Thue's 1914 paper: a translation

James F. Power

Department of Computer Science, National University of Ireland, Maynooth, Co. Kildare, Ireland.

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Introduction

Axel Thue published four papers directly relating to the theory of words and languages:

- two on patterns in infinite strings in 1906 and 1912 [1,2]
- and two on the more general problem of transformations in 1910 and 1914 [3,4]

Both the 1906 and 1912 papers have been translated and discussed extensively by Jean Berstel [Berstel, 1995], and are known, among other contributions, for their presentation of the Thue-Morse sequence. Thue's 1910 paper deals with transformations between trees, and is thus a more direct predecessor of his 1914 paper. It been discussed by Steinby and Thomas [Steinby and Thomas, 2000].

These notes are intended to accompany a reading of Thue's 1914 paper, which has not hitherto been discussed in detail. Thue's paper is mainly famous for proving an early example of an undecidable problem, cited prominently by Post [Post, 1947]. However, Post's paper principally makes use of the definition of *Thue systems*, described on the first two pages of Thue's paper, and does not depend on the more specific results in the remainder of Thue's paper.

Thus, Thue's paper has been "passed by reference" into the history of computing, based mainly on a small section of that work. A closer study of the remaining parts of that paper highlight a number of important themes in the history of computing: the transition from algebra to formal language theory, the analysis of the "computational power" (in a pre-1936 sense) of rules, and the development of algorithms to generate rule-sets.

Structure: This document is in three sections

- We present a brief overview of Thue's paper and a motivation for studying it (pp. 2-3). This is an extended abstract of a talk to be presented at the *International Conference on the History and Philosophy of Computing* (HaPoC 2013), 28-31 October, 2013, Ecole Normale Superieure, Paris.
- We provide some notes on the contents of the paper (pp. 4-10), which are intended to be read in conjunction with the paper (or its translation).
- The last section is a translation of Thue's paper, numbered as in Thue's *Selected Papers* [Nagell et al., 1977], pages 493-524.

[1] Axel Thue. Über unendliche Zeichenreihen. *Christiana Videnskabs-Selskabs Skrifter, I. Math.-naturv. Klasse*, 7, 1906

[2] Axel Thue. Über die gegenseitige Lage gleicher Teile gewisser Zeichenreihen. *Christiana Videnskabs-Selskabs Skrifter, I. Math.-naturv. Klasse*, 1, 1912

[3] Axel Thue. Die Lösung eines Spezialfalles eines generellen logischen Problems. *Christiana Videnskabs-Selskabs Skrifter, I. Math.-naturv. Klasse*, 8, 1910

[4] Axel Thue. Probleme über Veränderungen von Zeichenreihen nach gegebenen Regeln. *Christiana Videnskabs-Selskabs Skrifter, I. Math.-naturv. Klasse*, 10, 1914

An overview of Thue's paper

Rarely has any paper in the history of computing been given such a prestigious introduction as that given to Axel Thue's paper by Emil Post in 1947 [Post, 1947]:

> "Alonzo Church suggested to the writer that a certain problem of Thue [Thue, 1914] might be proved unsolvable ..."

However, only the first two pages of Thue's paper are directly relevant to Post's proof, and, in this abstract, I hope to shed some light on the remaining part, and to advocate its relevance for the history of computing.

Thue Systems Thue's 1914 paper is the last of four he published that directly relate to the theory of words and languages [Berstel, 1995, Steinby and Thomas, 2000]. In this 1914 paper, Thue introduces a system consisting of pairs of corresponding strings over a fixed alphabet:

$$\begin{array}{lllll} A_1, & A_2, & A_3, & \ldots, & A_n \\ B_1, & B_2, & B_3, & \ldots, & B_n, \end{array}$$

and poses the problem: given two arbitrary strings P and Q, can we get one from the other by replacing some substring A_i or B_i by its corresponding string? Post called these systems of "Thue type" and proved this problem to be recursively unsolvable.

Reception of Thue's Work Thue's earlier work was not widely cited but often rediscovered independently [Hedlund, 1967], and something similar seems to have happened with the 1914 paper.

For example, Thue is not among the 547 authors in Church's 1936 *Bibliography of Symbolic Logic* [Church, 1936], nor is Thue cited in Post's major work on tag systems, correspondence systems, or normal systems before 1947 [Post, 1943, 1946]. His work appears to have had no direct influence on the development of formal grammars by Chomsky in the 1950s [Chomsky, 1959, Scholz and Pullum, 2007]. Most subsequent references to Thue's paper (where they exist) note it only for providing a definition of Thue systems.

Thue's awareness Thue explicitly understood the general metamathematical context (that we now associate with *Hilbert's programme*), describing the problem as being of relevance to one of the "most fundamental problems that can be posed".

Further, he phrases the problem in terms that have become quite familiar in the post-1936 world:

> "... to find a method, where one can always calculate in a predictable number of operations, ..."

This language parallels that used in Hilbert's 10th problem in 1900 [Hilbert, 1902], and places Thue's work firmly in what we would now regard as computing, rather than pure algebra.

Foundations of Language Theory Having posed the general problem in §II of his paper, Thue then presents an early example of a proof of (what we would now call) *termination* and *local confluence* for a system where the rules are non-overlapping and non-increasing in size.

When reducing some string P, we must find some occurrence of A_i and replace it with B_i. A difficulty arises if there is an overlap: some substring CUD in P, such that A_i matches both CU and UD, and thus choosing one option will eliminate our ability to later choose the other.

In §IV, Thue presents the string U as a *common divisor* of CU and UD and then shows how we can apply **Euclid's algorithm** to derive a Thue system from this. Euclid's algorithm had been considerably generalised throughout the 19th century, but here the string U "measures" the strings CU and UD just as Euclid's lines measure each other (*Elements*, Book 10, proposition 3).

Thue derives another *algorithm* in §V which, given two strings P and Q will derive those strings equivalent to them, and gradually reduce them to a core set of irreducible strings, providing a solution to the word problem in a restricted case. He investigates variants of these presentations *based on their syntactic properties* in §VI and gives some examples in §VII.

We remark that from the identity $CU \equiv UD$ we can derive rules of the form $CU \rightarrow UD$, and that this template is precisely what Post termed *normal form* for his rewriting systems.

Thue's "completion" algorithm In §VIII of his paper Thue develops an algorithm to derive a system of equations from any given sequence R. This is interesting not just for its structure (the algorithm iterates until it reaches a fixed point) but also for its use of overlapping sequences as a generation mechanism.

Starting from some given identity sequence R we can identify all pairs where $R \equiv CU \equiv UD$, and then add the rules $C \leftrightarrow D$ to the Thue system. We can then apply these rules using R as a starting symbol to derive a further set of identity sequences $R_1, R_2, \ldots$. These, in turn, can be factored based on overlaps to provide a further set of rules $C_i \leftrightarrow D_i$ and so on. Since all R_i have the same length, as do all C_i and D_i, this process is guaranteed to terminate.

This is similar to, but *not* the Knuth-Bendix algorithm: there is no explicit concept of well-ordering, for example. However, it certainly contains many of the "basic features" of the algorithm as described by Buchberger [Buchberger, 1987], and could be considered, under restrictive conditions, as an embryonic version of it.

Notes to accompany the translation

Terminology: In the translation I have translated *Zeichenreihen* literally as "symbol sequence", even though Post had already interpreted it simply as "string". I do this mainly to retain fidelity with Thue's paper, where he on other occasions uses "sequence", "sub-sequence", and "null sequence" without using the word *Zeichen*.

- Section I (pg 493)

Here Thue briefly introduces the paper, noting that it follows from his earlier work on trees [Thue, 1910] and on sequences that *don't* contain overlapping sub-sequences [Thue, 1912]. These are the only two references in the paper even though e.g. the prior work by Dehn was clearly relevant [Dehn, 1911]. This seems to be a habit of Thue's: his 1912 paper only references [Thue, 1906], and the 1910 and 1906 papers appear to have no references at all.

With hindsight we can see Thue's work as fitting into the general format of Hilbert's programme and the work of Emil Post (see e.g. [DeMol, 2013]), and it is interesting to note that Thue explicitly understood his work as being of relevance to one of the "most fundamental problems". Yet Thue also limits *his* programme quite clearly: he will deal only with special cases of this problem.

- Section II (pg 493)

In this section Thue presents what Post termed a *Thue system* as a series of tuples of the form (A_k, B_k). Reading such a tuple as a rule $A_k \leftrightarrow B_k$ allowing the replacement of a sub-string A_k with the string B_k or vice versa, Thue defines the concept of *similar* sequences, and then *equivalent* sequences as the closure of this.

Thue does not explicitly allow for empty sequences in the A_i or B_i (or anywhere else in the paper) and, in the absence of an identity element, his *Problem (I)* is thus the word problem for semi-groups.

The two special cases he deals with are:

(a) Each A_k and B_k have the same length: thus, applying such a rule cannot change the length of the string, and there are only finitely many possibilities for permuting the symbols in these fixed-length strings.

(b) Each A_k is longer than its corresponding B_k: so, applying a rule forwards will basically shrink the string, which can only be done a finite number of times. Thue seems to slip into semi-Thue mode here, where he interprets the rules one-directionally as $A_k \rightarrow B_k$. He also explicitly refers to this as a *reduction*, and defines the term *irreducible* (also defined in his 1910 paper).

Either of these restrictions give us a system that is *terminating*. Thue proves by induction that if we also disallow overlaps among the A_k then the system must also be *confluent* (though he does not call it this), and thus the word problem is decidable in this case.

Notation: Thue distinguishes between:

$P \sim Q$	P can be transformed in 1 step into Q
$P = Q$	P can be transformed in 1 or more steps into Q
$P \equiv Q$	P is symbol-by-symbol identical to Q

- Section III (pg 497)

In this section Thue shifts the focus to systems based around some given *null sequence* R which can be deleted from, or inserted into, other strings. This allows him to redefine the terms *similar* and *equivalent* "in respect of R" in this new context.

In language theory terms, the null sequence here is not the empty sequence ϵ, but rather a *nullable* sequence. That is, for the null sequence R, we implicitly have the rule $R \leftrightarrow \epsilon$.

While it is not exactly explicit here, the introduction of a null sequence brings us from semi-groups to monoids, and Thue's *Problem (II)* is the word problem in this case.

