is a matrix in the tensor product of the tensor product where the lower case roman letters range over the dimension of the group. In order to find the representation matrix for the field $F^{abcd}_{\mu\nu}$ we now return to (3.52) and apply the gauge transformation (3.40) to yield

$$\begin{aligned} F^{abcd}_{\mu\nu} &\to (U^{-1})^k_{\ l}F^l_{\mu\nu}\{(T^k)^{ac}\delta^b_{\ d} + \delta^a_{\ c}(T^k)^{bd}\} \\ &= F^l_{\mu\nu}\{(U^{-1})^k_{\ l}(T^k)^{ac}\delta^b_{\ d} + \delta^a_{\ c}(U^{-1})^k_{\ l}(T^k)^{bd}\} \\ &= F^l_{\mu\nu}\{(U^{-1})^a_{\ m}(U^{-1})^c_{\ n}(T^l)^{mn}\delta^b_{\ d} + \delta^a_{\ c}(U^{-1})^b_{\ m}(U^{-1})^d_{\ n}(T^l)^{mn}\}\,, \end{aligned} \tag{3.53}$$

where in the last line we have used (3.51). Using the result $\delta^b_{\ d} = (U^{-1})^b_{\ g}(U^{-1})^d_{\ h}\delta^g_{\ h}$ and $\delta^a_{\ c} = (U^{-1})^a_{\ g}(U^{-1})^c_{\ h}\delta^g_{\ h}$ in (3.53) we get

$$\begin{aligned} F^{abcd}_{\mu\nu} \to F^l_{\mu\nu}\{&(T^l)^{mn}(U^{-1})^a_{\ m}(U^{-1})^c_{\ n}(U^{-1})^b_{\ g}(U^{-1})^d_{\ h}\delta^g_{\ h} \\ &+ (T^l)^{mn}(U^{-1})^a_{\ g}(U^{-1})^c_{\ h}\delta^g_{\ h}(U^{-1})^b_{\ m}(U^{-1})^d_{\ n}\} \\ = F^l_{\mu\nu}\{&(T^l)^{mn}(U^{-1})^a_{\ m}(U^{-1})^c_{\ n}(U^{-1})^b_{\ g}(U^{-1})^d_{\ h}\delta^g_{\ h} \\ &+ (T^l)^{gh}(U^{-1})^a_{\ m}(U^{-1})^c_{\ n}\delta^m_{\ n}(U^{-1})^b_{\ g}(U^{-1})^d_{\ h}\}\,, \end{aligned} \tag{3.54}$$

where in the second term we have replaced $m \leftrightarrow g$ and $n \leftrightarrow h$ as these are dummy variables. Taking the common factors from (3.54) we have

$$\begin{aligned}F^{abcd}_{\mu\nu} &\to F^{l}_{\mu\nu}\{(U^{-1})^{a}{}_{m}(U^{-1})^{b}{}_{g}(U^{-1})^{c}{}_{n}(U^{-1})^{d}{}_{h}\}\{(T^{l})^{mn}\delta^{g}{}_{h} + \delta^{m}{}_{n}(T^{l})^{gh}\} \\ &= F^{l}_{\mu\nu}\{(U^{-1})^{a}{}_{m}(U^{-1})^{b}{}_{g}(U^{-1})^{c}{}_{n}(U^{-1})^{d}{}_{h}\}\{(T^{l})^{mg}_{nh}\} \\ &= (U^{-1})^{abcd}_{mgnh}F^{mgnh}_{\mu\nu}\,.\end{aligned} \tag{3.55}$$

We can now build up the representation matrix for the transformation (3.55) which using (3.48) is given by

$$(T^{k})^{abcd}_{mgnh} = (T^{k})^{ab}_{mg}\delta^{c}{}_{n}\delta^{d}{}_{h} + \delta^{a}{}_{m}\delta^{b}{}_{g}(T^{k})^{cd}_{nh}\,. \tag{3.56}$$

We can therefore build up higher representations using the tensor product of the Lie algebra. These transformation rules will be used in the later sections for the dressed fields and in the construction of the gauge invariant field strength $F^{h}_{\mu\nu}$.

3.2.5 Product and Commutator in Lie Algebra

A key property that we have noticed from the above is that the product in the adjoint representation is equal to the commutator in the Lie algebra. This is a very useful property for us as it can be used to generate higher order representations. Suppose we consider the $(N^2-1)\times(N^2-1)$, matrix valued field in the adjoint representation, $A^{ab} := A^{c}(T^{c})_{ab}$, then the action of this matrix field on some $(N^2-1)\times 1$ column vector $\underline{B}$ can be represented as

$$\begin{aligned}(A\underline{B})^{a} := A^{ab}B^{b} = A^{c}(T^{c})_{ab}B^{b} = 2A^{c}B^{b}\mathrm{tr}\,([\tau^{c},\tau^{a}]\tau^{b}) = 2A^{c}B^{b}\mathrm{tr}\,(\tau^{a}[\tau^{b},\tau^{c}]) \\ = -2\ \mathrm{tr}\,([A,B]\tau^{a}) = [A,B]^{a}\,,\end{aligned} \tag{3.57}$$

where in addition to the cyclic property of the trace we have also used that $\mathrm{tr}\,([\tau^a,\tau^b]\tau^c) = \mathrm{tr}\,(\tau^a[\tau^b,\tau^c])$. Therefore we find that the product in the adjoint representation equals the commutator in the Lie algebra.

Equation (3.57) can now be generalised to the tensor product of the adjoint representation. If we now consider an $(N^2-1)^2 \times (N^2-1)^2$ tensor, $A^{ab}_{\ \ cd} := A^k(T^k)^{ab}_{\ \ cd}$ then using (3.48) the action of this tensor on some $(N^2-1)^2 \times 1$ matrix valued field $\underline{B}^{cd}$ is given by

$$\begin{aligned}(A\underline{B})^{ab} = A^{ab}_{\ \ cd}B^{cd} = A^k\{(T^k)^{ac}\delta^b_{\ d} + \delta^a_{\ c}(T^k)^{bd}\}B^{cd} &= A^{ac}B^{cb} + A^{bd}B^{ad} \\ &= A^{ac}B^{cb} - B^{ac}A^{cb} \\ &:= [A,B]^{ab}\,,\end{aligned} \tag{3.58}$$

where in the last line we have replaced index d by c as d is a dummy variable. Similarly for the higher order representations if we define a tensor valued field with eight indices, $A^{abcd}_{\ \ \ \ efgh} := A^k(T^k)^{abcd}_{\ \ \ \ efgh}$ then the action of this tensor on another tensor with four indices $\underline{B}^{efgh} := B^l(T^l)^{ef}_{\ \ gh}$ can be written as

$$\begin{aligned}(A\underline{B})^{abcd} &:= A^{abcd}_{\ \ \ \ efgh}B^{efgh} = A^k(T^k)^{abcd}_{\ \ \ \ efgh}B^l(T^l)^{ef}_{\ \ gh} \\ &= A^kB^l\{(T^k)^{ab}_{\ \ ef}\delta^c_{\ g}\delta^d_{\ h} + \delta^a_{\ e}\delta^b_{\ f}(T^k)^{cd}_{\ \ gh}\}(T^l)^{ef}_{\ \ gh} \\ &= A^kB^l\{(T^k)^{ab}_{\ \ ef}(T^l)^{ef}_{\ \ cd} + (T^l)^{ab}_{\ \ gh}(T^k)^{cd}_{\ \ gh}\}\,.\end{aligned} \tag{3.59}$$

In the second term of the last line we use the result $(T^k)^{cd}_{\ \ gh} = -(T^k)^{gh}_{\ \ cd}$ to get

$$\begin{aligned}(A\underline{B})^{abcd} &= A^kB^l\{(T^k)^{ab}_{\ \ ef}(T^l)^{ef}_{\ \ cd} - (T^l)^{ab}_{\ \ gh}(T^k)^{gh}_{\ \ cd}\} \\ &= (A)^{ab}_{\ \ ef}(B)^{ef}_{\ \ cd} - (B)^{ab}_{\ \ gh}(A)^{gh}_{\ \ cd} := [A,B]^{ab}_{\ \ cd}\,.\end{aligned} \tag{3.60}$$

Now equipped with the basic structures of Lie algebras we can discuss the idea of how we can construct gauge invariant objects using the dressing procedure.

3.3 Dressing Approach to Gauge Invariance

In this section we will study how gauge invariant charged particles can be constructed in gauge theories. We will use gauge invariance as the guiding principle in the construction of dressings [60, 61] appropriate for charged particles. We begin with the description of charged fields in the abelian theory and then later extend it to the non-abelian theory.

3.3.1 Charged Fields in Abelian and non-Abelian Theory

In general, if we talk about the properties of physical particles, we know they carry, in addition to mass and spin, additional quantum numbers such as isospin, electric or colour charge. In Chapter 2 we followed the normal route of, for example, identifying the electron with the Dirac spinor field ψ in QED. The local gauge transformations as we saw earlier in (2.5), for the matter field is

$$\psi(x) \to e^{-ie\theta(x)}\psi(x)\,. \tag{3.61}$$

So if e is switched off, then the Lagrangian fermion is locally gauge invariant. In the LSZ formalism it is assumed that at times long before and after any scattering process the fields entering or emerging from the vertex do not interact with each other any more *i.e.*, at times $t = -\infty$ and $t = +\infty$ the electrons are so far apart such that an interaction between them is negligible. However due to the long range nature of interactions (which implies that the potential between static charges slowly falls off as $\frac{e^2}{r}$) these interactions may not be ignored. Their negligence gives rise to IR divergences. What we can conclude from here is that since the coupling cannot vanish asymptotically the matter field ψ is not gauge invariant and is never physical. These matter fields do not create or annihilate charges as they are not gauge invariant in the remote past or future. Also as we cannot neglect interactions asymptotically, $\psi(x)$ is not a good asymptotic state [62–64]. Additionally one of the important requirements for a physical field as we have seen in Chapter 2 is that it must satisfy Gauss' law. The matter field $\psi(x)$ does not satisfy Gauss' law because Gauss' law generates local gauge transformations.

In order to dress an electron [65–67] we assume that we surround the electron with the electromagnetic cloud

$$\begin{aligned} \psi_D(x) &= \exp\Big(ie\frac{\partial_j A_j(x)}{\nabla^2}\Big)\psi(x) \\ &= h^{-1}(x)\psi(x)\,, \end{aligned} \tag{3.62}$$

which is locally gauge invariant and non-local (due to the factor $\frac{1}{\nabla^2}$). This dressing is more generally called Coulombic dressing. This choice of dressing has an appealing feature that the commutators of the electric and magnetic fields with (3.62) yield the electric and magnetic fields which we expect of a static charge. Individually, neither the matter field nor the dressing $h^{-1}(x)$ are gauge invariant but together, they describe a gauge invariant physical charge. This was first suggested by Dirac [68] who pointed out that the above dressing is a member of the set of composite fields:

$$\psi_f(x) \equiv \exp\Big(-ie\int d^4z f_\mu(x-z)A_\mu(z)\Big)\psi(x)\,, \tag{3.63}$$

which are locally gauge invariant for any f^μ so long as $\partial_\mu f^\mu(w) = \delta^{(4)}(w)$ holds. Now we want to apply this dressing approach to quarks where we replace the electromagnetic coupling constant 'ie' with 'g' and introduce colour indices for the gauge fields. In the non-abelian theory the physical fields must be invariant under local gauge transformations defined in (3.27) and (3.30). But these fields cannot be identified with the observables as they do not carry colour charges. To get a gauge invariant picture of the fermions in non-abelian theory we dress them as

$$\psi_h(x) = h^{-1}(x)\psi(x)\,, \tag{3.64}$$

which is the non-abelian analogue of ψ_D. In the above equation the dressing h^{-1}, transforms as

$$h^{-1}(x) \to h^{-1}(x)U(x)\,. \tag{3.65}$$

Now that we have all the ingredients needed we can now apply them in the constuction of physical gluonic configurations and as we shall see this will reveal an abelian gauge structure within the non-abelian gauge theory.

3.3.2 The Residual Abelian Gauge Structure

In a pure gauge theory the vector potential A^a_μ is the basic buliding block from which all other physical configurations are constructed. In order to build gauge invariant gluonic configurations we first construct a dressing field h^{-1} out of the gauge fields which transforms as (3.65). Using this dressing a gauge invariant gluonic field is given by

$$A^h_\mu := h^{-1}A_\mu h + \frac{1}{g}h^{-1}\partial_\mu h\,, \tag{3.66}$$

along with a physical field strength

$$F^h_{\mu\nu} := h^{-1}F_{\mu\nu}h\,. \tag{3.67}$$

There is a lot of freedom in the choice of the dressing and this reflects the specific physical situation being studied. In this thesis, we shall focus on the dressing that arises from requiring that $\partial^\mu A^h_\mu = 0$, and refer to this as the Landau dressing. For earlier work on this, see, [69, 70] and for a detailed discussion of how the dressings are related to gauge fixings we refer the reader to references [71–75]. We take space-time to be Euclidian so that the Laplacian $\Box = \partial^\mu \partial_\mu$ has a unique Green's function when acting on fields that vanish at infinity.

Using the perturbative expansion

$$v = g v_1 + g^2 v_2 + g^3 v_3 + \cdots \,, \tag{3.68}$$

the dressing h^{-1} can be expanded in powers of the coupling as

$$\begin{aligned} h^{-1} &= e^v = e^{g v_1 + g^2 v_2 + g^3 v_3 + \cdots} \\ &= 1 + g v_1 + g^2 \Big(\frac{1}{2} v_1 v_1 + v_2 \Big) + g^3 \Big(\frac{1}{6} v_1^3 + \frac{1}{2} v_1 v_2 + \frac{1}{2} v_2 v_1 + v_3 \Big) + \cdots \,, \end{aligned} \tag{3.69}$$

where $v = v^a \tau^a$. Similarly,

$$\begin{aligned} h = e^{-v} &= e^{-g v_1 - g^2 v_2 - g^3 v_3 + \cdots} \\ &= 1 - g v_1 + g^2 \Big(\frac{1}{2} v_1 v_1 - v_2 \Big) - g^3 \Big(\frac{1}{6} v_1^3 - \frac{1}{2} v_1 v_2 - \frac{1}{2} v_2 v_1 + v_3 \Big) + \cdots \,. \end{aligned} \tag{3.70}$$

Solving the Landau condition on the dressing leads to a perturbative solution. In (3.68) we find that (summarised in Appendix C)

$$v_n = \frac{1}{\Box} \partial_\mu \mathcal{A}^\mu_{n-1} \,, \tag{3.71}$$

where the first few terms are given by

$$\mathcal{A}^\mu_0 = A^\mu \,, \qquad \mathcal{A}^\mu_1 = [v_1, A^\mu] + \frac{1}{2} [\partial^\mu v_1, v_1] \,, \tag{3.72}$$

and

$$\mathcal{A}^\mu_2 = [v_2, A^\mu] + \frac{1}{2} [v_1, [v_1, A^\mu]] + \frac{1}{2} [\partial^\mu v_1, v_2] + \frac{1}{2} [\partial^\mu v_2, v_1] - \frac{1}{6} [v_1, [v_1, \partial^\mu v_1]] \,. \tag{3.73}$$

The resulting dressed potential (3.66) can then be written in the manifestly transverse form as

$$A^h_\mu = \left(g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Box}\right) \mathcal{A}^\nu \,, \tag{3.74}$$

where we have introduced a generalised potential $\mathcal{A}^\nu$ such that

$$\mathcal{A}^\nu = A^\nu + g\mathcal{A}^\nu_{\scriptscriptstyle 1} + g^2 \mathcal{A}^\nu_{\scriptscriptstyle 2} + \cdots \,. \tag{3.75}$$

This potential $\mathcal{A}^\nu$ plays the role in the non-abelian theory of the abelian potential in QED as it allows us to define a gauge invariant transverse field. Its identification reveals an abelian structure within the $SU(\mathrm{N})$ gauge theory under the gauge transformation property (3.33) which will be discussed next.

3.3.3 Gauge Transformations of the Vector Potential

Because of the fact that we have a transverse projector on $\mathcal{A}^\nu$ in (3.74), and that A^h_μ is gauge invariant at any order of perturbation theory we find $\mathcal{A}^\nu$ has a very simple gauge transformation property. To show this we note that at lowest order in coupling, $\mathcal{A}^\nu$ transforms as

$$\mathcal{A}_\nu \to (\mathcal{A}_\nu)^U = (A_\nu)^U + g(\mathcal{A}^\nu_{\scriptscriptstyle 1})^U + \cdots \,. \tag{3.76}$$

The vector potential, A^ν and the dressing, $v_{\scriptscriptstyle 1}$ transform as

$$A^\nu \to (A^\nu)^U = A^\nu + \partial^\nu \theta + g\Big([A^\nu, \theta] + \frac{1}{2}[\partial^\nu \theta, \theta]\Big) + \cdots \,, \tag{3.77}$$

and

$$v_{\scriptscriptstyle 1} \to v^U_{\scriptscriptstyle 1} = v_{\scriptscriptstyle 1} + \theta + g\frac{\partial^\nu}{\Box}\Big([A^\nu, \theta] + \frac{1}{2}[\partial^\nu \theta, \theta]\Big) + \cdots \,, \tag{3.78}$$

which when applied to (3.76) yields

$$\mathcal{A}^\nu \to (\mathcal{A}^\nu)^U = A^\nu + \partial^\nu\theta + g\Big([A^\nu,\theta] + \frac{1}{2}[\partial^\nu\theta,\theta] \\ + [v_{_1}+\theta, A^\nu + \partial^\nu\theta] + \frac{1}{2}[\partial^\nu(v_{_1}+\theta), v_{_1}+\theta]\Big) + \cdots \tag{3.79}$$

Further simplification leads to

$$\mathcal{A}^\nu \to \mathcal{A}^\nu + \partial^\nu\theta + \frac{1}{2}g\Big([v_{_1},\partial^\nu\theta] + [\partial^\nu v_{_1},\theta]\Big) + \cdots . \tag{3.80}$$

The above equation clearly illustrates the transformation of the generalised potential in the form

$$\mathcal{A}^\nu \to \mathcal{A}^\nu + \partial^\nu\Theta\,, \tag{3.81}$$

where, to lowest non-trivial order,

$$\Theta = \theta + \frac{1}{2}g[v_{_1},\theta]\,. \tag{3.82}$$

Thus from (3.74) we see that A^h_μ is gauge invariant. We will henceforth refer to $\mathcal{A}_\nu$ as an abelian potential but we stress that this is in a non-abelian Yang-Mills theory.

3.4 Physical Field Strength $\mathcal{F}_{\mu\nu}$

Now that we have constructed the abelian potential $\mathcal{A}_\mu$ (3.75) it is natural to define a physical field strength $\mathcal{F}_{\mu\nu}$ as

$$\begin{aligned}\mathcal{F}_{\mu\nu} &= \partial_\mu\mathcal{A}^\nu - \partial_\nu\mathcal{A}^\mu \\ &= \partial_\mu A^\nu - \partial_\nu A^\mu + g\big(\partial_\mu\mathcal{A}^\nu_{_1} - \partial_\nu\mathcal{A}^\mu_{_1}\big) + g^2\big(\partial_\mu\mathcal{A}^\nu_{_2} - \partial_\nu\mathcal{A}^\mu_{_2}\big) + \cdots\,,\end{aligned} \tag{3.83}$$

which is of abelian form but gauge invariant in the non-abelian theory due to (3.81). However, given the dressing, recall that the directly dressed field strength (3.67) can be written as the field strength of the dressed potential (3.66)

$$F^h_{\mu\nu} = \partial_\mu A^h_\nu - \partial_\nu A^h_\mu + g[A^h_\mu, A^h_\nu]\,. \tag{3.84}$$

In the free theory the two field strengths (3.83) and (3.84) agree but at higher order in the coupling they differ.

3.4.1 Relation between $\mathcal{F}_{\mu\nu}$ and $F^h_{\mu\nu}$

After finding the two fields strengths (3.83) and (3.84) in the previous section we will now investigate the relation between them. A natural expectation might be that they are the same. To check this we substitute the value of A^h_μ (3.74) into (3.84) to obtain:

$$\begin{aligned} F^h_{\mu\nu} &= \partial_\mu\Big\{\Big(g_{\nu\rho} - \frac{\partial_\nu\partial_\rho}{\Box}\Big)\mathcal{A}^\rho\Big\} - \partial_\nu\Big\{\Big(g_{\mu\rho} - \frac{\partial_\mu\partial_\rho}{\Box}\Big)\mathcal{A}^\rho\Big\} + g[A^h_\mu, A^h_\nu] \\ &= \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + g[A^h_\mu, A^h_\nu]\,. \end{aligned} \tag{3.85}$$

Using the definition (3.83) in (3.85) we can write the field strength factorisation as

$$F^h_{\mu\nu} = \mathcal{F}_{\mu\nu} + g[A^h_\mu, A^h_\nu]\,. \tag{3.86}$$

We thus find that the two field strengths are clearly not the same. These play a key role in constructing the mass like term that we will study in the next chapter.

3.5 Concluding Remarks

To recap this whole chapter we have demonstrated various properties of Lie algebras and derived new results that will be needed in the rest of the thesis. We have stressed on the role of the dressing from which other physical gluonic configurations are constructed. There has been lot of work on how the dressing principle can be used to construct quarks and gluons perturbatively [76,77]. Outside of perturbation theory this is not possible due to the Gribov ambiguity, see [78–80]. At small distances it is, however possible to perturbatively define the quarks and gluonic configurations [81–83]. As well as constructing the gauge invariant states, dressings can be used to calculate the inter-quark potential and to study the stability, creation and annihilation of particles in gauge theories [84].

Chapter 4

Gauge Invariant Mass terms

4.1 Introduction

The gauge theories (both abelian and non-abelian) have one of the striking features that the gauge fields in these theories are massless. Adding a naive mass term such as A^2 to the Lagrangian breaks gauge invariance. This is consistent with the degrees of freedom but leads to the difficulties in establishing renormalisability of the interacting theory for massive photons. However, not to be put off by this, there has been a lot of interesting debate on the construction of the gauge invariant mass terms that are generated in the gauge invariant expansions [85–88]. In the abelian theory there are various ways to obtain a gauge invariant mass term [89], however in non-abelian gauge theories it is not so straightforward. In a pure gauge theory indeed there has been a resurgence of interest in this over the past 10 years [90–97]. This can be traced in part to a seminal paper by Zwanziger [98] where the gauge invariance of this construction of mass operator was addressed. This mass term has been used up and exploited in several publications, see for example, [99–104].

In Zwanziger's description of the A^2 mass term, an expansion is introduced in terms of the powers of the field strength (following his notation):

$$
\begin{aligned}
|A^h|^2 = -\frac{1}{2}\Big(F_{\mu\nu}, \Big(\frac{1}{D^2}F_{\mu\nu}\Big)\Big) \\
+ \Big(\Big(\frac{1}{D^2}F_{\mu\nu}\Big), \Big[\frac{1}{D^2}D_\alpha F_{\alpha\mu}, \frac{1}{D^2}D_\beta F_{\beta\nu}\Big]\Big) \\
- \Big(\Big(\frac{1}{D^2}F_{\mu\nu}\Big), \Big[\frac{1}{D^2}D_\beta F_{\beta\rho}, \frac{1}{D^2}D_\rho F_{\mu\nu}\Big]\Big) ,
\end{aligned}
\tag{4.1}
$$

where gauge invariance is maintained order by order by the use of the inverse covariant Laplacian. Although Zwanziger introduced this expansion he did not offer a derivation or explain how it may be extended to higher orders. It is also not clear how unique the construction is.

In this chapter we will see how gauge invariant mass terms can be constructed in QED and the non-abelian Yang Mills theory. We shall try to write mass terms in terms of the field strength in both the theories. After this we shall introduce the gauge covariant Laplacian along with its inverse and summarise its main properties. In addition to the field strength decomposition introduced in the previous chapter there is another factorisation at play, that of the dressed field into transverse and longitudinal components which underlies various expansions seen in the literature. The difference between the decomposition and factorisation will later give us insight into Zwanziger's expansion.

4.2 Mass Terms in Abelian and non-Abelian theory

In QED the dressed field (3.66) is simply the transverse field A_μ^{T} where

$$
A_\mu^{T} = \left(g_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}\right) A^\nu . \tag{4.2}
$$

This can be used to construct, a gauge invariant mass-like term

$$\mathcal{M}^2 = \int \mathrm{d}^4x \; A^{T}_{\mu}(x) A^{T}_{\mu}(x) \,. \tag{4.3}$$

From equation (4.2) we can write the transverse vector potential as

$$A^{T}_{\mu}(x) = \frac{1}{\Box}\partial^\nu F_{\nu\mu}(x) \,. \tag{4.4}$$

Using this we have for (4.3)

$$\mathcal{M}^2 = \int \mathrm{d}^4x \; A^{T}_{\mu}(x) \frac{1}{\Box}\partial^\nu F_{\nu\mu}(x) \,. \tag{4.5}$$

Integrating by parts and using the fact that $F_{\nu\mu}$ is antisymmetric we end up with

$$\mathcal{M}^2 = -\int \mathrm{d}^4x \; \partial_\nu A^{T}_{\mu}(x) \Big(\frac{1}{\Box}F_{\nu\mu}\Big)(x) = -\frac{1}{2}\int \mathrm{d}^4x \; F_{\mu\nu}(x) \Big(\frac{1}{\Box}F^{\mu\nu}\Big)(x) \,, \tag{4.6}$$

which is the mass term generalisation for the abelian theory.

In much the same way, in a non-abelian theory a mass term can be written as

$$\mathcal{M}^2 = \int \mathrm{d}^4x \; A^{h\,a}_{\mu}(x) A^{h\,a}_{\mu}(x) = -\frac{1}{2}\int \mathrm{d}^4x \; \mathcal{F}^a_{\mu\nu}(x) \Big(\frac{1}{\Box}\mathcal{F}^{\mu\nu}\Big)^a(x) \,. \tag{4.7}$$

Here we see that the generalisation of the QED mass term (4.3) to the non-abelian theory is accomplished by replacing the QED field strength $F_{\mu\nu}$ by the physical field strength $\mathcal{F}_{\mu\nu}$ (3.83). Having seen the role of this field strength $\mathcal{F}_{\mu\nu}$, we now want to identify the role of the dressed field strength $F^h_{\mu\nu}$. Its role is not immediately obvious, and we will first need to discuss the approach of Zwanziger to the mass term.

Another possible generalisation of the right hand side of (4.6) which maintains gauge invariance is to use the non-abelian field strength $F_{\mu\nu}$ but replace the inverse Laplacian $1/\Box$ with the gauge covariant inverse $1/D^2$. This is essentially the first part of Zwanziger's approach to the mass term [98] as given by the first term of the expansion (4.1). What we will show in the next sections is how this is carried out in practice and then how the central role of the dressed field strength $F^h_{\mu\nu}$ emerges. This will then allow us to understand the relationship between the non-abelian mass term (4.7) and the gauge covariant Laplacian approach of Zwanziger.

4.3 The Gauge Covariant Inverse Laplacian

Before discussing how the inverse to the gauge covariant Laplacian is defined perturbatively, we first recall how the inverse to the normal Laplacian, $\Box$, is defined. Acting on an element $f(x)$, from a suitable class of test-functions we define

$$\frac{1}{\Box}f(x) := \int \mathrm{d}^4y\; K_0(x,y)f(y)\,, \tag{4.8}$$

where $K_0(x,y)$ is the Green's function for the Laplacian which in the Euclidian setting we can write as

$$K_0(x,y) = -4\pi^2\frac{1}{(x-y)^2}\,. \tag{4.9}$$

The Green's function satisfies $\Box_x K_0(x,y) = \delta^4(x-y)$ which ensures that (4.8) is the inverse Laplacian. Note that the subscript x in the Laplacian signifies which variable it acts upon.

To generalise this we consider the operator found by replacing the derivative ∂_μ by the covariant derivative $D_\mu = \partial_\mu + gA_\mu$. That is, we consider the gauge covariant Laplacian

$$D^2 := D^\mu D_\mu = \Box + g(\partial{\cdot}A + 2A{\cdot}\partial) + g^2A^2\,. \tag{4.10}$$

To construct the Green's function we need to identify the space of functions that this operator acts on. Given that this is a matrix operator it will act on vectors in the appropriate representation of the gauge group. To signify this we will write the functions as the column vector $\underline{f}$. The cases we will be interested in are when the vectors are in the fundamental, adjoint or tensor product of these representations but for the moment we will not specify the representation.

We now write

$$\frac{1}{D^2}\underline{f}(x) = \int \mathrm{d}^4y\; K(x,y)\underline{f}(y)\,, \tag{4.11}$$

where $K(x,y)$ is a matrix Green's function and require that

$$D_x^2 K(x,y) = \delta^4(x-y)\,. \tag{4.12}$$

This is solved perturbatively by letting $K(x,y)$ have the expansion

$$K(x,y) = K_0(x,y) + gK_1(x,y) + g^2K_2(x,y) + \cdots\,. \tag{4.13}$$

We then find that $K_0(x,y)$ is the free Green's function (4.9) times the identity matrix as expected while

$$\Box_x K_1(x,y) + (\partial{\cdot}A + 2A{\cdot}\partial_x)K_0(x,y) = 0\,, \tag{4.14}$$

and in general for $n \geq 2$

$$\Box_x K_n(x,y) + (\partial{\cdot}A + 2A{\cdot}\partial_x)K_{n-1}(x,y) + A^2K_{n-2}(x,y) = 0\,. \tag{4.15}$$

These equations can be solved in an iterative fashion resulting in

$$K_1(x,y) = \int \mathrm{d}^4z\;\left\{\partial_z^\rho K_0(x,z)A_\rho(z)K_0(z,y) - K_0(x,z)A_\rho(z)\partial_z^\rho K_0(z,y)\right\}\,, \tag{4.16}$$

$$K_2(x,y) = \int \mathrm{d}^4z \, \Big\{ \partial_z^\rho K_0(x,z) A_\rho(z) K_1(z,y) - K_0(x,z) A_\rho(z) \, \partial_z^\rho K_1(z,y) \\ - K_0(x,z) \, A_\rho(z) A_\rho(z) \, K_0(z,y) \Big\} \,, \tag{4.17}$$

and in general for $n \geq 2$

$$K_n(x,y) = \int \mathrm{d}^4z \, \Big\{ \partial_z^\rho K_0(x,z) A_\rho(z) K_{n-1}(z,y) - K_0(x,z) A_\rho(z) \partial_z^\rho K_{n-1}(z,y) \\ - K_0(x,z) A^2(z) K_{n-2}(z,y) \Big\} \,. \tag{4.18}$$

In the next subsection we will review some of the properties of the inverse covariant Laplacian $1/D^2$ and provide proofs of results that will be needed in the later part of thesis.

