# FACTORING FORMAL MAPS INTO REVERSIBLE OR INVOLUTIVE FACTORS

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ABSTRACT. An element $g$ of a group is called *reversible* if it is conjugate in the group to its inverse. An element is an *involution* if it is equal to its inverse. This paper is about factoring elements as products of reversibles in the group $\mathfrak{G}_n$ of formal maps of $(\mathbb{C}^n, 0)$, i.e. formally-invertible $n$-tuples of formal power series in $n$ variables, with complex coefficients. The case $n = 1$ was already understood [25].

Each product $F$ of reversibles has linear part $L(F)$ of determinant $\pm 1$. The main results are that for $n \geq 2$ each map $F$ with $\det(L(F)) = \pm 1$ is the product of $2 + 3 \cdot \text{ceiling}(\log_2 n)$ reversibles, and may also be factored as the product of $9 + 6 \cdot \text{ceiling}(\log_2 n)$ involutions (where the ceiling of $x$ is the smallest integer $\geq x$).

## 1. INTRODUCTION

1.1. It is an interesting fact that in many very large groups each element may be factored as the product of a small number of involutions. For instance, each permutation is the product of two involutions. Less trivially, Fine and Schweigert [11] showed that each homeomorphism of $\mathbb{R}$ onto itself is the composition at most four involutions, each one conjugate to the map $x \mapsto -x$.

A natural generalization of an involution is a reversible. An element $g$ of a group is called *reversible* if it is conjugate to its inverse, i.e. the conjugate $g^h := h^{-1}gh$ equals $g^{-1}$ for some $h$ from the group. We say that $h$ *reverses* $g$ or $h$ *is a reverser of* $g$, in this case. Furthermore, if the reverser $h$ can be chosen to be an involution (i.e. an element of order at most 2), then $g$ is called *strongly reversible*. (Note that some writers use the terminology "weakly reversible" and "reversible" instead of respectively "reversible" and "strongly reversible" used here. In finite group theory, the terms used are "real" and "strongly real".) A strongly reversible element is the product of two involutions. See [14, 15, 16]. If $g$ is reversed by an element of finite even order $2k$, then $g$ is the product of two elements of order $2k$. Indeed, it is easy to check that if $g$ is reversed by some element $h$, then it factors as $hf$, where $h^2 = f^{-2}$, so if $h$ has order $2k$, then so does $f$.

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Reversible maps have their origin in problems of classical dynamics, such as the harmonic oscilator, the $n$-body problem or billiards, and Birkhoff [3] was one of the first to realize their significance. He observed that a Hamiltonian system with Hamiltonian quadratic in the momentum (such as the $n$-body problem), and other interesting dynamical systems admit what are called "time reversal symmetries", i.e. transformations of the phase space that conjugate the dynamical system to its inverse.

In CR geometry reversible maps played important role in the celebrated work of Moser and Webster [23], arising as products of two involutions naturally associated to a CR singularity. Such a reversible map is called there "a discrete version of the Levi form" and plays a fundamental role in the proof of the convergence of the normal form for a CR singularity. More recently, this map has been used by Ahern and Gong [2] for so-called parabolic CR singularities.

The basic concept of reversible element makes sense in any group, and reversibility has been the focus of interest in many other application areas that involve some underlying group. For instance, reversible elements appear (sometimes under aliases) in connection with geometrical symmetries, special geodesics on Riemann surfaces, binary integral quadratic forms, quadratic correspondences, superposition of functions, approximation problems, toral automorphisms and foliations [7, 8, 10, 12, 24, 28]. For further references to some contexts in which reversible elements have played a part, and a short survey of factorization results involving reversibles, see [26].

From the point of view of group theory, the subgroup $R^\infty(G)$ generated by the reversible elements of a group $G$ is normal, and its isomorphism class is an isomorphism invariant of $G$. It has associated numerical invariants, which are very basic invariants of $G$, and their determination is a natural first step in the classification of $G$. One of these invariants is the supremum over all $g \in R^\infty(G)$ of the least number $k$ of reversible factors $r_j$ needed to represent $g$ as a product $r_1 \cdots r_k$. In the language of Klopsch and Lev [21], this is the "diameter" of $R^\infty$ with respect to the set $R(G)$ of reversibles.

The issue of factorization into reversibles (and involutions), and the number of factors needed has attracted attention in several group contexts — see for instance [5, 9, 17, 19, 22, 29].

In this paper we consider the group $\mathfrak{G}_n$ of formally-invertible maps in $n \geq 2$ complex variables, and we discuss the factorization of a given map as a product of reversibles, and as a product of involutions. We get an explicit upper bound in terms of $n$ for the above diameter, and also (when $n \geq 2$) for the (finite!) diameter of $R^\infty(\mathfrak{G}_n)$ with respect to the set of involutions.

In previous work the first author dealt with this problem for $n = 1$, and obtained the following results:

**Theorem 1.1.** [25] *Let $F \in \mathfrak{G}_1$. Then the following are equivalent:*
*(1) $F$ is a product of reversibles.*
*(2) $F(z) = \pm z + \mathrm{O}(z^2)$, i.e. $F(z) = \pm z +$terms in $z^2$ and higher powers of $z$.*
*(3) $F$ is the product of two reversibles.* □

**Theorem 1.2.** [25] *Let $F \in \mathfrak{G}_1$. Then the following are equivalent:*
*(1) $F$ is a product of involutions.*

*(2) For some* $a \in \mathbb{C}$, $F(z) = \pm z + az^2 \pm a^2 z^3 + \mathrm{O}(z^4)$.
*(3)* $F$ *is the product of four involutions.* □

Thus not every reversible series in one variable is the product of a finite number of involutions. It depends upon the conjugacy class of the series, modulo $z^4$. We shall see that the situation changes in higher dimensions.

In dimension 2, the authors previously showed the following:

**Theorem 1.3.** [27] *If* $F \in \mathfrak{G}_2$ *has linear part of determinant* 1*, then it may be factorized as the product of* 4 *reversible elements.*

1.2. **Results.** In this paper, we will show:

**Theorem 1.4.** *Let* $n \geq 2$ *and* $F \in \mathfrak{G}_n$ *have linear part of determinant* 1*. Let* $c = \mathrm{ceiling}(\log_2 n)$*. Then*
*(1)* $F$ *is the product of* $1 + 3c$ *reversibles.*
*(2)* $F$ *is the product of* $8 + 6c$ *involutions.* □

We also have:

**Corollary 1.5.** *Let* $n \geq 2$ *and* $F \in \mathfrak{G}_n$*. Let* $c = \mathrm{ceiling}(\log_2 n)$*. Then the following are equivalent:*
*(1)* $F$ *is a product of reversibles.*
*(2) The linear part of* $F$ *has determinant* $\pm 1$*.*
*(3)* $F$ *is the product of* $2 + 3c$ *reversibles.*
*(4)* $F$ *is the product of* $9 + 6c$ *involutions.*
*(5)* $F$ *is the product of* $3 + 6c$ *involutions and two reversible maps of order dividing* 4*.* □

Thus, for instance in dimension 2, *every* product of reversibles is also the product of at most 15 involutions.

1.3. **Outline.** In Section 2 we define terminology and notation, and develop some tools that will be used in the proofs of these results. We identify some interesting subgroups of $\mathfrak{G}_n$, and construct homomorphisms connecting them. In particular, we identify a subgroup $\mathfrak{C}_n$, the centraliser in $\mathfrak{G}_n$ of a matrix subgroup $D_n \leq \mathsf{GL}(n, \mathbb{C}) \leq \mathfrak{G}_n$, and we represent $\mathfrak{C}_n$ as the semidirect product of an abelian subgroup all of whose elements are reversible in $\mathfrak{G}_n$ and a subgroup (called $\mathfrak{K}_k$ or $\hat{\mathfrak{K}}_k$, depending on whether $n$ is even or odd) of $\mathfrak{C}_k$, where $k$ is roughly half of $n$. This structural information is summarized in the exact sequences shown in Figure 1 below. This allows us to carry out an induction, reducing the reversible factorization of elements of $\mathfrak{C}_n$ to the reversible factorization of $k$-dimensional maps, at the cost of one extra factor. Also, the subgroup $\mathfrak{C}_n$ has a representative of each so-called *generic* conjugacy class in $\mathfrak{G}_n$, and at the cost of an extra couple of factors, we can reduce the factorization of a general element of $\mathfrak{G}_n$ to the factorization of a generic element.

