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Noncommutativity and quantum structure of spacetime

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Abstract. Noncommutative geometry is a candidate for describing physics at the Planck scale. Although there are various proposals to formulate gravity theories in the noncommutative framework, not many exact solutions of noncommutative gravity theories are known. In this talk we describe a possible noncommutative analogue of the BTZ black hole, which is consistent with the geometric structure of the problem. The spacetime Poisson brackets that we obtain indicate the possibility of the quantization of time. We briefly discuss the issues of symmetries and particle statistics in such a spacetime.

1. Introduction

The description of physics at the Planck scale, where quantum aspects of gravity are expected to be manifest, remains an outstanding challenge in theoretical physics. It is plausible that strong quantum gravity effects associated with such high energy scales can affect the smooth structure of the spacetime manifold. There have been various proposals to replace the continuum picture of spacetime with a discrete atomistic structure at the Planck scale. In a related approach, it has been argued that Heisenberg's uncertainty principle together with Einstein's theory of general relativity leads to spacetime noncommutativity [1]. In this picture, the smooth manifold structure of the spacetime is replaced with a noncommutative algebra at high energies, and the associated dynamics is governed by a noncommutative generalization of gravity.

Most approaches to noncommutative gravity have involved deforming the commutative Einstein equations [2]. These approaches range from simply replacing point-wise products by Moyal star products in the Einstein-Hilbert action to the approach of Aschieri *et al.* [3] which preserves the diffeomorphism invariance of general relativity. In our approach [4], we write down Poisson brackets which are consistent with the geometry of some classical solution. In particular, we focus on the Banados-Teitelboim-Zanelli (BTZ) solution of $2+1$ gravity. The noncommutative counterpart of the BTZ solution is then obtained by 'quantization', with the deformation parameter being the noncommutativity parameter θ. The Poisson brackets and the resulting noncommutative algebra are not unique. In this regard, it may be desirable to impose the restriction that the isometry of the classical solution survives quantization and is implementable in any irreducible representation of the noncommutative algebra. In this talk we focus on the deformations with such additional constraints and show that they lead to spacetime noncommutativity where the time is quantized [5, 6].

Analysis of quantum field theory (QFT) on curved spacetime often reveals the features of the underlying geometry. It is thus desirable to have a formulation of QFT for a spacetime arising from noncommutative gravity. Here we take the preliminary steps for setting up QFT in such a background and briefly indicate the relevant issues.

2. Commutative BTZ

In this Section we shall briefly review certain features of the commutative BTZ solution that would be relevant for our analysis.

In terms of Schwarzschild-like coordinates (r, t, ϕ) the invariant measure for the BTZ black hole [7, 8] is expressed as

$$ds^2 = \Big(M - \frac{r^2}{\ell^2} - \frac{J^2}{4r^2}\Big)dt^2 + \Big(-M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2}\Big)^{-1} dr^2 + r^2\Big(d\phi - \frac{J}{2r^2}dt\Big)^2 , \qquad (1)$$

$$0 \le r < \infty \ , \ -\infty < t < \infty \ , \ 0 \le \phi < 2\pi \ ,$$

where M and J are the mass and spin, respectively, and $\Lambda = -1/\ell^2$ is the cosmological constant. For $0 < |J| < M\ell$, there are two horizons, the outer and inner horizons, corresponding respectively to $r = r_+$ and $r = r_-$, where

$$r_\pm^2 = \frac{M\ell^2}{2}\Big\{1 \pm \Big[1 - \Big(\frac{J}{M\ell}\Big)^2\Big]^{\frac{1}{2}}\Big\} \ . \qquad (2)$$

The two horizons coincide in the extremal case $|J| = M\ell > 0$, while the inner one disappears for $J = 0$, $M > 0$. The metric is diagonal in the coordinates (χ_+, χ_-, r), where

$$\chi_\pm = \frac{r_\pm}{\ell}t - r_\mp \phi \ , \qquad (3)$$

$$ds^2 = \frac{-(r^2 - r_+^2)d\chi_+^2 + (r^2 - r_-^2)d\chi_-^2}{r_+^2 - r_-^2} + \frac{\ell^2 r^2 dr^2}{(r^2 - r_+^2)(r^2 - r_-^2)} \ , \qquad (4)$$

which shows that χ_+ is the time-like coordinate in the region I) $r \ge r_+$, r is the time-like coordinate in the region II) $r_- \le r \le r_+$, and χ_- is the time-like coordinate in the region III) $0 \le r \le r_-$.