4.3.1 Properties of the Inverse Covariant Laplacian

It should be noted that the gauge covariant Laplacian D^2 in (4.11) has been shown to have a right inverse $1/D^2$ in the sense that

$$D^2 \Big(\frac{1}{D^2} \underline{f} \Big)(x) = \underline{f}(x) \,. \tag{4.19}$$

One can in an analogous way, see that the left inverse $1/\tilde{D}^2$ exists for D^2 that is,

$$\Big(\frac{1}{\tilde{D}^2} D^2 \underline{f} \Big)(x) = \underline{f}(x) \,. \tag{4.20}$$

Equations (4.19) and (4.20) can be shown by first starting with the construction of the right inverse that can be written from (4.11) and (4.12) in the form

$$D^2 \Big(\frac{1}{D^2} \underline{f} \Big)(x) = \int \mathrm{d}^4y \; D_x^2 K(x,y) \underline{f}(y) \,, \tag{4.21}$$

where $K(x,y)$ is defined in (4.13). Similarly we can define the left inverse $1/\tilde{D}^2$

$$\Big(\frac{1}{\tilde{D}^2}D^2\underline{f}\Big)(x) := \int \mathrm{d}^4y\ \tilde{K}(x,y)D^2\underline{f}(y) = \underline{f}(x)\,, \tag{4.22}$$

that can be expanded in a power series as in (4.13) to give

$$\tilde{K}(x,y) = \tilde{K}_0(x,y) + g\tilde{K}_1(x,y) + g^2\tilde{K}_2(x,y) + \cdots\,. \tag{4.23}$$

Proceeding in the same way as above we find for $n \geq 2$

$$\begin{aligned}\tilde{K}_n(x,y) = \int \mathrm{d}^4z\ \Big\{&\partial_z^\rho\tilde{K}_{n-1}(x,z)A_\rho(z)\tilde{K}_0(z,y) - \tilde{K}_{n-1}(x,z)A_\rho(z)\partial_z^\rho\tilde{K}_0(z,y)\\ &- \tilde{K}_{n-2}(x,z)A^2(z)\tilde{K}_0(z,y)\Big\}\,.\end{aligned} \tag{4.24}$$

An inductive proof will then show that order by order we have

$$K_n(x,y) = \tilde{K}_n(x,y)\,. \tag{4.25}$$

which we shall proceed to verify next.

When $n = 0,1$ it is straightforward to see that

$$K_0(x,y) = \tilde{K}_0(x,y) \quad \text{and} \quad K_1(x,y) = \tilde{K}_1(x,y)\,. \tag{4.26}$$

Assuming that the result holds true for the $(n-1)$-*th* term, that is, $K_{n-1}(x,y) = \tilde{K}_{n-1}(x,y)$, using this we should now be able to verify (4.25). We start with $K_n(x,y)$ (4.18) and use $K_{n-1}(z,y) = \tilde{K}_{n-1}(z,y)$ and $K_{n-2}(z,y) = \tilde{K}_{n-2}(z,y)$ to obtain

$$\begin{aligned}K_n(x,y) = \int \mathrm{d}^4z\ \Big\{&\partial_z^\rho\tilde{K}_0(x,z)A_\rho(z)\tilde{K}_{n-1}(z,y) - \tilde{K}_0(x,z)A_\rho(z)\partial_z^\rho\tilde{K}_{n-1}(z,y)\\ &- \tilde{K}_0(x,z)A^2(z)\tilde{K}_{n-2}(z,y)\Big\}\,.\end{aligned} \tag{4.27}$$

In the above equation we use the identification

$$\tilde{K}_{n-1}(z,y) = \int \mathrm{d}^4 w \, \Big\{ \partial_w^\mu \tilde{K}_{n-2}(z,w) A_\mu(w) \tilde{K}_0(w,y) - \tilde{K}_{n-2}(z,w) A_\mu(w) \partial_w^\mu \tilde{K}_0(w,y) - \tilde{K}_{n-3}(z,w) A^2(w) \tilde{K}_0(w,y) \Big\} . \tag{4.28}$$

and the result for $\tilde{K}_{n-2}(z,y)$ to obtain

$$\begin{aligned} K_n(x,y) = \int \mathrm{d}^4 z \, \mathrm{d}^4 w \, \Big\{ & \partial_z^\rho \tilde{K}_0(x,z) A_\rho(z) \Big(\partial_w^\mu \tilde{K}_{n-2}(z,w) A_\mu(w) \tilde{K}_0(w,y) \\ & - \tilde{K}_{n-2}(z,w) A_\mu(w) \partial_w^\mu \tilde{K}_0(w,y) - \tilde{K}_{n-3}(z,w) A^2(w) \tilde{K}_0(w,y) \Big) \Big\} \\ - \int \mathrm{d}^4 z \, \mathrm{d}^4 w \, \Big\{ & \tilde{K}_0(x,z) A_\rho(z) \partial_z^\rho \Big(\partial_w^\mu \tilde{K}_{n-2}(z,w) A_\mu(w) \tilde{K}_0(w,y) \\ & - \tilde{K}_{n-2}(z,w) A_\mu(w) \partial_w^\mu \tilde{K}_0(w,y) - \tilde{K}_{n-3}(z,w) A^2(w) \tilde{K}_0(w,y) \Big) \Big\} \\ - \int \mathrm{d}^4 z \, \mathrm{d}^4 w \, \Big\{ & \tilde{K}_0(x,z) A^2(z) \Big(\partial_w^\mu \tilde{K}_{n-3}(z,w) A_\mu(w) \tilde{K}_0(w,y) \\ & - \tilde{K}_{n-3}(z,w) A_\mu(w) \partial_w^\mu \tilde{K}_0(w,y) - \tilde{K}_{n-4}(z,w) A^2(w) \tilde{K}_0(w,y) \Big) \Big\} . \end{aligned} \tag{4.29}$$

Interchanging z and w for all the integrals in the above equation we have

$$\begin{aligned} K_n(x,y) = \int \mathrm{d}^4 z \, \mathrm{d}^4 w \, \Big\{ & \partial_w^\rho \tilde{K}_0(x,w) A_\rho(w) \Big(\partial_z^\mu \tilde{K}_{n-2}(w,z) A_\mu(z) \tilde{K}_0(z,y) \\ & - \tilde{K}_{n-2}(w,z) A_\mu(z) \partial_z^\mu \tilde{K}_0(z,y) - \tilde{K}_{n-3}(w,z) A^2(z) \tilde{K}_0(z,y) \Big) \Big\} \\ - \int \mathrm{d}^4 z \, \mathrm{d}^4 w \, \Big\{ & \tilde{K}_0(x,w) A_\rho(w) \partial_w^\rho \Big(\partial_z^\mu \tilde{K}_{n-2}(w,z) A_\mu(z) \tilde{K}_0(z,y) \\ & - \tilde{K}_{n-2}(w,z) A_\mu(z) \partial_z^\mu \tilde{K}_0(z,y) - \tilde{K}_{n-3}(w,z) A^2(z) \tilde{K}_0(z,y) \Big) \Big\} \\ - \int \mathrm{d}^4 z \, \mathrm{d}^4 w \, \Big\{ & \tilde{K}_0(x,w) A^2(w) \Big(\partial_z^\mu \tilde{K}_{n-3}(w,z) A_\mu(z) \tilde{K}_0(z,y) \\ & - \tilde{K}_{n-3}(w,z) A_\mu(z) \partial_z^\mu \tilde{K}_0(z,y) - \tilde{K}_{n-4}(w,z) A^2(z) \tilde{K}_0(z,y) \Big) \Big\} . \end{aligned} \tag{4.30}$$

Noting that $A_\mu(z)\tilde{K}_0(z,y)$ is common to the first, third and fifth line and that $A_\mu(z)\partial_z^\mu\tilde{K}_0(z,y)$ to the second, fourth and sixth line we have

$$\begin{aligned}K_n(x,y) = \int \mathrm{d}^4z\,\mathrm{d}^4w\, \Big\{\Big(&\partial_w^\rho\tilde{K}_0(x,w)A_\rho(w)\partial_z^\mu\tilde{K}_{n-2}(w,z) - \tilde{K}_0(x,w)A_\rho(w)\partial_w^\rho\partial_z^\mu\tilde{K}_{n-2}(w,z)\\ &- \tilde{K}_0(x,w)A^2(w)\partial_z^\mu\tilde{K}_{n-3}(w,z)\Big)A_\mu(z)\tilde{K}_0(z,y)\Big\}\\ - \int \mathrm{d}^4z\,\mathrm{d}^4w\, \Big\{\Big(&\partial_w^\rho\tilde{K}_0(x,w)A_\rho(w)\tilde{K}_{n-2}(w,z) - \tilde{K}_0(x,w)A_\rho(w)\partial_w^\rho\tilde{K}_{n-2}(w,z)\\ &- \tilde{K}_0(x,w)A^2(w)\tilde{K}_{n-3}(w,z)\Big)A_\mu(z)\partial_z^\mu\tilde{K}_0(z,y)\Big\}\\ - \int \mathrm{d}^4z\,\mathrm{d}^4w\, \Big\{\Big(&\partial_w^\rho\tilde{K}_0(x,w)A_\rho(w)\tilde{K}_{n-3}(z,w) - \tilde{K}_0(x,w)A_\rho(w)\partial_w^\rho\tilde{K}_{n-3}(w,z)\\ &- \tilde{K}_0(x,w)A^2(w)\tilde{K}_{n-4}(w,z)\Big)A^2(z)\tilde{K}_0(z,y)\Big\}\,.\end{aligned} \tag{4.31}$$

Taking the derivative ∂_z^μ common from the first braces

$$\begin{aligned}K_n(x,y) = \int \mathrm{d}^4z\,\mathrm{d}^4w\, \Big\{\partial_z^\mu\Big(&\partial_w^\rho\tilde{K}_0(x,w)A_\rho(w)\tilde{K}_{n-2}(w,z) - \tilde{K}_0(x,w)A_\rho(w)\partial_w^\rho\tilde{K}_{n-2}(w,z)\\ &- \tilde{K}_0(x,w)A^2(w)\tilde{K}_{n-3}(w,z)\Big)A_\mu(z)\tilde{K}_0(z,y)\Big\}\\ - \int \mathrm{d}^4z\,\mathrm{d}^4w\, \Big\{\Big(&\partial_w^\rho\tilde{K}_0(x,w)A_\rho(w)\tilde{K}_{n-2}(w,z) - \tilde{K}_0(x,w)A_\rho(w)\partial_w^\rho\tilde{K}_{n-2}(w,z)\\ &- \tilde{K}_0(x,w)A^2(w)\tilde{K}_{n-3}(w,z)\Big)A_\mu(z)\partial_z^\mu\tilde{K}_0(z,y)\Big\}\\ - \int \mathrm{d}^4z\,\mathrm{d}^4w\, \Big\{&\partial_w^\rho\tilde{K}_0(x,w)A_\rho(w)\tilde{K}_{n-3}(z,w) - \tilde{K}_0(x,w)A_\rho(w)\partial_w^\rho\tilde{K}_{n-3}(w,z)\\ &- \tilde{K}_0(x,w)A^2(w)\tilde{K}_{n-4}(w,z)\Big)A^2(z)\tilde{K}_0(z,y)\Big\}\,.\end{aligned} \tag{4.32}$$

Identifying the term inside the first and second braces with the term $K_{n-1}(x,z)$

and the term inside the last braces with $K_{n-2}(x,z)$ we obtain

$$K_n(x,y) = \int \mathrm{d}^4z \, \Big\{ \partial_z^\mu K_{n-1}(x,z) A_\mu(z) \tilde{K}_{\scriptscriptstyle 0}(z,y) - K_{n-1}(x,z) A_\mu(z) \partial_z^\mu \tilde{K}_{\scriptscriptstyle 0}(z,y) \\ - K_{n-2}(x,z) A^2(z) \tilde{K}_{\scriptscriptstyle 0}(z,y) \Big\} . \tag{4.33}$$

Using the result, $K_{n-1}(x,z) = \tilde{K}_{n-1}(x,z)$ and $K_{n-2}(x,z) = \tilde{K}_{n-2}(x,z)$ in the above equation we obtain

$$\begin{aligned} K_n(x,y) = \int \mathrm{d}^4z \, \mathrm{d}^4w \, \Big\{ &\partial_z^\mu \tilde{K}_{n-1}(x,z) A_\mu(z) \tilde{K}_{\scriptscriptstyle 0}(z,y) - \tilde{K}_{n-1}(x,z) A_\mu(z) \partial_z^\mu \tilde{K}_{\scriptscriptstyle 0}(z,y) \\ &- \tilde{K}_{n-2}(x,z) A^2(z) \tilde{K}_{\scriptscriptstyle 0}(z,y) \Big\} \\ = \tilde{K}_n(x,y) \, . & \end{aligned} \tag{4.34}$$

A related and very useful result is that

$$\mathrm{tr} \int \mathrm{d}^4x \, \Big\langle \underline{f}(x) \Big(\frac{1}{D^2} \, \underline{g} \Big)(x) \Big\rangle = \mathrm{tr} \int \mathrm{d}^4x \, \Big\langle \Big(\frac{1}{D^2} \underline{f} \Big)(x) \, \underline{g}(x) \Big\rangle \, . \tag{4.35}$$

For fields in the adjoint representation we introduce the appropriate colour indices such that $K(x,y) \to K^{ab}(x,y)$ and this identity is equivalent to the result that

$$K^{ab}(x,y) = K^{ba}(y,x) \, . \tag{4.36}$$

Note that when we are explicit about the matrix indices in the Green's function as in $K^{ab}(x,y)$ we mean that we have an expansion

$$K^{ab}(x,y) = \delta^{ab} K_{\scriptscriptstyle 0}(x,y) + g K_{\scriptscriptstyle 1}^{ab}(x,y) + g^2 K_{\scriptscriptstyle 2}^{ab}(x,y) + \cdots \, , \tag{4.37}$$

where, for example,

$$K_{{}_1}^{ab}(x,y) = \int \mathrm{d}^4 z \left\{ \partial_z^\rho K_{{}_0}^{ac}(x,z) A_\rho^{cd}(z) K_{{}_0}^{db}(z,y) - K_{{}_0}^{ac}(x,z) A_\rho^{cd}(z) \partial_z^\rho K_{{}_0}^{db}(z,y) \right\}, \tag{4.38}$$

and A_ρ^{cd} is the potential in the adjoint representation whence $A_\rho^{cd} = -A_\rho^e f^{cde}$. We will now use induction to show that (4.36) holds to all orders, that is,

$$K_n^{ab}(x,y) = K_n^{ba}(y,x) \,. \tag{4.39}$$

Starting with $n = 0$ it is quite obvious that

$$K_{{}_0}^{ab}(x,y) = K_{{}_0}^{ba}(y,x) \,, \tag{4.40}$$

because we are allowed to change the indices (ab) and the coordinates (x,y).

For $n = 1$ we need to show that $K_{{}_1}^{ab}(x,y) = K_{{}_1}^{ba}(y,x)$. We start with the left hand side defined in (4.38) and use (4.40) along with the property, $A_\rho^{cd}(z) = -A_\rho^{dc}(z)$, to find

$$K_{{}_1}^{ab}(x,y) = -\int \mathrm{d}^4 z \left\{ \partial_z^\rho K_{{}_0}^{ca}(z,x) A_\rho^{dc}(z) K_{{}_0}^{bd}(y,z) - K_{{}_0}^{ca}(z,x) A_\rho^{dc}(z) \partial_z^\rho K_{{}_0}^{bd}(y,z) \right\}. \tag{4.41}$$

Swapping the terms $\partial_z^\rho K_{{}_0}^{ca}(z,x)$ with $K_{{}_0}^{bd}(y,z)$ in the first integrand and similarly the terms $K_{{}_0}^{ca}(z,x)$ with $\partial_z^\rho K_{{}_0}^{bd}(y,z)$ in the second we get

$$\begin{aligned} K_{{}_1}^{ab}(x,y) &= -\int \mathrm{d}^4 z \left\{ K_{{}_0}^{bd}(y,z) A_\rho^{dc}(z) \partial_z^\rho K_{{}_0}^{ca}(z,x) - \partial_z^\rho K_{{}_0}^{bd}(y,z) A_\rho^{dc}(z) K_{{}_0}^{ca}(z,x) \right\} \\ &= K_{{}_1}^{ba}(y,x) \,. \end{aligned} \tag{4.42}$$

Next we verify that $K_{{}_2}^{ab}(x,y) = K_{{}_2}^{ba}(y,x)$. With the definition of $K_{{}_2}(x,y)$ already introduced in (4.17) but now carrying indices (ab) we start with the left hand side

$$\begin{aligned} K_{2}^{ab}(x,y) &= \int \mathrm{d}^4 z \, \Big\{ \partial^\rho_z K_{0}^{ac}(x,z) A^{cd}_\rho(z) K_{1}^{db}(z,y) - K_{0}^{ac}(x,z) A^{cd}_\rho(z) \; \partial^\rho_z K_{1}^{db}(z,y) \\ &\qquad - K_{0}^{ac}(x,z) \; A^{cd}_\rho(z) A^{de}_\rho(z) \; K_{0}^{eb}(z,y) \Big\} \\ &= \int \mathrm{d}^4 z \, \Big\{ - \partial^\rho_z K_{0}^{ca}(z,x) A^{dc}_\rho(z) K_{1}^{bd}(y,z) + K_{0}^{ca}(z,x) A^{dc}_\rho(z) \; \partial^\rho_z K_{1}^{bd}(y,z) \\ &\qquad - K_{0}^{ca}(z,x) \; A^{dc}_\rho(z) A^{ed}_\rho(z) \; K_{0}^{be}(y,z) \Big\} , \end{aligned} \tag{4.43}$$

where we have used (4.40) and (4.42) in addition to, $A^{cd}_\rho = -A^{dc}_\rho$. At this stage we make use of the inverse properties of the covariant Laplacian in the first two integrands to yield

$$\begin{aligned} K_{2}^{ab}(x,y) &= \int \mathrm{d}^4 z \, \Big\{ - \partial^\rho_z K_{1}^{ca}(z,x) A^{dc}_\rho(z) K_{0}^{bd}(y,z) + K_{1}^{ca}(z,x) A^{dc}_\rho(z) \; \partial^\rho_z K_{0}^{bd}(y,z) \\ &\qquad - K_{0}^{ca}(z,x) \; A^{dc}_\rho(z) A^{ed}_\rho(z) \; K_{0}^{be}(y,z) \Big\} . \end{aligned} \tag{4.44}$$

Rearranging the above equation we find

$$\begin{aligned} K_{2}^{ab}(x,y) &= \int \mathrm{d}^4 z \, \Big\{ \partial^\rho_z K_{0}^{bd}(y,z) A^{dc}_\rho(z) K_{1}^{ca}(z,x) - K_{0}^{bd}(y,z) A^{dc}_\rho(z) \; \partial^\rho_z K_{1}^{ca}(z,x) \\ &\qquad - K_{0}^{be}(y,z) \; A^{ed}_\rho(z) A^{dc}_\rho(z) \; K_{0}^{ca}(z,x) \Big\} , \end{aligned} \tag{4.45}$$

and hence

$$K_{2}^{ab}(x,y) = K_{2}^{ba}(y,x) \, . \tag{4.46}$$

To summarise we have seen that

$$K_{0}^{ab}(x,y) = K_{0}^{ba}(y,x) \, , \quad K_{1}^{ab}(x,y) = K_{1}^{ba}(y,x) \, , \quad K_{2}^{ab}(x,y) = K_{2}^{ba}(y,x) \, . \tag{4.47}$$

Assuming that the result holds true for the $(n-1)$-th term, that is, $K^{ab}_{n-1}(x,y) = K^{ba}_{n-1}(y,x)$, using this we should now be able to verify that

$$K^{ab}_{n}(x,y) = K^{ba}_{n}(y,x)\,, \tag{4.48}$$

where writing (4.18) with appropriate colour indices we have

$$\begin{aligned} K^{ab}_{n}(x,y) = \int \mathrm{d}^4 z\, \Big\{ &\partial^\rho_z K^{ac}_{0}(x,z) A^{cd}_\rho(z) K^{db}_{n-1}(z,y) - K^{ac}_{0}(x,z) A^{cd}_\rho(z) \partial^\rho_z K^{db}_{n-1}(z,y) \\ &- K^{ac}_{0}(x,z) A^{cd}_\rho(z) A^{de}_\rho(z) K^{eb}_{n-2}(z,y) \Big\}\,. \end{aligned} \tag{4.49}$$

Similarly the right hand side of (4.48) can be written as

$$\begin{aligned} K^{ba}_{n}(y,x) = \int \mathrm{d}^4 z\, \Big\{ &\partial^\rho_z K^{bd}_{0}(y,z) A^{dc}_\rho(z) K^{ca}_{n-1}(z,x) - K^{bd}_{0}(y,z) A^{dc}_\rho(z) \partial^\rho_z K^{ca}_{n-1}(z,x) \\ &- K^{be}_{0}(y,z) A^{ed}_\rho(z) A^{dc}_\rho(z) K^{ca}_{n-2}(z,x) \Big\}\,. \end{aligned} \tag{4.50}$$

To verify (4.48) we start with (4.49) and apply the following properties

$$K^{ac}_{0}(x,z) = K^{ca}_{0}(z,x), \quad K^{db}_{n-1}(z,y) = K^{bd}_{n-1}(y,z), \quad A^{cd}_\rho(z) = -A^{dc}_\rho(z)\,, \tag{4.51}$$

to get

$$\begin{aligned} K^{ab}_{n}(x,y) = \int \mathrm{d}^4 z\, \Big\{ &-\partial^\rho_z K^{ca}_{0}(z,x) A^{dc}_\rho(z) K^{bd}_{n-1}(y,z) + K^{ca}_{0}(z,x) A^{dc}_\rho(z) \partial^\rho_z K^{bd}_{n-1}(y,z) \\ &- K^{ca}_{0}(z,x) A^{dc}_\rho(z) A^{ed}_\rho(z) K^{be}_{n-2}(y,z) \Big\}\,. \end{aligned} \tag{4.52}$$

Using the properties of the inverse covariant Laplacian in (4.52) we obtain

$$K_n^{ab}(x,y) = \int \mathrm{d}^4 z \,\Big\{ -\partial_z^\rho K_{n-1}^{ca}(z,x) A_\rho^{dc}(z) K_0^{bd}(y,z) + K_{n-1}^{ca}(z,x) A_\rho^{dc}(z) \partial_z^\rho K_0^{bd}(y,z) \\ - K_{n-2}^{ca}(z,x) A_\rho^{dc}(z) A_\rho^{ed}(z) K_0^{be}(y,z) \Big\} \,. \tag{4.53}$$

Now swapping $K_0^{bd}(y,z)$ with $\partial_z^\rho K_{n-1}^{ca}(z,x)$ in the first integrand and $K_{n-1}^{ca}(z,x)$ with $\partial_z^\rho K_0^{bd}(y,z)$ in the second integrand we obtain

$$\begin{aligned} K_n^{ab}(x,y) &= \int \mathrm{d}^4 z \,\Big\{ -K_0^{bd}(y,z) A_\rho^{dc}(z) \partial_z^\rho K_{n-1}^{ca}(z,x) + \partial_z^\rho K_0^{bd}(y,z) A_\rho^{dc}(z) K_{n-1}^{ca}(z,x) \\ &\qquad - K_0^{be}(y,z) A_\rho^{ed}(z) A_\rho^{dc}(z) K_{n-2}^{ca}(z,x) \Big\} \\ &= K_n^{ba}(y,x) \,. \end{aligned} \tag{4.54}$$

We now need to clarify the gauge transformation properties of the inverse covariant Laplacian. In Chapter 3 we discussed the gauge transformation property (3.28) of the covariant derivative. Under a gauge transformation, the covariant Laplacian transforms as

$$D_x^2 \to U^{-1}(x) D_x^2 \, U(x) \,, \tag{4.55}$$

where $U(x)$ is the group element in the appropriate representation that we used to define the covariant derivative. From (4.55) and the fact that $K(x,y)$ is the Green's function for both the left and right inverse of the Laplacian action (4.19) and (4.20) it follows that the associated transformation of the Green's function $K(x,y)$ is

$$K(x,y) \to U^{-1}(x) K(x,y) U(y) \,. \tag{4.56}$$

In terms of the adjoint representation,

$$K^{ab}(x,y) \to (U^{-1})^a_c(x) K^{cd}(x,y) U^d_b(y) \,, \tag{4.57}$$

and for the tensor product of adjoint representation

$$K^{ab}_{cd}(x,y) \to (U^{-1})^{ab}_{\ ef}(x) K^{ef}_{gh}(x,y)(U)^{gh}_{\ cd}(y)\,. \tag{4.58}$$

This means that if we have a field $\underline{B}$ which transforms in the adjoint representation then so will $\frac{1}{D^2}\underline{B}$, *i.e.*,

$$\Big(\frac{1}{D^2}B\Big)^a(x) \to (U^{-1})^a_b(x)\Big(\frac{1}{D^2}B\Big)^b(x)\,. \tag{4.59}$$

While if we have a tensor product of the adjoint representation we get

$$\Big(\frac{1}{D^2}B\Big)^{ab}(x) \to (U^{-1})^{ab}_{\ cd}(x)\Big(\frac{1}{D^2}B\Big)^{cd}(x) = (U^{-1})^a_c(x)(U^{-1})^b_d(x)\Big(\frac{1}{D^2}B\Big)^{cd}(x)\,. \tag{4.60}$$

The terms on left side of (4.59) and (4.60) can be written as

$$\Big(\frac{1}{D^2}B\Big)^a(x) = \int \mathrm{d}^4y\; K^{ab}(x,y)B^b(y)\,, \tag{4.61}$$

and

$$\Big(\frac{1}{D^2}B\Big)^{ab}(x) = \int \mathrm{d}^4y\; K^{ab}_{\ cd}(x,y)B^{cd}(y)\,, \tag{4.62}$$

where for such gauge invariant fields $B^{ab} = -f^{abc}B^c$. It is not immediately obvious that the inverse covariant Laplacian preserves the Lie algebra structure displayed in these equations that is in order for

$$\Big(\frac{1}{D^2}B\Big)^{ab}(x) = -f^{abc}\Big(\frac{1}{D^2}B\Big)^c(x)\,, \tag{4.63}$$

we require

$$f^{ecd}K^{ab}_{\ cd}(x,y) = f^{abc}K^{ce}(x,y)\,. \tag{4.64}$$

Note that in the above equation, $K^{ab}_{cd}(x,y)$ are the fields in the tensor product of the adjoint representation

$$K^{ab}_{cd}(x,y) = \delta^a_c\delta^b_d K_0(x,y) + gK_1{}^{ab}_{cd}(x,y) + \cdots + g^n K_n{}^{ab}_{cd}(x,y)\,, \tag{4.65}$$

where to the lowest order,

$$\begin{aligned} K_1{}^{ab}_{cd}(x,y) = \int \mathrm{d}^4 z\, \Big\{ & \partial^\rho_z K_0(x,z)(A_\rho)^{ab}_{cd}(z)K_0(z,y) \\ & - K_0(x,z)(A_\rho)^{ab}_{cd}(z)\partial^\rho_z K_0(z,y)\Big\}\,, \end{aligned} \tag{4.66}$$

$$\begin{aligned} K_2{}^{ab}_{cd}(x,y) = \int \mathrm{d}^4 z\, \Big\{ & \partial^\rho_z K_0(x,z)(A_\rho)^{ab}_{c'd'}(z)K_1{}^{c'd'}_{cd}(z,y) \\ & - K_0(x,z)(A_\rho)^{ab}_{c'd'}(z)\ \partial^\rho_z K_1{}^{c'd'}_{cd}(z,y) \\ & - K_0(x,z)\ (A_\rho)^{ab}_{c'd'}(z)(A_\rho)^{c'd'}_{cd}(z)K_0(z,y)\Big\}\,, \end{aligned} \tag{4.67}$$

and in general for $n \geq 2$

$$\begin{aligned} K_n{}^{ab}_{cd}(x,y) = \int \mathrm{d}^4 z\, \Big\{ & \partial^\rho_z K_0(x,z)(A_\rho)^{ab}_{c'd'}(z)K_{n-1}{}^{c'd'}_{cd}(z,y) \\ & - K_0(x,z)(A_\rho)^{ab}_{c'd'}(z)\partial^\rho_z K_{n-1}{}^{c'd'}_{cd}(z,y) \\ & - K_0(x,z)(A_\rho)^{ab}_{c'd'}(z)(A_\rho)^{c'd'}_{e'f'}(z)K_{n-2}{}^{e'f'}_{cd}(z,y)\Big\}\,. \end{aligned} \tag{4.68}$$

Note that the potential in this representation is $(A_\rho)^{ab}_{cd} = (A_\rho)^e(T^e)^{ab}_{cd}$ with the representation matrix, $(T^e)^{ab}_{cd} = (T^e)^{ac}\delta^b{}_d + \delta^a{}_c(T^e)^{bd} = f^{aec}\delta^{bd} + f^{bed}\delta^{ac}$ defined in the previous chapter (3.58).

The fields $K^{ce}(x,y)$ in (4.64) are in the adjoint representation which have an expansion as (4.37) such that to the lowest order,

$$\begin{aligned} K_1^{ce}(x,y) = \int \mathrm{d}^4 z\, \Big\{ & \partial^\rho_z K_0(x,z)(A_\rho)^{ce}(z)K_0(z,y) \\ & - K_0(x,z)(A_\rho)^{ce}(z)\partial^\rho_z K_0(z,y)\Big\}\,, \end{aligned} \tag{4.69}$$

$$K_{_2}^{ce}(x,y) = \int \mathrm{d}^4 z\, \Big\{ \partial_z^\rho K_{_0}(x,z) A_\rho^{cd}(z) K_{_1}^{de}(z,y) - K_{_0}(x,z) A_\rho^{cd}(z) \partial_z^\rho K_{_1}^{de}(z,y) - K_{_0}(x,z) A_\rho^{cd}(z) A_\rho^{de}(z) K_{_0}(z,y) \Big\}\,, \tag{4.70}$$

and in general for $n \geq 2$

$$K_n^{ce}(x,y) = \int \mathrm{d}^4 z\, \Big\{ \partial_z^\rho K_{_0}(x,z) A_\rho^{cd}(z) K_{n-1}^{de}(z,y) - K_{_0}(x,z) A_\rho^{cd}(z) \partial_z^\rho K_{n-1}^{de}(z,y) - K_{_0}(x,z) A_\rho^{cd}(z) A_\rho^{df}(z) K_{n-2}^{fe}(z,y) \Big\}\,. \tag{4.71}$$

However from (4.64) if the fields are described in terms of $K_n{}^{ab}_{cd}(x,y)$ and $K_n^{ce}(x,y)$, it is to argue by induction that generally,

$$f^{ecd} K_n{}^{ab}_{cd}(x,y) = f^{abc} K_n^{ce}(x,y)\,, \tag{4.72}$$

or equivalently,

$$f^{ecd} (A_\rho)^{ab}_{\ \ cd} = f^{abc} (A_\rho)^{ce}\,. \tag{4.73}$$

Let us now show this by induction. We start with $n = 0$ such that left hand side of (4.72) is

$$f^{ecd} K_{_0}{}^{ab}_{cd}(x,y) = f^{ecd} \delta_c^a \delta_d^b K_{_0}(x,y) = f^{eab} K_{_0}(x,y)\,, \tag{4.74}$$

and the right hand side is

$$f^{abc} K_{_0}^{ce}(x,y) = f^{abc} \delta_e^c K_{_0}(x,y) = f^{eab} K_{_0}(x,y)\,. \tag{4.75}$$

For $n = 1$ we need to verify that $f^{ecd} K_{_1}{}^{ab}_{cd}(x,y) = f^{abc} K_{_1}^{ce}(x,y)$. Note that, for the proofs we will use the condensed notation and focus on the colour content whereby we ignore the irrelevant nested integral. For example, $K_{_1}{}^{ab}_{cd}(x,y)$ (4.66) has four indices which enter the right hand side as $(A_\rho)^{ab}_{\ \ cd}$. Suppressing the coordinates (x,y) we now start with the left hand side to find that the colour content of $f^{ecd}K_{1\ cd}^{\ ab}$ is given by

$$\begin{aligned} f^{ecd}(A_\rho)^{ab}_{\ \ cd} &= f^{ecd}A^g_\rho(f^{agc}\delta^{bd} + f^{bgd}\delta^{ac}) \\ &= (f^{ecb}f^{agc} + f^{ead}f^{bgd})A^g_\rho\,, \end{aligned} \tag{4.76}$$

where we have used (3.58) in the first line and only written the colour indices structures. The index d in (4.76) is a dummy variable so denoting it by a new variable c and using the Jacobi identity we obtain

$$f^{ecd}(A_\rho)^{ab}_{\ \ cd} = -f^{cab}f^{gce}A^g_\rho = f^{abc}A^{ce}_\rho\,. \tag{4.77}$$

For the right hand side we find form (4.69) that $K_1^{ce}(x,y)$ has two indices which enter as A^{ce}_ρ, that is, the colour content of $f^{abc}K_1^{ce}$ is given by

$$f^{abc}A^{ce}_\rho\,, \tag{4.78}$$

which is similar to (4.77).

