Efficient LMI-Based Quadratic Stabilization of Interval LPV Systems With Noisy Parameter Measures

Leopoldo Jetto and Valentina Orsini

Abstract—**The purpose of this note is to consider the quadratic stabilization of LPV systems in the realistic case where only Gaussian noisy parameter measures are available. Though neglected in the actual literature on LPV systems, this question is particular important because in all situations of a practical interest the parameter measurements (or estimates) are never exact. The assumed noisy nature of physical parameter readings requires a specifically developed approach consisting of mixed robust and LPV control methods. In the present case, an approach based on a vertex result on interval time varying (ITV) matrices is proposed. This allows the solvability conditions to be stated in terms of a set of LMI's, whose number is independent of the number of time-varying parameters.**

Index Terms—**LMIs, LPV systems, stabilization, stochastic norm.**

I. Introduction

Both in the quadratic [1]–[3] and nonquadratic approach, the practical tractability of the LPV control problem is attained giving up a general formulation and introducing additional constraints such linear fractional representation of the plant [3]–[5] affine parameter dependence [1], [6]–[8], multiconvexity arguments [7], [9]. Another common (and unrealistic) assumption of the above papers is that all the parameters values are exactly measured (or estimated) at all time instants. The difficulty of extending the above methods to parameter measures corrupted by Gaussian noise is mainly due to the loss of the original polytopic structure of the domain where the parameter readings take values. This problem has been noticed in [10] where, at the expense of some conservatism, an LMIs-based design method is proposed. In the above reference, the theoretical difficulty of dealing with noisy measures is overcome assuming a bounded observation noise and parameter readings still belonging to the original polytopic domain containing the exact parameter values.

This note considers the quadratic stabilization of discrete-time LPV systems in the realistic case where the parameter vector $\theta(\cdot)$ is measured under additive Gaussian noise.

This basic assumption unavoidably affects the controller synthesis method because physical parameters measures corrupted by Gaussian noise yield a stochastic dynamical matrix with theoretically unbounded elements. This requires to deal with the considered stabilization problem in a mixed framework: LPV control and robustness with respect to unbounded stochastic perturbations have to be simultaneously taken into account. To this purpose an observer-like controller is here used whose observer and feedback gains are designed on the basis of a unique, suitably defined, vertex matrix. In this way, robustly fixed gains are obtained, but the dependence on the time-varying plant parameters is maintained because the dynamical matrix of the controller contains the same dynamical matrix of the plant with noisy parameters. This allows both the robustness and the parameter-scheduling issues to be taken into account. To this purpose, the stabilization problem is faced modeling the dynamical time-varying matrix $A(\theta(\cdot))$ of the plant as an ITV matrix. This allows us to make a relaxed assumption on the elements of $A(\theta(\cdot))$, by simply requiring they be given by piecewise continuous functions of $\theta(\cdot)$. Constructive solvability conditions are established in terms of LMIs, which are derived exploiting a vertex result relative to the stabilization of a unique suitably defined extreme plant. A consequent salient feature of the resulting design procedure is that the number of LMIs is fixed and independent of the dimension of $\theta(\cdot)$. At the price of an acceptable degree of conservatism, this approach yields a synthesis method with a greatly enhanced numerical efficiency with respect to classical quadratic and nonquadratic methods. Using Bellman-Gronwall arguments [11] and the notion of stochastic Frobenius norm [12], the noisy measures problem is solved starting from a preliminary result relative to exact parameter measurements. The objectives of this note are pursued considering the stabilization problem of LPV systems with a time-varying dynamical matrix and constant sensor and actuator matrices (for practical control situations of this kind (see e.g., [1], [2], [10], and [13]). The note is organized in the following way. Some basic notations, preliminary results, and the problem statement are reported in Section II, the synthesis procedure of the controller is reported in Section III. Numerical examples and concluding remarks end the note.

II. Preliminaries

Notation

Given two matrices M and N, with elements $m_{i,j}$ and $n_{i,j}$ respectively, the notation $M \preceq N (M \succeq N)$ means $m_{i,j} \leq n_{i,j} (m_{i,j} \geq n_{i,j})$, i, $j = 1,\ldots,n$. If the symbol $\prec$, ($\succ$), is used, the strictly inequality holds. The notation $M \in [M^-, M^+]$ means that M is an interval matrix satisfying $M^- \preceq M \preceq M^+$. The matrix $\bar{M}$, whose elements are $\bar{m}_{i,j} = \max\left\{|m^-_{i,j}|, |m^+_{i,j}|\right\}$, $i, j = 1,\ldots,n$, is called the majorant matrix of M. Clearly, one has $M \preceq \bar{M}$ and if $M^- \succeq 0_n$, then $\bar{M} = M^+$. A time-varying matrix $M(\cdot)$ such that $M(\cdot) \in [M^-, M^+] \triangleq I_M$, is called an interval time-varying (ITV) matrix. The stochastic Frobenius norm of a matrix M with random elements is defined as $\left(\mathcal{E}\left(\|M\|^2_F\right)\right)^{1/2}$, where $\mathcal{E}$ denotes expectation and $\|M\|_F = \left(\sum^n_{i,j=1} |m_{i,j}|^2\right)^{1/2}$ is the classical Frobenius norm [12]. Analogously, the size of a random vector $x = [x_1,\ldots,x_n]^T$ is measured through $\left(\mathcal{E}\left(\|x\|^2_E\right)\right)^{1/2}$, where $\|x\|_E = \left(\sum^n_{i=1} |x_i|^2\right)^{1/2}$ is the classical euclidean norm. It is easy to see that if M and x are independent then $\left(\mathcal{E}\left(\|Mx\|^2_E\right)\right)^{1/2} \leq \left(\mathcal{E}\left(\|M\|^2_F\right)\right)^{1/2} \left(\mathcal{E}\left(\|x\|^2_E\right)\right)^{1/2}$.

The following Lemma [14] contains an improved version of a result given in [15] and [16] on the stability of the state transition matrix $\Phi_M(k, k_0)$ generated by an ITV dynamical matrix $M(\cdot) \in [M^-, M^+]$.

