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Block theory, branching rules, and centralizer algebras

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1. Introduction

This paper is one in a series [8–11] exploring the algebra kG^H, the centralizer in the group algebra kG of the subalgebra kH, where k is a field of characteristic not zero, G is a finite group, and H is a subgroup of G. All these papers search for theorems similar to Alperin's weight conjecture [2]. The immediate goal is to find results that relate information about blocks of kG^H or simple modules over kG^H to p-local information. The ultimate goal is to gain insight into Alperin's conjecture. See the introductions to [10] and [11] for a detailed description of the program.

In this paper, we obtain fairly complete information about the algebra kG^H when G is the symmetric group S_n, H is S_{n-1} identified with the subgroup of S_n consisting of all permutations fixing n, and k is a field of characteristic p with $p \neq 0$.

For comparison, first consider the algebra FG^H, where G is any finite group, H is any subgroup of G, and F is a field of characteristic 0 that is a splitting field for both G and H. For any irreducible character χ of G, let $e(\chi)$ be the primitive central idempotent of FG corresponding to χ. For any irreducible character ψ of H, let $f(\psi)$ be the primitive central idempotent of FH corresponding to ψ. A straightforward application of the Jacobson density theorem shows that

$$FG^H = \bigoplus_{\chi\in\mathrm{Irr}(G),\,\psi\in\mathrm{Irr}(H)} e(\chi)f(\psi)FG^H \cong \bigoplus_{\chi\in\mathrm{Irr}(G),\,\psi\in\mathrm{Irr}(H)} \mathrm{Mat}_{(\chi_H,\psi)}(F),$$

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where $(-,-)$ is the usual inner product of characters of H, and for any l, $\mathrm{Mat}_l(F)$ is the algebra of l by l matrices with entries in F. (See 2.1 in [11].) When we specialize to the case with $G = S_n$ and $H = S_{n-1}$, the classical branching rule gives us two further pieces of information. First, it provides a combinatorial way to determine whether $e(\chi)f(\psi)$ is 0; second, it tells us that when $e(\chi)f(\psi) \neq 0$, $(\chi_H, \psi) = 1$ so that $e(\chi)f(\psi)FG^H \cong F$. Thus $FS_n^{S_{n-1}}$ is isomorphic to the direct sum of m copies of F, where m is equal to the sum over all partitions λ of n of the number of distinct parts of λ.

Much of this paper is devoted to developing a similar understanding of the algebra $kS_n^{S_{n-1}}$ when k has characteristic p. In Theorem 2.2, we show that if e is a primitive central idempotent of kS_n and f is a primitive central idempotent of kS_{n-1} with $ef \neq 0$, then $efkS_n^{S_{n-1}}$ is a commutative local ring. Thus the decomposition

$$kS_n^{S_{n-1}} = \bigoplus_{e,f} efkS_n^{S_{n-1}}$$

is the block decomposition of the algebra $kS_n^{S_{n-1}}$, where e and f run through all primitive central idempotents of kS_n and kS_{n-1}, respectively. This result follows easily from the surprising fact that $kS_n^{S_{n-1}}$ is generated as an algebra by the centers of kS_n and kS_{n-1}. Since the algebra $kS_n^{S_{n-1}}$ is commutative, there is a bijection between its simple modules and its blocks.

We prove analogs of both consequences of the classical branching rule. In 3.2, we give a combinatorial way to determine whether the product ef is 0. As a consequence, we find a combinatorial parametrization of the blocks of $kS_n^{S_{n-1}}$. (See Corollary 3.3.) In Theorem 6.1, we prove an analog of the second consequence of the classical branching rule. We show that, when $ef \neq 0$, the isomorphism type of the algebra $efkS_n^{S_{n-1}}$ depends only on the unordered pair $\{w_e, w_f\}$, where w_e is the weight of e and w_f is the weight of f. This result is similar to the well-known fact that if two blocks of group algebras for symmetric groups have the same weight then their centers are isomorphic. The proof depends on a theorem of Enguehard [12], which states that there is a perfect isometry between any pair of blocks of symmetric groups that have equal weights.

In Sections 3 and 4, we explore the relationship between the blocks of $kS_n^{S_{n-1}}$ and the p-local structure of S_n. Green's theory gives a definition of defect groups in S_{n-1} for each block of $kS_n^{S_{n-1}}$. In Corollary 3.4, we find these defect groups. In Theorem 3.5, we show that an analog of Brauer's first main theorem holds; if $G = S_n$, $H = S_{n-1}$, and D is a p-subgroup of H, then there is a bijection between the blocks of kG^H with defect group D and the blocks of $kN_G(D)^{N_H(D)}$ with defect group D. Since both algebras are commutative, this theorem can be interpreted as a theorem counting simple kG^H-modules, a theorem that looks similar to the weight conjecture. These results rely on the fact that when e and f both have defect greater than zero, then it is possible to determine p-locally whether $ef = 0$. (See Proposition 3.1.)

In Section 4, we present a sort of lop-sided analog of the $k[G \times G]$-module approach to block theory. The $k[S_{n-1} \times S_n]$-module fkS_ne is indecomposable, since its endomorphism ring is isomorphic to the local ring $efkS_n^{S_{n-1}}$. We show that if D is a defect group in S_{n-1}

of ef, then the diagonal group $\delta(D)$ is a vertex of fkS_ne, and the Green correspondent of fkS_ne with respect to $(S_{n-1}\times S_n, N_{S_{n-1}}(D)\times N_{S_n}(D), \delta(D))$ is $\mathrm{Br}_D(f)kN_{S_n}(D)\,\mathrm{Br}_D(e)$, where Br_D denotes the Brauer map with respect to D.

Finally in Section 7, we obtain some results regarding the support of the block idempotents of $kS_n^{S_{n-1}}$. We show that when k has characteristic 2, then each idempotent in $kS_n^{S_{n-1}}$ lies in the k-span of the 2-regular S_{n-1}-orbit sums. The analogous result is false when the characteristic of k is odd. Section 7 builds on work in [17].

We will use several well known combinatorial notions. Let λ be a partition of a positive integer n. So λ is a nonincreasing sequence $\{\lambda_i\}$ of nonnegative integers whose sum is n. We let $l(\lambda)$ denote the number of nonzero terms in λ. Let t be an integer at least as big as $l(\lambda)$. The β-set of size t for λ is the strictly decreasing sequence $\lambda_1+t-1>\lambda_2+t-2>\cdots>\lambda_t$.

It is sometimes useful to represent a partition λ using an abacus with p-runners (for some $p>0$) and a β-set of size t. The runners are labelled (left to right) by $0,\dots,p-1$. The positions in the ith row of the abacus are labelled, left to right, from $p(i-1)$ to $pi-1$, for $i=1,2,\dots$. The abacus of λ is obtained by placing a bead at each position given by a β-number. The shape of the abacus is determined by $t \pmod p$. Conversely each abacus with p-runners represents the β-set of a partition.

Identify λ with the corresponding abacus with p-runners. The jth runner of λ can be thought of as a partition, labelled λ^j, represented by an abacus with one runner. The p-tuple $(\lambda^0,\dots,\lambda^{p-1})$ of such partitions is called the *p-quotient* of λ. The p-core of λ is another partition associated with λ. It is represented by the abacus with p-runners that is obtained by moving the beads on λ as far up as possible. So the p-core of λ depends only on the numbers of beads on each of the runners of λ. It is clear that λ is determined by its p-core and p-quotient.

2. Blocks

Finding the blocks of the algebra kG^H, when $G=S_n$ and $H=S_{n-1}$, will be easy after we have established that the $\mathbf{Z}$-algebra $\mathbf{Z}G^H$ is generated as an algebra by the center of $\mathbf{Z}G$ and the center of $\mathbf{Z}H$, where $\mathbf{Z}$ represents the rational integers. This will be proved in Proposition 2.1.

We need a standard $\mathbf{Z}$-basis for $\mathbf{Z}G^H$. Let $\lambda=((\lambda_1)^{n_1},\dots,(\lambda_r)^{n_r})$ be a partition of n, where $\lambda_1>\cdots>\lambda_r>0$ and $\sum_{i=1}^r n_i\lambda_i=n$. Suppose that σ is an element of S_n of cycle type λ, and that n occurs in a cycle of length λ_s when σ is written as a product of disjoint cycles. Consider the conjugation action of S_{n-1} on S_n. The orbit containing σ consists of all elements τ of S_n such that τ has cycle type λ and n occurs in τ in a cycle of length λ_s. Thus the orbits are labelled by pairs $(\lambda;\lambda_s)$, where λ is a partition of n and λ_s is one of the distinct parts of λ. For any such orbit, let $A(\lambda;\lambda_s)$ be the sum in $\mathbf{Z}S_n$ of all its elements. For example, in $(\mathbf{Z}S_3)^{S_2}$ we have $A((2,1);1)=(12)$ and $A((2,1);2)=(13)+(23)$. The set of all $A(\lambda;\lambda_s)$ is a basis for $\mathbf{Z}S_n^{S_{n-1}}$ as a $\mathbf{Z}$-module. It follows that the $\mathbf{Z}$-rank of $\mathbf{Z}S_n^{S_{n-1}}$ is $\sum_\lambda d(\lambda)$, where λ runs through all partitions of n and $d(\lambda)$ is the number of distinct parts of λ. Since there are $d(\lambda)+1$ ways of incrementing some part of λ by one, in order to produce a partition of $n+1$, induction shows that the $\mathbf{Z}$-rank of $\mathbf{Z}S_n^{S_{n-1}}$ is also given as $\sum_{i=0}^{n-1} p(i)$, where $p(i)$ is the number of partitions of i.