Hence,

$$f^{ecd}K_{1\ cd}^{\ ab}(x,y) = f^{abc}K_1^{ce}(x,y)\,. \tag{4.79}$$

Now let us show this also for $K_{2\ cd}^{\ ab}(x,y)$. Note that $K_{2\ cd}^{\ ab}(x,y)$ (4.67) has four indices which enter the right hand side in different ways. The first two terms on the right side contain $(A_\rho)^{ab}_{\ \ c'd'}K_{1\ cd}^{\ c'd'}$ up to a derivative on the second term. However, the third term involves the colour structure $(A_\rho)^{ab}_{\ \ c'd'}(A_\rho)^{c'd'}_{\ \ cd}$. To show (4.72) holds for $n=2$ we take into account both the combinations and show that they agree. Suppressing the coordinates we start with the left hand side of (4.72) for $n=2$ given by (4.67) and take the first combination for $f^{ecd}K_{2\,cd}^{\ ab}$ into account to give

$$f^{ecd}(A_\rho)^{ab}_{\ c'd'}K_{1\ cd}^{\ c'd'} = (A_\rho)^{ab}_{\ c'd'}f^{c'd'c}K_1^{ce} \,. \tag{4.80}$$

Note that we have only written the colour indices structures and used the result (4.79) for the last term. Now using (4.73) we find the colour content of $f^{ecd}K_{2\,cd}^{\ ab}$ is given by

$$f^{abc'}(A_\rho)^{c'c}K_1^{ce} = f^{c'ab}K_2^{c'e} \,, \tag{4.81}$$

where in the last step we have identified $(A_\rho)^{c'c}K_1^{ce}$ with the first combination in (4.70). Because c' is a dummy variable, we have

$$f^{ecd}K_{2\,cd}^{\ ab}(x,y) = f^{cab}K_2^{\ ce}(x,y) \,. \tag{4.82}$$

Now using the second combination arising from the third term in (4.70) and writing the relevant colour content for $f^{ecd}K_{2\,cd}^{\ ab}$ we obtain

$$f^{ecd}(A_\rho)^{ab}_{\ c'd'}(A_\rho)^{c'd'}_{\ cd} \,. \tag{4.83}$$

In the above equation the terms f^{ecd} and $(A_\rho)^{c'd'}_{\ cd}$ can be contracted using (4.73) as

$$f^{ecd}K_{2\,cd}^{\ ab} = (A_\rho)^{ab}_{\ c'd'}f^{c'd'c}(A_\rho)^{ce} = (A_\rho)^{c'c}f^{c'ab}(A_\rho)^{ce} \,. \tag{4.84}$$

In the last step we can identify the colour content of the terms $(A_\rho)^{c'c}(A_\rho)^{ce}$ with the combination from the third term in (4.70) $f^{c'ab}K_2^{c'e}$. Because c' is a dummy variable we obtain

$$f^{ecd}K_{2\,cd}^{\ ab} = f^{abc}K_2^{ce} \,. \tag{4.85}$$

Now that we have verified that the result holds for $n = 0, 1$ and 2, so in the inductive step we assume it is true for the $(n-1)$-th term, that is,

$$f^{ecd} K_{n-1}{}^{ab}_{cd}(x,y) = f^{abc} K^{ce}_{n-1}(x,y)\,, \tag{4.86}$$

and using this we can verify (4.72). As above $K_n{}^{ab}_{cd}(x,y)$ (4.68) has four indices which enter the right hand side in different ways. The first two terms on the right side contain $(A_\rho)^{ab}_{c'd'} K_{n-1}{}^{c'd'}_{cd}$ up to a derivative on the second term and the third term involves the colour structure $(A_\rho)^{ab}_{c'd'}(A_\rho)^{c'd'}_{e'f'} K_{n-2}{}^{e'f'}_{cd}$. We start with the left hand side of (4.72), suppress the coordinates (x,y) and use the first combination in $K_n{}^{ab}_{cd}(x,y)$ to write the colour content of $f^{ecd} K_n{}^{ab}_{cd}$ as

$$f^{ecd}(A_\rho)^{ab}_{c'd'} K_{n-1}{}^{c'd'}_{cd} = (A_\rho)^{ab}_{c'd'} f^{c'd'c} K^{ce}_{n-1}\,, \tag{4.87}$$

where we have used (4.86). The terms $(A_\rho)^{ab}_{c'd'} f^{c'd'c}$ in the above equation can be replaced using (4.73) as

$$f^{ecd} K_n{}^{ab}_{cd} = (A_\rho)^{c'c} f^{abc'} K^{ce}_{n-1} = f^{abc'} K^{c'e}_{n} = f^{abc} K^{ce}_{n}\,, \tag{4.88}$$

where we have replaced c' by c as c' is a dummy variable.

Next we shall verify (4.72) using the other combination. As above we now use the second combination in $K_n{}^{ab}_{cd}(x,y)$ (4.68) where the terms enter as a product $(A_\rho)^{ab}_{c'd'}(A_\rho)^{c'd'}_{e'f'} K_{n-2}{}^{e'f'}_{cd}$. We start with the left hand side of (4.72) and write the relevant colour content of $f^{ecd} K_n{}^{ab}_{cd}$ as

$$f^{ecd}(A_\rho)^{ab}_{c'd'}(A_\rho)^{c'd'}_{e'f'} K_{n-2}{}^{e'f'}_{cd} = (A_\rho)^{ab}_{c'd'}(A_\rho)^{c'd'}_{e'f'} f^{ce'f'} K^{ce}_{n-2}\,, \tag{4.89}$$

where again we have used the result (4.86) for $(n-2)$-th term. Using the property

(4.73) we find the above to be equal to

$$(A_\rho)^{ab}_{\ \ c'd'} K^{ce}_{n-2} (A_\rho)^{e'c} f^{c'd'e'} = f^{c'ab} (A_\rho)^{c'e'} (A_\rho)^{e'c} K^{c'e}_{n-2}\,, \tag{4.90}$$

where using (4.71) we identify the colour content of last term to be equivalent to $f^{c'ab} K^{ce}_n$. The result thus works to all orders by induction.

These properties allows us to use $K(x,y)$ as a dressing for fields defined at different points x and y. In particular if we consider the field strengths $F^a_{\mu\nu}$ at the two points, then working in the adjoint representation we will have the gauge invariant configuration:

$$\left\langle \underline{F}_{\mu\nu}(x), K(x,y)\underline{F}^{\mu\nu}(y)\right\rangle := F^a_{\mu\nu}(x) K^{ab}(x,y) F^{b\,\mu\nu}(y)\,. \tag{4.91}$$

Integrating this expression gives

$$-\frac{1}{2}\int \mathrm{d}^4x\; F^a_{\mu\nu}(x)\left(\frac{1}{D^2}F^{\mu\nu}\right)^{a}(x) := -\frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\; F^a_{\mu\nu}(x) K^{ab}(x,y) F^{b\,\mu\nu}(y)\,, \tag{4.92}$$

which is the gauge invariant generalisation of the abelian mass term (4.6) proposed in [98] by Zwanziger. Indeed the term on left hand side of (4.92) is the first term in the non-abelian expansion of the mass operator introduced by Zwanziger in equation (4.1). In the next sections we will make clear the relationship between this gauge invariant result (4.92) and the non-abelian mass term described in equation (4.7).

4.4 Role of Strings in Dressing

In this section we will briefly introduce strings which are often chosen to be the more useful candidates to construct gauge invariant composites of charged fields, for example the mesonic states. These constructs can be understood perturbatively by

Figure 4.1: Strings describing the positronium state (e^+e^-).

the path ordering. In general, at least in the perturbative regime, such dressings factorise into simple dressings for each individual fermion. This was suggested by Cahill and Stump [105,106] who stated that the dressing of a quark-antiquark system that corresponds to an ultra-heavy meson can be done through a cloud of glue such that the system

$$\bar{\psi}(x)\mathcal{K}(x,y)\psi(y)\,, \tag{4.93}$$

is gauge invariant. We now construct a positronium like state by attaching a string between the two fermions at points x and y as shown in Fig. 4.1.

$$\bar{\psi}(x)\exp\big(-ie\int_y^x A_\mu(z)dz^\mu\big)\psi(y)\,. \tag{4.94}$$

In fact, this state is clearly gauge invariant for an arbitrary contour taken from x to y. However, the contour dependence is difficult to interpret physically. We now make the usual decomposition of the vector potential into transverse and longitudinal components

$$A_\mu = A_\mu^T + \partial_\mu\Big(\frac{\partial\cdot A}{\Box}\Big)\,. \tag{4.95}$$

This follows from the fact that the dressed potential can be written in a gauge invariant form. Substituting the decomposition (4.95) into (4.94) we find that (4.94) factorises to give

$$\bar{\psi}(x)\exp\big(-iev_{\scriptscriptstyle 1}(x)\big)\exp\Big(-ie\int_y^x A^{\scriptscriptstyle T}_\mu(z)\;dz^\mu\Big)\exp\big(iev_{\scriptscriptstyle 1}(y)\big)\psi(y)\,, \tag{4.96}$$

which can be written in the form

$$\bar{\psi}(x)h(x)M(x,y)h^{-1}(y)\psi(y)\,, \tag{4.97}$$

where h^{-1} is the abelian dressing constructed out of the longitudinal components of the potential. In (4.97) $M(x,y)$ is a separately gauge invariant and contour dependent contribution to this dressing constructed from the transverse potential. We thus see that string type dressings can be factorised into the product of two separately gauge invariant states one of which is highly excited as the fields are localised along the string and this contains all of the contour dependence.

4.4.1 Factorising the Gauge Covariant Dressing

What we want to show now is that the dressing $K^{ab}(x,y)$ in (4.92) has a similar factorisation into the adjoint dressing needed to compensate for the field strength gauge transformations and a gauge invariant core. Before showing this we need to look in more detail at the dressing (3.67) used to make the gauge invariant field strength $F^h_{\mu\nu}$.

We have seen from equation (3.67) how the dressed field strength $F^h_{\mu\nu}$ is defined directly in the fundamental representation in terms of the fundamental dressing h. It is useful to see how this is defined in the adjoint representation. To do this we need to look at $F^h_{\mu\nu}$ in terms of its components, that is,

$$(F^h_{\mu\nu})^a := -2\,\mathrm{tr}\,(F^h_{\mu\nu}\tau^a) = (h^{-1})^{ab}F^b_{\mu\nu}\,, \tag{4.98}$$

where $(h^{-1})^{ab}$ is the dressing in the adjoint representation and can be written in terms of the fundamental dressing as,

$$(h^{-1})^{ab} = -2\,\mathrm{tr}\,(\tau^a h^{-1} \tau^b h)\,. \tag{4.99}$$

Equivalently, in the adjoint representation we can directly define

$$(h^{-1})^{ab} = (e^v)^{ab}\,, \tag{4.100}$$

where now $v = v^c T^c$ and $(T^a)_{bc} = -f_{abc}$ are the adjoint representation matrices of $SU(\mathrm{N})$. The adjoint dressing $(h^{-1})^{ab}$ in (4.99) now transforms under a gauge transformation as

$$\begin{aligned}(h^{-1})^{ab} \to -2\,\mathrm{tr}\,(\tau^a h^{-1} U \tau^b U^{-1} h) &= -2(h\,\tau^a h^{-1})_{ij}(U\tau^b U^{-1})_{ji} \\ &= -2(h\,\tau^a h^{-1})_{ij}(U\tau^b U^{-1})_{kl}\delta_{il}\delta_{jk} \\ &= 4(h\,\tau^a h^{-1}\tau^c)_{ii}(U\tau^b U^{-1}\tau^c)_{jj} \\ &= \{-2\,\mathrm{tr}\,(\tau^a h^{-1}\tau^c h)\}\{-2\,\mathrm{tr}\,(\tau^c U\tau^b U^{-1})\}\,.\end{aligned} \tag{4.101}$$

Note that in going from the second to the third line we have made use of (3.16). It is easy to check that this becomes the gauge transformation

$$(h^{-1})^{ab} \to (h^{-1})^{ac} U^{cb}\,, \tag{4.102}$$

where $U^{cb} = -2\,\mathrm{tr}\,(\tau^c U \tau^b U^{-1})$ is the adjoint representation of the transformation. Clearly we can mimic the gauge transformation property (4.56) of the gauge covariant Laplacian by the factorised dressing $h^{ac}(x)K_0(x,y)(h^{-1})^{cb}(y)$. What we now want to understand is how this factorisation emerges from the full dressing $K^{ab}(x,y)$.

That is, in analogy with the factorisation of the mesonic dressing (4.97), how is the reduction

$$-\frac{1}{2}\int \mathrm{d}^4x\; F^a_{\mu\nu}(x)\Big(\frac{1}{D^2}F^{\mu\nu}\Big)^a(x) \to -\frac{1}{2}\int \mathrm{d}^4x\; F^{h\,a}_{\mu\nu}(x)\Big(\frac{1}{\Box}F^{h\,\mu\nu}\Big)^a(x)\,, \tag{4.103}$$

achieved in terms of the transverse/longitudinal decomposition of the component fields.

To this end we identify

$$-\frac{1}{2}\int \mathrm{d}^4x\; F^a_{\mu\nu}(x)\Big(\frac{1}{D^2}F^{\mu\nu}\Big)^a(x) = -\frac{1}{2}\int \mathrm{d}^4x\; F^{h\,a}_{\mu\nu}(x)\Big(\frac{1}{\Box}F^{h\,\mu\nu}\Big)^a(x) + \mathcal{Q}\,, \tag{4.104}$$

where

$$\mathcal{Q} = -\tfrac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\; F^a_{\mu\nu}(x)\mathcal{Q}^{ab}(x,y)F^{\mu\nu\,b}(y)\,, \tag{4.105}$$

and

$$\mathcal{Q}^{ab}(x,y) = K^{ab}(x,y) - h^{ac}(x)K_0(x,y)(h^{-1})^{cb}(y)\,. \tag{4.106}$$

By construction the operator $\mathcal{Q}$ is gauge invariant as it is the difference of two gauge invariant terms. This means that we must have

$$\mathcal{Q}^{ab}(x,y) \to (U^{-1})^{ac}(x)\mathcal{Q}^{cd}(x,y)(U)^{db}(y)\,, \tag{4.107}$$

under a gauge transformation. Using the perturbative expansions for the various dressings we can write

$$\mathcal{Q}^{ab}(x,y) = g\mathcal{Q}_1^{ab}(x,y) + g^2\mathcal{Q}_2^{ab}(x,y) + g^3\mathcal{Q}_3^{ab}(x,y) + \cdots\,, \tag{4.108}$$

which induces an expansion in the operator $\mathcal{Q}$

$$\mathcal{Q} = g\mathcal{Q}_{1} + g^{2}\mathcal{Q}_{2} + g^{3}\mathcal{Q}_{3} + \cdots . \tag{4.109}$$

Note that this is not strictly an expansion in the coupling since the field strengths in the definition (4.105) will also induce powers of the coupling in the $\mathcal{Q}_i$ terms. From this it is important to note that it is the sum of all the terms in (4.109) that is gauge invariant, individual terms are not. To make this point clear let us expand both the terms of (4.106) perturbatively to order g^2 where we make use of (3.69) and (3.70) to yield

$$\begin{aligned}\mathcal{Q}^{ab}(x,y) &= \delta^{ab}K_{0}(x,y) + gK_{1}^{ab}(x,y) + g^{2}K_{2}^{ab}(x,y)\\ &\quad - \Big\{\delta^{ab}K_{0}(x,y) + g\Big(K_{0}(x,y)\big(v_{1}^{ab}(y) - v_{1}^{ab}(x)\big)\Big)\\ &\quad + g^{2}\Big(K_{0}(x,y)\big(\frac{1}{2}v_{1}^{ac}(x)v_{1}^{cb}(x) + \frac{1}{2}v_{1}^{ac}(y)v_{1}^{cb}(y)\big)\\ &\quad - v_{1}^{ac}(x)K_{0}(x,y)v_{1}^{cb}(y) + K_{0}(x,y)\big(v_{2}^{ab}(y) - v_{2}^{ab}(x)\big)\Big)\Big\} + \cdots .\end{aligned} \tag{4.110}$$

Note that in the above equation $K_{1}^{ab}(x,y)$ and $K_{2}^{ab}(x,y)$ are respectively linear and quadratic in the potential and contain both transverse and longitudinal components of the potential.

4.4.2 Decomposition of the Green's function $K(x,y)$

In this section we will find the decomposition of $K(x,y)$ perturbatively. We will first evaluate the transverse and longitudinal components to order g for $K_{1}(x,y)$ and later extend it to order g^2 for $K_{2}(x,y)$. The calculation of $K_{1}(x,y)$ is straightforward but calculating $K_{2}(x,y)$ is not trivial so we will present the details. Note however that we will initially drop the colour indices for simplicity but restore them in the later part of our calculations.

4.4.2.1 Decomposition of $K_1(x,y)$

In order to decompose $K_1(x,y)$ we substitute, $A_\rho(z) = A^T_\rho(z) + \partial_\rho v_1(z)$ into (4.16) to obtain:

$$K_1(x,y) = \int \mathrm{d}^4 z \, \Big\{ \partial^\rho_z K_0(x,z) \big(A^T_\rho(z) + \partial_\rho v_1(z) \big) K_0(z,y) \\ - K_0(x,z) \big(A^T_\rho(z) + \partial_\rho v_1(z) \big) \partial^\rho_z K_0(z,y) \Big\} . \tag{4.111}$$

After integrating by parts over z we obtain the desired decomposition

$$K_1(x,y) := K^T_1(x,y) + K^L_1(x,y) , \tag{4.112}$$

where

$$K^T_1(x,y) = \int \mathrm{d}^4 z \, \Big\{ \partial^z_\rho K_0(x,z) A^T_\rho(z) K_0(z,y) - K_0(x,z) A^T_\rho(z) \partial^\rho_z K_0(z,y) \Big\} , \tag{4.113}$$

is the transverse component and

$$K^L_1(x,y) = K_0(x,y) \big(v_1(y) - v_1(x) \big) , \tag{4.114}$$

the longitudinal component to order g respectively.

4.4.2.2 Decomposition of $K_2(x,y)$

In a similar way we now decompose $K_2(x,y)$ into transverse-transverse (TT), transverse-longitudinal (TL) and longitudinal-longitudinal (LL) components. To see how this works, we start with the definition of $K_2(x,y)$ (4.17) written in the form:

$$K_2(x,y) = \int d^4z \left\{\partial^z_\rho K_0(x,z)A_\rho(z)K_1(z,y) - K_0(x,z)A_\rho(z)\ \partial^z_\rho K_1(z,y) \right. \\ \left. - K_0(x,z)\ A_\rho(z)A_\rho(z)\ K_0(z,y)\right\}. \tag{4.115}$$

Making the substitutions $A_\rho(z) = A^T_\rho(z) + \partial_\rho v_1(z)$ and $K_1(z,y) = K^T_1(z,y) + K^L_1(z,y)$ with $K^L_1(z,y) = K_0(z,y)(v_1(y) - v_1(z))$ in (4.115) we obtain

$$K_2(x,y) = K^{TT}_2(x,y) + K^{TL}_2(x,y) + K^{LL}_2(x,y)\,. \tag{4.116}$$

In the above equation the TT components are given by

$$\begin{aligned} K^{TT}_2(x,y) &= \int d^4z \left\{\partial^z_\rho K_0(x,z)A^T_\rho(z)K^T_1(z,y) - K_0(x,z)A^T_\rho(z)\ \partial^z_\rho K^T_1(z,y)\right. \\ &\qquad \left. - K_0(x,z)\ A^T_\rho(z)A^T_\rho(z)\ K_0(z,y)\right\} \\ &= -\int d^4z \left\{2K_0(x,z)A^T_\rho(z)\ \partial^z_\rho K^T_1(z,y)\right. \\ &\qquad \left. + K_0(x,z)A^T_\rho(z)A^T_\rho(z)K_0(z,y)\right\}, \end{aligned} \tag{4.117}$$

where for the first line of (4.117) we have integrated by parts w.r.t. z and have used that $\partial^\rho_z A^T_\rho(z) = 0$. In a similar way the contribution from the TL and LL terms is:

$$\begin{aligned} K^{TL}_2(x,y) &= K^T_1(x,y)v_1(y) - v_1(x)K^T_1(x,y) \\ &\quad + \int d^4z \left\{\partial^\rho_z K_0(x,z)[v_1(z), A^T_\rho(z)]K_0(z,y)\right. \\ &\qquad \left. - K_0(x,z)[v_1(z), A^T_\rho(z)]\partial^\rho_z K_0(z,y)\right\}, \end{aligned} \tag{4.118}$$

and

$$K_{2}^{LL}(x,y) = v_{1}(x)v_{1}(x)K_{0}(x,y) - v_{1}(x)K_{0}(x,y)v_{1}(y) + \int \mathrm{d}^4 z \Big\{ (\partial^z_\rho K_{0}(x,z))K_{0}(z,y)v_{1}(z)(\partial^z_\rho v_{1}(z)) - K_{0}(x,z)\; (\partial^z_\rho K_{0}(z,y))v_{1}(z)(\partial^z_\rho v_{1}(z)) \Big\}\,. \tag{4.119}$$

We notice that the integrand in equation (4.119) is of the form $(\partial AC - A\partial C)B\partial B$ where $A = K_{0}(x,z)$, $B = v_{1}(z)$ and $C = K_{0}(z,y)$ and can thus be evaluated using the following identity:

$$\big\{(\partial A)C - A(\partial C)\big\}(B\partial B) = -\frac{1}{2}\big\{(\Box A)C - A(\Box C)\big\}B^2 - \frac{1}{2}\big\{(\partial A)C - A(\partial C)\big\}[\partial B, B]\,, \tag{4.120}$$

where $B(\partial B) = \frac{1}{2}\partial(B^2) + \frac{1}{2}[B, \partial B]$. Substituting (4.120) into (4.119) we finally write the LL components as

$$K_{2}^{LL}(x,y) = \frac{1}{2}v_{1}(x)v_{1}(x)K_{0}(x,y) + \frac{1}{2}K_{0}(x,y)v_{1}(y)v_{1}(y) - v_{1}(x)K_{0}(x,y)v_{1}(y) + \frac{1}{2}\int \mathrm{d}^4 z \Big\{ \partial^\rho_z K_{0}(x,z)K_{0}(z,y) - K_{0}(x,z)\partial^\rho_z K_{0}(z,y) \Big\}[v_{1}, \partial^\rho v_{1}](z)\,. \tag{4.121}$$

Above we have identified the mixed transverse/longitudinal contribution for the Green's function to various orders in the coupling. To order g in coupling from (4.113) and (4.114), this decomposition is straightforward such that the first term in the expansion (4.108) is given by

$$\mathcal{Q}_{1}^{ab}(x,y) = K_{1}^{ab}(x,y) - \big(v_{1}^{ab}(y) - v_{1}^{ab}(x)\big)K_{0}(x,y)\,, \tag{4.122}$$

which is simply the transverse part of $K_{1}^{ab}(x,y)$, see equation (4.16), as it should be to maintain gauge invariance at this order written in the form

$$\mathcal{Q}_{_1}^{ab}(x,y) = \int \mathrm{d}^4 z \left\{ \partial_z^\rho K_{_0}(x,z)(A_\rho^{^T})^{ab}(z) K_{_0}(z,y) - K_{_0}(x,z)(A_\rho^{^T})^{ab}(z)\partial_z^\rho K_{_0}(z,y) \right\}. \tag{4.123}$$

In contrast, the next term in the expansion (4.108) is not purely transverse and not immediately related to the next terms in the expansion of the Laplacian, that is,

$$\mathcal{Q}_{_2}^{ab}(x,y) \neq K_{_2}^{^{TT}ab}(x,y). \tag{4.124}$$

Indeed $\mathcal{Q}_{_2}^{ab}(x,y)$ will itself have a decomposition into transverse-transverse (TT), transverse-longitudinal (TL) and longitudinal-longitudinal (LL) components

$$\mathcal{Q}_{_2}^{ab}(x,y) = \mathcal{Q}_{_2}^{^{TT}ab}(x,y) + \mathcal{Q}_{_2}^{^{TL}ab}(x,y) + \mathcal{Q}_{_2}^{^{LL}ab}(x,y), \tag{4.125}$$

which reflects the fact that it is the sum of $g\mathcal{Q}_{_1}^{ab}(x,y) + g^2\mathcal{Q}_{_2}^{ab}(x,y)$ which now behaves properly under the gauge transformations (4.107) to this order. Therefore to order g^2 we need to calculate

$$\begin{aligned}\mathcal{Q}_{_2}^{ab}(x,y) = K_{_2}^{ab}(x,y) - \Big\{ & K_{_0}(x,y)\big(v_{_2}(y) - v_{_2}(x)\big) - v_{_1}(x) K_{_0}(x,y) v_{_1}(y) \\ & + K_{_0}(x,y)\Big(\frac{1}{2}v_{_1}(x)v_{_1}(x) + \frac{1}{2}v_{_1}(y)v_{_1}(y)\Big)\Big\}^{ab},\end{aligned} \tag{4.126}$$

where in addition to $K_{_2}^{ab}(x,y)$ calculated in terms of TT, TL and LL components, from (4.117), (4.118) and (4.121) there is also the contribution of T/L terms arising from $v_{_2}$ present in the above equation. Showing this whole set of calculations in one go would not be an easy task so we shall break the calculations into various steps.

4.4.3 Calculation of TL and LL components

In order to calculate the TL and LL components in (4.126) we first need to find the decomposition of v_2 into T/L components. Details of this decomposition are included in Appendix C.1. We integrate (C.14) to obtain the TL components

$$v_2^T(x) = \int d^4z\, K_0(x,z)[\partial_\rho v_1, A_\rho^T](z)\,. \tag{4.127}$$

In a similar manner the contribution to the LL components from (C.15) leads, upon integration, to the following terms

$$v_2^L(x) = \frac{1}{2}\int d^4z\, K_0(x,z)\partial_\rho^z[v_1, \partial_\rho v_1](z)\,. \tag{4.128}$$

Note that in (4.127) and (4.128), the superscripts T and L on v_2 denote specifically the transverse-longitudinal TL and longitudinal-longitudinal LL contribution from the dressing. Hence to order g^2 the total contribution to TL component is obtained by substituting (4.118) and (4.127) into (4.126) to yield:

$$\begin{aligned}\mathcal{Q}_2^{TL\,ab}(x,y) = &\Big\{K_1^T(x,y)v_1(y) - v_1(x)K_1^T(x,y)\Big\}^{ab}\\ &+\int d^4z\,\Big\{\Big(K_0(x,z) - K_0(y,z)\Big)K_0(x,y)[\partial_\rho v_1, A_\rho^T](z)\\ &\quad + \Big(\partial_\rho^z K_0(x,z)K_0(z,y) - K_0(x,z)\partial_\rho^z K_0(z,y)\Big)\times[v_1, A_\rho^T](z)\Big\}^{ab},\end{aligned} \tag{4.129}$$

and similarly the total contribution to LL component is found by substituting (4.121) and (4.128) into (4.126) to yield

$$\begin{aligned}\mathcal{Q}_2^{LL\,ab}(x,y) = \frac{1}{2}\int d^4z\,&\Big\{\Big(\partial_z^\rho K_0(x,z)K_0(z,y) - K_0(x,z)\partial_z^\rho K_0(z,y)\Big)\\ &- K_0(x,y)\big(\partial_\rho^z K_0(x,z) - \partial_\rho^z K_0(y,z)\Big)\times[v_1, \partial_\rho v_1](z)\Big\}^{ab}.\end{aligned} \tag{4.130}$$

We now return to (4.123) where using (4.105) we find the first term in the expansion

$$\mathcal{Q}_{1} = -\frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\ F^a_{\mu\nu}(x)\mathcal{Q}^{ab}_{1}(x,y)F^{\mu\nu\,b}(y)\,. \qquad (4.131)$$

It is now clear that $\mathcal{Q}_{1}$ is gauge invariant to lowest order in the coupling (where $F_{\mu\nu}\to F_{\mu\nu}$ and $A^{T}_{\mu}\to A^{T}_{\mu}$) but at higher order it changes.

In order to make this term in the expansion fully gauge invariant, Zwanziger essentially proposed the replacement

$$A^{T}_{\rho}(z) = \frac{1}{\Box}\partial_\beta(\partial^\beta A^\rho - \partial^\rho A^\beta)(z) \to \frac{1}{D^2}\Big(D_\beta F^{\beta\rho}\Big)(z)\,, \qquad (4.132)$$

so that the first term in the expansion of the operator $\mathcal{Q}$ becomes gauge invariant. At lowest order in the coupling this does not change $\mathcal{Q}_{1}$ but clearly it contributes new terms at higher order. To understand how this works and hence how to extend Zwanziger's result, we first need to make precise how the non-abelian mass term (4.7) is related to Zwanziger's term (4.92). This will be the topic we turn to next. Then we will see how to implement Zwanziger's resummation of that result to yield a term by term gauge invariant expansion of the mass (4.6).

To order g^2 the TL and LL components in (4.129) and (4.130) must be killed by the TL and LL extension of $\mathcal{Q}^{ab}_{1}(x,y)$. We call this expansion the Zwanziger expansion where we want each term to be fully gauge invariant and written in terms of the field strength. We need to understand the Zwanziger expansion and at all stages check the gauge transformations.