Overlaps The final few remarks of this section are of the utmost importance for the rest of the paper. Thue has already dealt (in §II) with the case where rules can't overlap, and he now addresses the case where they *can* overlap. In the context of §III, this means that we have some string in which R occurs twice as a sub-string, but in overlapping configurations. Calling this overlap U we get:

$$\begin{array}{ll} R \equiv CU & \\ & UD \equiv R \end{array}$$

where CUD is a sub-string of the current string. Thue derives the equivalence $C = D$ in respect of R here, and will make considerable use of this later.

- Section IV (pg 498)

Having established the importance of identities of the type $CU \equiv UD$, Thue investigates them further in this section.

First, Thue deals with a special case regarding power series (pp. 498-499). If we have $CU \equiv UD$ then C and D must have the same number of symbols. If, in addition, they both have the same number of symbols as U then we must actually have $CD \equiv DC$. Thue re-generalises this case slightly to consider situations where $XY \equiv YX$ and X and Y are of different lengths. In this case Thue proves that both strings must be composed of some common factor θ, with p copies in X, q copies in Y and thus $p + q$ copies in XY.

After dealing with a corollary (AA containing A), Thue now returns to the general case and sets up a kind of Euclidean algorithm for factoring overlapping strings. Starting with some string U_0, we factor this into a quotient C_1 and remainder U_1, and then follow the same process with the remainder.

This process can be seen in action in the diagram on page 500. Since we know $S \equiv CU$ then S must start with at least one C. But since $CU \equiv UD$, if U has more symbols than C then it must also start with a C. Hence $S \equiv CU$ must start with *two* C's. Repeating this process until what remains of U becomes shorter than C, we get $S \equiv C^n\alpha$: that is, C divides n times into S with remainder α. Following a similar process with the D's from the other end, we get the insight that C, D and U must be formed from regular patterns of α and β.

An important point here is that we can factor U as

$$U \equiv (\alpha\beta)^{n-1}\alpha \equiv \alpha(\beta\alpha)^{n-1}$$

But this has the same format as the identity we began with; i.e. it has the form $U \equiv C_1U_1 \equiv U_1D_1$, and we can presumably apply the same process, using U this time instead of S.

Note that U here is maximal and thus unique. If we hadn't demanded that U be maximal, we could possibly have stopped dividing by C at some earlier stage $m < n$ and then get a larger overlap

$$U \equiv (\alpha\beta)^{n-m}\alpha \equiv \alpha(\beta\alpha)^{n-m}$$

where $C \equiv (\alpha\beta)^m$ and $D \equiv (\beta\alpha)^m$.

In the final result of this section (pg 502) Thue shows that this process can work 'backwards'. Just as we can start with some M and derive N as the largest overlap with remainder X and Y, we can in a similar manner derive a string T for which M is the largest overlap with remainder X and Y.

- Section V (pg 503)

In this section Thue considers the relationship between a presentation in terms of some null sequence R (as in §III), and one in terms of a set of equations (as in §II). In particular he wants to know under what circumstances a set of equations (such as (1)) can *adequately* represent the null sequence. In algebraic terms, one might regard this as asking the question: when when can a set of equations in the presentation of a semi-group adequately model the presentation of a monoid (which could include equations of the form $R = 1$).

Two sequences that are provably equivalent according to the semi-group equations are called "parallel" and Thue uses the notation $P \doteqdot Q$. This notation is only used in this section and in examples 1 and 5 of §VII. In general, if R is the identity in a monoid, then we must have for any other element z that $zR = z = Rz$. For semi-group equations like (1) to model this, we must have the power to prove all equations of this kind; Thue splits this into two parts: the equations are

- *complete* (vollständiges) if we can prove $zR \doteqdot Rz$
- *perfect* (vollkommenes) if we can prove $RA \doteqdot RB$ implies $A \doteqdot B$

Thus we have an algorithm for dealing with sequences containing the null sequence. The theorem on page 504 sets this up, and the algorithm is presented on pg 505. Given some sequence P, a complete system allows us to "move around" any occurrence of the null sequence R in P, thus deriving all other sequences of the same length that differ only in the position of R. A perfect system then allows us to delete null sequences on the left, forming a new collection of (shorter) parallel sequences. Repeating this process we eventually get to a set of equivalent irreducible sequences. Given some other sequence Q, it is equivalent to P if it reduces down to the same set of irreducible sequences.

- Section VI (pg 506)

In this section Thue imposes some fairly severe restrictions on the format of the equations and shows that this helps determining the matching between sequences. He reintroduces the terminology from §III (no null sequences in §VI), and then in the main theorem on page 506 he restricts the format of allowable equations based on the first symbol on each side. Thus given some sequence P, if I apply a series of the equations to P, then I can guarantee that (at worst) each two applications will fix a leftmost symbol in P for the rest of the derivation.

For example, if I apply a rule of the form $A_i \rightarrow B_i$, then, if the leftmost symbol of P changes, it can only change from x_i to y_i. But since no other A_j or B_j starts with y_i, the leftmost symbol is effectively fixed for the rest of the derivation. (I could choose $B_i \rightarrow A_i$, but this just makes the first step redundant).

Alternatively, if I could apply a rule of the form $B_i \rightarrow A_i$ and change the leftmost symbol of P, it must change from y_i to x_i. But now if I wish to change the leftmost symbol again I must this time pick some rule of the form $A_i \rightarrow B_i$, and the argument from the previous paragraph then holds.

Thue presents this from a different perspective: if the leftmost part of two equivalent strings are equivalent, then so are the remaining rightmost parts. This is proved methodically on pages 506-509.

- Section VII (pg 510)

In this section Thue gives 5 examples of systems of equations that are complete and perfect.

The first example is a set of equations derived from a factoring of the null sequence R using the method of §IV. Two points worth noting here:

- just below the identities in (6) we are told that all the Y_i and X_r are different and
- just below the equivalences in (7) it is noted that X_1 begins with X_r (and, thus, so do all the left-hand-sides of the equations)

Taken together, this means that the equivalences in (7) have the format required for the main theorem in §VI. In particular, applying this theorem with R in place of C and D lets us conclude that whenever $RM = RN$ then $M = N$, meaning that (7) forms a perfect set of equivalences.

This approach is used (implicitly) to prove that the equation systems are perfect in examples 1, 2, 3 and 4. Note in examples 2, 3 and 4 that R is chosen so that it can be broken down into the usual "overlap" pattern

$$\begin{aligned} R &\equiv CU \\ & UD \equiv R \end{aligned}$$

which then yields equivalences of the form $C = D$.

Example 5 is a little different, since the equations are not constructed to be obviously perfect following the template of the others. Actually *proving* that the system is perfect requires considerable effort, stretching from page 514-516.

- Section VIII (pg 516)

Thue's "few remarks" here amount to an algorithm for constructing a system of equivalences starting from a given null sequence R.

Given some null sequence R, Thue shows how to generate an initial set of equations based on any overlaps (as in example 2 of the previous section). That is, we form the equation $C = D$ for each possible overlap U that satisfies $R \equiv CU \equiv UD$. This implies that C and D must have the same length, and they must have fewer symbols than R. Having derived a set of such equations, we can then apply them to R to derive another set of null sequences equivalent to R: these all have the same length as the original R. We can continue in this way, alternating between deriving new set of null sequences S_θ and new sets of equations E_θ. Since the length of all the Ps, Qs and Rs are bounded, the process must terminate, as noted on page 519.

In the following discussion (pp. 519-521) he shows how to use such a system δ and to 'minimise' these equations to a derived system ϵ. He then proves a series of theorems demonstrating the 'minimality' of this system.

In the final theorem starting on page 522 it may not be obvious that the eight cases listed exhaust all the possibilities for the configuration of the overlap between $M \equiv aR_zb$ and $N \equiv cR_\mu d$. The key here is to work out the overlap between M and N and to note that it (mostly) shrinks as we move through the cases.

Since both M and N are divided into three sub-strings, we can characterise the overlap by categorising the degree to which each sub-string is involved in it. For example, the maximal amount of overlap (assuming the strings aren't identical) would be given by the following configuration

$$\begin{array}{ccccccc}
\multicolumn{2}{c}{\overbrace{\qquad\quad}^{a}} & \multicolumn{2}{c}{\overbrace{\qquad\quad}^{R_z}} & \multicolumn{2}{c}{\overbrace{\qquad\quad}^{b}} & \\
C & e & f & g & h & i & D \\
 & \multicolumn{2}{c}{\underbrace{\qquad\quad}_{c}} & \multicolumn{2}{c}{\underbrace{\qquad\quad}_{R_\mu}} & \multicolumn{2}{c}{\underbrace{\qquad\quad}_{d}}
\end{array}$$

In this configuration, the overlap $U \equiv efghi$ involves *part* of a and d and *all* of the other four sub-strings. We can get the remaining configurations by sliding M to the left (or, equivalently, N to the right). Characterising the involvement of the six sub-strings as P, A or N for *part*, *all* or *none* respectively, we can actually track nine cases as we decrease the overlap. I found it useful to enumerate the first eight of these as follows:

	$U \equiv$	$M \equiv aR_zb$	$N \equiv cR_\mu d$
1	$efghi$	PAA	AAP
2	$efgb$	PAA	APN
3	$cefgb$	PAA	AAP
4	cfb	NPA	APN
5	fgh	NPA	APN
6	fb	NPA	PNN
7	cf	NNP	APN
8	f	NNP	PNN

The ninth case, which we could characterise as NNP versus AAP is actually impossible, since it would result in R_μ being a sub-string of R_z, and all the strings R are supposed to have the same length.

The eight cases Thue deals with are illustrated in Figure 1.

