Università Politecnica delle Marche

DIPARTIMENTO DI SCIENZE ECONOMICHE E SOCIALI

Asset Management with TEV and VaR Constraints: the Constrained Efficient Frontiers

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Ottobre 2013

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Abstract

It is well known that investors usually assign part of their funds to asset managers who are given the task of beating a benchmark portfolio. On the other hand, the risk management office could impose some restrictions to the asset managers' activity in order to mantain the overall portfolio risk under control. This situation could lead managers to select non efficient portfolios in the total return and absolute risk perspective.

In this paper we focus on portfolio efficiency when a tracking error volatility (TEV) constraint holds. First, we define the TEV Constrained-Efficient Frontier (ECTF), a set of TEV constrained portfolios that are mean-variance efficient. Second, we discuss the effects on such boundary when a VaR and/or a variance restriction is also added.

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Asset Management with TEV and VaR Constraints: the Constrained Efficient Frontiers[*]

Giulio Palomba   *Luca Riccetti*

1   Introduction

It is well known that investors usually assign part of their funds to asset managers who are given the task of beating a benchmark portfolio. On the other hand, the asset managers' activity is usually constrained by the risk management department in order to keep the overall portfolio risk close to that of the selected benchmark. A common practice is to restrict the portfolio Tracking Error Volatility (TEV), a measure of relative risk, defined as

$$T_0 = (\omega_P - \omega_B)'\Omega(\omega_P - \omega_B), \tag{1}$$

where $(\omega_P - \omega_B)$ is a n-dimensional vector containing the active portfolio weights, defined as the deviations from a benchmark portfolio B, while the $n \times n$ matrix Ω represents the covariance matrix of n risky asset returns and the scalar T_0 is a fixed value for TEV.

Unfortunately, equation (1) leads asset managers to select non efficient portfolios in σ_P, μ_P space, where σ_P and μ_P indicate the absolute risk and the expected portfolio return respectively. Roll (1992) shows that portfolios that optimize TEV are generally mean-variance suboptimal because they do not belong to the Mean-Variance Frontier (MVF) introduced by Markowitz (1952). Indeed, he defines the so-called "Mean-TEV Frontier" (MTF) that is the set of portfolios with a given expected return and the smallest TEV: this is a parabola in the σ_P^2, μ_P space, which is a horizontal translation to the right of the MVF. The horizontal distance is commonly known as the portfolio efficiency loss (of the benchmark) δ_B. It is a nonnegative distance because the benchmark belongs to the MTF, therefore can not lie to the left of the MVF. See Alexander and Baptista (2008) or Palomba and Riccetti (2012) for details.

In literature, some methods to mitigate the portfolio efficiency loss have been experimented, imposing different limits on the amount of risk that asset managers can take: Roll (1992) imposed a restriction on portfolio's beta,

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while Jorion (2003) added a constraint on portfolio variance into a TEV constrained asset allocation; subsequently, Alexander and Baptista (2008, 2010) proposed two different constrained asset allocation strategies: in the former contribution they imposed a Value-at-Risk (VaR) restriction by introducing the Constrained Mean-TEV Frontier (CMTF), a set of portfolios that satisfy the VaR constraint and have the smaller TEV when they are compared to other portfolios with the same expected return. In the latter, they have tried to minimise TEV by setting a target on the *ex-ante* portfolio Alpha, defined as the difference between the portfolio expected excess return and the benchmark's beta-adjusted expected excess return. Another interesting attempt is made by Bertrand (2010), who introduces the Iso-Information Ratio frontiers by fixing the manager's risk aversion, while the tracking error is allowed to vary. Finally, Palomba and Riccetti (2012) summarized some of these contributions by defininig the "Fixed VaR-TEV Frontier" (FVTF) as a set of portfolios which satisfy the VaR constraint and guarantee a TEV that does not exceed an *ex-ante* fixed value T_0. They also provide an accurate geometrical analysis and calculate the analytical solutions for all the intersections between various portfolio frontiers. Specifically, they discuss all interactions between the MVF (Markowitz, 1952), the MTF (Roll, 1992), the Constrained TEV Frontier or CTF (Jorion, 2003) and the Constrained VaR Frontier or CVF (Alexander and Baptista, 2008).

In this paper, we focus upon the TEV constrained frontiers, namely the CTF (Jorion, 2003) and the FVTF (Palomba and Riccetti, 2012). In particular, we draw attention to the subset of these boundaries which is efficient in terms of the variance (absolute risk) and the expected return. We also show that, in general, the efficient FVTF is a subset of the efficient CTF. Moreover, the imposition of restrictions to portfolio variance and VaR is discussed. All the analysis is conducted in variance-mean return space σ_P^2, μ_P, while all figures are illustrated in the standard σ_P, μ_P space. The field of investigation is restricted to the usual framework of unlimited short sales, quadratic utility function and normally distributed returns.

This paper proceeds as follows: section 2 contains a short review of the TEV and VaR constrained frontiers, while in section 3 we use the mean-variance dominance criterion to define three new constrained-efficient frontiers, we discuss their properties and also provide a short empirical analysis. Finally, section 4 concludes.

2   Constrained Frontiers

Jorion (2003) shows that a portfolio optimization with a constraint on maximum TEV leads asset managers to select a feasible portfolio lying inside the CTF, an elliptical boundary in the σ_P^2, μ_P space. The benchmark portfolio

usually lies inside this frontier.[1] When a TEV constraint is set, a reasonable method to avoid overly risky portfolios is to choose a point on the CTF with the same variance of the benchmark. This implies the condition $\mu_B < \mu_P < \mu_{J_1}$, where μ_B is the benchmark return and μ_{J_1} is the maximum return available on the CTF, because asset managers will select that portfolio lying in the upper part of the ellipse.

Alexander and Baptista (2008) focus on the imposition of a Value-at-Risk (VaR) constraint. The VaR is the θ-quantile of the portfolio distribution, where the confidence level is $0.5 < \theta < 1$; therefore, it is defined as the minimum loss that will be sustained with probability $1-\theta$. Under normality, this can be written $V_0 = z_\theta \sigma_P - \mu_P$, where z_θ is the critical value taken from the standardised normal distribution.[2] The VaR$= V_0$ is fixed by risk managers and corresponds to the intercept of the CVF

$$\mu_P = z_\theta \sigma_P - V_0, \tag{2}$$

a linear frontier in σ_P, μ_P space, where the slope is $z_\theta > 0$ and the intercept $(-V_0)$ should be positive. Portfolios that satisfy the VaR constraint lie to the left/above halfplane generated by the line represented by equation (2).

The CTF and the CVF, together with the hyperbolic functions MVF (Markowitz, 1952) and MTF(Roll, 1992), are plotted in Figure 1, in order to define the CMTF (Alexander and Baptista, 2008) and the FVTF (Palomba and Riccetti, 2012) in σ_P, μ_P space.[3]

Figure 1 (a) shows that the CMTF is composed by segment $\overline{M_1 R_1}$, arc $\overarc{R_1 R_2}$ and segment $\overline{R_2 M_2}$. This boundary is independent of any TEV restriction because it only depends on the VaR restriction. From the economic perspective, the following problems arise: the VaR constraint is independent of the benchmark and the benchmark itself can not satisfy the VaR constraint. In particular, portfolios on segment $\overline{M_1 K_1}$ (except K_1) and on segment $\overline{K_2 M_2}$ (except K_2) are not admissible because they do not satisfy the TEV restriction. For the same reason, there are many portfolios contained in the CTF surface area that do not satisfy the VaR constraint. Palomba and Riccetti (2012) address this problem by claiming that the VaR

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[1] Jorion (2003) also shows extreme cases in which the benchmark lies outside the CTF. This happens only for large TEV values, more precisely $T_0 > 2\delta_B$.

[2] All the contributions used in this framework (from Markowitz, 1952, onwards) are based on the strong hypothesis of normally distributed returns under which the portfolio variance and the expected return are the only relevant variables. Obviously this assumption can be removed or relaxed in favor of more general distributions. For example, assuming a t-Student distribution with (small) n d.o.f., the VaR line CVF is steeper than that obtained under normality (the quantile in equation (2) is $|t_{n,\theta}| > |z_\theta|$). This could be investigated in further resarch.

[3] In order to save space, Figure 1 provides the most intuitive graphical representation of the CMTF and the FVTF ("large bound" case). For a more general discussion see Alexander and Baptista (2008) and Palomba and Riccetti (2012). Moreover, Appendix A-1 contains a technical overview of all portfolio frontiers used in this paper.

Figure 1: Portfolio frontiers in the σ_P^2, μ_P space.