We will use $\mathbf{N}_n$ to denote the set containing the first n positive integers. For $i \geqslant 0$, we let V_i be the $\mathbf{Z}$-submodule of $\mathbf{Z}G$ that is generated by all permutations that fix i or more elements of $\mathbf{N}_n$, and set $U_i := V_i \cap \mathbf{Z}G^H$. We then have

$$0 = U_{n+1} \subseteq U_n \subseteq \cdots \subseteq U_1 \subseteq U_0 = \mathbf{Z}G^H.$$

Proposition 2.1. *Let* $\mathbf{Z}$ *be the ring of rational integers. Let* $G = S_n$ *and let* $H = S_{n-1}$*. The* $\mathbf{Z}$*-algebra* $\mathbf{Z}G^H$ *is generated by* $\mathbf{Z}G^G$ *and* $\mathbf{Z}H^H$*.*

Proof. Let $\mathcal{A}$ be the $\mathbf{Z}$-algebra generated by $\mathbf{Z}G^G$ and $\mathbf{Z}H^H$. We will prove by induction on $n-i$ that U_i is in $\mathcal{A}$. Since U_n contains only integer multiples of the identity, the start of the induction is trivial. For the rest of the proof, assume that $i < n$, and that $U_j \subseteq \mathcal{A}$ for all $j > i$. The $\mathbf{Z}$-module U_i is generated by U_{i+1} and the set of all $A(\lambda; \lambda_s)$ such that the multiplicity of 1 as a part of λ is i. We must show that $\mathcal{A}$ contains all such basis elements $A(\lambda; \lambda_s)$.

First, consider a partition μ of n of the form $\mu = ((\mu_1)^{n_1}, 1^{n-n_1\mu_1})$. The basis element $A(\mu; 1)$ is in $\mathbf{Z}H^H$, since it is the sum of all permutations in H of cycle type μ. The sum $A(\mu; \mu_1) + A(\mu; 1)$ is in $\mathbf{Z}G^G$, since it is the sum of all permutations in G of cycle type μ. It follows that $A(\mu; \mu_1)$ and $A(\mu; 1)$ are in $\mathcal{A}$.

Let $\lambda = ((\lambda_1)^{n_1}, \ldots, (\lambda_r)^{n_r})$ be a partition of n; assume that the multiplicity of 1 as a part of λ is i. We will show that for any $s \leqslant r$, the basis element $A(\lambda; \lambda_s)$ is in $\mathcal{A}$. Define partitions $\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(r-1)}$ as follows.

$$\begin{aligned}
\mu^{(1)} &= \big((\lambda_1)^{n_1}, 1^{n-n_1\lambda_1}\big),\\
\mu^{(2)} &= \big((\lambda_2)^{n_2}, 1^{n-n_2\lambda_2}\big),\\
&\ \ \vdots\\
\mu^{(r)} &= \big((\lambda_r)^{n_r}, 1^{n-n_r\lambda_r}\big).
\end{aligned}$$

(When $\lambda_r = 1$, which happens if $i > 0$, this last partition should really be written as (1^n) instead of $(1^{n_r}, 1^{n-n_r})$ as it is here.) Let a be the following product

$$a = \left(\prod_{j=1}^{s-1} A\big(\mu^{(j)}; 1\big)\right) A\big(\mu^{(s)}; \lambda_s\big) \left(\prod_{j=s+1}^{r} A\big(\mu^{(j)}; 1\big)\right).$$

The element a is in $\mathcal{A}$ since each factor in the product is in $\mathcal{A}$.

In the following paragraph, we will carefully examine the expansion of a to show that $a = A(\lambda; \lambda_s) + b$, where $b \in U_{i+1}$. Since $a \in \mathcal{A}$ and by induction assumption $U_{i+1} \subseteq \mathcal{A}$, it follows that $A(\lambda; \lambda_s) \in \mathcal{A}$.

Let t be a term in the expansion of a. The term t has the form $t_1 t_2 \cdots t_r$, where t_j is a term from $A(\mu^{(j)}; 1)$ when $j \neq s$ and is a term from $A(\mu^{(s)}; \lambda_s)$ when $j = s$. Each factor

t_j is a product of n_j disjoint λ_j-cycles. If the cycles appearing in t_1 through t_r are not all disjoint from each other, then the permutation t must fix more than i elements of $\mathbf{N}_n$; in this case $t \in V_{i+1}$. On the other hand, consider the sum u of all the terms t in the expansion of a such that the cycles appearing in t_1 through t_r *are* all disjoint; the terms in u are exactly the terms in $A(\lambda; \lambda_s)$. It follows that $a - A(\lambda; \lambda_s)$ is in V_{i+1}. Since $a - A(\lambda; \lambda_s)$ is also in $\mathbf{Z}G^H$, we have $a - A(\lambda; \lambda_s) \in U_{i+1}$, as we wanted. □

Next, we find the blocks of the algebra $kS_n{}^{S_{n-1}}$ for any field k. In Corollary 3.3 we will present a parametrization of these blocks by certain pairs of cores.

Theorem 2.2. *Let $G = S_n$ and let $H = S_{n-1}$. Let k be any field. Let e be a primitive central idempotent of kG and let f be a primitive central idempotent of kH. If $ef \neq 0$, then $efkG^H$ is a block of kG^H. Every block of kG^H arises in this way.*

Proof. Let $e_1, \ldots, e_r$ be the primitive central idempotents of kG; let $f_1, \ldots, f_s$ be the primitive central idempotents of kH. Then

$$kG^G = ke_1 \oplus \cdots \oplus ke_r \oplus J\big(kG^G\big)$$

and

$$kH^H = kf_1 \oplus \cdots \oplus kf_s \oplus J\big(kH^H\big).$$

By Proposition 2.1, kG^H is the generated as a k-algebra by kG^G and kH^H. It follows that

$$kG^H = \left(\bigoplus_{i,j} ke_i f_j\right) \oplus I,$$

where I is the ideal of kG^H generated by $J(kG^G)$ and $J(kH^H)$. The ideal I is nilpotent and kG^H/I is semisimple. It follows that $I = J(kG^H)$ and that every primitive idempotent of the commutative algebra kG^H is equal to $e_i f_j$ for some choice of i and j. □

In Section 6, we will show if $ef \neq 0$, then the isomorphism type of the algebra $efkS_n{}^{S_{n-1}}$ depends only on the unordered pair $\{w_e, w_f\}$, where w_e is the weight of e and w_f is the weight of f. In Section 5, we will obtain information about the dimension of $efkS_n{}^{S_{n-1}}$ as a vector space over k.

Proposition 2.1 could also have been proved using Jucys–Murphy elements. For any i with $2 \leqslant i \leqslant n$, let L_i be the sum in $\mathbf{Z}S_n$ of all the transpositions in S_i that are not in S_{i-1}. G.E. Murphy has shown that every element of the center of $\mathbf{Z}S_n$ can be written as a symmetric polynomial in $L_2, L_3, \ldots, L_n$ with coefficients in $\mathbf{Z}$.

Proposition 2.3. *Let $G = S_n$ and let $H = S_{n-1}$. Then $\mathbf{Z}G^H$ coincides with the $\mathbf{Z}$-polynomials in $L_2, L_3, \ldots, L_n$ that are symmetric in $L_2, L_3, \ldots, L_{n-1}$.*

Proof. Let $2 \leqslant u \leqslant n-1$. The transposition $(u-1, u)$ commutes with all L_i for which $2 \leqslant i \leqslant n$ and $i \neq u-1, u$. Also, it is easy to see that $(u-1, u)$ commutes with $L_{u-1}+L_u$ and $L_{u-1}L_u$. These two polynomials generate the ring of symmetric polynomials in L_u and L_{u+1} over any commutative ring. Moreover, the transpositions $(1,2), \ldots, (n-2, n-1)$ generate the group H. It follows that any $\mathbf{Z}$-polynomial in $L_2, \ldots, L_n$ that is symmetric in $L_2, \ldots, L_{n-1}$ lies in $\mathbf{Z}G^H$.

For the reverse inclusion, we modify the proof of Theorem 1.9 of [16]. Recall that V_i is the $\mathbf{Z}$-submodule of $\mathbf{Z}G$ that is generated by all permutations that fix i or more elements of $\mathbf{N}_n := \{1, \ldots, n\}$, and that $U_i = V_i \cap \mathbf{Z}G^H$.

Set $X([1^n]; 1) := 1$. Let $(\lambda; \lambda_s)$ be a H-orbit type, with $\lambda \neq [1^n]$. Then there is a positive integer r such that $\lambda_i > 1$ if and only if $1 \leqslant i \leqslant r$. Set $X(\lambda; \lambda_s)$ as the sum, in $\mathbf{Z}G$, of all distinct products of the form

$$(L_{i_1})^{\lambda_1 - 1}(L_{i_2})^{\lambda_2 - 1} \cdots (L_{i_r})^{\lambda_r - 1},$$

where $i_1, i_2, \ldots, i_r$ runs over sets of r distinct elements of $2, \ldots, n$, subject to $i_s = n$, if λ_s happens to be greater than 1. Note that $X(\lambda; \lambda_s)$ is an element of $\mathbf{Z}G^H$, by the first paragraph.

Now $A(\lambda; \lambda_s) \in U_i \backslash U_{i+1}$, where $i = n - \sum_{j=1}^r \lambda_j$. We claim that $A(\lambda; \lambda_s) \equiv X(\lambda; \lambda_s)$ modulo U_{i+1}. The proof is by reverse induction on i. The base case $i = n$ is trivial. So we may assume that $i < n$.

