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A covariant approach to Ashtekar's canonical gravity

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Abstract. A Lorentz and general coordinate covariant form of canonical gravity, using Ashtekar's variables, is investigated. A covariant treatment due to Crnkovic and Witten is used, in which a point in phase space represents a solution of the equations of motion and a symplectic functional 2-form is constructed, which is Lorentz and general coordinate invariant. The subtleties and difficulties due to the complex nature of Ashtekar's variables are addressed and resolved.

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1. Introduction

In 1986, Abhay Ashtekar [1] discovered a set of canonical variables for the gravitational field as described by the general theory of relativity. Ashtekar found that they led to a considerable simplification of the constraints associated with the Hamiltonian formulation of Einstein's theory. Indeed, Ashtekar's constraints are polynomials in the canonical variables. Ashtekar's canonical gravity is definite progress in the direction of a quantum theory of gravity since it gives rise to a closed constraint algebra [2].

Hamiltonian models of physical phenomena have always distinguished between time and space. The Hamiltonian of a dynamical system generates time translations, that is to say it determines the time evolution of the dynamical variables. Relativity regards time and space as being components of a single entity: spacetime. An equation, describing the way a physical quantity changes with time, does not look the same to all relativistic observers. In other words, an equation of this kind is not covariant. It is usual to develop the Hamilton mechanics of a relativistic field by specifying a spacetime foliated by spacelike hypersurfaces of constant time, and a Hamiltonian functional on this spacetime. However, this approach spoils covariance from the beginning because a time coordinate must be singled out, in order for the required foliation to make sense [3].

One way of viewing the role of canonical variables is that their initial values determine a solution of the Hamilton equations. In other words, there is a one-to-one correspondence between the canonical variables at any time t and the initial canonical variables [4]. Thus we can describe the phase space as the set of solutions of the Hamilton equations of motion. For a field theory, a knowledge of the initial canonical variables requires a knowledge of the field configuration and its time derivatives on a spacelike hypersurface, and a point in phase space is a solution of the Hamilton equations at a given time. The object of this

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paper is to describe Ashtekar's gravity in a manifestly covariant way. One possible way of achieving this goal is to use a simple construction due to Crnkovic and Witten.

The essence of the Crnkovic–Witten construction is the observation that a covariant theory must have an invariant symplectic form, and that each point in phase space represents a solution of the equations of motion. One can thus dispense with the Hamiltonian, and focus on the symplectic structure and the points of phase space as providing a covariant description of the dynamics in phase space. This idea has been successfully applied by Crnkovic and Witten [5] to the Yang–Mills field and to general relativity (using the 3-metric and the extrinsic curvature as canonical variables) where there is an additional complication due to gauge invariance. It has also been applied to general relativity in the Palatini formalism, using the metric and real connection as configuration space variables, in [6].

It is not immediately obvious how to implement the Crnkovic–Witten construction in the framework of Ashtekar's canonical gravity. In particular, the complex nature of the canonical variables leads to difficulties which will be addressed here. It will be shown that these difficulties can be overcome, and the Crnkovic–Witten construction can be applied successfully to give a covariant version of Ashtekar's theory.

2. Ashtekar's canonical gravity

In this section, we shall review Ashtekar's Hamiltonian formulation with a view to establishing our notation and conventions. Ashtekar's canonical variables are the inverse densitized triads E^{ai} and the Ashtekar connection A_{ai}, defined on a spacelike hypersurface Σ_t of constant time t. (Ashtekar's canonical variables can also be defined on a null hypersurface [7].) Here a and i are orthonormal and coordinate indices, respectively, ranging from 1 to 3. The metric signature is $-{+}{+}{+}$, and the completely anti-symmetric Levi-Civita tensor is taken to be $\varepsilon_{0123} = 1$. For a spacelike foliation, a set of orthonormal 1-forms is given by

$$e^0 = N\,\mathrm{d}t, \qquad e^a = h^a{}_i N^i\,\mathrm{d}t + h^a{}_i\,\mathrm{d}x^i, \tag{1}$$

where N and N^i are the lapse and shift functions, respectively. The dual basis vectors are

$$\beta_0 = \frac{1}{N}\left(\frac{\partial}{\partial t} - N^i\frac{\partial}{\partial x^i}\right), \qquad \beta_a = (h^{-1})_a{}^i\frac{\partial}{\partial x^i}. \tag{2}$$

Each β_a is spacelike, and the normal β_0 is timelike. The *densitized triads* are defined by

$$(E^{-1})_{ai} = \frac{1}{h}h_{ai}, \tag{3}$$

where h is the determinant of the matrix $[h_{ai}]$. The densitized triads are real valued on any coordinate patch provided that $h^0{}_i = 0$ [8]. This is the *time gauge* condition, which can be relaxed by allowing the densitized triads to become complex valued (see the appendix). The local group of local tangent space rotations, which preserves the time gauge condition, is the rotation group $SO(3)$. The *inverse densitized triads* E^{ai} satisfy

$$E^{ai}(E^{-1})_{aj} = \delta^i{}_j. \tag{4}$$

Let E be the determinant of the matrix $[E^{ai}]$. Now

$$E^{ai} = h(h^{-1})^{ai}, \qquad E = h^2. \tag{5}$$

We record the useful relations:

$$h_{ai} = \sqrt{E}(E^{-1})_{ai}, \qquad (h^{-1})^{ai} = \frac{E^{ai}}{\sqrt{E}}. \tag{6}$$

