Indian Journal of Radio & Space Physics
Vol. 15, June 1986, pp. 104-109

Variation in Ground Electric Field due to Various Forms of Intracloud Lightning Discharges

PRATAP SINGH, T S VERMA, K C MATHPAL & N C VARSHNEYA
Department of Physics, University of Roorkee, Roorkee 247 667

Received 19 *August* 1985; *revised received* 13 *January* 1986

The calculations for instantaneous and total changes in ground electric field due to various forms of intracloud lightning discharges are presented. To place in perspective the significance of the individual form of intracloud discharge, three possible cases, namely, propagation of positive, negative and complex streamers have been studied separately. The important aspects of thunderclouds such as spatially inhomogeneous charge distribution, redistribution of charge due to propagation of streamers, and concept of finite dimensions of streamers and thunderclouds have been taken into account. The present study points out the incapability of the electric field variation dependent hypothesis to distinguish between a negative and a complex streamer because both these streamers produce negative changes in ground electric field. Our theoretical findings suggest that a more convincing hypothesis to classify the nature of intracloud discharges should incorporate magnitudes of electric field variations along with the nature of electric field changes. Plots of total electric field change against distance of observing point show a reasonable agreement with the observed magnitude variations and electric field sign reversal distances at ground.

1 Introduction

Lightning occurs when some region of the atmosphere attains an electric charge sufficiently large that the electric field associated with the charge causes electric field breakdown of the air. The lightning discharges occurring entirely within a cloud are known as intracloud discharges. On the other hand, lightning also occurs between clouds, between a cloud and ground, and between a cloud and air. Moreover, now it is also believed that lightning discharges also occur in the atmosphere of non-terrestrial planets such as Venus and Jupiter[1]. Although intracloud discharges are the most common types of lightning discharges[2-4], investigations into the mechanism of intracloud discharges have been very rare in comparison to the understanding of the nature of cloud to ground lightning discharges. In the present study, emphasis has been focussed on the possible nature of intracloud discharges and their identification by variations in ground electric field.

The behaviour of ground electric field variation due to lightning discharge is assumed as one of the important sources to interpret the nature of intracloud lightning discharge propagation in clouds. Smith[5] investigated the nature of intracloud discharges with simultaneous electric field measurements at the ground and propounded that the intracloud discharge could be described as discharge of the vertically oriented dipole in which negative charge is frequently raised in clouds. On the basis of statistical study of the electrostatic field changes, Takagi[6] concluded, contrary to Smith, that the main process in cloud discharge involves a positive streamer moving downward to negatively charged region. Pierce[7] supported Smith conclusion, but later on Ogawa and Brook[8] showed that most frequently the discharge within the cloud would be initiated by a positive streamer from a centre in the upper P-region moving downward to the centre in the lower N-region of the cloud. Recently, Pradeep and Rai[4] reported that intracloud discharges transport negative charge upward from the lower region of cloud. It is evident that the views of the investigators regarding the process of intracloud lightning discharges are contradictory and still not clear. However, it is to be noted that views of all investigators coincide at a point that transport of negative charge from lower negative centre towards the upper positive charge centre produces a negative field change beneath the thundercloud and vice-versa.

The calculations for the electric field changes at the ground surface due to intracloud lightning discharge have been made by various investigators[8-10]. Most of the calculations have been made using a simple electrical dipole model with upper positive and lower negative point charges in clouds. These models are oversimplified because thunderclouds are believed to have spatially inhomogeneous charge distribution within them[11-15]. Moreover, as a streamer propagates upward or downward in the cloud, the charge density within the cloud will be perturbed because the streamer contains different density in its channel. Secondly, both lightning discharge and cloud are known to have finite dimensions. A more perfect model should incorporate all these aspects of thunderclouds. In the present study, calculations for ground electric field changes due to intracloud discharges have been performed considering above described prevailing conditions of thunderclouds. The computation has been made for the following three possible cases of intracloud discharges:

(i) when a discharge propagates from the centre of upper P-region towards the centre of lower N-region in the form of a positive streamer[6,8,10],

(ii) when a discharge propagates from the centre of lower N-region towards the centre of upper P-region in the form of a negative streamer[4,5,7],

(iii) when both positive and negative streamers propagate simultaneously[16,17] in the form of a complex streamer.