We need the notion of graphs associated with products of transpositions. Let σ be a permutation of N_n. Suppose that $\sigma = t_1 t_2 \cdots t_k$, where the t_i are transpositions. The graph associated with this factorization has vertices N_n, and one (undirected) edge $j - k$, for each (j, k) that occurs as one of the t_i. Note that the number k of transpositions is at least $n - l$, where l is the number of orbits of σ on $\mathbf{N}_n$. We call the factorization *minimal* if $k = n - l$. Clearly the factorization is minimal if and only if the graph is a tree. Moreover, in the minimal case the connected components of the graph correspond to the orbits of σ.

Consider now the occurrence of a partition σ in the expansion of a term $(L_{i_1})^{\lambda_1 - 1} \cdots (L_{i_r})^{\lambda_r - 1}$ of $X(\lambda; \lambda_s)$. This represents a factorization of σ into a product of

$$\sum_{j=1}^{r} (\lambda_j - 1) = n - l(\lambda)$$

transpositions (of a given kind). The number of elements of $\mathbf{N}_n$ that are not fixed by σ is at most $\sum_{j=1}^r \lambda_j = n - i$. So σ is an element of V_i. Suppose that $\sigma \notin V_{i+1}$. Then the connected components of the associated graph consist of i isolated vertices, and r star-shaped graphs, whose central vertices $i_1, \ldots, i_r$ have degrees $\lambda_1, \ldots, \lambda_r$, respectively. This graph is a tree. So the cycle lengths of σ coincide with the sizes of the components. It follows that σ is of cycle type λ. Moreover, the jth cycle of σ has largest symbol i_j. In particular, the cycle of σ that contains n has length λ_s. So σ is of H-orbit type $(\lambda; \lambda_s)$. Furthermore, σ does not occur in the expansion of any other summand of $X(\lambda; \lambda_s)$. Finally, a given λ_j-cycle occurs at most once in the expansion of $(L_{i_j})^{\lambda_j - 1}$. We conclude that σ occurs exactly once in the expansion of $(L_{i_1})^{\lambda_1 - 1}(L_{i_2})^{\lambda_2 - 1} \cdots (L_{i_r})^{\lambda_r - 1}$.

Partially order the set $\{A(\lambda; \lambda_s)\}$ according to where the elements occur in the filtration $U_{n+1} \subseteq U_n \subseteq \cdots \subseteq U_0 = \mathbf{Z}G^H$. Apply the same partial order to the set $\{X(\lambda; \lambda_s)\}$. The previous paragraph shows that the transition matrix between these ordered sets is unitriangular. Since $\mathbf{Z}G^H$ is freely generated as a $\mathbf{Z}$-module by the elements of $\{A(\lambda; \lambda_s)\}$, it is also freely generated by the elements of $\{X(\lambda; \lambda_s)\}$. This completes the proof. $\square$

Propositions 2.1 and 2.3 overlap with earlier work of Olshanski [20]. A convenient reference for Olshanski's theorem is the paper [19] of Okounkov and Vershik. Theorem 4.1 of [19] states that, if $\mathbf{C}$ denotes the complex numbers and l is any positive integer less than n, the algebra $(\mathbf{C}S_n)^{S_l}$ is generated by the center of $\mathbf{C}S_{n-l}$, the group that permutes the numbers $\{n-l+1, n-l+2, \ldots, n\}$, and the Jucys–Murphy elements $L_{n-l+1}, L_{n-l+2}, \ldots, L_n$. The proof they indicate seems to work when $\mathbf{C}$ is replaced with any commutative ring.

3. A first main theorem

As we saw in Theorem 2.2, the blocks of $kS_n^{S_{n-1}}$ are in bijection with pairs (e, f), where e is a primitive central idempotent of kS_n, f is a primitive central idempotent of kS_{n-1}, and $ef \neq 0$. We will see that it is possible to recognize p-locally whether such a product is 0. This leads to a version of Brauer's first main theorem for the algebra $kS_n^{S_{n-1}}$, and to a branching rule for blocks.

The Brauer map applied to block idempotents of the symmetric group has a very simple combinatorial interpretation, which we now describe. Any result about the symmetric group for which we do not give a precise reference may be found in [14].

Recall that irreducible complex characters of S_n are parameterized by partitions of n represented by Young diagrams. The characters corresponding to partitions λ and μ belong to the same block if and only if the Young diagrams corresponding to λ and μ have the same p-core. The number of rim p-hooks removed to obtain the p-core is the *weight* of the block. Of course $wp \leqslant n$, where w is the weight of a block.

Now let w be any positive integer with $wp \leqslant n$. Let D_w be a Sylow p-subgroup of S_{wp}, identified with a subgroup of S_n in the obvious way. The group D_w is a defect group of any block of S_n of weight w.

Let Ω be the set $\{wp+1, wp+2, \ldots, n\}$, and let S_Ω be the subgroup of S_n consisting of all permutations that act as the identity outside of Ω. Of course $S_\Omega \cong S_{n-wp}$. The centralizer of D_w is a direct product

$$C_{S_n}(D_w) = C_{S_{wp}}(D_w) \times S_\Omega \cong C_{S_{wp}}(D_w) \times S_{n-pw}.$$

The group $C_{S_{wp}}(D_w)$ contains a normal p-subgroup P such that $C_{C_{S_{wp}}(D_w)}(P) \subseteq P$. (See (2.6) in [6].) Hence the algebra $kC_{S_{wp}}(D_w)$ has just one block. Since $kC_{S_n}(D_w) \cong kC_{S_{wp}}(D_w) \otimes kS_{n-pw}$, it follows that the blocks of $kC_{S_n}(D_w)$ correspond one-to-one with the blocks of kS_{n-pw}.

Similarly,

$$N_{S_n}(D_w) = N_{S_{wp}}(D_w) \times S_\Omega \cong N_{S_{wp}}(D_w) \times S_{n-pw},$$

the algebra $kN_{S_{wp}}(D_w)$ has just one block, and the blocks of $kN_{S_n}(D_w)$ correspond one-to-one with the blocks of kS_{n-pw}. The central idempotents of the algebras $kN_{S_n}(D_w)$, $kC_{S_n}(D_w)$, and kS_Ω are the same and may be identified with the block idempotents of S_{n-pw}. Thus the blocks of $kN_{S_n}(D_w)$ and $kC_{S_n}(D_w)$ are parametrized by cores of Young diagrams corresponding to partitions of $n - pw$.

Now we turn to the Brauer map. Let Br_{D_w} be the k-linear map $kS_n \to kC_{S_n}(D_w)$ such that for any $\sigma \in S_n$, $\mathrm{Br}_{D_w}(\sigma) = \sigma$ if $\sigma \in C_{S_n}(D_w)$ and $\mathrm{Br}_{D_w}(\sigma) = 0$ if $\sigma \notin C_{S_n}(D_w)$. Let e be a block idempotent of S_n with weight v. If $v < w$, then $\mathrm{Br}_{D_w}(e) = 0$; if $v \geqslant w$, then $\mathrm{Br}_{D_w}(e)$ is the block idempotent of $kC_{S_n}(D_w)$ (or of $kN_{S_n}(D_w)$) that corresponds to the same p-core as e. When $\mathrm{Br}_{D_w}(e)$ is identified with a block of kS_{n-pw}, its weight is $v - w$.

The next proposition shows how to determine p-locally whether a product ef is 0, where e and f are block idempotents of kS_n and kS_{n-1}, respectively.

Proposition 3.1. *Let $G = S_n$ and let $H = S_{n-1}$. Let e be a primitive central idempotent of kG with defect group P. Let f be a primitive central idempotent of kH with defect group Q. Assume that P and Q have been chosen so that $P \subseteq Q$ or $Q \subseteq P$.* (*The description of defect groups above shows that this is always possible.*) *Let D be the smaller of P or Q. Then $ef \neq 0$ if and only if $\mathrm{Br}_D(e)\,\mathrm{Br}_D(f) \neq 0$.*

Proof. Since $\mathrm{Br}_D(ef) = \mathrm{Br}_D(e)\,\mathrm{Br}_D(f)$, one direction is trivial. We only need to prove that if $ef \neq 0$, then $\mathrm{Br}_D(e)\,\mathrm{Br}_D(f) \neq 0$.

Assume that $ef \neq 0$. Let w be the smaller of the weights of these blocks, so that $D = D_w$. Since $ef \neq 0$, there are irreducible characters χ and θ belonging to the blocks corresponding to e and f such that θ is a constituent of χ_H. As above, $\mathrm{Br}_D(e)$ and $\mathrm{Br}_D(f)$ correspond to blocks of S_{n-pw} and S_{n-pw-1}, blocks that have the same cores as e and f. We will exhibit irreducible characters ϕ and ψ belonging to these blocks such that ψ is a constituent of $\phi_{S_{n-pw-1}}$.

Let λ be the partition of n corresponding to χ and let μ be the partition of $n - 1$ corresponding to θ. So the Young digram for μ is obtained from λ by removing a node.

It will be convenient to represent a partition using an abacus with p-runners, labelled from 0 to $p - 1$. We will use the same letter to represent the abacus diagram as we use to represent the partition. Assume that the number of beads has been chosen so that μ is obtained from λ by moving one bead, labelled m, one position left, from runner i to runner $i - 1$. Consider the abacus representing λ. On runner i, let x be the number of beads above m, let v be the number of empty positions above m, and let t be the number of beads below m; on runner $i - 1$, let y be the number of beads above the empty space left of m, let u be the number of empty positions above that space, and let s be the number of beads below that space. Then

$$\mathrm{weight}(\mu) = \mathrm{weight}(\lambda) + t - s + u - v,$$

as we now show. The weight is the number of times a bead must be moved up one position to obtain a core. When m has been moved left, the beads on i that were below it can each move up one more position, increasing the weight by t. On the other hand, the beads on $i-1$ below m are now blocked by m and can move one position less; this decreases the weight by s. The other changes are accounted for by the bead m. When a core is obtained from μ, m moves u spaces; when a core is obtained from λ, m moves v spaces.

