Computational Security Subject to Source Constraints, Guesswork and Inscrutability

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***Abstract*—Guesswork forms the mathematical framework for quantifying computational security subject to brute-force determination by query. In this paper, we consider guesswork subject to a per-symbol Shannon entropy budget. We introduce inscrutability rate to quantify the asymptotic difficulty of guessing U out of V secret strings drawn from the string-source and prove that the inscrutability rate of any string-source supported on a finite alphabet $\mathcal{X}$, if it exists, lies between the per-symbol Shannon entropy constraint and $\log|\mathcal{X}|$. We show that for a stationary string-source, the inscrutability rate of guessing any fraction $(1-\epsilon)$ of the V strings for any fixed $\epsilon > 0$, as V grows, approaches the per-symbol Shannon entropy constraint (which is equal to the Shannon entropy rate for the stationary string-source). This corresponds to the minimum inscrutability rate among all string-sources with the same per-symbol Shannon entropy. We further prove that the inscrutability rate of any finite-order Markov string-source with hidden statistics remains the same as the unhidden case, i.e., the asymptotic value of hiding the statistics per each symbol is vanishing. On the other hand, we show that there exists a string-source that achieves the upper limit on the inscrutability rate, i.e., $\log|\mathcal{X}|$, under the same Shannon entropy budget.**

***Index Terms*—Brute-force attack; Guesswork; Inscrutability; Rényi entropy; Universal methods; Large deviations.**

I. Introduction

In recent years, data storage has experienced a shift toward cloud storage where data is stored in a diversity of sites, each hosted at multiple locations. Cloud service providers assume responsibility for availability, accessibility, and most important, the security, of the stored data. But how secure is the cloud? The vulnerabilities of the cloud storage services have been exploited in several recent incidents resulting in the compromise of very private data stored on the cloud. The security guarantees advertised by individual sites typically assume an isolated attack. However the actual vulnerability is to a coordinated attack, where an attacker with access to more than one site combines partial information to compromise overall security.

Guesswork, which forms the mathematical framework for quantifying computational security subject to brute-force determination by query, was first considered in a short paper by Massey [1] who demonstrated that the number of guesses expected of an attacker bears little relation to the Shannon entropy. Arikan [2] then proved that this guesswork grows exponentially with an exponent that is a specific Rényi entropy for iid processes. His result has been generalized to ergodic Markov chains [3] and a wide range of stationary sources [4], [5]. Arikan and Merhav [6] have also derived fundamental limits on guessing, subject to an allowable distortion. Sundaresan [7] considered guessing on iid processes with unknown statistics and showed that the growth rate of the average guesswork is related to a specific Rényi entropy. Finally in [8], Christiansen and Duffy showed that guesswork satisfies a large deviations principle, completely characterizing the rate function, and providing an approximation to the distribution of guesswork.

Recently, in [9], the idea of guesswork was extended beyond guessing a single secret string to a setup in which an attacker wishes to guess U out of V secret strings drawn independently from not necessarily identical string-sources. It was shown in [9] that when the individual string-sources are stationary, under some regularity conditions, guesswork satisfies a large deviations principle whose rate function is not necessarily convex. Further, it was shown that when all of the V strings are drawn independently from an identical stationary string-source, guesswork grows exponentially with an exponent that is the Rényi entropy rate of the string-source with parameter $(V-U+1)/(V-U+2)$.

In this paper, in a setup similar to [9], we consider V secret strings drawn independently from identical string-sources that are constrained to satisfy a given per-symbol Shannon entropy budget. Our contributions in this paper are summarized in the following:

- We show that the inscrutability rate of a constrained string-source, if it exists, lies between the per-symbol Shannon entropy constraint and the logarithm of the size of the support, i.e., $\log|\mathcal{X}|$.
- We consider guesswork on finite-memory stationary string-sources[1] with hidden statistics. We show that when the inquisitor does not know the statistics of a finite-memory string-source, he can devise a *universal* guessing strategy that is asymptotically optimal in the sense that it achieves the same inscrutability rate as the string-source with unhidden statistics.
- Finally, we establish that the upper bound on the inscrutability rate is tight by showing that there exists a string-source that achieves an inscrutability rate of $\log|\mathcal{X}|$ under the same Shannon entropy budget.