Lemma: If $\exists\gamma_{\bar{M}} \in (0, 1]$ such that $|\lambda_i\{\bar{M}\}| < \gamma_{\bar{M}}$, $i = 1,\ldots,n$, then, independently of the way the elements $m_{i,j}(\cdot)$ of an ITV dynamical matrix $M(\cdot)$ vary inside their respective intervals, the corresponding $\Phi_M(k, k_0)$ is exponentially $\gamma_{\bar{M}}$-stable, namely $\|\Phi_M(k, k_0)\| \leq m_M \gamma^{(k-k_0)}_{\bar{M}}$, for some $m_M > 0$, $\forall k_0$, $\forall k \geq k_0$. Moreover, if $\bar{M} = M^+$ or $\bar{M} = -M^-$, the condition is also necessary.

Consider the following discrete-time LPV system Σ

$$x(k+1) = A(\theta(k))x(k) + Bu(k), \quad x(0) = x_0 \tag{1}$$

$$y(k) = Cx(k) \tag{2}$$

where $u(\cdot) \in \mathbb{R}^m$ is the control input, $x(\cdot) \in \mathbb{R}^n$ is the state, $y(\cdot) \in \mathbb{R}^q$ is the output, $\theta(\cdot) = [\theta_1(\cdot),\ldots,\theta_p(\cdot)]^T$ is the vector composed of the time-varying parameters which are assumed to be

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The authors are with the Dipartimento di Ingegneria Informatica Gestionale e dell'Automazione, Università Politecnica delle Marche, Ancona, Italy (e-mail: L.Jetto@univpm.it, vorsini@univpm.it).

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measurable on-line according to $\tilde{\theta}(\cdot) = \theta(\cdot) + v(\cdot)$ where $v(\cdot)$ is a white Gaussian noise process with zero mean and Q covariance matrix ($v(\cdot) \sim N(0, Q)$). The following assumptions are made: A1) the true $\theta(\cdot)$ takes values in a compact set Θ; A2) the extremal vectors $\theta^- \triangleq [\theta_1^-, \ldots, \theta_p^-]^T$, and $\theta^+ \triangleq [\theta_1^+, \ldots, \theta_p^+]^T$ are "*a priori*" known; and A3) the elements $a_{i,j}(\tilde{\theta}(\cdot))$ of $A(\tilde{\theta}(\cdot))$ are piecewise continuous functions of $\tilde{\theta}$. The state transition matrix of Σ is denoted by $\Phi(\cdot, \cdot)$.

A. Problem Statement

Given a dynamic output feedback controller $\tilde{\Sigma}_c$, the feedback connection of Σ with $\tilde{\Sigma}_c$ is denoted by $\tilde{\Sigma}_f$. The state vector and the state transition matrix of $\tilde{\Sigma}_f$ are denoted by $x_f(\cdot)$ and $\tilde{\Phi}_f(\cdot, \cdot)$, respectively. The stabilization problem considered in this note consists in finding (if it exists) a dynamic output feedback controller $\tilde{\Sigma}_c$ scheduled by the noisy parameter measurements, yielding a quadratically exponentially $\tilde{\gamma}$-stable closed-loop system $\tilde{\Sigma}_f$. Owing to the stochastic nature of the uncertainty affecting the parameter readings, the quadratic stability of $\tilde{\Sigma}_f$ has to be intended in a mean square sense, namely for every initial state $x_f(k_0)$, $k_0 \geq 0$, there must exist constant scalars $\alpha > 0$ and $\tilde{\gamma} \in (0, 1)$, such that

$$\mathcal{E}\left(\|x_f(k)\|_E^2\right) \leq \alpha \mathcal{E}\left(\|x_f(k_0)\|_E^2\right) \tilde{\gamma}^{(k-k_0)} \quad \forall k \geq k_0, k_0 \geq 0. \tag{3}$$

The above stabilization problem is referred to as a quadratic mean-square stabilization problem (QMSSP).

III. Controller Design Procedure

A. Preliminary Step: The Noisy-Free Case

In this section, exact parameter measures are considered and the relative controller is denoted by Σ_c. The feedback connection of Σ with the controller Σ_c is denoted by Σ_f, and the state transition matrix of Σ_f is denoted by $\Phi_f(\cdot, \cdot)$. As $\theta(\cdot)$ takes values in the compact set Θ, assumption A3 implies that $A(\theta(\cdot))$ is a ITV matrix such that $A(\theta(\cdot)) \in [A^-, A^+] \triangleq I_A$, for suitably defined A^- and A^+. The following final assumption is now introduced: A4) the triplet $\bar{\Sigma} \equiv (C, \bar{A}, B)$, is reachable and observable.

The time-varying controller is assumed to have the same parameter dependence as the plant and constant input and output matrices. More precisely Σ_c is assumed to have the following observer-like based controller form:

$$z(k+1) = (A(\theta(k)) + LC)z(k) + Bu(k) - Ly(k) \tag{4}$$

$$u(k) = Kz(k) \tag{5}$$

where $z(\cdot) \in \mathbb{R}^n$ is the state of Σ_c. The feedback connection Σ_f of Σ_c with the ITV plant $\Sigma \equiv (C, A(\theta(\cdot)), B)$ is described by the pair $(C_f, A_f(\theta(\cdot)))$, with

$$A_f(\theta(\cdot)) = \begin{bmatrix} A(\theta(\cdot)) & BK \\ -LC & A(\theta(\cdot)) + LC + BK \end{bmatrix}, \quad C_f = [C \quad 0_{q,n}]. \tag{6}$$

Applying the transformation matrix $T = \begin{bmatrix} I_n & 0_n \\ I_n & -I_n \end{bmatrix}$, one has $\Sigma_f \equiv (\hat{C}_f, \hat{A}_f(\theta(\cdot)))$ with

$$\hat{A}_f(\theta(\cdot)) = \begin{bmatrix} A(\theta(\cdot)) + BK & -BK \\ 0_n & A(\theta(\cdot)) + LC \end{bmatrix}, \quad \hat{C}_f = [C \quad 0_{q,n}]. \tag{7}$$

The state transition matrices corresponding to $A_f(\theta(\cdot))$ and $\hat{A}_f(\theta(\cdot))$ are denoted by $\Phi_f(\cdot, \cdot)$ and $\hat{\Phi}_f(\cdot, \cdot)$, respectively. Consider now the following majorant, time-invariant matrix:

$$\widehat{\bar{A}}_f = \begin{bmatrix} \bar{A} + BK & -BK \\ 0_n & \bar{A} + LC \end{bmatrix}. \tag{8}$$