(a) The Constrained Mean-TEV Frontier

(b) The Fixed VaR-TEV Frontier
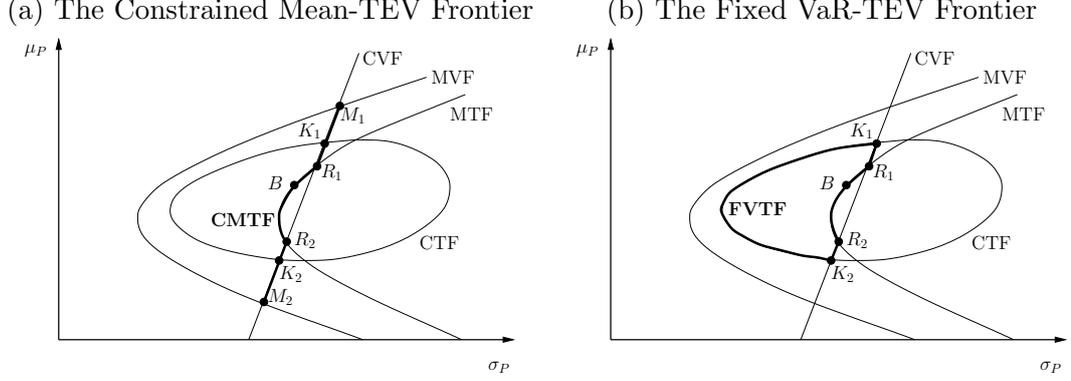
limit V_0 has to be compatible with the TEV restriction. From the geometrical point of view, V_0 must not be inferior to a threshold V_K that makes the straight line CVF tangent to the left of the CTF. Conversely, when the TEV and the VaR restrictions are not compatible ($V_0 < V_K$), the related frontiers CVF and CTF do not intersect.

Palomba and Riccetti (2012) face these problems, introducing the "Fixed VaR-TEV Frontier" (FVTF), a new boundary containing all the admissible portfolios lying to the left of the MTF, satisfy the VaR constraint and guarantee a TEV that does not exceed an ex-ante fixed value T_0. When this frontier operates, the TEV and the VaR restrictions are compatible. As Figure 1 (b) shows, this boundary is formed by the left arc $\overset{\frown}{K_1 K_2}$ on the CTF, the segments $\overline{K_1 R_1}$ and $\overline{R_2 K_2}$ on the CVF and the arc $\overset{\frown}{R_1 R_2}$ on the MTF.

An important property of such constrained frontiers is $\delta_P \leq \delta_B$ for any portfolio P, where the efficiency loss δ_P is the horizontal distance between a given portfolio P and the MVF in the σ_P^2, μ_P space.

Observing the FVTF, an interesting trade-off between relative and absolute risk emerges: an increase of the TEV expands the CTF surface area, thus augmenting the possibility of reducing the overall portfolio variance. As a consequence, each portfolio contained inside the FVTF has a reduced efficiency loss when it is compared with any portfolio lying on the CMTF. Other important properties of portfolios belonging to such constrained frontiers in σ_P, μ_P space derive from the following Definitions.

Definition 1 *Dominance principle: portfolio X dominates portfolio Y if and only if $\mu_X \geq \mu_Y$ and $\sigma_X \leq \sigma_Y$ and at least one strict inequality holds.*

Definition 2 *Portfolio X is said to be efficient when it is not dominated by other portfolios.*

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Accordingly, for any expected return $\mu_{K_2} \leq \mu_P < \mu_{K_1}$, where portfolios K_1 and K_2 are the intersections between the elliptical frontier and the linear boundary CVF, the following properties emerge:

- all portfolios belonging to the CMTF are not dominated by those belonging to the MTF,

- all portfolios belonging to the FVTF are not dominated by those belonging to the CMTF.

Briefly, when the asset manager has to face restrictions on maximum TEV and VaR which are not compatible, the fund has no way of existing because the VaR line CVF does not intersect the elliptical CTF; in this situation, the risk management has to modify at least one constraint assigned to the manager. Otherwise, when the maximum TEV and the VaR restrictions are compatible, portfolios lying on the CMTF are surely dominated. Finally, according to the dominance criterion, an asset manager could invest on portfolios contained in the efficient set of the MVF when a TEV limit has not been imposed.

3 The Constrained-Efficient Frontiers

In this section we use the dominance criterion already introduced in Definition 1, in order to focus our attention on the efficient subset of any given constrained portfolio frontier. Therefore the notion of portfolio efficiency is conditional to an existing TEV restriction. Specifically, we propose an analysis in which managers have to face several risk constraints by selecting only those admissible portfolios that are non dominated in terms of absolute risk and expected return.

Before starting the analysis, some notation has to be provided. Given n risky assets with expected returns μ and variance-covariance matrix Ω, the following constants are defined: $a = \iota'\Omega^{-1}\iota$, $b = \iota'\Omega^{-1}\mu$, $c = \mu'\Omega^{-1}\mu$ and $d = c - b^2/a$, where ι is a n-dimensional column vector in which each element is 1. The minimum variance portfolio of the "mean-variance frontier" C has expected return $\mu_C = b/a$ and variance $\sigma_C^2 = 1/a$. All these values are independent of managers' strategies because they are derived exclusively from the available data. The benchmark portfolio is $B \equiv (\sigma_B^2, \mu_B)$. In order to save space, we assume the parameter $\Delta_1 = \mu_B - \mu_C$ strictly positive. Under these conditions, the slope of the CTF horizontal axis is positive. Moreover, all the analysis will be carried out by imposing the TEV restriction $T_0 < \delta_B = \Delta_2 - \Delta_1^2/d$, where $\Delta_2 = \sigma_B^2 - \sigma_C^2$, which prevents any intersection between the CTF and the MVF.

3.1 The Efficient Constrained TEV Frontier (ECTF)

Definition 3 *The "Efficient Constrained TEV Frontier" (ECTF) is the set of portfolios that are on the "Constrained TEV Frontier" (CTF) and are not dominated in mean-variance terms.*

Figure 2: The Efficient Constrained TEV Frontier (ECTF)
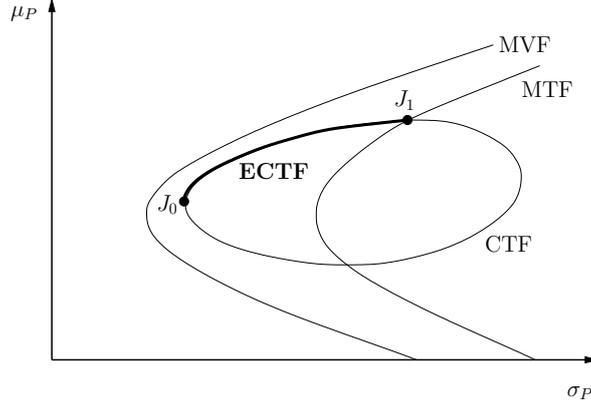
In practice, the ECTF corresponds to the upper arc $\widehat{J_0 J_1}$ on the CTF where the TEV assumes its maximum value T_0, as Figure 2 clearly shows. In the σ_P^2, μ_P space, the extremal portfolio

$$J_0 \equiv \left(\sigma_B^2 + T_0 - 2\sqrt{T_0 \Delta_2}, \mu_B - \Delta_1 \sqrt{T_0/\Delta_2} \right) \qquad (3)$$

is that of minimum variance, while the portfolio

$$J_1 \equiv \left(\sigma_B^2 + T_0 + 2\Delta_1 \sqrt{T_0/d}, \mu_B + \sqrt{dT_0} \right) \qquad (4)$$

corresponds to the intersection between the CTF and the MTF, which represents the portfolio with the highest expected return available. Analytical details about these portfolios are provided by Jorion (2003), while some useful properties regarding J_0 are provided in Appendix A-2.

It is worth noting that the TEV restriction enters equations (3) and (4), thus determining the position of the arc $\widehat{J_0 J_1}$. When $T_0 = 0$ the two portfolios collapse to the benchmark,[4] therefore $J_0 \equiv J_1 \equiv B$; otherwise, the expected return and the absolute risk of portfolio J_1 are always greater than those of portfolio J_0, independently of the horizontal slope of the CTF.

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[4]In short: the following relationships

- $\mu_{J_1} - \mu_{J_0} \geq 0 \Rightarrow \sqrt{T_0}[\sqrt{d} + \Delta_1/\sqrt{\Delta_2}] \geq 0$
- $\sigma_{J_1} - \sigma_{J_0} \geq 0 \Rightarrow 2\sqrt{T_0}[\Delta_1/\sqrt{d} + \sqrt{\Delta_2}] \geq 0$

are true by construction for any Δ_1.

