Extensions to a Lemma of Bernik with Applications in the area of Metric Diophantine Approximation.

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Contents

Notation Page

A list of frequently used notation is given to assist the reader. Throughout the list take $x \in \mathbb{R}$.

$[x]$ denotes the integer part of x.

$\{x\} = x - [x]$ denotes the fractional part of x.

$|.|$ denotes the usual Euclidean norm.

$||x|| := \min_{y\in\mathbb{Z}} |x - y|$ denotes the distance between x and the closest integer to x.

$|\mathbf{x}|_\infty := \max\{|x_1|, |x_2|, ..., |x_m|\}$ denotes the infinity norm of the vector $\mathbf{x} \in \mathbb{R}^m$.

$\mu(A)$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$.

$\mu_n(A)$ denotes, for $n > 1$, the n-dimensional Lebesgue measure of a measurable set $A \subset \mathbb{R}^n$.

$\mu_P(A)$ denotes the Haar measure of a measurable set $A \subset \mathbb{Q}_p$.

$\dim_{\mathcal{H}}(A)$ denotes the Hausdorff dimension of a set $A \subset \mathbb{R}$.

$a \ll b$ denotes the Vinogradov notation meaning that there exists some constant $c > 0$ such that $a < cb$.

$a \gg b$ denotes the Vinogradov notation meaning that there exists some constant $c > 0$ such that $a > cb$.

$a \asymp b$ denotes that $a \ll b$ and $a \gg b$.

$\mathcal{S}_P(\alpha)$ denotes the set of all numbers closer to the root α of P than any other root of P.

$P_n := \{P(x) \in \mathbb{Z}[x] : \deg(P(x)) = n\}$.

$P_n(Q) := \{P(x) \in \mathbb{Z}[x] : \deg(P(x)) = n, H(P) \leq Q\}$.

Abstract

This thesis is concerned with two extensions to a result of V. I Bernik [23] from 1983 which provides a quantitative description of the fact that two relatively prime polynomials in $\mathbb{Z}[x]$ cannot both have very small absolute values (in terms of their degrees and heights) in an interval unless that interval is extremely short. Bernik's result was presented for intervals in $\mathbb{R}$ and has the restriction that the polynomials being considered must have small modulus. In this thesis the result is extended to a cuboid in $\mathbb{R}^3$ and, in fact, it is clear from the proof that the result holds in $\mathbb{R}^n$. Furthermore the restriction that the polynomials must have small modulus is removed. This is the first extension of Bernik's result to consider polynomials of large modulus. Bernik's result is also extended to a parallelepiped in $\mathbb{R} \times \mathbb{C} \times \mathbb{Q}_p$. This is not the first extension of this kind but the method of proof used leads to a new and very useful proposition.

Acknowledgements

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Finally I would like to give special thanks to Dr. David Malone for agreeing to act as the internal examiner and to Prof. Simon Kristensen for agreeing to act as the external examiner. I am very grateful to you both.

Organisation of the Thesis

This thesis consists of five chapters:

- Chapter 1 is a background on the work that has been done in the area of metric Diophantine approximation. It is intended to give some necessary terminology and notation for the other four chapters and, more so, to motivate the reason we are interested in the questions that we consider in the other four chapters of the thesis.

- Chapter 2 gives a summary of the results and notation needed for the proofs in Chapter 3, Chapter 4 and Chapter 5.

- Chapter 3 is concerned with the extension of the previously mentioned result of V. I. Bernik [23] to a cuboid in $\mathbb{R}^3$. The proof given makes it clear that the result in fact holds in $\mathbb{R}^n$.

- Chapter 4 is concerned with the extension of the result of V. I. Bernik [23] to a parallelepiped in the space $\mathbb{R} \times \mathbb{C} \times \mathbb{Q}_p$. The proof used here is very similar to that of Chapter 3.

- Chapter 5 is concerned with an upcoming paper, [31], which considers the distribution of algebraic conjugate triples. In this chapter a brief summary of the main ideas of [31] are discussed and a brief overview of the proofs is given. It was this paper that motivated the extension of Bernik's Lemma to polynomials with very large modulus. A demonstration of why this extension was necessary for the proofs in [31] is also given in this chapter.

Chapter 1

BACKGROUND

1.1 Results In One Dimension

It all begins with Dirichlet! It had long been establish that the rationals are dense in the reals. In particular,

$$\text{for any } \epsilon > 0 \text{ and } \forall\ x \in \mathbb{R} \text{ there exist } p, q \in \mathbb{Z}, q > 0, \text{ s.t } \left| x - \frac{p}{q} \right| < \epsilon.$$

Thus, given any real number, a rational number can always be found such that the distance between the two numbers is as small as wanted. The next question to ask is, if instead one is given $x \in \mathbb{R}$, then for $q \in \mathbb{N}$, is there a limit on how small ϵ can be? Dirichlet [47] gave the first complete answer to this question.

Theorem 1.1 (**Dirichlet (1842)**).
Let x and Q be real numbers with $Q \geq 1$, then there exists a rational number $\frac{p}{q}$, with $1 \leq q \leq Q$, such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{qQ}.$$

Furthermore, if x is irrational, then there exist infinitely many rational numbers $\frac{p}{q}$ such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}. \tag{1.1}$$

Finally if $x = \frac{a}{b}$ is rational, then for any rational $\frac{p}{q} \neq \frac{a}{b}$ with $q > 0$ we have

$$\left| x - \frac{p}{q} \right| \geq \frac{1}{|b|q}.$$

Dirichlet's proof relied solely on the Pigeonhole Principle which states; if n balls are placed in m boxes (pigeonholes), and $m < n$, then at least one box must contain more than one ball. The proof runs as follows.

Proof of Dirichlet's Theorem.

Let t denote the integer part of Q; i.e. $t = [Q]$. If x is the rational $\frac{a}{b}$, with a and b integers and $1 \leq b \leq t$, then setting $p = a$ and $q = b$ the first inequality is obtained.

Otherwise, consider x to be irrational. Let $\{x\}$ denote the fractional part of x, i.e. $\{x\} = x - [x]$. Then the $t+2$ points $0, \{x\}, ..., \{tx\}, 1$ are pairwise distinct and so at least two of these points must lie in one of the $t+1$ intervals $[\frac{j}{t+1}, \frac{j+1}{t+1}]$, where $j = 0, ..., t$. Thus there exist integers k, l and m_k, m_l with $0 \leq k < l \leq t$ and

$$|(lx - m_l) - (kx - m_k)| \leq \frac{1}{t+1} < \frac{1}{Q}.$$

Setting $p := m_l - m_k$ and $q := l - k$ gives the inequality

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{qQ}.$$

Noticing that q satisfies $1 \leq q \leq t \leq Q$ completes the proof of the first assertion.

Now again suppose that x is irrational and let Q_0 be a positive integer. By the first assertion of the theorem, there exists an integer q with $1 \leq q \leq Q_0$ such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{qQ_0} \leq \frac{1}{q^2}$$

holds for some integer p. It may be assumed that q is the smallest integer between 1 and Q_0 with this property. By the first assertion of the theorem applied with $Q = \frac{1}{|x - \frac{p}{q}|}$, there exists a rational number $\frac{p'}{q'}$ with $1 \leq q' \leq \frac{1}{|x - \frac{p}{q}|}$ such that

$$\left| x - \frac{p'}{q'} \right| < \frac{1}{q'Q} = \frac{1}{q'}\left| x - \frac{p}{q} \right| < \frac{1}{q'Q_0} \text{ and } \left| x - \frac{p'}{q'} \right| < \frac{1}{q'^2}.$$

The choice of q ensures that $q' > q$ and so proceeding inductively an infinite sequence of distinct rational numbers satisfying (1.1) is obtained. This completes the proof of the second assertion.

Finally suppose $x = \frac{a}{b}$, then

$$\left| x - \frac{p}{q} \right| = \left| \frac{a}{b} - \frac{p}{q} \right| = \left| \frac{aq - bp}{bq} \right|$$

but $\frac{a}{b} \neq \frac{p}{q}$ implies that $aq - bp \neq 0$ and in particular $|aq - bp| \geq 1$. Thus

$$\left|x - \frac{p}{q}\right| \geq \left|\frac{1}{bq}\right| = \frac{1}{|b|q}.$$

□

So Dirichlet's Theorem gave the first complete result on how small ϵ could be. The next question to answer is, can Dirichlet's result be improved? This was answered by Liouville [70] and Hurwitz [59]. Define an algebraic number α to be of degree n if the minimal integer polynomial of α has degree n.

Theorem 1.2 (**Liouville (1844)**).
Let α be a real algebraic number of degree $n \geq 1$, then there exists a constant $c(\alpha) > 0$, depending only on α, such that

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{c(\alpha)}{q^n},$$

for all rational numbers $\frac{p}{q} \neq \alpha$.

So Liouville's Theorem shows, at least in the case of real algebraic numbers, that Dirichlet's Theorem can not be improved arbitrarily in terms of the exponent 2 in q^2 or the constant $c(\alpha)$. Furthermore, Liouville's Theorem established for the first time the existence of transcendental numbers. Liouville used the proof of his theorem to construct the first known example of a transcendental number, $l = \sum_{i=1}^{\infty} 10^{-i!}$.

Suppose that α is a real number such that for any $\nu \in \mathbb{R}$, $\nu > 0$, there exist $p, q \in \mathbb{N}$, $q > 1$, such that

$$0 < \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^\nu}$$

then α is called a **Liouville Number**. It is not difficult to show that all Liouville numbers are transcendental. However, the converse is not true. For example, in 1953 Mahler [73] showed that π was not a Liouville number by showing that

$$\left|\pi - \frac{p}{q}\right| > \frac{1}{q^{42}}$$

for all positive integers $p, q \geq 2$.

Theorem 1.3 (**Hurwitz (1891)**).
For every irrational number x there are infinitely many relatively prime integers p, q such that

$$\left|x - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}.$$

The hypothesis that x is irrational cannot be omitted. Moreover the constant $\sqrt{5}$ is the best possible. It can be shown that if $\sqrt{5}$ is replaced by any $\epsilon > \sqrt{5}$ then, choosing $x = \frac{1+\sqrt{5}}{2}$ (the golden ratio), there exist only finitely many relatively prime integers p, q such that the formula above holds. For a proof of the theorem see also [57].

From Hurwitz's Theorem it can be concluded that there exist numbers for which Dirichlet's Theorem cannot be improved arbitrarily. This leads to the following definitions of badly approximable and well approximable numbers. A number is said to be **badly approximable** if Dirichlet's Theorem cannot be improved arbitrarily for that number. More concretely, $x \in \mathbb{R}$ is **badly approximable** if

$$\inf_{q \in \mathbb{N}} q||qx|| > 0$$

where $||.||$ is used to denote the distance to the closest integer, i.e.

$$||x|| = \min_{y \in \mathbb{Z}} |x - y|.$$

In contrast a number is said to be **well approximable** if it is not badly approximable. The set of badly approximable numbers will be denoted in the following way:

$$\mathbf{Bad} := \{x \in \mathbb{R} : \inf_{q \in \mathbb{N}} q||qx|| > 0\}.$$

Much is now known about **Bad** (see for example [26] or [37] for a list of results and discussion). One particular result of note, which can be proved by use of continued fractions, is the following.

Lemma 1.4.
Every quadratic irrational is badly approximable.

A **quadratic irrational** is defined to be any number of the form $\frac{a+b\sqrt{c}}{d}$, where $a, b, c, d \in \mathbb{Z}$ with $b, c, d \neq 0$ and c is square-free. It is a widely held conjecture that the only algebraic irrationals that are in **Bad** are the quadratic irrationals unfortunately there is no evidence of this as of yet.

Given a set $A \subset \mathbb{R}$ denote by $\mu(A)$ the Lebesgue measure of A and by $\dim_{\mathcal{H}}(A)$ the Hausdorff dimension of A, which is defined in detail in Section 1.1.2. It has been shown that $\mu(\mathbf{Bad}) = 0$ and $\dim_{\mathcal{H}}(\mathbf{Bad}) = 1$, proofs of these facts will appear later.

Another very important and celebrated piece of the puzzle was provided by Roth [80].

Theorem 1.5 (Roth (1955)).
For any irrational algebraic number x and any $\epsilon > 0$ the inequality

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^{2+\epsilon}}$$

is satisfied for only a finite number of pairs $(p, q) \in \mathbb{N} \times \mathbb{Z}$.

For the proof of this Roth won the Fields Medal in 1958. Consider the reverse of the inequality in Roth's Theorem and let x and ϵ be fixed at they were in the theorem. Then, there must be a constant $c(x, \epsilon) > 0$ such that

$$\left| x - \frac{p}{q} \right| \geq \frac{c(x, \epsilon)}{q^{2+\epsilon}}$$

for all rational numbers $\frac{p}{q}$. The resemblance with Liouville's Theorem, Theorem 1.2, is now clear. The big difference however is that the right hand side of Roth's Theorem does not depend on the degree of x.

In conclusion it has been shown that **Bad** $\neq \emptyset$. For example the golden ratio is an element of **Bad**. Thus Dirichlet's Theorem cannot be improved by an arbitrary constant for every real number. However, since $\mu(\mathbf{Bad}) = 0$ the set of well approximable points is full, i.e. the set of points for which Dirichlet's Theorem can be improved is full.

The next set of motivational questions revolve around looking at the set of $x \in \mathbb{R}$ such that, for infinitely many $q \in \mathbb{N}$,

$$||qx|| < \psi(q)$$

where $\psi : \mathbb{N} \to [0, \infty)$. Such a function ψ which tends to zero as q tends to infinity is known as an **approximating function**. Any real number x for which the above inequality holds is said to be **ψ-approximable**.

Since it is clear that the function $||.||$ is invariant under translation by an integer it is very useful to note that the set of ψ-approximable points is invariant under translation by an integer. In particular it may be assumed without loss of generality that $x \in I = [0, 1]$ from this point on. Denote **the set of ψ-approximable points** by

$$\begin{aligned} W(\psi) :&= \{x \in I : x \text{ is } \psi\text{-approximable}\} \\ &= \{x \in I : ||qx|| < \psi(q) \text{ for infinitely many } q \in \mathbb{N}\}. \end{aligned}$$

Much has been shown about the set $W(\psi)$. A somewhat complete summary is now given.

1.1.1 Lebesgue Measure Results In One Dimension

First note that if $\psi(q) = 1/q$ then by Dirichlet's Theorem $W(\psi) = I$. Moving on the goal should clearly be to figure out under what conditions, if any, is $\mu(W(\psi)) = 0$ or $\mu(W(\psi)) = 1$. The first result of this kind is as follows, [63].

Theorem 1.6.
Let $\psi : \mathbb{N} \to [0, \infty)$ be a function. Furthermore assume that

$$\sum_{q=1}^{\infty} \psi(q) < \infty,$$

then $\mu(W(\psi)) = 0$.

The proof of this relies solely on the very useful Borel-Cantelli Lemma, or more specifically on the convergent part of the Borel-Cantelli Lemma which is simply known as the Cantelli Lemma.

Lemma 1.7 (Cantelli).
Let (Ω, μ) be a measure space with $\mu(\Omega)$ finite and let $A_j, j \in \mathbb{N}$, be a family of measurable sets. Let

$$A_\infty := \{\omega \in \Omega : \omega \in A_j \text{ for infinitely many } j \in \mathbb{N}\}$$

and suppose

$$\sum_{j=1}^{\infty} \mu(A_j) < \infty,$$

then

$$\mu(A_\infty) = 0.$$

Proof.
Note firstly that A_∞ can be written in 'lim-sup' form as

$$A_\infty = \bigcap_{N=1}^{\infty} \bigcup_{j=N}^{\infty} A_j.$$

Thus it follows that for each $N = 1, 2, ...$ the family $\{A_j : j \geq N\}$ is a cover for the set A_∞, so that

$$A_\infty \subseteq \bigcup_{j=N}^{\infty} A_j.$$

Hence, by the countable additivity property of a measure,

$$\mu(A_\infty) \leq \sum_{j=N}^{\infty} \mu(A_j).$$

Recall however that the sum $\sum_{j=1}^{\infty} \mu(A_j)$ converges by assumption and so the sum $\sum_{j=N}^{\infty} \mu(A_j)$ can be made arbitrarily small, thus $\mu(A_\infty) = 0$. □

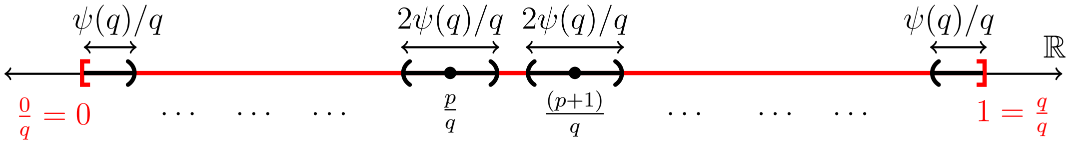
For each $N = 1, 2, ...$ the family $\{A_j : j \geq N\}$ will be called a **natural cover** for A_∞. The proof for Theorem 1.6 is now given.

Proof.

Begin by taking $\Omega = [0, 1]$ and μ to be Lebesgue measure. Now note that any point in the set $W(\psi)$ lies in infinitely many sets $B_{\psi(q)}(q)$, where

$$\begin{aligned} B_{\psi(q)}(q) &= \{x \in \Omega : ||qx|| < \psi(q)\} \\ &= \bigcup_{p=0}^{q} \left(\frac{p}{q} - \frac{\psi(q)}{q}, \frac{p}{q} + \frac{\psi(q)}{q} \right) \cap \Omega. \end{aligned}$$

Thus the family $\{B_{\psi(q)}(q) : q \in \mathbb{N}\}$ is a natural cover for $W(\psi)$. Pictorially the setup looks something like the following:



It is clear that each $B_{\psi(q)}(q)$ is a union of $q+1$ open intervals, $q-1$ of which have length $\frac{2\psi(q)}{q}$ and two of which have length $\frac{\psi(q)}{q}$, thus $|B_{\psi(q)}(q)| \leq 2\psi(q)$ (with equality when $\psi(q) \leq 1/2$). Hence, by assumption, the sum

$$\sum_{q} |B_{\psi(q)}(q)| \leq 2 \sum_{q} \psi(q) < \infty.$$

Thus, by Cantelli's Lemma,

$$\mu(W(\psi)) = 0.$$

□

Theorem 1.6 is in fact the convergent case of the following monumental theorem of A. Khintchine [63].

Theorem 1.8 (**Khintchine (1924)**).
Let $\psi : \mathbb{N} \to [0, \infty)$ *be a function. Then*

$$\sum_{q=1}^{\infty} \psi(q) < \infty \implies \mu(W(\psi)) = 0,$$

while if ψ *is monotonic then*

$$\sum_{q=1}^{\infty} \psi(q) = \infty \implies \mu(W(\psi)) = 1.$$

This is a wonderful zero-one law that completely describes the Lebesgue measure of the set $W(\psi)$. A point of note is that Khintchine originally proved the theorem under the stronger assumption that $q\psi(q)$ is monotonic. The introduction of the concept of regular systems in 1970 by A. Baker and W. M. Schmidt [3] allowed this to be replaced by the weaker assumption on ψ; see [9] also for additional details.

Khintchine's Theorem turns out in fact to be very delicate. What is meant by this is that, for example, if

$$\psi(q) = \frac{1}{q\log(q)}$$

the inequality

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2 \log(q)}$$

has, for almost all x, an infinite set of rational solutions $\frac{p}{q}$. But for

$$\psi(q) = \frac{1}{q(\log q)^2}$$

the inequality

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2 (\log q)^2}$$

has only a finite set of rational solutions. In particular

$$\mu(W(\psi)) = \begin{cases} 0 & \text{if } \psi(q) = \frac{1}{q(\log q)^2}. \\ 1 & \text{if } \psi(q) = \frac{1}{q\log q}. \end{cases}$$

This furthermore shows that Dirichlet's Theorem can, from a measure point of view, be improved by a logarithm but not by the square of a logarithm. It was previously claimed that the Lebesgue measure of the set **Bad** is zero. This will now be shown to be a simple corollary of Khintchine's Theorem.

Corollary 1.9.

$$\mu(\mathbf{Bad}) = 0.$$

Proof.
For any given $N \in \mathbb{N}$ define

$$A(N) := \left\{ x \in I : ||qx|| < \frac{1}{Nq} \text{ for i.m } q \in \mathbb{N} \right\}$$

where i.m stands for infinitely many. Since, for any $N \in \mathbb{N}$,

$$\sum_{q \in \mathbb{N}} \frac{1}{Nq} = \infty$$

Khintchine's Theorem implies that

$$\mu(A(N)) = 1.$$

Now define

$$A^*(N) := I \setminus A(N),$$

then

$$\mu(A^*(N)) = 0.$$

It is clear that the size of the sets $A^*(N)$ increases as N increases, but they remain null sets. Thus, since

$$\mathbf{Bad} \subset \bigcup_{N \in \mathbb{N}} A^*(N),$$

Bad is contained within a countable union of null sets implying that

$$\mu(\mathbf{Bad}) = 0.$$

□

The fact that monotonicity is essential for the divergent part of Khintchine's Theorem was shown by R. Duffin and A. Schaeffer [50]. They provided the first counterexample by construction of a non-monotonic approximation function $f(q)$ whose sum diverged, but for which $\mu(W(f)) = 0$. It was this that lead to considering the set

$$\tilde{W}(\psi) := \{x \in I : |qx - p| \leq \psi(q) \text{ for i.m } p \in \mathbb{Z} \;\&\; q \in \mathbb{N} \text{ with } \gcd(p, q) = 1\}.$$

Applying Cantelli's Lemma, in the same way as was done for the set $W(\psi)$, it is an easy exercise to show that for any approximating function ψ

$$\sum_{q=1}^{\infty} \frac{\phi(q).\psi(q)}{q} < \infty \implies \mu(\tilde{W}(\psi)) = 0$$

where ϕ denotes the Euler phi function. The associated divergent case is what is now known as the Duffin-Schaeffer Conjecture and remains an open problem. It runs as follows.

Conjecture 1.10 (**Duffin-Schaeffer (1941)**).
Given any approximation function ψ

$$\sum_{q=1}^{\infty} \frac{\phi(q).\psi(q)}{q} = \infty \implies \mu(\tilde{W}(\psi)) = 1.$$

1.1.2 Hausdorff Measure Results In One Dimension

Although very useful, Lebesgue measure certainly has a big deficiency in that it does not distinguish between two sets of Lebesgue measure zero. Clearly one would like some new machinery that does distinguish between two such null sets. In particular a finer type of measure is wanted. Named after Felix Hausdorff, who was the first to introduce the concept [58], Hausdorff measure is exactly the machinery needed. Hausdorff's work was based on previous work of Caratheodory [40] and his approach to Lebesgue measure. It has the advantage of being defined for any set, however, it also has a major disadvantage in that in many cases it can be difficult to calculate or even estimate.

A function $f : [0, \infty) \to [0, \infty)$ which is continuous, monotonic in a neighbourhood of 0 and has $f(0) = 0$ will be called a **dimension function**. The **diameter** of U for any non-empty $U \subset \mathbb{R}^n$ is defined as

$$\text{diam}(U) := \sup\{|x - y| : x, y \in U\},$$

i.e. the diameter of U is the greatest distance between any two points in U. Suppose $X \subset \mathbb{R}^n$ is non-empty and there exists a collection of subsets $\{U_i\}_{i\in\mathbb{N}}$ such that $0 < \text{diam}(U_i) \leq \rho$ for each $i \in \mathbb{N}$ and $X \subset \cup_{i\in\mathbb{N}} U_i$, this is called a $\boldsymbol{\rho}$**-cover** of X.

For any positive ρ, define

$$\mathcal{H}^f_\rho(X) := \inf\left\{ \sum_{i=0}^{\infty} f(\text{diam}(U_i)) : \{U_i\}_{i\in\mathbb{N}} \text{ is a } \rho\text{-cover of } X \right\}.$$

It is clear that $\mathcal{H}^f_\rho(X)$ increases as ρ decreases and therefore admits a (finite or infinite) limit as ρ tends to 0. In particular the quantity

$$\mathcal{H}^f(X) := \lim_{\rho \to 0^+} \mathcal{H}^f_\rho(X) = \sup_{\rho > 0} \mathcal{H}^f_\rho(X),$$

exists and is known as the **Hausdorff f-measure** of X.

A case of particular interest is that in which $f(r) = r^s$ for some $s \geq 0$. This is referred to as the **s-dimensional Hausdorff measure** and is more conveniently denoted by $\mathcal{H}^s$. It is interesting to note that if s is an integer then it can be shown that the s-dimensional Hausdorff measure is in fact proportional to the s-dimensional Lebesgue measure, which is denoted by μ_s. Furthermore, the constant of proportionality is known to be the inverse of the Lebesgue volume of the unit ball in dimension s, which is denote by ν_s. In particular it can be shown that

$$\mathcal{H}^s = \frac{\mu_s}{\nu_s}.$$

This is no trivial calculation, in fact getting the exact constant is quite difficult, details can be found in [75]. It is clear from this that indeed Hausdorff measure is a refinement of Lebesgue measure as wanted. In many cases due to the difficulty in calculating the Hausdorff measure, the Hausdorff dimension has been calculated instead. The concept of Hausdorff dimension can be explained as follows.

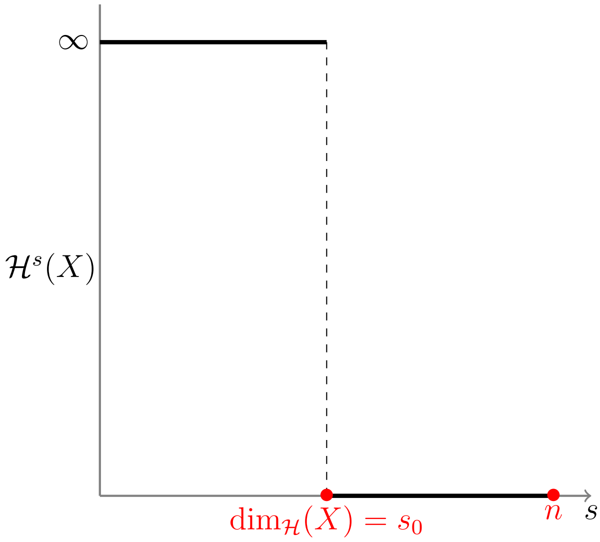
Suppose that $X \subset \mathbb{R}^n$ is non-empty and $\{U_i\}_{i \in \mathbb{N}}$ is a ρ-cover for X. Then, for $0 < \rho < 1$, $0 < s < t$, it is clear that

$$\text{diam}(U)^t \leq \rho^{t-s} \text{diam}(U)^s.$$

Furthermore, it follows by definition of $\mathcal{H}^f_\rho(X)$ that

$$\mathcal{H}^t_\rho(X) \leq \rho^{t-s} \mathcal{H}^s_\rho(X)$$

and so if $\mathcal{H}^t(X)$ is positive it must be that $\mathcal{H}^s(X)$ is infinite and if $\mathcal{H}^s(X)$ is finite then $\mathcal{H}^t(X)$ must be zero. Graphing $\mathcal{H}^s(X)$ against s, as seen below, gives a nice pictorial interpretation of this jump between ∞ to 0. It is the unique value $s = s_0$ at which $\mathcal{H}^s(X)$ takes this jump that is defined as the **Hausdorff dimension** and is denoted by $\dim_{\mathcal{H}}(X)$.



Formally written

$$\dim_{\mathcal{H}}(X) = \inf\{s \geq 0 : \mathcal{H}^s(X) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(X) = \infty\}$$

so that

$$\mathcal{H}^s(X) = \begin{cases} \infty & \text{if } s < \dim_{\mathcal{H}}(X) \\ 0 & \text{if } s > \dim_{\mathcal{H}}(X). \end{cases}$$

If $s = \dim_{\mathcal{H}}(X)$, then $\mathcal{H}^s(X)$ may be zero or infinite, or may satisfy

$$0 < \dim_{\mathcal{H}}(X) < \infty.$$

Some useful properties of Hausdorff dimension are the following (further details and proofs can all be found in [26] or [52]).

Theorem 1.11 (Properties of Hausdorff dimension).
Let $E, F, E_j \subset \mathbb{R}^n$ for $j, n \in \mathbb{N}$. Then,

(1) if $E \subset F$, $\dim_{\mathcal{H}}(E) \leq \dim_{\mathcal{H}}(F)$.

(2) $\dim_{\mathcal{H}}(E) \leq n$.

(3) the Hausdorff dimension of any countable set is zero and the Hausdorff dimension of any open set in $\mathbb{R}^n$ is n.

(4) if $\mu_n(E) > 0$, $\dim_{\mathcal{H}}(E) = n$.

(5) if $\dim_{\mathcal{H}}(E) < n$, $\mu_n(E) = 0$.

(6) $\dim_{\mathcal{H}}(\cup_{j=1}^{\infty} E_j) = \sup_{j\geq 1} \dim_{\mathcal{H}}(E_j)$.

Property 6 is called **countable stability** and can be extremely useful. In particular, it shows that instead of looking at the Hausdorff dimension of sets in $\mathbb{R}^n$ one can instead consider the easier problem of considering sets in I^n for a suitable interval I. It also shows something quite interesting in that sets which differ by only a countable set have the same Hausdorff dimension. Care must be taken however not to assume that knowing a set to be uncountable leads to an idea of the size of the Hausdorff dimension of the set. In particular there are null sets, such as the set of badly approximable numbers, which have maximal Hausdorff dimension ($\dim_{\mathcal{H}}(\mathbf{Bad}) = 1$) and there are uncountable sets, such as the set of Liouville numbers, which have zero Hausdorff dimension (again see [26] for further details and proof).

As mentioned before, calculating the Hausdorff dimension is, in general, much easier than finding the Hausdorff measure. In practice it is done in two steps by finding the upper bound and lower bound for $\dim_{\mathcal{H}}(X)$ separately and showing these are equal. The ability to find a natural cover for the set X is why finding the upper bound is usually quite simple. However, the same is unfortunately not true of finding the lower bound.

As an example of how Hausdorff measure is more refined than Lebesgue measure consider the set $W(\tau)$ defined by

$$W(\tau) := \{x \in I : ||qx|| < q^{-\tau} \text{ for infinitely many } q \in \mathbb{N}\}.$$

In particular $W(\tau) = W(\psi)$ with $\psi(q) = q^{-\tau}$. It is clear by Dirichlet's Theorem that

$$\tau \leq 1 \implies \mu(W(\tau)) = 1$$

and furthermore by Khintchine's Theorem

$$\tau > 1 \implies \sum_{q=1}^{\infty} \psi(q) = \sum_{q=1}^{\infty} q^{-\tau} < \infty \implies \mu(W(\tau)) = 0.$$

Although this is a nice result to have, since it completely classifies the set $W(\tau)$ for all $\tau \in \mathbb{R}$, it is also somewhat unsatisfactory. It does not require much thought to convince oneself that the size of $W(\tau)$ should decrease as τ increases. However, as far as Lebesgue measure is concerned, these sets are the same size.

This however is not the case when considering the Hausdorff dimension of the sets $W(\tau)$ for $\tau > 1$. The following theorem was first shown by Jarník in 1928 [60] and then independently and using a completely different method by Besicovitch in 1932 [33].

Theorem 1.12 (**Jarník-Besicovitch Theorem (1932)**).
Let $\tau > 1$. *Then*

$$\dim_{\mathcal{H}}(W(\tau)) = \frac{2}{\tau + 1}.$$

This indeed shows what intuition implies since as τ increases the set $W(\tau)$ does indeed decreases in some way. As mentioned previously, finding an upper bound for the Hausdorff dimension is simplified immensely upon finding a natural cover. This is emphasised in the proof of the upper bound for the Jarník-Besicovitch Theorem below. For details and proof of the lower bound see [26]. Before this, however, a Hausdorff version of the Borel-Cantelli Lemma, appropriately called the **Hausdorff-Cantelli** Lemma, is given [26].

Lemma 1.13 (**Hausdorff-Cantelli**).
Let $E \subset \mathbb{R}^n$, H_j *be a family of hypercubes and suppose*

$$E \subseteq \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \in H_j \textit{ for infinitely many } j \in \mathbb{N}\}.$$

If for some $s > 0$

$$\sum_{j=1}^{\infty} (\textit{diam}(H_j))^s < \infty,$$

then

$$\mathcal{H}^s(E) = 0$$

and thus

$$\dim_{\mathcal{H}}(E) \leq s.$$

Proof.
From the assumptions of the lemma

$$E \subseteq \bigcap_{N=1}^{\infty} \bigcup_{j=N}^{\infty} H_j.$$

Thus for each $N \in \mathbb{N}$ the family $\{H_j : j \geq N\}$ is a cover for E. Furthermore given $\delta > 0$, there exists an integer $N_1 = N_1(\delta)$ such that for all $j \geq N_1$, $\text{diam}(H_j) < \delta$ since $\sum_{j=1}^{\infty}(\text{diam}(H_j))^s < \infty$. Moreover, given any $\epsilon > 0$,

$$\sum_{j=N}^{\infty} (\text{diam}(H_j))^s < \epsilon$$

for N sufficiently large. Hence $\mathcal{H}^s_\delta(E) < \epsilon$. So letting $\delta \to 0$ the proof is complete. $\square$

Now recalling the set

$$W(\tau) := \{x \in I : ||qx|| < q^{-\tau} \text{ for infinitely many } q \in \mathbb{N}\}$$

define

$$B(q) = \{x \in I : ||qx|| < q^{-\tau}, q \in \mathbb{N}\}$$

for $\tau > 1$. It is then clear that the family $\{B(q) : q \in \mathbb{N}\}$ forms a natural cover for $W(\tau)$. Furthermore since $B(q)$ can be expressed as

$$B(q) = \bigcup_{p=0}^{q} \left(\frac{p}{q} - \frac{1}{q^{\tau+1}}, \frac{p}{q} + \frac{1}{q^{\tau+1}}\right) \cap I,$$

$B(q)$ is a union of intervals of length at most $\frac{2}{q^{\tau+1}}$. These intervals form a natural interval cover of $W(\tau)$ and each $x \in W(\tau)$ lies in infinitely many of these intervals. Letting $\mathcal{C}$ denote this natural interval cover, it is clear that

$$\sum_{C \in \mathcal{C}} (\text{diam}(C))^s \leq \sum_{q=1}^{\infty} \sum_{p=0}^{q} 2^s q^{-(\tau+1)s} \ll \sum_{q=1}^{\infty} q^{1-(\tau+1)s} < \infty$$

provided $1 - (\tau + 1)s < -1$; i.e. when $s > \frac{2}{\tau+1}$. Thus, the Hausdorff-Cantelli Lemma gives,

$$\dim_{\mathcal{H}}(W(\tau)) \leq \frac{2}{\tau + 1}$$

when $\tau > 1$ and so the upper bound for the Jarník-Besicovitch Theorem has been shown.

Clearly this Hausdorff dimension result is an improvement on the Lebesgue measure result since now a distinction can be made between sets that, up to now, had been considered the same. However a gap still remains. In particular the Jarník-Besicovitch Theorem shows that

$$\mathcal{H}^s(W(\tau)) = \begin{cases} \infty & \text{if } s < \dim_{\mathcal{H}}(W(\tau)) = \frac{2}{\tau+1} \\ 0 & \text{if } s > \dim_{\mathcal{H}}(W(\tau)) = \frac{2}{\tau+1}, \end{cases}$$

but what if $s = \frac{2}{\tau+1}$? No result yet presented gives any information about this. In 1931 Jarník [61] answered this question by providing a Hausdorff version of Khintchine's Theorem.

Theorem 1.14 (**Jarník's Theorem (1931)**).
Let ψ be an approximating function and let f be a dimension function such that $q \mapsto q^{-1}f(q)$ is a decreasing function tending to infinity as q tends to 0. Then

$$\mathcal{H}^f(W(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} qf\left(\frac{\psi(q)}{q}\right) < \infty \\ \infty & \text{if } \sum_{q=1}^{\infty} qf\left(\frac{\psi(q)}{q}\right) = \infty. \end{cases}$$

Tailoring it to the case of $f(q) = q^s$, Jarník's Theorem can be restated in the following way.

Theorem 1.15.
Let $\psi : \mathbb{N} \to [0, \infty)$ be a monotonic function and let $s \in (0, 1)$. Then

$$\mathcal{H}^s(W(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{1-s}\psi^s(q) < \infty \\ \infty & \text{if } \sum_{q=1}^{\infty} q^{1-s}\psi^s(q) = \infty. \end{cases}$$

This is a very nice and powerful result. Considering $\psi(q) = q^{-\tau}$ for $\tau > 1$ then Jarník's Theorem gives that $\dim_{\mathcal{H}}(W(\tau)) = \frac{2}{1+\tau}$. Furthermore, Jarník's Theorem answers that if $s = \frac{2}{\tau+1}$ then $\mathcal{H}^s(W(\tau)) = \infty$.

Similar to Khintchine's Theorem, Jarník's original statement included stronger assumptions on the monotonicity of ψ and furthermore included various other conditions placed on ψ and on s. The much clearer statement is due to Beresnevich, Dickinson and Velani [13] who removed the technical conditions and went a step further to combine the Khintchine and Jarník Theorems.

Theorem 1.16 (**The Khintchine-Jarník Theorem (2006)**).
Let $\psi : \mathbb{N} \to [0, \infty)$ be a monotonic function and let $s \in (0, 1]$. Then

$$\mathcal{H}^s(W(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{1-s}\psi^s(q) < \infty \\ H^s(I) & \text{if } \sum_{q=1}^{\infty} q^{1-s}\psi^s(q) = \infty. \end{cases}$$

It should be noted that there is nothing particularly spectacular about Theorem 1.16, which is just a combination of Theorem 1.8 and Theorem 1.15. The big improvement in [13] was the removal of various versions of the monotonicity condition of ψ.

1.1.3 The Mass Transference Principle

Finishing off the discussion of the one dimensional results a final interesting and surprising fact is presented. The **Mass Transference Principle**, which was developed by Beresnevich and Velani [14], was originally motivated by the desire to obtain a Hausdorff measure version of the Duffin-Schaeffer Conjecture. This extension to the conjecture will be discussed in detail in Section 1.2, Conjecture 1.24. Before the theorem can be stated, however, some relevant additional notation must first be introduced.

Given a dimension function f and a ball $B(\mathbf{x}, r)$ with radius $r > 0$ in $\mathbb{R}^n$ centred at $\mathbf{x} \in \mathbb{R}^n$, let B^f denote its dilate by f; that is,

$$B^f := B\left(\mathbf{x}, f(r)^{\frac{1}{n}}\right).$$

In the case when $f(r) = r^s$ for a given $s > 0$, write B^s instead of B^f. In particular note that $B^n = B$.

Theorem 1.17 (**Mass Transference Principle (2006)**).
Let $(B_i)_{i\geq 0}$ be a sequence of balls in $\Omega \subseteq \mathbb{R}^k$ with $r(B_i) \to 0$ as $i \to \infty$. Let f be a dimension function such that $r^{-k}f(r)$ is monotonic and suppose that for any ball B in Ω with $\mathcal{H}^k(B) > 0$,

$$\mathcal{H}^k\left(B \cap \limsup_{i\to\infty} B_i^f\right) = \mathcal{H}^k(B).$$

Then, for any ball B in Ω with $\mathcal{H}^\delta(B) > 0$,

$$\mathcal{H}^f\left(B \cap \limsup_{i\to\infty} B_i^k\right) = \mathcal{H}^f(B).$$

This theorem allows something very surprising to be shown. In particular it shows that

$$\text{Khintchine's Theorem} \implies \text{Jarník's Theorem}.$$

So, in fact, although the Hausdorff theory is a refinement of the Lebesgue theory, the mass transference principle shows that the Lebesgue theory underpins the Hausdorff theory in the sense that it allows a transfer of statements from the Lebesgue theory of limsup sets in $\mathbb{R}^n$ into statements in the Hausdorff theory. As another example of this the mass transference principle also shows that

$$\text{Dirichlet's Theorem} \implies \text{Jarník-Besicovitch Theorem}.$$

To see that Khintchine's Theorem implies Jarník's Theorem, consider first the case in which $\psi(r)/r \not\to 0$ as $r \to \infty$. Then it is clear that $W(\psi) = I$ and the result follows immediately. Next consider $\psi(r)/r \to 0$ as $r \to \infty$ and let $\Omega = I$, $\delta = 1$

and $f(r) = r^s$ for some $s \in (0,1)$. By the divergent sum assumption of Jarník's Theorem it may taken, for the chosen s, that

$$\sum q^{1-s}\psi(q)^s = \infty.$$

Next let

$$\phi(r) := r^{1-s}\psi(r)^s.$$

Then ϕ is an approximating function and

$$\sum \phi(q) = \infty.$$

Thus Khintchine's Theorem implies that

$$\mathcal{H}^1(B \cap W(\phi)) = \mathcal{H}^1(B \cap I)$$

for any ball B in $\mathbb{R}$ and so, by the mass transference principle,

$$\mathcal{H}^s(W(\psi)) = \mathcal{H}^s(I) = \infty.$$

This completes the proof of the divergence part of Jarník's Theorem. The convergence part, as shown previously, is straightforward. In fact, the mass transference principle is only ever used for divergence statements. Covering arguments are used for convergence statements. For more details and the proof that Dirichlet's Theorem implies the Jarník-Besicovitch Theorem see for example [20].

1.2 Higher Dimensional Results

Now that a summary of the one dimensional results has concluded, a discussion of some higher dimensional results begins. It is only natural to begin with Dirichlet's Theorem again. First the set $W(\psi)$ is brought into the higher dimensional framework by defining

$$W(m,n,\psi) := \{(\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_n}) \in (I^m)^n : ||\mathbf{q}\cdot\mathbf{x_i}|| < \psi(|\mathbf{q}|_\infty)\ \forall\ i = 1,2,...,n \\ \&\text{ for i.m } \mathbf{q} \in \mathbb{Z}^m\},$$

where $|.|_\infty$ denotes the infinity norm, that is for $\mathbf{q} = (q_1, q_2, ..., q_m)$

$$|\mathbf{q}|_\infty = \max\{|q_1|, |q_2|, ..., |q_m|\}.$$

From here the following two forms of Diophantine approximation can now be considered:

1) The set of **simultaneously ψ-approximable** vectors, denoted $\mathcal{S}_n(\psi)$, corresponding to $W(m,n,\psi)$ when $m = 1, n = n$. That is

$$\mathcal{S}_n(\psi) = W(1,n,\psi) := \{(x_1, x_2, ..., x_n) \in I^n : ||qx_i|| < \psi(|q|) \ \forall\ i = 1,2,...,n \\ \&\text{ for i.m } q \in \mathbb{Z}\}.$$

2) The set of **dually ψ-approximable** vectors, denoted $\mathcal{L}_m(\psi)$, corresponding to $W(m,n,\psi)$ when $m = m, n = 1$. That is

$$\mathcal{L}_m(\psi) = W(m,1,\psi) := \{\mathbf{x} \in I^m : ||\mathbf{q}.\mathbf{x}|| < \psi(|\mathbf{q}|_\infty) \ \text{ for i.m } \mathbf{q} \in \mathbb{Z}^m\}.$$

In the particular case when $\psi(q) = q^{-\tau}$ the sets $W(m,n,\psi), \mathcal{S}_n(\psi)$ and $\mathcal{L}_m(\psi)$ will be denoted by $W(m,n,\tau), \mathcal{S}_n(\tau)$ and $\mathcal{L}_m(\tau)$ respectively.

Dirichlet's Theorem can be generalised in the framework of either of these forms in the following ways:

Theorem 1.18 (**Generalised Dirichlet (Simultaneous Form)**).
Let $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n$ and $Q \in \mathbb{R}$, $Q \geq 1$. Then there exists $\mathbf{p} \in \mathbb{Z}^n$ and $q \in \mathbb{Z}$ with $1 \leq q \leq Q$ such that

$$|q\mathbf{x} - \mathbf{p}| < Q^{-\frac{1}{n}}.$$

Furthermore there are infinitely many $q \in \mathbb{N}$ such that

$$|\langle q\mathbf{x}\rangle| := \max\{||qx_1||, ||qx_2||, ..., ||qx_n||\} < q^{-\frac{1}{n}}.$$

Theorem 1.19 (**Generalised Dirichlet (Dual Form)**).
Let $\mathbf{x} = (x_1, ..., x_m) \in \mathbb{R}^m$ and $Q \in \mathbb{R}$, $Q \geq 1$. Then there exists $p \in \mathbb{Z}$ and $\mathbf{q} \in \mathbb{Z}^m$ with $1 \leq |\mathbf{q}|_\infty \leq Q$ such that

$$|\mathbf{q}.\mathbf{x} - p| < Q^{-m}.$$

Furthermore there are infinitely many $\mathbf{q} \in \mathbb{Z}^m$ *such that*

$$||\mathbf{q}.\mathbf{x}|| < |\mathbf{q}|_\infty^{-m}.$$

From the two generalised forms of Dirichlet's Theorem it is clear that any point in $\mathbb{R}^n$ is simultaneously (1/n)-approximable and dually n-approximable.

Just as was the case in the one dimensional framework the next logical step is to ask if the generalised Dirichlet Theorem can be improved. First the concept of badly approximable numbers is generalised in a natural way. The set of **n badly approximable linear forms in m variables**, denoted $\mathbf{Bad}(m, n)$, is defined as follows:

$$\mathbf{Bad}(m, n) := \left\{\mathbf{X} \in \mathbb{R}^{mn} : \inf_{\mathbf{q}\in\mathbb{Z}^m\setminus\{0\}} \min_{1\leq i\leq n} |\mathbf{q}|_\infty^{\frac{m}{n}} ||\mathbf{q}.\mathbf{x_i}|| > 0\right\}.$$

Clearly $\mathbf{Bad} = \mathbf{Bad}(1, 1)$ and so again by Hurwitz's Theorem the answer to our question is no, the generalised Dirichlet Theorem may not be improved for all $m, n \in \mathbb{N}$.

What if $mn > 1$? Unfortunately, as of yet, there is no answer to this. No mn-dimensional analogue of Hurwitz's Theorem has been found. Some results are known however. For example the soon to be discussed Khintchine-Groshev Theorem shows that $\mu_{mn}(\mathbf{Bad}(m, n)) = 0$ for $m, n \geq 1$ and Schmidt [84] improved on Jarník's result of $\dim_{\mathcal{H}}(\mathbf{Bad}) = 1$ to $\dim_{\mathcal{H}}(\mathbf{Bad}(m, n)) = mn$ for $mn > 1$. More results and discussion can be found in [83] or [20].

Khintchine's Theorem has a generalization to higher dimension due to the work of Groshev [56] (also see [86] Chapter 1, Section 5 for discussion). As was the case when working in one dimension there is no loss in generality in restricting $W(m, n, \psi)$ to the unit cube $I^{mn} = [0, 1]^{mn}$.

Theorem 1.20 (**Khintchine-Groshev Theorem (1938)**).
Let $\psi : \mathbb{N} \to [0, \infty)$ *and let* $m, n \in \mathbb{N}$ *with* $mn \geq 1$*. Then*

$$\mu_{mn}(W(m, n, \psi)) = \begin{cases} 0 & \text{if } \sum\limits_{q=1}^{\infty} q^{m-1}\psi^n(q) < \infty \\ 1 & \text{if } \sum\limits_{q=1}^{\infty} q^{m-1}\psi^n(q) = \infty \text{ and } \psi \text{ is monotonic.} \end{cases}$$

A significant improvement has been made to the Khintchine-Groshev Theorem in that the monotonicity condition has been removed for all but the case in which $m = n = 1$. As was discussed previously Duffin and Schaeffer provided a counterexample to dropping the monotonicity condition when $m = n = 1$. It was dropped in the case $m \geq 3$ by Schmidt [81] in 1960. In 1965 Gallagher [54] removed it in the case $n \geq 2$. The reader is also directed to [86] for a discussion of these proofs. It was not until 2010 that the final case of $m = 2, n = 1$ was dropped by Beresnevich and Velani [17].

As alluded to previously, in the same way that Khintchine's Theorem implied $\mu(\mathbf{Bad}) = 0$, the Khintchine-Groshev Theorem implies that $\mu_{mn}(\mathbf{Bad}(m,n)) = 0$ for $m, n \geq 1$. Also note that the set of simultaneous and dual ψ-approximations $\mathcal{S}_n(\psi)$ and $\mathcal{L}_m(\psi)$ are just special cases of the Khintchine-Groshev Theorem. In particular note that when $\tau > 1/n$, $\mu_n(\mathcal{S}_n(\tau)) = 0$ and when $\tau > m$, $\mu_m(\mathcal{L}_m(\tau)) = 0$.

The next step is to obtain a Hausdorff measure generalised statement. In particular does an analogue of the Jarník-Besicovitch Theorem exist? The first result of this kind is due to J. Bovey and M. Dodson [34] from 1986 who found the Hausdorff dimension of the set $W(m, n, \tau)$. In particular they showed the following.

Theorem 1.21.
Let $\tau > \frac{m}{n}$. Then

$$\dim_{\mathcal{H}}(W(m,n,\tau)) = (m-1)n + \frac{m+n}{\tau+1}.$$

In 1992 Dodson [48] extended this result to the set $W(m, n, \psi)$. First a definition is required. For an approximating function ψ, the **lower order** of $1/\psi$, denoted $\lambda(\psi)$, is given by

$$\lambda(\psi) = \liminf_{q\to\infty} \left(-\frac{\log(\psi(q))}{\log(q)} \right).$$

Note that if $\psi(q)$ decreases as q increases then $\lambda(\psi)$ is non-negative. $\lambda(\psi)$ can be thought of as an indicator of the behaviour of ψ near infinity.

Theorem 1.22 (Dodson (1992)).
Let $\psi : [0,\infty) \to [0,\infty)$ be a decreasing function (i.e. ψ is an approximating function). Then

$$\dim_{\mathcal{H}}(W(m,n,\psi)) = \begin{cases} (m-1)n + \frac{m+n}{\lambda(\psi)+1} & \text{if } \lambda(\psi) > \frac{m+n}{n}, \\ mn & \text{if } \lambda(\psi) \leq \frac{m+n}{n}. \end{cases}$$

As mentioned already this theorem can be considered as the generalised Jarník-Besicovitch Theorem and as the Hausdorff dimension version of the Khintchine-Groshev Theorem.

Dodson's result was later improved upon by Dickinson and Velani [45] who obtained a Jarník type result for systems of linear forms and in doing so obtained a Hausdorff measure version of the Khintchine-Groshev Theorem.

Theorem 1.23 (**Dickinson-Velani (1997)**).
Let $m, n \in \mathbb{N}$ and let f be a dimension function such that $q^{-mn}f(q) \to \infty$ as $q \to 0$ and $q^{-mn}f(q)$ is non-increasing. Let ψ be an approximating function such that $q^m\psi(q)^n \to 0$ as $q \to \infty$ and $q^m\psi(q)^n$ is non-increasing. Finally suppose that $q^{m(1+n)}\psi(q)^{-(m-1)n}f(\psi(q)/q)$ is non-increasing, then

$$\mathcal{H}^f(W(m,n,\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} f(\frac{\psi(q)}{q})\psi(q)^{-(m-1)n}q^{m(1+n)-1} < \infty \\ \infty & \text{if } \sum_{q=1}^{\infty} f(\frac{\psi(q)}{q})\psi(q)^{-(m-1)n}q^{m(1+n)-1} = \infty. \end{cases}$$

Note that the s-dimensional Hausdorff measure of $W(m, n, \psi)$ is obtained simply by letting $f(q) = q^s$. The result of Dickinson and Velani is a much better tool for distinguishing between sets than Dodson's Theorem, in which nothing can be said about the value of $\mathcal{H}^s(W(m, n, \psi))$ when $s = \dim_{\mathcal{H}}(W(m, n, \psi))$. As an example consider, for $\tau, \epsilon > 0$, the two functions

$$\psi_1(q) = q^{-\tau}(\log(q))^{-\frac{\tau}{m+n}} \quad \& \quad \psi_2(q) = q^{-\tau}(\log(q))^{-\frac{\tau}{m+n}(1+\epsilon)}.$$

Assuming that $\tau > \frac{m+n}{n}$ the Khintchine-Groshev Theorem implies

$$\mu_{mn}(W(m, n, \psi_1)) = \mu_{mn}(W(m, n, \psi_2)) = 0$$

and so is not enough to distinguish between the size of sets. Using Dodson's Theorem, and noting that the extra factor of ϵ has no effect on $\lambda(\psi)$, it is easily checked that

$$\dim_{\mathcal{H}}(W(m, n, \psi_1)) = \dim_{\mathcal{H}}(W(m, n, \psi_2)) = (m-1)n + \frac{m+n}{\tau+1}.$$

So again this is not enough to distinguish between the two sets. Finally using the Dickinson-Velani Theorem, not only can Dodson's result be obtained, but the sets can be distinguished between. It can now be shown that at $s = \dim_{\mathcal{H}}(W(m, n, \psi_1))$ $= \dim_{\mathcal{H}}(W(m, n, \psi_2))$

$$\mathcal{H}^s(W(m, n, \psi_1)) = \infty \text{ and } \mathcal{H}^s(W(m, n, \psi_2)) = 0.$$

For full details and more discussion on this example see [45].

The Duffin-Schaeffer Conjecture can also be brought into the higher dimensional framework in the following way. Let $\tilde{\mathcal{S}}_n(\psi)$ denote the set

$$\tilde{\mathcal{S}}_n(\psi) := \{(x_1, x_2, ..., x_n) \in I^n : \left| x_i - \frac{p_i}{q} \right| < \frac{\psi(q)}{q} \ \forall\ i = 1, 2, ..., n \text{ and for i.m}$$
$$\mathbf{p} = (p_1, ..., p_n) \in \mathbb{Z}^n \text{ and } q \in \mathbb{Z} \text{ with } (p_i, q) = 1\}.$$

Then the generalised Duffin-Schaeffer Conjecture can be stated in the following way.