II. Problem Setup and Related Work

Let $\mathcal{X} = \{a_1, \ldots, a_{|\mathcal{X}|}\}$ be a finite alphabet of size $|\mathcal{X}|$. Denote $x_k^{n+k-1} = x_k x_{k+1} \ldots x_{n+k-1} \in \mathcal{X}^n$ as a n-string

[1]This is a viable model for the case where the secret strings are chosen as chunks of a compressed file.

over $\mathcal{X}$. Further, let $x^n = x_1^n$ and for $i > n$, $x_i^n = \varnothing$, where $\varnothing$ denotes the null string. Let μ^n denote a probability measure on $\mathcal{X}^n$. We refer to $\{\mu^n\}_{n=1}^{\infty}$ as a string-source. We use the notation $\{\mu^n\}$ to denote $\{\mu^n\}_{n=1}^{\infty}$ as well. Note that the marginals of $\{\mu^n\}$ might be position dependent, i.e., $\sum_{x_n \in \mathcal{X}} \mu^n(x^n)$ is not necessarily equal to $\mu^{n-1}(x^{n-1})$. A string-source is said to be stationary if $\sum_{x_1,\ldots,x_k} \mu^{n+k}(x^{n+k}) = \mu^n(x_{k+1}^n)$. Let $X^n \in \mathcal{X}^n$ be a random n-string drawn from μ^n.

Some of the results in this paper are derived for finite-memory parametric string-sources.

Definition 1 (finite-memory parametric string-source): A finite-memory parametric string-source is parametrized with a d-dimensional parameter vector $\theta = (\theta_1, ..., \theta_d)$. Let $\Lambda \subset \mathbb{R}^d$ be a d-dimensional *open* set where the d parameters live. Then, μ_θ^n denotes a parametric probability measure defined by the parameter vector θ on n-strings. We assume that $\{\mu_\theta^n\}$ is a stationary string-source for all $\theta \in \Lambda$. We also assume that the source has a finite memory of at most h, i.e., the probability of observing each symbol at any position at most depends on the symbols in the previous h positions. We further assume that x_{-h+1}^0 is a run of length h of symbol a_0. We denote $\mathcal{P}_\Lambda$ as the *family* of parametric string-sources such that $\theta \in \Lambda$, i.e., $\mathcal{P}_\Lambda = \{\{\mu_\theta^n\} : \theta \in \Lambda\}$. See Appendix A for the regularity conditions on the parametric model.

The finite-memory parametric models include all iid and finite-memory Markov string-sources. The simplest parametric model is a binary iid string-source with $\mathcal{X} = \{0, 1\}$ and $\theta = P\{X_i = 1\}$ is the single source parameter, which lives in $\Lambda = (0, 1)$. Note that we exclude the boundaries. For example, $\mu_\theta(1, 1, 0) = \theta^2(1-\theta)$. Consider a binary (stationary) Markov source as another parametric model on $\mathcal{X} = \{0, 1\}$ with $d = 2$ parameters

$$(\theta_1, \theta_2) = (P\{X_i = 1|X_{i-1} = 0\}, P\{X_i = 1|X_{i-1} = 1\}),$$

that live in $\Lambda = (0, 1) \times (0, 1)$. For example, $\mu_\theta(1, 1, 0) = \theta_1\theta_2(1 - \theta_2)$ since we assume that $x_0 = 0$. Finally, consider order r Markov processes over alphabet $\mathcal{X}$. In this case, the source parameters are the *non-zero* transition probabilities given the previous r symbols, and hence, $d = |\mathcal{X}|^r(|\mathcal{X}| - 1)$.

Let $H^n(\mu^n)$ denote the Shannon entropy of a random n-string drawn from μ^n, i.e.,

$$H^n(\mu^n) = -E\left\{\log \mu^n(X^n)\right\} = \sum_{x^n \in \mathcal{X}^n} \mu^n(x^n) \log\left(\frac{1}{\mu^n(x^n)}\right).$$

Further, let $H(\{\mu^n\})$ be the Shannon entropy rate of the string-source (if it exists), i.e.,[2] $H(\{\mu^n\}) \triangleq \lim_{n\to\infty} \frac{1}{n} H^n(\mu^n)$.

Similar to [9], we consider V strings, denoted by $\mathbf{x}^{n,V} = (x^n(1), \ldots, x^n(V))$, that are drawn independently from an identical string-source $\{\mu^n\}$. This extends the guesswork problem to a multi-string system with V strings where an inquisitor wishes to identify the U out of V strings. The case where $V = U = 1$ corresponds to a single-string guesswork problem and has been studied extensively.