These subgroups and homomorphisms elaborate upon tools that were employed in our previous paper [27], in which we characterized the generic reversibles in dimension 2.

In considering involutive factors, we have to deal with the fact that not all one-dimensional maps $\chi \in \mathfrak{G}_1$ with multiplier 1 can be factored into involutions, so we have to find a way to factor the lift $H(\chi) \in \mathfrak{G}_2$ into involutions. Once we manage to do this, we can then start the induction at $n = 2$ and continue as before. This depends on the fact that the extra two or three reversible factors needed at each induction step are all *strongly reversible*, i.e. products of two involutions (see below).

In Section 3 we prove the two-dimensional results, and in Section 4 we prove the rest.

1.4. **Open Questions.** When we get into the detailed proofs, it will appear that for certain dimensions $n$ we can derive estimates for the number of reversible factors needed that are considerably smaller than the estimate in Theorem 1.4. For instance, we can do much better with $n = 96$ than $n = 97$. See Section 4 and Table 1 for details.

But we do not know sharp values for the number of reversible or involutive factors needed in any case of dimension greater than 1. It may even be the case that a universal number of factors suffices in all dimensions. Also, it remains open, even for one-variable maps, whether results such as these hold for convergent power series. These are interesting problems.

One might wonder whether the coefficient field $\mathbb{C}$ may be replaced by another in these results. In our arguments, the properties of $\mathbb{C}$ that we use are the fact that it has characteristic zero and is algebraically-closed. We have not investigated more general fields. The paper [25] gave a complete account of reversibility and factorization into reversibles in the one-dimensional formal map group for arbitrary coefficient fields of characteristic zero. As far as we know, there is little known about reversibility when the characteristic of the coefficient field is finite. One should mention that, thanks to Klopsch [6, p.16], [20] the involutions (and indeed the elements of finite order) have been identified for the so-called Nottingham groups (the one-dimensional case in which the coefficient field is finite), at least when the order of the field is odd.

## 2. Notation and Preliminaries

2.1. **Power Series Structures.** For $n \in \mathbb{N}$, let $\mathfrak{F}_n$ denote the ring of formal power series in $n$ (commuting) variables, with complex coefficients, and let $\mathfrak{F}_n^\times$ denote the multiplicative group of its invertible elements, i.e. those with nonzero constant term, and let $\mathfrak{M}_n$ denote the complementary set $\mathfrak{F}_n \sim \mathfrak{F}_n^\times$, the maximal ideal. Then an element of the set $\mathfrak{S}_n = (\mathfrak{M}_n)^n$ of $n$-tuples of elements of $\mathfrak{M}_n$ may be thought of as a formal map of $\mathbb{C}^n \to \mathbb{C}^n$, taking 0 to 0. Under formal composition, $\mathfrak{S}_n$ is a semigroup, with identity $\mathsf{id}_n(z) = z$. Let $\mathfrak{G}_n \subset \mathfrak{S}_n$ be the group $\mathfrak{G}_n$ of formally-invertible elements.

We remark that $\mathfrak{G}_n$ is isomorphic to the group of $\mathbb{C}$-algebra automorphisms of $\mathfrak{F}_n$. Indeed, if $g \in \mathfrak{G}_n$, then $f \mapsto f \circ g$ is an automorphism of $\mathfrak{F}_n$. Conversely, let $\Phi$ be any automorphism of $\mathfrak{F}_n$, and take $g = (\Phi(z_1), \cdots, \Phi(z_n))$. Then $\Phi$ must map the unique maximal ideal $\mathfrak{M}_n$ onto itself, and hence determines an automorphism of each quotient $\mathfrak{F}_n/(\mathfrak{M}_n^k)$. Since (the cosets of) $z_1,\ldots,z_n$ generate $\mathfrak{F}_n/(\mathfrak{M}_n^k)$, we have $\Phi(f) = f \circ g \mod \mathfrak{M}_n^k$ for each $f$. Since this holds for each $k \in \mathbb{N}$, we conclude that $\Phi$ is just $f \mapsto f \circ g$.

2.2. **The map $L$.** A typical element $F \in \mathfrak{S}_n$ takes the form

$$F(z) = (F_{,1}(z), \ldots, F_{,n}(z)) = (F_{,1}(z_1, \ldots, z_n), \ldots, F_{,n}(z_1, \ldots, z_n))$$

where each $F_{,j}(z)$ is a power series in $n$ variables having complex coefficients, and no constant term. We shall refer to such series $F$ as maps, even though they may be just 'formal', i.e. the series may fail to converge at any $z \neq 0$.

We usually write the formal composition of two maps $F, G \in \mathfrak{S}_n$ as $FG$. We also write the product of two complex numbers $a$ and $b$ as $ab$, but in cases where there might be some ambiguity we use $a \cdot b$. For $n$-tuples $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ (of various kinds) we also use $a \cdot b$ for the 'dot' product $a_1 \cdot b_1 + \cdots + a_n \cdot b_n$, and, a little more unusually, we will use $a \times b$ for the *coordinatewise product*:

$$(a_1, \ldots, a_n) \times (b_1, \ldots, b_n) := (a_1 b_1, \ldots, a_n b_n).$$

The series $F$ may be expressed as a sum

$$F = \sum_{k=1}^{\infty} L_k(F),$$

where $L_k(F)$ is homogeneous of degree $k$. We abbreviate $L_1(F)$ to $L(F)$. This term, the *linear part of* $F$, belongs to the algebra of $n \times n$ matrices.

An element $F$ of $\mathfrak{S}_n$ belongs to $\mathfrak{G}_n$ if and only its linear part $L(F)$ belongs to the general linear group $\mathsf{GL}(n, \mathbb{C})$.

We have the inclusion $\mathsf{GL}(n, \mathbb{C}) \to \mathfrak{G}_n$, and $L : \mathfrak{G}_n \to \mathsf{GL}(n, \mathbb{C})$ is a group homomorphism. We always identify $\mathsf{GL}(n, \mathbb{C})$ with its image in $\mathfrak{G}_n$.

The elements of the kernel of $L$ are said to be *tangent to the identity*.

2.3. **Elements of Finite Order.** We note the following [27, Lemma 2.1]:

**Lemma 2.1.** *Let $n \in \mathbb{N}$ and let $\mathfrak{H}$ be a subgroup of $\mathfrak{G}_n$ such that*
*(1) $L(F) \in \mathfrak{H}$ whenever $F \in \mathfrak{H}$, and*
*(2) $\mathfrak{H} \cap \ker L$ is closed under convex combinations, i.e. if $F_1, F_2 \in \mathfrak{H}$, $L(F_1) = L(F_2) = \mathsf{id}$ and $0 < \alpha < 1$, then $\alpha F_1 + (1-\alpha) F_2 \in \mathfrak{H}$.*
*Suppose $\Theta \in \mathfrak{H}$ has finite order. Then $\Theta$ is conjugated by an element of $\mathfrak{H} \cap \ker L$ to its linear part $L(\Theta)$.*

This applies to $\mathfrak{H} = \mathfrak{G}_n$, $\mathfrak{G}_n \cap \ker L$, $\mathfrak{G}_n \cap \ker(\det \circ L) = L^{-1}(\mathsf{SL}(n, \mathbb{C}))$ (and, more generally to $L^{-1}(H)$ for any subgroup $H \leq \mathsf{GL}(n, \mathbb{C})$), to the corresponding subgroups of biholomorphic germs (i.e. series that converge on a neighbourhood of the origin) and to other subgroups introduced below. It applies to the intersection of any two groups to which it applies.

In particular, in any $\mathfrak{H}$ to which the lemma applies, each involution is conjugate to one of the linear involutions in the group. In $\mathsf{GL}(n, \mathbb{C})$, a matrix is an involution if and only if it is diagonalizable with eigenvalues $\pm 1$.

Thus the involutions in $\mathfrak{G}_n$ are all conjugate to their linear parts, which are involutions in $\mathsf{GL}(n,\mathbb{C})$, and are classified up to conjugacy by the dimension of the eigenspace of the eigenvalue 1. Thus there are just $n$ conjugacy classes of proper involutions, and condition (4) in Corollary 1.5 says that for $n \geq 3$ one may represent any such $F$ as the product of at most $9 + 6 \cdot \text{ceiling}(\log_2 n)$ elements drawn from this small collection of classes.

We remark that there are also just a finite number of conjugacy classes in $\mathfrak{G}_n$ of maps of order dividing 4. The number is the number of ordered partitions of $n$ as a sum of 4 nonnegative integers, which equals $\binom{n+3}{3}$.