Conjecture 1.24 (**Generalised Duffin-Schaeffer Conjecture**).
Given any approximation function ψ

$$\sum_{q=1}^{\infty} \left(\frac{\phi(q).\psi(q)}{q} \right)^n = \infty \implies \mu_n(\tilde{\mathcal{S}}_n(\psi)) = 1.$$

It is clear that the case of $n = 1$ is just Conjecture 1.10 again, which as mentioned remains opened. In the case of $n \geq 2$ the conjecture however has been settled in the affirmative by A. D. Pollington and R. C. Vaughan [79].

Another very impressive application of the Mass Transference Principle is the development of a Hausdorff measure version of the Duffin-Schaeffer Conjecture. In particular Beresnevich and Velani [14] showed by use of the Mass Transference Principle that the Duffin-Schaeffer Conjecture for Lebesgue measure gives rise to the following analogous statement for Hausdorff measure.

Conjecture 1.25 (**Hausdorff Measure Duffin-Schaeffer Conjecture**).
Given an approximation function ψ *and a dimension function* f

$$\sum_{q=1}^{\infty} f\left(\frac{\psi(q)}{q}\right)\phi(q)^n = \infty \implies \mathcal{H}^f(\tilde{\mathcal{S}}_n(\psi)) = \mathcal{H}^f(I^k).$$

Clearly setting $f(q) = q^n$ immediately gives Conjecture 1.24 and so

$$\textit{Conjecture } 1.25 \implies \textit{Conjecture } 1.24.$$

What is more interesting is that Beresnevich and Velani [14] showed that

$$\textit{Conjecture } 1.24 \implies \textit{Conjecture } 1.25$$

and so in fact the conjectures are equivalent. Furthermore the result of Pollington and Vaughan combined with that of Beresnevich and Velani give that Conjecture 1.25 is true for $n \geq 2$.

Very recently D. Allen and V. Beresnevich [2] established a general form of the Mass Transference Principle for systems of linear forms. This allowed for a number of new applications including a general transference of Lebesgue measure Khintchine-Groshev type theorems to Hausdorff measure statements.

1.3 Diophantine Approximation On Manifolds

Up to this point all results were made significantly easier to prove due to the fact that only independent variables were being considered. In particular, sets have only been considered with points of the form $\mathbf{x} = (x_1, ..., x_n)$ in which the variables x_i and x_j were independent for all $i, j \in \{1, 2, ..., n\}, i \neq j$. One would like to obtain the same type of results but now for manifolds, such as curves and surfaces, in which the variables are now dependent.

For example, how does the Khintchine-Groshev Theorem change if instead of looking at $W(m, n, \psi)$ the set $\mathcal{M} \cap W(m, n, \psi) := W(\mathcal{M}, m, n, \psi)$ is considered where $\mathcal{M}$ is some manifold embedded in $\mathbb{R}^{mn}$. We consider manifolds presented in two forms, locally as a collection of equations, for example $\mathcal{M} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$, or alternatively as arising from some parametrisation map, for example take $f(x) = (x, x^2, x^3, ..., x^n)$ then $\mathcal{M} = f(U) \subset \mathbb{R}^n$ for some $U \subset \mathbb{R}$.

It is clear that for $\mathcal{M} \subset \mathbb{R}^{m_1 n_1}$ with $m_1 n_1 < mn$

$$\mu_{mn}(W(\mathcal{M}, m_1, n_1, \psi)) = 0.$$

In particular, the mn-dimensional Lebesgue measure of $W(\mathcal{M}, m_1, n_1, \psi)$ is zero regardless of what ψ is, therefore the induced Lebesgue measure, denoted by $\mu_{\mathcal{M}}$, will be used. The induced Lebesgue measure is defined as follows. Let $g : X \to Y$ be an invertible function, then for $A \subset Y$ define the **induced Lebesgue measure** on Y by $\mu(g^{-1}(A))$. For a very in depth account on this topic, the reader is referred to [[65], Chapter 9]. For a much more elementary account see [[26], Section 1.4]

Due to the difficult nature of the questions to be considered it is general practice to consider the intersection of a manifold with the simultaneous set $\mathcal{S}_n(\psi)$ and the dual set $\mathcal{L}_m(\psi)$ separately. The following are defined,

$$\mathcal{S}_n(\mathcal{M}, \psi) := \mathcal{M} \cap \mathcal{S}_n(\psi) \text{ and } \mathcal{L}_m(\mathcal{M}, \psi) := \mathcal{M} \cap \mathcal{L}_m(\psi).$$

Similarly define

$$\mathcal{S}_n(\mathcal{M}, \tau) := \mathcal{M} \cap \mathcal{S}_n(\tau) \text{ and } \mathcal{L}_m(\mathcal{M}, \tau) := \mathcal{M} \cap \mathcal{L}_m(\tau).$$

Note that if $\tau \leq 1/n$ then $\mathcal{S}_n(\mathcal{M}, \tau)$ has full measure and if $\tau \leq m$ then $\mathcal{L}_m(\mathcal{M}, \tau)$

has full measure. The dual set will first be considered as the results here are very similar to those of the classical dual set. Before continuing however the notion of an extremal manifold is introduced. A manifold $\mathcal{M}$ is said to be **extremal** if the set of well approximable points in $\mathcal{M}$ is relatively null, that is has $\mu_{\mathcal{M}}$-measure zero. Alternatively one can consider this as saying a manifold $\mathcal{M}$ is extremal if the sets $\mathcal{S}_n(\mathcal{M},\tau)$ and $\mathcal{L}_m(\mathcal{M},\tau)$ are such that $\mu_{\mathcal{M}}(\mathcal{S}_n(\mathcal{M},\tau)) = 0$ when $\tau > \frac{1}{n}$ or $\mu_{\mathcal{M}}(\mathcal{L}_m(\mathcal{M},\tau)) = 0$ when $\tau > m$.

In 1932 Mahler [72] conjectured that the Veronese curve

$$\mathcal{V}_n := \{(x, x^2, x^3, ..., x^n) : x \in \mathbb{R}\}$$

is extremal for all $n \in \mathbb{N}$. It was not until 1964 that Sprindžuk [85] proved this to be true. It is generally considered that it was this conjecture that began the investigation of Diophantine approximation on manifolds. Since Sprindžuk's result, manifolds satisfying a variety of analytic, geometric and number theoretic conditions have been shown to be extremal; see [26] for details and references.

In 1975, A. Baker [4] conjectured that for $n \geq 1$ the Veronese curve $\mathcal{V}_n$ is strongly extremal. A manifold $\mathcal{M}$ is **strongly extremal** if given any $v > n$, the set of points $\mathbf{x} = (x_1, ..., x_n) \in \mathcal{M}$ satisfying

$$|\mathbf{q}.\mathbf{x}| < \prod_{j=1}^{n}(|q_j| + 1)^{-\frac{v}{n}},$$

for infinitely many $\mathbf{q} \in \mathbb{Z}^n$ is null in $\mathcal{M}$. It is clear that a strongly extremal manifold is extremal. In 1984 V. I. Bernik [24] proved Baker's Conjecture, however, in 1980 Sprindžuk [87] had already extended Baker's Conjecture into a very general statement known as the Baker-Sprindžuk Conjecture.

Conjecture 1.26 (**Baker-Sprindzuk Conjecture (1980)**).
Let $f_1, ..., f_n$ be real analytic functions in $\mathbf{x} \in U$, U a domain in $\mathbb{R}^d$ which together with 1 *are linearly independent over* $\mathbb{R}$. *Then the manifold*

$$\mathcal{M} = \{\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), ..., f_n(\mathbf{x})) : \mathbf{x} \in U\}$$

is strongly extremal.

It was not until 1998 that the Baker-Sprindžuk Conjecture was shown to be true by Kleinbock and Margulis [64]. In fact, in their paper Kleinbock and Margulis not only prove the conjecture but also some generalizations. More precisely, consider a

d-dimensional submanifold

$$\mathcal{M} = \{\mathbf{f}(\mathbf{x}) : \mathbf{x} \in U\} \subset \mathbb{R}^n,$$

where U is an open subset of $\mathbb{R}^d$ and $\mathbf{f} = (f_1, ..., f_n)$ is a C^m embedding of U into $\mathbb{R}^n$. Recall that a function f is said to be of (differentiability) class C^k if the derivatives $f', f'', ..., f^{(k)}$ exist and are continuous. A manifold $\mathcal{M}$ is said to be C^k if its parametrisation map $\mathbf{f}$ is C^k.

For $l \leq m$, $\mathbf{y} = \mathbf{f}(\mathbf{x})$ is said to be an l-nondegenerate point of $\mathcal{M}$ if the space $\mathbb{R}^n$ is spanned by partial derivatives of $\mathbf{f}$ at $\mathbf{x}$ of order up to l. Furthermore $\mathbf{y}$ is said to be nondegenerate if it is l-nondegenerate for some l. This condition can be considered as an infinitesimal version of not lying in any proper affine hyperplane, i.e. of the linear independence of $1, f_1, ..., f_n$ over $\mathbb{R}$. In fact, if the functions f_i are analytic, it is easy to see that the linear independence of $1, f_1, ..., f_n$ over $\mathbb{R}$ in a domain U is equivalent to all points of $\mathcal{M} = \mathbf{f}(U)$ being nondegenerate. Thus Conjecture 1.26 would follow from the following statement of the Kleinbock-Margulis Theorem.

Theorem 1.27 (**Kleinbock-Margulis (1998)**).
Let $f_1, ..., f_n \in C^m(U)$, U an open subset of $\mathbb{R}^d$, be such that almost all points of $\mathcal{M} = \{\mathbf{f}(\mathbf{x}) : \mathbf{x} \in U\}$ are nondegenerate. Then M is strongly extremal.

Continuing with definitions, since the question at hand is the existence of a Khintchine-Groshev type theorem for either $\mathcal{S}_n(\mathcal{M}, \psi)$ or $\mathcal{L}_m(\mathcal{M}, \psi)$ the following terminology is given. A manifold $\mathcal{M}$ is said to be of **Khintchine type for convergence** if

$$\sum_{q=1}^{\infty} \psi(q)^n < \infty \implies \mu_{\mathcal{M}}(\mathcal{S}_n(\mathcal{M}, \psi)) = 0$$

and of **Khintchine type for divergence** if

$$\sum_{q=1}^{\infty} \psi(q)^n = \infty \text{ and } \psi \text{ being monotonic } \implies \mu_{\mathcal{M}}(\mathcal{M} \setminus \mathcal{S}_n(\mathcal{M}, \psi)) = 0.$$

Similarly a manifold $\mathcal{M}$ is said to be of **Groshev type for convergence** if

$$\sum_{q=1}^{\infty} q^{m-1}\psi(q) < \infty \implies \mu_{\mathcal{M}}(\mathcal{L}_m(\mathcal{M}, \psi)) = 0$$

and of **Groshev type for divergence** if

$$\sum_{q=1}^{\infty} q^{m-1}\psi(q) = \infty \text{ and } \psi \text{ being monotonic } \implies \mu_{\mathcal{M}}(\mathcal{M} \setminus \mathcal{L}_m(\mathcal{M}, \psi)) = 0.$$

All terminology is taken from [26] so see here for more details.

Under this terminology, a lot has been shown for the dual case but, unfortunately, not much for the simultaneous case. Most impressively Beresnevich [11] showed that any non-degenerate manifold is of Groshev type for convergence and Beresnevich, Bernik, Kleinbock and Margulis [12] showed that any non-degenerate manifold is also of Groshev type for divergence. This was an extension of work previously done by Bernik, Dickinson and Dodson [25] and Beresnevich, Bernik, Dickinson and Dodson [7] which together showed that any C^3 planar curve with non-zero curvature almost everywhere (equivalent to non-degenerate almost everywhere) is of Groshev type for convergence and divergence. That any non-degenerate manifold is of Groshev type for convergence was also shown independently and using a completely different method by Bernik, Kleinbock and Margulis [27]. For details on more results see [26] and [20].

Considering now the simultaneous case most results are only partial results. The exception being the case of planar curves. In 1979 Bernik [21] first showed that the Veronese curve of dimension 2, $\mathcal{V}_2$, i.e. the basic parabola $y = x^2$, is of Khintchine type for convergence. This result was extended in 2007 by Beresnevich, Dickinson and Velani [15] who showed

Theorem 1.28.
Any non-degenerate rational quadric in $\mathbb{R}^2$ is of Khintchine type for convergence and divergence.

Furthermore as a corollary to a theorem in the same paper (Corollary 1, [15]) Beresnevich, Dickinson and Velani showed that

Corollary 1.29.
Any C^3 non-degenerate planar curve is of Khintchine type for divergence.

In 2006 Vaughan and Velani [89] presented the following theorem.

Theorem 1.30.
Any C^2 non-degenerate planar curve is of Khintchine type for convergence.

Thus, using this theorem and the above corollary, a new corollary is produced.

Corollary 1.31.
Any C^3 non-degenerate planar curve is of Khintchine type for convergence and divergence.

Outside of the planar case the best known result is due to Beresnevich [18] who showed the following.

Theorem 1.32.
For $n \geq 2$, any analytic non-degenerate sub-manifold of $\mathbb{R}^n$ is of Khintchine type for divergence.

Some results have been shown for when a non-planar manifold is of Khintchine type for convergence. See [20] for details.

Moving now onto the Hausdorff theory and again beginning with the dual case. Results were first provided for particular curves with dimension at least 2. In 1978 R. C. Baker [6] gave the following result.

Theorem 1.33.
Let Γ be any planar curve which has non-zero curvature almost everywhere. Then for $\tau > 2$

$$\dim_{\mathcal{H}}(\mathcal{L}(\Gamma, m, \tau)) = 3/(\tau + 1).$$

In 1989 Dodson, Rynne and Vickers [49] showed the following.

Theorem 1.34.
Let $\mathcal{M}$ be any C^3 manifold in $\mathbb{R}^m$ of dimension $d \geq 2$ which is 2-curved except on a set of Hausdorff dimension $d-1$. Then for $\tau \geq m$

$$\dim_{\mathcal{H}}(\mathcal{L}(\mathcal{M}, m, \tau)) = d - 1 + \frac{m+1}{\tau+1}.$$

Bernik and Dodson [26] extended this to the case of any decreasing approximation function.

An analogue to the divergent part of Jarník's Theorem was established in 2006 by Beresnevich, Dickinson and Velani [Theorem 18, [13]] for any non-degenerate manifold. Their result ran as follows.

Theorem 1.35.
Let $\mathcal{M}$ be a non-degenerate manifold in $\mathbb{R}^m$ of dimension d. Let f be a dimension function such that $q^{-d}f(q) \to \infty$ as $q \to 0$ and $q^{-d}f(q)$ is decreasing. Furthermore, suppose that $q^{-(d-1)}f(q)$ is increasing. Let ψ be a real decreasing function. Then

$$\mathcal{H}^f(\mathcal{L}_m(\psi, \mathcal{M})) = \infty \quad \text{if} \quad \sum_{q=1}^{\infty} f\left(\frac{\psi(q)}{q}\right)\psi(q)^{-(d-1)}q^{n+d-1} = \infty.$$

Proving the convergent counterpart to this theorem remains open.

Moving onto the Hausdorff dimension results for the simultaneous case, very little is known here. In the planar case the following theorem was shown by Beresnevich, Dickinson and Velani [15].

Theorem 1.36.
Let Γ be any planar curve with non-zero curvature almost everywhere. Then for $\frac{1}{2} \leq \tau \leq 1$

$$\dim_{\mathcal{H}}(\mathcal{S}_2(\Gamma, \tau)) = \frac{2-\tau}{1+\tau}.$$

Obviously if $\tau \leq \frac{1}{2}$, Dirichlet's Theorem implies that the set has full measure.

When the case of $\tau > 1$ is considered something interesting happens. Unlike in the dual case, where a unified result could be found, different curves give different results. Two classic examples in which this occurs are presented:

1) Let Γ_n be the curve satisfying the equation $x^n + y^n = 1$ for $n \geq 2$. Using Wiles' Theorem [92], Dickinson [44] showed that $\mathcal{S}_n(\Gamma_n, \tau) = \emptyset$ for $n > 2$ and $\tau > n-1$. Thus $\dim_{\mathcal{H}}(\mathcal{S}_n(\Gamma_n, \tau)) = 0$ for $n > 2$ and $\tau > n-1$. In contrast, if $n = 2$ then there are infinitely many points on Γ_2. In particular, it was shown by Dickinson and Dodson [46] that $\dim_{\mathcal{H}}(\mathcal{S}_2(\Gamma_2, \tau)) = \frac{1}{\tau+1}$ for $\tau > 1$.

   It it worth mentioning that this was an improvement on a result of Melnichuk [76] who had shown that

$$\frac{1}{2(\tau+1)} \leq \dim_{\mathcal{H}}(\mathcal{S}_2(\Gamma_2, \tau) \leq \frac{1}{\tau+1}.$$

   Unfortunately Melnichuk's paper is in Russian and is not easily found so Dickinson and Dodson kindly reproduced his proof in their paper.

   It should also be pointed out that the result of Dickinson and Dodson is the first reasonably complete non-trivial result for the Hausdorff dimension of the set $\mathcal{S}_n(\mathcal{M}, \tau)$ for a smooth manifold $\mathcal{M}$ in $\mathbb{R}^n$ when τ is larger than the extremal value (i.e. the Dirichlet Bound) of $1/n$.

2) Let Γ be the curve representing the equation $y = x^2$ (the parabola) then it was shown by Beresnevich [10] that

$$\dim_{\mathcal{H}}(\mathcal{S}_2(\Gamma, \tau)) = \frac{1}{1+\tau}.$$

A very nice result of Budarina, Dickinson and Levesley [36] gives an answer for the Hausdorff dimension of the set of simultaneous τ-approximable points lying on integer polynomial curves. In particular define

$$\Gamma = \{(x, P_1(x), ..., P_{n-1}(x)) \in \mathbb{R}^n : P_j \in \mathbb{Z}[x]\}$$

and let $d_j = \deg(P_j)$ and $d = \max_{1 \le j \le n-1} \{d_j\}$. Then their result is as follows.

Theorem 1.37.
For $\tau \ge \max(d-1, 1)$ the Hausdorff dimension of $S_n(\Gamma, \tau)$ is

$$\dim_{\mathcal{H}}(S_n(\Gamma, \tau)) = \frac{2}{d(\tau+1)}.$$

This result has been recently extended to certain other graph varieties [88].

A perfect example of showing the strange and sensitive behaviour of the set $S_n(M, \tau)$ with regards to changing curves was shown by F. Adiceam [1]. Taking, for some $\alpha \in \mathbb{R}$,

$$\Gamma_\alpha := \{(x, P(x) + \alpha) \in \mathbb{R}^2 : P(x) \in \mathbb{Z}[x]\}$$

i.e. the 2-dimensional equivalent of the curve taken by Budarina, Dickinson and Levesley but translated vertically by a real number α, Adiceam showed the following.

Theorem 1.38.
Assume $d := \deg(P(x)) \ge 2$. If $\tau > d$, then

$$\mathcal{S}_2(\Gamma_\alpha, \tau) = \emptyset$$

for almost all $\alpha \in \mathbb{R}$.

Furthermore Adiceam showed that d is in fact optimal and obtained the following result on the Hausdorff dimension of $S_2(\Gamma_\alpha, \tau)$.

Theorem 1.39.
Assume $d \ge 2$. If $\tau \in (d-1, d]$, then

$$\dim_{\mathcal{H}}(\mathcal{S}_2(\Gamma_\alpha, \tau)) \le \frac{d-\tau}{\tau+1}$$

for almost all $\alpha \in \mathbb{R}$.

Adiceam's results are the first related to the study of the Hausdorff dimension of the set of well approximable points lying on a curve which is not defined by a polynomial with integer coefficients. The question of considering the curve translated horizontally remains open.

1.4 Approximation By Algebraic Numbers

As an extension to approximation by rational numbers, which is all that has been considered until this point, a vibrant area of research in Diophantine approximation is approximation by algebraic numbers. Recall that a real number α is said to be **algebraic** if it is the root of some non-zero polynomial $P(x)$ with rational coefficients. In particular, instead of considering the approximation of the real number x by the rational number $\frac{p}{q}$, the approximation of x by an algebraic number α is now considered. This immediately makes things much more complicated. Until now there was a natural way to measure the size of $\frac{p}{q}$, simply the $\max\{|p|,|q|\}$. Now however a new definition of size will be needed and more so it should coincide with $\max\{|p|,|q|\}$ when $\alpha = \frac{p}{q}$. To this end the measurement of height will be used and is defined as follows. The **height** of a polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$, denoted H(P), is given by

$$H(P) := \max_{0\leq i\leq n}\{|a_i|\}.$$

For α an algebraic number the height of α, denoted $H(\alpha)$, is the height of the minimal integer polynomial of α. Furthermore an algebraic number α is said to be of degree n if the minimal polynomial of α has degree n. For a very nice summary of some basic results on approximation by algebraic numbers see [[37], Chapter 2].

One of the first people to consider approximation by algebraic numbers was Mahler [71] who decided to use this idea as a method of classification. A real number is classed in one of two ways, it is either algebraic or it is not algebraic and is then called transcendental. This is a very simple and unsatisfying classification. The set of transcendental numbers is uncountable and preferably one would like a better way of classifying these. Mahler proposed a classification which would split the real numbers into four classes, one of which being the algebraic numbers and the other three classes would divide up the transcendental numbers. The method for determining which class a number belonged to was roughly based upon its algebraic approximation. In particular, Mahler's idea was based on classifying a real number ξ according to how accurate a non-zero integer polynomial evaluated at ξ approached 0. In other words Mahler considered the approximation of 0 by the values of polynomials $P(x) \in P_n(H)$ at the point ξ, where $P_n(H) := \{P(x) \in \mathbb{Z}[x] : \deg(P(x)) = n, H(P) \leq H\}$. For a fixed real number H Mahler defined

$$\omega_n(\xi, H) := \min\{|P(\xi)| : P(X) \in P_n(H), P(\xi) \neq 0\}.$$

Furthermore Mahler introduced the parameters

$$\omega_n(\xi) := \limsup_{H\to\infty} \frac{-\log(\omega_n(\xi, H))}{\log(H)} \quad \text{and}$$

$$\omega(\xi) := \limsup_{n\to\infty} \frac{\omega_n(\xi)}{n}.$$

In particular, $\omega_n(\xi)$ is the supremum of the set of real numbers $\omega > 0$ for which the inequality

$$|P(\xi)| < H^{-\omega}$$

has solutions, as H tends to infinity, by infinitely many polynomials $P \in P_n(H)$, i.e.

$$\omega_n(\xi) = \sup\{\omega > 0 : |P(\xi)| < H^{-\omega} \text{ for i.m } P \in P_n\}.$$

Under this notation, Mahler was able to divide the set of real numbers into four disjoint classes in the following way. A real number ξ is called

- an A-number, if $w(\xi) = 0$ (i.e. ξ is algebraic over $\mathbb{Q}$);
- an S-number, if $0 < w(\xi) < \infty$;
- a T-number, if $w(\xi) = \infty$ and $w_n(\xi) < \infty$ for any $n \geq 1$;
- a U-number, if $w(\xi) = \infty$ and $w_n(\xi) < \infty$ for all n large enough.

It is clear that the class of A-numbers consists of the algebraic numbers and consequently the transcendental numbers form the other three classes. As will be discussed in Section 1.5, Mahler [72] proved that almost all (in the sense of Lebesgue measure) real numbers ξ satisfy $\omega_n(\xi) \leq 4n$ for any $n \in \mathbb{N}$. Furthermore, in the same paper, Mahler conjectured that almost all real numbers ξ satisfy $\omega_n(\xi) \leq n$ for any $n \in \mathbb{N}$, this was shown to be true by Sprindzŭk [85] in 1965. Just a few of the many results that have been shown since Mahler's classification are now presented, proofs for each can be found in [37] or [85].

Proposition 1.40.
Let $n \in \mathbb{N}$ and $\xi \in \mathbb{R}$ not of algebraic degree at most n, then

$$\omega_n(\xi) \geq n.$$

In particular if ξ is transcendental, then

$$\omega(\xi) \geq 1.$$

Theorem 1.41.
Let ξ be an algebraic number of degree d and let n be a positive integer, then

$$\omega_n(\xi) = \min\{n, d-1\}.$$

Theorem 1.41 shows that the A-numbers are exactly the real algebraic numbers. From Proposition 1.40 and Theorem 1.41 it follows immediately follows that there do not exist real numbers ξ with $0 < \omega(\xi) < 1$.

Recall that a polynomial in two variables, with integer coefficients, is a sum of the form

$$P(X,Y) = \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} a_{i,j}X^iY^j$$

with $n_1, n_2 \in \mathbb{N}$ and integer coefficients $a_{i,j}$. Two complex numbers α and β are defined to be **algebraically dependent** if there is a non-zero polynomial $P(X,Y)$ in two variables, with integer coefficients, such that $P(\alpha,\beta) = 0$. Otherwise, α and β are defined to be **algebraically independent**. For example, the numbers e and $e-5$ are algebraically dependent, take $P(X,Y) = Y - X - 5$.

Theorem 1.42.
Any two algebraically dependent real numbers belong to the same class.

For more information on Mahler's Classification see [[37], Chapter 3] for an excellent description of these classes or alternatively [91].

A second classification of the real numbers was proposed by Koksma [66] in 1939. Koksma's classification is very similar to the classification of Mahler however, instead of looking at how accurate a non-zero integer polynomial at ξ approached 0, Koksma's idea was to consider the approximation of ξ by algebraic numbers.

Let $\mathbb{A}_n$ denote the set of algebraic numbers of degree n then, for a given real number $H \geq 1$, Koksma defined

$$\omega_n^*(\xi, H) := \min\{|\xi - \alpha| : \alpha \in \mathbb{A}_n, H(\alpha) \leq H, \alpha \neq \xi\}.$$

Furthermore Koksma introduced the parameters

$$\omega_n^*(\xi) := \limsup_{H\to\infty} \frac{-\log(H\omega_n^*(\xi,H))}{\log(H)} \quad \text{and}$$
$$\omega^*(\xi) := \limsup_{n\to\infty} \frac{\omega_n^*(\xi)}{n}.$$

In particular, $\omega_n^*(\xi)$ is the supremum of the set of real numbers $\omega > 0$ for which the inequality

$$0 < |\xi - \alpha| \leq H(\alpha)^{-\omega-1}$$

has solutions for infinitely many real algebraic numbers $\alpha \in \mathbb{A}_n$, i.e.

$$\omega_n^* := \sup\{\omega > 0 : |\xi - \alpha| \leq H(\alpha)^{-\omega-1} \text{ for i.m } \alpha \in \mathbb{A}_n\ \}.$$

It is an easy exercise to show that

$$\omega_n^*(\xi) = \omega_n^*(\xi + \frac{a}{b}) = \omega_n^*(\frac{a\xi}{b})$$

for any $\xi \in \mathbb{R}, n \in \mathbb{N}$ and $\frac{a}{b} \in \mathbb{Q} \setminus \{0\}$ (See Exercise 3.1, [37]). This exercise can be used to show that the function ω_n^* takes the same value at almost all real numbers, although this is not a trivial task; see [[37], Chapter 4] for details and proof.

Under this notation, Koksma was able to divide the set of real numbers into four disjoint classes in the following way. A real number ξ is called

- an A*-number, if $w^*(\xi) = 0$;
- an S*-number, if $0 < w^*(\xi) < \infty$;
- a T*-number, if $w^*(\xi) = \infty$ and $w_n^*(\xi) < \infty$ for any $n \geq 1$;
- a U*-number, if $w^*(\xi) = \infty$ and $w_n^*(\xi) < \infty$ for all n large enough.

Again see [[37],Chapter 3] or [91] for a more complete description and history on Koksma's classification.

Finding bounds on the values of ω_n^* is a big problem. In 1960 Wirsing [93] conjectured that for any $n \geq 1$ and any real number ξ, which is not of algebraic degree at most n, $\omega_n^* \geq n$. Put another way Wirsing's Conjecture can be stated as follows.

Conjecture 1.43 (Wirsing (1960)).
For $\xi \in \mathbb{R}$ not of algebraic degree at most n and any $\epsilon > 0$ there exists a constant $c = c(\xi, n, \epsilon) > 0$ such that

$$|\xi - \alpha| \leq cH(\alpha)^{-n-1+\epsilon},$$

for infinitely many $\alpha \in \mathbb{A}_n$.

In the same paper Wirsing proved that for any real number, not of algebraic degree at most n, $\omega_n^*(\xi) \geq \frac{(n+1)}{2}$. In particular he showed the following.

Theorem 1.44.
For $\xi \in \mathbb{R}$ not of algebraic degree $\leq n$ and any $\epsilon > 0$ there exists a constant $c = c(n, \epsilon) > 0$ such that

$$|\xi - \alpha| \leq cH(\alpha)^{-\frac{(n+3)}{2}}$$

for infinitely many $\alpha \in \mathbb{A}_n$.

This was improved on in 1967 by H. Davenport and W. M. Schmidt [42] for the case of $n = 2$ by replacing $\frac{(n+3)}{2}$ with 3. Even more, they proved this to be optimal in the case of approximation of a real number by a quadratic algebraic. To date this is the only completely solved case of Wirsing's Conjecture. See [[37],Chapter 3] for more details.

Since the publication of both Mahler and Koksma's classifications much work has gone into comparing the two. Here are just some of the results.

Proposition 1.45.
For any $n \in \mathbb{N}$ *and any* $\xi \in \mathbb{R}$

$$\omega_n(\xi) \geq \omega_n^*(\xi).$$

This is immediate from Wirsing's result however it had been established previous to Wirsing by Koksma [66]. It turns out that although bounding $\omega_n(\xi)$ below by $\omega_n^*(\xi)$ is in general easy, trying to bound $\omega_n^*(\xi)$ below by $\omega_n(\xi)$ is extremely difficult. The first result of this kind was obtained by Wirsing [93] who showed the following lower bounds.

Theorem 1.46 (Wirsing (1961))**.**
Let $n \in \mathbb{N}$ *and* $\xi \in \mathbb{R}$ *not be algebraic of degree at most* n*, then*

$$\begin{aligned} \omega_n^*(\xi) &\geq \omega_n(\xi) - n + 1, \\ \omega_n^*(\xi) &\geq \frac{\omega_n(\xi) + 1}{2}, \\ \omega_n^*(\xi) &\geq \frac{\omega_n(\xi)}{\omega_n(\xi) - n + 1}, \\ \textit{and finally} \qquad \omega_n^*(\xi) &\geq \frac{n}{4} + \frac{\sqrt{n^2 + 16n - 8}}{4}. \end{aligned}$$

From the third inequality of Theorem 1.46 and Proposition 1.45 the following is true.

Corollary 1.47.
Let $n \in \mathbb{N}$ *and* $\xi \in \mathbb{R}$ *not be algebraic of degree at most* n*, then*

$$\omega_n(\xi) = n \implies \omega_n^*(\xi) = n.$$

The following theorem analogous to Theorem 1.41 can also be shown to be true.

Theorem 1.48.
Let ξ be an algebraic number of degree d and let n be a positive integer, then

$$\omega_n^*(\xi) = \min\{n, d-1\}.$$

That is to say that the A*-numbers are exactly the real algebraic numbers. For more details on the relationships between Mahler and Koksma's classifications see [[37], Section 3.4] or [91]

Next, given an approximating function ψ, define the set

$$K_n(\psi) := \{\xi \in \mathbb{R} : |\xi - \alpha| < \psi(H(\alpha)) \text{ for i.m } \alpha \in \mathbb{A}_n\}.$$

It should be clear that, since the rationals are algebraic numbers of degree 1, $K_n(\psi)$ is a generalisation of $W(\psi)$. In the special case of $\psi(q) = q^{-(n+1)\tau}$ denote $K_n(\psi)$ by $K_n(\tau)$

One particularly nice result on the set $K_n(\tau)$ is due to A. Baker and W.M. Schmidt [3] who obtained an analogue to the Jarník-Besicovitch Theorem.

Theorem 1.49 (Baker-Schmidt Theorem (1970)).
Let $\tau \geq 1$, then

$$\dim_{\mathcal{H}}(K_n(\tau)) = \frac{1}{\tau}.$$

See [[37], Sec 5.6] for more details and discussion.

In 2006 Beresnevich, Dickinson and Velani [13] improved upon this by providing a complete measure theoretic description of $K_n(\psi)$ which implied the Baker-Schmidt Theorem and showed that $\mathcal{H}^{\frac{1}{\tau}}(K_n(\tau)) = \infty$. In particular they showed the following two theorems.

Theorem 1.50.
Let ψ be a real, positive decreasing function. Then

$$\mu(K_n(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} \psi(q) q^n < \infty \\ 1 & \text{if } \sum_{q=1}^{\infty} \psi(q) q^n = \infty. \end{cases}$$

Theorem 1.51.
Let f be a dimension function such that $q^{-1} f(q) \to \infty$ as $q \to 0$ and $q^{-1} f(q)$ is decreasing. Furthermore let ψ be a real, positive decreasing function. Then

$$\mathcal{H}^f(K_n(\psi)) = \begin{cases} 0 & \textit{if } \sum_{q=1}^{\infty} f(\psi(q))q^n < \infty \\ \infty & \textit{if } \sum_{q=1}^{\infty} f(\psi(q))q^n = \infty. \end{cases}$$

As previously mentioned, Theorem 1.51 not only implies the Baker-Schmidt Theorem but also shows that $H^{\frac{1}{\tau}}(K_n(\tau)) = \infty$.

It should be clear that a potentially useful tool in proving results regarding approximation by algebraic numbers would be results on the distance between certain classes of algebraic numbers or results on counting algebraic numbers within a certain domain. Define two algebraic numbers α and β to be **conjugate** if there exists an irreducible polynomial $P(x) \in \mathbb{Z}[x]$ such that $P(\alpha) = 0 = P(\beta)$. Furthermore define an **algebraic integer** to be an algebraic number that is a root of a polynomial with integer coefficients with leading coefficient 1. A natural question to then ask is:

How close can two algebraic conjugate numbers be?

This question has been considered in much detail over the last 55 years however complete answers are known only for the cases of algebraic numbers of degree 2 and degree 3. In particular define κ_n (respectively κ_n^*) to be the infimum of the set of κ such that

$$|\alpha_1 - \alpha_2| > H(\alpha_1)^{-\kappa}$$

for arbitrary algebraic conjugate numbers (respectively algebraic integers) α_1 and α_2 of degree n with $\alpha_1 \neq \alpha_2$ and with sufficiently large $H(\alpha_1)$. It is clear that $\kappa_n^* \leq \kappa_n$ for all n.

Mahler [74] gave the first result on this question in 1964 by showing that $\kappa_n \leq n-1$. Even now this is the best estimate obtained. It can be shown with relative ease that $\kappa_2 = 1$ (see [38]). Evertse [51] proved that $\kappa_3 = 2$ and Bugeaud and Mignotte [38] showed that $\kappa_2^* = 0$ and that $\kappa_3^* \geq \frac{3}{2}$. As mentioned before these are the only complete results. When $n > 3$ some partial results are known however.

Mignotte [77] proved that for all $n \geq 3$, $\kappa_n, \kappa_n^* \geq \frac{n}{4}$. This was improved by Bugeaud and Mignotte [38] who have shown that

$$\begin{aligned} \kappa_n &\geq \frac{n}{2} && \text{when } n \geq 4 \text{ is even,} \\ \kappa_n^* &\geq \frac{n-1}{2} && \text{when } n \geq 4 \text{ is even,} \\ \kappa_n &\geq \frac{n+2}{4} && \text{when } n \geq 5 \text{ is odd,} \\ \kappa_n^* &\geq \frac{n+2}{4} && \text{when } n \geq 5 \text{ is odd.} \end{aligned}$$

Improving on this again Bugeaud and Dujella [39] proved that for all $n \geq 4$

$$\kappa_n \geq \frac{n}{2} + \frac{n-2}{4(n-1)}.$$

Another improvement to Bugeaud and Mignotte's result was made by Beresnevich, Bernik and Götze [16] who showed that for any $n \geq 2$,

$$\min\{\kappa_n, \kappa^*_{n+1}\} \geq \frac{n+1}{3}.$$

In fact, in their paper they actually show that for sufficiently large Q there are at least $c(n)Q^{\frac{n+1}{3}}$ pairs of algebraic conjugate numbers of degree n (or algebraic conjugate integers of degree $n+1$) α_1 and α_2 with height $H(\alpha_1) \asymp Q$ such that

$$|\alpha_1 - \alpha_2| \asymp H(\alpha_1)^{-\frac{n+1}{3}}.$$

In contrast to the method of Bugeaud and Mignotte, which relied on finding explicit families of polynomials with clusters of roots, Beresnevich, Bernik and Götze used a completely different approach in which irreducible polynomials were implicitly tailored so that their derivatives assumed certain values. Further questions along this line of thinking that remain to be asked are how many pairs, triples, etc. of algebraic conjugate numbers of degree n lie within some domain of a Euclidean space. As a particular example consider the following question: How many pairs of algebraic conjugate numbers of degree n and bounded height Q lie within a rectangle of side lengths $Q^{-\mu_1}$ and $Q^{-\mu_2}$? Bernik, Götze and Kukso [29] gave an answer to this of $\gg Q^{n+1-\mu_1-\mu_2}$ provided $0 < \mu_1, \mu_2 < \frac{1}{2}$.

In an attempt to both improve and extend this result a recent result of V. I. Bernik, N. Budarina, D. Dickinson and S. Mc Guire [31] shows that the number of algebraic conjugate triples of height at most Q and degree at most n lying in the three-dimensional cube of side length $Q^{-\lambda}$, for $0 < \lambda < \frac{1}{3}$, is at least $Q^{-3\lambda+n+1}$. In Chapter 5 the main results of [31] are discussed and a brief overview of the proofs is given. In particular it is shown that the extension to Bernik's Lemma given in Chapter 3 is essential for the proof in [31].

1.5 Conjectures On The Set $L_n(w)$

Another set of interesting problems lie around considering the set $L_n(\omega)$ which is somewhat similar to the set $\mathcal{L}_m(\tau)$ discussed in Section 1.2. In particular let $P(x) = a_nx^n + ... + a_1x + a_0$ be an integer polynomial and again define $H(P) =$

$H := \max_{0 \le j \le n}\{|a_j|\}$ to be the height of the polynomial $P(x)$. Define the classes of polynomials P_n and $P_n(Q)$ by

$$P_n := \{P(x) \in \mathbb{Z}[x] : \deg(P(x)) = n\},$$
$$P_n(Q) := \{P(x) \in \mathbb{Z}[x] : \deg(P(x)) = n, H(P) \le Q\}.$$

We are interested in the following question: Given some $w \in \mathbb{R}$ how big is the set of $x \in \mathbb{R}$ for which $|P(x)| < H(P)^{-w}$ for infinitely many polynomials $P(x) \in P_n$. Denote this set by $L_n(w)$, i.e.

$$L_n(w) := \{x \in \mathbb{R} : |P(x)| < H(P)^{-w} \text{ for i.m } P \in P_n\}$$

where i.m stands for infinitely many. Note that $L_n(w)$ can be manipulated to represent a special case of the set $\mathcal{L}_m(\tau)$.

With regard to the Lebesgue measure of the set, $L_n(w)$, when $w \le n$ it can be shown to follow from the pigeonhole principle that $L_n(w) = \mathbb{R}$. In 1932, Mahler [72] conjectured that $\mu(L_n(w)) = 0$ when $w > n$, and indeed this was shown to be true by Sprindžuk [85] in 1964. Before Sprindžuk, however, there were some partial results. Mahler himself [72] proved the conjecture for $w > 4n$. This was improved by W.M. Schmidt [82] in 1961 to $w > 2n$. A further improvement was given by Volkmann [90] in 1962 who showed that the result was true for $w > \frac{4}{3}n$.

With regard to the Hausdorff dimension of the set, $\dim_{\mathcal{H}}(L_n(w))$, the first result is given by the Jarník-Besicovitch Theorem.

Theorem 1.52 (Jarník-Besicovitch Theorem for $L_n(w)$).
Let $w > 1$. Then

$$\dim_{\mathcal{H}}(L_1(w)) = \frac{2}{w+1}.$$

This was extended for $n > 1$ by A. Baker and W.M. Schmidt [3] in 1970.

Theorem 1.53.
For $n > 1$

$$\frac{n+1}{w+1} \le \dim_{\mathcal{H}}(L_n(w)) < \frac{2n+2}{w+1}.$$

In the same paper Baker and Schmidt also made the following conjecture.

Conjecture 1.54.
For $w > n$

$$\dim_{\mathcal{H}}(L_n(w)) = \frac{n+1}{w+1}.$$

Previous to the conjecture of Baker and Schmidt, in 1958 Kasch and Volkmann [62] had shown the following.

Theorem 1.55.

$$\dim_{\mathcal{H}}(L_2(w)) \leq \frac{3}{w+1}.$$

So Theorem 1.53 and Theorem 1.55 together prove the conjecture for $n = 2$.

In 1976, R.C. Baker [5] gave results in the cases of $n = 3$ and $n \geq 4$.

Theorem 1.56.
For $w > 3$

$$\dim_{\mathcal{H}}(L_3(w)) \leq \frac{4}{w+1}.$$

Furthermore when $n \geq 4$ if $w > (n^2 + n - 3)/3$

$$\dim_{\mathcal{H}}(L_n(w)) \leq \frac{n+1}{w+1}.$$

This result, together with Theorem 1.53, proves the conjecture for $n = 3$ and for $n \geq 4$ if $w > (n^2 + n - 3)/3$.

The conjecture was finally proven in the affirmative in 1983 by Bernik [23]. In his paper Bernik uses different methods, to those of R.C Baker, based on the following two lemmas from the same paper.

Lemma 1.57.
Let δ, η, μ be positive real numbers, let $s > 1$ be an integer and let $H_0(\delta, s)$ be a sufficiently large real number. Furthermore, let $P(x), T(x) \in \mathbb{Z}[x]$ be polynomials of degree s without common roots such that $\max(H(P), H(T)) = H^{\mu}$ where $H > H_0(\delta, s)$. Assume that the interval $I \subset (-s, s) \subset \mathbb{R}$ with $|I| = H^{-\eta}$. If there exists $\tau > 0$ such that for all $x \in I$

$$\max(|P(x)|, |T(x)|) < H^{-\tau},$$

then

$$\tau + \mu + 2\max(\tau + \mu - \eta, 0) < 2\mu s + \delta.$$

Lemma 1.57 can be thought of as a quantitative description of the fact that two relatively prime polynomials in $\mathbb{Z}[x]$ cannot both have very small absolute values (in terms of their degrees and heights) in an interval unless that interval is extremely short.

Lemma 1.58.
Suppose that $P(x) \in \mathbb{Z}[x]$ has degree at most l and height at most H^{λ}. Suppose that for all ω in some interval I the inequality $|P(\omega)| < H^{-\nu}$ holds, where $\nu > 3\lambda l$. Then

there exists a divisor $d(x)$ *of the polynomial* $P(x)$ *which is a power of a polynomial, irreducible over the rational field, with* $\deg(d(x)) = l_1$ *and* $H(d(x)) = H^{\lambda_1}$*, that satisfies, for all* $\omega \in I$*, the conditions*

$$|d(\omega)| < c(l)H^{-\nu+\lambda l},\ l_1 \le l,\ H^{\lambda_1} < c(l)H^{\lambda}.$$

In [23], and for many results since, Lemma 1.57 was a key tool in disproving the existence of certain cases by obtaining contradictions. Generally speaking, Lemma 1.57 is useful when dealing with problems that are concerned with small first derivatives since Lemma 1.57 also shows that two polynomials $P(x), T(x) \in P_n(Q)$ cannot be simultaneously small at a point as well as having simultaneously small derivatives at that point. See [28],[35] and [31] for just some of the many examples of Lemma 1.57 being used.

It turns out that Lemma 1.58, both in formulation and method of proof, is extremely similar to a lemma of Gel'fond [[55], Chapter 3, Section 4, Lemma 6].

In Chapter 3 an extension to Lemma 1.57 is shown that removes the restriction on the size of the polynomials and allows, for the first time, the possibility that $|P(x)|$ is very large. This improvement was motivated by the necessity for it in the proofs in [31].

In Chapter 4 an improvement on a previous extension of Lemma 1.57 to the space $\mathbb{R} \times \mathbb{C} \times \mathbb{Q}_p$ is given. Using this extension an example on determining results on the number of polynomials with bounded discriminants, in a very particular case, is shown. In particular a result of Beresnevich, Bernik and Götze [19] on counting polynomials with bounded discriminants is completed under certain constraints.

Chapter 2

Preliminary Results

This chapter is a summary of several definitions and lemmas that are necessary for the discussion and proofs in Chapter 3 and Chapter 4. Almost all of the following results and their proofs may be found in [85], [22], [23], [29] or [31]. Unless stated otherwise polynomials of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$$

with $a_i \in \mathbb{Z}$ for $i = 0, 1, ..., n$ will be considered. Recall that the height of the polynomial $P(x)$, denoted $H(P)$, is given by

$$H(P) := \max\{|a_0|, |a_1|, ..., |a_n|\}.$$

Recall the class of polynomials $P_n(Q)$ is defined by

$$P_n(Q) := \{P(x) \in \mathbb{Z}[x] : \deg(P(x)) = n, H(P) \leq Q\}.$$

If the condition $|a_n| \gg H(P)$ can be guaranteed then the following lemma can be very useful [85].

Lemma 2.1.
If there exists a number $c, 0 < c < 1$, such that the polynomial $P(x)$ satisfies the condition $|a_n| > cH$, where $H = H(P)$ is the height of the polynomial, then

$$\max_{i=1,2,...,n} |\alpha_i| \leq \frac{n}{c}$$

where the α_i denote the roots of $P(x)$.

Proof.
Let α be any zero of the polynomial $P(x)$. If $|\alpha| > 1$, then the equation

$$a_n\alpha = -a_{n-1} - ... - a_0\alpha^{-n+1}$$

implies

$$|\alpha| < \frac{|a_{n-1}| + ... + |a_0|}{|a_n|} \leq \frac{nH}{cH} = \frac{n}{c}.$$

Therefore $|\alpha| \leq \max(1, n/c) = n/c$, and the assertion follows immediately. □

Thus, under the condition $|a_n| \gg H(P)$, the previous lemma implies that all roots of $P(x) \in P_n(H)$ are bounded. In particular the condition $|a_n| \gg H(P)$ implies that the distance between any two roots is bounded. This will be very important for Chapter 5 and will be discussed further here. It is not, however, a condition we will consider in Chapter 3 or Chapter 4.

Suppose $\alpha_1, \alpha_2, ..., \alpha_n$ are the roots of the polynomial $P(x)$, then for each α_i the set $\mathcal{S}_P(\alpha_i)$ will denote the set of all real numbers x whose distance from α_i is not greater than their distance from any other root α_j, $i \neq j$. More precisely

$$\mathcal{S}_P(\alpha_i) = \{x \in \mathbb{R} : |x - \alpha_i| \leq |x - \alpha_j|,\ 1 \leq i \neq j \leq n\}.$$

The following notation will also be used,

$$\mathcal{S}_P(\alpha_i, \alpha_j, \alpha_k) = \mathcal{S}_P(\alpha_i) \times \mathcal{S}_P(\alpha_j) \times \mathcal{S}_P(\alpha_k).$$

Before moving on the following two identities are recalled. For $P \in P_n$,

$$|x - \alpha_i| = |P(x)| \Bigg(\prod_{\substack{j=1,...,n, \\ j \neq i}} |a_n||x - \alpha_j| \Bigg)^{-1}$$

$$\text{and} \qquad |P'(\alpha_i)| = \prod_{\substack{j=1,...,n, \\ j \neq i}} |a_n||\alpha_i - \alpha_j|.$$

Using these identities the following very useful lemma is easily proven; see [24] or [85].

Lemma 2.2.
Let $x \in \mathcal{S}_P(\alpha_i)$. Then

$$|x - \alpha_i| \leq n\frac{|P(x)|}{|P'(x)|} \qquad \textit{for} \quad P'(x) \neq 0,$$

$$|x-\alpha_i| \leq 2^{n-1}\frac{|P(x)|}{|P'(\alpha_i)|} \quad \mathit{for} \quad P'(\alpha_i) \neq 0$$

and

$$|x-\alpha_i| \leq \min_{2\leq j\leq n}\left(2^{n-j}|P(x)||P'(\alpha_i)|^{-1}\prod_{\substack{k=1\\k\neq i}}^{j}|\alpha_i-\alpha_k|\right)^{\frac{1}{j}} \quad \mathit{for} \quad P'(\alpha_i)\neq 0.$$

Proof.
Firstly notice that

$$P(x) = a_n(x-\alpha_1)...(x-\alpha_n)$$

implies that

$$P'(x) = a_n\sum_{j=1}^{n}\sum_{\substack{i=1,\\ i\neq j}}^{n}(x-\alpha_i)$$

and so

$$\frac{|P'(x)|}{|P(x)|} = \sum_{j=1}^{n}\frac{1}{|x-\alpha_j|}.$$

Since $x \in \mathcal{S}_P(\alpha_i)$ then by definition $|x-\alpha_i| \leq |x-\alpha_j|$ for $i \neq j$ and so, for x not a root,

$$|x-\alpha_i|\frac{|P'(x)|}{|P(x)|} = 1 + \sum_{\substack{j=1,\\ j\neq i}}^{n}\frac{|x-\alpha_i|}{|x-\alpha_j|} \leq n.$$

Rearranging gives the first inequality.

Next notice that for any $x \in \mathcal{S}_P(\alpha_i)$ and any $j \in \{1,...,n\}\setminus\{i\}$

$$|\alpha_i-\alpha_j| \leq |\alpha_i - x| + |x-\alpha_j| \leq 2|x-\alpha_j| \tag{2.1}$$

and so using this inequality and the above identities

$$|x-\alpha_i| = \frac{|P(x)|}{\prod\limits_{\substack{j=1,...,n,\\ j\neq i}}|a_n||x-\alpha_j|} \leq \frac{2^{n-1}|P(x)|}{\prod\limits_{\substack{j=1,...,n,\\ j\neq i}}|a_n||\alpha_i-\alpha_j|} = 2^{n-1}\frac{|P(x)|}{|P'(\alpha_i)|}.$$

Now for $i_1,...,i_j \in \{1,...,n\}\setminus\{i\}$ consider

$$|x-\alpha_i|^j \leq |x-\alpha_{i_1}||x-\alpha_{i_2}|...|x-\alpha_{i_j}|$$

$$= \frac{|P(x)|}{|a_n||x-\alpha_{i_{j+1}}|...|x-\alpha_{i_n}|}$$
$$\leq \frac{2^{n-j}|P(x)|}{|a_n||\alpha_i-\alpha_{i_{j+1}}|...|\alpha_i-\alpha_{i_n}|} \qquad \text{(by (2.1))}$$
$$= 2^{n-j}\frac{|P(x)|}{|P'(\alpha_i)|}|\alpha_i-\alpha_{i_1}|...|\alpha_i-\alpha_{i_j}|.$$

Rearranging finishes the proof. □

Note that when $j = n$ the third inequality of Lemma 2.2 gives

$$|x-\alpha_i| \leq \left(\frac{|P(x)|}{|a_n|}\right)^{\frac{1}{n}}.$$

This will be of particular use later on.

Another very important lemma for the work to come is Lemma 2.4 below. From this point on we will simply use H to denote the height of the polynomial P. Before a proof of this important lemma can be given we must first state the maximum modulus principle from complex analysis which is necessary for the proof.

Theorem 2.3 (**Maximum Modulus Principle**).
Suppose that f is analytic and nonconstant on a closed region R which is bounded by a simple closed curve C. Then the modulus $|f(z)|$ attains its maximum on C.

Lemma 2.4.
Let $P(x)$ be a polynomial of degree n and height H, with non-zero roots $\alpha_1, \alpha_2, ..., \alpha_n$. Then for any k-tuple of distinct roots $\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_k}$, $1 \leq i_1 < i_2 < ... < i_k \leq n$, $k \leq n$,

$$|\alpha_{i_1}\alpha_{i_2}...\alpha_{i_k}| < c(n)\frac{H}{|a_n|}.$$

where c(n) is a positive constant depending only on n.