4.5 The Role of the Dressed Field Strength

Having identified the role of the field strength $\mathcal{F}_{\mu\nu}$ (3.83) in defining a non-abelian mass term (4.7), and having defined the gauge covariant inverse Laplacian thus allowing for a precise definition of the Zwanziger term (4.92), we now connect these descriptions by identifying the common role played by the dressed field strength $F^h_{\mu\nu}$ (3.84).

Already we have seen in (4.104) that the factorisation of Zwanziger's gauge invariant expression (4.92) gives a term analogous to the mass term (4.7) but with the dressed field strength playing the role of the $\mathcal{F}_{\mu\nu}$ field strength. Now using the field strength factorisation equation (3.86) we shall find a similar decomposition to (4.104) in the non-abelian mass term (4.7).
Indeed we see that

$$-\frac{1}{2}\int \mathrm{d}^4x\ \mathcal{F}^a_{\mu\nu}(x)\Big(\frac{1}{\Box}\mathcal{F}^{\mu\nu}\Big)^a(x) = -\frac{1}{2}\int \mathrm{d}^4x\ F^{h\,a}_{\mu\nu}(x)\Big(\frac{1}{\Box}F^{h\,\mu\nu}\Big)^a(x) + \mathcal{P}\,, \quad (4.133)$$

where, using (3.86) the gauge invariant term $\mathcal{P}$ is given by

$$\begin{aligned}\mathcal{P} = \frac{g}{2}&\int \mathrm{d}^4x\,\mathrm{d}^4y\ ([A^h_\mu, A^h_\nu])^a(x)\,\delta^{ab}K_0(x,y)\,(F^h_{\mu\nu})^b(y)\\ &+\frac{g}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\ (F^h_{\mu\nu})^a(x)\,\delta^{ab}K_0(x,y)\,([A^h_\mu, A^h_\nu])^b(y)\\ &\quad-\frac{g^2}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\ ([A^h_\mu, A^h_\nu])^a(x)\,\delta^{ab}K_0(x,y)\,([A^h_\mu, A^h_\nu])^b(y)\,.\end{aligned} \quad (4.134)$$

As before we can introduce a perturbative expansion of this operator:

$$\mathcal{P} = g\mathcal{P}_1 + g^2\mathcal{P}_2 + g^3\mathcal{P}_3 + \cdots\,. \quad (4.135)$$

However, just as in the corresponding expansion (4.109) for $\mathcal{Q}$, it is useful not to strictly expand in the coupling but to allow in (4.135) for the field strength terms $F_{\mu\nu}$ to be kept together. This means, for example, that

$$\mathcal{P}_{_1} = \int \mathrm{d}^4x\, \mathrm{d}^4y\; F^a_{\mu\nu}(x)\, \delta^{ab} K_{_0}(x,y)\, [A^{^T}_\mu, A^{^T}_\nu]^b(y)\,. \tag{4.136}$$

We again stress that although $\mathcal{P}$ is gauge invariant, individual terms like $\mathcal{P}_{_1}$, $\mathcal{P}_{_2}$ are not. Equations (4.104) and (4.133) allow us to finally clarify the relation between the mass term (4.7) and Zwanziger's expression (4.92). Eliminating the common factor constructed out of the dressed field strength F^h we see that the mass term (4.7) can alternatively be written as

$$\mathcal{M}^2 = -\frac{1}{2} \int \mathrm{d}^4x\; F^a_{\mu\nu}(x) \Big(\frac{1}{D^2} F_{\mu\nu}\Big)^a(x) + \mathcal{P} - \mathcal{Q}\,, \tag{4.137}$$

where the operators $\mathcal{Q}$ and $\mathcal{P}$ are defined to all orders in perturbation theory by (4.105) and (4.134).

This succinct formula does not though fully describe Zwanziger's proposed expansion of the mass term as the individual terms in the operators $\mathcal{Q}$ and $\mathcal{P}$ are not gauge invariant. We shall see in the next chapter how to resum these expressions so as to maintain gauge invariance of each term in a new expansion of the operators $\mathcal{Q}$ and $\mathcal{P}$. This will allow us to generate Zwanziger's expression (4.1) and then go beyond it.

Chapter 5

Zwanziger's Expansion

5.1 Motivation

Having derived (4.137) we shall now see how to resum the expressions (4.109) and (4.135) so as to maintain gauge invariance for each term in the new expansions for the operators $\mathcal{Q}$ and $\mathcal{P}$. This will enable us to derive Zwanziger's expansion first to low orders and then for the next to next to leading order term. In Chapter 4 we successfully recovered the first term in Zwanziger's expression (4.1) which is quadratic in field strengths and represented by the first term of (4.137). In this chapter our starting point will be to recover the next term, *i.e.* that which is cubic in the field strengths. Later we will extend to higher orders to obtain the quartic terms in the field strength. We will note ambiguities in this construction.

5.2 Recovering Zwanziger's Expansion

We want to extend from (4.109) and (4.135) to the resummed expansions

$$\mathcal{Q} = g\mathcal{Y}_1 + g^2\mathcal{Y}_2 + g^3\mathcal{Y}_3 + \cdots , \tag{5.1}$$

and

$$\mathcal{P} = g\mathcal{Z}_1 + g^2\mathcal{Z}_2 + g^3\mathcal{Z}_3 + \cdots \,, \tag{5.2}$$

where each term in these expansions will be separately gauge invariant by construction. Note that in these expressions we are not formally expanding in the coupling but, as we will see, each term is more properly characterised by the power of the field strengths used to construct them with $\mathcal{Y}_1$ and $\mathcal{Z}_1$ both cubic in the field strengths.

5.2.1 $\mathcal{Q}$ to Order F^3

In order to recover Zwanziger's expression for the term $\mathcal{Y}_1$ we first need to rewrite (4.123) by introducing appropriate colour indices as

$$\begin{aligned}\mathcal{Q}_1^{ab}(x,y) = -\int \mathrm{d}^4 z\, \Big\{ &\partial_x^\rho \delta^{ac} K_0(x,z)(A_\rho^T)^{cd}(z)\delta^{db}K_0(z,y) \\ &- \delta^{ac}K_0(x,z)(A_\rho^T)^{cd}(z)\partial_y^\rho\delta^{db}K_0(z,y)\Big\}\,.\end{aligned} \tag{5.3}$$

Then, to impose gauge invariance, we need to make the replacements (4.132) and

$$\begin{aligned}&\partial_x^\rho\delta^{ac}K_0(x,z) \to \big(D_x^\rho K(x,z)\big)^{ac}, \quad \delta^{db}K_0(z,y)\to K^{db}(z,y) \\ &\delta^{ac}K_0(x,z)\to K^{ac}(x,z), \quad \partial_y^\rho\delta^{db}K_0(z,y)\to\big(D_y^\rho K(z,y)\big)^{db}\,.\end{aligned} \tag{5.4}$$

Applying these to (5.3) we find

$$\begin{aligned}\mathcal{Y}_1^{ab}(x,y) = -\int \mathrm{d}^4 z\, \Big\{ &\big(D_x^\rho K(x,z)\big)^{ac}\Big(\frac{1}{D^2}D_\beta F^{\beta\rho}\Big)^{cd}(z)K^{db}(z,y) \\ &- K^{ac}(x,z)\Big(\frac{1}{D^2}D_\beta F^{\beta\rho}\Big)^{cd}(z)\big(D_\rho^y K(z,y)\big)^{db}\Big\}\,.\end{aligned} \tag{5.5}$$

Using (4.131) the above equation becomes

$$\mathcal{Y}_{1} = \frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4z\,\Big\{F^a_{\mu\nu}(x)\big(D^\rho_x K(x,z)\big)^{ac}\Big(\frac{1}{D^2}D_\beta F^{\beta\rho}\Big)^{cd}(z)K^{db}(z,y)F^{\mu\nu b}(y) \\ - F^a_{\mu\nu}(x)K^{ac}(x,z)\Big(\frac{1}{D^2}D_\beta F^{\beta\rho}\Big)^{cd}(z)\big(D^y_\rho K(z,y)\big)^{db}F^{\mu\nu b}(y)\Big\}\,. \tag{5.6}$$

Using the properties of the inverse Laplacian given by (4.39) and integrating (5.6) with respect to both x and y we end up with

$$\mathcal{Y}_{1} = -\int \mathrm{d}^4z\,\frac{1}{D^2}\Big(D_\rho F^{\mu\nu}\Big)^c(z)\frac{1}{D^2}\Big(D_\beta F^{\beta\rho}\Big)^{cd}(z)\Big(\frac{1}{D^2}F^{\mu\nu}\Big)^d(z)\,. \tag{5.7}$$

Following (D.1) this can be written as

$$\mathcal{Y}_{1} = \int \mathrm{d}^4z\,\Big(\frac{1}{D^2}F^{\mu\nu}\Big)^d(z)\Big[\frac{1}{D^2}\big(D_\beta F^{\beta\rho}\big),\frac{1}{D^2}\big(D_\rho F^{\mu\nu}\big)\Big]^d(z)\,, \tag{5.8}$$

which is fully gauge invariant and agrees with the corresponding term in Zwanziger's expansion (4.1). Before proceeding further it is important to show that $\mathcal{Y}^{ab}_{1}(x,y)$ (5.5) has the following gauge transformation property to all orders,

$$\mathcal{Y}^{ab}_{1}(x,y) \to (U^{-1})^{ac}(x)\mathcal{Y}^{cd}_{1}(x,y)U^{db}(y)\,. \tag{5.9}$$

To show this we consider the first integrand in (5.5)

$$\mathcal{Y}^{ab}_{1}(x,y) = \Big(D^\rho_x K(x,z)\Big)^{ac}\Big(\frac{1}{D^2}D^\beta F_{\beta\rho}(z)\Big)^{cd}K^{db}(z,y)\,, \tag{5.10}$$

and apply the transformations (4.57) and (4.60) to yield

$$\mathcal{Y}^{ab}_{1}(x,y) \to \Big\{(U^{-1})^{ac'}(x)(D^\rho_x K(x,z))^{c'e'}(U)^{e'c}(z)\Big\}\Big\{(U^{-1})^{cd}_{mn}(z) \\ \times\Big(\frac{1}{D^2}D^\beta F_{\beta\rho}(z)\Big)^{mn}\Big\}\Big\{(U^{-1})^{df'}(z)K^{f'd'}(z,y)(U)^{d'b}(y)\Big\}\,. \tag{5.11}$$

In the above equation the term in the second braces can be written using (3.45) as $(U^{-1})^{cd}_{mn} = (U^{-1})^{cm}(U^{-1})^{dn}$. Since $(U)^{e'c}(U^{-1})^{cm} = \delta^{e'm}$ and $(U)^{nd}(U^{-1})^{df'} = \delta^{nf'}$, we find

$$\begin{aligned}
\mathcal{Y}_{1}^{ab}(x,y) &\to (U^{-1})^{ac'}(x)\Big(D_x^{\rho}K(x,z)\Big)^{c'e'}\delta^{e'm}\Big(\frac{1}{D^2}D^{\beta}F_{\beta\rho}(z)\Big)^{mn}\delta^{nf'}K^{f'd'}(z,y)(U)^{d'b}(y)\\
&= (U^{-1})^{ac'}(x)\Big(D_x^{\rho}K(x,z)\Big)^{c'e'}\Big(\frac{1}{D^2}D^{\beta}F_{\beta\rho}(z)\Big)^{e'f'}K^{f'd'}(z,y)(U)^{d'b}(y)\\
&= (U^{-1})^{ac'}(x)\mathcal{Y}_{1}^{c'd'}(x,y)(U)^{d'b}(y)\,,
\end{aligned} \tag{5.12}$$

which has the same transformation as (5.9). In the same way the second integrand in (5.5) transforms as (5.9).

5.2.2 $\mathcal{P}$ to Order F^3

Now that we have found the final expression for $\mathcal{Y}_1$ we return to (4.136) and follow the same route to find $\mathcal{Z}_1$. In (4.136) inserting colour indices in an appropriate way, we have

$$\mathcal{P}_1 = \int \mathrm{d}^4x\,\mathrm{d}^4y\; F^a_{\mu\nu}(x)\,\delta^{ab}K_0(x,y)\,[A^T_{\mu}, A^T_{\nu}]^b(y)\,, \tag{5.13}$$

where in addition to (4.132) we make the replacement $\delta^{ab}K_0(x,y) \to K^{ab}(x,y)$ to obtain

$$\mathcal{Z}_1 = \int \mathrm{d}^4x\,\mathrm{d}^4y\; F^a_{\mu\nu}(x)K^{ab}(x,y)\Big[\Big(\frac{1}{D^2}D^{\alpha}F_{\alpha\mu}\Big),\Big(\frac{1}{D^2}D^{\beta}F_{\beta\nu}\Big)\Big]^b(y)\,. \tag{5.14}$$

Using (4.36) and also the property (D.1) we integrate (5.14) w.r.t x to obtain:

$$\mathcal{Z}_1 = \int \mathrm{d}^4y\;\Big(\frac{1}{D^2}F_{\mu\nu}\Big)^b(y)\Big[\Big(\frac{1}{D^2}D^{\alpha}F_{\alpha\mu}\Big),\Big(\frac{1}{D^2}D^{\beta}F_{\beta\nu}\Big)\Big]^b(y)\,. \tag{5.15}$$

This is our final expression to order F^3 for $\mathcal{Z}_{_1}$. It is gauge invariant. Having obtained the values of $\mathcal{Y}_{_1}$ (5.8) and $\mathcal{Z}_{_1}$ (5.15) we can now write the expression for the mass term to order F^3. For this we substitute (5.8) and (5.15) into (4.137) to get

$$\begin{aligned}\mathcal{M}^2 = -\frac{1}{2}\int \mathrm{d}^4x\ F^a_{\mu\nu}(x)\Big(\frac{1}{D^2}F_{\mu\nu}\Big)^a(x)\\ +g\int \mathrm{d}^4x\ \Big(\frac{1}{D^2}F_{\mu\nu}\Big)^a(x)\Big[\Big(\frac{1}{D^2}D^\alpha F_{\alpha\mu}\Big),\Big(\frac{1}{D^2}D^\beta F_{\beta\nu}\Big)\Big]^a(x)\\ -g\int \mathrm{d}^4x\ \Big(\frac{1}{D^2}F_{\mu\nu}\Big)^a(x)\Big[\frac{1}{D^2}(D^\beta F_{\beta\rho}),\frac{1}{D^2}(D^\rho F_{\mu\nu})\Big]^a(x)+\cdots\end{aligned} \tag{5.16}$$

This is the expected expression for the mass term to order F^3 and was obtained by Zwanziger in [98], see equation (4.1) . Each line in the above expression is fully gauge invariant.

As far as we are aware, although this result has appeared in many places in the literature, no derivation has been presented. Note, though, that the expression (5.16) is not unique. We have made two choices which effect the results and that will lead to different expansions while still maintaining gauge invariance. The first was our choice of derivative used in the expression for $\mathcal{Q}^{ab}_{_1}(x,y)$ as seen in (5.3) as compared to (4.123). It is trivial to see that we could alternatively write

$$\tilde{\mathcal{Q}}^{ab}_{_1}(x,y) = -2\int \mathrm{d}^4z\ \delta^{ac}K_{_0}(x,z)(A^{^T}_\rho)^{cd}(z)\delta^{db}\partial^\rho_z K_{_0}(z,y)\,, \tag{5.17}$$

where we have simply integrated by parts and used the result that $\partial^\rho_x K_{_0}(x,z) = -\partial^\rho_z K_{_0}(x,z)$. But under the covariant trick $\partial^\rho_x K(x,z) \neq -\partial^\rho_z K(x,z)$ and hence we get a different gauge covariant expression following this route:

$$\tilde{\mathcal{Y}}_{_1} = \int \mathrm{d}^4z\ \Big(\frac{1}{D^2}F^{\mu\nu}\Big)^d(z)\Big[\Big(\frac{1}{D^2}D_\beta F^{\beta\rho}\Big),D_\rho\Big(\frac{1}{D^2}F^{\mu\nu}\Big)\Big]^d(z)\,. \tag{5.18}$$

Another choice arose from the identification (4.132) of the gauge covariant extension to the transverse field. We could alternatively identify

$$A^{T}_{\rho}(z) = \partial_\beta \frac{1}{\Box}(\partial^\beta A^\rho - \partial^\rho A^\beta)(z) \to D_\beta\Big(\frac{1}{D^2}F^{\beta\rho}\Big)(z)\,, \tag{5.19}$$

and this will lead to a new gauge invariant expression in the expansion of the mass term. Indeed following (5.19) we would now get

$$\tilde{\tilde{\mathcal{Y}}}_1 = \int \mathrm{d}^4 z\, \Big(\frac{1}{D^2}F^{\mu\nu}\Big)^d(z)\Big[D_\beta\Big(\frac{1}{D^2}F^{\beta\rho}\Big), \Big(\frac{1}{D^2}D_\rho F^{\mu\nu}\Big)\Big]^d(z)\,, \tag{5.20}$$

and combining these two choices we would have a term of the form

$$\tilde{\tilde{\tilde{\mathcal{Y}}}}_1 = \int \mathrm{d}^4 z\, \Big(\frac{1}{D^2}F^{\mu\nu}\Big)^d(z)\Big[D_\beta\Big(\frac{1}{D^2}F^{\beta\rho}\Big), D_\rho\Big(\frac{1}{D^2}F^{\mu\nu}\Big)\Big]^d(z)\,. \tag{5.21}$$

Indeed combinations of these would be equally valid.

In much the similar way these choices when applied to $\mathcal{P}_1$ (5.13) lead to the following possibilities:

$$\tilde{\mathcal{Z}}_1 = \int \mathrm{d}^4 y\, \Big(\frac{1}{D^2}F_{\mu\nu}\Big)^b(y)\Big[D^\alpha\Big(\frac{1}{D^2}F_{\alpha\mu}\Big), \Big(\frac{1}{D^2}D^\beta F_{\beta\nu}\Big)\Big]^b(y)\,, \tag{5.22}$$

$$\tilde{\tilde{\mathcal{Z}}}_1 = \int \mathrm{d}^4 y\, \Big(\frac{1}{D^2}F_{\mu\nu}\Big)^b(y)\Big[\Big(\frac{1}{D^2}D^\alpha F_{\alpha\mu}\Big), D^\beta\Big(\frac{1}{D^2}F_{\beta\nu}\Big)\Big]^b(y)\,, \tag{5.23}$$

and

$$\tilde{\tilde{\tilde{\mathcal{Z}}}}_1 = \int \mathrm{d}^4 y\, \Big(\frac{1}{D^2}F_{\mu\nu}\Big)^b(y)\Big[D^\alpha\Big(\frac{1}{D^2}F_{\alpha\mu}\Big), D^\beta\Big(\frac{1}{D^2}F_{\beta\nu}\Big)\Big]^b(y)\,. \tag{5.24}$$

The specific choice used in (5.8) and (5.15) reflects our aim to derive precisely Zwanziger's form of the expansion. However, other choices might have advantages in specific applications.

In Appendix D.2 we show that the expressions (4.132) and (5.19) are not the same and hence the possible extensions listed above are genuinely different gauge invariant expressions for the non-abelian mass term.

5.3 Calculation to Order F^4 of $\mathcal{Y}_2$ and $\mathcal{Z}_2$

We have so far seen how the mass term can be expressed in powers of the field strengths. Using this we have been able to recover Zwanziger's expression (4.1) in terms of the quadratic and cubic powers of the field strengths F^2, F^3 and also exposed the ambiguities in this expression. What we want to do now is, for the first time, construct the next terms in this expansion which will be quartic in the field strengths F^4. In our notation, from (4.137), (5.1) and (5.2), this will correspond to $\mathcal{Z}_2 - \mathcal{Y}_2$. Just as before we collect these gauge invariant operators once we have identified the transverse residue of the operator at the appropriate order. To calculate $\mathcal{Y}_2$ and $\mathcal{Z}_2$ we need to reinstate the higher order modifications we introduced by hand earlier in going from $\mathcal{Q}_1 \to \mathcal{Y}_1$ and $\mathcal{P}_1 \to \mathcal{Z}_1$. This means that we should view this process as the identification of, for example,

$$\begin{aligned} \mathcal{Q} &= g\mathcal{Q}_1 + g^2\mathcal{Q}_2 + \cdots \\ &= g\mathcal{Y}_1 + g\big(\mathcal{Q}_1 - \mathcal{Y}_1\big) + g^2\mathcal{Q}_2 + \cdots , \end{aligned} \tag{5.25}$$

so that now $g^2\mathcal{Y}_2$ is the gauge invariant extension of $g\big(\mathcal{Q}_1 - \mathcal{Y}_1\big) + g^2\mathcal{Q}_2$. This will only work if this term is only constructed out of the transverse field so that we can use the replacements such as (4.132). For the F^3 expression (5.3) this was relatively straightforward as we have seen that $\mathcal{Q}_1^{ab}(x, y)$ was purely transverse, but we have also seen that $\mathcal{Q}_2^{ab}(x, y)$ is not as it contains mixed transverse-longitudinal components (4.125). What we now need to ensure is that in the $O(g^2)$ parts of the combination $g\big(\mathcal{Q}_1 - \mathcal{Y}_1\big) + g^2\mathcal{Q}_2$ only the transverse components survive. That is,

$$g\big(\mathcal{Q}_1^{ab}(x,y) - \mathcal{Y}_1^{ab}(x,y)\big)^{TL} + g^2\mathcal{Q}_2^{TL\,ab}(x,y) = 0\,, \tag{5.26}$$

and

$$g\big(\mathcal{Q}_1^{ab}(x,y) - \mathcal{Y}_1^{ab}(x,y)\big)^{LL} + g^2\mathcal{Q}_2^{LL\,ab}(x,y) = 0\,, \tag{5.27}$$

leaving just the contribution from TT components. This is a strong statement of the underlying gauge invariance of this expansion since we have seen that $\mathcal{Y}_1$ is ambiguous. What we claim is that the ambiguities exposed in the previous section only contributes to the TT components.

In exactly the same way, for the second contribution to the mass term (4.137), we can write

$$\begin{aligned}\mathcal{P} &= g\mathcal{P}_1 + g^2\mathcal{P}_2 + g^3\mathcal{P}_3 + \cdots \\ &= g\mathcal{Z}_1 + g\big(\mathcal{P}_1 - \mathcal{Z}_1\big) + g^2\mathcal{Z}_2 + \cdots\,,\end{aligned} \tag{5.28}$$

where again $g^2\mathcal{Z}_2$ will then be identified with the gauge invariant extension of $g\big(\mathcal{P}_1 - \mathcal{Z}_1\big) + g^2\mathcal{P}_2$ assuming there are no residual LL or TL parts. The verification that only transverse fields survive is non-trivial and is the next topic.

5.4 Calculation of $\mathcal{Y}_2$

We have already seen there are ambiguities at the F^3 level, so to make precise how the calculations are performed to order F^4 we will adopt the expansion chosen by Zwanziger (5.8). In order to calculate $\mathcal{Y}_2$, the gauge invariant extension of $g\big(\mathcal{Q}_1 - \mathcal{Y}_1\big) + g^2\mathcal{Q}_2$, we consider the difference between each of the terms in (5.3) and the expression (5.5) by reinstating the higher order modifications. This will allow us to obtain the expression $g(\mathcal{Q}_{_1}^{ab}(x,y)-\mathcal{Y}_{_1}^{ab}(x,y))$. Once this has been achieved we then apply the decomposition technique to the fields in the obtained expression which generates new terms (in the potential and the Green's function) with mixed T/L components. These can now be isolated as TT, TL and LL components. Because we are interested in calculating the order g^2 contribution we need to take into account the contribution from the term $\mathcal{Q}_{_2}^{ab}(x,y)$ (4.125). However as we have seen earlier $\mathcal{Q}_{_2}^{ab}(x,y)$ is not purely transverse as it contains mixed T/L components given by (4.129) and (4.130). We need to remove these TL and LL components so that only TT fields survive. Equation (5.3) is our starting point where we use (4.132) to obtain

$$\begin{aligned}\mathcal{Q}_{_1}^{ab}(x,y) = -\int \mathrm{d}^4z\,\mathrm{d}^4w\,\Big\{&\partial_x^\rho\delta^{ac}K_{_0}(x,z)\big(K_{_0}(z,w)\partial^\beta f_{\beta\rho}(w)\big)^{cd}\delta^{db}K_{_0}(z,y)\\ &-\delta^{ac}K_{_0}(x,z)\big(K_{_0}(z,w)\partial^\beta f_{\beta\rho}(w)\big)^{cd}\partial_y^\rho\delta^{db}K_{_0}(z,y)\Big\}\,,\end{aligned} \tag{5.29}$$

where $K_{_0}(z,w)$ is the Green's functions for the inverse Laplacian. Similarly we write (5.5) as:

$$\begin{aligned}\mathcal{Y}_{_1}^{ab}(x,y) = -\int \mathrm{d}^4z\,\mathrm{d}^4w\,\Big\{&(D_x^\rho K(x,z))^{ac}\big(K(z,w)D^\beta F_{\beta\rho}(w)\big)^{cd}(z)K^{db}(z,y)\\ &-K^{ac}(x,z)\big(K(z,w)D^\beta F_{\beta\rho}(w)\big)^{cd}(z)\big(D_\rho^y K(z,y)\big)^{db}\Big\}\,.\end{aligned} \tag{5.30}$$

We now consider the difference between each of the terms in (5.29) and (5.30) by reinstating the following modifications

$$\begin{aligned}&\partial_x^\rho - D_x^\rho \to -gA_\rho(x),\quad \partial_w^\beta - D_w^\beta \to -gA_\beta(w),\quad \partial_y^\rho - D_y^\rho \to gA_\rho(y),\\ &\delta^{db}K_{_0}(z,y)-K^{db}(z,y)\to -gK_{_1}^{db}(z,y),\quad f_{\beta\rho}-F_{\beta\rho}\to -g[A_\beta,A_\rho],\\ &K_{_0}(z,w)-K(z,w)\to -gK_{_1}(z,w),\quad \delta^{ac}K_{_0}(x,z)-K^{ac}(x,z)\to -gK_{_1}^{ac}(x,z)\,.\end{aligned} \tag{5.31}$$

to obtain

$$
\begin{aligned}
g(\mathcal{Q}_{1}^{ab}(x,y) - \mathcal{Y}_{1}^{ab}(x,y)) = \\
\int \mathrm{d}^4 z \, \Big\{ & A_\rho(x) K_0(x,z) A_\rho^T(z) K_0(z,y) + K_0(x,z) A_\rho^T(z) A_\rho(y) K_0(z,y) \\
& + \partial_\rho^x K_1(x,z) A_\rho^T(z) K_0(z,y) - K_1(x,z) A_\rho^T(z) \partial_\rho^y K_0(z,y) \\
& + \partial_\rho^x K_0(x,z) A_\rho^T(z) K_1(z,y) - K_0(x,z) A_\rho^T(z) \partial_\rho^y K_1(z,y) \Big\}^{ab} \\
+ \int \mathrm{d}^4 z \, \mathrm{d}^4 w \, \Big\{ & \partial_\rho^x K_0(x,z) K_1(z,w) \partial^\beta f_{\beta\rho}(w) K_0(z,y) \\
& + K_0(x,z) K_1(z,w) \partial^\beta f_{\beta\rho}(w) \partial_\rho^z K_0(z,y) \\
& + \partial_\rho^x K_0(x,z) K_0(z,w) A_\beta(w) f_{\beta\rho}(w) K_0(z,y) \\
& + K_0(x,z) K_0(z,w) A_\beta(w) f_{\beta\rho}(w) \partial_\rho^z K_0(z,y) \\
& + \partial_\rho^x K_0(x,z) K_0(z,w) \partial_w^\beta [A_\beta, A_\rho](w) K_0(z,y) \\
& + K_0(x,z) K_0(z,w) \partial_w^\beta [A_\beta, A_\rho](w) \partial_\rho^z K_0(z,y) \Big\}^{ab} .
\end{aligned}
\tag{5.32}
$$

5.4.1 Cancellation of LL and TL terms

Having obtained (5.32) for the extended Zwanziger term we now want to extract the LL and TL terms from (5.32). We then add this contribution to (4.129) and (4.130) to verify that (5.26) and (5.27) hold. Note that in this procedure we expect all LL and TL components to vanish.