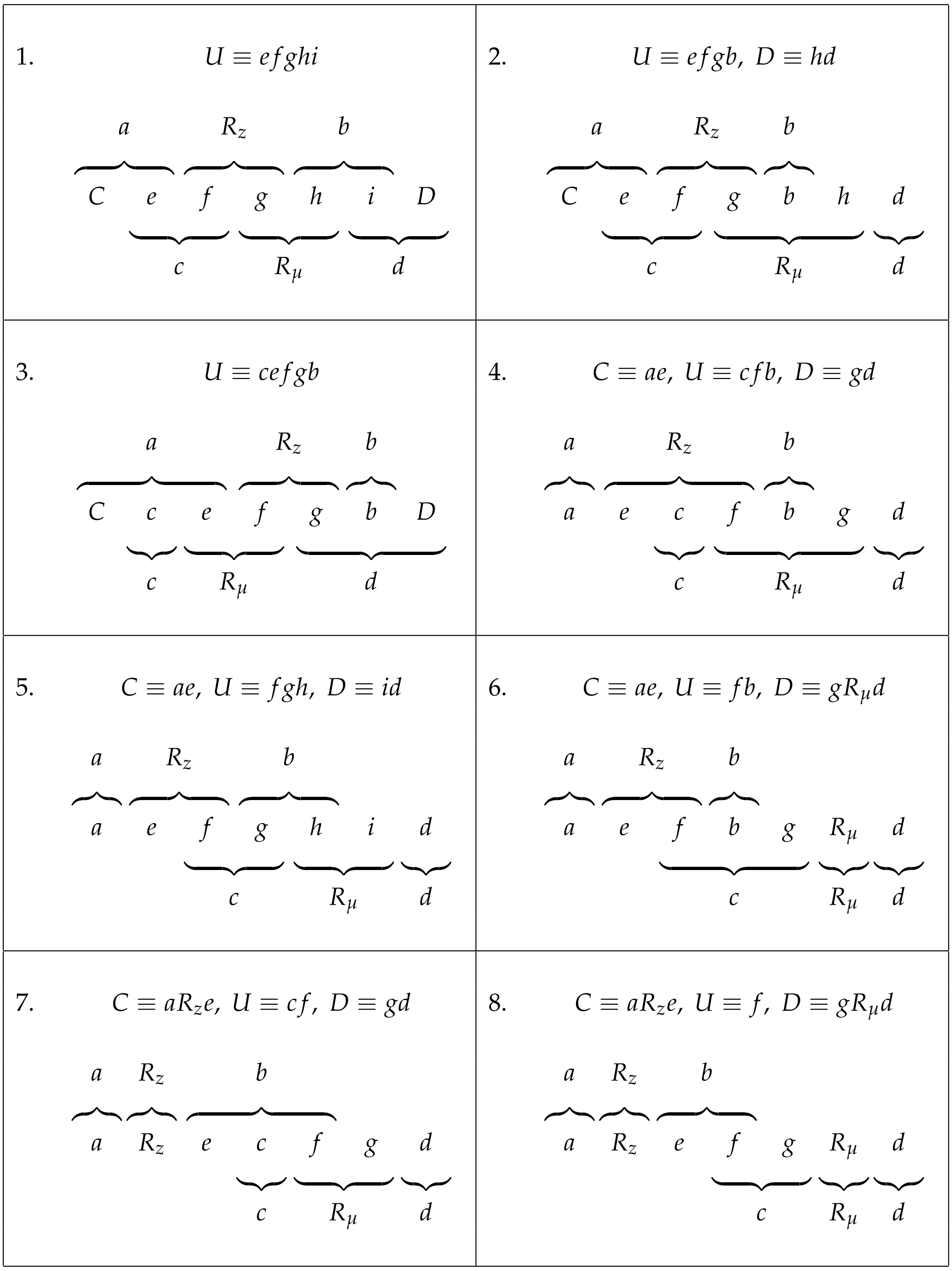

Figure 1: The eight cases of the final theorem in §VIII (pages 522-524)

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Axel Thue. Probleme über Veränderungen von Zeichenreihen nach gegebenen Regeln. *Christiana Videnskabs-Selskabs Skrifter, I. Math.-naturv. Klasse*, 10, 1914.

Problems concerning the transformation of symbol sequences according to given rules

Axel Thue

1914

This translation by James F. Power
<jpower@cs.nuim.ie>
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This document is a translation of Axel Thue's paper *Probleme über Veränderungen von Zeichenreihen nach gegebenen Reglen* (Kra. Videnskabs-Selskabets Skrifter. I. Mat. Nat.Kl. 1914. No. 10)

Based on the version published in his *Selected Mathematical Papers* as paper #28, pp. 493-524. Page numbers in this document follow that pagination.

Any margin notes (like this) are not part of the original paper, but typos noted here are typos in the original paper.

§ I

In a previous work[1] I have posed the general question whether two given concepts depicted as trees, but defined in different ways, must be equivalent to each other.

In this paper I will deal with a problem concerning the transformation of symbol sequences using rules. This problem, that in certain respects is a special case of one of the most fundamental problems that can be posed, is also of immediate significance for the general case. Since this task seems to be extensive and of the utmost difficulty, I must be satisfied with only treating the question in a piecewise and fragmentary manner.

In a previous year's work[2] I have already solved a special case concerning symbol sequences. On this occasion I will just settle some simple cases of the aforementioned general problem. I will not enter into a discussion here on the wider significance of investigations of this type.

§ II

We are given two series of symbol sequences:

$$\begin{matrix} A_1, & A_2, & A_3, & \ldots, & A_n \\ B_1, & B_2, & B_3, & \ldots, & B_n, \end{matrix}$$

where each symbol in each sequence A and in each sequence B is a symbol from some group of given symbols.

1 Die Lösung eines Spezialfalles eines generellen logischen Problems. (Christiana Videnskabsselskabs Skrifter, 1910.)

Selected Mathematical Papers pp. 273-310

2 Über die gegenseitige Lage gleicher Teile gewisser Zeichenreihen. (Christiana Videnskabsselskabs Skrifter, 1912.)

Selected Mathematical Papers pp. 413-478

For each value of k we will call A_k and B_k *corresponding sequences*.

If, given two arbitrary sequences P and Q, one can get one from the other by replacing some subsequence A or B by its corresponding sequence, then we say that P and Q are called *similar sequences* with reference to the corresponding sequences A and B. We indicate this by writing

$$P \sim Q$$

The sequences $\alpha A_h \beta$ and $\alpha B_h \beta$, where α and β denote symbol sequences, are thus (for example) similar sequences.

If any two two symbol sequences X and Y are procured in this way, then one can find a series of symbol sequences

$$C_1, C_2, \ldots, C_r$$

such that X and C_1, then C_r and Y, and finally C_h and C_{h+1} for each h are equivalent sequences, so then we have thus:

$$X \sim C_1 \sim C_2 \sim \ldots \sim C_r \sim Y,$$

then we say that X and Y are *equivalent sequences* in respect of the given sequences A and B.

We denote this by means of the equation

$$X = Y$$

When $P \sim Q$ we also have $P = Q$. Further, we have $A_k \sim B_k$ and $A_k = B_k$.

We can now pose the major general question (I):

Problem (I) *For any arbitrary given sequences A and B, to find a method, where one can always calculate in a predictable number of operations, whether or not two arbitrary given symbol sequences are equivalent in respect of the sequences A and B.*

This problem is easily solved in the following two cases (a) and (b).

(a) A_k and B_k contain equal number of symbols for each value of k.

Here, either two sequences X and Y are equivalent, or one can find sequences $C_1, C_2, \ldots, C_r$ such that:

$$C_0 \sim C_1 \sim C_2 \sim \ldots \sim C_r \sim C_{r+1},$$

where C_0 and C_{r+1} denote X and Y respectively, then any two of the sequences of C have equal number of symbols, but can be assumed to be different from each other.

r must consequently fall under a predictable limit, and the problem is thus solved.

(b) A_k contains more symbols than B_k for each value of k. The sequence A is in addition so constituted that any two arbitrary subsequences A_p and A_q must always lie completely outside each other for any values of p and q.

This seems to assume we're operating semi-Thue system $A_k \to B_k$

We assume, in other words, that no sequence in A can be a subsequence of another sequence in A, while further two arbitrary possible subsequences A_p and A_q are not allowed to have any common part.

We use the term *irreducible sequence* to refer to any sequence which contains no subsequence A.

Case (b) of the aforementioned problem can now be solved as follows:

Through repeated reductions of an arbitrary given symbol sequence S we can only get a single irreducible sequence. Here, in each reduction a subsequence A of the given sequence (or any sequence obtained from it via an previous reduction) is replaced by its corresponding sequence B.

That is, the system is confluent.

This statement must be correct in the case where S contains only a single symbol.

The proof proceeds by induction over the number of symbols in S.

It is also immediately apparent that if the statement is correct when the number of symbols in S is less than some number t, then it must also be correct when the number of symbols in S is equal to t.

In particular if S is irreducible, the case is immediately clear. It also follows when S only contains a single subsequence A, i.e.

$$S \equiv MA_kN,$$

where we use the symbol $\equiv$ to denote the identity. S can then only be reduced to the same irreducible sequence as MB_kN.

Since we know that A_k has more symbols than B_k, the inductive hypothesis applies to prove this case.

Finally we suppose that S is gradually reduced, in two different ways, to the two irreducible sequences P and Q respectively.

In the first reduction S is reduced by a single reduction to H, and then by a sequence of reductions to P. In the second reduction S is reduced by a single reduction to K, and then by a sequence of reductions to Q.

The case is immediately clear when the two first reductions are the same as each other, i.e. $H \equiv K$. If this is not the case, we can write:

$$\begin{aligned} S &\equiv MA_pLA_qN \\ H &\equiv MB_pLA_qN \\ K &\equiv MA_pLB_qN \end{aligned}$$

where one or more of the sequences M, L and N are allowed to be absent.

However H and K are so constituted that they contain fewer symbols than t, and can then be reduced to a single irreducible sequence MB_pLB_qN. That is,

$$P \equiv Q$$

Since any two similar rows, and even two equivalent rows, can be reduced in this way only to a single irreducible sequence, this proves the result.

Instead of problem (I), one can set up the still more general question:

Suppose P and Q signify two arbitrary symbol sequences, and that each symbol that occurs in them is different from those in the series A and B. Then, find a general method by which it is possible to decide whether any of the symbols of P and Q can be replaced by such symbol sequences, so that the symbol sequences P' and Q' obtained from P and Q in this way are equivalent.

We assume that symbols that are equal to one another are only replaced by sequences that are equal to one another.

The next few paragraphs are in a reduced font in Thue's paper, thus presumably a kind of sidenote.

We can also generalise problem (I) in another way.

Given two arbitrary symbol sequences P and Q, one can get one of them from the other by replacing a subsequence A' with another sequence B', where A' and B' have been obtained in such a way that one can write sequences in place of the symbols of two corresponding sequences A and B, such that A and B in this way turn to A' and B' respectively, then we can - with a new meaning of the words - define P and Q to be similar.

In a corresponding manner we can define equivalent sequences and pose the question: how can one always decide whether two sequences are equivalent, or whether in place of the sequences one can write two such sequences that in this way the sequences become equivalent.

We wish now to deal with an important special case of problem (I).

We wish to give a new definition of the concepts *similarity* and *equivalence*.

§ III

Let R signify an arbitrary given symbol sequence. Two arbitrary sequences are said to be *sequences similar to each other with respect to* R when we can get one from the other by removing a subsequence R.

The sequences

$$MRN \text{ and } MN,$$

where one of the arbitrary sequences M and N can of course be missing, are thus examples of similar sequences.

