Stability and Hopf bifurcation of controlled complex networks model with two delays *

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Abstract

This paper consider Hopf bifurcation of complex network have two independent delays. By analyzing the eigenvalue equations, the local stability of the system is studied. Taking delay as parameter, the change of system stability with time is studied and the emergence of inherent bifurcation is given. By changing the value of the delay, the bifurcation of a given system can be controlled. Numerical simulation results confirm the validity of the results found.

Keywords: Hopf bifurcation; Independent delays; Disease spreading networks; Numerical simulation.

1 Introduction

Over the past decade, the complex network that began with the famed "Kevin Bacon Six Degrees" has been found in some real networks, with in-depth studies in physics and mathematics[1]-[6]. The dynamics of these complex networks, especially the bifurcation, have become very important. Since bifurcation oscillations can be detrimental in some engineering applications, it has been recognized that bifurcation control has great potential in many technical disciplines. Therefore, it is very important to study the non-linear phenomena of complex networks, and more importantly, to control human beings [7]-[19].

In recent years, more and more researches have been done on time-delay systems in recent years because time-delay mathematical models can describe many kinds of real systems [20]-[25]. Time-delay systems

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and ordinary differential equations are very different, there may be complex bifurcation and chaos dynamic behavior. And time-delay complex networks have researched a wide range of researchers' interests since Watts and Strogatz's research[1], [26]-[34].

In 1999, Newman and Watts [5] studied the infected volume $V(t)$ grows as a sphere of radius t and surface $\Gamma_d t^{d-1}$, where Γ_d is a constant. And the total infected volume $V(t)$ is the sum of the primary volume $\Gamma_d \int_0^t \delta^{d-1} d\delta$ plus a contribution $V(t-\delta)$, which satisfies

$$V(t) = \Gamma_d \int_0^t \delta^{d-1}(1 + 2pV(t-\delta))d\delta. \tag{1}$$

In 2001, Yang [6] applied a time-delay Δ to Eq. (1), and further considered negative nonlinear interactions such as friction and slowdown due to congestion as in the cases of the Internet and traffic jam, which yielding

$$V(t) = \Gamma_d \int_0^t [1 + 2pV(t-\delta-\Delta) - \mu V^2(t-\delta-\Delta)]d\delta \tag{2}$$

However, the studies hasn't address the bifurcation behaviors of complex network models with two delays.

In 2004, X. Li et al [18] studied a complex network for disease spread with time delay:

$$\dot{V} = 1 + 2pV(t-\delta) - \mu(1+2p)V^2(t-\delta). \tag{3}$$

where V is the the total influenced volume, μ is the nonlinear interaction in the network and p is the probability of increasing the link between randomly chosen pairs of nodes, which means, in the cases of regular lattices with $p = 0$, small-world networks with $0 < p < 1$, and random networks with $p = 1$. In [34], a new hybrid control strategy is used for the system, and the Hopf bifurcation control problem of the model was studied.

This paper mainly concerns the effect of the delays τ_1, τ_2 on the stability and bifurcation behaviours of disease spreading in the N-W small-world networks. We generalize [18] by assuming τ_1, τ_2 for two different time delays in the system. The local stability of the system is studied at first. And then, taking delay as parameter, the conditions of Hopf bifurcation are given.

The remainder are as follows. First of all, in Section 2, we discusses Hopf bifurcation in the case of one delay ($p = 0$). Where we analyzes not only the stability but also about the Hopf bifurcation. The problem of Hopf bifurcation with two delays was studied in Section 3. To verify the theoretical analysis, numerical simulations are given in Section 4. Finally, we give the corresponding conclusions in Section 5.

2 The existence of Hopf bifurcation when $p = 0$

In this paper we generalize system(3) by assuming two different time delays $\tau_1 \geq 0, \tau_2 \geq 0$ in it. More precisely, our model is described by

$$\dot{V} = 1 + 2pV(t-\delta_1) - \mu(1+2p)V^2(t-\delta_2). \tag{4}$$

System (4) has equilibrium, which was noted as V^*, where

$$V^* = \frac{p + \sqrt{p^2 + \mu(1+2p)}}{\mu(1+2p)}.$$

Let $\tilde{V} = V - V^*$, system (4) becomes,

$$\dot{\tilde{V}}(t) = [1 + 2pV^* - \mu(1+2p)(V^*)^2] + 2p\tilde{V}(t-\delta_1) - 2\mu(1+2p)V^*\tilde{V}(t-\delta_2) - \mu(1+2p)\tilde{V}^2(t-\delta_2) \quad (5)$$

Note that $1 + 2pV^* - \mu(1+2p)(V^*)^2 = 0$ and $V^* = \frac{p+\sqrt{p^2+\mu(1+2p)}}{\mu(1+2p)}$, the linearized equation of system (4) at V^* becomes

$$\dot{\tilde{V}}(t) = 2p\tilde{V}(t-\delta_1) - 2\left[p + \sqrt{p^2 + \mu(1+2p)}\right]\tilde{V}(t-\delta_2) \quad (6)$$

And the linearized equation (6) has the following characteristic equation

$$\lambda + A_1 e^{-\lambda\delta_1} + A_2 e^{-\lambda\delta_2} = 0, \quad (7)$$

where

$$A_1 = -2p \leq 0 \quad \text{and} \quad A_2 = 2\left[p + \sqrt{p^2 + \mu(1+2p)}\right] > 0.$$

If $\delta_1 = 0$ and $\delta_2 = 0$, then $\lambda = -2\sqrt{p^2 + \mu(1+2p)} < 0$. V^* of system (4) is locally asymptotically stable. When the parameters δ_1 and δ_2 increase, V^* is still locally asymptotically stable if all the roots of Eq. (7) have negative real parts. It is unstable if Eq. (7) has at least one root with positive real part. Notice that $\lambda = 0$ is not a root of Eq. (7).

