# BOUNDARY SMOOTHNESS OF ANALYTIC FUNCTIONS 

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#### Abstract

We consider the behaviour of holomorphic functions on a bounded open subset of the plane, satisfying a Lipschitz condition with exponent $\alpha$, with $0<\alpha<1$, in the vicinity of an exceptional boundary point where all such functions exhibit some kind of smoothness. Specifically, we consider the relation between the abstract idea of a bounded point derivation on the algebra of such functions and the classical complex derivative evaluated as a limit of difference quotients. We obtain a result which applies, for example, when the open set admits an interior cone at the special boundary point.


## 1. Introduction

Let $U$ be a nonempty open subset of $\mathbb{C}$, and let $f: U \rightarrow \mathbb{C}$ be holomorphic on $U$. Suppose $0<\alpha<1$, and $f$ satisfies a Hölder, or Lipschitz condition with exponent $\alpha$ on $U$ : i.e. there exists $\kappa>0$ such that

$$
\begin{equation*}
|f(z)-f(w)| \leq \kappa|z-w|^{\alpha}, \forall z, w \in U \tag{1}
\end{equation*}
$$

Then $f$ has a unique continuous extension to $Y=\operatorname{clos}(U)$. This extension also satisfies the Lipschitz condition with exponent $\alpha$ on $Y$, with the same constant $\kappa$.

Let $b$ belong to the boundary of $U$. It may happen that all such $f$ are in some sense smoother at $b$ than a typical Hölder-continuous complexvalued function. That is, the additional assumption of analyticity on $U$ may force additional smoothness at $b$.

The strongest possible smoothness that might occur would be that all such $f$ are actually holomorphic on a certain neighbourhood of $b$.

[^0]In that case, $b$ is usually referred to as a removable singularity for $\operatorname{Lip} \alpha$ holomorphic functions on $U$. This phenomenon was first investigated by Dolzhenko, who showed [1] that $b$ is a removable singularity of this type if and only if there exists $r>0$ such that $\mathbb{B}(b, r) \sim U$ has zero $(1+\alpha)$-dimensional Hausforff measure, where $\mathbb{B}(b, r)$ denotes the closed disk having center $b$ and radius $r$. (It seems appropriate to mention here that author's interest in removable singularities, and in this whole area, was first aroused by Larry Zalcman's Monthly paper [9].)

It may also happen that more limited smoothness occurs at a boundary point $b$ that is not removable. In 4 Lord and the author considered the notion of a continuous point derivation at a boundary point, and gave a necessary and sufficient condition for the existence of such a nonzero derivation. The concept of continuous point derivation comes from the theory of commutative Banach algebras. If $A$ is a commutative Banach algebra with character space (maximal ideal space) $M(A)$, and $\phi \in M(A)$, then a continuous point derivation on $A$ at $\phi$ is a continuous linear functional $\partial: A \rightarrow \mathbb{C}$ such that the Leibniz rule

$$
\partial(f g)=\phi(f) \partial(g)+\partial(f) \phi(g)
$$

holds for all $f, g \in A$. In the present case, we considered the algebra $A=A_{\alpha}(U)$ of all holomorphic functions $f$ on $U$ that belong to the "little Lipschitz class", i.e. are not just $\operatorname{Lip} \alpha$ functions, but have the stronger property that for each $\epsilon>0$ there exists $\delta>0$ such that for all $z, w \in U$,

$$
|z-w|<\delta \Longrightarrow|f(z)-f(w)| \leq \epsilon|z-w|^{\alpha}
$$

The norm on $A$ is given by

$$
\|f\|_{A}=\sup _{U}|f|+\kappa(f),
$$

where $\kappa(f)$ is the least value that works in the inequality (11). The characters on this $A$ are precisely the evaluations $f \mapsto f(a)$, for $a \in Y$ (as follows from a result of Sherbert [8]), and we identify the point $a$ with the corresponding character. At points $a \in U$, the map $\partial$ : $f \mapsto f^{\prime}(a)$ is a nonzero continuous point derivation on $A$ at $a$. We gave a necessary and sufficient condition in order that there exist a nonzero continuous point derivation on $A$ at a given $b \in \operatorname{bdy} U$. This condition involved a set function known as lower $(1+\alpha)$-dimensional Hausdorff content, and denoted $M_{*}^{1+\alpha}$. The existence of the derivation is equivalent to the convergence of a Wiener-type series

$$
\sum_{n=0}^{\infty} 4^{n} M_{*}^{1+\alpha}\left(A_{n}(b) \sim U\right)
$$

where $A_{n}(b)$ denotes the annulus consisting of those $z$ with

$$
\frac{1}{2^{n+1}} \leq|z-b| \leq \frac{1}{2^{n}}
$$

For the definition of the content we refer the reader to [4]. We shall not use the content in the present paper, but we note that the above condition is explicit and practical, and allows one to determine by computation whether or not there exists such a point derivation at a given boundary point.