By A4, it is possible to find gains K and L such that $|\lambda_i\{\bar{A} + BK\}| < \rho_1$ and $|\lambda_i\{\bar{A} + LC\}| < \rho_2$, $i = 1, \ldots, n$, for arbitrarily fixed scalars $\rho_1, \rho_2 \in (0, 1]$. If, in addition, K and L are able to satisfy the following further requirements: $i)$ the positivity of the open-loop extreme matrix $\bar{A}$ must be preserved so that $\widehat{\bar{A}}_f \succeq 0_{2n}$, $ii)$ $|\hat{A}_f(\theta(\cdot))| \preceq \widehat{\bar{A}}_f$, $\forall \theta(\cdot) \in \Theta$, then the exponential λ-stability (for some $0 < \lambda < \rho \triangleq \max(\rho_1, \rho_2)$) of the time-varying closed-loop system Σ_f directly follows by the lemma of Section II. Next theorem states LMI conditions to find gains K and L yielding a matrix $\widehat{\bar{A}}_f$ with $|\lambda_i\{\widehat{\bar{A}}_f\}| < \rho$, $i = 1, \ldots, n$, and satisfying requirements $i)$ and $ii)$.

Theorem 1: In the noise-free case, the controller Σ_c given by (4) and (5) gives a quadratically exponentially λ-stable closed-loop system Σ_f, if there exist two matrices U_1 and U_2 and two diagonal matrices $S_1 \succ 0_n$ and $S_2 \succ 0_n$, such that the following LMI based conditions are satisfied:

$$\begin{bmatrix} S_1 & \frac{1}{\rho_1}(\bar{A}S_1 + BU_1)^T \\ \frac{1}{\rho_1}(\bar{A}S_1 + BU_1) & S_1 \end{bmatrix} > 0 \tag{9}$$

$$\begin{bmatrix} S_2 & \frac{1}{\rho_2}(\bar{A}^T S_2 + C^T U_2)^T \\ \frac{1}{\rho_2}(\bar{A}^T S_2 + C^T U_2) & S_2 \end{bmatrix} > 0 \tag{10}$$

$$\bar{A}S_1 + BU_1 \succeq 0_n, \bar{A}^T S_2 + C^T U_2 \succeq 0_n, -BU_1 \succeq 0_n \tag{11}$$

$$-\bar{A}S_1 - 2BU_1 - A^- S_1 \preceq 0_n \tag{12}$$

$$-\bar{A}^T S_2 - 2C^T U_2 - A^{-T} S_2 \preceq 0_n. \tag{13}$$

The gain matrices K and L of Σ_c are given by

$$K = U_1 S_1^{-1}, \quad L = (U_2 S_2^{-1})^T. \tag{14}$$

Proof: Putting $K \triangleq U_1 S_1^{-1}$ and applying the congruence transformation $W_1 = \operatorname{diag}[S_1^{-1}, S_1^{-1}]$, condition (9) can be rewritten as

$$\begin{bmatrix} S_1^{-1} & \frac{1}{\rho_1}(\bar{A} + BK)^T S_1^{-1} \\ \frac{1}{\rho_1} S_1^{-1}(\bar{A} + BK) & S_1^{-1} \end{bmatrix} > 0 \tag{15}$$

using the Schur complement and putting $P_1 \triangleq S_1^{-1}$, one has

$$P_1 - \frac{1}{\rho_1^2}\left(\bar{A} + BK\right)^T P_1 \left(\bar{A} + BK\right) > 0. \tag{16}$$

As $P_1 > 0$, inequality (16) is the classical discrete-time Lyapunov condition implying $|\lambda_i\{\bar{A} + BK\}| < \rho_1 \leq 1$. Moreover, as S_1 is diagonal and strictly positive and $U_1 S_1^{-1} = K$, the first of conditions (11) implies $\bar{A} + BK \succeq 0_n$. Putting $U_2 S_2^{-1} \triangleq L^T$ and arguing as before, it follows that (10) and the second of conditions (11) imply $|\lambda_i\{\bar{A} + LC\}| < \rho_2 \leq 1$ and $\bar{A} + LC \succeq 0_n$, respectively. The third of conditions (11) implies $-BK \succeq 0_n$ because S_1 is diagonal and strictly positive. By (8), it follows that (9)–(11) and (14) give $|\lambda_i\{\widehat{\bar{A}}_f\}| < \rho$, $\widehat{\bar{A}}_f \succeq 0_{2n}$. Moreover, by (12) and (13), one has $|A(\theta(\cdot)) + BK| \preceq \bar{A} + BK$, $|A(\theta(\cdot)) + LC| \preceq \bar{A} + LC$, $\forall A(\theta(\cdot)) \in [A^-, A^+]$. Hence, (7) and (8) imply: $|\hat{A}_f(\theta(\cdot))| \preceq \widehat{\bar{A}}_f$, $\forall A(\theta(\cdot)) \in [A^-, A^+]$. By the lemma of Section II, and the independence of condition (16) of the time-varying parameters, the above considerations directly imply the quadratic exponential λ-stability of $\hat{\Phi}_f(\cdot, \cdot)$, namely $\|\hat{\Phi}_f(k, k_0)\|_F \leq \hat{m}_f \lambda^{(k-k_0)}$, $\forall k_0, \forall k \geq k_0$, and for some $0 < \lambda < \rho$. The analogous property of $\Phi_f(\cdot, \cdot)$ follows from $A_f(\theta(\cdot)) = T^{-1}\hat{A}_f(\theta(\cdot))T$, which implies $\|\Phi_f(k, k_0)\|_F \leq m_f \lambda^{(k-k_0)}$, where $m_f = \hat{m}_f \|T\| \|T^{-1}\|$. $\triangle$

The above theorem deserves some remarks.

Remark 1: The requirement that $S_1 = P_1^{-1}$ and $S_2 = P_2^{-1}$ be diagonal is not restrictive. In fact if $\bar{A} + BK \succeq 0_n$, then $|\lambda_i\{\bar{A} + BK\}| < \rho_1 \leq 1$, if and only if the matrix P_1 satisfying (16) is diagonal [17]. An analogous consideration holds for $S_2 = P_2^{-1}$.

Remark 2: The use of two different scalars ρ_1 and ρ_2 in (9) and (10) introduces more flexibility in the synthesis procedure and allows the designer to fix the maximum allowed value of λ $(0 < \lambda < \rho)$ such that $\|\Phi_f(k, k_0)\|_F \leq m_f \lambda^{(k-k_0)}$. If the values $\rho_1 = \rho_2 = 1$ are chosen, the assumption of a reachable and observable $\bar{\Sigma} \equiv (C, \bar{A}, B)$ can be relaxed to that of input-output stabilizability.