When P and Q are similar sequences, we can indicate this by writing

$$P \sim Q.$$

If we are given two arbitrary sequences P and Q such that sequences $C_1, C_2, \ldots, C_r$ exist, where X and C_1, also C_r and Y, and finally C_h and C_{h+1} for each value of k, are similar sequences, so that

$$X \sim C_1 \sim C_2 \sim \cdots \sim C_r \sim Y$$

then we will call X and Y *equivalent sequences in respect of* R. We indicate this by writing

$$X = Y$$

Equivalent sequences can always be transformed into one another by removal and insertion of sequence R.

If we have $P \sim Q$ then we also have $P = Q$.

If we have $A = C$ and $B = C$ then we also have $A = B$.

If X and Y are two arbitrary equivalent sequences, then one can find such sequences H and K that

$$H_0 \sim H_1 \sim H_2 \sim \cdots \sim H_p,$$
$$K_0 \sim K_1 \sim K_2 \sim \cdots \sim K_q,$$

where $H_0 \equiv X$, $K_0 \equiv Y$ and $H_p \equiv K_q$, and one can always get from H_{r-1} to H_r and K_{s-1} to K by removing a null sequence R.

"Nullreihe" is first used here, but then defined at the end of this section

If we have

$$X \sim C_1 \sim C_2 \sim \cdots \sim C_m \sim Y$$

where e.g.

$$\begin{aligned} C_{t-1} &\equiv xRyz \\ C_t &\equiv xyz \\ C_{t+1} &\equiv xyRz \end{aligned}$$

then we also have

$$\cdots \sim C_{t-1} \sim xRyRz \sim C_{t+1} \sim \cdots$$

etc.

One can now state the major task (II):

Problem (II) *Given an arbitrary sequence R, to find a method where one can always decide in a finite number of investigations whether or not two arbitrary given sequences are equivalent with respect to R.*

The ultimate goal of our discussion now lies in giving the solutions for some examples of this task.

We have shown earlier that the case is clear when two subsequences of R can never have common part.

The difficulty arises when the opposite case occurs. If two subsequences of R can have a common part U then we can write

This pattern for overlapping sequences, $CU \equiv UD$, is a recurring motif in the remainder of the paper

$$R \equiv CU \equiv UD$$

or

$$C \sim C(UD) \equiv (CU)D \sim D$$

or

$$C = D$$

The sequence R is called a *null sequence*.

§ IV

By T^n, where T denotes and arbitrary symbol sequence, we wish to signify the construction of a sequence

$$TT \cdots T$$

from n copies of the sequence T.

We say that T^n is called a *Power series*.

If X and Y are two sequences so constituted that

$$XY \equiv YX$$

then there exists a sequence θ such that

$$X \equiv \theta^p, \qquad Y \equiv \theta^q$$

The case where X and Y contain equally many symbols is clearly true. For then

$$X \equiv Y$$

or

$$\theta \equiv X \equiv Y$$
$$p = q = 1$$

The case where XY contains only two symbols is thus clearly true.

However, if the case is true when XY has fewer than m symbols, then it must also be true when XY has exactly m symbols.

Suppose here that, for example, X is composed of more symbols than Y; then one has:

$$X \equiv YZ$$

or

$$(YZ)Y \equiv XY \equiv YX \equiv Y(YZ)$$

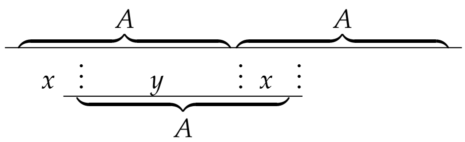
or

$$YZ \equiv ZY$$

or

$$Z \equiv \theta^\gamma, \quad Y \equiv \theta^\delta, \quad X \equiv \theta^{\gamma+\delta}$$

If a sequence AA is composed of an inner subsequence A, we can then write:



$$A \equiv xy \equiv yx$$

or consequently

$$x \equiv \theta^p, \quad y \equiv \theta^q, \quad A \equiv \theta^{p+q}.$$

Let U_0 now denote an arbitrary given symbol sequence. We can then define such sequences U, C and D such that:

This gives us a kind of Euclidean (GCD) algorithm for sequences, with each U_i as the quotient and C_i as the remainder.

$$\begin{array}{lcl} U_0 & \equiv & C_1U_1 \equiv U_1D_1 \\ U_1 & \equiv & C_2U_2 \equiv U_2D_2 \\ \multicolumn{3}{c}{\cdots\cdots\cdots\cdots\cdots\cdots} \\ U_{r-1} & \equiv & C_rU_r \equiv U_rD_r \end{array}$$

where U_p for any value of p is the largest sequence for which one can find such sequences C_p and D_p that

$$U_{p-1} \equiv C_pU_p \equiv U_pD_p,$$

where two subsequences U_r in the sequence can never have a common part.

If one has

$$U_0 \equiv CU \equiv UD,$$

then U must be equal to one of the sequences $U_1, U_2, \ldots, U_r$, as can be immediately seen.

If S denotes an arbitrary given symbol sequence and U the largest sequence for which one can find two sequences C and D such that

$$S \equiv CU \equiv UD$$

or

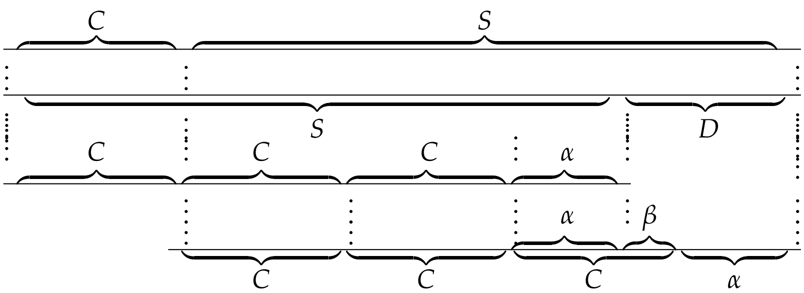
$$CS \equiv CUD \equiv SD$$

then first

$$S \equiv CC \cdots C\alpha \equiv C^n\alpha,$$

where the sequence α is either wholly missing, or is composed of fewer symbols than C.

Consequently there exists a sequence β such that



$$C \equiv \alpha\beta$$

$$D \equiv \beta\alpha$$

$$S \equiv C^n\alpha \equiv (\alpha\beta)^n\alpha \equiv \alpha\beta\alpha\beta\alpha \cdots \alpha\beta\alpha \equiv \alpha(\beta\alpha)^n \equiv \alpha D^n$$

$$U \equiv (\alpha\beta)^{n-1}\alpha \equiv \alpha(\beta\alpha)^{n-1}$$

C is the smallest sequence for which one can find a sequence D such that

$$CS \equiv SD.$$

If α contains at least one symbol, then we never have that

$$\alpha\beta \equiv \beta\alpha.$$

Otherwise we would get:

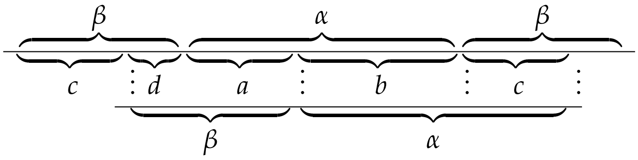
$$\beta S \equiv \beta\alpha(\beta\alpha)^n \equiv (\beta\alpha)^{n+1} \equiv (\alpha\beta)^{n+1} \equiv (\alpha\beta)^n\alpha\beta \equiv S\beta$$

where β is composed of fewer symbols than C.

Each of the sequences

$$\beta\alpha\beta \qquad \text{and} \qquad \alpha\beta\alpha$$

where α is composed of at least one symbol, contains only a single subsequence $\beta\alpha$ and a single subsequence $\alpha\beta$.



If we say that $\beta\alpha\beta$ contains an inner subsequence $\beta\alpha$, one can write

$$\alpha \equiv ab \equiv bc$$
$$\beta \equiv cd \equiv da$$

or

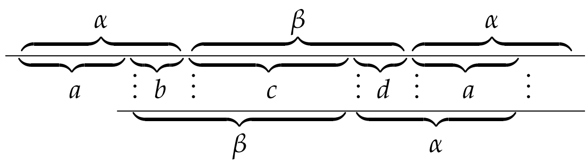
$$\begin{aligned} S &\equiv \alpha\beta\alpha\beta\alpha\cdots\alpha\beta\alpha \equiv (bc)(da)(bc)(da)(bc)\cdots(bc)(da)(bc) \\ &\equiv b(cd)(ab)(cd)(ab)\cdots(ab)(cd)(ab)c \equiv b\beta[\alpha\beta\alpha\cdots\alpha\beta\alpha c] \\ &\equiv [\alpha\beta\alpha\beta\alpha\cdots\alpha c]d\,ab \equiv [\alpha\beta\alpha\beta\alpha\cdots\alpha c]\beta b \end{aligned}$$

or

$$S \equiv b\beta W \equiv W\beta b.$$

However, $b\beta$ here would clearly have to be composed of fewer symbols than $\alpha\beta$, which is impossible. In this way it is also proven that $\beta\alpha\beta$ is not composed of an inner subsequence $\alpha\beta$.

Further, if $\alpha\beta\alpha$ is composed of an inner subsequence $\beta\alpha$, then we have:



$$\alpha \equiv ab \equiv da$$
$$\beta \equiv bc \equiv cd$$

or

$$\begin{aligned} S &\equiv (da)(bc)(da)(bc)(da)\cdots(da)(bc)(da) \\ &\equiv d(ab)(cd)(ab)(cd)\cdots(ab)(cd)a \\ &\equiv d[\alpha\beta\alpha\beta\alpha\cdots\alpha\beta\, a] \\ &\equiv [\alpha\beta\alpha\beta\alpha\cdots\alpha\beta\, a]b \end{aligned}$$

or

$$S \equiv dW \equiv Wb.$$

That d is composed of fewer symbols than $\alpha\beta$ is however impossible. In this way it is also proven that $\alpha\beta\alpha$ is not composed of an inner subsequence $\beta\alpha$.