The paper [4] has similar characterizations of the existence of higherorder continuous point derivations on $A$. It also has results about the "big Lip" algebra of all Lip $\alpha$ holomorphic functions on $U$. In the latter case, the character space is still $Y$, but the results are about weakstar continuous derivations - the big Lip algebra always has nonzero continuous point derivations at every point, but only the weak-star continuous ones are of any interest in connection with analytic function theory.
(For the avoidance of confusion, we note that what is here denoted $A_{\alpha}(U)$ was denoted $a(U)$ in [4], while the notation $A(U)$ was there used for the big Lip version.)

These results are rather abstract, and the purpose of the present paper is to relate them to concrete classical ideas of derivative. We are going to confine attention to the simplest case: the little lip algebra $A$, and (first-order) continuous point derivations. The question we address is the following:

Suppose the bounded open set $U$, a boundary point $b$, and $\alpha \in$ $(0,1)$ are given, and suppose there exists a nonzero continuous point derivation $\partial$ on $A=A_{\alpha}(U)$ at $b$. Can we evaluate $\partial f$ by a formula

$$
\partial f=c \lim _{n \uparrow \infty} \frac{f\left(z_{n}\right)-f(b)}{z_{n}-b},
$$

valid for every $f \in A$ ? (Here, as before, $f(b)$ denotes the value at $b$ of the unique continuous extension of $f$ to bdy $U$.)

We remark that a continuous point derivation at $b$ is uniquely determined by its value $c=\partial z$ at the identity function $z \mapsto z$. This follows using elementary algebra from the fact [4, Lemma 1.1] that the set of functions $f \in A$ that are holomorphic on a neighbourhood of $b$ is a dense subalgebra of $A$. We say that $\partial$ is normalised if $\partial z=1$. If $\partial$ is any nonzero derivation, then $\partial / c$ is normalised.

In the interest of further simplicity, we confine attention to the situation in which the boundary point is nicely accessible from $U$. We
say that $U$ has an interior cone at the boundary point $b$ if there is a segment $J$ ending at $b$ and a constant $t>0$ such that

$$
\operatorname{dist}(z, \mathbb{C} \sim U) \geq t|z-b|, \forall z \in J
$$

We call such a segment $J$ a nontangential ray to $b$.
We say that a sequence $\left(z_{n}\right)_{n}$ of points of $U$ converges non-tangentially to $b$, written $z_{n} \rightarrow_{\mathrm{nt}} b$, if there exists a constant $t>0$ such that

$$
\operatorname{dist}\left(z_{n}, \mathbb{C} \sim U\right) \geq t\left|z_{n}-b\right|, \forall z \in J
$$

Obviously, if $U$ has an interior cone at $b$, then any sequence converging to $b$ along a nontangential ray $J$ is converging nontangentially. However, the existence of a sequence converging nontangentially does not imply that $b$ lies on the boundary of a single connected component of $U$. Without going into details about Hausdorff content, we remark that for a closed ball $M_{*}^{\beta}(\mathbb{B}(a, r))=r^{\beta}$, that for a line segment $J$, $M_{*}^{\beta}(J)=0$ if $\beta>1$, and also that $M_{*}^{\beta}$ is countably subadditive. As a result, it is easy to construct many examples $U$ in which the complement of $U$ is a countable union of closed balls, line segments and the singleton $\{b\}$, and in which $A$ has a continuous point derivation at $b$. All you have to do is make sure that the sum of the $(1+\alpha)$-th powers of the radii of all the closed balls that meet $A_{n}(b)$ is no greater than $s_{n} / 4^{n}$, where $\sum_{n} s_{n}<+\infty$.

Our main result is the following:
Theorem 1.1. Let $0<\alpha<1$, let $U \subset \mathbb{C}$ be a bounded open set, let $b \in \operatorname{bdy} U, z_{n} \in U, z_{n} \rightarrow_{\mathrm{nt}} b$. Suppose $A=A_{\alpha}(U)$ admits a nonzero continuous point derivation at $b$. Let $\partial$ be the normalised derivation at b. Then for each $f \in A$, we have

$$
\frac{f\left(z_{n}\right)-f(b)}{z_{n}-b} \rightarrow \partial f .
$$

Corollary 1.2. Under the same hypotheses, if $U$ has an interior cone at $b$, and $J$ is a nontangential ray to $b$, then

$$
\partial f=\lim _{z \rightarrow b, z \in J} \frac{f(z)-f(b)}{z-b}, \forall f \in A .
$$

We would expect that these results could be extended to higher order derivations and to weak-star continuous derivations on the big Lip algebra.