5.4.1.1 Cancellation of TL components

As a first step we consider (5.32) and take into account those terms that lead to the contribution of TL fields. For this we decompose the potential and Green's function appearing in the integrands into transverse and longitudinal components and collect only terms with TL fields.

$$
\begin{aligned}
&g(\mathcal{Q}_{_1}^{T\,ab}(x,y) - \mathcal{Y}_{_1}^{ab}(x,y)) = \\
&\quad \int \mathrm{d}^4 z \, \Big\{ A_\rho^{L}(x) K_{_0}(x,z) A_\rho^{T}(z) K_{_0}(z,y) + K_{_0}(x,z) A_\rho^{T}(z) A_\rho^{L}(y) K_{_0}(z,y) \\
&\qquad\qquad + \partial_\rho^x K_{_1}^{L}(x,z) A_\rho^{T}(z) K_{_0}(z,y) - K_{_1}^{L}(x,z) A_\rho^{T}(z) \partial_\rho^y K_{_0}(z,y) \\
&\qquad\qquad + \partial_\rho^x K_{_0}(x,z) A_\rho^{T}(z) K_{_1}^{L}(z,y) - K_{_0}(x,z) A_\rho^{T}(z) \partial_\rho^y K_{_1}^{L}(z,y) \Big\}^{ab} \\
&\quad + \int \mathrm{d}^4 z \, \mathrm{d}^4 w \, \Big\{ \partial_\rho^x K_{_0}(x,z) K_{_1}^{L}(z,w) \partial^\beta f_{\beta\rho}(w) K_{_0}(z,y) \\
&\qquad\qquad + K_{_0}(x,z) K_{_1}^{L}(z,w) \partial^\beta f_{\beta\rho}(w) \partial_\rho^z K_{_0}(z,y) \\
&\qquad\qquad + \partial_\rho^x K_{_0}(x,z) K_{_0}(z,w) A_\beta^{L}(w) f_{\beta\rho}(w) K_{_0}(z,y) \\
&\qquad\qquad + K_{_0}(x,z) K_{_0}(z,w) A_\beta^{L}(w) f_{\beta\rho}(w) \partial_\rho^z K_{_0}(z,y) \\
&\qquad\qquad + \Big(\partial_\rho^x K_{_0}(x,z) K_{_0}(z,y) + K_{_0}(x,z) \partial_\rho^z K_{_0}(z,y) \Big) \\
&\qquad\qquad \times K_{_0}(z,w) \partial_w^\beta \Big([A_\beta^{T}, A_\rho^{L}](w) + [A_\beta^{L}, A_\rho^{T}](w) \Big) \Big\}^{ab} .
\end{aligned}
\tag{5.33}
$$

Recall that what we want to show is $g(\mathcal{Q}_{_1}^{ab} - \mathcal{Y}_{_1}^{ab}(x,y))^{TL} + g^2 \mathcal{Q}_{_2}^{TL\,ab}(x,y) = 0$, for the TL component. The terms in the first braces of (5.33) are straightforward as we simply replace $A_\rho^{L}(x) \to \partial_\rho v_{_1}(x)$ and $K_{_1}^{L}(x,z) \to K_{_0}(x,z)(v_{_1}(z) - v_{_1}(x))$. Therefore the first six terms in (5.33) after simplification leads to

$$
\begin{aligned}
&\big\{ v_{_1}(x) K_{_1}^{T}(x,y) - K_{_1}^{T}(x,y) v_{_1}(y) \big\} \\
&\quad + \int \mathrm{d}^4 z \, \Big(\partial_\rho^x K_{_0}(x,z) K_{_0}(z,y) - K_{_0}(x,z) \partial_\rho^y K_{_0}(z,y) \Big) \big[v_{_1}, A_\rho^{T} \big](z) \Big\}^{ab} .
\end{aligned}
\tag{5.34}
$$

After this reduction if we incorporate $\mathcal{Q}_{_2}^{TL\,ab}(x,y)$ (4.129) we find the first and the last line of (4.129) cancel with (5.34) leaving the contribution from the second line of (4.129). For now we consider the final six terms in the second braces of (5.33). In particular, the last two terms are given by

$$\int \mathrm{d}^4 z \, \mathrm{d}^4 w \, \Big\{ \Big(\partial^x_\rho K_0(x,z) K_0(z,w) K_0(z,y) + K_0(x,z) K_0(z,w) \partial^z_\rho K_0(z,y) \Big) \\ \times \partial^w_\beta \Big([A^T_\beta, \partial_\rho v_1](w) + [\partial_\beta v_1, A^T_\rho](w) \Big) \Big\}^{ab} . \tag{5.35}$$

Integrating using by parts w.r.t. w we find

$$- \int \mathrm{d}^4 z \, \mathrm{d}^4 w \, \Big\{ \Big(\partial^x_\rho K_0(x,z) \partial^w_\beta K_0(z,w) K_0(z,y) + K_0(x,z) \partial^w_\beta K_0(z,w) \partial^z_\rho K_0(z,y) \Big) \\ \times \Big([A^T_\beta, \partial_\rho v_1](w) + [\partial_\beta v_1, A^T_\rho](w) \Big) \Big\}^{ab} . \tag{5.36}$$

Using the following identity

$$[A^T_\beta, \partial_\rho v_1](w) + [\partial_\beta v_1, A^T_\rho](w) = \partial^w_\beta [v_1, A^T_\rho](w) + \partial^w_\rho [A^T_\beta, v_1](w) - [v_1, f_{\beta\rho}](w) , \tag{5.37}$$

in (5.36) we find

$$- \int \mathrm{d}^4 z \, \mathrm{d}^4 w \, \Big\{ \Big(\partial^x_\rho K_0(x,z) \partial^w_\beta K_0(z,w) K_0(z,y) + K_0(x,z) \partial^w_\beta K_0(z,w) \partial^z_\rho K_0(z,y) \Big) \\ \times \Big(\partial^w_\beta [v_1, A^T_\rho](w) + \partial^w_\rho [A^T_\beta, v_1](w) - [v_1, f_{\beta\rho}](w) \Big) \Big\}^{ab} . \tag{5.38}$$

We now integrate using by parts w.r.t. w that yields six terms

$$\int \mathrm{d}^4 z \, \mathrm{d}^4 w \, \Big\{ \Big(\partial^x_\rho K_0(x,z) \Box_w K_0(z,w) K_0(z,y) \\ + K_0(x,z) \Box_w K_0(z,w) \partial^z_\rho K_0(z,y) \Big) \times [v_1, A^T_\rho](w) \\ + \Big(\partial^x_\rho K_0(x,z) \partial^w_\beta \partial^w_\rho K_0(z,w) K_0(z,y) \\ + K_0(x,z) \partial^w_\beta \partial^w_\rho K_0(z,w) \partial^z_\rho K_0(z,y) \Big) \times [A^T_\beta, v_1](w) \\ + \Big(\partial^x_\rho K_0(x,z) \partial^w_\beta K_0(z,w) K_0(z,y) \\ + K_0(x,z) \partial^w_\beta K_0(z,w) \partial^z_\rho K_0(z,y) \Big) \times [v_1, f_{\beta\rho}](w) \Big) \Big\}^{ab} . \tag{5.39}$$

The first two lines of (5.39) can be simplified further to obtain

$$\int \mathrm{d}^4 z \left\{ \Big(\partial^x_\rho K_{_0}(x,z) K_{_0}(z,y) + K_{_0}(x,z) \partial^z_\rho K_{_0}(z,y) \Big) [v_{_1}, A^{^T}_\rho](z) \right\}^{ab}. \tag{5.40}$$

However the third and fourth line of (5.39) require more work. We first integrate by parts over w and replace $\partial^x_\rho K_{_0}(x,z) \to -\partial^z_\rho K_{_0}(x,z)$ and $\partial^w_\rho K_{_0}(z,w) \to -\partial^z_\rho K_{_0}(z,w)$. Then we integrate by parts over z to finally get

$$\int \mathrm{d}^4 z \left\{ \Big(K_{_0}(x,z) K_{_0}(x,y) - K_{_0}(x,y) K_{_0}(y,z) \Big) [A^{^T}_\beta, \partial^z_\beta v_{_1}](z) \right\}^{ab}, \tag{5.41}$$

which cancels with the second line of (4.129). Upto now we have seen that all the TL components of $\mathcal{Q}^{ab}_{_2}(x,y)$ (4.129) have vanished in contrast to the contribution from $g(\mathcal{Q}^{ab}_{_1} - \mathcal{Y}^{ab}_{_1}(x,y))^{^{TL}}$ leaving the contribution from (5.40), the last two lines of (5.39) and the first four terms in the second braces of (5.33). What we want to see now is the cancellation of these survivors among themselves. To achieve this we first consider the first two terms in second braces of (5.33) and introduce explicit colour indices in the form

$$\begin{aligned} \int \mathrm{d}^4 z\, \mathrm{d}^4 w \, \Big\{ & \big(\partial^x_\rho K_{_0}(x,z)\big)^{ac} \Big(K^{^L}_{_1}(z,w) \partial^\beta f_{\beta\rho}(w) \Big)^{cd} \big(K_{_0}(z,y)\big)^{db} \\ & + \big(K_{_0}(x,z)\big)^{ac} \Big(K^{^L}_{_1}(z,w) \partial^\beta f_{\beta\rho}(w) \Big)^{cd} \big(\partial^z_\rho K_{_0}(z,y)\big)^{db} \Big\}. \end{aligned} \tag{5.42}$$

The above equation contains the Green's function for the inverse Laplacian $(K^{^L}_{_1}(z,w)\partial^\beta f_{\beta\rho}(w))^{cd}$ that now contains fields in the tensor product of the adjoint representation with itself that is

$$\begin{aligned} \Big(K^{^L}_{_1}(z,w) \partial^\beta f_{\beta\rho}(w) \Big)^{cd} &= K^{^L}_{_1}(z,w)^{cd}_{ef} \, \big(\partial^\beta f_{\beta\rho}\big)^{ef}(w) \\ &= K_{_0}(z,w) \big[\big(v_{_1}(w) - v_{_1}(z)\big), \partial^\beta f_{\beta\rho}(w) \big]^{cd}, \end{aligned} \tag{5.43}$$

where we have made use of (3.58). Upon substituting this into (5.42) one finds

$$\begin{aligned}\int \mathrm{d}^4z\,\mathrm{d}^4w\,\Big\{&\partial^x_\rho K_0(x,z)K_0(z,w)\big[\big(v_1(w)-v_1(z)\big),\partial^\beta f_{\beta\rho}(w)\big]K_0(z,y)\\ &+K_0(x,z)K_0(z,w)\big[\big(v_1(w)-v_1(z)\big),\partial^\beta f_{\beta\rho}(w)\big]\partial^z_\rho K_0(z,y)\Big\}^{ab}.\end{aligned}\tag{5.44}$$

Expanding the commutators and using the definition $A^T_\rho(z)=\int \mathrm{d}^4w\; K_0(z,w)\partial^\beta f_{\beta\rho}(w)$ we integrate the above expression w.r.t. w to get

$$\begin{aligned}&\int \mathrm{d}^4z\,\mathrm{d}^4w\,\Big\{\Big(\partial^x_\rho K_0(x,z)K_0(z,y)+K_0(x,z)\partial^z_\rho K_0(z,y)\Big)K_0(z,w)\big[v_1,\partial^\beta f_{\beta\rho}\big](w)\Big\}^{ab}\\ &\quad-\int \mathrm{d}^4z\,\Big\{\Big(\partial^x_\rho K_0(x,z)K_0(z,y)+K_0(x,z)\partial^z_\rho K_0(z,y)\Big)\big[v_1,A^T_\rho\big](z)\Big\}^{ab}.\end{aligned}\tag{5.45}$$

The last line of the above equation cancels with (5.40) leaving the contribution from the last two lines of (5.39), the first line of (5.45) and the third and fourth lines in second braces of (5.33) which need to be simplified further. We now consider the third and fourth lines of (5.33) where introducing appropriate colour indices and using the longitudinal decomposition of the vector potential we have

$$\begin{aligned}\int \mathrm{d}^4z\,\mathrm{d}^4w\,\Big\{&\big(\partial^x_\rho K_0(x,z)\big)^{ac}\big(K_0(z,w)\partial^\beta v_1(w)f_{\beta\rho}(w)\big)^{cd}\big(K_0(z,y)\big)^{db}\\ &+\big(K_0(x,z)\big)^{ac}\big(K_0(z,w)\partial^\beta v_1(w)f_{\beta\rho}(w)\big)^{cd}\big(\partial^z_\rho K_0(z,y)\big)^{db}\Big\}.\end{aligned}\tag{5.46}$$

The above equation contains fields as products in the adjoint representation which upon using the property (3.57) becomes a commutator in the Lie algebra written in the form

$$\begin{aligned}\int \mathrm{d}^4z\,\mathrm{d}^4w\,\Big\{&\big(\partial^x_\rho K_0(x,z)\big)^{ac}K_0(z,w)\big[\partial^\beta v_1(w),f_{\beta\rho}(w)\big]^{cd}\big(K_0(z,y)\big)^{db}\\ &+\big(K_0(x,z)\big)^{ac}K_0(z,w)\big[\partial^\beta v_1(w),f_{\beta\rho}(w)\big]^{cd}\big(\partial^z_\rho K_0(z,y)\big)^{db}\Big\}.\end{aligned}\tag{5.47}$$

Integrating by parts over w results in the following four terms

$$
\begin{aligned}
-\int \mathrm{d}^4 z\, \mathrm{d}^4 w\, \Big\{ &\partial^x_\rho K_0(x,z) K_0(z,w) \big[v_1(w), \partial^\beta f_{\beta\rho}(w)\big] K_0(z,y) \\
&+ K_0(x,z) K_0(z,w) \big[v_1(w), \partial^\beta f_{\beta\rho}(w)\big] \partial^z_\rho K_0(z,y) \\
&+ \partial^x_\rho K_0(x,z) \partial^\beta_w K_0(z,w) \big[v_1(w), f_{\beta\rho}(w)\big] K_0(z,y) \\
&+ K_0(x,z) \partial^\beta_w K_0(z,w) \big[v_1(w), f_{\beta\rho}(w)\big] \partial^z_\rho K_0(z,y) \Big\}^{ab} ,
\end{aligned} \tag{5.48}
$$

where the first two lines cancel with the first integrand of (5.45) and the last two lines cancel with the last integrand of (5.39). Thus we have successfully been able to show the cancellation of the TL components. For the remainder of the calculations to be presented in this chapter we shall implement the same idea to show the cancellation of the LL terms (5.27).

5.4.1.2 Cancellation of LL components

Using the same strategy as for the TL components we will show that no LL components survive that is we will verify that (5.27) holds. To show this we return to (5.32) and extract only those terms that contribute to the LL components. They are found to be the last two lines of (5.32).

$$
\begin{aligned}
\int \mathrm{d}^4 z\, \mathrm{d}^4 w\, \Big\{ &\partial^x_\rho K_0(x,z) K_0(z,w) \partial^\beta_w [A^L_\beta, A^L_\rho](w) K_0(z,y) \\
&+ K_0(x,z) K_0(z,w) \partial^\beta_w [A^L_\beta, A^L_\rho](w) \partial^z_\rho K_0(z,y) \Big\}^{ab} .
\end{aligned} \tag{5.49}
$$

Substituting $A^L_\beta = \partial_\beta v_1$ and $A^L_\rho = \partial_\rho v_1$ into (5.49) and integrating the above equation by parts w.r.t. w we get

$$
\begin{aligned}
-\int \mathrm{d}^4 z\, \mathrm{d}^4 w\, \Big\{ &\partial^x_\rho K_0(x,z) \partial^\beta_w K_0(z,w) [\partial_\beta v_1, \partial_\rho v_1](w) K_0(z,y) \\
&+ K_0(x,z) \partial^\beta_w K_0(z,w) [\partial_\beta v_1, \partial_\rho v_1](w) \partial^z_\rho K_0(z,y) \Big\}^{ab} .
\end{aligned} \tag{5.50}
$$

Using the identity

$$[\partial_\beta v_{_1}, \partial_\rho v_{_1}](w) = \frac{1}{2}\Big(\partial_w^\beta [v_{_1}, \partial_\rho v_{_1}](w) + \partial_w^\rho [\partial_\beta v_{_1}, v_{_1}](w)\Big)\,, \tag{5.51}$$

$$\begin{aligned} \frac{1}{2}\int \mathrm{d}^4 z\, \Big\{ \big(-\partial_\rho^z K_{_0}(x,z) K_{_0}(z,y) + K_{_0}(x,z)\partial_\rho^z K_{_0}(z,y)\big)[v_{_1}, \partial_\rho v_{_1}](z) \\ + \big(-\partial_z^\rho K_{_0}(x,z) + \partial_z^\rho K_{_0}(y,z)\big) K_{_0}(x,y)[\partial_\rho v_{_1}, v_{_1}](z) \Big\}^{ab}. \end{aligned} \tag{5.52}$$

Now taking into account the longitudinal component, $\mathcal{Q}_{_2}^{LL\,ab}(x,y)$ (4.130) we find the terms are the same but carrying opposite signs. Hence $g(\mathcal{Q}_{_1}^{ab} - \mathcal{Y}_{_1}^{ab}(x,y))^{LL} + g^2 \mathcal{Q}_{_2}^{LL\,ab}(x,y) = 0$.

After the successful cancellation of the longitudinal fields the next task is to collect the transverse components and then reinstate the fields to get the contribution to order F^4.

5.4.2 Collection of TT components

In order to calculate the next order contribution to the TT components we need to reinstate the higher order modifications. Recall that what we want to calculate is the contribution from $g\big(\mathcal{Q}_{_1}^{ab}(x,y) - \mathcal{Y}_{_1}^{ab}(x,y)\big)^{TT} + g^2 \mathcal{Q}_{_2}^{TT\,ab}(x,y)$. The contribution from $g^2 \mathcal{Q}_{_2}^{TT\,ab}(x,y)$ has already been defined in the previous chapter and is given by (4.117). However the contribution from $g\big(\mathcal{Q}_{_1}^{ab}(x,y) - \mathcal{Y}_{_1}^{ab}(x,y)\big)^{TT}$ is obtained by making the following replacements in (5.29) as

$$\begin{aligned} &\partial_x^\rho \to -gA_\rho^{^T}(x), \quad \delta^{ac} K_{_0}(x,z) \to -gK_{_1}^{T\,ac}(x,z), \quad K_{_0}(z,w) \to -gK_{_1}^{^T}(z,w), \\ &\partial_w^\beta \to -gA_\beta^{^T}(w), \quad f_{\beta\rho} \to -g[A_\beta^{^T}, A_\rho^{^T}], \quad \delta^{db} K_{_0}(z,y) \to -gK_{_1}^{T\,db}(z,y)\,. \end{aligned} \tag{5.53}$$

Thus the total contribution to order F^4 for the TT components is

$$\begin{aligned}
g(\mathcal{Q}_{_1}^{T\,ab}(x,y) - \mathcal{Y}_{_1}^{ab}(x,y)) + g^2 \mathcal{Q}_{_2}^{TT\,ab}(x,y) = \\
\int \mathrm{d}^4 z \, \Big\{ 2A_\rho^{T\,ac}(x) K_{_0}^{ce}(x,z) A_\rho^{T\,ed}(z) K_{_0}^{db}(z,y) + 2\partial_\rho^x K_{_1}^{T\,ac}(x,z) A_\rho^{T\,cd}(z) K_{_0}^{db}(z,y) \\
- K_{_0}^{ac}(x,z)\; A_\rho^{T\,ce}(z) A_\rho^{T\,ed}(z)\; K_{_0}^{db}(z,y) \Big\} \\
+ 2 \int \mathrm{d}^4 z \, \mathrm{d}^4 w \, \Big\{ \partial_\rho^x K_{_0}^{ac}(x,z) (K_{_1}^{T}(z,w) \partial^\beta f_{\beta\rho}(w))^{cd} K_{_0}^{db}(z,y) \\
+ \partial_\rho^x K_{_0}^{ac}(x,z) (K_{_0}(z,w) A_\beta^{T}(w) f_{\beta\rho}(w))^{cd} K_{_0}^{db}(z,y) \\
+ \partial_\rho^x K_{_0}^{ac}(x,z) (K_{_0}(z,w) \partial_w^\beta [A_\beta^{T}, A_\rho^{T}])^{cd}(w) K_{_0}^{db}(z,y) \Big\} .
\end{aligned} \tag{5.54}$$

So far we have been able to identify all of the TT components involved in the calculation for $\mathcal{Y}_{_2}$. However, if we look at the structure of the integrands involved in (5.54) we see that these can still be simplified. The terms in first braces of (5.54) are straightforward so we will simplify them later in this section, however, for now we will simplify the first line in second braces given by

$$2 \int \mathrm{d}^4 z \, \mathrm{d}^4 w \, \Big\{ \partial_\rho^x K_{_0}^{ac}(x,z) (K_{_1}^{T}(z,w) \partial^\beta f_{\beta\rho}(w))^{cd} K_{_0}^{db}(z,y) \Big\} , \tag{5.55}$$

where

$$K_{_1}^{T}(z,w) = -2 \int \mathrm{d}^4 u \, \{ \partial_\lambda^z K_{_0}(z,u) A_\lambda^{T}(u) K_{_0}(u,w) \} . \tag{5.56}$$

Note that in (5.55), $K_{_1}^{T}(z,w)$ has four indices so using (D.2) and integrating w.r.t. w (5.55) becomes

$$-4 \int \mathrm{d}^4 z \, \mathrm{d}^4 u \, \Big\{ \partial_\rho^x K_{_0}^{ac}(x,z) K_{_0}^{ce}(z,u) [A_\lambda^{T}, \partial_\lambda A_\rho^{T}]^{ed}(u) K_{_0}^{db}(z,y) \Big\} . \tag{5.57}$$

where we have used the result that $\int \mathrm{d}^4 w \; K_{_0}(u,w) \partial^\beta f_{\beta\rho}(w) = A_\rho^{T}(u)$. After this simplification we now return to the last two lines in (5.54) where we use the property (3.57) to obtain

$$
\begin{aligned}
2\int \mathrm{d}^4 z\, \mathrm{d}^4 w\, \Big\{ &\partial^x_\rho K^{ac}_{0}(x,z) K^{ce}_{0}(z,w)[A^{T}_\beta, f_{\beta\rho}]^{ed}(w) K^{db}_{0}(z,y) \\
&+ \partial^x_\rho K^{ac}_{0}(x,z) K^{ce}_{0}(z,w)[A^{T}_\beta, \partial_\beta A^{T}_\rho]^{ed}(w) K^{db}_{0}(z,y) \Big\} .
\end{aligned}
\tag{5.58}
$$

Upon combining (5.57) and (5.58) we obtain

$$
-2\int \mathrm{d}^4 z\, \mathrm{d}^4 w\, \Big\{ \partial^x_\rho K^{ac}_{0}(x,z) K^{ce}_{0}(z,w)[A^{T}_\beta, \partial_\rho A^{T}_\beta]^{ed}(w) K^{db}_{0}(z,y) \Big\} . \tag{5.59}
$$

To summarise, the terms in second braces of (5.54) have been reduced to one term (5.59) and hence (5.54) is given by

$$
\begin{aligned}
g(\mathcal{Q}^{ab}_{1} - \mathcal{Y}^{ab}_{1}(x,y))^{TT} &+ g^2 \mathcal{Q}^{TT\,ab}_{2}(x,y) = \mathcal{A}(x,y) + \mathcal{B}(x,y) + \mathcal{C}(x,y) + \mathcal{D}(x,y) \\
= \int \mathrm{d}^4 z\, \Big\{ &2 A^{T\,ac}_\rho(x) K^{ce}_{0}(x,z) A^{T\,ed}_\rho(z) K^{db}_{0}(z,y) + 2\partial^x_\rho K^{T\,ac}_{1}(x,z) A^{T\,cd}_\rho(z) K^{db}_{0}(z,y) \\
&- K^{ac}_{0}(x,z)\; A^{T\,ce}_\rho(z) A^{T\,ed}_\rho(z)\; K^{db}_{0}(z,y) \Big\} \\
-2\int \mathrm{d}^4 z\, \mathrm{d}^4 w\, &\Big\{ \partial^x_\rho K^{ac}_{0}(x,z) K^{ce}_{0}(z,w)[A^{T}_\beta, \partial_\rho A^{T}_\beta]^{ed}(w) K^{db}_{0}(z,y) \Big\} .
\end{aligned}
\tag{5.60}
$$

Note that in above we have denoted the four terms by $\mathcal{A}(x,y), \mathcal{B}(x,y), \mathcal{C}(x,y), \mathcal{D}(x,y)$. As seen in Chapter 4 using (4.105) we can construct the operator $g^2\mathcal{Y}_2$ by sandwiching (5.60) between the two field strengths. This means that we should now calculate

$$
\begin{aligned}
\mathcal{Y}_2 &:= -\frac{1}{2}\int \mathrm{d}^4 x\, \mathrm{d}^4 y\; F^a_{\mu\nu}(x) \big\{ \mathcal{A}(x,y) + \mathcal{B}(x,y) + \mathcal{C}(x,y) + \mathcal{D}(x,y) \big\} F^{\mu\nu\,b}(y) \\
&= \mathcal{A} + \mathcal{B} + \mathcal{C} + \mathcal{D}
\end{aligned}
\tag{5.61}
$$

The calculation of (5.61) is non-trivial. We will consider all four terms of (5.61) and show in turn how we obtain our final expression. Note that in equation (5.60) terms like $K_{_1}^{T}(x,z)$ need further simplification. We start with the first term of (5.61)

$$\mathcal{A} = -\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4z\; F^a_{\mu\nu}(x)\big(A_\rho^{T\,ac}(x)K_{_0}^{ce}(x,z)A_\rho^{T\,ed}(z)K_{_0}^{db}(z,y)\big)F^{\mu\nu\,b}(y)\,, \tag{5.62}$$

and integrate w.r.t. y to yield

$$\mathcal{A} = -\int \mathrm{d}^4x\,\mathrm{d}^4z\; F^a_{\mu\nu}(x)\big(A_\rho^{T\,ac}(x)K_{_0}^{ce}(x,z)A_\rho^{T\,ed}(z)\big)\Big(\frac{1}{\Box}F^{\mu\nu}\Big)^d(z)\,. \tag{5.63}$$

Using property (3.57) the last two terms become a commutator

$$\mathcal{A} = -\int \mathrm{d}^4x\,\mathrm{d}^4z\; F^a_{\mu\nu}(x)A_\rho^{T\,ac}(x)K_{_0}^{ce}(x,z)\Big[A_\rho^{T},\frac{1}{\Box}F^{\mu\nu}\Big]^e(z)\,, \tag{5.64}$$

which when integrated w.r.t. z yields

$$\mathcal{A} = -\int \mathrm{d}^4x\; F^a_{\mu\nu}(x)A_\rho^{T\,ac}(x)\Big(\frac{1}{\Box}\Big[A_\rho^{T},\frac{1}{\Box}F^{\mu\nu}\Big]\Big)^c(x)\,. \tag{5.65}$$

Again using the property (3.57) we have

$$\mathcal{A} = -\int \mathrm{d}^4x\; F^a_{\mu\nu}(x)\Big[A_\rho^{T}(x),\frac{1}{\Box}\Big[A_\rho^{T},\frac{1}{\Box}F^{\mu\nu}\Big](x)\Big]^a\,. \tag{5.66}$$

At this order of coupling we can straight away rewrite the above result by using the substitution $1/\Box \to 1/D^2$ in addition to (4.132) to obtain

$$\mathcal{A} = -\int \mathrm{d}^4x\; F^a_{\mu\nu}(x)\Big[\Big(\frac{1}{D^2}D_\sigma F^{\sigma\rho}\Big),\frac{1}{D^2}\Big[\Big(\frac{1}{D^2}D_\tau F^{\tau\rho}\Big),\Big(\frac{1}{D^2}F^{\mu\nu}\Big)\Big]\Big]^a(x)\,. \tag{5.67}$$

Now we consider the second term $\mathcal{B}$ of (5.61) to obtain

$$\mathcal{B} = -\frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\Big\{F^a_{\mu\nu}(x)(2\partial^x_\rho K_{_1}^{T\,ac}(x,z)A_\rho^{T\,cd}(z)K_{_0}^{db}(z,y))F^{\mu\nu\,b}(y)\Big\}\,. \tag{5.68}$$

Using the definition of $K_1^{T\,ac}(x,z)$ as in (5.56) we have for (5.68)

$$\mathcal{B} = 2\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4w\,\mathrm{d}^4z\,\Big\{F^a_{\mu\nu}(x)\big(\partial^x_\rho K_0(x,w)A^T_\lambda(w)\partial^w_\lambda K_0(w,z)\big)^{ac}A^{T\,cd}_\rho(z) \times K_0^{db}(z,y)F^{\mu\nu b}(y)\Big\}\,. \tag{5.69}$$

Writing the colour indices explicitly leads to

$$\mathcal{B} = 2\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4w\,\mathrm{d}^4z\,\Big\{F^a_{\mu\nu}(x)(\partial^x_\rho)^{aa'}K_0^{a'b'}(x,w)A^{T\,b'c'}_\lambda(w) \times \big((\partial^w_\lambda K_0(w,z)\big)^{c'c}A^{T\,cd}_\rho(z)K_0^{db}(z,y)F^{\mu\nu b}(y)\Big\}\,, \tag{5.70}$$

where now the partial derivative ∂^x_ρ acts on the field strength $F^a_{\mu\nu}(x)$ to give

$$\mathcal{B} = -2\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4w\,\mathrm{d}^4z\,\Big\{K_0^{b'a'}(w,x)(\partial^x_\rho F_{\mu\nu})^{a'}(x)A^{T\,b'c'}_\lambda(w) \times (\partial^w_\lambda K_0(w,z))^{c'c}A^{T\,cd}_\rho(z)K_0^{db}(z,y)F^{\mu\nu b}(y)\Big\}\,. \tag{5.71}$$

Using the property (4.8) we integrate the above equation w.r.t. x and y to yield

$$\mathcal{B} = -2\int \mathrm{d}^4z\,\mathrm{d}^4w\,\Big\{\Big(\frac{1}{\Box}\partial_\rho F_{\mu\nu}\Big)^{b'}(w)A^{T\,b'c'}_\lambda(w)(\partial^w_\lambda K_0(w,z))^{c'c}A^{T\,cd}_\rho(z)\Big(\frac{1}{\Box}F^{\mu\nu}\Big)^d(z)\Big\}\,. \tag{5.72}$$

The last two terms in above become a commutator using (3.57)

$$\mathcal{B} = -2\int \mathrm{d}^4z\,\mathrm{d}^4w\,\Big\{\Big(\frac{1}{\Box}\partial_\rho F_{\mu\nu}\Big)^{b'}(w)A^{T\,b'c'}_\lambda(w)(\partial^w_\lambda K_0(w,z))^{c'c}\Big[A^T_\rho,\Big(\frac{1}{\Box}F^{\mu\nu}\Big)\Big]^c(z)\Big\}\,, \tag{5.73}$$

where now integrating the above equation with respect to z gives

$$\mathcal{B} = -2\int \mathrm{d}^4w\,\Big\{\Big(\frac{1}{\Box}\partial_\rho F_{\mu\nu}\Big)^{b'}(w)A^{T\,b'c'}_\lambda(w)\Big(\partial^w_\lambda\frac{1}{\Box}\Big[A^T_\rho,\frac{1}{\Box}F^{\mu\nu}\Big]\Big)^{c'}(w)\Big\}\,. \tag{5.74}$$

Again using the property (3.57) in above yields

$$\mathcal{B} = -2\int \mathrm{d}^4 w\,\Big\{\Big(\frac{1}{\Box}\partial_\rho F_{\mu\nu}\Big)^{b'}(w)\Big[A^{T}_{\lambda}, \partial^{w}_{\lambda}\frac{1}{\Box}\Big[A^{T}_{\rho}, \frac{1}{\Box}F^{\mu\nu}\Big]\Big]^{b'}(w)\Big\}\,. \tag{5.75}$$

Translating (5.75) in terms of covariant derivative and using (4.132) we obtain

$$\mathcal{B} = -2\int \mathrm{d}^4 x\,\Big(\frac{1}{D^2}D_\rho F^{\mu\nu}\Big)^{a}(x)\Big[\Big(\frac{1}{D^2}D_\sigma F^{\sigma\lambda}\Big), D_\lambda\frac{1}{D^2}\Big[\Big(\frac{1}{D^2}D_\alpha F^{\alpha\rho}\Big), \Big(\frac{1}{D^2}F^{\mu\nu}\Big)\Big]\Big]^{a}(x)\,. \tag{5.76}$$

In much the similar way the third term of (5.61) can be simplified to yield

$$\mathcal{C} = \frac{1}{2}\int \mathrm{d}^4 x\,\Big(\frac{1}{D^2}F^{\mu\nu}\Big)^{a}(x)\Big[\Big(\frac{1}{D^2}D_\sigma F^{\sigma\lambda}\Big), \Big[\Big(\frac{1}{D^2}D_\alpha F^{\alpha\rho}\Big), \Big(\frac{1}{D^2}F^{\mu\nu}\Big)\Big]\Big]^{a}(x)\,. \tag{5.77}$$

The fourth term of (5.61) is given by

$$\begin{aligned}
\mathcal{D} &= -\int \mathrm{d}^4 z\,\mathrm{d}^4 w\,\Big(\frac{1}{\Box}\partial_\rho F_{\mu\nu}\Big)^{c}(z)K^{ce}_{0}(z,w)[A^{T}_{\beta}, \partial_\rho A^{T}_{\beta}]^{ed}(w)\Big(\frac{1}{\Box}F_{\mu\nu}\Big)^{d}(z)\\
&= -\int \mathrm{d}^4 z\,\Big(\frac{1}{\Box}\partial_\rho F_{\mu\nu}\Big)^{c}(z)\Big(\frac{1}{\Box}[A^{T}_{\beta}, \partial_\rho A^{T}_{\beta}]\Big)^{cd}(z)\Big(\frac{1}{\Box}F_{\mu\nu}\Big)^{d}(z)\\
&= -\int \mathrm{d}^4 z\,\Big(\frac{1}{\Box}\partial_\rho F_{\mu\nu}\Big)^{c}(z)\Big[\frac{1}{\Box}[A^{T}_{\beta}, \partial_\rho A^{T}_{\beta}], \frac{1}{\Box}F_{\mu\nu}\Big]^{c}(z)\,.
\end{aligned} \tag{5.78}$$

Performing the substitutions in terms of covariant derivative yields

$$\mathcal{D} = -\int \mathrm{d}^4 x\,\Big(\frac{1}{D^2}D_\rho F^{\mu\nu}\Big)^{a}(x)\Big[\frac{1}{D^2}\Big[\Big(\frac{1}{D^2}D_\alpha F^{\alpha\beta}\Big), D_\rho\Big(\frac{1}{D^2}D_\tau F^{\tau\beta}\Big)\Big], \Big(\frac{1}{D^2}F^{\mu\nu}\Big)\Big]^{a}(x)\,. \tag{5.79}$$

Finally substituting the expressions $\mathcal{A}-\mathcal{D}$ back into (5.60) we obtain:

$$\begin{aligned}
\mathcal{Y}_{2} = &-\int \mathrm{d}^4 x\,F^{a}_{\mu\nu}(x)\Big[\Big(\frac{1}{D^2}D_\sigma F^{\sigma\rho}\Big), \frac{1}{D^2}\Big[\Big(\frac{1}{D^2}D_\tau F^{\tau\rho}\Big), \Big(\frac{1}{D^2}F^{\mu\nu}\Big)\Big]\Big]^{a}(x)\\
&-2\int \mathrm{d}^4 x\,\Big(\frac{1}{D^2}D_\rho F^{\mu\nu}\Big)^{a}(x)\Big[\Big(\frac{1}{D^2}D_\sigma F^{\sigma\lambda}\Big), D_\lambda\frac{1}{D^2}\Big[\Big(\frac{1}{D^2}D_\alpha F^{\alpha\rho}\Big), \Big(\frac{1}{D^2}F^{\mu\nu}\Big)\Big]\Big]^{a}(x)\\
&+\frac{1}{2}\int \mathrm{d}^4 x\,\Big(\frac{1}{D^2}F^{\mu\nu}\Big)^{a}(x)\Big[\Big(\frac{1}{D^2}D_\sigma F^{\sigma\lambda}\Big), \Big[\Big(\frac{1}{D^2}D_\alpha F^{\alpha\rho}\Big), \Big(\frac{1}{D^2}F^{\mu\nu}\Big)\Big]\Big]^{a}(x)\\
&-\int \mathrm{d}^4 x\,\Big(\frac{1}{D^2}D_\rho F^{\mu\nu}\Big)^{a}(x)\Big[\frac{1}{D^2}\Big[\Big(\frac{1}{D^2}D_\alpha F^{\alpha\beta}\Big), D_\rho\Big(\frac{1}{D^2}D_\tau F^{\tau\beta}\Big)\Big], \Big(\frac{1}{D^2}F^{\mu\nu}\Big)\Big]^{a}(x)\,,
\end{aligned} \tag{5.80}$$

which is gauge invariant as required.