We know analyze the pure imaginary roots in the system (4). Then analyze the conditions whrer crossing the imaginary axis.

If $p = 0$, then $A_1 = 0$, and Eq. (7) reduces to

$$\lambda + A_2 e^{-\lambda\delta_2} = 0. \quad (8)$$

Assume that $\lambda = \pm i\omega$ $(\omega > 0)$ are a pair of purely imaginary roots of Eq. (8). Then,

$$i\omega + A_2 e^{-i\omega\delta_2} = 0. \quad (9)$$

By separate the real and imaginary parts of Eq. (9), we have

$$\omega = A_2 \sin\omega\delta_2, \quad 0 = -A_2\cos\omega\delta_2. \quad (10)$$

From Eq. (10), we derive that there exists only one positive root $\omega_0 = A_2$, which is acquired at the critical values $\delta_2 = \delta_2^{(j)}$, $j = 0, 1, 2, ...,$ where

$$\delta_2^{(j)} = \frac{1}{\omega_0}\left(\frac{\pi}{2} + 2j\pi\right). \quad (11)$$

It is immediately seen that $\lambda = i\omega_0$ is a simple root of Eq. (8). Differentiating Eq. (8) with respect to δ_2, we obtain

$$\frac{d\lambda}{d\delta_2} = -\frac{\lambda^2}{1 + \delta_2\lambda},$$

leading to the transversality condition

$$\left.\frac{d\left(Re\lambda\right)}{d\delta_2}\right|_{\delta_2=\delta_2^{(j)}} = \frac{\omega_0^2}{1+\left[\delta_2^{(j)}\omega_0\right]^2} > 0.$$

As a result, the root of characteristic equation Eq. (8) near $\delta_2^{(j)}$ crosses the imaginary axis when δ_2 increases.

Summing up, we have:

Theorem 2.1. *If $\delta_2^{(0)}$ be defined as in Eq. (11). The equilibrium point V^* of system(4) is locally asymptotically stable when $\delta_2 \in [0, \delta_2^{(0)})$, and unstable when $\delta_2 > \delta_2^{(0)}$. And a Hopf bifurcation occurs when $\delta_2 = \delta_2^{(0)}$.*

3 Existence of Hopf bifurcation with two delays

If $0 < p \leq 1$, then $A_1 < 0$. For the given model, we will follow the method of M.J. Piotrowska [20] and X. Li, S. Ruan, J. Wei [21] to analysis the existence of Hopf bifurcation with two delays, and then uses the mathematical method developed by K. Gu, S., Niculescu, and J. Chen [22], by using a stability crossover curve.

First method

Following M.J. Piotrowska [20] and X. Li, S. Ruan, J. Wei [21] we rescale

$$\tilde{\lambda} = \frac{\lambda}{|A_1|}, \quad A = \frac{A_2}{|A_1|}, \quad \delta_1 = \frac{r_1}{|A_1|} \quad \text{and} \quad \delta_2 = \frac{r_2}{|A_1|},$$

so that the characteristic Eq. (7) takes the form

$$\tilde{\lambda} - e^{-\tilde{\lambda} r_1} + Ae^{-\tilde{\lambda} r_2} = 0 \tag{12}$$

If $\omega > 0$, suppose that $i\omega$ is one root of Eq. (12). Then, by separating the real and imaginary parts, then

$$\cos\omega r_1 = A\cos\omega r_2, \qquad \omega + \sin\omega r_1 = A\sin\omega r_2. \tag{13}$$

Then, we can arrive at

$$\sin\omega r_1 = \frac{A^2-\omega^2-1}{2\omega}, \tag{14}$$

From being $|\sin\omega r_1| \leq 1$, we obtain the inequalities $-2\omega \leq A^2-\omega^2-1 \leq 2\omega$, whose solutions is $A-1 \leq \omega \leq A+1$. Setting

$$g(\omega) = \frac{A^2-\omega^2-1}{2\omega},$$

we see that $g(A-1) = 1$, $g(A+1) = -1$, $g'(\omega) < 0$ and $g''(\omega) < 0$. A graphical inspection on the intersections of the functions $\sin\omega r_1$ and $g(\omega)$ yields if $r_1 > 0$, Eq. (14) has finite number of positive zeros $\omega_j,\ j = 1, 2, ..., m$ (see Fig. 1.) Moreover, for every $r_1 > 0$ and ω_j there exista r_2 satisfied $\cos\omega_j r_1 = A\cos\omega_j r_2$.