We have the following assumptions on the attacker and chooser:

- The length n of the chosen strings is known to the attacker.
- The chooser draws V strings *independently* from the string-source $\{\mu^n\}$.
- $\{\mu^n\}$ is known to the attacker. This assumption will be dropped for finite-memory parametric string-sources in Section IV.
- At each time, the attacker is allowed to pick one of the systems, say system i, and ask "Is $X^n(i) = y^n$?". He continues this process until he correctly guesses U of the V randomly drawn strings $\mathbf{x}^{n,V}$.
- In Sections III and V, we assume that the chooser is constrained to choose a string-source $\{\mu^n\} \in \Delta_{H_\mathcal{X}}$, where $\Delta_{H_\mathcal{X}}$ is the set of all string-sources supported on the finite alphabet $\mathcal{X}$ that satisfy a per-symbol entropy constraint of $H_\mathcal{X}$ for all $n \geq 1$. That is $(1/n)H^n(\mu^n) = H_\mathcal{X}$. We also assume that $H_\mathcal{X} > 0$.

In the single-string special case, it is straightforward to see that when the probability distribution μ^n is known to the attacker, the optimal strategy (that stochastically dominates any other strategy) would be to order all possible n-strings from the most likely outcome to the least likely (breaking ties arbitrarily), and then query the strings one by one from the top of the list until the correct password has been guessed.

In [9], it was proved that an asymptotically optimal strategy for the multi-string guesswork would be to round-robin the single-string optimal strategies. That is to query the most likely string of system 1 followed by the most likely string of system 2 and so forth till system V, before moving to the second most likely string of each system.

In the multi-string case, let $G_{\mu^n}(U, V, \mathbf{x}^{n,V})$ denote the number of queries required of an attacker to guess U out of V of sequences $\mathbf{x}^{n,V} = (x^n(1), \ldots, x^n(V))$ using the asymptotically optimal strategy described above. In the single-string case, we further use the short-hand $G_{\mu^n}(x^n)$ to denote $G_{\mu^n}(1, 1, \mathbf{x}^{n,1})$. We use the subscript μ^n in $G_{\mu^n}(\cdot)$ to emphasize that it is dependent on the specific string-source probability measure μ^n. The average guesswork $E\{G_{\mu^n}(U, V, \mathbf{X}^{n,V})\}$ quantifies the average number of guesses required of an attacker to identify U out of V of the secret strings, where the expectation is taken with respect to the iid copies of μ^n on each string.

Massey [1] demonstrated that the average guesswork in the single-string case is lower bounded by

$$E\{G_{\mu^n}(X^n)\} \geq (1/4)2^{H^n(\mu^n)} + 1.$$

The bound is tight up to a factor of $4/e$ for a geometric distribution (on an infinite support). Massey also proved that an upper bound on the average guesswork in terms of the Shannon entropy does not exist proving that average guesswork bears little relation to the Shannon entropy of the string-source in general.

In [2], Arikan considered an iid process and proved that the exponent of the average growth rate of the average guesswork is the specific Rényi entropy with parameter $\alpha = (1/2)$. In other words,

$$H_{1/2}^n(\mu^n) - \log(1 + \log|\mathcal{X}|) \leq \log E\{G_{\mu^n}(X^n)\} \leq H_{1/2}^n(\mu^n),$$

where $H_\alpha^n(\mu^n)$ is the Rényi entropy of order α ($\alpha > 0$, $\alpha \neq 1$)

[2] In this paper $\log(\cdot)$ always denotes the logarithm in base 2.

defined as

$$H_\alpha^n(\mu^n) = \frac{1}{1-\alpha}\log\left(\sum_{x^n\in\mathcal{X}^n}\mu^n(x^n)^\alpha\right).$$

Further, if it exists, the Rényi entropy rate of the string-source is defined as $H_\alpha(\{\mu^n\}) = \lim_{n\to\infty}\frac{1}{n}H_\alpha^n(\mu^n)$. Note that $H_\alpha(\{\mu^n\})$ if it exists converges to $H(\{\mu^n\})$ as $\alpha\to 1$.