5.5 Calculation of $\mathcal{Z}_2$

Using the same strategy as for $\mathcal{Y}_2$ we can in a similar way calculate $\mathcal{Z}_2$, the gauge invariant extension of $g\big(\mathcal{P}_1 - \mathcal{Z}_1\big) + g^2\mathcal{P}_2$. We have seen earlier that in order to obtain $\mathcal{Z}_1$, (5.14), we made use of the following replacements

$$(F^h_{\mu\nu})^a \to F^a_{\mu\nu}(x)\,, \quad A^h_\mu \to A^T_\mu\,, \quad A^h_\nu \to A^T_\nu\,, \tag{5.81}$$

in (4.134). Now that we have obtained $\mathcal{Z}_1$ we also have the choice of writing the commutator in the adjoint representation where using (3.57) we find

$$[A^h_\mu, A^h_\nu]^b = (A^h_\mu)^{bc}(A^h_\nu)^c\,. \tag{5.82}$$

Hence to lowest order (4.134) can be written as

$$\mathcal{P}_1 = \int \mathrm{d}^4x\,\mathrm{d}^4y\;(F^h_{\mu\nu})^a(x)\,\delta^{ab}K_0(x,y)\,(A^h_\mu)^{bc}(y)(A^h_\nu)^c(y)\,. \tag{5.83}$$

It is to be noted that we are not strictly expanding in the coupling but we are allowing for the field strength terms to be kept together. The above equation contains the field strength $(F^h_{\mu\nu})^a$ in the adjoint representation and more details on its expansion can be found in Appendix D.3. The next task is to reinstate the higher order modifications by making the following replacements in (5.83)

$$(F^h_{\mu\nu})^a \to -gF^c_{\mu\nu}(v_1)^{ca}\,, \quad (A^h_\nu)^c(y) \to g(A^{(1)T}_\nu)^c(y)\,, \quad (A^h_\mu)^{bc}(y) \to g(A^{(1)T}_\mu)^{bc}(y)\,, \tag{5.84}$$

to obtain

$$
\begin{aligned}
g\mathcal{P}_{1} = \int \mathrm{d}^4x\,\mathrm{d}^4y \Big\{ &- F^{d}_{\mu\nu}(x)(v_{1})^{da}(x)\,\delta^{ab}K_{0}(x,y)(A^{T}_{\mu})^{bc}(y)(A^{T}_{\nu})^{c}(y) \\
&+ F^{a}_{\mu\nu}(x)\,\delta^{ab}K_{0}(x,y)(A^{(1)T}_{\mu})^{bc}(y)(A^{T}_{\nu})^{c}(y) \\
&+ F^{a}_{\mu\nu}(x)\,\delta^{ab}K_{0}(x,y)(A^{T}_{\mu})^{bc}(y)(A^{(1)T}_{\nu})^{c}(y)\Big\}\,.
\end{aligned}
\tag{5.85}
$$

We will return to these terms later but for now we will consider (5.13) which can be written in the equivalent form as

$$
\mathcal{P}_{1} = \int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4w\,\mathrm{d}^4u \;\; F^{a}_{\mu\nu}\delta^{ab}K_{0}(x,y)\big(K_{0}(y,w)\partial^{\alpha}f_{\alpha\mu}(w)\big)^{bc}\big(K_{0}(y,u)\partial^{\beta}f_{\beta\nu}(u)\big)^{c}\,.
\tag{5.86}
$$

Using (4.61) and (4.62) we find

$$
\begin{aligned}
\mathcal{P}_{1} = \int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4w\,\mathrm{d}^4u \;\; &F^{a}_{\mu\nu}(x)\delta^{ab}K_{0}(x,y)\delta^{bc}_{de}K_{0}(y,w)\big(\partial^{\alpha}f_{\alpha\mu}\big)^{de}(w) \\
&\times \delta^{cf}K_{0}(y,u)\big(\partial^{\beta}f_{\beta\nu}\big)^{f}(u)\,.
\end{aligned}
\tag{5.87}
$$

We clearly see that the above equation is an expansion in coupling to lowest orders. However to calculate $\mathcal{Z}_{2}$ we need to reinstate the higher order modifications by making the following replacements

$$
\begin{aligned}
&\delta^{cf}K_{0}(y,u) \to -gK^{cf}_{1}(y,u)\,, \\
&\big(\partial^{\beta}f_{\beta\nu}\big)^{f}(u) \to -g\big(\partial^{\beta}[A_{\beta},A_{\nu}]^{f} + [A_{\beta},f_{\beta\nu}]^{f}\big)(u) \\
&\big(\partial^{\alpha}f_{\alpha\mu}\big)^{de}(w) \to -g\big(\partial^{\alpha}[A_{\alpha},A_{\mu}]^{de} + [A_{\alpha},f_{\alpha\mu}]^{de}\big)(w) \\
&K^{ab}_{0}(x,y) \to -gK^{ab}_{1}(x,y), \quad \delta^{bc}_{de}K_{0}(y,w) \to -gK_{1}{}^{bc}_{de}(y,w)\,,
\end{aligned}
\tag{5.88}
$$

to (5.87). It is the extension of the difference between gauge invariant expression (5.14) and (5.87) in addition to $g^2\mathcal{P}_2$ that will correspond to $g^2\mathcal{Z}_2$ that is,

$$
\begin{aligned}
g\big(\mathcal{P}_{1}-\mathcal{Z}_{1}\big) = &-\int \mathrm{d}^4x\,\mathrm{d}^4y\,\left\{F^a_{\mu\nu}(x)K_{1}^{ab}(x,y)(A^{T}_{\mu})^{bc}(y)(A^{T}_{\nu})^{c}(y)\right\} \\
&-\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4w\,\left\{F^a_{\mu\nu}(x)\delta^{ab}K_{0}(x,y)K_{1\,\,de}^{\,bc}(y,w)\big(\partial^\alpha f_{\alpha\mu}\big)^{de}(w)(A^{T}_{\nu})^{c}(y)\right\} \\
&-\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4u\,\left\{F^a_{\mu\nu}(x)\delta^{ab}K_{0}(x,y)(A^{T}_{\mu})^{bc}(y)K_{1}^{cf}(y,u)\big(\partial^\beta f_{\beta\nu}\big)^{f}(u)\right\} \\
&-\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4w\,\Big\{F^a_{\mu\nu}(x)\delta^{ab}K_{0}(x,y)\delta^{bc}_{de}K_{0}(y,w) \\
&\qquad\qquad \times\big(\partial^\alpha[A_\alpha,A_\mu]^{de}+[A_\alpha,f_{\alpha\mu}]^{de}\big)(w)(A^{T}_{\nu})^{c}(y)\Big\} \\
&-\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4u\,\Big\{F^a_{\mu\nu}(x)\delta^{ab}K_{0}(x,y)(A^{T}_{\mu})^{bc}(y)\delta^{cf}K_{0}(y,u) \\
&\qquad\qquad \times\big(\partial^\beta[A_\beta,A_\nu]^{f}+[A_\beta,f_{\beta\nu}]^{f}\big)(u)\Big\}\,.
\end{aligned}
\tag{5.89}
$$

Again we want to show that all the TL and LL terms from (5.89) when added to (5.85) cancel. The details of this cancellation are included in Appendix D.5. The surviving gauge invariant terms are then given by

$$
\begin{aligned}
&g\big(\mathcal{P}_{1}-\mathcal{Z}_{1}\big)+g^2\mathcal{P}_{2} = \mathcal{G}+\mathcal{H}+\mathcal{I}+\mathcal{J}+\mathcal{K} \\
&= -\int \mathrm{d}^4x\,\mathrm{d}^4y\,\Big\{F^a_{\mu\nu}(x)\delta^{ab}K_{0}(x,y)\frac{1}{\Box}\Big([A^{T}_{\alpha},\partial^\alpha A^{T}_{\mu}]^{bc}+[A^{T}_{\alpha},f_{\alpha\mu}]^{bc}\Big)(y)(A^{T}_{\nu})^{c}(y) \\
&\qquad + F^a_{\mu\nu}(x)\delta^{ab}K_{0}(x,y)(A^{T}_{\mu})^{bc}(y)\frac{1}{\Box}\Big([A^{T}_{\beta},\partial^\beta A^{T}_{\nu}]^{c}+[A^{T}_{\beta},f_{\beta\nu}]^{c}\Big) \\
&\qquad + F^a_{\mu\nu}(x)K_{1}^{T\,ab}(x,y)(A^{T}_{\mu})^{bc}(y)(A^{T}_{\nu})^{c}(y)\Big\} \\
&-\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4w\,\left\{F^a_{\mu\nu}(x)\delta^{ab}K_{0}(x,y)\Big(K_{1}^{T}(y,w)\partial^\alpha f_{\alpha\mu}(w)\Big)^{bc}(A^{T}_{\nu})^{c}(y)\right\} \\
&-\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4u\,\left\{F^a_{\mu\nu}(x)\delta^{ab}K_{0}(x,y)(A^{T}_{\mu})^{bc}(y)\Big(K_{1}^{T}(y,u)\partial^\beta f_{\beta\nu}(u)\Big)^{c}\right\}.
\end{aligned}
\tag{5.90}
$$

The commutators in the first line of (5.90) can be combined together to obtain

$$
\mathcal{G} = -\int \mathrm{d}^4x\,\mathrm{d}^4y\,\left\{F^a_{\mu\nu}(x)\delta^{ab}K_{0}(x,y)\frac{1}{\Box}\Big(2[A^{T}_{\alpha},\partial^\alpha A^{T}_{\mu}]^{bc}-[A^{T}_{\alpha},\partial^\mu A^{T}_{\alpha}]^{bc}\Big)(y)(A^{T}_{\nu})^{c}(y)\right\},
\tag{5.91}
$$

and using (3.57) we have

$$\begin{aligned}\mathcal{G} = -\int \mathrm{d}^4x\,\mathrm{d}^4y\,\Big\{&2F^a_{\mu\nu}(x)\delta^{ab}K_0(x,y)\Big[\frac{1}{\Box}\Big[A^T_\alpha,\partial^\alpha A^T_\mu\Big],A^T_\nu\Big]^b(y)\\&-F^a_{\mu\nu}(x)\delta^{ab}K_0(x,y)\Big[\frac{1}{\Box}\Big[A^T_\alpha,\partial^\mu A^T_\alpha\Big],A^T_\nu\Big]^b(y)\Big\}.\end{aligned} \tag{5.92}$$

The second integrand of (5.90) can be evaluated in a similar way to obtain

$$\begin{aligned}\mathcal{H} = -\int \mathrm{d}^4x\,\mathrm{d}^4y\,\Big\{&2F^a_{\mu\nu}(x)\delta^{ab}K_0(x,y)\Big[A^T_\mu,\frac{1}{\Box}\Big[A^T_\beta,\partial^\beta A^T_\nu\Big]\Big]^b(y)\\&-F^a_{\mu\nu}(x)\delta^{ab}K_0(x,y)\Big[A^T_\mu,\frac{1}{\Box}\Big[A^T_\beta,\partial^\beta A^T_\nu\Big]\Big]^b(y)\Big\}.\end{aligned} \tag{5.93}$$

For the third integrand we see that the last two terms become commutator and we also introduce the value of $K_1^{T\,ab}(x,y)$ to give

$$\mathcal{I} = 2\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4w\,\Big\{F^a_{\mu\nu}(x)\big(\partial^x_\lambda K_0(x,w)A^T_\lambda(w)K_0(w,y)\big)^{ab}[A^T_\mu,A^T_\nu]^b(y)\Big\}. \tag{5.94}$$

Writing the colour indices explicitly in the above equation leads to

$$\mathcal{I} = 2\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4w\,\Big\{F^a_{\mu\nu}(x)\partial^x_\lambda\delta^{ac}K_0(x,w)(A^T_\lambda)^{cd}(w)\delta^{db}K_0(w,y)[A^T_\mu,A^T_\nu]^b(y)\Big\}, \tag{5.95}$$

where now using the property (3.57) we obtain

$$\begin{aligned}\mathcal{I} &= -2\int \mathrm{d}^4w\,\Big\{\Big(\frac{1}{\Box}\partial_\lambda F_{\mu\nu}\Big)^c(w)(A^T_\lambda)^{cd}(w)\Big(\frac{1}{\Box}[A^T_\mu,A^T_\nu]\Big)^d(w)\Big\}\\&= -2\int \mathrm{d}^4w\,\Big\{\Big(\frac{1}{\Box}\partial_\lambda F_{\mu\nu}\Big)^c(w)\Big[A^T_\lambda,\Big[\frac{1}{\Box}[A^T_\mu,A^T_\nu]\Big]\Big]^d(w)\Big\}.\end{aligned} \tag{5.96}$$

Likewise we now do the expansion for the fourth integrand where we see that $K_1^T(y,w)$ carries colour indices written in the form

$$\mathcal{J} = -\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4w\,\Big\{F^a_{\mu\nu}(x)\delta^{ab}K_0(x,y)K^{T\,bc}_{1\ \ ed}(y,w)(\partial^\alpha f_{\alpha\mu})^{ed}(w)(A^T_\nu)^c(y)\Big\}. \tag{5.97}$$

Using the property (D.2) we obtain

$$\mathcal{J} = 2\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4w\,\mathrm{d}^4u\,\Big\{F^a_{\mu\nu}(x)\delta^{ab}K_0(x,y)\partial^y_\lambda K_0(y,u)(A^T_\lambda)^{bc}_{ed}(u)K_0(u,w) \times (\partial^\alpha f_{\alpha\mu})^{ed}(w)(A^T_\nu)^c(y)\Big\}. \tag{5.98}$$

Integrating the above equation w.r.t. w and u leads to

$$\mathcal{J} = 2\int \mathrm{d}^4x\,\mathrm{d}^4y\,\Big\{F^a_{\mu\nu}(x)\delta^{ab}K_0(x,y)\Big(\frac{1}{\Box}\Big[A^T_\lambda,\partial_\lambda A^T_\mu\Big]\Big)^{bc}(y)(A^T_\nu)^c(y)\Big\}. \tag{5.99}$$

Using (3.2.5) we finally get

$$\mathcal{J} = 2\int \mathrm{d}^4x\,\mathrm{d}^4y\,\Big\{F^a_{\mu\nu}(x)\delta^{ab}K_0(x,y)\Big[\frac{1}{\Box}\Big[A^T_\lambda,\partial_\lambda A^T_\mu\Big],A^T_\nu\Big]^b(y)\Big\}. \tag{5.100}$$

In a similar way the fifth integrand can be simplified and we obtain

$$\mathcal{K} = 2\int \mathrm{d}^4x\,\mathrm{d}^4y\,\Big\{F^a_{\mu\nu}(x)\delta^{ab}K_0(x,y)\Big[A^T_\mu(y),\frac{1}{\Box}\Big[A^T_\lambda,\partial_\lambda A^T_\nu\Big]\Big]^b(y)\Big\}. \tag{5.101}$$

Putting all the terms from $\mathcal{G}$ to $\mathcal{K}$ together we find that the first integrand of (5.92) and (5.93) cancels with (5.100) and (5.101). In addition to this, to order g^2 there is also the contribution from the term $\mathcal{P}_2$ defined in (4.134) such that the net contribution is given by

$$\begin{aligned} g\big(\mathcal{P}_1-\mathcal{Z}_1\big)+g^2\mathcal{P}_2 = &\int \mathrm{d}^4x\,\mathrm{d}^4y\,\Big\{F^a_{\mu\nu}(x)\delta^{ab}K_0(x,y)\Big[\frac{1}{\Box}\Big[A^T_\alpha,\partial_\mu A^T_\alpha\Big],A^T_\nu\Big]^b(y)\Big\} \\ &+\int \mathrm{d}^4x\,\mathrm{d}^4y\,\Big\{F^a_{\mu\nu}(x)\delta^{ab}K_0(x,y)\Big[A^T_\mu,\frac{1}{\Box}\Big[A^T_\beta,\partial_\beta A^T_\nu\Big]\Big]^b(y)\Big\} \\ &-2\int \mathrm{d}^4w\,\Big\{\Big(\frac{1}{\Box}\partial_\lambda F_{\mu\nu}\Big)^c(w)\Big[A^T_\lambda,\Big[\frac{1}{\Box}[A^T_\mu,A^T_\nu]\Big]\Big]^d(w)\Big\} \\ &-\frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\Big\{[A^T_\mu,A^T_\nu]^a(x)\,\delta^{ab}K_0(x,y)\,[A^T_\mu,A^T_\nu]^b(y)\Big\}. \end{aligned} \tag{5.102}$$

At this order we can simply perform the substitutions to the above result to get the contribution to order F^4 and obtain

$$
\begin{aligned}
\mathcal{Z}_2 = &\int \mathrm{d}^4x \, \Big(\frac{1}{D^2}F_{\mu\nu}\Big)^a(x)\Big[\frac{1}{D^2}\Big[\Big(\frac{1}{D^2}D_\sigma F^{\sigma\alpha}\Big), D^\mu\Big(\frac{1}{D^2}D_\tau F^{\tau\alpha}\Big)\Big], \Big(\frac{1}{D^2}D_\rho F^{\rho\nu}\Big)\Big]^a(x) \\
&+ \int \mathrm{d}^4x \, \Big(\frac{1}{D^2}F_{\mu\nu}\Big)^a(x)\Big[\Big(\frac{1}{D^2}D_\sigma F^{\sigma\mu}\Big), \frac{1}{D^2}\Big[\Big(\frac{1}{D^2}D_\rho F^{\rho\beta}\Big), D^\nu\Big(\frac{1}{D^2}D_\tau F^{\tau\beta}\Big)\Big]\Big]^a(x) \\
&- 2\int \mathrm{d}^4x \, \Big(\frac{1}{D^2}D_\lambda F^{\mu\nu}\Big)^a(x)\Big[\Big(\frac{1}{D^2}D_\alpha F^{\alpha\lambda}\Big), \frac{1}{D^2}\Big[\Big(\frac{1}{D^2}D_\sigma F^{\sigma\mu}\Big), \Big(\frac{1}{D^2}D_\rho F^{\rho\nu}\Big)\Big]\Big]^a(x) \\
&- \frac{1}{2}\int \mathrm{d}^4x \, \frac{1}{D^2}\Big[\Big(\frac{1}{D^2}D_\sigma F^{\sigma\mu}\Big), \Big(\frac{1}{D^2}D_\rho F^{\rho\nu}\Big)\Big]^a(x)\Big[\Big(\frac{1}{D^2}D_\tau F^{\tau\mu}\Big), \Big(\frac{1}{D^2}D_\alpha F^{\alpha\nu}\Big)\Big]^a(x)\,,
\end{aligned}
\tag{5.103}
$$

which is manifestly gauge invariant. Hence we see that the non-abelian mass term has an expansion in terms of the gauge invariant combinations of the field strengths,

$$
\mathcal{M}^2 = -\frac{1}{2}\int \mathrm{d}^4x \, F^a_{\mu\nu}(x)\Big(\frac{1}{D^2}F_{\mu\nu}\Big)^a(x) + g(\mathcal{Z}_1 - \mathcal{Y}_1) + g^2(\mathcal{Z}_2 - \mathcal{Y}_2) + \cdots\,, \tag{5.104}
$$

where $\mathcal{Z}_1 - \mathcal{Y}_1$ corresponds to Zwanziger's expression (4.1) and $\mathcal{Z}_2 - \mathcal{Y}_2$ are the new terms given by (5.80) and (5.103). These expansions are not unique and have ambiguities as discussed in Section 5.2. What we have presented here are the terms closest in form to the Zwanziger expansion (4.1) but other ways of representing these gauge invariant quantities exist.

Chapter 6

Conclusions and Outlook

In this thesis we have investigated the role of gauge invariance in understanding the physical configurations of gauge theories and how it can be used to construct gauge invariant objects. In the first chapter we gave a brief introduction to the standard model of particle physics, highlighting on the role of symmetries in any gauge theory. The mathematical formalism needed to describe these particles and their interactions is encoded by field theory. We then reviewed the role of symmetries in field theory primarily focussing on the internal symmetries which underly our descriptions of the forces of nature. They lead to conservation laws and to a variety of relations among Green's functions, referred to as Ward identities. These symmetries are important in particle physics. The consequences of gauge invariance were then discussed briefly in terms of Ward identities.

In Chapter 2 we have introduced gauge theories. For the abelian gauge theory we stated the Feynman rules (in Appendix A.1) that are derived from the QED Lagrangian and explained the regularisation procedure. To perform loop calculations in QED, we showed how using dimensional regularisation with $D = 4 \to 4-2\varepsilon$ the divergences are exposed as pole terms in ε. Dimensional regularisation is often preferred over other methods as it ensures gauge invariance and the validity of the

Ward identity to all orders of perturbation theory. To absorb the singularities and obtain an UV-finite Green's function we introduced the concept of renormalisation, preference being given to the on-shell scheme. Various examples on the use of the Ward identity were then discussed which included constraining the tensor structure of the vacuum polarisation of the theory. Regularisation independent consistency conditions initially proposed by Wu in QCD [49] which preserve gauge invariance were our next topic. We have seen that in order to maintain gauge invariance for any regularisation, Wu's conditions (2.54)-(2.59) must always hold. The Ward identity, $q^\mu \Pi_{\mu\nu}(q) = 0$, was verified in Wu's scheme where an arbitrary regularisation scheme was used. Later Wu's conditions were verified in both dimensional regularisation and Pauli-Villars. We noted the connection between translational invariance and Wu's integrals in dimensional regularisation. In the end it was shown that all of Wu's identities can be derived using translational invariance. Since gauge invariance is independent of the number of space-time dimensions, dimensional regularisation is by construction a gauge invariant regularisation. However, it has its own drawbacks. An extension of our work would be to test other methods which should work using Wu's approach. These can be applied, perhaps, to physical process in three dimensions where divergences get worse. The infrared problem which is not considered by Wu is another possible extension.

Non-abelian gauge theories were introduced in Chapter 3 where properties of Lie groups were discussed and various representations presented. Considering the gluonic fields, $F^a_{\mu\nu}$ in the fundamental representation we have shown how they transformed in the adjoint representation of the group. We then built up tensor product representation for the fields in the adjoint representation. We generalised this procedure to higher order and therefore, built up higher representations by taking tensor product of the tensor product. Some new identities were derived in Section 3.2.5

that were key for later chapters of the thesis. This was then followed by a description of how physical configurations can be constructed in a gauge theory using the dressing procedure as a guiding principle. Using the dressing approach we have explored a wider class of physical gluonic configurations in SU(N) gauge theories. We have seen that a physical particle corresponds to an appropriately dressed, gauge invariant field. This is a description of a matter field dressed by an electro-magnetic field. In particular, this dressing procedure explored an abelian gauge structure within the non-abelian gauge theory thus allowing for the succinct description of the mass term. Because of the fact that the non-abelian field strength is not gauge invariant we used two possible routes to construct gauge invariant expressions for the field strength. The first choice was to use the dressing procedure as discussed in Section 3.3 which allowed us to define the dressed field strength $F^h_{\mu\nu}$ (3.84) and the other was the construction of an alternative field strength $\mathcal{F}_{\mu\nu}$ (3.83) that was of an abelian form but gauge invariant in the non-abelian theory. It was the interplay between these two field strengths that played an important role in the construction of the mass term that was the core of our calculation.

In Chapter 4 we investigated how such mass terms can be constructed in SU(N) gauge theories. We have introduced Zwanziger's expansion [98] where gauge invariance is maintained order by order by the use of the inverse covariant Laplacian. The expansion (4.1) has been given by Zwanziger but no derivation or discussion was provided. We have presented some of the properties of the inverse covariant Laplacian and provided the proofs of the results in Section 4.3.1. The important role of dressings lies at the heart of our analysis and has a significance beyond this application. This enabled us to factorise the dressed mass into the product of two separate gauge invariant structures written in terms of field strengths (4.7). However Zwanziger's first term in (4.1) does not factorise out. So working in the adjoint representation we then identified the central role of the dressing and have shown how the dressed field $K^{ab}(x,y)$ (4.92) factorises into the product of two separate gauge invariant states (4.103). This then allowed us to decompose Zwanziger's expansion into the sum of an adjoint dressing and an operator $\mathcal{Q}$

$$-\frac{1}{2}\int \mathrm{d}^4x\; F^a_{\mu\nu}(x)\Big(\frac{1}{D^2}F^{\mu\nu}\Big)^{a}\!(x) = -\frac{1}{2}\int \mathrm{d}^4x\; F^{h\,a}_{\mu\nu}(x)\Big(\frac{1}{\Box}F^{h\,\mu\nu}\Big)^{a}\!(x) + \mathcal{Q}\,, \qquad (6.1)$$

which were separately gauge invariant. This last operator, as we have seen, had a perturbative expansion where the first order term $\mathcal{Q}^{ab}_1(x,y)$ (4.123) was purely transverse, however the second order term $\mathcal{Q}^{ab}_2(x,y)$ (4.126) was not, as it contained mixed transverse/longitudinal components. Using the field strength factorisation we have then shown how the non-abelian mass term can be decomposed into the sum of an adjoint dressing and the gauge invariant term $\mathcal{P}$ as

$$-\frac{1}{2}\int \mathrm{d}^4x\; \mathcal{F}^a_{\mu\nu}(x)\Big(\frac{1}{\Box}\mathcal{F}^{\mu\nu}\Big)^{a}(x) = -\frac{1}{2}\int \mathrm{d}^4x\; F^{h\,a}_{\mu\nu}(x)\Big(\frac{1}{\Box}F^{h\,\mu\nu}\Big)^{a}(x) + \mathcal{P}\,. \qquad (6.2)$$

These two descriptions were then connected by eliminating the common factor constructed to give the mass term

$$\mathcal{M}^2 = -\tfrac{1}{2}\int \mathrm{d}^4x\; F^a_{\mu\nu}(x)\Big(\frac{1}{D^2}F_{\mu\nu}\Big)^{a}(x) + \mathcal{P} - \mathcal{Q}\,, \qquad (6.3)$$

where the operators $\mathcal{P}$ and $\mathcal{Q}$ are calculated to the lowest orders in perturbation theory.

Using this succinct formula we then, in Chapter 5, recovered Zwanziger's expansion up to cubic in the powers of field strength. This however exposed ambiguities in the whole construction. In Section 5.3 we then derived for the first time, an explicit expression that was quartic in the field strengths that corresponded to $\mathcal{Z}_2 - \mathcal{Y}_2$ see equation (5.103) and (5.80). While this was relatively straightforward at the F^3 level however at the F^4 level the presence of mixed transverse/longitudinal components made this derivation non-trivial. Using the same procedure as for F^3 we identified the transverse residue of the operator at an appropriate order by ensuring that no TL or LL components survive. Reinstating the higher order modifications for both $\mathcal{Y}_2$ (5.80) and $\mathcal{Z}_2$ (5.103) and performing the substitutions at this order we arrived at our result.

The methods used in this thesis, we feel, can give us much insight into the construction of gauge invariant configurations that, we hope, can be applicable to a wider area. We may be able to use these ideas to construct dressings for hadronic states which may have a good overlap with the ground state.

Appendix A

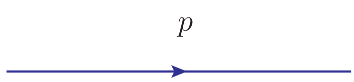
Calculation Techniques in QED

A.1 Feynman Diagrams

Feynman diagrams are a form of shorthand representation that tell us the calculation necessary for an interaction process. This is a key technique in field theory that helps in calculating scattering amplitudes [44] in various processes. Energy and momentum are conserved during every interaction and hence at every vertex. In a Feynman diagram, the propagator always occurs as an internal line. However the wave functions representing the physical particles are always represented by the external lines which are introduced later. We present here some of the Feynman rules for QED.

