Entanglement Spectrum of Composite Fermion States in Real Space

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We study the entanglement spectra of many particle systems in states which are closely related to products of Slater determinants or products of permanents, or combinations of the two. Such states notably include the Laughlin and Jain composite fermion states which describe most of the observed conductance plateaus of the fractional quantum Hall effect. We identify a set of 'Entanglement Wave Functions' (EWF), for subsets of the particles, which completely describe the entanglement spectra of such product wave functions, both in real space and in particle space. A subset of the EWF for the Laughlin and Jain states can be recognized as Composite Fermion states. These states provide an exact description of the low angular momentum sectors of the real space entanglement spectrum (RSES) of these trial wave functions and a physical explanation of the branches of excitations observed in the RSES of the Jain states.

In studying quantum states of many particle systems, it is often of interest to split the system into subsystems and focus on the reduced density matrix ρ_A of one of these subsystems. Various *entanglement spectra*[1] collect the eigenvalues and eigenstates of such reduced density matrices for different 'cuts' of the initial system into an A-subsystem and a B-subsystem. Two very natural cuts for an N particle system are the *particle cut*, for which the A-system simply consists of particles 1 through N_A (The B-system has the remaining particles) and the *real space cut*, where the A-system consists of particles within a certain spatial domain and the B-system has the particles in the complementary domain. These cuts lead to the particle entanglement spectrum (PES) and the real space entanglement spectrum (RSES). If the full system is in a state ψ, then we may write $\rho_A(Z_A, Z'_A) = \int_{D_B} \psi^*(Z_A, Z_B)\psi(Z'_A, Z_B)\, dZ_B$, where Z_A, $Z_{A'}$ and Z_B are sets of coordinates for particles in the A and B subsystems. For the particle cut, the domain D_B is all of space and Z_A and $Z_{A'}$ contain arbitrary positions. For the real space cut, D_B is the chosen domain for the particles in the B-system, while the positions Z_A and $Z_{A'}$can be restricted to lie in D_A, the complement of D_B. We should expected the PES and RSES to be adiabatically connected, as we can continuously grow the domains D_A and D_B for the A and B systems, allowing them to overlap and eventually to encompass all space. In the context of fractional quantum Hall (FQH) systems, the particle cut was studied in Refs. 2–4 and the real space cut in Refs. 5–7 and for filling $\nu = 1$ in Refs. 8 and 9. The eigenspaces of ρ_A at fixed angular momentum were indeed found the be of equal dimension for PES and RSES, though the eigenvalues are different.

The most ubiquitous states appearing in many particle physics are Slater determinants (for fermions) and permanents (for bosons). These have particularly simple reduced density matrices due to their factorization properties. For a Slater determinant $S_{\phi_1...\phi_N}(z_1, ..., z_N)$, involving N particles and N orbitals $\phi_1...\phi_N$, we have

$$S_{\phi_1...\phi_N}(z_1...z_N) = \frac{1}{N_A!N_B!}\sum_{\sigma\in S_N}\epsilon(\sigma)\epsilon(\sigma|_A)\epsilon(\sigma|_B) \times S_{\phi_{\sigma(1)}...\phi_{\sigma(N_A)}}(z_1...z_{N_A})S_{\phi_{\sigma(N_A+1)}...\phi_{\sigma(N)}}(z_{N_A+1}...z_N) \quad (1)$$

Here the ϵ-factors come from the usual antisymmetric tensors, that is, $\epsilon(\sigma)$ is the sign of the permutation $\sigma \in S_N$ and $\epsilon(\sigma|_A)$ and $\epsilon(\sigma|_B)$ give the sign of this permutation when it is restricted to $\{1...N_A\}$ and $\{N_A + 1...N\}$. Note that the summands depend only on the sets $\{\sigma(1)...\sigma(N_A)\}$ and the factor $\frac{1}{N_A!N_B!}$ can be removed if we change the sum to a sum over all the possible choices of N_A out of N orbitals. We see then that the Slater determinant splits into a sum of products of Slater determinants for the subsystems, with complementary occupation of the orbitals in the factors. For permanents one gets a similar result without the ϵ-factors. It is now easy to write the density matrix explicitly in terms of the determinants (or permanents) for the A-system and one finds that the eigenvalues of ρ_A are all equal and the eigenfunctions in the PES are precisely all the determinants (or permanents) that occur for the A-system. The RSES will usually be less trivial: in particular, the eigenstates of ρ_A which have the N_A particles of system A concentrated in the domain D_A will typically have the largest eigenvalues (or the lowest entanglement energies).