Proof.
Let

$$P(x) = |a_n|\prod_{i=1}^{n}(x-\alpha_i) = a_nx^n + ... + a_0.$$

Note that there is no issue in listing the roots $\alpha_1, ..., \alpha_n$ so that

$$0 < |\alpha_1| \leq |\alpha_2| \leq ... \leq |\alpha_{n_1}| \leq \frac{1}{2} < |\alpha_{n_1+1}| \leq ... \leq |\alpha_{n_2}| \leq 1 < |\alpha_{n_2+1}| \leq ... \leq |\alpha_n|.$$

Now notice that

$$|a_n||x-\alpha_{n_1+1}|....|x-\alpha_n| = \frac{|P(x)|}{|x-\alpha_1|...|x-\alpha_{n_1}|}$$

$$\implies |a_n| \max_{|x|=1} \prod_{i=n_1+1}^{n} (|x-\alpha_i|) \leq \max_{|x|=1} |P(x)| \prod_{i=1}^{n_1} \max_{|x|=1}(|x-\alpha_i|^{-1}).$$

Thus, by the maximum modulus principle,

$$|a_n||\alpha_{n_1+1}....\alpha_n| \leq (n+1)2^{n_1}H,$$

since

$$\max_{|x|=1} |P(x)| \leq (n+1)H \quad \text{and} \quad |x-\alpha_i| > \frac{1}{2} \quad \forall i \in \{1, ..., n_1\}.$$

Rearranging gives,

$$|\alpha_{n_2+1}....\alpha_n| \leq \frac{(n+1)2^{n_1}H}{|a_n||\alpha_{n_1+1}...\alpha_{n_2}|} \leq (n+1)2^{n_2}\frac{H}{|a_n|}.$$

Now, let $\alpha_{i_1}, ..., \alpha_{i_k}$ be some subset of roots of P. Certainly,

$$|\alpha_{i_1}...\alpha_{i_k}| \leq |\alpha_{n_2+1}...\alpha_n|,$$

since all roots α_i, with $i \leq n_2$, have modulus at most 1. By the above, it follows that

$$|\alpha_{i_1}...\alpha_{i_k}| \leq (n+1)2^{n_2}\frac{H}{|a_n|} \leq (n+1)2^{n}\frac{H}{|a_n|},$$

so the lemma holds with $c(n) = (n+1)2^n$. □

Lemma 2.5 ([23]).
Let $P(x) = a_n x^n + ... + a_1 x + a_0$ be an integer polynomial of height H^μ with roots $\alpha_1, ..., \alpha_n$. Furthermore assume $|a_n| = H^\gamma$, for $0 \leq \gamma \leq \mu$, then there exist a set of roots $\alpha_{i_1}, ..., \alpha_{i_k}$, with

$$|\alpha_{i_1}...\alpha_{i_k}| > c(n)H^{\mu-\gamma}.$$

In the proof of Lemma 2.5 below the definition of an elementary symmetric polynomial is used and so it is defined now. The elementary symmetric polynomials in n variables $\alpha_1, ..., \alpha_n$, denoted $e_k(\alpha_1, ..., \alpha_n)$ for $k = 0, 1, ..., n$, are defined by

$$\begin{aligned}
e_0(\alpha_1, \alpha_2, \ldots, \alpha_n) &= 1, \\
e_1(\alpha_1, \alpha_2, \ldots, \alpha_n) &= \sum_{1\leq j\leq n} \alpha_j, \\
e_2(\alpha_1, \alpha_2, \ldots, \alpha_n) &= \sum_{1\leq j<k\leq n} \alpha_j\alpha_k,
\end{aligned}$$

$$e_3(\alpha_1, \alpha_2, \ldots, \alpha_n) = \sum_{1\le j<k<l\le n} \alpha_j\alpha_k\alpha_l,$$

$$\vdots$$

$$e_n(\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha_1\alpha_2\cdots\alpha_n.$$

In particular, for $k \geq 0$ define

$$e_k(\alpha_1, \ldots, \alpha_n) = \sum_{1\le j_1<j_2<\cdots<j_k\le n} \alpha_{j_1}\cdots\alpha_{j_k}.$$

The elementary symmetric polynomials appear when a linear factorization of a monic polynomial is expanded:

$$\prod_{j=1}^{n}(x-\alpha_j) = x^n - e_1(\alpha_1,...,\alpha_n)x^{n-1} + e_2(\alpha_1,...,\alpha_n)x^{n-2} + ... + (-1)^n e_n(\alpha_1,...,\alpha_n)$$

where the values $\alpha_1, ..., \alpha_n$ are the roots of the monic polynomial and, up to a sign, the elementary symmetric polynomials represent the coefficents. In particular under the usual notation $a_i = e_{n-i}(\alpha_1, ..., \alpha_n)$ for $i = 0, ..., n$.

Proof.

If there exists at least one root α_j such that $|\alpha_j| > H^{\mu-\gamma}$ the proof is complete and so it is assumed that the roots of $P(x)$ are such that

$$|\alpha_1| \leq ... \leq |\alpha_n| \leq H^{\mu-\gamma}.$$

Let $|a_z| = \max_{i=0,\ldots,n}\{|a_i|\}$ then $|a_z| = H^{\mu}$. It can be shown, by use of the elementary symmetric polynomials, that for $0 \leq z \leq n-1$

$$\frac{a_z}{a_n} = (-1)^z \sum_{1\le i_1<i_2<\ldots<i_z\le n} (\alpha_{i_1}...\alpha_{i_z}).$$

Let the maximum of the terms in the sum be denoted by $\alpha_{j_1}...\alpha_{j_z}$ then

$$H^{\mu-\gamma} = \frac{|a_z|}{|a_n|} \leq c(n)|\alpha_{j_1}...\alpha_{j_z}|.$$

□

The following result of Mahler [74] gives a lower bound for the distance between two distinct roots of a polynomial.

Lemma 2.6.
Let $P \in P_n(Q)$. Then

$$|\alpha_i - \alpha_j| \gg Q^{-n+1}$$

for all roots α_i, α_j of P, $\alpha_i \neq \alpha_j$.

Another very useful concept is that of the resultant of two polynomials. This will be key in the proofs of Chapter 3 and Chapter 4. The **resultant** of two polynomials $P(x)$ and $T(x)$, denoted $R(P,T)$, is a polynomial expression of their coefficients. In particular, the resultant of the two polynomials

$$P(x) = a_n x^n + ... + a_1 x + a_0 \quad \text{and} \quad T(x) = b_m x^m + ... + b_1 x + b_0$$

with $a_n, b_m \neq 0$ is defined by

$$R(P,T) = a_n^m b_m^n \prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} (\alpha_i(P) - \alpha_j(T))$$

where $P(\alpha_i(P)) = 0$ for $i \in \{1, ..., n\}$ and $T(\alpha_j(T)) = 0$ for $j \in \{1, ..., m\}$. From the definition it is clear that $R(P,T) = 0$ if and only if $P(x)$ and $T(x)$ have a common root.

An alternative but equally useful definition of the resultant is the following. The resultant $R(P,T)$ is the determinant of the $(m+n) \times (m+n)$ Sylvester matrix $Syl(P,T)$ given by

$$\begin{pmatrix} a_n & a_{n-1} & a_{n-2} & \dots & 0 & 0 & 0 \\ 0 & a_n & a_{n-1} & \dots & 0 & 0 & 0 \\ 0 & 0 & a_n & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_1 & a_0 & 0 \\ 0 & 0 & 0 & \dots & a_2 & a_1 & a_0 \\ b_m & b_{m-1} & b_{m-2} & \dots & 0 & 0 & 0 \\ 0 & b_m & b_{m-1} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b_1 & b_0 & 0 \\ 0 & 0 & 0 & \dots & b_2 & b_1 & b_0 \end{pmatrix}.$$

It is evident from this definition of the resultant that if $P(x), T(x) \in \mathbb{Z}[x]$ then $R(P,T) \in \mathbb{Z}$. Furthermore this implies that if $P(x), T(x) \in \mathbb{Z}[x]$ and have no common roots then $|R(P,T)| \geq 1$. This is a key fact that will be made use of in the proofs in Chapter 3 and Chapter 4.

As already mentioned [Chapter 1, Lemma 1.57] in 1983 Bernik [23] produced the following result which has become a key tool in simplifying counting problems.

Lemma 1.57.
Let δ, η, μ be positive real numbers, let $s > 1$ be an integer and let $H_0(\delta, s)$ be a sufficiently large real number. Furthermore, let $P(x), T(x) \in \mathbb{Z}[x]$ be polynomials of degree s without common roots such that $\max(H(P), H(T)) = H^{\mu}$ where $H > H_0(\delta, s)$. Assume that the interval $I \subset (-s, s) \subset \mathbb{R}$ with $|I| = H^{-\eta}$. If there exists $\tau > 0$ such that for all $x \in I$

$$\max(|P(x)|, |T(x)|) < H^{-\tau},$$

then

$$\tau + \mu + 2\max(\tau + \mu - \eta, 0) < 2\mu s + \delta.$$

Note that the polynomials P and T being of the same degree s is not some inherent necessity for the proof. If instead P was of degree s_1 and T was of degree s_2 then the concluding inequality would become

$$\tau + \mu + 2\max(\tau + \mu - \eta, 0) < \mu(s_1 + s_2) + \delta.$$

The proof of Lemma 1.57 will not be included as it will be implicit from the generalised version of the lemma in Chapter 3. For now the only point that will be made is that the lemma refers to one interval, I, and one value, τ, which is defined to be positive. Bernik's result was extended to two intervals and two positive values in 1987 by Pereverzeva [78]. The proof remains almost identical to that of Bernik's Lemma with only one key difference. Instead of considering the points inside the two intervals I_1 and I_2 separately, the set of points $(x_1, x_2) \in I_1 \times I_2$ is considered. Furthermore one requires that this rectangle stays away from the $y = x$ line. In particular, moving forward the sets Π_n will be considered which are defined by

$$\Pi_n = I_1 \times I_2 \times I_3 \times ... \times I_n \subset \mathbb{R}^n,$$

with

$$\Pi_n \cap \{(x_1, x_2, x_3, ..., x_n) \in \mathbb{R}^n : |x_i - x_j| < \epsilon_0, 1 \leq i < j \leq n\} = \emptyset.$$

This strip of small measure is excluded so that the set of points $(x_1, ..., x_n) \in \mathbb{R}^n$ which are well approximated by points of the form $(\alpha, ..., \alpha) \in \mathbb{R}^n$ are not considered.

Now, with this new notation in mind, Pereverzeva's result can be stated.

Theorem 2.7.
Let $\delta, \mu, \eta_1, \eta_2$ all be positive real numbers and let $Q_0(\delta, s)$ be a sufficiently large real number. Furthermore, let $P(x), T(x) \in \mathbb{Z}[x]$ be polynomials of degree $s > 2$ without common roots such that $\max(H(P), H(T)) < Q^{\mu}$, where $Q > Q_0(\delta, s)$. Assume that the intervals $I_r \subset (-s, s) \subset \mathbb{R}$ with $|I_r| = Q^{-\eta_r}$ for $r = 1, 2$. If there exists $\tau_r > 0, r = 1, 2$ such that for all $(x_1, x_2) \in \Pi_2 = I_1 \times I_2$,

$$\max(|P(x_r)|, |T(x_r)|) < Q^{-\tau_r}, \quad r = 1, 2,$$

then

$$\sum_{r=1}^{2}(\tau_r + \mu + 2\max(\tau_r + \mu - \eta_r, 0)) < 2\mu s + \delta.$$

Pereverzeva's result can be extended easily to allow for three intervals and three positive values. This is generally how the lemma is quoted in literature however, as far as can be seen, no proof has ever been given for this and the reader is always just referred back to Pereverzeva's paper.

It can be noticed that in all the versions of Bernik's Lemma that have been considered so far the values, τ_i, have only ever been considered to be positive. To consider what happens if one or more of the values are negative is the aim of Chapter 3. It was the necessity for the allowance of these negative values in certain proofs of [31] that inspired this investigation into the allowance of negative values.

Before discussing our results however one last piece of notation is given. Let $P(x), T(x) \in \mathbb{Z}[x]$ be of degrees n_1 and n_2 respectively with $3 < n_1, n_2 \leq n$. Furthermore let $\alpha_1, \alpha_2, ..., \alpha_{n_1}$ be the roots of $P(x)$ and let $\beta_1, \beta_2, ..., \beta_{n_2}$ be the roots of $T(x)$. Define the intervals

$$\begin{aligned} \nu_i^r(P) &:= I_r \cap \mathcal{S}_{\mathcal{P}}(\alpha_i), \quad i = 1, ..., n_1, \ r = 1, 2, 3, \\ \nu_j^r(T) &:= I_r \cap \mathcal{S}_{\mathcal{T}}(\beta_j), \quad j = 1, ..., n_2, \ r = 1, 2, 3. \end{aligned} \tag{2.2}$$

Although it is possible that for some i and j, $\nu_i^r(P) = \emptyset, \nu_j^r(T) = \emptyset$, the following lemma guarantees that the sets are not empty for all i and j.

Lemma 2.8.
There exist at least one pair i and j, such that

$$|\nu_i^r(P)| \geq \frac{|I_r|}{n_1} \text{ and } |\nu_j^r(T)| \geq \frac{|I_r|}{n_2}$$

for each $r = 1, 2, 3$.

Proof.

Assume that for all $i = 1, ..., n_1$

$$|\nu_i^r(P)| < \frac{|I_r|}{n_1}$$

and note that since $\nu_i^r(P) = I_r \cap \mathcal{S}(\alpha_i)$,

$$\bigcup_{i=1}^{n_1} \nu_i^r(P) = I_r$$

and so

$$\left|\bigcup_{i=1}^{n_1} \nu_i^r(P)\right| = |I_r| < \frac{|\bigcup_{i=1}^{n_1} I_r|}{n_1} \leq \sum_{i=1}^{n_1} \frac{|I_r|}{n_1} = \frac{n_1|I_r|}{n_1} < |I_r|$$

which is a contradiction. Similarly for $|\nu_j^r(T)|$. □

We will denote one such pair of roots for which $\nu_i^r(P) \neq \emptyset, \nu_j^r(T) \neq \emptyset$ by α_1^r and β_1^r for each $r = 1, 2, 3$. Using this new notation, we move onto Chapter 3 and our first main result.

Chapter 3

Main Result 1

3.1 Introduction

In this chapter an extension to Bernik's Lemma, Lemma 1.57, is provided which removes, for the first time, the restriction on the size of the polynomials and allows for some of the values, τ_i, $i = 1, 2, 3$, to be negative. In particular, this allows $|P(x)|$ to be very large. The following lemma is the subject of a forthcoming paper [32].

Lemma 3.1.
Let $\delta, \mu, \eta_r \in \mathbb{R}^+$ for $r = 1, 2, 3$ and let $H_0(\delta, n)$ be a sufficiently large real number. Furthermore, let $P(x), T(x) \in \mathbb{Z}[x]$ be polynomials without common roots of degree n_1 and n_2 respectively with $3 \leq n_1, n_2 \leq n$ such that $\max(H(P), H(T)) = H^\mu$, where $H > H_0(\delta, n)$. Assume that the intervals $I_r \subset \mathbb{R}$ with $|I_r| = H^{-\eta_r}$ for $r = 1, 2, 3$. If there exist $\tau_1 > 0$ and $\tau_2, \tau_3 \in \mathbb{R}$ such that for all $(x_1, x_2, x_3) \in \Pi_3 \cap \mathcal{S}_P(\alpha_1^1, \alpha_1^2, \alpha_1^3) \cap \mathcal{S}_Q(\beta_1^1, \beta_1^2, \beta_1^3)$ with

$$\alpha_1^r \neq \alpha_1^{r'} \quad \textit{and} \quad \beta_1^r \neq \beta_1^{r'}, \quad \textit{for} \;\; 1 \leq r < r' \leq 3 \tag{3.1}$$

the inequality

$$\max(|P(x_r)|, |T(x_r)|) < H^{-\tau_r}, \quad 1 \leq r \leq 3,$$

holds, then

$$\sum_{r=1}^{3} (\tau_r + \mu + 2\max(\tau_r + \mu - \eta_r, 0)) < (n_1 + n_2)\mu + \delta.$$

The necessity of introducing the condition (3.1) is discussed in Remark 3.1 in Section 3.3. Furthermore it will become evident from the proof of Lemma 3.1 that there is nothing special about choosing to state the lemma for three values. In fact, it will be clear that the proof can be adapted for any k values with $2 \leq k \leq n$ provided that (3.1) holds; see Remark 3.2 in Section 3.3.

3.2 Proof Of Lemma 3.1

Proof.
Let $\alpha_1, \alpha_2, ..., \alpha_{n_1}$ be the roots of the polynomial $P(x)$ and $\beta_1, \beta_2, ..., \beta_{n_2}$ be the roots of the polynomial $T(x)$. Here n_1 and n_2 are the degrees of the polynomials $P(x)$ and $T(x)$, with $n_1 \leq n, n_2 \leq n$. We arrange the roots in increasing order according to the value of $\text{Re}(\alpha_i)$. If α_i is a complex root we order so that the roots for which $\text{Im}(\alpha_i) > 0$ appear first. Define $\nu_i^r(P)$ and $\nu_j^r(T)$ as in (2.2). Then, again, using Lemma 2.8 it can be shown that there exists at least one pair of roots for which $\nu_i^r(P) \neq \emptyset, \nu_j^r(T) \neq \emptyset$. One such pair of these root will be denoted by α_1^r and β_1^r for each $r = 1, 2, 3$. In particular, to ensure $\nu_i^r(P), \nu_j^r(T) \neq \emptyset$, from this point only the intervals

$$\nu^r(P) := I_r \cap \mathcal{S}(\alpha_1^r), \quad r = 1, 2, 3, \tag{3.2}$$

$$\nu^r(T) := I_r \cap \mathcal{S}(\beta_1^r), \quad r = 1, 2, 3. \tag{3.3}$$

will be considered.

Throughout the proof, it will be necessary to consider differences of the form $|x_r - \alpha_i^r|$ for some $x_r \in \nu^r(P)$. From this point on $x_r \in \nu^r(P)$ will be chosen so that $|x_r - \alpha_i^r| > \frac{1}{4}|\nu^r(P)|$. Similarly, when dealing with the roots of $T(x)$ choose $x_r \in \nu^r(T)$ such that $|x_r - \beta_i^r| > \frac{1}{4}|\nu^r(T)|$.

From the conditions of the lemma, moving forward it is assumed that

$$\alpha_1^r \neq \alpha_1^{r'} \text{ and } \beta_1^r \neq \beta_1^{r'}, \quad 1 \leq r < r' \leq 3.$$

Note if $\tau_1, \tau_2, \tau_3 > 0$ this assumption is not necessary as it holds automatically; see Remark 3.1 in Section 3.3.

Choose $\epsilon_0 > 0$ so that for $1 \leq r < r' \leq 3$, the following inequality holds:

$$\min(|\alpha_1^r - \alpha_1^{r'}|, |\beta_1^r - \beta_1^{r'}|) > \epsilon_0.$$

It is clear that such an ϵ_0 exists by (3.1). The roots of the polynomials $P(x)$ and $T(x)$ are then ordered in one of three ways depending on their distances from α_1^r

and β_1^r as follows. Define $a_r, b_r \in \mathbb{Z}$ such that for $r = 1, 2, 3$,

$$\begin{aligned}
|\alpha_1^r - \alpha_2^r| &\leq |\alpha_1^r - \alpha_3^r| \leq ... \leq |\alpha_1^r - \alpha_{a_r}^r| \\
&\leq \frac{\epsilon_0}{2} \leq |\alpha_1^r - \alpha_{a_r+1}^r| \leq ... \leq |\alpha_1^r - \alpha_{n_1}^r|, \\
|\beta_1^r - \beta_2^r| &\leq |\beta_1^r - \beta_3^r| \leq ... \leq |\beta_1^r - \beta_{b_r}^r| \\
&\leq \frac{\epsilon_0}{2} \leq |\beta_1^r - \beta_{b_r+1}^r| \leq ... \leq |\beta_1^r - \beta_{n_2}^r|.
\end{aligned}$$

Define the real numbers ρ_i^r, λ_j^r such that

$$\begin{aligned}
|\alpha_1^r - \alpha_i^r| &= H^{-\rho_i^r}, \quad i = 2, ..., n_1, \\
|\beta_1^r - \beta_j^r| &= H^{-\lambda_j^r}, \quad j = 2, ..., n_2.
\end{aligned}$$

Furthermore define

$$\begin{aligned}
l_i^r &= \rho_i^r + ... + \rho_{a_r}^r, \quad i = 2, ..., a_r, \\
\tilde{l}_i^r &= \rho_i^r + ... + \rho_{n_1}^r, \quad i = a_r + 1, ..., n_1, \\
m_j^r &= \lambda_j^r + ... + \lambda_{b_r}^r, \quad j = 2, ..., b_r, \\
\tilde{m}_j^r &= \lambda_j^r + ... + \lambda_{n_2}^r, \quad j = b_r + 1, ..., n_2.
\end{aligned}$$

For the polynomial $P(x) = a_{n_1}x^{n_1} + ... + a_1 x + a_0$ suppose that $|a_{n_1}| = H^{\gamma_1}$, $0 \leq \gamma_1 \leq \mu$.

Choose $s \in (0, \infty)$ such that $I_1, I_2, I_3 \subset (-s, s)$, i.e. s is chosen so that $H^{-\eta_r} \leq 2s$ for $r = 1, 2, 3$. For $x_1 \in \nu^1(P)$ the third inequality of Lemma 2.2 gives

$$\begin{aligned}
|x_1 - \alpha_1^1| &\leq \left(2^{n-n}|P(x_1)||P'(\alpha_1^1)|^{-1} \prod_{k=2}^{n} |\alpha_1^1 - \alpha_k^1|\right)^{\frac{1}{n}} \\
&= \left(\frac{|P(x_1)|}{|a_{n_1}| \prod_{k=2}^{n} |\alpha_1^1 - \alpha_k^1|} \prod_{k=2}^{n} |\alpha_1^1 - \alpha_k^1|\right)^{\frac{1}{n}} \\
&= (H^{-\tau_1}.H^{-\gamma_1})^{\frac{1}{n}} \leq H^{-\frac{\tau_1}{n}}
\end{aligned}$$

and so, since $\tau_1 > 0$, this is small and thus forces that α_1^1 is very close to the interval I_1. In particular there exists a small constant $\zeta \in \mathbb{R}^+$ such that $|\alpha_1^1| \leq s + \zeta$. Now recall α_1^2 is the closest root to $x_2 \in I_2$ and $\alpha_1^2 \neq \alpha_1^1$ so it must be that $|\alpha_1^2| < 2s + \zeta$, otherwise α_1^1 would be the closest root to x_2 . Similarly $|\alpha_1^3| < 2s + \zeta$.

By Lemma 2.5 there exists a set of roots $\alpha_{i_1}^1, ..., \alpha_{i_k}^1$ such that

$$|\alpha_{i_1}^1 ... \alpha_{i_k}^1| > c(n_1)H^{\mu - \gamma_1}. \tag{3.4}$$

From the set $\alpha_{i_1}^1, ..., \alpha_{i_k}^1$ remove all roots of modulus less than $2s + \max\{\frac{\epsilon_0}{2}, \zeta\}$. Then, (3.4) remains true by replacing $c(n_1)$ with some new constant $c_1(n_1)$. Moving forward fix the set of roots $\alpha_{i_1}^1, ..., \alpha_{i_k}^1$ to have modulus greater than $2s+\max\{\frac{\epsilon_0}{2}, \zeta\}$. Recalling that by assumption $|P(x_r)| < H^{-\tau_r}$ for $r = 1, 2, 3$, the second inequality of Lemma 2.2 gives

$$|x_1 - \alpha_1^1| \ll \left|\frac{P(x_1)}{P'(\alpha_1^1)}\right| = \frac{|P(x_1)|}{|a_{n_1}| \prod_{i=2}^{n_1} |\alpha_1^1 - \alpha_i^1|} < H^{-\tau_1 - \gamma_1 + l_1^1 + \tilde{l}_1^1}.$$

By the definition of α_1^1 none of the roots $\alpha_2^1, ..., \alpha_{a_1}^1$ can be included in the set $\alpha_{i_1}^1, ..., \alpha_{i_k}^1$. Furthermore since $|\alpha_1^1| \leq s + \zeta$,

$$|\alpha_{i_j}^1 - \alpha_1^1| \geq ||\alpha_{i_j}^1| - |\alpha_1^1|| \geq \frac{1}{2}|\alpha_{i_j}^1|, \quad j = 1, ..., k.$$

Therefore by Lemma 2.5

$$|\alpha_1^1 - \alpha_{a_1+1}^1| ... |\alpha_1^1 - \alpha_{n_1}^1| = H^{-\tilde{l}_1^1} \gg H^{\mu - \gamma_1}$$

and

$$|x_1 - \alpha_1^1| \ll H^{-\tau_1 - \mu + l_1^1}. \tag{3.5}$$

Similarly it can be shown that

$$|x_2 - \alpha_1^2| \ll H^{-\tau_2 - \mu + l_1^2}.$$

and

$$|x_3 - \alpha_1^3| \ll H^{-\tau_3 - \mu + l_1^3}.$$

More concisely, for $r = 1, 2, 3$,

$$|x_r - \alpha_1^r| \ll H^{-\tau_r - \mu + l_1^r}. \tag{3.6}$$

A similar argument is made for the polynomial

$$T(x) = b_{n_2} x^{n_2} + ... + b_1 x + b_0,$$

where it is taken that $|b_{n_2}| = H^{\gamma_2}$, $0 \leq \gamma_2 \leq \mu$. Thus the following inequality, analogous to (3.6), is obtained

$$|x_r - \beta_1^r| \ll H^{-\tau_r - \mu + m_1^r}. \tag{3.7}$$

Arguing as in the preparation for inequality (3.6) with the use of the third inequality of Lemma 2.2 and with $x_r \in \nu^r(P)$

$$|x_r - \alpha_1^r| \ll \min_{1 \le j \le a_r} H^{-\frac{\tau_r + \mu - l_j^r}{j}}. \tag{3.8}$$

Similarly for $x_r \in \nu^r(T)$

$$|x_r - \beta_1^r| \ll \min_{1 \le j \le b_r} H^{-\frac{\tau_r + \mu - m_j^r}{j}}. \tag{3.9}$$

For each $r = 1, 2, 3$ let the minimum on the right hand side of (3.8) be achieved at $j = j_r^\alpha$ and the minimum on the right hand side of (3.9) be achieved at $j = j_r^\beta$. From the definition of j_r^α for any i, $1 \le i \le a_r$,

$$H^{-\frac{\tau_r + \mu - l_{j_r^\alpha}^r}{j_r^\alpha}} \le H^{-\frac{\tau_r + \mu - l_i^r}{i}}. \tag{3.10}$$

This gives the inequality

$$i(\tau_r + \mu - l_{j_r^\alpha}^r) \ge j_r^\alpha(\tau_r + \mu - l_i^r). \tag{3.11}$$

Using (3.11) we claim the following.

Lemma 3.2.
When $2 \le i < j_r^\alpha$

$$\rho_i^r \ge \frac{\tau_r + \mu - l_{j_r^\alpha}^r}{j_r^\alpha} \tag{3.12}$$

and when $j_r^\alpha \le i \le a_r$

$$\rho_i^r \le \frac{\tau_r + \mu - l_i^r}{i}. \tag{3.13}$$

Proof.
Consider $2 \le i < j_r^\alpha$. Then, since $i < j_r^\alpha$,

$$\begin{aligned} l_i^r &= \rho_{i+1}^r + ... + \rho_{j_r^\alpha}^r + \rho_{j_r^\alpha+1}^r + ... + \rho_{a_r}^r \\ &= \rho_{i+1}^r + ... + \rho_{j_r^\alpha}^r + l_{j_r^\alpha}^r. \end{aligned}$$

But, by definition,

$$\rho_{i+1}^r \ge \rho_{i+2}^r \ge ... \ge \rho_{j_r^\alpha}^r$$

and so

$$l_i^r \le (j_r^\alpha - i)\rho_{i+1}^r + l_{j_r^\alpha}^r$$

$$\implies -l_i^r \geq -(j_r^\alpha - i)\rho_{i+1}^r - l_{j_r^\alpha}^r.$$

Thus using (3.11)

$$\begin{aligned}
&i(\tau_r + \mu - l_{j_r^\alpha}^r) \geq j_r^\alpha(\tau_r + \mu - l_i^r) \geq j_r^\alpha(\tau_r + \mu - (j_r^\alpha - i)\rho_{i+1}^r - l_{j_r^\alpha}^r)\\
\implies &i(\tau_r + \mu - l_{j_r^\alpha}^r) \geq j_r^\alpha(\tau_r + \mu) - j_r^\alpha(j_r^\alpha - i)\rho_{i+1}^r - j_r^\alpha(l_{j_r^\alpha}^r)\\
\implies &j_r^\alpha(j_r^\alpha - i)\rho_{i+1}^r \geq (j_r^\alpha - i)(\tau_r + \mu) - (j_r^\alpha - i)l_{j_r^\alpha}^r\\
\implies &\qquad \rho_{i+1}^r \geq \frac{\tau_r + \mu - l_{j_r^\alpha}^r}{j_r^\alpha}.
\end{aligned}$$

The argument for $j_r^\alpha \leq i \leq a_r$ is almost identical. □

From the definition of j_r^β for the polynomial $T(x)$ the following inequalities analogous to (3.12) and (3.13) are obtained,

$$\begin{aligned}
\lambda_j^r &\geq \frac{\tau_r + \mu - m_{j_r^\beta}^r}{j_r^\beta}, \quad j = 2, ..., j_r^\beta - 1, \qquad (3.14)\\
\lambda_j^r &\leq \frac{\tau_r + \mu - m_j^r}{j}, \quad j = j_r^\beta, ..., b_r.
\end{aligned}$$

By the way the interval $\nu^r(P)$ was defined for $r = 1, 2, 3$ Lemma 2.8 gives that $|\nu^r(P)| > c(n)H^{-\eta_r}$. Furthermore, recall that $x_r \in \nu_1^r(P)$ was chosen so that $|x_r - \alpha_1^r| > \frac{1}{4}|\nu^r(P)|$. Thus, by (3.8),

$$H^{-\frac{\tau_r + \mu - l_j^r}{j}} \geq H^{-\eta_r}.$$

Rearranging gives

$$\eta_r \geq \frac{\tau_r + \mu - l_j^r}{j}, \quad j = 1, ..., a_r. \qquad (3.15)$$

For the polynomial $T(x_r)$, the following analogous inequality to (3.15) can be obtained:

$$\eta_r \geq \frac{\tau_r + \mu - m_j^r}{j}, \quad j = 1, ..., b_r. \qquad (3.16)$$

From (3.10), for $j_r^\alpha \leq i \leq a_r$, inequality (3.13) becomes

$$\rho_i^r \leq \frac{\tau_r + \mu - l_{j_r^\alpha}^r}{j_r^\alpha}. \qquad (3.17)$$

The following analogous inequality to (3.17) can be obtained for the polynomial

$T(x_r)$ for $j_r^\beta \le j \le b_r$

$$\lambda_j^r \le \frac{\tau_r + \mu - m^r_{j_r^\beta}}{j_r^\beta}.$$

Moving forward it will be assumed without loss of generality that

$$\frac{\tau_r + \mu - m^r_{j_r^\beta}}{j_r^\beta} \ge \frac{\tau_r + \mu - l^r_{j_r^\alpha}}{j_r^\alpha}. \tag{3.18}$$

Next one proceeds to the evaluation of the differences between the roots of the polynomials $P(x_r)$ and $T(x_r)$. First an estimate for the difference

$$|\alpha_1^r - \beta_1^r| \le |\alpha_1^r - x_r'| + |x_r' - x_r''| + |x_r'' - \beta_1^r|$$

is obtained, where $x_r' \in \nu^r(P) \subset I_r$, $x_r'' \in \nu^r(T) \subset I_r$, so $|x_r' - x_r''| \le H^{-\eta_r}$. The differences $|x_r' - \alpha_1^r|$ and $|x_r'' - \beta_1^r|$ were calculated in (3.8) and (3.9) and so

$$|\alpha_1^r - \beta_1^r| \ll H^{-\frac{\tau_r+\mu-l^r_{j_r^\alpha}}{j_r^\alpha}} + H^{-\eta_r} + H^{-\frac{\tau_r+\mu-l^r_{j_r^\beta}}{j_r^\beta}}. \tag{3.19}$$

Using inequalities (3.15), (3.16), (3.18) and (3.19) gives

$$|\alpha_1^r - \beta_1^r| \ll H^{-\frac{\tau_r+\mu-l^r_{j_r^\alpha}}{j_r^\alpha}}. \tag{3.20}$$

From inequalities (3.14), (3.18) and (3.20) it follows that

$$\begin{aligned} |\alpha_1^r - \beta_j^r| &\le |\alpha_1^r - \beta_1^r| + |\beta_1^r - \beta_j^r| \\ &\le H^{-\frac{\tau_r+\mu-l^r_{j_r^\alpha}}{j_r^\alpha}} + H^{-\lambda_j^r} \\ &\le H^{-\frac{\tau_r+\mu-l^r_{j_r^\alpha}}{j_r^\alpha}} + H^{-\frac{\tau_r+\mu-m^r_{j_r^\beta}}{j_r^\beta}} \\ &\ll H^{-\frac{\tau_r+\mu-l^r_{j_r^\alpha}}{j_r^\alpha}} \end{aligned} \tag{3.21}$$

for $j = 1, ..., j_r^\beta$. Now from (3.21) and inequality (3.12) for all $1 \le i \le j_r^\alpha$ and $1 \le j \le j_r^\beta$

$$\begin{aligned} |\alpha_i^r - \beta_j^r| &\le |\alpha_i^r - \alpha_1^r| + |\alpha_1^r - \beta_j^r| \\ &\le H^{-\rho_i^r} + H^{-\frac{\tau_r+\mu-l^r_{j_r^\alpha}}{j_r^\alpha}} \\ &\le H^{-\frac{\tau_r+\mu-l^r_{j_r^\alpha}}{j_r^\alpha}} + H^{-\frac{\tau_r+\mu-l^r_{j_r^\alpha}}{j_r^\alpha}} \end{aligned}$$

$$\ll H^{-\frac{\tau_r+\mu-l^r_{j^\alpha_r}}{j^\alpha_r}}. \tag{3.22}$$

Finally, from inequality (3.22), it can be concluded that

$$\prod_{1\le i\le j^\alpha_r}\prod_{1\le j\le j^\beta_r}|\alpha^r_i-\beta^r_j| \ll H^{-j^\beta_r(\tau_r+\mu-l^r_{j^\alpha_r})}. \tag{3.23}$$

On the other hand, inequality (3.17) with $j^\alpha_r < i \le a_r$ and $1 \le j \le j^\beta_r$ gives

$$\begin{aligned}
|\alpha^r_i-\beta^r_j| &\ll |\alpha^r_i-\alpha^r_1|+|\alpha^r_1-\beta^r_1|+|\beta^r_1-\beta^r_j| \\
&\ll H^{-\rho^r_i}+H^{-\frac{\tau_r+\mu-l^r_{j^\alpha_r}}{j^\alpha_r}}+H^{-\lambda^r_j} \\
&\ll H^{-\rho^r_i}.
\end{aligned}$$

Hence

$$\prod_{j^\alpha_r<i\le a_r}\prod_{1\le j\le j^\beta_r}|\alpha^r_i-\beta^r_j| \ll H^{-j^\beta_r l^r_{j^\alpha_r}}. \tag{3.24}$$

From (3.23) and (3.24) it follows that

$$\prod_{1\le i\le a_r}\prod_{1\le j\le j^\beta_r}|\alpha^r_i-\beta^r_j| \ll H^{-j^\beta_r(\tau_r+\mu)}. \tag{3.25}$$

Since, by assumption, the polynomials $P(x)$ and $T(x)$ have no common roots we know, from our discussion in Chapter 2, that $|R(P,T)| \ge 1$. Using this and (3.25) the following inequality is obtained.

$$\begin{aligned}
1 &\le |R(P,T)| \\
&= |a_{n_1}|^{n_2}|b_{n_2}|^{n_1}\prod_{\substack{1\le i\le n_1\\ 1\le j\le n_2}}|\alpha_i-\beta_j| \\
&\le H^{\gamma_1 n_2+\gamma_2 n_1}\prod_{\substack{1\le i\le a_1\\ 1\le j\le j^\beta_1}}|\alpha^1_i-\beta^1_j|\prod_{\substack{1\le i\le a_2\\ 1\le j\le j^\beta_2}}|\alpha^2_i-\beta^2_j|\prod_{\substack{1\le i\le a_3\\ 1\le j\le j^\beta_3}}|\alpha^3_i-\beta^3_j| \\
&\quad\times\prod_{\mathcal{R}}|\alpha^r_i-\beta^r_j| \\
&\ll H^{\gamma_1 n_2+\gamma_2 n_1}\prod_{\substack{1\le i\le a_1\\ 1\le j\le j^\beta_1}}H^{-j^\beta_1(\tau_1+\mu)}\prod_{\substack{1\le i\le a_2\\ 1\le j\le j^\beta_2}}H^{-j^\beta_2(\tau_2+\mu)}\prod_{\substack{1\le i\le a_3\\ 1\le j\le j^\beta_3}}H^{-j^\beta_3(\tau_3+\mu)} \\
&\quad\times\prod_{\mathcal{R}}|\alpha^r_i-\beta^r_j|
\end{aligned} \tag{3.26}$$

where the set $\mathcal{R}$ is defined by

$$\mathcal{R} := \{(i,j) \in \{1,...,n_1\} \times \{1,...,n_2\} : |\alpha_1^{r'} - \alpha_i^{r''}| > \frac{\epsilon_0}{2} \text{ and } |\beta_1^{r'} - \beta_j^{r''}| > \frac{\epsilon_0}{2}$$
$$\text{for each } r' = 1,2,3 \text{ and } r'' \in \{1,2,3\}\}.$$

It is possible that $\mathcal{R} = \emptyset$ in which case recall that by definition $\gamma_1, \gamma_2 \leq \mu$ and so

$$\gamma_1 n_2 + \gamma_2 n_1 \leq \mu(n_1 + n_2).$$

If, however, $\mathcal{R} \neq \emptyset$, then using Lemma 2.4 the product

$$\prod_{\mathcal{R}} |\alpha_i^r - \beta_j^r|$$

is bounded by $H^{n_2(\mu-\gamma_1)+n_1(\mu-\gamma_2)+\delta}$. In either case inequality (3.26) can be rewritten as

$$1 \ll H^{(n_1+n_2)\mu+\delta-j_1^\beta(\tau_1+\mu)-j_2^\beta(\tau_2+\mu)-j_3^\beta(\tau_3+\mu)}.$$

Rearranging results in the inequality

$$j_1^\beta(\tau_1 + \mu) + j_2^\beta(\tau_2 + \mu) + j_3^\beta(\tau_3 + \mu) \leq (n_1 + n_2)\mu + \delta. \tag{3.27}$$

In the case of $j_1^\beta, j_2^\beta, j_3^\beta \geq 3$, (3.27) gives

$$(n_1+n_2)\mu \geq 3(\tau_1+\mu)+3(\tau_2+\mu)+3(\tau_3+\mu) \geq \sum_{r=1}^{3}(\tau_r+\mu+2\max(\tau_r+\mu-\eta_r,0))$$

which clearly proves Lemma 3.1. Thus, only the cases in which at least one of j_1^β, j_2^β or j_3^β is less than three need to be considered. To do this two arguments which depend on whether $j_r^\beta = 2$ or $j_r^\beta = 1$ will be used. Both arguments are now presented.

First consider when $j_1^\beta = 2$ and $j_2^\beta, j_3^\beta \geq 3$. If

$$\lambda_3^1 \geq \frac{\tau_1 + \mu - m^1_{j_1^\beta}}{j_1^\beta}, \tag{3.28}$$

then using (3.18) and (3.22) with $1 \leq i \leq j_1^\alpha$

$$|\alpha_i^1 - \beta_3^1| < |\alpha_i^1 - \beta_1^1| + |\beta_1^1 - \beta_3^1|$$

$$\ll H^{-\frac{\tau_1+\mu-l^1_{j_1^\alpha}}{j_1^\alpha}} + H^{-\lambda^1_3}$$
$$\ll H^{-\frac{\tau_1+\mu-l^1_{j_1^\alpha}}{j_1^\alpha}}$$

and so

$$\prod_{1\le i\le j_1^\alpha} |\alpha^1_i - \beta^1_3| \ll H^{-(\tau_1+\mu-l^1_{j_1^\alpha})}. \tag{3.29}$$

When $j_1^\alpha < i \le a_1$, by (3.20) and (3.28),

$$\begin{aligned}|\alpha^1_i - \beta^1_3| &< |\alpha^1_i - \alpha^1_1| + |\alpha^1_1 - \beta^1_1| + |\beta^1_1 - \beta^1_3| \\ &\ll H^{-\rho^1_i} + H^{-\frac{\tau_1+\mu-l^1_{j_1^\alpha}}{j_1^\alpha}} + H^{-\lambda^1_3} \\ &\ll H^{-\rho^1_i}.\end{aligned}$$

Thus

$$\prod_{j_1^\alpha < i\le a_1} |\alpha^1_i - \beta^1_3| \ll H^{-l^1_{j_1^\alpha}}. \tag{3.30}$$

Now from (3.29) and (3.30)

$$\prod_{1\le i\le a_1} |\alpha^1_i - \beta^1_3| \ll H^{-(\tau_1+\mu)}. \tag{3.31}$$

Using (3.25) and (3.31)

$$\begin{aligned}1 &\le |R(P,T))| \\ &\le H^{\gamma_1 n_2+\gamma_2 n_1} \prod_{\substack{1\le i\le a_1\\ 1\le j\le j_1^\beta}} |\alpha^1_i - \beta^1_j| \prod_{\substack{1\le i\le a_2\\ 1\le j\le j_2^\beta}} |\alpha^2_i - \beta^2_j| \prod_{\substack{1\le i\le a_3\\ 1\le j\le j_3^\beta}} |\alpha^3_i - \beta^3_j| \\ &\times \prod_{\mathcal{R}} |\alpha^r_i - \beta^r_j| \\ &= H^{\gamma_1 n_2+\gamma_2 n_1} \prod_{\substack{1\le i\le a_1\\ 1\le j\le j_1^\beta=2}} |\alpha^1_i - \beta^1_j| \prod_{1\le i\le a_1} |\alpha^1_i - \beta^1_3| \\ &\times \prod_{\substack{1\le i\le a_2\\ 1\le j\le j_2^\beta}} |\alpha^2_i - \beta^2_j| \prod_{\substack{1\le i\le a_3\\ 1\le j\le j_3^\beta}} |\alpha^3_i - \beta^3_j| \prod_{\mathcal{R}} |\alpha^r_i - \beta^r_j| \\ &\ll H^{(n_1+n_2)\mu+\delta-2(\tau_1+\mu)-(\tau_1+\mu)-j_2^\beta(\tau_2+\mu)-j_3^\beta(\tau_3+\mu)}\end{aligned}$$

which proves Lemma 3.1 for $j_2^\beta, j_3^\beta \ge 3$.

If (3.28) does not hold , i.e.

$$\lambda_3^1 < \frac{\tau_1 + \mu - m^1_{j_1^\beta}}{j_1^\beta}, \tag{3.32}$$

then, by (3.20) and (3.32), for any $j \geq 3$

$$|\alpha_1^1 - \beta_j^1| \leq |\alpha_1^1 - \beta_1^1| + |\beta_1^1 - \beta_j^1| \ll H^{-\frac{\tau_1+\mu-l^1_{j_1^\alpha}}{j_1^\alpha}} + H^{-\lambda_j^1} \ll H^{-\lambda_j^1}$$

and so

$$\prod_{3\leq j\leq b_1} |\alpha_1^1 - \beta_j^1| \ll H^{-m_2^1}. \tag{3.33}$$

Inequalities (3.25) and (3.33) together give

$$\prod_{1\leq i\leq n_1} \prod_{1\leq j\leq n_2} |\alpha_i^r - \beta_j^r| \ll H^{-2(\tau_1+\mu)-m_2^1-j_2^\beta(\tau_2+\mu)-j_3^\beta(\tau_3+\mu)+(n_1+n_2)\mu+\delta}. \tag{3.34}$$

From (3.16) $m_2^r \geq \tau_r + \mu - 2\eta$; therefore, the exponent in (3.34) can be replaced by

$$-(\tau_1+\mu) - 2(\tau_1+\mu-\eta) - j_2^\beta(\tau_2+\mu) - j_3^\beta(\tau_3+\mu) + (n_1+n_2)\mu + \delta,$$

which again leads to the proof of Lemma 3.1. The case in which $j_1^\beta = 2$ and $j_2^\beta, j_3^\beta \geq 3$ is now complete.

Now consider the case when $j_1^\beta = 1$ and $j_2^\beta, j_3^\beta \geq 3$. Assume

$$\lambda_2^1 > \frac{(\tau_1 + \mu - m^1_{j_1^\beta})}{j_1^\beta},$$
$$\lambda_3^1 > \frac{(\tau_1 + \mu - m^1_{j_1^\beta})}{j_1^\beta}. \tag{3.35}$$

Then, as in the case of inequality (3.31), one obtains

$$\prod_{1\leq i\leq a_1} \prod_{2\leq j\leq 3} |\alpha_i^1 - \beta_j^1| \ll H^{-2(\tau_1+\mu)}.$$

This together with (3.25) gives Lemma 3.1.

Next assume

$$\lambda_2^1 > \frac{(\tau_1 + \mu - m^1_{j_1^\beta})}{j_1^\beta} \qquad \text{and}$$

$$\lambda_3^1 \leq \frac{(\tau_1 + \mu - m^1_{j_1^\beta})}{j_1^\beta}. \tag{3.36}$$

In the same way (3.28) gave (3.31) the first inequality of (3.36) gives

$$\prod_{1 \leq i \leq a_1} |\alpha_i^1 - \beta_2^1| < c(s) H^{-(\tau_1+\mu)}. \tag{3.37}$$

The second inequality of (3.36), as has already been shown, leads to the inequality (3.33). Then (3.33) and (3.37) prove Lemma 3.1.

Finally assume

$$\begin{aligned} \lambda_2^1 &\leq \frac{(\tau_1 + \mu - m^1_{j_1^\beta})}{j_1^\beta}, \\ \lambda_3^1 &\leq \frac{(\tau_1 + \mu - m^1_{j_1^\beta})}{j_1^\beta}. \end{aligned} \tag{3.38}$$

In the same way (3.32) gave (3.33) the second inequality of (3.38) gives

$$\prod_{2 \leq j \leq b_1} |\alpha_1^1 - \beta_j^1| \ll H^{-m_1^1}. \tag{3.39}$$

Suppose furthermore

$$\rho_2^1 < \lambda_2^1$$

then when $2 \leq i \leq a_1$

$$\begin{aligned} |\alpha_i^1 - \beta_2^1| &\leq |\alpha_i^1 - \alpha_1^1| + |\alpha_1^1 - \beta_1^1| + |\beta_1^1 - \beta_2^1| \\ &\ll H^{-\rho_i^1} + H^{-\lambda_2^1} + H^{-\frac{\tau_1+\mu-l^1_{j_1^\alpha}}{j_1^\alpha}} \ll H^{-\rho_i^1}. \end{aligned}$$

Hence

$$\prod_{2 \leq i \leq a_1} |\alpha_i^1 - \beta_2^1| \ll H^{-l_1^1}. \tag{3.40}$$

If on the other hand

$$\rho_2^1 \geq \lambda_2^1$$

then for $2 \leq j \leq b_1$

$$\begin{aligned} |\alpha_2^1 - \beta_j^1| &\leq |\alpha_2^1 - \alpha_1^1| + |\alpha_1^1 - \beta_1^1| + |\beta_1^1 - \beta_j^1| \\ &\ll H^{-\rho_2^1} + H^{-\frac{\tau_1+\mu-l^1_{j_1^\alpha}}{j_1^\alpha}} + H^{-\lambda_j^1} \ll H^{-\lambda_j^1}. \end{aligned}$$

Hence

$$\prod_{2\le j\le b_1} |\alpha_2^1 - \beta_j^1| \ll H^{-m_1^1}. \tag{3.41}$$

Using (3.39), (3.40) and (3.41) in (3.25) gives

$$\begin{aligned}
1 &\le |R(P,T)| \\
&\le H^{\gamma_1 n_2+\gamma_2 n_1} \prod_{\substack{1\le i\le a_1\\ 1\le j\le j_1^\beta=1}} |\alpha_i^1-\beta_j^1| \prod_{2\le j\le b_1} |\alpha_1^1-\beta_j^1| \\
&\times \max\left\{\prod_{2\le i\le a_1} |\alpha_i^1-\beta_2^1|, \prod_{2\le j\le b_1} |\alpha_2^1-\beta_j^1|\right\} \\
&\times \prod_{\substack{1\le i\le a_2\\ 1\le j\le j_2^\beta}} |\alpha_i^2-\beta_j^2| \prod_{\substack{1\le i\le a_3\\ 1\le j\le j_3^\beta}} |\alpha_i^3-\beta_j^3| \prod_{\mathcal{R}} |\alpha_i^r-\beta_j^r| \\
&\ll H^{-(\tau_1+\mu)-m_1^1-\min(l_1^1,m_1^1)-j_2^\beta(\tau_2+\mu)-j_3^\beta(\tau_3+\mu)+\mu(n_1+n_2)+\delta}.
\end{aligned} \tag{3.42}$$

Now from (3.15) and (3.16)

$$\min(l_1^1, m_1^1) \ge \tau_1+\mu-\eta_1.$$

Thus inequality (3.42) can be rewritten as

$$\prod_{1\le i\le n_1}\prod_{1\le j\le n_2} |\alpha_i^r-\beta_j^r| \ll H^{-\tau_1-\mu-2\max(\tau_1+\mu-\eta_1,0)-j_2^\beta(\tau_2+\mu)-j_3^\beta(\tau_3+\mu)+\mu(n_1+n_2)+\delta},$$

which clearly proves Lemma 3.1 for $j_2^\beta, j_3^\beta \ge 3$. The case when $j_1^\beta = 1$ and $j_2^\beta, j_3^\beta \ge 3$ has now be completely considered.

It should be clear that there was nothing special about fixing $j_r^\beta \ge 3$ for $r = 2,3$ and choosing $j_1^\beta \le 2$ above. In particular the arguments above can be done for $j_r^\beta \le 2$ for any $r = 1,2,3$ by interchanging τ_1, l_1^1 and m_1^1 with τ_r, l_1^r and m_1^r. Choosing to show the argument for $r = 1$ was simply for demonstration purposes. Since these arguments are interchangeable in r the remaining cases (such as when $j_1^\beta = j_2^\beta = 1$ and $j_3^\beta = 3$ etc.) will just be combinations of the arguments above.

□

3.3 Remarks

***Remark* 3.1.**

One of the key differences between Bernik's original Lemma, Lemma 1.57, which allowed for only positive values, τ_i, and Lemma 3.1 is the additional assumption that

$$\alpha_1^r \neq \alpha_1^{r'} \text{ and } \beta_1^r \neq \beta_1^{r'}$$

for $1 \leq r < r' \leq 3$. Note if $\tau_1, \tau_2, \tau_3 > 0$ this assumption in not needed in Lemma 3.1 since by the definition of Π_3 for $x \in \mathcal{S}(\alpha_1^r)$ and $y \in \mathcal{S}(\alpha_1^{r'})$, $1 \leq r < r' \leq 3$, $|x-y| > \epsilon_0$ which in turn, by Lemma 2.2, implies that $|\alpha_1^r - \alpha_1^{r'}| > \frac{\epsilon_0}{2}$. Similarly, $|\beta_1^r - \beta_1^{r'}| > \frac{\epsilon_0}{2}$ for $1 \leq r < r' \leq 3$.

So provided $\tau_1, \tau_2, \tau_3 > 0$ the roots α_1^1, α_1^2 and α_1^3 are all different and the roots β_1^1, β_1^2 and β_1^3 are all different. However, if even one of the values, τ_r, is negative for example if $\tau_1, \tau_2 > 0$ and $\tau_3 < 0$ problems occur. It can no longer be guaranteed that $|\alpha_1^1 - \alpha_1^3| > \frac{\epsilon_0}{2}$ or that $|\alpha_1^2 - \alpha_1^3| > \frac{\epsilon_0}{2}$. In particular, it is now possible to have $\alpha_1^r = \alpha_1^{r'}$ for $r \neq r'$ and so Lemma 3.1 would no longer being true. For example suppose that $\alpha_1^1 = \alpha_1^3 \neq \alpha_1^2$. Then the expression for the resultant $R(P,T)$ would now only contain two products, in particular it would now have the form

$$|R(P,T)| \leq H^{\gamma_1 n_2 + \gamma_2 n_1} \prod_{\substack{1 \leq i \leq a_1 \\ 1 \leq j \leq j_1^\beta}} |\alpha_i^1 - \beta_j^1| \prod_{\substack{1 \leq i \leq a_2 \\ 1 \leq j \leq j_2^\beta}} |\alpha_1^2 - \beta_j^2|.$$

The expression obtained for the inequality in Lemma 3.1 would now only be an expression of the values τ_1 and τ_2, i.e.

$$\sum_{r=1}^{2} (\tau_r + \mu + 2\max(\tau_r + \mu - \eta_r, 0)) < (n_1 + n_2)\mu + \delta.$$

Thus it must be assumed from the outset that

$$\alpha_1^r \neq \alpha_1^{r'} \text{ and } \beta_1^r \neq \beta_1^{r'}$$

for $1 \leq r < r' \leq 3$.

***Remark* 3.2.**

It should be evident that the proof of Lemma 3.1 can easily be adapted for any k variables, with $2 \leq k \leq n$, provided

$$\alpha_1^r \neq \alpha_1^{r'} \quad \text{and} \quad \beta_1^r \neq \beta_1^{r'}, \quad \text{for} \quad 1 \leq r < r' \leq k.$$

When dealing with k variables all arguments will be identical to those made in the proof for $k = 3$ with the two exceptions. Firstly, the indices will now be of the form $j_4^\beta + ... + j_k^\beta$. Secondly, increasing the number of variables will of course increase the number of cases to be considered. However, one can again begin by considering the cases in which $j_1^\beta < 3$ while $j_r^\beta \geq 3$ for $r = 2, 3, ..., k$ and then simply work down through all other cases (such as $j_1^\beta, j_2^\beta < 3$ while $j_r^\beta \geq 3$ for $r = 3, ..., k$) in an identical fashion to above. The arguments will not change; however, the number of different products required to be bounded will certainly become larger depending on how many of the $j_r^\beta < 3$.

For an application of Lemma 3.1 see the proof of Lemma 5.5.