If one has

$$S \equiv PN \equiv NQ$$

where the number of symbols in P and Q are not less than the number in $\alpha\beta$ and $\beta\alpha$, then there is consequently a whole number m between 0 and n for which

So here we're deliberately selecting some N *shorter* than U

Thus $P \equiv (\alpha\beta)^{n-m}$ and $Q \equiv (\beta\alpha)^{n-m}$

$$N \equiv \alpha(\beta\alpha)^m \equiv (\alpha\beta)^m\alpha.$$

In order to find expressions for the sequence U that belongs to the sequence S, we now write:

$$\begin{array}{rcl} S & \equiv & \alpha_1(\beta_1\alpha_1)^{n_1} \\ \alpha_1\beta_1\alpha_1 & \equiv & \alpha_2(\beta_2\alpha_2)^{n_2} \\ \multicolumn{3}{c}{\dots\dots\dots\dots\dots\dots} \\ \alpha_{p-1}\beta_{p-1}\alpha_{p-1} & \equiv & \alpha_p(\beta_p\alpha_p)^{n_p} \end{array}$$

where α_{q-1} when $2 \leq q \leq p$, contains fewer symbols than $\beta_q\alpha_q$, while n_q for $1 \leq q \leq p-1$ is greater than 1.

Furthermore, let α_p for $n_p > 1$ be completely missing, while $\alpha_1(\beta_1\alpha_1)^{n_1-1}$ is the largest sequence that the two sequences of S, and $\alpha_q(\beta_q\alpha_q)^{n_q-1}$ is the largest sequence that the two sequences of $\alpha_{q-1}\beta_{q-1}\alpha_{q-1}$ can have in common. Depending on whether n_p is greater than or equal to 1, we can now treat α_p or β_p as S respectively, etc.

When the process terminates, either $n_p = 1$ and the last factoring is the trivial $\alpha_p\beta_p\alpha_p$, or $n_p > 1$ and the last factoring is to $\beta_p^{n_p}$.

That is, $\alpha_{q-1}\beta_{q-1}\alpha_{q-1}$ is the 'overlap' U_{q-1} on each line

If M denotes an arbitrary given sequence, and N denotes the largest sequence for which one can find sequences X and Y such that

$$M \equiv XN \equiv NY,$$

then

$$T \equiv XNY \equiv XM \equiv MY$$

is the shortest sequence for this largest sequence M where one can find sequences P and Q such that

$$T \equiv PM \equiv MQ.$$

One gets here that

$$P \equiv X, \quad Q \equiv Y.$$

§ V

Let it be the case that in respect of some null sequence R:

$$\left.\begin{array}{lcl} A_1 & = & B_1 \\ A_2 & = & B_2 \\ \multicolumn{3}{c}{\dotfill} \\ A_k & = & B_k \end{array}\right\} \qquad (1)$$

where two equivalent sequences A_h and B_h for each value of h are composed of the same number of symbols. One sees immediately that A_h and B_h are composed of equally many of each kind of symbol.

Same number of xs, same number of ys etc. Thus A_h is just a permutation of B_h.

One can write in place of a possible sub-sequence A_h or B_h of some sequence S the other of these equivalent sequences, so that the sequence T constructed in this way is equivalent to S in respect of R. We say that T is constructed from S through a *homogeneous transformation* according to system (1).

Two sequences S and T, equivalent in respect of R, which are also equivalent in respect of system (1) are called *parallel* sequences in respect of R and (1). We indicate this by writing

Corollary: Parallel sequences always contain the same number of symbols

$$S \doteqdot T.$$

If two sequences S and T are parallel to one another in respect of system (1), there thus exist such sequences $C_0, C_1, C_2, \cdots, C_r, C_{r+1}$ where C_0 and C_{r+1} denote S and T respectively, so that one can get one of the consecutive sequences C_m and C_{m+1} from the other by exchanging a possible sub-sequence A_h with the corresponding sequence B_h.

When one can not derive any of the equivalences (1) from the others through homogeneous transformation we say that the equivalences (1) are independent of one another.

Given the sequences

$$zR \text{ and } Rz$$

where R denotes the null sequence, if for any symbol z one can always transform them into one another through homogeneous transformation by the system (1), so that

$$zR \doteqdot Rz$$

then we say that (1) forms a *complete*[5] system of equivalences.

[5] vollständiges

Each sub-sequence R of an arbitrary sequence S can thus through (1) be moved arbitrarily in the sequence S without changing the order of the remaining symbols of S.

In this case we have the following theorem.

If one can get a sequence α from a sequence A, and a sequence β from a sequence B by removing a sequence R, and meanwhile one can transform α and β into one another by successive homogeneous transformations according to a complete system of equivalences, then the sequences A and B have this same property.

We indeed get that e.g.

$$A \doteqdot R\alpha \doteqdot R\beta \doteqdot B.$$

If a system of equivalences derived from a null sequence R has the property that $A \doteqdot B$ whenever $RA \doteqdot RB$, then we say that the system is perfect[6] in respect of R.

[6] vollkommenes

A complete and perfect system of equivalences in respect of a null sequence R thus has the property that, in respect of the system it is always the case that

$$CR \doteqdot RC,$$

where C denotes an arbitrary sequence, meanwhile, when $RA \doteqdot RB$ we always have $A \doteqdot B$.

THEOREM. If one can get a sequence α from a sequence A, and a sequence β from a sequence B by removing a sequence R, and meanwhile in respect of a complete and perfect system of equivalences in respect of R

$$A \doteqdot B,$$

then we also have

$$\alpha \doteqdot \beta.$$

Then:

$$R\alpha \doteqdot A \doteqdot B \doteqdot R\beta$$

or

$$\alpha \doteqdot \beta.$$

If

$$R \equiv CU \equiv UD,$$

or

$$CR \equiv RD,$$

so then in respect of a complete and perfect system of equivalences in respect of a null sequence R we always have

$$C \neq D.$$

For

$$RC \neq CR \equiv RD.$$

If one has found a complete and perfect system of equivalences in respect of a null sequence R, then we can immediately see how in this way our problem (II) is easily solved.

Namely, if S denotes an arbitrary sequence, then one can set up a series of sequence systems

$$N_0, N_1, N_2, \cdots, N_r$$

that for each value of p all sequences of N_p are parallel, while S is equal to one of the sequences of N_0. Further the series N can be so chosen that no sequence of N_r contains a sub-sequence R, while it is possible to obtain for any value of $p > 0$ a sequence of N_{p+1} from a sequence of N_p through removal of a sub-sequence R.

Finally, the series N is so chosen that every sequence parallel to a sequence of the series N is contained in the series N.

Having removed then from an arbitrary sequence of a series N_p a possible sequence R, one can obtain in this way for any value of $p < r$ one of the sequences in the series N_{p+1}.

We say now that N_r forms an irreducible sequence system belonging to S.

Our problem (II) is now completed through the remark that similar sequences, and thus equivalent sequences, must have the same irreducible sequence system.

For a complete and perfect system of equivalences, a null sequence R must also be parallel to equivalent sequences with equally many symbols in respect of the aforementioned system.

We can, however, decide for certain whether or not two sequences are parallel in a calculable number of steps.

§ VI

Let there be given the two series of symbol sequences

$$A_1, A_2, \cdots, A_k$$
$$B_1, B_2, \cdots, B_k$$

where A_p and B_p for each value of p are - as before - corresponding sequences.

Two arbitrary sequences S and T are called equivalent in respect of the k pairs of corresponding sequences A_p and B_p when there exist such sequences $C_0, C_1, C_2, \cdots, C_r, C_{r+1}$, where C_0 and C_{r+1} denote S and T respectively, that one can obtain C_{q+1} from C_q for each value of q through the exchange of a subsequence A or B for its corresponding sequence.

We represent this, as before, through the equivalence

$$S = T$$

C_q and C_{q+1} are called, as before, equivalent sequences, and we write

$$C_q \sim C_{q+1}$$

.

We have here the equivalences:

$$\left.\begin{array}{l} A_1 = B_1 \\ A_2 = B_2 \\ \dots\dots\dots \\ A_k = B_k \end{array}\right\} \qquad (2)$$

Theorem. For arbitrary values of p and q, let A_p and B_p always start with different symbols on the left, and also for any two of the sequences in B, so we can write:

$$\begin{array}{l} x_1 P_1 \equiv A_1 = B_1 \equiv y_1 Q_1 \\ x_2 P_2 \equiv A_2 = B_2 \equiv y_2 Q_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ x_k P_k \equiv A_k = B_k \equiv y_k Q_k, \end{array}$$

where x and y are such symbols that each y is different from each of the other symbols y and x. If C, M, D and N are any such sequences that in respect of system (2):

$$CM = DN$$

where

$$C = D$$

then it is also the case that

$$M = N.$$

We only need to show here that if

$$zM = zN$$

for any symbol z, then M and N must always be equivalent. For convenience, we will prove the following more comprehensive theorem:

Let zM and zN, where z denotes a single symbol, be two sequences that are equivalent in respect of system (2), i.e. we are given such sequences $E_1, E_2, \cdots, E_p$ that

$$zM \sim E_1 \sim E_2 \sim \cdots \sim E_p \sim zN, \tag{3}$$

then one can find such sequences $F_1, F_2, \cdots, F_q$ that

$$M \sim F_1 \sim F_2 \sim \cdots \sim F_q \sim N, \tag{4}$$

where the number of F-sequences q is not greater than the number of E-sequences p.

This theorem is clearly true when

First, three base cases where the derivation is 0, 1 or 2 steps

$$zM \equiv zN \quad \text{i.e.} \quad M \equiv N.$$

Further, also when

$$zM \sim zN \quad \text{i.e.} \quad M \sim N.$$

Finally, the theorem must also be true when

If this happens we must have applied a rule forwards and then backwards, as two different rules must change the first letter.

$$zM \sim E_1 \sim zN,$$

because here $M \equiv N$.

We wish now to assume that the theorem is true when $1 \leq p < n$. We will then prove that the theorem is true when $p = n$. We can then write:

Now three inductive cases...

$$zM \sim z_1C_1 \sim z_2C_2 \sim \cdots \sim z_{n-1}C_{n-1} \sim z_nC_n \sim zN, \tag{5}$$

where each z denotes a single symbol. If z is different from each x and y one has:

Case 1: If z is different from each x and y then the equations in (2) won't allow you to change z

$$z \equiv z_1 \equiv z_2 \equiv \cdots \equiv z_n$$

or

$$M \sim C_1 \sim C_2 \sim \cdots \sim C_{n-1} \sim C_n \sim N.$$

If one of the symbols $z_1, z_2, \cdots, z_n$ e.g. z_r equals z then the theorem is also quite clear: Then we have

$$zM \sim z_1C_1 \sim z_2C_2 \sim \cdots \sim z_{r-1}C_{r-1} \sim z_rC_r$$
$$z_rC_r \sim z_{r+1}C_{r+1} \sim z_{r+2}C_{r+2} \sim \cdots \sim z_nC_n \sim zN,$$

then there exist such sequences α and β that

by the inductive hypothesis

$$M \sim \alpha_1 \sim \alpha_2 \sim \cdots \sim \alpha_s \sim C_r$$
$$C_r \sim \beta_1 \sim \beta_2 \sim \cdots \sim \beta_t \sim N,$$

where

$$s + t \leq n - 1.$$

We thus need only consider now the case where z denotes and x or a y, while each of the symbols $z_1, z_2, \cdots, z_n$ in (5) is different from z.