The torsion-free, metric-compatible connection 1-forms ω_{AB} are given by

$$\omega_{AB} = \tfrac{1}{2}\left[(i_A i_B\,\mathrm{d}e_C)e^C - i_A\,\mathrm{d}e_B + i_B\,\mathrm{d}e_A\right], \tag{7}$$

where $A, B, \ldots$ range from 0 to 3, i_A is the interior derivative along β_A, and d is the exterior derivative. The *Ashtekar connection*, A_{ai}, can then be defined in terms of the components of the connection 1-forms, ω_{AB}:

$$A_{ai} = \omega_{0ai} - \tfrac{1}{2}\mathrm{i}\varepsilon_{abc}\omega^{bc}{}_i. \tag{8}$$

The curvature 2-forms, R_{AB}, are given by

$$R_{AB} = \mathrm{d}\omega_{AB} + \omega_{AC}\wedge\omega^C{}_B, \qquad R_{BA} = -R_{AB}. \tag{9}$$

They satisfy the Hodge duality relations:

$$*R_{AB} = \tfrac{1}{2}\varepsilon_{ABCD}R^{CD}, \qquad **R_{AB} = -R_{AB}. \tag{10}$$

The self-dual curvature 2-forms, ${}^+R_{AB}$, are then given by

$${}^+R_{AB} = R_{AB} - \mathrm{i}*R_{AB}, \qquad *{}^+R_{AB} = \mathrm{i}\,{}^+R_{AB}. \tag{11}$$

In the absence of torsion, we have the identity:

$$R_{AB}\wedge e^B = 0. \tag{12}$$

Thus, for a vacuum gravitational field, the action density 4-form can be written as:

$$\begin{aligned}\mathcal{L}\,\mathrm{d}^4x &= \tfrac{1}{2}R_{AB}\wedge *e^{AB}\\ &= \tfrac{1}{2} * R_{AB}\wedge e^{AB} + \tfrac{1}{2}\mathrm{i}R_{AB}\wedge e^{AB}\\ &= \tfrac{1}{2}\mathrm{i}\,{}^+R_{AB}\wedge e^{AB},\end{aligned} \tag{13}$$

where e^{AB} stands for $e^A\wedge e^B$. Now (11) tells us that

$${}^+R^{bc} = \mathrm{i}\varepsilon^{abc}\,{}^+R_{0a}, \tag{14}$$

and therefore,

$$\mathcal{L}\,\mathrm{d}^4x = \mathrm{i}^+R_{0a}\wedge\left(e^{0a} + \tfrac{1}{2}\mathrm{i}\varepsilon^{abc}e_{bc}\right). \tag{15}$$

Writing

$$F_a ={}^+ R_{0a}, \qquad \Lambda^a = e^{0a} + \tfrac{1}{2}\mathrm{i}\varepsilon^{abc}e_{bc}, \tag{16}$$

we have

$$\mathcal{L}\,\mathrm{d}^4x = \mathrm{i}F_a\wedge\Lambda^a. \tag{17}$$

It is straightforward to obtain the important relations:

$$\begin{aligned}F_a &= \tfrac{1}{2}F_{a\mu\nu}\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\\ &= \left(\dot{A}_{ai} - \partial_i A_{a0} - \mathrm{i}\varepsilon_{abc}A^b{}_0A^c{}_i\right)\mathrm{d}t\wedge\mathrm{d}x^i + \tfrac{1}{2}\left(\partial_i A_{aj} - \partial_j A_{ai} - \mathrm{i}\varepsilon_{abc}A^b{}_iA^c{}_j\right)\mathrm{d}x^i\wedge\mathrm{d}x^j,\end{aligned} \tag{18}$$

$$\begin{aligned}\Lambda^a &= \tfrac{1}{2}\Lambda^a{}_{\mu\nu}\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\\ &= \tfrac{1}{2}\left(\varepsilon^{abc}\mathcal{N}E_b{}^jE_c{}^k + 2\mathrm{i}E^{aj}N^k\right)\varepsilon_{ijk}\,\mathrm{d}t\wedge\mathrm{d}x^i + \tfrac{1}{2}\mathrm{i}\varepsilon_{ijk}E^{ai}\,\mathrm{d}x^j\wedge\mathrm{d}x^k,\end{aligned} \tag{19}$$

with the help of (1), (6), (8), and (11). Here, we have used the symbol $\mathcal{N}$ for $N/\sqrt{E}$, and the Greek letters $\mu, \nu, \ldots$ for coordinate indices, ranging from 0 to 3. It follows that

$$\mathcal{L}\,\mathrm{d}^4x = \big[A_{ai}\dot{E}^{ai} - A_{a0}\big(\partial_i E^{ai} - \mathrm{i}\varepsilon^{abc}A_{bi}E_c{}^i\big) + \tfrac{1}{2}\mathrm{i}\mathcal{N}\varepsilon^{abc}F_{aij}E_b{}^iE_c{}^j + N^iF_{aij}E^{aj}\big]\,\mathrm{d}t\wedge\mathrm{d}^3x. \tag{20}$$