Definition 2 (inscrutability): The inscrutability of identifying U out of V of the V random n-strings $\mathbf{X}^{n,V}$, denoted by $S^n(U,V,\mu^n)$ is defined as

$$S^n(U,V,\mu^n) \triangleq \log E\{G_{\mu^n}(U,V,\mathbf{X}^{n,V})\}.$$

The inscrutability rate of a string-source, denoted by $S(U,V,\{\mu^n\})$, if it exists, is defined as

$$S(U,V,\{\mu^n\}) \triangleq \lim_{n\to\infty}\frac{1}{n}S^n(U,V,\mu^n).$$

In particular, it can be concluded from Arikan's result that for an iid string-source $\{\mu^n\}$ the inscrutability rate for $U = V = 1$ is

$$S(1,1,\{\mu^n\}) = H_{1/2}(\{\mu^n\}).$$

Arikan's result was later generalized to ergodic Markov chains [3] and a wide class of stationary sources [4], [5], for which the inscrutability rate can be related to the specific Rényi entropy rate with parameter $(1/2)$ under those setups as well. Recently, the authors in [9] derived the inscrutability rate for arbitrary U and V as the specific Rényi entropy rate with parameter $(V-U+1)/(V-U+2)$. That is

$$S(U,V,\{\mu^n\}) = H_{(V-U+1)/(V-U+2)}(\{\mu^n\}). \tag{1}$$

In particular, it can be deduced from this result that in the large system limit when $V\to\infty$, if U/V stays bounded away from 1 the inscrutability rate converges to the specific Shannon entropy rate. This is stated in the following proposition.

Proposition 1: If U scales with V in such a way that $U/V < (1-\delta)$ for some $\delta > 0$, then

$$\lim_{V\to\infty} S(U,V,\{\mu^n\}) = H(\{\mu^n\}).$$

Proof: This is an immediate consequence of (1). ∎

The authors in [9] further showed that the guesswork $G_{\mu^n}(U,V,\mathbf{X}^{n,V})$ satisfies a large deviations principle and identified its rate function which is stated in Lemma 4 of [9].

III. Minimum Inscrutability String-Source with Constrained Shannon Entropy

In this section, we consider a multi-string system with secret strings drawn *independently* from the string-source $\{\mu^n\}$. We assume that $\{\mu^n\} \in \Delta_{H_\mathcal{X}}$. First, we identify the string-source in $\Delta_{H_\mathcal{X}}$, denoted by $\{\underline{\mu}^n\}$, that achieves the smallest inscrutability for all $n \geq 1$.

Theorem 2: For any $1 \leq U \leq V$, the inscrutability of identifying U out of V secret strings chosen from any string-source $\{\mu^n\} \in \Delta_{H_\mathcal{X}}$ is bounded from below by

$$S^n(U,V,\mu^n) \geq S^n(U,V,\underline{\mu}^n), \tag{2}$$

where $\underline{\mu}^n$ is a truncated geometric distribution on the support $\mathcal{X}^n$ that satisfies the per-symbol entropy constraint. Further, the inscrutability rate exists for the string-source $\{\underline{\mu}^n\}$ and is equal to the per-symbol Shannon entropy constraint. That is

$$S(U,V,\{\underline{\mu}^n\}) = \lim_{n\to\infty}\frac{1}{n}S^n(U,V,\underline{\mu}^n) = H_\mathcal{X}. \tag{3}$$

See Appendix B for the proof.

By considering Proposition 1 and Theorem 2, when $\{\mu^n\}$ is a finite-memory parametric string-source, if U scales with V such that $U/V < (1-\delta)$, then

$$\lim_{V\to\infty} S(U,V,\{\mu^n\}) = S(U,V,\{\underline{\mu}^n\}) = H(\{\mu^n\}).$$

This shows, as V grows large, the inscrutability rate of any finite-memory parametric string-source with a given Shannon entropy rate approaches the lowest limit of the inscrutability rate. Observe that inscrutability rate is defined as the asymptotic limit as $n\to\infty$ of the per-symbol inscrutability and in the above statement the limits as $n\to\infty$ and $V\to\infty$ are not interchangeable.

IV. Inscrutability of Finite-Memory Parametric String-Sources with Hidden Statistics

In this section, we investigate the impact of hiding the string-source statistics on the inscrutability of identifying U out of V secret strings drawn independently from a parametric string-source $\{\mu_\theta^n\}$. To do so, we need a *universal* guessing strategy that does not use the string-source statistics.

