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New York J. Math. **19** (2013) 13–22.**Natural maps between CAT(0)
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ABSTRACT. It is shown that certain natural maps between the ideal, Gromov, and end boundaries of a complete CAT(0) space can fail to be either injective or surjective. Additionally the natural map from the Gromov boundary to the end boundary of a complete CAT(−1) space can fail to be either injective or surjective.

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1. Introduction

In [2], a new class of metric spaces called *rough* CAT(0) spaces are introduced, and their interior geometry is studied. The boundary geometry is then defined and studied in [3]. By *interior geometry*, we mean the geometry of the space itself, whereas we use the term *boundary* in the sense of a boundary at infinity. Specifically we define a new notion of boundary at infinity for such spaces X that we call the *bouquet boundary* $\partial_B X$.

Rough CAT(0) spaces include both of the well-known classes of CAT(0) spaces and Gromov hyperbolic spaces, and it is proved in [3] that $\partial_B X$ coincides with the ideal boundary $\partial_I X$ if X is a complete CAT(0) space, and it coincides with the Gromov boundary $\partial_G X$ if X is a Gromov hyperbolic space. With a view to proving that $\partial_B X$ is nonempty when X is a reasonable unbounded space, $\partial_B X$ is also related to the end boundary $\partial_E X$, and in fact

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it is shown in [3] that we have the following commutative diagram of natural maps between these notions of boundary at infinity:

$$\begin{array}{ccccc}
 \partial_I X & \xhookrightarrow{i_I} & \partial_B X & \xrightarrow{\mu} & \partial_G X \\
 & & \downarrow \eta & \swarrow \phi & \\
 & & \partial_E X & &
 \end{array}$$

The results in [3] are mainly positive: it is shown that i_I is injective, or bijective if X is complete CAT(0), and that μ is bijective if X is Gromov hyperbolic. Furthermore conditions for η to be surjective or injective are also given.

In this note, we instead concentrate on negative results in the context of complete CAT(0) spaces. In this context, i_I is a natural bijection, so we will not even define the bouquet boundary: instead we identify it with the well-known ideal boundary and study $\nu := \mu \circ i_I$ instead of μ . Writing $\nu := \mu \circ i_I$ and $\epsilon := \eta \circ i_I$, and omitting the identification, we get the following simpler commutative diagram:

$$\begin{array}{ccc}
 \partial_I X & \xrightarrow{\nu} & \partial_G X \\
 \downarrow \epsilon & & \swarrow \phi \\
 \partial_E X & &
 \end{array}$$

It turns out that none of the natural maps in this last diagram are necessarily bijective. In fact, we construct counterexamples in this note that allow us to state the following theorem.

Theorem 1.1. *There exists a complete CAT(0) space X such that:*

- (a) $\nu : \partial_I X \rightarrow \partial_G X$ is neither injective nor surjective.
- (b) $\phi : \partial_G X \rightarrow \partial_E X$ is neither injective nor surjective.
- (c) $\epsilon : \partial_I X \rightarrow \partial_E X$ is neither injective nor surjective.

Lastly, $\partial_I Y$, $\partial_G Y$, and $\partial_E Y$ may be empty even if Y is an unbounded complete CAT(0) space.

Constructions were given in [4, Section 3] for spaces in which ν fails to be injective or surjective. However those spaces were far from being CAT(0). It is perhaps a little surprising that such counterexamples also exist in the class of complete CAT(0) spaces.

By the results of [3], $\partial_I X$, $\partial_B X$, and $\partial_G X$ can all be identified if X is both Gromov hyperbolic and complete CAT(0), and so in particular if X is a complete CAT(-1) space. Thus ν is a bijection in this case. However the map ϕ may still be badly behaved, as the following result indicates.

Theorem 1.2. *There exists a complete CAT(-1) space X such that*

$$\phi : \partial_G X \rightarrow \partial_E X$$

is neither injective nor surjective. Also $\partial_G Y$ and $\partial_E Y$ may be empty even if Y is an unbounded complete CAT(-1) space.

After some preliminaries in Section 2, we give counterexamples and prove the above results in Section 3.

2. Preliminaries

Throughout this section, we suppose (X, d) is a metric space. We say that X is *proper* if every closed ball in X is compact.

