On the fractal nature of Penrose tiling

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An earliest preoccupation of man has been to find ways of partitioning infinite space into regions having a finite number of distinct shapes and yielding beautiful patterns called tiling. Archaeological edifices, everyday objects of use like baskets, carpets, textiles, etc. and many biological systems such as beehives, onion peels and spider webs also exhibit a variety of tiling. Escher's classical paintings have not only given a new dimension to the artistic value of tiling but also aroused the curiosity of mathematicians. The generation of aperiodic tiling with five-fold rotational symmetry by Penrose in 1974 and the more recent production of decorated pentagonal tiles by Rosemary Grazebrook have heightened the interest in the subject among artists, engineers, biologists, crystallographers and mathematicians[1–5]. In spite of its long history, the subject of tiling is still evolving. In this communication, we propose a novel algorithm for the growth of a Penrose tiling and relate it to the equally fascinating subject of fractal geometry pioneered by Mandelbrot[6]. The algorithm resembles those for generation of fractal objects such as Koch's recursion curve, Peano curve, etc. and enables consideration of the tiling as cluster growth as well. Thus it clearly demonstrates the dual nature of a Penrose tiling as a natural and a non-random fractal.

THE aperiodic tilings have many interesting properties which can be illustrated with respect to the most discussed quasi-periodic tiling: the Penrose tiling[7]. These can be *infinite in number*. The tilings exhibit *local isomorphism* which ensures that every finite region in any one tiling is contained somewhere inside every other and that too infinitely many times. The tilings can be generated from one another by the methods of inflation or deflation. For example, in deflation a cluster of tiles is subdivided into smaller pieces following specific procedures. Performing such operations iteratively, one can generate an aperiodic tiling with a much larger number of smaller tiles. These procedures endow the tiling with the property of *self-similarity*. These properties have suggested, right from the time of the discovery of Penrose tilings, that the tiling is fractal in nature[2]. If so, it should be possible to generate the tiling by an algorithm that is characteristic of fractal generation and consisting of the three steps of initiation, generation and cascade at all scales. Till date, there are no methods of generating a Penrose tiling in a fashion similar to the generation of a non-random fractal object. We postulate, in this communication, such an algorithm and arrive at the Hausdorff fractal dimension.

Let us consider a figure in the form of two straight edges AB and BC of equal length, a, and intersecting each other at an angle of 108° (Figure 1 *a*). The point of intersection, B, is called a vertex. Divide the side AB in the ratio $\tau : 1$, as seen from A, and replace a short segment with an isosceles triangle having sides BE and ED of length a/τ. The triangle should be erected such that