Chapter 4

Main Result 2

4.1 Introduction

As mentioned previously, in this chapter a second extension to Bernik's Lemma, Lemma 1.57, is presented which leads to a very useful proposition. This proposition is used to prove a particular case in a result on determining the number of polynomials with bounded discriminants. In particular, in Section 4.4, the upper bound is obtained to a result of Beresnevich, Bernik and Götze in [19] for a very specific case. Previous to this, only a lower bound had been found; see Section 4.4. The following results are the subject of a forthcoming paper [32].

To begin, some of the notation used in Chapter 3 needs to be adjusted and some necessary results need to be presented. Considering the roots of the polynomial P define $\alpha_1(P), ..., \alpha_{n_1}(P)$ to be the real roots and $\beta_1(P), ..., \beta_{\frac{n_2}{2}}(P)$ to be the non-real roots located in the upper-half plane. The set of non-real roots located in the lower-half plane will be denoted $\beta_{\frac{n_2}{2}+1}(P), ..., \beta_{n_2}(P)$. It is clear that each non-real root in the lower-half plane is just the complex conjugate of one of the non-real roots in the upper-half plane. With this in mind the non-real roots in the lower-half plane are labelled so that $\overline{\beta}_i(P) = \beta_{\frac{n_2}{2}+i}(P)$ for $i = 1, ..., \frac{n_2}{2}$. Clearly $n_1+n_2 = n$. Furthermore, define the roots of P in $\mathbb{Q}_p$ as $\gamma_1(P), ..., \gamma_{n_3}(P)$ with $n_3 \leq n$. The roots of a second polynomial T are similarly spilt into the sets $\{\alpha_i(T)\}$, $\{\beta_j(T)\}$ and $\{\gamma_k(T)\}$ where $1 \leq i \leq m_1$, $1 \leq j \leq m_2$ with $m_1 + m_2 = n$ and $1 \leq k \leq m_3 \leq n$.

For each real root, $\alpha_i(P)$, the set $\mathcal{S}^1(\alpha_i(P))$ will be defined by

$$\mathcal{S}^1(\alpha_i(P)) = \{x \in \mathbb{R} : |x - \alpha_i(P)| = \min_{l=1,...,n_1} |x - \alpha_l(P)|\}.$$

In a similar fashion analogues for $\mathbb{C}$ and $\mathbb{Q}_p$ are defined in the obvious way as follows

$$\mathcal{S}^2(\beta_j(P))=\{z \in \mathbb{C}^+ : |z-\beta_j(P)| = \min_{l=1,\dots,\frac{n_2}{2}} |z-\beta_l(P)|\},$$
$$\mathcal{S}^2(\overline{\beta}_j(P))=\{z \in \mathbb{C}^- : |z-\overline{\beta}_j(P)| = \min_{l=1,\dots,\frac{n_2}{2}} |z-\overline{\beta}_l(P)|\},$$
$$\mathcal{S}^3(\gamma_k(P))=\{\omega \in \mathbb{Q}_p : |\omega-\gamma_k(P)|_p = \min_{l=1,\dots,n_3} |\omega-\gamma_l(P)|_p\},$$

where $|.|_p$ denotes the p-adic norm, $\mathbb{C}^+ := \{z \in \mathbb{C} : \mathrm{Im}(z) \geq 0\}$ and $\mathbb{C}^- := \{z \in \mathbb{C} : \mathrm{Im}(z) \leq 0\}$. Clearly if $z \in \mathbb{C}^+$ and $z \in \mathcal{S}^2(\beta_j(P))$ for some $j \in 1,...,\frac{n_2}{2}$ then $\overline{z} \in \mathcal{S}^2(\overline{\beta}_j(P))$.

In Lemma 4.1, z will be taken in a disk in $\mathbb{C}$ and differences of the form $|z-\beta_j|$ for some $z \in \mathbb{C}$ and some non-real root β_j will be estimated. If it can be assumed that $z \in \mathcal{S}^2(\beta_j(P))$ (i.e. z is such that $\mathrm{Im}(z) > 0$) and that $j \in \{1,...,\frac{n_2}{2}\}$ then, as will be seen, estimating $|z-\beta_j|$ will be simplified greatly. This is the reason for considering the sets $\mathcal{S}^2(\beta_j(P))$ and $\mathcal{S}^2(\overline{\beta}_j(P))$ separately. Furthermore, by symmetry, estimating $|z-\beta_j|$ will give an estimate for $|\overline{z}-\overline{\beta}_j|$. Differences of the form $|z-\overline{\beta}_j|$ and $|\overline{z}-\beta_j|$ will also have to be considered but unfortunately nothing is known about these and so Lemma 2.4 will be used to estimate these, just as was done in Chapter 3.
The following notation will also be used:

$$\mathcal{S}(\alpha_i(P), \beta_j(P), \gamma_k(P)) = \mathcal{S}^1(\alpha_i(P)) \times \mathcal{S}^2(\beta_j(P)) \times \mathcal{S}^3(\gamma_k(P)).$$

Let $I \subset \mathbb{R}$ be an interval, $C \subset \mathbb{C}$ be a disk and $K \subset \mathbb{Q}_p$ be a cylinder, and define the parallelepiped $\Omega = I \times C \times K \subset \mathbb{R} \times \mathbb{C} \times \mathbb{Q}_p$. Fix $\delta_1 > 0$. Any complex number z lying in C with $|\mathrm{Im}(z)| < \delta_1$ will be excluded. As long as δ_1 is an arbitrary small number, this can be done without loss of generality. Later in this chapter will appear inequalities of the form $|z-\beta| < Q^{-v}$; from this, with the condition $|\mathrm{Im}(z)| \geq \delta_1$, one obtains $|\mathrm{Im}(\beta)| \geq \frac{\delta_1}{2}$ i.e. $\beta \notin \mathbb{R}$. In particular, this implies that $|\beta_i - \overline{\beta}_j| > \delta_1$, and for any real root α_i, $|\alpha_i - \beta_j| = |\alpha_i - \overline{\beta}_j| > \delta_1$.

Let $\mu_P(A)$ be the Haar measure of a measurable set $A \subset \mathbb{Q}_p$. We are now in a position to present our second main result.

Lemma 4.1.

Let $\delta, \eta_r \in \mathbb{R}^+$, $r = 1, 2, 3$ and let $Q_0(\delta, s)$ be a sufficiently large real number. Furthermore, let $P, T \in \mathcal{P}_n(Q)$ be polynomials of degree n without common roots such that $\max(H(P), H(T)) = Q$*, where $Q > Q_0(\delta, n)$. Take $\Omega = I \times C \times K \subset \mathbb{R} \times \mathbb{C} \times \mathbb{Q}_p$ with $\mu(I) = Q^{-\eta_1}$, diam$(C) = Q^{-\eta_2}$, $\mu_p(K) = Q^{-\eta_3}$. If there exist $\tau_1, \tau_2, \tau_3 > 0$ such* *that for all* $(x, z, \omega) \in \Omega \cap \mathcal{S}(\alpha_1(P), \beta_1(P), \gamma_1(P)) \cap \mathcal{S}(\alpha_1(T), \beta_1(T), \gamma_1(T))$

$$\max(|P(x)|, |T(x)|) < Q^{-\tau_1},$$
$$\max(|P(z)|, |T(z)|) < Q^{-\tau_2},$$
$$\max(|P(\omega)|_p, |T(\omega)|_p) < Q^{-\tau_3},$$

and for $J(x) = P(x)$ *or* $T(x)$

$$\tau_1 + 1 \geq q_1(J) + \rho_2(J),$$
$$\tau_2 + 1 \geq r_1(J) + \lambda_2(J),$$
$$\tau_3 \geq s_1(J) + \sigma_2(J),$$

then

$$\tau_1 + 2\tau_2 + \tau_3 + 3 + 2\Bigg(\sum_{j=1}^{n_1-1} \max(\tau_1 + 1 - j\eta_1, 0) + 2\sum_{j=1}^{\frac{n_2}{2}-1} \max(\tau_2 + 1 - j\eta_2, 0) + \sum_{j=1}^{n_3-1} \max(\tau_3 - j\eta_3, 0)\Bigg) < 2n + \delta.$$

Suppose $(x, z, \omega) \in \mathcal{S}(\alpha_1(P), \beta_1(P), \gamma_1(P)) \cap \mathcal{S}(\alpha_1(T), \beta_1(T), \gamma_1(T))$. The other roots are then ordered according to their distance from $\alpha_1(J)$, $\beta_1(J)$ and $\gamma_1(J)$, where $J(x) = P(x)$ or $T(x)$, as follows:

$$|\alpha_1(J) - \alpha_2(J)| \leq |\alpha_1(J) - \alpha_3(J)| \leq ... \leq |\alpha_1(J) - \alpha_{n_1}(J)|,$$
$$|\beta_1(J) - \beta_2(J)| \leq |\beta_1(J) - \beta_3(J)| \leq ... \leq |\beta_1(J) - \beta_{\frac{n_2}{2}}(J)|,$$
$$|\gamma_1(J) - \gamma_2(J)|_p \leq |\gamma_1(J) - \gamma_3(J)|_p \leq ... \leq |\gamma_1(J) - \gamma_{n_3}(J)|_p.$$

Note that the set of differences $|\beta_1(P) - \beta_i(P)|$ are only taken up as far as $i = \frac{n_2}{2}$. This is because $|\beta_1(P) - \beta_i(P)| = |\overline{\beta}_1(P) - \overline{\beta}_i(P)|$ and so only $i \leq \frac{n_2}{2}$ need to be considered since any resulting calculations, as already discussed, will be the same for $\frac{n_2}{2} < i \leq n_2$. This is a common technique see, for example, [28].

Define the real numbers $\rho_i(J), \lambda_i(J), \sigma_i(J)$ such that

$$|\alpha_1(J) - \alpha_i(J)| = Q^{-\rho_i(J)}, \quad i = 2, ..., n_1,$$
$$|\beta_1(J) - \beta_i(J)| = Q^{-\lambda_i(J)}, \quad i = 2, ..., \frac{n_2}{2},$$
$$|\gamma_1(J) - \gamma_i(J)|_p = Q^{-\sigma_i(J)}, \quad i = 2, ..., n_3.$$

Furthermore define

$$
\begin{aligned}
q_i(J) &= \rho_{i+1}(J) + ... + \rho_{n_1}(J), \quad i = 1, ..., n_1 - 1,\\
r_i(J) &= \lambda_{i+1}(J) + ... + \lambda_{\frac{n_2}{2}}(J), \quad i = 1, ..., \frac{n_2}{2} - 1,\\
s_i(J) &= \sigma_{i+1}(J) + ... + \sigma_{n_3}(J), \quad i = 1, ..., n_3 - 1.
\end{aligned}
$$

Under this notation it will be shown that Lemma 4.1 is, in fact, a simple corollary to the following proposition.

Proposition 4.2.

Let $\delta, \eta_r \in \mathbb{R}^+$, $r = 1, 2, 3$ and let $Q_0(\delta, n)$ be a sufficiently large real number. Furthermore, let $P, T \in \mathcal{P}_n(Q)$ be polynomials of degree n without common roots such that $\max(H(P), H(T)) = Q$, where $Q > Q_0(\delta, n)$. Take $\Omega = I \times C \times K \subset \mathbb{R} \times \mathbb{C} \times \mathbb{Q}_p$ with $\mu(I) = Q^{-\eta_1}$, diam$(C) = Q^{-\eta_2}$, $\mu_p(K) = Q^{-\eta_3}$. If there exist $\tau_1, \tau_2, \tau_3 > 0$ such that for all $(x, z, \omega) \in \Omega \cap \mathcal{S}(\alpha_1(P), \beta_1(P), \gamma_1(P)) \cap \mathcal{S}(\alpha_1(T), \beta_1(T), \gamma_1(T))$

$$
\begin{aligned}
\max(|P(x)|, |T(x)|) &< Q^{-\tau_1},\\
\max(|P(z)|, |T(z)|) &< Q^{-\tau_2},\\
\max(|P(\omega)|_p, |T(\omega)|_p) &< Q^{-\tau_3},
\end{aligned}
$$

and for $J(x) = P(x)$ or $T(x)$

$$
\begin{aligned}
\tau_1 + 1 &\geq q_1(J) + \rho_2(J),\\
\tau_2 + 1 &\geq r_1(J) + \lambda_2(J),\\
\tau_3 &\geq s_1(J) + \sigma_2(J), \qquad (4.1)
\end{aligned}
$$

then

$$
\tau_1 + 2\tau_2 + \tau_3 + 3 + 2\left(\sum_{i=1}^{n_1-1} q_i(J) + 2 \sum_{j=1}^{\frac{n_2}{2}-1} r_j(J) + \sum_{k=1}^{n_3-1} s_k(J) \right) \leq 2n + \delta
$$

Although Proposition 4.2 is more powerful than Lemma 4.1 it is more difficult to use; see Section 4.4 for details.

In Lemma 3.1 no assumptions such as (4.1) were made as when only dealing with real intervals, I_r, inequalities of the form of (4.1) were shown to hold always under the assumptions of the lemma.

With regard to the complex inequality of (4.1), it is almost certain that, under the assumptions of Proposition 4.2, this holds always also since the argument should follow the real argument made in Lemma 3.1. With regard to the p-adic inequality of (4.1), it is not clear whether this will always hold, under the assumptions of Proposition 4.2. However, it is not difficult to show that there are infinitely many cases for which it does. As an example of a polynomial for which Proposition 4.2 can be applied to consider $P(z) = 2z^4 + z^3 - 2z^2 + 2z - 1$ with $I = [\frac{1}{4}, \frac{1}{2}]$, $C = \{z \in \mathbb{C} : |z - \frac{1+i}{4}| \leq \frac{1}{4}\}$ and $K \subseteq \{z \in \mathbb{Q}_{19} : |P(z)|_{19} < \frac{1}{19}\} \cup \{4\}$. One can easily check that in this case $\rho_2 = -\log_2(\sqrt{5})$, $\lambda_2 = -\log_2(\frac{\sqrt{7}}{2})$, $\sigma_2 = 0$, $\tau_1 = -\log_2(\frac{77}{128})$, $\tau_2 = -\log_2(\frac{61}{100})$ and $\tau_3 = 1$. Thus (4.1) can be seen to hold.

4.2 Additional Results

Some very useful inequalities are now presented, which are just extensions to those found in Lemma 2.2. The proofs for the real inequalities below can be found in [22] and for the complex and p-adic inequalities in [69].

Lemma 4.3.
Let $P \in P_n(Q)$. Then for $x \in \mathcal{S}^1(\alpha_1)$, $z \in \mathcal{S}^2(\beta_1)$ and $w \in \mathcal{S}^3(\gamma_1)$ the inequalities

$$\begin{aligned}
|x - \alpha_1| &\leq n\frac{|P(x)|}{|P'(x)|} && \text{for} \quad P'(u) \neq 0, \\
|z - \beta_1| &\leq n\frac{|P(z)|}{|P'(z)|} && \text{for} \quad P'(z) \neq 0, \\
|w - \gamma_1|_p &\leq n\frac{|P(w)|_p}{|P'(w)|_p} && \text{for} \quad P'(w) \neq 0, \\
|x - \alpha_1| &\leq 2^{n-1}\frac{|P(x)|}{|P'(\alpha_1)|} && \text{for} \quad P'(\alpha_1) \neq 0, \\
|z - \beta_1| &\leq 2^{n-1}\frac{|P(z)|}{|P'(\beta_1)|} && \text{for} \quad P'(\beta_1) \neq 0, \\
|w - \gamma_1|_p &\leq 2^{n-1}\frac{|P(w)|_p}{|P'(\gamma_1)|_p} && \text{for} \quad P'(\gamma_1) \neq 0,
\end{aligned}$$

hold, together with

$$|x - \alpha_1| \leq \min_{2 \leq j \leq n} \left(2^{n-j}|P(x)||P'(\alpha_1)|^{-1} \prod_{k=2}^{j} |\alpha_1 - \alpha_k| \right)^{\frac{1}{j}} \quad \text{for} \quad P'(\alpha_1) \neq 0,$$

$$|z-\beta_1| \le \min_{2\le j\le n}\left(2^{n-j}|P(z)||P'(\beta_1)|^{-1}\prod_{k=2}^{j}|\beta_1-\beta_k|\right)^{\frac{1}{j}} \quad for \quad P'(\beta_1)\neq 0,$$

$$|w-\gamma_1|_p \le \min_{2\le j\le n}\left(2^{n-j}|P(w)||P'(\gamma_1)|^{-1}\prod_{k=2}^{j}|\gamma_1-\gamma_k|_p\right)^{\frac{1}{j}} \quad for \quad P'(\gamma_1)\neq 0.$$

4.3 Proof Of Lemma 4.1

Under the assumptions of Lemma 4.1, it was shown in [23] that $q_j(P) \ge \tau_1 + 1 - j\eta_1$. In fact, this inequality was shown to be true as part of the proof in Section 3.2, see (3.15). In [28], it was shown that $r_j(P) \ge \tau_2 + 1 - j\eta_2$ and $s_j(P) \ge \tau_3 - j\eta_3$. Thus, under the assumptions of Lemma 4.1, the following system of inequalities can be taken to hold,

$$\begin{aligned} q_j(P) &\ge \tau_1 + 1 - j\eta_1,\\ r_j(P) &\ge \tau_2 + 1 - j\eta_2,\\ s_j(P) &\ge \tau_3 - j\eta_3. \end{aligned} \tag{4.2}$$

It is clear that using (4.2), Lemma 4.1 follows immediately from Proposition 4.2.

Now the proposition is proved.

Proof of proposition.

All the following calculations are analogous to those carried out in [23], [28] and Section 3.2. Let $P(x) = a_n x^n + ... + a_0$ and $T(x) = b_n x^n + ... + b_0$. Furthermore define $\mathcal{K}(\alpha_i, \beta_j) = |\alpha_i(P) - \beta_j(P)||\alpha_i(P) - \beta_j(T)||\alpha_i(T) - \beta_j(P)||\alpha_i(T) - \beta_j(T)|$ and note that since, by assumption, P and T have no common roots

$$\begin{aligned} 1 &\le |R(P,T)||R(P,T)|_p\\ &\le |a_n|^n|b_n|^n \prod_{1\le i\le j\le n_1}|\alpha_i(P)-\alpha_j(T)| \prod_{1\le i\le j\le n_2}|\beta_i(P)-\beta_j(T)|\\ &\quad\times \prod_{1\le i\le j\le n_3}|\gamma_i(P)-\gamma_j(T)|_p \times \prod_{\substack{1\le i\le n_1\\ 1\le j\le n_2}}\mathcal{K}(\alpha_i,\beta_j). \end{aligned} \tag{4.3}$$

Here the basic property of the p-adic norm that for any $a \in \mathbb{Z}$ one always has $1 \le |a||a|_p$ is being used.

As was done in Chapter 3, suppose that $|a_n| = Q^{\zeta_1}$, for $0 \le \zeta_1 \le 1$, and

$|b_n| = Q^{\zeta_2}$, for $0 \le \zeta_2 \le 1$. Furthermore recall that $\beta_{i+\frac{n_2}{2}}(P) := \overline{\beta}_i(P)$ and note

$$\begin{aligned}\prod_{1\le i\le j\le n_2} |\beta_i(P) - \beta_j(T)| &= \prod_{1\le i\le j\le \frac{n_2}{2}} |\beta_i(P) - \beta_j(T)| \prod_{1\le i\le j\le \frac{n_2}{2}} |\overline{\beta}_i(P) - \overline{\beta}_j(T)| \\ &\times \prod_{\substack{1\le i\le \frac{n_2}{2}\\ 1\le j\le \frac{n_2}{2}}} |\beta_i(P) - \overline{\beta}_j(T)| \prod_{\substack{1\le i\le \frac{n_2}{2}\\ 1\le j\le \frac{n_2}{2}}} |\overline{\beta}_i(P) - \beta_j(T)| \\ &= \Bigg(\prod_{1\le i\le j\le \frac{n_2}{2}} |\beta_i(P) - \beta_j(T)| \prod_{\substack{1\le i\le \frac{n_2}{2}\\ 1\le j\le \frac{n_2}{2}}} |\beta_i(P) - \overline{\beta}_j(T)| \Bigg)^2\end{aligned}$$

since it is clear that $|\beta_i(P)-\beta_j(T)| = |\overline{\beta}_i(P)-\overline{\beta}_j(T)|$ and $|\beta_i(P)-\overline{\beta}_j(T)| = |\overline{\beta}_i(P)-\beta_j(T)|$. Nothing is known about the distances $|\beta_i(P)-\overline{\beta}_j(T)|$ and, in fact, these could be very large. Similarly, nothing is known about the distances $|\alpha_i - \beta_j|$ which again could be very large. Using Lemma 2.4 however, the differences $|\beta_i(P) - \overline{\beta}_j(T)|$ and $|\alpha_i-\beta_j|$ can be bounded, just as was done in Chapter 3 on the set $\mathcal{R}$, see (3.26). This can be done since for each i, j, the triangle inequality gives that $|\beta_i(P) - \overline{\beta}_j(T)| \le 2\max\{|\beta_i(P)|, |\overline{\beta}_j(T)|\}$, and so for some $0 \le f_1, ..., f_{\frac{n_2}{2}}, g_1, ..., g_{\frac{n_2}{2}} \le \frac{n_2}{2}$

$$\begin{aligned}\prod_{1\le i\le j\le \frac{n_2}{2}} |\beta_i(P) - \overline{\beta}_j(T)| &\le 2^{\frac{n_2^2}{4}} |\beta_1(P)|^{f_1} ... |\beta_{\frac{n_2}{2}}(P)|^{f_{\frac{n_2}{2}}} |\overline{\beta}_1(T)|^{g_1} ... |\overline{\beta}_{\frac{n_2}{2}}(T)|^{g_{\frac{n_2}{2}}} \\ &< 2^{\frac{n_2^2}{4}} \left(\frac{H(P)}{|a_n|}\right)^{\frac{n_2}{2}} \left(\frac{H(T)}{|b_n|}\right)^{\frac{n_2}{2}} \le 2^{\frac{n_2^2}{4}} Q^{\frac{n}{2}(1-\zeta_1)+\frac{n}{2}(1-\zeta_2)} \\ &< c_1(n) Q^{\frac{n}{2}(1-\zeta_1)+\frac{n}{2}(1-\zeta_2)}\end{aligned}$$

for some constant $c_1(n) > 0$.

The same argument can be made for the differences $|\alpha_i-\beta_j|$, so that $\prod |\alpha_i-\beta_j| < c_2(n)Q^{\frac{n}{2}(1-\zeta_1)+\frac{n}{2}(1-\zeta_2)}$, for some constant $c_2(n) > 0$. Thus, for Q sufficiently large, $Q^\delta > c_1(n)c_2(n)$ and $R(P,T)$ can be rewritten as

$$\begin{aligned}1 &\le |R(P,T)||R(P,T)|_p \\ &\le |a_n|^n |b_n|^n \prod_{1\le i\le j\le n_1} |\alpha_i(P) - \alpha_j(T)| \prod_{1\le i\le j\le n_2} |\beta_i(P) - \beta_j(T)| \\ &\quad \times \prod_{1\le i\le j\le n_3} |\gamma_i(P) - \gamma_j(T)|_p \times \prod_{\substack{1\le i\le n_1\\ 1\le j\le n_2}} \mathcal{K}(\alpha_i, \beta_j) \\ &\le |a_n|^n |b_n|^n Q^{n(1-\zeta_1)+n(1-\zeta_2)+\delta} \prod_{1\le i\le j\le n_1} |\alpha_i(P) - \alpha_j(T)| \prod_{1\le i\le j\le \frac{n_2}{2}} |\beta_i(P) - \beta_j(T)|^2\end{aligned}$$

$$\times \prod_{1\le i\le j\le n_3} |\gamma_i(P)-\gamma_j(T)|_p$$
$$\le Q^{2n+\delta} \prod_{1\le i\le j\le n_1} |\alpha_i(P)-\alpha_j(T)| \prod_{1\le i\le j\le \frac{n_2}{2}} |\beta_i(P)-\beta_j(T)|^2$$
$$\times \prod_{1\le i\le j\le n_3} |\gamma_i(P)-\gamma_j(T)|_p \tag{4.4}$$

Now the proof revolves around bounding each of the products of (4.4). We will now consider the real, complex and p-adic roots one by one. When considering each, we order the roots so that

$$\begin{aligned} q_1(T) &\le q_1(P) \\ r_1(T) &\le r_1(P) \\ s_1(T) &\le s_1(P). \end{aligned} \tag{4.5}$$

First consider the differences of the real roots and note that for $x \in \mathcal{S}^1(\alpha_1(P))$ by Lemma 2.2

$$|x-\alpha_1(P)| \le 2^{n-1}\frac{|P(x)|}{|P'(\alpha_1)|} \ll Q^{-\tau_1-1+q_1(P)}.$$

Similarly for $y \in \mathcal{S}^1(\alpha_1(T))$

$$|y-\alpha_1(T)| \ll Q^{-\tau_1-1+q_1(T)}.$$

So for $x \in \mathcal{S}^1(\alpha_1(P))$ and $y \in \mathcal{S}^1(\alpha_1(T))$, using (4.2) and (4.5), it is seen that

$$\begin{aligned} |\alpha_1(P)-\alpha_1(T)| &\le |\alpha_1(P)-x| + |x-y| + |y-\alpha_1(T)| \\ &\ll Q^{-\tau_1-1+q_1(P)} + Q^{-\eta_1} + Q^{-\tau_1-1+q_1(T)} \\ &\ll Q^{-\tau_1-1+q_1(P)}. \end{aligned}$$

This gives

$$\begin{aligned} \prod_{2\le j\le n_1} |\alpha_1(P)-\alpha_j(T)| &\le \prod_{2\le j\le n_1}\Big(|\alpha_1(P)-\alpha_1(T)| + |\alpha_1(T)-\alpha_j(T)|\Big) \\ &\ll \prod_{2\le j\le n_1}\Big(Q^{-\tau_1-1+q_1(P)} + Q^{-\rho_j(T)}\Big). \end{aligned}$$

Recall that (4.1) gives $\tau_1 + 1 - q_1(P) \ge \rho_2(P)$. Using this gives

$$\prod_{2\le j\le n_1} |\alpha_1(P)-\alpha_j(T)| \ll \prod_{2\le j\le n_1} (Q^{-\rho_2(P)} + Q^{-\rho_j(T)})$$

$$\ll \prod_{2\leq j\leq n_1} Q^{\max(-\rho_2(P),-\rho_j(T))}. \tag{4.6}$$

Now consider

$$\begin{aligned}\prod_{2\leq i\leq n_1} |\alpha_i(P)-\alpha_1(T)| &\leq \prod_{2\leq i\leq n_1}\Big(|\alpha_i(P)-\alpha_1(P)|+|\alpha_1(P)-\alpha_1(T)|\Big)\\ &\ll \prod_{2\leq i\leq n_1}\Big(Q^{-\rho_i(P)}+Q^{-\tau_1-1+q_1(P)}\Big)\\ &\leq \prod_{2\leq i\leq n_1}\Big(Q^{-\rho_i(P)}+Q^{-\rho_2(P)}\Big)\\ &\ll \prod_{2\leq i\leq n_1} Q^{-\rho_i(P)} = Q^{-q_1(P)}. \end{aligned} \tag{4.7}$$

Combining (4.6) and (4.7) gives

$$\prod_{2\leq j\leq n_1}|\alpha_1(P)-\alpha_j(T)|\prod_{2\leq i\leq n_1}|\alpha_i(P)-\alpha_1(T)| \ll \prod_{2\leq j\leq n_1} Q^{\max(-\rho_2(P),-\rho_j(T))}Q^{-q_1(P)}.$$

If there exists $\phi\in\mathbb{Z}$ with $2\leq\phi\leq n_1$ such that

$$\begin{aligned} -\rho_2(P) < -\rho_j(T) \quad \forall \;\; j\in[2,\phi]\\ \text{and} \quad -\rho_2(P) \geq -\rho_j(T) \quad \forall \;\; j\in(\phi,n_1] \end{aligned}$$

then

$$\begin{aligned}\prod_{2\leq j\leq n_1} Q^{\max(-\rho_2(P),-\rho_j(T))} &= Q^{-\rho_2(T)-\ldots-\rho_\phi(T)-(n_1-\phi)\rho_2(P)}\\ &\leq Q^{-\rho_2(T)-\ldots-\rho_\phi(T)-\rho_{\phi+1}(P)-\ldots-\rho_{n_1}(P)} \leq Q^{-q_1(P)}.\end{aligned}$$

If, on the other hand, no such ϕ exists, i.e. $-\rho_2(P)\geq-\rho_j(T)$ for all $j\in[2,n_1]$ then

$$\prod_{2\leq j\leq n_1} Q^{\max(-\rho_2(P),-\rho_j(T))} = Q^{-(n_1-1)\rho_2(P)} \leq Q^{-\rho_2(P)-\rho_3(P)-\ldots-\rho_{n_1}(P)} = Q^{-q_1(P)}.$$

In either case

$$\prod_{2\leq j\leq n_1}|\alpha_1(P)-\alpha_j(T)|\prod_{2\leq i\leq n_1}|\alpha_i(P)-\alpha_1(T)| \ll Q^{-2q_1(P)}. \tag{4.8}$$

Carrying out almost identical calculations for i, some fixed element of $\{3,...,n_1\}$, while j runs from $i+1$ to n_1 and for j, some fixed element $\{3,...,n_1\}$, while i runs from $j+1$ to n_1 leads to results of the same form. For example

$$\begin{aligned}\prod_{3\leq j\leq n_1} |\alpha_2(P)-\alpha_j(T)| \leq \prod_{3\leq j\leq n_1} \Big(&|\alpha_2(P)-\alpha_1(P)| + |\alpha_1(P)-\alpha_1(T)| \\ &+ |\alpha_1(T)-\alpha_j(T)|\Big) \\ \ll \prod_{2\leq j\leq n_1} &\Big(Q^{-\rho_2(P)} + Q^{-\rho_2(P)} + Q^{-\rho_j(T)}\Big) \\ \ll \prod_{2\leq j\leq n_1} &Q^{\max(-\rho_2(P),-\rho_j(T))}.\end{aligned}$$

Similarly

$$\prod_{3\leq i\leq n_1} |\alpha_i(P)-\alpha_2(T)| \ll \prod_{2\leq i\leq n_1} Q^{\max(-\rho_i(P),-\rho_2(T))}.$$

Arguing in an identical fashion to (4.8) it is clear that

$$\prod_{3\leq i\leq n_1} |\alpha_i(P)-\alpha_2(T)| \prod_{3\leq j\leq n_1} |\alpha_2(P)-\alpha_j(T)| \ll Q^{-2q_2(P)}.$$

More generally, using the same approach,

$$\prod_{k\leq i\leq n_1} |\alpha_i(P)-\alpha_{k-1}(T)| \prod_{k\leq j\leq n_1} |\alpha_{k-1}(P)-\alpha_j(T)| \ll Q^{-2q_{k-1}(P)}.$$

The final case that needs considering is when $i=j\geq 2$. Note

$$\begin{aligned}|\alpha_i(P)-\alpha_i(T)| &\leq |\alpha_i(P)-\alpha_1(P)| + |\alpha_1(P)-\alpha_1(T)| + |\alpha_1(T)-\alpha_i(T)| \\ &\ll Q^{-\rho_i(P)} + Q^{-\rho_2(P)} + Q^{-\rho_i(T)} \ll Q^{\max(-\rho_i(P),-\rho_i(T))}. \qquad (4.9)\end{aligned}$$

So finally

$$\begin{aligned}\prod_{1\leq i\leq j\leq n_1} |\alpha_i(P)-\alpha_j(T)| = &\prod_{\substack{1\leq i\leq j\leq n_1\\ i\neq j}} |\alpha_i(P)-\alpha_j(T)| \prod_{i=1}^{n} |\alpha_i(P)-\alpha_i(T)| \\ &\ll Q^{-2(q_1(P)+q_2(P)+\ldots+q_{n_1-1}(P))} \prod_{i=1}^{n} |\alpha_i(P)-\alpha_i(T)| \\ &\ll Q^{-2(q_1(P)+q_2(P)+\ldots+q_{n_1-1}(P))} Q^{-\tau_1-1+q_1(P)} \\ &\times \prod_{i=2}^{n_1} Q^{\max(-\rho_i(P),-\rho_i(T))}\end{aligned}$$

$$\leq Q^{-2(q_1(P)+q_2(P)+\ldots+q_{n_1-1}(P))}Q^{-\tau_1-1+q_1(P)}Q^{-q_1(P)}$$
$$= Q^{-(\tau_1+1+2(q_1(P)+q_2(P)+\ldots+q_{n_1-1}(P)))}. \qquad (4.10)$$

Identical calculations are carried out when considering the differences between the complex roots. Using Lemma 2.2, along with equations (4.2) and (4.5), it is clear that

$$|\beta_1(P) - \beta_1(T)| \ll Q^{-\tau_2-1+r_1(P)}.$$

Using this and (4.1) it can be shown that

$$\prod_{2\leq k\leq i\leq \frac{n_2}{2}} |\beta_i(P) - \beta_{k-1}(T)| \prod_{2\leq k\leq j\leq \frac{n_2}{2}} |\beta_{k-1}(P) - \beta_l(T)| \ll Q^{-2r_{k-1}(P)}.$$

Furthermore, by the same method used to obtain (4.9), it can be shown that

$$|\beta_i(P) - \beta_i(T)| \ll Q^{\max(-\lambda_i(P),-\lambda_i(T))}.$$

So finally

$$\prod_{1\leq i\leq j\leq \frac{n_2}{2}} |\beta_i(P) - \beta_j(T)| \ll Q^{-(\tau_2+1+2(r_1(P)+r_2(P)+\ldots+r_{\frac{n_2}{2}-1}(P)))}.$$

Thus

$$\prod_{1\leq i\leq j\leq \frac{n_2}{2}} |\beta_i(P) - \beta_j(T)|^2 \ll Q^{-2(\tau_2+1+2(r_1(P)+\ldots+r_{\frac{n_2}{2}-1}(P)))}. \qquad (4.11)$$

Finally, the p-adic case is considered. Again almost identical calculations are carried out. Using Lemma 2.2, along with equations (4.2) and (4.5), it is clear that

$$|\gamma_1(P) - \gamma_1(T)|_p \ll Q^{-\tau_3+s_1(P)}.$$

Using this and (4.1) it can be shown that

$$\prod_{2\leq k\leq i\leq n_3} |\gamma_i(P) - \gamma_{k-1}(T)|_p \prod_{2\leq k\leq j\leq n_3} |\gamma_{k-1}(P) - \gamma_l(T)|_p \ll Q^{-2s_{k-1}(P)}.$$

Furthermore it can be shown that

$$|\gamma_i(P) - \gamma_i(T)|_p \ll Q^{\max(-\sigma_i(P),-\sigma_i(T))}.$$

So finally

$$\prod_{1\leq i\leq j\leq n_3} |\gamma_i(P)-\gamma_j(T)|_p \ll Q^{-(\tau_3+2(s_1(P)+s_2(P)+\ldots+s_{n_3}(P)))}. \tag{4.12}$$

Using (4.10), (4.11) and (4.12) in (4.4) gives

$$\begin{aligned}1 &\leq |R(P,T)||R(P,T)|_p\\ &\leq Q^{2n+\delta}Q^{-(\tau_1+1+2(q_1(P)+\ldots+q_{n_1-1}(P)))}Q^{-2(\tau_2+1+2(r_1(P)+\ldots+r_{\frac{n_2}{2}-1}(P)))}\\ &\quad\times Q^{-(\tau_3+2(s_1(P)+\ldots+s_{n_3-1}(P)))}\end{aligned}$$

so that

$$\tau_1+2\tau_2+\tau_3+3+2\left(\sum_{i=1}^{n_1-1}q_i(P)+2\sum_{j=1}^{\frac{n_2}{2}-1}r_j(P)+\sum_{k=1}^{n_3-1}s_k(P)\right)\leq 2n+\delta$$

as required. □

4.4 Examples

Let $P(x)\in\mathcal{P}_n(Q)$ with roots $\alpha_1,\ldots,\alpha_n$ then the **discriminant** of $P(x)$, denoted $D(P)$, is defined to be

$$D(P) := \frac{(-1)^{\frac{n(n-1)}{2}}R(P,P')}{a_n}.$$

Recall $R(P,P')$ denotes the resultant of the polynomial $P(x)$ and its derivative $P'(x)$, as was defined in Chapter 2. From this definition, it is not difficult to rewrite the discriminant in terms of its roots,

$$D(P) = a_n^{2n-2}\prod_{1\leq i<j\leq n}(\alpha_i-\alpha_j)^2.$$

It can also be shown that $D(P)$ is expressible as the determinant of a $(2n-1)\times(2n-1)$ Sylvester matrix; see [19] for details. This, in particular, implies that $D(P)\in\mathbb{Z}$. Define $\mathcal{P}_n^v(Q)$ for $0\leq v\leq n-1$ as follows:

$$\mathcal{P}_n^v(Q) := \{P(x)\in\mathcal{P}_n(Q) : 1\leq |D(P)| < Q^{2n-2-2v}\}.$$

Letting $\#U$ represent the cardinality of some set U, we are interested in finding bounds for $\#\mathcal{P}_n^v(Q)$. Let f, g be real valued functions, both defined on some unbounded subset of the real positive numbers, such that $g(x)$ is strictly positive for all large enough values of x. Then one writes

$$f(x) = o(g(x)) \text{ as } x \to \infty$$

(read "f(x) is little-o of g(x)") if for all $c > 0$ there exists a constant $N(c) := N$ such that

$$|f(x)| \le cg(x), \ \forall \ x \ge N.$$

For example, $1/x = o(1)$. As $g(x)$ is non-zero, or at least becomes non-zero beyond a certain point, the relation $f(x) = o(g(x))$ is equivalent to

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0.$$

In 2010, Koleda [67] obtained both upper and lower bounds for the cardinality of $\mathcal{P}_n^v(Q)$ in the case $n = 3$ and $0 \le v < 3/5$. In particular, it was shown that for $0 \le v < 3/5$ and c_1, a positive constant which depends only on n and is independent of Q,

$$\#P_3(Q, v) = c_1 Q^{4 - \frac{5}{3}v}(1 + o(1)).$$

In 2013 Koleda and Korlukova [68] showed that for $0 \le v < \frac{1}{2}$,

$$\#P_2(Q, v) = \lambda Q^{3-2v}(1 + o(1)), \ \lambda = 20(1 + \ln 2).$$

It was shown by Beresnevich, Bernik and Götze [19] in 2016 that for $0 \le v \le n - 1$,

$$\#\mathcal{P}_n^v(Q) \gg Q^{n+1-\frac{n+2}{n}v}.$$

Using Proposition 4.2 it will now be shown that the upper bound is in fact of the same order of the lower bound in a very particular case. The result is believed to hold true in general and the proof of this will be the subject of future work.

Consider Proposition 4.2 in the one-dimensional setting. Just as was done in [19], only the unit interval $[-\frac{1}{2}, \frac{1}{2}]$ will be considered since all results may be extended to any arbitrary interval in $\mathbb{R}$; see [8] for appropriate techniques. Begin by assuming that the upper bound is not of the same order as the lower bound, in particular, assume that

$$\#\mathcal{P}_n^v(Q) \gg Q^{n+1-\frac{n+2}{n}v+\epsilon}.$$

Then there must exist an interval I of size $Q^{-\rho_2(P)}$ containing a root of $P(x)$ such that

$$\#\mathcal{P}_n^v(Q, I) \gg Q^{n+1-\frac{n+2}{n}v-\rho_2(P)+\epsilon},$$

where $\mathcal{P}_n^v(Q,I)$ is the set of polynomials $P \in \mathcal{P}_n^v(Q)$ which have a root in the interval I. If not, then, $\#\mathcal{P}_n^v(Q,I) \ll Q^{n+1-\frac{n+2}{n}v-\rho_2(P)+\epsilon}$ in all $Q^{\rho_2(P)}$ subintervals I but this contradicts the assumption that $\#\mathcal{P}_n^v(Q) \gg Q^{n+1-\frac{n+2}{n}v+\epsilon}$. Note that choosing the interval to be of size $Q^{-\rho_2(P)}$ is for convenience only.

Let $m = n+1-\frac{n+2}{n}v-\rho_2(P)$. Proposition 4.2 will now be used to contradict the assumption that $\#\mathcal{P}_n^v(Q,I) \gg Q^{m+\epsilon}$ in the case that $m > 0$.

Using Taylor series, it is not difficult to show that on the interval I

$$|P(x)| \ll Q^{1-q_1(P)-\rho_2(P)}.$$

To see this recall that we can express

$$P(x) = P(\alpha_1) + P'(\alpha_1)(x-\alpha_1) + \sum_{j=2}^{n} \frac{P^{(j)}(\alpha_1)(x-\alpha_1)^j}{j!}$$

where α_1 is taken to be the root of P in the interval of size $Q^{-\rho_2(P)}$. Now an estimate is obtained for each of these terms. Clearly $P(\alpha_1) = 0$. Since $x, \alpha_1 \in I$, $|x-\alpha_1| < Q^{-\rho_2(P)}$. Now recall

$$|P'(\alpha_1)| = |a_n| \prod_{i=2}^{n} |\alpha_1 - \alpha_i| \leq Q^{1-\rho_2(P)-\ldots-\rho_n(P)} = Q^{1-q_1(P)}.$$

Thus

$$|P'(\alpha_1)||x-\alpha_1| \leq Q^{1-q_1(P)-\rho_2(P)}.$$

Using the trivial estimate $|P^{(j)}(\alpha_1)| \leq n^{j+1}Q$

$$|P^{(j)}(\alpha_1)||x-\alpha_1|^j \ll Q^{1-j\rho_2(P)}.$$

So, up to a constant, that depends on n only,

$$|P(x)| \ll Q^{1-q_1(P)-\rho_2(P)}.$$

In [19], it was shown that if

$$\rho_2(P) + 2\rho_3(P) + \ldots + (n-1)\rho_n(P) \geq v,$$

then $P(x) \in \mathcal{P}_n^v(Q)$ (see how equation 40 was obtained in [19] for details). Moving forward we will assume that $\rho_2(P) + 2\rho_3(P) + \ldots + (n-1)\rho_n(P) \geq v$.

Consider first the case in which $m \in \mathbb{N}$ and for $1 \le l \le n$ define the set

$$M(a_n, ..., a_l; Q^{1-q_1(P)-\rho_2(P)}) := \{P \in \mathcal{P}_n(Q) : |P(x)| \ll Q^{1-q_1(P)-\rho_2(P)} \text{ and } a_j(P) = a_j, j = l, ..., n\}.$$

So $M(a_n, ..., a_l; Q^{1-q_1(P)-\rho_2(P)})$ is the set of polynomials in $P_n(Q)$ with the $n-l+1$ coefficients $a_l, ..., a_n$ equal that satisfy $|P(x)| \ll Q^{1-q_1(P)-\rho_2(P)}$. Sets of the form $M(a_n, ..., a_l; Q^{1-q_1(P)-\rho_2(P)})$ will be discussed in much more detail in Chapter 5. Now, fix $P_0(x) \in M(a_n, ..., a_{m+1}; Q^{1-q_1(P)-\rho_2(P)})$. For each $P_j(x) \in M(a_n, ..., a_{m+1}; Q^{1-q_1(P)-\rho_2(P)})$ construct the polynomials

$$R_j(x) = P_j(x) - P_0(x)$$

with

$$|R_j(x)| \ll Q^{1-q_1(P)-\rho_2(P)} \text{ and } \deg(R_j(x)) = n - m = \frac{n+2}{n}v + \rho_2(P) - 1.$$

If there exist at least two $R_j(x)$ without common roots, then by Proposition 4.2 with

$$\tau_1 + 1 := \tau + 1 = q_1(P) + \rho_2(P) + \frac{\epsilon}{2}$$

one has

$$\tau_1 + 1 + 2(q_1(P) + ... + q_{n-1}(P)) < 2\deg(R_j) + \delta.$$

Suppose $\rho_2(P) = v$ and $\delta < \frac{\epsilon}{2}$. Then by the definition of m

$$1 \le m = n + 1 - 2v - \frac{2v}{n} < n + 1 - 2v,$$

i.e. $v < \frac{n}{2}$ and by Proposition 4.2

$$\begin{aligned}\tau_1 + 1 + 2\sum_{i=1}^{n-1} q_i(P) =& \frac{\epsilon}{2} + 3q_1(P) + \rho_2(P) + 2\sum_{i=2}^{n-1} q_i(P) \\ =& \frac{\epsilon}{2} + \rho_2(P) + 3(\rho_2(P) + ... + \rho_n(P)) \\ &+ 2(\rho_3(P) + ... + \rho_n(P)) + ... + 2(\rho_n(P)) \\ =& \frac{\epsilon}{2} + 4\rho_2(P) + 5\rho_3(P) + 7\rho_4(P) + ... + (2n-1)\rho_n(P) \\ >& 2\rho_2(P) + 2(\rho_2(P) + 2\rho_3(P) + ... + (n-1)\rho_n(P)) + \delta \\ >& 2\rho_2(P) + 2v + \frac{4}{n}v - 2 + \delta = 2\deg(R_j) + \delta\end{aligned}$$

since $v \leq \rho_2(P)+2\rho_3(P)+...+(n-1)\rho_n(P)$. Thus a contradiction to Proposition 4.2 is obtained.

If $m \notin \mathbb{N}$ define $\tilde{m} \in \mathbb{N}$ such that $m+1 > \tilde{m} > m$. Then construct polynomials $R_j(x) = P_j(x) - P_0(x)$ with

$$|R_j(x)| \ll Q^{1-q_1(P)-\rho_2(P)} \text{ and } \deg(R_j(x)) = n - \tilde{m} < n - m = \frac{n+2}{n}v + \rho_2(P) - 1$$

and the proof follows as it did before. So it can be seen that provided there exist at least two $R_j(x)$ without common roots and $m = n+1-\frac{n+2}{n}v - \rho_2(P) > 0$ then

$$\#\mathcal{P}_n^v(Q) \ll Q^{n+1-\frac{n+2}{n}v}.$$

4.4.1 Remarks

Remark 4.1.

By using Proposition 4.2 in the above, a contradiction was obtained. It will now be shown that, although easier to use, Lemma 4.1 is weaker than Proposition 4.2 since it does not guarantee a contradiction. To see this first recall that by Lemma 4.1

$$\tau + 1 + 2\sum_{j=1}^{n}(\tau + 1 - j\eta) < 2(\deg(R_j(x)) + \delta = 2\rho_2(P) + 2v + \frac{4}{n}v - 2 + \delta.$$

Since an interval I of size $Q^{-\rho_2(P)}$ is being considered, take $\eta = \rho_2(P)$. Then

$$\tau + 1 + 2\sum_{j=1}^{n}(\tau + 1 - j\eta) \leq \tau + 1 + 2(n)(\tau + 1 - \rho_2(P))$$

since $-\rho_2(P) \geq -2\rho_2(P) \geq ... \geq -n\rho_2(P)$. Now suppose $\tau = 8$, $v = 9 = \rho_2(P)$ and $n = 12$ then

$$\tau + 1 + 2\sum_{j=1}^{n}(\tau + 1 - j\eta) \leq 9 < 37 + \delta = 2\rho_2(P) + 2v + \frac{4}{n}v - 2 + \delta.$$

and so it is seen that Lemma 4.1 does not give the contradiction required where as, as shown above, Proposition 4.2 does.

Remark 4.2.

It is clear that there is a lot left to do in order to completely prove

$$\#\mathcal{P}_n^v(Q) \ll Q^{n+1-\frac{n+2}{n}v}.$$

In particular, the case in which there does not exist two polynomials $R_j(x)$ without common roots must be considered. Completing this proof will be the subject of future work. It would appear that we can follow a similar method of proof to that used in [31] when dealing with reducible polynomials, with some slight modifications. Also the case when $m = n + 1 - \frac{n+2}{n}v - \rho_2(P) \leq 0$ must be dealt with.

Chapter 5

The Distribution of Algebraic Conjugate Triples

5.1 Introduction

In this chapter the main results of [31] will be discussed and a brief overview of the proofs will be given. In particular, an example of how Lemma 3.1 is used in this paper will be presented. A very rough draft of the paper has been attached at the end of this thesis to assist the readers' understanding. This paper is still a work in progress and most definitely still contains typos and incomplete sections. It is, however, complete enough to fulfil its purpose of giving the ideas behind the proofs used.

In [31] the following question is considered: Given three real intervals of equal size how many polynomials $P \in P_n(Q)$ pass through all three intervals. That is to say how many polynomials have a root in all three intervals? Define a point $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ to be an **algebraic conjugate triple** if there exist $P \in Z[x]$ such that $P(\alpha_i) = 0$ for $i = 1, 2, 3$. Then, the above question is equivalent to considering the distribution of algebraic conjugate triples in cubes in $\mathbb{R}^3$. Unfortunately the results in [31] only hold for cubes which do not contain rational points with small denominator. The reason for this is made clear in the paper and will be commented on later. It is unclear whether the reason for the results not holding for cubes, containing rational points with small denominator, is due to the method of proof used or whether the results do not hold for such boxes.

The paper is set up in the following way. Fix $\epsilon_0, \epsilon_1 \in \mathbb{R}^+$ to be sufficiently small and let $\lambda \in \mathbb{R}$ be such that $0 < \lambda < \frac{1}{3}$. Choose $Q_0 \in \mathbb{R}$ large enough so that $Q_0^{-\epsilon_1} < \frac{\epsilon_0}{2}$. Throughout the paper other conditions on Q_0 are determined. Next consider the set of $Q_0^{\lambda+2\epsilon}$ rational points $\frac{p}{q} \in [0,1]$ with $q < Q_0^{\frac{\lambda}{2}+\epsilon}$. Define

A_λ to be the union of the intervals centred at these rational points such that if $|q| < \epsilon_0^{-\frac{1}{3}}$ the interval has length $2\epsilon_0$ and if $|q| \geq \epsilon_0^{-\frac{1}{3}}$ the interval has length $|q|^{-3}$. Let $I_1, I_2, I_3 \subset [0,1]$ be intervals of length $Q^{-\lambda}$. For $Q > Q_0$ define the box

$$\Pi_\lambda^3(Q) := I_1 \times I_2 \times I_3 = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \subset [\epsilon_0, 1-\epsilon_0]^3 \subset \mathbb{R}^3$$

with the conditions

$$\Pi_\lambda^3(Q) \cap \{(x_1, x_2, x_3) \in [0,1]^3 : |x_i - x_j| < \epsilon_0, 1 \leq i < j \leq 3\} = \emptyset$$

and

$$\Pi_\lambda^3(Q) \cap A_\lambda = \emptyset.$$

In [31] it is shown that the number of algebraic conjugate triples of height at most Q and degree at most n lying in $\Pi_\lambda^3(Q)$, for $0 < \lambda < \frac{1}{3}$, is at least $Q^{-3\lambda+n+1}$. In particular the following theorem is proved.

Theorem 5.1.
The cardinality of the set of algebraic conjugate triples with $P \in P_n(Q)$ lying in $\Pi_\lambda^3(Q)$ is $\gg Q^{-3\lambda+n+1}$.

In fact it can be shown that Theorem 5.1 follows from Theorem 5.2 below, the proof of which constitutes most of the paper. For $\delta_0 \in \mathbb{R}^+$ denote by $L_n(\delta_0, Q)$ the set of $(x_1, x_2, x_3) \in \Pi_\lambda^3(Q)$ for which the system

$$|P(x_i)| < Q^{-\tau_i}, \ |P'(x_i)| > \delta_0 Q, \ 1 \leq i \leq 3, \tag{5.1}$$

has a solution $P \in P_n(Q)$ with $\sum_{i=1}^{3} \tau_i = n-2$, $\tau_i > 0$. Then the following theorem is shown to hold.

Theorem 5.2.
For any real number $s \in (0,1)$, there exists a constant $\delta_0 > 0$ such that

$$\mu(L_n(\delta_0, Q)) > s\mu(\Pi_\lambda^3(Q))$$

for sufficiently large Q.

It is also shown in [31] that Theorem 5.1 cannot be improved arbitrarily. In particular it can be shown that there exist boxes $\Pi = I_1 \times I_2 \times I_3$ with $I_i = Q^{-g_i}$ and $g_1 + g_2 + g_3 > 1$ for which Theorem 5.1 will not hold, see [29] and [[31], Page 2] for explicit examples.

To see why Theorem 5.1 follows from Theorem 5.2 is not difficult. Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ be a triple of roots of the polynomial $P \in P_n(Q)$ such that $\boldsymbol{x} = (x_1, x_2, x_3) \in S(\alpha_1, \alpha_2, \alpha_3) \cap L_n(\delta_0, Q)$. Then, by Lemma 2.2,

$$|x_i - \alpha_i| \leq n\frac{|P(x_i)|}{|P'(x_i)|} \leq n\delta_0^{-1}Q^{-\tau_i-1}. \tag{5.2}$$

Let K be the maximum number of algebraic conjugate triples $\boldsymbol{\rho_l} := (\rho_{l,1}, \rho_{l,2}, \rho_{l,3})$ where $\rho_{l,i} \in \mathbb{A}_n$ for $1 \leq l \leq K$, $i = 1, 2, 3$, such that the cuboids $B(\boldsymbol{\rho_l})$ do not intersect, where

$$B(\boldsymbol{\rho_l}) := \{\boldsymbol{x} \in \Pi_\lambda^3(Q) : |x_i - \rho_{l,i}| < n\delta_0^{-1}Q^{-\tau_i-1},\ i = 1, 2, 3\}.$$

Furthermore define the cuboids

$$\tilde{B}(\boldsymbol{\rho_l}) := \{\boldsymbol{x} \in \Pi_\lambda^3(Q) : |x_i - \rho_{l,i}| < 2n\delta_0^{-1}Q^{-\tau_i-1},\ i = 1, 2, 3\}.$$

Note that if $\boldsymbol{\alpha} = \boldsymbol{\rho_l}$ for some $l \in \{1, ..., K\}$ then, by (5.2), it is clear that $\boldsymbol{x} \in \tilde{B}(\boldsymbol{\rho_l}) = \tilde{B}(\boldsymbol{\alpha})$. If $\boldsymbol{\alpha} \neq \boldsymbol{\rho_l}$ for any $1 \leq l \leq K$ then, by definition of $B(\boldsymbol{\rho_l})$, there exists $\boldsymbol{\rho_l}$ such that, for $i = 1, 2, 3$,

$$|\alpha_i - \rho_{l,i}| < n\delta_0^{-1}Q^{-\tau_i-1}.$$

Thus, by (5.2) and use of the triangle inequality,

$$|x_i - \rho_{l,i}| < 2n\delta_0^{-1}Q^{-\tau_i-1}$$

and so $\boldsymbol{x} \in \tilde{B}(\boldsymbol{\rho_l})$.