2.4. **Linear reversibles.** Reversibility is preserved by homomorphisms, so a map $F \in \mathfrak{G}_n$ is reversible only if $L(F)$ is reversible in $\mathsf{GL}(n,\mathbb{C})$. Classification of linear reversible maps is simple. Suppose $F \in \mathsf{GL}(n,\mathbb{C})$ is reversible. Since the Jordan normal form of $F^{-1}$ consists of blocks of the same size as $F$ with inverse eigenvalues, the eigenvalues of $F$ that are not $\pm 1$ must split into groups of pairs $\lambda, \lambda^{-1}$. Furthermore, we must have the same number of Jordan blocks of each size for $\lambda$ as for $\lambda^{-1}$. Vice versa, if the eigenvalues of $F$ are either $\pm 1$ or split into groups of pairs $\lambda, \lambda^{-1}$ with the same number of Jordan blocks of each size, then both $F$ and $F^{-1}$ have the same Jordan normal form and are therefore conjugate to each other.

2.5. **The Groups $D \leq \mathsf{GL}(2,\mathbb{C})$ and $D_n \in \mathsf{GL}(n,\mathbb{C})$.** In particular, a linear map is reversible in $\mathsf{GL}(2,\mathbb{C})$ if and only if it is an involution or is conjugate to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ or to a matrix of the form

$$\tau_\mu = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}, \tag{2.1}$$

for some $\mu \in \mathbb{C}^\times$. Thus each reversible $F \in \mathfrak{G}_2$ is conjugate in $\mathfrak{G}_2$ (by a linear conjugacy) to a map having one of these types as its linear part.

The collection of maps $\tau_\mu$, defined in (2.1) forms an abelian subgroup of $\mathfrak{G}_2$, which we denoted by $D$ in [27]. The element (2.1) has infinite order precisely when $\mu$ is not a root of unity, and this is what we regarded as the generic situation when $n = 2$.

We now extend this notation to higher dimensions.

When $n = 2m \geq 2$ is even, we denote by $D_n$ the set of maps $T \in \mathfrak{G}_n$ of the form

$$T(z) = (T_1(z_1, z_2), \ldots, T_m(z_{n-1}, z_n)),$$

where each $T_j \in D$.

When $n = 2m + 1 \geq 3$ is odd, we denote by $D_n$ the set of maps $T \in \mathfrak{G}_n$ of the form

$$T(z) = (T_1(z_1, z_2), \ldots, T_m(z_{n-2}, z_{n-1}), z_n),$$

where each $T_j \in D$, i.e. $T = T' \times \mathsf{id}_1$, where $T' \in D_{n-1}$ and $\mathsf{id}_1$ is the identity map of $\mathbb{C}$.

In either case ($n = 2m$ or $n = 2m + 1$), $D_n$ is a subgroup of $\mathfrak{G}_n$, isomorphic to the $m$-fold cartesian product $D^m$.

An element $T \in D_n$ is called *generic* if the associated $T_j = \tau(\mu_j)$, where there is no "resonance" relation

$$\mu_1^{r_1} \cdots \mu_m^{r_m} = 1$$

with each $r_j \in \mathbb{Z}$, except the trivial relation with all $r_j = 0$. If $T$ is generic, then in particular no $\mu_j$ is a root of unity. One could rephrase the condition as stating that the $\mu_j$ generate a free abelian subgroup of $\mathbb{C}^\times$ of rank $m$.

We shall make use of the classical Poincaré-Dulac Theorem [18, Section 4.8, Theorem 4.22], and we state it here in our language, for the reader's convenience:

**Theorem 2.2** (Poincaré-Dulac)**.** *Each map $F \in \mathfrak{G}_n$ is conjugate in $\mathfrak{G}_n$ to a map in the centralizer in $\mathfrak{G}_n$ of the linear part $L(F)$.* □

In case $L(F)$ is a generic member of $D_n$ we shall see shortly (cf. Lemma 2.3) that the centralizer of $L(F)$ in $\mathfrak{G}_n$ coincides with the centralizer of the whole subgroup $D_n$ in $\mathfrak{G}_n$.

2.6. **The Group $\mathfrak{C}_n = C_{D_n}(\mathfrak{G}_n)$.** In what follows, we shall usually have to distinguish odd and even $n \geq 2$. When $z \in \mathbb{C}^n$ with $n = 2m$ or $n = 2m+1$, we define

$$p(z) = (z_1 z_2, \ldots, z_{2m-1} z_{2m}),$$

and we set

$$\pi(z) := \begin{cases} p & , \quad n = 2m, \\ (p, z_n) & , \quad n = 2m+1, \end{cases}$$

Both $p$ and $\pi$ depend (implicitly) on $n$.

It is convenient, when dealing with $\mathfrak{G}_n$ for a given $n \geq 2$, to denote by $k$ the number

$$k = \begin{cases} m & , \quad n = 2m, \\ m+1 & , \quad n = 2m+1. \end{cases}$$

Thus $m$ is the floor of $n/2$, and $k$ is its ceiling. We shall assume this relation between $n$, $m$ and $k$, always.

The map $\pi$ sends $\mathbb{C}^n$ onto $\mathbb{C}^k$. A right inverse for $\pi$ is the map $\epsilon = \epsilon_n \colon \mathbb{C}^k \to \mathbb{C}^n$, given by

$$\epsilon(t_1, \ldots, t_m) = (t_1, 1, \ldots, t_m, 1)$$

when $n = 2m$, and

$$\epsilon(t_1, \ldots, t_m, t_k) = (t_1, 1, \ldots, t_m, 1, t_k)$$

when $n = 2m+1$.

We note, for future reference, that these maps preserve the coordinatewise product:

$$\pi(a \times b) = \pi(a) \times \pi(b), \ \forall a, b \in \mathbb{C}^n$$

and

$$\epsilon(a \times b) = \epsilon(a) \times \epsilon(b), \ \forall a, b \in \mathbb{C}^k.$$

**Lemma 2.3.** *Let $n \geq 2$, and $F \in \mathfrak{G}_n$. Then the following are equivalent:*
*(1) $F$ commutes with each element of $D_n$.*
*(2) For some generic $\Lambda \in D_n$, $F$ commutes with $\Lambda$.*
*(3)*
*If $n = 2m$ is even, then $F$ takes the form*

$$F(z) = (z_1\varphi_1(p), \dots, z_n\varphi_n(p)),$$

*for some $\varphi_j \in \mathfrak{F}_m^\times$.*
*If $n = 2m+1$ is odd, then $F$ takes the form*

$$F(z) = (z_1\varphi_1(p, z_n), \dots, z_{2m}\varphi_{2m}(p, z_n), z_n\varphi_n(p, z_n) + p_1\zeta_1(p) + \dots + p_m\zeta_m(p))$$

*for some $\varphi_j \in \mathfrak{F}_{m+1}^\times$ and some $\zeta_k \in \mathfrak{F}_m$.*

*Proof.* The only nontrivial implication is (2) $\Longrightarrow$ (3).

Suppose (2) holds, and fix $F \in \mathfrak{G}_n$ commuting with some generic $\Lambda \in D_n$.

Case 1°: $n = 2m$, even.

Then $\Lambda = \operatorname{diag}(\lambda_1, 1/\lambda_1, \dots, \lambda_m, 1/\lambda_m)$, and there is no nontrivial resonance relation $\prod \lambda_j^{\kappa_j} = 1$.

We may write $F_{,1}$, the first component of $F$, in the form

$$z_1\psi_1(z_1, \dots, z_n) + z_2\psi_2(z_2, \dots, z_n) + \dots + z_n\psi_n(z_n),$$

where $\psi_j \in \mathfrak{F}_{n-j+1}$ (by gathering all monomial terms that involve $z_1$ into the term $z_1\psi_1(z_1, \dots, z_n)$, all the terms that involve $z_2$ but not $z_1$ into the next, and so on). Equating the first components in the two sides of the equation $F\Lambda = \Lambda F$ gives

$$\begin{aligned} & \lambda_1 z_1\psi_1(\Lambda z) + \frac{z_2}{\lambda_1}\psi_2\left(\frac{z_2}{\lambda_1}, \lambda_2 z_3, \dots\right) + \dots + \frac{z_n}{\lambda_m}\psi_n\left(\frac{z_n}{\lambda_m}\right) \\ = \quad & \lambda_1 z_1\psi_1(z_1, \dots, z_n) + \lambda_1 z_2\psi_2 + \dots + \lambda_1 z_n\psi_n(z_n). \end{aligned}$$

Now, equating the coefficients of each monomial on the two sides, and using nonresonance, gives that $\psi_2 = \dots = \psi_n = 0$ and $\psi_1(\Lambda z) = \psi_1(z)$, so that $\psi_1(z)$ depends only on $p$. Thus the first component of $F$ has the desired form.