In broad outline, the methods we shall use are adapted from those used to prove a similar result about bounded analytic functions in [5]. In that paper, we used duality ideas from functional analysis, the Riesz Representation Theorem, and the Cauchy transform. In order
to transfer the methods to the Lipschitz algebra one has to overcome various technical problems.

The methods we use in this paper, using duality and abstractions from functional analysis, are termed nonconstructive, when contrasted with explicit methods that involve direct use of the Cauchy integral formula applied to individual functions. It should also be possible to approach the proof in a constructive spirit. This would involve the explicit use of Hausdorff contents, and it is to some extent a matter of taste (perhaps influenced by familiarity with various techniques) which might be regarded as preferable. We leave the constructive approach for another day.

As will be seen, we extend here the arsenal of techniques for dealing with spaces of holomorphic functions in Lipschitz classes. We expect that these techniques will prove useful in dealing with other problems in the same area, such as the behaviour of functions near a special boundary point which may not be accessible by nontangential approach, and various approximation problems..

Throughout the paper, $0<\alpha<1, U \subset \mathbb{C}$ is a bounded open set, $Y=\operatorname{clos}(U)$, and $b \in X=\operatorname{bdy} U$.

## 2. Extensions and Distributions

2.1. We have already remarked that the elements of $A=A_{\alpha}(U)$ extend uniquely to continuous functions on $Y=\operatorname{clos}(U)$. It is in fact obvious that this extension imbeds $A$ isometrically as a closed subalgebra of $\operatorname{lip}(\alpha, Y)$.

We are interested in point derivations, and point derivations annihilate the constant functions (because $\partial 1=\partial\left(1^{2}\right)=2 \cdot 1 \cdot \partial 1$ ), so we are really more interested in the quotient space $A / \mathbb{C}$ modulo constants, and the 'pure' Lipschitz seminorm $\|f\|^{\prime}=\kappa(f)$ from (11). We note that the extension to $Y$ also preserves this pure seminorm.

Our first significant point is that the restriction of the extension to the boundary $X$ is also isometric on both the norm and the pure seminorm. That the sup norm is preserved is the classical maximum principle. But we also have:

Lemma 2.1. Let $f \in \operatorname{Lip}(\alpha, Y)$ be holomorphic on $U$. Then

$$
\sup \left\{\frac{|f(z)-f(w)|}{|z-w|^{\alpha}}: z, w \in Y\right\}=\sup \left\{\frac{|f(z)-f(w)|}{|z-w|^{\alpha}}: z, w \in X\right\}
$$

Proof. For each fixed $w_{0} \in Y$, Let $\Sigma\left(w_{0}\right)$ be the Riemann surface of $\log \left(z-w_{0}\right)$, and let $p: \Sigma \rightarrow \mathbb{C} \sim\left\{w_{0}\right\}$ be the covering map, so that $\log \left(Z-w_{0}\right)$ is a well-defined holomorphic function on $\Sigma\left(w_{0}\right)$, with real
part $\log \left|p(Z)-w_{0}\right|$ and imaginary part $\arg \left(Z-w_{0}\right)$. Then $\left(Z-w_{0}\right)^{\alpha}$, interpreted as

$$
\exp \left(\alpha \log \left|p(Z)-w_{0}\right|+i \alpha \arg \left(Z-w_{0}\right)\right)
$$

is also holomorphic on $\Sigma\left(w_{0}\right)$, and

$$
g(Z):=\frac{f(p(Z))-f\left(w_{0}\right)}{\left(Z-w_{0}\right)^{\alpha}}
$$

is a well-defined function on $\Sigma\left(w_{0}\right) \cap p^{-1}(Y)$, with absolute value that depends only on the projection $p(Z) \in \mathbb{C}$. It is holomorphic on $p^{-1}(U)$.

Since $\frac{|f(z)-f(w)|}{|z-w|^{\alpha}}$ is continuous on $Y \times Y$, it attains its supremum, say $m$, at some point $\left(z_{0}, w_{0}\right) \in Y \times Y$. If $z_{0} \notin X$, then we have a contradiction to the maximum principle unless $\left|f(p(Z))-f\left(w_{0}\right)\right|$ is identically equal to $m\left|p(Z)-w_{0}\right|^{\alpha}$ on the connected component of each preimage of $z_{0}$ in $p^{-1}(U)$, and hence at some point of $p^{-1}(X)$. Thus we may assume $z_{0} \in X$. Similarly, we may assume $w_{0} \in X$.