Remark 3: As shown in the proof of Theorem 1, condition (12) implies $|A(\theta(\cdot)) + BK| \preceq \bar{A} + BK, \forall A(\theta(\cdot)) \in [A^-, A^+]$, whence $-\bar{A} - 2BK \preceq A^-$. If $\bar{A} = -A^-$, the previous inequality can be rewritten as $A^- - 2BK \preceq A^-$, which cannot be satisfied by any $BK \neq 0$ because, as shown in the proof of Theorem 1, the third of condition (11) implies $-BK \succeq 0_n$. In this case, it is enough to define a new plant Σ_1 with dynamical matrix $A_1(\theta(\cdot)) \triangleq -A(\theta(\cdot))$, and the same B and C matrices. In this way, one obtains $\bar{A}_1 = A_1^+$, and the same design procedure given in Theorem 1 can be applied to Σ_1 replacing $\bar{A}$ and A^- with $\bar{A}_1$ and A_1^- respectively. If a stabilizing pair of gains (K_1, L_1) for Σ_1 is found, the original plant Σ is stabilized by a controller Σ_c with gains $K = -K_1$, and $L = -L_1$.

Remark 4: The vertex result of Theorem 1 makes reference to the unique majorant matrix $\bar{A}$ and to A^-. As a consequence, the number of LMIs results to be independent of the number p of uncertain parameters. This is surely an advantage from the computational point of view, especially for large p. Nevertheless if $\bar{A} \neq A^+$ and $\bar{A} \neq -A^-$, then $\bar{A}$ is a fictitious vertex which does not really belong either to the interval I_A where $A(\cdot)$ takes values or to $-I_A$. Hence, the stabilization of $\bar{A}$ is not necessary for stabilizing the ITV matrix $A(\theta(\cdot)) \in [A^-, A^+]$. As the gains K and L of Σ_c are computed with reference to $\bar{A}$, this could be a source of conservatism in the case of a large distance between $\bar{A}$ and A^+ and/or between $\bar{A}$ and $-A^-$. A possible remedy to this inconvenience is described beneath.

Assume there exists a (possibly null) matrix G such that one of the two following sets of LMIs are satisfied:

$$A^+ + BGC \succeq 0_n, \text{ and } A^+ + BGC \succeq -(A^- + BGC) \tag{17}$$

or

$$A^- + BGC \preceq 0_n, \text{ and} - (A^- + BGC) \succeq A^+ + BGC. \tag{18}$$

If such a matrix G exists, it can be seen as an internal static output gain giving the pre-compensated plant $\Sigma_p \equiv (C, A_p(\cdot), B)$, where $A_p(\cdot) \triangleq A(\cdot) + BGC \in [A_p^-, A_p^+] \triangleq I_{A_p}$. If matrix G satisfies condition (17), the new extremal matrix is given by $\bar{A}_p = A_p^+ = A^+ + BGC$, while $\bar{A}_p = -A_p^- = -(A^- + BGC)$ if G satisfies (18). Hence, the pre-compensated plant Σ_p is such that the corresponding vertex matrix $\bar{A}_p$ belongs either to the interval I_{A_p} where $A_p(\cdot)$ takes values or to $-I_{A_p} \triangleq \left[-A_p^+, -A_p^-\right]$ and the above mentioned source of conservatism is avoided. It is easy to see that by assumption A4) also the extremal triplet $\bar{\Sigma}_p \equiv (C, \bar{A}_p, B)$ of the pre-compensated plant Σ_p is reachable and observable.

In conclusion, for systems $\Sigma \equiv (C, A(\cdot), B)$ for which $\bar{A} \neq A^+$ and $\bar{A} \neq -A^-$, and the set of conditions (9)–(13) is not satisfied, an improved design procedure of the stabilizing Σ_c (if any) consists of the two following steps: 1) find an internal static output feedback G solving the set of LMIs (17) or (18) and 2) apply the same design procedure given in Theorem 1 (respectively, Remark 3) to the pre-compensated plant Σ_p if G satisfies (17) [respectively, (18)].

Remark 5: On the basis of Remarks 3 and 4, it can be stated that the present method is particularly advisable for systems for which $\bar{A} = A^+$ or $\bar{A} = -A^-$ or conditions (17) or (18) can be satisfied by properly choosing matrix G. An important class of systems implicitly satisfying conditions (17) is clearly given by positive systems [17] for which one has $A^- \succeq 0_n$. In this case conditions (17) are automatically satisfied for null G and $\bar{A}_p = A_p^+ = A^+$. Analogously, for negative ITV matrices $A(\cdot)$ one has $A^+ \preceq 0_n$ so that (18) are automatically satisfied for null G and $\bar{A}_p = -A_p^- = -A^-$. From the computational point of view, the independence of solvability conditions on the dimension of $\theta(\cdot)$ makes the method particularly appealing for LPV systems with a large number of time-varying parameters.

B. The Noisy Case

The above results are now exploited to deal with the main problem of noisy parameter measurements. In particular, defining $\Delta A(\tilde{\theta}(\cdot), \theta(\cdot)) \triangleq A(\tilde{\theta}(\cdot)) - A(\theta(\cdot))$, conditions will be stated on $\left(\mathcal{E}\left(\|\Delta A(\tilde{\theta}(\cdot), \theta(\cdot))\|_F^2\right)\right)^{1/2}$ ensuring that Σ is stabilized by a controller $\tilde{\Sigma}_c$ given by (4) and (5), with the same gains K and L of Σ_c, but with $A(\theta(\cdot))$ replaced by $A(\tilde{\theta}(\cdot))$. To evidence differences and analogies between Σ_c and $\tilde{\Sigma}_c$, in this section the two controllers are denoted by $\Sigma_c(A(\theta(\cdot)), K, L)$ and $\tilde{\Sigma}_c(A(\tilde{\theta}(\cdot)), K, L)$ respectively. If in (4), $A(\theta(\cdot))$ is replaced by $A(\tilde{\theta}(\cdot))$, the matrix $A_f(\theta(\cdot))$ given by (6) becomes

$$\begin{aligned} A_f(\tilde{\theta}(\cdot), \theta(\cdot)) &= \begin{bmatrix} A(\theta(\cdot)) & BK \\ -LC & A(\tilde{\theta}(\cdot)) + LC + BK \end{bmatrix} \\ &= A_f(\theta(\cdot)) + \Delta A_f(\tilde{\theta}(\cdot), \theta(\cdot)) \end{aligned} \tag{19}$$

where

$$\Delta A_f(\tilde{\theta}(\cdot), \theta(\cdot)) \triangleq \begin{bmatrix} 0_n & 0_n \\ 0_n & \Delta A(\tilde{\theta}(\cdot), \theta(\cdot)) \end{bmatrix}. \tag{20}$$