In the text these three cases will be referred to as propagation of positive streamer, negative streamer and complex streamer, respectively. Here, it may be pointed out that sufficient literature is not available on electric field variation when a complex streamer takes place in thunderclouds.

2 Thundercloud Model

As a result of charge separation, positive and negative charges are accumulated in the upper and lower regions of thundercloud, respectively. In the present computation, we consider a cloud of cylindrical shape of radius D and having vertical charge distribution within it. Following Mathpal and Varshneya[12], Singh *et al.*[13-15], the charge distribution inside a thundercloud may be represented by

$$\begin{aligned}\rho(\mathbf{r}_{P'}) = \rho(Z', t) &= \rho_0 \quad \text{for } -(L+H) \leqslant Z' \leqslant -L \\ &= \frac{\rho_0 Z'}{L} \quad \text{for } -L \leqslant Z' \leqslant L \\ &= -\rho_0 \quad \text{for } L \leqslant Z'(L+H) \qquad \ldots(1)\end{aligned}$$

where ρ_0 is the average charge density of each polarity, $\mathbf{r}_{P'}$ the position vector at point P′, Z' the vertical position of point P′, t the time, L the vertical length of the charging zone on each side from the cloud centre, and H the vertical length of both lower main negative and upper main positive charge regions. In the intervening space, charge varies linearly with height. A schematic diagram of a thundercloud is shown in Fig. 1.

As stated earlier, the vertical charge distribution in cloud is disturbed due to initiation of positive or negative or complex streamer because a streamer propagates in the intervening space of thundercloud with the charge density from where it starts. Moreover, the thickness of the streamer initiating region also decreases with an increase in the length of streamer. First, we present a detailed formulation for ground electric field at a point P on the ground for complex streamer propagation. The charge distribution inside a thundercloud when both positive and negative streamers having radii R_s propagate simultaneously from the centre of upper and lower regions of thundercloud (complex streamer case), respectively, may be given in following forms:

(i) Charge distribution for the cloud region having streamers (viz. region between $+R_s$ and $-R_s$, Fig. 1)

$$\begin{aligned}\rho(\mathbf{r}_{P'}) = \rho(Z', t) &= \rho_0 \quad \text{for } Z_1 \leqslant Z' \leqslant (L+Z_2) \\ &= \frac{\rho_0 Z'}{L} \quad \text{for } -Z_1 \leqslant Z' \leqslant Z \\ &= -\rho_0 \\ &\quad \text{for } -(L+Z_2') \leqslant Z' \leqslant -Z_1 \qquad \ldots(2)\end{aligned}$$

(ii) Charge distribution for the region other than streamers (viz. region between $+R_s$ and $+D$ on one side and between $-R_s$ and $-D$ on the other side, Fig. 1)

$$\begin{aligned}\rho(\mathbf{r}_{P'}) = \rho(Z', t) &= \rho_0 \quad \text{for } L \leqslant Z' \leqslant (L+Z_2) \\ &= \rho_0 \frac{Z'}{L} \quad \text{for } -L \leqslant Z' \leqslant L \\ &= -\rho_0 \\ &\quad \text{for } -(L+Z_2') \leqslant Z' \leqslant -L \qquad \ldots(3)\end{aligned}$$

where Z_1 is the vertical position of the tip of streamer from the origin of the co-ordinate system, and Z_2 the corresponding thickness of P- and N-regions of cloud. Initially at $t=0$, $Z_1 = L$ and $Z_2 = Z_2' = H$.