Thus we see that the Ashtekar connection A_{ai} are the momenta conjugate to the inverse densitized triads E^{ai} and that Ashtekar's Hamiltonian is

$$H = \int_{\Sigma_t}\mathrm{d}^3x\,\big[A_{a0}\big(\partial_i E^{ai} - \mathrm{i}\varepsilon^{abc}A_{bi}E_c{}^i\big) - \tfrac{1}{2}\mathrm{i}\mathcal{N}\varepsilon^{abc}F_{aij}E_b{}^iE_c{}^j - N^iF_{aij}E^{aj}\big], \tag{21}$$

for a spacelike hypersurface Σ_t. This is the form of the Hamiltonian given in [9].

The general theory of relativity has a phase space structure analogous to that of the $SU(2)$ Yang–Mills field, where local $SO(3)$ tangent space rotations or, more generally, Lorentz transformations play the role of gauge transformations in Yang–Mills theory. General coordinate invariance and Lorentz invariance require the introduction of redundant canonical variables. This leads to constraints expressing the resulting interdependence of the canonical variables.

In Ashtekar's formulation, the constraints take the polynomial form:

$$\begin{aligned}\frac{\delta H}{\delta A_{a0}} &= \partial_i E^{ai} - \mathrm{i}\varepsilon^{abc}A_{bi}E_c{}^i = 0,\\ \frac{\delta H}{\delta\mathcal{N}} &= -\tfrac{1}{2}\mathrm{i}\varepsilon^{abc}F_{aij}E_b{}^iE_c{}^j = 0,\\ \frac{\delta H}{\delta N^i} &= -F_{aij}E^{aj} = 0,\end{aligned} \tag{22}$$

everywhere on the hypersurface Σ_t.

Ashtekar [10] has shown that these secondary constraints are first class. We see that the Hamiltonian is a linear combination of the constraints. It is therefore first class and weakly zero. It is important to recall that the Yang–Mills Hamiltonian is not weakly zero in general. This reflects a dynamical difference between the Yang–Mills field and the gravitational field.

3. Crnkovic–Witten theory

Let us review the Crnkovic–Witten construction in the case of the scalar field. We begin with the action of the scalar field in flat spacetime:

$$\begin{aligned}S &= \int_M \mathrm{d}^4x\,\mathcal{L},\\ \mathcal{L} &= \tfrac{1}{2}\big(\partial_\mu\phi\partial^\mu\phi - V(\phi)\big).\end{aligned} \tag{23}$$

Crnkovic and Witten's idea involves the introduction of a *symplectic current* at each spacetime point x:

$$J_\mu(x) = \delta\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\right)\wedge\delta\phi(x) = \delta\partial_\mu\phi(x)\wedge\delta\phi(x), \tag{24}$$

where δ stands for the functional exterior derivative of forms on the phase space of the scalar field [5]. Now

$$\begin{aligned}\delta J_\mu(x) &= \delta(\delta\partial_\mu\phi(x))\wedge\delta\phi(x) - \delta\partial_\mu\phi(x)\wedge\delta(\delta\phi(x))\\ &= \partial_\mu\delta(\delta\phi(x)) = 0.\end{aligned} \tag{25}$$

This means that J_μ is closed as a functional 2-form. Furthermore,

$$
\begin{aligned}
\partial^\mu J_\mu(x) &= \delta\big(\partial^\mu \partial_\mu \phi(x)\big) \wedge \delta\phi(x) + \delta\partial_\mu\phi(x) \wedge \delta\partial^\mu\phi(x) \\
&= -V''(\phi)\delta\phi(x) \wedge \delta\phi(x) + \delta\partial_\mu\phi(x) \wedge \delta\partial^\mu\phi(x) \\
&= -\delta\partial_\mu\phi(x) \wedge \delta\partial^\mu\phi(x) = 0, \qquad (26)
\end{aligned}
$$

with the help of the equation of motion

$$\partial_\mu \partial^\mu \phi + V'(\phi) = 0. \qquad (27)$$

Stokes' theorem then implies that

$$\int_N \mathrm{d}^4 x\, \partial^\mu J_\mu(x) = \int_{\partial N} \mathrm{d}\sigma^\mu(x)\, J_\mu(x) = 0, \qquad (28)$$

where N is a submanifold of M with boundary ∂N. Suppose $\partial N = \Sigma_{t_1} \cup \Sigma_{t_2} \cup \Sigma$, where $\Sigma_{t_1}, \Sigma_{t_2}$ are spacelike hypersurfaces of constant time, and $\mathrm{d}\sigma^\mu J_\mu$ vanishes everywhere on the hypersurface Σ. Then

