NON-NEGATIVE VERSUS POSITIVE SCALAR CURVATURE

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Abstract: *We show that results about spaces or moduli spaces of positive scalar curvature metrics proved using index theory can typically be extended to non-negative scalar curvature metrics. We illustrate this by providing explicit generalizations of some classical results concerning moduli spaces of positive scalar curvature metrics. We also present the first examples of manifolds with infinitely many path-components of Ricci non-negative metrics in both the compact and non-compact cases.*

§0 Introduction

There has been much recent activity concerning the topology of spaces and moduli spaces of Riemannian metrics satisfying some form of curvature condition on a fixed manifold. Such curvature conditions include positive scalar curvature, positive Ricci curvature, non-negative and negative sectional curvature. For some recent results in this direction see, for example, [BHSW], [BERW], [HSS], [CS], [CM], [Wal1], [Wal2], [Wr], [BH], [DKT], [FO1-3] and the book [TW].

In this paper all manifolds under consideration will be closed unless otherwise stated, and we will always assume that spaces of metrics are equipped with the smooth topology.

The principal theme in this paper is the comparison of (moduli) spaces of non-negative scalar curvature metrics with (moduli) spaces of positive scalar curvature metrics. We begin by observing that most of the results concerning (moduli) spaces of positive scalar curvature metrics are established using the index theory of Dirac operators. We will present some of the relevant details concerning this in §2, however it will suffice for our present purposes to note that one of the key results which makes index theory such an important tool in this context is the classical theorem of Lichnerowicz. In order to state this, let us first recall that if (M, g) is a Riemannian spin manifold, we can consider the Dirac operator D defined by Atiyah and Singer acting on the space of sections of any spinor bundle over M. A harmonic spinor on M is a section of a spinor bundle belonging to $\ker D$. Conversely we say that M has no harmonic spinors if the kernel of the Dirac operator is trivial for all spinor bundles over M, that is to say, zero is not an eigenvalue of the Dirac operator.

The Lichnerowicz theorem provides a link between harmonic spinors and positive scalar curvature:

Theorem 0.1. *([LM; II 8.9]) If (M, g) is a closed Riemannian spin manifold with either positive scalar curvature or non-negative scalar curvature which is positive at some point, then M admits no harmonic spinors.*

Our first result provides an extension of the Lichnerowicz theorem to scalar flat metrics:

Theorem 0.2. *If (M, g) is a closed Riemannian spin manifold with positive scalar curvature and g' is any metric with non-negative scalar curvature in the same path-component of non-negative scalar curvature metrics as g, then (M, g') admits no harmonic spinors.*

The significance of this result is that it essentially says that from the point-of-view of index theory, there is no difference between working with metrics of positive scalar curvature and metrics of non-negative scalar curvature, provided the relevant path-component of non-negative scalar curvature metrics contains a positive scalar curvature metric.

The manifolds appearing in Theorem 0.2 should be contrasted with the so-called *strongly scalar flat* manifolds, studied for example in [De] and [Fu]. These are manifolds which do not admit positive scalar curvature, but admit a scalar-flat metric (which is then necessarily Ricci-flat).

Theorem 0.2 is an elementary corollary of the following result, which should be compared, for example, with Theorems 1.2 and 1.3.

Theorem 0.3. *Let M be a closed spin manifold and suppose g_t, $t \in [0, T]$, is a smooth path of nonnegative scalar curvature metrics. If g_0 admits a parallel spinor (and so is Ricci-flat), then g_t is Ricci-flat for all $t \in [0, T]$. If furthermore $\pi_1(M) = 0$, then g_t also admits a parallel spinor for all $t \in [0, T]$.*

A principal aim of this paper is to derive some consequences of Theorem 0.2 for (moduli) spaces of non-negative scalar curvature metrics. We will address this issue in two main ways: by generalizing a well-known result about moduli spaces of positive scalar curvature metrics, and by providing some new examples involving spaces of Ricci non-negative metrics.