Denote

$$r_2^0 = \min\left\{r_2^j : j = 1, 2, ..., m\right\}, \tag{15}$$

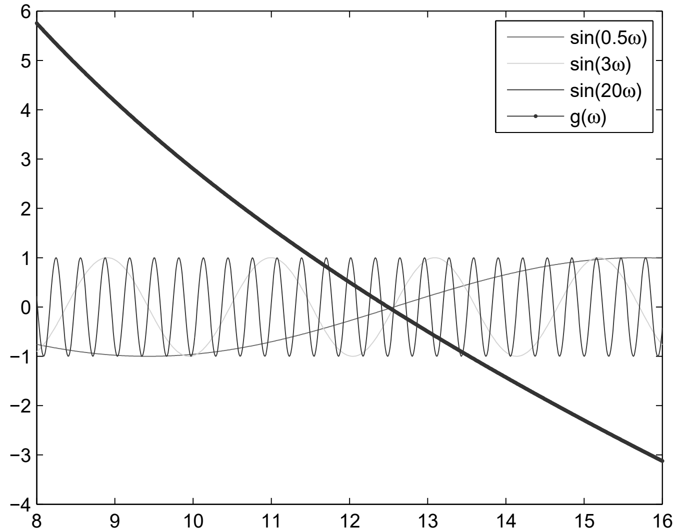

Figure 1: $g(\omega), sin(r_1\omega)$ versus ω with $\mu = 0.3$, $p = 0.05$.

where

$$r_2^j = \min\left\{r_2 > 0 : \cos\omega_j r_1 = A\cos\omega_j r_2\right\},$$

and $\omega_0 = \omega_j$ for $j = 1, 2, ..., m$ such that $r_2^j = r_2^0$. We now examine the conditions that guarantee that the root of the Eq. (12) moves from left to the right half plane. Let $\tilde{\lambda}(r_2)$ be the root of Eq. (12), then we can get some $r_2 = r_2^0$ for which $Re[\tilde{\lambda}(r_2^0)] = 0$ and $Im[\tilde{\lambda}(r_2^0)] = \omega_0$. We calculate

$$\frac{d\tilde{\lambda}}{dr_2} = -r_1 e^{-\tilde{\lambda}r_1}\frac{d\tilde{\lambda}}{dr_2} + Ae^{-\tilde{\lambda}r_2}\left(r_2\frac{d\tilde{\lambda}}{dr_2} + \tilde{\lambda}\right),$$

and, taking into account Eq. (13), we obtain

$$\left(\frac{d\tilde{\lambda}}{dr_2}\right)^{-1} = \frac{1 + r_1\tilde{\lambda}}{\tilde{\lambda}Ae^{-\tilde{\lambda}r_2}} + \frac{r_1 - r_2}{\tilde{\lambda}}.$$

Then,

$$sign\left[\frac{d\left(Re\tilde{\lambda}\right)}{dr_2}\right]_{r_2=r_2^0} = sign\left[Re\left(\frac{d\tilde{\lambda}}{dr_2}\right)^{-1}_{r_2=r_2^0}\right] = sign\left[\sin\omega_0 r_2^0 + r_1\omega_0\cos\omega_0 r_2^0\right].$$

For given $r_1 > 0$, when both $\sin\omega_0 r_2^0$ and $\cos\omega_0 r_2^0$ are positive, then $[d(Re\tilde{\lambda})/dr_2]_{r_2=r_2^0} > 0$, and the purely imaginary roots $\pm i\omega_0$ of Eq. (12) move to the right half-plane when r_2 increases. On the other hand, when both $\sin\omega_0 r_2^0$ and $\cos\omega_0 r_2^0$ are negative, then $[d(Re\tilde{\lambda})/dr_2]_{r_2=r_2^0} < 0$, and the pure imaginary roots $\pm i\omega_0$ of Eq. (12) moves to the left half-plane. Setting

$$\Omega_1 = \left[0, \frac{\pi}{2}\right] \quad \text{and} \quad \Omega_2 = \left[\pi, \frac{3\pi}{2}\right], \tag{16}$$

if $\omega_0 r_2^0 \in \Omega_1$, then $[d(Re\tilde{\lambda})/dr_2]_{r_2=r_2^0} > 0$, while if $\omega_0 r_2^0 \in \Omega_2$, then $[d(Re\tilde{\lambda})/dr_2]_{r_2=r_2^0} < 0$.

Let $\omega_0 r_2^0 \in \Omega_1 \cup \Omega_2$. Then $\lambda = \pm i\omega_0$ are simple roots of Eq. (12). If $\lambda = i\omega_0$ is a repeated root, then $1 + r_1 e^{-i\omega_0 r_1} - r_2^0 A e^{-i\omega_0 r_2^0} = 0$. From Eq. (12), we know $e^{-i\omega_0 r_1} = i\omega_0 + Ae^{-i\omega_0 r_2^0}$. Thus, we find that $-\omega_0 r_1 = (r_2^0 - r_1)A \sin\omega_0 r_2^0$ and $1 = (r_2^0 - r_1)A\cos\omega_0 r_2^0$. As a result, $-\omega_0 r_2^0 = (\sin\omega_0 r_2^0)/(\cos\omega_0 r_2^0)$. If $\omega_0 \tau_2^0 \in \Omega$, then the left hand side of this identity is negative, and the right hand side is positive.

Collecting together all the previous results, we are able to formulate the following theorem

Theorem 3.1. *Let* $\delta_2^0 = r_2^0/|A_1|$, *with* r_2^0 *defined by Eq.* (15), Ω_1 *and* Ω_2 *defined as in Eq.* (16)

1) *The equilibrium point* V^* *of* (4) *is locally asymptotically stable for* $\delta_2 \in [0,\, \delta_2^0)$.

2) *If* $\omega_0 \delta_2^0 |A_1| \in \Omega_1$, *Eq.* (4) *exist Hopf bifurcation near* V^* *when* $\delta_2 = \delta_2^0$.