$$\int_{\Sigma_{t_1}} \mathrm{d}\sigma^\mu(x)\, J_\mu(x) = \int_{\Sigma_{t_2}} \mathrm{d}\sigma^\mu(x)\, J_\mu(x), \qquad (29)$$

where $\mathrm{d}\sigma^\mu$ is chosen to point in the same temporal direction on both Σ_{t_1} and Σ_{t_2}. This means that the closed functional 2-form

$$
\begin{aligned}
\Omega &= \int_{\Sigma_t} \mathrm{d}\sigma^\mu(x)\, J_\mu(x) \\
&= \int_{\Sigma_t} \mathrm{d}\sigma^\mu(x)\, \delta\partial_\mu\phi(x) \wedge \delta\phi(x) \qquad (30)
\end{aligned}
$$

is independent of the choice of Σ_t. When we perform a Lorentz transformation $\Sigma_t \to \Sigma_{t'}$ and $\Omega \to \Omega'$, where

$$\Omega' = \int_{\Sigma_{t'}} \mathrm{d}\sigma^\mu(x')\, J_\mu(x') = \int_{\Sigma_t} \mathrm{d}\sigma^\mu(x)\, J_\mu(x) = \Omega. \qquad (31)$$

We conclude that Ω is a Lorentz invariant symplectic form on the phase space of the scalar field, and that it is possible to formulate the Hamiltonian theory of the scalar field in a manifestly covariant way. The Lorentz invariance of the symplectic form allows us to choose a spacelike hypersurface Σ_t, such that

$$\mathrm{d}\sigma^0(x) = \mathrm{d}^3 x, \qquad \mathrm{d}\sigma^i(x) = 0 \qquad (32)$$

for all $x \in \Sigma_t$ and, hence, we obtain the standard symplectic form

$$\Omega = \int_{\Sigma_t} \mathrm{d}^3 x\, \delta\dot{\phi}(x) \wedge \delta\phi(x) = \int_{\Sigma_t} \mathrm{d}^3 x\, \delta\left(\frac{\delta\mathcal{L}}{\delta\dot{\phi}(x)}\right) \wedge \delta\phi(x), \qquad (33)$$

where $\dot{\phi}$ is the momentum canonically conjugate to ϕ.

Next we consider the construction of a Lorentz invariant and gauge invariant symplectic form on the phase space of the $SU(2)$ Yang–Mills field, A_μ, in flat spacetime. In this case, the action is

$$S = -\tfrac{1}{4} \int_M \mathrm{d}^4 x\, \mathrm{tr}(F_{\mu\nu} F^{\mu\nu}),$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \qquad (34)$$

The symplectic current is taken to be

$$J_\mu = \operatorname{tr}\delta\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu A_\nu)}\right)\wedge\delta A^\nu = \operatorname{tr}\delta F_{\mu\nu}\wedge\delta A^\nu, \tag{35}$$

where δ is the functional exterior derivative of forms on the Yang–Mills phase space [5]. This symplectic current is closed, since

$$\delta(\delta F_{\mu\nu}) = 0, \qquad \delta(\delta A^\mu) = 0, \tag{36}$$

and therefore,

$$\delta J_\mu = \operatorname{tr}\delta(\delta F_{\mu\nu})\wedge\delta A^\nu - \operatorname{tr}\delta F_{\mu\nu}\wedge\delta(\delta A^\nu) = 0. \tag{37}$$

On introducing a basis, say $\{T^a\}$, for the $SU(2)$ Lie algebra, we have

$$\begin{aligned}\nabla_\mu A_{a\nu} &= \partial_\mu A_{a\nu} + [A_\mu, A_\nu]_a \\ &= \partial_\mu A_{a\nu} - \varepsilon_{abc}A^b{}_\mu A^c{}_\nu.\end{aligned} \tag{38}$$

Thus

$$\delta F_{a\mu\nu} = \nabla_\mu\delta A_{a\nu} - \nabla_\nu\delta A_{a\mu},$$

or

$$\delta F_{\mu\nu} = \nabla_\mu\delta A_\nu - \nabla_\nu\delta A_\mu. \tag{39}$$

As a result of a gauge transformation, $A_{a\mu}\to A'_{a\mu}$, with

$$A'_{a\mu} = A_{a\mu} + \partial_\mu\lambda_a + [A_\mu, \lambda]_a \tag{40}$$

for some infinitesimal real-valued function λ on spacetime. We have

$$\delta A'_{a\mu} = \delta A_{a\mu} + [\delta A_\mu, \lambda]_a, \tag{41}$$

$$\delta F'_{a\mu\nu} = \delta F_{a\mu\nu} + [\delta F_{\mu\nu}, \lambda]_a. \tag{42}$$

The symplectic current transforms according to

$$\begin{aligned}J'_\mu &= \delta F'_{a\mu\nu}\wedge\delta A'^{a\nu} \\ &= J_\mu - \varepsilon_{abc}\lambda^c\left(\delta F^b{}_{\mu\nu}\wedge\delta A^{a\nu} + \delta F^a{}_{\mu\nu}\wedge\delta A^{b\nu}\right) + \mathrm{O}(\lambda^2) \\ &= J_\mu + \mathrm{O}(\lambda^2).\end{aligned} \tag{43}$$