If z were an x, e.g.

Case 2: start with an x

$$z \equiv x_r$$

then we have in (5)

$$z_1 \equiv z_2 \equiv \cdots \equiv z_n \equiv y_r$$

z_1 in particular being different from z, i.e.

$$z_1 \equiv y_r.$$

If one has further that $z_k \equiv y_r$ while z_{k+1} is different from y_r then we have

$$z_{k+1} \equiv x_r \equiv z$$

which is clearly impossible.

We thus obtain here

$$x_rM \sim y_rC_1 \sim y_rC_2 \sim \cdots \sim y_rC_n \sim x_rN$$

or

$$x_rP_rM' \sim y_rQ_rM' \sim \cdots \sim y_rQ_rN' \sim x_rP_rN'$$

where

$$P_rM' \equiv M$$
$$P_rN' \equiv N.$$

However, since here

$$y_rQ_rM' \sim y_rC_2 \sim \cdots \sim y_rC_{n-1} \sim y_rQ_rN' \sim y_rQ_rN'$$

then one can find such sequences γ that

$$M' \sim \gamma_1 \sim \gamma_2 \sim \cdots \sim \gamma_\nu \sim N',$$

where

$$\nu \leq n - 2,$$

and thus

$$M \equiv P_r M' \sim P_r \gamma_1 \sim P_r \gamma_2 \sim \cdots \sim P_r \gamma_\nu \sim P_r N' \equiv N.$$

Finally, if z were equal to a y, e.g. y_r, then we can get from (5),

Case 3: start with a y; much the same as case 2.

$$y_r M \sim x_r C_1 \sim z_2 C_2 \sim \cdots \sim z_{n-1} C_{n-1} \sim x_r C_n \sim y_r N$$

or

$$y_r Q_r M' \sim x_r P_r M' \sim z_2 C_2 \sim \cdots \sim z_{n-1} C_{n-1} \sim x_r P_r N' \sim y_r Q_r N'$$

where

$$Q_r M' \equiv M, \quad Q_r N' \equiv N.$$

Since

$$x_r P_r M' \sim z_2 C_2 \sim \cdots \sim z_{n-1} C_{n-1} \sim x_r P_r N'$$

then there exists such sequences δ that

$$M' \sim \delta_1 \sim \delta_2 \sim \cdots \sim \delta_\mu \sim N',$$

where

$$\mu \leq n - 2,$$

and thus

$$M \equiv Q_r M' \sim Q_r \delta_1 \sim Q_r \delta_2 \sim \cdots \sim Q_r \delta_\mu \sim Q_r N' \equiv N.$$

In this way our theorem is proven.

Let T denote an arbitrary sequence such that for each value of a symbol z it is always the case that

$$zT \equiv Tz.$$

Further, let T' denote an arbitrary sequence equivalent to T. If then

$$T' \equiv abc \cdots gh,$$

where $a, b, c, \cdots, g, h$ are single symbols, then we would have

$$T' \equiv abc \cdots gh = bc \cdots gha.$$

For

$$a(abc \cdots gh) \equiv aT' = aT = Ta = T'a = (abc \cdots gh)a \equiv a(bc \cdots gha)$$

or

$$abc \cdots gh = bc \cdots gha.$$

Thus if T contains n symbols, then n arbitrary consecutive symbols of the sequence TT form a sequence equivalent to T.

We will now demonstrate some null sequences R for which one can find a perfect and complete system of equivalences.

§ VII

Example 1.

Let R be a null sequence defined using the following relations:

$$\left.\begin{aligned} R \equiv X_0 \equiv (X_1Y_1)^{n_1}X_1 &\equiv X_1(Y_1X_1)^{n_1} \\ X_1 \equiv (X_2Y_2)^{n_2}X_2 &\equiv X_2(Y_2X_2)^{n_2} \\ X_2 \equiv (X_3Y_3)^{n_3}X_3 &\equiv X_3(Y_3X_3)^{n_3} \\ \cdots\cdots\cdots\cdots\cdots&\cdots\cdots\cdots\cdots \\ X_{r-1} \equiv (X_rY_r)^{n_r}X_r &\equiv X_r(Y_rX_r)^{n_r} \end{aligned}\right\} \qquad (6)$$

where $Y_1, Y_2, \cdots, Y_r$ and X_r denote different individual symbols, while r and each n signify an arbitrary positive whole number.

These differences are important: it means (7) will then fit the format required for the theorem in §VI

Here we thus have

$$\begin{aligned} R \quad &\equiv (X_1Y_1)^{n_1}X_1 \equiv (X_1Y_1)^{n_1}(X_2Y_2)^{n_2}X_2 \equiv \cdots \\ &\cdots \equiv (X_1Y_1)^{n_1}(X_2Y_2)^{n_2}\cdots(Y_pX_p)^{n_p}X_p \equiv \cdots \end{aligned}$$

where

$$1 \leq p \leq r.$$

Further we also have:

$$\begin{aligned} R \quad &\equiv X_1(Y_1X_1)^{n_1} \equiv X_2(Y_2X_2)^{n_2}(Y_1X_1)^{n_1} \equiv \cdots \\ &\cdots \equiv X_p(X_pY_p)^{n_p}\cdots(Y_2X_2)^{n_2}(Y_1X_1)^{n_1} \end{aligned}$$

where

$$1 \leq p \leq r.$$

We get now e.g.

$$\begin{aligned} R \equiv X_1Y_1 \vdots (X_1Y_1)^{n_1-1}X_1 \vdots \\ \vdots (X_1Y_1)^{n_1-1}X_1 \vdots Y_1X_1 \equiv R \end{aligned}$$

or

$$X_1Y_1 = Y_1X_1.$$

More generally, one obtains for each relevant value of $q > 0$

$$\begin{aligned} R \equiv (X_1Y_1)^{n_1}\cdots(X_qY_q)^{n_q}X_{q+1}Y_{q+1}[(X_{q+1}Y_{q+1})^{n_{q+1}-1}X_{q+1}] \\ [(X_{q+1}Y_{q+1})^{n_{q+1}-1}X_{q+1}]Y_{q+1}X_{q+1}(Y_qX_q)^{n_q}\cdots(Y_1X_1)^{n_1} \equiv R \end{aligned}$$

or one gets the equivalence:

$$(X_1Y_1)^{n_1}\cdots(X_qY_q)^{n_q}X_{q+1}Y_{q+1} = Y_{q+1}X_{q+1}(Y_qX_q)^{n_q}\cdots(Y_1X_1)^{n_1}$$

We thus have the following r equivalences in respect of R:

$$\left.\begin{array}{c} X_1Y_1 = Y_1X_1 \\ (X_1Y_1)^{n_1}X_2Y_2 = Y_2X_2(Y_1X_1)^{n_1} \\ (X_1Y_1)^{n_1}(X_2Y_2)^{n_2}X_3Y_3 = Y_3X_3(Y_2X_2)^{n_2}(Y_1X_1)^{n_1} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (X_1Y_1)^{n_1}\cdots(X_{r-1}Y_{r-1})^{n_{r-1}}X_rY_r = Y_rX_r(Y_{r-1}X_{r-1})^{n_{r-1}}\cdots(Y_1X_1)^{n_1} \end{array}\right\} \quad (7)$$

We remark here that X_1 and R begin on the left with X_r.

and X_r is different from any Y_i

We add to (7) all possible equivalences:

$$R\delta = \delta R$$

where δ is not equal to any of the symbols $Y_1, Y_2, \cdots, Y_r$ and X_r, so the system formed in this way, which we will call H, is a perfect system.

It's *perfect* because it fits the format required for the main theorem of §VI, and we can apply this theorem with $C \equiv D \equiv R$

We will now prove that H is also a complete system in respect of R, or that in respect of H it is always the case that

$$Rz \doteqdot zR,$$

when z denotes a single arbitrary symbol.

We however need only prove the case where z is equal to one of the symbols $Y_1, Y_2, \cdots, Y_r$ or X_r.

We will however first prove that in respect of H or (7):

$$(X_1Y_1)^{n_1}\cdots(X_qY_q)^{n_q}[X_{q+1}Y_{q+1}]^m \doteqdot [Y_{q+1}X_{q+1}]^m(Y_qX_q)^{n_q}\cdots(Y_1X_1)^{n_1} \quad (8)$$

where m is arbitrary.

The theorem is valid according to (7) for $m = 0$, $q = 1$. But if the theorem is valid for $q = h$, $m = 0$ and for $q = h$, $m = k$, so it is also valid according to (7) for $q = h$, $m = k + 1$.

For

Some typos in the following:
- added superscript n_h in line 2
- changed Y_{h+1} to X_{h+1} in line 3

$$\begin{aligned} &(X_1Y_1)^{n_1}\cdots(X_hY_h)^{n_h}[X_{h+1}Y_{h+1}]^{m+1} \equiv \\ &\equiv (X_1Y_1)^{n_1}\cdots(X_hY_h)^{n_h}[X_{h+1}Y_{h+1}]^m X_{h+1}Y_{h+1} \doteqdot \\ &\doteqdot [Y_{h+1}X_{h+1}]^m(Y_hX_h)^{n_h}\cdots(Y_1X_1)^{n_1}X_{h+1}Y_{h+1} \doteqdot \\ &\doteqdot [Y_{h+1}X_{h+1}]^m(X_1Y_1)^{n_1}\cdots(X_hY_h)^{n_h}X_{h+1}Y_{h+1} \doteqdot \\ &\doteqdot [Y_{h+1}X_{h+1}]^m Y_{h+1}X_{h+1}(Y_hX_h)^{n_h}\cdots(Y_1X_1)^{n_1} \equiv \\ &\quad [Y_{h+1}X_{h+1}]^{m+1}(Y_hX_h)^{n_h}\cdots(Y_1X_1)^{n_1} \end{aligned}$$

Thus (8) is also valid for $q = h, m = n$ or for $q = h + 1, m = 0$.

In this way is (8) proven.