As a result, we may regard $A / \mathbb{C}$ as a subspace of $\operatorname{lip}(\alpha, X) / \mathbb{C}$. Thus, by the Hahn-Banach Theorem, each continuous linear functional $T \in$ $A^{*}$ that annihilates the constants has an isometric extension to the whole of $\operatorname{lip}(\alpha, X) / \mathbb{C}$, hence (by a standard method) may be represented by a Borel-regular measure $\mu$ on $X \times X$ having no mass on the diagonal and total mass equal to $\|T\|^{\prime}$ (the dual norm on $A^{*}$ to $\| \dot{\|}^{\prime}$ ), via a formula

$$
T f=\int_{X \times X} \frac{f(z)-f(w)}{\left(|z-w|^{\alpha}\right.} d \mu(z, w), \forall f \in A .
$$

In particular, if we assume that $\partial$ is a continuous point derivation on $A$, then it has a representation of this kind.
2.2. Extensions. Let lip $\alpha$ denote, for short, the global space $\operatorname{lip}(\alpha, \mathbb{C})$ of bounded $\operatorname{lip} \alpha$ functions on $\mathbb{C}$. Each $f \in A$ may be extended (in many different ways) to an element of $\operatorname{lip} \alpha$, without increasing its pure norm $\|f\|^{\prime}$ or supremum. (In fact it is not difficult to check that if $\omega(r)$ is any concave upper envelope for the modulus of continuity of $f$, then $f$ may be extended so that its modulus of continuity remains bounded by $\omega$. For instance, this may be seen by applying the method used to prove Kirszbraun's Theorem in [2].) Thus the restriction map to $U$ (or $Y$, or $X$ ) makes $A$ isometric to a quotient space of

$$
\tilde{A}=\{f \in \operatorname{lip} \alpha: f \text { is holomorphic on } U\} .
$$

We shall find it convenient to work with globally-defined functions in the sequel.
2.3. Distributions. Let $\mathcal{D}$ denote the space of test functions (i.e. $C^{\infty}$ functions having compact support), and let $\mathcal{D}^{\prime}$ denote its dual, the Schwartz distribution space. If $\mu$ is any complex measure on $X \times X$, having no mass on the diagonal, then we may define a distribution $T_{1} \in \mathcal{D}^{\prime}$ by setting

$$
\begin{equation*}
\left\langle\phi, T_{1}\right\rangle=\int_{X \times X} \frac{\phi(z)-\phi(w)}{|z-w|^{\alpha}} d \mu(z, w), \forall \phi \in \mathcal{D} . \tag{2}
\end{equation*}
$$

This distribution $T_{1}$ will not, in general, be representable by integration against a locally-integrable funtion or a measure, but will extend continuously to an element of $(\operatorname{lip} \alpha)^{*}$. It is a bit 'wilder' than a measure. (It may be represented in the form

$$
T_{1}=\nu_{0}+\frac{\partial}{\partial x} \nu_{1}+\frac{\partial}{\partial y} \nu_{2},
$$

where the $\nu_{j}$ are compactly-supported measures, but we shall not use this representation, as it carries less information than the fact that $T_{1}$ acts continuously on $\operatorname{lip} \alpha$.) We denote the extension of $T_{1}$ to $\operatorname{lip} \alpha$ by the same notation $T_{1}$, and write its value as $T_{1}(f)$ or $\left\langle f, T_{1}\right\rangle$ for any $f \in \operatorname{lip} \alpha$.

Since $\left\langle\phi, T_{1}\right\rangle$ is unaffected if $\phi$ is altered away from $X$, it is clear that $T_{1}$ has support in $X$. Thus we can also define $\left\langle\phi, T_{1}\right\rangle$ for any function $\phi$ defined and $C^{\infty}$ on a neighbourhood of $X$ to be $\left\langle\tilde{\phi}, T_{1}\right\rangle$ where $\tilde{\phi}$ is any element of $\mathcal{E}$ (the space of globally-defined $C^{\infty}$ functions) that agrees with $\phi$ near $X$. For instance, $\left\langle\frac{1}{z-a}, T_{1}\right\rangle$ makes sense, for $a \notin X$. Similarly, $\left\langle f, T_{1}\right\rangle$ makes sense whenever $f$ is defined on some neighbourhood of $X$ and satisfies a little-lip $\alpha$ condition there.
2.4. Cauchy Transforms. The main idea behind what follows is that although $T_{1}$ is wilder than a measure, it is still tame enough to allow us to treat it almost as though it were a measure. Specifically, the Cauchy transform of $T_{1}$ (which we are about to define) is representable by integration against a locally-integrable function. This fact was already noted and exploited in [6].