Therefore for every $\boldsymbol{x} \in L_n(\delta_0, Q)$ there exists an algebraic conjugate triple $\boldsymbol{\rho_l}$ such that $\boldsymbol{x} \in \tilde{B}(\boldsymbol{\rho_l})$, i.e.

$$L_n(\delta_0, Q) \subseteq \bigcup_{l=1}^{K} \tilde{B}(\boldsymbol{\rho_l}).$$

Using this gives the inequality

$$s\mu(\Pi_\lambda^3(Q)) < \mu(L_n(\delta_0, Q)) \leq \sum_{l=1}^{K} \mu(\tilde{B}(\boldsymbol{\rho_l})) < K(2^6 n^3 \delta_0^{-3} Q^{\sum_{i=1}^{3}(-\tau_i-1)}).$$

Hence, since $\sum_{i=1}^{3}(-\tau_i - 1) = -n - 1$,

$$K \gg Q^{n+1}\mu(\Pi_\lambda^3(Q)).$$

Throughout the proof of Theorem 5.2 it is often necessary to consider sets of polynomials satisfying conditions similar to (5.1) which might be reducible. Note that if $P \in P_n(Q)$ is reducible, i.e. $P(x) = R(x)T(x)$, and P satisfies (5.1) then

$$\prod_{i=1}^{3} |P(x_i)| < Q^{-(n-2)}.$$

Suppose that $\deg R = n_R$ and $\deg T = n_T$ with $n_R + n_T = n$ then it must be that either

$$\prod_{i=1}^{3} |R(x_i)| < Q^{-(n_R-1)}$$

or

$$\prod_{i=1}^{3} |T(x_i)| < Q^{-(n_T-1)}.$$

Now note that if $R(x)$ is linear, i.e. $R(x) = bx + a$, and $|b| < \epsilon_0^{-\frac{1}{3}}$ then by the definition of $\Pi^3_\lambda(Q)$ the distance of x_i from the rational $\frac{a}{b}$ is at least ϵ_0. In particular, for $i = 1, 2, 3$,

$$|bx_i + a| > \epsilon_0 |b|.$$

Thus it is clear that

$$|T(x_i)| = \frac{|P(x_i)|}{|R(x_i)|} < \frac{Q^{-\tau_i}}{\epsilon_0 |b|} < \frac{Q^{-\tau_i}}{\epsilon_0}.$$

In order to prove Theorem 5.2 it is first necessary to prove Theorem 5.3 below. Let $J_n(Q)$ be the set of $(x_1, x_2, x_3) \in \Pi^3_\lambda(Q)$ for which the system

$$|P(x_i)| < \frac{Q^{-\tau_i}}{\epsilon_0}, \; i = 1, 2, 3 \tag{5.3}$$

has a solution for $P \in P_n(Q)$ with $\sum_{i=1}^{3} \tau_i = n - 1$ and such that if $n = 1$ then $H(P) > \epsilon_0^{-\frac{1}{3}}$.

Theorem 5.3.
For any real number $s \in (0, 1)$

$$\mu(J_n(Q)) < s\mu(\Pi^3_\lambda(Q))$$

for Q sufficiently large.

Note that Theorem 5.3, in comparison to Theorem 5.2, allows for the range $-1 \leq \tau_i < 0$ and there is no condition on the derivative.

Several lemmas are used in the proofs of Theorem 5.2 and Theorem 5.3. Three of the most important of these have already be seen; the distance bounding lemma, Lemma 2.2, Mahler's result on the distance between the roots of polynomials, Lemma 2.6, and the generalisation of Bernik's Lemma from Chapter 3, Lemma 3.1. Apart from these there is one other main lemma that is used in the proofs of Theorem 5.2 and Theorem 5.3 which will now be discussed.

Considering the following set eases the complexity of the counting arguments used in [31]. Define

$$M(a_n, ..., a_l; \eta) := \{P \in P_n(Q) : |P(d_i)| < \eta \text{ and } a_j(P) = a_j, j = l, ..., n\}$$

where d_i denotes the centre of the interval I_i. So $M(a_n, ..., a_l; \eta)$ is the set of polynomials in $P_n(Q)$, with the $n-l+1$ coefficients $a_l, ..., a_n$ equal, that satisfy $|P(d_i)| < \eta$. It is worth noting that since $|x_i - x_j| > \epsilon_0$ then $|d_i - d_j| > \frac{\epsilon_0}{2}$.

Lemma 5.4.
Let $\eta \in \mathbb{R}^+$. Then,

$$\#M(\eta, a_n, ..., a_{n-k+1}; \eta) \ll \max(1, \eta^{n-k+1}).$$

The following proof is due to Dickinson [31].

Proof.
Suppose that $P_1, P_2, ..., P_t \in M(a_n, ..., a_{n-k+1}; \eta)$. Construct the difference polynomials $R_i = P_i - P_1$ of degree at most $n-k$ which satisfy

$$|R_i(d_j)| < 2\eta$$

for $i = 2, ..., t$. Hence

$$R_i(d_j) = \theta_i \eta$$

for $j = k, ..., n$ and $|\theta_i| < 2$. For each i this is a set of $n-k+1$ simultaneous equations with unknowns $a_l(R_i)$ for $l = 0, ..., n-k$. Given that, for $j \neq k$, $|d_j - d_k| > \frac{\epsilon_0}{2}$ we obtain using Cramer's rule that $|a_l(R_i)| \ll \eta$ for all $i = 2, ..., t$. Thus

$$|a_l(P_i) - a_l(P_1)| \ll \eta$$

and

$$\#M(a_n, ..., a_{n-k+1}; \eta) \ll \max(1, \eta^{n-k+1}).$$

$\square$

From this point forward it will be assumed that $|a_n(P)| \gg H(P)$. This can be done without loss of generality, see [[31], Page 7, Remark 1] for a full explanation of this.

The proof of Theorem 5.3 involves using an induction argument. The base cases of the induction argument (when $n = 1, 2, 3$ in Theorem 5.3) correspond to Lemma 7, Lemma 8 and Lemma 9 of the paper. The case in which $n = 1$, i.e. the linear case, is quite simple. However, it is the reason that the set A_λ had to be removed from $\Pi^3_\lambda(Q)$; see [31] for details. The proof of Theorem 5.3 for $n = 2$ is given below so as to give an explicit example of an application of Lemma 3.1. The proof below varies slightly from that given in the paper in that the calculations are given in more detail here. The case of $n = 3$ is very important as in many ways it forms the proof of Theorem 5.2 and Theorem 5.3 in terms of the techniques developed in the proof.

Define

$$B(P) := \{(x_1, x_2, x_3) \in \Pi^3_\lambda(Q) : |P(x_i)| < Q^{-\tau_i}\}.$$

Then, by Lemma 2.2, the following upper bound is obtained on the size of $B(P)$. For $n = 1, 2, 3$:

$$\mu(B(P)) \ll \prod_{i=1}^{n} \min(Q^{-\lambda}, Q^{-\tau_i}|P'(\alpha_i)|^{-1}).$$

It should be clear that

$$\mu(J_n(Q)) \ll \sum_{P \in P_n(Q)} \mu(B(P))$$

and so evaluating $\sum_{P \in P_n(Q)} \mu(B(P))$ completes the proof. In a way the whole proof of Theorem 5.3 boils down to finding a small enough upper bound on $\mu(B(P))$.

An overview of the proofs and a general discussion of the cases $n = 1, 2$ and 3 is now given.

In the linear case (Lemma 7 of the paper) the assumption of $|P(x_i)| < Q^{-\tau_i}$ is used to get the following bound for $x_3 \in I_3$:

$$\left| x_3 - \frac{a}{b} \right| < \frac{Q^{-\tau_3}}{|b|},$$

where $P(x) = bx + a$. It can then be shown, by definition of $\Pi^3_\lambda(Q)$, that this bound will only hold if $|b| \geq Q^{\frac{\lambda}{2}+\epsilon}$. This is why rationals with small denominators were removed. Since it can also be shown, under the assumptions of the lemma, that

$|b| \leq Q^{-\tau_1+\epsilon}$, one now has that $\tau_1 < -\frac{\lambda}{2}$. With this bound on τ_1 it is not difficult to show that the measure of the set of $x_3 \in I_3$, which lie within $Q^{3\tau_1}$ of any rational $\frac{p}{q}$ with $|q| \leq Q^{-\tau_1+\epsilon}$, is $\ll Q^{-\lambda-\epsilon}$. Thus

$$\mu(J_1(Q)) < Q^{-3\lambda-\epsilon}.$$

The proof for the cases $n = 2$ and $n = 3$ can be considered to comprise of two main cases. The first main case is when a separation between the roots can be guaranteed and the second main case is when a separation cannot be guaranteed. To show an explicit example in which Lemma 3.1 is used, the proof of Theorem 5.3 in the case of $n = 2$ is now presented. In particular the following lemma is proved.

Lemma 5.5.
For fixed $\kappa \in (0, 1)$ define $J_2(Q) \subset \Pi^3_\lambda(Q)$ to be the set of points $(x_1, x_2, x_3) \in \Pi^3_\lambda(Q)$ such that

$$|P(x_i)| < \frac{Q^{-\tau_i}}{\epsilon_0}$$

holds with $\sum_{i=1}^{3} \tau_i = 1$ for some $P \in P_2(Q)$. Then for sufficiently large Q

$$\mu(J_2(Q)) < \kappa\mu(\Pi^3_\lambda(Q)).$$

Following this, an overview of how the proof for the case $n = 3$ works will be given, a lot of which mirrors the proof for the case $n = 2$ now presented.

Proof of Lemma 5.5.
Throughout it will be assumed that Q is chosen sufficiently large so that $\log(Q) < Q^\epsilon$ and that $2\epsilon < 1-3\epsilon$. Note that for any quadratic polynomial $P(x) = a_2x^2+a_1x+a_0$ the only way for $|P(x_i)| < Q^{-\tau_i}$ to hold with $|x_i - x_j| > \epsilon_0$ for $1 \leq i < j \leq 3$ is with $Q^{-\tau_1} \ll |a_2| \leq Q^{-\tau_1+\epsilon}$ and $\tau_1 < \epsilon$.

To see this note that since $|P(x_i)| < Q^{-\tau_i} \leq Q^{-\tau_1}$ one has that

$$-2Q^{-\tau_1} \leq P(x_i) - P(x_j) \leq 2Q^{-\tau_1}.$$

Assume without loss of generality that $x_i > x_j$ so that $\epsilon_0 < x_i - x_j < 1$. Now consider the following required inequalities:

$$-2Q^{-\tau_1} \leq a_2(x_i^2 - x_j^2) + a_1(x_i - x_j), \quad a_2(x_i^2 - x_j^2) + a_1(x_i - x_j) \leq 2Q^{-\tau_1}.$$

If $a_2, a_1 \geq 0$ then

$$\begin{aligned}2Q^{-\tau_1} \geq a_2(x_i^2 - x_j^2) + a_1(x_i - x_j) &= |a_2||x_i^2 - x_j^2| + |a_1||x_i - x_j| \\ &> |a_2|\epsilon_0^2 + |a_1|\epsilon_0 \geq |a_2|\epsilon_0^2\end{aligned}$$

thus

$$|a_2| \ll Q^{-\tau_1+\epsilon}.$$

If $a_2, a_1 \leq 0$ then the argument is the same and again the requirement that $|a_2| \ll Q^{-\tau_1+\epsilon}$ is found.
If $a_2 \geq 0, a_1 \leq 0$ then

$$\begin{aligned}2Q^{-\tau_1} \geq a_2(x_i^2 - x_j^2) + a_1(x_i - x_j) &= a_2(x_i^2 - x_j^2) - |a_1|(x_i - x_j) \\ &> |a_2|\epsilon_0^2 - |a_1| \geq |a_2|\epsilon_0^2 - Q\end{aligned}$$

thus

$$|a_2| \ll \max\{Q^{-\tau_1+\epsilon}, Q^{1+\epsilon}\}$$

which is certainly satisfied if $|a_2| \ll Q^{-\tau_1+\epsilon}$.
If $a_2 \leq 0, a_1 \geq 0$ then the argument is the same and again the requirement that $|a_2| \ll \max\{Q^{-\tau_1+\epsilon}, Q^{1+\epsilon}\}$ is found.
So it is now clear that it must be assumed that $|a_2| \ll Q^{-\tau_1+\epsilon}$. Note $\tau_1 < \epsilon$ as otherwise $|a_2| \leq Q^0$ which is a contradiction.

It may be assumed that $|a_2| \gg Q^{-\tau_1}$ since if not $|a_2| < cQ^{-\tau_1}$, $\forall\ c \in \mathbb{R}^+$, and so in particular we can say that $|a_2| < \frac{1}{3}Q^{-\tau_1}$ and thus, for each $i = 1, 2, 3$, $|P(x_i)| = |a_2x_i^2 + a_1x_i + a_0| \leq |a_2|(1)^2 + |a_2|(1) + |a_2| \leq Q^{-\tau_1}$, since $x_1 \in I_1 \subset (0, 1)$. This is then a stronger set of inequalities than (5.3).

First consider $\tau_1 < \epsilon$ and $\tau_2, \tau_3 > 0$ and assume that $\tau_1 - \tau_i < -2\epsilon$ for $i = 2, 3$. Then, from the third inequality of Lemma 2.2, one has that

$$|x_i - \alpha_i| < Q^{-\frac{\tau_i}{2}}|a_2|^{-\frac{1}{2}} \ll Q^{\frac{(\tau_1-\tau_i)}{2}} < Q^{-\epsilon}$$

and so the two roots α_1 and α_2 of $P(x)$ satisfy

$$|\alpha_1 - \alpha_2| > \frac{\epsilon_0}{2}.$$

Therefore $|P'(\alpha_i)| \gg |a_2| \gg Q^{-\tau_1}$.

Define the set $B(P)$ as

$$B(P) := \{(x_1, x_2, x_3) \in \Pi_\lambda^3(Q) : |P(x_i)| < Q^{-\tau_i}, i = 1, 2, 3\}.$$

Then by Lemma 2.2

$$\mu(B(P)) \leq Q^{-\lambda-\tau_2-\tau_3}|a_2|^{-2} \ll Q^{-\lambda-\tau_2-\tau_3+2\tau_1}.$$

Thus

$$\begin{aligned}\mu(J_2(Q)) &= \mu\Bigg(\bigcup_{P\in P_2(Q^{-\tau_1+\epsilon})} B(P)\Bigg)\\ &\ll Q^{-\lambda+2\tau_1-\tau_2-\tau_3} \sum_{P\in P_2(Q^{-\tau_1+\epsilon})} 1\\ &\ll Q^{-\lambda+2\tau_1-\tau_2-\tau_3}Q^{-3\tau_1+3\epsilon}\\ &\ll Q^{-1-\lambda+3\epsilon} = Q^{2\lambda-1+3\epsilon}Q^{-3\lambda}\\ &< \kappa\mu(\Pi^3_\lambda(Q))\end{aligned}$$

since recall $0 < \lambda < \frac{1}{3}$.

Suppose now that $\tau_1 - \tau_i \geq -2\epsilon$. It is then clear that $\tau_1 - \tau_i \geq -\lambda + \epsilon$. Recall from (5.8) that

$$|P(d_i)| \ll Q^{-\tau_i} + |a_2|Q^{-\lambda} \ll Q^{-\tau_i} + Q^{-\tau_1-\lambda+\epsilon}.$$

Thus

$$|P(d_i)| \ll Q^{-\tau_i} \ll 1 \quad \text{for } i = 2, 3$$

and so by Lemma 5.4

$$\#M(a_2, a_1; Q^{-\tau_1}) \ll \max(1, Q^{-\tau_1}) \ll Q^{-\tau_1}$$

since $-\tau_1 > 0$. By Lemma 2.2

$$\mu(B(P)) \ll Q^{-\lambda-\tau_2-\tau_3}|a_2|^{-2}.$$

Thus

$$\begin{aligned}\mu(J_2(Q)) &= \mu\Bigg(\bigcup_{\substack{P\in M(a_2,a_1;Q^{-\tau_1})\\ |a_2|,|a_1|\leq Q^{-\tau_1+\epsilon}}} B(P)\Bigg)\\ &\ll Q^{-\lambda}Q^{-\tau_2-\tau_3}Q^{-\tau_1}Q^{-\tau_1+\epsilon} \sum_{|a_2|\leq Q^{-\tau_1+\epsilon}} |a_2|^{-2}\\ &\ll Q^{-\lambda-1} < \kappa\mu(\Pi^3_\lambda(Q)).\end{aligned}$$

Next consider $\tau_1, \tau_2 \le \epsilon$ which implies that $\tau_3 \ge 1 - 2\epsilon$. To do this three cases will be considered:

1) $\lambda \le \tau_2 - \tau_1 < 1 - 3\epsilon$.

2) $\tau_2 - \tau_1 \ge 1 - 3\epsilon$.

3) $\tau_2 - \tau_1 < \lambda$.

Case 1

Consider
$$-\tau_1 - \lambda \ge -\tau_i \quad \text{for } i = 2, 3. \tag{5.4}$$
Then, by (5.8),
$$|P(d_i)| \ll Q^{-\tau_1 - \lambda} \quad \text{for } i = 2, 3$$
and so, by Lemma 5.4,
$$\#M(a_2, a_1; Q^{-\tau_1 - \lambda}) \ll \max(1, Q^{-\lambda - \tau_1}).$$
By Lemma 2.2
$$\mu(B(P)) \ll Q^{-2\lambda} \times \frac{Q^{-\tau_3}}{|a_2|}.$$

If $\lambda + \tau_1 \le 0$
$$\begin{aligned}\sum_{\substack{P \in M(a_2, a_1; Q^{-\tau_1 - \lambda}) \\ |a_2|, |a_1| \le Q^{-\tau_1 + \epsilon}}} \mu(B(P)) &\ll \sum_{|a_2|, |a_1| \le Q^{-\tau_1 + \epsilon}} Q^{-\lambda - \tau_1} Q^{-2\lambda - \tau_3} |a_2|^{-1} \\ &\le \log(Q) Q^{-3\lambda - 2\tau_1 - \tau_3 + \epsilon} \\ &\ll Q^{-3\lambda - \tau_1 + \tau_2 - 1 + 2\epsilon} < Q^{-3\lambda - \epsilon}\end{aligned}$$

and $\mu(J_2(Q)) \le \kappa \Pi^3_\lambda(Q)$.

If $\lambda + \tau_1 > 0$
$$\begin{aligned}\sum_{\substack{P \in M(a_2, a_1; Q^{-\tau_1 - \lambda}) \\ |a_2|, |a_1| \le Q^{-\tau_1 + \epsilon}}} \mu(B(P)) &\ll \sum_{|a_2|, |a_1| \le Q^{-\tau_1 + \epsilon}} Q^{-2\lambda - \tau_3} |a_2|^{-1} \\ &\le \log(Q) Q^{-2\lambda - \tau_1 - \tau_3 + \epsilon} \\ &\ll Q^{-2\lambda - 1 + \tau_2 + 2\epsilon} < Q^{-3\lambda - \epsilon}\end{aligned}$$

and $\mu(J_2(Q)) \le \kappa \Pi^3_\lambda(Q)$.

Case 2

If $\tau_2 - \tau_1 \geq 1 - 3\epsilon$ then $\tau_2 - \tau_1 \geq 2\epsilon$ and, by the third inequality of Lemma 2.2,

$$|x_i - \alpha_i| < Q^{-\frac{\tau_i}{2}} |a_2|^{-\frac{1}{2}} \ll Q^{\frac{(\tau_1 - \tau_i)}{2}} < Q^{-\epsilon}$$

and so the two roots α_1 and α_2 of $P(x)$ are distinct and satisfy

$$|\alpha_1 - \alpha_2| > \frac{\epsilon_0}{2}.$$

Therefore $|P'(\alpha_i)| \gg |a_2| \gg Q^{-\tau_1}$. By Lemma 2.2

$$\mu(B(P)) \leq Q^{-\lambda-\tau_2-\tau_3} |a_2|^{-2} \ll Q^{-\lambda-\tau_2-\tau_3+2\tau_1}.$$

Thus

$$\begin{aligned}
\mu(J_2(Q)) &= \mu\left(\bigcup_{P \in P_2(Q^{-\tau_1+\epsilon})} B(P) \right) \\
&\ll \sum_{|a_2|,|a_1|,|a_0| \leq Q^{-\tau_1+\epsilon}} Q^{-\lambda+2\tau_1-\tau_2-\tau_3} \\
&\leq Q^{-\lambda-\tau_1-\tau_2-\tau_3+3\epsilon} \\
&\leq Q^{-\lambda-1+3\epsilon} < \kappa\mu(\Pi^3_\lambda(Q)).
\end{aligned}$$

Case 3

Consider $\tau_2 - \tau_1 < \lambda$ and suppose that root separation can be guaranteed. In particular, suppose $\epsilon < \tau_2 - \tau_1 < \lambda$. Then, as was done in Case 2, by Lemma 2.2

$$\mu(B(P)) \ll Q^{-\lambda} \times \frac{Q^{-\tau_2}}{|a_2|} \times \frac{Q^{-\tau_3}}{|a_2|}.$$

Thus

$$\begin{aligned}
\sum_{P \in P_2(Q^{-\tau_1+\epsilon})} \mu(B(P)) &\ll Q^{-\lambda-\tau_2-\tau_3} \sum_{P \in P_2(Q^{-\tau_1+\epsilon})} \frac{1}{|a_2|^2} \\
&\ll Q^{-\lambda-\tau_2-\tau_3} Q^{-2\tau_1+2\epsilon} \sum_{|a_2| \leq Q^{-\tau_1+\epsilon}} \frac{1}{|a_2|^2} \\
&\ll Q^{-\lambda-\tau_2-\tau_3-2\tau_1+2\epsilon+\tau_1-\epsilon} = Q^{-\lambda-1+\epsilon}.
\end{aligned}$$

and $\mu(J_2(Q)) \leq \kappa \Pi^3_\lambda(Q)$.

What remains is to consider when root separation can not be guaranteed, i.e.

when $0 \leq \tau_2 - \tau_1 < \epsilon$. This is split into two subcases:

1) $|\alpha_1 - \alpha_2| \leq Q^{\frac{\tau_1 - 1}{4}}$

2) $|\alpha_1 - \alpha_2| > Q^{\frac{\tau_1 - 1}{4}}$

First consider subcase 1 and define $u_i(P)$ to be the real number such that

$$\min_{j=1,2}(Q^{-\tau_i}|P^{(j)}(\alpha_i)|^{-1})^{\frac{1}{j}} = Q^{-u_i(P)}$$

where it can be shown that $\tau_i/2 \leq u_i(P) \leq \tau_i + 1$ which is a finite range independent of Q.

To see this note that

$$\begin{aligned} Q^{-u_i(P)} &\leq (Q^{-\tau_i}|P^{(2)}(\alpha_i)|^{-1})^{\frac{1}{2}} \\ &= Q^{-\frac{\tau_i}{2}}|2a_2|^{-\frac{1}{2}} \leq Q^{-\frac{\tau_i}{2}}. \end{aligned}$$

Furthermore, using the trivial estimate $|P^{(j)}(x)| \leq 2^{j+1}Q$,

$$\begin{aligned} Q^{-u_i(P)} &= \min_{j=1,2}(Q^{-\tau_i}|P^{(j)}(\alpha_i)|^{-1})^{\frac{1}{j}} \\ &\geq \min_{j=1,2}(Q^{-\frac{\tau_i}{j}}|2^{j+1}Q|^{-\frac{1}{j}}) \\ &\gg \min_{j=1,2}(Q^{-\frac{\tau_i+1}{j}}) \geq Q^{-\tau_i - 1}. \end{aligned}$$

The reason for defining the quantity $u_i(P)$ is so that the intervals $[\tau_i/2, \tau_i+1]$ can now be divided into smaller intervals of size ϵ_1 which allows the set of polynomials $P \in P_2(Q)$ with $u_i \leq u_i(P) \leq u_i + \epsilon_1$, for sufficiently small ϵ_1, to be considered. This is done on multiple occasions throughout [31]. This set is denoted $P_2(Q, u_1, u_2, u_3)$.

Consider the set $P_2(Q, u_1, u_2, u_3)$ and divide $\Pi^3_\lambda(Q)$ into smaller boxes of side lengths $Q^{-\lambda}, Q^{-\lambda}, Q^{-u_3+\gamma}$ where γ is chosen small enough that $u_3 - \gamma > 0$. If there is at most one polynomial in each box then the total measure of the set

$$D(P) = \{(x_1, x_2, x_3) \in \Pi^3_\lambda(Q) : |P(x_i)| < Q^{-\tau_i}, P \in P_2(Q^{-\tau_1})\}$$

is at most

$$(Q^{-2\lambda}Q^{-u_3})(Q^{-\lambda+u_3-\gamma}) = Q^{-3\lambda-\gamma},$$

since, by Lemma 2.2, one has that $|P(x_3)| < Q^{-\tau_3}$ on an interval $|x_3 - \alpha_3| < Q^{-u_3}$.

Now suppose two polynomials belong to one box and take $t_3 = \frac{\tau_3}{|\epsilon - \tau_1|}$ and $\eta_3 = \frac{u_3}{|\epsilon - \tau_1|}$, then by the generalised Bernik's Lemma, Lemma 3.1,

$$\begin{aligned}4+\delta &> \frac{\tau_3}{|\epsilon-\tau_1|}+1+2\bigg(\frac{\tau_3}{|\epsilon-\tau_1|}+1-\frac{u_3}{|\epsilon-\tau_1|}\bigg)\\ \implies \delta \geq |\epsilon-\tau_1|\delta &> \tau_3-|\epsilon-\tau_1|+2\tau_3-2u_3.\end{aligned}$$

Note that the values τ_1 and τ_2 were not considered in Lemma 3.1 since in this subcase it cannot be guaranteed that $\alpha_1 \neq \alpha_2$.

Let $\rho \in \{1, 2\}$ be such that

$$\begin{aligned}\min_{j=1,2}(Q^{-\tau_i}|P^{(j)}(\alpha_i)|^{-1})^{\frac{1}{j}} &= (Q^{-\tau_i}|P^{(\rho)}(\alpha_i)|^{-1})^{\frac{1}{\rho}}\\ &= Q^{-u_i(P)} \leq Q^{-u_i}.\end{aligned}$$

If $\rho = 1$ then

$$\begin{aligned}Q^{-u_3} &\geq \bigg(\frac{Q^{-\tau_3}}{|P'(\alpha_1)|}\bigg)\\ &= \frac{Q^{-\tau_3}}{|a_2||\alpha_1-\alpha_2|}\\ &\gg \frac{Q^{-\tau_3}}{Q^{-\tau_1}.Q^{\frac{\tau_1-1}{4}}}\\ &= Q^{-\tau_3+\frac{3\tau_1+1}{4}}\\ \implies u_3 &\leq \tau_3-\frac{3\tau_1+1}{4}.\end{aligned}$$

If $\rho = 2$

$$\begin{aligned}Q^{-u_3} \geq (Q^{-\tau_3}|P^2(\alpha_1)|^{-1})^{\frac{1}{2}} \gg Q^{-\frac{\tau_3+1}{2}}\\ \implies u_3 \leq \frac{\tau_3+1}{2} \leq \tau_3-\frac{3\tau_1+1}{4}.\end{aligned}$$

Therefore, by Lemma 3.1,

$$\begin{aligned}\delta &> \tau_3-|\epsilon-\tau_1|+2\tau_3-2(\tau_3-\frac{3\tau_1+1}{4})\\ &= \tau_3-\epsilon+\tau_1+\frac{3\tau_1+1}{2}\\ &= \tau_3-\epsilon+\frac{1}{2}+\frac{5\tau_1}{2}\\ &= \tau_3-\epsilon+\frac{1}{2}+5\bigg(\frac{1-\tau_3-\epsilon}{4}\bigg) && (\text{since } \tau_2-\tau_1<\epsilon \implies \tau_1 > \tfrac{1-\tau_3-\epsilon}{2})\\ &= \frac{7-\tau_3-9\epsilon}{4} > \frac{1}{2}., && (\text{since } \tau_3 \leq 3)\end{aligned}$$

Hence $\delta > \frac{1}{2}$ which is a contradiction.

Now consider the second subcase, that is when $|\alpha_1 - \alpha_2| > Q^{\frac{\tau_1 - 1}{4}}$. Using Lemma 2.2 one has

$$|x_3 - \alpha_1| \leq \frac{|P(x_3)|}{|a_2||\alpha_1 - \alpha_2|} \ll Q^{-\tau_3 + \frac{3\tau_1+1}{4}}.$$

Define $B(P) = \{x_3 \in \Pi^1_\lambda(Q) : |P(x_3)| < Q^{-\tau_3}\}$. Then, by Lemma 2.2,

$$\mu(B(P)) \leq Q^{-\tau_3 + \frac{3\tau_1+1}{4}}.$$

Thus

$$\begin{aligned}
\mu(J_2(Q)) &\leq Q^{-2\lambda} \sum_{P \in P_2(Q^{-\tau_1+\epsilon})} \mu(B(P)) \\
&\ll Q^{-2\lambda} Q^{-\tau_3 + \frac{3\tau_1+1}{4}} \sum_{P \in P_2(Q^{-\tau_1+\epsilon})} 1 \\
&\ll Q^{-2\lambda - \tau_3 + \frac{3\tau_1+1}{4}} Q^{-3\tau_1 + 3\epsilon}.
\end{aligned}$$

Finally note that

$$\begin{aligned}
-2\lambda - \tau_3 + \frac{3\tau_1+1}{4} - 3\tau_1 + 3\epsilon &= -2\lambda - \tau_3 + \frac{1}{4} - \frac{9\tau_1}{4} + 3\epsilon \\
&= -2\lambda - \tau_3 + \frac{1}{4} - \frac{9}{4}\left(\frac{1-\tau_3}{2}\right) + 3\epsilon \\
&= -2\lambda + \frac{\tau_3 - 7}{8} + 3\epsilon \\
&\leq -2\lambda - \frac{1}{2} + 3\epsilon,
\end{aligned}$$

thus $\mu(J_2(Q)) \leq \kappa \Pi^3_\lambda(Q)$.

□

The overview of the proof of Theorem 5.3 in the case of $n = 3$ (Lemma 9 of [31]) is now given. As already mentioned, the vast majority of this will mirror the proof of Lemma 5.5, but will simply contain more cases. The first main case is again when a separation between the roots can be guaranteed (This is Case 1 in the proof of Lemma 9 in [[31], Page 11]). Guaranteeing separation can be done in a few different ways, the simplest being to assume that either $\tau_i > 0$ or $|a_n| > Q^{-\tau_i+\epsilon}$ for $i = 1, 2, 3$. Recall that in Chapter 3, Remark 3.1, it was shown that $\tau_i > 0$ for all $i = 1, 2, 3$ gave $|\alpha_i - \alpha_j| > \frac{\epsilon_0}{2}$. Now that it is known that $|\alpha_i - \alpha_j| > \frac{\epsilon_0}{2}$ a lower bound can be obtained on the first derivative. In particular

$$|P'(\alpha_i)| = |a_n| \prod_{j=1, j\neq i}^{n} |\alpha_i - \alpha_j| \geq \frac{\epsilon_0^n}{2^n}|a_n|. \tag{5.5}$$

Then, by Lemma 2.2,

$$|x_i - \alpha_i| \ll Q^{-\tau_i}|a_n|^{-1}. \tag{5.6}$$

Thus

$$\mu(B(P)) \ll \prod_{i=1}^{3} \min(Q^{-\lambda}, Q^{-\tau_i}|a_n|^{-1}) \ll Q^{-\tau_1-\tau_2-\tau_3}|a_n|^{-3} = Q^{1-n}|a_n|^{-3}. \tag{5.7}$$

In order to evaluate $\sum_{P\in P_n(Q)} \mu(B(P))$ Lemma 5.4 will be used, however, a bound for $P(d_i)$ is first required where, recall, d_i is the center of the interval I_i. To do this note that if $x_i \in I_i$ is such that $|P(x_i)| < Q^{-\tau_i}$ then, by the Mean Value Theorem,

$$|P(d_i)| < Q^{-\tau_i} + |P'(\omega)||I_i| \ll Q^{-\tau_i} + |a_n|Q^{-\lambda}, \tag{5.8}$$

for some $\omega \in (x_i, d_i)$ (or (d_i, x_i)). So in the case of $n = 3$, which is being considered,

$$|P(d_i)| \ll \max(Q^{-\tau_i}, |a_3|Q^{-\lambda}).$$

Different cases are now considered depending on the different possible outcomes for $\max(Q^{-\tau_i}, |a_3|Q^{-\lambda})$ for each $i = 1, 2, 3$. A bound is then obtained on the cardinality of the set $M(a_3, .., a_i; \max(Q^{-\tau_i}, |a_3|Q^{-\lambda}))$.

An example of such an assumption would be to consider the case when $Q^{-\tau_3} \leq Q^{-\tau_2} \leq |a_3|Q^{-\lambda} < Q^{-\tau_1}$. Then for $i = 2, 3$, $|P(d_i)| \ll |a_3|Q^{-\lambda}$ and so by Lemma 5.4

$$\#M(a_3, a_2; |a_3|Q^{-\lambda}) \ll \max(1, |a_3|^2 Q^{-2\lambda}).$$

This then gives

$$\begin{aligned} \sum_{P\in P_3(Q)} \mu(B(P)) &= \sum_{\substack{|a_i|<Q, i=2,3 \\ P\in M(a_3,a_2;|a_3|Q^{-\lambda})}} \mu(B(P)) \\ &\ll \sum_{|a_i|<Q, i=2,3} \max(1, |a_3|^2 Q^{-2\lambda}) Q^{-2}|a_3|^{-3}. \end{aligned}$$

If $|a_3|Q^{-\lambda} \leq 1$

$$\sum_{\substack{|a_i|<Q^{\lambda}, i=2,3 \\ P \in M(a_3,a_2;|a_3|Q^{-\lambda})}} \mu(B(P)) \ll \sum_{|a_i|<Q^{\lambda}, i=2,3} Q^{-2}|a_3|^{-3} \ll Q^{2\lambda-2}$$
$$\ll Q^{-3\lambda}Q^{5\lambda-2} \ll s\mu(\Pi_{\lambda}^{3}(Q)).$$

If $|a_3|Q^{-\lambda} > 1$

$$\sum_{\substack{|a_i|<Q, i=2,3 \\ P \in M(a_3,a_2;|a_3|Q^{-\lambda})}} \mu(B(P)) \ll \sum_{|a_i|<Q, i=2,3} \log(Q)Q^{-2-2\lambda}|a_3|^{-1}$$
$$\ll Q^{\epsilon-1-2\lambda} \ll s\mu(\Pi_{\lambda}^{3}(Q)).$$

So in either case

$$\mu(J_3(Q)) \leq \sum_{P \in P_3(Q)} \mu(B(P)) \ll s\mu(\Pi_{\lambda}^{3}(Q))$$

as required. What remains is to consider the other variations such as $Q^{-\tau_3} \leq |a_3|Q^{-\lambda} < Q^{-\tau_2} \leq Q^{-\tau_1}$. The calculations in each of these are carried out in an identical fashion as the one just shown. This completes the first main case.

The second main case is when separation between the roots $\alpha_1, \alpha_2, \alpha_3$ cannot be guaranteed, or at best separation between only two of the roots can be guaranteed (This is Case 2 in the proof of Lemma 9 in [[31], Page 12]). In other words when both $\tau_i < \epsilon$ and $|a_n| < Q^{-\tau_i+\epsilon}$ for at least one $i = 1, 2, 3$. It can be assumed that this occurs at least for $i = 1$. Bounding $\mu(B(P))$ is now a much more difficult task and, unlike in the first main case, much care must be taken. The main idea going forward is to consider the possible size of the value τ_2 separately and, in particular, how it compares to the size of τ_1. As an example of one particular case which is considered in the paper, let $\tau_1 < \epsilon$ and $|a_3| < Q^{-\tau_1+\epsilon}$. The value τ_2 is then considered in two distinct placings, when $0 < \tau_2 \leq \tau_1 + \epsilon$ ([[31], Page 12, Subcase 2a]) and when $\tau_1 \leq \tau_2 \leq 0$ ([[31], Page 14, Subcase 2b]). More conditions on τ_1, τ_2 and τ_3 are considered within these separate cases where necessary. We will now aim to describe this process in more detail.

The size of the value τ_2 is considered in two parts.

Part 1. Assume $\tau_2 \geq 0$. In this subcase the size of $|P'(\alpha_j)|$ where $j = 2$ or $j = 3$ is considered. Recall that if a lower bound on the first derivative can be found for $j = 2, 3$, like in (5.5), then an upper bound can be found on the distance $|x_j - \alpha_j|$ for $j = 2, 3$, similar to (5.6), and so finally an upper bound can be found for $\mu(B(P))$, just as was done in (5.7). From here, as described above, a bound on $\mu(J_n(Q))$ can be obtained by evaluating $\sum_{P\in P_n(Q)} \mu(B(P))$ in a similar fashion to that shown in the first main case above. Thus, when a lower bound can be obtained for $|P'(\alpha_j)|$ for both $j = 2$ and $j = 3$, the proof is completed in an identical fashion to the example shown above by evaluating $\sum_{P\in P_n(Q)} \mu(B(P))$.

When a lower bound cannot be obtained for $|P'(\alpha_j)|$ for at least one of $j = 2$ or $j = 3$ it is not difficult to check that the method shown above is not good enough to prove Theorem 5.3 in this case. Instead a proof by contradiction is now used. To do this $\Pi^3_\lambda(Q)$ is divided into smaller boxes of carefully chosen side lengths. These sidelengths vary depending on where the value of τ_2 being considered can lie. However, they are always chosen so as to ensure that Lemma 3.1 gives a contradiction under the assumption that more than one polynomial belongs to any of these smaller boxes. A polynomial P is said to belong to a box M if there exists $(x_1, x_2, x_3) \in M$ such that (5.3) holds. From here working out $\mu(J_n(Q))$ is a simple task. An example of the process just described was demonstrated in the proof of Lemma 5.5 and can be found in [[31], Pages 13-14].

Part 2. Assume $\tau_2 < 0$. This is again split into two cases. The first subcase is when it can be guaranteed that $\tau_2 \neq \tau_1$ ([[31], Page 14, Subcase 2bi]). The method of proof here is almost identical to that in the first subcase when τ_2 was positive.

The second case is when it cannot be guaranteed that $\tau_2 \neq \tau_1$ ([[31], Page 15, Subcase 2bii]). This is by far the most tricky of all cases. The main problem here is that if $\tau_2 = \tau_1$ then nothing is known about the separation of the roots α_1 and α_2. Up to this point it could always be guaranteed that $\alpha_1 \neq \alpha_2$ which meant that Lemma 3.1 could be used since condition (3.1) was satisfied. However, now if $\alpha_1 = \alpha_2 = \alpha$ then Lemma 3.1 may only be applied for two of the roots α and α_3 which significantly weakens the inequality to

$$\sum_{r=2}^{3}(\tau_r + \mu + 2\max(\tau_r + \mu - \eta_r, 0)) < 2n\mu + \delta.$$

This means that trying to use the same idea as before, of splitting up $\Pi^3_\lambda(Q)$ into smaller boxes and choosing the side lengths to ensure that at most one polynomial belongs to each of these smaller boxes, is much more difficult. Furthermore there is no information about $P'(\alpha_3)$. In order to deal with these issues the value of τ_1 is considered when $\tau_1 < -\lambda+\epsilon$ and when $\tau_1 \geq -\lambda+\epsilon$ separately. Splitting the value of τ_1 in this way was, of course, chosen carefully to suit the calculations, see [[31], Page 15, Subcase 2bii] for full details. Considering the two intervals ($\tau_1 \in (-1, -\lambda + \epsilon)$

and $\tau_1 \in (-\lambda+\epsilon, 0)$) separately, methods similar to those described above, involving Lemma 2.2, Lemma 3.1 and Lemma 5.4, are adapted to complete the proof. Details will not be given of this as too much notation is needed and the reader is instead directed to [[31], Page 16-17, Subcase 2bii], however, a similar process was explicitly shown in the proof of Lemma 5.5.

Once the base case of the induction argument has been proved, the induction hypothesis is taken to be that Theorem 5.3 is true for $3 < m \leq n-1$. What remains is to prove Theorem 5.3 is true for $m = n$. To do this three subsections are considered which depend on the signs of the values τ_1 and τ_2. In particular, subsection 1 considers $\tau_i \geq 0$ for $i = 1, 2, 3$, subsection 2 considers $\tau_1 < 0$ and $\tau_i \geq 0$ for $i = 2, 3$ and subsection 3 considers $\tau_i < 0$ for $i = 1, 2$ and $\tau_3 \geq 0$. In each subsection the range of values that $|P'(\alpha_i)|$ can take at each of the roots is partitioned into $n-1$ different classes which are denoted by T_j^i where $i \in \{1, 2, 3\}$ and $j \in \{1, ..., n-1\}$. An example of how these classes T_j^i look is the following,

$$T_j^i := \{\alpha_i : Q^{\frac{1-\tau_i}{2}+\frac{(j-2)\tau_i}{n-1}} < |P'(\alpha_i)| < Q^{\frac{1-\tau_i}{2}+\frac{(j-1)\tau_i}{n-1}}\}.$$

The complete partitioning of the range of values that $|P'(\alpha_i)|$ can take at each of the roots is given in the Appendices. This complete partitioning can also be found at the end of section 2.0 in ([[31], Pages 18-19].

In the first subsection, having $\tau_i > 0$ for all $i = 1, 2, 3$ guarantees root separation, as described before, and so the range of values $|P'(\alpha_i)|$ are partitioned for $i = 1, 2, 3$. This is partition A from the first Appendix. In the second subsection, having $\tau_i > 0$ for $i = 2, 3$, means that root separation can only be guaranteed between α_2 and α_3, and so the range of values $|P'(\alpha_i)|$ are partitioned for $i = 2, 3$ only. This is partition B from the first Appendix. Finally, in the third subsection, since only $\tau_3 > 0$, the range of values $|P'(\alpha_3)|$ are partitioned only. This is partition C from the first Appendix.

Theorem 5.3 is then shown to be true in each subsection by three propositions. These three propositions correspond to what happens when the first derivatives at the roots α_i, $i = 1, 2, 3$, are "large", "small" or "mixed". The first derivatives are said to be "large" if $\alpha_i \in T_j^i$ for all $i = 1, 2, 3$ and for some $j \in \{3, ..., n-1\}$. The first derivatives are said to be "small" if $\alpha_i \in T_1^i \cup T_2^i$ for all $i = 1, 2, 3$. The first derivatives are said to be "mixed" if $\alpha_i \in T_1^i \cup T_2^i$ for $i = 1, 2$ and $\alpha_3 \in T_j^3$ for some $j \in \{3, ..., n-1\}$. Note that there will be no mixed case to consider for the third subsection as only the root α_3 was considered here. The propositions alluded to here are explicitly stated in Appendix A, Appendix B and Appendix C. They can also be found, along with a discussion, in sections 2.1, 2.2 and 2.3 of [31]. In the

Appendices, the notation used matches that of the paper.

Each proposition is shown to be true using arguments almost identical to those described above to prove the base cases of the induction argument. Together the propositions give measure estimates for the set of points $(x_1, x_2, x_3) \in \Pi^3_\lambda(Q) \cap \mathcal{S}_P(\alpha_1, \alpha_2, \alpha_3)$ with the roots α_1, α_2 and α_3 lying in various combinations of these sets T^i_j. It was the third of the subsections, when $\tau_1, \tau_2 < 0$, that inspired the investigation into Lemma 3.1.

One additional idea is required for the case of large first derivatives. This is to split $\Pi^3_\lambda(Q)$ into appropriately chosen cuboids which are then split into essential and inessential domains. Doing this, with the induction hypothesis, is enough to prove the proposition in the case of large first derivatives (See discussion in [[31], Page 20, Section 2.1]). It was Sprindžuk [85] who in 1965 first introduced the method of essential and inessential domains. For the sake of completeness the following description is given. Let $\mathcal{P}$ be a set of polynomials satisfying certain conditions and $\sigma(P)$ be a set of points (defined for each $P \in \mathcal{P}$) which meet certain conditions. A set $\sigma(P)$ is called **essential** if

$$\mu\left(\sigma(P) \bigcap \bigcup_{T \in \mathcal{P}} \sigma(T)\right) < \frac{\mu(\sigma(P))}{2}.$$

A set that is not essential is called **inessential**. Roughly speaking a set $\sigma(P) \subset I \subset \mathbb{R}$ is called essential if (in terms of its Lebesgue measure) more than half of $\sigma(P)$ is free from points from any other set $\sigma(T)$.

A rough description of how the method of essential and inessential domains is used in the paper is now given for partition A from the first Appendix (Again, see discussion in [[31], Page 20, Section 2.1]).

For a polynomial $P \in P_n(Q)$ define the set

$$\sigma_1(P) := \{(x_1, x_2, x_3) \in \Pi^3_\lambda(Q) \cap S_P(\alpha_1, \alpha_2, \alpha_3) : |x_i - \alpha_i| < Q^{-\tau_i}|P'(\alpha_i)|^{-1}\}.$$

Define $v_i = \frac{2\tau_i}{n-1}$ and $l_i = \frac{(n-l+1)v_i}{2}$ for $i = 1, 2, 3$ such that $\sum_{i=1}^{3} v_i = 2$, $\sum_{i=1}^{3} l_i = n - l + 1$ and $l_i \leq \tau_i$. Finally, define

$$\tilde{\sigma}_1(P) := \{(x_1, x_2, x_3) \in \Pi^3_\lambda(Q) \cap S_P(\alpha_1, \alpha_2, \alpha_3) : |x_i - \alpha_i| < cQ^{-l_i}|P'(\alpha_i)|^{-1}\}.$$

It is easily checked that

$$\mu(\sigma_1(P)) < c^{-1}\mu(\tilde{\sigma}_1(P)).$$

See Appendix A and [31] for details.

The next step is to fix the $(l-2)$-tuple $\mathbf{b}_l = (a_n, ..., a_{n+3-l})$ consisting of coefficients of the polynomials $P(x) \in P_n(Q)$ and define $P_n(Q, \mathbf{b}_l)$ to be the subclass of polynomials $P \in P_n(Q)$ with the same $(l-2)$-tuple $\mathbf{b}_l$. It is then clear that if $P \in P_n(Q, \mathbf{b}_l)$ and $\tilde{\sigma}_1(P)$ is essential, then

$$\sum_{\substack{P \in P_n(Q,\mathbf{b}_l) \\ \tilde{\sigma}_1(P) \text{ essential}}} \mu(\tilde{\sigma}_1(P)) \leq 2^3 \mu(\Pi_\lambda^3(Q))$$

since $\mu(\tilde{\sigma}_1(P)) < \mu(\Pi_\lambda^3(Q))$. Thus as $\#\{\mathbf{b}_l\} \leq (2Q)^{l-2}$ a bound for $\mu(J_n(Q))$ is found,

$$\begin{aligned} \mu(J_n(Q)) &\leq \sum_{\mathbf{b}_l} \sum_{\substack{P \in P_n(Q,\mathbf{b}_l) \\ \tilde{\sigma}_1(P) \text{ essential}}} \mu(\sigma_1(P)) \leq c^{-1} Q^{-l+2} \sum_{\mathbf{b}_l} \sum_{\substack{P \in P_n(Q,\mathbf{b}_l) \\ \tilde{\sigma}_1(P) \text{ essential}}} \mu(\tilde{\sigma}_1(P)) \\ &\leq c^{-1} Q^{-l+2} 2^3 (2Q)^{l-2} \mu(\Pi_\lambda^3(Q)) < s\mu(\Pi_\lambda^3(Q).) \end{aligned}$$

In the case that $\tilde{\sigma}_1(P)$ is inessential the set $\tilde{\sigma}_1(P,T) := \tilde{\sigma}_1(P) \cap \tilde{\sigma}_1(T)$ is considered. Note that since $\tilde{\sigma}_1(P)$ is inessential,

$$\mu(\tilde{\sigma}_1(P,T)) \geq \frac{\mu(\tilde{\sigma}_1(P))}{2}.$$

In the set $\tilde{\sigma}_1(P,T)$ a Taylor expansion allows an upper bound to be found on $|P(x_i)|$ which with the induction hypothesis proves the proposition, see [[31], Page 21, Section 2.1] for full details. In particular this proves Proposition A.1. The proofs for Proposition A.2 and Proposition A.3 vary only slightly in that the sets $\sigma_1(P)$ and $\tilde{\sigma_1}(P)$ vary depending on the partition being considered, see [[31], Page 22, Section 2.2] and [[31], Page 26, Section 2.3] for full details. Full details of how these sets are defined in the paper are given in Appendix A. Very similar arguments to those just described can be found in [29] and [30].

In the case of small first derivatives the method of proof is very similar to that described in the base cases. This method, described below, was explicitly shown in Case 3 of the proof of Lemma 5.5, so the reader should use this as a reference to the method. The cube $\Pi_\lambda^3(Q)$ is first split into smaller cuboids of carefully chosen side lengths so that, as described before, it is ensured by Lemma 3.1 that at most one polynomial belongs to each box. Before using Lemma 3.1, however, bounds need to be found for $|P(x_i)|$ inside these smaller cubes. In particular, we want to have inequalities of the form $|P(x_i)| < Q^{-t_i}$ for $i = 1, 2, 3$, where t_i takes over the place of τ_i inside the smaller cuboids. This is done by using a Taylor expansion on $|P(x_i)|$

inside the smaller cuboids. Using these newly found t_i in place of the τ_i in Lemma 3.1 gives the contradiction we set up for and guarantees at most one polynomial belongs to each box. This guarantee, along with the induction hypothesis, is enough to finish the measure argument and prove the proposition in the case of small first derivatives. In particular this proves Proposition B.1, Proposition B.2 and Proposition B.3.

The proof in the case of the mixed first derivatives is similar to that of the small first derivatives.

It should be noted that the proofs of subsection 1 (when all $\tau_i > 0$) and subsection 2 (when only $\tau_1 < 0$) were possible without the generalisation of Bernik's Lemma, Lemma 3.1, as Lemma 1.57 was enough. However, Lemma 3.1 was crucial for subsection 3 (when only $\tau_3 > 0$). With the proof of all three subsections the induction argument is complete and Theorem 5.3 is proved.

The remainder of the paper is proving Theorem 5.2. This is done by tailoring the arguments described above for the proof of Theorem 5.3 to suit the conditions of Theorem 5.2. In fact the proof is almost exactly the same as that of Theorem 5.3 in the case where $\tau_i > 0$ for $i = 1, 2, 3$ except that $\tau_1 + \tau_2 + \tau_3 = n - 2$. See proposition 10 and proposition 11 in [31] for full details.

The induction hypothesis for Theorem 5.2 is that the following is true for $3 \leq m \leq n - 1$: For fixed $\kappa \in (0, 1)$, let $\delta_0 \in \mathbb{R}^+$ and define $B_m(Q, \delta_0) \subset \Pi^3_\lambda(Q)$ to be the set of points $(x_1, x_2, x_3) \in \Pi^3_\lambda(Q)$ such that

$$|P(x_i)| < \frac{Q^{-\tau_i}}{\epsilon_0} \text{ and } |P'(x_i)| < \delta_0 Q$$

holds with $\sum_{i=1}^{3} \tau_i = m - 2$ for $P \in P_m(Q)$. Then, for sufficiently large, Q

$$\mu(B_m(Q, \delta_0)) < \kappa\mu(\Pi^3_\lambda(Q)).$$

Again what remains is to prove Theorem 5.2 is true for the case of $m = n$. This is done with only one partition this time and the arguments are identical in nature to those made throughout the proof of Theorem 5.3.

Appendices

The partitions and propositions for the induction hypothesis used in the proof of Theorem 5.3 are given here in full detail. Recall the induction hypothesis is that Theorem 5.3 holds for $1 \le m \le n-1$.