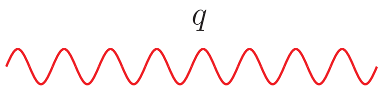
Feynman rules:

(1) Fermion (Dirac) Propagator:



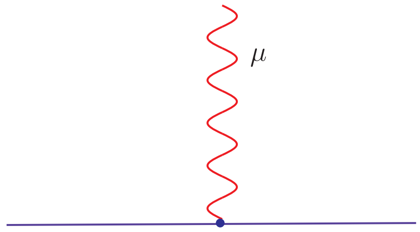
$$\longrightarrow \quad \frac{i}{\not{p} - m + i\epsilon}$$

(2) Photon Propagator (Feynman gauge):



$$\longrightarrow \quad \frac{-ig^{\mu\nu}}{q^2 + i\epsilon}$$

(3) Vertex:



$$\longrightarrow \quad ie\gamma^{\mu}$$

(4) For every closed fermionic loop multiply by a factor of -1.

(5) Integrate over every independent momentum with the measure $\frac{d^4p}{2\pi^4}$.

(6) Include symmetry factors.

These Feynman diagrams are used to calculate the cross-sections of the various interactions generally represented by the elements of S-matrix.

A.2 S-Matrix Formulation

In this formalism, $|i\rangle$ denotes an initial state long before the scattering occurs, $t_i = -\infty$. The particles do not interact. Long after the scattering has occured, $t_f = \infty$, the particles result into many different final states $|f\rangle$. The S-Matrix by definition relates state at $t = -\infty$ to that at $t = \infty$ by the simple relation:

$$|\phi(t = \infty)\rangle = \hat{S}|\phi(t = -\infty)\rangle\,. \tag{A.1}$$

The transition probability that after scattering the system is in state $|f\rangle$ at $t = \infty$ is given by:

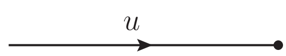
$$|S_{fi}|^2 = |\langle f|\phi(t=\infty)\rangle|^2 = |\langle f|\hat{S}|\phi(t=-\infty)\rangle|^2 \,. \tag{A.2}$$

To pass from Feynman diagram to S-Matrix elements there is a simple algorithm. One does the following in the order given:

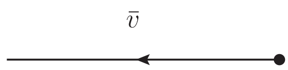
- Amputate all external legs, that is remove all external propagators (or equivalently, multiply each external leg by the inverse propagator).

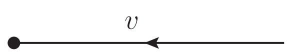
- Put all the external momenta onshell.

Then on the fermionic legs that have been cut off, for particles with momentum p and spin s, multiply by

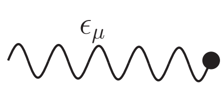
- $u(p,s)$ for an incoming electron $\longrightarrow$ 

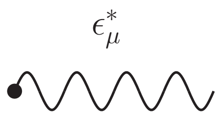
- $\bar{u}(p,s)$ for an outgoing electron $\longrightarrow$ 

- $\bar{v}(p,s)$ for an incoming positron $\longrightarrow$ 

- $v(p,s)$ for an outgoing positron $\longrightarrow$ 

On the amputated photon legs, for photons with momentum k and helicity λ, multiply by

- $\epsilon_\mu(k,\lambda)$ for an incoming photon $\longrightarrow$ 

- $\epsilon^*_\mu(k,\lambda)$ for an outgoing photon $\longrightarrow$ 

These rules provide iM and the transition amplitude is, $S_{fi} = -i(2\pi)^4\delta^4(p_f - p_i)M$ $= \big($ sum of all connected, amputated Feynman diagrams with p_i incoming and p_f outgoing $\big)$.

The probability of transition from initial state to final state is $|S_{fi}|^2$. Thus Feynman diagrams depicted earlier are used for computing amplitude M_{fi} for an arbitrary process in QED. In order to calculate unpolarized cross-section for QED processes the general procedure is:

1. Draw the diagram of desired process.

2. Use the Feynman rules to write amplitude M_{fi}

3. Square the amplitude M^2 and average or sum over spins using the completeness relations.

$$\sum_s u^s(p)\bar{u}^s(p) = \not{p} + m\,. \tag{A.3}$$

$$\sum_s v^s(p)\bar{v}^s(p) = \not{p} - m\,. \tag{A.4}$$

4. Evaluate the traces using trace theorems; collect terms and simplify.

5. Specialize to a particular frame of reference and draw a picture of kinematic variables in that frame. Express all 4-momentum vectors in terms of E and θ.

6. Plug M^2 into cross section formula $i.e.$ $\frac{d\sigma}{d\Omega}$ and then calculate $d\sigma$ by integration.

A.3 Gamma Matrix Algebra

When we evaluate integrals in D dimensions we encounter expressions with the gamma matrices γ^μ involved and the Clifford algebra is then D dimensional.

- $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad \gamma^\mu\gamma_\mu = D, \quad \gamma^\mu\gamma^\nu = g^{\mu\nu}, \quad \not{a}\not{a} = a^2, \quad \not{a}\not{b} + \not{b}\not{a} = 2a.b$

- $\gamma^\mu \not{a}\not{b}\not{c}\gamma_\mu = -2\not{c}\not{b}\not{a} + (4-D)\not{a}\not{b}\not{c}$

- $\gamma^\mu \not{a}\not{b}\gamma_\mu = 4a.b - (4-D)\not{a}\not{b}, \quad \gamma^\mu \not{a}\gamma_\mu = (2-D)\not{a}\,.$

One can also calculate the traces of γ matrices

- $\mathrm{tr}\,(\mathbf{1}) = 4, \quad \mathrm{tr}\,(\text{any odd no. of } \gamma' s) = 0, \quad \mathrm{tr}\,(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}, \quad \mathrm{tr}\,(\not{a}\not{b}) = 4a.b$

- $\mathrm{tr}\,(\not{a}\not{b}\not{c}\not{d}) = 4\Big((a.b)(c.d) - (a.c)(b.d) + (a.d)(b.c)\Big)\,.$

The definition of γ^5 in 4-dimension is

- $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ with $[\gamma^5, \gamma^\nu] = 0$.

In $D \neq 4$ dimensions, the definition of γ^5 is more complicated, but in this thesis we did not need to use γ^5.

A.4 Analysis of Dimensions

We are working in momentum space in four dimensions and can determine the dimension of a field by analysing the free action

$$S = \int \mathrm{d}^4x\, \mathcal{L}\,, \tag{A.5}$$

where dimension of Lagrangian $[\mathcal{L}] = 4$ and the dimension of coupling constant $[e] = 0$. However in D dimension since $[\mathcal{L}] = D$ and using $[\not{\partial}] = 1$, $[m] = 1$ we find, $[\psi] = \frac{D-1}{2}$ and $[A] = \frac{D-2}{2}$.

Considering the interaction term $\mathcal{L}_{int} = \bar{\psi}e\gamma^\mu\psi A_\mu$ one can easily find the dimension of coupling constant in D dimension which varies as

$$[e] \to \mu^\varepsilon e, \quad \varepsilon = \frac{4-D}{2}\,, \tag{A.6}$$

where e is dimensionless and μ^ε is the new scale[1] that carries all the information about dimensions. In D dimensions the vertex is modified as $ie\gamma_\mu \to ie\mu^\varepsilon\gamma_\mu$.

[1]The arbitrary mass parameter μ is introduced to maintain the coupling constant dimensions of the integrals.

Appendix B

One-Loop Integrals

B.1 Feynman Parameterisation

To evaluate the loop integrals that contain the products of denominators of the form $\mathcal{D} = (p^2 - m^2).[(p+q)^2 - m^2]$ Feynman introduced the following identities to combine these products into one single denominator

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA + (1-x)B]^2}\,, \tag{B.1}$$

$$\frac{1}{ABC} = 2\int_0^1 dx \quad x\int_0^1 dy \frac{1}{[(A-B)xy + (B-C)x + C]^3}\,, \tag{B.2}$$

where the integration parameters x and y are called Feynman parameters.

All the momentum integrals are calculated in Euclidean space as it makes sure that all the poles disappear in the propagators. The type of integrals that we would be encountering will be of the form:

$$\begin{aligned} I^r(a) &= \int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2 - a^2 + i\epsilon)^r} \\ &= \frac{1}{(2\pi)^D}\int dk^0 \int d^{D-1}\mathbf{k} \frac{1}{(k_0^2 - \mathbf{k}^2 - a^2 + i\epsilon)^r}\,. \end{aligned} \tag{B.3}$$

To do this integration we will use integration in the plane of the complex variable k_0. The deformation of the contour corresponds to the so called Wick's rotation *i.e.*

$$k_0 \to ik_E^0 , \qquad \mathbf{k} = \mathbf{k}_E , \quad \int_{-\infty}^{\infty} dk^0 \to i \int_{-\infty}^{\infty} dk_E^0 . \tag{B.4}$$

which is shown in the corresponding diagram of Fig. B.1.

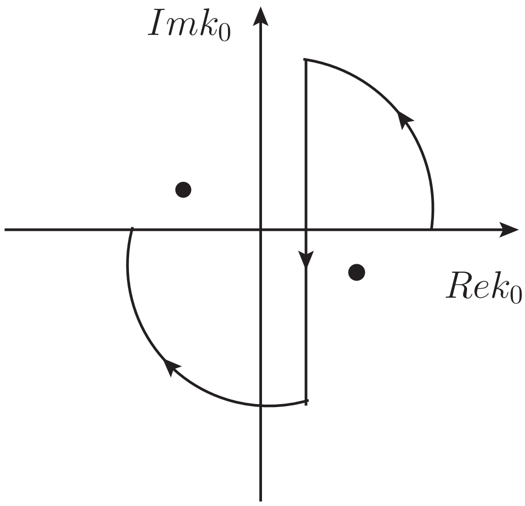


Figure B.1: The k_0 integration plane

Going from Minkowski space to Euclidean space the integral becomes:

$$\begin{aligned} I^r(a) &= \frac{i}{(2\pi)^D} \int_{-\infty}^{\infty} dk_E^0 \int d^{D-1}\mathbf{k}_E \frac{1}{({k_E^0}^2 - \mathbf{k}_E^2 - a^2 + i\epsilon)^r} \\ &= \frac{i}{(2\pi)^D} \int_{-\infty}^{\infty} dk_E^0 \int d^{D-1}\mathbf{k}_E \frac{(-1)^r}{({k_E^0}^2 + \mathbf{k}_E^2 + a^2 - i\epsilon)^r} , \end{aligned} \tag{B.5}$$

where $k_E = (k_E^0, \mathbf{k}_E) \Longrightarrow k_E^2 = {k^0}_E^2 + \mathbf{k}_E^2$ (*i.e.* while going from Minkowski space to Eucledian space we end up with all positive diagonal terms).

$$\begin{aligned} I^r(a) &= \frac{i}{(2\pi)^D} \int d^D k_E \frac{(-1)^r}{(k_E^2 + a^2 - i\epsilon)^r} \\ &= \frac{i}{(2\pi)^D} (-1)^r \int dk_E \quad k_E^{D-1} d\Omega_{D-1} \frac{1}{(k_E^2 + a^2 - i\epsilon)^r} , \end{aligned} \tag{B.6}$$

where $d\Omega_{D-1}$ is the solid angle. Making use of the relation between solid angle Ω and gamma Γ functions

$$\int d\Omega_{D-1} = \frac{2(\pi)^{\frac{D}{2}}}{\Gamma(\frac{D}{2})}\,, \tag{B.7}$$

we end up with[1]:

$$\begin{aligned} I^r(a) &= \frac{1}{(2\pi)^D}\int d^Dk \frac{1}{(k^2-a^2+i\epsilon)^r} \\ &= (-1)^r \frac{i}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(r-\frac{D}{2})}{\Gamma(r)} (a^2-i\epsilon)^{\frac{D}{2}-r}\,. \end{aligned} \tag{B.8}$$

B.2 Derivation of Formula for Wu's Identity

In this appendix we will show how to obtain equation (2.85) that can be used to generate all of the Wu's identities. To start with we first rewrite Wu's identites (2.49), (2.51) in dimensional regularisation as

$$\begin{aligned} \int \frac{d^Dk}{(2\pi)^D} \frac{k_\mu k_\nu}{(k^2-M^2)^2} &= \frac{1}{2} g_{\mu\nu} \int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2-M^2)}\,, \\ \int \frac{d^Dk}{(2\pi)^D} \frac{k_\mu k_\nu}{(k^2-M^2)^3} &= \frac{1}{4} g_{\mu\nu} \int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2-M^2)^2}\,. \end{aligned} \tag{B.9}$$

The above equations follow some pattern so the equation for nth term can be written as:

$$\begin{aligned} \int \frac{d^Dk}{(2\pi)^D} \frac{k_\mu k_\nu}{(k^2-M^2)^n} &= \frac{1}{X} g_{\mu\nu} \int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2-M^2)^{n-1}} \\ I^n_{\mu\nu} &= \frac{1}{X} g_{\mu\nu} I^{n-1}\,, \end{aligned} \tag{B.10}$$

where we have identified the above integrals with the Wu's identities. In (B.10) X is a constant that we want to evaluate for which we contract (B.10) by $g^{\mu\nu}$ on both

[1] In going from (B.6) to (B.8) we have dropped the subscript E over the momentum k_E.

sides to obtain:

$$g^{\mu\nu}\int\frac{d^Dk}{(2\pi)^D}\,\frac{k_\mu k_\nu}{(k^2-M^2)^n}=\frac{D}{X}\int\frac{d^Dk}{(2\pi)^D}\,\frac{1}{(k^2-M^2)^{n-1}}\,, \tag{B.11}$$

where $D=g_{\mu\nu}g^{\mu\nu}$. Adding and subtracting M^2 in the above equation we obtain:

$$\int\frac{d^Dk}{(2\pi)^D}\left\{\frac{1}{(k^2-M^2)^{n-1}}+\frac{M^2}{(k^2-M^2)^n}\right\}=\frac{D}{X}\int\frac{d^Dk}{(2\pi)^D}\,\frac{1}{(k^2-M^2)^{n-1}}\,. \tag{B.12}$$

Let now $\mathcal{R}$ be an arbitrary integral defined in the form:

$$\mathcal{R}^n=\int\frac{d^Dk}{(2\pi)^D}\,\frac{1}{(k^2-M^2)^n}\,. \tag{B.13}$$

Substituting the above equation into into (B.12) yields:

$$\Big(1-\frac{D}{X}\Big)\mathcal{R}^{n-1}=-M^2\mathcal{R}^n\,. \tag{B.14}$$

Using (B.8) we evaluate integrals $\mathcal{R}^{n-1}$ and $\mathcal{R}^n$ to obtain

$$\mathcal{R}^{n-1}=(-1)^{n-1}\frac{i}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(n-1-\frac{D}{2})}{\Gamma(n-1)}(M^2)^{\frac{D}{2}-n+1}\,, \tag{B.15}$$

and

$$\mathcal{R}^n=(-1)^n\frac{i}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(n-\frac{D}{2})}{\Gamma(n)}(M^2)^{\frac{D}{2}-n}\,. \tag{B.16}$$

Substituting the integrals (B.15) and (B.16) into (B.14) and using the properties of gamma function we obtain

$$1-\frac{D}{X}=\frac{n-1-\frac{D}{2}}{n-1}\,, \tag{B.17}$$

which upon simplification yields:

$$X = 2(n-1)\,. \tag{B.18}$$

Hence (B.10) becomes:

$$I^n_{\mu\nu} = \frac{1}{2(n-1)} g_{\mu\nu} I^{n-1}\,, \tag{B.19}$$

which is similar to (2.85) as discussed in Chapter 2.

Appendix C

The Dressed Potential

Using the equation for the dressed potential (3.66) and imposing the Landau condition $\partial^\mu A_\mu^h = 0$, one can calculate the power of the perturbative expansion to various orders in coupling. To start with we write the dressed vector potential in the form,

$$A_\mu^h = h^{-1} A_\mu h + \frac{1}{g} h^{-1} \partial_\mu h \,. \tag{C.1}$$

Substituting the values of h^{-1} and h from (3.69) and (3.70) we have,

$$\begin{aligned}
A_\mu^h = &\big\{1 + gv_1 + g^2\big(\tfrac{1}{2}v_1^2 + v_2\big) + g^3\big(\tfrac{1}{6}v_1^3 + \tfrac{1}{2}v_1v_2 + \tfrac{1}{2}v_2v_1 + v_3\big)\big\} \\
&\qquad \times A_\mu\big\{1 - gv_1 + g^2\big(\tfrac{1}{2}v_1^2 - v_2\big) - g^3\big(\tfrac{1}{6}v_1^3 - \tfrac{1}{2}v_1v_2 - \tfrac{1}{2}v_2v_1 + v_3\big)\big\} \\
&+ \frac{1}{g}\big\{1 + gv_1 + g^2\big(\tfrac{1}{2}v_1^2 + v_2\big) + g^3\big(\tfrac{1}{6}v_1^3 + \tfrac{1}{2}v_1v_2 + \tfrac{1}{2}v_2v_1 + v_3\big)\big\} \\
&\qquad \times g\big\{ - \partial_\mu v_1 + g\partial_\mu\big(\tfrac{1}{2}v_1^2 - v_2\big) - g^2\partial_\mu\big(\tfrac{1}{6}v_1^3 - \tfrac{1}{2}v_1v_2 - \tfrac{1}{2}v_2v_1 + v_3\big)\big\} \\
= &A_\mu - \partial_\mu v_1 + g\big\{[v_1, A_\mu] + \tfrac{1}{2}[\partial_\mu v_1, v_1] - \partial_\mu v_2\big\} \\
&\qquad + g^2\big\{\tfrac{1}{2}v_1^2 A_\mu + v_2 A_\mu + \tfrac{1}{2}A_\mu v_1^2 - A_\mu v_2 - v_1 A_\mu v_1 - \tfrac{1}{6}\partial_\mu(v_1 v_1^2) \\
&\qquad\quad + \tfrac{1}{2}(\partial_\mu v_1)v_2 + \tfrac{1}{2}v_1(\partial_\mu v_2) + \tfrac{1}{2}(\partial_\mu v_2)v_1 + \tfrac{1}{2}v_2(\partial_\mu v_1) - \partial_\mu v_3 \\
&\qquad\qquad + \tfrac{1}{2}v_1\partial_\mu v_1^2 - v_1(\partial_\mu v_2) - \tfrac{1}{2}v_1^2(\partial_\mu v_1^2) - v_2(\partial_\mu v_1)\big\} \,.
\end{aligned} \tag{C.2}$$

Using the identity,

$$\partial_\mu(AB) = (\partial_\mu A)B + A(\partial_\mu B)\,, \tag{C.3}$$

in (C.2) we find A^h_μ reduces to,

$$\begin{aligned}
A^h_\mu = A_\mu - \partial_\mu v_1 &+ g\big\{[v_1, A_\mu] + \tfrac{1}{2}[\partial_\mu v_1, v_1] - \partial_\mu v_2\big\} \\
&+ g^2\big\{\tfrac{1}{2}v_1^2 A_\mu + v_2 A_\mu + \tfrac{1}{2}A_\mu v_1^2 - A_\mu v_2 - v_1 A_\mu v_1 - \tfrac{1}{6}v_1^2(\partial_\mu v_1) \\
&\quad + \tfrac{1}{2}(\partial_\mu v_1)v_2 - \tfrac{1}{2}v_1(\partial_\mu v_2)\tfrac{1}{6}(\partial_\mu v_1)v_1^2\tfrac{1}{6}v_1(\partial_\mu v_1)v_1 + \tfrac{1}{2}(\partial_\mu v_2)v_1 \\
&\quad - \tfrac{1}{2}v_2(\partial_\mu v_1) + \tfrac{1}{2}v_1(\partial_\mu v_1)v_1 - \partial_\mu v_3\big\} \\
= A_\mu - \partial_\mu v_1 &+ g\big\{[v_1, A_\mu] + \tfrac{1}{2}[\partial_\mu v_1, v_1] - \partial_\mu v_2\big\} \\
&+ g^2\big\{\tfrac{1}{2}v_1[v_1, A_\mu] - \tfrac{1}{2}[v_1, A_\mu]v_1 + [v_2, A_\mu] + \tfrac{1}{2}[\partial_\mu v_2, v_1] \\
&\quad + \tfrac{1}{2}[\partial_\mu v_1, v_2] - \tfrac{1}{6}(\partial_\mu v_1)v_1^2 + \tfrac{1}{3}v_1(\partial_\mu v_1)v_1 - \tfrac{1}{6}v_1^2(\partial_\mu v_1) - \partial_\mu v_3\big\}\,.
\end{aligned} \tag{C.4}$$

Further simplification gives,

$$\begin{aligned}
A^h_\mu = A_\mu - \partial_\mu v_1 &+ g\big\{[v_1, A_\mu] + \tfrac{1}{2}[\partial_\mu v_1, v_1] - \partial_\mu v_2\big\} \\
&+ g^2\big\{[v_2, A_\mu] + \tfrac{1}{2}[v_1, [v_1, A_\mu]] + \tfrac{1}{2}[\partial_\mu v_1, v_2] \\
&\quad + \tfrac{1}{2}[\partial_\mu v_2, v_1] - \tfrac{1}{6}[v_1, [v_1, \partial_\mu v_1]] - \partial_\mu v_3\big\}\,.
\end{aligned} \tag{C.5}$$

Imposing Landau condition, $\partial_\mu A^h_\mu = 0$ in (C.5) yields

$$\begin{aligned}
0 = \partial\cdot A - \Box v_1 &+ g\Big\{\partial^\mu\Big([v_1, A_\mu] + \frac{1}{2}[\partial_\mu v_1, v_1]\Big) - \Box v_2\Big\} \\
&+ g^2\big\{\partial^\mu\big([v_2, A_\mu] + \tfrac{1}{2}[v_1, [v_1, A_\mu]] + \tfrac{1}{2}[\partial_\mu v_1, v_2] \\
&\quad + \tfrac{1}{2}[\partial_\mu v_2, v_1] - \tfrac{1}{6}[v_1, [v_1, \partial_\mu v_1]]\big) - \Box v_3\big\}\,.
\end{aligned} \tag{C.6}$$

We can solve for v_1,v_2 and v_3 to obtain

$$\begin{aligned} v_1 &= \tfrac{1}{\Box}\partial \cdot A \\ v_2 &= \tfrac{1}{\Box}\partial^\mu\big([v_1, A_\mu] + \tfrac{1}{2}[\partial_\mu v_1, v_1]\big) \\ v_3 &= \tfrac{1}{\Box}\partial^\mu\big([v_2, A_\mu] + \tfrac{1}{2}[v_1, [v_1, A_\mu]] + \tfrac{1}{2}[\partial_\mu v_1, v_2] \\ &\qquad + \tfrac{1}{2}[\partial_\mu v_2, v_1] - \tfrac{1}{6}[v_1, [v_1, \partial_\mu v_1]]\big)\,, \end{aligned} \tag{C.7}$$

and in general for $n \geq 1$ we find

$$v_{_n} = \frac{1}{\Box}\partial_\mu \mathcal{A}^\mu_{_{n-1}}\,. \tag{C.8}$$

This is the perturbative expansion for the dressing used in Chapter 3.

C.1 Decomposition of v_2

In this section we show the decomposition of v_2 into transverse (4.127) and longitudinal (4.128) components that has been used in Chapter 4. In the abelian theory it is easy to understand the decomposition of the vector potential into transverse and longitudinal components, however in the non-abelian theory this decomposition is not straightforward due to the presence of coupling constant g to various orders. In the non-abelian theory, the transverse vector potential is given by

$$A^h_\mu(x) = A_\mu(x) - \partial^\mu v_{_1}(x) + g\big(\mathcal{A}^\mu_{_1}(x) - \partial^\mu v_{_2}(x)\big) + \cdots\,. \tag{C.9}$$

In the above equation, we find to order g, a contribution from two terms namely

$$\mathcal{A}^\mu_{_1}(x) = [v_{_1}, A^\mu](x) + \frac{1}{2}[\partial^\mu v_{_1}, v_{_1}](x)\,, \tag{C.10}$$

and

$$v_2(x) = \frac{1}{\Box}(\partial \cdot \mathcal{A}_1)(x)\,. \tag{C.11}$$

Note however that in (C.10), $\mathcal{A}^\mu_1$ carries mixed longitudinal and transverse components in the vector potential A_μ that can be decomposed as

$$\mathcal{A}^\mu_1(x) = \mathcal{A}^{\mu^T}_1(x) + \mathcal{A}^{\mu^L}_1(x)\,, \tag{C.12}$$

where

$$\mathcal{A}^{\mu^T}_1(x) = [v_1, A^{\mu^T}](x) \qquad \text{and} \qquad \mathcal{A}^{\mu^L}_1(x) = \frac{1}{2}[v_1, \partial^\mu v_1](x)\,, \tag{C.13}$$

are respectively the transverse and longitudinal component of $\mathcal{A}^\mu_1$. Using this we can find the longitudinal and transverse contribution for v_2 defined in (C.11) such that the transverse component is

$$v^T_2(x) = \frac{1}{\Box}\partial \cdot \mathcal{A}^T_1(x) = \frac{1}{\Box}[\partial_\mu v_1, A^{\mu^T}](x)\,, \tag{C.14}$$

and

$$v^L_2(x) = \frac{1}{\Box}\partial \cdot \mathcal{A}^L_1(x) = \frac{1}{2\Box}\partial_\mu[v_1, \partial_\mu v_1](x)\,, \tag{C.15}$$

is the longitudinal component. One can explore how these decompositions can be carried out to higher orders but for the calculations presented in this thesis we shall only consider the decomposition to order g^2.

Appendix D

Properties of Lie Algebra

This appendix explains various definitions and the properties used in the calculations presented in Chapters 4 and 5.

D.1 Product and Commutator in Lie Algebra

We can set up a dictionary between our construction of mass term and Zwanziger via the modification

$$\left(\frac{1}{D^2}B\right)^{ab}(x)C^b(x) = \left[\left(\frac{1}{D^2}B\right)(x), C(x)\right]^a, \tag{D.1}$$

$$\begin{aligned}\left(\frac{1}{D^2}B\right)^{ab}_{\ \ cd}(x)\left(\frac{1}{D^2}C\right)^{cd}(x)D^b(x) &= \left[\left(\frac{1}{D^2}B\right)(x), \left(\frac{1}{D^2}C\right)(x)\right]^{ab}D^b(x) \\ &= \left[\left[\left(\frac{1}{D^2}B\right)(x), \left(\frac{1}{D^2}C\right)(x)\right], D(x)\right]^a,\end{aligned} \tag{D.2}$$

and

$$\begin{aligned}\left(\frac{1}{D^2}B\right)^{ab}(x)\left(\frac{1}{D^2}C\right)^{bc}(x)D^c(x) &= \left(\frac{1}{D^2}B\right)^{ab}(x)\left[\left(\frac{1}{D^2}C\right)(x), D(x)\right]^b \\ &= \left[\left(\frac{1}{D^2}B\right)(x), \left[\left(\frac{1}{D^2}C\right)(x), D(x)\right]\right]^a.\end{aligned} \tag{D.3}$$

These relations have been used explicitly in deriving the Zwanziger's next order leading term.

D.2 Discussion of Ambiguities in Transverse Field

Here we show that the expressions (4.132) and (5.19) are not unique. We start with (4.132) and use the perturbative expansion to order g

$$\begin{aligned}
\frac{1}{D^2}\Big(D_\beta F^{\beta\rho}\Big)(x) :=& \int \mathrm{d}^4y\ K(x,y)(D_\beta F^{\beta\rho})(y) \\
=& \int \mathrm{d}^4y\ \Big\{K_0(x,y)\partial^y_\beta F^{\beta\rho}(y) \\
&+ g\big(K_1(x,y)\partial^y_\beta F^{\beta\rho}(y) + K_0(x,y)A_\beta(y)F^{\beta\rho}(y)\big)\Big\} \\
=& \int \mathrm{d}^4y\ \Big\{-\partial^y_\beta K_0(x,y)F^{\beta\rho}(y) \\
&+ g\big(-\partial^y_\beta K_1(x,y)F^{\beta\rho}(y) + K_0(x,y)A_\beta(y)F^{\beta\rho}(y)\big)\Big\}\,.
\end{aligned} \tag{D.4}$$

In the similar way (5.19) can be expanded perturbatively to order g to yield

$$\begin{aligned}
D_\beta\Big(\frac{1}{D^2}F^{\beta\rho}\Big)(x) :=& D^x_\beta \int \mathrm{d}^4y\ K(x,y)F^{\beta\rho}(y) \\
=& \int \mathrm{d}^4y\ \Big\{\partial^x_\beta K_0(x,y)F^{\beta\rho}(y) \\
&+ g\big(A_\beta(x)K_0(x,y)F^{\beta\rho}(y) + \partial^x_\beta K_1(x,y)F^{\beta\rho}(y)\big)\Big\}\,.
\end{aligned} \tag{D.5}$$

Now we look at the difference between (D.4) and (D.5) to obtain:

$$\begin{aligned}
&\frac{1}{D^2}\Big(D_\beta F^{\beta\rho}\Big)(x) - D_\beta\Big(\frac{1}{D^2}F^{\beta\rho}\Big)(x) \\
&= g\int \mathrm{d}^4y\ \Big\{K_0(x,y)\big(A_\beta(y)-A_\beta(x)\big)F^{\beta\rho}(y)\Big\} \\
&\quad + g\int \mathrm{d}^4y\,\mathrm{d}^4z\ \Big\{K_0(x,z)\partial^z_\lambda K_0(z,y) - \partial^z_\lambda K_0(x,z)K_0(z,y)\Big\} \\
&\qquad \times \partial^z_\beta A_\lambda(z)F^{\beta\rho}(y) + O(g^2)\,.
\end{aligned} \tag{D.6}$$

This is clearly not zero.