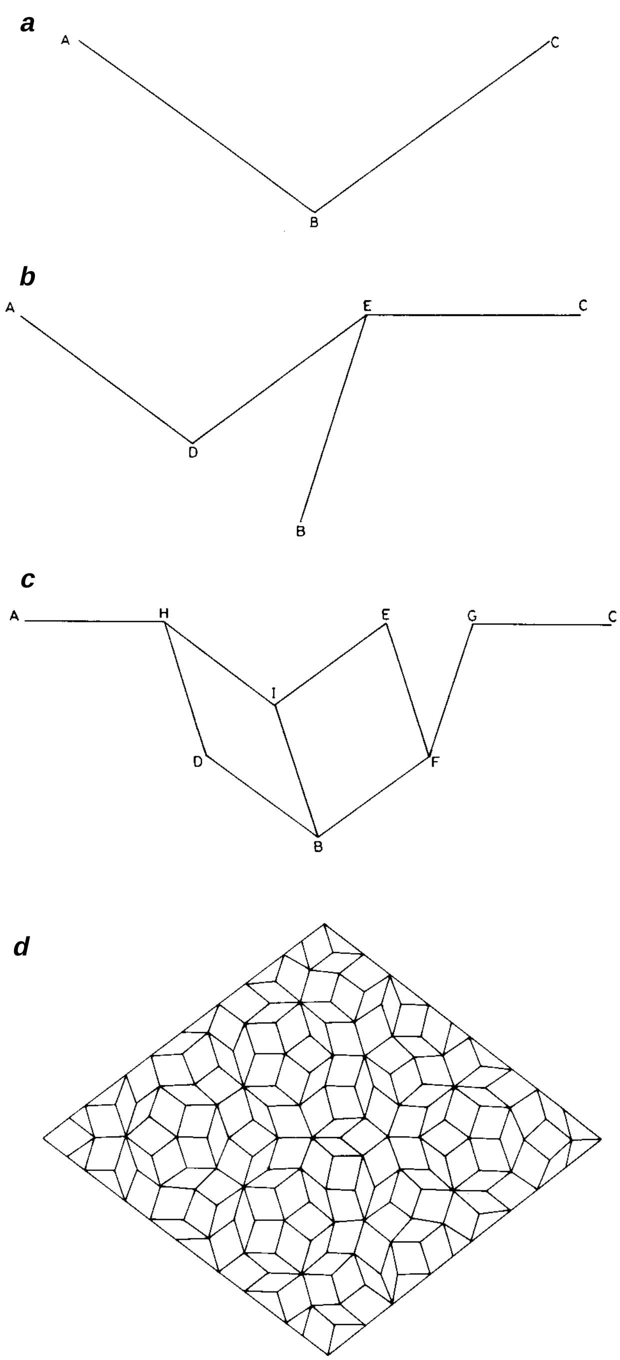

Figure 1. Proposed algorithm for the generation of a Penrose tiling. ***a***, the initiator; ***b***, the first generation ($n = 1$); ***c***, the second generation ($n = 2$); and ***d***, Penrose tiling in a thick rhomb: $n = 7$.

*For correspondence. (e-mail: dnml@csnml.ren.nic.in) the angle at B is divided in the ratio 2 : 1 and $\angle EDB = \angle EBD = 72°$ each. τ is the golden mean and has the value of $(1 + \sqrt{5})/2$. Next join the point E to C which lies at a distance of a/τ as well. We may now erase all other lines which do not have a length a/τ (Figure 1 *b*). These steps generate three line segments of equal length (DE, EB and EC) and two new vertices D and E with $\angle ADE$ and $\angle BEC$ being 108°. It may be noted that one of these angles is exterior to the apex of the isosceles triangle erected, while the other is exterior to its base. These two angles are once again divided as before by erecting two isosceles triangles of side a/τ^2 such that two adjacent angles of 36° each are formed at the apex and two adjacent angles of 72° form at the base of the previously erected isosceles triangle. All vertices at a distance of a/τ^2 will then be joined and all segments whose length is not a/τ^2 will be erased (Figure 1 *c*). The two isosceles triangles erected in this operation are EFG and DHI. It may be seen that the rhombi BDHI and BIEF are the thin and thick rhombi used by Penrose to generate an aperiodic tiling. The newly formed vertices with an included angle of 108° are at F, G, H and I. The angle EIB is also 108°. The procedure described above can be repeated to divide all the 108° angles by erecting isosceles triangles of side $1/\tau$ of the side on which they are being erected. We also join the newly formed vertices to already existing vertices at a distance equal to the side of the triangle erected. The procedure can be repeated *ad infinitum*. With each complete operation of the algorithm, the edges in the pattern are reduced by a factor of τ. The pattern of a Penrose tiling begins to emerge as we proceed and all the known distinct vertices appear when the length scale reaches $a/(\tau^6)$. Such a complete Penrose tiling shown in Figure 1 *d*, is obtained after reflecting the structure derived at $n = 7$ in a mirror placed along AC. The ratio of the edge length of a tile (a_n) to the edge of the rhomb tiled (a) will always be $\bar{\tau}^n$, where n represents the number of times the scale length has been reduced by a factor of τ. All further discussions refer to the tiling within this rhomb. Many others have repeatedly stressed the self-similar nature of the Penrose tiling. The present algorithm is the first to employ an iterative procedure commonly employed to generate deterministic fractals. This novel algorithm which has an *initiator* (lines at 108° to each other), a *generator* (erection of isosceles triangles with sides $1/\tau$ times the previous length scale and erasing all sides of other length scales) and *cascading* is typical of procedures which generate patterns with fractal dimensions. This procedure also obviates the need for assembling tiles according to matching rules of Penrose. In addition, it permits us to utilize various procedures developed for determining the fractal dimension of patterns arising from such an algorithm.