Now considering a volume element $d^3 r_{P'}$ at a point P′ which has position vector $\mathbf{r}_{P'}$ (Fig. 1), the potential function at point P on the ground surface due to this charge element may be given by

$$d\Phi(\mathbf{r}_P) = \frac{\rho(\mathbf{r}_{P'}) d^3 r_{P'}}{4\pi\varepsilon_0 |\mathbf{r}_P - \mathbf{r}_{P'}|} \qquad \ldots(4)$$

where ε_0 is permittivity of the space within the thundercloud. Expanding $1/|\mathbf{r}_P - \mathbf{r}_{P'}|$ in terms of spherical harmonics, Eq. (4) leads to[12]

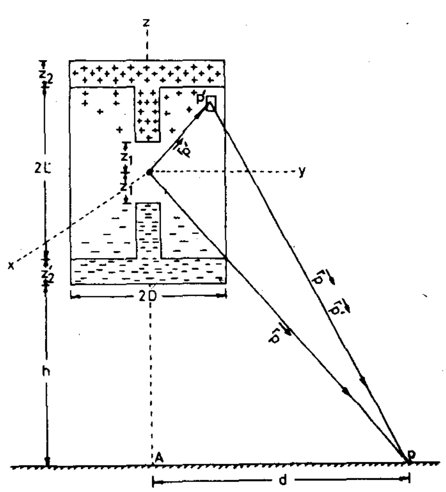

Fig. 1—Thundercloud size and charge distribution within it

$$\Phi=\frac{2}{\varepsilon_0}\sum_l\sum_m\frac{Y_{lm}(\theta,\varphi)q_{lm}}{[(2l+1)(r_p)^{l+1}]} \quad \ldots(5)$$

where multipole moments q_{lm} are given by

$$q_{lm}=\int Y^*_{lm}(\theta,\varphi')(r_{P'})^l\rho(\mathbf{r}_{P'})d^3r_{P'} \quad \ldots(6)$$

The parameter l must be zero or a positive integer and integer m can have values $-l$, $-(l-1)\ldots-1$, 0, $1\ldots(l-1)$, l. If one considers only up to dipole terms ($l=1$), Eq. (5) may be written as

$$\Phi=\frac{2}{\varepsilon_0}\left[\frac{q_{00}Y_{00}}{r_p}+\frac{(q_{l-1}Y_{l-1}+q_{10}Y_{10}+q_{11}Y_{11})}{3r_p^2}\right] \quad \ldots(7)$$

On evaluating we find that all q_{lm} terms except q_{10} vanish and thus Eq. (7) reduces to

$$\Phi=\frac{2}{\varepsilon_0}\left[\frac{q_{10}Y_{10}}{3r_p^2}\right] \quad \ldots(8)$$

Substituting values of q_{10} and Y_{10} from Appendix I, one may obtain

$$\Phi=\frac{\rho_0}{\varepsilon_0 r_P^2}\left[R_s^2\left\{\frac{2Z_1^3}{3L}+\frac{(L+Z_2)^2+(L+Z_2')^2}{2}-Z_1^2\right\}+(D^2-R_s^2)\left\{\frac{2L^3}{3L}+\frac{(L+Z_2)^2+(L+Z_2')^2}{2}-L^2\right\}\right]\cos\theta \quad \ldots(9)$$

On the other hand, Fig. 1 shows that

$$r_P\cos\theta=Z \text{ and } r_P=(Z^2+d^2)^{1/2} \quad \ldots(10)$$

where d is the distance between observation point on the ground and a point on the ground right below the centre of cloud base, and Z the vertical position of the origin of co-ordinate system from the ground and is given by (Fig. 1)

$$Z=h+H+L$$

where h is the height of cloud base from the ground. Substituting $\cos\theta$ in terms of Z and d in Eq. (9), we obtain

$$\Phi=\frac{\rho_0}{\varepsilon_0}\left[R_s^2\left\{\frac{2Z_1^3}{3L}+\frac{(L+Z_2)^2+(L+Z_2')^2}{2}-Z_1^2\right\}+(D^2-R_s^2)\left\{\frac{2L^3}{3L}+\frac{(L+Z_2)^2+(L+Z_2')^2}{2}-L^2\right\}\right]\times\frac{Z}{(Z^2+d^2)^{3/2}} \quad \ldots(11)$$