A similar argument shows that each other component takes the form in (3), so (3) holds.

Case 2°: $n = 2m+1$, odd. This time $\Lambda = \operatorname{diag}(\lambda_1, 1/\lambda_1, \dots, \lambda_m, 1/\lambda_m, 1)$, and again there is no nontrivial resonance relation between the $\lambda_j$, $j = 1,\dots,m$.

Focussing, as before, on the first component in the identity $F\Lambda = \Lambda F$, we have

$$\begin{aligned} & \lambda_1 z_1\psi(\Lambda z) + \frac{z_2}{\lambda_1}\psi_2\left(\frac{z_2}{\lambda_1}, \lambda_2 z_3, \dots\right) + \dots + \frac{z_{2m}}{\lambda_m}\psi_n\left(\frac{z_{2m}}{\lambda_m}, z_n\right) + z_n\psi_n(z_n) \\ = \quad & \lambda_1 z_1\psi_1(z_1, \dots, z_n) + \lambda_1 z_2\psi_2 + \dots + \lambda_1 z_{2m}\psi_{2m}(z_{2m}, z_n) + \lambda_1 z_n\psi_n(z_n). \end{aligned}$$

Identifying terms, as before, we see that it proceeds just as in the even case (with $z_n$ as an added parameter), for $\psi_1,\dots,\psi_{2m}$, and find that $\psi_1$ depends only on $p$ and $z_n$, and that $\psi_2 = \dots = \psi_{2m} = 0$. Finally, $\psi_n(z_n) = 0$, since $\lambda_1 \neq 1$, so that the first component of $F$ takes the desired form.

A similar argument looks after all the components except the last.

Writing $z = (z', z_n)$, with $z' \in \mathbb{C}^{2m}$, we may write the $n$-th component of $F$ in the form

$$F_{,n}(z) = z' \cdot G(z') + z_n \psi_n(z),$$

where $\psi_n \in \mathfrak{F}_n$ and $G(z') \in (\mathfrak{F}_{2m})^{2m}$ is a $2m$-vector of power series in $2m$ variables, and $\cdot$ here denotes the dot product. (This $\psi_n$ is not the one used in the argument about the first component, the one that turned out to be zero.) The last component of the identity $F\Lambda = \Lambda F$ then yields

$$(\Lambda' z') \cdot G(\Lambda' z') + z_n \psi_n(\Lambda z) = z' \cdot G(z') + z_n \psi_n(z),$$

where $\Lambda'$ denotes $\mathrm{diag}(\lambda_1, 1/\lambda_1, \ldots, \lambda_m, 1/\lambda_m)$. This tells us that $\psi_n(z)$ depends only on $p$ and $z_n$, and that $z' \cdot G(z')$ depends only on $p$, and hence takes the form $p \cdot \zeta(p)$, for some $m$-tuple $\zeta \in (\mathfrak{F}_m)^m$. Thus (3) holds. □

*Remark* 2.4. We note that by condition (3) of the lemma, each $F \in \mathfrak{C}_n$ has a diagonal linear part $L(F)$, because the terms $p \cdot \zeta(p)$ that occur in the odd case are at least quadratic, so that in all cases $L(F) = \mathrm{diag}(\varphi_1(0), \ldots, \varphi_n(0))$.

**Definition 2.5.** We denote by $\mathfrak{C}_n$ the group of all maps $F \in \mathfrak{G}_n$ that satisfy any of the equivalent conditions of Lemma 2.3.

2.7. **The Functions $M$, $\hat{M}$ and the Involution $J$.** In terms of the coordinatewise product, in the even case $n = 2m$ we may represent the $F(z)$ in condition (3) of the lemma more compactly as $z \times \varphi(p)$. We also denote this map $F$ by $M(\varphi)$. Thus $M = M_n$ is a bijection from $(\mathfrak{F}_m^\times)^{2m}$ onto $\mathfrak{C}_{2m}$. It is *not*, however, a homomorphism from the abelian product group structure of $(\mathfrak{F}_m^\times)^{2m}$.

For $n = 2m+1$ odd, we note from the proof of the lemma that for $F \in \mathfrak{C}_n$, the last component $F(z)_{,n}$ takes the form $\psi(\pi(z)) = \psi(p, z_n)$, where $\psi \in \mathfrak{M}_k$ is completely unrestricted, except that it must have a nonzero coefficient on the monomial $z_n$. We denote the set of such $\psi$ by $\hat{\mathfrak{M}}_k$, and we refer to them as *admissible* elements of $\mathfrak{M}_k$. Thus $F$ takes the form

$$F(z) = (z_1 \varphi_1(p, z_n), \ldots, z_{2m} \varphi_{2m}(p, z_n), \psi(p, z_n)),$$

with $\varphi \in (\mathfrak{F}_k^\times)^{2m}$ and $\psi \in \hat{\mathfrak{M}}_k$. We denote this $F$ by $M_n(\varphi, \psi) = \hat{M}(\varphi, \psi)$. As before, $M_n = \hat{M}$ is a bijection from its domain onto $\mathfrak{C}_n$.

Denoting $j(z') = (z', 0)$ for $z' \in \mathbb{C}^{2m}$, we may write

$$\hat{M}(\varphi, \psi)(z', z_n) = j(z' \times \varphi(p, z_n)) + \psi(p, z_n) e_n,$$

where $e_n$ denotes the last vector of the standard basis of $\mathbb{C}^n$.

Notice that $M_n$ has a rather different kind of domain, depending on the parity of $n$.

We shall also use the notation $J$ for the involutive element of $\mathfrak{G}_n$ defined by

$$J(z) = \begin{cases} (z_2, z_1, z_4, z_3, \ldots, z_{2m}, z_{2m-1}) & , \quad n = 2m, \\ (z_2, z_1, z_4, z_3, \ldots, z_{2m}, z_{2m-1}, z_n) & , \quad n = 2m+1. \end{cases}$$

Observe that $J$ reverses every $\Lambda \in D_n$, i.e. $J^{-1} \Lambda J = \Lambda^{-1}$.

2.8. **The Groups $\mathfrak{K}_m$ and $\hat{\mathfrak{K}}_k$.** For $m \in \mathbb{N}$, we denote by $\mathfrak{K}_m$ the set of elements $F \in \mathfrak{G}_m$ that take the form

$$F(t) = (t_1\varphi_1(t), \dots, t_m\varphi_m(t)),$$

with each $\varphi_j \in \mathfrak{F}_m^\times$. One readily checks that $\mathfrak{K}_m$ is a subgroup of $\mathfrak{G}_m$.
We use the notation $N(\varphi) = N(\varphi_1, \dots, \varphi_m)$ to denote $F$ of the above form. Using the coordinate-wise product, we also write $F(t) = t \times \varphi(t)$ and $N(\varphi) = \mathsf{id}_m \times \varphi$.

For $k = m + 1$, we denote by $\hat{\mathfrak{K}}_k$ the set of elements $F \in \mathfrak{G}_k$ that (with $t = (t', t_k)$) take the form

$$F(t) = (t_1\varphi_1(t), \dots, t_m\varphi_m(t), t_k\varphi_k(t) + t' \cdot \zeta(t')),$$

with each $\varphi_j \in \mathfrak{F}_k^\times$ and $\zeta \in \mathfrak{F}_m$. We remark that *every* series $g(t) \in \mathfrak{M}_{m+1}$ with $g(0) = 0$ may be written in the form $t_k\varphi_k(t) + t' \cdot \zeta(t')$ for some $\varphi_k \in \mathfrak{F}_k$ and $\zeta \in \mathfrak{F}_m$, so that the form of the last component $F_{,k}$ is restricted only by the requirement that the coefficient of the monomial $t_k$ be nonzero. This requirement is obviously needed for the invertibility of $F$. Thus $\hat{\mathfrak{K}}_k$ consists of the maps of the form

$$F(t) = (t_1\varphi_1(t), \dots, t_m\varphi_m(t), \psi(t)),$$

with $\varphi \in (\mathfrak{F}_k^\times)^m$ and $\psi \in \hat{\mathfrak{M}}_k$ (i.e. $\psi$ admissible). It is routine to check that $\hat{\mathfrak{K}}_k$ is a subgroup of $\mathfrak{G}_k$.