The Cauchy transform of $\phi \in \mathcal{D}$ is its convolution

$$
\hat{\phi}:=\phi *\left(\frac{1}{\pi z}\right)
$$

with the fundamental solution of $\frac{\partial}{\partial \bar{z}}$. In other words,

$$
\hat{\phi}(z)=\frac{1}{\pi} \int \frac{\phi(\zeta)}{z-\zeta} d m(\zeta)
$$

for all $z \in \mathbb{C}$, where $m$ denotes area measure. This function belongs to the space $\mathcal{E}$, and satisfies

$$
\frac{\partial \hat{\phi}}{\partial z}=\phi
$$

For distributions $T$ having compact support, we define

$$
\langle\phi, \hat{T}\rangle=-\langle\hat{\phi}, T\rangle, \forall \phi \in \mathcal{D}
$$

If $T_{1}$ is given by Equation (2), then consider

$$
H(a)=\frac{1}{\pi} \int \frac{z-w}{(z-a)(w-a)|z-w|^{\alpha}} d \mu(z, w)
$$

for $a \in \mathbb{C}$. This is well-defined whenever

$$
\tilde{H}(a)=\int \frac{|w-z|^{1-\alpha}}{|z-a| \cdot|w-a|} d|\mu|(z, w)<\infty
$$

which happens almost everywhere with respect to area measure, and $\tilde{H}$ is locally-integrable, as is seen by an application of Fubini's Theorem. Also $|H(a)| \leq \tilde{H}(a)$ for all such $a$. Another Fubini calculation yields

$$
\left\langle\phi, \widehat{T}_{1}\right\rangle=\int_{\mathbb{C}} \phi \cdot H d m
$$

for all $\phi \in \mathcal{D}$. Thus $H$ represents $\widehat{T_{1}}$. Based on this, we sometimes write $\widehat{T}_{1}(a)$ for $H(a)$. Note that

$$
\widehat{T}_{1}(a)=H(a)=\left\langle\frac{1}{\pi(a-z)}, T_{1}\right\rangle
$$

whenever $a \notin X$.
Note that these facts do not depend on the relation of $T_{1}$ to a derivation, but only on its representability in the form (2) for some measure $\mu$ on $X \times X$.

Note also, for future reference, that if $T$ actually represents a normalized point derivation at a point $b$ of $X$, then $H(a)=\frac{1}{\pi(b-a)^{2}}=$ $\mathrm{o}\left(1 /|a|^{2}\right)$ as $a \rightarrow \infty$.

## 3. Estimates

3.1. The product $g \cdot T_{1}$. The dual of any Banach algebra is naturally a module over the algebra. In the present situation, $\operatorname{lip} \alpha$ acts on $(\operatorname{lip} \alpha)^{*}$, so given $g \in \operatorname{lip} \alpha$ and $T_{1}$ as in Equation (22), we may define a new element $g \cdot T_{1}$ of $(\operatorname{lip} \alpha)^{*}$ by setting

$$
\left\langle\phi, g \cdot T_{1}\right\rangle=\left\langle g \cdot \phi, T_{1}\right\rangle, \quad \forall \phi \in \mathcal{D}
$$

We remark that $\left\langle 1, g \cdot T_{1}\right\rangle=\left\langle g, T_{1}\right\rangle \neq 0$, in general, so we cannot represent $g T_{1}$ by a measure as in Equation (2). However, writing

$$
\phi(z) g(z)-\phi(w) g(w)=(\phi(z)-\phi(w)) \cdot g(z)+\phi(w) \cdot(g(z)-g(w))
$$

a short calculation gives

$$
\left\langle\phi, g \cdot T_{1}\right\rangle=\int_{X \times X} \frac{\phi(z)-\phi(w)}{|z-w|^{\alpha}} d \mu^{\prime}(z, w)+\int_{X} \phi(w) d \lambda(w),
$$

where $\mu^{\prime}$ is the measure on $X \times X$ such that

$$
\mu^{\prime}(E)=\int_{E} g(z) d \mu(z, w)
$$

whenever $E \subset X \times X$ is a Borel set, and $\lambda$ is the measure on $X$ such that

$$
\lambda(E)=\int_{E \times X} \frac{g(z)-g(w)}{|z-w|^{\alpha}} d \mu(z, w)
$$

whenever $E \subset X$ is Borel, i.e. $\lambda$ is the first-coordinate marginal of the measure

$$
\frac{g(z)-g(w)}{|z-w|^{\alpha}} \cdot \mu(z, w)
$$

(a bounded multiple of $\mu$ ). So we may write $g \cdot T_{1}=S_{1}+S_{2}$, where $S_{1} \perp \mathbb{C} 1$ is represented (as in Equation (2)) by the measure $\mu^{\prime}=g(z) \cdot \mu$ on $X \times X$, and $S_{2}$ is represented by the measure $\lambda$ on $X$.