Thus the symplectic current is an $SU(2)$ singlet. This allows us to write

$$\begin{aligned}\partial^\mu J_\mu &= \nabla^\mu J_\mu \\ &= \operatorname{tr}\nabla^\mu\delta F_{\mu\nu}\wedge\delta A^\nu + \operatorname{tr}\delta F_{\mu\nu}\wedge\nabla^\mu\delta A^\nu.\end{aligned} \tag{44}$$

The equations of motion

$$\nabla^\mu F_{\mu\nu} = \partial^\mu F_{\mu\nu} + [A^\mu, F_{\mu\nu}] = 0, \tag{45}$$

imply that

$$\nabla^\mu\delta F_{\mu\nu} = -[\delta A^\mu, F_{\mu\nu}]. \tag{46}$$

Also

$$\begin{aligned}\operatorname{tr}\nabla^\mu\delta F_{\mu\nu}\wedge\delta A^\nu &= \varepsilon_{abc}F^c{}_{\mu\nu}\delta A^{a\nu}\wedge\delta A^{b\mu} \\ &= 0.\end{aligned} \tag{47}$$

Next we consider

$$\operatorname{tr}\delta F_{\mu\nu}\wedge\nabla^{\mu}\delta A^{\nu}=\tfrac{1}{2}\operatorname{tr}\delta F_{\mu\nu}\wedge(\nabla^{\mu}\delta A^{\nu}-\nabla^{\nu}\delta A^{\mu})$$
$$=\tfrac{1}{2}\operatorname{tr}\delta F_{\mu\nu}\wedge\delta F^{\mu\nu}=0. \tag{48}$$

Combining (47) and (48), we see that

$$\partial^{\mu}J_{\mu}=0 \tag{49}$$

by (44). It follows that the closed functional 2-form, Ω, given by

$$\Omega=\int_{\Sigma_t}\mathrm{d}\sigma^{\mu}\,J_{\mu}=\int_{\Sigma_t}\mathrm{d}\sigma^{\mu}\,\operatorname{tr}\delta F_{\mu\nu}\wedge\delta A^{\nu}, \tag{50}$$

is Lorentz invariant. Thus we have constructed a Lorentz invariant and gauge invariant symplectic form, Ω, on the $SU(2)$ Yang–Mills phase space.

We can obtain the standard $SU(2)$ Yang–Mills symplectic form by a suitable choice of the spacelike hypersurface Σ_t:

$$\Omega=\int_{\Sigma_t}\mathrm{d}^3x\,\operatorname{tr}\delta E_i\wedge\delta A^i, \tag{51}$$

where $E_i=F_{0i}$ is the momentum canonically conjugate to A^i.

4. A symplectic form for Ashtekar's canonical gravity

The inverse densitized triads and the Ashtekar connection act as symplectic coordinates in the phase space of Ashtekar's canonical gravity. We wish to put a symplectic form on Ashtekar's phase space in a manner consistent with the Crnkovic–Witten construction. An extra difficulty here, over and above the problem of gauge invariance, is the complex nature of Ashtekar's canonical variables. Denoting the functional exterior derivative of forms on Ashtekar's phase space by δ, we have

$$\delta A_{ai}=\frac{\delta\omega_{0ai}}{\delta E^{bj}}\delta E^{bj}+\frac{\delta\omega_{0ai}}{\delta\dot{E}^{bj}}\delta\dot{E}^{bj}-\tfrac{1}{2}\mathrm{i}\varepsilon_{acd}\frac{\delta\omega^{cd}{}_{i}}{\delta E^{bj}}\delta E^{bj}, \tag{52}$$

where the shorthand notation

$$\frac{\delta\omega_{0ai}}{\delta E^{bj}}=\int_{\Sigma_t}\frac{\delta\omega_{0ai}(x)}{\delta E^{bj}(y)}\,\mathrm{d}^3y,$$

is understood.

We require the symplectic form,

$$\Omega=\int_{\Sigma_t}\mathrm{d}^3x\,\delta E^{ai}\wedge\delta A_{ai}$$
$$=-\int_{\Sigma_t}\mathrm{d}^3x\left(\frac{\delta A_{ai}}{\delta E^{bj}}\delta E^{bj}\wedge\delta E^{ai}+\frac{\delta A_{ai}}{\delta\dot{E}^{bj}}\delta\dot{E}^{bj}\wedge\delta E^{ai}\right), \tag{53}$$

to be real valued in order to have a unique symplectic structure on Ashtekar's phase space. A complex-valued symplectic form would give rise to two real symplectic structures. Moreover, a real-valued symplectic form produces real-valued Poisson brackets.