As a consequence, defining $x_f(\cdot) \triangleq [x(\cdot), z(\cdot)]^T$, the closed-loop system $\tilde{\Sigma}_f$ is given by

$$x_f(k+1) = (A_f(\theta(k)) + \Delta A_f(\tilde{\theta}(k), \theta(k)))x_f(k). \tag{21}$$

Assume that, following the procedure given in the previous section, a controller $\Sigma_c(A(\theta(\cdot)), K, L)$ quadratically stabilizing Σ in the noise-free case has been found. The purpose of the following theorem is to prove that there exist a bound δ such that $\tilde{\Sigma}_c(A(\tilde{\theta}(\cdot)), K, L)$ solves the QMSSP for Σ, provided that $\left(\mathcal{E}\|\Delta A_f(\tilde{\theta}(\cdot), \theta(\cdot))\|_F^2\right)^{1/2} = \left(\mathcal{E}\|\Delta A(\tilde{\theta}(\cdot), \theta(\cdot))\|_F^2\right)^{1/2} \leq \delta$.

Theorem 2: Assume there exists a controller $\Sigma_c(A(\theta(\cdot)), K, L)$ yielding a quadratically exponentially λ-stable Σ_f in the noise-free case. Then, $\tilde{\Sigma}_c(A(\tilde{\theta}(\cdot)), K, L)$ also solves the QMSSP for Σ in the noisy case provided that $\left(\mathcal{E}\|\Delta A(\tilde{\theta}(\cdot), \theta(\cdot))\|_F^2\right)^{1/2} \leq \delta$, with

$$\delta < \frac{1-\lambda}{m_f} \tag{22}$$

and the closed-loop system $\tilde{\Sigma}_f$ is quadratically exponentially mean-square $\tilde{\gamma}$-stable with $\tilde{\gamma} \triangleq \tilde{\lambda}^2$, and $\tilde{\lambda} = \lambda + m_f\delta < 1$.

Proof: By (21), one has

$$\begin{aligned} x_f(k) = \Phi_f(k, k_0)x_f(k_0) & \\ + \sum_{j=k_0}^{k-1} \Phi_f(k, j+1)\Delta A_f(\tilde{\theta}(j), \theta(j))\, x_f(j)&, \\ k_0 \geq 0.& \end{aligned} \tag{23}$$

The whiteness of the parameter measurement noise $v(\cdot)$ implies that $\Delta A_f(\tilde{\theta}(j), \theta(j))$ and $x_f(j)$ are independent. Hence, recalling the properties of the stochastic Frobenius norm and that by Theorem 1 one has $\|\Phi_f(k,\bar{k})\|_F \le m_f \lambda^{(k-\bar{k})}$, (23) implies

$$\left(\mathcal{E}\left(\|x_f(k)\|_E^2\right)\right)^{1/2} \le m_f\lambda^{(k-k_0)}\left(\mathcal{E}\left(\|x_f(k_0)\|_E^2\right)\right)^{1/2} + \sum_{j=k_0}^{k-1}\left[m_f\lambda^{(k-j-1)}\left(\mathcal{E}\left(\|\Delta A_f(\tilde{\theta}(j),\theta(j))\|_F^2\right)\right)^{1/2} \times \left(\mathcal{E}\left(\|x_f(j)\|_E^2\right)\right)^{1/2}\right]$$

multiplying both sides by λ^{-k} and defining $\lambda^{-k}\left(\mathcal{E}\left(\|x_f(k)\|_E^2\right)\right)^{1/2} \triangleq x'(k)$, one obtains

$$x'(k) \le m_f x'(k_0) + \sum_{j=k_0}^{k-1} m_f \lambda^{-1}\delta x'(j)$$

where $\left(\mathcal{E}\left(\|\Delta A_f(\tilde{\theta}(j),\theta(j))\|_F^2\right)\right)^{1/2} \le \delta$.

The above inequality is in a form to which the discrete Bellman-Gronwall Lemma can be applied, obtaining $x'(k) \le m_f x'(k_0)\prod_{j=k_0}^{k-1}\left(1+m_f\lambda^{-1}\delta\right)$, which implies

$$\left(\mathcal{E}\left(\|x_f(k)\|_E^2\right)\right)^{1/2} \le m_f\left(\mathcal{E}\left(\|x_f(k_0)\|_E^2\right)\right)^{1/2}\cdot(\lambda+m_f\delta)^{(k-k_0)}. \quad (24)$$

If $\left(\mathcal{E}\left(\|\Delta A_f(\tilde{\theta}(k),\theta(k)\|_F^2\right)\right)^{1/2} \le \delta < (1-\lambda)/m_f, \forall k \in \mathbb{Z}^+$, it follows that $\forall x_f(k_0)$, $\left(\mathcal{E}\left(\|x_f(\cdot)\|_E^2\right)\right)^{1/2}$ is upperly bounded by a function monotonically decreasing according to

$$\left(\mathcal{E}\left(\|x_f(k)\|_E^2\right)\right)^{1/2} \le m_f\left(\mathcal{E}\left(\|x_f(k_0)\|_E^2\right)\right)^{1/2}\tilde{\lambda}^{(k-k_0)} \quad \forall k \ge k_0, k_0 \ge 0 \quad (25)$$

where $\tilde{\lambda} \triangleq \lambda + m_f\delta < 1$. Inequality (3) directly follows with $\alpha = m_f^2$, and $\tilde{\gamma} = \tilde{\lambda}^2$. △