The classic positive scalar curvature result we will generalize involves the Kreck-Stolz s-invariant. The s-invariant is the principal tool for studying path-connectedness of moduli spaces of positive scalar curvature metrics. This was developed and first applied in [KS]. The s-invariant is defined for spin manifolds M^{4n-1} ($n \geq 2$) with vanishing real Pontrjagin classes and positive scalar curvature, and is an invariant of the path-component in the space of positive scalar curvature metrics. Moreover, if $H^1(M; \mathbb{Z}_2) = 0$ (which means the spin structure on M is uniquely determined by the orientation), and g is a positive scalar curvature metric on M, then $|s(M, g)| \in \mathbb{Q}$ is an invariant of the path-component of the *moduli* space of positive scalar curvature metrics on M containing g.

Using Theorem 0.2 we can establish:

Theorem 0.4. *For a closed spin manifold (M, g) of dimension $4k - 1$ ($k \geq 2$) with positive scalar curvature and vanishing real Pontrjagin classes, the Kreck-Stolz s-invariant is an invariant of the path-component of non-negative scalar curvature metrics containing g. If in addition $H^1(M; \mathbb{Z}_2) = 0$, $|s|$ is an invariant of the path-component containing $[g]$ in the moduli space of non-negative scalar curvature metrics.*

We can rephrase Theorem 0.4 in the following way. Let U_1, U_2 be path-components of (the moduli space of) positive scalar curvature metrics on M, and denote the corresponding path-components for non-negative scalar curvature metrics by $\bar{U}_1$, $\bar{U}_2$. If U_1 and U_2 are distinguished by their s-invariants, then

$$\bar{U}_1 \cap \bar{U}_2 = \emptyset.$$

From Theorem 0.4 we immediately obtain the following result, which is the non-negative scalar curvature analogue of [KS; Corollary 2.15]:

Corollary 0.5. *Given any M as in Theorem 0.4 with $H^1(M; \mathbb{Z}_2) = 0$, the moduli space of non-negative scalar curvature metrics on M has infinitely many path-components.*

In focusing on the s-invariant it should not be forgotten that one can re-visit many other results for (moduli) spaces of positive scalar curvature metrics established using index theory, and making the required adjustments re-state these as results about non-negative scalar curvature. For example, one could do this with the theorems about the higher homotopy groups of the (observer moduli) space of positive scalar curvature metrics established recently in [HSS], as these results rely on the invertibility of a family of Dirac operators, and this is governed by the existence or otherwise of harmonic spinors. As a sample result (extending [HSS; Theorem 1.1]) we have

Theorem 0.6. *Given $k \in \mathbb{N} \cup \{0\}$, there is an $N(k) \in \mathbb{N}$ such that for each $n \geq N(k)$ and each closed spin manifold M^{4n-k-1} admitting a metric g_0 with positive scalar curvature, the homotopy group $\pi_k(\mathrm{Riem}_{scal\geq 0}, g_0)$, where $\mathrm{Riem}_{scal\geq 0}$ denotes the space of non-negative scalar curvature metrics on M, contains elements of infinite order if $k \geq 1$, and infinitely many different elements if $k = 0$. Their images under the Hurewicz homomorphism in $H_k(\mathrm{Riem}_{scal\geq 0})$ still have infinite order.*

One could similarly generalize to non-negative scalar curvature the classic results of Hitchin on the non-triviality of $\pi_0(\mathrm{Riem}_{scal>0})$ and $\pi_1(\mathrm{Riem}_{scal>0})$ for spin manifolds in dimensions 0 and 1, respectively 0 and 7 modulo 8. (See [Hi] for the full details, or for a synopsis explaining the dependence of these results on the invertibility of the Dirac operator, see [LM; IV.7].) The same can also be said for the more recent results of Crowley-Schick ([CS]), as these build on the ideas of Hitchin.

We also use Theorem 0.2 to derive some new examples involving Ricci non-negative metrics. We remark that the following theorem presents merely one set of examples among many that are possible. Details of the Bott manifold B^8 appearing in this theorem are given in §2.