Further, differentiating Eq. (11) with respect to Z, one may get electric field, E_c, due to a complex streamer at point P on the ground, i.e.

$$E_c=\frac{\partial\Phi}{\partial Z} \quad \ldots(12)$$

which leads to

$$E_c=\frac{\rho_0}{\varepsilon_0}\left[R_s^2\left\{\frac{2Z_1^3}{3L}+\frac{(L+Z_2)^2+(L+Z_2')^2}{2}-Z_1^2\right\}+(D^2-R_s^2)\left\{\frac{2L^3}{3L}+\frac{(L+Z_2)^2+(L+Z_2')^2}{2}-L^2\right\}\right]\times\frac{(d^2-2Z^2)}{(d^2+Z^2)^{5/2}} \quad \ldots(13)$$

where $Z_2=H-\frac{R_s^2}{D^2}\left[\frac{Z_1^2}{2L}+\frac{L}{2}-Z_1\right]$ $\ldots(14)$

and $Z_2'=H-\frac{R_s^2}{D^2}\left[\frac{Z_1^2}{2L}+\frac{L}{2}+Z_1\right]$ $\ldots(15)$

Such expressions for Z_2 and Z_2' are explained in Appendix II.

Further, similar equation for ground electric field for positive and negative streamers may be obtained by the same pattern. For the case when positive streamer propagates electric field, E_P will be given by

$$E_P=\frac{\rho_0}{4\varepsilon_0}\left[R_s^2\left\{(L+H)^2-L^2+\frac{2(Z_1^2+L^3)}{3L}+(L+Z_2)^2-Z_1^2\right\}+(D^2-R_s^2)\times\left\{(L+H)^2-\frac{2}{3}L^2+(L+Z_2)^2\right\}\right]\times\frac{(d^2-2Z^2)}{(d^2+Z^2)^{5/2}} \quad \ldots(16)$$

and electric field, E_N, when negative streamer propagates will be given by

$$E_N=\frac{\rho_0}{4\varepsilon_0}\left[R_s^2\left\{\frac{L^2}{6}-\frac{Z_1^2}{2}-\frac{Z_1^3}{3L}\right\}+D^2\left\{\frac{(L+H)^2}{2}-\frac{L^2}{3}+\frac{(L+Z_2')^2}{2}\right\}\right]\times\frac{(d^2-2Z^2)}{(d^2+Z^2)^{5/2}} \quad \ldots(17)$$

If Q is the total positive (or negative) charge within upper (or lower) charged region of thundercloud, then Q will be given by

$$Q=\int_0^D\int_0^{L+H}\int_0^{2\pi}\rho(\mathbf{r}_{P'})d\rho'dZ'd\varphi' \quad \ldots(18)$$

using initial charge distribution described by Eq. (1), one may get

$$Q = \rho_0 \pi D^2 (H + L/2)$$

or

$$\rho_0 = \frac{Q}{\pi D^2 (H + L/2)} \quad \ldots (19)$$

Let for a given set of involved parameters, the values of E for a particular streamer length, 0, 100, 200, 300,...m be E_0, E_1, E_2, $E_3 \ldots$ Vm^{-1}. Following Khastgir and Shah[10], the instantaneous field changes at streamer length 100, 200, 300...m are obtained from $E_1 - E_0$, $E_2 - E_1$, $E_3 - E_2$ and corresponding total changes from initiation of the streamer are obtained from $E_1 - E_0$, $E_2 - E_0$, $E_3 - E_0 \ldots$ Vm^{-1}.