We use the notation

$$\hat{N}(\varphi, \psi) = \hat{N}(\varphi_1, \dots, \varphi_k, \psi_1, \dots, \psi_m)$$

to denote $F$ of the above form, and we may also write $F(t', t_k) = j(t' \times \varphi(t)) + \psi(t)e_k$ and $\hat{N}(\varphi, \psi) = j \circ (\mathsf{id}_k \times \varphi) + \psi e_k$, where $e_k$ stands for the vector $(0, \dots, 0, 1) \in \mathbb{C}^k$.

2.9. **The Homomorphisms $P$, $H$, and $\Phi$.** If $n = 2m = 2k$ is even, then to $F = M(\varphi) \in \mathfrak{C}_n$ we associate the $k$ variable map $P_n(F) \in \mathfrak{G}_k$ defined by

$$P(F)(t) = t \times \pi(\varphi(t)) = (t_1 \cdot \varphi_1(t) \cdot \varphi_2(t), \dots, t_m \cdot \varphi_{2m-1}(t) \cdot \varphi_{2m}(t)).$$

If $n = 2m + 1 = 2k - 1$ is odd, then to $F = \hat{M}(\varphi, \psi) \in \mathfrak{C}_n$ we associate the $k$ variable map

$$P(F)(t) = (t', 1) \times \pi(\varphi(t), \psi(t)) = (t_1 \cdot \varphi_1(t) \cdot \varphi_2(t), \dots, t_m \cdot \varphi_{2m-1}(t) \cdot \varphi_{2m}(t), \psi(t)),$$

where $t = (t', t_k)$.

We have the basic semiconjugation property:

**Lemma 2.6.** *For each $F \in \mathfrak{C}_n$, we have*

$$P(F) \circ \pi = \pi \circ F. \tag{2.2}$$

*Moreover, property* (2.2) *determines $P(F)$ uniquely.* □

Using the maps $p$ and $\pi$, we may rewrite the definition of $P$ as:

$$\begin{aligned} P(M(\varphi)) &= N(\pi \circ \varphi) = N(p \circ \varphi), && \text{if } n = 2m, \\ P(\hat{M}(\varphi, \psi)) &= \hat{N}(\pi \circ (\varphi, \psi)) = \hat{N}(p \circ \varphi, \psi), && \text{if } n = 2m + 1, \end{aligned} \tag{2.3}$$

where $\varphi$ runs through $(\mathfrak{F}_k^\times)^{2m}$ and in the odd case $\psi$ runs through $\hat{\mathfrak{M}}_k$.

**Lemma 2.7.** *$P : \mathfrak{C}_n \to \mathfrak{G}_k$ is a group homomorphism.*

*Proof.* This follows without further calculation from the uniqueness in (2.2) and the associativity of composition: If $F, G \in \mathfrak{C}$, then $P(F)P(G)\pi = P(F)\pi G = \pi FG$, so $P(FG) = P(F)P(G)$. □

However, it is useful to note the explicit formulas for compositions of maps in the images of $M$ and $N$, which are readily proved by direct calculation:

**Lemma 2.8.** *Let $\varphi$,$\varphi'$ and $\varphi'' \in (\mathfrak{F}_m^\times)^{2m}$. Let $\chi' = P(M(\varphi'))$. Then the following are equivalent:*
*(1)$M(\varphi'') = M(\varphi)M(\varphi')$.*
*(2) $\varphi''(t) = \varphi'(t) \times \varphi(\chi'(t))$.*
*(3) $\varphi'' = \varphi' \times \varphi \circ (\mathsf{id}_m \times \pi \circ \varphi')$.* □

**Lemma 2.9.** *Let $\lambda$,$\lambda'$ and $\lambda'' \in (\mathfrak{F}_m^\times)^m$. Let $\chi' = N(\lambda')$. Then the following are equivalent:*
*(1)$N(\lambda'') = N(\lambda)N(\lambda')$.*
*(2) $\lambda''(t) = \lambda'(t) \times \lambda(\chi'(t))$.*
*(3) $\lambda'' = \lambda' \times \lambda \circ (\mathsf{id}_m \times \lambda')$.*

Similarly, for $\hat{M}$ and $\hat{N}$, we have:

**Lemma 2.10.** *Let $\varphi$,$\varphi'$ and $\varphi'' \in (\mathfrak{F}_{m+1}^\times)^{2m+1}$, and $\psi$,$\psi'$ and $\psi'' \in \hat{\mathfrak{M}}_{m+1}$. Let $\chi' = P(\hat{M}(\varphi', \psi'))$. Then the following are equivalent:*
*(1)$\hat{M}(\varphi'', \psi'') = \hat{M}(\varphi, \psi)\hat{M}(\varphi', \psi')$.*
*(2) $\varphi''(t) = \varphi'(t) \times \varphi(\chi'(t))$ and $\psi''(t) = \psi(\chi'(t))$.*
*(3) $\varphi'' = \varphi' \times \varphi \circ (\mathsf{id}_m \times \pi \circ \varphi')$ and $\psi'' = \psi \circ \chi'$.* □

**Lemma 2.11.** *Let $\lambda$,$\lambda'$ and $\lambda'' \in (\mathfrak{F}_m^\times)^m$ and $\psi$,$\psi'$ and $\psi'' \in \hat{\mathfrak{M}}_{m+1}$. Let $\chi' = \hat{N}(\varphi', \psi')$. Then the following are equivalent:*
*(1)$\hat{N}(\lambda'', \psi'') = \hat{N}(\lambda, \psi)\hat{N}(\lambda', \psi')$.*
*(2) $\lambda''(t) = \lambda'(t) \times \lambda(\chi'(t))$ and $\psi''(t) = \psi(\chi'(t))$.*
*(3) $\lambda'' = \lambda' \times \lambda \circ (\mathsf{id}_m \times \lambda')$ and $\psi'' = \psi \circ \chi'$.*

The fact that $P$ is a homomorphism is obtained again in the even case by precomposing $\pi$ with the equation in part (3) of Lemma 2.8 and using $P(M(\varphi)) = \mathsf{id}_m \times \pi \circ \varphi$. In fact, $\pi(a \times b) = \pi(a) \times \pi(b)$, so (3) gives

$$\pi \circ \varphi'' = (\pi \circ \varphi') \times (\pi \circ \varphi) \circ (\mathsf{id}_m \times \pi \circ \varphi').$$

A similar argument applies in the odd case, using Lemmas 2.10 and 2.11.

The kernel of $P$ is the set of maps $F$ of the form

$$F(z) = \begin{cases} M(\varphi) & , \quad n = 2m = 2k, \\ M(\varphi, 1) & , \quad n = 2m+1 = 2k-1 \end{cases}$$

where $\varphi \in (\mathfrak{F}_k^\times)^{2m}$ and $\varphi_{2j-1}(t) = 1/\varphi_{2j}^{-1}(t)$ for $j = 1, \ldots, m$. The group $\ker P$ is abelian.

For $n = 2m$, the map

$$\Phi : \begin{cases} (\mathfrak{F}_k^\times)^m & \to \quad \mathfrak{C}_n, \\ \qquad \varphi & \mapsto \quad M\left(\varphi_1, 1/\varphi_1, \ldots, \varphi_m, 1/\varphi_m\right)\right), \end{cases}$$

is a group isomomorphism onto $\ker P$.

For $n = 2m + 1$, the corresponding isomorphism onto $\ker P$ is

$$\Phi : \begin{cases} (\mathfrak{F}_k^\times)^m & \to \quad \mathfrak{C}_n, \\ \qquad \varphi & \mapsto \quad M\left(\varphi_1, 1/\varphi_1, \ldots, \varphi_m, 1/\varphi_m, 1)\right), \end{cases}$$

Note that in each case the image of $\Phi$ consists of reversible elements. All are reversed by $J$.

Clearly, when $n = 2m$, the image of $P$ lies in $\mathfrak{K}_m$, and when $n = 2m + 1$, the image lies in $\hat{\hat{\mathfrak{K}}}_k$. To see that these are the exact images of $P$ in the respective cases, we define right inverse maps:

For $n = 2m$, we define $H : \mathfrak{K}_m \to \mathfrak{C}_n$ by

$$H(N(\lambda))(z) = M(\epsilon \circ \lambda) = M(\lambda_1, 1, \lambda_2, 1, \ldots, \lambda_m, 1). \tag{2.4}$$

From the definition of $P$, we see that $P(M(\varphi)) = N(\pi \circ \varphi)$ so obviously $P(H(N(\lambda))) = N(\pi\epsilon\lambda) = N(\lambda)$, as required.