We write $A \wedge B$ for the minimum of two numbers A, B .

An *h -short segment* from x to y , $x, y \in X$, is a path of length at most $d(x, y) + h$, $h \geq 0$. A *geodesic segment* is a 0-short segment. X is a *length space* if there is an h -short segment between each pair $x, y \in X$ for every $h > 0$, and a *geodesic space* if there is a geodesic segment between each pair $x, y \in X$.

A *geodesic ray* in X is a path $\gamma : [0, \infty) \rightarrow X$ such that each initial segment $\gamma|_{[0, t]}$ of γ is a geodesic segment. The *ideal boundary* $\partial_1 X$ of X is the set of equivalence classes of geodesic rays in X , where two geodesic rays γ_1, γ_2 are said to be equivalent if $d(\tilde{\gamma}_1(t), \tilde{\gamma}_2(t))$ is uniformly bounded for all $t \geq 0$, where $\tilde{\gamma}_i$ is the unit speed reparametrization of γ_i , $i = 1, 2$.

We refer the reader to [1, Part II] for the theory of CAT(κ) spaces for $\kappa \in \mathbf{R}$. In particular, we note that a smooth Riemannian manifold is CAT(κ) if and only if it is simply connected and has sectional curvature $\leq \kappa$. Also the ideal boundary $\partial_1 X$ of a complete CAT(0) space can be identified with the set of unit speed geodesic rays from some origin $o \in X$ [1, II.8.2]: this identification is independent of the choice of origin. If X is a simply connected smooth Riemannian n -manifold of sectional curvature ≤ 0 , then $\partial_1 X$ is homeomorphic to \mathbf{S}^{n-1} .

As discussed in the introduction, we can identify the ideal boundary and the bouquet boundary of [3] in the context of complete CAT(0) spaces. The fact that these are not the same in rough CAT(0) spaces, or even in incomplete CAT(0) spaces, and that the bouquet boundary is better behaved than the ideal boundary, is discussed in detail in [3, Section 4], so we will not discuss it further here. We simply identify these notions for complete CAT(0) spaces.

We refer the reader to [7], [5], [9], or [1, Part III.H] for the theory of Gromov hyperbolic spaces. We use the nongeodesic definition: a metric space (X, d) is *δ -hyperbolic*, $\delta \geq 0$, if

$$\langle x, z \rangle_w \geq \langle x, y \rangle_w \wedge \langle y, z \rangle_w - \delta, \quad x, y, z, w \in X,$$

where $\langle x, z \rangle_w$ is the Gromov product defined by

$$2 \langle x, y \rangle_w = d(x, w) + d(y, w) - d(x, y).$$

A *Gromov sequence* in a metric space X is a sequence (x_n) in X such that $\langle x_m, x_n \rangle_o \rightarrow \infty$ as $m, n \rightarrow \infty$. If $x = (x_n)$ and $y = (y_n)$ are two such sequences, we write $(x, y) \in E$ if $\langle x_m, y_n \rangle_o \rightarrow \infty$ as $m, n \rightarrow \infty$. Then E is a reflexive symmetric relation on the set of Gromov sequences in X , so its transitive closure, which we denote by \sim , is an equivalence relation on the set of Gromov sequences in X . The *Gromov boundary* $\partial_G X$ is the set of equivalence classes $[(x_n)]$ of Gromov sequences.

The relation E above is an equivalence relation if X is Gromov hyperbolic, but this is not true in general metric spaces [4, 1.5]. Gromov sequences and the Gromov boundary have mainly been considered in Gromov hyperbolic spaces, but they have also been defined as above in general metric spaces [4]. The Gromov boundary is independent of the choice of basepoint o .

The natural map $\nu : \partial_1 X \rightarrow \partial_G X$ is induced by the map $f(\lambda) = (x_n)$ that takes a geodesic ray λ parametrized by arclength to a sequence of points (x_n) , where $x_n = \lambda(t_n)$ and (t_n) is any sequence of numbers tending to infinity.

A CAT(-1) space is both CAT(0) and Gromov hyperbolic; see II.1.12 and III.H.1.2 of [1]. Hence by Theorems 4.20 and 5.15 of [3], we can identify $\partial_1 X$ with $\partial_G X$ if X is a complete CAT(-1) space. A related result is Lemma III.H.3.1 of [1], which says that we can identify $\partial_1 X$ with $\partial_G X$ if X is a proper geodesic Gromov hyperbolic space.