The concept of constrained-efficiency could be extended from the mean-variance efficiency to the mean-VaR or the mean-CVaR (Conditional Value-at-Risk) efficiency. Indeed, the mean-VaR efficient frontier is a proper subset of the mean-CVaR efficient frontier which, in turn, is a proper subset of the efficient set of the MVF (for a detailed discussion, see Alexander, 2009).

3.2 The Efficient Constrained TEV-Variance Frontier (ECTVF)

Definition 4 *The "Efficient Constrained TEV-Variance Frontier" (here-after ECTVF) is the subset of the ECTF in which all portfolios have a variance not superior to a maximum threshold (σ_0^2).*

Let S_0 be a reference portfolio in the σ_P^2, μ_P space, which lies inside the CTF and for which the overall portfolio variance must belong to the interval $\sigma_{J_0}^2 \leq \sigma_0^2 \leq \sigma_{J_1}^2$, therefore

$$\sigma_B^2 + T_0 - 2\sqrt{T_0 \Delta_2} \leq \sigma_0^2 \leq \sigma_B^2 + T_0 + 2\Delta_1 \sqrt{T_0/d}. \tag{5}$$

This condition is crucial to have a constraint on portfolio variance. For instance, the restriction $\sigma_0^2 < \sigma_{J_0}^2$ does not permit to satisfy the condition TEV$\leq T_0$, while choosing a portfolio with $\sigma_0^2 > \sigma_{J_1}^2$ does not produce any restriction to the ECTF. Geometrically, the variance constraint corresponds to a vertical line crossing the CTF in two portfolios, namely S_2 and the TEV efficient $S_1 \equiv (\sigma_0^2, \mu_{S_1})$. In practice, the ECTVF consists of the arc starting from portfolio J_0 to S_1. Clearly, when $\sigma_0^2 = \sigma_{J_1}^2$ the ECTVF and the ECTF coincide. For simplicity and without any loss of generality, Figure 3 shows the case of $S_0 \equiv B$ for which $\sigma_0^2 = \sigma_B^2$ is the variance associated to the benchmark.[5]

3.3 Efficient Fixed VaR-TEV Frontier (EFVTF)

Definition 5 *The "Efficient Fixed VaR-TEV Frontier" (EFVTF) is the subset of the ECTF in which all portfolios have a VaR not superior to a maximum threshold (V_0).*

The EFVTF is the subset of the ECTF which lies to the left of the straight line CVF; in practice, it is given by the intersection between the FVTF and the ECTF. In this context, the condition

$$V_K \leq V_0 \leq \max\{V_{J_0}, V_{J_1}\} \tag{6}$$

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[5]For example, Jorion (2003) shows that, if the restriction $\sigma_0 = \sigma_B$ is imposed, the expected return on the CTF is

$$\mu_{S_1} = \mu_B - T_0 \frac{\Delta_1}{2\Delta_2} + \sqrt{T_0 \left(d - \frac{\Delta_1^2}{\Delta_2}\right)\left(1 - \frac{T_0}{4\Delta_2}\right)}.$$

Figure 3: ECTVF with variance restriction $\sigma_0 = \sigma_B$
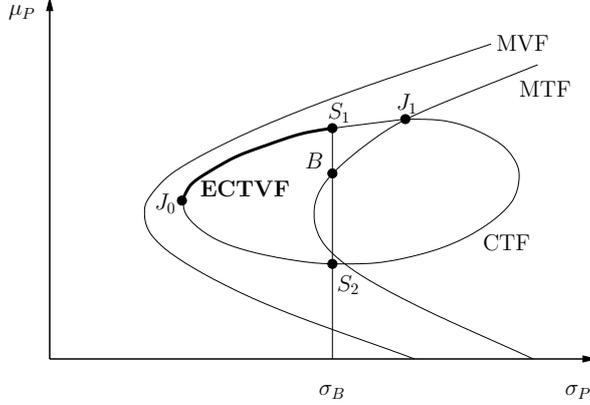
must hold. Indeed, the VaR is restricted to be not inferior the V_K value for which the CVF is tangent to the ECTF. On the other hand, $\max\{V_{J_0}, V_{J_1}\}$ represents the VaR bound beyond which the frontiers EFVTF and ECTF coincide.

The determination of the maximum value between V_{J_0} and V_{J_1} depends on the slope of the straight line CVF. Since the straight line passing through J_0 and J_1 has the slope

$$\hat{z} = \frac{\mu_{J_1} - \mu_{J_0}}{\sigma_{J_1} - \sigma_{J_0}}, \tag{7}$$

where σ_{J0} and σ_{J1} are the standard errors associated to portfolios J_0 and J_1, the maximum is V_{J_1} when $z_\theta > \hat{z}$ and the maximum is V_{J_0} otherwise. Figure 4 illustrates all the scenarios about the EFVTF related to the condition (6) when $z_\theta > \hat{z}$.

Figure 4 (a) shows the restriction $V_0 = V_K$ for which the EFVTF collapses to tangency portfolio K. Palomba and Riccetti (2012) define this restriction as the medium bound.

Figures 4 (b) and (c) exhibit the cases in which $V_K < V_0 \leq V_{J_0}$ and $V_{J_0} < V_0 < V_{J_1}$, where V_{J_0} is the VaR associated to the minimum variance porftolio J_0, and V_{J_1} is the VaR of the straight line CVF passing through the maximum expected return portfolio J_1. In the former case the CVF intersects the ECTF in portfolios K_1 and K_2. This is the special situation in which portfolio J_0 does not belong to the EFVTF, which is defined by the arc $\widehat{K_2K_1}$. Obviously, when $V_0 = V_{J_0}$, portfolios K_2 and J_0 coincide and the EFVTF is the arc $\widehat{J_0K_1}$. In the latter case, portfolio K_1 is the only intersection between the CVF and the ECTF and the minimum variance portfolio J_0 belongs to the EFVTF. These scenarios are the intermediate, the maximum and the large bound case introduced by Palomba and Riccetti (2012). See also Appendix A-1 for details. In this context, we define the situation with two intersections between the CVF and the ECTF ($V_0 \leq V_{J_0}$)

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Figure 4: The EFVTF when $z_\theta > \hat{z}$

(a) medium bound: $K \in$ ECTF

(b) low intermediate bound
EFVTF $\subset$ ECTF

(c) high intermediate bound
EFVTF $\subset$ ECTF

(d) large bound: EFVTF $\equiv$ ECTF
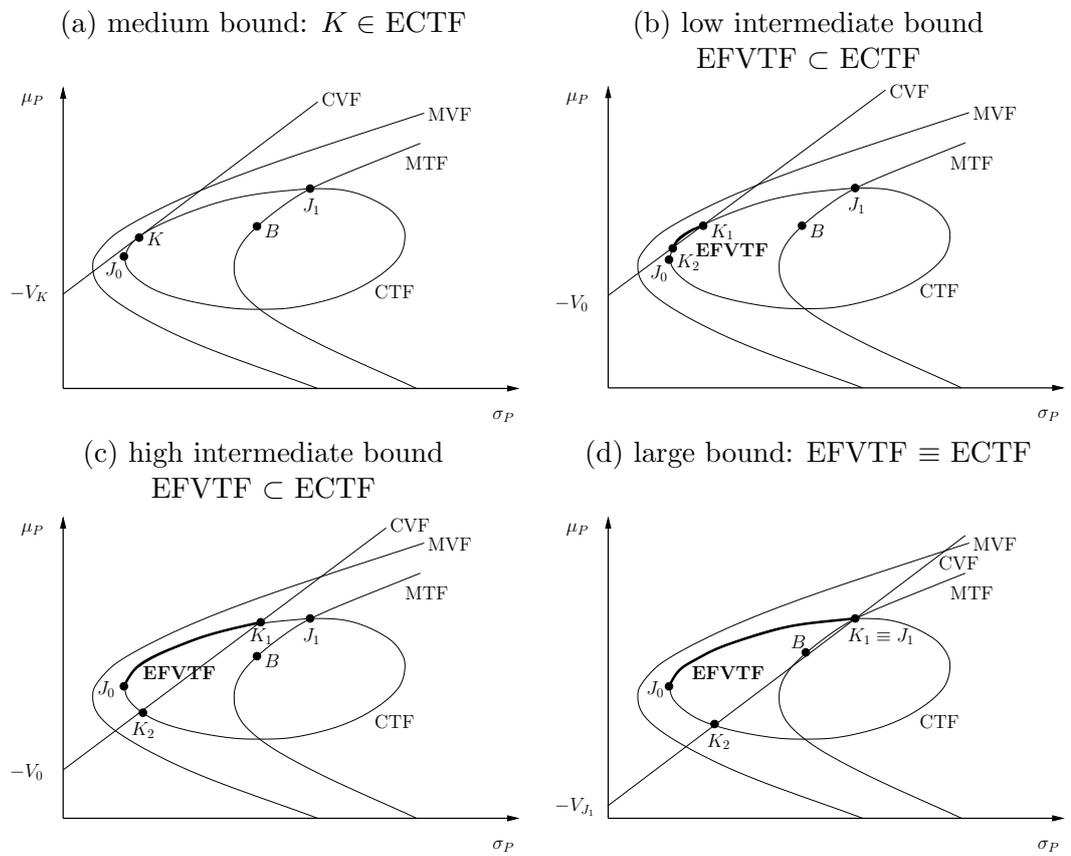
as the *low* intermediate bound, while the situation with a single intersection $(V_0 > V_{J_0})$ corresponds to the *high* intermediate bound.