Partition A: $\tau_i \ge 0$ for $i = 1, 2, 3$. Let $v_i = \frac{2\tau_i}{n-1}$. We say that $\alpha_i \in T_j^i$ if

$$\begin{array}{llll}
T_1^i: & & |P'(\alpha_i)| \le Q^{\frac{1-\tau_i}{2}}, & \\
T_2^i: & Q^{\frac{1-\tau_i}{2}} & < |P'(\alpha_i)| \le Q^{\frac{1-\tau_i}{2}+\frac{\tau_i}{2(n-1)}}, & \\
T_l^i: & Q^{\frac{1-\tau_i}{2}+\frac{(l-2)\tau_i}{2(n-1)}} & < |P'(\alpha_i)| \le Q^{\frac{1-\tau_i}{2}+\frac{(l-1)\tau_i}{2(n-1)}}, & 2 \le l \le n-2, \\
T_{n-1}^i: & Q^{\frac{1-v_i}{2}} = Q^{\frac{1-\tau_i}{2}+\frac{(n-3)\tau_i}{2(n-1)}} & < |P'(\alpha_i)|. &
\end{array}$$

Partition B: $\tau_1 < 0$, $\tau_i \ge 0$ for $i = 2, 3$. Let $v_1 = \tau_1$ and $v_i = \frac{(2-\tau_1)\tau_i}{n-1-\tau_1}$. We say that $\alpha_i \in T_j^i$ for $i = 2, 3$ if

$$\begin{array}{llll}
T_1^i: & & |P'(\alpha_i)| \le Q^{\frac{1-\tau_i}{2}}, & \\
T_2^i: & Q^{\frac{1-\tau_i}{2}} & < |P'(\alpha_i)| \le Q^{\frac{1-\tau_i}{2}+\frac{\tau_i}{2(n-1-\tau_1)}}, & \\
T_l^i: & Q^{\frac{1-\tau_i}{2}+\frac{(l-2)\tau_i}{2(n-1-\tau_1)}} & < |P'(\alpha_i)| \le Q^{\frac{1-\tau_i}{2}+\frac{(l-1)\tau_i}{2(n-1-\tau_1)}}, & 2 \le l \le n-2, \\
T_{n-1}^i: & Q^{\frac{1-v_i}{2}} = Q^{\frac{1-\tau_i}{2}+\frac{(n-3)\tau_i}{2(n-1-\tau_1)}} & < |P'(\alpha_i)|. &
\end{array}$$

Partition C: $\tau_1, \tau_2 < 0$, $\tau_3 \ge 0$. Let $v_1 = \tau_1, v_2 = \tau_2$ and $v_3 = \frac{(2-\tau_1-\tau_2)\tau_3}{n-1-\tau_1-\tau_2} = 2-\tau_1-\tau_2$. We say that $\alpha_3 \in T_j^3$ if

$$\begin{array}{llll}
T_1^3: & & |P'(\alpha_i)| \le Q^{\frac{1-\tau_3}{2}}, & \\
T_2^3: & Q^{\frac{1-\tau_3}{2}} & < |P'(\alpha_i)| \le Q^{\frac{1-\tau_3}{2}+\frac{1}{2}}, & \\
T_l^3: & Q^{\frac{1-\tau_3}{2}+\frac{(l-2)}{2}} & < |P'(\alpha_i)| \le Q^{\frac{1-\tau_3}{2}+\frac{(l-1)}{2}}, & 2 \le l \le n-2, \\
T_{n-1}^3: & Q^{\frac{1-v_3}{2}} = Q^{\frac{1-\tau_3}{2}+\frac{(n-3)}{2}} & < |P'(\alpha_i)|. &
\end{array}$$

Note that in each case $\sum_{i=1}^{3} v_i = 2$.

A Large derivatives

For Partition A with $l = \{3, ..., n-1\}$ define $J_{n,A_1}(Q,l)$ to be the set of $(x_1, x_2, x_3) \in \Pi^3_\lambda(Q) \cap S_P(\alpha_1, \alpha_2, \alpha_3)$ for which (5.3) holds for some $P \in P_n(Q)$ with

$$\alpha_3 \in T^3_l,\ \alpha_1, \alpha_2 \in \cup_{m=l}^{n-1} T^i_m.$$

For Partition B with $l = \{3, ..., n-1\}$ define $J_{n,B_1}(Q,l)$ to be the set of $(x_1, x_2, x_3) \in \Pi^3_\lambda(Q) \cap S_P(\alpha_1, \alpha_2, \alpha_3)$ for which the system

$$|P(x_i)| < Q^{-\tau_i},\ i = 1, 2, 3 \text{ and } \alpha_2, \alpha_3 \in \bigcup_{m=l}^{n-1} T^i_m,$$

has a solution $P \in P_n(Q)$.

For Partition C with $l = \{3, ..., n-1\}$ define $J_{n,C_1}(Q,l)$ to be the set of $(x_1, x_2, x_3) \in \Pi^3_\lambda(Q) \cap S_P(\alpha_1, \alpha_2, \alpha_3)$ for which the system

$$|P(x_i)| < Q^{-\tau_i},\ i = 1, 2, 3 \text{ and } \alpha_3 \in \bigcup_{m=l}^{n-1} T^3_m,$$

has a solution $P \in P_n(Q)$.

Proposition A.1 (Proposition 1 of the paper)**.**
For sufficiently large Q

$$\mu(J_{n,A_1}(Q)) < \kappa\mu(\Pi^3_\lambda(Q)).$$

Proposition A.2 (Proposition 2 of the paper)**.**
For sufficiently large Q

$$\mu(J_{n,B_1}(Q)) < \kappa\mu(\Pi^3_\lambda(Q)).$$

Proposition A.3 (Proposition 3 of the paper)**.**
For sufficiently large Q

$$\mu(J_{n,C_1}(Q)) < \kappa\mu(\Pi^3_\lambda(Q)).$$

Let $c > 1$ be a constant to be chosen later.

Let $J_{n,A_1}(Q) = \cup_{l=3}^{n-1} J_{n,A_1}(Q,l)$ and for a polynomial $P \in P_n(Q)$ define the set

$$\sigma_{A_1}(P) := \{(x_1, x_2, x_3) \in \Pi^3_\lambda(Q) \cap S_P(\alpha_1, \alpha_2, \alpha_3) : |x_i - \alpha_i| < Q^{-\tau_i}|P'(\alpha_i)|^{-1}\}.$$

Define the numbers $l_A^i = \frac{(n-l+1)v_i}{2}$ and notice that

$$\sum_{i=1}^{3} l_A^i = n - l + 1, \quad l_A^i \le \tau_i, \ i = 1, 2, 3.$$

Finally define the set

$$\sigma_{Al}(P) := \{(x_1, x_2, x_3) \in \Pi_\lambda^3(Q) \cap S_P(\alpha_1, \alpha_2, \alpha_3) : |x_i - \alpha_i| < cQ^{-l_A^i} |P'(\alpha_i)|^{-1}\}.$$

Let $J_{n,B_1}(Q) = \cup_{l=3}^{n-1} J_{n,B_1}(Q, l)$ and

$$J_1 = \{x_1 \in I_1 : |x_1 - \alpha_1| < \min(Q^{-u_1}, Q^{-\lambda})\}.$$

For a polynomial $P \in P_n(Q)$ define the set

$$\sigma_{A_1}(P) := J_1 \times \{(x_2, x_3) \in \Pi_\lambda^2(Q) \cap S_P(\alpha_2, \alpha_3) : |x_i - \alpha_i| < Q^{-\tau_i} |P'(\alpha_i)|^{-1}\}.$$

Define the numbers $l_B^i = \frac{(n-l+1-\tau_1)v_i}{2-\tau_1}$ for $i = 2, 3$ and notice that

$$\tau_1 + l_B^2 + l_B^3 = n - l + 1, \quad l_B^i \le \tau_i, \ i = 2, 3.$$

Finally define the set

$$\sigma_{Bl}(P) := J_1 \times \{(x_2, x_3) \in \Pi_\lambda^2(Q) \cap S_P(\alpha_2, \alpha_3) : |x_i - \alpha_i| < cQ^{-l_B^i} |P'(\alpha_i)|^{-1}\}.$$

Let $J_{n,C_1}(Q) = \cup_{l=3}^{n-1} J_{n,C_1}(Q, l)$ and for $i = 1, 2$, let

$$J_i = \{x_i \in I_i : |x_i - \alpha_i| < \min(Q^{-u_i}, Q^{-\lambda})\}.$$

For a polynomial $P \in P_n(Q)$ define the set

$$\sigma_{C_1}(P) := J_1 \times J_2 \times \{x_3 \in \Pi_\lambda(Q) \cap S_P(\alpha_3) : |x_3 - \alpha_3| < Q^{-\tau_3} |P'(\alpha_3)|^{-1}\}.$$

Define the numbers $l_C^3 = \frac{(n-l+1-\tau_1-\tau_2)\tau_3}{n-1-\tau_1-\tau_2}$ and notice that

$$\tau_1 + \tau_2 + l_C^3 = n - l + 1, \quad l_C^3 \le \tau_3.$$

Finally define the set

$$\sigma_{Cl}(P) := J_1 \times J_2 \times \{x_3 \in \Pi_\lambda(Q) \cap S_P(\alpha_3) : |x_3 - \alpha_3| < cQ^{-l_C^3}|P'(\alpha_3)|^{-1}\}.$$

B Small derivatives

For Partition A define $J_{n,A_2}(Q)$ to be the set of $(x_1, x_2, x_3) \in \Pi^3_\lambda(Q) \cap S_P(\alpha_1, \alpha_2, \alpha_3)$ for which (5.3) holds for some $P \in P_n(Q)$ with $\alpha_i \in T^i_1 \cup T^i_2,\ i = 1, 2, 3.$

For Partition B define $J_{n,B_2}(Q)$ to be the set of $(x_1, x_2, x_3) \in \Pi^3_\lambda(Q) \cap S_P(\alpha_1, \alpha_2, \alpha_3)$ for which (5.3) holds for some $P \in P_n(Q)$ with $\alpha_i \in T^i_1 \cup T^i_2,\ i = 2, 3.$

For Partition C define $J_{n,C_2}(Q)$ to be the set of $(x_1, x_2, x_3) \in \Pi^3_\lambda(Q) \cap S_P(\alpha_1, \alpha_2, \alpha_3)$ for which (5.3) holds for some $P \in P_n(Q)$ with $\alpha_3 \in T^3_1 \cup T^3_2$.

Proposition B.1 (Proposition 4 of the paper)**.**
For sufficiently large Q

$$\mu(J_{n,A_2}(Q)) < \kappa\mu(\Pi^3_\lambda(Q)).$$

Proposition B.2 (Proposition 5 of the paper)**.**
For sufficiently large Q

$$\mu(J_{n,B_2}(Q)) < \kappa\mu(\Pi^3_\lambda(Q)).$$

Proposition B.3 (Proposition 6 of the paper)**.**
For sufficiently large Q

$$\mu(J_{n,C_2}(Q)) < \kappa\mu(\Pi^3_\lambda(Q)).$$

C Mixed derivatives

For Partition A define $J_{n,A_3}(Q,l)$ to be the set of $(x_1,x_2,x_3) \in \Pi^3_\lambda(Q) \cap S_P(\alpha_1,\alpha_2,\alpha_3)$ for which (5.3) holds for some $P \in P_n(Q)$ with $\alpha_i \in T^i_1 \cup T^i_2,\ i = 1,2$ and $\alpha_3 \in T^3_l$ for some $l > 2$.

For Partition B define $J_{n,B_3}(Q)$ to be the set of $(x_1,x_2,x_3) \in \Pi^3_\lambda(Q) \cap S_P(\alpha_1,\alpha_2,\alpha_3)$ for which (5.3) holds for some $P \in P_n(Q)$ with $\alpha_2 \in T^2_1 \cup T^2_2$ and $\alpha_3 \in T^3_l$ for some $l > 2$.

Proposition C.1 (Proposition 7 of the paper)**.**
For sufficiently large Q

$$\mu(J_{n,A_3}(Q,l)) < \kappa\mu(\Pi^3_\lambda(Q)).$$

Proposition C.2 (Proposition 8 of the paper)**.**
For sufficiently large Q

$$\mu(J_{n,B_3}(Q)) < \kappa\mu(\Pi^3_\lambda(Q)).$$

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THE DISTRIBUTION OF ALGEBRAIC CONJUGATE POINTS

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Abstract. In this paper it is proved that the number of algebraic conjugate triples of height at most Q and degree at most n lying in a three-dimensional box of sidelength $Q^{-\lambda}$, $0 < \lambda < 1/3$ is at least $Q^{n+1-3\lambda}$. This question is a natural extension of problems in number theory connected with rational points lying in certain domains. The proof uses ideas from metric Diophantine approximation.

Given an interval on the real line, how many integer polynomials of given height and degree will contain a root in that interval? This is obviously equivalent to considering the distribution of algebraic numbers of given height and degree and is indeed a well known problem. In this paper we generalise the question as follows. Given three equal intervals on the real line, how many integer polynomials of given height and degree pass through all three intervals, i.e. contain a root in all three intervals. Again, this is obviously equivalent to considering the distribution of algebraic conjugate triples in cubes in $\mathbb{R}^3$.

The distribution of algebraic conjugate pairs was considered in [8]. Unfortunately that paper contains an omission in the proof concerning reducible polynomials which will be covered in this paper. We also hope that the proof contained in this paper will be more adaptable to proving further results; in particular it may extend to more than three dimensions and it may also be possible to consider more general rectangular boxes rather than just cubes.

As will be seen below our proof only gives results for certain generic boxes. In particular we will omit boxes containing rational points of small denominator. We are uncertain as to whether this is a lack in our proof or that the result does not actually hold for such boxes. Particular details will be given after the statements of the theorems.

Fix $\epsilon_0, \epsilon_1 > 0$ as sufficiently small real numbers and let $\lambda \in \mathbb{R}$ be such that $0 < \lambda < 1/3$. Choose Q_0 large enough so that $Q_0^{-\epsilon_1} < \epsilon_0/2$. (Other conditions on Q_0 will be determined throughout the paper.) Consider the set of $Q_0^{\lambda+2\varepsilon}$ rational points $p/q \in [0,1]$ with $q < Q_0^{\lambda/2+\varepsilon}$. Let A_λ be the union of intervals centred at these points such that if $|q| < \epsilon_0^{-1/3}$ the length of the interval is $2\epsilon_0$ and if $|q| \geq \epsilon_0^{-1/3}$ the length of the interval is $|q|^{-3}$. Let I_1, I_2, I_3 be intervals contained in $[0,1]$ of length $Q^{-\lambda}$ for $0 < \lambda < 1/3$. Then, for $Q > Q_0$ the box $\Pi^3_\lambda(Q)$ is considered where

$$\Pi^3_\lambda(Q) = I_1 \times I_2 \times I_3 = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \subset [\epsilon_0, 1-\epsilon_0]^3 \subset \mathbb{R}^3$$

such that

$$\Pi^3_\lambda(Q) \cap \{(x_1, x_2, x_3) \in [0,1]^3 : |x_i - x_j| \leqslant \epsilon_0, 1 \leq i < j \leq 3\} = \emptyset \quad (1)$$

1

and

$$\Pi_\lambda^3(Q) \cap A_\lambda^3 = \emptyset. \tag{2}$$

Suppose that

$$|I_j| = b_j - a_j = Q^{-\lambda},\ 1 \leq j \leq 3.$$

Thus, if $(x, y, z) \in \Pi_\lambda^3(Q)$ then neither x, y or z are "close" to a rational with small denominator.

A point $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ is called an algebraic conjugate point if there exists $P(x) = a_n x^n + \ldots a_1 x + a_0 \in \mathbb{Z}[x]$ such that $P(\alpha_i) = 0$ for $i = 1, 2, 3$. The height of P will be denoted $H(P)$ and $H(P) = \max_{i=0,\ldots,n} |a_i|$. Define the class of polynomials $\mathcal{P}_n(Q)$ by

$$\mathcal{P}_n(Q) = \{P \in \mathbb{Z}[x] : \deg P = n, H(P) \leqslant Q\}. \tag{3}$$

The following theorem will be proved.

Theorem 1. *The cardinality of the set of algebraic conjugate points with $P \in \mathcal{P}_n(Q)$ lying in $\Pi_\lambda^3(Q)$ is*

$$\gg Q^{n+1-3\lambda}.$$

This theorem will follow easily from the theorem below, the proof of which constitutes most of the paper. PUT IT HOW IT FOLLOWS

Let $\delta_0 \in \mathbb{R}^+$. Denote by $L_n(\delta_0, Q)$ the set of $(x_1, x_2, x_3) \in \Pi_\lambda^3(Q)$ for which the system

$$|P(x_i)| < Q^{-\tau_i},\ |P'(x_i)| > \delta_0 Q,\ 1 \leq i \leq 3, \tag{4}$$

has a solution $P \in \mathcal{P}_n(Q)$ with $\sum_{i=1}^3 \tau_i = n - 2$, $\tau_i > 0$.

Theorem 2. *For any real number κ, $0 < \kappa < 1$, there exists $\delta_0 > 0$ such that*

$$\mu(L_n(\delta_0, Q)) > \kappa\mu(\Pi_\lambda^3(Q))$$

for sufficiently large Q.

This theorem cannot be arbitrarily improved as is shown in the following example. Let $P \in \mathcal{P}_3(X)$ where X is a fixed real number and assume that the roots $\beta_1, \beta_2, \beta_3$ of P lie in $[0, 1]^3$.

Consider the box $\Pi = [\beta_1 - Q^{-\mu_1}, \beta_1 + Q^{-\mu_1}] \times [\beta_2 - Q^{-\mu_2}, \beta_2 + Q^{-\mu_2}] \times [\beta_3 - Q^{-\mu_3}, \beta_3 + Q^{-\mu_3}]$ where $\mu_1 + \mu_2 + \mu_3 = 3 + \eta$ for some $\eta > 0$, $\mu_i > 0, i = 1, 2, 3$. Let $T \in \mathcal{P}_3(Q)$ be a second polynomial with no roots in common with P, with a triple of roots lying in Π and with all roots lying in some bounded interval $[-c, c]$ then the following contradiction

$$1 \leq |R(T, P)| \ll Q^3 Q^{-\mu_1 - \mu_2 - \mu_3} < Q^{-\eta}$$

is obtained where $R(T, P)$ is the resultant of T and P and the implied constant depends on X and c. Note that if $a_n(T) = H(T)$ then the roots of T will all lie in such a bounded interval as will be seen in Lemma 1.

More generally,

Theorem 3 (Roy and Waldschmidt [10]). *There exists a constant $c > 0$ and real numbers x_1, x_2, x_3 such that*

$$\max_{1\le i\le 3} |x_i - \alpha_i| \ge cH(\alpha)^{-3n^{1/3}}$$

for any choice of 3 distinct conjugates $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ of an algebraic number α of degree between 3 and n.

They actually proved a more general theorem for any number of disctict conjugates.

A different approach was taken by Beresnevich, Bernik and Götze (2010) who investigated how many polynomials of given degree and height have close real roots. Let $\mathcal{A}_n(Q,\mu)$ be the set of real algebraic numbers α_1 of degree n with height $H(\alpha) \asymp Q$ which have a real algebraic conjugate α_2 satisfying

$$|\alpha_1 - \alpha_2| \asymp Q^{-\mu}.$$

Theorem 4 (Beresnevich, Bernik and Götze [1]). *For any $0 < \mu \le (n+1)/3$ and any interval $J \subset [-1/2, 1/2]$*

$$\#(\mathcal{A}_n(Q,\mu) \cap J) \ge \frac{1}{2} Q^{n+1-2\mu} |J|.$$

It is expected that the upper bound for $\#(\mathcal{A}_n(Q,\mu)$ is of similar order. The main theorems proved in this paper do not address the question of close real roots as this region is specifically omitted in the definition of $\Pi^3_\lambda(Q)$; so in the event that the upper bound in both cases holds the results are non contradictory.

Throughout the proof of the main theorem we will need to consider other polynomials satisfying similar conditions which might be reducible. If $P \in \mathcal{P}_n(Q)$ is reducible so $P(x) = R(x)T(x)$ and P satisfies (4) then $\prod_{i=1}^3 |P(x_i)| < Q^{-(n-2)}$. Suppose that $\deg R = n_R$ and $\deg T = n_T$ with $n_R + n_T = n$. Then, it can be readily verified that either $\prod_{i=1}^3 |R(x_i)| < Q^{-(n_R-1)}$ or $\prod_{i=1}^3 |T(x_i)| < Q^{-(n_T-1)}$. Note that if $R(x) = bx + a$ is linear and $|b| < \epsilon_0^{-1/3}$ then the distance of x_i from the rational a/b is at least ϵ_0 from the definition of $\Pi^3_\lambda(Q)$. In this case $|bx_i + a| > \epsilon_0 |b|$ for $i = 1, 2, 3$. Thus the system

$$|T(x_i)| < \frac{Q^{-\tau_i}}{\epsilon_0 |b|} < \frac{Q^{-\tau_i}}{\epsilon_0}$$

is satisfied. In order to prove Theorem 2 it will first be necessary to prove the following very similar result. Let $J_n(Q)$ be the set of $(x_1, x_2, x_3) \in \Pi^3_\lambda(Q)$ for which the system

$$|P(x_i)| < \epsilon_0^{-1} Q^{-\tau_i}, i = 1, 2, 3 \tag{5}$$

has a solution $P \in \mathcal{P}_n(Q)$ for any triple (τ_1, τ_2, τ_3) with $\sum_{i=1}^3 \tau_i = n - 1$ and such that if $n = 1$ then $H(P) > \epsilon_0^{-1/3}$.

Theorem 5. *For any real number $0 < \kappa < 1$*

$$\mu(J_n(Q)) < \kappa \mu(\Pi^3_\lambda(Q))$$

for Q sufficiently large.

Notice that in this theorem the range $-1 \le \tau_i < 0$ is included and there is no condition on the derivative. This was the theorem omitted in [8]. From the discussion above it should be clear that when reducible polynomials are considered in the proof of Theorem 2 then it is not necessary to consider linear polynomials of "small" height.

1. Auxiliary statements

This section contains several lemmas that will be used in the proofs of Theorems 2 and 5. Throughout the proof the following notation will be used repeatedly. If α is a root of P then

$$S_P(\alpha) = \{x \in \mathbb{R} : |x-\alpha| = \min_{\beta : P(\beta)=0} |x-\beta|\}.$$

Clearly for each P, each $x \in \mathbb{R}$ belongs to at least one of these sets. We will also use the notation

$$S_P(\alpha_1,\alpha_2,\alpha_3) = S_P(\alpha_1) \times S_P(\alpha_2) \times S_P(\alpha_3).$$

For each polynomial $P \in \mathcal{P}_n(Q)$ three distinct roots are chosen which for convenience are labelled $\alpha_1(P), \alpha_2(P), \alpha_3(P)$. Where there is no confusion we will write $\alpha_1, \alpha_2, \alpha_3$.

Lemma 1 (see [11]). *Let $P(x) = a_n x^n + \ldots a_0$. If $|a_n| \gg H(P)$ then for any i, $1 \leqslant i \leqslant n$ there exists a constant $c(n) > 0$ such that*

$$|\alpha_i| < c(n);$$

i.e. the roots of P are bounded.

Lemma 2 ([11, 2]). *Let $x \in S_P(\alpha)$. Then*

$$|x-\alpha| \leqslant n\frac{|P(x)|}{|P'(x)|} \quad \textit{for} \ \ P'(x) \neq 0,$$

$$|x-\alpha| \leqslant 2^{n-1}|P(x)||P'(\alpha)|^{-1} \quad \textit{for} \ \ P'(\alpha) \neq 0, \tag{6}$$

and

$$|x-\alpha| \leqslant \min_{2\leqslant j\leqslant n}(2^{n-j}|P(x)||P'(\alpha)|^{-1}\prod_{k=2}^{j}|\alpha-\alpha_k|)^{\frac{1}{j}} \quad \textit{for} \ \ P'(\alpha) \neq 0$$

where $\alpha_2,\ldots,\alpha_n$ are the other roots of P. This is equivalent to

$$|x-\alpha|^j \leqslant \min_{2\leqslant j\leqslant n}(2^{n-j}j!\binom{n-1}{j-1}|P(x)||P^{(j)}(\alpha)|^{-1}). \tag{7}$$

Note that as there exists at least one j such that $|P^{(j)}(\alpha)| \gg |a_n|$ this implies that

$$|x-\alpha|^j \leqslant \min_{2\leqslant j\leqslant n}(2^{n-j}j!\binom{n-1}{j-1}|a_n|^{-1}|P(x)|). \tag{8}$$

There are several implications that can be drawn from this lemma. Suppose that either $\tau_i > 0$ or $|a_n| > Q^{-\tau_i+\varepsilon}$ for each i. If $x_i \in S_P(\alpha_i)$ for some $P \in \mathcal{P}_n(Q)$ then, from (7), there exists $\eta > 0$ such that

$$|x_i - \alpha_i| < |a_n|^{-1/n} Q^{-\tau_i/n} < Q^{-\eta}.$$

Thus as $|x_i - x_j| > \varepsilon_0$ we have $|\alpha_i - \alpha_j| > \varepsilon_0/2$ for $i, j = 1, 2, 3, i \neq j$. In particular if $n = 3$ then

$$|P'(\alpha_i)| = |a_3| \prod_{j=1}^{3} |\alpha_i - \alpha_j| > |a_3|\varepsilon_0^2/4. \tag{9}$$

Similarly, if $n = 4$ then consider the remaining root α_4. If $|\alpha_1 - \alpha_4| \leq \varepsilon_0/4$ then $|\alpha_i - \alpha_4| \geq \varepsilon_0/4$, $i = 2, 3$. Hence $|P'(\alpha_i)| \gg |a_4|$ for $i = 2, 3$. A similar argument will show that when $n = 5$ at least one of the roots satisfies $|P'(\alpha_i)| \gg |a_5|$.

More generally using the last inequality in the lemma this gives (using the fact that at least two of the root differences are large)

$$\begin{aligned} |x_i - \alpha_i| &\leq \varepsilon_0^{-2} \left(|P(x_i)||P'(\alpha_i)|^{-1} \prod_{k=2}^{n-2} |\alpha_i - \alpha_k| \right)^{1/(n-2)} \\ &\leq 2^{-2} Q^{-\frac{\tau_i}{n-2}} |a_n|^{-1/(n-2)} \varepsilon_0^{-2}. \end{aligned} \tag{10}$$

Define $u_i(P)$ as the real number such that

$$\min_{j=1,\ldots,n} \left(Q^{-\tau_i} |P^{(j)}(\alpha_i)|^{-1} \right)^{1/j} = Q^{-u_i(P)}. \tag{11}$$

It is not difficult to show that $\tau_i/n \leq u_i(P) \leq \tau_i + 1$ which is a finite range independent of Q. We divide the intervals $[\tau_i/n, \tau_i+1]$ into small intervals of length ε_1 and let $\mathcal{P}_n(Q, u_1, u_2, u_3)$ be the set of polynomials $P \in \mathcal{P}_n(Q)$ such that $u_i \leq u_i(P) \leq u_i + \varepsilon_1$ and ε_1 is chosen sufficiently small.

The next lemma is a version of a lemma originally proved by Bernik in [3]. In the original statements it was necessary that $\tau_i > 0$. This version of the lemma, which includes the possibility that $\tau_i \leq 0$ was proved very recently in [7].

Lemma 3. *Fix $\delta > 0$ and $Q_0(\delta)$. Suppose that $\eta_1, \eta_2, \eta_3 \in \mathbb{R}^+$ and let $P_1, P_2 \in \mathcal{P}_n(Q)$ where $Q > Q_0(\delta)$. Further suppose that P_1, P_2 have no roots in common. Let J_1, J_2 and J_3 denote intervals with lengths $|J_i| = Q^{-\eta_i}$. If there exist real numbers t_i, $i = 1, 2, 3$ such that for all $(x_1, x_2, x_3) \in J_1 \times J_2 \times J_3 \cap S_{P_l}(\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_j(P_l) \neq \alpha_k(P_l)$, $j \neq k$, $l = 1, 2$, $j, k \in \{1, 2, 3\}$*

$$\max(|P_1(x_i)|, |P_2(x_i)|) < Q^{-t_i}, \quad 1 \leq i \leq 3,$$

then

$$\max_{j=1,2,3} (\sum_{i=1}^{j} (t_i + 2\max(t_i + 1 - \eta_i, 0) + 1) < 2n + \delta. \tag{12}$$

This lemma will be used repeatedly throughout the proof to obtain contradictions. When it is known that the roots are separated the lemma can be used with $j = 3$ in (12). If it is only known that two of the roots are different then we will use $j = 2$. If nothing is known about the root separations then only the case $j = 1$ can be used. NOT QUITE RIGHT - SORT OUT!

In what follows it is often necessary to compare the value of the derivative of P at $x \in S_P(\alpha)$ with the derivative of P at α; the following lemma gives a general result.

Lemma 4. *Let $\omega_1, \omega_2 \in \mathbb{R}$ and $\omega_1 \geq 2\omega_2 + 1$ and $\omega_1 > 0$. Let $x \in S_P(\alpha)$ for some $P \in P_n(Q)$ and suppose that $|P(x)| < Q^{-\omega_1}$. If $|P'(x)| > 2n^3 Q^{-\omega_2}$ then*

$$|P'(x)|/2 < |P'(\alpha)| < 2|P'(x)|. \tag{13}$$

On the other hand, if $|P'(x)| \leq 2n^3 Q^{-\omega_2}$ then

$$|P'(\alpha)| < 2^{n+1} n^2 Q^{-\omega_2}. \tag{14}$$

Proof. Suppose that $x \in S_P(\alpha)$ for some $P \in P_n(Q)$ and that $|P(x)| < Q^{-\omega_1}$. Then, by Lemma 2

$$|x - \alpha| < nQ^{-\omega_1} |P'(x)|^{-1}.$$

By Taylor's formula

$$P'(x) = \sum_{j=1}^{n} ((j-1)!)^{-1} P^{(j)}(\alpha)(x - \alpha)^{j-1}.$$

First suppose that $|P'(x)| > 2n^3 Q^{-\omega_2}$. Then

$$|x - \alpha| < 2^{-1} n^{-2} Q^{\omega_2 - \omega_1}.$$

Estimating each term in the Taylor series for $2 \leq j \leq n$ by using the trivial estimate $|P^{(j)}(x)| \leq n^{j+1} Q$ and the fact that $x \in [0, 1]$ gives

$$(j-1)!^{-1} |P^{(j)}(\alpha)||x - \alpha|^{j-1} < (j-1)!^{-1} n^{j+1} Q (2^{-1} n^{-2} Q^{\omega_2 - \omega_1})^{j-1} \leq nQ^{-\omega_2},$$

which implies $\sum_{j=2}^{n} |(j-1)!^{-1} P^{(j)}(\alpha)(x - \alpha)^{j-1}| < n(n-1)Q^{-\omega_2}$. Thus,

$$|P'(x)|/2 < |P'(\alpha)| < 2|P'(x)|.$$

Now suppose that $|P'(x)| \leq 2n^3 Q^{-\omega_2}$. Then, again by Lemma 2 and using the Taylor series for P' it follows that $|P'(\alpha)| \leq |P'(x)| + \sum_{j=2}^{n} |(j-1)!^{-1} P^{(j)}(\alpha)(x - \alpha)^{j-1}| < 2^{n+1} n^2 Q^{-\omega_2}$. □

Next, we prove a short lemma on counting polynomials. Denote by d_i the centre of the interval I_i. Note that as $|x_i - x_j| > \varepsilon_0$ then $|d_i - d_j| > \epsilon_0/2$. For $\eta > 0$ define

$$M(a_n, \ldots, a_l; \eta) = \{P \in \mathcal{P}_n(Q) : a_j(P) = a_j, |P(d_j)| < \eta, j = l, \ldots, n\}.$$

Lemma 5.

$$\#M(a_n, \ldots, a_{n-k+1}) \ll \max(1, \eta^{n-k+1}).$$

Proof. Suppose that $P_1, \dots, P_t \in M(a_n, \dots, a_{n-k+1}; \eta)$ and construct the difference polynomials $R_i = P_i - P_1$ of degree at most $n-k$ which satisfy

$$|R_i(d_j)| < 2\eta$$

for $i = 2, \dots, t$. Hence $R_i(d_j) = \theta_i \eta$ for $j = k, \dots, n$ and $|\theta_i| < 2$. For each i this is a set of $n-k+1$ simultaneous equations with unknowns $a_l(R_i)$ for $l = 0, \dots, n-k$. Given that $|d_j - d_k| > \varepsilon_0/2$, $j \neq k$ we obtain using Cramer's rule that $|a_l(R_i)| \ll \eta$ for all $i = 2, \dots, t$. Thus

$$|a_l(P_i) - a_l(P_1)| \ll \eta$$

and

$$\#M(a_n, \dots, a_{n-k+1}; \eta) \ll \max(1, \eta^{n-k+1})$$

as required. □

Finally, we include a lemma by Mahler [9] regarding the distance between roots of polynomials.

Lemma 6. *Let $P \in \mathcal{P}_n(Q)$. Then $|\alpha_i - \alpha_j| \gg Q^{-n+1}$ for all roots α_i, α_j of P, $\alpha_i \neq \alpha_j$.*

Remark 1. *Before begining the proofs we first explain why we need only consider* leading *polynomials; that is those polynomials $P \in \mathbb{Z}[x]$ of degree n with $a_n(P) \gg H(P)$. It was shown in* [11] *that if a polynomial P does not satisfy $|a_n| \gg H(P)$ then a transformation $\overline{P}(x) = P(x+m)$ for some $0 \le m \le n$ can be performed followed by an inversion to obtain $\tilde{P}(x) = x^n \overline{P}(1/x)$. This new polynomial $\tilde{P}(x) = \sum_{i=0}^{n} b_i x^i$ satisfies $|b_n| \gg H(\tilde{P}) \asymp H(P)$. It can be readily verified that if $|x-y| < \eta$, $x, y \in I_i$, then*

$$|\tilde{x} - \tilde{y}| < \frac{\eta}{|x-m||y-m|} < \frac{\eta}{\epsilon_0^2}$$

where $\tilde{x} = (x-m)^{-1}$ and $\tilde{y} = (y-m)^{-1}$. Thus, by Lemma 2 these transformations preserve measures (up to a constant) of sets which satisfy inequalities of the form (4)*. Therefore, without loss of generality it will be assumed from now on that $|a_n(P)| \gg H(P)$.*

2. Reducible Polynomials — Proof of Theorem 5

Throughout we will say that the polynomial P *belongs to a set* S if there exists $(x_1, x_2, x_3) \in S$ such that (5) holds. Fix ε and ε_1 such that

$$Q^{-\varepsilon^2/4} < \kappa_1 \quad \text{and } \varepsilon_1 < \varepsilon/16n. \tag{15}$$

We begin by proving three lemmas which prove Theorem 5 for $P \in \mathcal{P}_n(Q)$ with $n = 1, 2$ and 3. The very easy linear case is done first. Although this case is easy it is also the reason that intervals around rational points with small denominators are excluded from $\Pi^3_\lambda(Q)$.

Lemma 7. *Fix $0 < \kappa < 1$. Define $J_1(Q) \subset \Pi^3_\lambda(Q)$ to be the set of points $(x_1, x_2, x_3) \in \Pi^3_\lambda(Q)$ for which the inequalities*

$$|P(x_i)| < \epsilon_0^{-1} Q^{-\tau_i}$$

hold for any triple (τ_1, τ_2, τ_3) with $\sum_{i=1}^3 \tau_i = 0$ for some $P \in \mathcal{P}_1(Q)$ with $H(P) > \epsilon_0^{-1/3}$. Then for sufficiently large Q

$$\mu(J_1(Q)) < \kappa\mu(\Pi^3_\lambda(Q)).$$

Proof. Without loss of generality suppose that $\tau_1 \leq \tau_2 \leq \tau_3$ and note that for a linear polynomial $P(x) = bx + a$ to satisfy $|P(x_i)| < Q^{-\tau_i}$ with $|x_i - x_j| > \epsilon_0$ for $1 \leq i < j \leq 3$ it is necessary that $|b| \leq Q^{-\tau_1+\varepsilon}$ (so $\tau_1 \leq 0$) and $\tau_1 \leq \tau_2 \leq \tau_1 + \varepsilon < \varepsilon$.

Consider the inequality, with $x \in I_3$,

$$|x - a/b| < Q^{-\tau_3}/|b|. \tag{16}$$

It is not difficult to show that this implies $|x - a/b| \leq |b|^{-3}$. Note that from the definition of $\Pi^3_\lambda(Q)$ every $x \in I_3$ satisfies

$$|x - a/b| > |b|^{-3}$$

when $|b| < Q^{\lambda/2+\varepsilon}$ so that (16) cannot be satisfied for these rationals. Thus we suppose that $|b| \geq Q^{\lambda/2+\varepsilon}$ (which implies that $\tau_1 < -\lambda/2$). The distance between two rationals p/q and p'/q' with $|q|, |q'| \leq Q^{-\tau_1+\varepsilon}$ is at least $Q^{2\tau_1-2\varepsilon}$. Thus the number of such rationals lying in I_3 is at most $\max(1, Q^{-\lambda-2\tau_1+2\varepsilon}) = Q^{-\lambda-2\tau_1+2\varepsilon}$. The measure of the set of $x_3 \in I_3$ which lie within $Q^{3\tau_1}$ of such a rational is at most

$$Q^{-\lambda-2\tau_1+2\varepsilon+3\tau_1} \ll Q^{-\lambda} Q^{\tau_1+2\varepsilon} \ll Q^{-\lambda-\varepsilon}$$

as $\tau_1 < -\lambda/2$.

The set of (τ_1, τ_2, τ_3) satisfying $\sum_{i=1}^3 \tau_i = 0$ with $-1 \leq \tau_i$ clearly has finite volume. Integrating over this set completes the proof of the lemma.

□

Now we consider the quadratic case. The proof of this lemma will involve the first demonstration of how Lemma 3 is used throughout the remainder of the paper.

Lemma 8. *Fix $0 < \kappa < 1$. Define $J_2(Q) \subset \Pi^3_\lambda(Q)$ to be the set of points $(x_1, x_2, x_3) \in \Pi^3_\lambda(Q)$ for which the inequalities*

$$|P(x_i)| < \epsilon_0^{-1} Q^{-\tau_i}$$

hold for any triple (τ_1, τ_2, τ_3) with $\sum_{i=1}^3 \tau_i = 1$ for some $P \in \mathcal{P}_2(Q)$. Then for sufficiently large Q

$$\mu(J_2(Q)) < \kappa\mu(\Pi^3_\lambda(Q)).$$

Proof. Given Lemma 7 it may be assumed that P is irreducible. Again, suppose without loss of generality that $\tau_1 \leq \tau_2 \leq \tau_3$ and note that for a quadratic polynomial $P(x) = a_2x^2 + a_1x + a_0$ to satisfy $|P(x_i)| < Q^{-\tau_i}$ with $|x_i - x_j| > \epsilon_0$ for $i = 1, 2, 3$ it is necessary that

$Q^{-\tau_1} \le |a_2| \le Q^{-\tau_1+\varepsilon}$ and $\tau_1 < 0$. Let $B(P)$ be the set of points $(x_1, x_2, x_3) \in \Pi^3_\lambda(Q)$ such that $|P(x_i)| < Q^{-\tau_i}$. A number of cases will be considered below and in each case it will be shown that the appropriate measure being considered is at most $Q^{-\varepsilon}Q^{-3\lambda} = Q^{-\varepsilon}\mu(\Pi^3_\lambda(Q))$. Putting each of the cases together will prove the result. If the polynomials are reducible then there exists a linear polynomial satisfying the inequalities in Lemma 7. Therefore it can be assumed that the polynomials are irreducible.

Suppose that $\tau_2 \ge \tau_1+\varepsilon$. Then, from Lemma 2, for $i = 2, 3$, $|x_i - \alpha_i| < Q^{-\tau_i/2}|a_2|^{-1/2} \ll Q^{(\tau_1-\tau_i)/2} < Q^{-\varepsilon}$. Again, as $|x_2 - x_3| > \epsilon_0$ this means that the two roots α_2 and α_3 of P are distinct and satisfy $|\alpha_2 - \alpha_3| > \epsilon_0/2$. (The "third" root α_1 will equal either α_2 or α_3). Therefore $|P'(\alpha_i)| \gg |a_2| \gg Q^{-\tau_1}$, $i = 2, 3$. Thus, by Lemma 2

$$\mu(B(P)) \le Q^{-\lambda}Q^{-\tau_2-\tau_3}|a_2|^{-2} \le Q^{-\lambda}Q^{-\tau_2-\tau_2+2\tau_1}$$

so that

$$\begin{aligned}\mu\left(\bigcup_{P\in\mathcal{P}_2(Q^{-\tau_1+\varepsilon})} B(P)\right) &\ll Q^{-\lambda-\tau_2-\tau_3+2\tau_1}\sum_{P\in\mathcal{P}_2(Q^{-\tau_1+\varepsilon})} 1\\ &\ll Q^{-\lambda-\tau_1-\tau_2-\tau_3+3\varepsilon} = Q^{-1-\lambda+3\varepsilon} = Q^{-3\lambda}Q^{-1/3+2\varepsilon}\end{aligned}$$

for sufficiently large Q as $0 < \lambda < 1/3$.

Now suppose that $-1 + 6\varepsilon \le \tau_1 \le \tau_2 \le \tau_1 + \varepsilon$. Using the Mean Value theorem the inequality

$$|P(d_i)| < Q^{-\tau_i} + |P'(\xi)||I_i| \ll Q^{-\tau_i} + |a_2|Q^{-\lambda} \ll \max(Q^{-\tau_i}, |a_2|Q^{-\lambda})$$

is obtained (where d_i is the centre of the interval I_i). From this it should be clear that $|P(x_i)| \ll Q^{-\tau_i}$ for all $x_i \in I_i$, $i = 1, 2$. As

$$|a_2d_3^2 + a_1d_3 + a_0| < Q^{-\tau_1+\varepsilon-\lambda}$$

it should be clear that $\#M(a_2, a_1; Q^{-\tau_1+\varepsilon-\lambda}) \ll \max(Q^{-\tau_1+\varepsilon-\lambda}, 1)$. By Lemma 6 we know that the distance between the two roots of P satisfies $|\alpha_1 - \alpha_2| \ge (Q^{-\tau_1+\varepsilon})^{-1} = Q^{\tau_1-\varepsilon}$. Thus $|P'(\alpha_i)| \gg |a_2|Q^{\tau_1-\varepsilon} \gg Q^{-\varepsilon}$. Then, by Lemma 2

$$\mu(B(P)) \ll Q^{-2\lambda}Q^{-\tau_3}|P'(\alpha_3)|^{-1} \ll Q^{-2\lambda}Q^{-1+2\tau_1+2\varepsilon}.$$

Thus

$$\mu\left(\bigcup_{P\in\mathcal{P}_2(Q^{-\tau_1})} B(P)\right) \ll \sum_{|a_i|\le Q^{-\tau_1+\varepsilon}, i=1,2}\ \sum_{P\in M(a_2,a_1;Q^{-\tau_1+\varepsilon-\lambda})} \mu(B(P))$$

$$\ll \begin{cases} Q^{-2\lambda}Q^{-1+2\tau_1+2\varepsilon}Q^{-3\tau_1+3\varepsilon-\lambda} = Q^{-3\lambda}Q^{-1-\tau_1+5\varepsilon} < Q^{-3\lambda-\varepsilon} & \text{if } -1+6\varepsilon \le \tau_1 < -\lambda+\varepsilon. \\ Q^{-2\lambda}Q^{-1+2\tau_1+2\varepsilon}Q^{-2\tau_1+2\varepsilon} = Q^{-3\lambda}Q^{-1+\lambda+4\varepsilon} & \text{if } \tau_1 \ge -\lambda+\varepsilon. \end{cases}$$

This leaves the case $\tau_1 \le \tau_2 \le \tau_1+\varepsilon$, $\tau_1 \le -1+6\varepsilon$ with $|a_2| \ll Q^{-\tau_1+\varepsilon}$. First the "large" derivative case is considered. Suppose that $x_3 \in S_P(\alpha_3)$ and that $|P'(\alpha_3)| \gg Q^{-1-\tau_1+5\varepsilon}$;

then $\mu(B(P)) \ll Q^{-2\lambda}Q^{-\tau_3+1+\tau_1-5\varepsilon}$. The set of (x_1,x_2,x_3) satisfying (5) for some $P \in \mathcal{P}_2(Q^{-\tau_1})$ with this derivative condition is at most

$$\begin{aligned}\sum_{P\in\mathcal{P}_2(Q^{-\tau_1+\varepsilon})} \mu(B(P)) &\ll \sum_{|a_i|<Q^{-\tau_1+\varepsilon}} \sum_{P\in M(a_2,a_1;Q^{-\tau_1+\varepsilon-\lambda})} Q^{-2\lambda-\tau_3+1+\tau_1-5\varepsilon} \\ &\ll Q^{-3\lambda}Q^{-\tau_3+1+\tau_1-5\varepsilon-3\tau_1+3\varepsilon} \ll Q^{-3\lambda-\varepsilon}.\end{aligned}$$

Moving to the small derivative case consider the set of $P \in \mathcal{P}_2(Q^{-\tau_1}, u_1, u_2, u_3)$ satisfying $|P'(\alpha_3)| \leq Q^{-1-\tau_1+5\varepsilon}$ where u_i is defined in (11). Divide $\Pi^3_\lambda(Q)$ into smaller boxes of sidelengths $\min(Q^{-u_1}, Q^{-\lambda})$, $\min(Q^{-\lambda}, Q^{-u_2})$ and $Q^{-u_3+\gamma}$ such that $u_3-\gamma>0$. Lemma 3 will now be used to show there cannot exist two polynomials P_1 and P_2 satisfying (5) together with the derivative condition. Suppose two such polynomials exists. It should be clear that $|P_i''(\alpha_j)| = 2|a_2| \ll Q^{-\tau_1+\varepsilon}$. Using the Taylor expansion of P about α_3 it is not difficult to show that $|P_i(x_3)| < Q^{-\tau_3}$ on an interval of length Q^{-u_3}, $i=1,2$. Thus there exist two polynomials P_1 and P_2 of height at most $Q^{-\tau_1+\varepsilon}$ which satisfy $|P_i(x_3)| < (Q^{-\tau_1+\varepsilon})^{-\tau_3/(-\tau_1+\varepsilon)}$ on an interval of length $(Q^{-\tau_1+\varepsilon})^{-u_3/(-\tau_1+\varepsilon)}$. Putting $\eta_3 = u_3/(\varepsilon-\tau_1)$ and $t_3 = \tau_3/(\varepsilon-\tau_1)$, Lemma 3 gives that

$$\frac{\tau_3}{(\varepsilon-\tau_1)} + 1 + 2\left(\frac{\tau_3}{(\varepsilon-\tau_1)} + 1 - \frac{u_3}{(\varepsilon-\tau_1)}\right) < 4+\delta$$

for all $\delta>0$. If the minimum in the definiton of u_3, is at $j=1$ then $u_3 \leq \tau_3 - 1 - \tau_1 + 5\varepsilon$ and

$$\frac{\tau_3}{(\varepsilon-\tau_1)} + 1 + 2\left(\frac{\tau_3}{(\varepsilon-\tau_1)} + 1 - \frac{u_3}{(\varepsilon-\tau_1)}\right) - 4 \geq 3+\tau_1-12\varepsilon > 1$$

which is a contradiction. If, on the other hand, the minimum in the definition of u_3 is at $j=2$ then $u_3 \leq (\tau_3+1)/2$ and

$$\frac{\tau_3}{(\varepsilon-\tau_1)} + 1 + 2\left(\frac{\tau_3}{(\varepsilon-\tau_1)} + 1 - \frac{u_3}{(\varepsilon-\tau_1)}\right) - 4 \geq 2\tau_3+\tau_1-1-\varepsilon$$

which is again a contradiction. Thus at most one polynomial belongs to each box. Hence, by Lemma 2 the total measure of the set of (x_1,x_2,x_3) satisfying (5) together with the small derivative condition for some $P \in \mathcal{P}_2(Q^{-\tau_1}, u_1, u_2, u_3)$ is at most

$$Q^{-2\lambda-u_3}Q^{-2\lambda+u_3-\gamma} \ll Q^{-3\lambda-\gamma}.$$

As there are at most a finite number of tuples (u_1,u_2,u_3) the measure of the set of (x_1,x_2,x_3) satisfying (5) together with the small derivative condition for some $P \in \mathcal{P}_2(Q^{-\tau_1})$ is at most $Q^{-\gamma}Q^{-3\lambda}$.

Adding up the measures over all cases gives that $\mu(J_2(Q)) < \kappa\mu(\Pi^3_\lambda(Q))$ as required.

The set of (τ_1,τ_2,τ_3) satisfying $\sum_{i=1}^3 \tau_i = 1$ with $-1 \leq \tau_i$ clearly has finite volume. Integrating over this set completes the proof of the lemma.

□

The next lemma for $n=3$ is very important and, together with the previous two lemmas, will be the base of an induction argument. In many ways the proof of this lemma is the proof of Theorems 2 and 5 writ small and all of the techniques used in the proof will also be used in the proofs of those theorems.

Lemma 9. *Fix $0<\kappa<1$. Define $J_3(Q)\subset\Pi_\lambda^3(Q)$ to be the set of points $(x_1,x_2,x_3)\in\Pi_\lambda^3(Q)$ for which the inequalities*

$$|P(x_i)|<\epsilon_0^{-1}Q^{-\tau_i}$$

hold for any triple (τ_1,τ_2,τ_3) with $\sum_{i=1}^3\tau_i=2$ for some $P\in\mathcal{P}_3(Q)$. Then for sufficiently large Q

$$\mu(J_3(Q))<\kappa\mu(\Pi_\lambda^3(Q)).$$

Proof. Suppose throughout that $\tau_1\le\tau_2\le\tau_3$. It will also be assumed that $0<\varepsilon<\frac{1-3\lambda}{8}$ and that Q is chosen sufficiently large so that $Q^{-\varepsilon/3}<\min(\kappa,\epsilon_0/2)$ and $\log Q<Q^{\varepsilon}$.

There are two main cases and the first one is not difficult.

Case 1. Let $P\in\mathcal{P}_3(Q)$ and define

$$B(P)=\{\mathbf{x}\in\Pi_\lambda^3(Q):|P(x_i)|<Q^{-\tau_i}\}.$$

Suppose that $\tau_1>\varepsilon$ or $|a_3|>Q^{-\tau_i+\varepsilon}$ then for $x_i\in S_P(\alpha_i)$ from Lemma (2)

$$|x_i-\alpha_i|<(|P(x_i)||a_3|^{-1})^{1/3}<|a_3|^{-1/3}Q^{-\tau_i/3}<Q^{-\varepsilon/3}.$$

Thus $\alpha_i(P)\ne\alpha_j(P)$, $i\ne j$ and

$$|\alpha_i-\alpha_j|>\epsilon_0/2;$$

i.e. the three roots are well separated. Since $|P'(\alpha_i)|=|a_3|\prod_{j=1;j\ne i}^k|\alpha_i-\alpha_j|$ it should be clear that

$$|P'(\alpha_i)|>2^{-2}\epsilon_0^2|a_3|\ \text{ for }\ i=1,2,3.$$

From Lemma 2 therefore, for $x_i\in S_P(\alpha_i)$,

$$|x_i-\alpha_i|<4|P(x_i)||P'(\alpha_i)|^{-1}<16Q^{-\tau_i}\epsilon_0^{-2}|a_3|^{-1}$$

and

$$|B(P)|\ll\prod_{i=1}^3\min(Q^{-\lambda},Q^{-\tau_i}|a_3|^{-1})\ll Q^{-\tau_1-\tau_2-\tau_3}|a_3|^{-3}=Q^{-2}|a_3|^{-3}. \tag{17}$$

Note that it can be readily verified that the minima in the products above cannot be $Q^{-\lambda}$ for all three of them.

Let d_i be the centre of the interval I_i. By the Mean Value Theorem, as $|P(x_i)|<Q^{-\tau_i}$,

$$|P(d_i)|<Q^{-\tau_i}+|P'(\xi)||I_i|\ll Q^{-\tau_i}+|a_3|Q^{-\lambda}\ll\max(Q^{-\tau_i},|a_3|Q^{-\lambda}) \tag{18}$$

for some ξ lying between x_i and d_i. First suppose that $Q^{-\tau_2}\le|a_3|Q^{-\lambda}$. Then from Lemma 5 $\#M(a_3,a_2;|a_3|Q^{-\lambda})\ll\max(1,|a_3|^2Q^{-2\lambda})$. This gives

$$\sum_{|a_i|\le Q,i=2,3}\ \sum_{P\in M(a_3,a_2;|a_3|Q^{-\lambda})}\mu(B(P))\ll\sum_{|a_i|\le Q,i=2,3}Q^{-2}|a_3|^{-3}\max(1,|a_3|^2Q^{-2\lambda})$$

$$\ll \begin{cases} \sum_{|a_i|\le Q, i=2,3} Q^{-2-2\lambda}|a_3|^{-1} \ll Q^{-1-2\lambda}\log Q \ll Q^{-3\lambda}Q^{\lambda-1+\varepsilon}, & \text{if } |a_3|Q^{-\lambda} > 1 \\ \sum_{|a_i|\le Q^{\lambda}, i=2,3} Q^{-2}|a_3|^{-3} \ll Q^{\lambda-2} \ll Q^{-3\lambda}Q^{4\lambda-2} & \text{else.} \end{cases}$$

Now suppose that $|a_3|Q^{-\lambda} \le Q^{-\tau_2}$ then $P(x_i) \ll Q^{-\tau_i}$ for all $x_1 \in I_1$ and $x_2 \in I_2$ and from (17)

$$|B(P)| \ll Q^{-2\lambda-\tau_3}|a_3|^{-1}.$$

Also implied are the facts that $|a_1|, |a_2|, |a_3| \le \min(Q, Q^{\lambda-\tau_2})$ and $\tau_2 \le \lambda$. From Lemma 5 therefore, using $\tau_1 \le \tau_2$

$$\sum_{|a_i|\le\min(Q,Q^{\lambda-\tau_2}), i=1,2,3} \; \sum_{P\in M(a_3,a_2,a_1;|a_3|Q^{-\lambda})} \mu(B(P)) \le \sum_{|a_i|\le\min(Q,Q^{\lambda-\tau_2}), i=1,2,3} Q^{-2\lambda-\tau_3}|a_3|^{-1}\max(|a_3|Q^{-\lambda}, 1)$$

$$\le \begin{cases} Q^{-3\lambda}Q^{3\lambda-3\tau_2-\tau_3} \le Q^{-3\lambda}Q^{2\lambda-1} & \text{for } |a_3| > Q^{\lambda}, 0 \le \lambda-\tau_2 \le 1 \\ Q^{-3\lambda}Q^{3-\tau_3} \le Q^{-3\lambda}Q^{1+\tau_1+\tau_2} \le Q^{-3\lambda}Q^{2\lambda-1} & \text{for } |a_3| > Q^{\lambda}, \lambda-\tau_2 \ge 1 \\ Q^{-2\lambda-\tau_3}Q^{2\lambda}\log Q \le Q^{-3\lambda}Q^{-2+\tau_1+\tau_2+3\lambda+\varepsilon} \le Q^{-3\lambda}Q^{5\lambda-2+\varepsilon}, & |a_3| \le Q^{\lambda}. \end{cases}$$

The last inequality follows from the fact that $\tau_1 \le \tau_2 \le \lambda$.