By an *end* of a metric space X (with basepoint o), we mean a sequence (U_n) of components of $X \setminus \bar{B}_n$, where $\bar{B}_n = \overline{B(o, n)}$ for fixed $o \in X$ and $U_{n+1} \subset U_n$ for all $n \in \mathbf{N}$. We do not require \bar{B}_n to be compact. We denote by $\partial_E X$ the collection of ends of X and call it the *end boundary* of X .

Ends with respect to different basepoints are compatible under set inclusion: defining U_n, V_n for all $n \in \mathbf{N}$ to be components of $X \setminus \overline{B(o, n)}$ and $X \setminus \overline{B(o', n)}$, respectively, it is clear that U_n is a subset of a unique V_m whenever $n - m > d(o, o')$. This compatibility gives rise to a natural bijection between ends with respect to different basepoints, allowing us to identify them and treat the end boundary as being independent of the basepoint.

Suppose X is a complete CAT(0) space. If we map a geodesic ray λ from o parametrized by arclength to the end (U_n) , where U_n is the component of $X \setminus \bar{B}_n$ containing $\lambda(n+1)$, then this map induces the natural map $\epsilon : \partial_1 X \rightarrow \partial_E X$; see [3, Theorem 4.24]. The natural map $\phi : \partial_G X \rightarrow \partial_E X$ is also induced by the map taking a Gromov sequence (x_n) to the unique end “containing” it, in the sense that for each $m \in \mathbf{N}$ there exists $N \in \mathbf{N}$ such that x_n lies in some component U_m of $X \setminus \bar{B}_m$ for all $n \geq N$; see [3, Proposition 5.19].

3. Examples

Euclidean \mathbf{R}^n for $n > 1$ is all we need to consider to prove that ν and ϵ can fail to be injective. As is well known, the ideal boundary of \mathbf{R}^n (with cone topology attached) is homeomorphic to \mathbf{S}^{n-1} , the sphere of dimension

$n - 1$. By contrast the end boundary of \mathbf{R}^n is clearly a singleton set, and it turns out that $\partial_G \mathbf{R}^n$ is also a singleton set.

Let us indicate how to prove this last fact. First since the natural map ν exists, $\partial_G \mathbf{R}^n$ is nonempty. We now appeal to Theorem 2.2 of [4] which states that ν is surjective if X is a proper geodesic space. In fact, as is clear from the proof of that result, ν is induced by the map that takes a geodesic ray $\gamma : [0, \infty) \rightarrow X$ parametrized by arclength to the Gromov sequence $(\gamma(t_n))_{n=1}^\infty$, where (t_n) is any sequence of nonnegative numbers with limit infinity. Since the ideal boundary of a complete CAT(0) space can be viewed as the set of geodesic rays from any fixed origin, it follows that we get representatives of all points in $\partial_G \mathbf{R}^2$ by considering only the Gromov sequences $x^t := (na_t)_{n=1}^\infty$, where $a_t = (\cos t, \sin t) \in \mathbf{R}^2$, $t \in \mathbf{R}$. A straightforward calculation, or an appeal to Lemma 3.2, shows that $(x^t, x^s) \in E$ for all pairs t, s , except when $|t-s|$ is an odd multiple of π , i.e. except when x^t and x^s are tending to infinity in opposite directions. But in the exceptional case, we have $(x^t, x^{t+\pi/2}) \in E$ and $(x^{t+\pi/2}, x^s) \in E$, so all Gromov sequences are equivalent.

The exceptional case \mathbf{R}^n for $n = 1$ is easily analyzed: ν and ϵ are bijective, and the cardinalities of $\partial_I X$, $\partial_G X$, and $\partial_E X$ are all 2.

We now generalize this result to arbitrary Hilbert spaces of dimension greater than 1. Note that the above method of proof fails in this context since infinite dimensional Hilbert spaces are not proper. Although these infinite dimensional examples are not necessary ingredients in the proofs of our main results, the method of proof will be useful for later examples that are needed.