Denoting the total variation of a measure $\mu$ by $\|\mu\|$, we note for future reference that

$$
\|\lambda\| \leq \kappa(g) \cdot\|\mu\| .
$$

Let us call $S_{1}$ the main part of $g \cdot T_{1}$ and $S_{2}$ the residual part of $g \cdot T_{1}$.
3.2. Estimate. We are aiming for an estimate for the growth of the Cauchy transform of $g \cdot T_{1}$ as we approach a boundary point nontangentially. The main step is an estimate for the Cauchy transform $\hat{S}_{1}$ of the main part.
Lemma 3.1. Fix a measure $\mu$ on $X \times X$. Let $b \in X$ and let $g \in \operatorname{lip} \alpha$ have $g(b)=0$. Let $S_{1}$ be the distribution given by

$$
\left\langle\phi, S_{1}\right\rangle=\int_{X \times X} \frac{\phi(z)-\phi(w)}{|z-w|^{\alpha}} g(z) d \mu(z, w), \quad \forall \phi \in \mathcal{D} .
$$

Fix $t$ with $0<t<1$. Then there is a constant $c$ that depends only on $t$ such that

$$
\left|\hat{S}_{1}(a)\right| \leq \frac{c \kappa(g) \cdot\|\mu\|}{|a-b|}
$$

for all $a \in \mathbb{C} \sim X$ with $\operatorname{dist}(a, X) \geq t|a-b|$.

Proof. We may assume without loss in generality that $\kappa(g)=1$. Then $|g(z)| \leq|z-b|^{\alpha}$ for all $z \in \mathbb{C}$.

Assume $\operatorname{dist}(a, X) \geq t|a-b|$.
Let $z \in X$. If $|z-b| \leq 2|a-b|$, then

$$
|z-b| \leq \frac{2 \operatorname{dist}(a, X)}{t} \leq \frac{2|z-a|}{t}
$$

whereas if $|z-b|>2|a-b|$, then

$$
|z-b| \leq|z-a|+|a-b|<|z-a|+\frac{1}{2}|z-b|
$$

so $|z-b|<2|z-a|$. Thus in either case $|z-b| \leq 2|z-a| / t$. Hence $|g(z)| \leq c|z-a|^{\alpha}$ for all $z \in X$, where $c$ depends only on $t$. Henceforth we shall use $c$ to denote a constant, which may differ at each occurrence, depending only on $t$.

Let

$$
K(z, w)=\frac{|w-z|^{1-\alpha}}{|z-a| \cdot|w-a|}
$$

Then

$$
\left|\hat{S}_{1}(a)\right| \leq \int_{X \times X} K(z, w)|g(z)| d|\mu|(z, w) .
$$

We have $|z-w|^{1-\alpha} \leq|z-a|^{1-\alpha}+|w-a|^{1-\alpha}$, so for $z, w \in X$ we have

$$
K(z, w) \leq \frac{1}{|z-a| \cdot|w-a|^{\alpha}}+\frac{1}{|z-a|^{\alpha} \cdot|w-a|}
$$

so

$$
K(z, w)|g(z)| \leq \frac{c}{|z-a|^{1-\alpha} \cdot|w-a|^{\alpha}}+\frac{c}{|w-a|} \leq \frac{c}{\operatorname{dist}(a, X)} .
$$

The desired result follows.
Lemma 3.2. If $T_{1}$ is given by Equation (2), and $0<t<1$, then there is a constant $c$ depending only on $t$ such that

$$
\left|\widehat{g \cdot T_{1}}(a)\right| \leq \frac{c \kappa(g)\|\mu\|}{\operatorname{dist}(a, X)}
$$

whenever $\operatorname{dist}(a, X) \geq t|a-b|$.
Proof. Let $S_{1}$ and $S_{2}$ be the parts of $g \cdot T_{1}$ and $\lambda$ represent $S_{2}$, as in the last section. Since $\lambda$ is a measure supported on $X$, we have

$$
|\hat{\lambda}(a)| \leq \frac{\|\lambda\|}{\operatorname{dist}(a, X)} \leq \frac{\|\mu\| \cdot \kappa(g)}{\operatorname{dist}(a, X)}
$$

whenever $\operatorname{dist}(a, X) \geq t|a-b|$. Combining this with the last lemma, we get

$$
\left|\widehat{g \cdot T_{1}}(a)\right| \leq\left|\hat{S}_{1}(a)\right|+\left|\hat{S}_{2}(a)\right| \leq \frac{c \kappa(g)\|\mu\|}{\operatorname{dist}(a, X)}
$$

as required.
3.3. Estimate for $\hat{T}_{1}$. By a similar (slightly simpler) argument we obtain the following.

Lemma 3.3. Let $T_{1}$ be given by Equation (2). Then for each $t$ with $0<t<1$ there exists a constant $c$, depending only on $t$, such that

$$
\left|\hat{T}_{1}(a)\right| \leq \frac{c\|\mu\|}{\operatorname{dist}(a, X)^{1+\alpha}},
$$

whenever $\operatorname{dist}(a, X) \geq t|a-b|$.