3 Computation and Discussion

Reynolds and Neill[18] indicated the mean height between positive and negative charge centres involved in intracloud discharges to be approximately equal to 2 km. Further, the vertical extent and mean altitude of intracloud discharges have been reported to be equal to about 2 km and 5 km respectively[3]. But little information regarding the diameters of the streamers is available in literature. Some investigators[19] used the diameter of streamer to be equal to 2 km. Therefore, in the present calculations we use $L = 1$ km, $H = 0.25$ km, $D = 0.8$ km and $R_s = 0.125$ km for determining numerical values of instantaneous and total change in ground electric field. An average value of negative charge, $Q = -40$ C, has been used following the reports of several investigators[16,20].

Theoretically, computed curves representing instantaneous change in ground electric field with streamer length, L_s, at various points on the ground surface have been shown in Fig. 2. Fig. 2 also illustrates the instantaneous change in electric field due to positive, negative and complex streamers propagation. It is seen that the instantaneous change in electric field decreases as the distance of the observing point increases. The variation patterns of the ground electric field calculated in the present study are compared with the patterns reported by other researchers. It is found that the behaviour of electric field changes for positive and negative streamers is similar, i.e. positive and negative streamers cause positive and negative changes in electric field, respectively[4,5,8,10]. But a further examination of Fig. 2 reveals the most surprising result of the present study. It is found that negative changes in electric field are also obtained due to the propagation of a complex streamer. However, the changes caused by a complex streamers are higher in magnitude in comparison to changes caused by a negative streamer of the same radius. But, it is difficult to predict from field change records whether negative changes in electric field are due to initiation of a negative streamer or due to a complex streamer. Thus, it is pointed out that the present theory, which envisages that negative changes in ground electric field are produced only by a negative streamer, does not seem perfectly plausible. Further, it is suggested that a more satisfactory hypothesis should account for the magnitudes of the electric field variations due to different forms of discharges.

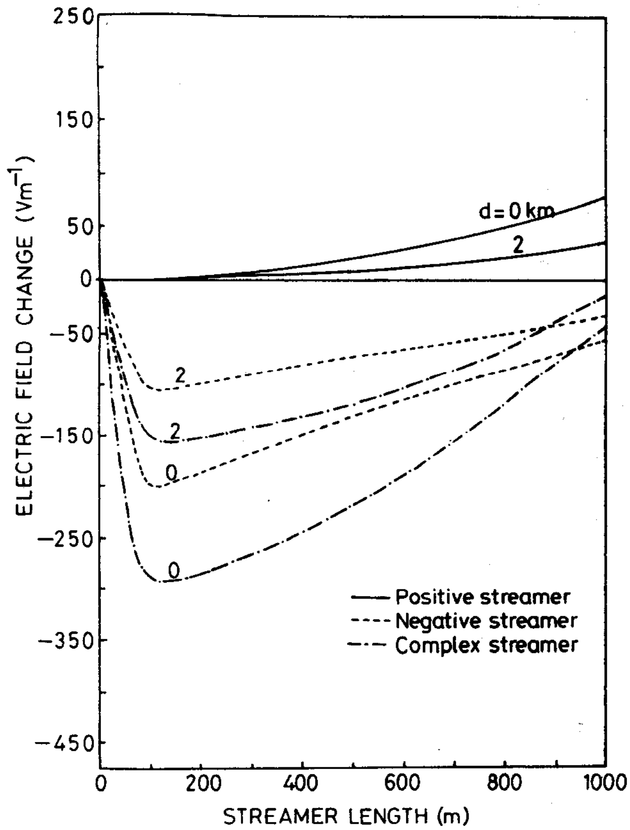

Fig. 2—Instantaneous changes in ground electric field with streamer length of positive, negative and complex streamers