The low energy states of strongly correlated systems are typically superpositions of many Slater determinants or permanents and have much more complicated and interesting entanglement spectra. Nevertheless, many trial wave functions for such systems, notably the Laughlin and Jain wave functions for FQH states, are products of Slater determinants, or projections of such products. Wave functions of this type have factorizations similar to Eq. (1) and their PES and RSES can be completely de-

scribed by a set of *entanglement wavefunctions* (EWF), which are themselves products of Slater determinants. To show this explicitly, we consider a general product of Slater determinants $\psi(z_1...z_N) = \prod_{i=1}^m S^{(i)}(z_1...z_N)$. Here, each of the $S^{(i)}$ is a slater determinant involving a set of N orbitals $\phi_1^{(i)}, \ldots, \phi_N^{(i)}$. It is clear from the expansion in Eq. (1) that we can write

$$\psi(Z_A, Z_B) = \sum_k s_k \prod_{i=1}^m S^{(k_i^A)}(Z_A) \prod_{i=1}^m S^{(k_i^B)}(Z_B).$$

Here the summation variable k runs through all splittings of the sets of orbitals $\{\phi_1^{(i)}, \ldots, \phi_N^{(i)}\}$ into subsets of N_A and N_B orbitals associated with the A and B subsystem. We may write $k = (k_1^A, k_1^B, ..., k_m^A, k_m^B)$, where k_i^A is the subset of $\{\phi_1^{(i)}, \ldots, \phi_N^{(i)}\}$ consisting of the N_A orbitals which are assigned to the A-system and k_i^B contains the remaining N_B orbitals in $\{\phi_1^{(i)} \ldots, \phi_N^{(i)}\}$. The constants s_k are the combined signatures of permutations that appear and hence $s_k \in \{1, -1\}$. The entanglement wave functions are the products $\xi_k(Z_A) := \prod_{i=1}^m S^{(k_i^A)}(Z_A)$ and $\zeta_k(Z_B) := \prod_{i=1}^m S^{(k_i^B)}(Z_B)$ appearing in this factorization. We may now write

$$\rho_A(Z_A, Z_A') = \sum_{k,l} Q_{k,l}\, \xi_k(Z_A) \xi_l^*(Z_A'), \text{ with}$$
$$Q_{kl} = s_k s_l \int_{D_B} \zeta_k(Z_B) \zeta_l^*(Z_B)\, dZ_B. \tag{2}$$

If we write the eigenvalues and eigenvectors of ρ_A as λ_p and g_p, so $\int \rho_A(Z_A, Z_A') g_p(Z_A')\, dZ_A' = \lambda_p g_p(Z_A)$, then we see that, for any g_p with $\lambda_p \neq 0$, we have

$$g_p(Z_A) = \sum_k \eta_{pk} \xi_k(Z_A), \tag{3}$$

with

$$\eta_{pk} = \frac{1}{\lambda_p} \sum_l Q_{kl} \int_{D_A} \xi_l^*(Z_A') g_p(Z_A')\, dZ_A'. \tag{4}$$