We now get according to (7) and (8)

Typo: added superscript n_1 in the second line

$$\begin{array}{lllllll} Y_qR & \equiv & Y_qX_q(Y_qX_q)^{n_q}\cdots(Y_1X_1)^{n_1} & \doteqdot & (X_1Y_1)^{n_1}\cdots(X_qY_q)^{n_q}X_qY_q & \equiv & RY_q \\ X_rR & \equiv & X_r(X_1Y_1)^{n_1}\cdots(X_rY_r)^{n_r}X_r & \doteqdot & X_r(Y_rX_r)^{n_r}\cdots(Y_1X_1)^{n_1}X_r & \equiv & RX_r \end{array}$$

Thus H is also a complete system in respect of R.

Our problem (II) is accordingly solved by this means for the given null sequence R.

The above theory keeps its validity if $Y_1, Y_2, \cdots, Y_r$ and X_r are sequences provided that they cannot overlap with one another.

Example 2.

$$R \equiv ab\,bc\,ab,$$

where a, b and c denote single symbols.

$$\begin{array}{l} R \equiv abbc[ab] \\ \qquad\qquad [ab]bcab \equiv R \end{array}$$

or

$$abbc = bcab$$

or

$$\begin{array}{l} R \equiv abbca[b] \\ \qquad\qquad [b]cabab = R \end{array}$$

or

$$abbca = cabab.$$

In respect of the system

As before, this system of equivalences fits the format required for the main theorem in §VI, and is thus perfect.

$$\left.\begin{array}{lll} abbc & = & bcab \\ abbca & = & cabab \end{array}\right\}$$

we get however

$$\begin{array}{l} aR \equiv \quad a\,[abbc]\,ab = ab\,[cabab] = ab\,[abbc]\,a = abbcaba \quad \equiv Ra \\ \qquad bR \equiv b\,[abbca]\,b = [bcab]\,abb = abbcabb \equiv Rb \\ \qquad cR \equiv cab\,[bcab] = [cabab]\,bc = abbcabc \equiv Rc. \end{array}$$

Example 3.

Let

$$R \equiv ABABA,$$

where A and B are such sequences that ABA is the largest sequence that the two sequences of R have in common.

Further let

$$ABA \equiv UUU\cdots U \equiv U^n$$

where U is not a power sequence, and where U contains more symbols than A.

Thus we have here

$$U \equiv AX \equiv YA$$

or

$$B \equiv XU^{n-2}Y.$$

Since $AB = BA$, we get

$$R = BAABA \equiv XU^{n-2}YAAXU^{n-2}YA \equiv XU^{2n-1}$$
$$R = ABAAB \equiv AXU^{n-2}YAAXU^{n-2}Y \equiv U^{2n-1}Y$$

or

$$X = Y.$$

If X contains more symbols than A, or

$$U \equiv ACA$$

then we get

$$AC = CA.$$

If A and C here represent single different symbols, or sequences which cannot have an overlap with each other, then our problem (II) is solved through these latest equivalences.

Example 4.

$$R \equiv x^n y x^n$$

where x and y are single symbols.

We get

$$x^n y = y x^n$$
$$R = x^{2n} y = y x^{2n}$$
$$R = yx[x^{2n-1}]$$
$$[x^{2n-1}]xy = R$$

or

$$xy = yx$$

which is sufficient.

Example 5.

Let

$$R \equiv x^n y x^n y x^n \dots x^n y x^n \equiv x^n (y x^n)^p \equiv (x^n y)^p x^n$$
$$n > 1, p > 1.$$

Here we have first

$$x^n y = y x^n.$$

Thus we can always pull all the y's to the left (moving any x^n to the right) or vice versa.

Second one thus gets:

$$R = y^p x[x^{(p+1)n-1}]$$
$$[x^{(p+1)n-1}]xy^p = R$$

or

$$y^p x = xy^p.$$

We will now show that the equivalences:

$$\left.\begin{aligned} x^n y &= yx^n \\ y^p x &= xy^p \end{aligned}\right\} \qquad \text{(K)}$$

form a complete and perfect system K in respect of R.

First it is certainly

Proving completeness is the easy part.

Typo: changed $=$ to $\doteqdot$ at the end of the first equation

$$xR \doteqdot xx^{(p+1)n}y^p \doteqdot x^{(p+1)n}y^p x \doteqdot Rx$$
$$yR \doteqdot yy^p x^{(p+1)n} \doteqdot y^p x^{(p+1)n} y \doteqdot Ry.$$

Second we will show the following:
If S and T are such sequences that, in respect of the System K,

This is just the definition of a perfect system being spelt out

$$zS \doteqdot zT,$$

so that one can thus find a sequences E where

$$zS \sim E_1 \sim E_2 \sim \cdots \sim E_r \sim zT,$$

where z is an arbitrary symbol denoting x or y, then in respect of K we would also have

$$S \doteqdot T.$$

There is then such a sequence F that

$$S \sim F_1 \sim F_2 \sim \cdots \sim F_r \sim T.$$

The proof will be by induction over
(a) the length of a derivation and
(b) the number of symbols in S.

Through the figure

$$X \sim Y$$

we will indicate here that one can get Y from X by exchanging a subsequence $x^n y$ or yx^n or $y^p x$ or xy^p for its corresponding sequence.

The theorem is valid now first when $zS \sim zT$. Then clearly $S \sim T$.

Second the theorem is also valid when

For the base cases, note that both the equivalences in K must change the leftmost symbol.

$$zS \sim E \sim zT$$

Then clearly

$$S \equiv T.$$

Third, the theorem is valid when both S and T denote just a single symbol. Then clearly

$$S \equiv T.$$

We assume now in advance that the theorem is always true when both S and T are composed of at most m symbols. Further we assume that the theorem remains true when both S and T contain $m+1$ symbols, and where the number of E-sequences r is not greater than $n > 1$.

Inductive hypothesis

We then need only to prove that the theorem remains true when both S and T are composed of $m+1$ symbols, while in the derivation

$$zS \sim E_1 \sim E_2 \sim \cdots \sim E_r \sim zT$$

the number r of E-sequences is equal to $n+1$.

If it is the case that e.g.

$$z \equiv x$$

so we thus have

$$xS \sim z_1C_1 \sim z_2C_2 \sim \cdots \sim z_{n+1}C_{n+1} \sim xT.$$

If here e.g.

$$z_k \equiv x$$

so we get

$$S \doteqdot C_k \doteqdot T.$$

In the opposite case one gets however

i.e. no z_k is x

$$xS \sim yC_1 \sim yC_2 \sim \cdots \sim yC_{n+1} \sim xT.$$

If here either

$$S \equiv x^{n-1}yS', \qquad T \equiv x^{n-1}yT'$$

or

$$S \equiv y^pS', \qquad T \equiv y^pT'$$

then one gets respectively

$$xS \equiv x^nyS' \sim y(x^nS') \sim \cdots \sim y(x^nT') \sim x^nyT' \equiv xT$$
$$xS \equiv xy^pS' \sim y(y^{p-1}xS') \sim \cdots \sim y(y^{p-1}xT') \sim xy^pT' \equiv xT$$

In both cases we get

$$S' \doteqdot T'$$

or

$$S \doteqdot T.$$

We need then only to consider the case where e.g.

Since $\sim$ is symmetric this covers the other "two" cases

$$S \equiv x^{n-1}yS_1, \qquad T \equiv y^pT_1.$$

We get then:

$$xS \equiv x^nyS_1 \sim y(x^nS_1) \sim \cdots \sim y(y^{p-1}xT_1) \sim xy^pT_1 \equiv xT$$

or

$$x^nS_1 \mathrel{⧧} y^{p-1}xT_1.$$

We get here the alternatives:

$$y^{p-1}xT_1 \sim \cdots \sim y^{p-1}x(x^{n-1}T_2) \sim \cdots \sim x^nS_1$$
$$y^{p-1}xT_1 \sim \cdots \sim y^{p-1}x(y^pT_2) \sim y^{2p-1}xT_2 \cdots \sim x^nS_1.$$

In the first alternative

$$T_1 \mathrel{⧧} x^{n-1}T_2$$
$$x^ny^{p-1}T_2 \mathrel{⧧} x^nS_1$$

or

$$y^{p-1}T_2 \mathrel{⧧} S_1$$

or

Typo: changed $\equiv$ to $\mathrel{⧧}$ at the end of this equation

$$S \equiv x^{n-1}yS_1 \mathrel{⧧} x^{n-1}y^pT_2 \mathrel{⧧} y^px^{n-1}T_2 \mathrel{⧧} y^pT_1 \equiv T.$$

In the second alternative

$$y^{p-1}xT_1 \sim \cdots \sim y^{qp-1}x(x^{n-1}T_3) \sim \cdots \sim x^nS_1$$

or

$$xT_1 \mathrel{⧧} y^{p(q-1)}x^nT_3$$
$$T_1 \mathrel{⧧} x^{n-1}y^{p(q-1)}T_3$$
$$S_1 \mathrel{⧧} y^{qp-1}T_3$$

or

$$S \equiv x^{n-1}yS_1 \mathrel{⧧} x^{n-1}y^{qp}T_3 \mathrel{⧧} y^px^{n-1}y^{(q-1)p}T_3 \mathrel{⧧} y^pT_1 \equiv T.$$

In this way the theorem is proved.

§ VIII

Finally we wish to make a few remarks.

If R denotes an arbitrary null sequence, then there exists three series of symbol sequences

$$P_1, P_2, \ldots P_m \qquad (\alpha)$$
$$Q_1, Q_2, \ldots Q_m \qquad (\beta)$$
$$R_1, R_2, R_3, \ldots R_n \qquad (\gamma)$$

with the following properties:

1. P_r and Q_r are - for each value of r - equivalent to each other in respect of R, and each of these sequences contains fewer symbols than R.

2. All sequences $R_1, R_2, \ldots R_n$, each of which denote a null sequence, are equivalent to one another in respect of the equivalences

$$\left.\begin{array}{c} P_1 = Q_1 \\ P_2 = Q_2 \\ \cdots\cdots\cdots \\ P_m = Q_m \end{array}\right\} \qquad (\delta)$$

3. For each r the series (γ) contains two sequences R_p and R_q such that

$$\begin{aligned} R_p &\equiv P_r U \\ & \quad UQ_r \equiv R_q \end{aligned}$$

where U denotes a symbol sequence.

4. If for two arbitrary sequences R_p and R_q of the series (γ) there exist such symbol sequences C, D and U that

$$\begin{aligned} R_p &\equiv CU \\ & \quad UD \equiv R_q \end{aligned}$$

then the equivalence

$$C = D$$

forms one of the equivalences of (δ).

One sees immediately that all the (γ)-sequences contain equally many symbols, and similarly for P_r and Q_r for each value of r.

We will now show how one can gradually form the sequences in (γ) and the equivalences in (δ).