Working in the time gauge and using (6) and (7), it is straightforward to show that

$$\int_{\Sigma_t} \mathrm{d}^3x \, \frac{\delta\omega_{0ai}}{\delta E^{bj}} \delta E^{bj} \wedge \delta E^{ai}$$

$$= \frac{1}{2\mathcal{N}} \Big[\big((E^{-1})_{bi} (E^{-1})_{ak} + (E^{-1})^c{}_i (E^{-1})_{ck} \delta_{ab} \big) \partial_j N^k - \big((E^{-1})_{bi} (E^{-1})^c{}_j (E^{-1})_{ck}$$
$$+ (E^{-1})_{bk} (E^{-1})^c{}_i (E^{-1})_{cj} \big) \big(\dot{E}_a{}^k + E_a{}^\ell \partial_\ell N^k - N^\ell \partial_\ell E_a{}^k \big)$$
$$- (E^{-1})_{ai} (E^{-1})_{cj} (E^{-1})_{bk} \big(N^\ell \partial_\ell E^{ck} - \dot{E}^{ck} \big) \Big] \delta E^{bj} \wedge \delta E^{ai}, \tag{54}$$

$$\int_{\Sigma_t} \mathrm{d}^3x \, \frac{\delta\omega_{0ai}}{\delta \dot{E}^{bj}} \delta \dot{E}^{bj} \wedge \delta E^{ai}$$

$$= \frac{1}{2\mathcal{N}} \big[(E^{-1})^c{}_i (E^{-1})_{cj} \eta_{ab} + (E^{-1})_{bi} (E^{-1})_{aj} - (E^{-1})_{ai} (E^{-1})_{bj} \big] \delta \dot{E}^{bj} \wedge \delta E^{ai}. \tag{55}$$

Following Henneaux *et al* [8], we can write

$$\varepsilon_{acd} \omega^{cd}{}_i = \frac{\delta}{\delta E^{ai}} \int_{\Sigma_t} \mathrm{d}^3x \, G,$$

where

$$G = \varepsilon^{jk\ell} h_{bj} \partial_k h^b{}_\ell. \tag{56}$$

Then

$$\int_{\Sigma_t} \mathrm{d}^3x \, \varepsilon_{acd} \frac{\delta\omega^{cd}{}_i}{\delta E^{bj}} \delta E^{bj} \wedge \delta E^{ai} = \frac{\delta}{\delta E^{bj}} \left(\frac{\delta}{\delta E^{ai}} \int_{\Sigma_t} \mathrm{d}^3x \, G \right) \delta E^{bj} \wedge \delta E^{ai}$$
$$= -\frac{\delta}{\delta E^{bj}} \left(\frac{\delta}{\delta E^{ai}} \int_{\Sigma_t} \mathrm{d}^3x \, G \right) \delta E^{bj} \wedge \delta E^{ai} = 0. \tag{57}$$

Thus the complex part of the symplectic form, Ω, is zero in the time gauge.

Now we must show that the symplectic form is real valued for all other gauges, apart from the time gauge. When we go from a time gauge hypersurface to a more general hypersurface, the group of local symmetries enlarges from the rotation group $SO(3)$ to the Lorentz group $SO(3,1)$ (see the appendix).

Ashtekar's action can be written as:

$$S = \int_M \mathcal{L} \, \mathrm{d}^4x,$$

where

$$\mathcal{L} \, \mathrm{d}^4x = \mathrm{i} F_a \wedge \Lambda^a, \tag{58}$$

as in (17) and (20). The symplectic form (53) can be then be written as:

$$\Omega = \int_{\Sigma_t} \mathrm{d}^3x \, \delta \left(\frac{\delta \mathcal{L}}{\delta \dot{A}_{ai}} \right) \wedge \delta A^{ai}. \tag{59}$$

The analogy with the Hamiltonian formulation of the $SU(2)$ Yang–Mills field suggests that we ought to postulate a functional 2-form on Ashtekar's phase space, with the vector density on spacetime:

$$J_\mu = \delta\left[\frac{\delta\mathcal{L}}{\delta(\partial_\mu A_{ai})}\right] \wedge \delta A^{ai}$$

$$= \delta\left[\frac{\delta\mathcal{L}}{\delta(\partial_\mu A_{a\nu})}\right] \wedge \delta A^{a\nu}$$

$$= \mathrm{i}\,\mathrm{tr}\,\delta\Lambda_{\mu\nu} \wedge \delta A^\nu, \tag{60}$$

as a symplectic current. Here, the trace tr relates to a representation of $sl(2,\mathbb{C})$, the Lie algebra of $SO(3,1)$. We can associate an $SO(3,1)$ covariant derivative D_μ with the connection A_μ such that

$$\delta F_{\mu\nu} = D_\mu \delta A_\nu - D_\nu \delta A_\mu. \tag{61}$$

Under a local $SO(3,1)$ transformation, $A_\mu \to A_\mu + \partial_\mu\lambda + [A_\mu, \lambda]$, where λ is a real-valued function on spacetime. It is found that $\delta A_\mu \to \delta A_\mu + [\delta A_\mu, \lambda]$ and $\delta\Lambda_{\mu\nu} \to \delta\Lambda_{\mu\nu} + [\delta\Lambda_{\mu\nu}, \lambda]$. It follows that the symplectic current is an $SO(3,1)$ singlet. Thus we can write:

$$\partial^\mu J_\mu = D^\mu J_\mu \tag{62}$$

$$= \mathrm{i}\,\mathrm{tr}\, D^\mu \delta\Lambda_{\mu\nu} \wedge \delta A^\nu + \mathrm{i}\,\mathrm{tr}\,\delta\Lambda_{\mu\nu} \wedge D^\mu \delta A^\nu. \tag{63}$$