(Moreover, for each $\chi \in \mathfrak{K}_m$, $H(\chi)$ is the unique element $F \in \mathfrak{C}_n$ with the properties $F(z)_{,j} = z_j$ for all even $j$ and $F \circ \epsilon = \epsilon \circ \chi$.)

For $n = 2k - 1$, we define $H = H_n : \hat{\hat{\mathfrak{K}}}_k \to \mathfrak{C}_n$ by

$$H(\hat{N}(\lambda, \psi))(z) = \hat{M}(\epsilon \circ (\lambda, \psi))(z) = \hat{M}(\lambda_1, 1, \lambda_2, 1, \ldots, \lambda_m, 1, \psi)(z). \tag{2.5}$$

In this case, the definition of $P$ amounts to

$$P(\hat{M}(\varphi, \psi)) = \hat{N}(p \circ \varphi, \psi),$$

so again $P(H(N(\lambda, \psi))) = N(\lambda, \psi)$, as required.

(Moreover, for each $\chi \in \hat{\hat{\mathfrak{K}}}_k$, $H_n(\chi)$ is the unique element $F \in \mathfrak{C}_n$ with the properties $F(z)_{,j} = z_j$ for all even $j$ and $F \circ \epsilon = \epsilon \circ \chi$.)

So we have proved:

**Lemma 2.12.** *The image of $P$ is $\mathfrak{K}_m$ if $n = 2m$ and is $\hat{\hat{\mathfrak{K}}}_k$ if $n = 2k - 1$.* □

Next we have:

**Lemma 2.13.** *For each $n \geq 2$, $H_n$ is a group homomorphism.*

*Proof.* We give the explicit version, taking the cases separately.

1°: $n = 2m$. Fix two elements $\chi$,$\chi' \in \mathfrak{K}_m$, and let $\chi'' = \chi\chi'$. There are unique elements $\lambda$,$\lambda'$ and $\lambda'' \in (\mathfrak{F}_m^\times)^m$ such that $\chi = N(\lambda)$, $\chi' = N(\lambda')$, and $\chi'' = N(\lambda'')$. By Lemma 2.8 and the definition of $H$, it suffices to show that

$$\epsilon \circ \lambda'' = \epsilon \circ \lambda' \times \epsilon \circ \varphi \circ (\mathsf{id}_m \times \pi \circ \epsilon \circ \varphi')$$

But this is immediate from Lemma 2.9 and the fact that $\pi \circ \epsilon = \mathsf{id}_m$.

$$
\begin{array}{ccccccccc}
(1) & \rightarrow & (\mathfrak{F}_m^{\times})^m & \underset{\Phi}{\rightarrow} & \mathfrak{C}_{2m} & \overset{H}{\underset{P}{\leftrightarrows}} & \mathfrak{K}_m & \rightarrow (1) & \\
 & & & & \cap & & \cap & & \\
 & & & & \mathfrak{G}_{2m} & & \mathfrak{G}_m & & \\
 & & & & & & & & \\
(1) & \rightarrow & (\mathfrak{F}_{m+1}^{\times})^m & \underset{\Phi}{\rightarrow} & \mathfrak{C}_{2m+1} & \overset{H}{\underset{P}{\leftrightarrows}} & \hat{\mathfrak{K}}_{m+1} & \rightarrow & (1) \\
 & & & & \cap & & \cap & & \\
 & & & & \mathfrak{G}_{2m+1} & & \mathfrak{G}_{m+1} & &
\end{array}
$$

**Figure 1:** Exact Sequences

2°: $n = 2m + 1$. Fix $\chi$,$\chi' \in \hat{\mathfrak{K}}_k$, and let $\chi'' = \chi\chi'$. There are unique elements $\lambda$,$\lambda'$ and $\lambda'' \in (\mathfrak{F}_k^{\times})^m$ and $\psi$,$\psi'$ and $\psi'' \in \hat{\mathfrak{M}}_k$ such that $\chi = \hat{N}(\lambda, \psi)$, $\chi' = \hat{N}(\lambda', \psi')$, and $\chi'' = \hat{N}(\lambda'', \psi'')$. By the definition of $H$, we have to show that

$$\hat{M}(\epsilon \circ \lambda'', \psi'') = \hat{M}(\epsilon \circ \lambda, \psi)\hat{M}(\epsilon \circ \lambda', \psi').$$

By Lemma 2.10, in view of the fact that

$$P(\hat{M}(\epsilon \circ \lambda', \psi')) = \hat{N}(\pi \circ \epsilon \circ (\lambda', \psi')) = \hat{N}(\lambda', \psi') = \chi',$$

this amounts to showing that

$$\epsilon \circ \lambda'' = \epsilon \circ \lambda' \times \epsilon \circ \lambda \circ \chi' \text{ and } \psi'' = \psi \circ \chi'.$$

But this is immediate from Lemma 2.11. □

**Corollary 2.14.** *Let $n \geq 2$. The group $\mathfrak{C}_n$ is the semidirect product of $\operatorname{im}\Phi$ and $\operatorname{im} H$. Each $F \in \mathfrak{C}_n$ has a unique factorization in the form $H(\chi)\Phi(\varphi)$, with $\chi \in \operatorname{im} P$ and $\varphi \in (\mathfrak{F}_k^{\times})^m$.*

*Proof.* This follows from the facts that $\operatorname{im}\Phi = \ker P$ is normal and that the homomorphism $H$ is a right inverse for $P$. □

However, $\mathfrak{C}_n$ is not the *direct* product of $\operatorname{im} H$ and $\operatorname{im}\Phi$.

The structural results of this subsection are summarized in Figure 1

## 3. Proof of Theorem 1.4 in Dimension 2

The case $n = 2$ of our main theorem serves as the foundation layer for an inductive proof of the general case, and now we lay this down.

For the reader's convenience, We include the short proof of our previously-published Theorem 1.3 (which is the same as part (1) of Theorem 1.4 in case $n = 2$).

*Proof.* Let $F \in \mathfrak{G}_2$ have $\det L(F) = 1$. We have to show that $F$ it may be factorized as $F = g_1 g_2 g_3 g_4$, where each $g_j$ is reversible in $\mathfrak{G}$.

In fact, if $\det L(F) = 1$, then multiplying by some (reversible) $\Lambda \in D$ (possibly the identity) we can arrange that $L(F\Lambda)$ is conjugate to an infinite-order element of $D$. Then by Poincaré-Dulac, $F\Lambda$ is conjugate (say by $K \in \mathfrak{G}$) to some element of $\mathfrak{C}$, so $(F\Lambda)^K$ may be factored as $H(\chi)\Phi(\varphi)$, where $\chi(t) = t + \text{HOT}$. Now $\Phi(\varphi)$ is reversible, and we know [25, Theorem 9(2)] that $\chi$ is the product of two reversibles in $\mathfrak{G}_1$, so $H(\chi)$ is the product of two reversibles, say $H(\chi_1)$ and $H(\chi_2)$. Thus

$$F^K = H(\chi_1)H(\chi_2)\Phi(\varphi)(\Lambda^{-1})^K$$

is the product of four reversibles, and conjugating with $K^{-1}$ we obtain the result. □

*Proof of Theorem 1.4 part (2) when $n = 2$.* Let $F \in \mathfrak{G}_2$ with $\det L(F) = +1$. With the notation in the last proof, $\Lambda^K$ and $\Phi(\varphi)$ are strongly-reversible, since $\Lambda$ and $\Phi(\varphi)$ are reversed by the involution $J$. Thus it suffices to prove that $H(\chi)$ is the product of 10 involutions, whenever $\chi(t) = t + \text{HOT} \in \mathfrak{G}_1$.

Now we may factor $\chi$ as $\chi_1\chi_2$, where these take the form

$$\begin{array}{rcl} \chi_1(t) & = & t + \alpha t^2 + \alpha^2 t^3, \\ \chi_2(t) & = & t + \beta t^3 + \text{HOT}. \end{array}$$

(possibly with $\alpha = 0$ or $\beta = 0$). The map $\chi_1$ is the identity or is conjugate to the map $f_1$, given by (cf. [25])

$$f_1(t) = \frac{t}{1-t} = t + t^2 + t^3 + t^4 + \text{HOT}, \tag{3.1}$$

reversed by $t \mapsto -t$, hence is the product of two involutions in $\mathfrak{G}_1$, and hence so is $H(\chi_1)$. Thus, since $H(\chi) = H(\chi_1)H(\chi_2)$, it suffices to show that $S = H(\chi_2)$ is the product of 8 involutions.