Figure 4 (d) shows the large bound case $V_0 = V_{J_1}$ under which the straight line CVF passes through the portfolio J_1 and the EFVTF and the ECTF coincide; the same situation is also available when $V_0 > V_{J_1}$ because the VaR constraint is not binding for the ECTF.

Clearly, under the condition $z_\theta < \hat{z}$, the VaR restrictions change as follows:

- low intermediate bound ($V_K < V_0 \leq V_{J_1}$): the EFVTF is the arc $\widehat{K_1 K_2}$ or the arc $\widehat{J_1 K_2}$ in the special case $V_0 = V_{J_1}$;

- high intermediate bound ($V_{J_1} < V_0 < V_{J_0}$): the EFVTF is the arc $\widehat{J_1 K_2}$;

- large bound ($V_0 \geq V_{J_0}$): EFTVF $\equiv$ ECTF.

3.4   Relationships between efficient frontiers

Once a TEV constraint is set, the restrictions to the overall portfolio variance or to the VaR necessarily depend on the choice of the reference portfolio S_0.

Let such portfolio be non dominated, therefore $S_0 \in$ ECTF. The intersection between the linear constraints $\sigma_P = \sigma_0$ and $\mu_P = z_\theta \sigma_P - V_0$ lies on the arc $\widehat{J_0 J_1}$: this implies that the reference portfolio could be $S_0 \equiv K_1$ or, alternatively, $S_0 \equiv K_2$, where K_1 and K_2 are the intersections between the ECTF and the linear boundary CVF; more precisely, K_1 is the one lying to the right of the tangency portfolio K, while K_2 belongs to the arc $\widehat{J_0 K}$. As a consequence, the reference portfolio could be K, J_0 or J_1 when the VaR is set to the extremal values of the $[V_K, \max\{V_{J_0}, V_{J_1}\}]$ interval. All the scenarios are presented in Table 1.

If $S_0 \notin$ ECTF, the constraint on the overall variance is generally more stringent than that on the VaR, because it eliminates the arc $\widehat{S_1 K_1}$ from the feasible portfolios belonging to the efficient CTF (see Figure 5 (a)). This happens because the slope $z_\theta < \infty$ by construction, while the straight line $\sigma_P^2 = \sigma_0^2$ is vertical in the σ_P^2, μ_P space. This implies that ECTVF $\subset$ EFVTF $(\widehat{J_0 S_1} \subset \widehat{J_0 K_1})$.

Figures 5 (b)-(c) show that a low intermediate VaR bound and the condition $z_\theta < \hat{z}$ represent two exceptions to this situation, but they produce the same effect. In both cases the set of feasible porfolios is restricted because the VaR constraint eliminates the arc $\widehat{J_0 K_2}$. Thus, the frontiers are overlapping and the relationship EFVTF $\cap$ ECTVF $= \widehat{K_2 S_1}$ holds.

A common practice used in the field of asset allocation is to set the benchmark as the reference portfolio in order to impose some restrictions on the risk measures (see, for instance, Jorion, 2003; Bertrand, 2010). Nevertheless, the imposition of $S_0 \equiv B$, which does not belong to the ECTF,

Figure 5: Variance and VaR constraints when the reference portfolio $S_0 \notin$ ECTF

(a) General setup

(b) Low intermediate VaR bound

(c) Slope $z_\theta < \hat{z}$
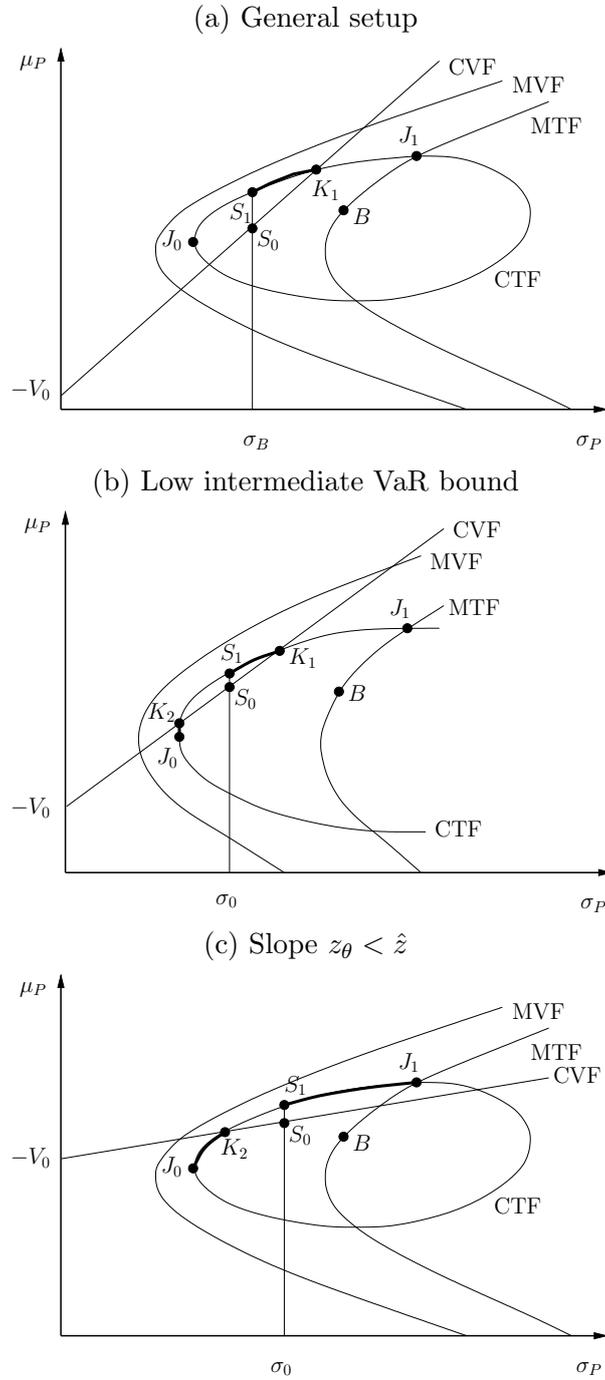
Table 1: Relationships between efficient frontiers when $S_0 \in \text{ECTF}$

bound	ref. portfolio	relationship
medium/low intermediate	$S_0 \equiv K_1$	$\text{EFVTF} \subset \text{ECTVF}$ $\left(\widehat{K_2 K_1} \subset \widehat{J_0 K_1} \right)$
	$S_0 \equiv K$ or $S_0 \equiv K_2$	$\text{EFVTF} \cap \text{ECTVF} = S_0$ $\left(\widehat{S_0 K_1} \cap \widehat{J_0 S_0} \right)$
high intermediate/large with $z_\theta \geq \hat{z}$	$S_0 \equiv K_1$	$\text{EFVTF} \equiv \text{ECTVF} \subset \text{ECTF}$ $\left(\widehat{J_0 K_1} \subset \widehat{J_0 J_1} \right)$
	$S_0 \equiv J_1$	$\text{EFVTF} \equiv \text{ECTVF} \equiv \text{ECTF}$
high intermediate/large with $z_\theta < \hat{z}$	$S_0 \equiv K_2$	$\text{EFVTF} \cap \text{ECTVF} = K_2$ $\left(\widehat{K_2 J_1} \cap \widehat{J_0 K_2} \right)$
	$S_0 \equiv J_0$	$J_0 \equiv \text{ECTVF} \subset \text{EFVTF} \equiv \text{ECTF}$ $\left(J_0 \in \widehat{J_0 J_1} \right)$

Note: K is the tangency portfolio between the ECTF and the CVF, K_1 is the intersection between the ECTF and the CVF with $\mu_{K_1} > \mu_K$ and K_2 is the intersection between the ECTF and the CVF with $\mu_{K_2} < \mu_K$.

does not correspond to a passive management strategy, in which the benchmark portfolio is replicated. The implied restrictions on VaR and portfolio variance produce one of the following scenarios:

1. if $V_{J_0} < V_B < V_{J_1}$, it follows that $\sigma_B^2 < \sigma_{J_1}^2$. The CVF intersects the ECTF, so the arc $\widehat{K_1 J_1}$ is cut off. The variance constraint $\sigma_0^2 = \sigma_B^2$ is more stringent because it cuts off the arc $\widehat{S_1 J_1}$. This is the situation plotted in Figure 6 (a), in which $\text{ECTVF} \subset \text{EFVTF} \subset \text{ECTF}$.