Fig. 3 illustrates the total change in electric field and its sign reversal pattern with distance for streamer length of 1 km. All the three possible intracloud discharge forms have been studied. We obtain total change in electric field to be equal to 1.4, −0.71 and −1.01 kVm^{-1} respectively for positive, negative and complex streamers of length 1 km at $d = 2$ km. Mackerras[21] reported total negative change in electric field of the order of −1.4 kVm^{-1}. Pradeep and Rai[4] recorded total negative field change to be approximately equal to −1 kVm^{-1} and total positive field change less than 3 kVm^{-1}. The obtained magnitudes of the electric field variations are in good accord with the observed values. Fig. 3 also exhibits the well known behaviour of the sign dependence of field change upon distance, i.e. a positive/negative streamer will cause a positive/negative field change on the ground close to thundercloud and a negative/

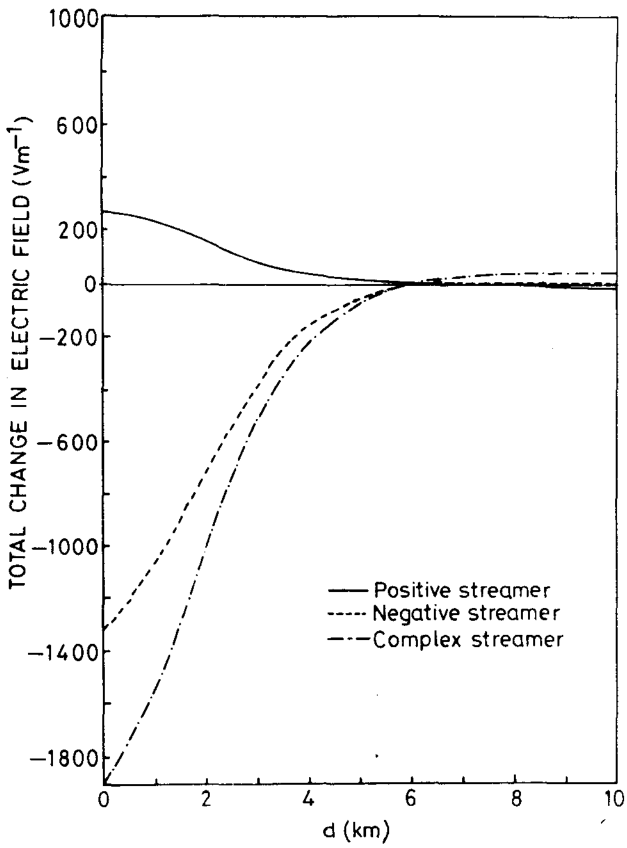

Fig. 3—Total change in ground electric field due to positive, negative and complex streamers at various observing points at the ground surface

positive field change beyond the reversal distance. Mackerras[21] reported the sign reversal distance in the range of 6-9 km. From the present calculation we get this distance to be approximately equal to 6 km, which is in good agreement with the observed values[21].

4 Conclusion

The study reveals that both negative and complex streamers produce negative changes in ground electric field. These negative changes are due to negative or complex streamers which could not be identified by the present theory. However, positive streamers may be identified very well by the present hypothesis by measuring variations in electric field at the ground. This study suggests that a more satisfactory theory should incorporate the magnitudes of electric field variations and behaviour of ground electric field changes due to intracloud discharges. The major obstacle in the field is the dearth of field experiments. A more detailed study to classify the nature of intracloud discharges needs simultaneous records of streamer propagation by high speed oscillographs and measurements of magnitude variations in ground electric field due to recorded discharges. Such measurements may resolve some of the existing uncertainties which this article has attempted to identify.