Theorem 0.7. *If K^4 denotes the K3 surface, B^8 the Bott manifold, and Σ^{4n-1}, is any homotopy $(4n-1)$-sphere $(n \geq 2)$ which bounds a parallelisable manifold, then both $\Sigma \times K^4$ and $\Sigma \times B^8$ have infinitely many path-components of non-negative Ricci curvature metrics.*

To the best of the author's knowledge, Theorem 0.5 is the first result of any kind concerning the topology of the space of Ricci non-negative metrics. It should be noted that we cannot use Theorem 0.3 to establish these examples as the real Pontrjagin classes here are not all zero, and so the s-invariant is not defined. The important thing here is that although the manifolds above are known to admit metrics which have both positive scalar and non-negative Ricci curvature, none are known to admit metrics with strictly positive Ricci curvature. There are no known obstructions to positive Ricci curvature for these manifolds (the manifolds admit positive scalar curvature, and also have finite fundamental group and thus comply with Myers' Theorem). Nevertheless, the author is tempted to conjecture that no Ricci positive metrics exist.

The examples in Theorem 0.7 give rise to the first examples of complete non-compact manifolds with infinitely many path-components of Ricci non-negative metrics.

Theorem 0.8. *If M denotes any of the manifolds in Theorem 0.7, $M \times \mathbb{R}$ has infinitely many path-components of complete Ricci non-negative metrics.*

It should be noted that by the Cheeger-Gromoll splitting theorem ([CG]) the examples in Theorem 0.8 do not admit any complete metrics of positive Ricci curvature.

This paper is laid out as follows. In §1 below we consider harmonic spinors and prove Theorem 0.3, deriving Theorem 0.2. In §2 we briefly review some index theory and prove Theorems 0.4, 0.7 and 0.8.

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§1 Harmonic spinors

Our main task in this section is to provide a proof for Theorem 0.3, from which Theorem 0.2 can be derived as an elementary corollary. Before proving this, however, we will gather together some relevant results from the literature.

An elementary corollary of the proof of the Lichnerowicz theorem (Theorem 0.1) is the following fact which we will need later:

Lemma 1.1. *([LM; II 8.10]) On a closed spin manifold with identically vanishing scalar curvature, every harmonic spinor is globally parallel.*

With this in mind Theorem 0.2 can be rephrased to assert that under the given hypotheses (M, g') cannot admit any globally parallel spinors.

The existence of a parallel spinor on a compact Riemannian spin manifold has certain well-known consequences, the most important of which for our purposes is that it implies that the manifold is actually Ricci flat. More than this, the next result shows that there are no positive scalar curvature metrics arbitrarily close-by.

Theorem 1.2. *([DWW; Theorem 4.2 and Remark]) If (M, g) is a closed Riemannian spin manifold with a non-trivial parallel spinor, then there is no path of metrics g_t with $g_0 = g$ such that $\mathrm{scal}(g_t) > 0$ for all $t > 0$. Moreover, there is no path of non-negative scalar curvature metrics g_s with $g_0 = g$ containing a sequence of positive scalar curvature metrics g_{s_n} where $s_n \to 0^+$ as $n \to \infty$.*

We will also need the following result from the same paper which concerns simply-connected manifolds:

Theorem 1.3. *([DWW; Theorem 3.4]) Let (M, g) be a simply-connected closed Riemannian spin manifold with a non-trivial parallel spinor. Then there exists a neighbourhood U of g in the space of smooth Riemannian metrics on M, such that any metric with non-negative scalar curvature in U must admit a parallel spinor, and consequently be Ricci-flat.*

It should be noted that the existence of a parallel spinor for some metric does not exclude the possibility that the manifold admits metrics of positive scalar curvature. For example Calabi-Yau manifolds are known to admit both positive scalar curvature metrics as well as Ricci-flat metrics with parallel spinors. Although the existence of a parallel spinor implies the metric is Ricci-flat, it is unknown whether there exist any Ricci-flat metrics without a parallel spinor.

The existence of a parallel spinor on a compact Riemannian spin manifold places restrictions on the holonomy group of that manifold. For a discussion about these points and detailed references, see for example section 1 of [AKWW]. Although we will not use holonomy arguments directly, the above results from [DWW] depend in part on such matters. (One might also compare the results in [Wang].) Holonomy is central to the paper [AKWW], from which we will need the following theorem:

Theorem 1.4. *([AKWW; Corollary 3]) Let (M, g_0) be a closed Riemannian spin manifold which admits a parallel spinor on its universal cover. If g_t, $t \in [0, T]$, is a smooth family of Ricci-flat metrics on M extending g_0, then the pull-back of g_t to the universal cover admits a parallel spinor for all $t \in [0, T]$, and the dimension of the space of parallel spinors is independent of t.*