Since $y+u=v+x$ (both are the number of filled and empty positions above m), it follows that

$$\mathrm{weight}(\lambda)-\mathrm{weight}(\mu)=(y+s)-(x+t). \tag{1}$$

There are two cases we must consider. First, assume that $Q=D$. Then the idempotent $\mathrm{Br}_D(f)$ corresponds to a block of S_{n-pw-1} of defect zero. Let ψ be the unique irreducible character in this block, and let δ be the corresponding p-core of $n-pw-1$. Now δ is also the p-core of μ. So its $i-1$th runner contains $y+s+1$ beads, and its ith runner contains $x+t$ beads. But $\mathrm{weight}(\lambda)\geqslant\mathrm{weight}(\mu)$, as $Q=D$. It follows from (1) that $y+s+1>x+t$. In other words, in the abacus representing δ, runner $i-1$ has more beads than runner i. Since all the beads are as far up as they can go, there is an empty space to the right of the lowest bead on runner $i-1$. Moving that bead right gives an abacus representing a partition of $n-pw$. Let θ be the corresponding irreducible character of S_{n-pw}. Then by construction, ψ is a constituent of the restriction of θ to S_{n-pw-1}. Also, θ has the same p-core as λ. So θ belongs to the block corresponding to $\mathrm{Br}_D(e)$. This completes the first case of the proof.

The proof in the second case, when $P=D$, is similar. All the abacus diagrams are mirror images of the abacus diagrams in the first case. □

Proposition 3.1 has the following combinatorial interpretation. As mentioned in the introduction, this should be thought of as part of a branching rule for blocks.

Corollary 3.2. *Let k be a field of characteristic p. Let e be a primitive central idempotent of kS_n, corresponding to the p-core γ. Let f be a primitive central idempotent of kS_{n-1}, corresponding to the p-core δ. Then $ef\neq 0$ if and only if δ is the core of a partition obtained from γ by removing a node or γ is the core of a partition obtained from δ by adding a node.*

Combining Theorem 2.2 and Proposition 3.1, we obtain a parametrization of the blocks of $kS_n^{S_{n-1}}$.

Corollary 3.3. *Let $G=S_n$ and let $H=S_{n-1}$. Let k be a field of characteristic p. The blocks of kG^H are in bijection with the set of all pairs (γ,δ) of p-cores that satisfy the following three conditions*:

(1) *γ is a partition of a non-negative integer l such that $l\leqslant n$ and $l\equiv n \pmod p$.*
(2) *δ is a partition of a non-negative integer m such that $m\leqslant n-1$ and $m\equiv (n-1) \pmod p$.*

(3) *δ is the core of a partition obtained from γ by removing a node, or γ is the core of a partition obtained from δ by adding a node.*

Whenever k is a field of characteristic p, G is a finite group, and H is a subgroup of G, Green's theory defines defect groups in H for primitive idempotents of the algebra kG^H. The defect groups of the primitive idempotent ε are the minimal subgroups D of H such that $\varepsilon = \mathrm{Tr}_D^H(\alpha)$ for some element $\alpha \in kG^D$. Since the defect groups of ε are also the maximal p-subgroups D of H such that $\mathrm{Br}_D(\varepsilon) \neq 0$, Proposition 3.1 can be used to find the defect groups of the primitive idempotents of $kS_n^{S_{n-1}}$. (Section 2 of [4] gives a good approach to Green's theory in the form we need it.)

Corollary 3.4. *Let $G = S_n$ and let $H = S_{n-1}$. Let e be a primitive central idempotent of kG with defect group P. Let f be a primitive central idempotent of kH with defect group Q. Assume that P and Q have been chosen so that $P \subseteq Q$ or $Q \subseteq P$. Let D be the smaller of P or Q. Assume that $ef \neq 0$. Then the group D is a defect group in H of the primitive idempotent ef of kG^H.*

Proof. Let E be a defect group in H of ef. By (2.6) in [4], since $\mathrm{Br}_D(ef) \neq 0$, we have $D \subseteq_H E$. We will be finished when we have shown that $|E| \leqslant |D|$.

First, assume that $D = Q$. There is an element $\alpha \in kH^D$ such that $f = \mathrm{Tr}_D^H(\alpha)$. Then $ef = e\,\mathrm{Tr}_D^H(\alpha) = \mathrm{Tr}_D^H(e\alpha)$. So $E \subseteq_H D$.

Next, assume that $D = P$. There is an element $\beta \in kG^D$ such that $e = \mathrm{Tr}_D^G(\beta)$. So $e = \sum_x \mathrm{Tr}_{D^x \cap H}^H(\beta^x)$, where x runs through representatives for (D, H)-double cosets in G. It follows that $ef \in \sum_x \mathrm{Tr}_{D^x \cap H}^H(kG^{D^x \cap H})$. Since ef is primitive, Rosenberg's lemma implies that there is an x such that $ef \in \mathrm{Tr}_{D^x \cap H}^H(kG^{D^x \cap H})$. Therefore there is an x such that $E \subseteq_H D^x \cap H$. Hence $|E| \leqslant |D|$, as we wanted. □

Proposition 3.1 also allows us to obtain an analog of Brauer's First Main Theorem. (Compare this to Theorem 6.2 in [11].)

Theorem 3.5. *Let $G = S_n$ and let $H = S_{n-1}$. Let k be a field of characteristic p. Let D be a p-subgroup of H. The Brauer map Br_D gives a bijection from the set of all primitive idempotents of kG^H with defect group D to the set of all primitive idempotents of $kN_G(D)^{N_H(D)}$ with defect group D.*

Proof. This follows easily from 3.1, 3.4, the description of $N_G(D)$ before 3.1 and the combinatorial interpretation of the Brauer map from the discussion preceding Proposition 3.1. □

4. Modules over $k[H \times G]$

One approach to blocks of group algebras views blocks of kG as right modules over the algebra $k[G \times G]$, with the action $a(x, y) = x^{-1}ay$ for all x and y in G. For example, the book [3] takes this point of view throughout. In this section, we explore a similar approach to blocks of kG^H.

Let e be a primitive central idempotent of kG, and let f be a primitive central idempotent of kH. Consider the right $k[H \times G]$-module $fkGe$. If $a \in efkG^H$, then multiplication from the left by a is a $k[H \times G]$-module endomorphism of $fkGe$ that sends fe to a. Since any $k[H \times G]$-module endomorphism of $fkGe$ is determined by the image of fe, and since the image of fe must be an element of $efkG^H$, it follows that every element of $\mathrm{End}_{k[H\times G]}(fkGe)$ arises in this way. If we write endomorphisms of right modules on the left, then the resulting map $efkG^H \to \mathrm{End}_{k[H\times G]}(fkGe)$ is an algebra isomorphism. When $G = S_n$ and $H = S_{n-1}$, Theorem 2.2 shows that $efkG^H$ has just one idempotent, so $fkGe$ is an indecomposable $k[H \times G]$-module. The next theorem determines the vertex and Green correspondent of this module. Together with results from Section 6 of [11], it suggests that there may be a sort of lopsided block theory similar to the usual symmetric one.

Theorem 4.1. *Let $G = S_n$ and let $H = S_{n-1}$. Let k be a field of characteristic p. Let e be a primitive central idempotent of kG with defect group P and let f be a primitive central idempotent of kH with defect group Q. Assume that P and Q have been chosen so that $P \subseteq Q$ or $Q \subseteq P$. Let D be the smaller of P or Q. Assume that $ef \neq 0$.*

(1) *The diagonal group $\delta(D)$ is a vertex of the indecomposable right $k[H \times G]$-module $fkGe$.*
(2) *Let $\mathcal{F}$ denote the Green correspondence with respect to $(H \times G, N_H(D) \times N_G(D), \delta(D))$. Then*

$$\mathcal{F}(fkGe) = \mathrm{Br}_D(f)kN_G(D)\,\mathrm{Br}_D(e).$$

Before we can give a proof of Theorem 4.1, we need the following result. It holds quite generally, not just for the symmetric group. The proof is similar to the proof in the classical case when $H = G$, but some care must be taken.

Proposition 4.2. *Let G be any finite group, let H be a subgroup of G, and let k be a field of characteristic p. Assume that D is a p-subgroup of H. Let e and f be primitive central idempotents of kG and kH, respectively. Then the $k[N_H(D) \times N_G(D)]$-module $\mathrm{Br}_D(f)kN_G(D)\,\mathrm{Br}_D(e)$ is isomorphic to a direct summand of the restriction $(fkGe)_{k[N_H(D)\times N_G(D)]}$.*

Note that this proposition does not say that $\mathrm{Br}_D(f)kN_G(D)\,\mathrm{Br}_D(e)$ is nonzero.

For the proof of Proposition 4.2, we will need the following lemma. It has appeared implicitly in the literature. See [1], for instance. If S is a subset of the group G, then S^+ denotes the sum in the group algebra kG of all the elements of S.

Lemma 4.3. *Let P be a normal p-subgroup of a finite group G and let H be a subgroup of G that contains P. Suppose that C is an H-orbit in $G \setminus C_G(P)$. Then $C^+ \in J(kG^H)$.*

Proof. Let x be an element of G that is not in $C_G(P)$. Let X be the orbit of x under the conjugation action of P. Assume that M is a simple kG-module. Since $P \lhd G$ and k has characteristic p, every element of P acts as the identity on M. It follows that if x and y are in X and m is in M, then $xm = ym$. Since $x \notin C_G(P)$, p divides the order of X. Therefore X^+ acts as 0 on M. This holds for every simple kG-module M, so $X^+ \in J(kG)$.