2. if $V_B \geq \max\{V_{J_0}, V_{J_1}\}$ and $\sigma_B^2 < \sigma_{J_1}^2$ a restriction on the ECTF is also produced. The plot is in Figure 6 (b). The condition on VaR guarantees that the straight line CVF passing through the benchmark lies to the right of portfolios J_0 and J_1. In this context, the condition on portfolio variance is sufficient to obtain $\text{ECTVF} \subset \text{EFVTF} \equiv \text{ECTF}$;

3. if $\sigma_B^2 \geq \sigma_{J_1}^2$, it follows that $V_B \geq \max\{V_{J_0}, V_{J_1}\}$. This is a very special case in which the restrictions based on the benchmark coordinates do not affect the ECTF, therefore $\text{ECTVF} \equiv \text{EFVTF} \equiv \text{ECTF}$. This is the situation in Figure 6 (c).

Focussing on the expected returns and the slope of the CVF, under the

Figure 6: Reference portfolio $S_0 \equiv B$

(a) $V_{J_0} < V_B < V_{J_1}$ $\Rightarrow$ ECTVF $\subset$ EFVTF $\subset$ ECTF
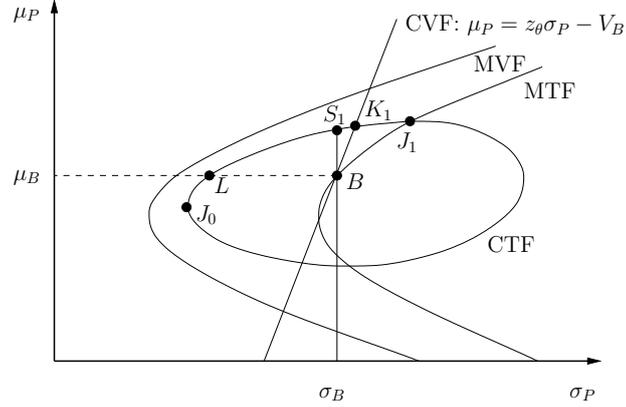
(b) $V_B \geq \max\{V_{J_0}, V_{J_1}\}$ and $\sigma_B^2 < \sigma_{J_1}^2$ $\Rightarrow$ ECTVF $\subset$ EFVTF $\equiv$ ECTF
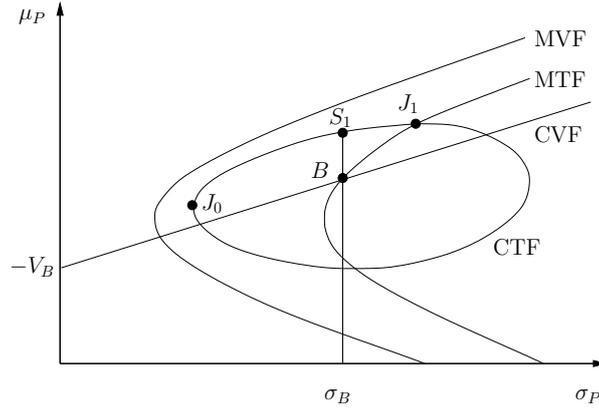
(c) $\sigma_B^2 \geq \sigma_{J_1}^2$ $\Rightarrow$ ECTVF $\equiv$ EFVTF $\equiv$ ECTF

13
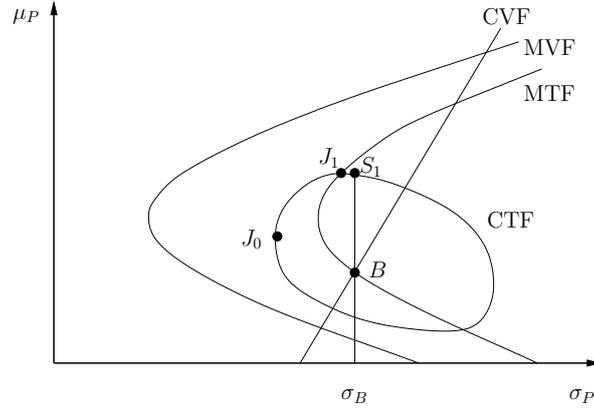
assumption $V_B > V_{J_0}$, the above conditions can be summarized as follows:[6]

$$
\begin{cases}
1. & z_\theta > z_\gamma \quad \text{and} \quad \mu_C < \dfrac{\mu_{J_1} + \mu_B}{2} \\[3mm]
2. & z_\theta \leq z_\gamma \quad \text{and} \quad \mu_C < \dfrac{\mu_{J_1} + \mu_B}{2} \\[3mm]
3. & \mu_C \geq \dfrac{\mu_{J_1} + \mu_B}{2}
\end{cases}
\tag{8}
$$

where μ_C is the expected return of the minimum variance portfolio on the MVF and $z_\gamma = (\mu_{J_1} - \mu_B)/(\sigma_{J_1} - \sigma_B)$ is the slope of the theoretical straight line passing through portfolios B and J_1. From the conditions in (8), one can observe that:

(a) when $\Delta_1 > 0$ (the CTF has positive slope), surely $\mu_C \leq \dfrac{\mu_{J_1} + \mu_B}{2}$;

(b) when the managers confidence level is low, it follows that $z_\theta < \sqrt{d}$ (see Alexander and Baptista, 2008, or Palomba and Riccetti, 2012 for details). This always implies $V_B > V_{J_1}$;

(c) when $T_0 > 0$, the restriction $\sigma_{J_1}^2 = \sigma_B^2$ indicates that the straight line passing through portfolios B and J_1 is vertical in σ_P, μ_P space. Moreover, the condition $\mu_C \geq (\mu_{J_1} + \mu_B)/2$ is sufficient to obtain $V_{J_1} < V_B$ because the straight line passing through B and J_1 is vertical or has a negative slope in σ_P, μ_P space, while the CVF slope z_θ is greater than zero by construction.

Analytical details about all these relationships are provided in Appendix A-4.

3.5 VaR and variance bounds in presence of a TEV constraint

The aim of this section is to show how it is possible to set a constraint to the overall portfolio variance or to the VaR in presence of a TEV restriction already imposed to asset managers. In particular, we are interested to examine how the constrained portfolio variance $\sigma_{S_0}^2 \in [\sigma_{J_0}^2, \sigma_{J_1}^2]$ and the constrained VaR $V_0 \in [V_K, \max\{V_{J_0}, V_{J_1}\}]$ can restrict the number of feasible portfolios lying on the efficient CTF. In doing so, we focus on the following portfolios:

1. J_1 is that of maximum expected return or Information Ratio (IR= $(\mu_P - \mu_B)/T_0$, see for instance Lee, 2000);

---

[6] In practice, in our discussion the condition $V_B < V_{J_0}$ is avoided because it corresponds to a very special situation in which the straight line CVF is approximately horizontal in σ_P, μ_P space.

2. J_0 is that of minimum absolute risk;

3. K is that for which the VaR is minimised;

4. L is that of minimum efficiency loss, hence $\delta_L = \min\limits_{P}\{\delta_P\}$, where $\mu_{J_0} \leq \mu_P \leq \mu_{J_1}$. As shown in Appendix A-3, this portfolio has coordinates

$$L \equiv (\sigma_B^2 + T_0 - 2\sqrt{\delta_B T_0}, \mu_B). \tag{9}$$

Figure 7 illustrates the most general case in which $\Delta_1 > 0$. The following hierarchies apply:[7]

$$\begin{cases} \mu_{J_0} < \mu_B < \mu_K < \mu_{J_1} \\ \sigma_{J_0} < \sigma_L < \sigma_K < \sigma_{J_1} \\ V_K < V_L < \min\{V_{J_0}, V_{J_1}\} < \max\{V_{J_0}, V_{J_1}\}. \end{cases} \tag{10}$$