Proposition 3.1. *Suppose first that X is a Hilbert space of dimension greater than 1. Then $\partial_I X$ has cardinality at least that of the continuum, while $\partial_G X$ and $\partial_E X$ are singleton sets. Thus the natural maps ν and ϵ are not injective.*

Before proving Proposition 3.1, we first prove a simple but useful lemma. In this lemma, and in the proof of Proposition 3.1, (\cdot, \cdot) is the inner product in a Hilbert space X , $|\cdot|$ is the associated norm, and

$$\angle(u, v) = \cos^{-1}((u, v)/|u||v|)$$

is the angle between two nonzero vectors u, v in X .

Lemma 3.2. *Suppose X is a Hilbert space, and that $u, v \in X \setminus \{0\}$ are such that $\angle(u, v) \leq \alpha$ for some $0 < \alpha < \pi$. Then $\langle u, v \rangle_0 \geq k(|u| \wedge |v|)$, where $k = k(\alpha) > 0$.*

Proof. Writing $a := |u - v|$, $b := |u|$, and $c := |v|$, we assume without loss of generality that $b \leq c$. By the cosine rule, $a^2 = b^2 + c^2 - 2bc \cos \alpha$. Thus $2\langle u, v \rangle_0 \geq cf(t)$, where $t = b/c$ and $f(t) = t + 1 - \sqrt{1 + t^2 - 2t \cos \alpha}$. Now $t \mapsto \sqrt{1 + t^2} + rt$ is convex for all $|r| \leq 2$, so it follows by calculus that $f(t) \geq kt$, where $k = f(1) > 0$. \square

Proof of Proposition 3.1. Since X is a complete CAT(0) space, its ideal boundary can be identified with the set R of unit speed geodesic rays from the origin. For a Hilbert space, this latter set is naturally bijective to the sphere $S := \partial B(0, 1)$ via the identification of $\gamma \in R$ with $\gamma(1)$. In particular, the cardinality of $\partial_1 X$ is at least that of the continuum.

Now $(nu)_{n=1}^\infty$ is a Gromov sequence whenever $u \in S$, so certainly $\partial_G X$ is nonempty. Suppose that $x = (x_n)$ and $y = (y_n)$ are Gromov sequences. Pick $u_1, u_2, u_3 \in S$ so that $\angle(u_i, u_j) \geq 2\pi/3$ for each pair of distinct indices i, j ; this can be even be done by picking these points on a single great circle in S . For $i = 1, 2, 3$, let C_i be the cone of points $u \in X \setminus \{0\}$ such that $\angle(u, u_i) < \pi/3$, so that these three cones are pairwise disjoint. Thus, by taking subsequences if necessary, we may assume that both of the sequences x and y avoid one of these three cones C_i . Letting $z = (-nu_i)_{n=1}^\infty$, and applying Lemma 3.2 with $\alpha = 2\pi/3$, we see that (x, z) and (z, y) both lie in E , and so $x \sim y$. Thus all Gromov sequences are equivalent, as required. \square

Finding examples where ν and ϵ fail to be surjective appears to be more difficult than finding examples where they fail to be injective. The key will be to consider a suitable metric subspace of the infinite dimensional Hilbert space ℓ^2 given by the following definition.

Definition 3.3. The *Hilbert flying saucer* is $X := \bigsqcup_{i=1}^\infty Y_i \subset \ell^2$, where Y_i is the following closed disk of codimension $i - 1$ in ℓ^2 :

$$Y_i := \bigsqcup x = (x_j)_{j=1}^\infty : \|x\| \leq i, x_j = 0 \text{ for all } j < i .$$

We attach to X the induced length metric d . We next prove that ν can fail to be surjective.