It follows that if C is as in the statement of the lemma, then $C^+ \in J(kG)$. Since $J(kG) \cap kG^H$ is a nilpotent two-sided ideal of kG^H, $J(kG) \cap kG^H \subseteq J(kG^H)$. The result follows. $\square$

Proof of Proposition 4.2. Let $e_1 := \mathrm{Br}_D(e)$ and let $f_1 := \mathrm{Br}_D(f)$. The Brauer map Br_D restricts to a k-algebra homomorphism $kG^D \to kC_G(D)$. It follows that $\mathrm{Br}_D(fe) = f_1e_1$. So we may write

$$fe = f_1e_1 + a + b,$$

where $a \in k[N_G(D) \setminus C_G(D)]$ and $b \in k[G \setminus N_G(D)]$. Since $e, f, e_1, f_1 \in kG^{N_H(D)}$, and $k[G \setminus N_G(D)]$ is stable under $N_H(D)$-conjugation, a lies in $kN_G(D)^{N_H(D)}$.

Now Lemma 4.3 implies that $a \in J(kN_G(D)^{N_H(D)})$. It follows that $f_1e_1 + af_1e_1$ is a unit in $f_1e_1kN_G(D)^{N_H(D)}$. Let u be its inverse. Note that $u \in kN_G(D)^{N_H(D)}$.

Let $\phi : f_1kN_G(D)e_1 \to fkGe$ be the map given by $\phi(x) = fxe = fex$. Let $\pi : kG \to kN_G(D)$ be the projection onto $kN_G(D)$ with kernel $k[G \setminus N_G(D)]$. Let $\psi : kN_G(D) \to kN_G(D)$ be the map given by $\psi(y) = uy$. All three of these maps are $k[N_H(D) \times N_G(D)]$-module homomorphisms. Let $x \in f_1kN_G(D)e_1 = f_1e_1N_G(D)$. Then $\pi\phi(x) = \pi(fex) = (f_1e_1 + a)x = (f_1e_1 + af_1e_1)x$. It follows that $\psi\pi\phi(x) = u(f_1e_1 + af_1e_1)x = x$. Therefore $f_1kN_G(D)e_1$ is isomorphic to a direct summand of the restriction of $fkGe$ to $N_H(D) \times N_G(D)$. $\square$

Proof of Theorem 4.1. First, we will show that $\delta(D)$ is a vertex of the module $\mathrm{Br}_D(f)kN_G(D)\,\mathrm{Br}_D(e)$. Let $e_1 = \mathrm{Br}_D(e)$, $f_1 = \mathrm{Br}_D(f)$, $G_1 = N_G(D)$, and $H_1 = N_H(D)$. Let P_1 be a defect group of e_1 and let Q_1 be a defect group of f_1. Note that $D = Q_1$ and $D \lhd P_1$, or $D = P_1$ and $D \lhd Q_1$. By III8.7 in [13] (a result due to Green), the $k[H_1 \times G_1]$-module $f_1kG_1e_1$ is $Q_1 \times P_1$-projective. Therefore there is an indecomposable summand of the restriction $(f_1kG_1e_1)_{Q_1\times P_1}$ that shares a vertex with $(f_1kG_1e_1)_{H_1\times G_1}$. By III8.3 and III8.1 in [13], it follows that there is a $z \in G_1$ such that $\delta(Q_1^z \cap P_1)$ is a vertex of $(f_1kG_1e_1)_{H_1\times G_1}$. Conjugating this vertex by (z^{-1}, z^{-1}), we see that $\delta(Q_1 \cap zP_1z^{-1})$ is also a vertex. Recall that $D = Q_1$ or $D = P_1$. In the case with $D = Q_1$, we have $Q_1 \lhd G_1$ and $Q_1 \subseteq P_1$; hence $\delta(Q_1^z \cap P_1) = \delta(D)$. In the case with $D = P_1$, we have $P_1 \lhd G_1$ and $P_1 \subseteq Q_1$; hence $\delta(Q_1 \cap zP_1z^{-1}) = \delta(D)$. Thus $\delta(D)$ is a vertex of the $k[Q_1 \times P_1]$-module $f_1kG_1e_1$.

By Proposition 4.2, the $k[H_1 \times G_1]$-module $f_1kG_1e_1$ is a direct summand of the restriction $(fkGe)_{H_1\times G_1}$. Since $\delta(D)$ is a vertex of $f_1kG_1e_1$, it follows from the Burry–Carlson–Puig theorem (3.12.3 in [5]) that $\delta(D)$ is a vertex of $(fkGe)_{H\times G}$ and that $\mathcal{F}(fkGe) = f_1kG_1e_1$. $\square$

5. The dimension of a block

We will prove in this section that the dimension of a block of $kS_n^{S_{n-1}}$ depends only on the unordered pair of weights of the associated blocks of kS_{n-1} and kS_n. This result is an immediate consequence of Theorem 6.1, which shows that the isomorphism type of a block of $kS_n^{S_{n-1}}$ depends only on the unordered pair of weights of the associated blocks of kS_{n-1} and kS_n. We give a direct proof here, as it is conceptually simpler, and also as it allows us to derive a formula for the dimension of a block.

We begin by recalling some facts about the blocks of kS_n. It is known that the dimension $k_p(w)$ of a p-block of a symmetric group depends only on its weight w and on the prime p. Let $p(n)$ denote the number of partitions of n. The partition generating function is

$$P(x) = \sum_{i=0}^{\infty} p(n)x^n = \prod_{i\geqslant 0}\left(1 - x^i\right)^{-1}. \tag{2}$$

For any positive integer l, let $K(l;x)$ be the power series $P(x)^l$. J.B. Olsson showed in [21] that the dimension of a p-block of a symmetric group that has weight w is the coefficient $k_p(w)$ of x^w in the power series $K(p;x)$. For convenience, we set $k_p(w) = 0$, for all $w < 0$.

Lemma 5.1. *Let (w_1, w_2) be a pair of nonnegative integers, and let k be a field of characteristic 2. Then there is a unique positive integer n such that kS_n has a block idempotent e of weight w_1, kS_{n-1} has a block idempotent f of weight w_2, and $ef \neq 0$.*

Proof. Set $n_t = t(t+1)/2$, for $t \geqslant 0$. Let λ_t be the partition $[t, t-1, \ldots, 2, 1]$ of n_t. Then λ_t is a 2-core, every 2-core has this form, and Nakayama's conjecture shows that the 2-blocks of S_n are indexed by

$$\{\lambda_t \mid t \geqslant 0, n - n_t \text{ is positive and even}\}.$$

Set $n = t_{w_1-w_2-2} + 2w_1 = t_{w_2-w_1+1} + 2w_1$, and notice that $n - 1 = t_{w_1-w_2} + 2w_2 = t_{w_2-w_1-1} + 2w_2$. It follows that kS_n has a block idempotent e of weight w_1 and kS_{n-1} has a block idempotent f of weight w_2. If $w_1 - w_2 - 1 > 0$, then f has core $\lambda_{w_1-w_2}$. Removing a node from $\lambda_{w_1-w_2}$ and take cores, we obtain the core $\lambda_{w_1-w_2-2}$ of e. If $w_1 - w_2 - 1 \leqslant 0$, then e has core $\lambda_{w_2-w_1+1}$. Removing a node from $\lambda_{1+w_2-w_1}$ and take cores, we obtain the core $\lambda_{w_2-w_1-1}$ of e. So in all cases $ef \neq 0$.

Now suppose that n is any positive integer such that kS_n has a primitive central idempotent e of weight w_1, kS_{n-1} has a primitive central idempotent f of weight w_2, and $ef \neq 0$. Let γ be the core of e. Equation (1) shows that γ can be represented on an abacus with two runners, such that the difference in the number of beads on runners 0 and 1 is $w_1 - w_2 - 1$. If $w_1 - w_2 - 1 > 0$, then $\gamma = \lambda_{w_1-w_2-1}$, while if $w_1 - w_2 - 1 \leqslant 0$, then $\gamma = \lambda_{w_2-w_1+1}$. In all cases n, and hence e and f, are determined by $n = |\gamma| + 2w_1$. $\square$

We let $B(w_1, w_2)$ denote the block $efkS_n^{S_{n-1}}$ given by the above lemma, and we let $k_2(w_1, 1_2)$ denote the dimension of this block as a vector space over k.

Let e be a central primitive idempotent in kS_n. The irreducible characters in the corresponding block of S_n are indexed by the partitions of n that have the same p-core as e. We shall say that any such partition belongs to (or is in) e. As each partition is determined by its p-core and p-quotient, the partitions belonging to e are indexed by their p-quotients.

Now suppose that f is a central primitive idempotent in kS_{n-1}, with $ef \neq 0$. If λ is a partition in e, and μ is a partition in f, and μ can be obtained by removing a node from λ, we shall say that (λ, μ) belongs to ef, or to the block $efkS_n^{S_{n-1}}$. By the discussion in Section 1, the dimension of $efkS_n^{S_{n-1}}$ equals the number of pairs (λ, μ) that belong to ef.