References

1 Lanzerotti L J, Rinnert K, Krider E P, Uman M A, Dehmel G, Gliem F O & Axford W I, *Proceedings in atmospheric electricity* (A Dupak Publishing, Virginia) 1983, 411.
2 Uman M A, *Lightning* (McGraw Hill, New York), 1969, 47.
3 Teer T L & Few A A, *J Geophys Res (USA)*, **79** (1974) 3436.
4 Pradeep P & Rai J, *Indian J Radio & Space Phys*, **14** (1985) 55.
5 Smith L G, *Q J R Meteorol Soc (GB)*, **83** (1957) 103.
6 Takagi M, *Proc Res Inst Atmos Nagoya Univ (Japan)*, **8B** (1961) 1.
7 Pierce E T, *Monograph on radio noise of terrestrial origin* (Elsevier, Amsterdam) 1962, 55.
8 Ogawa T & Brook M, *J Geophys Res (USA)*, **769** (1964) 5141.
9 Pierce E T, *Q J R Meteorol Soc (GB)*, **81** (1955) 211.
10 Khastgir S R & Saha S K, *J Atmos & Terr Phys (GB)*, **34** (1972) 115.
11 Moore C B & Vonnegut B, *Lightning*, Vol. 1, edited by R H Golde, (Academic, New York) 1977, 51.
12 Mathpal K C & Varshneya N C, *J Meteorol Soc Jpn (Japan)*, **61** (1983) 909.
13 Singh P, Verma T S & Varshneya N C, *J Meteorol Soc Jpn (Japan)*, **62** (1984) 775.
14 Singh P, Verma T S & Varshneya N C, *Natl Acad Sci Lett (India)*, **8** (1985) 25.
15 Singh P, Verma T S & Varshneya N C, *Indian J Radio & Space Phys*, **14** (1985) 126.
16 Ishikawa H, *Proc Res Inst Atmos Nagoya Univ (Japan)*, **8B** (1961) 1.
17 Brook M & Ogawa T, *Lightning*, Vol. 1, edited by R H Golde (Academic Press, New York), 1977, 191.
18 Reynolds S E & Neill N, *J Meteorol (USA)*, **12** (1955) 1.
19 Pathak P P, Rai J & Varshneya N C, *Ann Geophys (France)*, **36** (1980) 613.
20 Mathpal K C, Varshneya N C & Dass N, *Rev Geophys Space Phys (USA)*, **18** (1980) 361.
21 Mackerras D J, *J Geophys Res (USA)*, **73** (1968) 1175.
22 Jackson J D, *Classical electrodynamics* (John Wiley, New York) 1975, 84.

Appendix I

We have

$$q_{lm} = \int Y^*_{lm}(\theta', \varphi)(r_{P'})^l \rho(\mathbf{r}_{P'}) d^3 r_{P'} \quad \ldots(1.1)$$

Substituting $l = 1$ and $m = 0$ in Eq. (1.1) and expressing $d^3 r_{P'}$ in terms of cylindrical coordinates (ρ', Z', φ'), we get

$$q_{10} = \iiint Y^*_{10}(\theta', \varphi') r_{P'} \rho(\mathbf{r}_{P'}) d\rho' dZ' d\varphi' \quad \ldots(1.2)$$

Further

$Y^*_{10}(\theta', \varphi')$ may be expressed as[22]

$$Y^*_{10}(\theta', \varphi') = \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta \quad \ldots(1.3)$$

Thus

$$q_{10}=\left(\frac{3}{4\pi}\right)^{1/2}\left[\int_0^D \rho'\mathrm{d}\rho'\int_{-(L+Z_2')}^{(L+Z_2)} r_{P'}\cos\theta' \times \rho(\mathbf{r}_{P'})\mathrm{d}Z'\int_0^{2\pi}\mathrm{d}\varphi'\right] \quad \ldots(1.4)$$

Putting $\cos\theta' = Z'/r_{P'}$ and integrating last term of Eq. (1.4), one may get

$$q_{10}=(3\pi)^{1/2}\int_0^D \rho'\mathrm{d}\rho' \times \int_{-(L+Z_2')}^{(L+Z_2)} Z'\rho(\mathbf{r}_{P'})\mathrm{d}Z' \quad \ldots(1.5)$$