Theorem 3.4. *The Hilbert flying saucer X is a complete CAT(0) space. Moreover $\partial_1 X$ is empty, while $\partial_G X$ and $\partial_E X$ are singleton sets. Thus the natural maps ν and ϵ are not surjective.*

Proof. We define o to be the origin in ℓ^2 so that $o \in Y_i$ for all i . Completeness is easy, since each Y_i is closed in ℓ^2 and a finite number of the sets Y_i cover any given compact set. To show that X is CAT(0), it suffices to show that $X_i := \bigsqcup_{j=1}^i Y_j$ is CAT(0), where again we attach the induced length metric. We establish this fact inductively. Since any ball in ℓ^2 is CAT(0), it follows that Y_i is CAT(0) for all i . In particular X_1 is CAT(0). Suppose now that X_i is CAT(0) for some i . Now $Z_i := X_i \cap Y_{i+1}$ is a convex and closed (hence complete) subset of ℓ^2 , and X_{i+1} is obtained by gluing X_i and Y_{i+1} along Z_i . But gluing two CAT(0) spaces along a pair of isometric complete convex spaces gives another CAT(0) space [1, II.11.1], so X_{i+1} is CAT(0), completing the inductive step. Thus X is CAT(0).

Suppose for the sake of contradiction that $\gamma : [0, \infty) \rightarrow X$ is a geodesic ray parametrized by arclength with $\gamma(0) = o$. Writing $\gamma(1) = x = (x_i)$, we have $x_j \neq 0$ for some $j \in \mathbf{N}$. But $\|\gamma(j+1)\| = j+1$, where $\|\cdot\|$ is the ℓ^2 -norm, so $\gamma(j+1) = y = (y_i)$, where $y_i = 0$ for all $i \leq j$. But, as a subset

of ℓ^2 , X is star-shaped with respect to o , so the ℓ^2 line segment is the unique X -geodesic from o to y , and this does not pass through x .

Using o as the basepoint, it is clear that X has a single end, so it remains to prove that $\partial_{\mathbb{G}}X$ is a singleton set. Let us denote by e_i the unit vector in the i th coordinate direction. We denote distance in X by $d(\cdot, \cdot)$ and the ℓ^2 norm by $|\cdot|$. Although in general we know only that $d(u, v) \geq |u - v|$, the ℓ^2 line segment from u to v is contained in X if either $v = 0$ or $|v| = |u|$ (in the latter case because points on the line segment have norm no larger than $|u|$), and so $d(u, v) = |u - v|$ in both of these cases.

Let $x_n = 2^{n-1/2}(e_{2^n} + e_{2^n+1})$, and so $|x_n| = 2^n$, $n \in \mathbf{N}$. Also $\angle(x_i, x_j) = \pi/2$ for any pair of distinct indices i, j . It follows from Lemma 3.2 that (x_n) is a Gromov sequence in ℓ^2 . In fact, it is also a Gromov sequence in X . To see this, note that if $i \leq j$ and we write $x_j^{\square} = 2^{i-1/2}(e_{2^i} + e_{2^i+1})$, then $d(x_i, x_j^{\square}) = 2^{i+1/2}$ and so $\langle x_i, x_j^{\square} \rangle_0 = 2^i k$, where $k := 1 - 1/\sqrt{2}$. Now $d(x_i, x_j) \leq d(x_i, x_j^{\square}) + d(x_j^{\square}, x_j)$ and $d(0, x_j) = d(0, x_j^{\square}) + d(x_j^{\square}, x_j)$, so $\langle x_i, x_j \rangle_0 \geq 2^i k$ for all $i \leq j$. Thus x is a Gromov sequence in X , and $\partial_{\mathbb{G}}X$ is nonempty.

It remains to prove that all Gromov sequences are equivalent, so suppose $x = (x_n)$ and $y = (y_n)$ are a pair of Gromov sequences. Without loss of generality, we assume that $x_n, y_n \neq 0$ for all $n \in \mathbf{N}$. The idea of this proof is similar to that of Proposition 3.1: we pick a Gromov sequence $z = (z_n)$, where $z_n = nu_n \in X$ and u_n lies in the unit sphere S of ℓ^2 , such that the sequences x and y avoid a cone around each of the points z_n , and it will then follow that both (x, z) and (z, y) are elements of the relation E . Unlike the earlier proof, the requirement that $z_n \in X$ means that u_n must depend on n , and this means that we will need to iterate a countable number of times the process of taking subsequences of x and y . When taking subsequences for the n th time, we will insist that the first n entries in the subsequences of stage $n - 1$ are retained at stage n : this ensures that the diagonal subsequences associated with this process for x and y are subsequences of the n th iterated subsequences of x and y , respectively, for each $n \in \mathbf{N}$.