Lemma 5.2. *For all integers w_1, w_2, the blocks $B(w_1, w_2)$ and $B(w_2, w_1)$ have the same dimension.*

Proof. Let (λ, μ) be a pair of partitions belonging to $B(w_1, w_2)$. Represent λ and μ on abacus diagrams with two runners and t beads, for some $t > 0$. Let $\hat{\lambda}$ be the partition obtained by transposing the runners of λ, and let $\hat{\mu}$ be the partition obtained by transposing the runners of μ. Now $\hat{\lambda}$ belongs to a block of weight w_1, and $\hat{\mu}$ belongs to a block of weight w_2. Since μ is obtained by removing a node, say node i, from λ, the ith β-number of μ is one less than the ith β-number of λ. The construction ensures that the ith β-number of $\hat{\mu}$ is one more than the ith β-number of $\hat{\lambda}$. So $\hat{\mu}$ is obtained by adding a node to $\hat{\lambda}$. It follows that $(\hat{\mu}, \hat{\lambda})$ belongs to $B(w_2, w_1)$. The association

$$(\lambda, \mu) \leftrightarrow (\hat{\mu}, \hat{\lambda})$$

is involutary. We conclude that the dimension of $B(w_1, w_2)$ is equal to the dimension of $B(w_2, w_1)$. $\square$

We can now prove the main result of this section.

Theorem 5.3. *Let $G = S_n$, $H = S_{n-1}$ and let k be a field of characteristic p. Let e be a block idempotent of kG, of weight w_e, and let f be a block idempotent of kH, of weight w_f. Assume that $ef \neq 0$. Then the dimension of the block $efkG^H$ depends only on the unordered pair of weights $\{w_e, w_f\}$.*

Proof. Choose $t > 0$, such that if partitions are represented on abacus diagrams with p runners and t beads, and (λ, μ) is a pair of partitions in ef, then μ is obtained from λ by moving a bead on runner 1 into an empty position due left on runner 0.

Let i be an integer with $0 \leqslant i \leqslant w_e$. Fix a $(p-2)$-tuple $(\rho^2, \dots, \rho^{p-1})$ of partitions with $\sum_{j=2}^{p-1} |\rho^j| = w_e - i$. Let (λ, μ) be a pair of partitions in ef such that $\lambda^j = \rho^j$, for $j = 2, \dots, p-1$. Then $|\lambda^0| + |\lambda^1| = i$. The first two runners of λ represent a certain partition, call it $\tilde{\lambda}$, on an abacus with two runners. Similarly the first two runners of μ represent a partition $\tilde{\mu}$. Now $\tilde{\lambda}$ has weight i. So $\tilde{\mu}$ has weight $i - (w_e - w_f)$, using Eq. (1) from the proof of Proposition 3.1. It is clear that as (λ, μ) range over all such pairs, $(\tilde{\lambda}, \tilde{\mu})$ range over all pairs of partitions belonging to the block $B(i, i - (w_e - w_f))$.

The number of $(p-2)$-tuples $(\rho^2, \ldots, \rho^{p-1})$ equals $k_{p-2}(w_e - i)$. Each $(p-2)$-tuple gives rise to $k_2(i, i-(w_e - w_f))$ different pairs (λ, μ) belonging to ef. As i varies between 0 and w_e, we obtain every pair of partitions in ef exactly once. It follows that

$$\dim\bigl(efkG^H\bigr) = \sum_{i=0}^{w_e} k_{p-2}(w_e - i)k_2\bigl(i, i-(w_e - w_f)\bigr). \tag{3}$$

Substituting $j = i - (w_e - w_f)$ in these equations, and using Lemma 5.2, we see that we can transpose w_e and w_f, without changing the dimension. The theorem follows. $\square$

Equation (3) in the proof of Theorem 5.3 can be interpreted in terms of power series. Define $k_p(n,m) := \dim(efkG^H)$, whenever e has weight n and f has weight m. Form the following power series in the variables x and y.

$$K(p; x, y) := \sum_{n \geqslant 0, m \geqslant 0} k_p(n,m)x^n y^m.$$

Now recall (from Eq. (2) at the beginning of this section) that $K(l; x)$ is the power series $P(x)^l$. Let $K(l; xy)$ be its evaluation at xy. Then Eq. (3) can be interpreted as showing that, as power series in x and y,

$$K(p; x, y) = K(p-2; xy) \times K(2; x, y).$$

Note that there is no comma between x and y in $K(p-2; xy)$.

We end the section by giving a precise formula for $k_2(w_1, w_2)$. This, together with Theorem 5.3, can be used to compute $k_p(w_1, w_2)$, for any pair of nonnegative integers w_1, w_2, and any prime p.

Proposition 5.4. *Suppose that $w_1 \geqslant w_2$. Then the dimension $k_2(w_1, w_2)$ of the block $B(w_1, w_2)$ is given by* (*the finite sum*)

$$k_2(w_1, w_2) = \sum_{i=0}^{\infty} \bigl((w_1 - w_2) + (2i+1)\bigr)k_2\bigl(w_2 - i(w_1 - w_2) - i(i+1)\bigr).$$

Proof. Let $B(w_1, w_2) = e_1 e_2 k S_n^{S_{n-1}}$, where k is a field of characteristic 2, e_1 is a central primitive idempotent in kS_n, of weight w_1, and e_2 is a central primitive idempotent in kS_{n-1}, of weight w_2. Proposition 3.1 shows that the core of e_1 is the core of a partition obtained by adding a node to the core of e_2. Let (λ_1, λ_2) be a pair of partitions in $B(w_1, w_2)$. Represent λ_1 on an abacus with two runners, with the number t of beads chosen so that λ_2 is obtained from λ_1 by moving one bead on runner 1 into an empty space to its left on runner 0. Let a, respectively, b, denote the number of beads on runners 0 and 1 of λ_2. Then (1) shows that $w_1 - w_2 = a - b - 1$.

Suppose that there exists a central primitive idempotent e_3 in kS_{n+1}, different to e_1, such that $e_3 e_2 \neq 0$. Let (λ_3, λ_4) be a pair of partitions in $e_3 e_2$. Represent λ_3 on an abacus with two runners and t beads. Using Proposition 3.1, we see that λ_4 is obtained from λ_3 by moving a bead on runner 0 into an empty position one space up and to its right on runner 1. In particular, λ_3 has $a+1$ beads on runner 0 and $b-1$ beads on runner 1. Let w_3 be the weight of e_3. Modifying the proof of (1), it can be shown that $w_3 - w_2 = (b-1) - (a+1) + 2 = b - a$. Combining this with the equality at the end of the previous paragraph, we see that $w_3 = w_2 - (w_1 - w_2) - 1$. Notice that $w_3 < w_2$.

The 2-core of e_1 is the 2-core of a block idempotent e_4 in kS_{n-2} that has weight $w_1 - 1$, while the 2-core of e_3 is the 2-core of a block idempotent e_5 in kS_{n-2} that has weight $w_3 - 1$ (if the idempotents exist). Moreover, we have $e_2e_4 \neq 0$ and $e_2e_5 \neq 0$, and e_4, e_5 are the only block idempotents in kS_{n-2} that do not annihilate e_2.

We now count irreducible characters. There are $k_2(w_2)$ partitions that belong to e_2. Let μ be one such. The previous paragraphs show that adding a node to μ produces a partition λ such that (λ, μ) belongs to $B(w_1, w_2)$ or to $B(w_2 - (w_1 - w_2) - 1, w_2)$, while removing a node from μ produces a partition ρ such that (μ, ρ) belongs to $B(w_2, w_1 - 1)$ or to $B(w_2, w_2 - (w_1 - w_2) - 2)$. But (as is well known) μ has one more addable node than removable node. It follows (on using Lemma 5.2) that

$$k_2(w_1, w_2) = k_2(w_2) + k_2(w_1 - 1, w_2) + k_2\bigl(w_2, w_2 - (w_1 - w_2) - 2\bigr) \\ - k_2\bigl(w_2, w_2 - (w_1 - w_2) - 1\bigr).$$

If we apply this formula to the $(w_1 - w_2) + 1$ numbers $k_2(w_1 - i, w_2)$, for $i = 0, \ldots, (w_1 - w_2)$, then add and cancel equal terms of opposite sign, we obtain

$$k_2(w_1, w_2) = (w_1 - w_2 + 1)k_2(w_2) + k\bigl(w_2, w_2 - (w_1 - w_2) - 2\bigr).$$

Applying this formula repeatedly to the summand on the extreme right, we obtain the statement of the proposition. $\square$

6. The isomorphism type of a block

We show in this section that the isomorphism type of the blocks of $kS_n^{S_{n-1}}$ depends only on the unordered pair of weights of the associated blocks of kS_{n-1} and kS_n. In fact, we prove the corresponding statement over a suitable valuation ring of characteristic zero.

We need the notion of perfect isometries between blocks of finite groups. Our primary reference is [7]. For this discussion, let G_1 and G_2 be arbitrary finite groups, and let $(F, \mathcal{O}, k)$ be a splitting p-modular system for both G_1 and G_2. So $F = \mathrm{Frac}(\mathcal{O})$ has characteristic 0, $\mathcal{O}$ is a discrete valuation ring with maximal ideal (π), and $\mathcal{O}/(\pi) = k$. For $i = 1, 2$, let e_i be an idempotent in $Z\mathcal{O}G_i$. We use $\mathcal{R}_F(G_i, e_i)$ to denote the Grothendieck group of the abelian category of left e_iFG_i modules. The elements of $\mathcal{R}_F(G_i, e_i)$ are formal $\mathbf{Z}$-linear combinations of the irreducible characters of e_iFG_i, and the group operation is formal addition of characters.