There is one final result from the literature which we will need in the proof Theorem 0.3, and this is the basic structure theorem for Ricci-flat metrics (see [CG] or [FW; 4.1]):

Theorem 1.5. *(The Ricci-flat structure theorem.) If (M, g) is a closed Ricci-flat manifold, then there is a finite normal Riemannian covering $\pi : (\bar{M}, \bar{g}) \times (T^q, h_{fl}) \to (M, g)$ where $(\bar{M}, \bar{g})$ is a simply-connected Ricci-flat manifold and (T^q, h_{fl}) is the q-torus equipped with a flat metric.*

Proof of Theorem 0.3. We first note that admitting a parallel spinor, or equivalently the Dirac operator having a zero eigenvalue, is a closed condition. The same is also true for the Ricci-flat condition.

In the case where M is simply-connected, the argument is straightforward. As a consequence of closedness, there exists $c \in [0, T]$ such that g_t admits a parallel spinor for all $t \in [0, c]$, and c is maximal with respect to this property. (Note that by Theorem 1.4 we could equivalently define c by replacing the parallel spinor condition with Ricci-flatness.) If $c = T$ there is nothing to show, so suppose that $c < T$. By Theorem 1.3 there is a $\delta > 0$ such that every g_t with $t \in (c - \delta, c + \delta)$ admits a parallel spinor and is in particular Ricci-flat. This contradicts the maximality of $c < T$, showing that in fact $c = T$.

If M is not simply-connected we must proceed more carefully. The essential difficulty is that the universal cover will be non-compact if the fundamental group is infinite, so we cannot simply lift the above argument to the universal cover.

We first we allow the metric path g_t to evolve under the Ricci flow for some short time interval $[0, s_0]$ to obtain a new smooth path of metrics h_t. More precisely we obtain

a homotopy of metrics $H(t, s)$ where $H(t, 0) = g_t$; $H(t, s_0) = h_t$ and for each $t_0 \in [0, T]$, $H(t_0, s)$ is the Ricci flow of g_{t_0}. By [Br; 2.18], h_t has strictly positive scalar curvature if and only if g_t is not Ricci-flat. Of course, $g_t = h_t = H(t, s)$ for all s if g_t is Ricci-flat.

We will work initially with the path h_t. As g_0 is Ricci-flat we have $h_0 = g_0$, and h_0 has a parallel spinor. Let $c \in [0, T]$ now be the maximal value for which h_t is Ricci-flat for all $t \in [0, c]$. Again we will assume that $c < T$ and derive a contradiction.

By the structure theorem (Theorem 1.5), for each t there is a finite normal Riemannian covering

$$\pi_t : (\bar{M}, \bar{h}_t) \times (T^q, h_{fl,t}) \to (M, h_t)$$

where $(\bar{M}, \bar{h}_t)$ is simply-connected and Ricci-flat, and $(T^q, h_{fl,t})$ is the q-torus equipped with a flat metric. We will assume the spin structure on the torus is chosen so that it admits a parallel spinor.

Let $\theta : \bar{M} \times \mathbb{R}^q \to \bar{M} \times T^q$ be the product map which is the identity on the first factor and a standard universal cover of the torus on the second. Thus the composition $\pi_t \circ \theta$ is a universal covering map of M for each t. We will consider the pull-back metrics $(\pi_0 \circ \theta)^*(h_c)$ and $(\pi_c \circ \theta)^*(h_c)$ on $\bar{M} \times \mathbb{R}^q$. Observe that these metrics create Riemannian universal covers

$$\pi_0 \circ \theta : (\bar{M} \times \mathbb{R}^q, (\pi_0 \circ \theta)^*(h_c)) \to (M, h_c);$$
$$\pi_c \circ \theta : (\bar{M} \times \mathbb{R}^q, (\pi_c \circ \theta)^*(h_c)) \to (M, h_c).$$

By a basic topological result on covering spaces (see for example [Ha; 1.37]) we see that our two universal covering maps are related by a diffeomorphism of the universal covers, and moreover since the covering maps are local isometries, we see that this diffeomorphism is in fact an isometry.