Hence, we see that the states g_p in the entanglement spectrum can be written as linear combinations of the EWF. Substituting the expression for $g_p(Z_A)$ from Eq. (3) into the expression for η_{pk}, we obtain

$$\sum_l M_{il} \eta_{kl} = \lambda_k \eta_{ki}, \tag{5}$$

with

$$M_{il} = \sum_j Q_{ij} \int_{D_A} \xi_j^*(Z_A') \xi_l(Z_A')\, dZ_A'. \tag{6}$$

Thus, the nonzero eigenvalues λ_p of ρ_A are also eigenvalues of a finite dimensional matrix, M, given in terms of the overlap matrix of the EWF. In fact, there is a one to one correspondence between eigenvectors η_k of M with nonzero eigenvalues and corresponding eigenvectors $\sum_i \eta_{ki} \xi_i$ of ρ_A (in particular, the rank of M is equal to the rank of ρ_A). Hence the entanglement spectrum can be described, completely, in terms of the EWF: the nonzero eigenvalues of ρ_A are given by (5) and the corresponding eigenvectors by (3), where the η_p are eigenvectors of M.

All calculations above work equally well no matter what the domain of integration D_B is, so the EWF play the same role both for the PES and for the RSES with any choice of domains for the A and B systems. One may also use products of permanents instead of Slater determinants or even mix permanents and determinants. Often, the state ψ and the orbitals $\phi_j^{(i)}$ are eigenstates of some symmetry, e.g. momentum p or angular momentum L, so that $p_\psi = \sum_{i,j} p_{\phi_j^{(i)}}$ or $L_\psi = \sum_{i,j} L_{\phi_j^{(i)}}$. The corresponding EWF are then also eigenfunctions of the symmetry and if the choice of domain D_B respects the symmetry, ρ_A is block diagonal with blocks labeled by eigenvalues of the symmetry and the calculation above can be performed within every block, using only the EWF with the appropriate eigenvalue of the symmetry.

Usually the number of EWF ξ_i is larger than the number of eigenstates g_p of ρ_A, but the ξ_i are not linearly independent. We will now argue that we should expect that the full space spanned by the ξ_i is needed to describe the ES. From Eq. (6), we see that any linear relation between the EWF gives rise to a zero eigenvector of M: if $\sum_i \eta_i \xi_i = 0$ then $M\eta = 0$. Hence the rank of M is less than or equal to the dimension of the space spanned by the EWF. If the rank of M is in fact smaller, there must be a vector η such that $M\eta = 0$ but $\Xi_\eta = \sum_i \eta_i \xi_i \neq 0$. In this case, Ξ_η is orthogonal to $\sum_j Q_{ij}^* \xi_j$ for all i. We now show that Ξ_η with these properties do not exist if the ξ_i are holomorphic functions of the z_i. In Ref. 7, three of us showed that, for holomorphic states, the rank of ρ_A does not depend on the domain D_A. Actually the states considered were not quite holomorphic — the property we used is that they are determined by their values on any open subset of their domain. This allows for example FQH trial wave functions built from orbitals in the lowest Landau level, which are holomorphic up to nonzero geometrical factors (gaussians for a system on a disk). We will use a straightforward generalization of this result. Instead of having a single domain D_A for all particles of the A-system, we can allow individual (open) domains $D_{A,i}$ in space for each of these particles. The rank of ρ_A and hence M still does not depend on the choice of these domains. Consider domains $D_{A,i}$ which are very small neighborhoods of points of space. We take D_B to be the full space. We now get, for $p = (z_1^A, z_2^A, ..., z_{N_A}^A)$,

$$\sum_j Q_{ij} \int_{D_A} \xi_j^*(Z_A)\, \Xi_\eta(Z_A)\, dZ_A \approx \mathcal{A} \sum_j Q_{ij} \xi_j^*(p) \sum_l \eta_l \xi_l(p)$$

Here, $\mathcal{A}$ is the product of the areas of the $D_{A,i}$. The ex-

pression above should be zero by the property that Ξ_η is orthogonal to $\sum_j Q^*_{ij}\xi_j$. However, $\sum_j Q_{ij}\xi^*_j$ is antiholomorphic and clearly nonzero for almost all p (or else M would be zero). Hence we find that $\sum_l \eta_l \xi_l(p) = 0$ for almost all p and hence $\sum_l \eta_l \xi_l = 0$, since this expression is holomorphic. Indeed, no $\Xi_\eta \neq 0$ exists. The rank of M thus equals the number of independent EWF and we conclude that, when the EWF are holomorphic, the entanglement spectrum spans exactly the same vector space as the EWF.