Let $I:\mathcal{R}_F(G_1,e_1)\to\mathcal{R}_F(G_2,e_2)$ be a group homomorphism, and $R:\mathcal{R}_F(G_2,e_2)\to\mathcal{R}_F(G_1,e_1)$ be its adjoint, with respect to the usual inner products on Ze_1FG_1 and Ze_2FG_2. Set

$$\mu=\sum_{\chi\in\operatorname{Irr}(e_1FG_1)}\chi\otimes I(\chi).$$

Then μ is a $\mathbf{Z}$-linear combination of characters of $(e_1FG_1,\ e_2FG_2)$-bimodules. The map I is an isometry if, for each $\chi\in\operatorname{Irr}(e_1FG_1)$, there is a sign $\varepsilon(\chi)\in\{\pm1\}$, such that $\varepsilon(\chi)I(\chi)\in\operatorname{Irr}(e_2FG_2)$. Broué calls such an isometry *perfect* if the associated μ satisfies, for all $g_1\in G_1$ and $g_2\in G_2$

- $(\mu(g_1,g_2)/|C_G(g_1)|)\in\mathcal{O}$ and $(\mu(g_1,g_2)/|C_G(g_2)|)\in\mathcal{O}$.
- If $\mu(g_1,g_2)\neq0$, then g_1 is p-regular if and only if g_2 is p-regular.

Let $I:\mathcal{R}_F(G_1,e_1)\to\mathcal{R}_F(G_2,e_2)$ be a perfect isometry. The set $\{e(\chi)\mid\chi$ an irreducible character of $e_1FG_1\}$ form a basis for Ze_1FG_1. It follows from [7, 1.5] that the F-algebra isomorphism $Ze_1FG_1\cong Ze_2FG_2$, induced by $e(\chi)\to e(\varepsilon(\chi)I(\chi))$, restricts to an $\mathcal{O}$-algebra isomorphism, denoted Λ, from $Ze_1\mathcal{O}G_1$ to $Ze_2\mathcal{O}G_2$. Explicitly, if $z\in Z(e_1\mathcal{O}G_1)$, we have $z=\sum_\chi\omega_\chi(z)e(\chi)$, where $\omega_\chi(z)=\chi(z)/\chi(1)$ is the central character associated with χ. Then

$$\Lambda(z)=\sum_\chi\omega_\chi(z)e\bigl(\varepsilon(\chi)I(\chi)\bigr).$$

M. Enguehard [12] has shown that there are many perfect isometries between blocks of finite symmetric groups which have the same weight. We will need some detailed information about how these isometries arise.

Let κ and μ be the cores of two p-blocks A and B of symmetric groups that have the same weights. Enguehard shows that there is a perfect isometry from A to B induced by

$$\lambda\mapsto\alpha(\lambda)\Psi(\lambda),$$

where λ runs through all partitions associated with A, whenever α and Ψ are maps with the following properties:

(1) Ψ maps all partitions with core κ bijectively onto all partitions with core μ.
(2) Associated with each partition λ with core κ, there is a bijective map Φ_λ between the hooks of λ whose length is divisible by p and the corresponding hooks of $\Psi(\lambda)$.
(3) Φ_λ preserves hook lengths.
(4) Ψ sends the partition obtained by removing a hook h from λ to the partition obtained by removing $\Phi_\lambda(h)$ from $\Psi(\lambda)$. Here h is any hook of length divisible by p.
(5) $\alpha(\lambda)$ is a sign $\{+1,-1\}$ such that, for each hook h of length divisible by p,

$$\operatorname{LegPar}(h)\operatorname{LegPar}\bigl(\Phi_\lambda(h)\bigr)=\alpha(\lambda)\alpha(\lambda-h),$$

where $\mathrm{LegPar}(j)$ is the parity of the leg length of the hook j, and $\lambda - h$ is the partition obtained by removing h from λ.

Represent the p-cores κ and μ on abacus diagrams with p runners. Represent all partitions with p-core κ or μ on abacus diagrams, so that the diagram for κ or μ is obtained when the beads are pushed up as far as possible on all runners.

Morris and Olsson [15] define a sign $\delta = \delta_p$ associated with each abacus diagram. Its main property is that if μ is obtained from λ by moving a bead up one position on a runner, then

$$\mathrm{LegPar}(h) = \delta(\mu)\delta(\lambda).$$

Each partition with p-core κ, respectively μ, is represented by the p-quotient associated to its abacus diagram. Identify partitions and p-quotients. Let σ be any permutation of $\{0, 1, \dots, p-1\}$. Then σ acts on the partitions with p-core κ by permuting the corresponding p-quotients. For λ a partition with p-core κ, define

$$\Psi(\lambda) = \text{the partition with } p\text{-core } \mu \text{ and } p\text{-quotient } \sigma(\lambda),$$

and

$$\alpha(\lambda) = \delta(\lambda)\delta\big(\Psi(\lambda)\big).$$

For each λ, let Φ_λ be the obvious identification of the hooks of λ of length divisible by p with the corresponding hooks of $\Psi(\lambda)$.

It is easily checked that Ψ, Φ_λ, and α satisfy all the requirements of Enguehard. In particular, this gives $p!$ different perfect isometries between two blocks of symmetric groups of the same weight.

Theorem 6.1. *Let $G_1 = S_n$, $H_1 = S_{n-1}$ and let $(F, \mathcal{O}, k)$ be a p-modular system. Let e_1 be a block idempotent in $\mathcal{O}G_1$, of weight w_e, and let f_1 be a block idempotent in $\mathcal{O}H_1$, of weight w_f. Assume that $e_1 f_1 \neq 0$. Then the isomorphism type of the block $e_1 f_1 \mathcal{O}G_1^{H_1}$ depends only on the unordered pair of weights $\{w_e, w_f\}$.*

Proof. Let $G_2 = S_m$, $H_2 = S_{m-1}$, where m is any positive integer. Let e_2 be a block idempotent in $\mathcal{O}G_2$, and let f_2 be a block idempotent in $\mathcal{O}H_2$. Assume that $e_2 f_2 \neq 0$. Suppose that the ordered pair of weights of e_2 and f_2 is (w_e, w_f) or (w_f, w_e). We will show that $e_1 f_1 \mathcal{O}G_1^{H_1} \cong e_2 f_2 \mathcal{O}G_2^{H_2}$. First we consider the case that e_2 has weight w_e, and f_2 has weight w_f.

Let $i = 1, 2$. The $\mathcal{O}$-algebra

$$T_i := Ze_i\mathcal{O}G_i \otimes_{\mathcal{O}} Zf_i\mathcal{O}FH_i$$

is embedded in the semisimple algebra $T_i \otimes_{\mathcal{O}} F \cong Ze_i FG_i \otimes_F Zf_i FH_i$. Let I_i be the ideal of $T_i \otimes_{\mathcal{O}} F$ that is generated by the primitive idempotents

$$\{e(\chi) \otimes e(\psi) \mid \chi \in \operatorname{Irr}(e_i FG_i), \psi \in \operatorname{Irr}(f_i FH_i), \text{ and } e(\chi)e(\psi) = 0\}.$$

It follows from Proposition 2.1 that, as $\mathcal{O}$-algebras,

$$e_i f_i \mathcal{O} G_i^{H_i} \cong T_i / I_i \cap T_i.$$

Represent the partitions in e_i and f_i on abacus diagrams with p runners and t_i beads, where t_i is chosen so that if (λ, μ) is a pair of partitions in $e_i f_i$, then μ is obtained from λ by moving a bead on runner 1 into an empty position due left on runner 0.

As described in the paragraphs preceding this theorem, Enguehard has shown that we can choose isomorphisms $\Lambda_e : Ze_1 FG_1 \to Ze_2 FG_2$ and $\Lambda_f : Zf_1 FH_1 \to Zf_2 FH_2$ such that $\Lambda_e(Ze_1 \mathcal{O} G_1) = Ze_2 \mathcal{O} G_2$ and $\Lambda_e(Zf_1 \mathcal{O} H_1) = Zf_2 \mathcal{O} H_2$. These maps induce an isomorphism $\Lambda_e \otimes \Lambda_f : T_1 \otimes F \to T_2 \otimes F$ that restricts to an $\mathcal{O}$-algebra isomorphism $T_1 \to T_2$. To show that $e_1 f_1 \mathcal{O} G_1^{H_1} \cong e_2 f_2 \mathcal{O} G_2^{H_2}$, it is now only necessary to show that Enguehard's isomorphisms can be chosen so that $\Lambda_e \otimes \Lambda_f(I_1) = I_2$.

Choose Λ_e so that at the level of partitions it sends a partition λ_1 in e_1 to the unique partition λ_2 in e_2 that has the same p-quotient as λ_1. (In terms of the discussion preceding this theorem, choose the permutation σ to be the identity.) Similarly, choose Λ_f so that it induces a p-quotient preserving bijection between the partitions in f_1 and the partitions in f_2.

Let $\lambda_1 \leftrightarrow \lambda_2$ be corresponding partitions in e_1, respectively, e_2, and let $\mu_1 \leftrightarrow \mu_2$ be corresponding partitions in f_1, respectively, f_2. Let $i = 1$ or 2. The first two runners of λ_i represent a certain partition $\hat{\lambda}$ on an abacus with two runners. By Eq. (1) in the proof of Proposition 3.1, $w_e - w_f$ determines the relative number of beads on the first two runners of λ_i; therefore $\hat{\lambda}$ does not depend on i. Similarly the first two runners of μ_i determine a partition $\hat{\mu}$. Now (λ_i, μ_i) belongs to $e_i f_i$ if and only if $\hat{\mu}$ can be obtained from $\hat{\lambda}$ by deleting a node. It follows that $\Lambda_e \otimes \Lambda_f(I_1) = I_2$. Hence $T_1/T_1 \cap I_1 \cong T_2/T_2 \cap I_2$.