Since h_0 admits a parallel spinor we see immediately that $(\bar{M} \times \mathbb{R}^q, (\pi_0 \circ \theta)^*(h_0))$ also admits a parallel spinor. Applying Theorem 1.4 to the path of metrics $(\pi_0 \circ \theta)^*(h_t)$ then shows that $(\bar{M} \times \mathbb{R}^q, (\pi_0 \circ \theta)^*(h_c))$ admits a parallel spinor. Using the above isometry we deduce that $(\bar{M} \times \mathbb{R}^q, (\pi_c \circ \theta)^*(h_c))$ also admits a parallel spinor.

Now by Theorem 1.5, we have

$$(\bar{M} \times T^q, \pi_c^*(h_c)) = (\bar{M} \times T^q, \bar{h}_c + h_{fl,c}),$$

and therefore

$$(\bar{M} \times \mathbb{R}^q, (\pi_c \circ \theta)^*(h_c)) = (\bar{M} \times \mathbb{R}^q, \bar{h}_c + h_{Eucl}),$$

where h_{Eucl} is a Euclidean metric. It is well known that a Riemannian product manifold admits a parallel spinor if and only if all the individual factors do (see for example [Fu; Lemma 3.1] or [L; Theorem 2.5]). We therefore deduce that $(\bar{M}, \bar{h}_c)$ admits a parallel spinor. In turn we then see that $(\bar{M} \times T^q, \pi_c^*(h_c))$ must likewise admit a parallel spinor.

Pull the path of metrics h_t back to a smooth path of metrics on $\bar{M} \times T^q$ via the map π_c. Clearly $\pi_c^*(h_t)$ is Ricci-flat or has positive scalar curvature according to whether h_t is Ricci-flat or has positive scalar curvature. As $\pi_c^*(h_c)$ admits a parallel spinor, by Theorem 1.2 along the path $\pi_c^*(h_t)$ there is no sequence of positive scalar curvature metrics $\pi_c^*(h_{t_i})$ with $t_i \to c^+$ as $i \to \infty$. However by definition of c such a sequence must exist, which gives the desired contradiction. We therefore conclude that $c = T$, and so the path h_t is Ricci-flat for all $t \in [0, T]$.

It remains to show that the original path g_t is Ricci-flat: this follows automatically from the Ricci-flatness of h_t in conjunction with [Br; 2.18]. □

As an immediate corollary we obtain:

Proof of Theorem 0.2. The Lichnerowicz theorem (Theorem 0.1) shows that g' cannot admit a harmonic spinor except possibly in the case that g' is scalar flat. In this case Lemma 1.1 shows that (M, g') admits a harmonic spinor if and only if it admits a parallel spinor. By Theorem 0.3, if (M, g') is scalar flat and admits a parallel spinor, then there cannot be a smooth path of non-negative scalar curvature metrics g_t on M such that $g_0 = g$ (which has positive scalar curvature) and $g_1 = g'$. But by assumption g and g' belong to the same path-component of non-negative scalar curvature metrics on M. Hence (M, g') cannot admit a parallel spinor, and hence cannot admit a harmonic spinor as claimed. □

§2 Index theory and the s-invariant

Given a closed Riemannian spin manifold (X^{4k}, g), the Atiyah-Singer index theorem (see for example [LM; II 6.3]) asserts that for the Atiyah-Singer Dirac operator D^+ we have

$$\mathrm{ind}D^+(X^{4k}, g) = \hat{A}(X).$$

Of course the right-hand side of this equation is a smooth topological invariant of X, and thus the index is independent of the metric g. When combined with the Lichnerowicz theorem (Theorem 0.1) it is straightforward to deduce that if X admits a positive scalar curvature metric (or a non-negative scalar curvature metric which is positive at a point) then $\hat{A}(X) = 0$.