We now apply the above results to the Laughlin and Jain FQH states. These can be written in the form

$$\psi_{LJ}(z_1...z_N) = \mathbf{P}\left[\chi_n(z_1...z_N)(\chi_1(z_1...z_N))^p\right]. \qquad (7)$$

Here, the z_i are complex coordinates on a two dimensional space (usually the plane, sphere or torus) and χ_n is the Slater determinant describing n fully filled Landau levels of orbitals on this space. Note that the number of orbitals in the Landau levels is adjusted to the number of particles. The single Landau level of the χ_1 factors has N orbitals, while the n Landau levels in the χ_n factor have on average N/n orbitals. One may think of these as Landau levels for composite fermions[10], which experience a reduced magnetic flux, reducing the number of orbitals available. The operator $\mathbf{P}$ is the orthogonal projection onto the lowest Landau level for the full system. Eq. (7) gives an excellent description of the ground state of FQH systems at filling $\nu = \frac{n}{pn+1}$. Excited states above this ground state can be described by modifying the first Slater determinant (χ_n) so that it leaves some of the orbitals in the n Landau levels unoccupied and/or includes some occupied orbitals in the $(n+1)^{\text{th}}$ Landau level. The factor $(\chi_1)^p$ is often written in the form $\prod_{i<j}(z_i - z_j)^p$. It is not involved in excitations and can be thought of as a device to attach flux to the composite fermions. We call all states of the form just described, with or without excitations, composite fermion (CF) states. The Laughlin states are the case $n = 1$.

CF states are usually not products of Slater determinants, unless $\mathbf{P}$ acts trivially, which happens for the Laughlin ground states and some of their excitations. However, since the orthogonal projection $\mathbf{P}$ factors into single particle projections, we can still write

$$\psi_{CF} = \sum_k s_k \mathbf{P}\left(\prod_{i=1}^{p+1} S^{(k_i^A)}(Z_A)\right)\mathbf{P}\left(\prod_{i=1}^{p+1} S^{(k_i^B)}(Z_B)\right),$$

where the $p+1$ Slater determinants are now built from subsets of the orbitals involved in χ_n and in the p factors of χ_1. We see then that it makes sense to define the EWF for the wave functions in Eq. (7) to be projected products of Slater determinants, $\xi_k(Z_A) := \mathbf{P}\left(\prod_{i=1}^{p+1} S^{(k_i^A)}(Z_A)\right)$ and $\zeta_k(Z_B) := \mathbf{P}\left(\prod_{i=1}^{p+1} S^{(k_i^B)}(Z_B)\right)$. With this definition, we can still apply the results (2)–(6). In fact, FQH systems have symmetries which allow for a refined version of the construction above. E.g. for the system on a sphere, the angular momentum component L_z is conserved, even when the system is split into complementary A and B domains, as long as the domain boundary is a horizontal circle (e.g. the equator of the sphere). Since $\mathbf{P}$ commutes with L_z, ψ_{CF} and the EWF are eigenfunctions of L_z and ρ_A is block diagonal with blocks $\rho_A^{L_z^A}$ labeled by the total angular momentum L_z^A of the particles in the A-system. We then get the results (2) through (6) for each block.