Figure 7: Relevant portfolios on the ECTF
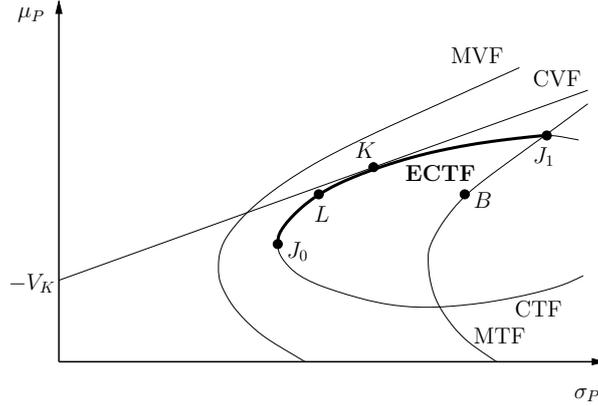
The tangency portfolio K represents the point in which the ECTF can be split into arcs $\widehat{J_0 K}$ and $\widehat{K J_1}$. On the one hand, all along the arc $\widehat{J_0 K}$, a trade off between VaR and portfolio variance emerges, because stringent VaR restrictions allow managers to increase the overall portfolio variance, while the more the portfolio variance is constrained, the more the VaR restriction is relaxed. This result is independent of the slope of the straight line CVF. Focussing on the extremal portfolios of such arc, Figure 7 clearly shows that J_0 is that of maximum VaR and minimum variance and return, while K is that of minimum VaR, but it is associated to the maximum variance and expected return. On the other hand, all along the arc $\widehat{K J_1}$, the classical trade off between expected return and measures of risk emerges. In this situation, a VaR restriction implies a restriction on the overall portfolio

---

[7]The relationship $\mu_K < \mu_B$ is possible in certain special cases with the slope $z_\theta \to \infty$. Moreover, when $\Delta_1 < 0$, it follows that $\mu_B < \mu_{J_0}$, therefore $L \notin$ ECTF.

variance and viceversa. The extremal position J_1 is available only when the constraints on the absolute risk measures are absent or non binding.

If managers' aim is to minimise the portfolio efficiency loss on the ECTF, they can invest on portfolio L. It allows them to replicate the benchmark return and also to reduce the measures of risk because $\sigma_L^2 < \sigma_B^2$ and $V_L < V_B$. Furthermore, from the risk perpective, this choice is often convenient because the restrictions on the VaR and the variance have been set close to their minimum values.

3.6 An empirical example

In this section we provide a short example about the TEV efficient portfolio frontier in presence of a constraint on the overall portfolio variance or on the VaR. In doing so, we use the same data and the same restrictions appearing in section 5 of Palomba and Riccetti (2012). Accordingly, we use the quarterly returns of the 50 stocks composing the DJ Eurostoxx 50 index over the period which ranges from the first quarter of 2003 to the fourth quarter of 2010. The benchmark portfolio is the Standard & Poor 500 Composite index, which has coordinates $B \equiv (72.423, 1.484)$ in σ_P^2, μ_P space, while the constraints on TEV and VaR are $T_0 = 20$ and $V_0 = 15$ respectively. Table 2 shows the results we obtained in our analysis by setting the variance constraint $\sigma_0^2 = 49$.

Table 2: Results

Portfolios:	portfolios of interest				intersections	
	J_0	J_1	L	K	K_1	S_1
Expected Return	1.376	4.567	1.484	2.299	4.192	3.262
Variance	37.888	94.330	37.921	40.365	68.058	49.000
Std. Error	6.155	9.712	6.158	6.353	8.250	7.000
Sharpe Ratio	0.224	0.470	0.241	0.362	0.508	0.466
Alpha	-0.108	3.083	0.000	0.815	2.708	1.778
TEV	20.000	20.000	20.000	20.000	20.000	20.000
Information Ratio	-0.005	0.154	0.000	0.041	0.135	0.089
Efficiency Loss	2.637	37.130	2.629	3.169	15.655	5.955
VaR	12.944	18.028	12.842	12.481	15.000	13.023

CVF slope (z_θ): 2.326 ($\theta = 0.99$)
Line passing through portfolios J_0 and J_1 ($\hat{z} = 0.89707$, $\theta = 0.81516$)

Summary:
This is the HIGH INTERMEDIATE BOUND (EFVTF $\subset$ ECTF)
ECTF: arc $J_0 \equiv (37.888, 1.376) - J_1 \equiv (94.330, 4.567)$
ECTVF: arc $J_0 \equiv (37.888, 1.376) - S_1 \equiv (49.000, 3.262)$
EFVTF: arc $J_0 \equiv (37.888, 1.376) - K_1 \equiv (68.058, 4.192)$
ECTVF $\subset$ EFVTF $\subset$ ECTF

The example basically provides the scenario already plotted in Figure 5 (a) in which:

(a) there is only one intersection between the ECTF and the straight line CVF, provided by portfolio K_1. Accordingly, this is the high intermediate bound. The other intersection between the CVF and the elliptical boundary CTF is $K_2 \equiv (42.432, 0.154)$, which lies below portfolio J_0;

(b) the constraint on the variance portfolio is more stringent than that imposed to the VaR. This implies that the intersection K_1 lies to the right of S_1. Since both restrictions cut off a portion of the arc $\overset{\frown}{J_0 J_1}$ containing J_1, the relationship ECTVF $\subset$ EFVTF $\subset$ ECTF holds;

(c) the restrictions $V_0 = 15$ and $\sigma_0^2 = 49$ do not exclude J_0, K, L from the constrained set of TEV efficient portfolios;

(d) portfolios K_1 and S_1 become those with the maximum expected return/IR under $V_0 = 15$ and $\sigma_0^2 = 49$ respectively.

Finally, using the benchmark as the reference portfolio, the variance and the VaR restrictions become $\sigma_B^2 = 72.423$ and $V_B = 18.314$. This corresponds to the large bound situation (portfolio K_1 lies to the right of portfolio J_1), where $S_1 \equiv (72.423, 4.311)$ and ECTVF$= \overset{\frown}{J_0 S_1}$. The relationship among the three TEV efficient frontiers is ECTVF $\subset$ ECTF $\equiv$ EFVTF.

4 Concluding remarks

This paper analyses the situation in which the risk management imposes a maximum TEV constraint to asset managers. Accordingly, Jorion (2003) introduces an elliptical frontier in the traditional mean-variance space containing all the constant TEV portfolios.

Nevertheless, many portfolios belonging to the surface area of this frontier are overly risky. Jorion (2003) addresses this problem by adding a variance constraint, while Palomba and Riccetti (2012), starting from the work of Alexander and Baptista (2008), insert also a Value-at-Risk (VaR) constraint, thus defining the "Fixed VaR-TEV Frontier" (FVTF), a portfolio boundary which satisfies the VaR constraint and guarantees a TEV that does not exceed an *ex-ante* fixed value T_0. Provided that the above mentioned frontiers are closed and bounded sets, we focus on their efficient subset by using the mean-variance dominance criterion; accordingly, we define the "Efficient Constrained TEV Frontier"(ECTF) as the subset on the Jorion's elliptical boundary containing non dominated portfolios. Moreover, the imposition of a maximum variance restriction reduces the ECTF to the "Efficient Constrained TEV-Variance Frontier" (ECTVF), while the "Efficient Fixed VaR-TEV Frontier" (EFVTF) arises when a VaR limit is imposed.

It is well known that investors aim to maximise their utility which is a function of the portfolio mean and variance. Therefore they can be interested in maximising the expected return of their portfolios or in minimising some measures of risk. Once a maximum TEV is fixed, the risk management can improve the managed portfolio performance by setting additional constraints on the overall portfolio variance or the VaR. In the usual space with coordinates given by the portfolio standard deviation and the expected return, these constraints correspond to a straight line which splits the plane in two halfplanes: the former restriction produces a vertical line for a fixed standard deviation, while the latter restriction is represented by the linear boundary "Constrained VaR frontier" (CVF), whose slope $z_\theta > 0$ is the θ-quantile of the portfolio distribution that depends on the managers confidence level $0.5 < \theta < 1$. In general, focussing on the TEV efficient portfolios, this property makes the VaR constraint preferable to a variance constraint. In practice, once a reference portfolio is set (usually this is the benchmark), it produces a less stringent effect because the number of portfolios satisfying the variance restriction is not superior to the number of the VaR constrained portfolios.

In this framework, the compatibility between the TEV-efficiency and the restrictions on risk measures is crucial. In other words, at least one variance/VaR constrained portfolio must exist or, alternatively, at least one portfolio belonging to the ECTF must lie on the admissible halfplane.

Finally, all the results of our analysis are obtained when the returns are assumed normally distributed and short sales are allowed. Relaxing these assumptions could make the entire approach more realistic. This could be the goal of further research.

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Appendix

A-1 Portfolio frontiers

In this section we summarize the analytical definitions in σ_P^2, μ_P space of all portfolio frontiers used in our analysis.