It was shown that the manifold of the BTZ black hole solution is the quotient space of the universal covering space of AdS^3 by some elements of the group of isometries of AdS^3. The connected component of the latter is $SO(2,2)$. Say AdS^3 is spanned by coordinates (t_1, t_2, x_1, x_2) parameterizing R^4, satisfying

$$-t_1^2 - t_2^2 + x_1^2 + x_2^2 = -\ell^2 \ . \qquad (5)$$

Alternatively, one can introduce 2×2 real unimodular matrices

$$g = \frac{1}{\ell}\begin{pmatrix} t_1 + x_1 & t_2 + x_2 \\ -t_2 + x_2 & t_1 - x_1 \end{pmatrix} , \qquad \det g = 1 \ , \qquad (6)$$

belonging to the defining representation of $SL(2, R)$. The isometries correspond to the left and right actions on g,

$$g \to h_L g h_R \ , \qquad h_L, h_R \in SL(2, R) \ . \qquad (7)$$

Since (h_L, h_R) and $(-h_L, -h_R)$ give the same action, the connected component of the isometry group for AdS^3 is $SL(2, R) \times SL(2, R)/Z_2 \approx SO(2, 2)$.

The BTZ black-hole is obtained by discrete identification of points on the universal covering space of AdS^3. This insures periodicity in ϕ, $\phi \sim \phi + 2\pi$. The condition is

$$g \sim \tilde{h}_L g \tilde{h}_R \;, \tag{8}$$

where $(\tilde{h}_L, \tilde{h}_R)$ are certain elements of $SO(2,2)$. $\tilde{h}_L$ and $\tilde{h}_R$ can be expressed as diagonal $SL(2,R)$ matrices

$$\tilde{h}_L = \begin{pmatrix} e^{\pi(r_+-r_-)/\ell} & \\ & e^{-\pi(r_+-r_-)/\ell} \end{pmatrix} \;, \qquad \tilde{h}_R = \begin{pmatrix} e^{\pi(r_++r_-)/\ell} & \\ & e^{-\pi(r_++r_-)/\ell} \end{pmatrix} \;. \tag{9}$$

For $0 < |J| < M\ell$, the universal covering space of AdS^3 is covered by three types of coordinate patches which are bounded by the two horizons at $r = r_+$ and $r = r_-$. For all three coordinate patches, g can be decomposed according to

$$g = \begin{pmatrix} e^{\frac{1}{2\ell}(\chi_+-\chi_-)} & \\ & e^{-\frac{1}{2\ell}(\chi_+-\chi_-)} \end{pmatrix} g^{(0)}(r) \begin{pmatrix} e^{\frac{1}{2\ell}(\chi_++\chi_-)} & \\ & e^{-\frac{1}{2\ell}(\chi_++\chi_-)} \end{pmatrix} \;, \tag{10}$$

where $g^{(0)}(r)$ is an $SO(2)$ matrix which only depends on r and the coordinate patch. The periodicity condition for ϕ easily follows from (8). The identification (8) breaks the $SO(2,2)$ group of isometries to a two-dimensional subgroup $\mathcal{G}_{BTZ}$, consisting of only the diagonal matrices in $\{h_L\}$ and $\{h_R\}$. $\mathcal{G}_{BTZ}$ is the isometry group of the BTZ black hole, and from (10) is associated with translations in χ_+ and χ_-, or equivalently t and ϕ, on $r =$ constant surfaces.

3. Noncommutative BTZ

In this Section we shall find the nontrivial Poisson brackets among the $SL(2,R)$ matrix elements which describe a BTZ black hole.

For generic spin, $0 < |J| < M\ell$ (and $M > 0$), we shall search for Poisson brackets for the matrix elements of g which are polynomial of lowest order. They should be consistent with the quotienting (8), as well as with the unimodularity condition and, of course, with the Jacobi identity. For convenience we write the $SL(2,R)$ matrix as

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \;, \qquad \alpha\delta - \beta\gamma = 1 \;. \tag{11}$$

Under the quotienting (8):

$$\begin{aligned} \alpha &\sim e^{\,2\pi r_+/\ell}\,\alpha \\ \beta &\sim e^{-2\pi r_-/\ell}\,\beta \\ \gamma &\sim e^{\,2\pi r_-/\ell}\,\gamma \\ \delta &\sim e^{-2\pi r_+/\ell}\,\delta \;. \end{aligned} \tag{12}$$