## 4. Proof of Theorem

Suppose $A=A_{\alpha}(U)$ admits a nonzero continuous point derivation at $b$, and let $\partial$ be the normalised derivation at that point. Given a function $f \in A$, we use the same symbol $f$ to denote some global extension in $\operatorname{lip} \alpha$. Note that the extension is uniquely-determined on $X$, but not outside $Y$. None of the quantities we will consider depend on which extension is taken.

As we have seen, there is a measure $\mu$ on $X \times X$, having no mass on the diagonal, such that

$$
\partial f=\int_{X \times X} \frac{f(z)-f(w)}{\left(|z-w|^{\alpha}\right.} d \mu(z, w), \forall f \in A .
$$

Let $T_{1}$ be the distribution defined by Equation (2). Then $T_{1}$ has support in $X$, and extends continuously to an element of lip $\alpha^{*}$.

Let $\mathcal{A}$ denote the set of all $f \in \operatorname{lip} \alpha$ that are holomorphic on $U$ and on a neighbourhood of $b$. As remarked earlier, $\mathcal{A}$ is dense in $A$.
$\mathcal{E}^{\prime}$ is a module over $\mathcal{E}$, via the multiplication operation $(\lambda, T) \mapsto \lambda \cdot T$ defined by

$$
\langle\phi, \lambda \cdot T\rangle=\langle\lambda \phi, T\rangle
$$

for all $\phi, \lambda \in \mathcal{E}$ and $T \in \mathcal{E}^{\prime}$. So we may define $T_{0}=(z-b) \cdot T_{1}$, where $(z-b)$ denotes the function $z \mapsto(z-b)$.

We calculate that for $f \in \mathcal{A}$ we have

$$
\left\langle f, T_{0}\right\rangle=\left\langle(z-b) f, T_{1}\right\rangle=f(b)
$$

and hence by continuity this also holds for all $f \in A$, i.e. the distribution $T_{0}$ represents evaluation at $b$ on $A$. Next,

$$
\frac{\partial}{\partial \bar{z}}\left((z-b) \cdot \hat{T}_{1}\right)=(z-b) \cdot T_{1}=T_{0}=\frac{\partial}{\partial \bar{z}} \hat{T}_{0}
$$

so by Weyl's Lemma (cf. [7, p.72] or [3, Theorem 4.4.1, p. 110])

$$
\hat{T}_{0}=(z-b) \cdot \hat{T}_{1}+h
$$

where $h$ is an entire function. Also, if $\phi \in \mathcal{D}$ vanishes on a neighbourhood of $Y$, then

$$
\left\langle\phi, \hat{T}_{0}\right\rangle=\hat{\phi}(b)=\left\langle\phi, \frac{1}{\pi(b-z)}\right\rangle .
$$

Thus $\hat{T}_{0}(z)=\frac{1}{\pi(b-z)}$ off $Y$. In particular, $\hat{T}_{0}(z)$ tends to 0 as $z \rightarrow \infty$. Now we also have

$$
\hat{T}_{1}(z)=\mathrm{O}\left(\frac{1}{|z|^{2}}\right)
$$

as $z \rightarrow \infty$, hence $h(z) \rightarrow 0$ as $z \rightarrow \infty$, whence $h$ is identically zero, and

$$
\hat{T}_{0}=(z-b) \hat{T}_{1}
$$

Next, define $T=-\pi(z-b) \cdot T_{0}=-\pi(z-b)^{2} \cdot T_{1}$. Then $T$ annihilates $A$, and by a similar argument to that above we see that

$$
\hat{T}=-\pi(z-b)^{2} \hat{T}_{1}+k
$$

for some entire $k$. Now the fact that $T$ annihilates $A$ forces $\hat{T}=0$ off $Y$, so we get

$$
\hat{T}=1-\pi(z-b)^{2} \cdot \hat{T}_{1}
$$

Now suppose $\left(z_{n}\right)_{n} \subset U$ and $z_{n} \rightarrow_{\mathrm{nt}} b$. By Lemma 3.3,

$$
\begin{equation*}
\hat{T}\left(z_{n}\right)-1=-\pi\left(z_{n}-b\right)^{2} \cdot \hat{T}_{1}\left(z_{n}\right)=\mathrm{O}\left(\left|z_{n}-b\right|^{\alpha}\right) \rightarrow 0 \tag{3}
\end{equation*}
$$

as $n \uparrow \infty$. In particular, $\hat{T}\left(z_{n}\right) \rightarrow 1$, so we may choose $N_{1} \in \mathbb{N}$ such that $\left|\hat{T}\left(z_{n}\right)\right|>\frac{1}{2}$ for $n>N_{1}$.