We note now that a subset of the EWF for the Laughlin and Jain states are themselves CF states. We will call these states the CF-EWF. The CF-EWF are those EWF for which the p factors which come from the $(\chi_1)^p$ factor of ψ_{CF} all have precisely the N_A orbitals with the lowest angular momenta filled. This means that the product of these factors can be written as $\prod_{i<j=1}^{N_A}(z_i - z_j)^p$ and the factors are thus the usual flux attachment factor for the particles of the A-system. Clearly, in the expansion of ψ_{CF}, terms in which the A-system is in a CF-EWF have the B-system in a similar state where the N_B highest angular momenta are filled in the Slater determinants that come from the p factors of χ_1. The CF-EWF are of particular interest when studying the low angular momentum sectors of the ES. For the RSES, low angular momentum corresponds with low entanglement energy, since states with low angular momentum L_z^A have the highest probability density to find the particles $1...N_A$ in the domain D_A (i.e. near the north pole of the sphere) and hence also the highest probability density for finding particles $N_A + 1...N$ in D_B. It is natural to guess that the CF-EWF give a good description of the low-energy part of the RSES of the Jain states from the point of view that the RSES should be similar to the edge spectrum of the parent state and this was our motivation for introducing these states very briefly in Ref. 7, without reference to the full set of EWF. We may in fact expect that, at low angular momenta, the CF-EWF span the same space as the full set of EWF. This intuition comes from considering localized excitations of CF-states. These are coherent states built from the excitations we have described. A local excitation which exists for any FQH state is a Laughlin quasihole at position w, which can be introduced by multiplying the wave function by a factor $\prod_j(z_j - w)$. This excitation can be viewed as a superposition of Slater determinants with non-minimal angular momentum either at $\nu = n$ or at $\nu = 1$. However, in the case of $\nu = 1$ we can invert the construction; if we introduce a higher angular momentum orbital in one of the χ_1 related factors of the EWF, this can always be thought of as a superposition of Laughlin quasiparticles. As a result, we could have created the higher angular momentum state also by introducing higher angular momentum orbitals in the $\nu = n$ factor of the EWF. It thus seems

L_z^A	-64	-63	-62	-61	-60	-59	-58	-57	-56	-55	-54	-53
ES	1	1	2	3	5	7	11	15	22	29	40	52
EWF	1	1	2(3)	3(5)	5(11)	7(18)	11(34)	15(55)	22(95)	29(148)	40(238)	52(360)
CF-EWF	1	1	2	3	5	7	11	15	22	28	38	48

L_z^A	-61	-60	-59	-58	-57	-56	-55	-54
ES	1	4	9	20	40	72	121	194
EWF	1	4 (5)	9 (15)	20 (40)	40 (97)	72 (212)	121 (435)	194 (843)
CF-EWF	1	4	9	20	40(42)	70(76)	115(131)	176(212)

TABLE I: ES, EWF and CF-EW countings. Numbers of independent states given. Total numbers of states in brackets, if different. *Top:* $\nu = 1/2$ Laughlin state for $N = 16$. *Bottom:* $\nu = 2/3$ Jain state for $N = 18$.

naively that the CF-EWF should fully describe the ES of the system. However, it is clear that the correspondence just described must break down beyond some maximal L_z^A; once the N_A orbitals in the χ_n factor of the EWF are precisely the largest angular momentum orbitals in the n CF LLs, the only way to raise the angular momentum further is to change the orbitals in the χ_1 factors. Still, we expect that the set of angular momenta where the spaces of EWF and CF-EWF coincide increases with N (with N_A/N fixed while taking this limit), so that eventually, for any given excess in L_z^A above the minimal L_z^A, the RSES in the limit $N \to \infty$ (with N_A/N nonzero) may be obtained from the CF-EWF.

We now present a comparison between the CF-EWF construction and the RSES of CF states on the sphere calculated using the method introduced in Ref. 7. In Table I we show the numbers EWF and CF-EWF (both the total numbers and the numbers of independent states), for different L_z^A sectors, for the $\nu = \frac{1}{2}$ Laughlin and $\nu = \frac{2}{3}$ Jain states on the sphere. We also give the number of independent states in the entanglement spectra for these states. As expected, the numbers of independent EWF match perfectly with the counting of the ES and the number of independent CF-EWF matches the EWF and ES at low angular momenta. We note that at low angular momenta, the CF-EWF are all linearly independent. In fact for the $\nu = \frac{1}{2}$ state, they are independent at any L_z^A and it is not difficult to see that this is true for the CF-EWF of any state where the projection $\mathbf{P}$ acts trivially. More generally, we conjecture that the CF-EWF at any given excess in L_z^A above the minimal L_z^A become linearly independent when $N \to \infty$ (keeping $0 < N_A/N < 1$ in the limit).