3) *If* $\omega_0 \delta_2^0 |A_1| \in \Omega_2$, *no Hopf bifurcation exist at* $\delta_2 = \delta_2^0$ *for the given system (2).*

Second method

Following K. Gu, S., Niculescu, and J. Chen [22], we rewrite Eq. (7) as

$$q_0(\lambda) + q_1(\lambda)e^{-\lambda\delta_1} + q_2(\lambda)e^{-\lambda\delta_2} = 0, \tag{17}$$

where

$$q_0(\lambda) = \lambda, \quad q_1(\lambda) = -2p < 0, \quad q_2(\lambda) = 2\left[p + \sqrt{p^2 + \mu(1+2p)}\right] > 0.$$

In order to use these of Gu et al. [22] the following conditions need to be checked:

I) $deg[q_0(\lambda)] \geq \max\{deg[q_1(\lambda)], deg[q_2(\lambda)]\}$;

II) $q_0(0) + q_1(0) + q_2(0) \neq 0$;

III) the three polynomials $q_0(\lambda), q_1(\lambda)$ and $q_2(\lambda)$ do not have common roots;

IV) $\lim\limits_{\lambda\to\infty} \left(\left|\dfrac{q_1(\lambda)}{q_0(\lambda)}\right| + \left|\dfrac{q_2(\lambda)}{q_0(\lambda)}\right| \right) < 1.$

Condition I) holds since $deg[q_0(\lambda)] = 1$ and $deg[q_1(\lambda)]$=$deg[q_2(\lambda)]] = 0$; condition II) is also satisfied as $q_0(0)+q_1(0)+q_2(0) = 2\sqrt{p^2 + \mu(1+2p)} > 0$; condition III) is immediate; concerning condition IV) it follows being the limit as $\lambda \to \infty$ equal to zero.

To characterize the point where has purely imaginary roots, we divide Eq. (17) by $q_0(\lambda)$, and get

$$a(i\omega, \delta_1, \delta_2) = 1 + a_1(\lambda)e^{-\lambda\delta_1} + a_2(\lambda)e^{-\lambda\delta_2} = 0, \tag{18}$$

in which

$$a_1(\lambda) = -\frac{2p}{\lambda}, \qquad a_2(\lambda) = \frac{2\left[p + \sqrt{p^2 + \mu(1+2p)}\right]}{\lambda}.$$

For $\lambda = i\omega$, $\omega > 0$, the three terms in

$$1 + a_1(i\omega)e^{-i\omega\delta_1} + a_2(i\omega)e^{-i\omega\delta_2} = 0,$$

are vectors in the complex plane with the sum of zero. The last two contain all directions by adjusting δ_1, δ_2. As a result, we can replace the investigation on crossing the imaginary axis into a geometric problem of a triangle. Since the length of each edge of a triangle cannot exceed the sum of the lengths of the remaining two edges, we have that ω must satisfy the three conditions

$$|a_1(i\omega)| + |a_2(i\omega)| \geq 1, \qquad -1 \leq |a_1(i\omega)| - |a_2(i\omega)| \leq 1. \tag{19}$$

Hence, we obtain

$$2\sqrt{p^2 + \mu(1+2p)} \leq \omega \leq 4p + 2\sqrt{p^2 + \mu(1+2p)}. \tag{20}$$

Set

$$\Omega = \left[2\sqrt{p^2 + \mu(1+2p)}, 4p + 2\sqrt{p^2 + \mu(1+2p)}\right].$$

Having determined all ω satisfying the triangle conditions (19), the delays δ_1 and δ_2 such that $a(i\omega, \delta_1, \delta_2) = 0$ can be calculated using ω and the orientation of the vectors. By the law of cosines, the internal angles $\theta_1, \theta_2 \in [0, \pi]$ of the triangle are computed by

$$\begin{aligned}\theta_1(\omega) &= \cos^{-1}\left(\frac{1 + |a_1(i\omega)|^2 - |a_2(i\omega)|^2}{2\,|a_1(i\omega)|}\right) \\ &= \cos^{-1}\left(\frac{\omega^2 - 4[p^2 + \mu(1+2p)] - 8p\sqrt{p^2 + \mu(1+2p)}}{4p\omega}\right)\end{aligned}$$

and

$$\begin{aligned}\theta_2(\omega) &= \cos^{-1}\left(\frac{1 + |a_2(i\omega)|^2 - |a_1(i\omega)|^2}{2\,|a_2(i\omega)|}\right) \\ &= \cos^{-1}\left(\frac{\omega^2 + 4[p^2 + \mu(1+2p)] + 8p\sqrt{p^2 + \mu(1+2p)}}{4\omega\left[p + \sqrt{p^2 + \mu(1+2p)}\right]}\right).\end{aligned}$$

Since the triangle can be located above and under the real axis, the following two equations must hold for δ_1 and δ_2,

$$\left\{\arg[a_1(i\omega)e^{-i\omega\delta_1}] + 2m\pi\right\} \pm \theta_1(\omega) = \pi, \qquad \left\{\arg[a_2(i\omega)e^{-i\omega\delta_2}] + 2n\pi\right\} \mp \theta_2(\omega) = \pi.$$