In order to prove Theorems 0.4 and 0.7 we will need to investigate the index on manifolds with boundary. We will begin by recalling the index theorem of Atiyah-Patodi-Singer:

Theorem 2.1. *([APS]) Let (W, g_W) be a compact even dimensional Riemannian spin manifold with non-empty boundary M, where the metric g_W is a product $dt^2 + g_M$ in a neighbourhood of the boundary. Consider the Atiyah-Singer Dirac operator D^+ on W acting on the subspace of spinor bundle sections for which the restriction to M belongs to the span of the negative eigenspaces of the operator induced on M. Then the index of this (restricted domain) Dirac operator on W is given by*

$$\mathrm{ind}D^+(W, g_W) = \int_W \hat{A}(\{p_i(W, g_W)\}) - \frac{h(M, g_M) + \eta(M, g_M)}{2},$$

where $\hat{A}$ denotes the $\hat{A}$-polynomial in the Pontrjagin forms $p_i(W, g_W)$, h is the dimension on the space of harmonic spinors on the boundary M, and η is the eta-invariant of the Dirac operator on M.

At first glance, the index would appear to depend on both the topology and the global geometry of W. In actual fact, the metric-dependence of the index only involves the metric

in a neighbourhood of the boundary. To see this we consider metrics g_W and g'_W on W which both take the form $dt^2 + g_M$ in near the boundary. It follows easily from Theorem 2.1 that
$$\text{ind}D^+(W, g_W) - \text{ind}D^+(W, g'_W) = \text{ind}D^+(W \cup -W, g_W \cup g'_W)$$
$$= \hat{A}(W \cup -W),$$

where we have used the Atiyah-Singer index theorem for closed manifolds to derive the last line. Now $\hat{A}$ is a ring homomorphism from oriented bordism to the rationals, so the $\hat{A}$-genus of any null bordant oriented manifold is zero. As it is well-known that 'double' manifolds such as $W \cup -W$ are null bordant in this way, the result follows.

As it will be important for later arguments, we note that if $\text{scal}(g_W) > 0$, then just as in the case of closed manifolds, we have $\text{ind}D^+(W, g_W) = 0$.

Now consider the behaviour of the index formula in Theorem 2.1 under smooth deformations of the metric. The Pontrjagin forms depend smoothly on the metric, and hence the integral term in the formula varies smoothly. The eigenvalues of the boundary Dirac operator also vary smoothly. (This behaviour is well known, however see [No] for a detailed account.) Notice that the boundary conditions imposed on the domain of the Dirac operator will certainly be stable over the metric deformation *provided no boundary Dirac operator has zero eigenvalues.* If this is the case then h will be fixed over the deformation, as h can only vary if an eigenvalue hits zero or becomes non-zero, and the resulting variation is then clearly by an integer amount. Recall that the eta-invariant is also defined in terms of these eigenvalues. The eta *function* $\eta(z)$ is defined for complex numbers z to be

$$\eta(z) = \sum_{\lambda \neq 0} (\text{sign } \lambda)|\lambda|^{-z}$$

wherever the series on the right-hand side is convergent, where the sum is over all non-zero eigenvalues. It can be shown that this function is defined and holomorphic on the half-space consisting of all z with suitably large real part. There is a unique meromorphic extension of $\eta(z)$ across the whole complex plane, and the eta-invariant is defined to be the value of this extension at 0. It can further be shown that the eta-invariant is always finite. As the index is always an integer, it follows immediately from the index formula that if the eta invariant jumps during the metric deformation, this must also be by an integer amount, and such behaviour can only occur if an eigenvalue hits or departs from zero. In summary we have:

Observation 2.2. *Given the hypotheses of Theorem 2.1, if we consider a smooth path of metrics g_t on W (products near the boundary) such that the Dirac operator on M has no zero eigenvalues for any t, then each term in the index formula varies smoothly during the deformation. In particular, as the index and h are integers, these remain constant throughout the deformation.*

In [KS; Remark 2.2] it is stated that if the boundary metric g_M has positive scalar curvature, the index of (W, g_W) depends only on the path-component of g_M in the space of positive scalar curvature metrics on M. To see this, consider any two metrics on W which restrict to metrics in the same path-component of positive scalar curvature metrics

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on M. These metrics can be joined by a path of metrics on W for which the boundary restrictions all have positive scalar curvature. By the Lichnerowicz theorem (Theorem 0.1 above) the Dirac operator on M has no zero eigenvalues for any metric in this path, and therefore the index remains constant by Observation 2.2. Conversely if two positive scalar curvature metrics on M extended to W yield different indices, these metrics must lie in different path-components of the space of positive scalar curvature metrics on M. This basic idea is the starting point for the derivation of the s-invariant, as well as for many other results about the topology of (moduli) spaces of positive scalar curvature metrics.