Now we turn to the other case, when e_2 has weight w_f and f_2 has weight w_e. Only a small modification must be made to the proof. Isomorphisms $\Lambda_e : Ze_1 FG_1 \to Zf_2 FH_2$ and $\Lambda_f : Zf_1 FH_1 \to Ze_2 FG_2$ must be chosen so that on the level of partitions, Λ_e and Λ_f exchange the first two permutations in each p-quotient. (In terms of the discussion preceding this theorem, the permutation σ is the transposition $(0, 1)$.) These isomorphisms induce an isomorphism

$$Ze_1 \mathcal{O} G_1 \otimes Zf_1 \mathcal{O} H_1 \to Zf_2 \mathcal{O} H_2 \otimes Ze_2 \mathcal{O} G_2$$

that sends the kernel of the epimorphism

$$Ze_1 \mathcal{O} G_1 \otimes Zf_1 \mathcal{O} H_1 \to e_1 f_1 \mathcal{O} G_1^{H_1}$$

to the kernel of the epimorphism

$$Zf_2 \mathcal{O} H_2 \otimes Ze_2 \mathcal{O} G_2 \to e_2 f_2 \mathcal{O} G_2^{H_2}.$$

Therefore $e_1 f_1 \mathcal{O}G_1^{H_1} \cong e_2 f_2 \mathcal{O}G_2^{H_2}$. $\square$

7. Blocks and p-regular orbits

We end the paper with some results on the support of the block idempotents of kG^H. Many of these are analogues of results in [17].

If x is an element of kG and if g is an element of G, then $(x \mid g)$ will denote the coefficient with which g occurs in x. Call a H-orbit in G a p-regular orbit if its elements have orders coprime to p.

Proposition 7.1. *Let $G = S_n$, $H = S_{n-1}$ and let k be a field of characteristic 2. Then the 2-regular H-orbit sums form a unital subalgebra of kG^H.*

Proof. Let K and L be 2-regular H-orbits in G and let $g \in G$ be 2-singular. We need to show that $(K^+L^+ \mid g) = 0$.

Now g has at least one even length orbit $\mathbf{O}$ on $\mathbf{N}_n$. We let $S(\mathbf{O})$ and $S(\mathbf{N}_n \backslash \mathbf{O})$ be the pointwise stabilizers of $\mathbf{N}_n \backslash \mathbf{O}$, respectively $\mathbf{O}$, in G. So $S(\mathbf{O}) \times S(\mathbf{N}_n \backslash \mathbf{O})$ is a Young subgroup of G.

Since $g \in S(\mathbf{O}) \times S(\mathbf{N}_n \backslash \mathbf{O})$, we may write, uniquely, $g = cd$, where $c \in S(\mathbf{O})$ and $d \in S(\mathbf{N}_n \backslash \mathbf{O})$. Let u be the 2-part of c, and let s and t be the 2-part, respectively, $2'$-part, of $|\mathbf{O}|$. As c is an $|\mathbf{O}|$-cycle, u is a product of t cycles, each of length s. Thus

$$C_G(u) = Z_s \wr S_t \times S(\mathbf{N}_n \backslash \mathbf{O}).$$

It is not hard to show that, as s and t are coprime, $Z_s \wr S_t = \Delta(Z_s) \times W$, where $\Delta(Z_s)$ is the diagonal subgroup (see Proposition 22 of [17]). It follows that $\langle u \rangle$ has a normal complement, $W \times S(\mathbf{N}_n \backslash \mathbf{O})$ in $C_G(u)$.

Suppose that $\mathbf{O}$ can be chosen so that $n \notin \mathbf{O}$. The Brauer homomorphism $\mathrm{Br}_{\langle u \rangle}$ is the algebra morphism $kG^{\langle u \rangle} \to kC_G(u)$, such that $\mathrm{Br}_{\langle u \rangle}(K^+) = (K \cap C_G(u))^+$, for each $\langle u \rangle$-orbit K in G. Since $n \notin \mathbf{O}$, we have c and hence u are elements of H. So $kG^H \subseteq kG^{\langle u \rangle}$. It follows that

$$\bigl(K^+L^+ \mid g\bigr) = \bigl(\mathrm{Br}_{\langle u \rangle}\bigl(K^+\bigr)\,\mathrm{Br}_{\langle u \rangle}\bigl(L^+\bigr) \mid g\bigr) = \bigl((K \cap C_G(u))^+\bigl(L \cap C_G(u)\bigr)^+ \mid g\bigr).$$

But $K \cap C_G(u)$ and $L \cap C_G(u)$ consist of p-regular elements. So they are contained in the normal complement to $\langle u \rangle$ in $C_G(u)$. We conclude that $(K^+L^+ \mid g) = 0$ in this case.

We may suppose then that g has a single even length orbit on $\mathbf{N}_n$. All 2-regular permutations have sign $+1$, while each even length cycle has sign -1. It follows that each product in K^+L^+ has sign $+1$, while g has sign -1. We conclude that $(K^+L^+ \mid g) = 0$. This completes the proof. $\square$

The proposition is false if p is odd. In fact, $G = S_3$ and $p = 3$ furnishes a counterexample. Let k be a field of characteristic 3. Now $\{(1,2)\}$, $\{(1,3),(2,3)\}$ are

3-regular orbits of S_2 on S_3, while $\{(1,2,3),(1,3,2)\}$ is an orbit whose elements have order 3. However the following equality holds in $kS_3^{S_2}$:

$$(1,2) \times \big((1,3) + (2,3)\big) = (1,2,3) + (1,3,2).$$

Corollary 7.2. *Let $G = S_n$, $H = S_{n-1}$ and let k be a field of characteristic 2. Then the k-span of the 2-regular H-orbit sums coincides with the set of squares of elements of kG^H.*

Proof. Recall that kG^H is the subalgebra of kG that is generated by $Z(kG)$ and $Z(kH)$. The binomial theorem modulo 2 shows that every square of an element of kG^H is a k-linear sum of terms of the form $(K^+)^2(L^+)^2$, where K is a conjugacy class of G and L is a conjugacy class of H. Now [17, Corollary 6] shows that $(K^+)^2$ and $(L^+)^2$ are sums of 2-regular H-orbits. It then follows from Proposition 7.1 that $(K^+)^2(L^+)^2$ lies in the k-span of the 2-regular H-orbit sums.

Now suppose that $(\lambda;\lambda_s)$ is a 2-regular H-orbit type. Then from its definition, and the binomial theorem modulo 2, $X(\lambda;\lambda_s) = X(\mu,\mu_s)^2$, where μ is the partition of n with $\mu_i = (\lambda_i + 1)/2$, for $i = 1,\dots,l(\lambda)$, and with $\mu_i = 1$ or 0, for $i > l(\lambda)$. The $X(\lambda;\lambda_s)$ are linearly independent. It follows that the dimension of the subspace of squares in kG^H is greater than or equal to the number of 2-regular H-orbits. The result now follows from the previous paragraph. □

Corollary 7.3. *Let $G = S_n$, $H = S_{n-1}$ and let k be a field of characteristic 2. Then each idempotent in kG^H lies in the k-span of the 2-regular H-orbit sums.*

Proof. By Theorem 2.2, the primitive idempotents of kG^H have the form ef, where e is a block idempotent of kG, and f is a block idempotent of kH. Now e lies in the span of the 2-regular conjugacy class sums of G, and f lies in the span of the 2-regular conjugacy class sums of H, by a well known result of Osima (see [18, 3.6.22]). The corollary is then an immediate consequence of Proposition 7.1. □

The analogous result is false when p is odd. Let k be a field of characteristic 3. Then the support of each primitive idempotent of $kS_5^{S_4}$ includes 3-singular S_4-orbit sums.

Our final theorem can be used to lift Corollary 7.3 to characteristic zero.

Theorem 7.4. *Let $(F, \mathcal{O}, k)$ be a 2-modular system, and let $G = S_n$, and $H = S_{n-1}$. Then each primitive idempotent in $\mathcal{O}G^H$ is of the form $e - f$, where e is an idempotent in $Z(\mathcal{O}G)$ and f is an idempotent in $Z(\mathcal{O}H)$.*

Proof. Let e_t denote the block idempotent, of $\mathcal{O}G$ or of $\mathcal{O}H$, if it exists, that corresponds to the 2-core $\lambda_t = [t, t-1,\dots,2,1]$. Recall that $n_t = t(t+1)/2$. Let s be the integer determined by $n_s \leqslant n < n_{s+1}$. For notational convenience, set $i - 2$ to be 1, if $i = 0$.

It follows from the branching theorem for blocks, Corollary 3.2, that the primitive idempotents of $\mathcal{O}G^H$ are

$$\{e_{i-2}e_i \mid i = 0, \ldots, s\}.$$

Now $e_s = e_{s-2}e_s$ and $e_{s-1} = e_{s-3}e_{s-1}$ and $e_i = e_{i-2}e_i + e_i e_{i+2}$, for $i = 0, \ldots, s-2$. We can use these equations to express $e_{i-2}e_i$ in terms of the e_j, for $i = 0, \ldots, s$:

$$e_{i-2}e_i = \sum_{j=0}^{\lfloor \frac{s-i}{4} \rfloor} e_{i+4j} - \sum_{j=0}^{\lfloor \frac{s-i-2}{4} \rfloor} e_{i+4j+2}.$$

Depending on the parity of n and the parity of i, this shows that $e_{i-2}e_i$ is of the form $e - f$ or $f - e$, where e is an idempotent in $Z(\mathcal{O}S_n)$ and f is an idempotent in $Z(\mathcal{O}S_{n-1})$. However, we can change between these forms using the equality $f - e = (1-e) - (1-f)$. This completes the proof. □

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