Considering that the streamer channel and the region with channel have different charge distributions, Eq. (1.5) can be broken up as

$$q_{10}=(3\pi)^{1/2}\left[\int_0^{R_s}\rho'\mathrm{d}\rho'\int_{-(L+Z_2')}^{(L+Z_2)} Z'\rho(\mathbf{r}_{P'})\mathrm{d}Z' + \int_{R_s}^{D}\rho'\mathrm{d}\rho'\int_{-(L+Z_2')}^{(L+Z_2)} Z'\rho(\mathbf{r}_{P'})\mathrm{d}Z'\right]$$

$$=(3\pi)^{1/2}\left[\int_0^{R_s}\rho'\mathrm{d}\rho'\int_{-(L+Z_2')}^{-Z_1} Z'\rho(\mathbf{r}_{P'})\mathrm{d}Z' + \int_{-Z_1}^{Z_1} Z'\rho(\mathbf{r}_{P'})\mathrm{d}Z' + \int_{Z_1}^{L+Z_2} Z'\rho(\mathbf{r}_{P'})\mathrm{d}Z' + \int_{R_s}^{D}\rho'\mathrm{d}\rho'\int_{-(L+Z_2')}^{-L} Z'\rho(\mathbf{r}_{P'})\mathrm{d}Z' + \int_{-L}^{L} Z'\rho(\mathbf{r}_{P'})\mathrm{d}Z' + \int_{L}^{(L+Z_2)} Z'\rho(\mathbf{r}_{P'})\mathrm{d}Z'\right] \quad \ldots(1.6)$$

Substituting regionwise density from Eqs (2) and (3) in Eq. (1.6), one may get

$$q_{10}=(3\pi)^{1/2}\rho_0\left[R_s^2\left\{\frac{2Z_1^3}{3L}+\frac{(L+Z_2)^2+(L+Z_2')^2}{2} - Z_1^2\right\}+(D^2-R_s^2)\left\{\frac{2L^3}{3L}+\frac{(L+Z_2)^2+(L+Z_2')^2}{2}-L^2\right\}\right] \quad \ldots(1.7)$$

Appendix II

Consider a positive streamer of radius R_s initiates from the centre of upper P-region. Let at any instant Z_1 is the vertical position of the tip of streamer from the origin of the co-ordinate system and Z_2 the corresponding thickness of P-region. Assuming that the streamer has the same charge density as that of P-region and the charge of P-region is gradually decreasing with the increase in the streamer length, one may state:

(Charge in P-region + Charge in streamer) at instant t = Initial Charge in P-region + Charge in the space of streamer) or

$$\pi D^2 Z_2\rho_0+\int_0^{R_s}\rho'\mathrm{d}\rho'\int_{Z_1}^{L}\rho_0\mathrm{d}Z'\int_0^{2\pi}\mathrm{d}\varphi'$$

$$=\pi D^2 H\rho_0+\int_0^{R_s}\rho'\mathrm{d}\rho'\int_{Z_1}^{L}\rho(\mathbf{r}_{P'})\mathrm{d}Z'\int_0^{2\pi}\mathrm{d}\varphi' \quad \ldots(2.1)$$

or

$$\pi D^2 Z_2\rho_0+\pi R_s^2(L-Z_1)\rho_0 = \pi D^2\rho_0 H+\rho_0\pi R_s^2(L^2-Z_1^2)/(2L) \quad \ldots(2.2)$$

This equation leads to

$$Z_2=H-\frac{R_s^2}{D^2}\left[\frac{Z_1^2}{2L}+\frac{L}{2}-Z_1\right] \quad \ldots(2.3)$$

Similarly one may obtain

$$Z_2'=H-\frac{R_s^2}{D^2}\left[\frac{Z_1^2}{2L}+\frac{L}{2}+Z_1\right] \quad \ldots(2.4)$$