A. Classical frontiers

MVF - Mean-variance Frontier (Markowitz, 1952):

$$\sigma_P^2 = \sigma_C^2 + \frac{1}{d}(\mu_P - \mu_C)^2 \tag{A-1}$$

MTF - Mean-TEV Frontier (Roll, 1992):

$$\sigma_P^2 = \sigma_B^2 + \frac{1}{d}(\mu_P - \mu_B)^2 + 2\frac{\Delta_1}{d}(\mu_P - \mu_B) \tag{A-2}$$

CTF - Constrained TEV Frontier (Jorion, 2003):

$$d(\sigma_P^2 - \sigma_B^2 - T_0)^2 + 4\Delta_2(\mu_P - \mu_B)^2 +$$
$$-4\Delta_1(\sigma_P^2 - \sigma_B^2 - T_0)(\mu_P - \mu_B) - 4d\delta_B T_0 = 0 \tag{A-3}$$

CVF - Constrained VaR Frontier (Alexander and Baptista, 2008):

$$\sigma_P^2 = \left(\frac{\mu_P + V_0}{z_\theta}\right)^2 \tag{A-4}$$

B. VaR depending frontiers

CMTF - Constrained Mean-TEV Frontier (Alexander and Baptista, 2008): see, for instance, Figure 1 (a). The form of this boundary depends on the restriction imposed to the VaR, specifically

- small bound ($V_0 < V_M$), the CMTF is an empty set;
- minimum bound ($V_0 = V_M$), the CMTF is the tangency portfolio M between the CVF and the MVF,
- intermediate bound ($V_M < V_0 < V_R$), the CMTF is the segment $\overset{\frown}{M_1 M_2}$, where M_1 and M_2 are the intersection between the CVF and the MVF;
- maximum bound ($V_0 = V_R$), the CMTF is the segment $\overset{\frown}{M_1 M_2}$, tangent to the MTF in portfolio R;
- large bound ($V_0 > V_R$), the CMTF is formed by three consecutive sets, namely the segment $\overline{M_1 R_1}$, the arc $\overset{\frown}{R_1 R_2}$ and the segment $\overline{M_2 R_2}$, where R_1 and R_2 are the intersections between the CVF and the MTF.

FVTF - Fixed VaR-TEV Frontier (Palomba and Riccetti, 2012): see, for instance, Figure 1 (b). The form of this boundary depends upon the restriction imposed to the VaR, specifically

- small bound ($V_0 < V_M$), minimum bound ($V_0 = V_M$) and strong bound ($V_M < V_0 < V_K$), the FVTF is an empty set;
- medium bound ($V_0 = V_K$), the FVTF is the tangency portfolio K between the CVF and the CTF;
- intermediate bound ($V_K < V_0 < V_R$), the FVTF is formed by the left arc $\overset{\frown}{K_1 K_2}$ on the CTF and by the segment $\overline{K_1 K_2}$ on the CVF, where K_1 and K_2 are the intersections between the CVF and the CTF;
- maximum bound ($V_0 = V_R$), the FVTF is formed by the left arc $\overset{\frown}{K_1 K_2}$ on the CTF and by the segment $\overline{K_1 K_2}$ on the CVF which is tangent to the MTF in portfolio R;
- large bound ($V_R < V_0 < \max\{V_{J_1}, V_{J_2}\}$), the FVTF is usually formed by the following consecutive sets: the left arc $\overset{\frown}{K_1 K_2}$, the segment $\overline{K_2 R_2}$, the arc $\overset{\frown}{R_1 R_2}$, and the segment $\overline{K_1 R_1}$.
  Portfolio $J_2 \equiv \left(\sigma_B^2 + T_0 - 2\Delta_1 \sqrt{T_0/d}, \mu_B + \sqrt{dT_0} \right)$ is the lower intersection between the CTF and the MTF;
- larger bound ($V_0 = \max\{V_{J_1}, V_{J_2}\}$) and no bound ($V_0 > \max\{V_{J_1}, V_{J_2}\}$), the FVTF is formed by two consecutive (left) arcs, namely $\overset{\frown}{K_1 K_2}$ and $\overset{\frown}{J_1 J_2}$.

C. Efficient frontiers

ECTF - Efficient Constrained TEV Frontier: it is a subset of the CTF and is formed by the left arc $\overset{\frown}{J_0 J_1}$, as Figure 2 shows;

ECTVF - Efficient Constrained TEV-Variance Frontier: it is a subset of the ECTF for which the portfolio variance does not exceed a fixed bound σ_0^2 (see Figure 3);

EFVTF - Efficient Fixed VaR-TEV Frontier: it is a subset of the ECTF for which the portfolio VaR does not exceed a fixed bound V_0 (see Figure 4).

A-2   The minimum variance portfolio on the CTF

The portfolio J_0 defined in equation (3) is the minimum variance on the CTF and it is derived by Jorion (2003) by minimising equation (A-3) respect to σ_P.

   The absolute risk associated to this portfolio is never lower than that of the minimum variance portfolio on the MVF (portfolio C), while its expected return depends on the benchmark and on the restriction T_0.

Proof   Jorion (2003) shows that a necessary condition for obtaining the CTF is

$$4T_0\Delta_2 - y^2 \geq 0,$$

where $y = (\sigma_P^2 - \sigma_B^2 - T_0)$. After some algebra, it is easy to show that

$$\sigma_B^2 + T_0 - 2\sqrt{T_0\Delta_2} \leq \sigma_P^2 \leq \sigma_B^2 + T_0 - 2\sqrt{T_0\Delta_2},$$

where the minimum is the variance of the portfolio J_0, as documented by equation (3). If σ_C^2 is the minimum variance portfolio available in σ_P^2, μ_P space, it is possible to express the variance of J_0 as $\sigma_{J_0}^2 = \sigma_C^2 + \phi_V$, where ϕ_V is non negative by construction. Using this definition one can obtain

$$\begin{aligned}
\phi_V &= \sigma_{J_0}^2 - \sigma_C^2 \\
&= \sigma_B^2 + T_0 - 2\sqrt{T_0\Delta_2} - \sigma_C^2 \\
&= \Delta_2 + T_0 - 2\sqrt{T_0\Delta_2} \\
&= (\sqrt{\Delta_2} - \sqrt{T_0})^2.
\end{aligned}$$

The variance is always greater than σ_C^2; this equality holds also when $T_0 > \Delta_2$ which corresponds to the large TEV constraint $T_0 > \delta_B$.

   The expected return of portfolio J_0 can be greater, less or equal to that of the minimum variance portfolio on the MVF, hence

$$\begin{aligned}
\phi_M &= \mu_{J_0} - \mu_C \\
&= \mu_B - \Delta_1\sqrt{T_0/\Delta_2} - \mu_C \\
&= \Delta_1\left[1 - \sqrt{T_0/\Delta_2}\right].
\end{aligned}$$

For any $\Delta_1 > 0$, $\mu_{J_0} - \mu_C \geq 0$ when $T_0 \leq \Delta_2$, while $\mu_{J_0} - \mu_C < 0$ otherwise. ∎

A-3   The minimum efficiency loss portfolio in the CTF

The aim of this section is to show that portfolio $L \in$ CTF in equation (9) is that of minimum efficiency loss in σ_P^2, μ_P space, where the efficiency loss δ_P is the horizontal distance between a given portfolio P and another that has the same expected return and is situated on the MVF.

Proof   Using the results of Palomba and Riccetti (2012), the efficiency loss of any portfolio lying on the CTF in σ_P^2, μ_P space is provided by the difference

$$\begin{aligned}
\delta_P &= \sigma_P^2 - \sigma_{P*}^2 \\
&= \sigma_B^2 + T_0 + \frac{2}{d}\left\{\Delta_1(\mu_P - \mu_B) - \sqrt{d\delta_B[dT_0 - (\mu_P - \mu_B)^2]}\right\} - \sigma_C^2 - \frac{1}{d}(\mu_P - \mu_C)^2,
\end{aligned}$$

where portfolios P and P^* have the same expected return μ_P and lie on the CTF and on the MVF. The minimum efficiency loss portfolio is obtaining via the first order condition

$$\frac{\partial \delta_P}{\partial \mu_P} \geq 0$$

$$\frac{2}{d}\left\{ \Delta_1 + \frac{d\delta_B(\mu_P - \mu_B)}{\sqrt{d\delta_B[dT_0 - (\mu_P - \mu_B)^2]}} \right\} - \frac{2}{d}(\mu_P - \mu_C) \geq 0$$

$$(\mu_B - \mu_P)\left[1 - \frac{\sqrt{d\delta_B}}{\sqrt{dT_0 - (\mu_P - \mu_B)^2}} \right] \geq 0.$$

Clearly, this derivative is zero when $\mu_P = \mu_B$, since

$$\left[1 - \frac{\sqrt{d\delta_B}}{\sqrt{dT_0 - (\mu_P - \mu_B)^2}} \right] < 0 \quad \Rightarrow \quad \left[\frac{\sqrt{dT_0 - (\mu_P - \mu_B)^2} - \sqrt{d\delta_B}}{\sqrt{dT_0 - (\mu_P - \mu_B)^2}} \right] < 0.$$

In particular:

- the numerator is negative because

$$\sqrt{dT_0 - (\mu_P - \mu_B)^2} - \sqrt{d\delta_B} < 0 \quad \Rightarrow \quad (\mu_P - \mu_B)^2 > d(T_0 - \delta_B).$$

  When the MVF and the CTF do not intersect, $T_0 < \delta_B$, therefore the above condition is always satisfied since $d(T_0 - \delta_B)$ is negative;

- the denominator is positive because

$$dT_0 - (\mu_P - \mu_B)^2 > 0 \quad \Rightarrow \quad \mu_B - \sqrt{dT_0} < \mu_P < \mu_B + \sqrt{dT_0}.$$

  This corresponds to the relationship $\mu_P \in (\mu_{J_2}, \mu_{J_1})$, where the portfolios J_2 and J_1 are respectively those of minimum and maximum expected return on the CTF (see Jorion, 2003, and/or Figure 1).