Note that the round-robin of single-string optimal strategies is an asymptotically optimal strategy for the multi-string system [9], and hence, we only need to find an asymptotically optimal single-string guessing strategy. The guessing strategy does not require the knowledge of the string-source $\{\mu_\theta^n\}$. For now, we assume that the inquisitor only knows the space in which the parametric source lives, i.e., he knows Λ.

A. Universal Type-Size Guessing Strategy

We shall provide a guessing strategy for parametric string-sources using the method of types (see [10]). The universal guessing strategy that will be described here coincides with Arikan and Merhav's universal guessing strategy on iid processes in [6] and also bears great similarity with Kosut and Sankar's universal type-size coding (universal compression without prefix constraint) on iid processes in [11]. The type class of sequence x^n is defined as

$$T_\Lambda(x^n) = \{y^n\in\mathcal{X}^n : \mu_\theta^n(y^n) = \mu_\theta^n(x^n)\ \forall\theta\in\Lambda\}. \tag{4}$$

Further, $|T_\Lambda(x^n)|$ denotes the size of the type class of x^n, i.e., the total number of sequences with the same type as x^n.

Single-string universal guessing strategy:

- We order all sequences based on the size of their corresponding type classes in an ascending fashion and break ties arbitrarily.
- We let $G_\star(x^n)$ be the order in which the sequence x^n appears in the above list. Clearly, the sequence x^n may appear before y^n only if $|T_\Lambda(x^n)| \leq |T_\Lambda(y^n)|$.

Our main result on the universal type-size guessing strategy described above is the following.

Theorem 3: Let Λ denote the simplex of iid probability vectors over finite alphabet $\mathcal{X}$. Let $G_{\mu_\theta^n}$ be an optimal non-universal guessing strategy for parametric source with parameter vector θ, such that in $G_{\mu_\theta^n}$ ties are broken in favor of

guessing sequences with smaller type-sizes first and if there is a tie in the size of the type the tie is broken arbitrarily. Then for any individual sequence x^n, the universal guessing function $G_\star(x^n)$ obeys:

$$\frac{G_\star(x^n)}{G_{\mu_\theta^n}(x^n)} = O(n^{|\mathcal{X}|}). \tag{5}$$

See Appendix C for the proof.

Theorem 3 has two main implications. The first is on the large deviations principle for the multi-string system with unknown statistics. This result was expected in light of the analysis of the single-string universal strategies in [7], and the recent results on the large deviations for multi-string guesswork [9].

Corollary 4: The sequence $\{1/n \log G_\star(U, V, \mathbf{x}^{n,V})\}$ satisfies a large deviations principle with the same rate function $I_{\text{opt}}(U, V, t)$ of any optimal non-universal guessing strategy $G_{\mu^n}(U, V, \mathbf{x}^{n,V})$, where I_{opt} is defined in Lemma 4 of [9].

Proof: By invoking Theorem 3, observe that

$$\frac{1}{n} \log G_\star(x^n) \leq \frac{1}{n} \log G_{\mu_\theta^n}(x^n) + |\mathcal{X}| \frac{\log n}{n} + O\left(\frac{1}{n}\right). \tag{6}$$

Therefore, $G_\star$ satisfies LDP with the same rate function as $G_{\mu_\theta^n}$ for any θ, where the rate function is derived in [8]. Since $G_\star$ is optimal for individual system, then the round-robin of $G_\star$ is also asymptotically optimal for multi-system case and it would satisfy LDP with the specific rate function $I_{\text{opt}}(U, V, t)$ in light of Lemma 4 of [9]. ∎

Let the inscrutability rate of the universal type-size guessing strategy be defined as

$$S_\star(U, V, \{\mu^n\}) = \lim_{n\to\infty} \frac{1}{n} \log E\{G_\star(U, V, \mathbf{X}^{n,V})\}.$$

Here, we also obtain the multi-string counterpart of Sundaresan's Theorem 16 of [7] on the growth rate of the average universal guesswork.

Corollary 5: The inscrutability rate of the universal type-size guessing strategy is given by

$$S_\star(U, V, \{\mu^n\}) = H_{(V-U+1)/(V-U+2)}(\{\mu^n\}). \tag{7}$$

This is straightforward by putting together Corollary 4 of this paper and Corollary 1 of [9].

This establishes that the inscrutability rates for a finite-memory parametric sources with hidden and unhidden statistics is the same.