Let

$$S_1, S_2, \cdots, S_k$$

denote k series of symbol sequences R

$$\begin{array}{llll} R_1^1, & R_2^1, & \cdots, & R_{n_1}^1 \\ R_1^2, & R_2^2, & \cdots, & R_{n_2}^2 \\ \cdots & \cdots & \cdots & \cdots \\ R_1^k, & R_2^k, & \cdots, & R_{n_k}^k \end{array}$$

where each R_x^y denotes a single symbol sequence, while

$$R_1^\theta, R_2^\theta, \cdots, R_{n_\theta}^\theta$$

for each θ is said to denote the series S_θ.

Further, we signify by

$$E_1, E_2, \cdots, E_h$$

h systems of equivalences

$$\begin{array}{ccc} P_1^1 = Q_1^1 & & P_1^h = Q_1^h \\ P_2^1 = Q_2^1 & \cdots & P_2^h = Q_2^h \\ \cdots\cdots\cdots & & \cdots\cdots\cdots \\ P_{m_1}^1 = Q_{m_1}^1 & & P_{m_h}^h = Q_{m_h}^h \end{array}$$

where each P and each Q denotes a single symbol sequence, and where E_θ for each value of θ is said to represent the system

$$\begin{array}{c} P_1^\theta = Q_1^\theta \\ P_2^\theta = Q_2^\theta \\ \cdots\cdots\cdots \\ P_{m_\theta}^\theta = Q_{m_\theta}^\theta. \end{array}$$

The series S_1 only contains the null sequence R.

For each value of θ we form E_θ from S_θ and further $S_{\theta+1}$ from E_θ as follows:

First, if the system

$$S_1, S_2, \cdots, S_\theta$$

contains two such sequences R_p and R_q that

$$\begin{aligned} R_p &\equiv CU \\ & \quad\; UD \equiv R_q \end{aligned}$$

where C, D and U denote single symbols or sequences, then

$$C = D$$

is equal to one of the equivalences from E_θ.

For each equivalence

$$P_r^\theta = Q_r^\theta$$

from E_θ that are opposite for each value of r in the group, the series $S_1, S_2, \cdots, S_\theta$ contains such sequences R_p and R_q that

$$\begin{aligned} R_p &\equiv P_r^\theta U \\ & \quad\; UQ_r^\theta \equiv R_q \end{aligned}$$

where U denotes a single symbol or a sequence. In this way E_θ is completely defined.

Finally, let $S_{\theta+1}$ be formed from all of those unique sequences $R_{\theta+1}$, that are equivalent to all R-sequences in the series $S_1, S_2, \cdots, S_\theta$ in respect of the equivalences of the system $E_1, E_2, \cdots, E_\theta$.

One sees immediately then that S_θ is contained in $S_{\theta+1}$ and that E_θ is contained in $E_{\theta+1}$.

One can however choose θ so large that

$$S_{\theta+1} \equiv S_\theta$$

and thus also

$$E_{\theta+1} \equiv E_\theta.$$

In this way our claim is proven.

From the system (δ) we can now choose a system (ε) of equivalences independent from each other

$$\left.\begin{array}{l} A_1 = B_1 \\ A_2 = B_2 \\ \cdots\cdots\cdots \\ A_k = B_k \end{array}\right\} \qquad (\varepsilon)$$

that one can derive each equivalence in (δ) from (ε) while one can thus derive no equivalence in (ε) from the others.

(ε) can be so chosen that the number k of these equivalences is minimised. Further, one can choose (ε) so that none of these equivalences can be replaced by another with fewer symbols.

In (ε) it is never the case that

$$A_r \equiv B_r$$

and further we never have simultaneously

$$A_r \equiv A_s$$
$$B_r \equiv B_s$$

or

$$A_r \equiv B_s$$
$$B_r \equiv A_s.$$

THEOREM. The system (ε) contains no equivalences of the form

$$TX = TY$$

where e.g. X starts the left of one of the sequences R_x of the (γ)-sequences, i.e.

$$XW \equiv R_x.$$

Since the named equivalence must also occur in (δ), there are such sequences R_y, R_z and R_μ in (γ) that:

$$\begin{aligned} R_y &\equiv TXU \\ &\qquad UTY \equiv R_z \\ R_\mu &\equiv UTX \\ &\qquad XW \equiv R_x \end{aligned}$$

or the equivalence

$$UT = W$$

is contained in (δ), or

$$\begin{aligned} R_x \equiv XW = XUT& \\ UTY \equiv R_z&. \end{aligned}$$

However (ε) then contains the equivalence

$$X = Y,$$

from which one can clearly derive

$$TX = TY.$$

Impossible, since by the definition of (ε) you can't derive one of its equations from any of the others.

Theorem. The system (ε) contains no equivalence of the form:

$$SX = SY,$$

where S forms the right end of a (γ)-sequence.

Since the named equivalence must also occur in (δ), there are such sequences R_x, R_y, R_z and R_μ in (γ) that:

$$\begin{aligned} R_y &\equiv SXU \\ &\qquad USY \equiv R_z \\ R_x &\equiv KS \\ &\qquad SXU \equiv R_y \end{aligned}$$

or one obtains the equivalence $K = XU$ which is thus contained in (δ). Finally

$$\begin{aligned} R_x &\equiv KS = XUS = R_\mu \\ R_z &\equiv \quad USY. \end{aligned}$$

However (δ) then contains the equivalence

$$X = Y,$$

which is impossible.

Theorem. If

$$R_x \equiv PU$$

and

$$UQ \equiv R_y,$$

where R_x and R_y denote two sequences from (γ), while thus

$$P = Q$$

forms one of the equivalences of (δ), then we have in respect of (δ) or in respect of (ε)

$$PR = RP$$
$$QR = RQ$$
$$UR = RU$$

where R denotes an arbitrary sequence of (γ).

Then

$$PR = PR_y \equiv PUQ \equiv R_xQ = R_xP = RP$$
$$UR = UR_x \equiv UPU = UQU \equiv R_yU = RU.$$

Further, one gets in respect of (ε)

$$PU \equiv R_x = R_y \equiv UQ = UP.$$

Theorem. If one of the sequences P and Q in (δ), which we shall represent with C, has the form

$$C \equiv NM,$$

where M forms the starting left side of a sequence R_x of the (γ)-sequences, then we have for each arbitrary sequence R of the (γ)-sequences in respect of (δ):

$$NR = RN$$
$$MR = RM.$$

For if D is the sequence corresponding to C in (δ), there are clearly such sequences R_y and R_z in (γ) that

$$R_y \equiv NMU$$
$$UD \equiv R_z$$

or

Typo: changed equals to equivalence

$$R_z = UNM$$
$$MW \equiv R_x$$

or

$$UN = W$$

or finally

$$NR = NR_x \equiv NMW = NMUN \equiv R_yN = RN.$$

Further

$$MR = MR_z \equiv MUNM = MWM \equiv R_xM = RM.$$

THEOREM. Let R_x, R_y, R_z and R_μ denote four arbitrary different or not different null sequences R of the system (γ). Further M and N denote two sequences which are obtained through insertion of R_z and R_μ in R_x and R_y respectively. That is,

$$M \equiv aR_zb$$
$$N \equiv cR_\mu d$$

where

Typo: changed equals to equivalence

$$R_x \equiv ab$$
$$R_y \equiv cd.$$

If there are then such sequences C, U and D that

$$M \equiv CU$$
$$UD \equiv N$$

then either

$$C = D$$

forms one of the equivalences of (δ), or one can obtain sequences from C and D through removal of subsequences R of the systems (γ) that are equivalent in respect of (δ).

In cases 1-5 below we get $C = D$, while in cases 6-8 C and D differ by some R.

Letting the symbol $=$ represent equivalence in respect of (δ), we can distinguish the following cases:

These 8 cases cover all possible configurations of the overlap between aR_zb and $cR_\mu d$

1.

$$a \equiv Ce \qquad c \equiv ef$$
$$b \equiv hi \qquad d \equiv iD$$
$$R_z \equiv fg$$
$$gh \equiv R_\mu$$

or

$$f = h$$
$$R_x \equiv Cehi = Cefi$$
$$efiD \equiv R_y$$

or

$$C = D.$$

2.

$$a \equiv Ce \qquad c \equiv ef$$
$$D \equiv hd$$
$$R_z \equiv fg$$
$$gbh \equiv R_\mu$$

or

$$f = bh$$
$$R_x \equiv Ceb$$
$$ebD \equiv ebhd = efd \equiv R_y$$

or

$$C = D.$$

3.
$$a \equiv Cce, \qquad d \equiv gbD$$
$$R_\mu \equiv ef$$
$$fg \equiv R_z$$
or
$$e = g$$
$$R_x \equiv Cceb = Ccgb$$
$$cgbD \equiv R_y$$
or
$$C = D.$$

4.
$$C \equiv ae, \qquad D \equiv gd$$
$$R_z \equiv ecf$$
$$fbg \equiv R_\mu$$
or
$$ec = bg$$
$$R_x \equiv ab$$
$$bgf = ecf \equiv R_z$$
or
$$a = gf$$
$$R_\mu \equiv fbg = fec$$
$$cd \equiv R_y$$
or
$$fe = d$$
or
$$C \equiv ae = gfe = gd \equiv D.$$

5.
$$b \equiv gh, \qquad c \equiv fg$$
$$C \equiv ae, \qquad D \equiv id$$
$$R_z \equiv ef$$
$$fgd \equiv R_y$$
or
$$e = gd$$
$$R_x \equiv agh$$
$$hi \equiv R_\mu$$
or
$$ag = i$$
or
$$C \equiv ae = agd = id \equiv D.$$

6.
$$c \equiv fbg$$
$$C \equiv ae, \qquad D \equiv gR_\mu d$$
$$R_z \equiv ef$$
$$fbgd \equiv R_y$$

or

$$e = bgd$$

or

$$C \equiv ae = abgd = R_x gd$$
$$D \equiv gR_\mu d.$$

7.

$$b \equiv ecf$$
$$C \equiv aR_z e, \qquad D \equiv gd$$
$$R_z \equiv aecf$$
$$fg \equiv R_\mu$$

or

$$aec = g$$

or

$$D \equiv gd = aecd = aeR_x$$
$$C \equiv aR_z e.$$

8.

$$b \equiv ef, \qquad c \equiv fg$$
$$C \equiv aR_x e, \qquad D \equiv gR_\mu d$$
$$R_x \equiv aef$$
$$fgd \equiv R_y$$

or

$$ae = gd.$$

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Axel Thue.