According to these results, the efficiency loss δ_P reaches its minimum value when $\mu_P = \mu_B$ which is the expected return of portfolio L. By substituting $\mu_P = \mu_B$ in equation (A-3) the following portfolio variance

$$\sigma_L^2 = \sigma_B^2 + T_0 - 2\sqrt{\delta_B T_0}$$

is obtained; the corresponding minimum efficiency loss is

$$\begin{aligned}
\delta_L &= \sigma_C^2 + \frac{1}{d}(\mu_B - \mu_C)^2 - \sigma_B^2 - T_0 + 2\sqrt{\delta_B T_0} \\
&= \frac{\Delta_1^2}{d} - \Delta_2 - T_0 + 2\sqrt{\delta_B T_0} \\
&= (\sqrt{\delta_B} - \sqrt{T_0})^2.
\end{aligned} \tag{A-5}$$

∎

A-4 VaR constraint through the benchmark

This section is dedicated to equation (8) and its properties.

Proof of equation (8) Given $\sigma_{J_1}^2 > \sigma_B^2$, that guarantees a positive slope for the straight line passing through portfolios B and J_1, it follows that

$$\sigma_{J_1}^2 - \sigma_B^2 > 0$$

$$\sqrt{T_0}\left[\sqrt{T_0} + 2\frac{\Delta_1}{\sqrt{d}}\right] > 0$$

$$\sqrt{\frac{T_0}{d}}\left[\sqrt{dT_0} + 2\Delta_1\right] > 0$$

$$\sqrt{\frac{T_0}{d}}\left[\mu_{J_1} - \mu_B + 2(\mu_B - \mu_C)\right] > 0$$

$$\sqrt{\frac{T_0}{d}}\left[\mu_{J_1} + \mu_B - 2\mu_C\right] > 0. \tag{A-6}$$

Since $\sqrt{T_0/d} \geq 0$, the condition $\sigma_{J_1}^2 - \sigma_B^2 > 0$ is satisfied if and only if

$$\mu_{J_1} + \mu_B - 2\mu_C > 0 \qquad \Rightarrow \qquad \mu_C < \frac{\mu_{J_1} + \mu_B}{2}. \tag{A-7}$$

This implicitly demonstrates the third equation in (8). Moreover, when this inequality holds, from equation (2) we obtain the condition

$$V_{J_1} \leq V_B \qquad \Rightarrow \qquad z_\theta \sigma_{J_1} - \mu_{J_1} \leq z_\theta \sigma_B - \mu_B \qquad \Rightarrow \qquad z_\theta \leq \frac{\mu_{J_1} - \mu_B}{\sigma_{J_1} - \sigma_B}.$$

This demonstrates the second relationship in equation (8). Clearly, the first equation is obtained imposing the opposite inequality $V_B < V_{J_1}$.

∎

As we claimed in section 3.4, some useful results emerge from equation (8).

(a) when $\Delta_1 > 0$, surely $\mu_C < \dfrac{\mu_{J_1} + \mu_B}{2}$.

> **Proof** It is well known that the condition $\Delta_1 > 0$ implies $\mu_B > \mu_C$, while the imposition of a TEV restriction $T_0 > 0$ makes the return of portfolio J_1 greater than that of the benchmark. Accordingly, the following hierarchy arises
>
> $$\mu_C < \mu_B < \frac{\mu_{J_1} + \mu_B}{2} < \mu_{J_1};$$
>
> otherwise, if $T_0 = 0$, it follows that $\mu_{J_1} = \mu_B$ and the above hierarchy collapses into $\mu_B > \mu_C$. ∎

(b) when the managers confidence level is low, that is $z_\theta < \sqrt{d}$, it follows that $V_B > V_{J_1}$;

> **Proof** Both portfolios B and J_1 belong to the hyperbola MTF and the expected return μ_{J_1} is greater than μ_B for any positive T_0. When the condition $\sigma_{J_1}^2 - \sigma_B^2 \geq 0$ is also satisfied, any straight line passing through those portfolios has a positive slope in σ_P, μ_P space and, by construction, it is steeper than the asymptotic slope

of the MTF ($\sqrt{d}$).

Moreover, in general it is possible to demonstrate also that the relationship

$$\frac{\mu_{J_1} - \mu_B}{\sigma_{J_1} - \sigma_B} > \sqrt{d}$$

is always true. Indeed, using some algebra:

$$\frac{\mu_{J_1} - \mu_B}{\sigma_{J_1} - \sigma_B} > \sqrt{d}$$

$$\frac{\sqrt{dT_0}}{\sigma_{J_1} - \sigma_B} > \sqrt{d}$$

$$\sqrt{T_0} + \sigma_B > \sigma_{J_1}$$

$$T_0 + \sigma_B^2 + 2\sqrt{T_0}\sigma_B > \sigma_B^2 + T_0 + 2\Delta_1\sqrt{T_0/d}$$

$$\sigma_B > \frac{\Delta_1}{\sqrt{d}}$$

$$\mu_C + \sqrt{d}\sigma_B > \mu_B \tag{A-8}$$

The left side of equation (A-8) returns the equation of the asymptote of the MVF evaluated in $\sigma_P = \sigma_B$; as demonstrated by Alexander and Baptista (2008), and as Figure A-1 clearly shows, this line lies above the MTF, while the benchmark lies on the MTF, therefore the relationship $\mu_C + \sqrt{d}\sigma_B > \mu_B$ is always true.
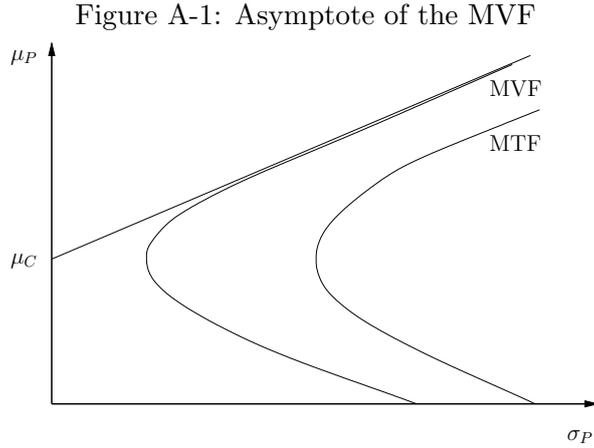
Figure A-1: Asymptote of the MVF

■

(c) when $T_0 > 0$, the restriction $\sigma_{J_1}^2 = \sigma_B^2$ indicates that the straight line passing through portfolios B and J_1 is vertical in σ_P, μ_P space. Moreover, the condition $\mu_C \geq (\mu_{J_1} + \mu_B)/2$ is sufficient to obtain $V_{J_1} < V_B$ because the straight line passing through B and J_1 is vertical or has a negative slope in σ_P, μ_P space, while the CVF slope z_θ is greater than zero by construction.

Proof Observing equation (4), setting $T_0 = 0$ determines $J_1 \equiv B$, therefore the solution $\sigma_{J_1}^2 = \sigma_B^2$ is trivial. Otherwise, it is possible to obtain the same equality

also when the TEV restriction is $T_0 > 0$. In particular, this is the situation

$$\mu_C = \frac{\mu_{J_1} + \mu_B}{2}. \tag{A-9}$$

When this condition holds, it follows that $J_1 \equiv (\sigma_B, 2\mu_C - \mu_B)$, $\Delta_1 = \mu_B - \mu_C < 0$ and $V_{J_1} = z_\theta \sigma_B + \mu_B - 2\mu_C < z_\theta \sigma_B - \mu_B = V_B$. ∎

A-5 Supplementary material

The programming routines for the analysis carried out in this paper can be found at

http://utenti.dea.univpm.it/palomba/TEV-VaR.html