IV. Numerical Examples

Example 1: This example is a more involved version of the example reported in [18]. Consider the following LPV system $\Sigma = (C, A(\theta(\cdot)), B)$:

$$C = [1 \quad 1]$$
$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix}$$
$$A(\theta(\cdot)) = \begin{bmatrix} -2+0.2\theta_1^2(\cdot)+\theta_2(\cdot) & 1.2+0.1\theta_1(\cdot)\theta_4(\cdot) \\ 0.1+\theta_3(\cdot) & -0.1+\theta_5(\cdot) \end{bmatrix}$$

where $\theta_1(\cdot) \in [-3,-2.5], \theta_2(\cdot) \in [0.3,0.5], \theta_3(\cdot) \in [0.5,0.6]$, $\theta_4(\cdot) \in [2.5,3], \theta_5(\cdot) \in [0.25,0.5]$. The two parameters $\theta_2(\cdot)$ and $\theta_5(\cdot)$ are assumed to be observed under additive, mutually uncorrelated white Gaussian observation noises. According to the notation of Section II, this corresponds to a covariance matrix $Q = \mathrm{diag}\left[0,\sigma_2^2,0,0,\sigma_5^2\right]$. The first step of the synthesis method is to apply the procedure given in Theorem 1. The ITV $A(\theta(\cdot))$ is such that $A(\theta(\cdot)) \in [A^-, A^+] = \begin{bmatrix} [-0.45, 0.3] & [0.3, 0.575] \\ [0.6, 0.7] & [0.15, 0.4] \end{bmatrix}$, and assumption A4 is satisfied. Choosing $\rho_1 = 0.95$, $\rho_2 = 0.8$, the procedure of Theorem 1 can be applied, but the LMIs (9)–(13) are not feasible. This is probably due to the fact that $\bar{A} \neq A^+$ and $\bar{A} \neq -A^-$, because $\left|a_{i,j}^-\right| \le \left|a_{i,j}^+\right|, i,j = 1,2$ save $\left|a_{1,1}^-\right| > \left|a_{1,1}^+\right|$. Applying the procedure given in Remark 4, the LMI (17) is satisfied choosing $G = [0.45, 0]^T$, which gives $A_p^- \triangleq A^- + BGC = \begin{bmatrix} 0 & 0.75 \\ 0.6 & 0.15 \end{bmatrix}$, $\bar{A}_p = A_p^+ \triangleq A^+ + BGC = \begin{bmatrix} 0.75 & 1.025 \\ 0.7 & 0.4 \end{bmatrix}$. The set of LMIs (9)–(13) applied to $(C, \bar{A}_p, B)$ results to be feasible. By (14), the gain matrices K and L of $\Sigma_c(A_p(\theta(\cdot)), K, L)$ are $K = \begin{bmatrix} -0.2659 & -0.6952 \\ -1.8564 & -1.0288 \end{bmatrix}$, $L = \begin{bmatrix} -0.3719 \\ -0.2727 \end{bmatrix}$, and the following closed-loop eigenvalues are obtained: $\lambda_i\{\bar{A}_p + BK\} = \{0.813; -0.0318\}$, $\lambda_i\{\bar{A}_p + LC\} = \{0.7957; -0.2903\}$. In the noise free-case, the closed-loop system Σ_f with the positivized plant results to be quadratically exponentially λ-stable with $\|\Phi_f(k,k_0)\|_F \le m_f\lambda^{(k-k_0)} = 90\cdot 0.813^{(k-k_0)}, \forall k_0, \forall k \ge k_0$. In the case of noisy measures of $\theta_2(\cdot)$ and $\theta_5(\cdot)$, Theorem 2 states that the QMSSP is solvable by $\tilde{\Sigma}_c(A(\tilde{\theta}(\cdot)), K, L)$ if inequality (22) is satisfied, namely if $\left(\mathcal{E}\left(\|\Delta A(\tilde{\theta}(\cdot),\theta(\cdot))\|_F^2\right)\right)^{1/2} \le \delta < 0.0021$. As $\Delta A(\tilde{\theta}(\cdot),\theta(\cdot)) = \mathrm{diag}[v_2(\cdot), v_5(\cdot)]$, one has $\mathcal{E}(\|\Delta A(\tilde{\theta}(\cdot),\theta(\cdot))\|_F^2) = \mathcal{E}\left(v_2(\cdot)^2 + v_5(\cdot)^2\right) = \sigma_2^2 + \sigma_5^2$. Hence, the QMSSP admits a solution if $\left(\sigma_2^2+\sigma_5^2\right)^{1/2} < 0.0021$.

Example 2: The widely investigated case-study of a DC motor speed control is now considered. Assuming the rotational speed and electric current as state variables and the voltage as input, the continuous-time state space form is described by the triplet

$$C_C = [1 \quad 0]$$
$$B_C = \begin{bmatrix} 0 \\ \frac{1}{L_m} \end{bmatrix}$$
$$A_C = \begin{bmatrix} \frac{-b}{J_m} & \frac{K_m}{J_m} \\ \frac{-K_m}{L_m} & \frac{-R_m}{L_m} \end{bmatrix}$$

where $b = 0.1$ Nms, is the damping ratio of the mechanical system, $J_m = 0.01\ \mathrm{kg}\cdot\mathrm{m}^2/\mathrm{s}^2$ is the moment of inertia of the rotor, $K_m = 0.01\mathrm{N}\cdot\mathrm{m}/\mathrm{A}$ is the electromotive force constant, $R_m = 1\,\Omega$ is the electric resistance, $L_m = 0.5$ H is the electric inductance (the numerical values of parameters have been taken from [19]). An LPV system is obtained assuming a time-varying resistance $R_m(\cdot) \triangleq \theta(\cdot) \in [1,5]$ with $R_m(\cdot)$ measured under a scalar additive white noise $v(\cdot) \sim N(0,Q)$. Approximating the derivative with Euler method and choosing a sampling period $T_s = 0.1$, results in the following discretized triplet:

$$C \triangleq C_C = [1 \quad 0]$$
$$B \triangleq T_S B_C = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix}$$
$$\cdot A(\theta(\cdot)) \triangleq I + T_S A_C \in [A^-, A^+] = \begin{bmatrix} 0 & 0.1 \\ -0.002 & [0, 0.8] \end{bmatrix}$$