Noting that

$$\arg[a_1(i\omega)] = \arg\left[\frac{2p}{\omega}i\right] = \frac{\pi}{2}, \qquad \arg[a_2(i\omega)] = \arg\left[-\frac{2\left[p + \sqrt{p^2 + \mu(1+2p)}\right]}{\omega}i\right] = \frac{3\pi}{2},$$

we derive the following values of the delays

$$\delta_1^{m\pm}(\omega) = \frac{\arg[a_1(i\omega)] + (2m-1)\pi \pm \theta_1(\omega)}{\omega}, \qquad \delta_2^{n\mp}(\omega) = \frac{\arg[a_2(i\omega)] + (2n-1)\pi \mp \theta_2(\omega)}{\omega},$$

that is

$$\delta_1^{m\pm}(\omega) = \frac{1}{\omega}\left[-\frac{\pi}{2} + 2m\pi \pm \theta_1(\omega)\right], \qquad \delta_2^{n\mp}(\omega) = \frac{1}{\omega}\left[\frac{\pi}{2} + 2n\pi \mp \theta_2(\omega)\right],$$

where $m = m_0^{\pm}, m_0^{\pm} + 1, m_0^{\pm} + 2, \ldots$, and $n = n_0^{\mp}, n_0^{\mp} + 1, n_0^{\mp} + 2, \ldots$, with $n_0^+, n_0^-, m_0^+, m_0^-$ ($n_0^+ \leq n_0^-$ and $m_0^+ \geq m_0^-$) the smallest possible integers for which the corresponding delays $\delta_1^{m_0\pm}, \delta_2^{n_0\mp}$ are non-negative.

Proposition 1. *Assume that* (20) *holds. The stability switching curve is described by* $C_1(m,n) \cup C_2(m,n)$, *where*

$$C_1(m,n) = \left\{\delta_1^{m+}(\omega), \delta_2^{n-}(\omega)\right\} \quad \textit{and} \quad C_2(m,n) = \left\{\delta_1^{m-}(\omega), \delta_2^{n+}(\omega)\right\}.$$

$C_1(m,n)$ *and* $C_2(m,n)$ *have the same starting point, and* $C_1(m,n)$ *and* $C_2(m,n)$ *have the same end point.*

Proof. The geometric characteristics of the stability switching curve is determined by the behavior of $C^+(m,n)$ and $C^-(m,n)$ at the initial and end points of Ω. Notice first that

$$\theta_1(2\sqrt{p^2+\mu(1+2p)}) = \cos^{-1}(-1) = \pi, \qquad \theta_1(4p+2\sqrt{p^2+\mu(1+2p)}) = \cos^{-1}(1) = 0$$

and

$$\theta_2(2\sqrt{p^2+\mu(1+2p)}) = \cos^{-1}(1) = 0, \qquad \theta_2(4p+2\sqrt{p^2+\mu(1+2p)}) = \cos^{-1}(1) = 0$$

Therefore, the starting points of $C_1(m,n)$ and $C_2(m,n)$ are given by

$$\delta_1^{m\pm}(\omega) = \frac{1}{\omega}\left[-\frac{\pi}{2} + 2m\pi\right], \qquad \delta_2^{n\mp}(\omega) = \frac{1}{\omega}\left[\frac{\pi}{2} + 2n\pi\right],$$

$$\left\{\delta_1^{m+}(2\sqrt{p^2+\mu(1+2p)}), \delta_2^{n-}(2\sqrt{p^2+\mu(1+2p)})\right\} =$$
$$\left\{\frac{1}{2\sqrt{p^2+\mu(1+2p)}}\left(\frac{\pi}{2} + 2m\pi\right), \frac{1}{2\sqrt{p^2+\mu(1+2p)}}\left(\frac{\pi}{2} + 2n\pi\right)\right\}$$

and

$$\left\{\delta_1^{m-}(2\sqrt{p^2+\mu(1+2p)}), \delta_2^{n+}(2\sqrt{p^2+\mu(1+2p)})\right\} =$$
$$\left\{\frac{1}{2\sqrt{p^2+\mu(1+2p)}}\left(-\frac{3\pi}{2} + 2m\pi\right), \frac{1}{2\sqrt{p^2+\mu(1+2p)}}\left(\frac{\pi}{2} + 2n\pi\right)\right\},$$

respectively, while for the end points of $C_1(m,n)$ and $C_2(m,n)$ we have

$$\left\{\delta_1^{m+}(4p+2\sqrt{p^2+\mu(1+2p)}), \delta_2^{n-}(4p+2\sqrt{p^2+\mu(1+2p)})\right\} =$$
$$\left\{\frac{1}{4p+2\sqrt{p^2+\mu(1+2p)}}\left(-\frac{\pi}{2} + 2m\pi\right), \frac{1}{4p+2\sqrt{p^2+\mu(1+2p)}}\left(\frac{\pi}{2} + 2n\pi\right)\right\}$$

and

$$\left\{\delta_1^{m-}(4p+2\sqrt{p^2+\mu(1+2p)}), \delta_2^{n+}(4p+2\sqrt{p^2+\mu(1+2p)})\right\} =$$
$$\left\{\frac{1}{4p+2\sqrt{p^2+\mu(1+2p)}}\left(-\frac{\pi}{2} + 2m\pi\right), \frac{1}{4p+2\sqrt{p^2+\mu(1+2p)}}\left(\frac{\pi}{2} + 2n\pi\right)\right\},$$

respectively. The conclusion holds. □

4 Numerical examples

In this section, by using numerical simulations, we will verify the correctness of the above results. These numerical simulations agree well with our theoretical results.

In order to facilitate comparison, we first select the parameters as $\mu = 0.3$ $p = 0.05$ and $\tau_1^* = 1$, and then we have

$$\omega_0 = 11.6100, \ \tau_2^* = 1.3245.$$

The dynamical behavior is illustrated in Fig. 2 and Fig 3. From Fig. 2 and Fig. 3, one can find that trajectories converge to the equilibrium point when $\tau_2 < \tau_2^*$, that is, V^* is stable. If τ_2 is increased to pass τ_2^*, V^* is un stable and a Hopf bifurcation occurs.