Using Theorem 0.2 we can adjust the arguments in the above paragraph to incorporate non-negative scalar curvature metrics:

Proposition 2.3. *Consider manifolds W and M as in Theorem 2.1. Given a path g_t of non-negative scalar curvature metrics on M, $t \in [0,1]$, suppose that for some $t_0 \in [0,1]$ the metric g_{t_0} has positive scalar curvature. If $\bar{g}_t$ is any smooth path of metrics on W which extend g_t (and take the form of a product near the boundary), then $\mathrm{ind}D^+(W, \bar{g}_t)$ is independent of t.*

Proof. By Theorem 0.2 no metric in the path g_t has any harmonic spinors, so in particular the Dirac operator on M has no zero eigenvalues for any $t \in [0,1]$. By Observation 2.2 the index of $(W, \bar{g}_t)$ is constant. $\qquad\square$

Proof of Theorem 0.4. By checking the arguments in [KS] it is easily verified that the invariance properties of s are a direct consequence of the invariance of the index on path-components of positive scalar curvature boundary metrics as discussed above. The result now follows immediately from Proposition 2.3. $\qquad\square$

Turning our attention to examples, we consider two families of products, one involving a K3 surface K^4, and the other involving a Bott manifold B^8 as a factor. Recall that K^4 can be defined by

$$K^4 := \{(z_0, z_1, z_2, z_3) \,|\, z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\} \subset \mathbb{C}P^3.$$

The Bott manifold can be constructed by forming the boundary connected sum of 28 copies of the manifold constructed by plumbing the tangent disk bundle of S^4 to itself according to the E_8-graph. Thus resulting object has boundary S^7, and this can then be made into a smooth closed manifold B^8 by gluing in a disc D^8. Together with $\mathbb{H}P^2$, B^8 generates $\Omega_8^{spin} \cong \mathbb{Z} \oplus \mathbb{Z}$. We have $\hat{A}(K^4) = -2$ and $\hat{A}(B^8) = 1$, and so neither manifold admits a positive scalar curvature metric. However both are known to support Ricci flat metrics (see for example [B; page 128] for K^4 and [J] for B^8).

We also consider the set of homotopy spheres which bound parallelisable manifolds in dimensions $4n - 1$, $(n \geq 2)$. Although finite for each n, this family grows more than exponentially with dimension. The moduli space of positive Ricci curvature metrics for each of these spheres was shown to have infinitely many path-components in [Wr]. This result was established by exhibiting an infinite family of Ricci positive metrics on each sphere, and showing that these metrics can be distinguished by their s-invariants.

Proof of Theorem 0.7. It suffices to consider $\Sigma^{4n-1} \times K^4$ for some choice of homotopy sphere Σ^{4n-1} bounding a parallelisable manifold, as the argument in all other cases is identical.

In [Wr] it was shown that we can find a sequence of Ricci positive metrics g_i on Σ such that $s(\Sigma, g_i) \neq s(\Sigma, g_j)$ whenever $i \neq j$, so g_i and g_j belong to different path-components of the moduli space of positive scalar curvature metrics on Σ. For each i there is a parallelisable bounding manifold W_i for Σ such that g_i extends to a positive scalar curvature metric $\bar{g}_i$ over W_i (product near the boundary).

The W_i are constructed by plumbing D^{2n}-bundles over S^{2n}. If we consider the oriented union $W_i \cup_\Sigma (-W_j)$, it is established for example in [Ca; page 73] that $\hat{A}(W_i \cup (-W_j))$ is a non-zero multiple of the difference of signatures $\text{sig}(W_i) - \text{sig}(W_j)$. As noted in [Wr; §2], for $i \neq j$ we have $\text{sig}(W_i) \neq \text{sig}(W_j)$, and thus $\hat{A}(W_i \cup (-W_j)) \neq 0$. As the $\hat{A}$-genus is multiplicative for products and $\hat{A}(K^4) \neq 0$, we deduce that

$$\hat{A}((W_i \times K^4) \cup (-W_j \times K^4)) \neq 0.$$

Let g_K denote a Ricci flat metric on K^4, and consider the product metrics $g_i + g_K$. These have non-negative Ricci curvature and positive scalar curvature. By the above, these metrics can be extended to positive scalar curvature metrics $\bar{g}_i + g_K$ on $W_i \times K^4$, so

$$\text{ind}D^+(W_i \times K^4, \bar{g}_i + g_K) = \text{ind}D^+(W_j \times K^4, \bar{g}_j + g_K) = 0.$$