B. Universal Bayesian Guessing Strategy

In this section, we present a Bayesian viewpoint on universal guesswork. The Bayesian construction assumes the least-favorable Jeffreys' prior in the context of universal compression (see [12]). Let $\mathcal{I}(\theta)$ be the Fisher information matrix associated with the parameter vector θ, i.e.,

$$\mathcal{I}_{i,j}(\theta) \triangleq \lim_{n\to\infty} \frac{1}{n \log e} \mathbf{E}\left\{ \frac{\partial^2}{\partial\theta_i \partial\theta_j} \log\left(\frac{1}{\mu_\theta^n(X^n)} \right) \right\}. \tag{8}$$

We assume that the source is ergodic such that the above limit exists. Let Jeffreys' prior, denoted by p_Λ, be

$$p_\Lambda(\theta) \triangleq \frac{|\mathcal{I}(\theta)|^{\frac{1}{2}}}{\int_\Lambda |\mathcal{I}(\lambda)|^{\frac{1}{2}} d\lambda}. \tag{9}$$

Let μ_Λ^n denote the mixture distribution with Jeffreys' prior:

$$\mu_\Lambda^n(x^n) = \int_\Lambda \mu_\theta^n(x^n) p_\Lambda(\theta) d\theta. \tag{10}$$

Let $G_{\mu_\Lambda^n}$ be the optimal procedure for the distribution μ_Λ^n.

Theorem 6: $G_{\mu_\Lambda^n}$ and $G_\star$ are asymptotically equivalent, and hence are both asymptotically optimal.

The proof follows the same lines of Theorem 6 of [13].

C. Extension to Finite-Memory Markov Sources

Next, we generalize universal multi-string guesswork to finite-memory sources. The Bayesian universal guessing described in this paper is general and readily applicable to finite-memory sources, such as, finite-state machines [14] and context trees [15]. The type-size guessing can also be generalized by considering more general notion of types (see [16]). Let h denote the memory of the source, i.e., height of the largest suffix in the context tree that represents the source [15]. Note that for an iid source, $h = 0$. The next theorem generalizes Theorem 3 to this more general class of stationary sources.

Theorem 7: For a finite-memory source with d-dimensional parameter vector and a context tree of depth h, let $G_\star$ be the guesswork function of the universal type-size guessing strategy. Further, let $G_{\mu_\theta^n}$ be the guesswork of an optimal non-universal guessing strategy for parametric source with parameter vector θ, such that in $G_{\mu_\theta^n}$ ties are broken in favor of guessing sequences with smaller type-sizes first and if there is a tie in the size of the type the tie is broken arbitrarily. Then, for any individual sequence x^n we have

$$\frac{G_\star(x^n)}{G_{\mu_\theta^n}(x^n)} = O((h+1)n^{d+1}). \tag{11}$$

See Appendix D for the proof.

Thus far, we assumed that the source parameters of a finite memory source were unknown. Next, we further extend the universal guessing strategy to twice universal finite-memory sources, where in addition to the source statistics being unknown to the inquisitor, the (finite) source model is also unknown (cf. [17] for a formal definition).

Let $h : \mathbb{N} \to \mathbb{N}$ be any function such that $h(n) = o(\log n)$ and $h(n) = \omega(1)$. For any $n \geq 1$, let the unknown source model be described by a Markov source of order $h(n)$, which defines a parametric source with $d(n) = (|\mathcal{X}| - 1)|\mathcal{X}|^{h(n)}$ parameters. Let $\Lambda_{d(n)}$ denote the space of parameter vectors for the model. Note that using this strategy we will surely asymptotically overestimate the number of unknown source parameters as the number of source parameters is growing unboundedly. On the other hand, we will show that even with this model we can achieve a guesswork exponent that is of the right order. Let $\mu_{\Lambda_{d(n)}}^n$ be defined similar to (10). We use h instead of $h(n)$ when it is clear from the context. Let $G_{\mu_\Lambda^{n\,h}}$ be the order in which x^n appears when the sequences are sorted based on $\mu_{\Lambda_{d(n)}}^n$ in a decreasing fashion. Then, by invoking Theorem 7 and considering the growth rate of $h(n)$ we have

$$\frac{1}{n} \log \left(\frac{G_{\mu_{\Lambda_{d(n)}}^n}(x^n)}{G_{\mu_\theta^n}(x^n)} \right) = o(1).$$

Note that a universal strategy could have been achieved by using type-size coding using the twice universal types

defined in [17], which would be asymptotically equivalent to the aforementioned Bayesian strategy. Note that a similar statement holds for the type-size guesswork, denoted by $G_{\star\star}$. This will lead to the following.