For $n \in \mathbf{N}$, let S_n be the intersection of S with the plane generated by $e_{2^{n-1}}$ and e_{2^n} : thus (S_n) is a sequence of pairwise orthogonal great circles on S . Pick five vectors $v_{1,i}$, $i = 1, \dots, 5$, in $S_1 \cup S_2$ such that $\angle(v_{1,i}, v_{1,j}) \geq \pi/2$ for all $1 \leq i \leq 5$: we could for instance pick four such vectors in S_1 and an arbitrary vector in S_2 . Thus the cones $C_{1,i}$ of points $w \in \ell^2 \setminus \{0\}$ such that $\angle(v_{1,i}, w) < \pi/4$ are pairwise disjoint, and so at least three of them must be disjoint from the set $\{x_1, y_1\}$. Of these three, at least one contains infinitely many x_n and infinitely many y_n . By taking subsequences $x^1 = (x_n^1)$, $y^1 = (y_n^1)$, of x and y , respectively, and letting u_1 be one of the vectors $v_{1,i}$, we get that $\angle(u_1, w) \geq \pi/4$ whenever $w = x_n^1$ or $w = y_n^1$ for any $n \in \mathbf{N}$. We assume, as we may, that $x_1^1 = x_1$ and $y_1^1 = y_1$.

For the second stage, we pick seven vectors $v_{2,i}$, $i = 1, \dots, 7$, in $S_3 \cup S_4$ such that $\angle(v_{2,i}, v_{2,j}) \geq \pi/2$ for all $1 \leq i < j \leq 7$: we could for instance

pick four such vectors in S_3 and three in S_4 . At least three of the seven associated cones fail to intersect the set $\{x_1^1, x_2^2, y_1^1, y_2^1\}$. It follows that by taking subsequences $x^2 = (x_n^2)$, $y^2 = (y_n^2)$, of x^1 and y^1 , respectively, and letting u_2 be one of the vectors $v_{2,i}$, we get that $\angle(u_2, w) \geq \pi/4$ whenever $w = x_n^2$ or $w = y_n^2$ for any $n \in \mathbf{N}$. We assume, as we may, that $x_n^2 = x_n^1$ and $y_n^2 = y_n^1$ for $n = 1, 2$.

We proceed in this manner, picking vectors $v_{m,i}$, $i = 1, \dots, 2m + 3$, at the m th stage from the next few circles S_i that we have not used yet in this construction, in such a way that $\angle(v_{m,i}, v_{m,j}) \geq \pi/2$ for all $1 \leq i < j \leq 2m + 3$. We use as many circles as are needed to ensure that this can be done: for $m \in \{2p - 1, 2p\}$, $p \in \mathbf{N}$, it suffices to use $p + 1$ circles. Thus for $m = 3$, we pick nine vectors from $S_5 \cup S_6 \cup S_7$, for $m = 4$ we pick eleven vectors from $S_8 \cup S_9 \cup S_{10}$, etc. Carrying out the construction as before, we get a unit vector $u_m \in S_M$ for some $M \geq m$, and subsequences $x^m = (x_n^m)$, $y^m = (y_n^m)$ of x^{m-1} and y^{m-1} , respectively, such that $\angle(u_m, w) \geq \pi/4$ whenever $w = x_n^m$ or $w = y_n^m$ for some $n \in \mathbf{N}$, and such that $x_n^m = x_n^{m-1}$ and $y_n^m = y_n^{m-1}$ whenever $n \leq m$.

Defining $x_n^\square = x_n^n$, $y_n^\square = y_n^n$ for $n \in \mathbf{N}$, we get subsequences $x^\square = (x_n^\square)$, $y^\square = (y_n^\square)$ of x, y , respectively, and a sequence $u = (u_n)$ on S such that $\angle(u_n, w) \geq \pi/4$ whenever $w \in \{x_m, y_m\}$ and $n, m \in \mathbf{N}$, and such that $\angle(u_m, u_n) = \pi/2$ whenever $m \neq n$. Letting $z_n = nu_n$, it follows from the fact that $u_m \in S_M$ for some $M \geq m$ that $z_n \in X$ for all $n \in \mathbf{N}$. Writing $z = (z_n)$, and arguing as we did in the proof that $\partial_G X$ is nonempty, it follows that z is a Gromov sequence, and that both (x, z) and (z, y) lie in E . We leave the details to the reader. \square