All quadratic combinations of matrix elements scale differently, except for $\alpha\delta$ and $\beta\gamma$, which are invariant under (12). Lowest order polynomial expressions for the Poisson brackets of α, β, γ and δ which are preserved under (12) are quadratic and have the form:

$$\begin{array}{lll} \{\alpha,\beta\} = c_1\alpha\beta & \{\alpha,\gamma\} = c_2\alpha\gamma & \{\alpha,\delta\} = f_1(\alpha\delta,\beta\gamma) \\ \{\beta,\delta\} = c_3\beta\delta & \{\gamma,\delta\} = c_4\gamma\delta & \{\beta,\gamma\} = f_2(\alpha\delta,\beta\gamma) \end{array} \;, \tag{13}$$

where c_{1-4} are constants and $f_{1,2}$ are functions. They are constrained by

$$c_1 + c_2 = c_3 + c_4$$

$$\begin{aligned} f_1(\alpha\delta,\beta\gamma) &= (c_1+c_2)\beta\gamma \\ f_2(\alpha\delta,\beta\gamma) &= (c_2-c_4)\alpha\delta\ , \end{aligned} \tag{14}$$

after demanding that det$g$ is a Casimir of the algebra. From (14) there are three independent constants c_{1-4}. Further restrictions on the constants come from the Jacobi identity, which leads to the following two possibilities:

$$\mathrm{A}:\quad c_2 = c_4 \qquad \text{and} \qquad \mathrm{B}:\quad c_2 = -c_1\ .$$

Both cases define two-parameter families of Poisson brackets. Say we call c_2 and c_3 the two independent parameters. The two cases are connected by an $SO(2,2)$ transformation. Case A goes to case B when

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \to g' = \begin{pmatrix} \beta & -\alpha \\ \delta & -\gamma \end{pmatrix} = g h_R^{(0)}\ , \qquad h_R^{(0)} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\ , \tag{15}$$

along with

$$c_3 \to c_2\ , \qquad c_2 \to c_3\ . \tag{16}$$

In terms of the embedding coordinates, this corresponds to $(t_1, t_2, x_1, x_2) \to (t_2, -t_1, x_2, -x_1)$.

There are three types of coordinate patches in the generic case of $M > 0$ and $0 < |J| < M\ell$, and their boundaries are the two horizons. Denote them again by: I) $r \geq r_+$, II) $r_- \leq r \leq r_+$ and III) $0 \leq r \leq r_-$. The corresponding maps to $SL(2,R)$ are given by (10), with
I) $r \geq r_+$,

$$g^{(0)}(r) = g_I^{(0)}(r) = \frac{1}{\sqrt{r_+^2 - r_-^2}} \begin{pmatrix} \sqrt{r^2 - r_-^2} & \sqrt{r^2 - r_+^2} \\ \sqrt{r^2 - r_+^2} & \sqrt{r^2 - r_-^2} \end{pmatrix}\ , \tag{17}$$

II) $r_- \leq r \leq r_+$,

$$g^{(0)}(r) = g_{II}^{(0)}(r) = \frac{1}{\sqrt{r_+^2 - r_-^2}} \begin{pmatrix} \sqrt{r^2 - r_-^2} & -\sqrt{r_+^2 - r^2} \\ \sqrt{r_+^2 - r^2} & \sqrt{r^2 - r_-^2} \end{pmatrix}\ , \tag{18}$$

III) $0 \leq r \leq r_-$,

$$g^{(0)}(r) = g_{III}^{(0)}(r) = \frac{1}{\sqrt{r_+^2 - r_-^2}} \begin{pmatrix} \sqrt{r_-^2 - r^2} & -\sqrt{r_+^2 - r^2} \\ \sqrt{r_+^2 - r^2} & -\sqrt{r_-^2 - r^2} \end{pmatrix}\ . \tag{19}$$