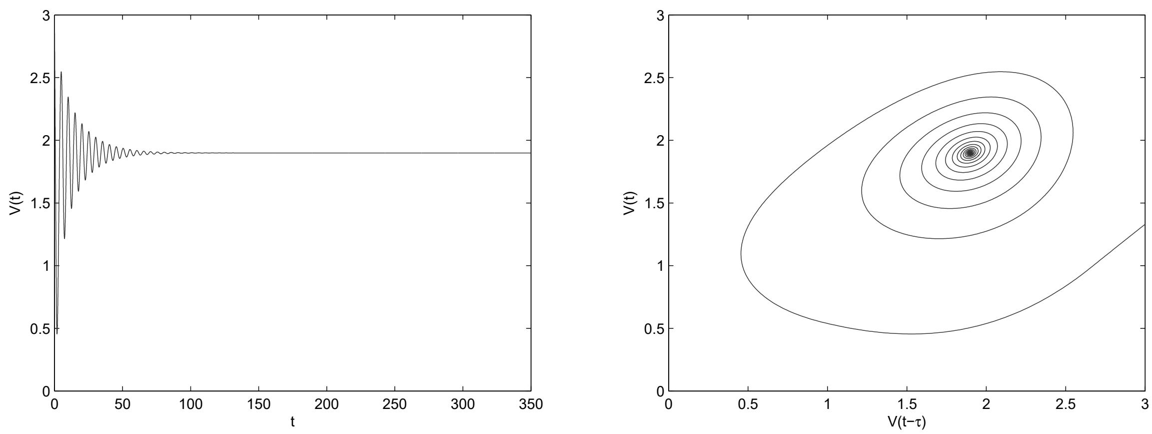

Figure 2: Phase portrait of model Eq. (3) for $\tau_1 = 1, \ \tau_2 = 1.2$.

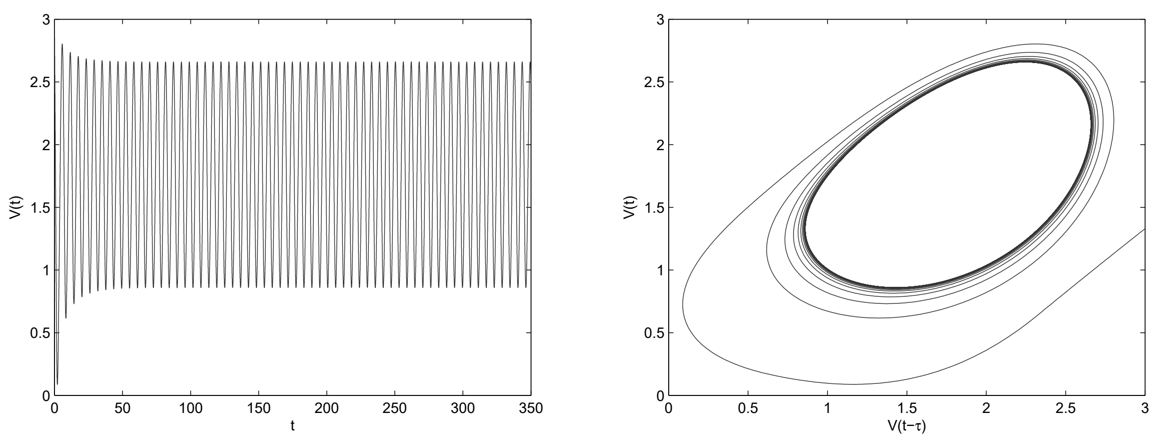

Figure 3: Waveform plot and phase portrait of model Eq. (3) for $\tau_1 = 1, \ \tau_2 = 1.4$. Here initial value is $V(0) = 3$.

At the same time, from Figs 3 and 4, periodic solutions bifurcate from V^* is stable.

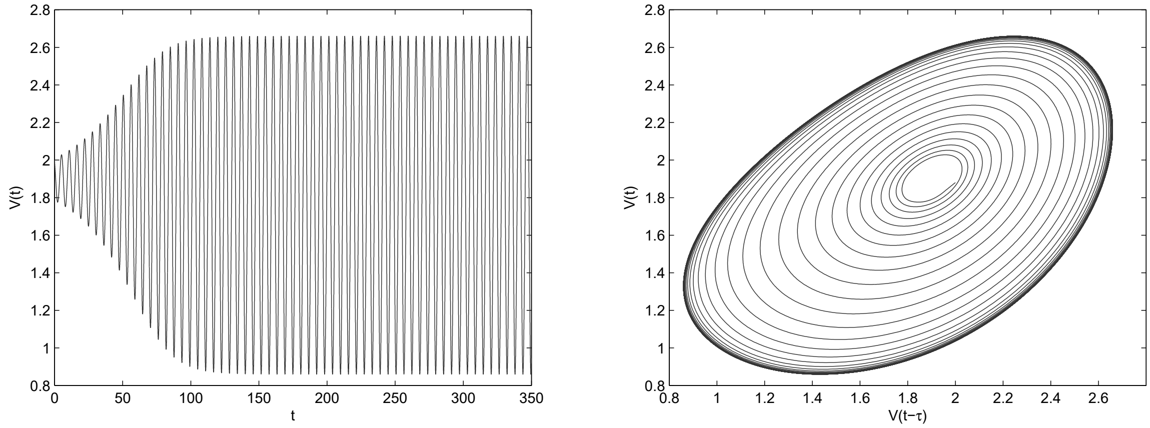

Figure 4: Waveform plot and phase portrait of model Eq. (3) for $\tau_1 = 1,\ \tau_2 = 1.4$. Here initial value is $V(0) = 2$.