which satisfies assumption A4. This first step of the synthesis is accomplished following the procedure given in Theorem 1. Choosing $\rho_1 = 0.9$, $\rho_2 = 0.81$, the LMIs (9)–(13) result to be feasible, and (14) gives the following gain matrices: $K = [0, -1.0824]$, $L = [0.1873, 0.0445]^T$. In the noise free-case, the closed-loop system Σ_f results to be quadratically exponentially λ-stable with $\|\Phi_f(k,k_0)\|_F \le m_f\lambda^{(k-k_0)} = 12\cdot 0.8075^{(k-k_0)}, \forall k_0, \forall k \ge k_0$, where $\lambda = 0.8075$ is the closed-loop eigenvalue with maximum modulus. In the case of noisy measures of $R_m(\cdot) \triangleq \theta(\cdot)$, Theorem 2 states that the QMSSP is solvable by $\tilde{\Sigma}_c(A(\tilde{\theta}(\cdot)), K, L)$ if inequality (22) is satisfied, namely if $\left(\mathcal{E}\left(\|\Delta A(\tilde{\theta}(\cdot),\theta(\cdot))\|_F^2\right)\right)^{1/2} \le \delta < 0.016$. All the elements of $\Delta A(\tilde{\theta}(\cdot),\theta(\cdot))$ are null save the entry (2, 2) which is equal to $-T_s v(\cdot)/L_m$. It follows that $\mathcal{E}(\|\Delta A(\tilde{\theta}(\cdot),\theta(\cdot))\|_F^2) = \mathcal{E}\left((T_s v(\cdot)/L_m)^2\right) < (0.016)^2$, if $\mathcal{E}(v(\cdot)^2) = Q \triangleq \sigma_v^2 < 0.0064$.

Example 3: The purpose of this example is just to make some comparisons with usual synthesis method for LPV systems with exact parameter measures. A first comparison has been made with the method described in [8, Theor. 4], where a parameter dependent state feedback control law is proposed. The following dynamical matrix is considered in [8], with reference to the stability analysis problem:

$$A(\theta(\cdot)) = \begin{bmatrix} 0.8 & -0.25 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0.03 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \theta(\cdot) \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} [0.8 \quad -0.5 \quad 0 \quad 1]. \tag{26}$$

To deal with synthesis, the following input and measure matrices have been assumed here: $B^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $C = I_4$. The set of LMIs in [8, Theor. 4] results to be feasible for $|\theta(\cdot)| \leq 10^4$. For the present approach, as $C = I_4$, the set of conditions (9)–(13) reduces to (9), the first of (11) and (12). It is easy to see that in this case one has $\bar{A} \neq A^+$ and $\bar{A} \neq -A^-$, $\forall \theta(\cdot) \in \mathbb{R}$. All the same, assuming $\rho = 0.8$, the above conditions result to be feasible in the larger domain $|\theta(\cdot)| \leq 10^5$. The gain matrix K corresponding to $|\theta(\cdot)| = 10^5$ stabilizes all the $A(\theta(\cdot)) + BK$, with $|\theta(\cdot)| \leq 10^5$, and is given by $K = \begin{bmatrix} -0.7486 & 0.0153 & 0 & -0.9539 \\ 0.1104 & 0.0633 & -1 & 0.0976 \end{bmatrix}$. The eigenvalues of $\lambda_i\{\bar{A} + BK\}$ result to be $\{0.2; -0.4893; 0.5535; 0.0848\}$.

The following measure matrix $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, has been chosen to deal with a non accessible state vector. Assuming $\rho = 0.8$, the set of conditions (8)–(12) applied to $(C, \bar{A}, B)$ results again to be feasible for $|\theta(\cdot)| \leq 10^5$. By (14), the gain matrices K and L of $\Sigma_c(A(\theta(\cdot)), K, L)$ are $K = \begin{bmatrix} -0.7257 & 0 & 0 & -0.9182 \\ 0 & 0 & -1 & 0 \end{bmatrix}$, $L^T = \begin{bmatrix} -0.7251 & -0.8780 & 6.78 \cdot 10^3 & 0.0658 \\ 6.89 \cdot 10^{-7} & 1.41 \cdot 10^{-6} & -0.0955 & -1 \end{bmatrix}$, and the following closed-loop eigenvalues are obtained: $\lambda_i\{\bar{A} + BK\} = \{0.2; -0.4642; 0.5385, 0\}$, $\lambda_i\{\bar{A} + LC\} = \{0.7303; -0.0074; -0.2718 \pm 0.1943i\}$.

In this case, the comparison has been made with the mixed H_2/H_∞ gain scheduled controller for LFT plants described in [4]. According to representation (7) in [4], the LFT structure of the considered LPV plant particularized for the stabilization problem with the controlled variable $z(\cdot)$ coincident with the output variable $y(\cdot)$, and null disturbance matrices B_1, $D_{\Delta 1}$, D_{11}, D_{21}, is given by

$$\begin{bmatrix} A & B_\Delta & B_1 & B_2 \\ C_\Delta & D_{\Delta\Delta} & D_{\Delta 1} & D_{\Delta 2} \\ C_1 & D_{1\Delta} & D_{11} & D_{12} \\ C_2 & D_{2\Delta} & D_{21} & 0 \end{bmatrix} = \left[\begin{array}{cccc|c|c|cc} 0.8 & -0.25 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0.03 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ \hline 0.8 & -0.5 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$w_\Delta(k) = \Delta(k) z_\Delta(k)$$

where $\Delta(k)$ evolves in the polytopic set $\mathcal{P}_\Delta \triangleq \mathbf{co}\{\Delta_1, \Delta_2\}$, with $\Delta_1 = -10^5$, $\Delta_2 = 10^5$ and $\mathbf{co}$ stands for convex hull. The LPV output feedback controller has the following LFT structure:

$$\begin{bmatrix} x_K(k+1) \\ u(k) \\ z_K(k) \end{bmatrix} = \begin{bmatrix} A_K & B_{K1} & B_{K\Delta} \\ C_{K1} & D_{K11} & D_{K1\Delta} \\ C_{K\Delta} & D_{K\Delta 1} & D_{K\Delta\Delta} \end{bmatrix} \begin{bmatrix} x_K(k) \\ y(k) \\ w_K(k) \end{bmatrix}$$
$$w_K(k) = \Delta_K(k) z_K(k).$$