Conventionally, the estimation of the fractal dimension of a non-random fractal based on an iterative scheme of the above three steps is given as,

$$D_f = \operatorname*{lt}_{l \to 0} \frac{\log(N)}{\log(1/l)},$$

where N is the number of edges of length l which occur within the fractal object. The discovery of quasi-crystals has conclusively demonstrated that aperiodic packing of atoms occurs in nature. In a Penrose tiling, the closest distance of separation of two atoms can only be along the short diagonal of a thin rhomb. For the present algorithm, if the edge length of a rhomb is taken as the reference, when n tends to infinity, the edge length tends to zero and the rhomb collapses to a point. Thus it would not be appropriate to evaluate the fractal dimension of the tiling in the limit when n tends to infinity. Therefore, when one considers the Penrose tiling from a crystallographic point of view, a limit has to be placed on the value of n, the number of recursions of the algorithm. The recursions have to be limited only to the extent of achieving distances of separation of the order of atomic spacings between the vertices in the tiling. From the point of view of the present algorithm, one achieves atomic distances of separation from a starting separation, a, of 1 cm in 39 recursions. Hence, the formula for D_f can be modified as,

$$D_f = \operatorname*{lt}_{n \to n'} \frac{\log(N)}{\log(\tau^{n-1})},$$

where n' is the physical limit on the number of recursions. One can obtain the edge length and the number of vertices generated in a tiling in terms of the Fibonacci numbers. These have been estimated as:

$$N_{\text{vertices}} = F_{2n+1} + 4F_n,$$

$$N_{\text{edges}} = 2N_{\text{vertices}} - 8F_n + \frac{A_n}{2},$$

where F represents the respective Fibonacci numbers in a sequence starting with zero. A_n is an even integer dependent on n. Based on an analysis using the edge length in the above formula for D_f, we arrive at a value of the Hausdorff dimension as 1.974 using the size of n' as 39. This value is close to the fractal dimension obtained for the growth of two-dimensional clusters of atoms. However, it may be noted that limit of D_f as $n \to \infty$ exists and its value is 2, implying a non-fractal space filling structure.

Quasicrystals which incorporate aperiodic tilings, like Penrose tiling, are known to grow by the clustering process. Recently Lord *et al.*[5] have shown that the clustering process is very similar to the way quasicrystals actually grow. A cluster is conceived to grow by accretion of successive shells around an initial seed, such as a 12- or 13-

atom icosahedron. The clusters grow as they bond to each other by sharing the atoms. One of the most commonly discussed cluster growth models is the diffusion-limited aggregation (DLA) of atoms[8–10]. DLA exhibits random branching during growth and can be shown as a non-deterministic fractal, governed by the sticking probability. Notwithstanding this randomness, DLA manifests, irrespective of the magnitude of the sticking probability, some important regularities in an average sense[10], e.g. (i) the most probable value of the angle between neighbouring branches is found to be 36°, and (ii) Fibonacci numbers are known to occur in DLA clusters. In Penrose tilings also, these two parameters, viz. the angle of 36° and Fibonacci numbers, govern the pattern generation. However unlike those in DLA, the angles in a tiling are exact multiples of 36° and the number of tiles, edges and vertices are precise functions of the Fibonacci numbers as shown above. It remains to be seen as to how a deterministic algorithm for Penrose tiling, like the one generated in the present study, and diffusion controlled probabilistic DLA algorithm of Whitten and Sander[9] bear these important similarities.

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