Our next step is choosing appropriate values of τ_2 to control the networks. If $\tau_2 > 1.3245$, a Hopf bifurcation occurs as well know. As τ_2 increase and $\omega_0\delta_2^0\,|A_1| \in \Omega_1$, the amplitude of the periodic solution bifurcate from V^* become larger (Fig. 5).

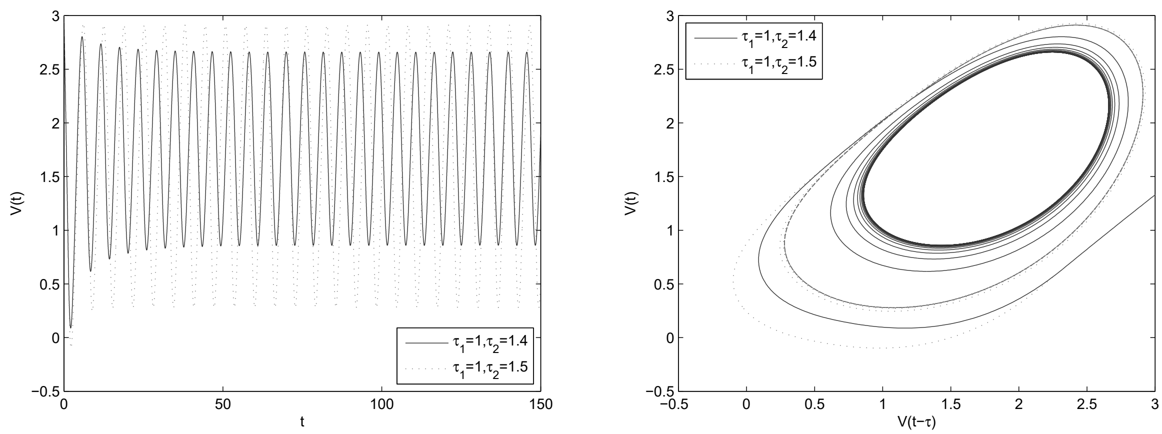

Figure 5: Waveform plot and phase portrait of model Eq. (3) for τ_2 increase. Here initial value is $V(0) = 2$.

At the same time, we can using μ and p as control parameters to control the networks (Fig. 6).

5 Conclusions and discussion

In this paper, we study the stability and Hopf bifurcation of a complex network model with two delays, and discuss the critical parameter values for bifurcation in detail. By choosing time delayτ_2, μ and p parameter, the bifurcation behavior of the system can be well changed and the bifurcation can be controlled effectively. The numerical results verify the correctness of our theoretical results.

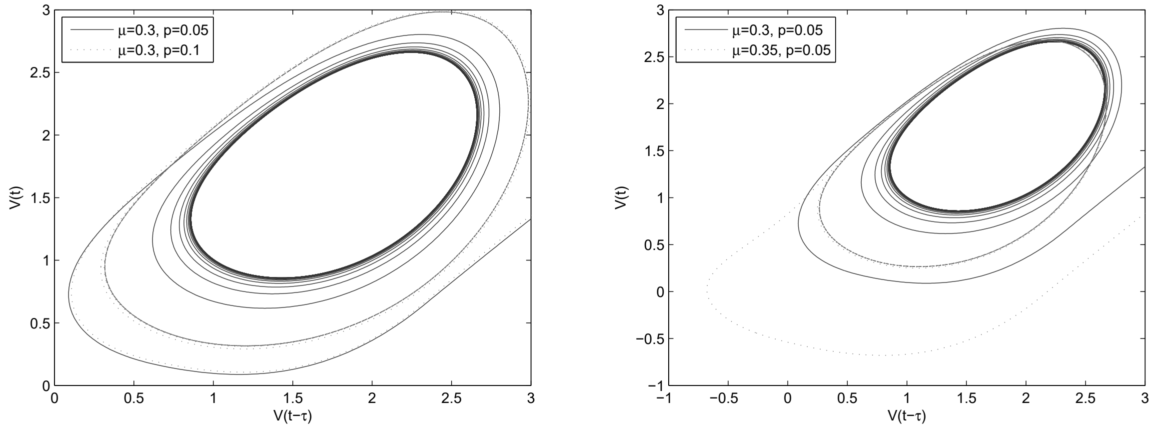

Figure 6: Phase portrait of model Eq. (3) for different μ and p. Here the initial value is $\tau_1 = 1$, $\tau_2 = 1.4$, and $V(0) = 3$.

Note that a network model with distributed delays is more general than that with discrete delays. Our further study is propose to generalize the delayed model considered in this paper as

$$\dot{V}(t) = 1 + 2p \int_{-\infty}^{t} V(r)g(t-r,m)dr - \mu(1+2p)\left[\int_{-\infty}^{t} V(r)g(t-r,n)dr\right]^2,$$

where $g(\cdot)$ is a gamma distribution.i.e.

$$g(a,l) = \left(\frac{l+1}{T}\right)^{l+1} \frac{a^l e^{-\frac{l+1}{T}a}}{l!}.$$

These are beyond the scope of the present paper and will be further investigated elsewhere in the near future.

References

[1] D. J. Watts, S. H. Strogatz, Colletive dynamics of ‘small-world’ networks, Nature 393 (1998) 440-442.

[2] R. Albert, A. L. Barab*á*sr, Statistical mechanics of complex networks, Reviews of Modern Physics 74 (2002) 47-97.

[3] M. E. J. Newman, The structure and function of complex networks, SIAM Review 45 (2003) 167-256.