Case 2. Suppose that $\tau_1 \le \varepsilon$ and $|a_3| < Q^{-\tau_1+\varepsilon}$. Without loss of generality it may be concluded that $|a_3| \ge Q^{-\tau_1}$ as otherwise a better inequality is obtained in (5). There are two subcases depending on the size of τ_2.

Subcase 2a. Suppose that $0 < \tau_2 \le \tau_1 + \varepsilon$. If $\tau_1 > -\varepsilon$ the number of polynomials with $H(P) = |a_3| < Q^{-\tau_1+\varepsilon} \le Q^{2\varepsilon}$ is at most $8Q^{8\varepsilon}$ and by Lemma 2

$$\mu(B(P)) \le Q^{-2\lambda}(Q^{-\tau_3}|a_3|^{-1})^{1/3}.$$

Summing up these measures gives

$$\sum_{P\in\mathcal{P}_3(Q^{2\varepsilon})} \mu(B(P)) \ll Q^{-2\lambda+8\varepsilon-\tau_3/3} \ll Q^{-3\lambda}Q^{\lambda-2/3+10\varepsilon}.$$

Note that if $0 < \tau_2 < \tau_1 + \varepsilon$ then $-\varepsilon < \tau_1 \le \varepsilon$.

Thus we need only consider $\tau_1 < -\varepsilon$. From Lemma 2

$$|x_i - \alpha_i| < Q^{(\tau_1-\tau_i)/3} < Q^{-\varepsilon}$$

for $i = 2, 3$. Hence, $|\alpha_2 - \alpha_3| > \epsilon_0/2$ as $|x_2 - x_3| > \epsilon_0$. If $|\alpha_1 - \alpha_2| < Q^{-\varepsilon}$ then $|\alpha_3 - \alpha_1| > \epsilon_0$. Therefore, for either $i = 2$ or $i = 3$ we know that $|P'(\alpha_i)| \gg |a_3| \gg Q^{-\tau_1}$. Suppose that $|P'(\alpha_i)| > Q^{-\tau_1}$ and $|P'(\alpha_j)| = Q^{v(P)}$ where either $i = 2$ and $j = 3$ or vice versa. From Lemma 6 $|\alpha_1 - \alpha_j| \gg (Q^{-\tau_1+\varepsilon})^{-2}$ so that

$$Q^{v(P)} = |P'(\alpha_j)| \gg Q^{-\tau_1}Q^{2\tau_1-2\varepsilon} = Q^{\tau_1-\varepsilon}$$

giving $v(P) > \tau_1 - \varepsilon$.

Note that if $|a_3| < Q^{-\tau_1+\varepsilon}$ then $|P(x)| < Q^{-\tau_1}$ for all $x \in I_1$. Thus

$$\mu(B(P)) \le Q^{-\lambda}\min(Q^{-\lambda}, |a_3|^{-1}Q^{-\tau_i})\min(Q^{-\lambda}, Q^{-\tau_j - v(P)}) \le Q^{-\lambda-\tau_2-\tau_3-v(P)}|a_3|^{-1}. \quad (19)$$

Consider first those polynomials with "large derivative" $v(P) > v = \max(\tau_1 - \varepsilon, -2-2\tau_1+4\varepsilon)$. For these polynomials from Lemma 5, as $\tau_2 > 0$,

$$\sum_{|a_3|,|a_2|\le Q^{-\tau_1+\varepsilon}} \sum_{P\in M(a_3,a_2;|a_3|Q^{-\lambda})} \mu(B(P)) \le \begin{cases} \displaystyle\sum_{|a_3|,|a_2|\le Q^{-\tau_1+\varepsilon}} Q^{-3\lambda-\tau_2-\tau_3-v}|a_3| & \text{if } |a_3| > Q^{\lambda} \\ \displaystyle\sum_{|a_3|,|a_2|\le Q^{-\tau_1+\varepsilon}} Q^{-\lambda-\tau_2-\tau_3-v}|a_3|^{-1} & \text{else} \end{cases}$$

$$\le \begin{cases} Q^{-3\lambda}Q^{-\tau_1-\tau_2-\tau_3-2\tau_1-v+3\varepsilon} < Q^{-3\lambda-\varepsilon} & \text{if } |a_3| \ge Q^{\lambda}, \\ Q^{-\lambda-2-v+\varepsilon}\log Q \le Q^{-3\lambda-2+3\lambda+2\varepsilon} \le Q^{-3\lambda}Q^{-1+2\varepsilon} & \text{else.} \end{cases}$$

The second inequality uses $v > \tau_1 - \varepsilon$.

Now consider those polynomials with "small derivative" $v(P) \le v = -2-2\tau_1+4\varepsilon$. First we restrict to considering polynomials in $\mathcal{P}_n(Q, u_1, u_2, u_3)$ (where u_i is defined in (11)) so that Lemma 3 can be used.

Divide $\Pi^3_\lambda(Q)$ into smaller boxes of sidelengths $Q^{-\lambda}$, Q^{-u_2}, and $Q^{-u_3+\gamma}$ for some small $\gamma > 0$ chosen so that $u_3 - \gamma > 0$. Using Lemma 2 and expanding P in a Taylor series about α_2 and α_3 respectively it is not difficult to show that $|P(x_2)| < Q^{-\tau_2}$ and $|P(x_3)| < Q^{-\tau_3}$ on intervals

$$|x_2 - \alpha_2| < Q^{-u_2} \text{ and } |x_3 - \alpha_3| < Q^{-u_3}$$

respectively. Suppose that two polynomials belong to one small box. As $\alpha_2 \ne \alpha_3$ it is known that either $|P'(\alpha_3)| = Q^v$ and $|P'(\alpha_2)| \gg Q^{-\tau_1}$ or $|P'(\alpha_3)| \gg Q^{-\tau_1}$ and $|P'(\alpha_2)| = Q^v$ with $v > \tau_1 - \varepsilon$. In Lemma 3 we use $\eta_i = u_i/(\varepsilon - \tau_1)$ and $t_i = \tau_i/(\varepsilon - \tau_1)$ for $i = 2, 3$. Then

$$\frac{\tau_2}{(\varepsilon-\tau_1)} + \frac{\tau_3}{(\varepsilon-\tau_1)} + 2 + 2\left(\frac{\tau_2}{(\varepsilon-\tau_1)} + 1 - \frac{u_2}{(\varepsilon-\tau_1)} + \frac{\tau_3}{(\varepsilon-\tau_1)} + 1 - \frac{u_3}{(\varepsilon-\tau_1)}\right) < 6 + \delta \quad (20)$$

for all $\delta > 0$. If the minimum in the definitions of u_2, u_3 is at $j = 1$ then

$$u_2 + u_3 \le \tau_2 + \tau_3 + (\varepsilon - \tau_1) + v.$$

Then, from (20)

$$\tau_2 + \tau_3 + 2\tau_1 - 2v < \delta.$$

Using $v = -2 - 2\tau_1 + 4\varepsilon$ gives that the LHS is at least

$$6 + 5\tau_1 - 8\varepsilon \ge 1 - 8\varepsilon$$

which is a contradiction.

If the minimum in the definitions of u_2, u_3 is at $j \ge 2$ then $u_i \le \frac{\tau_i + (\varepsilon - \tau_1)}{2}$ and again from (20)

$$4 = 2(\tau_1 + \tau_2 + \tau_3) < \delta$$

which is obviously a contradiction.

Now consider the mixed cases: $u_i \le \tau_i + v$ and $u_j \le (\tau_j + (\varepsilon - \tau_1))/2$. From (20)

$$2 - 2v + \tau_j < \delta$$

which is a contradiction as $v < 4\varepsilon$. The second mixed case has $u_i \le \tau_i + (\varepsilon - \tau_1)$ and $u_j \le (\tau_j + (\varepsilon - \tau_1))/2$. Again, from (20) this implies that

$$2 + 2\tau_1 + \tau_j < \delta$$

which is a contradiction as $\tau_j > 0$.

Thus, there is at most one polynomial in each box and the total measure of the set of $(x_1, x_2, x_3) \in \Pi^3_\lambda(Q)$ satisfying $|P(x_i)| < Q^{-\tau_i}$ for some $P \in \mathcal{P}_3(Q^{-\tau_1+\varepsilon}, u_1, u_2, u_3)$ is at most $Q^{-\lambda-u_2-u_3}Q^{-2\lambda+u_2+u_3-\gamma} \ll Q^{-3\lambda-\gamma}$. Remembering that there are only finitely many triples (u_1, u_2, u_3) proves that the total measure of the set of (x_1, x_2, x_3) satisfying $|P(x_i)| < Q^{-\tau_i}$ for some $P \in \mathcal{P}_3(Q^{-\tau_1+\varepsilon})$ is at most $Q^{-3\lambda-\gamma+\varepsilon}$.

Subcase 2b. Suppose from now on that $\tau_1 \le \tau_2 \le 0$. For each polynomial P define $v(P)$ so that $|P'(\alpha_3)| = Q^{v(P)}$.

Again there are two subcases:

Subcase 2bi: $Q^{-\tau_2} \ll Q^{-\tau_1-\lambda}$.

As $\tau_2 < 0$ this implies that $\tau_1 < -\lambda$ and that $\tau_2 > \tau_1+\lambda$. Thus, (19) holds. As $|a_3| > Q^{-\tau_1}$ we also have $|a_3| > Q^\lambda$ so from Lemma 5, $\#M(a_3, a_2; |a_3|Q^{-\lambda}) \ll \max(|a_3|^2Q^{-2\lambda}, 1) = |a_3|^2Q^{-2\lambda}$. Partition the polynomials as follows:

$$\begin{aligned} T_1 :\ \ v(P) &\le -2 - 2\tau_1 + 4\varepsilon \\ T_2 :\ \ v(P) &\ge -2 - 2\tau_1 + 4\varepsilon. \end{aligned}$$

For $P \in T_2$ the measure of the set of points satisfying $|P(x_3)| < Q^{-\tau_3}$ is at most

$$\begin{aligned} \sum_{|a_i|<Q^{-\tau_1+\varepsilon}, i=2,3}\ \sum_{P\in M(a_3,a_2;|a_3|Q^{-\lambda})} \mu(B(P)) &\ll \sum_{|a_i|<Q^{-\tau_1+\varepsilon}, i=2,3} Q^{-\lambda-\tau_2-\tau_3-v(P)}|a_3|Q^{-2\lambda} \\ &\ll Q^{-3\lambda-2-2\tau_1+2\varepsilon-v(P)} < Q^{-3\lambda-2\varepsilon} \end{aligned}$$

from (19).

Now consider those polynomials in $T_1 \cap \mathcal{P}_n(Q, u_1, u_2, u_3)$. Divide $\Pi^3_\lambda(Q)$ into smaller cuboids of sidelengths $Q^{-\lambda}$, $\min(Q^{-\lambda}, Q^{-u_2})$ and $Q^{-u_3-\gamma}$ where γ is chosen so that $u_3+\gamma > \lambda$. Suppose that two of these polynomials belong to the same box and use Lemma 3 with $\eta_2 = u_2/(\varepsilon - \tau_1)$, $\eta_3 = u_3/(\varepsilon - \tau_1)$, $t_2 = \tau_2/(\varepsilon - \tau_1)$ and $t_3 = \tau_3/(\varepsilon - \tau_1)$. If $j = 1$ in the definition of u_3 (see (11)) then $u_3 \le \tau_3 - 2 - 2\tau_1 + 4\varepsilon$ and

$$\frac{\tau_3}{(\varepsilon-\tau_1)} + \frac{\tau_2}{(\varepsilon-\tau_1)} + 2 + 2\left(1 + \frac{2+2\tau_1-4\varepsilon}{(\varepsilon-\tau_1)}\right) + 2\max\left(\frac{\tau_2}{(\varepsilon-\tau_1)} + 1 - \frac{u_2}{(\varepsilon-\tau_1)}, 0\right) < 6+\delta$$

for all $\delta > 0$. This implies that (using 0 in the maximum)

$$\tau_3 + \tau_2 + 6\tau_1 + 4 - 10\varepsilon < \delta$$

for all $\delta > 0$; i.e. that

$$6 + 5\tau_1 - 10\varepsilon < \delta$$

for all $\delta > 0$ which is clearly a contradiction.

If, on the other hand, $j > 1$ in the definition of u_3 then $u_3 \leq \frac{\tau_3+\tau_1+\varepsilon}{2}$ and as above from Lemma 3

$$\frac{\tau_3}{(\varepsilon-\tau_1)} + \frac{\tau_2}{(\varepsilon-\tau_1)} + 2 + 2\left(\frac{\tau_3}{(\varepsilon-\tau_1)} + 1 - \frac{\tau_3+\tau_1+\varepsilon}{2(\varepsilon-\tau_1)}\right) < 6+\delta$$

for all $\delta > 0$. This implies that

$$2 + \tau_3 - \varepsilon < \delta$$

for all $\delta > 0$ which is again a contradiction.

Thus, there is at most one polynomial belonging to each of these smaller cuboids and the total measure of the set of points satisfying (5) is at most $Q^{-3\lambda-\gamma}$.

Subcase 2bii: $Q^{-\tau_1-\lambda} \ll Q^{-\tau_2}$.

Here we have $\tau_2 - \tau_1 < \lambda$. In this situation we have no information about $P'(\alpha_3)$ and $|P(x_i)| < Q^{-\tau_i}$ for all $x_i \in I_i$, $i = 1, 2$. From Lemma 5 $\#M(a_3, a_2, a_1; |a_3|Q^{-\lambda}) \leq \max(Q^{-\tau_1+\varepsilon-\lambda}, 1)$. The two possibilities will be considered separately. First suppose that $Q^{-\tau_1+\varepsilon-\lambda} \geq 1$ so that $\tau_1 < -\lambda + \varepsilon$.

As before partition the polynomials as follows:

$$\begin{aligned} T_1: \quad v(P) &\leq -2 - 2\tau_1 + \lambda + 5\varepsilon \\ T_2: \quad v(P) &\geq -2 - 2\tau_1 + \lambda + 5\varepsilon. \end{aligned}$$

Let $P \in T_2$ then from Lemma 2

$$\mu(B(P)) \leq Q^{-2\lambda-\tau_3-v(P)}$$

and

$$\begin{aligned} \sum_{|a_i|\leq Q^{-\tau_1+\varepsilon}, i=1,2,3} \ \sum_{P\in M(a_3,a_2,a_1;|a_3|Q^{-\lambda})} \mu(B(P)) &\ll Q^{-2\lambda-\tau_3-(-2-2\tau_1+\lambda+5\varepsilon)} Q^{-\tau_1+\varepsilon-\lambda} \sum_{|a_i|\leq Q^{-\tau_1+\varepsilon}, i=1,2,3} 1 \\ &\ll Q^{-3\lambda} Q^{3\tau_1-4\varepsilon} Q^{-3\tau_1+3\varepsilon} \ll Q^{-3\lambda-\varepsilon}. \end{aligned}$$

Now consider those polynomials in $T_1 \cap \mathcal{P}_n(Q, u_1, u_2, u_3)$. Divide $\Pi^3_\lambda(Q)$ into smaller cuboids of sidelengths $Q^{-\lambda}$, $Q^{-\lambda}$ and $Q^{-u_3-\gamma}$ where γ is chosen so that $u_3 + \gamma > \lambda$. Suppose that two of these polynomials belong to the same box.

If, $j > 1$ in the definition of u_3 then $u_3 \leq \frac{\tau_3-\tau_1}{2}$ and from Lemma 3 with $t_3 = \tau_3/(\varepsilon - \tau_1)$ and $\eta_3 = u_3/(\varepsilon - \tau_1)$,

$$\frac{\tau_3}{(\varepsilon-\tau_1)} + 1 + 2\left(\frac{\tau_3}{(\varepsilon-\tau_1)} + 1 - \frac{\tau_3-\tau_1}{2(\varepsilon-\tau_1)}\right) < 6+\delta$$

for all $\delta > 0$. This implies that

$$2\tau_3 + 4\tau_1 - 3\varepsilon < \delta$$

so that

$$4-2\tau_2+2\tau_1-3\varepsilon<\delta$$

for all $\delta>0$ which is a contradiction.

Suppose therefore that $j=1$ in the definition of u_3. If $\tau_2>\tau_1+\varepsilon$ then the roots α_2 and α_3 are separated and we can use Lemma 3 with $\eta_2=\lambda/(\varepsilon-\tau_1)$, $t_2=\tau_2/(\varepsilon-\tau_1)$, $\eta_3=u_3/(\varepsilon-\tau_1)$ and $t_3=\tau_3/(\varepsilon-\tau_1)$. If $j=1$ in the definition of u_3 (see (11)) then $u_3\le\tau_3+v(P)$ and

$$\frac{\tau_2}{(\varepsilon-\tau_1)}+\frac{\tau_3}{(\varepsilon-\tau_1)}+2+2\left(\frac{\tau_3}{(\varepsilon-\tau_1)}+1-\frac{u_3}{(\varepsilon-\tau_1)}\right)+2\max\left(\frac{\tau_2-\lambda}{(\varepsilon-\tau_1)}+1,0\right)<6+\delta$$

for all $\delta>0$. This implies that

$$2+\tau_1-2v(P)-2\varepsilon<\delta$$

giving that

$$6+5\tau_1-2\lambda-12\varepsilon<\delta$$

for all $\delta>0$ which is a contradiction as $\tau_1\ge-1$ and $\lambda<1/3$.

If, finally for this case, $\tau_2\le\tau_1+\varepsilon$ then we know nothing about how the roots α_2 and α_3 are distributed and we can only use Lemma 3 for the case of the single variable x_3 with $\eta_3=u_3/(\varepsilon-\tau_1)$ and $t_3=\tau_3/(\varepsilon-\tau_1)$ to obtain

$$\frac{\tau_3}{(\varepsilon-\tau_1)}+1+2\left(\frac{\tau_3}{(\varepsilon-\tau_1)}+1-\frac{u_3}{(\varepsilon-\tau_1)}\right)<6+\delta$$

for all $\delta>0$. This implies, as $u_3\le\tau_3+v$, that

$$\tau_3+3\tau_1-2v(P)-3\varepsilon<\delta$$

giving that

$$6+5\tau_1-2\lambda-14\varepsilon<\delta$$

for all $\delta>0$ which is a contradiction as $\tau_1\ge-1$ and $\lambda<1/3$.

Thus, there is at most one polynomial belonging to each of these smaller cuboids and the total measure of the set of points satisfying (5) is at most $Q^{-3\lambda-\gamma}$.

Now suppose $Q^{-\tau_1+\varepsilon-\lambda}\le1$ and again partition the polynomials into sets T_1, T_2 such that

$$\begin{array}{lll} T_1: & v(P) & \le-\tau_3+4\lambda+\varepsilon\\ T_2: & v(P) & \ge-\tau_3+4\lambda+\varepsilon \end{array}.$$

First consider those polynomials in T_2. Then from Lemma 2

$$\mu(B(P))\le Q^{-2\lambda-\tau_3-v(P)}.$$

Thus

$$\sum_{|a_i|\le Q^\lambda,i=1,2,3}\ \sum_{P\in M(a_3,a_2,a_1;|a_3|Q^{-\lambda})}\mu(B(P))\le Q^{-2\lambda}Q^{-\tau_3+\tau_3-4\lambda-\varepsilon}Q^{3\lambda}\le Q^{-3\lambda-\varepsilon}.$$

Now consider those polynomials in $\mathcal{P}_n(Q, u_1, u_2, u_3) \cap T_1$ and divide $\Pi^3_\lambda(Q)$ into smaller cuboids of sidelengths $Q^{-\lambda}$, $Q^{-\lambda}$ and $Q^{-u_3-\gamma}$ where γ is chosen so that $u_3+\gamma > \lambda$. Suppose that two such polynomials belong to the same box and use Lemma 3 with $\eta_3 = u_3/(\varepsilon - \tau_1)$ and $t_3 = \tau_3/(\varepsilon - \tau_1)$. If $j = 1$ then $u_3 \le \tau_3 - \tau_3 + 4\lambda + \varepsilon = 4\lambda + \varepsilon$. Then

$$\frac{\tau_3}{(\varepsilon - \tau_1)} + 1 + 2\left(\frac{\tau_3}{(\varepsilon - \tau_1)} + 1 - \frac{4\lambda + \varepsilon}{(\varepsilon - \tau_1)}\right) < 6 + \delta$$

for all $\delta > 0$. This implies that

$$3\tau_3 + 3\tau_1 - 8\lambda - 5\varepsilon < \delta$$

which is a contradiction as $\tau_3 \ge 2$, $\tau_1 > -1$ and $\lambda < 1/3$.

If on the other hand $j \ge 2$ then $u_3 \le \frac{\tau_3 - \tau_1}{2}$ which leads to the inequality

$$\frac{\tau_3}{(\varepsilon - \tau_1)} + 1 + 2\left(\frac{\tau_3}{(\varepsilon - \tau_1)} + 1 - \frac{(\tau_3 - \tau_1)}{2(\varepsilon - \tau_1)}\right) < 6 + \delta$$

for all $\delta > 0$ so that

$$2\tau_3 + 4\tau_1 - 3\varepsilon < \delta.$$

This gives

$$4 - 2\tau_2 + 2\tau_1 - 3\varepsilon < \delta$$

which again is a contradiction as $0 > \tau_2 \ge \tau_1 \ge -\lambda + \varepsilon$.

Thus, there is at most one polynomial belonging to each of the small cuboids and the total measure of the set of points satisfying (5) is at most $Q^{-3\lambda-\gamma}$.

Putting together all of the inqualities for all cases yields that for Q sufficiently large $\mu(J_3(Q)) < \kappa\mu(\Pi^3_\lambda(Q))$.

The region defined by $-1 \le \tau_1 \le (n-1)/3$, $-1 \le \tau_2 \le (n-1)/3$ and $\tau_3 = n - 1 - \tau_1 - \tau_2$ clearly has finite volume. Integrating over this region proves the lemma.

□

Induction hypothesis.

Define $J_m(Q)$ to be the set of points $(x, y, z) \in \Pi^3_\lambda(Q)$ satifying (5) for some $P \in \mathcal{P}_m(Q)$ and any triple (τ_1, τ_2, τ_3) such that $\sum_{i=1}^3 \tau_i = m - 1$, $-1 \le \tau_1 \le \tau_2 \le \tau_3$. We now begin a proof by induction. The induction hypothesis is that for $1 \le m \le n - 1$ and $0 < \kappa < 1$

$$\mu(J_m(Q)) < \kappa\mu(\Pi^3_\lambda(Q)), \tag{21}$$

for Q sufficiently large. The base cases are Lemmas 7, 8 and 9.

First we consider $|\tau_1| < \varepsilon$ and deal with those polynomials in $\mathcal{P}_n(Q^\varepsilon)$.

Lemma 10. *Suppose that $|\tau_1| < \varepsilon$. The measure of the set of points $(x_1, x_2, x_3) \in \Pi^3_\lambda(Q)$ for which (5) holds for at least one $P \in \mathcal{P}_n(Q^\varepsilon)$ is at most $Q^{-3\lambda-\varepsilon}$.*

Proof. There are at most $3Q^{(n+1)\varepsilon}$ polynomials in $\mathcal{P}_n(Q^\varepsilon)$. Let $B(P)$ be the set of points $(x_1, x_2, x_3) \in \Pi^3_\lambda(Q)$ satisfying (5) for some $P \in \mathcal{P}_n(Q^\varepsilon)$. Then from Lemma 2

$$\mu(B(P)) \leq \prod_{i=1}^{3} \min(Q^{-\lambda,(Q^{-\tau_i}|a_n|^{-1})^{1/n})} \leq Q^{-\lambda} \min(Q^{-\lambda}, Q^{-\tau_2/n})Q^{-\tau_3/n}.$$

Thus, if $\tau_2 \leq n\lambda$ then

$$\mu(B(P)) \leq Q^{-2\lambda}Q^{-(n-1-\tau_1-\tau_2)/n} \leq Q^{-3\lambda}Q^{-1+\frac{1}{n}+2\lambda+\varepsilon} \leq Q^{-3\lambda-(n+2)\varepsilon}$$

for $n \geq 4$, $\lambda < 1/3$ and Q sufficiently large. Similarly, if $\tau_2 \geq n\lambda$ then the same inequality

$$\mu(B(P)) \leq Q^{-\lambda}Q^{-(n-1-\tau_1)/n} \leq Q^{-3\lambda-(n+2)\varepsilon}$$

also holds. Thus

$$\sum_{P\in\mathcal{P}_n(Q^\varepsilon)} \mu(B(P)) \ll Q^{(n+1)\varepsilon}Q^{-3\lambda-(n+2)\varepsilon} < Q^{-3\lambda-\varepsilon}.$$

□

From here on it will be assumed that one or both of $\varepsilon < |\tau_1|$ or $H(P) \geq Q^\varepsilon$ hold.

There follow three subsections depending on sizes of the derivatives of the polynomials at certain roots. Each of these subsections will contain three (or more propositions) depending on the respective signs of τ_1 and τ_2. The proofs in each section are very similar and will therefore only be done completely in the first case.

First the three partitions are detailed.

Partition A — $\tau_i \geq 0$ for $i = 1, 2, 3$. Let $v_i = 2\tau_i/(n-1)$. We say that $\alpha_i \in T^i_j$ if

$$\begin{array}{llcll}
T^i_1: & & & |P'(\alpha_i)| \leq Q^{\frac{1-\tau_i}{2}}, \\
T^i_2: & Q^{\frac{1-\tau_i}{2}} & < & |P'(\alpha_i)| < Q^{\frac{1-\tau_i}{2}+\frac{\tau_i}{2(n-1)}}, \\
T^i_l: & Q^{\frac{1-\tau_i}{2}+\frac{(l-2)\tau_i}{2(n-1)}} & < & |P'(\alpha_i)| \leq Q^{\frac{1-\tau_i}{2}+\frac{(l-1)\tau_i}{2(n-1)}}, & 2 \leq l \leq n-2, \\
T^i_{n-1}: & Q^{\frac{1-v_i}{2}} = Q^{\frac{1-\tau_i}{2}+\frac{(n-3)\tau_i}{2(n-1)}} & < & |P'(\alpha_i)| & .
\end{array}$$

Partition B — $\tau_1 < 0$, $\tau_i \geq 0$ for $i = 2, 3$. Let $v_1 = \tau_1$ and $v_i = \frac{(2-\tau_1)\tau_i}{n-1-\tau_1}$. We say that $\alpha_i \in T^i_j$, $i = 2, 3$ if

$$\begin{array}{llcll}
T^i_1: & & & |P'(\alpha_i)| \leq Q^{\frac{1-\tau_i}{2}}, \\
T^i_2: & Q^{\frac{1-\tau_i}{2}} & < & |P'(\alpha_i)| < Q^{\frac{1-\tau_i}{2}+\frac{\tau_i}{2(n-1-\tau_1)}}, \\
T^i_l: & Q^{\frac{1-\tau_i}{2}+\frac{(l-2)\tau_i}{2(n-1-\tau_1)}} & < & |P'(\alpha_i)| \leq Q^{\frac{1-\tau_i}{2}+\frac{(l-1)\tau_i}{2(n-1-\tau_1)}}, & 2 \leq l \leq n-2, \\
T^i_{n-1}: & Q^{\frac{1-v_i}{2}} = Q^{\frac{1-\tau_i}{2}+\frac{(n-3)\tau_i}{2(n-1-\tau_1)}} & < & |P'(\alpha_i)| & .
\end{array}$$

Partition C — $\tau_1, \tau_2 < 0$. Let $v_i = \tau_i$ for $i = 1, 2$ and $v_3 = \frac{(2-\tau_1-\tau_2)\tau_3}{n-1-\tau_1-\tau_2} = 2 - \tau_1 - \tau_2$. We say that $\alpha_3 \in T_j^3$ if

$$\begin{array}{llcll} T_1^3: & & & |P'(\alpha_3)| & \leq & Q^{\frac{1-\tau_3}{2}}, \\ T_2^3: & Q^{\frac{1-\tau_3}{2}} & < & |P'(\alpha_3)| & < & Q^{\frac{1-\tau_3}{2}+\frac{1}{2}}, \\ T_l^3: & Q^{\frac{1-\tau_3}{2}+\frac{(l-2)}{2}} & < & |P'(\alpha_3)| & \leq & Q^{\frac{1-\tau_3}{2}+\frac{(l-1)}{2}},\ 2 \leq l \leq n-2, \\ T_{n-1}^3: & Q^{\frac{1-v_3}{2}} = Q^{\frac{1-\tau_3}{2}+\frac{(n-3)}{2}} & < & |P'(\alpha_3)| & & . \end{array}$$

Note that in each case $\sum_{i=1}^3 v_i = 2$.

Throughout the proof Taylor series are used to estimate the values of $|P(x)|$. Almost always the value of $P^{(j)}(x)$ is taken as trivially satisfying $|P^{(j)}(x)| \ll Q$. The v_i have been chosen so that the estimate for the first derivative in the Taylor series and the estimate for the second derivative are almost the same.

2.1. **Large derivative.** Let $c > 1$ be a constant to be chosen later.

For Partition A with $l \in \{3, \dots, n-1\}$ define the set $J_{n,A1}(Q, l)$ of $(x_1, x_2, x_3) \in \Pi_\lambda^3(Q) \cap S_P(\alpha_1, \alpha_2, \alpha_3)$ for which (5) holds with

$$\alpha_3 \in T_l^3,\ \alpha_1, \alpha_2 \in \cup_{m=l}^{n-1} T_m^i.$$

Let $J_{n,A1}(Q) = \cup_{l=3}^{n-1} J_{n,A1}(Q, l)$ and for a polynomial $P \in \mathcal{P}_n(Q)$ define the set

$$\sigma_{A1}(P) := \{(x_1, x_2, x_3) \in \Pi_\lambda^3(Q) \cap S_P(\alpha_1, \alpha_2, \alpha_3) : |x_i - \alpha_i| < Q^{-\tau_i} |P'(\alpha_i)|^{-1}\}.$$

Define the numbers l_A^i, $i = 1, 2, 3$ by $l_A^i = \frac{(n-l+1)v_i}{2}$ and notice that

$$\sum_{i=1}^{3} l_A^i = n - l + 1,\ l_A^i \leq \tau_i, i = 1, 2, 3. \tag{22}$$

Also define the set

$$\sigma_{Al}(P) := \{(x_1, x_2, x_3) \in \Pi_\lambda^3(Q) \cap S_P(\alpha_1, \alpha_2, \alpha_3) : |x_i - \alpha_i| < cQ^{-l_A^i} |P'(\alpha_i)|^{-1}\}.$$

For Partition B with $l \in \{3, \dots, n-1\}$ define the set $J_{n,B1}(Q, l)$ of $(x_1, x_2, x_3) \in \Pi_\lambda^3(Q) \cap S_P(\alpha_1, \alpha_2, \alpha_3)$ for which the system

$$|P(x_i)| < Q^{-\tau_i},\ i = 1, 2, 3;\ \alpha_2, \alpha_3 \in \bigcup_{m=l}^{n-1} T_m^i,$$

has a solution $P \in \mathcal{P}_n(Q)$. Let $J_{n,B1}(Q) = \cup_{l=3}^{n-2} J_{n,B1}(Q, l)$ and

$$J_1 = \{x_1 \in I_1 : |x_1 - \alpha_1| < \min(Q^{-u_1}, Q^{-\lambda})\}.$$

For a polynomial $P \in \mathcal{P}_n(Q)$ define the set

$$\sigma_{B1}(P) = J_1 \times \{(x_2, x_3) \in \Pi_\lambda^2(Q) \cap S_P(\alpha_2, \alpha_3) : |x_i - \alpha_i| < Q^{-\tau_i} |P'(\alpha_i)|^{-1}, i = 2, 3\}.$$

Define the numbers l_B^i, $i = 2, 3$ as $l_B^i = \frac{(n-l+1-\tau_1)v_i}{2-\tau_1}$ and notice that

$$\tau_1 + l_B^2 + l_B^3 = n - l + 1,\ \ l_B^i \le \tau_i, i = 2, 3. \tag{23}$$

Further define

$$\sigma_{Bl}(P) = J_1 \times \{(x_2, x_3) \in \Pi_\lambda^2(Q) \cap S_P(\alpha_2, \alpha_3) : |x_i - \alpha_i| < cQ^{-l_B^i}|P'(\alpha_i)|^{-1}\}.$$

For Partition C with $l \in \{3, \dots, n-1\}$ define the set $J_{n,C1}(Q, l)$ of $(x_1, x_2, x_3) \in \Pi_\lambda^3(Q) \cap S_P(\alpha_1, \alpha_2, \alpha_3)$ for which the system

$$|P(x_i)| < Q^{-\tau_i},\ \ i = 1, 2, 3; \alpha_3 \in \cup_{m=l}^{n-1} T_m^3,$$

has a solution $P \in \mathcal{P}_n(Q)$. Let $J_{n,C1}(Q) = \cup_{l=3}^{n-2} J_{n,C1}(Q, l)$ and for $i = 1, 2$, let

$$J_i = \{x_i \in I_i : |x_i - \alpha_i| < \min(Q^{-u_i}, Q^{-\lambda})\}.$$

For a polynomial $P \in \mathcal{P}_n(Q)$ define the set

$$\sigma_{C1}(P) = J_1 \times J_2 \times \{x_3 \in \Pi_\lambda(Q) \cap S_P(\alpha_3) : |x_3 - \alpha_3| < Q^{-\tau_3}|P'(\alpha_3)|^{-1}\}.$$

Define the number l_C^3 as $l_C^3 = \frac{(n-l+1-\tau_1-\tau_2)\tau_3}{n-1-\tau_1-\tau_2}$ and notice that

$$\tau_1 + \tau_2 + l_C^3 = n - l + 1, l_C^3 \le \tau_3. \tag{24}$$

Furthermore, define

$$\sigma_{Cl}(P) = J_1 \times J_2 \times \{x_3 \in \Pi_\lambda^3(Q)_\lambda(Q) \cap S_P(\alpha_3) : |x_3 - \alpha_3| < cQ^{-l_C^3}|P'(\alpha_3)|^{-1}\}.$$

The idea used in each of the following propostions is that if the polynomial is small on $\sigma_{D1}(P)$ then it is also small on $\sigma_{Dl}(P)$ (where D represents A, B or C). It is not difficult to show that $Q^{-l_{Di}}|P'(\alpha_i)|^{-1} < Q^{-\lambda}$ and also, $\mu(\sigma_{D1}(P)) = c^{-3}Q^{-l+2}\mu(\sigma_{Dl}(P)) \le c^{-1}Q^{-l+2}\mu(\sigma_{Dl}(P))$.

Fix the $(l-2)$–tuple of coefficients $\mathbf{b}_l = (a_n, \dots, a_{n+3-l})$ and let the subclass of polynomials $P \in \mathcal{P}_n(Q)$ with the same $(l-2)$–tuple $\mathbf{b}_l$ be denoted by $\mathcal{P}_n(Q, \mathbf{b}_l)$. The sets $\sigma_{Dl}(P)$ will be divided into essential and inessential domains for $P \in \mathcal{P}_n(Q, \mathbf{b}_l)$. A set $\sigma_{Dl}(P)$ is called *essential* if $\mu(\sigma_{Dl}(P) \cap \sigma_{Dl}(\tilde{P})) < \mu(\sigma_{Dl}(P))/2$ for all $\tilde{P} \in \mathcal{P}_n(Q, \mathbf{b}_l)$ with $\tilde{P} \ne P$. Otherwise, it is called *inessential*.

Consider, the essential sets $\sigma_{Dl}(P)$ for each partition. By definition, and because $\mu(\sigma_{Dl}(P)) < \mu(\Pi_\lambda^3(Q))$,

$$\sum_{\substack{P \in \mathcal{P}_n(Q,\mathbf{b}_l) \\ \sigma_{Dl}(P)\text{essential}}} \mu(\sigma_{Dl}(P)) \le 2^3 \mu(\Pi_\lambda^3(Q)).$$

Using this, it follows that

$$\begin{aligned} \sum_{\mathbf{b}_l} \sum_{\substack{P \in \mathcal{P}_n(Q,\mathbf{b}_l) \\ \sigma_{Dl}(P)\text{essential}}} \mu(\sigma_{D1}(P)) &\le \sum_{\mathbf{b}_l} c^{-1}Q^{-l+2} \sum_{\substack{P \in \mathcal{P}_n(Q,\mathbf{b}_l) \\ \sigma_{Dl}(P)\text{essential}}} \mu(\sigma_{Dl}(P)) \\ &\le 2^3 2^{l-2} c^{-1} \mu(\Pi_\lambda^3(Q)) \le \kappa_1 \mu(\Pi_\lambda^3(Q)) \end{aligned} \tag{25}$$

for c chosen appropriately.

For convenience we introduce the following notation. If $\sigma_{Dl}(P)$ is inessential there exists $\tilde{P} \in \mathcal{P}_n(Q, \mathbf{b}_l)$, $P \neq \tilde{P}$ such that $\mu(\sigma_{Dl}(P, \tilde{P})) := \mu(\sigma_{Dl}(P) \cap \sigma_{Dl}(\tilde{P})) \geq \mu(\sigma_{Dl}(P))/2$.

Proposition 1. *For sufficiently large Q*

$$\mu(J_{n,A1}(Q)) < \kappa\mu(\Pi_\lambda^3(Q)).$$

Proof. Given (25) we need only consider the inessential sets $\sigma_{Al}(P)$. It can be readily verified that on $\sigma_{Al}(P, \tilde{P})$

$$|P'(\alpha_i)||x_i - \alpha_i| \leq cQ^{-l_i} \text{ and } |P^{(j)}(\alpha_i)||x_i - \alpha_i|^j \leq Q(cQ^{-l_A^i}Q^{\frac{\tau_i - 1}{2} - \frac{(l-2)\tau_i}{2(n-1)}})^j < cQ^{-l_A^i} \text{ for } j \geq 2.$$

Thus, using the Taylor expansion of P about α_i, on $\sigma_l(P, \tilde{P})$, $|P(x_i)| < ncQ^{-l_A^i}$.

Put $R(t) = P(t) - \tilde{P}(t)$ so that $\deg R \leq n + 2 - l$ and $H(R) \leq 2Q$. Then, in $\sigma_{Al}(P, \tilde{P})$ we have

$$|R(x_i)| < 4ncQ^{-l_A^i}, H(R) \leq 2Q, \tag{26}$$

Hence, by the induction hypothesis, given (22), the set of (x_1, x_2, x_3) lying in at least one inessential domain has measure at most $\kappa_1\mu(\Pi_\lambda^3(Q))$. This, together with (25) gives $\mu(J_{n,A1}(Q, l)) < 2\kappa_1\mu(\Pi_\lambda^3(Q))$. Therefore, by choosing κ_1 appropriately

$$\mu(J_{n,A1}(Q)) < \kappa\mu(\Pi_\lambda^3(Q)).$$

□

Proposition 2. *For sufficiently large Q*

$$\mu(J_{n,B1}(Q)) < \kappa\mu(\Pi_\lambda^3(Q)).$$

Proof. The proof follows that of Proposition 1.

Again, only the inessential sets need be considered and as before it can be readily verified that for $i = 2, 3$ on $\sigma_{Bl}(P, \tilde{P})$

$$|P^{(j)}(\alpha_i)||x_i - \alpha_i|^j \leq cQ^{-l_B^i}$$

so that on $\sigma_{Bl}(P, \tilde{P})$, $|P(x_i)| < Q^{-l_B^i}$, also $|P(x_1)| < Q^{-\tau_1}$.

Use precisely the same argument as in Proposition 1 to obtain polynomials R such that $\deg R \leq n + 2 - l$ with $H(R) \leq 2Q$ and such that on $\sigma_{Bl}(P, \tilde{P})$

$$|R(x_1)| < 2Q^{-\tau_1}, \quad |R(x_i)| < 4Q^{-l_B^i}, H(R) \leq 2Q, i = 2, 3.$$

As before, given (23)

$$\mu(J_{n,B1}(Q)) < \kappa\mu(\Pi_\lambda^3(Q))$$

for $n \geq 4$. □

Finally we deal with Partition C.

Proposition 3. *For sufficiently large Q*

$$\mu(J_{n,C1}(Q)) < \kappa\mu(\Pi^3_\lambda(Q)).$$

Proof. As before, it can be readily verified that on $\sigma_{Cl}(P,\tilde{P})$

$$|P^{(j)}(\alpha_3)||x_3-\alpha_3|^j \le cQQ^{-jl^3_C}Q^{j(\frac{\tau_3-1}{2}-\frac{(l-2)\tau_3}{2(n-1-\tau_1-\tau_2)})} < cQ^{-l^3_C}.$$

Follow the argument as in Proposition 1 to obtain a polynomial $R(t)=P(t)-\tilde{P}(t)$ with $\deg R \le n+2-l$ and $H(R)\le 2Q$ such that on $\sigma_{Cl}(P,\tilde{P})$

$$|R(x_3)| < 4Q^{-l^3_C}, \quad |R(x_i)| < 2Q^{-\tau_i},\ i=1,2.$$

Again, given (24),

$$\mu(J_{n,C1}(Q)) < \kappa\mu(\Pi^3_\lambda(Q)), n\ge 4.$$

□

2.2. **Small Derivative.** This section deals with the case when the derivatives of the polynomials at the roots are small. As in the previous section there are three propostions, one for each partition, and the arguments are very similar. In each case Lemma 3 is used to obtain a contradiction. By the induction assumption and Lemmas 7, 8 and 9 it may be assumed that the polynomials are irreducible. We will consider polynomials $P\in\mathcal{P}_n(Q,u_1,u_2,u_3)$ where $u_i\le u_i(P)\le u_i+\varepsilon$ and $u_i(P)$ is defined in (11).

For Partition A define the set $J_{n,A2}(Q)$ of $\mathbf{x}\in\Pi^3_\lambda(Q)\cap S_P(\alpha_1,\alpha_2,\alpha_3)$ for which (5) holds for some $P\in\mathcal{P}_n(Q)$ with $\alpha_i\in T^i_1\cup T^i_2$, $i=1,2,3$. Similarly, for Partition B define the set $J_{n,B2}(Q)$ of $\mathbf{x}\in\Pi^3_\lambda(Q)\cap S_P(\alpha_1,\alpha_2,\alpha_3)$ for which (5) holds for some $P\in\mathcal{P}_n(Q)$ with $\alpha_i\in T^i_1\cup T^i_2$, $i=2,3$.

For Partition A using Lemma 2, Lemma 10 and following an argument similar to that for obtaining (10) and to that in Lemma 9 we know that $|\alpha_i-\alpha_j|\ge\epsilon_0/2$ for $i,j\in\{1,2,3\}$, $i\ne j$. Thus by Lemma 2

$$|x_i-\alpha_i|\le Q^{-u_i(P)} \ll Q^{-\frac{\tau_i}{n-2}}|a_n|^{-1/(n-2)}$$

and, as $\sum_{i=1}^3\tau_i=n-1$ and $0<\lambda<1/3$, there exists at least one $i\in\{1,2,3\}$ for which

$$\min_{j=1,\dots,n}(|a_n|Q^{\tau_i})^{-1/j} < Q^{-\lambda}$$

and so at least one i for which $u_i(P)>\lambda$. For convenience suppose that $i=3$.

For Partition B using Lemma 2 and Lemma 10 and following a similar argument to the one above it can be shown know that $|\alpha_2-\alpha_3|\ge\epsilon_0/2$ and that $|\alpha_1-\alpha_i|>\epsilon_0/2$ for at least one of $i=2,3$. Thus by Lemma 2

$$|x_i-\alpha_i|\le Q^{-u_i(P)} \ll Q^{-\frac{\tau_i}{n-1}}|a_n|^{-1/(n-1)}$$

and, as $\sum_{i=1}^3\tau_i=n-1$ and $0<\lambda<1/3$, there exists at least one $i\in\{1,2,3\}$ for which

$$\min_{j=1,\dots,n}(|a_n|Q^{\tau_i})^{-1/j} < Q^{-\lambda}$$

and so at least one i for which $u_i(P) > \lambda$. Again for convenience suppose that $i = 3$.

For Partition C define the set $J_{n,C2}(Q)$ of $\mathbf{x} \in \Pi^3_\lambda(Q) \cap S_P(\alpha_1, \alpha_2, \alpha_3)$ for which (5) holds for some $P \in \mathcal{P}_n(Q)$ with $\alpha_3 \in T^3_1 \cup T^3_2$. It should be clear from Lemma 2 that since $\tau_3 = n - 1 - \tau_1 - \tau_2 \geq n - 1$ the inequality

$$|x_3 - \alpha_3| \leq Q^{-u_3(P)} \ll Q^{-\frac{\tau_3}{n}} |a_n|^{-1/n} \leq Q^{-1+1/n} < Q^{-\lambda} \tag{27}$$

holds implying that $u_3(P) > \lambda$.

In each case therefore a number $\gamma > 0$ can be chosen such that $u_3(P) - \gamma > \lambda$. Then, $\Pi^3_\lambda(Q)$ is divided into smaller boxes M_i with side lengths $\min(Q^{-u_1}, Q^{-\lambda})$, $\min(Q^{-u_2}, Q^{-\lambda})$, $Q^{-u_3-\gamma}$. There are $Q^{-3\lambda} Q^{u_3-\gamma} \max(Q^{u_1}, Q^\lambda) \max(Q^{u_2}, Q^\lambda)$ such boxes. Suppose that at most one P belongs to each box. Then, the set of points $\mathbf{x} \in \Pi^3_\lambda(Q)$ satisfying (5) for some $P \in \mathcal{P}_n(Q, u_1, u_2, u_3)$ has measure at most

$$Q^{-3\lambda} Q^{u_3-\gamma} \max(Q^{u_2}, Q^\lambda) \max(Q^{u_1}, Q^\lambda) Q^{-u_3} \min(Q^{-\lambda}, Q^{-u_2}) \min(Q^{-\lambda}, Q^{-u_1}) \leq Q^{-\gamma} \mu(\Pi^3_\lambda(Q)).$$

There are a finite number of sets $\mathcal{P}_n(Q, u_1, u_2, u_3)$ so the measure of the set of points $\mathbf{x} \in \Pi^3_\lambda(Q)$ satisfying (5) for some $P \in \mathcal{P}_n(Q)$ is at most $Q^{-\gamma} \mu(\Pi^3_\lambda(Q))$.

Proposition 4. *For each $\kappa > 0$*

$$\mu(J_{n,A2}(Q)) \leq \kappa \mu(\Pi^3_\lambda(Q))$$

for sufficiently large Q.

Proof. It will be shown that at most one polynomial $P \in \mathcal{P}_n(Q, u_1, u_2, u_3)$ with $\alpha_i(P) \in T^i_1 \cup T^i_2$ for $i = 1, 2, 3$ belongs to each box. Assume there are two such polynomials, P_1 and P_2, $P_1 \neq P_2$. Using Taylor series, it can be readily verified that on M,

$$|P_j(x_3)| \leq Q^{-\tau_3 + n\gamma + n\varepsilon_1}, \quad |P_j(x_i)| \leq Q^{-\tau_i + n\varepsilon_1} \tag{28}$$

for $i = 1, 2$, $j = 1, 2$ and sufficiently large Q. For P representing either P_1 or P_2 let j_i be such that

$$Q^{-\frac{\tau_i}{j_i}} |P^{(j)}(\alpha_i)|^{-1/j_i} = \min_{j=1,\ldots,n} Q^{-\frac{\tau_i}{j}} |P^{(j)}(\alpha_i)|^{-1/j};$$

i.e. j_i is the minimum in Lemma 2. If $j_i \geq 2$ then $u_i \leq \frac{1+\tau_i}{2}$. If $j_i = 1$ then

$$Q^{-u_i} = Q^{-\tau_i} |P'(\alpha_i)|^{-1} \geq Q^{-\frac{(\tau_i+1)}{2} - \frac{\tau_i}{2(n-1)}}$$

so

$$u_i \leq \frac{\tau_i + 1}{2} + \frac{\tau_i}{2(n-1)}.$$

Now, Lemma 3 is used with $\eta_i = u_i$, $i = 1, 2$ and $\eta_3 = u_3 - \gamma$, $t_i = \tau_i - n\varepsilon_1$ for $i = 1, 2$ and $t_3 = \tau_3 - n\gamma - n\varepsilon_1$

Then,

$$\sum_{i=1}^{3} (t_i + 1 + 2\max(t_i + 1 - \eta_i, 0)) =$$

$$\tau_1 + \tau_2 + \tau_3 + 3 - 3n\varepsilon_1 - n\gamma + 2(\tau_1 + \tau_2 + \tau_3 + 3 - n\gamma - 3n\varepsilon_1 - u_1 - u_2 - u_3 + \gamma)$$

$$\geq 2(n-1)+5-9n\varepsilon_1-3n\gamma+2\gamma = 2n+3-9n\varepsilon_1-3n\gamma+2\gamma.$$

From Lemma 3 we then have

$$2n+3-9n\varepsilon_1-3n\gamma+2\gamma \leq 2n+\delta$$

for all $\delta > 0$ which is clearly a contradiction. Thus at most one polynomial $P \in \mathcal{P}_n(Q)$ belongs to each M_i and

$$\mu(J_{n,A2}(Q)) \leq Q^{-3\lambda-\gamma}.$$

□

Next consider Partition B.

Proposition 5. *For each $\kappa > 0$*

$$\mu(J_{n,B2}(Q)) \leq \kappa\mu(\Pi^3_\lambda(Q))$$

for sufficiently large Q.

Proof. Again it is only necessary to show that at most one polynomial belongs to each small box. Exactly as before assume that $P_1, P_2 \in \mathcal{P}_n(Q, u_1, u_2, u_3)$ belong to some M say, with $P_1 \neq P_2$ to obtain that on M,

$$|P_j(x_2)| < Q^{-\tau_2+n\varepsilon_1}, \quad |P_j(x_3)| \leq Q^{-\tau_3+n\varepsilon_1+n\gamma}$$

for sufficiently large Q. Following the argument above use Lemma 3 with $\eta_2 = u_2$, $\eta_3 = u_3 - \gamma$, $t_2 = \tau_2 - n\varepsilon_1$ and $t_3 = \tau_3 - n\varepsilon_1 - n\gamma$. As before

$$u_i \leq \frac{\tau_i+1}{2} + \frac{\tau_i}{2(n-1-\tau_1)}, i = 2, 3.$$

Thus,

$$\sum_{i=2}^{3}(t_i+1+2\max(t_i+1-\eta_i, 0)) =$$

$$\tau_2+\tau_3+2-2n\varepsilon_1-n\gamma+2(\tau_2+\tau_3+2-n\gamma-2n\varepsilon_1+\gamma-u_2-u_3)$$

$$\geq 2(n-1)-2\tau_1+3-6n\varepsilon_1-3n\gamma+2\gamma = 2n+1-2\tau_1-6n\varepsilon_1-3n\gamma+2\gamma.$$

From Lemma 3 we then have

$$2n+1-2\tau_1-6n\varepsilon_1-3n\gamma+2\gamma \leq 2n+\delta$$

for all $\delta > 0$ which is clearly a contradiction as $\tau_1 < 0$. Thus at most one polynomial $P \in \mathcal{P}_n(Q)$ belongs to each M_i.

Hence

$$\mu(J_{n,B2}(Q)) \leq Q^{-3\lambda-\gamma}.$$

□

Remark 2. *Note that this proof is valid for any τ_1 satisfying*

$$1-2\tau_1-6n\varepsilon_1-3n\gamma+2\gamma \leq \delta;$$

i.e. for example if $0 \leq \tau_1 \leq 1/4$. This will be used later in Proposition 7.

Finally Partition C is considered.

Proposition 6. *For each* $\kappa > 0$

$$\mu(J_{n,C2}(Q)) \leq \kappa\mu(\Pi^3_\lambda(Q))$$

for sufficiently large Q.