It is easy to find a complete $\text{CAT}(-1)$ space X in which $\phi : \partial_G X \rightarrow \partial_E X$ is not injective. Indeed the hyperbolic plane $X = \mathbf{H}^2$ is one such example. Since we can identify $\partial_I X$ with $\partial_G X$ in this case (because X is complete $\text{CAT}(-1)$, or alternatively because X is a proper geodesic Gromov hyperbolic space: see Section 2), $\partial_G X$ can be identified with the unit circle, and so its cardinality is that of the continuum. On the other hand, it is clear that $\partial_E X$ is a singleton set. To prove that ϕ may fail to be surjective even if X is complete $\text{CAT}(-1)$, we need to work a bit harder.

Example 3.5. Let X be the Hilbert flying saucer, but let us replace our original metric d by the conformally distorted length metric d^\square given infinitesimally at a point with polar coordinates $(t, \theta) \in [0, \infty) \times \partial B(0, 1)$ by ds , where $ds^2 = dt^2 + \sinh^2(t)d\theta^2$. Then (X, d^\square) has a single end, since d^\square -balls around 0 coincide with d -balls.

Each of the sets Y_i in Definition 3.3 is $\text{CAT}(-1)$ with respect to d^\square in fact any geodesic triangle in Y_i is contained in an isometric copy of a hyperbolic plane. Since we obtain (X, d^\square) by gluing a succession of spaces X_i and Y_{i+1} along a pair of isometric complete convex spaces, the resulting space (X, d^\square) is $\text{CAT}(-1)$. It is also clearly complete. Thus $\partial_I X$ can be identified with

$\partial_G X$. But the d - and d^\square -geodesic paths from 0 to $x \in X$ coincide as sets, so again $\partial_I X$ is empty.

We are now ready to prove our main theorems.

Proof of Theorem 1.1. If we glue a pair of disjoint complete CAT(0) spaces X and Y by identifying a single point in X with another point in Y , then we get another CAT(0) space which we denote $X + Y$ according to the basic gluing theorem II.11.1 of [1]. It is also easy to see that $X + Y$ is complete and that the ideal, Gromov, and end boundaries of $X + Y$ can be identified with the disjoint union of the corresponding boundaries in X and Y . The maps ν , ϕ , and ϵ for $X + Y$ are also obtained by taking the “disjoint union” of the corresponding maps for X and Y , e.g. ν_{X+Y} is defined by $\nu_{X+Y}(x) = \nu_X(x)$ for all $x \in \partial_I X$ and $\nu_{X+Y}(y) = \nu_Y(y)$ for all $y \in \partial_I Y$. It follows that if we have separate spaces where each of the maps ν , ϕ , and ϵ fails to be injective or surjective, then by gluing all of these spaces at a single point, we get a space where all three of these maps fails to be both injective and surjective.

Now ν and ϵ fail to be injective in a Hilbert space of dimension larger than 1, and they fail to be surjective in the Hilbert flying saucer, according to Proposition 3.1 and Theorem 3.4. As for ϕ , it fails to be injective in the hyperbolic plane, and it fails to be surjective in a hyperbolic version of the Hilbert flying saucer (Example 3.5). Since all of these spaces are complete CAT(0) spaces, we can glue them to get a complete CAT(0) space that fails to have any of these injectivity or surjectivity properties.

Finally, let X be the subset of \mathbf{R}^2 consisting of the union of the line segments from $(0, 0)$ to $(n, 1)$ for all $n \in \mathbf{N}$, and attach the Euclidean length metric to X . Then X is an unbounded tree but it has no end, so $\partial_E X$, $\partial_G X$, and $\partial_I X$ are all empty. \square

The proof of Theorem 1.2 is very similar. The hyperbolic version of the Hilbert flying saucer in Example 3.5 is complete CAT(-1), so when we glue it at a single point to the hyperbolic plane, the resulting space is also complete CAT(-1). By the properties of the individual space, we see that ϕ for the glued space fails to be either injective or surjective. Finally the example in the last paragraph of the previous proof is CAT($-\infty$), so it also works for Theorem 1.2.

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