For $i \neq j$ suppose the metrics $g_i + g_K$ and $g_j + g_K$ belong to the same path-component of non-negative scalar curvature metrics on $\Sigma \times K^4$, i.e. there is a path h_t, $t \in [0,1]$, with $\text{scal}(h_t) \geq 0$, $h_0 = g_i + g_K$ and $h_1 = g_j + g_K$. Let $\bar{h}_t$ be any path of metrics on $W_i \times K^4$ which extend h_t (and take the form of a product near the boundary). Applying Proposition 2.3 to the path $\bar{h}_t$ we deduce that

$$\text{ind}D^+(W_i \times K^4, \bar{h}_1) = 0.$$

Following the argument presented after Theorem 2.1 then yields

$$\begin{aligned} 0 =& \text{ind}D^+(W_i \times K^4, \bar{h}_1) - \text{ind}D^+(W_j \times K^4, \bar{g}_j + g_K) \\ =& \hat{A}((W_i \times K^4) \cup (-W_j \times K^4)) \\ \neq& 0, \end{aligned}$$

and we have a contradiction. Thus $g_i + g_K$ and $g_j + g_K$ cannot belong to the same path-component of non-negative scalar curvature metrics, and hence must belong to different path components of Ricci non-negative metrics. $\qquad\square$

As remarked in the introduction, one can replace the homotopy spheres in Theorem 0.7 with other manifolds. For example one could use the infinite family of 7-dimensional Einstein manifolds $M_{k,l}$ considered in [KS], which were shown to have infinitely many path-components of Ricci positive metrics in [KS], and infinitely many path components of non-negative sectional curvature metrics in [KPT].

Proof of Theorem 0.8. With M as in Theorem 0.7, consider a smooth family of Ricci non-negative metrics g_t on $M \times \mathbb{R}$. The manifold $(M \times \mathbb{R}, g_t)$ has two ends, and by a standard argument (see for example [P; page 260]) must contain a line. Thus by the Cheeger-Gromoll splitting theorem the manifold must split as a Riemannian product between a Riemannian manifold diffeomorphic to M and a line, though topologically this splitting might not coincide with the given product structure.

At any given point in $M \times \mathbb{R}$ it is clear that as the metric g_t changes smoothly with t, we can choose a smoothly varying vector tangent to the unique line through that point. Collectively we obtain a smoothly varying one-dimensional distribution on $M \times \mathbb{R}$, together with a smoothly varying distribution of orthogonal complements.

The Riemannian product structure provided by the splitting theorem is given by the integral submanifolds to these distributions, together with their induced metrics. In particular this means that we obtain a smooth path of Ricci non-negative metrics on M. (In detail, choose any point in $M \times \mathbb{R}$ and consider the codimension one integral submanifold M_{t_0} through this point for some time $t = t_0$. Of course M_{t_0} is diffeomorphic to M. The compactness of M_{t_0} ensures that there is an $\epsilon > 0$ such that for any $t \in (t_0 - \epsilon, t_0 + \epsilon)$ there is a canonical diffeomorphism between M_{t_0} and M_t given by mapping each point in M_{t_0} to the unique nearest point in M_t, measured with respect to some fixed choice of background metric. Thus there exists a smooth path of diffeomorphisms $M \to M_t$, and pulling back the metrics induced by g_t on M_t gives the desired path of Ricci non-negative metrics on M.)

Finally, we note that if two Ricci non-negative metrics $ds^2 + h_0$ and $ds^2 + h_1$ on $M \times \mathbb{R}$ belong to the same path-component of Ricci non-negative metrics, then h_0 and h_0 must belong to the same path-component of Ricci non-negative metrics on M. The result now follows immediately from Theorem 0.7. □

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