Since

$$D^\mu \Lambda_{\mu\nu} = 0, \tag{64}$$

we have

$$D^\mu \delta\Lambda_{\mu\nu} = -[\delta A^\mu, \Lambda_{\mu\nu}]$$

and

$$\mathrm{i}\,\mathrm{tr}\, D^\mu \delta\Lambda_{\mu\nu} \wedge \delta A^\nu = -\mathrm{i}\,\mathrm{tr}[\delta A^\mu, \Lambda_{\mu\nu}] \wedge \delta A^\nu$$

$$= \tfrac{1}{2}\mathrm{i}\,\mathrm{tr}[\delta A^\mu, \delta A^\nu] \wedge \Lambda_{\mu\nu} = 0. \tag{65}$$

When we vary the action with respect to the orthonormal 1-forms, while keeping the connection 1-forms fixed, the equations of motion imply:

$$F^{\mu\nu}\delta\Lambda_{\mu\nu} = 0. \tag{66}$$

Consequently,

$$\mathrm{i}\,\mathrm{tr}\,\delta\Lambda_{\mu\nu} \wedge D^\mu \delta A^\nu = \tfrac{1}{2}\mathrm{i}\,\mathrm{tr}\,\delta\Lambda_{\mu\nu} \wedge \delta F^{\mu\nu} = -\tfrac{1}{2}\mathrm{i}\,\mathrm{tr}\,\delta(F^{\mu\nu}\delta\Lambda_{\mu\nu}) = 0. \tag{67}$$

It is clear from (65) and (67) that the divergence of the symplectic current in (63) vanishes, and it follows that the closed functional 2-form

$$\Omega = \int_{\Sigma_t} \mathrm{d}\sigma^\mu\, J_\mu = \mathrm{i}\int_{\Sigma_t} \mathrm{d}\sigma^\mu\, \mathrm{tr}\,\delta\Lambda_{\mu\nu} \wedge \delta A^\nu, \tag{68}$$

is a Lorentz invariant and general coordinate invariant symplectic form on Ashtekar's phase space. In particular, since the imaginary part of Ω vanishes in the time gauge, and Ω is Lorentz invariant, then Ω is *real valued* in *any* local Lorentz frame, even one in which the inverse densitized triads are complex valued.

A further question, which must be addressed, is the convergence of the integral in (68) when Σ_t is non-compact since, unlike the case of a scalar field or a Yang–Mills field, one cannot in general assume that E^{ai} and A_{ai} vanish outside a compact region of Σ_t. This question is analysed in detail in [6], where the sufficient conditions on the asymptotic form of the 3-metric g_{ij} are derived to ensure the convergence of the relevant integrals when Σ_t is non-compact, though the complex variables E^{ai} and A_{ai} are not considered there. Having proven that Ω is real when E^{ai} and A_{ai} are used as canonical variables, we only need to transcribe the conditions on g_{ij} used in [6] to equivalent conditions on E^{ai} and A_{ai} in Ashtekar's formalism and this will ensure that (68) converges.

5. Conclusions

We have described Ashtekar's canonical gravity in a manifestly covariant way by using a construction due to Crnkovic and Witten. This construction had worked for the ADM formulation of general relativity, so we hoped it might work for Ashtekar's formulation, at least in the time gauge. The only obstacles to be overcome were gauge invariance and the complex nature of Ashtekar's canonical variables.

Gauge invariance was incorporated into the symplectic form we put on Ashtekar's phase space, along with Lorentz invariance, as in the work of Crnkovic and Witten. Using a result in a paper by Henneaux *et al*, we showed that the symplectic form is *real valued* in the time gauge, thereby giving rise to a *unique* symplectic structure on Ashtekar's phase space, as well as real-valued Poisson brackets.

It remained to show that the symplectic form is real valued for all other gauges, in addition to the time gauge, when the canonical variables are all $sl(2,\mathbb{C})$ valued. This was accomplished using the analogy with the Hamiltonian formulation of the $SU(2)$ Yang–Mills field. As a result, we know that the Crnkovic–Witten construction can be applied to Ashtekar's canonical gravity.