Fig. 1 shows the RSES of the $\nu = 2/5$ Jain state for $N = 36$ and $N = 38$ particles. Observe that the RSES exhibits branches of excitations and that it depends on the N_A parity (similar results have been obtained in studies of edge states in ref.[11]). We find that, in the large N_A limit, the RSES counting is given by $2, 4, 12, 24, 44, 50, 92, ..$ for even N_A and by $1, 4, 9, 20, 42, ..$ for odd N_A. The counting within each branch is independent of the parity and given by, $1, 2, 5, 10, 20, \ldots$, which

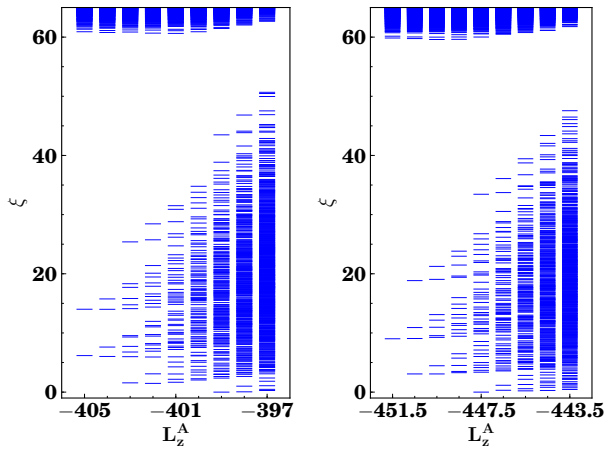

FIG. 1: Low L_z^A part of the RSES for the $\nu = 2/5$ Jain state, for $(N, N_A) = (36, 18)$ (*left*) and for $(N, N_A) = (38, 19)$ (*right*). Eigenvalues of ρ_A are calculated up to a single overall scale. This sets the zero of entanglement energy ξ. We normalize so that the lowest level shown has $\xi = 0$.

E_{CF} \ L_z^A	-405	-404	-403	-402	-401	-400	-399	-398
7	-	-	-	-	-	-	1	2
8	-	-	1	2	5	10	20	36
9	1	2	5	10	20	36	65	110
10	1	2	5	10	20	36	65	110
11	-	-	1	2	5	10	20	36
12	-	-	-	-	-	-	1	2
Total	2	4	12	24	50	92	172	296

E_{CF} \ L_z^A	-451.5	-450.5	-449.5	-448.5	-447.5	-446.5	-445.5	-444.5
8	-	-	-	-	1	2	5	10
9	-	1	2	5	10	20	36	65
10	1	2	5	10	20	36	65	110
11	-	1	2	5	10	20	36	65
12	-	-	-	-	1	2	5	10
Total	1	4	9	15	42	80	147	260

TABLE II: Numbers of independent CF-EWF vs. E_{CF} for the $\nu = 2/5$ Jain state. *Top:* $N = 36$ and $N_A = 18$, *Bottom:* $N = 38$ and $N_A = 19$.

corresponds to the counting of a $U(1) \times U(1)$ conformal field theory, or two non-interacting chiral edge bosons.

This counting can be understood from the point of view of the CF-EWF. The Slater determinants in the EWF which relate to χ_2 can be characterized by the numbers n_0 and $n_1 = N_A - n_0$ of orbitals filled in the first and second CF LLs, or alternatively in terms of the CF kinetic energy E_{CF}. We will take $E_{CF} = \sum_l l n_l$, so we work in units of the CF cyclotron energy and set the zero point energy for particles in the lowest CF LL to zero. In Table II, we give the number of linearly independent

CF-EWF in terms of the CF kinetic energy, E_{CF}, for the first L_z^A sectors. Observe that the counting within a row, given by the number of independent CF-EWF with the same kinetic energy, matches the counting of the RSES branches. Moreover, if we interpret every row as a branch we recover exactly the same multi-branch structure observed in Fig. 1, with the expected counting in the large N_A limit.