Corollary 8: For any $1 \leq U \leq V$, the inscrutability rate of a finite-memory Markov source with unknown order and unknown parameters is the specific Rényi entropy rate:

$$S_{\star\star}(U, V, \{\mu^n\}) = H_{(V-U+1)/(V-U+2)}(\{\mu^n\}),$$

where $S_{\star\star}(U, V, \{\mu^n\})$ is the inscrutability rate of guessing U out of V secret strings chosen from an unknown Markov string-source with unknown finite order.

Observe that when the number of unknown parameters grows, the class of probability distributions that can be described becomes richer. On the other hand, the cost of universality grows linearly with the number of unknown parameters. Although our results show that the cost of universality is asymptotically negligible, this overhead can be quite large for moderate problem sizes when the class of distributions is fairly complex. This is analogous to the cost of universal compression that can be quite large for small to moderate sequence lengths while universal compression is known to asymptotically achieve the Shannon entropy.

V. Maximum Inscrutability String-Source with Constrained Shannon Entropy

Thus far, we showed that with a constrained Shannon entropy budget, choosing strings independently from a stationary string-source, corresponds to the minimum inscrutability rate against adversarial attacks. Furthermore, if the string-source is finite-memory parametric, hiding the string-source statistics is not asymptotically a remedy in the sense that it does not decrease the inscrutability rate. A natural question is whether there exists a string source in $\Delta_{H_\mathcal{X}}$ that has a larger inscrutability rate than the Shannon entropy rate. This is answered in the following theorem.

Theorem 9: For any $1 \leq U \leq V$, the inscrutability of identifying U out of V strings drawn independently from $\{\mu^n\} \in \Delta_{H_\mathcal{X}}$ is bounded from above by

$$S^n(U, V, \mu^n) \leq S^n(U, V, \overline{\mu}^n), \tag{12}$$

where $\overline{\mu}^n$ is such that all symbols but one are uniform and the probability measure is distributed between the most probable symbol and the rest of the uniform symbols such that the Shannon entropy budget $H_\mathcal{X}$ is satisfied. Further, the inscrutability rate exists for the string-source $\{\overline{\mu}^n\}$ and is equal to $\log|\mathcal{X}|$. That is

$$S(U, V, \{\overline{\mu}^n\}) = \lim_{n\to\infty} \frac{1}{n} S^n(U, V, \overline{\mu}^n) = \log|\mathcal{X}|. \tag{13}$$

See Appendix E for the proof.

Theorem 9 indeed reveals that given any non-zero Shannon entropy budget $H_\mathcal{X}$, the inscrutability rate of the string-source $\{\overline{\mu}^n\}$ is equal to that of a uniform distribution on the entire support set, which needs a larger entropy budget $\log|\mathcal{X}|$ per symbol. In light of Theorems 2 and 9, if the inscrutability rate exists for a string-source $\{\mu^n\} \in \Delta_{H_\mathcal{X}}$, then for all $1 \leq U \leq V$ it satisfies:

$$H_\mathcal{X} \leq S(U, V, \{\mu^n\}) \leq \log|\mathcal{X}|. \tag{14}$$

VI. Conclusion

In this paper, we considered guesswork in a multi-string setting where V secret strings are independently drawn from a string-source with constrained Shannon entropy budget. We showed that the asymptotic computational security of the system against an inquisitor, who knows the (statistics of the) string-source and wishes to correctly guess U out of V secret strings, is related to a quantity defined as inscrutability. We also established that with a constrained Shannon entropy budget, the inscrutability rate of finite-memory parametric string-sources approaches that of the source with minimum inscrutability rate as V grows. Furthermore, we also proved that hiding the statistics of any finite-memory string-source does not provide larger inscrutability rate, i.e., the per-symbol gain of hiding the statistics of a finite-memory string-source is asymptotically vanishing. Finally, we showed that there exists a string-source with the same entropy budget that asymptotically provides maximum inscrutability rate of $\log|\mathcal{X}|$ achievable on a support of size $|\mathcal{X}|$.

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