Appendix

A general orthonormal basis can be obtained from one adapted to a spacelike hypersurface, Σ_t, as follows. Denoting four-dimensional orthonormal and coordinate indices by $A, B, \ldots$ and $\mu, \nu, \ldots$, respectively, let

$$h^A{}_\mu = \begin{bmatrix} N & 0 \\ h^a{}_j N^j & h^a{}_i \end{bmatrix} \tag{A1}$$

be a tetrad with $h^0 = N\,\mathrm{d}x^0$ normal to Σ_t. This choice of tetrad is compatible with the time gauge condition. Here N is the lapse function, N^i are the shift functions and $h^a{}_i$ is an orthonormal triad on Σ_t satisfying

$$h^a{}_i h_{aj} = g_{ij}, \qquad (h^{-1})_a{}^i h^a{}_j = \delta^i{}_j, \tag{A2}$$

where g_{ij} is the three-dimensional metric on Σ_t. An arbitrary Lorentz boost, tangent to Σ_t, with 3-velocity v^a is given by

$$L(v)^A{}_B = \begin{bmatrix} \gamma & -\gamma v_b \\ -\gamma v^a & \delta^a{}_b + \dfrac{\gamma^2}{1+\gamma} v^a v_b \end{bmatrix},$$

where

$$\gamma = (1 - v^a v_a)^{-1/2}. \tag{A3}$$

So an arbitrary tetrad is of the form:

$$e^A{}_\mu = L(v)^A{}_B h^B{}_\mu. \tag{A4}$$

Note, however, that

$$e^a{}_i = h^a{}_i + \frac{\gamma^2}{1+\gamma} v^a v_b h^b{}_i \tag{A5}$$

are not an orthonormal *triad* because $e^a{}_i e_{aj} \neq g_{ij}$. Let e be the determinant of the matrix, $[e_{ai}]$. Defining the inverse densitized triads E^{ai} by

$$E^{ai} = e(e^{-1})^{ai} - \mathrm{i}\varepsilon^{ijk} e^a{}_j e^b{}_k v_b, \tag{A6}$$

Ashtekar's Lagrangian takes the form:

$$L = \int_{\Sigma_t} \mathrm{d}^3x \left[A_{ai}\dot{E}^{ai} - A_{a0}\big(\partial_i E^{ai} - \mathrm{i}\varepsilon^{abc} A_{bi} E_c{}^i\big) + \tfrac{1}{2}\mathrm{i}\mathcal{N}\varepsilon^{abc} F_{aij} E_b{}^i E_c{}^j + N^i F_{aij} E^{aj}\right], \tag{A7}$$

where $\mathcal{N} = \gamma N/e$. This shows that E^{ai} and A_{ai} are canonically conjugate, with $\mathcal{N}$, N^i and A_{a0} behaving as Lagrange multipliers for the secondary constraints. It is straightforward to verify that the complex inverse densitized triads satisfy

$$E^{ai} E_a{}^j = g g^{ij}, \qquad E^{ai} E_b{}^j g_{ij} = g \delta^a{}_b, \tag{A8}$$

where g is the determinant of the matrix $[g_{ij}]$, and so they can be regarded, in a sense, as a complex orthonormal triad density. The effect of an infinitesimal Lorentz boost on the canonical variables is easily calculated. Using

$$\delta L^A{}_B(0) = \begin{bmatrix} 0 & -\delta v_b \\ -\delta v^a & 0 \end{bmatrix}, \tag{A9}$$

we find

$$\delta_v A^a{}_i = -(\partial_i \delta v^a - \mathrm{i}\varepsilon^{abc} A_{bi} \delta v_c), \tag{A10}$$

$$\delta_v E^{ai} = \mathrm{i}\varepsilon^{abc} E_b{}^i \delta v_c, \tag{A11}$$

which is to be compared with the effect of an infinitesimal tangent space rotation on Σ_t, parameterized by $\delta\theta^a$,

$$\delta_\theta A^a{}_i = \mathrm{i}(\partial_i \delta\theta^a - \mathrm{i}\varepsilon^{abc} A_{bi} \delta\theta_c), \tag{A12}$$

$$\delta_\theta E^{ai} = \varepsilon^{abc} E_b{}^i \delta\theta_c. \tag{A13}$$

As an extra check that these variables are canonically conjugate, it is instructive to prove that Lorentz transformations leave the Poisson brackets unchanged. It is easy to verify that an infinitesimal boost leaves the Poisson bracket unchanged as, of course, do infinitesimal rotations. As boosts and rotations form a group, we can simply exponentiate and deduce that *finite* Lorentz transformations also leave the Poisson bracket invariant. Hence,

$$\{E^{ai}, A_{bj}\} = \delta^a{}_b \delta^i{}_j \tag{A14}$$

must hold for the complex E^{ai} with $v^a \neq 0$. In conclusion, it has been shown that it is not necessary to match the choice of an orthonormal frame to the foliation of spacetime in Ashtekar's canonical gravity. The inverse densitized triads are now complex valued, but there are still conditions on them, since the imaginary part only has three degrees of freedom, v^a, rather than the nine which would be necessary for a complex 3×3 matrix. The Ashtekar connection, A_{ai}, becomes $sl(2, \mathbb{C})$ valued. The infinitesimal $sl(2, \mathbb{C})$ gauge transformations are given above in (A10) and (A11). Finally, this appendix is equivalent to the work of Ashtekar *et al* in [11], where the results are formulated in spinor notation.

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