Note that because we are considering two CF LL's, the CF-EWF contributing to a single branch, i.e. the CF-EWF with the same kinetic energy, correspond to states in which the composite fermions are excited only within the same CF LL (inter-LL CF excitations), otherwise we change E_{CF} and we move to another branch. Because inter-LL excitations produce a $U(1)$ counting per LL, in the large N_A limit, this explains the $U(1)\times U(1)$ counting observed in the $\nu = 2/5$ RSES branches.

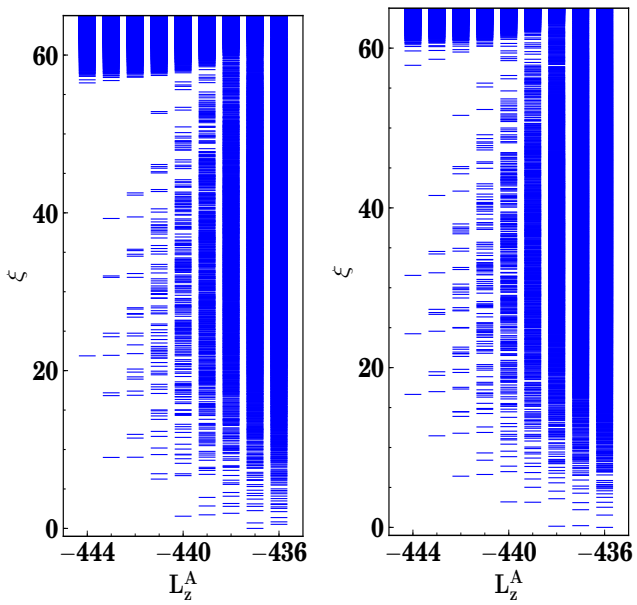

FIG. 2: RSES counting for the $\nu = 3/7$ Jain state for $(N, N_A) = (42, 21)$ (*left*) and for $(N, N_A) = (39, 19)$ (*right*).

E_{CF} \ L_z^A	-515.5	-514.5	-513.5	-512.5	-511.5
19	-	-	-	-	1
20	-	-	-	2	6
21	-	1	3	9	24
22	-	2	6	18	44
23	1	3	9	24	57
24	-	2	6	18	44
25	-	1	3	9	24
26	-	-	-	2	6
27	-	-	-	-	1
Total	1	9	27	82	207

E_{CF} \ L_z^A	-444	-443	-442	-441	-440
17	-	-	-	-	1
18	-	-	1	3	10
19	-	1	4	12	31
20	1	3	10	25	61
21	1	4	12	31	73
22	1	3	10	25	61
23	-	1	4	12	31
24	-	-	1	3	10
25	-	-	-	-	1
Total	3	12	42	111	279

TABLE III: number of independent CF-EWF vs. E_{CF} for the $\nu = 3/7$ Jain state. *Left:* $N = 42$ and $N_A = 21$. *Right:* $N = 39$ and $N_A = 19$

For the $\nu = 3/7$ Jain state we also find a RSES with many branches of excitations (see Fig. 2). As shown in Table III, for $N_A \bmod 3 = 0$, the spectrum presents two types of branches with countings $1, 3, 9, 24, ..$ and $2, 6, 18, ...$ For any other value of N_A, there are two type of branches, with countings $1, 3, 10, 25, ..$ and $1, 4, 12, 31, ...$ Again, these countings are as predicted by the CF kinetic energy of the CF-EWF. Note that now we are considering three CF LL in our construction and as a result, we have CF-EWF with the same E_{CF} but with different numbers of CF's in the CF LL's (e.g. for $N_A = 3$, we may have 1 CF in each CF LL or 3 CFs in the middle CF LL) Therefore the counting of the branches for large N_A exceeds the counting that would be expected from 3 independent chiral boson edge modes.

Explicit generating functions for the numbers of independent CF states with given E_{CF} and L_z will be presented in a forthcoming work [12]. There we will also present results on the RSES of Jain states with reverse flux attachment. The EWF and CF-EWF can also be applied to these states, but the resulting countings are more difficult to obtain, since the LLL projection $\mathbf{P}$ now has a nontrivial effect, even as $N_A \to \infty$.

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