Proof. Following the above arguments and assuming that $P_1, P_2 \in \mathcal{P}_n(Q, u_1, u_2, u_3)$ belong to some small box M say, it can be readily verified that on M, (28) holds for sufficiently large Q. If $\alpha_3 \in T^3_1$ then

$$u_3 \leq \frac{\tau_3 + 1}{2}$$

and if $\alpha_3 \in T^3_2$ then

$$u_3 \leq \frac{\tau_3 + 1}{2} + \frac{1}{2}$$

Suppose first that $H(P) \geq Q^{-\tau_1+\varepsilon}$. This implies, by Lemma 2 that if $(x_1, x_2, x_3) \in S_P(\alpha_1, \alpha_2, \alpha_3)$ then $|\alpha_i - \alpha_j| > \epsilon_0/2$, $i, j \in \{1, 2, 3\}$, $i \neq j$; i.e. the roots are different meaning that Lemma 3 can be used for the three variables (x_1, x_2, x_3) with $\eta_3 = u_3 - \gamma$, $t_3 = \tau_3 - n\gamma - n\varepsilon_1$ and $t_i = \tau_i - n\varepsilon_1$ for $i = 1, 2$. We have, using zero in the maximum for $i = 1$ and 2,

$$\begin{aligned} 2n + \delta \geq \sum_{i=1}^{3}(t_1 + 1) + 2\sum_{i=1}^{3}\max(t_i + 1 - \eta_i, 0) \ &\geq \ \sum_{i=1}^{3}\tau_i + 3 - 3n\varepsilon_1 - n\gamma + 2(\tau_3 + 1 - n\gamma - n\varepsilon_1 + \gamma \\ &\geq \ 2n + 1 - \tau_1 - \tau_2 - 3n\gamma - 5n\varepsilon_1 + 2\gamma. \end{aligned}$$

for all $\delta > 0$ which is a contradiction as $\tau_1, \tau_2 < 0$.

Remark 3. *Just as Remark 2 this proof works in exactly the same way for small positive* τ_1 *and* τ_2*, for instance* $\tau_1 + \tau_2 \leq 1/4$.

Now suppose that $H(P) \leq Q^{-\tau_1+\varepsilon}$; as it is unknown whether for $(x_1, x_2, x_3) \in S_P(\alpha_1, \alpha_2, \alpha_3)$ we have $\alpha_i \neq \alpha_j$, $i \neq j$ it is only possible to use Lemma 3 for the variable x_3 with $\eta_3 = \frac{u_3-\gamma}{\varepsilon-\tau_1}$ and $t_3 = \frac{\tau_3-n\gamma-n\varepsilon_1}{\varepsilon-\tau_1}$. From that lemma the inequality

$$t_3 + 1 + 2(t_3 + 1 - \eta_3) = \frac{1}{\varepsilon - \tau_1}(2n - 4 - 5\tau_1 - 2\tau_2 - 3n\gamma - 3n\varepsilon_1 + 2\gamma + 3\varepsilon) < 2n + \delta$$

must hold for all $\delta > 0$. Rearranging gives

$$(2n - 5)\tau_1 + 2n(1 - \varepsilon) - 4 - 2\tau_2 - 3n\gamma - 3n\varepsilon_1 + 2\gamma + 3\varepsilon < \delta(\varepsilon - \tau_1) < \delta(1 + \varepsilon)$$

so that

$$1 - 2n\varepsilon - 2\tau_2 - 3n\gamma - 3n\varepsilon_1 + 2\gamma + 3\varepsilon < \delta(1 + \varepsilon)$$

for all $\delta > 0$ which is clearly a contradiction.

Thus, in each box M there are at most two polynomials in $\mathcal{P}_n(Q, u_1, u_2, u_3)$ (one with "large" height and one with "small" height) which implies that

$$\mu(J_{n,C2}(Q)) \leq Q^{-3\lambda-\gamma}$$

as required. □

2.3. **Mixed derivatives.** In this section the case when the derivatives are "mixed" for Partitions A and B are considered. To that end for Partition A define the set $J_{n,A3}(Q,l)$ of $\mathbf{x} \in \Pi^3_\lambda(Q) \cap S_P(\alpha_1,\alpha_2,\alpha_3)$ for which (5) holds for some $P \in \mathcal{P}_n(Q)$ with $\alpha_i \in T^i_1 \cup T^i_2$ for $i = 1,2$ and $\alpha_3 \in T^3_l$ for some $l > 2$.

Proposition 7. *For sufficiently large* Q

$$\mu(J_{n,A3}(Q,l)) < \kappa\mu(\Pi^3_\lambda(Q)).$$

Proof. Suppose that $\mathbf{x} \in J_{n,A3}(Q,l)$. Then, there exists $P \in \mathcal{P}_n(Q)$ such that by Lemma 2

$$\begin{aligned} |x_1 - \alpha_1| &\leq Q^{-u_1(P)}, \\ |x_2 - \alpha_2| &\leq Q^{-u_2(P)}, \\ |x_3 - \alpha_3| &\leq Q^{-\tau_3 - \frac{(1-\tau_3)}{2} - \frac{(l-2)\tau_3}{2(n-1)}}. \end{aligned}$$

As in the Proposition 4 the polynomials are restricted to $P \in \mathcal{P}_n(Q,u_1,u_2,u_3)$ with, for $i = 1,2$, $u_i \leq \frac{\tau_i+1}{2} + \frac{\tau_i}{2(n-1)}$ as $\alpha_i \in T^i_1 \cup T^i_2$. From Remark 2 it may be assumed without loss of generality that $\frac{(2l-3)\tau_3}{2(n-1)} > 4\varepsilon$. Also, by Lemma 10, as in Proposition 4, it may be assumed that $|\alpha_i - \alpha_j| > \epsilon_0$ for $i,j \in \{1,2,3\}$, $i \neq j$. Divide $\Pi^3_\lambda(Q)$ into smaller cuboids of sidelengths $\min(Q^{-\lambda}, Q^{-u_1}), \min(Q^{-\lambda}, Q^{-u_2})$ and

$$Q^{-\frac{(n-l)\tau_3}{n-1} - \frac{(1-\tau_3)}{2} - \frac{(l-1)\tau_3}{2(n-1)} - \frac{1}{2}\max(\{\frac{(2l-3)\tau_3}{2(n-1)}\},\varepsilon/2)} = Q^{-w_3};$$

it is easy to show that $w_3 > \lambda$. Define

$$\theta = \frac{(2l-3)\tau_3}{2(n-1)} - \frac{3}{4}\max\left(\left\{\frac{(2l-3)}{2(n-1)}\tau_3\right\}, \varepsilon/2\right)$$

where the fractional part $\{\ \}$ is defined to be 1 if $\frac{(2l-3)\tau_3}{2(n-1)} \in \mathbb{N}$. First note that if $\theta < 1$ then, using Remark 2,

$$\frac{\varepsilon}{2} \leq \frac{(2l-3)\tau_3}{2(n-1)} = \left\{\frac{(2l-3)\tau_3}{2(n-1)}\right\} < 1 \tag{29}$$

which implies that $\theta > 0$.

If there are at most Q^θ polynomials in $\mathcal{P}_n(Q,u_1,u_2,u_3)$ belonging to each M_i then the measure of the set of $\mathbf{x} \in \Pi^3_\lambda(Q)$ satisfying the conditions in the proposition is at most

$$\begin{aligned} \mu(\Pi^3_\lambda(Q))Q^{\theta - \tau_3 - \frac{(1-\tau_3)}{2} - \frac{(l-2)\tau_3}{2(n-1)}}Q^{w_3} &= Q^{\left[\frac{(2l-3)\tau_3}{2(n-1)}\right] + \frac{3}{4}\max\left(\left\{\frac{(2l-3)\tau_3}{2(n-1)}\right\},\varepsilon/2\right) - \frac{(2l-3)\tau_3}{2(n-1)}}\mu(\Pi^3_\lambda(Q)) \\ &= Q^{-\frac{1}{4}\max\left(\left\{\frac{(2l-3)\tau_3}{2(n-1)}\right\},\varepsilon/2\right)}\mu(\Pi^3_\lambda(Q)) \\ &\leq Q^{-\varepsilon/8}\mu(\Pi^3_\lambda(Q)) \leq \kappa_1\mu(\Pi^3_\lambda(Q)) \end{aligned} \tag{30}$$

from (15).

Otherwise in at least one box, M say, there exist at least Q^θ polynomials. It is not difficult to show using Taylor series that in M each of these polynomials satisfy

$$|P(x_i)| < Q^{-\tau_i + n\varepsilon_1}, i = 1, 2 \text{ and } |P(x_3)| < Q^{-\frac{(n-l)\tau_3}{n-1} - \frac{1}{2}\max\left(\{\frac{(2l-3)\tau_3}{2(n-1)}\}, \varepsilon/2\right)}. \tag{31}$$

Suppose first that $\theta < 1$ and note (29). Also note that for $l \geq 3$ the sum of the powers in (31) satisfies

$$\begin{aligned} \tau_1 + \tau_2 + \frac{(n-l)\tau_3}{n-1} + \frac{1}{2}\frac{(2l-3)\tau_3}{2(n-1)} &= n - 1 - \frac{(2l-1)}{4(n-1)}\tau_3 \\ &\geq n - 1 - \frac{2l-1}{2(2l-3)} > n - 2 \end{aligned}$$

so that if P is reducible then there exists a polynomial S with $\deg S \leq n-1$, satisfying $S(x_i) < Q^{-t_i}$ with $t_1 + t_2 + t_3 > n - 1$. The measure of set of $(x_1, x_2, x_3) \in \Pi^3_\lambda(Q)$ which satisfy these conditions is at most $\kappa\mu(\Pi^3_\lambda(Q))$ by the induction assumption. On the other hand, if there are at least two irreducible P_i then using Lemma 3 with $t_i = \tau_i - n\varepsilon_1$, $\eta_i = u_i$ for $i = 1, 2$, and

$$t_3 = \frac{(n-l)\tau_3}{n-1} + \frac{1}{2}\frac{(2l-3)\tau_3}{2(n-1)}, \quad \eta_3 = w_3$$

we have,

$$\sum_{i=1}^{3}(t_i + 1 + 2(t_i + 1 - \eta_i)) = 2n + 3 - 6n\varepsilon_1 - \frac{3}{4}\frac{(2l-3)\tau_3}{n-1} < 2n + \delta$$

for all $\delta > 0$. Hence, from (29)

$$\frac{3}{2} - 6n\varepsilon_1 \leq \delta$$

which is a contradiction. Thus there is at most one irreducible polynomial R belonging to each box and the measure of the set of (x_1, x_2, x_3) satisfying the inequalities for this polynomial is as required.

Now suppose that $\theta > 1$. Then, by the pigeonhole principle, there are at least $Q^{\frac{1}{4}\max\left(\left\{\frac{(2l-3)\tau_3}{2(n-1)}\right\}, \varepsilon/2\right)}$ of the polynomials $P_1, \ldots, P_k$ which have the same $[\theta]$ coefficients $a_n, \ldots, a_{n-[\theta]+1}$. Consider the new polynomials $R_j = P_j - P_1$ with $\deg R \leq n - [\theta]$ so that on M say

$$|R(x_i)| < 2Q^{-\tau_i + n\varepsilon_1}, i = 1, 2 \text{ and } |R(x_3)| < 2Q^{-\frac{(n-l)\tau_3}{n-1} - \frac{1}{2}\max\left(\left\{\frac{(2l-3)\tau_3}{2(n-1)}\right\}, \varepsilon/2\right)}.$$

It is easily verified that

$$\begin{gathered} \tau_1 + \tau_2 + \frac{(n-l)\tau_3}{n-1} + \frac{1}{2}\max\left(\left\{\frac{(2l-3)\tau_3}{2(n-1)}\right\}, \varepsilon/2\right) \\ = n - 1 - [\theta] - \frac{1}{2}\max\left(\left\{\frac{(2l-3)\tau_3}{2(n-1)}\right\}, \varepsilon/2\right) - \frac{\tau_3}{2(n-1)} - \{\theta\} \geq n - [\theta] - 2. \end{gathered}$$

Thus, if R_i is reducible then there exists a polynomial S_i which satisfies

$$S(x_i) \leq Q^{-t_i}$$

with $\deg S = n_S$ and $\sum_{i=1}^{3} t_i \geq n_S - 1$. Therefore, by the induction assumption, the set of x satisfying these inequalities has measure at most $\kappa \Pi_\lambda^3(Q)$.

On the other hand suppose that two of the R_j are irreducible. Again, Lemma 3 is used with $t_i = \tau_i - n\varepsilon_1$ for $i = 1, 2$, $t_3 = \frac{(n-l)\tau_3}{(n-1)} + \frac{1}{2}\max\left(\left\{\frac{(2l-3)\tau_3}{2(n-1)}\right\}, \varepsilon/2\right)$, $\eta_i = u_i$ for $i = 1, 2$ and $\eta_3 = w_3$. Hence,

$$\sum_{i=1}^{3}(t_i + 1 + 2(1 + t_i - \eta_i)) = 2n + 3 - 6n\varepsilon_1 + \frac{(3-2l)\tau_3}{n-1} + \frac{1}{2}\max\left(\left\{\frac{(2l-3)\tau_3}{2(n-1)}, \varepsilon/2\right\}\right)$$

By Lemma 3 this is at most $2n - 2[\theta] + \delta$ for all $\delta > 0$ which implies that

$$3 - 2\{\theta\} - 6n\varepsilon_1 + \frac{1}{2}\max\left(\left\{\frac{(2l-3)\tau_3}{2(n-1)}, \varepsilon/2\right\}\right) < \delta$$

for all δ which is clearly a contradiction. Thus in each M_i there is at most one irreducible polynomial $R_i = P_i - P_1$. This, together with (30) implies that the set of (x_1, x_2, x_3) satisfying (5) has measure at most $\kappa\mu(\Pi_\lambda^3(Q))$. □

Precisely, the same argument can be made if $\alpha_1 \in T_1^1 \cup T_2^2$, $\alpha_2 \in T_{l_2}^2$ and $\alpha_3 \in T_{l_3}^3$ with $l_2, l_3 \geq 3$. Using Remark 3 it may be assumed that

$$\frac{(2l_2-3)\tau_2}{2(n-1)} + \frac{(2l_3-3)\tau_3}{2(n-1)} \geq 4\varepsilon.$$

Let

$$\theta = \frac{(2l_2-3)\tau_2}{2(n-1)} + \frac{(2l_3-3)\tau_3}{2(n-1)} - \frac{3}{4}\left(\max\left(\left\{\frac{(2l_2-3)\tau_2}{2(n-1)}\right\}, \varepsilon/4\right) + \max\left(\left\{\frac{(2l_3-3)\tau_3}{2(n-1)}\right\}, \varepsilon/4\right)\right).$$

There exists $P \in \mathcal{P}_n(Q)$ such that by Lemma 2

$$\begin{aligned} |x_1 - \alpha_1| &\leq Q^{-u_1(P)}, \\ |x_2 - \alpha_2| &\leq Q^{-\tau_2 - \frac{(1-\tau_2)}{2} - \frac{(l_2-2)\tau_3}{2(n-1)}}. \\ |x_3 - \gamma_1| &\leq Q^{-\tau_3 - \frac{(1-\tau_3)}{2} - \frac{(l_3-2)\tau_3}{2(n-1)}}. \end{aligned}$$

Divide $\Pi_\lambda^3(Q)$ into smaller cuboids of sidelengths Q^{-u_1},

$$Q^{-\frac{(n-l_2)\tau_2}{n-1} - \frac{(1-\tau_2)}{2} - \frac{(l_2-1)\tau_2}{2(n-1)} - \frac{1}{2}\max\left(\left\{\frac{(2l_2-3)\tau_3}{2(n-1)}\right\}, \varepsilon/4\right)} = Q^{-w_2}.$$

and

$$Q^{-\frac{(n-l_3)\tau_3}{n-1} - \frac{(1-\tau_3)}{2} - \frac{(l_3-1)\tau_3}{2(n-1)} - \frac{1}{2}\max\left(\left\{\frac{(2l_3-3)\tau_3}{2(n-1)}\right\}, \varepsilon/4\right)} = Q^{-w_3}.$$

Again, it is easy to show that $w_i > \lambda$, $i = 2, 3$. Following this the proof is exactly the same taking the appropriate η_i and t_i.

For Partition B define the set $J_{n,B3}(Q,l)$ of $\mathbf{x} \in \Pi^3_\lambda(Q) \cap S_P(\alpha_1, \alpha_2, \alpha_3)$ for which (5) holds for some $P \in \mathcal{P}_n(Q)$ with $\alpha_2 \in T^2_1 \cup T^2_2$ and $\alpha_3 \in T^3_l$ for some $l > 2$.

Proposition 8. *For sufficiently large* Q

$$\mu(J_{n,B3}(Q,l)) < \kappa\mu(\Pi^3_\lambda(Q)).$$

Proof. This proof is very similar to the previous one so some of the details will be omitted. As before the polynomials are restricted to $P \in \mathcal{P}_n(Q, u_1, u_2, u_3)$ with $u_2 \le \frac{\tau_2+1}{2} + \frac{\tau_2}{2(n-1-\tau_1)}$. Suppose that $\mathbf{x} \in J_{n,B3}(Q,l)$. Then, there exists $P \in \mathcal{P}_n(Q)$ such that by Lemmas 2

$$\begin{aligned} |x_1 - \alpha_1| &\le Q^{-u_1(P)}, \\ |x_2 - \alpha_2| &\le Q^{-u_2(P)}, \\ |x_3 - \alpha_1| &\le Q^{-\tau_3 - \frac{(1-\tau_3)}{2} - \frac{(l-2)\tau_3}{2(n-1-\tau_1)}}. \end{aligned}$$

Case 1 For this case suppose that

$$\left\{ \frac{(2l-3)\tau_3}{n-1-\tau_1} \right\} > \varepsilon.$$

Let $0 < h_1 < 1$ and, for the class $P \in \mathcal{P}_n(Q, u_1, u_2, u_3)$, divide $\Pi^3_\lambda(Q)$ into smaller boxes of sidelengths $\min(Q^{-u_1}, Q^{-\lambda}), \min(Q^{-u_2}, Q^{-\lambda})$ and

$$Q^{-\frac{(n-l-\tau_1)\tau_3}{n-1-\tau_1} - \frac{(1-\tau_3)}{2} - \frac{(l-1)\tau_3}{2(n-1-\tau_1)} - h_1\left\{\frac{(2l-3)}{2(n-1-\tau_1)}\tau_3\right\}} = Q^{-w_3};$$

it is not difficult to show that $w_3 > \lambda$. Let $0 < h_2 < 1$ and define

$$\theta = \frac{(2l-3)\tau_3}{2(n-1-\tau_1)} - h_2\left\{\frac{(2l-3)}{2(n-1-\tau_1)}\tau_3\right\} = \left[\frac{(2l-3)\tau_3}{2(n-1-\tau_1)}\right] + (1-h_2)\left\{\frac{(2l-3)\tau_3}{2(n-1-\tau_1)}\right\} \tag{32}$$

where as before $\{\ \} \in (0,1]$. Note that $\{\theta\} \le 1 - h_2$.

If there are at most θ polynomials in $\mathcal{P}_n(Q, u_1, u_2, u_3)$ belonging to each M_i then the argument is precisely the same as in the previous proposition; i.e. the measure of the set of $\mathbf{x} \in \Pi^3_\lambda(Q)$ satisfying the conditions in the proposition is at most

$$\begin{aligned} \mu(\Pi^3_\lambda(Q))Q^{\theta - \tau_3 - \frac{(1-\tau_3)}{2} - \frac{(l-2)\tau_3}{2(n-1-\tau_1)}}Q^{w_3} &= Q^{(h_1-h_2)\left\{\frac{(2l-3)\tau_3}{2(n-1)}\right\}}\mu(\Pi^3_\lambda(Q)) \\ &\le Q^{-\varepsilon^2/4}\mu(\Pi^3_\lambda(Q)) \le \kappa_1\mu(\Pi^3_\lambda(Q)) \end{aligned} \tag{33}$$

from (15) provided that $h_2 - h_1 > \varepsilon/4$.

Otherwise in at least one box M, say there exist at least Q^θ polynomials. Using Taylor series on M the inequalities

$$|P(x_i)| < Q^{-\tau_i - n\varepsilon_1}, i = 1, 2 \quad \text{and} \quad |P(x_3)| < Q^{-\frac{(n-l-\tau_1)\tau_3}{n-1-\tau_1} - h_1\left\{\frac{(2l-3)}{2(n-1-\tau_1)}\tau_3\right\}} \tag{34}$$

are readily obtained for each of these polynomials.

First suppose that $\theta < 1$. From Remark 2 it may be assumed without loss of generality that

$$\frac{\varepsilon}{<} \frac{(2l-3)\tau_3}{2(n-1-\tau_1)} = \left\{ \frac{(2l-3)\tau_3}{2(n-1-\tau_1)} \right\} < 1. \tag{35}$$

From Lemma 10 it may also be assumed that $|\tau_1| > \varepsilon$. Following the proof of the previous proposition and assuming that there are two such polynomials Lemma 3 is used with

$$t_2 = \tau_2 - n\varepsilon_1,\ \eta_2 = u_2,\ t_3 = \frac{(n-l-\tau_1)\tau_3}{n-1-\tau_1} + h_1 \frac{(2l-3)}{4(n-1-\tau_1)}\tau_3, \eta_3 = w_3. \tag{36}$$

Putting these together gives that

$$\sum_{i=2}^{3}(t_i + 1 + 2\max((t_i+1-\eta_i),0) \geq 2n+1-2\tau_1 - 3n\varepsilon_1 + (h_1-2)\frac{(2l-3)\tau_3}{2(n-1-\tau_1)}. \tag{37}$$

From Lemma 3 and (35) this implies that

$$h_1 - 1 - 2\tau_1 - 3n\varepsilon_1 < \delta$$

for all $\delta > 0$ which is a contradiction provided that $h_1 > 1 + 2\tau_1 + 3n\varepsilon_1 + \varepsilon$.

Now we turn to the case $\theta > 1$ so that $[\theta] \geq 1$. Then at least $Q^{(1-h_2)\left\{\frac{(2l-3)\tau_3}{2(n-1-\tau_1)}\right\}}$ polynomials $P_1, \ldots, P_k$, have the same $[\theta]$ coefficients $a_n, \ldots, a_{n-[\theta]+1}$. Consider the new polynomials $R_j = P_j - P_1$. so that on M say,

$$|R(x_i)| < 2Q^{-\tau_i + n\varepsilon_1}, i = 1,2 \quad \text{and} \quad |R(x_3)| < 2Q^{-\frac{(n-l-\tau_1)\tau_3}{(n-1-\tau_1)} - h_1\left\{\frac{(2l-3)}{2(n-1-\tau_1)}\tau_3\right\}}.$$

Note that $\deg R \leq n - [\theta]$ and

$$\tau_1 + \tau_2 + \frac{(n-l-\tau_1)\tau_3}{(n-1-\tau_1)} + h_1\left\{\frac{(2l-3)}{2(n-1-\tau_1)}\tau_3\right\}$$

$$= n - 1 - [\theta] - \left\{\frac{(2l-3)\tau_3}{2(n-1-\tau_1)}\right\} + h_1\left\{\frac{(2l-3)\tau_3}{2(n-1-\tau_1)}\right\} - \frac{\tau_3}{2(n-1-\tau_1)} \geq n - 2 - [\theta].$$

Therefore, if R_i is reducible then there exists a polynomial S_i which satisfies

$$S(x_i) \leq Q^{-t_i}$$

with $\deg S = n_S$ and $\sum_{i=1}^{3} t_i \geq n_S - 1 \leq n - [\theta] - 1$. Thus by the induction assumption the set of x satisfying these inequalities has measure at most $\kappa \Pi_\lambda^3(Q)$.

Now suppose that there exist two of the R_j which are irreducible. As before,

$$\sum_{i=2}^{3}(t_i+1+2\max((t_i+1-\eta_i),0) = 2n+1-2\tau_1+h_1\frac{(2l-3)\tau_3}{2(n-1-\tau_1)} - 3n\varepsilon_1 + \frac{(3-2l)\tau_3}{n-1-\tau_1} < 2(n-[\theta])+\delta$$

for all $\delta > 0$. This implies that

$$1 - 2\tau_1 - 3n\varepsilon_1 + (h_1 - 2h_2)\left\{\frac{(2l-3)\tau_3}{n-1-\tau_1}\right\} - 2\{\theta\} < \delta$$

for all $\delta > 0$. Using (32) gives

$$1 - 2\tau_1 - 3n\varepsilon_1 + (h_1 - 2)\left\{\frac{(2l-3)\tau_3}{n-1-\tau_1}\right\} < \delta$$

which again is a contradiction for $h_1 > 1 + 2\tau_1 + 3n\varepsilon_1 + \varepsilon$. Thus h_1 and h_2 are chosen to satisfy $1 > h_1 > 1 + 2\tau_1 + 3n\varepsilon_1 + \varepsilon$ and $1 > h_2 > \varepsilon/4 + h_1$. This is possible from (15) and the fact that $|\tau_1| > \varepsilon$.

Case 2 Now suppose that

$$\left\{\frac{(2l-3)\tau_3}{n-1-\tau_1}\right\} \le \varepsilon.$$

The argument is the same as that in Case 1 with different values for θ, w_3 and hence t_3. For the class $P \in \mathcal{P}_n(Q, u_1, u_2, u_3)$, divide $\Pi^3_\lambda(Q)$ into smaller boxes of sidelengths $\min(Q^{-u_1}, Q^{-\lambda}), \min(Q^{-u_2}, Q^{-\lambda})$ and

$$Q^{-\frac{(n-l-\tau_1)\tau_3}{n-1-\tau_1} - \frac{(1-\tau_3)}{2} - \frac{(l-1)\tau_3}{2(n-1-\tau_1)} + 2\varepsilon} = Q^{-w_3};$$

it is not difficult to show that $w_3 > \lambda$. Also define

$$\theta = \left[\frac{(2l-3)\tau_3}{2(n-1-\tau_1)}\right] + \varepsilon \tag{38}$$

so that $\{\theta\} = \varepsilon$. It may be assumed, using Lemma 10, that $[\theta] \ge 1$.

If there are at most θ polynomials in $\mathcal{P}_n(Q, u_1, u_2, u_3)$ belonging to each M_i then as before the measure of the set of $\mathbf{x} \in \Pi^3_\lambda(Q)$ satisfying the conditions in the proposition is at most

$$\mu(\Pi^3_\lambda(Q))Q^{\theta - \tau_3 - \frac{(1-\tau_3)}{2} - \frac{(l-2)\tau_3}{2(n-1-\tau_1)}}Q^{w_3} = Q^{-\varepsilon}\mu(\Pi^3_\lambda(Q)) \tag{39}$$

from (15).

Otherwise in at least one box M, say there exist at least Q^θ polynomials. Using Taylor series on M the inequalities

$$|P(x_i)| < Q^{-\tau_i - n\varepsilon_1}, i = 1, 2 \quad \text{and} \quad |P(x_3)| < Q^{-\frac{(n-l-\tau_1)\tau_3}{n-1-\tau_1} + 4\varepsilon} \tag{40}$$

are readily obtained for each of these polynomials. The estimate for $|P(x_3)|$ comes from the term $|P''(\alpha_3)||x - \alpha_3|^2 \ll Q^{1-2w_3}$ which is the largest term in the Taylor series.

Then at least Q^ε polynomials $P_1, \dots, P_k$, have the same $[\theta]$ coefficients $a_n, \dots, a_{n-[\theta]+1}$. Consider the new polynomials $R_j = P_j - P_1$. so that on M say,

$$|R(x_i)| < 2Q^{-\tau_i + n\varepsilon_1}, i = 1, 2 \quad \text{and} \quad |R(x_3)| < 2Q^{-\frac{(n-l-\tau_1)\tau_3}{(n-1-\tau_1)} + 4\varepsilon}.$$

Note that $\deg R \le n - [\theta]$ and

$$\tau_1 + \tau_2 + 2n\varepsilon_1 + \frac{(n-l-\tau_1)\tau_3}{(n-1-\tau_1)} - 4\varepsilon$$

$$> n - 1 - [\theta] + 2n\varepsilon_1 - 5\varepsilon - \frac{1}{2} + \frac{\tau_2}{n-1-\tau_1} > n - 2 - [\theta].$$

Therefore, if R_i is reducible then there exists a polynomial S_i which satisfies

$$S(x_i) \leq Q^{-t_i}$$

with $\deg S = n_S$ and $\sum_{i=1}^{3} t_i \geq n_S - 1 \leq n - [\theta] - 1$ and by the induction assumption the set of x satisfying these inequalities has measure at most $\kappa \Pi_\lambda^3(Q)$.

Now suppose that there exist two of the R_j which are irreducible. Take $t_2 = \tau_2 - n\varepsilon_1$, $\eta_2 = u_2$, $t_3 = \frac{(n-l-\tau_1)\tau_3}{n-1-\tau_1} - 4\varepsilon$, and $\eta_3 = w_3$. Then, from Lemma 3

$$\sum_{i=2}^{3}(t_i + 1 + 2\max((t_i + 1 - \eta_i), 0) = 2n + 1 - 2\tau_1 - 3n\varepsilon_1 - 8\varepsilon + \frac{(3-2l)\tau_3}{n-1-\tau_1} < 2(n - [\theta]) + \delta$$

for all $\delta > 0$. This implies that

$$1 - 2\tau_1 - 3n\varepsilon_1 - 8\varepsilon - 2\{\theta\} = 1 - 2\tau_1 - 3n\varepsilon_1 - 10\varepsilon < \delta$$

for all $\delta > 0$ which is clearly a contradiction.

Thus in each M_i there are at most two irreducible polynomials $R_i = P_i - P_1$ and the set of (x_1, x_2, x_3) satisfying the inequalities in the theorem has measure at most $Q^{-\gamma}\mu(\Pi_\lambda^3(Q))$ from (39). □

To complete the proof of the theorem, as in each of the base cases, note that the set of triples (τ_1, τ_2, τ_3) satisfying $\sum_{i=1}^{3} \tau_i = n - 1$ with $\tau_i \geq -1$ has finite volume. Thus, integrating over all such triples and choosing Q sufficiently large completes the proof of the theorem.

3. Proof of main Theorem

This proof almost exactly follows the proof of Theorem 5 for Partition A except that $\tau_1 + \tau_2 + \tau_3 = n - 2$.

The theorem concerns one fixed triple (τ_1, τ_2, τ_3) with $\tau_i > 0$ for $i = 1, 2, 3$. Choose ε such that $\varepsilon < \min_{i=1,2,3} \tau_i/2$ and choose Q large enough that $Q^{-\varepsilon} < \epsilon_0/2$. This will imply that if $(x_1, x_2, x_3) \in \Pi_\lambda^3(Q) \cap S_P(\alpha_1, \alpha_2, \alpha_3)$ satisfies (4) then $|\alpha_i - \alpha_j| > \epsilon_0/2$ for $i, j \in \{1, 2, 3\}$, $i \neq j$. For convenience ε is also chosen to satisfy

$$\min_{i=1,2,3} \frac{(2l-3)\tau_i}{2(n-2)} > 4\varepsilon. \tag{41}$$

Note that it is impossible for $P \in \mathcal{P}_1(Q) \cup \mathcal{P}_2(Q)$ with $\tau_i > 0$, $i = 1, 2, 3$ to satisfy (4). Thus as the base of another induction argument the cubic case is considered; i.e. $P \in \mathcal{P}_3(Q)$.

Lemma 11. *Fix $0 < \kappa < 1$. Let $\delta_0 \in \mathbb{R}^+$. Define the set $B(Q, \delta_0)$ as those $\mathbf{x} \in \Pi_\lambda^3(Q)$ such that*

$$|P(x_i)| < Q^{-\tau_i}, \quad |P'(x_1)| < \delta_0 Q, \quad \sum_{i=1}^{3} \tau_i = 1.$$

Then, there exists $\delta_0 > 0$ *such that*

$$\mu(B(Q,\delta_0)) < \kappa\mu(\Pi_\lambda^3(Q)).$$

Proof. This proof is almost the same as that for Case 1 in Lemma 9. From Remark 1 it may be assumed that $|a_n| \gg H(P)$.

Suppose $P \in \mathcal{P}_3(Q)$, $\tau_3 \geq \tau_2 \geq \tau_1 > \varepsilon$ or $|a_n| > Q^\varepsilon$ and let

$$B(P) = \{\mathbf{x} \in \Pi_3^\lambda(Q) : |P(x_i)| < Q^{-\tau_i}, |P'(x_1)| < \delta_0 Q\}.$$

Then, exactly as in (17) the inequality

$$|B(P)| \leq \prod_{i=1}^{3} \min(Q^{-\lambda}, Q^{-\tau_i}|a_3|^{-1}) \tag{42}$$

is obtained. Also note that since

$$|P'(\alpha_i)| > 2^{-2}\epsilon_0^2|a_3| \quad \text{for} \quad i = 1,\ldots,3$$

and $|P'(\alpha_1)| < \delta_0 Q$ this implies that $|a_3| \ll \delta_0 Q$.

Consider $P \in \mathcal{P}_3(Q) \cap M(a_3; |a_3|Q^{-\lambda})$ with $|a_3|Q^{-\lambda} > Q^{-\tau_1}$. Then $|P(d_i)| < |a_3|Q^{-\lambda}$ by (18) for $i = 1, 2, 3$. From Lemma 5 $\#M(a_3) \ll \max(1, |a_3|^3 Q^{-3\lambda})$. Hence, using (42),

$$|B(P)| \leq Q^{-1}|a_3|^{-3}$$

and

$$\sum_{a_3 \leq \delta_0 Q} \sum_{P \in M(a_3;|a_3|Q^{-\lambda})} \mu(B(P)) \ll \sum_{|a_3| \leq \delta_0 Q} \max(1, |a_3|^3 Q^{-3\lambda}) \prod_{i=1}^{3} \min(Q^{-\lambda}, Q^{-\tau_i}|a_3|^{-1}).$$

As $|a_3|Q^{-\lambda} > Q^{-\tau_i}$ we have

$$\sum_{|a_3| \leq \delta_0 Q} \sum_{P \in M(a_3;|a_3|Q^{-\lambda})} \mu(B(P)) \ll \begin{cases} \displaystyle\sum_{|a_3| \leq \delta_0 Q} Q^{-1}Q^{-3\lambda} \leq \delta_0 \mu(\Pi_\lambda^3(Q)), & \text{if } |a_3|Q^{-\lambda} > 1 \\ \displaystyle\sum_{|a_3| \leq Q^\lambda} Q^{-1}|a_3|^{-3} \leq Q^{-1+3\lambda}\mu(\Pi_\lambda^3(Q)), & \text{else.} \end{cases}$$

Next, suppose $Q^{-\tau_2} \leq |a_3|Q^{-\lambda} \leq Q^{-\tau_1}$. Clearly, $1 \leq |a_3| \leq \min(Q, Q^{\lambda-\tau_1})$ so $\tau_1 \leq \lambda$. In this case $|P(y)| < Q^{-\tau_1}$ for all $y \in I_1$ and $|P(d_i)| < |a_3|Q^{-\lambda}$ for $i = 2, 3$.

As $\tau_1 > 0$, $|a_3|Q^{-\lambda} < 1$ and $\#M(a_3; |a_3|Q^{-\lambda}) \ll 1$. Hence, Using (42),

$$\sum_{|a_3| < Q^{\lambda-\tau_1}} \sum_{P \in M(a_3;|a_3|Q^{-\lambda})} \mu(B(P)) \leq \sum_{|a_3| < Q^{\lambda-\tau_1}} Q^{-\lambda}|a_3|^{-2}Q^{-\tau_2-\tau_3} \leq Q^{-3\lambda}Q^{-1+\tau_1+2\lambda} \leq Q^{-1+3\lambda}\mu(\Pi_\lambda^3(Q)).$$

Finally consider polynomials satisfying $Q^{-\tau_3} \leq |a_3|Q^{-\lambda} \leq Q^{-\tau_2}$. Thus $|a_3| \leq Q^{\lambda-\tau_2}$ so $\tau_1 \leq \tau_2 \leq \lambda$ and $\tau_3 = 1 - \tau_2 - \tau_3 \geq 1 - 2\lambda$. Here $|P(y)| < Q^{-\tau_i}$ for all $y \in I_i$

for $i = 1, 2$ and $|P(d_3)| < |a_3|Q^{-\lambda}$. As before, since $\tau_1 > 0$, $\#M(a_3; |a_3|Q^{-\lambda}) \ll 1$ and $\mu(B(P)) \le Q^{-2\lambda-\tau_3}|a_3|^{-1}$ so from (42),

$$\sum_{a_3 \le Q^{\lambda-\tau_2}} \sum_{P \in M(a_3;|a_3|Q^{-\lambda})} \mu(B(P)) \ll Q^{-2\lambda-\tau_3} \log Q \ll Q^{\lambda-\tau_3}\mu(\Pi^3_\lambda(Q)) \log Q \ll Q^{-1+3\lambda} \log Q \mu(\Pi^3_\lambda(Q)).$$

This leaves the case $\tau_1 \le \varepsilon$ and $|a_n| < Q^\varepsilon$. From Lemma 6 for $P \in \mathcal{P}_3(Q^\varepsilon)$, the root differences $|\alpha_i - \alpha_j| \ge Q^{-2\varepsilon}$

Putting together the three inequalities yields that for Q sufficiently large there exists $\delta_0 > 0$ such that $\mu(J_3(Q)) < \kappa\mu(\Pi^3_\lambda(Q))$.

□

Note that in the proof of Theorem 1 this is the only place where δ_0 appears explicitly.

There now begins a proof by induction with the induction hypothesis being that for $3 \le m \le n-1$ there exists $\delta_0 > 0$ such that the set of $\mathbf{x} \in \Pi^3_\lambda(Q)$ for which there exists $P \in \mathcal{P}_m(Q)$ satisfying

$$|P(x_i)| < Q^{-\tau_i}, \quad |P'(x_1)| < \delta_0 Q$$

with $\sum_{i=1}^3 \tau_i > m-2$ has measure at most

$$\kappa\mu(\Pi^3_\lambda(Q)).$$

For $m = 3$ this is Lemma 11.

3.0.1. *Partitioning the roots.* Let $v_i = \frac{\tau_i}{n-2}$ so that $v_1 + v_2 + v_3 = 1$ and $\tau_i \ge v_i > 0$. Each of the roots of a polynomial P will lie in one of the following sets.

$$\begin{array}{llcrcl}
T^i_1: & & & |P'(\alpha_i)| & \le & Q^{\frac{1-\tau_i}{2}}, \\
T^i_2: & Q^{\frac{1-\tau_i}{2}} & < & |P'(\alpha_i)| & < & Q^{\frac{1-\tau_i}{2}+\frac{\tau_i}{2(n-2)}}, \\
T^i_l: & Q^{\frac{1-\tau_i}{2}+\frac{(l-2)\tau_i}{2(n-2)}} & < & |P'(\alpha_i)| & \le & Q^{\frac{1-\tau_i}{2}+\frac{(l-1)\tau_i}{2(n-2)}},\ 2 \le l \le n-2, \\
T^i_{n-1}: & Q^{\frac{1-v_i}{2}} = Q^{\frac{1-\tau_i}{2}+\frac{(n-3)\tau_i}{2(n-2)}} & < & |P'(\alpha_i)| & \le & \delta_0 Q \\
T^i_n: & \delta_0 q & < & |P'(\alpha_i)| & &
\end{array}$$

The rest of the proof consists of three propositions which are equivalent to Propositions 1 4 and 7 involves measure estimates for the set of points $(x_1, x_2, x_3) \in \Pi^3_\lambda(Q) \cap S_P(\alpha_1, \alpha_2, \alpha_3)$ with α_i lying in various combinations of the sets T^i_l. Note that in $\cup_{j=2}^n T^i_j$ from Lemma 4, $|P'(\alpha_i)| \asymp |P'(x_i)|$ for $x_i \in S_P(\alpha_i)$.

For $l \in \{3, \ldots, n-2\}$ define the set $J_1(Q, l)$ of $\mathbf{x} \in \Pi^3_\lambda(Q) \cap S_P(\alpha_1, \alpha_2, \alpha_3)$ for which the system

$$|P(x_i)| < Q^{-\tau_i},\ i = 1, 2, 3; \quad |P'(x_1)| < \delta_0 Q, \quad \alpha_1 \in T^\alpha_l,\ \alpha_i \in \cup_{m=l}^n T^i_m, i = 2, 3$$

has a solution $P \in \mathcal{P}_n(Q)$. Let $J_1(Q) = \cup_{l=3}^{n-2} J_1(Q, l)$.

Proposition 9. *For sufficiently large Q and any $\kappa > 0$ there exists δ_0 such that*

$$\mu(J_1(Q)) < \kappa\mu(\Pi^3_\lambda(Q)).$$

Proof. This proof is exactly the same as that of Proposition 1 with the numbers $l_i = (n-l)v_i$ replacing l_{Ai}; the set

$$\sigma_1(P) := \{\mathbf{x} \in \Pi^3_\lambda(Q) \cap S_P(\alpha_1, \alpha_2, \alpha_3) : |x_i - \alpha_i| < cQ^{-\tau_i}|P'(\alpha_i)|^{-1}\}$$

replacing σ_{A1} and the set

$$\sigma_l(P) := \{\mathbf{x} \in \Pi^3_\lambda(Q) \cap S_P(\alpha_1, \alpha_2, \alpha_3) : |x_i - \alpha_i| < cQ^{-l_i}|P'(\alpha_i)|^{-1}\}$$

replacing σ_{Al}. Except for the fact that $\sum_{i=1}^3 \tau_i = n-2$, $\sum_{i=1}^3 l_i = n-l$ the proof is the same in it entirety and will not be repeated. Obviously Lemma 11 is used instead of Lemma 9.

□

Define the set $J_2(Q)$ of $\mathbf{x} \in \Pi^3_\lambda(Q) \cap S_P(\alpha_1, \alpha_2, \alpha_3)$ for which the inequalities

$$|P(x_i)| < Q^{-\tau_i}, \quad |P'(x_1)| < \delta_0 Q \tag{43}$$

hold for some $P \in \mathcal{P}_n(Q)$ with $\alpha_i \in T^i_1 \cup T^i_2$.

Proposition 10. *For each $\kappa > 0$ there exists $\delta_0 > 0$ such that*

$$\mu(J_2(Q) \leq \kappa\mu(\Pi^3_\lambda(Q)).$$

Proof. This proof follows that of Proposition 4; in particular the preamble for Section 2.2 Partition A is precisely the same and will not be repeated. The only difference is that Theorem 5 is used to deal with the reducible polynomials.

To show that at most one polynomial in $\mathcal{P}_n(Q, u_1, u_2, u_3)$ belongs to each box follow the proof of Proposition 4 the only difference being that

$$u_i \leq \frac{\tau_i + 1}{2} + \frac{\tau_i}{2(n-2)}.$$

Using this in Lemma 3 as before gives a contradiction. □

Define the set $J_3(Q)$ of $\mathbf{x} \in \Pi^3_\lambda(Q) \cap S_P(\alpha_1, \alpha_2, \alpha_3)$ for which the inequalities

$$|P(x_i)| < Q^{-\tau_i}, \quad |P'(x_1)| < \delta_0 Q \tag{44}$$

hold for some $P \in \mathcal{P}_n(Q)$ with $\alpha_i \in T^i_1 \cup T^i_2$ for $i = 1, 2$ and $\alpha_3 \in T^3_l$ for $l \geq 3$.

Proposition 11. *For sufficiently large Q*

$$\mu(J_3(Q)) < \kappa\mu(\Pi^3_\lambda(Q)).$$

Proof. Suppose that $\mathbf{x} \in J_3(Q, l)$. Then, there exists $P \in \mathcal{P}_n(Q)$ such that by Lemma 2

$$\begin{aligned} |x_1 - \alpha_1| &\leq Q^{-u_1(P)}, \\ |x_2 - \alpha_2| &\leq Q^{-u_2(P)}, \\ |x_3 - \alpha_3| &\leq Q^{-\tau_3 - \frac{(1-\tau_3)}{2} - \frac{(l-2)\tau_3}{2(n-2)}}. \end{aligned}$$

As in Proposition 4 the polynomials are restricted to $P \in \mathcal{P}_n(Q, u_1, u_2, u_3)$ with, for $i = 1, 2$, $u_i \leq \frac{\tau_i+1}{2} + \frac{\tau_i}{2(n-2)}$. The proof follows that of Proposition 7 and so many of the details will be omitted. From Theorem 5 it may be assumed without loss of generality that the polynomials are irreducible.

Case 1. Suppose that

$$\frac{(2l-3)\tau_3}{2(n-2)} > 1 \quad \text{and} \quad \left\{\frac{(2l-3)\tau_3}{2(n-2)}\right\} > 4\varepsilon.$$

Divide $\Pi^3_\lambda(Q)$ into smaller boxes of sidelengths $\min(Q^{-\lambda}, Q^{-u_1}), \min(Q^{-\lambda}, Q^{-u_2})$ and

$$Q^{-\frac{(n-l-1)\tau_3}{n-2} - \frac{(1-\tau_3)}{2} - \frac{(l-1)\tau_3}{2(n-2)} - \frac{1}{2}\left\{\frac{(l-3)\tau_3}{2(n-2)}\right\}} = Q^{-w_3};$$

it is easy to show that $w_3 > \lambda$. Define

$$\theta = \frac{(2l-3)\tau_3}{2(n-2)} - \frac{3}{4}\left\{\frac{(2l-3)}{2(n-2)}\tau_3\right\} = \left[\frac{(2l-3)\tau_3}{2(n-2)}\right] + \frac{1}{4}\left\{\frac{(2l-3)\tau_3}{2(n-2)}\right\}$$

where the fractional part $\{\ \}$ is defined to be 1 if $\frac{(2l-3)\tau_3}{2(n-2)} \in \mathbb{N}$.

If there are at most Q^θ polynomials in $\mathcal{P}_n(Q, u_1, u_2, u_3)$ belonging to each M_i then the measure of the set of $\mathbf{x} \in \Pi^3_\lambda(Q)$ satisfying the conditions in the proposition is at most

$$\mu(\Pi^3_\lambda(Q))Q^{\theta-\tau_3-\frac{(1-\tau_3)}{2}-\frac{(l-2)\tau_3}{2(n-2)}}Q^{w_3} =\leq Q^{-\varepsilon}\mu(\Pi^3_\lambda(Q)). \tag{45}$$

Otherwise in at least one box, M say, there exist at least Q^θ polynomials. Label these polynomials $P_1, \ldots, P_k$. It is not difficult to show using Taylor series that in M

$$|P(x_i)| < Q^{-\tau_i+n\varepsilon_1}, i = 1, 2 \quad \text{and} \quad |P(x_3)| < Q^{-\frac{(n-l-1)\tau_3}{n-2} - \frac{1}{2}\left\{\frac{(2l-3)\tau_3}{2(n-2)}\right\}}. \tag{46}$$

As $\theta > 1$ there are at least $Q^{\frac{1}{2}\left\{\frac{(2l-3)\tau_3}{2(n-2)}\right\}}$ of the polynomials $P_1, \ldots, P_k$ which have the same $[\theta]$ coefficients $a_n, \ldots, a_{n-[\theta]+1}$. Consider the new polynomials $R_j = P_j - P_1$ with $\deg R \leq n - [\theta]$ so that on M say

$$|R(x_i)| < 2Q^{-\tau_i+n\varepsilon_1}, i = 1, 2 \quad \text{and} \quad |R(x_3)| < 2Q^{-\frac{(n-l-1)\tau_3}{n-2} - \frac{1}{2}\left\{\frac{(2l-3)\tau_3}{2(n-2)}\right\}}.$$

The sum of the powers in the above inequalities is

????

Suppose that two of the R_j are irreducible and use Lemma 3 with $t_i = \tau_i - n\varepsilon_1$ for $i = 1, 2$, $t_3 = \frac{(n-l-1)\tau_3}{(n-2)} + \frac{1}{2}\left\{\frac{(2l-3)\tau_3}{2(n-2)}\right\}$, $\eta_i = u_i$ for $i = 1, 2$ and $\eta_3 = w_3$. It is not difficult to show that

$$\sum_{i=1}^{3}(t_i + 1 + 2(1 + t_i - \eta_i)) = 2n + 1 - 6n\varepsilon_1 + \frac{(3-2l)\tau_3}{n-2} + \frac{1}{2}\left\{\frac{(2l-3)\tau_3}{2(n-2)}\right\}.$$

By Lemma 3 this is at most $2n - 2[\theta] + \delta$ which implies that

$$1 - 6n\varepsilon_1 < \delta$$

for all δ which is clearly a contradiction. Thus in each M_i there is at most one irreducible polynomial $R_i = P_i - P_1$.

First suppose that $\theta < 1$ and note that this implies

$$4\varepsilon \leq \frac{(2l-3)\tau_3}{2(n-2)} < 1.$$

If, there are at least two irreducible P_i then using Lemma 3 with $t_i = \tau_i - n\varepsilon_1$, $\eta_i = u_i$ for $i = 1, 2$, and

$$t_3 = \frac{(n-l-1)\tau_3}{n-2} + \frac{1}{2}\frac{(2l-3)\tau_3}{2(n-2)}, \quad \eta_3 = w_3$$

we haveSTART HERE

$$\sum_{i=1}^{3}(t_i+1+2(t_i+1-\eta_i)) = 2n+2+\frac{(3-2l)\tau_3}{n-2} - 6n\varepsilon_1 - \{\frac{(2l-3)\tau_3}{2(n-2)}\} + \frac{1}{2}\frac{(2l-3)\tau_3}{2(n-2)} < 2n+\delta$$

for all $\delta > 0$. Hence,

$$2 + \frac{(3-2l)\tau_3}{n-2} + \frac{1}{2}\frac{(l-3)\tau_3}{2(n-2)} - 6n\varepsilon_1 - \frac{(2l-3)\tau_3}{2(n-2)} \leq \delta$$

so that

$$\frac{1}{4} - 6n\varepsilon_1 \leq 1 + \frac{1}{2}\frac{(l-3)\tau_3}{2(n-2)} - \max\left(\{\frac{(l-3)\tau_3}{2(n-2)}\}, 4\varepsilon\right) - 6n\varepsilon_1 < \delta$$

which is a contradiction. Therefore at most one of the polynomials is irreducible.

Thus there is at most one irreducible polynomial R belonging to each box and the measure of the set of (x_1, x_2, x_3) satisfying the inequalities for this polynomial is small.

Note that

$$\tau_1 + \tau_2 + \frac{(n-l)\tau_3}{n-2} + \frac{1}{2}\left\{\frac{(l-3)\tau_3}{2(n-2)}\right\}$$

$$= n - 2 - [\theta] - \frac{1}{2}\left\{\frac{(l-3)\tau_3}{2(n-2)}\right\} - \frac{\tau_3}{2(n-2)} - \{\theta\} \geq n - [\theta] - 2.$$

Thus, if R_i is reducible then there exists a polynomial S_i which satisfies

$$S(x_i) \leq Q^{-t_i}$$

with $\deg S = n_S$ and $\sum_{i=1}^{3} t_i \geq n_S - 1$. Therefore, by the induction assumption, the set of x satisfying these inequalities has measure at most $\kappa\Pi_\lambda^3(Q)$.

Finally, together with (30) this implies that the set of (x_1, x_2, x_3) satisfying (5) has measure at most $\kappa\mu(\Pi_\lambda^3(Q))$. $\square$

Precisely, the same argument can be made if $\alpha_1 \in T_1^1 \cup T_2^2$, $\alpha_2 \in T_{l_2}^2$ and $\alpha_3 \in T_{l_3}^3$ with $l_2, l_3 \geq 3$. Using Remark 3 it may be assumed that SORT OUT

$$\frac{(l_2-3)\tau_2}{2(n-2)} + \frac{(l_3-3)\tau_3}{2(n-2)} \geq 4\varepsilon.$$

Let

$$\theta = \frac{(2l_2-3)\tau_2}{2(n-2)} + \frac{(2l_3-3)\tau_3}{2(n-2)} - \frac{3}{4}\left\{\frac{(l_2-3)\tau_2}{2(n-2)}\right\} + \left\{\frac{(l_3-3)\tau_3}{2(n-2)}\right\}.$$

There exists $P \in \mathcal{P}_n(Q)$ such that by Lemma 2

$$\begin{aligned} |x_1 - \alpha_1| &\leq Q^{-u_1(P)}, \\ |x_2 - \alpha_2| &\leq Q^{-\tau_2 - \frac{(1-\tau_2)}{2} - \frac{(l_2-2)\tau_3}{2(n-2)}}. \\ |x_3 - \gamma_1| &\leq Q^{-\tau_3 - \frac{(1-\tau_3)}{2} - \frac{(l_3-2)\tau_3}{2(n-2)}}. \end{aligned}$$

Divide $\Pi_\lambda^3(Q)$ into smaller cuboids of sidelengths Q^{-u_1},

$$Q^{-\frac{(n-l_2)\tau_2}{n-2} - \frac{(1-\tau_2)}{2} - \frac{(l_2-1)\tau_2}{2(n-2)} - \frac{1}{2}\{\frac{(2l_2-3)\tau_3}{2(n-2)}\}} = Q^{-w_2}.$$

and

$$Q^{-\frac{(n-l_3)\tau_3}{n-2} - \frac{(1-\tau_3)}{2} - \frac{(l_3-1)\tau_3}{2(n-2)} - \frac{1}{2}\{\frac{(l_3-3)\tau_3}{2(n-2)}\}} = Q^{-w_3}.$$

Again, it is easy to show that $w_i > \lambda$, $i = 2, 3$. Following this the proof is exactly the same taking the appropriate η_i and t_i.

Finally we consider the mixed case.

Define the set $J_{n,4}(Q, l)$ of $\mathbf{x} \in \Pi_\lambda^3(Q) \cap S_P(\alpha_1, \alpha_2, \alpha_3)$ for which the inequalities

$$|P(x_i)| < Q^{-\tau_i}, \tag{47}$$

hold for some $P \in \mathcal{P}_n(Q)$ with $\alpha_i \in T_1^i \cup T_2^i$ for $i = 1, 2$ and $\alpha_3 \in T_l^3$ for some $l > 2$. Also suppose that $v_3 = \frac{\tau_3}{n-2} > \frac{1}{l-2}$.

Proposition 12. *For sufficiently large Q*

$$J_{n,4}(Q, l) < \kappa\mu(\Pi_\lambda^3(Q)).$$

Proof. TO BE COMPLETED

□

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