If $\beta = 0$, then $\chi_2$ is the product of 4 involutions in view of [25, Theorem 9], so we consider the case $\beta \neq 0$. By a conjugation in $\mathfrak{G}_1$, we may take $\beta = 1$, so $S$ takes the form

$$S(z) = \big(z_1(1 + p^2 + \text{HOT}), z_2\big)\,.$$

Take the map

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}(2, \mathbb{C}).$$

A calculation yields that, up to terms of degree 5,

$$T^{-1}ST(z) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} 1+A & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} az_1 + bz_2 \\ cz_1 + dz_2 \end{pmatrix} = \begin{pmatrix} z_1 + dB \\ z_2 - cB \end{pmatrix},$$

where $\det T = 1$, $A = (az_1 + bz_2)^2(cz_1 + dz_2)^2$ and $B = (az_1 + bz_2)^3(cz_1 + dz_2)^2$.

Let $\Lambda_1 = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$. Then $\Lambda_1 T^{-1} S T$ equals

$$\left(2z_1\left(1 + ad(a^2d^2 + 6abcd + 3b^2c^2)p^2\right), \frac{1}{2}z_2\left(1 - bc(b^2c^2 + 6abcd + 3a^2d^2)p^2\right)\right) \tag{3.2}$$

plus non-resonant terms of order 5. By the Poincaré-Dulac Theorem, $\Lambda_1 T^{-1} S T$ is conjugate to the map $S_1$ obtained by removing all non-resonant terms. A calculation shows that $S_1$ equals (3.2) up to terms of degree 5 in $z$. We now choose $T$ such that

$$ad(a^2d^2 + 6abcd + 3b^2c^2) = bc(b^2c^2 + 6abcd + 3a^2d^2), \tag{3.3}$$

or, substituting $ad = bc + 1$,

$$(bc+1)((bc+1)^2 + 6bc(bc+1) + 3b^2c^2) = bc(b^2c^2 + 6bc(bc+1) + 3(bc+1)^2), \tag{3.4}$$

which simplifies to

$$6b^2c^2 + 6bc + 1 = 0 \tag{3.5}$$

and clearly has a solution.

Then $S_1(z)$ factors as $H(\chi)\Phi(\varphi)$ with $\chi(t) = P(S_1)(t) = t + O(t^4)$. Hence by [25, Theorem 9], $\chi$ and therefore $H(\chi)$ is the product of four involutions. Thus $S_1$ is the product of 6 involutions. Thus $\Lambda_1 T^{-1} S T$ is the product of 6 involutions, so $S$ is the product of 8. This concludes the proof. □

Each product $F = f_1 \cdots f_n$ of reversible $f_j$'s has $\det L(F) = \pm 1$, so (multiplying if necessary by a suitable linear involution) it follows from Theorem 1.3 that each product of reversibles reduces to the product of five. It also follows that the elements that are products of reversibles are precisely those with $\det L(F) = \pm 1$. Thus the case $n = 2$ of Corollary 1.5 is immediate.

## 4. Proof of Theorem 1.4 in Dimension $n > 2$

We will actually prove a more refined result, in which the number of factors required depends in a more complicated way on the dimension $n$.

First, we introduce notation for the number of factors needed, in various situations:

For $n \geq 2$, let $r_1(n)$ denote the least $r \in \mathbb{N}$ such that each $F \in \mathfrak{G}_n$ having $\det L(F) = 1$ may be expressed as the product of $r$ reversible elements of $\mathfrak{G}_n$. Similarly, let $r_d(n)$ be the least number of reversible factors from $\mathfrak{G}_n$ required for the factorization of each $F \in \mathfrak{G}_n$ having $L(F) \in D_n$. Finally, let $r_c(n)$ be the least number of reversible factors from $\mathfrak{G}_n$ required for the factorization of each $F \in \mathfrak{C}_n$ having $L(F) \in D_n$.

It is obvious that

$$r_c(n) \leq r_d(n) \leq r_1(n) \tag{4.1}$$

whenever $n \geq 2$.

**Lemma 4.1.** *Let $n \geq 2$. Each diagonal matrix $T \in \mathsf{SL}(n, \mathbb{C})$ may be factored as the product of two diagonal matrices $T_1 T_2$, where $T_1 \in D_n$ and there is a permutation matrix $\sigma$ such that the conjugate $T_2^\sigma$ belongs to $D_n$.*

*Proof.* Let $T = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, and note that $\lambda_1 \cdots \lambda_n = 1$.

Take

$$T_1 = \mathrm{diag}(\lambda_1, 1/\lambda_1, \lambda_1\lambda_2\lambda_3, 1/(\lambda_1\lambda_2\lambda_3), \ldots),$$

and

$$T_2 = \mathrm{diag}(1, \lambda_1\lambda_2, 1/(\lambda_1\lambda_2), \lambda_1\lambda_2\lambda_3\lambda_4, 1/(\lambda_1\lambda_2\lambda_3\lambda_4), \ldots).$$

If $n$ is odd, then the last entry in $T_1$ is 1, so $T_1 \in D_n$. Since $T_2$ is conjugated into $D_n$ by the permutation $(1n)$ that swaps the coordinates $z_1$ and $z_n$, we are done, in this case.

If $n$ is even, then $T_1 \in D_n$, and $T_2$ has both first and last entries equal to 1, so it is conjugated into $D_n$ by the $n$-cycle $(12 \ldots n)$ that rotates the last coordinate back into first position, and shifts the others down. □

**Lemma 4.2.** *Each element of $D_n$ may be factored as the product of two generic elements of $D_n$.*

*Proof.* Let $T \in D_n$. Then $T = \Phi(\alpha)$ for some $\alpha \in (\mathbb{C}^\times)^m$. Choose $\lambda \in (\mathbb{C}^\times)^m$ such that $\lambda_j$ is multiplicatively independent of $\alpha_1,\ldots,\alpha_m,\lambda_1,\ldots,\lambda_{j-1}$, for each $j$. Take $T_1 = \Phi(\alpha \times \lambda)$ and $T_2 = \Phi(\lambda_1^{-1}, \ldots, \lambda_m^{-1})$. Then each $T_j$ is a generic element of $D_n$, and $T = T_1T_2$. □

**Lemma 4.3.** *Let $n \geq 2$. Each diagonal matrix $T \in \mathsf{SL}(n, \mathbb{C})$ may be factored as the product of three diagonal matrices $T_1T_2T_3$, where $T_1 \in D_n$ and there is a permutation matrix $\sigma$ such that the conjugates $T_2^\sigma$ and $T_3^\sigma$ belong to $D_n$, and $T_3^\sigma$ is a generic element.*

*Proof.* Let $T_1T_2'$ be the factorization and $\sigma$ the permutation given by Lemma 4.1, and apply Lemma 4.2 to $(T_2')^\sigma$. □

**Lemma 4.4.** *Let $n \geq 3$. Then $r_1(n) \leq r_c(n) + 2$.*

*Proof.* Fix $F \in \mathfrak{G}_n$ with $\det L(F) = 1$.

By using a linear conjugation, if need be, we may assume that $L(F)$ is in Jordan canonical form, so that the diagonal elements multiply to 1.

Write $L(F) = T + N$, where $T$ is diagonal and $N$ is strictly upper triangular. Applying the last lemma, we can write $T = T_1T_2T_3$, where $T_1 \in D_n$ and both $T_2$ and $T_3$ are diagonal, and conjugate by the same permutation $\sigma$ of coordinates to elements of $D_n$, with $T_3^\sigma$ generic. Let $F_1 = (T_1T_2)^{-1}F$. Then $L(F_1)$ is upper triangular, with the same diagonal as $T_3$.

The eigenvalues of $L(F_1)$ are its diagonal elements, and are distinct, so we may conjugate $L(F_1)$ to $T_3$ by using an element of $\mathsf{GL}(n, \mathbb{C})$. Applying the same conjugation to $F_1$, we conjugate $F_1$ to a map $F_2$ with $L(F_2) = T_3$. Applying Poincaré-Dulac, we can conjugate $F_2$ to a map $F_3$ that commutes with $T_3$, without changing the linear part, so $L(F_3) = T_3$. Then $F_3^\sigma$ commutes with $T_3^\sigma$, and hence belongs to $\mathfrak{C}_n$, and has $L(F_3^\sigma) = T_3^\sigma$.