D.3 Field Strength $(F^h_{\mu\nu})^a$ in the Adjoint Representation

In this section we derive the equation used in the calculations for the derivation of $\mathcal{Z}_2$ introduced in Chapter 5.

$$\begin{aligned}(F^h_{\mu\nu})^a := -2\ \mathrm{tr}\,(\tau^a F^h_{\mu\nu}) &= -2\ \mathrm{tr}\left\{\tau^a\big(F_{\mu\nu} + g[v_{_1}, F_{\mu\nu}]\big)\big)\right\} \\ &= F^a_{\mu\nu} - 2g\ \mathrm{tr}\left(\tau^a[v_{_1}, F_{\mu\nu}]\right) \\ &= F^a_{\mu\nu} - 2gv^b_{_1}F^c_{\mu\nu}\ \mathrm{tr}\left(\tau^a[\tau^b,\tau^c]\right) \\ &= F^a_{\mu\nu} - gF^c_{\mu\nu}(v_{_1})^{ca}\,. \end{aligned} \tag{D.7}$$

D.4 Calculation of $A^{(1)T}_\mu$

Using (4.4) we can write the dressed vector potential to all orders as

$$A^h_\mu = \frac{1}{\Box}\partial^\nu \mathcal{F}_{\nu\mu}\,, \tag{D.8}$$

where substituting $\mathcal{F}_{\nu\mu} = \partial_\nu\mathcal{A}_\mu - \partial_\mu\mathcal{A}_\nu = f_{\nu\mu} + g(\partial_\nu\mathcal{A}^\mu_{_1} - \partial_\mu\mathcal{A}^\nu_{_1})$ in (D.8) we obtain to order g

$$\begin{aligned} A^h_\mu &= \frac{1}{\Box}\partial^\nu\left\{f_{\nu\mu} + g(\partial_\nu\mathcal{A}^\nu_{_1} - \partial_\mu\mathcal{A}^\nu_{_1})\right\} \\ &= \frac{1}{\Box}\partial^\nu f_{\nu\mu} + g\Big(\frac{1}{\Box}\partial^\nu f^{(1)}_{\nu\mu}\Big)\,. \end{aligned} \tag{D.9}$$

The dressed vector potential can also be written in a manifestly gauge invariant way to order g as

$$A^h_\mu = A^{T}_\mu + gA^{(1)T}_\mu\,. \tag{D.10}$$

Comparing (D.9) and (D.10) we get

$$A_\mu^{(1)T} = \frac{1}{\Box}\Big(\partial^\nu f_{\nu\mu}^{(1)}\Big) = \frac{1}{\Box}\partial^\nu\Big(\partial_\nu \mathcal{A}_{_1}^\mu - \partial_\mu \mathcal{A}_{_1}^\nu\Big)\,, \tag{D.11}$$

where $\mathcal{A}_{_1}^\mu = [v_{_1}, A^\mu] + \frac{1}{2}[\partial^\mu v_{_1}, v_{_1}]$. In order to calculate $A_\mu^{(1)T}$ we need to evaluate $\partial_\nu \mathcal{A}_{_1}^\mu$ and $\partial_\mu \mathcal{A}_{_1}^\nu$ given by

$$\partial_\mu \mathcal{A}_{_1}^\nu = [\partial_\mu v_{_1}, A^\nu] + [v_{_1}, \partial_\mu A^\nu] + \tfrac{1}{2}[\partial_\mu \partial^\nu v_{_1}, v_{_1}] + \tfrac{1}{2}[\partial^\nu v_{_1}, \partial_\mu v_{_1}]\,, \tag{D.12}$$

and

$$\partial_\nu \mathcal{A}_{_1}^\mu = [\partial_\nu v_{_1}, A^\mu] + [v_{_1}, \partial_\nu A^\mu] + \tfrac{1}{2}[\partial_\nu \partial^\mu v_{_1}, v_{_1}] + \tfrac{1}{2}[\partial^\mu v_{_1}, \partial_\nu v_{_1}]\,. \tag{D.13}$$

Substituting (D.12) and (D.13) into (D.11) we have

$$A_\mu^{(1)T} = \frac{1}{\Box}\partial^\nu\Big([\partial_\nu v_{_1}, A^\mu] - [\partial_\mu v_{_1}, A^\nu] + [v_{_1}, f_{\nu\mu}] + [\partial^\mu v_{_1}, \partial_\nu v_{_1}]\Big)\,. \tag{D.14}$$

In the above equation, decomposing A^μ into transverse and longitudinal components we obtain

$$\begin{aligned} A_\mu^{(1)T} &= \frac{1}{\Box}\partial^\nu\Big([\partial_\nu v_{_1}, A_\mu^T] + [\partial_\nu v_{_1}, \partial_\mu v_{_1}] - [\partial_\mu v_{_1}, A_\nu^T] + [v_{_1}, f_{\nu\mu}]\Big) \\ &= \frac{1}{\Box}\Big\{[\Box v_{_1}, A_\mu^T] + [\partial_\alpha v_{_1}, \partial^\alpha A_\mu^T] + [\Box v_{_1}, \partial_\mu v_{_1}] + [\partial_\alpha v_{_1}, \partial_\alpha \partial_\mu v_{_1}] \\ &\qquad - [\partial_\alpha \partial_\mu v_{_1}, A_\alpha^T] + [\partial_\alpha v_{_1}, f_{\alpha\mu}] + [v_{_1}, \partial^\alpha f_{\alpha\mu}]\Big\}\,, \end{aligned} \tag{D.15}$$

which is used in Chapter 5.

D.5 Cancellation of LL and TL components

In this appendix we will show that all the TL and LL terms from (5.89) when added to (5.85) cancel. To begin with we shall now work on each integrand of (5.85) and (5.89) and identify the TL and LL components. We hope to be able to show that $g(\mathcal{P}_1 - \mathcal{Z}_1) + g^2\mathcal{Z}_2$ has no LL or TL contribution so that the remainder is just TT and hence can be identified as the next gauge invariant term in the expansion of mass term in addition to $\mathcal{P}_2$.

We start with (5.85) and substitute (D.15) into the second line of (5.85) to obtain the T/L decomposition

$$\begin{aligned}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\Big\{F^a_{\mu\nu}\delta^{ab}K_0(x,y)\frac{1}{\Box}\Big([\Box v_1, A^T_\mu]^{bc} + [\partial_\alpha v_1, \partial^\alpha A^T_\mu]^{bc} + [\Box v_1, \partial_\mu v_1]^{bc}\\ + [\partial_\alpha v_1, \partial_\alpha\partial_\mu v_1]^{bc} - [\partial_\alpha\partial_\mu v_1, A^T_\alpha]^{bc} + [\partial_\alpha v_1, f_{\alpha\mu}]^{bc}\\ + [v_1, \partial^\alpha f_{\alpha\mu}]^{bc}\Big)(y)(A^T_\nu)^c(y)\Big\}\,.\end{aligned} \tag{D.16}$$

We now consider the fourth line of (5.89). From the colour indices involved in the expression our suspicion is that the fourth line of (5.89) and (D.16) should cancel. For this we integrate the fourth line of (5.89) with respect to w and obtain

$$-\int \mathrm{d}^4x\,\mathrm{d}^4y\,\Big\{F^a_{\mu\nu}(x)\delta^{ab}K_0(x,y)\frac{1}{\Box}\Big(\partial^\alpha[A_\alpha, A_\mu]^{bc} + [A_\alpha, f_{\alpha\mu}]^{bc}\Big)(y)(A^T_\nu)^c(y)\Big\}\,, \tag{D.17}$$

The above equation contains commutators which needs to be simplified. These commutators have the structure

$$\begin{aligned}\partial^\alpha[A_\alpha, A_\mu] + [A_\alpha, f_{\alpha\mu}] = [A^T_\alpha, \partial^\alpha A^T_\mu] + [\Box v_1, A^T_\mu] + [\partial_\alpha v_1, \partial^\alpha A^T_\mu] + [A^T_\alpha, \partial^\alpha\partial_\mu v_1]\\ + [\Box v_1, \partial_\mu v_1] + [\partial_\alpha v_1, \partial_\alpha\partial_\mu v_1] + [A^T_\alpha, f_{\alpha\mu}] + [\partial_\alpha v_1, f_{\alpha\mu}]\,.\end{aligned} \tag{D.18}$$

Note that in the above equation the commutators $[A^T_\alpha, \partial^\alpha A^T_\mu]$ and $[A^T_\alpha, f_{\alpha\mu}]$ contain TT components while the remaining ones consist of the LL and TL components. Substituting (D.18) into (D.17) gives

$$\begin{aligned} -\int \mathrm{d}^4x\, \mathrm{d}^4y\, \Big\{ F^a_{\mu\nu}(x)\delta^{ab}K_0(x,y)\frac{1}{\Box}\Big([A^T_\alpha, \partial^\alpha A^T_\mu]^{bc} + [\Box v_1, A^T_\mu]^{bc} \\ + [\partial_\alpha v_1, \partial^\alpha A^T_\mu]^{bc} + [A^T_\alpha, \partial^\alpha \partial_\mu v_1]^{bc} + [\Box v_1, \partial_\mu v_1]^{bc} \\ + [\partial_\alpha v_1, \partial_\alpha \partial_\mu v_1]^{bc} + [A^T_\alpha, f_{\alpha\mu}]^{bc} + [\partial_\alpha v_1, f_{\alpha\mu}]^{bc}\Big)(y)(A^T_\nu)^c(y)\Big\}. \end{aligned} \tag{D.19}$$

Combining (D.16) and (D.19) we find most of the terms cancel giving the overall contribution

$$\begin{aligned} \int \mathrm{d}^4x\, \mathrm{d}^4y\, \Big\{ F^a_{\mu\nu}(x)\delta^{ab}K_0(x,y) \\ \times \frac{1}{\Box}\Big([v_1, \partial^\alpha f_{\alpha\mu}]^{bc} - [A^T_\alpha, \partial^\alpha A^T_\mu]^{bc} - [A^T_\alpha, f_{\alpha\mu}]^{bc}\Big)(y)(A^T_\nu)^c(y)\Big\}. \end{aligned} \tag{D.20}$$

In the above equation the first term carries TL components and the other two terms are TT. The TL components need to be removed as we want the terms to be gauge invariant. We will return to this equation later, what we want to show now is the cancellation of the terms arising from the last line of (5.85) with the terms from the last line of (5.89) after making suitable decompositions. Putting both the integrands together and applying the same methodology as in (D.20) we find most of the longitudinal terms cancel leaving the following terms

$$\begin{aligned} \int \mathrm{d}^4x\, \mathrm{d}^4y\, \Big\{ F^a_{\mu\nu}(x)\delta^{ab}K_0(x,y)(A^T_\mu)^{bc}(y) \\ \times \frac{1}{\Box}\Big([v_1, \partial^\beta f_{\beta\nu}]^c - [A^T_\beta, \partial^\beta A^T_\nu]^c - [A^T_\beta, f_{\beta\nu}]^c\Big)\Big\}. \end{aligned} \tag{D.21}$$

Similar to (D.20) we find here that the first term has TL components and the last two terms TT components. We hope to remove the extra longitudinal fields by taking into account the contribution from the other left over terms, that is, the first line of (5.85) and the first three lines of (5.89). To accomplish this we return to (5.85) and consider the first line, but there is not much to be done here as it is already in its simplified form. However we want to remove this term which we suspect will be canceled by the first line of (5.89) which contains the field $K_1^{ab}(x,y)$. Applying the decomposition $K_1^{ab}(x,y) = K_1^{T\,ab}(x,y) + \delta^{ac}K_0(x,y)(v_1^{cb}(y) - v_1^{cb}(x))$ in the first line of (5.89) we obtain

$$\begin{aligned} -\int \mathrm{d}^4x\,\mathrm{d}^4y\,\Big\{ & F^a_{\mu\nu}K_1^{T\,ab}(x,y)(A^T_\mu)^{bc}(y)(A^T_\nu)^c(y) \\ & + F^a_{\mu\nu}\delta^{ad}K_0(x,y)\big(v_1^{db}(y)\big)(A^T_\mu)^{bc}(y)(A^T_\nu)^c(y) \\ & - F^a_{\mu\nu}\delta^{ac}K_0(x,y)\big(v_1^{cb}(x)\big)(A^T_\mu)^{bc}(y)(A^T_\nu)^c(y)\Big\}\,. \end{aligned} \tag{D.22}$$

The last line of this equation cancels with the first line of (5.85) leaving a contribution from the first two lines of (D.22).

We still need to simplify the second and third lines of (5.89) which would then allow us to cancel the remaining longitudinal/transverse terms. Starting with the second line of (5.89) and decomposing the $K_1{}^{bc}_{de}(y,w)$ into the transverse and longitudinal components

$$\begin{aligned} -\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4w\,\Big\{ & F^a_{\mu\nu}(x)\delta^{ab}K_0(x,y)\Big(K_1^T(y,w)\partial^\alpha f_{\alpha\mu}(w)\Big)^{bc}(A^T_\nu)^c(y) \\ & + F^a_{\mu\nu}(x)\delta^{ab}K_0(x,y)K_0(y,w)\big[v_1(w),\partial^\alpha f_{\alpha\mu}(w)\big]^{bc}(A^T_\nu)^c(y) \\ & - F^a_{\mu\nu}(x)\delta^{ab}K_0(x,y)K_0(y,w)\big[v_1(y),\partial^\alpha f_{\alpha\mu}(w)\big]^{bc}(A^T_\nu)^c(y)\Big\}\,, \end{aligned} \tag{D.23}$$

where $K_1(y,w) = K_1^T(y,w) + K_0(y,w)\big(v_1(w) - v_1(y)\big)$ as in (5.43). In order to compare with the other equations we rewrite the second and third integrand of

(D.23) as

$$-\int \mathrm{d}^4x\,\mathrm{d}^4y\,\Big\{F^a_{\mu\nu}\delta^{ab}K_{0}(x,y)\frac{1}{\Box}\big[v_{1},\partial^\alpha f_{\alpha\mu}\big]^{bc}(y)(A^{T}_{\nu})^c(y) - F^a_{\mu\nu}\delta^{ab}K_{0}(x,y)\big[v_{1},A^{T}_{\mu}\big]^{bc}(y)(A^{T}_{\nu})^c(y)\Big\}\,. \tag{D.24}$$

The first line of this equation cancels with the first term of (D.20). Using the same analysis the third line of (5.89) when expanded contributes to the following

$$\begin{aligned}-\int \mathrm{d}^4x\,\mathrm{d}^4y\,\Big\{&F^a_{\mu\nu}(x)\delta^{ab}K_{0}(x,y)(A^{T}_{\mu})^{bc}(y)\frac{1}{\Box}\big[v_{1},\partial^\beta f_{\beta\nu}\big]^c(y)\\ &- F^a_{\mu\nu}(x)\delta^{ab}K_{0}(x,y)(A^{T}_{\mu})^{bc}(y)\big[v_{1},A^{T}_{\nu}\big]^c(y)\Big\}\\ -\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4u\;&F^a_{\mu\nu}(x)\delta^{ab}K_{0}(x,y)(A^{T}_{\mu})^{bc}(y)\big(K^{T}_{1}(y,u)\partial^\beta f_{\beta\nu}(u)\big)^c\,.\end{aligned} \tag{D.25}$$

Here the first line cancels with the first term of (D.21). We now collect all the remaining terms from (D.22), (D.24), (D.25) to yield

$$\begin{aligned}-\int \mathrm{d}^4x\,\mathrm{d}^4y\,\Big\{&F^a_{\mu\nu}(x)\delta^{ac}K_{0}(x,y)\big(v^{cb}_{1}(y)\big)(A^{T}_{\mu})^{bc}(y)(A^{T}_{\nu})^c(y)\\ &- F^a_{\mu\nu}(x)\delta^{ab}K_{0}(x,y)\big[v_{1},A^{T}_{\mu}\big]^{bc}(y)(A^{T}_{\nu})^c(y)\\ &- F^a_{\mu\nu}(x)\delta^{ab}K_{0}(x,y)(A^{T}_{\mu})^{bc}(y)\big[v_{1},A^{T}_{\nu}\big]^c(y)\Big\}\,.\end{aligned} \tag{D.26}$$

It is not quite obvious that they will cancel. However, working in the adjoint representation as we have seen earlier the commutator in the Lie algebra becomes the product in the adjoint representation which allows us to write

$$[v_{1},A^{T}_{\nu}]^c = (v_{1})^{cd}(A^{T}_{\nu})^d\,,\qquad [v_{1},A^{T}_{\mu}]^{bc} = (v_{1})^{bd}(A^{T}_{\mu})^{dc} - (A^{T}_{\mu})^{bd}(v_{1})^{dc}\,. \tag{D.27}$$

Inserting these into (D.26) we find all the terms cancel. We have thus been successful in achieving our task of cancelling the LL and TL components.

Bibliography

[1] F. Halzen and A. D. Martin, *Quarks and Leptons: An introductory course in modern particle physics* (New York, USA: Wiley 396p, 1984).

[2] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field theory* (Addison-Wesley Publishing Company, 1996).

[3] T.Muta, *Foundations of Quantum Chromodynamics: An Introduction to Perturbative Methods in Gauge Theories* (World Scientific Publishing, 1998).

[4] W. N. Cottingham and D. A. Greenwood, (Cambridge, UK: Univ. Pr. 272 p, 2007), *An introduction to the standard model of particle physics*.

[5] I. J. R. Aitchison and A. J. G. Hey, *Gauge theories in particle physics: A practical introduction. Vol. 2: Non-Abelian gauge theories: QCD and the electroweak theory* (Bristol, UK: IOP 454 p, 2004).

[6] S. Eidelman *et al.*, Phys. Lett. **B592**, 1 (2004), *Review of particle physics*, Particle Data Group.

[7] J. Chyla, *Quarks, partons and Quantum Chromodynamics* (PRA-HEP-04-01).

[8] J. L. Rosner, Am. J. Phys. **71**, 302 (2003), hep-ph/0206176, *Resource letter: The Standard model and beyond*.

[9] J. Collins, *Foundations of perturbative QCD* (Cambridge, UK: Univ. Pr. 624 p, 2011).

[10] A. Pais, *Inward Bound* (Oxford University Press, Oxford, 1986).

[11] *The Particle Adventure (The Fundamentals of Matter and Force)* (Particle Data Group, 2009).

[12] D. J. Griffiths, *Introduction to elementary particles* (New York, USA: Wiley 392p, 1987).

[13] L. Alvarez-Gaume *et al.*, (2005), hep-th/0510040, *Introductory Lectures on Quantum Field Theory.*

[14] R. J. Crewther, (1995), hep-th/9505152, *Introduction to quantum field theory.*

[15] M. Dine, (2000), hep-ph/0010035, *The utility of quantum field theory.*

[16] L. H. Ryder, *Quantum Field theory - Second Edition* (Cambridge University Press, 1996).

[17] R. Jackiw, (1996), hep-th/9602122, *The Unreasonable effectiveness of quantum field theory.*

[18] R. Jackiw, Proc.Nat.Acad.Sci. **95**, 12776 (1998), *Field theory: Why have some physicists abandoned it?*

[19] W. Buchmuller and C. Ludeling, (2006), hep-ph/0609174, *Field theory and standard model.*

[20] T. Kashiwa and Y. Takahashi, (1994), hep-th/9401097, *Gauge invariance in quantum electrodynamics.*

[21] I. Bialynicki-Birula, (Prepared for 4th Winter School of Theoretical Physics - Functional Methods in Quantum Field Theory and Statistical Mechanics, Karpacz, Poland, 17 Feb - 2 Mar, 1967), *Gauge invariance in quantum electrodynamics.*

[22] I. J. R. Aitchison and A. J. G. Hey, *Gauge theories in particle physics: A practical introduction* (Bristol, UK: HILGER 571p, 1989).

[23] I. J. R. Aitchison and A. J. G. Hey, *Gauge theories in particle physics: A practical introduction. Vol. 1: From relativistic quantum mechanics to QED* (Bristol, UK: IOP 406 p, 2003).

[24] J. L. Friar and S. Fallieros, Phys. Rev. **C46**, 2393 (1992), *Gauge invariance and the nonrelativistic Ward-Takahashi identity.*

[25] Y. Takahashi, Nuovo Cim. **6**, 371 (1957), *On the generalized Ward identity.*

[26] A. A. Slavnov, Int.J.Mod.Phys. **A25**, 903 (2010), *Slavnov-Taylor identities.*

[27] Y.-L. Wu, Mod. Phys. Lett. **A19**, 2191 (2004), hep-th/0311082, *Symmetry-preserving loop regularization and renormalization of QFTs.*

[28] M. Danos, Found. Phys. **27**, 995 (1997), hep-th/9702096, *Ward-Takahashi identities and Noether's theorem in quantum field theory.*

[29] D. D. Tong, *Quantum Field Theory* (University of Cambridge Part III Mathematical Tripos, 2006).

[30] M. Kaku, *Quantum Field Theory - A modern Introduction* (Oxford University Press, 1993).

[31] R. P. Feynman, Phys. Rev. **80**, 440 (1950), *Mathematical formulation of the quantum theory of electromagnetic interaction.*

[32] D. I. Kazakov, (2009), 0901.2208, *Radiative Corrections, Divergences, Regularization, Renormalization, Renormalization Group and All That in Examples in Quantum Field Theory.*

[33] A. Das, *Lectures on Quantum Field Theory* (John Wiley and Sons ltd, 2008).

[34] W. Pauli and F. Villars, Rev. Mod. Phys. **21**, 434 (1949), *On the Invariant Regularization in Relativistic Quantum Theory.*

[35] G. 't Hooft, Nucl.Phys. **B33**, 173 (1971), *Renormalisation of Massless Yang-Mills Fields.*

[36] G. 't Hooft and M. Veltman, NATO Adv.Study Inst.Ser.B Phys. **4**, 177 (1974), *DIAGRAMMAR.*

[37] G. 't Hooft, AIP Conf.Proc. **957**, 154 (2007), 0707.4568, *Emergent Quantum Mechanics and Emergent Symmetries.*

[38] F. J. Dyson, Phys. Rev. **75**, 1736 (1949), *The S matrix in quantum electrodynamics.*

[39] J. Sun, Hadronic J. **21**, 583 (1998), hep-th/9906199, *On the ultraviolet divergence in QED.*

[40] G. 't Hooft, Nucl.Phys. **B61**, 455 (1973), *Dimensional regularization and the renormalization group.*

[41] S. J. Brodsky *et al.*, Phys.Rev. **D58**, 116006 (1998), hep-ph/9801330, *An Analytic extension of the MS-bar renormalization scheme.*

[42] A. A. Slavnov, Theor. Math. Phys. **10**, 99 (1972), *Ward Identities in Gauge Theories.*

[43] R. Jackiw, (1997), hep-th/9709090, *Evolution of Ward-Takahashi relations and how I used them.*

[44] D. J. Miller, *Introductry QED and QCD* (University of Glasgow-RAL Summer School, 2007).

[45] A. Grozin, (2005), hep-ph/0508242, *Lectures on QED and QCD.*

[46] V. N. Gribov and J. Nyiri, Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol. **13**, 1 (2000), *Quantum electrodynamics: Gribov lectures on theoretical physics.*

[47] J. C. Ward, Phys. Rev. **78**, 182 (1950), *An Identity in Quantum Electrodynamics.*

[48] D. Perkins, (1982), *Introduction to high energy physics.*

[49] Y.-L. Wu, Int. J. Mod. Phys. **A18**, 5363 (2003), hep-th/0209021, *Symmetry principle preserving and infinity free regularization and renormalization of quantum field theories and the mass gap.*

[50] Y.-L. Ma and Y.-L. Wu, Int. J. Mod. Phys. **A21**, 6383 (2006), hep-ph/0509083, *Anomaly and anomaly-free treatment of QFTs based on symmetry-preserving loop regularization.*

[51] B. C. Hall, (2000), math-ph/0005032, *An Elementary Introduction to Groups and Representations.*

[52] S. Sternberg, *Group theory and physics* (Cambridge University Press, New York 429 p, 1994).

[53] C. Chevalley and S. Eilenberg, Ark.Mat.Astron.Fys. **63**, 85 (1948), *Cohomology Theory of Lie Groups and Lie Algebras.*

[54] V. G. Ivancevic and T. T. Ivancevic, (2011), 1104.1106, *Lecture Notes in Lie Groups.*

[55] A. Nyffeler, (Saha Institute of Nuclear Physics, Kolkata, India,, 2011), Basics of perturbative QCD (Lectures at Advanced School on Radiative Corrections for the LHC).

[56] H. F. Jones, (Bristol, UK: Hilger 287 p, 1990), *Groups, representations and physics.*

[57] K. Huang, *Quantum field theory: From operators to path integrals* (New York, USA: Wiley 426 p., 1998).

[58] D. Lichtenberg, (1978), *Unitary Symmetry and Elementary Particles.*

[59] P. Cvitanovic, *Group Theory, Birdtracks, Lies, and Exceptional Groups* (Princeton University Press July, 2008).

[60] R. Horan *et al.*, Pramana **51**, 317 (1998), hep-th/9810089, *Charges in gauge theories.*

[61] E. Bagan *et al.*, Annals Phys. **282**, 471 (2000), hep-ph/9909257, *Charges from dressed matter: Construction.*

[62] R. Horan *et al.*, (2000), hep-th/0002206, *Asymptotic dynamics in quantum field theory: When does the coupling switch off?*

[63] R. Horan *et al.*, J. Math. Phys. **41**, 4437 (2000), hep-th/9909044, *Asymptotic dynamics in quantum field theory.*

[64] E. Bagan *et al.*, (1996), hep-ph/9609330, *Constituent quarks from QCD: Perturbation theory and the infra-red.*

[65] M. Lavelle and D. McMullan, (1993), hep-ph/9306331, *On quark confinement.*

[66] E. Bagan *et al.*, (2001), hep-th/0110033, *Charged particles: A builder's guide.*

[67] E. Bagan *et al.*, (1999), hep-th/0003021, *Charged matter: Physics and renormalisation.*

[68] P. A. M. Dirac, Can. J. Phys. **33**, 650 (1955), *Gauge invariant formulation of quantum electrodynamics.*

[69] J. Cornwall, Phys.Rev. **D10**, 500 (1974), *Spontaneous Symmetry Breaking Without Scalar Mesons II.*

[70] J. Cornwall *et al.*, *Quantum Field Theory: a modern introduction* (Cambridge University Press, New York, 2011).

[71] D. McMullan, SIGMA **3**, 035 (2007), hep-th/0702228, *Charges in gauge theories.*

[72] A. Ilderton *et al.*, JHEP **03**, 044 (2007), hep-th/0701168, *Colour, copies and confinement.*

[73] E. Bagan *et al.*, Nucl. Phys. Proc. Suppl. **74**, 353 (1999), hep-th/9809025, *The glue around quarks and the interquark potential.*

[74] E. Bagan *et al.*, Nucl. Phys. Proc. Suppl. **54A**, 208 (1997), hep-ph/9609333, *How do constituent quarks arise in QCD? Perturbation theory and the infrared.*

[75] M. Lavelle and D. McMullan, Phys. Lett. **B436**, 339 (1998), hep-th/9805013, *Colour charges and the anti-screening contribution to the interquark potential.*

[76] E. Bagan *et al.*, Phys. Lett. **B477**, 396 (2000), hep-th/0003087, *Electrons and photons: Fact not fiction.*

[77] E. Bagan *et al.*, (1999), hep-th/0004091, *Effective quarks and their interactions.*

[78] V. Gribov, Nucl.Phys. **B139**, 1 (1978), *Quantization of Nonabelian Gauge Theories.*

[79] V. Gribov, (1991), *Possible solution of the problem of quark confinement.*

[80] D. Zwanziger, Nucl.Phys. **B192**, 259 (1981), *Covariant Quantization of Gauge Fields Without Gribov Ambiguity.*

[81] D. McMullan, Commun. Math. Phys. **160**, 431 (1994), *Constrained quantization, gauge fixing and the Gribov ambiguity.*

[82] K. Langfeld *et al.*, PoS **CONFINEMENT8**, 047 (2008), 0812.2418, *Solution of the Gribov problem from gauge invariance.*

[83] T. Heinzl *et al.*, Phys. Rev. **D76**, 114510 (2007), 0705.2718, *Coulomb gauge Gribov copies and the confining potential.*

[84] A. Ilderton *et al.*, (2009), 0907.4071, *Stability, creation and annihilation of charges in gauge theories.*

[85] J. M. Cornwall, Phys. Rev. **D26**, 1453 (1982), *Dynamical Mass Generation in Continuum QCD.*

[86] R. Jackiw and S.-Y. Pi, Phys.Lett. **B403**, 297 (1997), hep-th/9703226, *Seeking an even parity mass term for 3-D gauge theory.*

[87] E. Stueckelberg, Helv.Phys.Acta **30**, 209 (1957), *Theory of the radiation of photons of small arbitrary mass.*

[88] M. A. L. Capri *et al.*, Phys. Rev. **D72**, 105016 (2005), hep-th/0510240, *A study of the gauge invariant, nonlocal mass operator* $Tr \int d^4x F_{\mu\nu}(D^2)^{-1}F_{\mu\nu}$ *in Yang-Mills theories.*

[89] A. M. Polyakov, Chur, Switzerland: Harwood (1987) 301 P. (Contemporary Concepts In Physics, 3) (1987), *Gauge fields and strings.*

[90] D. Dudal *et al.*, PoS **QCD-TNT09**, 012 (2009), 0911.0082, *Aspects of the Gribov-Zwanziger framework.*

[91] C. S. Fischer and D. Zwanziger, Phys. Rev. **D72**, 054005 (2005), hep-ph/0504244, *Infrared behaviour and running couplings in interpolating gauges in QCD.*

[92] J. A. Gracey, Phys. Lett. **B651**, 253 (2007), 0706.1440, *One loop renormalization of the non-local gauge invariant operator* $min_U(A^U_\mu)^2$ *in QCD.*

[93] D. Dudal *et al.*, Braz. J. Phys. **37**, 232 (2007), hep-th/0609184, *A local non-Abelian gauge invariant action stemming from the nonlocal operator* $F_{\mu\nu}(D^2)^{-1}F_{\mu\nu}$.

[94] J. Greensite *et al.*, Phys. Rev. **D71**, 114507 (2005), hep-lat/0504008, *Localized eigenmodes of covariant Laplacians in the Yang- Mills vacuum.*

[95] J. Greensite *et al.*, Phys. Rev. **D74**, 094507 (2006), hep-lat/0606008, *Peculiarities in the spectrum of the adjoint scalar kinetic operator in Yang-Mills theory.*

[96] M. A. L. Capri *et al.*, Phys. Rev. **D74**, 045008 (2006), hep-th/0605288, *Quantum properties of a non-Abelian gauge invariant action with a mass parameter.*

[97] D. Dudal *et al.*, Nucl.Phys.Proc.Suppl. **174**, 201 (2007), hep-th/0611114, *Dimension two gluon condensates in a variety of gauges and a gauge invariant Yang-Mills action with a mass.*

[98] D. Zwanziger, Nucl. Phys. **B345**, 461 (1990), *Quantization of gauge fields, classical gauge invariance and gluon confinement.*

[99] D. Dudal *et al.*, Phys. Lett. **B555**, 126 (2003), hep-th/0212182, *The anomalous dimension of the composite operator in the Landau gauge.*

[100] D. Dudal *et al.*, JHEP **01**, 044 (2004), hep-th/0311194, *Dynamical gluon mass generation from* $\langle A_\mu^2 \rangle$ *in linear covariant gauges.*

[101] H. Verschelde *et al.*, Phys.Lett. **B516**, 307 (2001), hep-th/0105018, *The Nonperturbative groundstate of QCD and the local composite operator* A_μ^2*.*

[102] T. Murakami *et al.*, Nucl. Phys. Proc. Suppl. **129**, 718 (2004), hep-lat/0309163, *Non-perturbative propagators and dimension 2 condensate in Yang-Mills theory.*

[103] F. Gubarev *et al.*, Phys.Rev.Lett. **86**, 2220 (2001), hep-ph/0010057, *On the significance of the vector potential squared.*

[104] F. Gubarev and V. I. Zakharov, Phys.Lett. **B501**, 28 (2001), hep-ph/0010096, *On the emerging phenomenology of* $\langle (A_\mu^a)^2_{min} \rangle$*.*

[105] K. E. Cahill *et al.*, Phys. Rev. **D20**, 540 (1979), *Gauge invariance and the quark - anti-quark static potential.*

[106] K. E. Cahill and D. R. Stump, Phys. Rev. **D20**, 2096 (1979), *On the Wilson loop formula.*