Consider any point $a \in U$ with $\hat{T}(a) \neq 0$, and define

$$
R_{a}=\frac{1}{\pi \hat{T}(a)(a-z)} \cdot T=\frac{(z-b)^{2}}{\hat{T}(a)(z-a)} \cdot T_{1}
$$

i.e.

$$
\left\langle\phi, R_{a}\right\rangle=\frac{1}{\hat{T}(a)}\left\langle\frac{\phi(z)}{\pi(a-z)}, T\right\rangle, \forall \phi \in \mathcal{D}
$$

This is a well-defined distribution since $\phi(z) /(z-a)$ is $C^{\infty}$ near $X$, and hence near the support of $T$. Also $R_{a}$ is supported on $X$, and represents
$a$ on $A$, since for $f \in \mathcal{A}$ we may write $f(z)=f(a)+(z-a) g(z)$ for a $g \in \mathcal{A}$, and get

$$
\left\langle f, R_{a}\right\rangle=\frac{1}{\hat{T}(a)}\left\langle\frac{f(a)}{\pi(a-z)}+\frac{g(z)}{\pi}, T\right\rangle=f(a)-\frac{\langle g, T\rangle}{\pi \hat{T}(a)}=f(a) .
$$

Thus the functional

$$
f \mapsto \frac{f(a)-f(b)}{a-b}-\partial f
$$

is represented on $A$ by the distribution

$$
D_{a}=\frac{R_{a}-T_{0}}{a-b}-T_{1}
$$

Hence $D_{z_{n}}(f) \rightarrow 0$ as $n \uparrow \infty$, for all $f \in \mathcal{A}$. To prove the theorem, we have to show that this also holds for all $f$ in the closure $A$ of $\mathcal{A}$. To do this, it suffices to show that the functionals $D_{z_{n}}$ are uniformly bounded on $A$, for $n \geq N_{1}$, i.e that

$$
\left|D_{z_{n}}(f)\right| \leq c \kappa(f)
$$

for some constant $c>0$, for all $f \in A$ and all $n>N_{1}$.
Fix an arbitrary $f \in \operatorname{lip} \alpha$, holomorphic on $U$. Take $g(z)=f(z)-$ $g(b)$, so $D_{a}(f)=D_{a}(g), \kappa(f)=\kappa(g)$ and $g(b)=0$.

Let

$$
L_{a}=\hat{T}(a) R_{a}=\frac{(z-b)^{2}}{z-a} \cdot T_{1}=\frac{1}{\pi(a-z)} \cdot T
$$

Then $D_{a}=E_{a}+F_{a}$, where

$$
\begin{aligned}
E_{a} & =\frac{L_{a}-T_{0}}{a-b}-T_{1}, \\
F_{a} & =\frac{R_{a}-L_{a}}{a-b} .
\end{aligned}
$$

For $a \in U$, we calculate

$$
\begin{gathered}
L_{a}-T_{0}=-\frac{T}{\pi(z-a)}+\frac{T}{\pi(z-b)}=\frac{b-a}{\pi(z-a)(z-b)} \cdot T, \\
\frac{L_{a}-T_{0}}{a-b}=-\frac{T}{\pi(z-a)(z-b)}=\left(\frac{z-b}{z-a}\right) \cdot T_{1}, \\
E_{a}=\left(\frac{a-b}{z-a}\right) \cdot T_{1},
\end{gathered}
$$

so

$$
E_{a}(g)=(a-b) \cdot\left\langle\frac{g(z)}{z-a}, T_{1}\right\rangle=(a-b) \cdot \widehat{g \cdot T_{1}}(a)
$$

Thus Lemma 3.2 gives

$$
\left|E_{z_{n}} g\right| \leq\left|z_{n}-b\right| \cdot \frac{c \kappa(g) \cdot\|\mu\|}{\operatorname{dist}\left(z_{n}, X\right)} \leq c \kappa(f)
$$

Next,

$$
R_{a}-L_{a}=(\hat{T}(a)-1) T_{a}=\frac{\hat{T}(a)-1}{\hat{T}(a)} \cdot R_{a}
$$

Since $g \in A$ and $g(b)=0$, we have

$$
\left|R_{a}(g)\right|=|g(a)| \leq \kappa(g)|b-a|^{\alpha} .
$$

Then for $a=z_{n}$ with $n \geq N_{1}$, we get

$$
\left|\left(R_{a}-T_{a}\right)(g)\right| \leq 2 c|a-b|^{1-\alpha} \kappa(g)|a-b|^{\alpha}=c|a-b|,
$$

so

$$
\left|F_{z_{n}}(g)\right| \leq c, \text { for } n \geq N_{1}
$$

Thus $D_{z_{n}}(g)$ is indeed bounded, as required. This concludes the proof.

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