The LMIs relative to the LPV control problem with H_∞ performance (see [4, App. A]) applied to the considered stabilization problem are not satisfied in the domain $|\theta(k)| \leq 10^5$, while a feasible solution is obtained considering the smaller domain $|\theta(k)| \leq 11 \cdot 10^3$. In this case, the vertices of the polytopic set $\mathcal{P}_\Delta$ are $\Delta_1 = -11 \cdot 10^3$ and $\Delta_2 = 11 \cdot 10^3$. The matrices of the LPV controller are

$$A_K = \begin{bmatrix} -1.4580 & 4.8284 & 0.0011 & 1.13 \cdot 10^{-6} \\ -96.69 & 320.23 & 0.07 & 7 \cdot 10^{-5} \\ 331.94 & -1.1 \cdot 10^3 & -2.6336 & -0.0027 \\ -3.7 \cdot 10^5 & 1.22 \cdot 10^6 & 2.72 \cdot 10^3 & 2.7795 \end{bmatrix}$$
$$D_{K11} = \begin{bmatrix} -319.78 & 0.0183 \\ 255.64 & -1.0147 \end{bmatrix}$$
$$B_{K1} = \begin{bmatrix} 0.0003 & -1 \cdot 10^{-8} \\ 0.0172 & -9 \cdot 10^{-7} \\ -0.0590 & 3 \cdot 10^{-5} \\ 65.82 & -0.0378 \end{bmatrix}$$
$$C_{K1} = \begin{bmatrix} 1.78 \cdot 10^6 & -5.96 \cdot 10^6 & -1.30 \cdot 10^3 & -1.4035 \\ -1.43 \cdot 10^6 & 4.76 \cdot 10^6 & 1.04 \cdot 10^3 & 1.1223 \end{bmatrix}$$
$$D_{K1\Delta} = \begin{bmatrix} 386.74 \\ -309.05 \end{bmatrix}$$
$$D_{K\Delta 1} = [-1.47 \cdot 10^3 \quad 0.0017]$$
$$B_{K\Delta} = \begin{bmatrix} -3 \cdot 10^{-4} \\ -0.0208 \\ 0.5243 \\ -1.40 \cdot 10^3 \end{bmatrix}$$
$$C_{K\Delta} = [8.31 \cdot 10^6 \quad -2.75 \cdot 10^7 \quad -53.75 \quad 0.0308]$$
$$D_{K\Delta\Delta} = -6.0511.$$

According to [4, eq. (30)], one has $\Delta_K(k) \triangleq \Delta_K(\Delta) = \mathbf{co}\left\{-9.7487 \cdot 10^{-5}, 9.7487 \cdot 10^{-5}\right\}$.

V. Conclusion

This note has considered the QMSSP for discrete-time LPV in the more general case of noisy parameter measures. This imposes an approach to the synthesis problem where both robust and LPV control elements have to be simultaneously used. To this purpose, an ITV matrix based approach has been adopted. The solvability conditions have been established in terms of LMIs which only involve the extremal matrices A^- and $\bar{A}$. This makes the method also appealing from the numerical point of view because the set of LMIs to be checked is independent of the number of time-varying parameters and all the calculations can be performed off-line. Another feature of the synthesis procedure is its independence from the way the physical parameters enter the dynamical matrix. The exponential stability degree of the closed-loop system can be controlled through parameters ρ_1 and ρ_2. The class of systems for which the present approach is particularly advisable has been evidenced in Remark 5. The applicability of the method has been illustrated by numerical examples. In particular, Example 3 shows that the method may be also competitive when applied in a different context.

Under some additional assumptions on the parameter variability, the present method can be also applied relaxing some conditions on the plant dynamics. For example, very large parameter variations, possibly preventing the quadratic approach, can be also accepted subdividing the whole parameter set as $\mathcal{P} = \cup_{i=1}^{\ell}\mathcal{P}_i$ and assuming $\theta(\cdot) \in \mathcal{P}_i$ for a sufficiently long time interval T_i. This assumption also allows the hypothesis of constant sensor and actuator matrices to be relaxed to that of piece-wise constant matrices. The case of bounded observation noise follows as a particular case of the present approach.

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Assessing Asymptotic Stability of Linear Continuous Time-Varying Systems by Computing the Envelope of all Trajectories

Germain Garcia, Pedro L. D. Peres, and Sophie Tarbouriech

***Abstract*—In this note, necessary and sufficient numerical conditions for asymptotic stability and for uniform asymptotic stability of linear continuous time-varying systems are derived. For a given set of initial conditions, a tube containing all the trajectories of the system is constructed in the state space. At each instant of time, there exists an initial condition inside the set such that the resulting trajectory attains the border of the tube. Based on the above formulation, necessary and sufficient conditions for asymptotic stability and for uniform asymptotic stability are expressed through the solution of a linear differential Lyapunov equation. The conditions can deal with the stability of periodic systems as well. One of the main characteristics of the proposed necessary and sufficient conditions is that the only assumption on the dynamical matrix of the time-varying system is continuity. Examples from the literature illustrate the superiority of the proposed conditions when compared to other methods.**

***Index Terms*—Continuous-time systems, stability of linear systems, time-varying systems.**

I. Introduction

The stability of linear continuous time-varying systems has been investigated in numerous papers [1]–[7]. Although from a theoretical point of view there exist necessary and sufficient conditions in the literature [8]–[10], a lot of effort has been dedicated to the search for numerically tractable necessary and sufficient conditions (see [11] and references therein). In many cases, only sufficient conditions are obtained, as for instance in the methods based on the analysis of the eigenvalues of a time-invariant system [12], [13].

As it is well known, even when the eigenvalues of the system have strictly negative real parts for all instants of time the time-varying system can be unstable (see for instance the second example in this note). On the other hand, an asymptotically stable linear time-varying system can exhibit a system matrix with eigenvalues that have strictly positive real parts [12]. This gives an idea of the difficulty of assessing the stability of a linear time-varying system.

Other techniques use the Lyapunov theory, for instance, by associating to the time-varying original system time-invariant piecewise approximations from which sufficient conditions for stability are derived [1], [5], [7]. This is the case of the recently published paper [11], where classical Lyapunov equations are solved for a sequence of discrete points inside the time interval of interest. Associating a quadratic Lyapunov function to each point of the grid, a tube is constructed in the state space. For the selected set of initial conditions, all the trajectories of the original system lie strictly inside the tube. The dynamic matrix

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G. Garcia and S. Tarbouriech are with the CNRS, LAAS, F-31077 Toulouse, France, and the Université de Toulouse; UPS, INSA, INP, ISAE, LAAS, F-31077 Toulouse, France (e-mail: garcia@laas.fr; tarbour@laas.fr).

P. L. D. Peres is with the School of Electrical and Computer Engineering, University of Campinas—UNICAMP, 13083-970, Campinas, SP, Brazil (e-mail: peres@dt.fee.unicamp.br).

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