[4] X. F. Wang, Complex networks: topology, dynamics and synchronization, Int. J. Bifur. Chaos 12 (2002) 885-916.

[5] M. E. J. Newman, D. J. Watts, Renormalization group analysis of the small-world network model, Phys. Lett. A 263 (1999) 341-346.

[6] X. S. Yang, Chaos in small-world networks, Phys. Rev. E 63 (2001) 046206.

[7] G. Chen, J. L. Moiola, H. O. Wang, Bifurcation control: Theories, methods, and applications, Int. J. Bifur. Chaos 10 (2000) 511-548.

[8] D. W. Berns, J. L. Moiola, G. Chen, Feedback control of limit cycle amplitudes from a frequency domain approach, Automatica 34 (1998) 1567-1573.

[9] E. Ott, C. Grebogi, J. A. Yorke, Controlling chaos, Phys. Rev. Lett. 64 (1990) 1196-1199.

[10] D. J. Hill, I. A. Hiskens, Y. Wang, Robust adaptive or nonlinear control for modern power systems, Proc. 32nd IEEE Conf. Decision and Control, San Antonio, TX, (1993) 2335-2340.

[11] Z. Chen, P. Yu, Hopf bifurcation control for an internet congestion model, Int. J. Bifur. Chaos 15 (2005) 2643-2651.

[12] Q. Zhu, Razumikhin-type theorem for stochastic functional differential equations with Lévy noise and Markov switching, Internat. J. Control 90 (2017) 1703-1712.

[13] Z. Liu, K. W. Chung, Hybrid control of bifurcation in continuous nonlinear dynamical systems, Int. J. Bifur. Chaos 15 (2005) 3895-3903.

[14] X. S. Luo, G. Chen, B. H. Wang, J. Q. Fang, Hybrid control of period-doubling bifurcation and chaos in discrete nonlinear dynamical systems, Chaos Solit. Fract. 18 (2003) 775-783.

[15] Y. Guo, Global stability analysis for a class of Cohen-Grossberg neural network models, Bull. Korean Math. Soc. 49 (2012) 1193-1198.

[16] R. Song, Q. Zhu, Stability of linear stochastic delay differential equations with infinite Markovian switchings, Internat. J. Robust Nonlinear Control 28 (2018) 825-837.

[17] X. Li, Y. Y. Jin, G. Chen, Complexity and synchronization of the World Trade Web, Physica A 328 (2003) 287-296.

[18] X. Li, G. Chen, C. Li, Stability and bifurcation of disease spreading in complex networks, Int. J. Systems Science 35 (2004) 527-536.

[19] C. Li, G. Chen, X. Liao, J. Yu, Hopf bifurcation in an Internet congestion control model, Chaos, Solitons & Fractals 19 (2004) 853-862.

[20] M. J. Piotrowska, A remark on the ODE with two discrete delays, J. Math. Anal. Appl., 329 (2007), 664-676.

[21] X. Li, S. Ruan, J. Wei, Stability and bifurcation in delay-differential equations with two delays, J. Math. Anal. Appl., 236 (1999), 254-280.

[22] K. Gu, S. L. Niculescu, J. Chen, On stability crossing curves for general systems with two delays, Journal of Mathematic Analysis and Application, 311 (2005), 231-252.

[23] M. E. Bleich, J. E. S. Socolar, Stability of periodic orbits controlled by time-delay feedback, Phys. Lett. A 210 (1996) 87-94.

[24] Q. Zhu, S. Song, T. Tang, Mean square exponential stability of stochastic nonlinear delay systems, Internat. J. Control 90 (2017) 2384-2393.

[25] R. Raja, Q. Zhu, S. Senthilraj, R. Samidurai, Improved stability analysis of uncertain neutral type neural networks with leakage delays and impulsive effects, Appl. Math. Comput. 266 (2015) 1050-1069.

[26] X. Zhou, Q. Zhu, Robust finite-time state estimation for uncertain discrete-time Markovian jump neural networks with two delay components, Neurocomputing 283 (2018) 64-72.

[27] Y. Guo, Mean square global asymptotic stability of stochastic recurrent neural networks with distributed delays, Appl. Math. Comput. 215 (2009) 791-795.

[28] Q. Zhu, X. Li, X. Yang, Exponential stability for stochastic reaction-diffusion BAM neural networks with time-varying and distributed delays, Appl. Math. Comput. 217 (2011) 6078-6091.

[29] Q. Zhu, X. Li, Exponential and almost sure exponential stability of stochastic fuzzy delayed Cohen-Grossberg neural networks, Fuzzy Sets and Syst. 203 (2012) 74-94.

[30] M. Xiao, W.X. Zheng, J. Cao, Hopf bifurcation of an (n+1)-neuron bidirectional associative memory neural network model with delays. IEEE Trans Neural Netw Learn Syst 24, 118-132, 2013.

[31] M. Xiao, W.X. Zheng, J. Cao, Bifurcation and control in a neural network with small and large delays. Neural Networks 44, 132-142, 2013.

[32] Z. Cheng, J. Cao, Bifurcation and stability analysis of a neural network model with distributed delays. Nonlinear Dynamics 48, 363-373, 2006.

[33] Z. Cheng, Y. Wang, J. Cao, Stability and Hopf bifurcation of a neural network model with distributed delays and strong kernel, Nonlinear Dyn 86 (2016) 323-335.

[34] Z. Cheng, J. Cao, Hybrid control of Hopf bifurcation in complex networks with delays, Neurocomputing 73 (2010) 3139-3146.

[35] J. Hale, Theory of Functional Differential Equations, Springer-Verlag, New York, 1977.