Using the maps (17-19), we can write the Poisson brackets for the various cases in terms of the Schwarzschild-like coordinates (r, t, ϕ). The results are the same in all three coordinate patches. For the two-parameter families A one gets:

$$\begin{aligned} \{\phi, t\} &= \frac{\ell^3}{2} \frac{c_3 - c_2}{r_+^2 - r_-^2} \\ \{r, \phi\} &= -\frac{\ell r_+(c_3 + c_2)}{2r} \frac{r^2 - r_+^2}{r_+^2 - r_-^2} \\ \{r, t\} &= -\frac{\ell^2 r_-(c_3 + c_2)}{2r} \frac{r^2 - r_+^2}{r_+^2 - r_-^2}\ , \end{aligned} \tag{20}$$

for the two-parameter families B one gets:

$$\{\phi, t\} = \frac{\ell^3}{2} \frac{c_3 - c_2}{r_+^2 - r_-^2}$$

$$\begin{aligned}\{r,\phi\} &= -\frac{\ell r_-(c_2+c_3)}{2r}\frac{r^2-r_-^2}{r_+^2-r_-^2}\\ \{r,t\} &= -\frac{\ell^2 r_+(c_2+c_3)}{2r}\frac{r^2-r_-^2}{r_+^2-r_-^2}\ .\end{aligned}\tag{21}$$

These Poisson brackets are invariant under the action of the isometry group $\mathcal{G}_{BTZ}$ of the BTZ black hole. The first bracket agrees in both cases. The latter two brackets vanish at the outer horizon $r = r_+$ for case A, and at the inner horizon $r = r_-$ for case B. A central element of the Poisson algebra can be constructed out of the Schwarzschild coordinates for both cases. It is given by

$$\rho_\pm = (r^2 - r_\pm^2)\exp\left\{-\frac{2\kappa\chi_\pm}{\ell}\right\}\ , \qquad c_2 \neq c_3\ , \tag{22}$$

where the upper and lower signs correspond to case A and B, respectively,

$$\kappa = \frac{c_3+c_2}{c_3-c_2}\ , \tag{23}$$

and $\chi_\pm$ were defined in (3). The $\rho_\pm$ = constant surfaces define symplectic leaves which are topologically R^2 for generic values of the parameters (more specifically, $c_2 \neq \pm c_3$). We can coordinatize them by χ_+ and χ_-. One then has a trivial Poisson algebra in the coordinates $(\chi_+, \chi_-, \rho_\pm)$:

$$\{\chi_+,\chi_-\} = \frac{\ell^2}{2}(c_3-c_2)\ , \qquad \{\rho_\pm,\chi_+\} = \{\rho_\pm,\chi_-\} = 0\ . \tag{24}$$

The action of the $\mathcal{G}_{BTZ}$ transforms one symplectic leaf to another, except for the case $c_2 = -c_3$ in which we shall focus from now on.

In passing to the noncommutative theory, the operator associated with $\rho_\pm$ is central in the quantum algebra and proportional to the identity in any irreducible representation. Irreducible representations then select $\rho_\pm$ = constant surfaces and the isometry group $\mathcal{G}_{BTZ}$ maps between different irreducible representations, and thus cannot be implemented as inner transformations. However, for the special case of $c_2 = -c_3$, the parameter κ vanishes ($\kappa = 0$) and the irreducible representation is preserved under the action of $\mathcal{G}_{BTZ}$. In this case, the radial coordinate is in the center of the algebra, r = constant defines $R\times S^1$ symplectic leaves, and they are invariant under the action of $\mathcal{G}_{BTZ}$. The coordinates ϕ and t parametrizing any such surface are canonically conjugate:

$$\{\phi,t\} = \frac{c_3\ell^3}{r_+^2-r_-^2}\ , \qquad \{\phi_\pm, r\} = \{t,r\} = 0\ . \tag{25}$$

The Poisson brackets can be interpreted in terms of a twist [9] in the decomposition of g given in (10), where the twist is with respect to the first and third matrices. In passing to the noncommutative theory, we need to define a deformation of the commutative algebra generated by $t, e^{i\phi}$ and r. Call the corresponding quantum operators $\hat t, e^{i\hat\phi}$ and $\hat r$, respectively. Their commutation relations are

$$[e^{i\hat\phi},\hat t] = \theta e^{i\hat\phi}\ , \qquad [\hat r,\hat t] = [\hat r, e^{i\hat\phi}] = 0\ , \tag{26}$$

where from (25) the constant θ is linearly related to $\ell^3/(r_+^2 - r_-^2)$. There are now two central elements in the algebra: i) $\hat r$ and ii) $e^{-2\pi i\hat t/\theta}$. From i), irreducible representations select the $R\times S^1$ symplectic leaves. The action of $\mathcal{G}_{BTZ}$ does not take you out of any particular irreducible representation, and in this sense we can say that the isometry of the classical solution survives quantization.