Now $F_3^\sigma$ is the product of $r_c(n)$ reversibles, hence so are $F_3$, $F_2$ and $F_1$. Since $T_1$ and $T_2$ are reversible, $F$ is the product of $2 + r_c(n)$ reversibles. □

**Lemma 4.5.** *Let $n \geq 3$. Then $r_d(n) \leq r_c(n) + 1$.*

*Proof.* Fix $F \in \mathfrak{G}_n$ with $L(F) \in D_n$. By Lemma 4.2, we may factor $L(F) = T_1 T_2$, where each $T_j \in D_n$ is generic. Taking $F_1 = T_1^{-1} F$, we have $L(F_1) = T_2$, and applying Poincaré-Dulac we can conjugate $F_1$ to an element $F_2$ of $\mathfrak{C}_n$ having $L(F_2) = T_2$. Since $F_2$ is the product of $r_c(n)$ reversibles, so is $F_1$, and hence $F = T_1 F_1$ is the product of $1 + r_c(n)$. □

**Lemma 4.6.** *Let $m \geq 1$. Then*
*(1) $r_c(2m) \leq 1 + r_d(m)$, and*
*(2) $r_c(2m+1) \leq 1 + r_1(m+1)$.*

*Proof.* (1) Let $n = 2m$. Fix $F \in \mathfrak{C}_n$, with $L(F) \in D_n$. Then $\chi = P(F_1)$ belongs to $\mathfrak{G}_m$ and is tangent to the identity, so it may be factored as the product of $r_d(m)$ reversibles.

By Corollary 2.14, we can factor $F$ as $H(\chi)\Phi(\varphi)$, for some $\varphi \in (\mathfrak{F}_m^\times)^m$, and we know that $\Phi(\varphi)$ is reversed by $J$, so $F$ is the product of $1 + r_d(m)$ reversibles. Thus $r_c(n) \leq 1 + r_d(m)$.

(2) Let $n = 2m + 1 = 2k - 1$. Fix $F \in \mathfrak{C}_n$, with $L(F) \in D_n$. Then this time $\chi = P(F_1) \in \mathfrak{G}_k$ may fail to be tangent to the identity, or even to belong to $D_k$, but still has $\det L(\chi) = 1$, so it may be factored as the product of $r_1(k)$ reversibles. Proceeding as before, we get $r_c(n) \leq 1 + r_1(k)$, as required. □

**Corollary 4.7.** *If $n \geq 2$, then $r_1(n) \leq 1 + 3 \cdot \text{ceiling}(\log_2 n)$.*

*Proof.* We proceed inductively, starting at $n = 2$.

For $n = 2$, Theorem 1.3 tells us that $r_1(n) \leq 4 = 1 + 3 \cdot \text{ceiling}(\log_2 n)$.

Fix $n > 2$, and assume that for every $n' < n$, we have $r_1(n') \leq 1 + 3 \cdot \text{ceiling}(\log_2 n')$.

Then with $k$ as usual, Lemmas 4.4 and 4.6 and inequalities 4.1 yield

$$r_1(n) \leq 2 + r_c(n) \leq 3 + r_1(k) \leq 4 + 3 \cdot \text{ceiling}(\log_2 k)$$

so, since $\text{ceiling}(\log_2 k)$ is one less than $\text{ceiling}(\log_2 n)$, we have $r_1(n) \leq 1 + 3 \cdot \text{ceiling}(\log_2 n)$, and the induction step is complete. □

This Corollary has the same content as Theorem 1.4, part (1), so that is now proven.

*Proof of Theorem 1.4, Part (2).* Denote the minimal number of involutive factors needed to express each member of the classes corresponding to $r_1$, $r_d$ and $r_c$, respectively, by $i_1$, $i_d$ and $i_c$, respectively. Observing that the elements of $D_n$ and of $\operatorname{im}\Phi$ are strongly-reversible, and reviewing the proofs of Lemmas 4.4 and 4.6, we obtain the following estimates:

$$\begin{aligned} i_c(n) &\leq i_d(n) \leq i_1(n) \\ i_1(n) &\leq i_c(n) + 4 \\ i_d(n) &\leq i_c(n) + 2 \\ i_c(2m) &\leq 2 + i_d(m) \\ i_c(2m+1) &\leq 2 + i_1(m+1), \end{aligned}$$

whenever $m, n \in \mathbb{N}$ and the terms on both sides are defined (i.e. we say nothing about $i_1(1)$, $i_d(1)$ or $i_c(1)$ ). We can now carry out an induction to estimate $c_1(n)$, and each induction step adds 6 to the number of involutions that will suffice.

At the lowest level, when $n = 2$, Theorem 1.3 part (2) tells us that $14 = 8 + 6 \cdot \text{ceiling}(\log_2 n)$ involutions suffice, so induction gives the result, since $\text{ceiling}(\log_2 n)$ increases by 1 at each step. □

*Proof of Corollary 1.5.* The equivalence of (1), (2) and (3) follows from the theorem and the fact that each reversible, and hence each product of reversibles has determinant $\pm 1$.

Closer analysis of the proof of the theorem given above reveals that each $F$ with $\det L(F) = 1$ may also be represented as the product of $6 \cdot \text{ceiling}(\log_2 n)$ involutions and one special map that is a homomorphic image of an element $\chi \in \mathfrak{G}_1$ having multiplier $+1$. (The homomorphism is the composition of repeated $H$ and inner automorphisms.) Examining the detail in the proof of Theorem 1.1, one finds that $\chi$ is the product of two reversibles, one strongly reversible, and the other reversed by an element of order dividing 4. (The theorem is Theorem 9 of [O], and the proof is on pp. 18-19 of that paper. The map is denoted $f$, instead of $\chi$. Three cases are considered. In case 1°, $f$ is factored as $gh$, where $g$ is conjugate to $z + z^2 + z^3$, which is strongly reversible, and $h$ is $\mathsf{id}$ or is conjugate to $z + z^3 + \frac{3}{2}z^5$, which is reversed by $z \mapsto iz$. In case 2° — note that there is a misprint: this case is $p > 2$, not $p \geq 2$ —, $f$ is the product of two maps conjugate to $z + z^2 + z^3$. Finally, in case 3°, $f = gh$, where $g$ is conjugate to the aforementioned $z + z^3 + \frac{3}{2}z^5$ and $h$ is conjugate to $z + z^4 + 2z^7$, and hence is strongly reversible.) Thus $\chi$, and hence the special map, are each the product of two involutions and two reversible maps of degree dividing 4, so that $F$ is the product of $2 + 6 \cdot \text{ceiling}(\log_2 n)$ involutions and two reversible maps of degree dividing 4. □

| $n$ | $r_1(n)$ | $r_d(n)$ | $r_c(n)$ | $n$ | $r_1(n)$ | $r_d(n)$ | $r_c(n)$ |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 2 | 2 | 9 | 13 | 12 | 11 |
| 2 | 4 | 4 | 3 | 10 | 12 | 11 | 10 |
| 3 | 7 | 6 | 5 | 11 | 12 | 11 | 10 |
| 4 | 7 | 6 | 5 | 12 | 11 | 10 | 9 |
| 5 | 10 | 9 | 8 | 13 | 13 | 12 | 11 |
| 6 | 9 | 8 | 7 | 14 | 12 | 11 | 10 |
| 7 | 10 | 9 | 8 | 15 | 12 | 11 | 10 |
| 8 | 9 | 8 | 7 | 16 | 11 | 10 | 9 |

TABLE 1.

*Remark* 4.8. The inequalities in Lemmas 4.4 and 4.6 may be used to derive estimates for $r_1(n)$ that are often considerably smaller than the estimate $1+3\cdot\text{ceiling}(\log_2 n)$. These estimates depend on the parity of the terms in the chain of links $n' \to k'$ connecting $n$ to 2. For instance, from the chain

$$96 \to 48 \to 24 \to 12 \to 6 \to 3 \to 2$$

one obtains $r_1(96) \leq 14$, in contrast to the estimate $r_1(97) \leq 20$ obtained from the chain

$$97 \to 49 \to 25 \to 13 \to 7 \to 4 \to 2.$$

The best estimates are obtained for powers of 2:

$$c_1(2^n) \leq 2 + 2n.$$

Table 1 gives the best estimates obtainable from these Lemmas for the first few $n$.

We do not know sharp values for $r_1(n)$ or $r_d(n)$, in any case of dimension greater than 1.

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