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With regard to the central element ii) $e^{-2\pi i\hat{t}/\theta}$, one can identify it with $e^{i\chi}$ in an irreducible representation. The spectrum of $\hat{t}$ is then discrete [5, 6]:

$$n\theta - \frac{\chi\theta}{2\pi}\ , \ \ n \in Z\ . \tag{27}$$

This implies that if the above description admits a Hamiltonian formulation, then the corresponding time-independent Hamiltonian operator would be conserved only modulo $2\pi/\theta$.

4. QFT on noncommutative BTZ

In this Section we give a very brief outline of the issues involved in setting up a QFT in the noncommutative BTZ spacetime.

In the commutative case, the steps involved in setting up a QFT include the mode expansion of the field and imposing commutation relations between the creation and annihilation operators depending on the field statistics and, finally, representing the operators of the theory in a suitable Hilbert space. It is assumed that the symmetries of the theory should be implemented in such a way that the statistics be superselected. Finally, the discrete spacetime symmetries should also be properly implemented in the quantum theory.

In the case of a Moyal plane, which is endowed with a star product, one way to implement the symmetries is by using the Drinfeld twist. In the presence of a finite noncommutative parameter, the coproduct for any continuous symmetry, e.g. Poincare symmetry, must be twisted so that it is compatible with the rule of star multiplication. This in turn implies that the flip operator, which defines the statistics, should be twisted as well. As a result, the algebra of the creation and annihilation operators differs from that in the commutative case. In addition, in the Moyal case, parity cannot be implemented as an automorphism and hence is not a good symmetry. These issues have been discussed extensively in the literature [10, 11, 12, 13].

In our formulation, the deformation of the BTZ black hole is described by a noncommutative cylinder. As we have seen, the spectrum of the time operator is quantized in such a case. In order to set up a consistent QFT for this noncommutative spacetime, the issues mentioned above must be addressed properly, which is presently under investigation.

References

[1] Doplicher S, Fredenhagen K and Roberts J E 1994 *Phys. Lett.* B **331** 39; *do.* 1995 *Commun. Math. Phys.* **172** 187 (*Preprint* arXiv:hep-th/0303037)

[2] Madore J and Mourad J 1994 *Int. J. Mod. Phys.* D **3** 221

[3] Aschieri P, Blohmann C, Dimitrijevic M, Meyer F, Schupp P and Wess J 2005 *Class. Quant. Grav.* **22** 3511; Aschieri P, Dimitrijevic M, Meyer F and Wess J 2006 *Class. Quant. Grav.* **23** 1883

[4] Dolan B, Gupta Kumar S and Stern A 2007 *Class. Quant. Grav.* **24** 1647 (*Preprint* arXiv:hep-th/0611233)

[5] Chaichian M, Demichev A, Presnajder P and Tureanu A 2001 *Eur. Phys. J.* C **20** 767; *do.* 2001 *Phys. Lett.* B **515** 426

[6] Balachandran A P, Govindarajan T R, Martins A G and Teotonio-Sobrinho P 2004 *JHEP* **0411** 068

[7] Banados M, Teitelboim C and Zanelli J 1992 *Phys. Rev. Lett.* **69** 1849; Banados M, Henneaux M, Teitelboim C and Zanelli J 1993 *Phys. Rev.* D **48** 1506

[8] Carlip S 1995 *Class. Quant. Grav.* **12** 2853

[9] Connes A and Landi G 2001 *Commun. Math. Phys.* **221** 141

[10] Oeckl R 2000 *Nucl. Phys.* B **581** 559 (*Preprint* arXiv:hep-th/0003018)

[11] Chaichian M, Nishijima K and Tureanu A 2003 *Phys. Lett.* **B568** 146 (*Preprint* arXiv:hep-th/0209008)

[12] Balachandran A P, Mangano G, Pinzul A and Vaidya S 2006 *Int. J. Mod. Phys.* **A21** 3111 (*Preprint* arXiv:hep-th/0508002)

[13] Balachandran A P, Govindarajan T R, Mangano G, Pinzul A, Qureshi B A and Vaidya S 2007 *Phys. Rev.* D **75** 045009 (*Preprint* arXiv:hep-th/0608179)